## Topics In Analysis

Kuttler See Real and Abstract Analysis First. This is merely a collection of topics February 4, 2024

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## Chapter 1

## Preface

This is not a book. It is a collection of topics. If you are not interested in something past Page 1000, you should not read it here. Instead, see "Real and Abstract Analysis" or "Analysis of Functions of Complex and Many Variables" two books on this web page. If you are reading something after page 2000 and want to review some elementary topic, I often have a link so you can quickly go to it. That is the only reason elementary topics are present in this conglomeration of topics. You should not read this book to learn standard topics in Advanced calculus or Linear Algebra for example. Also, the presentation tends to be ad hoc and not as well developed as it is elsewhere.

## Part I

## Review Of Advanced Calculus

This is a collection of topics and the presentation is not done as well as in Real and Abstract Analysis. If you are not interested in something after Page 1000, you will probably find it done better in this other book. You should avoid this pile of topics if at all possible. I have elementary material in it only for the convenience of the reader who wants a quick review. It has not been as well maintained as the other books either.

## Chapter 2

## Set Theory

### 2.1 Basic Definitions

A set is a collection of things called elements of the set. For example, the set of integers, the collection of signed whole numbers such as $1,2,-4$, etc. This set whose existence will be assumed is denoted by $\mathbb{Z}$. Other sets could be the set of people in a family or the set of donuts in a display case at the store. Sometimes parentheses, $\}$ specify a set by listing the things which are in the set between the parentheses. For example the set of integers between -1 and 2 , including these numbers could be denoted as $\{-1,0,1,2\}$. The notation signifying $x$ is an element of a set $S$, is written as $x \in S$. Thus, $1 \in\{-1,0,1,2,3\}$. Here are some axioms about sets. Axioms are statements which are accepted, not proved.

1. Two sets are equal if and only if they have the same elements.
2. To every set $A$, and to every condition $S(x)$ there corresponds a set, $B$, whose elements are exactly those elements $x$ of $A$ for which $S(x)$ holds.
3. For every collection of sets there exists a set that contains all the elements that belong to at least one set of the given collection.
4. The Cartesian product of a nonempty family of nonempty sets is nonempty.
5. If $A$ is a set there exists a set, $\mathscr{P}(A)$ such that $\mathscr{P}(A)$ is the set of all subsets of $A$. This is called the power set.

These axioms are referred to as the axiom of extension, axiom of specification, axiom of unions, axiom of choice, and axiom of powers respectively.

It seems fairly clear you should want to believe in the axiom of extension. It is merely saying, for example, that $\{1,2,3\}=\{2,3,1\}$ since these two sets have the same elements in them. Similarly, it would seem you should be able to specify a new set from a given set using some "condition" which can be used as a test to determine whether the element in question is in the set. For example, the set of all integers which are multiples of 2. This set could be specified as follows.

$$
\{x \in \mathbb{Z}: x=2 y \text { for some } y \in \mathbb{Z}\}
$$

In this notation, the colon is read as "such that" and in this case the condition is being a multiple of 2.

Another example of political interest, could be the set of all judges who are not judicial activists. I think you can see this last is not a very precise condition since there is no way to determine to everyone's satisfaction whether a given judge is an activist. Also, just because something is grammatically correct does not mean it makes any sense. For example consider the following nonsense.
$S=\{x \in$ set of dogs $:$ it is colder in the mountains than in the winter $\}$.
So what is a condition?

We will leave these sorts of considerations and assume our conditions make sense. The axiom of unions states that for any collection of sets, there is a set consisting of all the elements in each of the sets in the collection. Of course this is also open to further consideration. What is a collection? Maybe it would be better to say "set of sets" or, given a set whose elements are sets there exists a set whose elements consist of exactly those things which are elements of at least one of these sets. If $\mathscr{S}$ is such a set whose elements are sets,

$$
\cup\{A: A \in \mathscr{S}\} \text { or } \cup \mathscr{S}
$$

signify this union.
Something is in the Cartesian product of a set or "family" of sets if it consists of a single thing taken from each set in the family. Thus $(1,2,3) \in\{1,4, .2\} \times\{1,2,7\} \times\{4,3,7,9\}$ because it consists of exactly one element from each of the sets which are separated by $\times$. Also, this is the notation for the Cartesian product of finitely many sets. If $\mathscr{S}$ is a set whose elements are sets,

$$
\prod_{A \in S} A
$$

signifies the Cartesian product.
The Cartesian product is the set of choice functions, a choice function being a function which selects exactly one element of each set of $\mathscr{S}$. You may think the axiom of choice, stating that the Cartesian product of a nonempty family of nonempty sets is nonempty, is innocuous but there was a time when many mathematicians were ready to throw it out because it implies things which are very hard to believe, things which never happen without the axiom of choice.
$A$ is a subset of $B$, written $A \subseteq B$, if every element of $A$ is also an element of $B$. This can also be written as $B \supseteq A$. $A$ is a proper subset of $B$, written $A \subset B$ or $B \supset A$ if $A$ is a subset of $B$ but $A$ is not equal to $B, A \neq B . A \cap B$ denotes the intersection of the two sets, $A$ and $B$ and it means the set of elements of $A$ which are also elements of $B$. The axiom of specification shows this is a set. The empty set is the set which has no elements in it, denoted as $\emptyset$. $A \cup B$ denotes the union of the two sets, $A$ and $B$ and it means the set of all elements which are in either of the sets. It is a set because of the axiom of unions.

The complement of a set, (the set of things which are not in the given set ) must be taken with respect to a given set called the universal set which is a set which contains the one whose complement is being taken. Thus, the complement of $A$, denoted as $A^{C}$ ( or more precisely as $X \backslash A$ ) is a set obtained from using the axiom of specification to write

$$
A^{C} \equiv\{x \in X: x \notin A\}
$$

The symbol $\notin$ means: "is not an element of". Note the axiom of specification takes place relative to a given set. Without this universal set it makes no sense to use the axiom of specification to obtain the complement.

Words such as "all" or "there exists" are called quantifiers and they must be understood relative to some given set. For example, the set of all integers larger than 3. Or there exists an integer larger than 7 . Such statements have to do with a given set, in this case the integers. Failure to have a reference set when quantifiers are used turns out to be illogical even though such usage may be grammatically correct. Quantifiers are used often enough
that there are symbols for them. The symbol $\forall$ is read as "for all" or "for every" and the symbol $\exists$ is read as "there exists". Thus $\forall \forall \exists \exists$ could mean for every upside down $A$ there exists a backwards $E$.

DeMorgan's laws are very useful in mathematics. Let $\mathscr{S}$ be a set of sets each of which is contained in some universal set, $U$. Then

$$
\cup\left\{A^{C}: A \in \mathscr{S}\right\}=(\cap\{A: A \in \mathscr{S}\})^{C}
$$

and

$$
\cap\left\{A^{C}: A \in \mathscr{S}\right\}=(\cup\{A: A \in \mathscr{S}\})^{C}
$$

These laws follow directly from the definitions. Also following directly from the definitions are:

Let $\mathscr{S}$ be a set of sets then

$$
B \cup \cup\{: A \in \mathscr{S}\}=\cup\{B \cup A: A \in \mathscr{S}\}
$$

and: Let $\mathscr{S}$ be a set of sets show

$$
B \cap \cup\{A: A \in \mathscr{S}\}=\cup\{B \cap A: A \in \mathscr{S}\}
$$

Unfortunately, there is no single universal set which can be used for all sets. Here is why: Suppose there were. Call it $S$. Then you could consider $A$ the set of all elements of $S$ which are not elements of themselves, this from the axiom of specification. If $A$ is an element of itself, then it fails to qualify for inclusion in $A$. Therefore, it must not be an element of itself. However, if this is so, it qualifies for inclusion in $A$ so it is an element of itself and so this can't be true either. Thus the most basic of conditions you could imagine, that of being an element of, is meaningless and so allowing such a set causes the whole theory to be meaningless. The solution is to not allow a universal set. As mentioned by Halmos in Naive set theory, "Nothing contains everything". Always beware of statements involving quantifiers wherever they occur, even this one.

### 2.2 The Schroder Bernstein Theorem

It is very important to be able to compare the size of sets in a rational way. The most useful theorem in this context is the Schroder Bernstein theorem which is the main result to be presented in this section. The Cartesian product is discussed above. The next definition reviews this and defines the concept of a function.

Definition 2.2.1 Let $X$ and $Y$ be sets.

$$
X \times Y \equiv\{(x, y): x \in X \text { and } y \in Y\}
$$

A relation is defined to be a subset of $X \times Y$. A function, $f$, also called a mapping, is a relation which has the property that if $(x, y)$ and $\left(x, y_{1}\right)$ are both elements of the $f$, then $y=y_{1}$. The domain of $f$ is defined as

$$
D(f) \equiv\{x:(x, y) \in f\}
$$

written as $f: D(f) \rightarrow Y$.

It is probably safe to say that most people do not think of functions as a type of relation which is a subset of the Cartesian product of two sets. A function is like a machine which takes inputs, $x$ and makes them into a unique output, $f(x)$. Of course, that is what the above definition says with more precision. An ordered pair, $(x, y)$ which is an element of the function or mapping has an input, $x$ and a unique output, $y$, denoted as $f(x)$ while the name of the function is $f$. "mapping" is often a noun meaning function. However, it also is a verb as in " $f$ is mapping $A$ to $B$ ". That which a function is thought of as doing is also referred to using the word "maps" as in: $f$ maps $X$ to $Y$. However, a set of functions may be called a set of maps so this word might also be used as the plural of a noun. There is no help for it. You just have to suffer with this nonsense.

The following theorem which is interesting for its own sake will be used to prove the Schroder Bernstein theorem.

Theorem 2.2.2 Let $f: X \rightarrow Y$ and $g: Y \rightarrow X$ be two functions. Then there exist sets $A, B, C, D$, such that

$$
\begin{gathered}
A \cup B=X, C \cup D=Y, A \cap B=\emptyset, C \cap D=\emptyset, \\
f(A)=C, g(D)=B .
\end{gathered}
$$

The following picture illustrates the conclusion of this theorem.


Proof: Consider the empty set, $\emptyset \subseteq X$. If $y \in Y \backslash f(\emptyset)$, then $g(y) \notin \emptyset$ because $\emptyset$ has no elements. Also, if $A, B, C$, and $D$ are as described above, $A$ also would have this same property that the empty set has. However, $A$ is probably larger. Therefore, say $A_{0} \subseteq X$ satisfies $\mathscr{P}$ if whenever $y \in Y \backslash f\left(A_{0}\right), g(y) \notin A_{0}$.

$$
\mathscr{A} \equiv\left\{A_{0} \subseteq X: A_{0} \text { satisfies } \mathscr{P}\right\}
$$

Let $A=\cup \mathscr{A}$. If $y \in Y \backslash f(A)$, then for each $A_{0} \in \mathscr{A}, y \in Y \backslash f\left(A_{0}\right)$ and so $g(y) \notin A_{0}$. Since $g(y) \notin A_{0}$ for all $A_{0} \in \mathscr{A}$, it follows $g(y) \notin A$. Hence $A$ satisfies $\mathscr{P}$ and is the largest subset of $X$ which does so. Now define

$$
C \equiv f(A), D \equiv Y \backslash C, B \equiv X \backslash A
$$

It only remains to verify that $g(D)=B$.
Suppose $x \in B=X \backslash A$. Then $A \cup\{x\}$ does not satisfy $\mathscr{P}$ and so there exists $y \in$ $Y \backslash f(A \cup\{x\}) \subseteq D$ such that $g(y) \in A \cup\{x\}$. But $y \notin f(A)$ and so since $A$ satisfies $\mathscr{P}$, it follows $g(y) \notin A$. Hence $g(y)=x$ and so $x \in g(D)$ and this proves the theorem.

Theorem 2.2.3 (Schroder Bernstein) If $f: X \rightarrow Y$ and $g: Y \rightarrow X$ are one to one, then there exists $h: X \rightarrow Y$ which is one to one and onto.

Proof: Let $A, B, C, D$ be the sets of Theorem2.2.2 and define

$$
h(x) \equiv\left\{\begin{array}{cc}
f(x) & \text { if } x \in A \\
g^{-1}(x) & \text { if } x \in B
\end{array}\right.
$$

Then $h$ is the desired one to one and onto mapping.
Recall that the Cartesian product may be considered as the collection of choice functions.

Definition 2.2.4 Let $I$ be a set and let $X_{i}$ be a set for each $i \in I . f$ is a choice function written as

$$
f \in \prod_{i \in I} X_{i}
$$

if $f(i) \in X_{i}$ for each $i \in I$.
The axiom of choice says that if $X_{i} \neq \emptyset$ for each $i \in I$, for $I$ a set, then

$$
\prod_{i \in I} X_{i} \neq \emptyset
$$

Sometimes the two functions, $f$ and $g$ are onto but not one to one. It turns out that with the axiom of choice, a similar conclusion to the above may be obtained.

Corollary 2.2.5 If $f: X \rightarrow Y$ is onto and $g: Y \rightarrow X$ is onto, then there exists $h: X \rightarrow Y$ which is one to one and onto.

Proof: For each $y \in Y, f^{-1}(y) \equiv\{x \in X: f(x)=y\} \neq \emptyset$. Therefore, by the axiom of choice, there exists $f_{0}^{-1} \in \prod_{y \in Y} f^{-1}(y)$ which is the same as saying that for each $y \in Y$, $f_{0}^{-1}(y) \in f^{-1}(y)$. Similarly, there exists $g_{0}^{-1}(x) \in g^{-1}(x)$ for all $x \in X$. Then $f_{0}^{-1}$ is one to one because if $f_{0}^{-1}\left(y_{1}\right)=f_{0}^{-1}\left(y_{2}\right)$, then

$$
y_{1}=f\left(f_{0}^{-1}\left(y_{1}\right)\right)=f\left(f_{0}^{-1}\left(y_{2}\right)\right)=y_{2} .
$$

Similarly $g_{0}^{-1}$ is one to one. Therefore, by the Schroder Bernstein theorem, there exists $h: X \rightarrow Y$ which is one to one and onto.

Definition 2.2.6 A set $S$, is finite if there exists a natural number $n$ and a map $\theta$ which maps $\{1, \cdots, n\}$ one to one and onto $S$. $S$ is infinite if it is not finite. A set $S$, is called countable if there exists a map $\theta$ mapping $\mathbb{N}$ one to one and onto $S$.(When $\theta$ maps a set $A$ to a set $B$, this will be written as $\theta: A \rightarrow B$ in the future.) Here $\mathbb{N} \equiv\{1,2, \cdots$ the natural numbers. $S$ is at most countable if there exists a map $\theta: \mathbb{N} \rightarrow S$ which is onto.

The property of being at most countable is often referred to as being countable because the question of interest is normally whether one can list all elements of the set, designating a first, second, third etc. in such a way as to give each element of the set a natural number. The possibility that a single element of the set may be counted more than once is often not important.

Theorem 2.2.7 If $X$ and $Y$ are both at most countable, then $X \times Y$ is also at most countable. If either $X$ or $Y$ is countable, then $X \times Y$ is also countable.

Proof: It is given that there exists a mapping $\eta: \mathbb{N} \rightarrow X$ which is onto. Define $\eta(i) \equiv x_{i}$ and consider $X$ as the set $\left\{x_{1}, x_{2}, x_{3}, \cdots\right\}$. Similarly, consider $Y$ as the set $\left\{y_{1}, y_{2}, y_{3}, \cdots\right\}$. It follows the elements of $X \times Y$ are included in the following rectangular array.

$$
\begin{array}{ccccc}
\left(x_{1}, y_{1}\right) & \left(x_{1}, y_{2}\right) & \left(x_{1}, y_{3}\right) & \cdots & \leftarrow \text { Those which have } x_{1} \text { in first slot. } \\
\left(x_{2}, y_{1}\right) & \left(x_{2}, y_{2}\right) & \left(x_{2}, y_{3}\right) & \cdots & \leftarrow \text { Those which have } x_{2} \text { in first slot. } \\
\left(x_{3}, y_{1}\right) & \left(x_{3}, y_{2}\right) & \left(x_{3}, y_{3}\right) & \cdots & \leftarrow \text { Those which have } x_{3} \text { in first slot. }
\end{array}
$$

Follow a path through this array as follows.


Thus the first element of $X \times Y$ is $\left(x_{1}, y_{1}\right)$, the second element of $X \times Y$ is $\left(x_{1}, y_{2}\right)$, the third element of $X \times Y$ is $\left(x_{2}, y_{1}\right)$ etc. This assigns a number from $\mathbb{N}$ to each element of $X \times Y$. Thus $X \times Y$ is at most countable.

It remains to show the last claim. Suppose without loss of generality that $X$ is countable. Then there exists $\alpha: \mathbb{N} \rightarrow X$ which is one to one and onto. Let $\beta: X \times Y \rightarrow \mathbb{N}$ be defined by $\beta((x, y)) \equiv \alpha^{-1}(x)$. Thus $\beta$ is onto $\mathbb{N}$. By the first part there exists a function from $\mathbb{N}$ onto $X \times Y$. Therefore, by Corollary 2.2.5, there exists a one to one and onto mapping from $X \times Y$ to $\mathbb{N}$. This proves the theorem.

Theorem 2.2.8 If $X$ and $Y$ are at most countable, then $X \cup Y$ is at most countable. If either $X$ or $Y$ are countable, then $X \cup Y$ is countable.

Proof: As in the preceding theorem,

$$
X=\left\{x_{1}, x_{2}, x_{3}, \cdots\right\}
$$

and

$$
Y=\left\{y_{1}, y_{2}, y_{3}, \cdots\right\}
$$

Consider the following array consisting of $X \cup Y$ and path through it.


Thus the first element of $X \cup Y$ is $x_{1}$, the second is $x_{2}$ the third is $y_{1}$ the fourth is $y_{2}$ etc.
Consider the second claim. By the first part, there is a map from $\mathbb{N}$ onto $X \times Y$. Suppose without loss of generality that $X$ is countable and $\alpha: \mathbb{N} \rightarrow X$ is one to one and onto. Then define $\beta(y) \equiv 1$, for all $y \in Y$, and $\beta(x) \equiv \alpha^{-1}(x)$. Thus, $\beta$ maps $X \times Y$ onto $\mathbb{N}$ and this shows there exist two onto maps, one mapping $X \cup Y$ onto $\mathbb{N}$ and the other mapping $\mathbb{N}$ onto $X \cup Y$. Then Corollary 2.2.5 yields the conclusion. This proves the theorem.

### 2.3 Equivalence Relations

There are many ways to compare elements of a set other than to say two elements are equal or the same. For example, in the set of people let two people be equivalent if they have the same weight. This would not be saying they were the same person, just that they weighed the same. Often such relations involve considering one characteristic of the elements of a set and then saying the two elements are equivalent if they are the same as far as the given characteristic is concerned.

Definition 2.3.1 Let $S$ be a set. $\sim$ is an equivalence relation on $S$ if it satisfies the following axioms.

1. $x \sim x$ for all $x \in S$. (Reflexive)
2. If $x \sim y$ then $y \sim x$. (Symmetric)
3. If $x \sim y$ and $y \sim z$, then $x \sim z$. (Transitive)

Definition 2.3.2 $[x]$ denotes the set of all elements of $S$ which are equivalent to $x$ and $[x]$ is called the equivalence class determined by $x$ or just the equivalence class of $x$.

With the above definition one can prove the following simple theorem.
Theorem 2.3.3 Let $\sim$ be an equivalence class defined on a set, $S$ and let $\mathscr{H}$ denote the set of equivalence classes. Then if $[x]$ and $[y]$ are two of these equivalence classes, either $x \sim y$ and $[x]=[y]$ or it is not true that $x \sim y$ and $[x] \cap[y]=\emptyset$.

### 2.4 Partially Ordered Sets

Definition 2.4.1 Let $\mathscr{F}$ be a nonempty set. $\mathscr{F}$ is called a partially ordered set if there is a relation, denoted here by $\leq$, such that

$$
\begin{gathered}
x \leq x \text { for all } x \in \mathscr{F} \\
\text { If } x \leq y \text { and } y \leq z \text { then } x \leq z
\end{gathered}
$$

$\mathscr{C} \subseteq \mathscr{F}$ is said to be a chain if every two elements of $\mathscr{C}$ are related. This means that if $x, y \in \mathscr{C}$, then either $x \leq y$ or $y \leq x$. Sometimes a chain is called a totally ordered set. $\mathscr{C}$ is said to be a maximal chain if whenever $\mathscr{D}$ is a chain containing $\mathscr{C}, \mathscr{D}=\mathscr{C}$.

The most common example of a partially ordered set is the power set of a given set with $\subseteq$ being the relation. It is also helpful to visualize partially ordered sets as trees. Two points on the tree are related if they are on the same branch of the tree and one is higher than the other. Thus two points on different branches would not be related although they might both be larger than some point on the trunk. You might think of many other things which are best considered as partially ordered sets. Think of food for example. You might find it difficult to determine which of two favorite pies you like better although you may be able to say very easily that you would prefer either pie to a dish of lard topped with whipped cream and mustard. The following theorem is equivalent to the axiom of choice. For a discussion of this, see the appendix on the subject.

Theorem 2.4.2 (Hausdorff Maximal Principle) Let $\mathscr{F}$ be a nonempty partially ordered set. Then there exists a maximal chain.

## Chapter 3

## Continuous Functions Of One Variable

There is a theorem about the integral of a continuous function which requires the notion of uniform continuity. This is discussed in this section. Consider the function $f(x)=\frac{1}{x}$ for $x \in(0,1)$. This is a continuous function because, it is continuous at every point of $(0,1)$. However, for a given $\varepsilon>0$, the $\delta$ needed in the $\varepsilon, \delta$ definition of continuity becomes very small as $x$ gets close to 0 . The notion of uniform continuity involves being able to choose a single $\delta$ which works on the whole domain of $f$. Here is the definition.

Definition 3.0.1 Let $f: D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a function. Then $f$ is uniformly continuous iffor every $\varepsilon>0$, there exists a $\delta$ depending only on $\varepsilon$ such that if $|x-y|<\delta$ then $|f(x)-f(y)|<$ $\varepsilon$.

It is an amazing fact that under certain conditions continuity implies uniform continuity.

Definition 3.0.2 A set, $K \subseteq \mathbb{R}$ is sequentially compact if whenever $\left\{a_{n}\right\} \subseteq K$ is a sequence, there exists a subsequence, $\left\{a_{n_{k}}\right\}$ such that this subsequence converges to a point of $K$.

The following theorem is part of the Heine Borel theorem.
Theorem 3.0.3 Every closed interval, $[a, b]$ is sequentially compact.
Proof: Let $\left\{x_{n}\right\} \subseteq[a, b] \equiv I_{0}$. Consider the two intervals $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$ each of which has length $(b-a) / 2$. At least one of these intervals contains $x_{n}$ for infinitely many values of $n$. Call this interval $I_{1}$. Now do for $I_{1}$ what was done for $I_{0}$. Split it in half and let $I_{2}$ be the interval which contains $x_{n}$ for infinitely many values of $n$. Continue this way obtaining a sequence of nested intervals $I_{0} \supseteq I_{1} \supseteq I_{2} \supseteq I_{3} \cdots$ where the length of $I_{n}$ is $(b-a) / 2^{n}$. Now pick $n_{1}$ such that $x_{n_{1}} \in I_{1}, n_{2}$ such that $n_{2}>n_{1}$ and $x_{n_{2}} \in I_{2}, n_{3}$ such that $n_{3}>n_{2}$ and $x_{n_{3}} \in I_{3}$, etc. (This can be done because in each case the intervals contained $x_{n}$ for infinitely many values of $n$.) By the nested interval lemma there exists a point, $c$ contained in all these intervals. Furthermore,

$$
\left|x_{n_{k}}-c\right|<(b-a) 2^{-k}
$$

and so $\lim _{k \rightarrow \infty} x_{n_{k}}=c \in[a, b]$. This proves the theorem.
Theorem 3.0.4 Let $f: K \rightarrow \mathbb{R}$ be continuous where $K$ is a sequentially compact set in $\mathbb{R}$. Then $f$ is uniformly continuous on $K$.

Proof: If this is not true, there exists $\varepsilon>0$ such that for every $\delta>0$ there exists a pair of points, $x_{\delta}$ and $y_{\delta}$ such that even though $\left|x_{\delta}-y_{\delta}\right|<\delta,\left|f\left(x_{\delta}\right)-f\left(y_{\delta}\right)\right| \geq \varepsilon$. Taking a succession of values for $\delta$ equal to $1,1 / 2,1 / 3, \cdots$, and letting the exceptional pair of points for $\delta=1 / n$ be denoted by $x_{n}$ and $y_{n}$,

$$
\left|x_{n}-y_{n}\right|<\frac{1}{n},\left|f\left(x_{n}\right)-f\left(y_{n}\right)\right| \geq \varepsilon .
$$

Now since $K$ is sequentially compact, there exists a subsequence, $\left\{x_{n_{k}}\right\}$ such that $x_{n_{k}} \rightarrow$ $z \in K$. Now $n_{k} \geq k$ and so

$$
\left|x_{n_{k}}-y_{n_{k}}\right|<\frac{1}{k}
$$

Consequently, $y_{n_{k}} \rightarrow z$ also. ( $x_{n_{k}}$ is like a person walking toward a certain point and $y_{n_{k}}$ is like a dog on a leash which is constantly getting shorter. Obviously $y_{n_{k}}$ must also move toward the point also. You should give a precise proof of what is needed here.) By continuity of $f$

$$
0=|f(z)-f(z)|=\lim _{k \rightarrow \infty}\left|f\left(x_{n_{k}}\right)-f\left(y_{n_{k}}\right)\right| \geq \varepsilon
$$

an obvious contradiction. Therefore, the theorem must be true.
The following corollary follows from this theorem and Theorem 3.0.3.
Corollary 3.0.5 Suppose $I$ is a closed interval, $I=[a, b]$ and $f: I \rightarrow \mathbb{R}$ is continuous. Then $f$ is uniformly continuous.

### 3.1 Exercises

1. A function, $f: D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz continuous or just Lipschitz for short if there exists a constant, $K$ such that

$$
|f(x)-f(y)| \leq K|x-y|
$$

for all $x, y \in D$. Show every Lipschitz function is uniformly continuous.
2. If $\left|x_{n}-y_{n}\right| \rightarrow 0$ and $x_{n} \rightarrow z$, show that $y_{n} \rightarrow z$ also.
3. Consider $f:(1, \infty) \rightarrow \mathbb{R}$ given by $f(x)=\frac{1}{x}$. Show $f$ is uniformly continuous even though the set on which $f$ is defined is not sequentially compact.
4. If $f$ is uniformly continuous, does it follow that $|f|$ is also uniformly continuous? If $|f|$ is uniformly continuous does it follow that $f$ is uniformly continuous? Answer the same questions with "uniformly continuous" replaced with "continuous". Explain why.

### 3.2 Theorems About Continuous Functions

In this section, proofs of some theorems which have not been proved yet are given.
Theorem 3.2.1 The following assertions are valid

1. The function, af $+b g$ is continuous at $x$ when $f$, $g$ are continuous at $x \in D(f) \cap D(g)$ and $a, b \in \mathbb{R}$.
2. If and $f$ and $g$ are each real valued functions continuous at $x$, then $f g$ is continuous at $x$. If, in addition to this, $g(x) \neq 0$, then $f / g$ is continuous at $x$.
3. If $f$ is continuous at $x, f(x) \in D(g) \subseteq \mathbb{R}$, and $g$ is continuous at $f(x)$, then $g \circ f$ is continuous at $x$.
4. The function $f: \mathbb{R} \rightarrow \mathbb{R}$, given by $f(x)=|x|$ is continuous.

Proof: First consider 1.) Let $\varepsilon>0$ be given. By assumption, there exist $\delta_{1}>0$ such that whenever $|x-y|<\delta_{1}$, it follows $|f(x)-f(y)|<\frac{\varepsilon}{2(|a|+|b|+1)}$ and there exists $\boldsymbol{\delta}_{2}>0$ such that whenever $|x-y|<\boldsymbol{\delta}_{2}$, it follows that $|g(x)-g(y)|<\frac{\varepsilon}{2(|a|+|b|+1)}$. Then let $0<\boldsymbol{\delta} \leq \min \left(\boldsymbol{\delta}_{1}, \delta_{2}\right)$. If $|x-y|<\delta$, then everything happens at once. Therefore, using the triangle inequality

$$
\begin{gathered}
|a f(x)+b f(x)-(a g(y)+b g(y))| \\
\leq|a||f(x)-f(y)|+|b||g(x)-g(y)| \\
<|a|\left(\frac{\varepsilon}{2(|a|+|b|+1)}\right)+|b|\left(\frac{\varepsilon}{2(|a|+|b|+1)}\right)<\varepsilon .
\end{gathered}
$$

Now consider 2.) There exists $\delta_{1}>0$ such that if $|y-x|<\delta_{1}$, then

$$
|f(x)-f(y)|<1
$$

Therefore, for such $y$,

$$
|f(y)|<1+|f(x)|
$$

It follows that for such $y$,

$$
\begin{aligned}
\mid f g(x) & -f g(y)|\leq|f(x) g(x)-g(x) f(y)|+|g(x) f(y)-f(y) g(y)| \\
& \leq|g(x)||f(x)-f(y)|+|f(y)||g(x)-g(y)| \\
& \leq(1+|g(x)|+|f(y)|)[|g(x)-g(y)|+|f(x)-f(y)|] .
\end{aligned}
$$

Now let $\varepsilon>0$ be given. There exists $\boldsymbol{\delta}_{2}$ such that if $|x-y|<\delta_{2}$, then

$$
|g(x)-g(y)|<\frac{\varepsilon}{2(1+|g(x)|+|f(y)|)}
$$

and there exists $\delta_{3}$ such that if $|x-y|<\delta_{3}$, then

$$
|f(x)-f(y)|<\frac{\varepsilon}{2(1+|g(x)|+|f(y)|)}
$$

Now let $0<\boldsymbol{\delta} \leq \min \left(\boldsymbol{\delta}_{1}, \boldsymbol{\delta}_{2}, \boldsymbol{\delta}_{3}\right)$. Then if $|x-y|<\boldsymbol{\delta}$, all the above hold at once and so

$$
\begin{gathered}
|f g(x)-f g(y)| \leq \\
(1+|g(x)|+|f(y)|)[|g(x)-g(y)|+|f(x)-f(y)|] \\
<(1+|g(x)|+|f(y)|)\left(\frac{\varepsilon}{2(1+|g(x)|+|f(y)|)}+\frac{\varepsilon}{2(1+|g(x)|+|f(y)|)}\right)=\varepsilon
\end{gathered}
$$

This proves the first part of 2.) To obtain the second part, let $\delta_{1}$ be as described above and let $\delta_{0}>0$ be such that for $|x-y|<\delta_{0}$,

$$
|g(x)-g(y)|<|g(x)| / 2
$$

and so by the triangle inequality,

$$
-|g(x)| / 2 \leq|g(y)|-|g(x)| \leq|g(x)| / 2
$$

which implies $|g(y)| \geq|g(x)| / 2$, and $|g(y)|<3|g(x)| / 2$.
Then if $|x-y|<\min \left(\boldsymbol{\delta}_{0}, \boldsymbol{\delta}_{1}\right)$,

$$
\begin{aligned}
& \qquad \left.\frac{f(x)}{g(x)}-\frac{f(y)}{g(y)} \right\rvert\,
\end{aligned}=\left\lvert\, \frac{\left|\frac{f(x) g(y)-f(y) g(x)}{g(x) g(y)}\right|}{} \begin{aligned}
& \leq \frac{|f(x) g(y)-f(y) g(x)|}{\left(\frac{|g(x)|^{2}}{2}\right)} \\
& \\
& =\frac{2|f(x) g(y)-f(y) g(x)|}{|g(x)|^{2}} \\
& \leq \frac{2}{|g(x)|^{2}}[|f(x) g(y)-f(y) g(y)+f(y) g(y)-f(y) g(x)|] \\
& \leq \frac{2}{|g(x)|^{2}}[|g(y)||f(x)-f(y)|+|f(y)||g(y)-g(x)|] \\
& \leq \frac{2}{|g(x)|^{2}}\left[\frac{3}{2}|g(x)||f(x)-f(y)|+(1+|f(x)|)|g(y)-g(x)|\right] \\
& \leq \frac{2}{|g(x)|^{2}}(1+2|f(x)|+2|g(x)|)[|f(x)-f(y)|+|g(y)-g(x)|] \\
& \equiv M[|f(x)-f(y)|+|g(y)-g(x)|]
\end{aligned}\right.
$$

where $M$ is defined by

$$
M \equiv \frac{2}{|g(x)|^{2}}(1+2|f(x)|+2|g(x)|)
$$

Now let $\delta_{2}$ be such that if $|x-y|<\delta_{2}$, then

$$
|f(x)-f(y)|<\frac{\varepsilon}{2} M^{-1}
$$

and let $\delta_{3}$ be such that if $|x-y|<\boldsymbol{\delta}_{3}$, then

$$
|g(y)-g(x)|<\frac{\varepsilon}{2} M^{-1}
$$

Then if $0<\delta \leq \min \left(\delta_{0}, \delta_{1}, \delta_{2}, \delta_{3}\right)$, and $|x-y|<\delta$, everything holds and

$$
\left|\frac{f(x)}{g(x)}-\frac{f(y)}{g(y)}\right| \leq M[|f(x)-f(y)|+|g(y)-g(x)|]
$$

$$
<M\left[\frac{\varepsilon}{2} M^{-1}+\frac{\varepsilon}{2} M^{-1}\right]=\varepsilon
$$

This completes the proof of the second part of 2.)
Note that in these proofs no effort is made to find some sort of "best" $\delta$. The problem is one which has a yes or a no answer. Either is it or it is not continuous.

Now consider 3.). If $f$ is continuous at $x, f(x) \in D(g) \subseteq \mathbb{R}^{p}$, and $g$ is continuous at $f(x)$, then $g \circ f$ is continuous at $x$. Let $\varepsilon>0$ be given. Then there exists $\eta>0$ such that if $|y-f(x)|<\eta$ and $y \in D(g)$, it follows that $|g(y)-g(f(x))|<\varepsilon$. From continuity of $f$ at $x$, there exists $\delta>0$ such that if $|x-z|<\delta$ and $z \in D(f)$, then $|f(z)-f(x)|<\eta$. Then if $|x-z|<\delta$ and $z \in D(g \circ f) \subseteq D(f)$, all the above hold and so

$$
|g(f(z))-g(f(x))|<\varepsilon .
$$

This proves part 3.)
To verify part 4.), let $\varepsilon>0$ be given and let $\delta=\varepsilon$. Then if $|x-y|<\delta$, the triangle inequality implies

$$
\begin{aligned}
|f(x)-f(y)| & =\| x|-|y|| \\
& \leq|x-y|<\delta=\varepsilon
\end{aligned}
$$

This proves part 4.) and completes the proof of the theorem.
Next here is a proof of the intermediate value theorem.
Theorem 3.2.2 Suppose $f:[a, b] \rightarrow \mathbb{R}$ is continuous and suppose $f(a)<c<f(b)$. Then there exists $x \in(a, b)$ such that $f(x)=c$.

Proof: Let $d=\frac{a+b}{2}$ and consider the intervals $[a, d]$ and $[d, b]$. If $f(d) \geq c$, then on $[a, d]$, the function is $\leq c$ at one end point and $\geq c$ at the other. On the other hand, if $f(d) \leq c$, then on $[d, b] f \geq 0$ at one end point and $\leq 0$ at the other. Pick the interval on which $f$ has values which are at least as large as $c$ and values no larger than $c$. Now consider that interval, divide it in half as was done for the original interval and argue that on one of these smaller intervals, the function has values at least as large as $c$ and values no larger than $c$. Continue in this way. Next apply the nested interval lemma to get $x$ in all these intervals. In the $n^{\text {th }}$ interval, let $x_{n}, y_{n}$ be elements of this interval such that $f\left(x_{n}\right) \leq c, f\left(y_{n}\right) \geq c$. Now $\left|x_{n}-x\right| \leq(b-a) 2^{-n}$ and $\left|y_{n}-x\right| \leq(b-a) 2^{-n}$ and so $x_{n} \rightarrow x$ and $y_{n} \rightarrow x$. Therefore,

$$
f(x)-c=\lim _{n \rightarrow \infty}\left(f\left(x_{n}\right)-c\right) \leq 0
$$

while

$$
f(x)-c=\lim _{n \rightarrow \infty}\left(f\left(y_{n}\right)-c\right) \geq 0
$$

Consequently $f(x)=c$ and this proves the theorem.
Lemma 3.2.3 Let $\phi:[a, b] \rightarrow \mathbb{R}$ be a continuous function and suppose $\phi$ is $1-1$ on $(a, b)$. Then $\phi$ is either strictly increasing or strictly decreasing on $[a, b]$.

Proof: First it is shown that $\phi$ is either strictly increasing or strictly decreasing on $(a, b)$.

If $\phi$ is not strictly decreasing on $(a, b)$, then there exists $x_{1}<y_{1}, x_{1}, y_{1} \in(a, b)$ such that

$$
\left(\phi\left(y_{1}\right)-\phi\left(x_{1}\right)\right)\left(y_{1}-x_{1}\right)>0 .
$$

If for some other pair of points, $x_{2}<y_{2}$ with $x_{2}, y_{2} \in(a, b)$, the above inequality does not hold, then since $\phi$ is $1-1$,

$$
\left(\phi\left(y_{2}\right)-\phi\left(x_{2}\right)\right)\left(y_{2}-x_{2}\right)<0 .
$$

Let $x_{t} \equiv t x_{1}+(1-t) x_{2}$ and $y_{t} \equiv t y_{1}+(1-t) y_{2}$. Then $x_{t}<y_{t}$ for all $t \in[0,1]$ because

$$
t x_{1} \leq t y_{1} \text { and }(1-t) x_{2} \leq(1-t) y_{2}
$$

with strict inequality holding for at least one of these inequalities since not both $t$ and $(1-t)$ can equal zero. Now define

$$
h(t) \equiv\left(\phi\left(y_{t}\right)-\phi\left(x_{t}\right)\right)\left(y_{t}-x_{t}\right) .
$$

Since $h$ is continuous and $h(0)<0$, while $h(1)>0$, there exists $t \in(0,1)$ such that $h(t)=0$. Therefore, both $x_{t}$ and $y_{t}$ are points of $(a, b)$ and $\phi\left(y_{t}\right)-\phi\left(x_{t}\right)=0$ contradicting the assumption that $\phi$ is one to one. It follows $\phi$ is either strictly increasing or strictly decreasing on $(a, b)$.

This property of being either strictly increasing or strictly decreasing on $(a, b)$ carries over to $[a, b]$ by the continuity of $\phi$. Suppose $\phi$ is strictly increasing on $(a, b)$, a similar argument holding for $\phi$ strictly decreasing on $(a, b)$. If $x>a$, then pick $y \in(a, x)$ and from the above, $\phi(y)<\phi(x)$. Now by continuity of $\phi$ at $a$,

$$
\phi(a)=\lim _{x \rightarrow a+} \phi(z) \leq \phi(y)<\phi(x) .
$$

Therefore, $\phi(a)<\phi(x)$ whenever $x \in(a, b)$. Similarly $\phi(b)>\phi(x)$ for all $x \in(a, b)$. This proves the lemma.

Corollary 3.2.4 Let $f:(a, b) \rightarrow \mathbb{R}$ be one to one and continuous. Then $f(a, b)$ is an open interval, $(c, d)$ and $f^{-1}:(c, d) \rightarrow(a, b)$ is continuous.

Proof: Since $f$ is either strictly increasing or strictly decreasing, $f(a, b)$ is an open interval, $(c, d)$. Assume $f$ is decreasing. Now let $x \in(a, b)$. Why is $f^{-1}$ is continuous at $f(x)$ ? Since $f$ is decreasing, if $f(x)<f(y)$, then $y \equiv f^{-1}(f(y))<x \equiv f^{-1}(f(x))$ and so $f^{-1}$ is also decreasing. Let $\varepsilon>0$ be given. Let $\varepsilon>\eta>0$ and $(x-\eta, x+\eta) \subseteq(a, b)$. Then $f(x) \in(f(x+\eta), f(x-\eta))$. Let $\delta=\min (f(x)-f(x+\eta), f(x-\eta)-f(x))$. Then if

$$
|f(z)-f(x)|<\delta,
$$

it follows

$$
z \equiv f^{-1}(f(z)) \in(x-\eta, x+\eta) \subseteq(x-\varepsilon, x+\varepsilon)
$$

so

$$
\left|f^{-1}(f(z))-x\right|=\left|f^{-1}(f(z))-f^{-1}(f(x))\right|<\varepsilon
$$

This proves the theorem in the case where $f$ is strictly decreasing. The case where $f$ is increasing is similar.

## Chapter 4

## The Riemann Stieltjes Integral

The integral originated in attempts to find areas of various shapes and the ideas involved in finding integrals are much older than the ideas related to finding derivatives. In fact, Archimedes ${ }^{1}$ was finding areas of various curved shapes about 250 B.C. using the main ideas of the integral. What is presented here is a generalization of these ideas. The main interest is in the Riemann integral but if it is easy to generalize to the so called Stieltjes integral in which the length of an interval, $[x, y]$ is replaced with an expression of the form $F(y)-F(x)$ where $F$ is an increasing function, then the generalization is given. However, there is much more that can be written about Stieltjes integrals than what is presented here. A good source for this is the book by Apostol, [4].

### 4.1 Upper And Lower Riemann Stieltjes Sums

The Riemann integral pertains to bounded functions which are defined on a bounded interval. Let $[a, b]$ be a closed interval. A set of points in $[a, b],\left\{x_{0}, \cdots, x_{n}\right\}$ is a partition if

$$
a=x_{0}<x_{1}<\cdots<x_{n}=b .
$$

Such partitions are denoted by $P$ or $Q$. For $f$ a bounded function defined on $[a, b]$, let

$$
\begin{aligned}
M_{i}(f) & \equiv \sup \left\{f(x): x \in\left[x_{i-1}, x_{i}\right]\right\}, \\
m_{i}(f) & \equiv \inf \left\{f(x): x \in\left[x_{i-1}, x_{i}\right]\right\} .
\end{aligned}
$$

Definition 4.1.1 Let $F$ be an increasing function defined on $[a, b]$ and let $\Delta F_{i} \equiv F\left(x_{i}\right)-$ $F\left(x_{i-1}\right)$. Then define upper and lower sums as

$$
U(f, P) \equiv \sum_{i=1}^{n} M_{i}(f) \Delta F_{i} \text { and } L(f, P) \equiv \sum_{i=1}^{n} m_{i}(f) \Delta F_{i}
$$

respectively. The numbers, $M_{i}(f)$ and $m_{i}(f)$, are well defined real numbers because $f$ is assumed to be bounded and $\mathbb{R}$ is complete. Thus the set $S=\left\{f(x): x \in\left[x_{i-1}, x_{i}\right]\right\}$ is bounded above and below.

In the following picture, the sum of the areas of the rectangles in the picture on the left is a lower sum for the function in the picture and the sum of the areas of the rectangles in the picture on the right is an upper sum for the same function which uses the same partition. In these pictures the function, $F$ is given by $F(x)=x$ and these are the ordinary upper and lower sums from calculus.

[^0]

What happens when you add in more points in a partition? The following pictures illustrate in the context of the above example. In this example a single additional point, labeled $z$ has been added in.


Note how the lower sum got larger by the amount of the area in the shaded rectangle and the upper sum got smaller by the amount in the rectangle shaded by dots. In general this is the way it works and this is shown in the following lemma.

Lemma 4.1.2 If $P \subseteq Q$ then

$$
U(f, Q) \leq U(f, P), \text { and } L(f, P) \leq L(f, Q)
$$

Proof: This is verified by adding in one point at a time. Thus let

$$
P=\left\{x_{0}, \cdots, x_{n}\right\}
$$

and let

$$
Q=\left\{x_{0}, \cdots, x_{k}, y, x_{k+1}, \cdots, x_{n}\right\}
$$

Thus exactly one point, $y$, is added between $x_{k}$ and $x_{k+1}$. Now the term in the upper sum which corresponds to the interval $\left[x_{k}, x_{k+1}\right]$ in $U(f, P)$ is

$$
\begin{equation*}
\sup \left\{f(x): x \in\left[x_{k}, x_{k+1}\right]\right\}\left(F\left(x_{k+1}\right)-F\left(x_{k}\right)\right) \tag{4.1.1}
\end{equation*}
$$

and the term which corresponds to the interval $\left[x_{k}, x_{k+1}\right]$ in $U(f, Q)$ is

$$
\begin{gather*}
\sup \left\{f(x): x \in\left[x_{k}, y\right]\right\}\left(F(y)-F\left(x_{k}\right)\right) \\
+\sup \left\{f(x): x \in\left[y, x_{k+1}\right]\right\}\left(F\left(x_{k+1}\right)-F(y)\right)  \tag{4.1.2}\\
\equiv M_{1}\left(F(y)-F\left(x_{k}\right)\right)+M_{2}\left(F\left(x_{k+1}\right)-F(y)\right)
\end{gather*}
$$

All the other terms in the two sums coincide. Now

$$
\sup \left\{f(x): x \in\left[x_{k}, x_{k+1}\right]\right\} \geq \max \left(M_{1}, M_{2}\right)
$$

and so the expression in 4.1.2 is no larger than

$$
\begin{aligned}
& \sup \left\{f(x): x \in\left[x_{k}, x_{k+1}\right]\right\}\left(F\left(x_{k+1}\right)-F(y)\right) \\
& +\sup \left\{f(x): x \in\left[x_{k}, x_{k+1}\right]\right\}\left(F(y)-F\left(x_{k}\right)\right) \\
= & \sup \left\{f(x): x \in\left[x_{k}, x_{k+1}\right]\right\}\left(F\left(x_{k+1}\right)-F\left(x_{k}\right)\right),
\end{aligned}
$$

the term corresponding to the interval, $\left[x_{k}, x_{k+1}\right]$ and $U(f, P)$. This proves the first part of the lemma pertaining to upper sums because if $Q \supseteq P$, one can obtain $Q$ from $P$ by adding in one point at a time and each time a point is added, the corresponding upper sum either gets smaller or stays the same. The second part about lower sums is similar and is left as an exercise.

Lemma 4.1.3 If $P$ and $Q$ are two partitions, then

$$
L(f, P) \leq U(f, Q)
$$

Proof: By Lemma 4.1.2,

$$
L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q)
$$

Definition 4.1.4

$$
\begin{aligned}
& \bar{I} \equiv \inf \{U(f, Q) \text { where } Q \text { is a partition }\} \\
& \underline{I} \equiv \sup \{L(f, P) \text { where } P \text { is a partition }\} .
\end{aligned}
$$

Note that $\underline{I}$ and $\bar{I}$ are well defined real numbers.
Theorem 4.1.5 $\underline{I} \leq \bar{I}$.
Proof: From Lemma 4.1.3,

$$
\underline{I}=\sup \{L(f, P) \text { where } P \text { is a partition }\} \leq U(f, Q)
$$

because $U(f, Q)$ is an upper bound to the set of all lower sums and so it is no smaller than the least upper bound. Therefore, since $Q$ is arbitrary,

$$
\begin{aligned}
\underline{I} & =\sup \{L(f, P) \text { where } P \text { is a partition }\} \\
& \leq \inf \{U(f, Q) \text { where } Q \text { is a partition }\} \equiv \bar{I}
\end{aligned}
$$

where the inequality holds because it was just shown that $\underline{I}$ is a lower bound to the set of all upper sums and so it is no larger than the greatest lower bound of this set. This proves the theorem.

Definition 4.1.6 A bounded function $f$ is Riemann Stieltjes integrable, written as

$$
f \in R([a, b])
$$

if

$$
\underline{I}=\bar{I}
$$

and in this case,

$$
\int_{a}^{b} f(x) d F \equiv \underline{I}=\bar{I}
$$

When $F(x)=x$, the integral is called the Riemann integral and is written as

$$
\int_{a}^{b} f(x) d x
$$

Thus, in words, the Riemann integral is the unique number which lies between all upper sums and all lower sums if there is such a unique number.

Recall the following Proposition which comes from the definitions.
Proposition 4.1.7 Let $S$ be a nonempty set and suppose $\sup (S)$ exists. Then for every $\delta>0$,

$$
S \cap(\sup (S)-\delta, \sup (S)] \neq \emptyset
$$

If $\inf (S)$ exists, then for every $\delta>0$,

$$
S \cap[\inf (S), \inf (S)+\delta) \neq \emptyset
$$

This proposition implies the following theorem which is used to determine the question of Riemann Stieltjes integrability.
Theorem 4.1.8 A bounded function $f$ is Riemann integrable if and only if for all $\varepsilon>0$, there exists a partition $P$ such that

$$
\begin{equation*}
U(f, P)-L(f, P)<\varepsilon . \tag{4.1.3}
\end{equation*}
$$

Proof: First assume $f$ is Riemann integrable. Then let $P$ and $Q$ be two partitions such that

$$
U(f, Q)<\bar{I}+\varepsilon / 2, L(f, P)>\underline{I}-\varepsilon / 2
$$

Then since $\underline{I}=\bar{I}$,

$$
U(f, Q \cup P)-L(f, P \cup Q) \leq U(f, Q)-L(f, P)<\bar{I}+\varepsilon / 2-(\underline{I}-\varepsilon / 2)=\varepsilon
$$

Now suppose that for all $\varepsilon>0$ there exists a partition such that 4.1.3 holds. Then for given $\varepsilon$ and partition $P$ corresponding to $\varepsilon$

$$
\bar{I}-\underline{I} \leq U(f, P)-L(f, P) \leq \varepsilon
$$

Since $\varepsilon$ is arbitrary, this shows $\underline{I}=\bar{I}$ and this proves the theorem.
The condition described in the theorem is called the Riemann criterion .
Not all bounded functions are Riemann integrable. For example, let $F(x)=x$ and

$$
f(x) \equiv\left\{\begin{array}{l}
1 \text { if } x \in \mathbb{Q}  \tag{4.1.4}\\
0 \text { if } x \in \mathbb{R} \backslash \mathbb{Q}
\end{array}\right.
$$

Then if $[a, b]=[0,1]$ all upper sums for $f$ equal 1 while all lower sums for $f$ equal 0 . Therefore the Riemann criterion is violated for $\varepsilon=1 / 2$.

### 4.2 Exercises

1. Prove the second half of Lemma 4.1.2 about lower sums.
2. Verify that for $f$ given in 4.1 .4 , the lower sums on the interval $[0,1]$ are all equal to zero while the upper sums are all equal to one.
3. Let $f(x)=1+x^{2}$ for $x \in[-1,3]$ and let $P=\left\{-1,-\frac{1}{3}, 0, \frac{1}{2}, 1,2\right\}$. Find $U(f, P)$ and $L(f, P)$ for $F(x)=x$ and for $F(x)=x^{3}$.
4. Show that if $f \in R([a, b])$ for $F(x)=x$, there exists a partition, $\left\{x_{0}, \cdots, x_{n}\right\}$ such that for any $z_{k} \in\left[x_{k}, x_{k+1}\right]$,

$$
\left|\int_{a}^{b} f(x) d x-\sum_{k=1}^{n} f\left(z_{k}\right)\left(x_{k}-x_{k-1}\right)\right|<\varepsilon
$$

This sum, $\sum_{k=1}^{n} f\left(z_{k}\right)\left(x_{k}-x_{k-1}\right)$, is called a Riemann sum and this exercise shows that the Riemann integral can always be approximated by a Riemann sum. For the general Riemann Stieltjes case, does anything change?
5. Let $P=\left\{1,1 \frac{1}{4}, 1 \frac{1}{2}, 1 \frac{3}{4}, 2\right\}$ and $F(x)=x$. Find upper and lower sums for the function, $f(x)=\frac{1}{x}$ using this partition. What does this tell you about $\ln (2)$ ?
6. If $f \in R([a, b])$ with $F(x)=x$ and $f$ is changed at finitely many points, show the new function is also in $R([a, b])$. Is this still true for the general case where $F$ is only assumed to be an increasing function? Explain.
7. In the case where $F(x)=x$, define a "left sum" as

$$
\sum_{k=1}^{n} f\left(x_{k-1}\right)\left(x_{k}-x_{k-1}\right)
$$

and a "right sum",

$$
\sum_{k=1}^{n} f\left(x_{k}\right)\left(x_{k}-x_{k-1}\right)
$$

Also suppose that all partitions have the property that $x_{k}-x_{k-1}$ equals a constant, $(b-a) / n$ so the points in the partition are equally spaced, and define the integral to be the number these right and left sums get close to as $n$ gets larger and larger. Show that for $f$ given in 4.1.4, $\int_{0}^{x} f(t) d t=1$ if $x$ is rational and $\int_{0}^{x} f(t) d t=0$ if $x$ is irrational. It turns out that the correct answer should always equal zero for that function, regardless of whether $x$ is rational. This is shown when the Lebesgue integral is studied. This illustrates why this method of defining the integral in terms of left and right sums is total nonsense. Show that even though this is the case, it makes no difference if $f$ is continuous.

### 4.3 Functions Of Riemann Integrable Functions

It is often necessary to consider functions of Riemann integrable functions and a natural question is whether these are Riemann integrable. The following theorem gives a partial answer to this question. This is not the most general theorem which will relate to this question but it will be enough for the needs of this book.

Theorem 4.3.1 Let $f, g$ be bounded functions and let

$$
f([a, b]) \subseteq\left[c_{1}, d_{1}\right], g([a, b]) \subseteq\left[c_{2}, d_{2}\right]
$$

Let $H:\left[c_{1}, d_{1}\right] \times\left[c_{2}, d_{2}\right] \rightarrow \mathbb{R}$ satisfy,

$$
\left|H\left(a_{1}, b_{1}\right)-H\left(a_{2}, b_{2}\right)\right| \leq K\left[\left|a_{1}-a_{2}\right|+\left|b_{1}-b_{2}\right|\right]
$$

for some constant $K$. Then if $f, g \in R([a, b])$ it follows that $H \circ(f, g) \in R([a, b])$.
Proof: In the following claim, $M_{i}(h)$ and $m_{i}(h)$ have the meanings assigned above with respect to some partition of $[a, b]$ for the function, $h$.

Claim: The following inequality holds.

$$
\begin{gathered}
\left|M_{i}(H \circ(f, g))-m_{i}(H \circ(f, g))\right| \leq \\
K\left[\left|M_{i}(f)-m_{i}(f)\right|+\left|M_{i}(g)-m_{i}(g)\right|\right] .
\end{gathered}
$$

Proof of the claim: By the above proposition, there exist $x_{1}, x_{2} \in\left[x_{i-1}, x_{i}\right]$ be such that

$$
H\left(f\left(x_{1}\right), g\left(x_{1}\right)\right)+\eta>M_{i}(H \circ(f, g)),
$$

and

$$
H\left(f\left(x_{2}\right), g\left(x_{2}\right)\right)-\eta<m_{i}(H \circ(f, g)) .
$$

Then

$$
\begin{gathered}
\left|M_{i}(H \circ(f, g))-m_{i}(H \circ(f, g))\right| \\
<2 \eta+\left|H\left(f\left(x_{1}\right), g\left(x_{1}\right)\right)-H\left(f\left(x_{2}\right), g\left(x_{2}\right)\right)\right| \\
<2 \eta+K\left[\left|f\left(x_{1}\right)-f\left(x_{2}\right)\right|+\left|g\left(x_{1}\right)-g\left(x_{2}\right)\right|\right] \\
\leq 2 \eta+K\left[\left|M_{i}(f)-m_{i}(f)\right|+\left|M_{i}(g)-m_{i}(g)\right|\right] .
\end{gathered}
$$

Since $\eta>0$ is arbitrary, this proves the claim.
Now continuing with the proof of the theorem, let $P$ be such that

$$
\sum_{i=1}^{n}\left(M_{i}(f)-m_{i}(f)\right) \Delta F_{i}<\frac{\varepsilon}{2 K}, \sum_{i=1}^{n}\left(M_{i}(g)-m_{i}(g)\right) \Delta F_{i}<\frac{\varepsilon}{2 K}
$$

Then from the claim,

$$
\sum_{i=1}^{n}\left(M_{i}(H \circ(f, g))-m_{i}(H \circ(f, g))\right) \Delta F_{i}
$$

$$
<\sum_{i=1}^{n} K\left[\left|M_{i}(f)-m_{i}(f)\right|+\left|M_{i}(g)-m_{i}(g)\right|\right] \Delta F_{i}<\varepsilon
$$

Since $\varepsilon>0$ is arbitrary, this shows $H \circ(f, g)$ satisfies the Riemann criterion and hence $H \circ(f, g)$ is Riemann integrable as claimed. This proves the theorem.

This theorem implies that if $f, g$ are Riemann Stieltjes integrable, then so is $a f+$ $b g,|f|, f^{2}$, along with infinitely many other such continuous combinations of Riemann Stieltjes integrable functions. For example, to see that $|f|$ is Riemann integrable, let $H(a, b)=|a|$. Clearly this function satisfies the conditions of the above theorem and so $|f|=H(f, f) \in R([a, b])$ as claimed. The following theorem gives an example of many functions which are Riemann integrable.

Theorem 4.3.2 Let $f:[a, b] \rightarrow \mathbb{R}$ be either increasing or decreasing on $[a, b]$ and suppose $F$ is continuous. Then $f \in R([a, b])$.

Proof: Let $\varepsilon>0$ be given and let

$$
x_{i}=a+i\left(\frac{b-a}{n}\right), i=0, \cdots, n
$$

Since $F$ is continuous, it follows from Corollary 3.0 .5 on Page 38 that it is uniformly continuous. Therefore, if $n$ is large enough, then for all $i$,

$$
F\left(x_{i}\right)-F\left(x_{i-1}\right)<\frac{\varepsilon}{f(b)-f(a)+1}
$$

Then since $f$ is increasing,

$$
\begin{aligned}
U(f, P)-L(f, P) & =\sum_{i=1}^{n}\left(f\left(x_{i}\right)-f\left(x_{i-1}\right)\right)\left(F\left(x_{i}\right)-F\left(x_{i-1}\right)\right) \\
& \leq \frac{\varepsilon}{f(b)-f(a)+1} \sum_{i=1}^{n}\left(f\left(x_{i}\right)-f\left(x_{i-1}\right)\right) \\
& =\frac{\varepsilon}{f(b)-f(a)+1}(f(b)-f(a))<\varepsilon
\end{aligned}
$$

Thus the Riemann criterion is satisfied and so the function is Riemann Stieltjes integrable. The proof for decreasing $f$ is similar.

Corollary 4.3.3 Let $[a, b]$ be a bounded closed interval and let $\phi:[a, b] \rightarrow \mathbb{R}$ be Lipschitz continuous and suppose $F$ is continuous. Then $\phi \in R([a, b])$. Recall that a function, $\phi$, is Lipschitz continuous if there is a constant, $K$, such that for all $x, y$,

$$
|\phi(x)-\phi(y)|<K|x-y|
$$

Proof: Let $f(x)=x$. Then by Theorem 4.3.2, $f$ is Riemann Stieltjes integrable. Let $H(a, b) \equiv \phi(a)$. Then by Theorem 4.3.1 $H \circ(f, f)=\phi \circ f=\phi$ is also Riemann Stieltjes integrable. This proves the corollary. In fact, it is enough to assume $\phi$ is continuous, although this is harder. This is the content of the next theorem which is where the difficult theorems about continuity and uniform continuity are used. This is the main result on the existence of the Riemann Stieltjes integral for this book.

Theorem 4.3.4 Suppose $f:[a, b] \rightarrow \mathbb{R}$ is continuous and $F$ is just an increasing function defined on $[a, b]$. Then $f \in R([a, b])$.

Proof: By Corollary 3.0.5 on Page 38, $f$ is uniformly continuous on $[a, b]$. Therefore, if $\varepsilon>0$ is given, there exists a $\delta>0$ such that if $\left|x_{i}-x_{i-1}\right|<\delta$, then $M_{i}-m_{i}<\frac{\varepsilon}{F(b)-F(a)+1}$. Let

$$
P \equiv\left\{x_{0}, \cdots, x_{n}\right\}
$$

be a partition with $\left|x_{i}-x_{i-1}\right|<\delta$. Then

$$
\begin{aligned}
U(f, P)-L(f, P) & <\sum_{i=1}^{n}\left(M_{i}-m_{i}\right)\left(F\left(x_{i}\right)-F\left(x_{i-1}\right)\right) \\
& <\frac{\varepsilon}{F(b)-F(a)+1}(F(b)-F(a))<\varepsilon
\end{aligned}
$$

By the Riemann criterion, $f \in R([a, b])$. This proves the theorem.

### 4.4 Properties Of The Integral

The integral has many important algebraic properties. First here is a simple lemma.
Lemma 4.4.1 Let $S$ be a nonempty set which is bounded above and below. Then if $-S \equiv$ $\{-x: x \in S\}$,

$$
\begin{equation*}
\sup (-S)=-\inf (S) \tag{4.4.5}
\end{equation*}
$$

and

$$
\begin{equation*}
\inf (-S)=-\sup (S) \tag{4.4.6}
\end{equation*}
$$

Proof: Consider 4.4.5. Let $x \in S$. Then $-x \leq \sup (-S)$ and so $x \geq-\sup (-S)$. It follows that $-\sup (-S)$ is a lower bound for $S$ and therefore, $-\sup (-S) \leq \inf (S)$. This $\operatorname{implies} \sup (-S) \geq-\inf (S)$. Now let $-x \in-S$. Then $x \in S$ and so $x \geq \inf (S)$ which implies $-x \leq-\inf (S)$. Therefore, $-\inf (S)$ is an upper bound for $-S$ and so $-\inf (S) \geq \sup (-S)$. This shows 4.4.5. Formula 4.4.6 is similar and is left as an exercise.

In particular, the above lemma implies that for $M_{i}(f)$ and $m_{i}(f)$ defined above

$$
M_{i}(-f)=-m_{i}(f) \text { and } m_{i}(-f)=-M_{i}(f)
$$

Lemma 4.4.2 If $f \in R([a, b])$ then $-f \in R([a, b])$ and

$$
-\int_{a}^{b} f(x) d F=\int_{a}^{b}-f(x) d F
$$

Proof: The first part of the conclusion of this lemma follows from Theorem 4.3.2 since the function $\phi(y) \equiv-y$ is Lipschitz continuous. Now choose $P$ such that

$$
\int_{a}^{b}-f(x) d F-L(-f, P)<\varepsilon
$$

Then since $m_{i}(-f)=-M_{i}(f)$,

$$
\varepsilon>\int_{a}^{b}-f(x) d F-\sum_{i=1}^{n} m_{i}(-f) \Delta F_{i}=\int_{a}^{b}-f(x) d F+\sum_{i=1}^{n} M_{i}(f) \Delta F_{i}
$$

which implies

$$
\varepsilon>\int_{a}^{b}-f(x) d F+\sum_{i=1}^{n} M_{i}(f) \Delta F_{i} \geq \int_{a}^{b}-f(x) d F+\int_{a}^{b} f(x) d F
$$

Thus, since $\varepsilon$ is arbitrary,

$$
\int_{a}^{b}-f(x) d F \leq-\int_{a}^{b} f(x) d F
$$

whenever $f \in R([a, b])$. It follows

$$
\int_{a}^{b}-f(x) d F \leq-\int_{a}^{b} f(x) d F=-\int_{a}^{b}-(-f(x)) d F \leq \int_{a}^{b}-f(x) d F
$$

and this proves the lemma.
Theorem 4.4.3 The integral is linear,

$$
\int_{a}^{b}(\alpha f+\beta g)(x) d F=\alpha \int_{a}^{b} f(x) d F+\beta \int_{a}^{b} g(x) d F
$$

whenever $f, g \in R([a, b])$ and $\alpha, \beta \in \mathbb{R}$.
Proof: First note that by Theorem 4.3.1, $\alpha f+\beta g \in R([a, b])$. To begin with, consider the claim that if $f, g \in R([a, b])$ then

$$
\begin{equation*}
\int_{a}^{b}(f+g)(x) d F=\int_{a}^{b} f(x) d F+\int_{a}^{b} g(x) d F \tag{4.4.7}
\end{equation*}
$$

Let $P_{1}, Q_{1}$ be such that

$$
U\left(f, Q_{1}\right)-L\left(f, Q_{1}\right)<\varepsilon / 2, U\left(g, P_{1}\right)-L\left(g, P_{1}\right)<\varepsilon / 2 .
$$

Then letting $P \equiv P_{1} \cup Q_{1}$, Lemma 4.1.2 implies

$$
U(f, P)-L(f, P)<\varepsilon / 2, \text { and } U(g, P)-U(g, P)<\varepsilon / 2
$$

Next note that

$$
m_{i}(f+g) \geq m_{i}(f)+m_{i}(g), M_{i}(f+g) \leq M_{i}(f)+M_{i}(g)
$$

Therefore,

$$
L(g+f, P) \geq L(f, P)+L(g, P), U(g+f, P) \leq U(f, P)+U(g, P)
$$

For this partition,

$$
\begin{aligned}
\int_{a}^{b}(f+g)(x) d F & \in[L(f+g, P), U(f+g, P)] \\
& \subseteq[L(f, P)+L(g, P), U(f, P)+U(g, P)]
\end{aligned}
$$

and

$$
\int_{a}^{b} f(x) d F+\int_{a}^{b} g(x) d F \in[L(f, P)+L(g, P), U(f, P)+U(g, P)] .
$$

Therefore,

$$
\begin{gathered}
\left|\int_{a}^{b}(f+g)(x) d F-\left(\int_{a}^{b} f(x) d F+\int_{a}^{b} g(x) d F\right)\right| \leq \\
U(f, P)+U(g, P)-(L(f, P)+L(g, P))<\varepsilon / 2+\varepsilon / 2=\varepsilon
\end{gathered}
$$

This proves 4.4.7 since $\varepsilon$ is arbitrary.
It remains to show that

$$
\alpha \int_{a}^{b} f(x) d F=\int_{a}^{b} \alpha f(x) d F
$$

Suppose first that $\alpha \geq 0$. Then

$$
\begin{aligned}
& \int_{a}^{b} \alpha f(x) d F \equiv \sup \{L(\alpha f, P): P \text { is a partition }\}= \\
& \alpha \sup \{L(f, P): P \text { is a partition }\} \equiv \alpha \int_{a}^{b} f(x) d F
\end{aligned}
$$

If $\alpha<0$, then this and Lemma 4.4.2 imply

$$
\begin{aligned}
& \int_{a}^{b} \alpha f(x) d F=\int_{a}^{b}(-\alpha)(-f(x)) d F \\
= & (-\alpha) \int_{a}^{b}(-f(x)) d F=\alpha \int_{a}^{b} f(x) d F
\end{aligned}
$$

This proves the theorem.
In the next theorem, suppose $F$ is defined on $[a, b] \cup[b, c]$.
Theorem 4.4.4 If $f \in R([a, b])$ and $f \in R([b, c])$, then $f \in R([a, c])$ and

$$
\begin{equation*}
\int_{a}^{c} f(x) d F=\int_{a}^{b} f(x) d F+\int_{b}^{c} f(x) d F \tag{4.4.8}
\end{equation*}
$$

Proof: Let $P_{1}$ be a partition of $[a, b]$ and $P_{2}$ be a partition of $[b, c]$ such that

$$
U\left(f, P_{i}\right)-L\left(f, P_{i}\right)<\varepsilon / 2, i=1,2
$$

Let $P \equiv P_{1} \cup P_{2}$. Then $P$ is a partition of $[a, c]$ and

$$
U(f, P)-L(f, P)
$$

$$
\begin{equation*}
=U\left(f, P_{1}\right)-L\left(f, P_{1}\right)+U\left(f, P_{2}\right)-L\left(f, P_{2}\right)<\varepsilon / 2+\varepsilon / 2=\varepsilon . \tag{4.4.9}
\end{equation*}
$$

Thus, $f \in R([a, c])$ by the Riemann criterion and also for this partition,

$$
\begin{aligned}
\int_{a}^{b} f(x) d F+\int_{b}^{c} f(x) d F & \in\left[L\left(f, P_{1}\right)+L\left(f, P_{2}\right), U\left(f, P_{1}\right)+U\left(f, P_{2}\right)\right] \\
& =[L(f, P), U(f, P)]
\end{aligned}
$$

and

$$
\int_{a}^{c} f(x) d F \in[L(f, P), U(f, P)]
$$

Hence by 4.4.9,

$$
\left|\int_{a}^{c} f(x) d F-\left(\int_{a}^{b} f(x) d F+\int_{b}^{c} f(x) d F\right)\right|<U(f, P)-L(f, P)<\varepsilon
$$

which shows that since $\varepsilon$ is arbitrary, 4.4.8 holds. This proves the theorem.
Corollary 4.4.5 Let $F$ be continuous and let $[a, b]$ be a closed and bounded interval and suppose that

$$
a=y_{1}<y_{2} \cdots<y_{l}=b
$$

and that $f$ is a bounded function defined on $[a, b]$ which has the property that $f$ is either increasing on $\left[y_{j}, y_{j+1}\right]$ or decreasing on $\left[y_{j}, y_{j+1}\right]$ for $j=1, \cdots, l-1$. Then $f \in R([a, b])$.

Proof: This follows from Theorem 4.4.4 and Theorem 4.3.2.
The symbol, $\int_{a}^{b} f(x) d F$ when $a>b$ has not yet been defined.

Definition 4.4.6 Let $[a, b]$ be an interval and let $f \in R([a, b])$. Then

$$
\int_{b}^{a} f(x) d F \equiv-\int_{a}^{b} f(x) d F
$$

Note that with this definition,

$$
\int_{a}^{a} f(x) d F=-\int_{a}^{a} f(x) d F
$$

and so

$$
\int_{a}^{a} f(x) d F=0
$$

Theorem 4.4.7 Assuming all the integrals make sense,

$$
\int_{a}^{b} f(x) d F+\int_{b}^{c} f(x) d F=\int_{a}^{c} f(x) d F
$$

Proof: This follows from Theorem 4.4.4 and Definition 4.4.6. For example, assume

$$
c \in(a, b)
$$

Then from Theorem 4.4.4,

$$
\int_{a}^{c} f(x) d F+\int_{c}^{b} f(x) d F=\int_{a}^{b} f(x) d F
$$

and so by Definition 4.4.6,

$$
\begin{aligned}
\int_{a}^{c} f(x) d F & =\int_{a}^{b} f(x) d F-\int_{c}^{b} f(x) d F \\
& =\int_{a}^{b} f(x) d F+\int_{b}^{c} f(x) d F
\end{aligned}
$$

The other cases are similar.
The following properties of the integral have either been established or they follow quickly from what has been shown so far.

$$
\begin{gather*}
\text { If } f \in R([a, b]) \text { then if } c \in[a, b], f \in R([a, c]),  \tag{4.4.10}\\
\int_{a}^{b} \alpha d F=\alpha(F(b)-F(a))  \tag{4.4.11}\\
\int_{a}^{b}(\alpha f+\beta g)(x) d F=\alpha \int_{a}^{b} f(x) d F+\beta \int_{a}^{b} g(x) d F  \tag{4.4.12}\\
\int_{a}^{b} f(x) d F+\int_{b}^{c} f(x) d F=\int_{a}^{c} f(x) d F  \tag{4.4.13}\\
\int_{a}^{b} f(x) d F \geq 0 \text { if } f(x) \geq 0 \text { and } a<b  \tag{4.4.14}\\
\left|\int_{a}^{b} f(x) d F\right| \leq\left|\int_{a}^{b}\right| f(x)|d F| \tag{4.4.15}
\end{gather*}
$$

The only one of these claims which may not be completely obvious is the last one. To show this one, note that

$$
|f(x)|-f(x) \geq 0,|f(x)|+f(x) \geq 0
$$

Therefore, by 4.4.14 and 4.4.12, if $a<b$,

$$
\int_{a}^{b}|f(x)| d F \geq \int_{a}^{b} f(x) d F
$$

and

$$
\int_{a}^{b}|f(x)| d F \geq-\int_{a}^{b} f(x) d F
$$

Therefore,

$$
\int_{a}^{b}|f(x)| d F \geq\left|\int_{a}^{b} f(x) d F\right|
$$

If $b<a$ then the above inequality holds with $a$ and $b$ switched. This implies 4.4.15.

### 4.5 Fundamental Theorem Of Calculus

In this section $F(x)=x$ so things are specialized to the ordinary Riemann integral. With these properties, it is easy to prove the fundamental theorem of calculus ${ }^{2}$. Let $f \in R([a, b])$. Then by 4.4.10 $f \in R([a, x])$ for each $x \in[a, b]$. The first version of the fundamental theorem of calculus is a statement about the derivative of the function

$$
x \rightarrow \int_{a}^{x} f(t) d t
$$

Theorem 4.5.1 Let $f \in R([a, b])$ and let

$$
F(x) \equiv \int_{a}^{x} f(t) d t
$$

Then if $f$ is continuous at $x \in(a, b)$,

$$
F^{\prime}(x)=f(x)
$$

Proof: Let $x \in(a, b)$ be a point of continuity of $f$ and let $h$ be small enough that $x+h \in$ $[a, b]$. Then by using 4.4.13,

$$
h^{-1}(F(x+h)-F(x))=h^{-1} \int_{x}^{x+h} f(t) d t
$$

Also, using 4.4.11,

$$
f(x)=h^{-1} \int_{x}^{x+h} f(x) d t
$$

Therefore, by 4.4.15,

$$
\begin{aligned}
\mid h^{-1}(F(x+h)- & F(x))-f(x)\left|=\left|h^{-1} \int_{x}^{x+h}(f(t)-f(x)) d t\right|\right. \\
& \leq\left|h^{-1} \int_{x}^{x+h}\right| f(t)-f(x)|d t|
\end{aligned}
$$

Let $\varepsilon>0$ and let $\delta>0$ be small enough that if $|t-x|<\delta$, then

$$
|f(t)-f(x)|<\varepsilon
$$

Therefore, if $|h|<\delta$, the above inequality and 4.4.11 shows that

$$
\left|h^{-1}(F(x+h)-F(x))-f(x)\right| \leq|h|^{-1} \varepsilon|h|=\varepsilon
$$

Since $\varepsilon>0$ is arbitrary, this shows

$$
\lim _{h \rightarrow 0} h^{-1}(F(x+h)-F(x))=f(x)
$$

[^1]and this proves the theorem.
Note this gives existence for the initial value problem,
$$
F^{\prime}(x)=f(x), F(a)=0
$$
whenever $f$ is Riemann integrable and continuous. ${ }^{3}$
The next theorem is also called the fundamental theorem of calculus.
Theorem 4.5.2 Let $f \in R([a, b])$ and suppose there exists an antiderivative for $f, G$, such that
$$
G^{\prime}(x)=f(x)
$$
for every point of $(a, b)$ and $G$ is continuous on $[a, b]$. Then
\[

$$
\begin{equation*}
\int_{a}^{b} f(x) d x=G(b)-G(a) . \tag{4.5.16}
\end{equation*}
$$

\]

Proof: Let $P=\left\{x_{0}, \cdots, x_{n}\right\}$ be a partition satisfying

$$
U(f, P)-L(f, P)<\varepsilon
$$

Then

$$
\begin{aligned}
G(b)-G(a) & =G\left(x_{n}\right)-G\left(x_{0}\right) \\
& =\sum_{i=1}^{n} G\left(x_{i}\right)-G\left(x_{i-1}\right) .
\end{aligned}
$$

By the mean value theorem,

$$
\begin{aligned}
G(b)-G(a) & =\sum_{i=1}^{n} G^{\prime}\left(z_{i}\right)\left(x_{i}-x_{i-1}\right) \\
& =\sum_{i=1}^{n} f\left(z_{i}\right) \Delta x_{i}
\end{aligned}
$$

where $z_{i}$ is some point in $\left[x_{i-1}, x_{i}\right]$. It follows, since the above sum lies between the upper and lower sums, that

$$
G(b)-G(a) \in[L(f, P), U(f, P)]
$$

and also

$$
\int_{a}^{b} f(x) d x \in[L(f, P), U(f, P)]
$$

Therefore,

$$
\left|G(b)-G(a)-\int_{a}^{b} f(x) d x\right|<U(f, P)-L(f, P)<\varepsilon .
$$

Since $\varepsilon>0$ is arbitrary, 4.5 .16 holds. This proves the theorem.

[^2]The following notation is often used in this context. Suppose $F$ is an antiderivative of $f$ as just described with $F$ continuous on $[a, b]$ and $F^{\prime}=f$ on $(a, b)$. Then

$$
\int_{a}^{b} f(x) d x=F(b)-\left.F(a) \equiv F(x)\right|_{a} ^{b}
$$

Definition 4.5.3 Let $f$ be a bounded function defined on a closed interval $[a, b]$ and let $P \equiv\left\{x_{0}, \cdots, x_{n}\right\}$ be a partition of the interval. Suppose $z_{i} \in\left[x_{i-1}, x_{i}\right]$ is chosen. Then the sum

$$
\sum_{i=1}^{n} f\left(z_{i}\right)\left(x_{i}-x_{i-1}\right)
$$

is known as a Riemann sum. Also,

$$
\|P\| \equiv \max \left\{\left|x_{i}-x_{i-1}\right|: i=1, \cdots, n\right\}
$$

Proposition 4.5.4 Suppose $f \in R([a, b])$. Then there exists a partition, $P \equiv\left\{x_{0}, \cdots, x_{n}\right\}$ with the property that for any choice of $z_{k} \in\left[x_{k-1}, x_{k}\right]$,

$$
\left|\int_{a}^{b} f(x) d x-\sum_{k=1}^{n} f\left(z_{k}\right)\left(x_{k}-x_{k-1}\right)\right|<\varepsilon .
$$

Proof: Choose $P$ such that

$$
U(f, P)-L(f, P)<\varepsilon
$$

and then both $\int_{a}^{b} f(x) d x$ and $\sum_{k=1}^{n} f\left(z_{k}\right)\left(x_{k}-x_{k-1}\right)$ are contained in $[L(f, P), U(f, P)]$ and so the claimed inequality must hold. This proves the proposition.

It is significant because it gives a way of approximating the integral.
The definition of Riemann integrability given in this chapter is also called Darboux integrability and the integral defined as the unique number which lies between all upper sums and all lower sums which is given in this chapter is called the Darboux integral. The definition of the Riemann integral in terms of Riemann sums is given next.

Definition 4.5.5 A bounded function, $f$ defined on $[a, b]$ is said to be Riemann integrable if there exists a number, I with the property that for every $\varepsilon>0$, there exists $\delta>0$ such that if

$$
P \equiv\left\{x_{0}, x_{1}, \cdots, x_{n}\right\}
$$

is any partition having $\|P\|<\delta$, and $z_{i} \in\left[x_{i-1}, x_{i}\right]$,

$$
\left|I-\sum_{i=1}^{n} f\left(z_{i}\right)\left(x_{i}-x_{i-1}\right)\right|<\varepsilon .
$$

The number $\int_{a}^{b} f(x) d x$ is defined as $I$.
Thus, there are two definitions of the Riemann integral. It turns out they are equivalent which is the following theorem of of Darboux.

Theorem 4.5.6 A bounded function defined on $[a, b]$ is Riemann integrable in the sense of Definition 4.5 .5 if and only if it is integrable in the sense of Darboux. Furthermore the two integrals coincide.

The proof of this theorem is left for the exercises in Problems 10-12. It isn't essential that you understand this theorem so if it does not interest you, leave it out. Note that it implies that given a Riemann integrable function $f$ in either sense, it can be approximated by Riemann sums whenever $\|P\|$ is sufficiently small. Both versions of the integral are obsolete but entirely adequate for most applications and as a point of departure for a more up to date and satisfactory integral. The reason for using the Darboux approach to the integral is that all the existence theorems are easier to prove in this context.

### 4.6 Exercises

1. Let $F(x)=\int_{x^{2}}^{x^{3}} \frac{t^{5}+7}{t^{7}+87 t^{6}+1} d t$. Find $F^{\prime}(x)$.
2. Let $F(x)=\int_{2}^{x} \frac{1}{1+t^{4}} d t$. Sketch a graph of $F$ and explain why it looks the way it does.
3. Let $a$ and $b$ be positive numbers and consider the function,

$$
F(x)=\int_{0}^{a x} \frac{1}{a^{2}+t^{2}} d t+\int_{b}^{a / x} \frac{1}{a^{2}+t^{2}} d t
$$

Show that $F$ is a constant.
4. Solve the following initial value problem from ordinary differential equations which is to find a function $y$ such that

$$
y^{\prime}(x)=\frac{x^{7}+1}{x^{6}+97 x^{5}+7}, y(10)=5
$$

5. If $F, G \in \int f(x) d x$ for all $x \in \mathbb{R}$, show $F(x)=G(x)+C$ for some constant, $C$. Use this to give a different proof of the fundamental theorem of calculus which has for its conclusion $\int_{a}^{b} f(t) d t=G(b)-G(a)$ where $G^{\prime}(x)=f(x)$.
6. Suppose $f$ is Riemann integrable on $[a, b]$ and continuous. (In fact continuous implies Riemann integrable.) Show there exists $c \in(a, b)$ such that

$$
f(c)=\frac{1}{b-a} \int_{a}^{b} f(x) d x
$$

Hint: You might consider the function $F(x) \equiv \int_{a}^{x} f(t) d t$ and use the mean value theorem for derivatives and the fundamental theorem of calculus.
7. Suppose $f$ and $g$ are continuous functions on $[a, b]$ and that $g(x) \neq 0$ on $(a, b)$. Show there exists $c \in(a, b)$ such that

$$
f(c) \int_{a}^{b} g(x) d x=\int_{a}^{b} f(x) g(x) d x
$$

Hint: Define $F(x) \equiv \int_{a}^{x} f(t) g(t) d t$ and let $G(x) \equiv \int_{a}^{x} g(t) d t$. Then use the Cauchy mean value theorem on these two functions.
8. Consider the function

$$
f(x) \equiv\left\{\begin{array}{l}
\sin \left(\frac{1}{x}\right) \text { if } x \neq 0 \\
0 \text { if } x=0
\end{array} .\right.
$$

Is $f$ Riemann integrable? Explain why or why not.
9. Prove the second part of Theorem 4.3.2 about decreasing functions.
10. Suppose $f$ is a bounded function defined on $[a, b]$ and $|f(x)|<M$ for all $x \in[a, b]$. Now let $Q$ be a partition having $n$ points, $\left\{x_{0}^{*}, \cdots, x_{n}^{*}\right\}$ and let $P$ be any other partition.
Show that

$$
|U(f, P)-L(f, P)| \leq 2 M n| | P| |+|U(f, Q)-L(f, Q)|
$$

Hint: Write the sum for $U(f, P)-L(f, P)$ and split this sum into two sums, the sum of terms for which $\left[x_{i-1}, x_{i}\right]$ contains at least one point of $Q$, and terms for which $\left[x_{i-1}, x_{i}\right]$ does not contain any points of $Q$. In the latter case, $\left[x_{i-1}, x_{i}\right]$ must be contained in some interval, $\left[x_{k-1}^{*}, x_{k}^{*}\right]$. Therefore, the sum of these terms should be no larger than $|U(f, Q)-L(f, Q)|$.
11. $\uparrow$ If $\varepsilon>0$ is given and $f$ is a Darboux integrable function defined on $[a, b]$, show there exists $\delta>0$ such that whenever $\|P\|<\delta$, then

$$
|U(f, P)-L(f, P)|<\varepsilon .
$$

12. $\uparrow$ Prove Theorem 4.5.6.

## Chapter 5

## Some Important Linear Algebra

This chapter contains some important linear algebra as distinguished from that which is normally presented in undergraduate courses consisting mainly of uninteresting things you can do with row operations.

### 5.1 Subspaces Spans And Bases

Definition 5.1.1 Let $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{p}\right\}$ be vectors in $\mathbb{F}^{n}$. A linear combination is any expression of the form

$$
\sum_{i=1}^{p} c_{i} \mathbf{x}_{i}
$$

where the $c_{i}$ are scalars. The set of all linear combinations of these vectors is called $\operatorname{span}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{n}\right)$. If $V \subseteq \mathbb{F}^{n}$, then $V$ is called a subspace if whenever $\alpha, \beta$ are scalars and $\mathbf{u}$ and $\mathbf{v}$ are vectors of $V$, it follows $\alpha \mathbf{u}+\beta \mathbf{v} \in V$. That is, it is "closed under the algebraic operations of vector addition and scalar multiplication". A linear combination of vectors is said to be trivial if all the scalars in the linear combination equal zero. A set of vectors is said to be linearly independent if the only linear combination of these vectors which equals the zero vector is the trivial linear combination. Thus $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{n}\right\}$ is called linearly independent if whenever

$$
\sum_{k=1}^{p} c_{k} \mathbf{x}_{k}=\mathbf{0}
$$

it follows that all the scalars, $c_{k}$ equal zero. A set of vectors, $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{p}\right\}$, is called linearly dependent if it is not linearly independent. Thus the set of vectors is linearly dependent if there exist scalars, $c_{i}, i=1, \cdots, n$, not all zero such that $\sum_{k=1}^{p} c_{k} \mathbf{x}_{k}=\mathbf{0}$.

Lemma 5.1.2 $A$ set of vectors $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{p}\right\}$ is linearly independent if and only if none of the vectors can be obtained as a linear combination of the others.

Proof: Suppose first that $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{p}\right\}$ is linearly independent. If

$$
\mathbf{x}_{k}=\sum_{j \neq k} c_{j} \mathbf{x}_{j}
$$

then

$$
\mathbf{0}=1 \mathbf{x}_{k}+\sum_{j \neq k}\left(-c_{j}\right) \mathbf{x}_{j}
$$

a nontrivial linear combination, contrary to assumption. This shows that if the set is linearly independent, then none of the vectors is a linear combination of the others.

Now suppose no vector is a linear combination of the others. Is $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{p}\right\}$ linearly independent? If it is not there exist scalars, $c_{i}$, not all zero such that

$$
\sum_{i=1}^{p} c_{i} \mathbf{x}_{i}=\mathbf{0}
$$

Say $c_{k} \neq 0$. Then you can solve for $\mathbf{x}_{k}$ as

$$
\mathbf{x}_{k}=\sum_{j \neq k}\left(-c_{j}\right) / c_{k} \mathbf{x}_{j}
$$

contrary to assumption. This proves the lemma.
The following is called the exchange theorem.

Theorem 5.1.3 (Exchange Theorem) Let $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{r}\right\}$ be a linearly independent set of vectors such that each $\mathbf{x}_{i}$ is in $\operatorname{span}\left(\mathbf{y}_{1}, \cdots, \mathbf{y}_{s}\right)$. Then $r \leq s$.

Proof: Define span $\left\{\mathbf{y}_{1}, \cdots, \mathbf{y}_{s}\right\} \equiv V$, it follows there exist scalars, $c_{1}, \cdots, c_{s}$ such that

$$
\begin{equation*}
\mathbf{x}_{1}=\sum_{i=1}^{s} c_{i} \mathbf{y}_{i} \tag{5.1.1}
\end{equation*}
$$

Not all of these scalars can equal zero because if this were the case, it would follow that $\mathbf{x}_{1}=0$ and so $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{r}\right\}$ would not be linearly independent. Indeed, if $\mathbf{x}_{1}=\mathbf{0}, 1 \mathbf{x}_{1}+$ $\sum_{i=2}^{r} 0 \mathbf{x}_{i}=\mathbf{x}_{1}=\mathbf{0}$ and so there would exist a nontrivial linear combination of the vectors $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{r}\right\}$ which equals zero.

Say $c_{k} \neq 0$. Then solve (5.1.1) for $\mathbf{y}_{k}$ and obtain

$$
\mathbf{y}_{k} \in \operatorname{span}(\mathbf{x}_{1}, \overbrace{\mathbf{y}_{1}, \cdots, \mathbf{y}_{k-1}, \mathbf{y}_{k+1}, \cdots, \mathbf{y}_{s}}^{\mathrm{s} \text { vel vectors here }}) .
$$

Define $\left\{\mathbf{z}_{1}, \cdots, \mathbf{z}_{s-1}\right\}$ by

$$
\left\{\mathbf{z}_{1}, \cdots, \mathbf{z}_{s-1}\right\} \equiv\left\{\mathbf{y}_{1}, \cdots, \mathbf{y}_{k-1}, \mathbf{y}_{k+1}, \cdots, \mathbf{y}_{s}\right\}
$$

Therefore, span $\left\{\mathbf{x}_{1}, \mathbf{z}_{1}, \cdots, \mathbf{z}_{s-1}\right\}=V$ because if $\mathbf{v} \in V$, there exist constants $c_{1}, \cdots, c_{s}$ such that

$$
\mathbf{v}=\sum_{i=1}^{s-1} c_{i} \mathbf{z}_{i}+c_{s} \mathbf{y}_{k}
$$

Now replace the $\mathbf{y}_{k}$ in the above with a linear combination of the vectors,

$$
\left\{\mathbf{x}_{1}, \mathbf{z}_{1}, \cdots, \mathbf{z}_{s-1}\right\}
$$

to obtain

$$
\mathbf{v} \in \operatorname{span}\left\{\mathbf{x}_{1}, \mathbf{z}_{1}, \cdots, \mathbf{z}_{s-1}\right\}
$$

The vector $\mathbf{y}_{k}$, in the list $\left\{\mathbf{y}_{1}, \cdots, \mathbf{y}_{s}\right\}$, has now been replaced with the vector $\mathbf{x}_{1}$ and the resulting modified list of vectors has the same span as the original list of vectors, $\left\{\mathbf{y}_{1}, \cdots, \mathbf{y}_{s}\right\}$.

Now suppose that $r>s$ and that

$$
\operatorname{span}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{l}, \mathbf{z}_{1}, \cdots, \mathbf{z}_{p}\right)=V
$$

where the vectors, $\mathbf{z}_{1}, \cdots, \mathbf{z}_{p}$ are each taken from the set, $\left\{\mathbf{y}_{1}, \cdots, \mathbf{y}_{s}\right\}$ and $l+p=s$. This has now been done for $l=1$ above. Then since $r>s$, it follows that $l \leq s<r$ and so $l+1 \leq r$. Therefore, $\mathbf{x}_{l+1}$ is a vector not in the list, $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{l}\right\}$ and since

$$
\operatorname{span}\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{l}, \mathbf{z}_{1}, \cdots, \mathbf{z}_{p}\right\}=V
$$

there exist scalars, $c_{i}$ and $d_{j}$ such that

$$
\begin{equation*}
\mathbf{x}_{l+1}=\sum_{i=1}^{l} c_{i} \mathbf{x}_{i}+\sum_{j=1}^{p} d_{j} \mathbf{z}_{j} . \tag{5.1.2}
\end{equation*}
$$

Now not all the $d_{j}$ can equal zero because if this were so, it would follow that $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{r}\right\}$ would be a linearly dependent set because one of the vectors would equal a linear combination of the others. Therefore, (5.1.2) can be solved for one of the $\mathbf{z}_{i}$, say $\mathbf{z}_{k}$, in terms of $\mathbf{x}_{l+1}$ and the other $\mathbf{z}_{i}$ and just as in the above argument, replace that $\mathbf{z}_{i}$ with $\mathbf{x}_{l+1}$ to obtain

$$
\operatorname{span}(\mathbf{x}_{1}, \cdots \mathbf{x}_{l}, \mathbf{x}_{l+1}, \overbrace{\mathbf{z}_{1}, \cdots \mathbf{z}_{k-1}, \mathbf{z}_{k+1}, \cdots, \mathbf{z}_{p}}^{\mathrm{p}-1 \text { vectors here }})=V .
$$

Continue this way, eventually obtaining

$$
\operatorname{span}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{s}\right)=V
$$

But then $\mathbf{x}_{r} \in \operatorname{span}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{s}\right)$ contrary to the assumption that $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{r}\right\}$ is linearly independent. Therefore, $r \leq s$ as claimed.

Definition 5.1.4 A finite set of vectors, $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{r}\right\}$ is a basis for $\mathbb{F}^{n}$ if

$$
\operatorname{span}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{r}\right)=\mathbb{F}^{n}
$$

and $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{r}\right\}$ is linearly independent.
Corollary 5.1.5 Let $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{r}\right\}$ and $\left\{\mathbf{y}_{1}, \cdots, \mathbf{y}_{s}\right\}$ be two bases ${ }^{1}$ of $\mathbb{F}^{n}$. Then $r=s=n$.
Proof: From the exchange theorem, $r \leq s$ and $s \leq r$. Now note the vectors,

$$
\mathbf{e}_{i}=\overbrace{(0, \cdots, 0,1,0 \cdots, 0)}^{1 \text { is in the } i^{\text {th }} \text { slot }}
$$

for $i=1,2, \cdots, n$ are a basis for $\mathbb{F}^{n}$. This proves the corollary.
Lemma 5.1.6 Let $\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{r}\right\}$ be a set of vectors. Then $V \equiv \operatorname{span}\left(\mathbf{v}_{1}, \cdots, \mathbf{v}_{r}\right)$ is a subspace.

[^3]Proof: Suppose $\alpha, \beta$ are two scalars and let $\sum_{k=1}^{r} c_{k} \mathbf{v}_{k}$ and $\sum_{k=1}^{r} d_{k} \mathbf{v}_{k}$ are two elements of $V$. What about

$$
\alpha \sum_{k=1}^{r} c_{k} \mathbf{v}_{k}+\beta \sum_{k=1}^{r} d_{k} \mathbf{v}_{k} ?
$$

Is it also in $V$ ?

$$
\alpha \sum_{k=1}^{r} c_{k} \mathbf{v}_{k}+\beta \sum_{k=1}^{r} d_{k} \mathbf{v}_{k}=\sum_{k=1}^{r}\left(\alpha c_{k}+\beta d_{k}\right) \mathbf{v}_{k} \in V
$$

so the answer is yes. This proves the lemma.
Definition 5.1.7 A finite set of vectors, $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{r}\right\}$ is a basis for a subspace, $V$ of $\mathbb{F}^{n}$ if $\operatorname{span}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{r}\right)=V$ and $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{r}\right\}$ is linearly independent.

Corollary 5.1.8 Let $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{r}\right\}$ and $\left\{\mathbf{y}_{1}, \cdots, \mathbf{y}_{s}\right\}$ be two bases for $V$. Then $r=s$.
Proof: From the exchange theorem, $r \leq s$ and $s \leq r$. Therefore, this proves the corollary.
Definition 5.1.9 Let $V$ be a subspace of $\mathbb{F}^{n}$. Then $\operatorname{dim}(V)$ read as the dimension of $V$ is the number of vectors in a basis.

Of course you should wonder right now whether an arbitrary subspace even has a basis. In fact it does and this is in the next theorem. First, here is an interesting lemma.

Lemma 5.1.10 Suppose $\mathbf{v} \notin \operatorname{span}\left(\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}\right)$ and $\left\{\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}\right\}$ is linearly independent. Then $\left\{\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}, \mathbf{v}\right\}$ is also linearly independent.

Proof: Suppose $\sum_{i=1}^{k} c_{i} \mathbf{u}_{i}+d \mathbf{v}=\mathbf{0}$. It is required to verify that each $c_{i}=0$ and that $d=$ 0 . But if $d \neq 0$, then you can solve for $\mathbf{v}$ as a linear combination of the vectors, $\left\{\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}\right\}$,

$$
\mathbf{v}=-\sum_{i=1}^{k}\left(\frac{c_{i}}{d}\right) \mathbf{u}_{i}
$$

contrary to assumption. Therefore, $d=0$. But then $\sum_{i=1}^{k} c_{i} \mathbf{u}_{i}=0$ and the linear independence of $\left\{\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}\right\}$ implies each $c_{i}=0$ also. This proves the lemma.

Theorem 5.1.11 Let $V$ be a nonzero subspace of $\mathbb{F}^{n}$. Then $V$ has a basis.
Proof: Let $\mathbf{v}_{1} \in V$ where $\mathbf{v}_{1} \neq 0$. If $\operatorname{span}\left\{\mathbf{v}_{1}\right\}=V$, stop. $\left\{\mathbf{v}_{1}\right\}$ is a basis for $V$. Otherwise, there exists $\mathbf{v}_{2} \in V$ which is not in span $\left\{\mathbf{v}_{1}\right\}$. By Lemma 5.1.10 $\left\{\mathbf{v}_{1}, \mathbf{v}_{2}\right\}$ is a linearly independent set of vectors. If span $\left\{\mathbf{v}_{1}, \mathbf{v}_{2}\right\}=V$ stop, $\left\{\mathbf{v}_{1}, \mathbf{v}_{2}\right\}$ is a basis for $V$. If $\operatorname{span}\left\{\mathbf{v}_{1}, \mathbf{v}_{2}\right\} \neq V$, then there exists $\mathbf{v}_{3} \notin \operatorname{span}\left\{\mathbf{v}_{1}, \mathbf{v}_{2}\right\}$ and $\left\{\mathbf{v}_{1}, \mathbf{v}_{2}, \mathbf{v}_{3}\right\}$ is a larger linearly independent set of vectors. Continuing this way, the process must stop before $n+1$ steps because if not, it would be possible to obtain $n+1$ linearly independent vectors contrary to the exchange theorem. This proves the theorem.

In words the following corollary states that any linearly independent set of vectors can be enlarged to form a basis.

Corollary 5.1.12 Let $V$ be a subspace of $\mathbb{F}^{n}$ and let $\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{r}\right\}$ be a linearly independent set of vectors in $V$. Then either it is a basis for $V$ or there exist vectors, $\mathbf{v}_{r+1}, \cdots, \mathbf{v}_{s}$ such that $\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{r}, \mathbf{v}_{r+1}, \cdots, \mathbf{v}_{s}\right\}$ is a basis for $V$.

Proof: This follows immediately from the proof of Theorem 59.16.4. You do exactly the same argument except you start with $\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{r}\right\}$ rather than $\left\{\mathbf{v}_{1}\right\}$.

It is also true that any spanning set of vectors can be restricted to obtain a basis.
Theorem 5.1.13 Let $V$ be a subspace of $\mathbb{F}^{n}$ and suppose $\operatorname{span}\left(\mathbf{u}_{1} \cdots, \mathbf{u}_{p}\right)=V$ where the $\mathbf{u}_{i}$ are nonzero vectors. Then there exist vectors, $\left\{\mathbf{v}_{1} \cdots, \mathbf{v}_{r}\right\}$ such that $\left\{\mathbf{v}_{1} \cdots, \mathbf{v}_{r}\right\} \subseteq$ $\left\{\mathbf{u}_{1} \cdots, \mathbf{u}_{p}\right\}$ and $\left\{\mathbf{v}_{1} \cdots, \mathbf{v}_{r}\right\}$ is a basis for $V$.

Proof: Let $r$ be the smallest positive integer with the property that for some set,

$$
\left\{\mathbf{v}_{1} \cdots, \mathbf{v}_{r}\right\} \subseteq\left\{\mathbf{u}_{1} \cdots, \mathbf{u}_{p}\right\}, \operatorname{span}\left(\mathbf{v}_{1} \cdots, \mathbf{v}_{r}\right)=V
$$

Then $r \leq p$ and it must be the case that $\left\{\mathbf{v}_{1} \cdots, \mathbf{v}_{r}\right\}$ is linearly independent because if it were not so, one of the vectors, say $\mathbf{v}_{k}$ would be a linear combination of the others. But then you could delete this vector from $\left\{\mathbf{v}_{1} \cdots, \mathbf{v}_{r}\right\}$ and the resulting list of $r-1$ vectors would still span $V$ contrary to the definition of $r$. This proves the theorem.

### 5.2 An Application To Matrices

The following is a theorem of major significance.
Theorem 5.2.1 Suppose $A$ is an $n \times n$ matrix. Then $A$ is one to one if and only if $A$ is onto. Also, if $B$ is an $n \times n$ matrix and $A B=I$, then it follows $B A=I$.

Proof: First suppose $A$ is one to one. Consider the vectors, $\left\{A \mathbf{e}_{1}, \cdots, A \mathbf{e}_{n}\right\}$ where $\mathbf{e}_{k}$ is the column vector which is all zeros except for a 1 in the $k^{t h}$ position. This set of vectors is linearly independent because if

$$
\sum_{k=1}^{n} c_{k} A \mathbf{e}_{k}=\mathbf{0}
$$

then since $A$ is linear,

$$
A\left(\sum_{k=1}^{n} c_{k} \mathbf{e}_{k}\right)=\mathbf{0}
$$

and since $A$ is one to one, it follows

$$
\sum_{k=1}^{n} c_{k} \mathbf{e}_{k}=\mathbf{0}^{2}
$$

which implies each $c_{k}=0$. Therefore, $\left\{A \mathbf{e}_{1}, \cdots, A \mathbf{e}_{n}\right\}$ must be a basis for $\mathbb{F}^{n}$ because if not there would exist a vector, $\mathbf{y} \notin \operatorname{span}\left(A \mathbf{e}_{1}, \cdots, A \mathbf{e}_{n}\right)$ and then by Lemma 5.1.10, $\left\{A \mathbf{e}_{1}, \cdots, A \mathbf{e}_{n}, \mathbf{y}\right\}$ would be an independent set of vectors having $n+1$ vectors in it, contrary to the exchange theorem. It follows that for $\mathbf{y} \in \mathbb{F}^{n}$ there exist constants, $c_{i}$ such that

$$
\mathbf{y}=\sum_{k=1}^{n} c_{k} A \mathbf{e}_{k}=A\left(\sum_{k=1}^{n} c_{k} \mathbf{e}_{k}\right)
$$

showing that, since $\mathbf{y}$ was arbitrary, $A$ is onto.
Next suppose $A$ is onto. This means the span of the columns of $A$ equals $\mathbb{F}^{n}$. If these columns are not linearly independent, then by Lemma 5.1.2 on Page 61, one of the columns is a linear combination of the others and so the span of the columns of $A$ equals the span of the $n-1$ other columns. This violates the exchange theorem because $\left\{\mathbf{e}_{1}, \cdots, \mathbf{e}_{n}\right\}$ would be a linearly independent set of vectors contained in the span of only $n-1$ vectors. Therefore, the columns of $A$ must be independent and this equivalent to saying that $A \mathbf{x}=\mathbf{0}$ if and only if $\mathbf{x}=\mathbf{0}$. This implies $A$ is one to one because if $A \mathbf{x}=A \mathbf{y}$, then $A(\mathbf{x}-\mathbf{y})=\mathbf{0}$ and so $\mathbf{x}-\mathbf{y}=\mathbf{0}$.

Now suppose $A B=I$. Why is $B A=I$ ? Since $A B=I$ it follows $B$ is one to one since otherwise, there would exist, $\mathbf{x} \neq \mathbf{0}$ such that $B \mathbf{x}=\mathbf{0}$ and then $A B \mathbf{x}=A \mathbf{0}=\mathbf{0} \neq I \mathbf{x}$. Therefore, from what was just shown, $B$ is also onto. In addition to this, $A$ must be one to one because if $A \mathbf{y}=\mathbf{0}$, then $\mathbf{y}=B \mathbf{x}$ for some $\mathbf{x}$ and then $\mathbf{x}=A B \mathbf{x}=A \mathbf{y}=\mathbf{0}$ showing $\mathbf{y}=\mathbf{0}$. Now from what is given to be so, it follows $(A B) A=A$ and so using the associative law for matrix multiplication,

$$
A(B A)-A=A(B A-I)=0
$$

But this means $(B A-I) \mathbf{x}=\mathbf{0}$ for all $\mathbf{x}$ since otherwise, $A$ would not be one to one. Hence $B A=I$ as claimed. This proves the theorem.

This theorem shows that if an $n \times n$ matrix, $B$ acts like an inverse when multiplied on one side of $A$ it follows that $B=A^{-1}$ and it will act like an inverse on both sides of $A$.

The conclusion of this theorem pertains to square matrices only. For example, let

$$
A=\left(\begin{array}{ll}
1 & 0  \tag{5.2.3}\\
0 & 1 \\
1 & 0
\end{array}\right), B=\left(\begin{array}{ccc}
1 & 0 & 0 \\
1 & 1 & -1
\end{array}\right)
$$

Then

$$
B A=\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right)
$$

but

$$
A B=\left(\begin{array}{ccc}
1 & 0 & 0 \\
1 & 1 & -1 \\
1 & 0 & 0
\end{array}\right)
$$

### 5.3 The Mathematical Theory Of Determinants

### 5.3.1 The Function sgn

The following Lemma will be essential in the definition of the determinant.
Lemma 5.3.1 There exists a function, $\operatorname{sgn}_{n}$ which maps each ordered list of numbers from $\{1, \cdots, n\}$ to one of the three numbers, 0,1 , or -1 which also has the following properties.

$$
\begin{gather*}
\operatorname{sgn}_{n}(1, \cdots, n)=1  \tag{5.3.4}\\
\operatorname{sgn}_{n}\left(i_{1}, \cdots, p, \cdots, q, \cdots, i_{n}\right)=-\operatorname{sgn}_{n}\left(i_{1}, \cdots, q, \cdots, p, \cdots, i_{n}\right) \tag{5.3.5}
\end{gather*}
$$

In words, the second property states that if two of the numbers are switched, the value of the function is multiplied by -1 . Also, in the case where $n>1$ and $\left\{i_{1}, \cdots, i_{n}\right\}=\{1, \cdots, n\}$ so that every number from $\{1, \cdots, n\}$ appears in the ordered list, $\left(i_{1}, \cdots, i_{n}\right)$,

$$
\begin{gather*}
\operatorname{sgn}_{n}\left(i_{1}, \cdots, i_{\theta-1}, n, i_{\theta+1}, \cdots, i_{n}\right) \equiv \\
(-1)^{n-\theta} \operatorname{sgn}_{n-1}\left(i_{1}, \cdots, i_{\theta-1}, i_{\theta+1}, \cdots, i_{n}\right) \tag{5.3.6}
\end{gather*}
$$

where $n=i_{\theta}$ in the ordered list, $\left(i_{1}, \cdots, i_{n}\right)$.
Proof: Define $\operatorname{sign}(x)=1$ if $x>0,-1$ if $x<0$ and 0 if $x=0$. If $n=1$, there is only one list and it is just the number 1 . Thus one can define $\operatorname{sgn}_{1}(1) \equiv 1$. For the general case where $n>1$, simply define

$$
\operatorname{sgn}_{n}\left(i_{1}, \cdots, i_{n}\right) \equiv \operatorname{sign}\left(\prod_{r<s}\left(i_{s}-i_{r}\right)\right)
$$

This delivers either $-1,1$, or 0 by definition. What about the other claims? Suppose you switch $i_{p}$ with $i_{q}$ where $p<q$ so two numbers in the ordered list $\left(i_{1}, \cdots, i_{n}\right)$ are switched. Denote the new ordered list of numbers as $\left(j_{1}, \cdots, j_{n}\right)$. Thus $j_{p}=i_{q}$ and $j_{q}=i_{p}$ and if $r \notin\{p, q\}, j_{r}=i_{r}$. See the following illustration

$$
\begin{array}{cccccccc}
\frac{i_{1}}{1} & \frac{i_{2}}{2} & \cdots & \frac{i_{p}}{p} & \cdots & \frac{i_{q}}{q} & \cdots & \frac{i_{n}}{n} \\
\frac{i_{1}}{1} & \frac{i_{2}}{2} & \cdots & \frac{i_{q}}{p} & \cdots & \frac{i_{p}}{q} & \cdots & \frac{i_{n}}{n} \\
\frac{j_{1}}{1} & \frac{j_{2}}{2} & \cdots & \frac{j_{p}}{p} & \cdots & \frac{j_{q}}{q} & \cdots & \frac{j_{n}}{n}
\end{array}
$$

Then

$$
\begin{gathered}
\operatorname{sgn}_{n}\left(j_{1}, \cdots, j_{n}\right) \equiv \operatorname{sign}\left(\prod_{r<s}\left(j_{s}-j_{r}\right)\right) \\
=\operatorname{sign}((\begin{array}{c}
\text { both } p, q \\
\left(i_{p}-i_{q}\right)
\end{array} \overbrace{\prod_{p<j<q}\left(i_{j}-i_{q}\right) \prod_{p<j<q}\left(i_{p}-i_{j}\right)}^{\text {one of } p, q} \prod_{r<s, r, s \notin\{p, q\}}^{\text {neither } p \text { nor } q}\left(i_{s}-i_{r}\right))
\end{gathered}
$$

The last product consists of the product of terms which were in the un-switched product $\prod_{r<s}\left(i_{s}-i_{r}\right)$ so produces no change in sign, while the two products in the middle both introduce $q-p-1$ minus signs. Thus their product produces no change in sign. The first factor is of opposite sign to the $i_{q}-i_{p}$ which occured in $\operatorname{sgn}_{n}\left(i_{1}, \cdots, i_{n}\right)$. Therefore, this switch introduced a minus sign and

$$
\operatorname{sgn}_{n}\left(j_{1}, \cdots, j_{n}\right)=-\operatorname{sgn}_{n}\left(i_{1}, \cdots, i_{n}\right)
$$

Now consider the last claim. In computing $\operatorname{sgn}_{n}\left(i_{1}, \cdots, i_{\theta-1}, n, i_{\theta+1}, \cdots, i_{n}\right)$ there will be the product of $n-\theta$ negative terms

$$
\left(i_{\theta+1}-n\right) \cdots\left(i_{n}-n\right)
$$

and the other terms in the product for computing $\operatorname{sgn}_{n}\left(i_{1}, \cdots, i_{\theta-1}, n, i_{\theta+1}, \cdots, i_{n}\right)$ are those which are required to compute $\operatorname{sgn}_{n-1}\left(i_{1}, \cdots, i_{\theta-1}, i_{\theta+1}, \cdots, i_{n}\right)$ multiplied by terms of the form $\left(n-i_{j}\right)$ which are nonnegative. It follows that

$$
\operatorname{sgn}_{n}\left(i_{1}, \cdots, i_{\theta-1}, n, i_{\theta+1}, \cdots, i_{n}\right)=(-1)^{n-\theta} \operatorname{sgn}_{n-1}\left(i_{1}, \cdots, i_{\theta-1}, i_{\theta+1}, \cdots, i_{n}\right)
$$

It is obvious that if there are repeats in the list the function gives 0 .

Lemma 5.3.2 Every ordered list of distinct numbers from $\{1,2, \cdots, n\}$ can be obtained from every other such ordered list by a finite number of switches. Also, $\operatorname{sgn}_{n}$ is unique.

Proof: This is obvious if $n=1$ or 2. Suppose then that it is true for sets of $n-1$ elements. Take two ordered lists of numbers, $P_{1}, P_{2}$. Make one switch in both to place $n$ at the end. Call the result $P_{1}^{n}$ and $P_{2}^{n}$. Then using induction, there are finitely many switches in $P_{1}^{n}$ so that it will coincide with $P_{2}^{n}$. Now switch the $n$ in what results to where it was in $P_{2}$.

To see $\operatorname{sgn}_{n}$ is unique, if there exist two functions, $f$ and $g$ both satisfying 5.3.4 and 5.3.5, you could start with $f(1, \cdots, n)=g(1, \cdots, n)=1$ and applying the same sequence of switches, eventually arrive at $f\left(i_{1}, \cdots, i_{n}\right)=g\left(i_{1}, \cdots, i_{n}\right)$. If any numbers are repeated, then 5.3.5 gives both functions are equal to zero for that ordered list.

Definition 5.3.3 When you have an ordered list of distinct numbers from $\{1,2, \cdots, n\}$, say

$$
\left(i_{1}, \cdots, i_{n}\right)
$$

this ordered list is called a permutation. The symbol for all such permutations is $S_{n}$. The number $\operatorname{sgn}_{n}\left(i_{1}, \cdots, i_{n}\right)$ is called the sign of the permutation.

A permutation can also be considered as a function from the set

$$
\{1,2, \cdots, n\} \text { to }\{1,2, \cdots, n\}
$$

as follows. Let $f(k)=i_{k}$. Permutations are of fundamental importance in certain areas of math. For example, it was by considering permutations that Galois was able to give a criterion for solution of polynomial equations by radicals, but this is a different direction than what is being attempted here.

In what follows sgn will often be used rather than $\operatorname{sgn}_{n}$ because the context supplies the appropriate $n$.

### 5.4 The Determinant

Definition 5.4.1 Let $f$ be a function which has the set of ordered lists of numbers from $\{1, \cdots, n\}$ as its domain. Define

$$
\sum_{\left(k_{1}, \cdots, k_{n}\right)} f\left(k_{1} \cdots k_{n}\right)
$$

to be the sum of all the $f\left(k_{1} \cdots k_{n}\right)$ for all possible choices of ordered lists $\left(k_{1}, \cdots, k_{n}\right)$ of numbers of $\{1, \cdots, n\}$. For example,

$$
\sum_{\left(k_{1}, k_{2}\right)} f\left(k_{1}, k_{2}\right)=f(1,2)+f(2,1)+f(1,1)+f(2,2) .
$$

### 5.4.1 The Definition

Definition 5.4.2 Let $\left(a_{i j}\right)=A$ denote an $n \times n$ matrix. The determinant of $A$, denoted by $\operatorname{det}(A)$ is defined by

$$
\operatorname{det}(A) \equiv \sum_{\left(k_{1}, \cdots, k_{n}\right)} \operatorname{sgn}\left(k_{1}, \cdots, k_{n}\right) a_{1 k_{1}} \cdots a_{n k_{n}}
$$

where the sum is taken over all ordered lists of numbers from $\{1, \cdots, n\}$. Note it suffices to take the sum over only those ordered lists in which there are no repeats because if there are, $\operatorname{sgn}\left(k_{1}, \cdots, k_{n}\right)=0$ and so that term contributes 0 to the sum.

### 5.4.2 Permuting Rows Or Columns

Let $A$ be an $n \times n$ matrix, $A=\left(a_{i j}\right)$ and let $\left(r_{1}, \cdots, r_{n}\right)$ denote an ordered list of $n$ numbers from $\{1, \cdots, n\}$. Let $A\left(r_{1}, \cdots, r_{n}\right)$ denote the matrix whose $k^{\text {th }}$ row is the $r_{k}$ row of the matrix $A$. Thus

$$
\begin{equation*}
\operatorname{det}\left(A\left(r_{1}, \cdots, r_{n}\right)\right)=\sum_{\left(k_{1}, \cdots, k_{n}\right)} \operatorname{sgn}\left(k_{1}, \cdots, k_{n}\right) a_{r_{1} k_{1}} \cdots a_{r_{n} k_{n}} \tag{5.4.7}
\end{equation*}
$$

and

$$
A(1, \cdots, n)=A
$$

Proposition 5.4.3 Let

$$
\left(r_{1}, \cdots, r_{n}\right)
$$

be an ordered list of numbers from $\{1, \cdots, n\}$. Then

$$
\begin{align*}
& \operatorname{sgn}\left(r_{1}, \cdots, r_{n}\right) \operatorname{det}(A) \\
= & \sum_{\left(k_{1}, \cdots, k_{n}\right)} \operatorname{sgn}\left(k_{1}, \cdots, k_{n}\right) a_{r_{1} k_{1}} \cdots a_{r_{n} k_{n}}  \tag{5.4.8}\\
= & \operatorname{det}\left(A\left(r_{1}, \cdots, r_{n}\right)\right) \tag{5.4.9}
\end{align*}
$$

Proof: Let $(1, \cdots, n)=(1, \cdots, r, \cdots s, \cdots, n)$ so $r<s$.

$$
\begin{equation*}
\operatorname{det}(A(1, \cdots, r, \cdots, s, \cdots, n))= \tag{5.4.10}
\end{equation*}
$$

$$
\sum_{\left(k_{1}, \cdots, k_{n}\right)} \operatorname{sgn}\left(k_{1}, \cdots, k_{r}, \cdots, k_{s}, \cdots, k_{n}\right) a_{1 k_{1}} \cdots a_{r k_{r}} \cdots a_{s k_{s}} \cdots a_{n k_{n}}
$$

and renaming the variables, calling $k_{s}, k_{r}$ and $k_{r}, k_{s}$, this equals

$$
\begin{gather*}
=\sum_{\left(k_{1}, \cdots, k_{n}\right)} \operatorname{sgn}\left(k_{1}, \cdots, k_{s}, \cdots, k_{r}, \cdots, k_{n}\right) a_{1 k_{1}} \cdots a_{r k_{s}} \cdots a_{s k_{r}} \cdots a_{n k_{n}} \\
=\sum_{\left(k_{1}, \cdots, k_{n}\right)}-\operatorname{sgn}(k_{1}, \cdots, \overbrace{k_{r}, \cdots, k_{s}}^{\text {These got switched }}, \cdots, k_{n}) a_{1 k_{1}} \cdots a_{s k_{r}} \cdots a_{r k_{s}} \cdots a_{n k_{n}} \\
=-\operatorname{det}(A(1, \cdots, s, \cdots, r, \cdots, n)) \tag{5.4.11}
\end{gather*}
$$

Consequently,

$$
\begin{gathered}
\operatorname{det}(A(1, \cdots, s, \cdots, r, \cdots, n))= \\
-\operatorname{det}(A(1, \cdots, r, \cdots, s, \cdots, n))=-\operatorname{det}(A)
\end{gathered}
$$

Now letting $A(1, \cdots, s, \cdots, r, \cdots, n)$ play the role of $A$, and continuing in this way, switching pairs of numbers,

$$
\operatorname{det}\left(A\left(r_{1}, \cdots, r_{n}\right)\right)=(-1)^{p} \operatorname{det}(A)
$$

where it took $p$ switches to obtain $\left(r_{1}, \cdots, r_{n}\right)$ from $(1, \cdots, n)$. By Lemma 5.3.1, this implies

$$
\operatorname{det}\left(A\left(r_{1}, \cdots, r_{n}\right)\right)=(-1)^{p} \operatorname{det}(A)=\operatorname{sgn}\left(r_{1}, \cdots, r_{n}\right) \operatorname{det}(A)
$$

and proves the proposition in the case when there are no repeated numbers in the ordered list, $\left(r_{1}, \cdots, r_{n}\right)$.However, if there is a repeat, say the $r^{\text {th }}$ row equals the $s^{\text {th }}$ row, then the reasoning of 5.4.10 -5.4.11 shows that $\operatorname{det} A\left(r_{1}, \cdots, r_{n}\right)=0$ and also $\operatorname{sgn}\left(r_{1}, \cdots, r_{n}\right)=0$ so the formula holds in this case also.

Observation 5.4.4 There are $n!$ ordered lists of distinct numbers from $\{1, \cdots, n\}$.
To see this, consider $n$ slots placed in order. There are $n$ choices for the first slot. For each of these choices, there are $n-1$ choices for the second. Thus there are $n(n-1)$ ways to fill the first two slots. Then for each of these ways there are $n-2$ choices left for the third slot. Continuing this way, there are $n$ ! ordered lists of distinct numbers from $\{1, \cdots, n\}$ as stated in the observation.

### 5.4.3 A Symmetric Definition

With the above, it is possible to give a more symmetric description of the determinant from which it will follow that $\operatorname{det}(A)=\operatorname{det}\left(A^{T}\right)$.

Corollary 5.4.5 The following formula for $\operatorname{det}(A)$ is valid.

$$
\begin{gather*}
\operatorname{det}(A)=\frac{1}{n!} \\
\sum_{\left(r_{1}, \cdots, r_{n}\right)} \sum_{\left(k_{1}, \cdots, k_{n}\right)} \operatorname{sgn}\left(r_{1}, \cdots, r_{n}\right) \operatorname{sgn}\left(k_{1}, \cdots, k_{n}\right) a_{r_{1} k_{1}} \cdots a_{r_{n} k_{n}} \tag{5.4.12}
\end{gather*}
$$

And also $\operatorname{det}\left(A^{T}\right)=\operatorname{det}(A)$ where $A^{T}$ is the transpose of $A$. (Recall that for $A^{T}=\left(a_{i j}^{T}\right)$, $a_{i j}^{T}=a_{j i}$.)

Proof: From Proposition 5.4.3, if the $r_{i}$ are distinct,

$$
\operatorname{det}(A)=\sum_{\left(k_{1}, \cdots, k_{n}\right)} \operatorname{sgn}\left(r_{1}, \cdots, r_{n}\right) \operatorname{sgn}\left(k_{1}, \cdots, k_{n}\right) a_{r_{1} k_{1}} \cdots a_{r_{n} k_{n}}
$$

Summing over all ordered lists, $\left(r_{1}, \cdots, r_{n}\right)$ where the $r_{i}$ are distinct, (If the $r_{i}$ are not distinct, $\operatorname{sgn}\left(r_{1}, \cdots, r_{n}\right)=0$ and so there is no contribution to the sum.)

$$
\begin{gathered}
n!\operatorname{det}(A)= \\
\sum_{\left(r_{1}, \cdots, r_{n}\right)} \sum_{\left(k_{1}, \cdots, k_{n}\right)} \operatorname{sgn}\left(r_{1}, \cdots, r_{n}\right) \operatorname{sgn}\left(k_{1}, \cdots, k_{n}\right) a_{r_{1} k_{1}} \cdots a_{r_{n} k_{n}} .
\end{gathered}
$$

This proves the corollary since the formula gives the same number for $A$ as it does for $A^{T}$.

### 5.4.4 The Alternating Property Of The Determinant

Corollary 5.4.6 If two rows or two columns in an $n \times n$ matrix $A$, are switched, the determinant of the resulting matrix equals $(-1)$ times the determinant of the original matrix. If $A$ is an $n \times n$ matrix in which two rows are equal or two columns are equal then $\operatorname{det}(A)=0$. Suppose the $i^{\text {th }}$ row of $A$ equals $\left(x a_{1}+y b_{1}, \cdots, x a_{n}+y b_{n}\right)$. Then

$$
\operatorname{det}(A)=x \operatorname{det}\left(A_{1}\right)+y \operatorname{det}\left(A_{2}\right)
$$

where the $i^{\text {th }}$ row of $A_{1}$ is $\left(a_{1}, \cdots, a_{n}\right)$ and the $i^{\text {th }}$ row of $A_{2}$ is $\left(b_{1}, \cdots, b_{n}\right)$, all other rows of $A_{1}$ and $A_{2}$ coinciding with those of $A$. In other words, det is a linear function of each row A. The same is true with the word "row" replaced with the word "column".

Proof: By Proposition 5.4 .3 when two rows are switched, the determinant of the resulting matrix is $(-1)$ times the determinant of the original matrix. By Corollary 5.4.5 the same holds for columns because the columns of the matrix equal the rows of the transposed matrix. Thus if $A_{1}$ is the matrix obtained from $A$ by switching two columns,

$$
\operatorname{det}(A)=\operatorname{det}\left(A^{T}\right)=-\operatorname{det}\left(A_{1}^{T}\right)=-\operatorname{det}\left(A_{1}\right) .
$$

If $A$ has two equal columns or two equal rows, then switching them results in the same matrix. Therefore, $\operatorname{det}(A)=-\operatorname{det}(A)$ and $\operatorname{so} \operatorname{det}(A)=0$.

It remains to verify the last assertion.

$$
\begin{aligned}
& \operatorname{det}(A) \equiv \sum_{\left(k_{1}, \cdots, k_{n}\right)} \operatorname{sgn}\left(k_{1}, \cdots, k_{n}\right) a_{1 k_{1}} \cdots\left(x a_{k_{i}}+y b_{k_{i}}\right) \cdots a_{n k_{n}} \\
&=x \sum_{\left(k_{1}, \cdots, k_{n}\right)} \operatorname{sgn}\left(k_{1}, \cdots, k_{n}\right) a_{1 k_{1}} \cdots a_{k_{i}} \cdots a_{n k_{n}} \\
&+y \sum_{\left(k_{1}, \cdots, k_{n}\right)} \operatorname{sgn}\left(k_{1}, \cdots, k_{n}\right) a_{1 k_{1}} \cdots b_{k_{i}} \cdots a_{n k_{n}} \\
& \equiv x \operatorname{det}\left(A_{1}\right)+y \operatorname{det}\left(A_{2}\right) .
\end{aligned}
$$

The same is true of columns because $\operatorname{det}\left(A^{T}\right)=\operatorname{det}(A)$ and the rows of $A^{T}$ are the columns of $A$.

### 5.4.5 Linear Combinations And Determinants

Linear combinations have been discussed already. However, here is a review and some new terminology.

Definition 5.4.7 A vector $\mathbf{w}$, is a linear combination of the vectors $\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{r}\right\}$ if there exists scalars, $c_{1}, \cdots c_{r}$ such that $\mathbf{w}=\sum_{k=1}^{r} c_{k} \mathbf{v}_{k}$. This is the same as saying

$$
\mathbf{w} \in \operatorname{span}\left(\mathbf{v}_{1}, \cdots, \mathbf{v}_{r}\right)
$$

The following corollary is also of great use.
Corollary 5.4.8 Suppose $A$ is an $n \times n$ matrix and some column (row) is a linear combination of $r$ other columns (rows). Then $\operatorname{det}(A)=0$.

Proof: Let $A=\left(\begin{array}{lll}\mathbf{a}_{1} & \cdots & \mathbf{a}_{n}\end{array}\right)$ be the columns of $A$ and suppose the condition that one column is a linear combination of $r$ of the others is satisfied. Then by using Corollary 5.4.6 the determinant of $A$ is zero if and only if the determinant of the matrix $B$, which has this special column placed in the last position, equals zero. Thus $\mathbf{a}_{n}=\sum_{k=1}^{r} c_{k} \mathbf{a}_{k}$ and so

$$
\operatorname{det}(B)=\operatorname{det}\left(\begin{array}{llllll}
\mathbf{a}_{1} & \cdots & \mathbf{a}_{r} & \cdots & \mathbf{a}_{n-1} & \sum_{k=1}^{r} c_{k} \mathbf{a}_{k}
\end{array}\right)
$$

By Corollary 5.4.6

$$
\operatorname{det}(B)=\sum_{k=1}^{r} c_{k} \operatorname{det}\left(\begin{array}{llllll}
\mathbf{a}_{1} & \cdots & \mathbf{a}_{r} & \cdots & \mathbf{a}_{n-1} & \mathbf{a}_{k}
\end{array}\right)=0
$$

because there are two equal columns. The case for rows follows from the fact that $\operatorname{det}(A)=$ $\operatorname{det}\left(A^{T}\right)$.

### 5.4.6 The Determinant Of A Product

Recall the following definition of matrix multiplication.
Definition 5.4.9 If $A$ and $B$ are $n \times n$ matrices, $A=\left(a_{i j}\right)$ and $B=\left(b_{i j}\right), A B=\left(c_{i j}\right)$ where

$$
c_{i j} \equiv \sum_{k=1}^{n} a_{i k} b_{k j} .
$$

One of the most important rules about determinants is that the determinant of a product equals the product of the determinants.

Theorem 5.4.10 Let $A$ and $B$ be $n \times n$ matrices. Then

$$
\operatorname{det}(A B)=\operatorname{det}(A) \operatorname{det}(B)
$$

Proof: Let $c_{i j}$ be the $i j^{t h}$ entry of $A B$. Then by Proposition 5.4.3,

$$
\left.\begin{array}{c}
\operatorname{det}(A B)= \\
=\sum_{\left(k_{1}, \cdots, k_{n}\right)} \operatorname{sgn}\left(k_{1}, \cdots, k_{n}\right) c_{1 k_{1}} \cdots c_{n k_{n}} \\
=\sum_{\left(k_{1}, \cdots, k_{n}\right)} \operatorname{sgn}\left(k_{1}, \cdots, k_{n}\right)\left(\sum_{r_{1}} a_{1 r_{1}} b_{r_{1} k_{1}}\right) \cdots\left(\sum_{r_{n}} a_{n r_{n}} b_{r_{n} k_{n}}\right) \\
=\sum_{\left(r_{1} \cdots, r_{n}\right)} \operatorname{sgn}\left(k_{1}, \cdots, k_{n}\right) b_{r_{1} k_{1}} \cdots b_{r_{n} k_{n}}\left(a_{1 r_{1}} \cdots a_{n r_{n}}\right) \\
\operatorname{sgn}\left(r_{1} \cdots, r_{n}\right)
\end{array}\right) a_{1 r_{1}} \cdots a_{n r_{n}} \operatorname{det}(B)=\operatorname{det}(A) \operatorname{det}(B) . I .
$$

### 5.4.7 Cofactor Expansions

Lemma 5.4.11 Suppose a matrix is of the form

$$
M=\left(\begin{array}{cc}
A & *  \tag{5.4.13}\\
\mathbf{0} & a
\end{array}\right)
$$

or

$$
M=\left(\begin{array}{cc}
A & \mathbf{0}  \tag{5.4.14}\\
* & a
\end{array}\right)
$$

where $a$ is a number and $A$ is an $(n-1) \times(n-1)$ matrix and $*$ denotes either a column or a row having length $n-1$ and the 0 denotes either a column or a row of length $n-1$ consisting entirely of zeros. Then $\operatorname{det}(M)=a \operatorname{det}(A)$.

Proof: Denote $M$ by $\left(m_{i j}\right)$. Thus in the first case, $m_{n n}=a$ and $m_{n i}=0$ if $i \neq n$ while in the second case, $m_{n n}=a$ and $m_{i n}=0$ if $i \neq n$. From the definition of the determinant,

$$
\operatorname{det}(M) \equiv \sum_{\left(k_{1}, \cdots, k_{n}\right)} \operatorname{sgn}_{n}\left(k_{1}, \cdots, k_{n}\right) m_{1 k_{1}} \cdots m_{n k_{n}}
$$

Letting $\theta$ denote the position of $n$ in the ordered list, $\left(k_{1}, \cdots, k_{n}\right)$ then using Lemma 5.3.1, $\operatorname{det}(M)$ equals

$$
\sum_{\left(k_{1}, \cdots, k_{n}\right)}(-1)^{n-\theta} \operatorname{sgn}_{n-1}\left(k_{1}, \cdots, k_{\theta-1}, \stackrel{\theta}{k_{\theta+1}}, \cdots, \stackrel{n-1}{k_{n}}\right) m_{1 k_{1}} \cdots m_{n k_{n}}
$$

Now suppose 5.4.14. Then if $k_{n} \neq n$, the term involving $m_{n k_{n}}$ in the above expression equals zero. Therefore, the only terms which survive are those for which $\theta=n$ or in other words, those for which $k_{n}=n$. Therefore, the above expression reduces to

$$
a \sum_{\left(k_{1}, \cdots, k_{n-1}\right)} \operatorname{sgn}_{n-1}\left(k_{1}, \cdots k_{n-1}\right) m_{1 k_{1}} \cdots m_{(n-1) k_{n-1}}=a \operatorname{det}(A) .
$$

To get the assertion in the situation of 5.4.13 use Corollary 5.4.5 and 5.4.14 to write

$$
\operatorname{det}(M)=\operatorname{det}\left(M^{T}\right)=\operatorname{det}\left(\left(\begin{array}{cc}
A^{T} & \mathbf{0} \\
* & a
\end{array}\right)\right)=a \operatorname{det}\left(A^{T}\right)=a \operatorname{det}(A)
$$

In terms of the theory of determinants, arguably the most important idea is that of Laplace expansion along a row or a column. This will follow from the above definition of a determinant.

Definition 5.4.12 Let $A=\left(a_{i j}\right)$ be an $n \times n$ matrix. Then a new matrix called the cofactor matrix, $\operatorname{cof}(A)$ is defined by $\operatorname{cof}(A)=\left(c_{i j}\right)$ where to obtain $c_{i j}$ delete the $i^{\text {th }}$ row and the $j^{\text {th }}$ column of $A$, take the determinant of the $(n-1) \times(n-1)$ matrix which results, (This is called the $i j^{\text {th }}$ minor of $A$.) and then multiply this number by $(-1)^{i+j}$. To make the formulas easier to remember, $\operatorname{cof}(A)_{i j}$ will denote the $i j^{\text {th }}$ entry of the cofactor matrix.

The following is the main result. Earlier this was given as a definition and the outrageous totally unjustified assertion was made that the same number would be obtained by expanding the determinant along any row or column. The following theorem proves this assertion.

Theorem 5.4.13 Let $A$ be an $n \times n$ matrix where $n \geq 2$. Then

$$
\begin{equation*}
\operatorname{det}(A)=\sum_{j=1}^{n} a_{i j} \operatorname{cof}(A)_{i j}=\sum_{i=1}^{n} a_{i j} \operatorname{cof}(A)_{i j} . \tag{5.4.15}
\end{equation*}
$$

The first formula consists of expanding the determinant along the $i^{\text {th }}$ row and the second expands the determinant along the $j^{\text {th }}$ column.

Proof: Let $\left(a_{i 1}, \cdots, a_{i n}\right)$ be the $i^{\text {th }}$ row of $A$. Let $B_{j}$ be the matrix obtained from $A$ by leaving every row the same except the $i^{t h}$ row which in $B_{j}$ equals

$$
\left(0, \cdots, 0, a_{i j}, 0, \cdots, 0\right)
$$

Then by Corollary 5.4.6,

$$
\operatorname{det}(A)=\sum_{j=1}^{n} \operatorname{det}\left(B_{j}\right)
$$

Denote by $A^{i j}$ the $(n-1) \times(n-1)$ matrix obtained by deleting the $i^{\text {th }}$ row and the $j^{\text {th }}$ column of $A$. Thus $\operatorname{cof}(A)_{i j} \equiv(-1)^{i+j} \operatorname{det}\left(A^{i j}\right)$. At this point, recall that from Proposition 5.4.3, when two rows or two columns in a matrix $M$, are switched, this results in multiplying the determinant of the old matrix by -1 to get the determinant of the new matrix. Therefore, by Lemma 5.4.11,

$$
\begin{aligned}
\operatorname{det}\left(B_{j}\right) & =(-1)^{n-j}(-1)^{n-i} \operatorname{det}\left(\left(\begin{array}{cc}
A^{i j} & * \\
\mathbf{0} & a_{i j}
\end{array}\right)\right) \\
& =(-1)^{i+j} \operatorname{det}\left(\left(\begin{array}{cc}
A^{i j} & * \\
\mathbf{0} & a_{i j}
\end{array}\right)\right)=a_{i j} \operatorname{cof}(A)_{i j}
\end{aligned}
$$

Therefore,

$$
\operatorname{det}(A)=\sum_{j=1}^{n} a_{i j} \operatorname{cof}(A)_{i j}
$$

which is the formula for expanding $\operatorname{det}(A)$ along the $i^{\text {th }}$ row. Also,

$$
\begin{aligned}
\operatorname{det}(A) & =\operatorname{det}\left(A^{T}\right)=\sum_{j=1}^{n} a_{i j}^{T} \operatorname{cof}\left(A^{T}\right)_{i j} \\
& =\sum_{j=1}^{n} a_{j i} \operatorname{cof}(A)_{j i}
\end{aligned}
$$

which is the formula for expanding $\operatorname{det}(A)$ along the $i^{\text {th }}$ column.

### 5.4.8 Formula For The Inverse

Note that this gives an easy way to write a formula for the inverse of an $n \times n$ matrix.
Theorem 5.4.14 $A^{-1}$ exists if and only if $\operatorname{det}(A) \neq 0$. If $\operatorname{det}(A) \neq 0$, then $A^{-1}=\left(a_{i j}^{-1}\right)$ where

$$
a_{i j}^{-1}=\operatorname{det}(A)^{-1} \operatorname{cof}(A)_{j i}
$$

for $\operatorname{cof}(A)_{i j}$ the $i j^{\text {th }}$ cofactor of $A$.
Proof: By Theorem 5.4.13 and letting $\left(a_{i r}\right)=A$, if $\operatorname{det}(A) \neq 0$,

$$
\sum_{i=1}^{n} a_{i r} \operatorname{cof}(A)_{i r} \operatorname{det}(A)^{-1}=\operatorname{det}(A) \operatorname{det}(A)^{-1}=1
$$

Now consider

$$
\sum_{i=1}^{n} a_{i r} \operatorname{cof}(A)_{i k} \operatorname{det}(A)^{-1}
$$

when $k \neq r$. Replace the $k^{t h}$ column with the $r^{t h}$ column to obtain a matrix $B_{k}$ whose determinant equals zero by Corollary 5.4.6. However, expanding this matrix along the $k^{t h}$ column yields

$$
0=\operatorname{det}\left(B_{k}\right) \operatorname{det}(A)^{-1}=\sum_{i=1}^{n} a_{i r} \operatorname{cof}(A)_{i k} \operatorname{det}(A)^{-1}
$$

Summarizing,

$$
\sum_{i=1}^{n} a_{i r} \operatorname{cof}(A)_{i k} \operatorname{det}(A)^{-1}=\delta_{r k}
$$

Using the other formula in Theorem 5.4.13, and similar reasoning,

$$
\sum_{j=1}^{n} a_{r j} \operatorname{cof}(A)_{k j} \operatorname{det}(A)^{-1}=\delta_{r k}
$$

This proves that if $\operatorname{det}(A) \neq 0$, then $A^{-1}$ exists with $A^{-1}=\left(a_{i j}^{-1}\right)$, where

$$
a_{i j}^{-1}=\operatorname{cof}(A)_{j i} \operatorname{det}(A)^{-1}
$$

Now suppose $A^{-1}$ exists. Then by Theorem 5.4.10,

$$
1=\operatorname{det}(I)=\operatorname{det}\left(A A^{-1}\right)=\operatorname{det}(A) \operatorname{det}\left(A^{-1}\right)
$$

so $\operatorname{det}(A) \neq 0$.
The next corollary points out that if an $n \times n$ matrix $A$ has a right or a left inverse, then it has an inverse.

Corollary 5.4.15 Let $A$ be an $n \times n$ matrix and suppose there exists an $n \times n$ matrix $B$ such that $B A=I$. Then $A^{-1}$ exists and $A^{-1}=B$. Also, if there exists $C$ an $n \times n$ matrix such that $A C=I$, then $A^{-1}$ exists and $A^{-1}=C$.

Proof: Since $B A=I$, Theorem 5.4.10 implies

$$
\operatorname{det} B \operatorname{det} A=1
$$

and so $\operatorname{det} A \neq 0$. Therefore from Theorem 5.4.14, $A^{-1}$ exists. Therefore,

$$
A^{-1}=(B A) A^{-1}=B\left(A A^{-1}\right)=B I=B
$$

The case where $C A=I$ is handled similarly.
The conclusion of this corollary is that left inverses, right inverses and inverses are all the same in the context of $n \times n$ matrices.

Theorem 5.4.14 says that to find the inverse, take the transpose of the cofactor matrix and divide by the determinant. The transpose of the cofactor matrix is called the adjugate or sometimes the classical adjoint of the matrix $A$. It is an abomination to call it the adjoint although you do sometimes see it referred to in this way. In words, $A^{-1}$ is equal to one over the determinant of $A$ times the adjugate matrix of $A$.

### 5.4.9 Cramer's Rule

In case you are solving a system of equations, $A \mathbf{x}=\mathbf{y}$ for $\mathbf{x}$, it follows that if $A^{-1}$ exists,

$$
\mathbf{x}=\left(A^{-1} A\right) \mathbf{x}=A^{-1}(A \mathbf{x})=A^{-1} \mathbf{y}
$$

thus solving the system. Now in the case that $A^{-1}$ exists, there is a formula for $A^{-1}$ given above. Using this formula,

$$
x_{i}=\sum_{j=1}^{n} a_{i j}^{-1} y_{j}=\sum_{j=1}^{n} \frac{1}{\operatorname{det}(A)} \operatorname{cof}(A)_{j i} y_{j} .
$$

By the formula for the expansion of a determinant along a column,

$$
x_{i}=\frac{1}{\operatorname{det}(A)} \operatorname{det}\left(\begin{array}{ccccc}
* & \cdots & y_{1} & \cdots & * \\
\vdots & & \vdots & & \vdots \\
* & \cdots & y_{n} & \cdots & *
\end{array}\right)
$$

where here the $i^{\text {th }}$ column of $A$ is replaced with the column vector $\left(y_{1} \cdots, y_{n}\right)^{T}$, and the determinant of this modified matrix is taken and divided by $\operatorname{det}(A)$. This formula is known as Cramer's rule.

### 5.4.10 Upper Triangular Matrices

Definition 5.4.16 A matrix $M$, is upper triangular if $M_{i j}=0$ whenever $i>j$. Thus such a matrix equals zero below the main diagonal, the entries of the form $M_{i i}$ as shown.

$$
\left(\begin{array}{cccc}
* & * & \cdots & * \\
0 & * & \ddots & \vdots \\
\vdots & \ddots & \ddots & * \\
0 & \cdots & 0 & *
\end{array}\right)
$$

A lower triangular matrix is defined similarly as a matrix for which all entries above the main diagonal are equal to zero.

With this definition, here is a simple corollary of Theorem 5.4.13.
Corollary 5.4.17 Let $M$ be an upper (lower) triangular matrix. Then $\operatorname{det}(M)$ is obtained by taking the product of the entries on the main diagonal.

### 5.5 The Cayley Hamilton Theorem*

Definition 5.5.1 Let A be an $n \times n$ matrix. The characteristic polynomial is defined as

$$
q_{A}(t) \equiv \operatorname{det}(t I-A)
$$

and the solutions to $q_{A}(t)=0$ are called eigenvalues. For A matrix and $p(t)=t^{n}+$ $a_{n-1} t^{n-1}+\cdots+a_{1} t+a_{0}$, denote by $p(A)$ the matrix defined by

$$
p(A) \equiv A^{n}+a_{n-1} A^{n-1}+\cdots+a_{1} A+a_{0} I .
$$

The explanation for the last term is that $A^{0}$ is interpreted as I, the identity matrix.

The Cayley Hamilton theorem states that every matrix satisfies its characteristic equation, that equation defined by $q_{A}(t)=0$. It is one of the most important theorems in linear algebra ${ }^{3}$. The proof in this section is not the most general proof, but works well when the field of scalars is $\mathbb{R}$ or $\mathbb{C}$. The following lemma will help with its proof.

Lemma 5.5.2 Suppose for all $|\lambda|$ large enough,

$$
A_{0}+A_{1} \lambda+\cdots+A_{m} \lambda^{m}=0
$$

where the $A_{i}$ are $n \times n$ matrices. Then each $A_{i}=0$.
Proof: Multiply by $\lambda^{-m}$ to obtain

$$
A_{0} \lambda^{-m}+A_{1} \lambda^{-m+1}+\cdots+A_{m-1} \lambda^{-1}+A_{m}=0
$$

Now let $|\lambda| \rightarrow \infty$ to obtain $A_{m}=0$. With this, multiply by $\lambda$ to obtain

$$
A_{0} \lambda^{-m+1}+A_{1} \lambda^{-m+2}+\cdots+A_{m-1}=0
$$

Now let $|\lambda| \rightarrow \infty$ to obtain $A_{m-1}=0$. Continue multiplying by $\lambda$ and letting $\lambda \rightarrow \infty$ to obtain that all the $A_{i}=0$.

With the lemma, here is a simple corollary.
Corollary 5.5.3 Let $A_{i}$ and $B_{i}$ be $n \times n$ matrices and suppose

$$
A_{0}+A_{1} \lambda+\cdots+A_{m} \lambda^{m}=B_{0}+B_{1} \lambda+\cdots+B_{m} \lambda^{m}
$$

for all $|\lambda|$ large enough. Then $A_{i}=B_{i}$ for all $i$. If $A_{i}=B_{i}$ for each $A_{i}, B_{i}$ then one can substitute an $n \times n$ matrix $M$ for $\lambda$ and the identity will continue to hold.

Proof: Subtract and use the result of the lemma. The last claim is obvious by matching terms.

With this preparation, here is a relatively easy proof of the Cayley Hamilton theorem.
Theorem 5.5.4 Let $A$ be an $n \times n$ matrix and let $q(\lambda) \equiv \operatorname{det}(\lambda I-A)$ be the characteristic polynomial. Then $q(A)=0$.

Proof: Let $C(\lambda)$ equal the transpose of the cofactor matrix of $(\lambda I-A)$ for $|\lambda|$ large. (If $|\lambda|$ is large enough, then $\lambda$ cannot be in the finite list of eigenvalues of $A$ and so for such $\lambda,(\lambda I-A)^{-1}$ exists.) Therefore, by Theorem 5.4.14

$$
C(\lambda)=q(\lambda)(\lambda I-A)^{-1}
$$

Say

$$
q(\lambda)=a_{0}+a_{1} \lambda+\cdots+\lambda^{n}
$$

[^4]Note that each entry in $C(\lambda)$ is a polynomial in $\lambda$ having degree no more than $n-1$. For example, you might have something like

$$
\begin{gathered}
C(\lambda)=\left(\begin{array}{ccc}
\lambda^{2}-6 \lambda+9 & 3-\lambda & 0 \\
2 \lambda-6 & \lambda^{2}-3 \lambda & 0 \\
\lambda-1 & \lambda-1 & \lambda^{2}-3 \lambda+2
\end{array}\right) \\
=\left(\begin{array}{ccc}
9 & 3 & 0 \\
-6 & 0 & 0 \\
-1 & -1 & 2
\end{array}\right)+\lambda\left(\begin{array}{ccc}
-6 & -1 & 0 \\
2 & -3 & 0 \\
1 & 1 & -3
\end{array}\right)+\lambda^{2}\left(\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right)
\end{gathered}
$$

Therefore, collecting the terms in the general case,

$$
C(\lambda)=C_{0}+C_{1} \lambda+\cdots+C_{n-1} \lambda^{n-1}
$$

for $C_{j}$ some $n \times n$ matrix. Then

$$
C(\lambda)(\lambda I-A)=\left(C_{0}+C_{1} \lambda+\cdots+C_{n-1} \lambda^{n-1}\right)(\lambda I-A)=q(\lambda) I
$$

Then multiplying out the middle term, it follows that for all $|\lambda|$ sufficiently large,

$$
\begin{gathered}
a_{0} I+a_{1} I \lambda+\cdots+I \lambda^{n}=C_{0} \lambda+C_{1} \lambda^{2}+\cdots+C_{n-1} \lambda^{n} \\
-\left[C_{0} A+C_{1} A \lambda+\cdots+C_{n-1} A \lambda^{n-1}\right] \\
=-C_{0} A+\left(C_{0}-C_{1} A\right) \lambda+\left(C_{1}-C_{2} A\right) \lambda^{2}+\cdots+\left(C_{n-2}-C_{n-1} A\right) \lambda^{n-1}+C_{n-1} \lambda^{n}
\end{gathered}
$$

Then, using Corollary 5.5.3, one can replace $\lambda$ on both sides with $A$. Then the right side is seen to equal 0 . Hence the left side, $q(A) I$ is also equal to 0 .

### 5.5.1 An Identity of Cauchy

Theorem 5.5.5 Both the left and the right sides in the following yield the same polynomial in the variables $a_{i}, b_{i}$ for $i \leq n$.

$$
\prod_{i, j}\left(a_{i}+b_{j}\right)\left|\begin{array}{ccc}
\frac{1}{a_{1}+b_{1}} & \cdots & \frac{1}{a_{1}+b_{n}}  \tag{5.5.16}\\
\vdots & & \vdots \\
\frac{1}{a_{n}+b_{1}} & \cdots & \frac{1}{a_{n}+b_{n}}
\end{array}\right|=\prod_{j<i}\left(a_{i}-a_{j}\right)\left(b_{i}-b_{j}\right)
$$

Proof: The theorem is true if $n=2$. This follows from some computations. Suppose it
is true for $n-1, n \geq 3$.

$$
\begin{aligned}
& \left|\begin{array}{cccc}
\frac{1}{a_{1}+b_{1}} & \frac{1}{a_{1}+b_{2}} & \cdots & \frac{1}{a_{1}+b_{n}} \\
\vdots & \vdots & \ldots & \vdots \\
\frac{1}{a_{n-1}+b_{1}} & \frac{1}{a_{n-1}+b_{2}} & & \frac{1}{a_{n-1}+b_{n}} \\
\frac{1}{a_{n}+b_{1}} & \frac{1}{a_{n}+b_{2}} & \cdots & \frac{1}{a_{n}+b_{n}}
\end{array}\right| \\
& \frac{a_{n}-a_{1}}{\left(a_{1}+b_{1}\right)\left(b_{1}+a_{n}\right)} \\
& \vdots \\
& \frac{a_{n}-a_{1}}{\left(a_{1}+b_{2}\right)\left(b_{2}+a_{n}\right)} \\
& \frac{a_{n}-1}{\left(a_{n-1}+b_{1}\right)\left(a_{n}+b_{1}\right)} \\
& \frac{1}{a_{n}+b_{1}}
\end{aligned}
$$

Continuing to use the multilinear properties of determinants, this equals

$$
\left|\begin{array}{cccc}
\frac{1}{\left(a_{1}+b_{1}\right)\left(b_{1}+a_{n}\right)} & \frac{1}{\left(a_{1}+b_{2}\right)\left(b_{2}+a_{n}\right)} & \cdots & \frac{1}{\left(a_{1}+b_{n}\right)\left(a_{n}+b_{n}\right)} \\
\vdots & \vdots & \cdots & \vdots \\
\frac{1}{\left(a_{n-1}+b_{1}\right)\left(a_{n}+b_{1}\right)} & \frac{1}{\left(b_{2}+a_{n}\right)\left(b_{2}+a_{n-1}\right)} & & \frac{1}{\left(a_{n}+b_{n}\right)\left(b_{n}+a_{n-1}\right)} \\
\frac{1}{a_{n}+b_{1}} & \frac{1}{a_{n}+b_{2}} & \cdots & \frac{1}{a_{n}+b_{n}}
\end{array}\right| \prod_{k=1}^{n-1}\left(a_{n}-a_{k}\right)
$$

and this equals

$$
\left|\begin{array}{cccc}
\frac{1}{\left(a_{1}+b_{1}\right)} & \frac{1}{\left(a_{1}+b_{2}\right)} & \cdots & \frac{1}{\left(a_{1}+b_{n}\right)} \\
\vdots & \vdots & \cdots & \vdots \\
\frac{1}{\left(a_{n-1}+b_{1}\right)} & \frac{1}{\left(b_{2}+a_{n-1}\right)} & & \frac{1}{\left(b_{n}+a_{n-1}\right)} \\
1 & 1 & \cdots & 1
\end{array}\right| \frac{\prod_{k=1}^{n-1}\left(a_{n}-a_{k}\right)}{\prod_{k=1}^{n}\left(a_{n}+b_{k}\right)}
$$

Now take -1 times the last column and add to each previous column. Thus it equals

$$
\left|\begin{array}{cccc|}
\frac{b_{n}-b_{1}}{\left(a_{1}+b_{1}\right)\left(a_{1}+b_{n}\right)} & \frac{b_{n}-b_{2}}{\left(a_{1}+b_{2}\right)\left(a_{1}+b_{n}\right)} & \cdots & \frac{1}{\left(a_{1}+b_{n}\right)} \\
\vdots & \vdots & \cdots & \vdots \\
\frac{b_{n}-b_{1}}{\left(b_{1}+a_{n-1}\right)\left(b_{n}+a_{n-1}\right)} & \frac{b_{n}-b_{2}}{\left(b_{2}+a_{n-1}\right)\left(b_{n}+a_{n-1}\right)} & & \frac{1}{\left(a_{n-1}+b_{n}\right)}
\end{array}\right| \begin{aligned}
& \prod_{k=1}^{n-1}\left(a_{n}-a_{k}\right) \\
& \prod_{k=1}^{n}\left(a_{n}+b_{k}\right) \\
& 0
\end{aligned}
$$

Now continue simplifying using the multilinear property of the determinant.

$$
\left|\begin{array}{cccc}
\frac{1}{\left(a_{1}+b_{1}\right)} & \frac{1}{\left(a_{1}+b_{2}\right)} & \cdots & 1 \\
\vdots & \vdots & \cdots & \vdots \\
\frac{1}{\left(b_{1}+a_{n-1}\right)} & \frac{1}{\left(b_{2}+a_{n-1}\right)} & & 1 \\
0 & 0 & \cdots & 1
\end{array}\right| \frac{\prod_{k=1}^{n-1}\left(a_{n}-a_{k}\right)}{\prod_{k=1}^{n}\left(a_{n}+b_{k}\right)} \frac{\prod_{k=1}^{n-1}\left(b_{n}-b_{k}\right)}{\prod_{k=1}^{n-1}\left(a_{k}+b_{n}\right)}
$$

Now, expanding along the bottom row, what has just resulted is

$$
\left|\begin{array}{ccc}
\frac{1}{a_{1}+b_{1}} & \cdots & \frac{1}{a_{1}+b_{n-1}} \\
\vdots & \cdots & \vdots \\
\frac{1}{a_{n-1}+b_{1}} & \cdots & \frac{1}{a_{n-1}+b_{n-1}}
\end{array}\right| \frac{\prod_{k=1}^{n-1}\left(a_{n}-a_{k}\right)}{\prod_{k=1}^{n}\left(a_{n}+b_{k}\right)} \frac{\prod_{k=1}^{n-1}\left(b_{n}-b_{k}\right)}{\prod_{k=1}^{n-1}\left(a_{k}+b_{n}\right)}
$$

By induction this equals

$$
\begin{gathered}
\frac{\prod_{k=1}^{n-1}\left(a_{n}-a_{k}\right)}{\prod_{k=1}^{n}\left(a_{n}+b_{k}\right)} \frac{\prod_{k=1}^{n-1}\left(b_{n}-b_{k}\right)}{\prod_{k=1}^{n-1}\left(a_{k}+b_{n}\right)} \frac{\prod_{j<i \leq n-1}\left(a_{i}-a_{j}\right)\left(b_{i}-b_{j}\right)}{\prod_{i, j \leq n-1}\left(a_{i}+b_{j}\right)} \\
=\frac{\prod_{j<i \leq n}\left(a_{i}-a_{j}\right)\left(b_{i}-b_{j}\right)}{\prod_{i, j \leq n}\left(a_{i}+b_{j}\right)}
\end{gathered}
$$

### 5.6 Block Multiplication Of Matrices

Consider the following problem

$$
\left(\begin{array}{ll}
A & B \\
C & D
\end{array}\right)\left(\begin{array}{ll}
E & F \\
G & H
\end{array}\right)
$$

You know how to do this. You get

$$
\left(\begin{array}{cc}
A E+B G & A F+B H \\
C E+D G & C F+D H
\end{array}\right)
$$

Now what if instead of numbers, the entries, $A, B, C, D, E, F, G$ are matrices of a size such that the multiplications and additions needed in the above formula all make sense. Would the formula be true in this case? I will show below that this is true.

Suppose $A$ is a matrix of the form

$$
A=\left(\begin{array}{ccc}
A_{11} & \cdots & A_{1 m}  \tag{5.6.17}\\
\vdots & \ddots & \vdots \\
A_{r 1} & \cdots & A_{r m}
\end{array}\right)
$$

where $A_{i j}$ is a $s_{i} \times p_{j}$ matrix where $s_{i}$ is constant for $j=1, \cdots, m$ for each $i=1, \cdots, r$. Such a matrix is called a block matrix, also a partitioned matrix. How do you get the block $A_{i j}$ ? Here is how for $A$ an $m \times n$ matrix:

$$
\overbrace{\left(\begin{array}{lll}
0 & I_{s_{i} \times s_{i}} & 0
\end{array}\right)}^{s_{i} \times m} A \overbrace{\left(\begin{array}{c}
0  \tag{5.6.18}\\
I_{p_{j} \times p_{j}} \\
0
\end{array}\right)}^{n \times p_{j}} .
$$

In the block column matrix on the right, you need to have $c_{j}-1$ rows of zeros above the small $p_{j} \times p_{j}$ identity matrix where the columns of $A$ involved in $A_{i j}$ are $c_{j}, \cdots, c_{j}+p_{j}$ and in the block row matrix on the left, you need to have $r_{i}-1$ columns of zeros to the left of the $s_{i} \times s_{i}$ identity matrix where the rows of $A$ involved in $A_{i j}$ are $r_{i}, \cdots, r_{i}+s_{i}$. An important observation to make is that the matrix on the right specifies columns to use in the block and the one on the left specifies the rows used. There is no overlap between the blocks of $A$. Thus the identity $n \times n$ identity matrix corresponding to multiplication on the right of $A$ is of the form

$$
\left(\begin{array}{ccc}
I_{p_{1} \times p_{1}} & & 0 \\
& \ddots & \\
0 & & I_{p_{m} \times p_{m}}
\end{array}\right)
$$

these little identity matrices don't overlap. A similar conclusion follows from consideration of the matrices $I_{s_{i} \times s_{i}}$.

Next consider the question of multiplication of two block matrices. Let $B$ be a block matrix of the form

$$
\left(\begin{array}{ccc}
B_{11} & \cdots & B_{1 p}  \tag{5.6.19}\\
\vdots & \ddots & \vdots \\
B_{r 1} & \cdots & B_{r p}
\end{array}\right)
$$

and $A$ is a block matrix of the form

$$
\left(\begin{array}{ccc}
A_{11} & \cdots & A_{1 m}  \tag{5.6.20}\\
\vdots & \ddots & \vdots \\
A_{p 1} & \cdots & A_{p m}
\end{array}\right)
$$

and that for all $i, j$, it makes sense to multiply $B_{i s} A_{s j}$ for all $s \in\{1, \cdots, p\}$. (That is the two matrices, $B_{i s}$ and $A_{s j}$ are conformable.) and that for fixed $i j$, it follows $B_{i s} A_{s j}$ is the same size for each $s$ so that it makes sense to write $\sum_{s} B_{i s} A_{s j}$.

The following theorem says essentially that when you take the product of two matrices, you can do it two ways. One way is to simply multiply them forming $B A$. The other way is to partition both matrices, formally multiply the blocks to get another block matrix and this one will be $B A$ partitioned. Before presenting this theorem, here is a simple lemma which is really a special case of the theorem.

Lemma 5.6.1 Consider the following product.

$$
\left(\begin{array}{l}
\mathbf{0} \\
I \\
\mathbf{0}
\end{array}\right)\left(\begin{array}{lll}
\mathbf{0} & I & \mathbf{0}
\end{array}\right)
$$

where the first is $n \times r$ and the second is $r \times n$. The small identity matrix $I$ is an $r \times r$ matrix and there are $l$ zero rows above $I$ and $l$ zero columns to the left of $I$ in the right matrix. Then the product of these matrices is a block matrix of the form

$$
\left(\begin{array}{lll}
\mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & I & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}\right)
$$

Proof: From the definition of the way you multiply matrices, the product is

$$
\left(\begin{array}{c}
\mathbf{0} \\
I \\
\mathbf{0}
\end{array}\right) \mathbf{0}
$$

which yields the claimed result. In the formula $\mathbf{e}_{j}$ referrs to the column vector of length $r$ which has a 1 in the $j^{t h}$ position. This proves the lemma.

Theorem 5.6.2 Let $B$ be a $q \times p$ block matrix as in 5.6 .19 and let $A$ be a $p \times n$ block matrix as in 5.6.20 such that $B_{i s}$ is conformable with $A_{s j}$ and each product, $B_{i s} A_{s j}$ for $s=1, \cdots, p$ is of the same size so they can be added. Then BA can be obtained as a block matrix such that the $i j^{\text {th }}$ block is of the form

$$
\begin{equation*}
\sum_{s} B_{i s} A_{s j} \tag{5.6.21}
\end{equation*}
$$

Proof: From 5.6.18

$$
B_{i s} A_{s j}=\left(\begin{array}{lll}
\mathbf{0} & I_{r_{i} \times r_{i}} & \mathbf{0}
\end{array}\right) B\left(\begin{array}{c}
\mathbf{0} \\
I_{p_{s} \times p_{s}} \\
\mathbf{0}
\end{array}\right)\left(\begin{array}{lll}
\mathbf{0} & I_{p_{s} \times p_{s}} & \mathbf{0}
\end{array}\right) A\left(\begin{array}{c}
\mathbf{0} \\
I_{q_{j} \times q_{j}} \\
\mathbf{0}
\end{array}\right)
$$

where here it is assumed $B_{i s}$ is $r_{i} \times p_{s}$ and $A_{s j}$ is $p_{s} \times q_{j}$. The product involves the $s^{\text {th }}$ block in the $i^{\text {th }}$ row of blocks for $B$ and the $s^{\text {th }}$ block in the $j^{\text {th }}$ column of $A$. Thus there are the same number of rows above the $I_{p_{s} \times p_{s}}$ as there are columns to the left of $I_{p_{s} \times p_{s}}$ in those two inside matrices. Then from Lemma 5.6.1

$$
\left(\begin{array}{c}
\mathbf{0} \\
I_{p_{s} \times p_{s}} \\
\mathbf{0}
\end{array}\right)\left(\begin{array}{lll}
\mathbf{0} & I_{p_{s} \times p_{s}} & \mathbf{0}
\end{array}\right)=\left(\begin{array}{ccc}
\mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & I_{p_{s} \times p_{s}} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}\right)
$$

Since the blocks of small identity matrices do not overlap,

$$
\sum_{s}\left(\begin{array}{ccc}
\mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & I_{p_{s} \times p_{s}} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}\right)=\left(\begin{array}{ccc}
I_{p_{1} \times p_{1}} & & 0 \\
& \ddots & \\
0 & & I_{p_{p} \times p_{p}}
\end{array}\right)=I
$$

and so

$$
\begin{gathered}
\sum_{s} B_{i s} A_{s j}= \\
\\
\sum_{s}\left(\begin{array}{lll}
\mathbf{0} & I_{r_{i} \times r_{i}} & \mathbf{0}
\end{array}\right) B\left(\begin{array}{c}
\mathbf{0} \\
I_{p_{s} \times p_{s}} \\
\mathbf{0}
\end{array}\right)\left(\begin{array}{lll}
\mathbf{0} & I_{p_{s} \times p_{s}} & \mathbf{0}
\end{array}\right) A\left(\begin{array}{c}
\mathbf{0} \\
I_{q_{j} \times q_{j}} \\
\mathbf{0}
\end{array}\right) \\
\left.\mathbf{0} \begin{array}{lll}
I_{r_{i} \times r_{i}} & \mathbf{0}
\end{array}\right) B I A\left(\begin{array}{c}
\mathbf{0} \\
I_{q_{j} \times q_{j}} \\
\mathbf{0}
\end{array}\right)=\left(\begin{array}{lll}
\mathbf{0} & I_{r_{i} \times r_{i}} & \mathbf{0}
\end{array}\right) B A\left(\begin{array}{c}
\mathbf{0} \\
I_{q_{j} \times q_{j}} \\
\mathbf{0}
\end{array}\right)
\end{gathered}
$$

Hence the $i j^{\text {th }}$ block of $B A$ equals the formal multiplication according to matrix multiplication,

$$
\sum_{s} B_{i s} A_{s j} .
$$

This proves the theorem.
Example 5.6.3 Let an $n \times n$ matrix have the form

$$
A=\left(\begin{array}{cc}
a & \mathbf{b} \\
\mathbf{c} & P
\end{array}\right)
$$

where $P$ is $n-1 \times n-1$. Multiply it by

$$
B=\left(\begin{array}{cc}
p & \mathbf{q} \\
\mathbf{r} & Q
\end{array}\right)
$$

where $B$ is also an $n \times n$ matrix and $Q$ is $n-1 \times n-1$.

You use block multiplication

$$
\left(\begin{array}{cc}
a & \mathbf{b} \\
\mathbf{c} & P
\end{array}\right)\left(\begin{array}{cc}
p & \mathbf{q} \\
\mathbf{r} & Q
\end{array}\right)=\left(\begin{array}{cc}
a p+\mathbf{b r} & a \mathbf{q}+\mathbf{b} Q \\
p \mathbf{c}+P \mathbf{r} & \mathbf{c q}+P Q
\end{array}\right)
$$

Note that this all makes sense. For example, $\mathbf{b}=1 \times n-1$ and $\mathbf{r}=n-1 \times 1$ so $\mathbf{b r}$ is a $1 \times 1$. Similar considerations apply to the other blocks.

Here is an interesting and significant application of block multiplication. In this theorem, $p_{M}(t)$ denotes the characteristic polynomial, $\operatorname{det}(t I-M)$. Thus the zeros of this polynomial are the eigenvalues of the matrix, $M$.

Theorem 5.6.4 Let $A$ be an $m \times n$ matrix and let $B$ be an $n \times m$ matrix for $m \leq n$. Then

$$
p_{B A}(t)=t^{n-m} p_{A B}(t),
$$

so the eigenvalues of $B A$ and $A B$ are the same including multiplicities except that $B A$ has $n-m$ extra zero eigenvalues.

Proof: Use block multiplication to write

$$
\begin{aligned}
& \left(\begin{array}{cc}
A B & 0 \\
B & 0
\end{array}\right)\left(\begin{array}{cc}
I & A \\
0 & I
\end{array}\right)=\left(\begin{array}{cc}
A B & A B A \\
B & B A
\end{array}\right) \\
& \left(\begin{array}{cc}
I & A \\
0 & I
\end{array}\right)\left(\begin{array}{cc}
0 & 0 \\
B & B A
\end{array}\right)=\left(\begin{array}{cc}
A B & A B A \\
B & B A
\end{array}\right)
\end{aligned}
$$

Therefore,

$$
\left(\begin{array}{cc}
I & A \\
0 & I
\end{array}\right)^{-1}\left(\begin{array}{cc}
A B & 0 \\
B & 0
\end{array}\right)\left(\begin{array}{cc}
I & A \\
0 & I
\end{array}\right)=\left(\begin{array}{cc}
0 & 0 \\
B & B A
\end{array}\right)
$$

Since the two matrices above are similar it follows that $\left(\begin{array}{cc}0 & 0 \\ B & B A\end{array}\right)$ and $\left(\begin{array}{cc}A B & 0 \\ B & 0\end{array}\right)$ have the same characteristic polynomials. Therefore, noting that $B A$ is an $n \times n$ matrix and $A B$ is an $m \times m$ matrix,

$$
t^{m} \operatorname{det}(t I-B A)=t^{n} \operatorname{det}(t I-A B)
$$

and so $\operatorname{det}(t I-B A)=p_{B A}(t)=t^{n-m} \operatorname{det}(t I-A B)=t^{n-m} p_{A B}(t)$. This proves the theorem.

### 5.7 Exercises

1. Show that matrix multiplication is associative. That is, $(A B) C=A(B C)$.
2. Show the inverse of a matrix, if it exists, is unique. Thus if $A B=B A=I$, then $B=A^{-1}$.
3. In the proof of Theorem 5.4.14 it was claimed that $\operatorname{det}(I)=1$. Here $I=\left(\delta_{i j}\right)$. Prove this assertion. Also prove Corollary 5.4.17.
4. Let $\mathbf{v}_{1}, \cdots, \mathbf{v}_{n}$ be vectors in $\mathbb{F}^{n}$ and let $M\left(\mathbf{v}_{1}, \cdots, \mathbf{v}_{n}\right)$ denote the matrix whose $i^{\text {th }}$ column equals $\mathbf{v}_{i}$. Define

$$
d\left(\mathbf{v}_{1}, \cdots, \mathbf{v}_{n}\right) \equiv \operatorname{det}\left(M\left(\mathbf{v}_{1}, \cdots, \mathbf{v}_{n}\right)\right)
$$

Prove that $d$ is linear in each variable, (multilinear), that

$$
\begin{equation*}
d\left(\mathbf{v}_{1}, \cdots, \mathbf{v}_{i}, \cdots, \mathbf{v}_{j}, \cdots, \mathbf{v}_{n}\right)=-d\left(\mathbf{v}_{1}, \cdots, \mathbf{v}_{j}, \cdots, \mathbf{v}_{i}, \cdots, \mathbf{v}_{n}\right) \tag{5.7.22}
\end{equation*}
$$

and

$$
\begin{equation*}
d\left(\mathbf{e}_{1}, \cdots, \mathbf{e}_{n}\right)=1 \tag{5.7.23}
\end{equation*}
$$

where here $\mathbf{e}_{j}$ is the vector in $\mathbb{F}^{n}$ which has a zero in every position except the $j^{\text {th }}$ position in which it has a one.
5. Suppose $f: \mathbb{F}^{n} \times \cdots \times \mathbb{F}^{n} \rightarrow \mathbb{F}$ satisfies 5.7 .22 and 5.7 .23 and is linear in each variable. Show that $f=d$.
6. Show that if you replace a row (column) of an $n \times n$ matrix $A$ with itself added to some multiple of another row (column) then the new matrix has the same determinant as the original one.
7. If $A=\left(a_{i j}\right)$, show $\operatorname{det}(A)=\sum_{\left(k_{1}, \cdots, k_{n}\right)} \operatorname{sgn}\left(k_{1}, \cdots, k_{n}\right) a_{k_{1} 1} \cdots a_{k_{n} n}$.
8. Use the result of Problem 6 to evaluate by hand the determinant

$$
\operatorname{det}\left(\begin{array}{cccc}
1 & 2 & 3 & 2 \\
-6 & 3 & 2 & 3 \\
5 & 2 & 2 & 3 \\
3 & 4 & 6 & 4
\end{array}\right)
$$

9. Find the inverse if it exists of the matrix,

$$
\left(\begin{array}{ccc}
e^{t} & \cos t & \sin t \\
e^{t} & -\sin t & \cos t \\
e^{t} & -\cos t & -\sin t
\end{array}\right)
$$

10. Let $L y=y^{(n)}+a_{n-1}(x) y^{(n-1)}+\cdots+a_{1}(x) y^{\prime}+a_{0}(x) y$ where the $a_{i}$ are given continuous functions defined on a closed interval, $(a, b)$ and $y$ is some function which
has $n$ derivatives so it makes sense to write $L y$. Suppose $L y_{k}=0$ for $k=1,2, \cdots, n$. The Wronskian of these functions, $y_{i}$ is defined as

$$
W\left(y_{1}, \cdots, y_{n}\right)(x) \equiv \operatorname{det}\left(\begin{array}{ccc}
y_{1}(x) & \cdots & y_{n}(x) \\
y_{1}^{\prime}(x) & \cdots & y_{n}^{\prime}(x) \\
\vdots & & \vdots \\
y_{1}^{(n-1)}(x) & \cdots & y_{n}^{(n-1)}(x)
\end{array}\right)
$$

Show that for $W(x)=W\left(y_{1}, \cdots, y_{n}\right)(x)$ to save space,

$$
W^{\prime}(x)=\operatorname{det}\left(\begin{array}{ccc}
y_{1}(x) & \cdots & y_{n}(x) \\
y_{1}^{\prime}(x) & \cdots & y_{n}^{\prime}(x) \\
\vdots & & \vdots \\
y_{1}^{(n)}(x) & \cdots & y_{n}^{(n)}(x)
\end{array}\right)
$$

Now use the differential equation, $L y=0$ which is satisfied by each of these functions, $y_{i}$ and properties of determinants presented above to verify that

$$
W^{\prime}+a_{n-1}(x) W=0
$$

Give an explicit solution of this linear differential equation, Abel's formula, and use your answer to verify that the Wronskian of these solutions to the equation, $L y=0$ either vanishes identically on $(a, b)$ or never.
11. Two $n \times n$ matrices, $A$ and $B$, are similar if $B=S^{-1} A S$ for some invertible $n \times n$ matrix, $S$. Show that if two matrices are similar, they have the same characteristic polynomials.
12. Suppose the characteristic polynomial of an $n \times n$ matrix, $A$ is of the form

$$
t^{n}+a_{n-1} t^{n-1}+\cdots+a_{1} t+a_{0}
$$

and that $a_{0} \neq 0$. Find a formula $A^{-1}$ in terms of powers of the matrix, $A$. Show that $A^{-1}$ exists if and only if $a_{0} \neq 0$.
13. In constitutive modeling of the stress and strain tensors, one sometimes considers sums of the form $\sum_{k=0}^{\infty} a_{k} A^{k}$ where $A$ is a $3 \times 3$ matrix. Show using the Cayley Hamilton theorem that if such a thing makes any sense, you can always obtain it as a finite sum having no more than $n$ terms.

### 5.8 Shur's Theorem

Every matrix is related to an upper triangular matrix in a particularly significant way. This is Shur's theorem and it is the most important theorem in the spectral theory of matrices.

Lemma 5.8.1 Let

$$
\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{n}\right\}
$$

be a basis for $\mathbb{F}^{n}$. Then there exists an orthonormal basis for $\mathbb{F}^{n}$,

$$
\left\{\mathbf{u}_{1}, \cdots, \mathbf{u}_{n}\right\}
$$

which has the property that for each $k \leq n$,

$$
\operatorname{span}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{k}\right)=\operatorname{span}\left(\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}\right)
$$

Proof: Let $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{n}\right\}$ be a basis for $\mathbb{F}^{n}$. Let $\mathbf{u}_{1} \equiv \mathbf{x}_{1} /\left|\mathbf{x}_{1}\right|$. Thus for $k=1$, $\operatorname{span}\left(\mathbf{u}_{1}\right)=$ $\operatorname{span}\left(\mathbf{x}_{1}\right)$ and $\left\{\mathbf{u}_{1}\right\}$ is an orthonormal set. Now suppose for some $k<n, \mathbf{u}_{1}, \cdots, \mathbf{u}_{k}$ have been chosen such that $\left(\mathbf{u}_{j} \cdot \mathbf{u}_{l}\right)=\delta_{j l}$ and span $\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{k}\right)=\operatorname{span}\left(\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}\right)$. Then define

$$
\begin{equation*}
\mathbf{u}_{k+1} \equiv \frac{\mathbf{x}_{k+1}-\sum_{j=1}^{k}\left(\mathbf{x}_{k+1} \cdot \mathbf{u}_{j}\right) \mathbf{u}_{j}}{\left|\mathbf{x}_{k+1}-\sum_{j=1}^{k}\left(\mathbf{x}_{k+1} \cdot \mathbf{u}_{j}\right) \mathbf{u}_{j}\right|} \tag{5.8.24}
\end{equation*}
$$

where the denominator is not equal to zero because the $\mathbf{x}_{j}$ form a basis and so

$$
\mathbf{x}_{k+1} \notin \operatorname{span}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{k}\right)=\operatorname{span}\left(\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}\right)
$$

Thus by induction,

$$
\mathbf{u}_{k+1} \in \operatorname{span}\left(\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}, \mathbf{x}_{k+1}\right)=\operatorname{span}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{k}, \mathbf{x}_{k+1}\right)
$$

Also, $\mathbf{x}_{k+1} \in \operatorname{span}\left(\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}, \mathbf{u}_{k+1}\right)$ which is seen easily by solving 5.8.24 for $\mathbf{x}_{k+1}$ and it follows

$$
\operatorname{span}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{k}, \mathbf{x}_{k+1}\right)=\operatorname{span}\left(\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}, \mathbf{u}_{k+1}\right)
$$

If $l \leq k$,

$$
\begin{aligned}
\left(\mathbf{u}_{k+1} \cdot \mathbf{u}_{l}\right) & =C\left(\left(\mathbf{x}_{k+1} \cdot \mathbf{u}_{l}\right)-\sum_{j=1}^{k}\left(\mathbf{x}_{k+1} \cdot \mathbf{u}_{j}\right)\left(\mathbf{u}_{j} \cdot \mathbf{u}_{l}\right)\right) \\
& =C\left(\left(\mathbf{x}_{k+1} \cdot \mathbf{u}_{l}\right)-\sum_{j=1}^{k}\left(\mathbf{x}_{k+1} \cdot \mathbf{u}_{j}\right) \delta_{l j}\right) \\
& =C\left(\left(\mathbf{x}_{k+1} \cdot \mathbf{u}_{l}\right)-\left(\mathbf{x}_{k+1} \cdot \mathbf{u}_{l}\right)\right)=0
\end{aligned}
$$

The vectors, $\left\{\mathbf{u}_{j}\right\}_{j=1}^{n}$, generated in this way are therefore an orthonormal basis because each vector has unit length.

The process by which these vectors were generated is called the Gram Schmidt process. Recall the following definition.

Definition 5.8.2 An $n \times n$ matrix, $U$, is unitary if $U U^{*}=I=U^{*} U$ where $U^{*}$ is defined to be the transpose of the conjugate of $U$.

Theorem 5.8.3 Let A be an $n \times n$ matrix. Then there exists a unitary matrix, $U$ such that

$$
\begin{equation*}
U^{*} A U=T \tag{5.8.25}
\end{equation*}
$$

where $T$ is an upper triangular matrix having the eigenvalues of $A$ on the main diagonal listed according to multiplicity as roots of the characteristic equation.

Proof: Let $\mathbf{v}_{1}$ be a unit eigenvector for $A$. Then there exists $\lambda_{1}$ such that

$$
A \mathbf{v}_{1}=\lambda_{1} \mathbf{v}_{1},\left|\mathbf{v}_{1}\right|=1
$$

Extend $\left\{\mathbf{v}_{1}\right\}$ to a basis and then use Lemma 5.8.1 to obtain $\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{n}\right\}$, an orthonormal basis in $\mathbb{F}^{n}$. Let $U_{0}$ be a matrix whose $i^{t h}$ column is $\mathbf{v}_{i}$. Then from the above, it follows $U_{0}$ is unitary. Then $U_{0}^{*} A U_{0}$ is of the form

$$
\left(\begin{array}{llll}
\lambda_{1} & * & \cdots & * \\
0 & & & \\
\vdots & & A_{1} & \\
0 & & &
\end{array}\right)
$$

where $A_{1}$ is an $n-1 \times n-1$ matrix. Repeat the process for the matrix, $A_{1}$ above. There exists a unitary matrix $\widetilde{U}_{1}$ such that $\widetilde{U}_{1}^{*} A_{1} \widetilde{U}_{1}$ is of the form

$$
\left(\begin{array}{llll}
\lambda_{2} & * & \cdots & * \\
0 & & & \\
\vdots & & A_{2} & \\
0 & & &
\end{array}\right)
$$

Now let $U_{1}$ be the $n \times n$ matrix of the form

$$
\left(\begin{array}{cc}
1 & \mathbf{0} \\
\mathbf{0} & \widetilde{U}_{1}
\end{array}\right)
$$

This is also a unitary matrix because by block multiplication,

$$
\begin{aligned}
\left(\begin{array}{cc}
1 & \mathbf{0} \\
\mathbf{0} & \widetilde{U}_{1}
\end{array}\right)^{*}\left(\begin{array}{cc}
1 & \mathbf{0} \\
\mathbf{0} & \widetilde{U}_{1}
\end{array}\right) & =\left(\begin{array}{cc}
1 & \mathbf{0} \\
\mathbf{0} & \widetilde{U}_{1}^{*}
\end{array}\right)\left(\begin{array}{cc}
1 & \mathbf{0} \\
\mathbf{0} & \widetilde{U}_{1}
\end{array}\right) \\
& =\left(\begin{array}{cc}
1 & \mathbf{0} \\
\mathbf{0} & \widetilde{U}_{1}^{*} \widetilde{U}_{1}
\end{array}\right)=\left(\begin{array}{cc}
1 & \mathbf{0} \\
\mathbf{0} & I
\end{array}\right)
\end{aligned}
$$

Then using block multiplication, $U_{1}^{*} U_{0}^{*} A U_{0} U_{1}$ is of the form

$$
\left(\begin{array}{lllll}
\lambda_{1} & * & * & \cdots & * \\
0 & \lambda_{2} & * & \cdots & * \\
0 & 0 & & & \\
\vdots & \vdots & & A_{2} & \\
0 & 0 & & &
\end{array}\right)
$$

where $A_{2}$ is an $n-2 \times n-2$ matrix. Continuing in this way, there exists a unitary matrix, $U$ given as the product of the $U_{i}$ in the above construction such that

$$
U^{*} A U=T
$$

where $T$ is some upper triangular matrix. Since the matrix is upper triangular, the characteristic equation is $\prod_{i=1}^{n}\left(\lambda-\lambda_{i}\right)$ where the $\lambda_{i}$ are the diagonal entries of $T$. Therefore, the $\lambda_{i}$ are the eigenvalues.

What if $A$ is a real matrix and you only want to consider real unitary matrices?

Theorem 5.8.4 Let A be a real $n \times n$ matrix. Then there exists a real unitary matrix, $Q$ and a matrix $T$ of the form

$$
T=\left(\begin{array}{ccc}
P_{1} & \cdots & *  \tag{5.8.26}\\
& \ddots & \vdots \\
0 & & P_{r}
\end{array}\right)
$$

where $P_{i}$ equals either a real $1 \times 1$ matrix or $P_{i}$ equals a real $2 \times 2$ matrix having two complex eigenvalues of $A$ such that $Q^{T} A Q=T$. The matrix, $T$ is called the real Schur form of the matrix $A$.

Proof: Suppose

$$
A \mathbf{v}_{1}=\lambda_{1} \mathbf{v}_{1},\left|\mathbf{v}_{1}\right|=1
$$

where $\lambda_{1}$ is real. Then let $\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{n}\right\}$ be an orthonormal basis of vectors in $\mathbb{R}^{n}$. Let $Q_{0}$ be a matrix whose $i^{t h}$ column is $\mathbf{v}_{i}$. Then $Q_{0}^{*} A Q_{0}$ is of the form

$$
\left(\begin{array}{llll}
\lambda_{1} & * & \cdots & * \\
0 & & & \\
\vdots & & A_{1} & \\
0 & & &
\end{array}\right)
$$

where $A_{1}$ is a real $n-1 \times n-1$ matrix. This is just like the proof of Theorem 5.8.3 up to this point.

Now in case $\lambda_{1}=\alpha+i \beta$, it follows since $A$ is real that $\mathbf{v}_{1}=\mathbf{z}_{1}+i \mathbf{w}_{1}$ and that $\overline{\mathbf{v}}_{1}=$ $\mathbf{z}_{1}-i \mathbf{w}_{1}$ is an eigenvector for the eigenvalue, $\alpha-i \beta$. Here $\mathbf{z}_{1}$ and $\mathbf{w}_{1}$ are real vectors. It is clear that $\left\{\mathbf{z}_{1}, \mathbf{w}_{1}\right\}$ is an independent set of vectors in $\mathbb{R}^{n}$. Indeed, $\left\{\mathbf{v}_{1}, \overline{\mathbf{v}}_{1}\right\}$ is an independent set and it follows span $\left(\mathbf{v}_{1}, \overline{\mathbf{v}}_{1}\right)=\operatorname{span}\left(\mathbf{z}_{1}, \mathbf{w}_{1}\right)$. Now using the Gram Schmidt theorem in $\mathbb{R}^{n}$, there exists $\left\{\mathbf{u}_{1}, \mathbf{u}_{2}\right\}$, an orthonormal set of real vectors such that $\operatorname{span}\left(\mathbf{u}_{1}, \mathbf{u}_{2}\right)=$ $\operatorname{span}\left(\mathbf{v}_{1}, \overline{\mathbf{v}}_{1}\right)$. Now let $\left\{\mathbf{u}_{1}, \mathbf{u}_{2}, \cdots, \mathbf{u}_{n}\right\}$ be an orthonormal basis in $\mathbb{R}^{n}$ and let $Q_{0}$ be a unitary matrix whose $i^{\text {th }}$ column is $\mathbf{u}_{i}$. Then $A \mathbf{u}_{j}$ are both in $\operatorname{span}\left(\mathbf{u}_{1}, \mathbf{u}_{2}\right)$ for $j=1,2$ and so $\mathbf{u}_{k}^{T} A \mathbf{u}_{j}=0$ whenever $k \geq 3$. It follows that $Q_{0}^{*} A Q_{0}$ is of the form

$$
\left(\begin{array}{cccc}
* & * & \cdots & * \\
* & * & & \\
0 & & & \\
\vdots & & A_{1} & \\
0 & & &
\end{array}\right)
$$

where $A_{1}$ is now an $n-2 \times n-2$ matrix. In this case, find $\widetilde{Q}_{1}$ an $n-2 \times n-2$ matrix to put $A_{1}$ in an appropriate form as above and come up with $A_{2}$ either an $n-4 \times n-4$ matrix or an $n-3 \times n-3$ matrix. Then the only other difference is to let

$$
Q_{1}=\left(\begin{array}{ccccc}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & & & \\
\vdots & \vdots & & \widetilde{Q}_{1} & \\
0 & 0 & & &
\end{array}\right)
$$

thus putting a $2 \times 2$ identity matrix in the upper left corner rather than a one. Repeating this process with the above modification for the case of a complex eigenvalue leads eventually to 5.8 .26 where $Q$ is the product of real unitary matrices $Q_{i}$ above. Finally,

$$
\lambda I-T=\left(\begin{array}{ccc}
\lambda I_{1}-P_{1} & \cdots & * \\
& \ddots & \vdots \\
0 & & \lambda I_{r}-P_{r}
\end{array}\right)
$$

where $I_{k}$ is the $2 \times 2$ identity matrix in the case that $P_{k}$ is $2 \times 2$ and is the number 1 in the case where $P_{k}$ is a $1 \times 1$ matrix. Now, it follows that $\operatorname{det}(\lambda I-T)=\prod_{k=1}^{r} \operatorname{det}\left(\lambda I_{k}-P_{k}\right)$. Therefore, $\lambda$ is an eigenvalue of $T$ if and only if it is an eigenvalue of some $P_{k}$. This proves the theorem since the eigenvalues of $T$ are the same as those of $A$ because they have the same characteristic polynomial due to the similarity of $A$ and $T$.

Definition 5.8.5 When a linear transformation, $A$, mapping a linear space, $V$ to $V$ has a basis of eigenvectors, the linear transformation is called non defective. Otherwise it is called defective. An $n \times n$ matrix, $A$, is called normal if $A A^{*}=A^{*} A$. An important class of normal matrices is that of the Hermitian or self adjoint matrices. An $n \times n$ matrix, $A$ is self adjoint or Hermitian if $A=A^{*}$.

The next lemma is the basis for concluding that every normal matrix is unitarily similar to a diagonal matrix.

Lemma 5.8.6 If $T$ is upper triangular and normal, then $T$ is a diagonal matrix.
Proof: Since $T$ is normal, $T^{*} T=T T^{*}$. Writing this in terms of components and using the description of the adjoint as the transpose of the conjugate, yields the following for the $i k^{t h}$ entry of $T^{*} T=T T^{*}$.

$$
\sum_{j} t_{i j} t_{j k}^{*}=\sum_{j} t_{i j} \overline{t_{k j}}=\sum_{j} t_{i j}^{*} t_{j k}=\sum_{j} \overline{t_{j i}} t_{j k}
$$

Now use the fact that $T$ is upper triangular and let $i=k=1$ to obtain the following from the above.

$$
\sum_{j}\left|t_{1 j}\right|^{2}=\sum_{j}\left|t_{j 1}\right|^{2}=\left|t_{11}\right|^{2}
$$

You see, $t_{j 1}=0$ unless $j=1$ due to the assumption that $T$ is upper triangular. This shows $T$ is of the form

$$
\left(\begin{array}{cccc}
* & 0 & \cdots & 0 \\
0 & * & \cdots & * \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & *
\end{array}\right)
$$

Now do the same thing only this time take $i=k=2$ and use the result just established. Thus, from the above,

$$
\sum_{j}\left|t_{2 j}\right|^{2}=\sum_{j}\left|t_{j 2}\right|^{2}=\left|t_{22}\right|^{2}
$$

showing that $t_{2 j}=0$ if $j>2$ which means $T$ has the form

$$
\left(\begin{array}{ccccc}
* & 0 & 0 & \cdots & 0 \\
0 & * & 0 & \cdots & 0 \\
0 & 0 & * & \cdots & * \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & *
\end{array}\right)
$$

Next let $i=k=3$ and obtain that $T$ looks like a diagonal matrix in so far as the first 3 rows and columns are concerned. Continuing in this way it follows $T$ is a diagonal matrix.
Theorem 5.8.7 Let A be a normal matrix. Then there exists a unitary matrix, $U$ such that $U^{*} A U$ is a diagonal matrix.

Proof: From Theorem 5.8.3 there exists a unitary matrix, $U$ such that $U^{*} A U$ equals an upper triangular matrix. The theorem is now proved if it is shown that the property of being normal is preserved under unitary similarity transformations. That is, verify that if $A$ is normal and if $B=U^{*} A U$, then $B$ is also normal. But this is easy.

$$
\begin{aligned}
B^{*} B & =U^{*} A^{*} U U^{*} A U=U^{*} A^{*} A U \\
& =U^{*} A A^{*} U=U^{*} A U U^{*} A^{*} U=B B^{*}
\end{aligned}
$$

Therefore, $U^{*} A U$ is a normal and upper triangular matrix and by Lemma 5.8.6 it must be a diagonal matrix. This proves the theorem.
Corollary 5.8.8 If $A$ is Hermitian, then all the eigenvalues of $A$ are real and there exists an orthonormal basis of eigenvectors.

Proof: Since $A$ is normal, there exists unitary, $U$ such that $U^{*} A U=D$, a diagonal matrix whose diagonal entries are the eigenvalues of $A$. Therefore, $D^{*}=U^{*} A^{*} U=U^{*} A U=$ $D$ showing $D$ is real.

Finally, let

$$
U=\left(\begin{array}{llll}
\mathbf{u}_{1} & \mathbf{u}_{2} & \cdots & \mathbf{u}_{n}
\end{array}\right)
$$

where the $\mathbf{u}_{i}$ denote the columns of $U$ and

$$
D=\left(\begin{array}{ccc}
\lambda_{1} & & 0 \\
& \ddots & \\
0 & & \lambda_{n}
\end{array}\right)
$$

The equation, $U^{*} A U=D$ implies

$$
\begin{aligned}
A U & =\left(\begin{array}{llll}
A \mathbf{u}_{1} & A \mathbf{u}_{2} & \cdots & A \mathbf{u}_{n}
\end{array}\right) \\
& =U D=\left(\begin{array}{llll}
\lambda_{1} \mathbf{u}_{1} & \lambda_{2} \mathbf{u}_{2} & \cdots & \lambda_{n} \mathbf{u}_{n}
\end{array}\right)
\end{aligned}
$$

where the entries denote the columns of $A U$ and $U D$ respectively. Therefore, $A \mathbf{u}_{i}=\lambda_{i} \mathbf{u}_{i}$ and since the matrix is unitary, the $i j^{t h}$ entry of $U^{*} U$ equals $\delta_{i j}$ and so

$$
\delta_{i j}=\overline{\mathbf{u}}_{i}^{T} \mathbf{u}_{j}=\overline{\mathbf{u}_{i}^{T} \overline{\mathbf{u}}_{j}}=\overline{\mathbf{u}_{i} \cdot \mathbf{u}_{j}}
$$

This proves the corollary because it shows the vectors $\left\{\mathbf{u}_{i}\right\}$ form an orthonormal basis.

Corollary 5.8.9 If A is a real symmetric matrix, then $A$ is Hermitian and there exists a real unitary matrix, $U$ such that $U^{T} A U=D$ where $D$ is a diagonal matrix.

Proof: This follows from Theorem 5.8.4 and Corollary 5.8.8.

### 5.9 The Right Polar Decomposition

The right polar decomposition involves writing a matrix as a product of two other matrices, one which preserves distances and the other which stretches and distorts. First here are some lemmas.

Lemma 5.9.1 Let A be a Hermitian matrix such that all its eigenvalues are nonnegative. Then there exists a Hermitian matrix, $A^{1 / 2}$ such that $A^{1 / 2}$ has all nonnegative eigenvalues and $\left(A^{1 / 2}\right)^{2}=A$.

Proof: Since $A$ is Hermitian, there exists a diagonal matrix $D$ having all real nonnegative entries and a unitary matrix $U$ such that $A=U^{*} D U$. Then denote by $D^{1 / 2}$ the matrix which is obtained by replacing each diagonal entry of $D$ with its square root. Thus $D^{1 / 2} D^{1 / 2}=D$. Then define

$$
A^{1 / 2} \equiv U^{*} D^{1 / 2} U
$$

Then

$$
\left(A^{1 / 2}\right)^{2}=U^{*} D^{1 / 2} U U^{*} D^{1 / 2} U=U^{*} D U=A
$$

Since $D^{1 / 2}$ is real,

$$
\left(U^{*} D^{1 / 2} U\right)^{*}=U^{*}\left(D^{1 / 2}\right)^{*}\left(U^{*}\right)^{*}=U^{*} D^{1 / 2} U
$$

so $A^{1 / 2}$ is Hermitian. This proves the lemma.
There is also a useful observation about orthonormal sets of vectors which is stated in the next lemma.

Lemma 5.9.2 Suppose $\left\{\mathbf{x}_{1}, \mathbf{x}_{2}, \cdots, \mathbf{x}_{r}\right\}$ is an orthonormal set of vectors. Then if $c_{1}, \cdots, c_{r}$ are scalars,

$$
\left|\sum_{k=1}^{r} c_{k} \mathbf{x}_{k}\right|^{2}=\sum_{k=1}^{r}\left|c_{k}\right|^{2}
$$

Proof: This follows from the definition. From the properties of the dot product and using the fact that the given set of vectors is orthonormal,

$$
\begin{gathered}
\left|\sum_{k=1}^{r} c_{k} \mathbf{x}_{k}\right|^{2}=\left(\sum_{k=1}^{r} c_{k} \mathbf{x}_{k}, \sum_{j=1}^{r} c_{j} \mathbf{x}_{j}\right) \\
=\sum_{k, j} c_{k} \overline{c_{j}}\left(\mathbf{x}_{k}, \mathbf{x}_{j}\right)=\sum_{k=1}^{r}\left|c_{k}\right|^{2} .
\end{gathered}
$$

This proves the lemma.
Next it is helpful to recall the Gram Schmidt algorithm and observe a certain property stated in the next lemma.

Lemma 5.9.3 Suppose $\left\{\mathbf{w}_{1}, \cdots, \mathbf{w}_{r}, \mathbf{v}_{r+1}, \cdots, \mathbf{v}_{p}\right\}$ is a linearly independent set of vectors such that $\left\{\mathbf{w}_{1}, \cdots, \mathbf{w}_{r}\right\}$ is an orthonormal set of vectors. Then when the Gram Schmidt process is applied to the vectors in the given order, it will not change any of the $\mathbf{w}_{1}, \cdots, \mathbf{w}_{r}$.

Proof: Let $\left\{\mathbf{u}_{1}, \cdots, \mathbf{u}_{p}\right\}$ be the orthonormal set delivered by the Gram Schmidt process. Then $\mathbf{u}_{1}=\mathbf{w}_{1}$ because by definition, $\mathbf{u}_{1} \equiv \mathbf{w}_{1} /\left|\mathbf{w}_{1}\right|=\mathbf{w}_{1}$. Now suppose $\mathbf{u}_{j}=\mathbf{w}_{j}$ for all $j \leq k \leq r$. Then if $k<r$, consider the definition of $\mathbf{u}_{k+1}$.

$$
\mathbf{u}_{k+1} \equiv \frac{\mathbf{w}_{k+1}-\sum_{j=1}^{k+1}\left(\mathbf{w}_{k+1}, \mathbf{u}_{j}\right) \mathbf{u}_{j}}{\left|\mathbf{w}_{k+1}-\sum_{j=1}^{k+1}\left(\mathbf{w}_{k+1}, \mathbf{u}_{j}\right) \mathbf{u}_{j}\right|}
$$

By induction, $\mathbf{u}_{j}=\mathbf{w}_{j}$ and so this reduces to $\mathbf{w}_{k+1} /\left|\mathbf{w}_{k+1}\right|=\mathbf{w}_{k+1}$. This proves the lemma.
This lemma immediately implies the following lemma.
Lemma 5.9.4 Let $V$ be a subspace of dimension $p$ and let $\left\{\mathbf{w}_{1}, \cdots, \mathbf{w}_{r}\right\}$ be an orthonormal set of vectors in $V$. Then this orthonormal set of vectors may be extended to an orthonormal basis for $V$,

$$
\left\{\mathbf{w}_{1}, \cdots, \mathbf{w}_{r}, \mathbf{y}_{r+1}, \cdots, \mathbf{y}_{p}\right\}
$$

Proof: First extend the given linearly independent set $\left\{\mathbf{w}_{1}, \cdots, \mathbf{w}_{r}\right\}$ to a basis for $V$ and then apply the Gram Schmidt theorem to the resulting basis. Since $\left\{\mathbf{w}_{1}, \cdots, \mathbf{w}_{r}\right\}$ is orthonormal it follows from Lemma 5.9.3 the result is of the desired form, an orthonormal basis extending $\left\{\mathbf{w}_{1}, \cdots, \mathbf{w}_{r}\right\}$. This proves the lemma.

Here is another lemma about preserving distance.
Lemma 5.9.5 Suppose $R$ is an $m \times n$ matrix with $m>n$ and $R$ preserves distances. Then $R^{*} R=I$.

Proof: Since $R$ preserves distances, $|R \mathbf{x}|=|\mathbf{x}|$ for every $\mathbf{x}$. Therefore from the axioms of the dot product,

$$
\begin{aligned}
& |\mathbf{x}|^{2}+|\mathbf{y}|^{2}+(\mathbf{x}, \mathbf{y})+(\mathbf{y}, \mathbf{x}) \\
= & |\mathbf{x}+\mathbf{y}|^{2} \\
= & (R(\mathbf{x}+\mathbf{y}), R(\mathbf{x}+\mathbf{y})) \\
= & (R \mathbf{x}, R \mathbf{x})+(R \mathbf{y}, R \mathbf{y})+(R \mathbf{x}, R \mathbf{y})+(R \mathbf{y}, R \mathbf{x}) \\
= & |\mathbf{x}|^{2}+|\mathbf{y}|^{2}+\left(R^{*} R \mathbf{x}, \mathbf{y}\right)+\left(\mathbf{y}, R^{*} R \mathbf{x}\right)
\end{aligned}
$$

and so for all $\mathbf{x}, \mathbf{y}$,

$$
\left(R^{*} R \mathbf{x}-\mathbf{x}, \mathbf{y}\right)+\left(\mathbf{y}, R^{*} R \mathbf{x}-\mathbf{x}\right)=0
$$

Hence for all $\mathbf{x}, \mathbf{y}$,

$$
\operatorname{Re}\left(R^{*} R \mathbf{x}-\mathbf{x}, \mathbf{y}\right)=0
$$

Now for a $\mathbf{x}, \mathbf{y}$ given, choose $\alpha \in \mathbb{C}$ such that

$$
\alpha\left(R^{*} R \mathbf{x}-\mathbf{x}, \mathbf{y}\right)=\left|\left(R^{*} R \mathbf{x}-\mathbf{x}, \mathbf{y}\right)\right|
$$

Then

$$
\begin{aligned}
0 & =\operatorname{Re}\left(R^{*} R \mathbf{x}-\mathbf{x}, \bar{\alpha} \mathbf{y}\right)=\operatorname{Re} \alpha\left(R^{*} R \mathbf{x}-\mathbf{x}, \mathbf{y}\right) \\
& =\left|\left(R^{*} R \mathbf{x}-\mathbf{x}, \mathbf{y}\right)\right|
\end{aligned}
$$

Thus $\left|\left(R^{*} R \mathbf{x}-\mathbf{x}, \mathbf{y}\right)\right|=0$ for all $\mathbf{x}, \mathbf{y}$ because the given $\mathbf{x}, \mathbf{y}$ were arbitrary. Let $\mathbf{y}=R^{*} R \mathbf{x}-\mathbf{x}$ to conclude that for all $\mathbf{x}$,

$$
R^{*} R \mathbf{x}-\mathbf{x}=\mathbf{0}
$$

which says $R^{*} R=I$ since $\mathbf{x}$ is arbitrary. This proves the lemma.
With this preparation, here is the big theorem about the right polar decomposition.
Theorem 5.9.6 Let $F$ be an $m \times n$ matrix where $m \geq n$. Then there exists a Hermitian $n \times n$ matrix, $U$ which has all nonnegative eigenvalues and an $m \times n$ matrix, $R$ which preserves distances and satisfies $R^{*} R=I$ such that

$$
F=R U .
$$

Proof: Consider $F^{*} F$. This is a Hermitian matrix because

$$
\left(F^{*} F\right)^{*}=F^{*}\left(F^{*}\right)^{*}=F^{*} F
$$

Also the eigenvalues of the $n \times n$ matrix $F^{*} F$ are all nonnegative. This is because if $\mathbf{x}$ is an eigenvalue,

$$
\lambda(\mathbf{x}, \mathbf{x})=\left(F^{*} F \mathbf{x}, \mathbf{x}\right)=(F \mathbf{x}, F \mathbf{x}) \geq 0
$$

Therefore, by Lemma 5.9.1, there exists an $n \times n$ Hermitian matrix, $U$ having all nonnegative eigenvalues such that

$$
U^{2}=F^{*} F .
$$

Consider the subspace $U\left(\mathbb{F}^{n}\right)$. Let $\left\{U \mathbf{x}_{1}, \cdots, U \mathbf{x}_{r}\right\}$ be an orthonormal basis for $U\left(\mathbb{F}^{n}\right) \subseteq$ $\mathbb{F}^{n}$. Note that $U\left(\mathbb{F}^{n}\right)$ might not be all of $\mathbb{F}^{n}$. Using Lemma 5.9.4, extend to an orthonormal basis for all of $\mathbb{F}^{n}$,

$$
\left\{U \mathbf{x}_{1}, \cdots, U \mathbf{x}_{r}, \mathbf{y}_{r+1}, \cdots, \mathbf{y}_{n}\right\}
$$

Next observe that $\left\{F \mathbf{x}_{1}, \cdots, F \mathbf{x}_{r}\right\}$ is also an orthonormal set of vectors in $\mathbb{F}^{m}$. This is because

$$
\begin{aligned}
\left(F \mathbf{x}_{k}, F \mathbf{x}_{j}\right) & =\left(F^{*} F \mathbf{x}_{k}, \mathbf{x}_{j}\right)=\left(U^{2} \mathbf{x}_{k}, \mathbf{x}_{j}\right) \\
& =\left(U \mathbf{x}_{k}, U^{*} \mathbf{x}_{j}\right)=\left(U \mathbf{x}_{k}, U \mathbf{x}_{j}\right)=\delta_{j k}
\end{aligned}
$$

Therefore, from Lemma 5.9.4 again, this orthonormal set of vectors can be extended to an orthonormal basis for $\mathbb{F}^{m}$,

$$
\left\{F \mathbf{x}_{1}, \cdots, F \mathbf{x}_{r}, \mathbf{z}_{r+1}, \cdots, \mathbf{z}_{m}\right\}
$$

Thus there are at least as many $\mathbf{z}_{k}$ as there are $\mathbf{y}_{j}$. Now for $\mathbf{x} \in \mathbb{F}^{n}$, since

$$
\left\{U \mathbf{x}_{1}, \cdots, U \mathbf{x}_{r}, \mathbf{y}_{r+1}, \cdots, \mathbf{y}_{n}\right\}
$$

is an orthonormal basis for $\mathbb{F}^{n}$, there exist unique scalars,

$$
c_{1} \cdots, c_{r}, d_{r+1}, \cdots, d_{n}
$$

such that

$$
\mathbf{x}=\sum_{k=1}^{r} c_{k} U \mathbf{x}_{k}+\sum_{j=r+1}^{n} d_{k} \mathbf{y}_{k}
$$

Define

$$
\begin{equation*}
R \mathbf{x} \equiv \sum_{k=1}^{r} c_{k} F \mathbf{x}_{k}+\sum_{j=r+1}^{n} d_{k} \mathbf{z}_{k} \tag{5.9.27}
\end{equation*}
$$

Then also there exist scalars $b_{k}$ such that

$$
U \mathbf{x}=\sum_{k=1}^{r} b_{k} U \mathbf{x}_{k}
$$

and so from 5.9.27, applied to $U \mathbf{x}$ in place of $\mathbf{x}$

$$
R U \mathbf{x}=\sum_{k=1}^{r} b_{k} F \mathbf{x}_{k}=F\left(\sum_{k=1}^{r} b_{k} \mathbf{x}_{k}\right)
$$

Is $F\left(\sum_{k=1}^{r} b_{k} \mathbf{x}_{k}\right)=F(\mathbf{x})$ ?

$$
\begin{aligned}
& \left(F\left(\sum_{k=1}^{r} b_{k} \mathbf{x}_{k}\right)-F(\mathbf{x}), F\left(\sum_{k=1}^{r} b_{k} \mathbf{x}_{k}\right)-F(\mathbf{x})\right) \\
& =\left(\left(F^{*} F\right)\left(\sum_{k=1}^{r} b_{k} \mathbf{x}_{k}-\mathbf{x}\right),\left(\sum_{k=1}^{r} b_{k} \mathbf{x}_{k}-\mathbf{x}\right)\right) \\
& =\left(U^{2}\left(\sum_{k=1}^{r} b_{k} \mathbf{x}_{k}-\mathbf{x}\right),\left(\sum_{k=1}^{r} b_{k} \mathbf{x}_{k}-\mathbf{x}\right)\right) \\
& =\left(U\left(\sum_{k=1}^{r} b_{k} \mathbf{x}_{k}-\mathbf{x}\right), U\left(\sum_{k=1}^{r} b_{k} \mathbf{x}_{k}-\mathbf{x}\right)\right) \\
& =\left(\sum_{k=1}^{r} b_{k} U \mathbf{x}_{k}-U \mathbf{x}, \sum_{k=1}^{r} b_{k} U \mathbf{x}_{k}-U \mathbf{x}\right)=0
\end{aligned}
$$

Therefore, $F\left(\sum_{k=1}^{r} b_{k} \mathbf{x}_{k}\right)=F(\mathbf{x})$ and this shows

$$
R U \mathbf{x}=F \mathbf{x}
$$

From 5.9.27 and Lemma 5.9.2 $R$ preserves distances. Therefore, by Lemma 5.9.5 $R^{*} R=I$. This proves the theorem.

Here is a useful fact from Linear algebra.

Lemma 5.9.7 Suppose $\operatorname{det}(A)=0$. Then for all sufficiently small nonzero $\varepsilon, \operatorname{det}(A+\varepsilon I) \neq$ 0.

Proof: First suppose $A$ is a $p \times p$ matrix. Suppose also that $\operatorname{det}(A)=0$. Thus, the constant term of $\operatorname{det}(\lambda I-A)$ is 0 . Consider $\varepsilon I+A \equiv A_{\varepsilon}$ for small real $\varepsilon$. The characteristic polynomial of $A_{\varepsilon}$ is

$$
\operatorname{det}\left(\lambda I-A_{\varepsilon}\right)=\operatorname{det}((\lambda-\varepsilon) I-A)
$$

This is of the form

$$
(\lambda-\varepsilon)^{p}+a_{p-1}(\lambda-\varepsilon)^{p-1}+\cdots+(\lambda-\varepsilon)^{m} a_{m}
$$

where the $a_{j}$ are the coefficients in the characteristic equation for $A$ and $m$ is the largest such that $a_{m} \neq 0$. The constant term of this characteristic polynomial for $A_{\varepsilon}$ must be nonzero for all positive $\varepsilon$ small enough because it is of the form

$$
(-1)^{m} \varepsilon^{m} a_{m}+(\text { higher order terms in } \varepsilon)
$$

which shows that $\varepsilon I+A$ is invertible for all $\varepsilon$ small enough but nonzero.

## Chapter 6

## Multi-variable Calculus

### 6.1 Continuous Functions

In what follows, $\mathbb{F}$ will denote either $\mathbb{R}$ or $\mathbb{C}$. It turns out it is more efficient to not make a distinction. However, the main interest is in $\mathbb{R}$ so if you like, you can think $\mathbb{R}$ whenever you see $\mathbb{F}$.

### 6.2 Open And Closed Sets

Eventually, one must consider functions which are defined on subsets of $\mathbb{F}^{n}$ and their properties. The next definition will end up being quite important. It describe a type of subset of $\mathbb{F}^{n}$ with the property that if $\mathbf{x}$ is in this set, then so is $\mathbf{y}$ whenever $\mathbf{y}$ is close enough to $\mathbf{x}$. In all of this, for $\mathbf{x}$ a vector, $|\mathbf{x}|$ is given by $(\mathbf{x}, \mathbf{x})^{1 / 2}$ where this denotes the square root of the inner product of the vector with itself as described earlier. Then the distance between the vectors $\mathbf{x}$ and $\mathbf{y}$ is defined as $|\mathbf{x}-\mathbf{y}|$.

Definition 6.2.1 Let $U \subseteq \mathbb{F}^{n}$. $U$ is an open set if whenever $\mathbf{x} \in U$, there exists $r>0$ such that $B(\mathbf{x}, r) \subseteq U$. More generally, if $U$ is any subset of $\mathbb{F}^{n}, \mathbf{x} \in U$ is an interior point of $U$ if there exists $r>0$ such that $\mathbf{x} \in B(\mathbf{x}, r) \subseteq U$. In other words $U$ is an open set exactly when every point of $U$ is an interior point of $U$.

If there is something called an open set, surely there should be something called a closed set and here is the definition of one.

Definition 6.2.2 A subset, $C$, of $\mathbb{F}^{n}$ is called a closed set if $\mathbb{F}^{n} \backslash C$ is an open set. They symbol, $\mathbb{F}^{n} \backslash C$ denotes everything in $\mathbb{F}^{n}$ which is not in $C$. It is also called the complement of $C$. The symbol, $S^{C}$ is a short way of writing $\mathbb{F}^{n} \backslash S$.

To illustrate this definition, consider the following picture.
$\mathbf{x}$
$B(\mathbf{x}, r)$$\vdots \quad U$

You see in this picture how the edges are dotted. This is because an open set, can not include the edges or the set would fail to be open. For example, consider what would happen if you picked a point out on the edge of $U$ in the above picture. Every open ball centered at that point would have in it some points which are outside $U$. Therefore, such a point would violate the above definition. You also see the edges of $B(\mathbf{x}, r)$ dotted suggesting
that $B(\mathbf{x}, r)$ ought to be an open set. This is intuitively clear but does require a proof. This will be done in the next theorem and will give examples of open sets. Also, you can see that if $\mathbf{x}$ is close to the edge of $U$, you might have to take $r$ to be very small.

It is roughly the case that open sets don't have their skins while closed sets do. Here is a picture of a closed set, $C$.


Note that $\mathbf{x} \notin C$ and since $\mathbb{F}^{n} \backslash C$ is open, there exists a ball, $B(\mathbf{x}, r)$ contained entirely in $\mathbb{F}^{n} \backslash C$. If you look at $\mathbb{F}^{n} \backslash C$, what would be its skin? It can't be in $\mathbb{F}^{n} \backslash C$ and so it must be in $C$. This is a rough heuristic explanation of what is going on with these definitions. Also note that $\mathbb{F}^{n}$ and $\emptyset$ are both open and closed. Here is why. If $\mathbf{x} \in \emptyset$, then there must be a ball centered at $\mathbf{x}$ which is also contained in $\emptyset$. This must be considered to be true because there is nothing in $\emptyset$ so there can be no example to show it false ${ }^{1}$. Therefore, from the definition, it follows $\emptyset$ is open. It is also closed because if $\mathbf{x} \notin \emptyset$, then $B(\mathbf{x}, 1)$ is also contained in $\mathbb{F}^{n} \backslash \emptyset=\mathbb{F}^{n}$. Therefore, $\emptyset$ is both open and closed. From this, it follows $\mathbb{F}^{n}$ is also both open and closed.

Theorem 6.2.3 Let $\mathbf{x} \in \mathbb{F}^{n}$ and let $r \geq 0$. Then $B(\mathbf{x}, r)$ is an open set. Also,

$$
D(\mathbf{x}, r) \equiv\left\{\mathbf{y} \in \mathbb{F}^{n}:|\mathbf{y}-\mathbf{x}| \leq r\right\}
$$

is a closed set.
Proof: Suppose $\mathbf{y} \in B(\mathbf{x}, r)$. It is necessary to show there exists $r_{1}>0$ such that $B\left(\mathbf{y}, r_{1}\right) \subseteq B(\mathbf{x}, r)$. Define $r_{1} \equiv r-|\mathbf{x}-\mathbf{y}|$. Then if $|\mathbf{z}-\mathbf{y}|<r_{1}$, it follows from the above triangle inequality that

$$
\begin{aligned}
|\mathbf{z}-\mathbf{x}| & =|\mathbf{z}-\mathbf{y}+\mathbf{y}-\mathbf{x}| \\
& \leq|\mathbf{z}-\mathbf{y}|+|\mathbf{y}-\mathbf{x}| \\
& <r_{1}+|\mathbf{y}-\mathbf{x}|=r-|\mathbf{x}-\mathbf{y}|+|\mathbf{y}-\mathbf{x}|=r
\end{aligned}
$$

[^5]Note that if $r=0$ then $B(\mathbf{x}, r)=\emptyset$, the empty set. This is because if $\mathbf{y} \in \mathbb{F}^{n},|\mathbf{x}-\mathbf{y}| \geq 0$ and so $\mathbf{y} \notin B(\mathbf{x}, 0)$. Since $\emptyset$ has no points in it, it must be open because every point in it, (There are none.) satisfies the desired property of being an interior point.

Now suppose $\mathbf{y} \notin D(\mathbf{x}, r)$. Then $|\mathbf{x}-\mathbf{y}|>r$ and defining $\delta \equiv|\mathbf{x}-\mathbf{y}|-r$, it follows that if $\mathbf{z} \in B(\mathbf{y}, \boldsymbol{\delta})$, then by the triangle inequality,

$$
\begin{aligned}
|\mathbf{x}-\mathbf{z}| & \geq|\mathbf{x}-\mathbf{y}|-|\mathbf{y}-\mathbf{z}|>|\mathbf{x}-\mathbf{y}|-\delta \\
& =|\mathbf{x}-\mathbf{y}|-(|\mathbf{x}-\mathbf{y}|-r)=r
\end{aligned}
$$

and this shows that $B(\mathbf{y}, \delta) \subseteq \mathbb{F}^{n} \backslash D(\mathbf{x}, r)$. Since $\mathbf{y}$ was an arbitrary point in $\mathbb{F}^{n} \backslash D(\mathbf{x}, r)$, it follows $\mathbb{F}^{n} \backslash D(\mathbf{x}, r)$ is an open set which shows from the definition that $D(\mathbf{x}, r)$ is a closed set as claimed.

A picture which is descriptive of the conclusion of the above theorem which also implies the manner of proof is the following.


### 6.3 Continuous Functions

With the above definition of the norm in $\mathbb{F}^{p}$, it becomes possible to define continuity.
Definition 6.3.1 A function $\mathbf{f}: D(\mathbf{f}) \subseteq \mathbb{F}^{p} \rightarrow \mathbb{F}^{q}$ is continuous at $\mathbf{x} \in D(\mathbf{f})$ if for each $\varepsilon>0$ there exists $\delta>0$ such that whenever $\mathbf{y} \in D(\mathbf{f})$ and

$$
|\mathbf{y}-\mathbf{x}|<\delta
$$

it follows that

$$
|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})|<\varepsilon .
$$

$\mathbf{f}$ is continuous if it is continuous at every point of $D(\mathbf{f})$.
Note the total similarity to the scalar valued case.

### 6.3.1 Sufficient Conditions For Continuity

The next theorem is a fundamental result which will allow us to worry less about the $\varepsilon \delta$ definition of continuity.

Theorem 6.3.2 The following assertions are valid.

1. The function, af $+b \mathbf{g}$ is continuous at $\mathbf{x}$ whenever $\mathbf{f}, \mathbf{g}$ are continuous at $\mathbf{x} \in D(\mathbf{f}) \cap$ $D(\mathbf{g})$ and $a, b \in \mathbb{F}$.
2. If $\mathbf{f}$ is continuous at $\mathbf{x}, \mathbf{f}(\mathbf{x}) \in D(\mathbf{g}) \subseteq \mathbb{F}^{p}$, and $\mathbf{g}$ is continuous at $\mathbf{f}(\mathbf{x})$, then $\mathbf{g} \circ \mathbf{f}$ is continuous at $\mathbf{x}$.
3. If $\mathbf{f}=\left(f_{1}, \cdots, f_{q}\right): D(\mathbf{f}) \rightarrow \mathbb{F}^{q}$, then $\mathbf{f}$ is continuous if and only if each $f_{k}$ is a continuous $\mathbb{F}$ valued function.
4. The function $f: \mathbb{F}^{p} \rightarrow \mathbb{F}$, given by $f(\mathbf{x})=|\mathbf{x}|$ is continuous.

The proof of this theorem is in the last section of this chapter. Its conclusions are not surprising. For example the first claim says that $(a \mathbf{f}+b \mathbf{g})(\mathbf{y})$ is close to $(a \mathbf{f}+b \mathbf{g})(\mathbf{x})$ when $\mathbf{y}$ is close to $\mathbf{x}$ provided the same can be said about $\mathbf{f}$ and $\mathbf{g}$. For the second claim, if $\mathbf{y}$ is close to $\mathbf{x}, \mathbf{f}(\mathbf{x})$ is close to $\mathbf{f}(\mathbf{y})$ and so by continuity of $\mathbf{g}$ at $\mathbf{f}(\mathbf{x}), \mathbf{g}(\mathbf{f}(\mathbf{y}))$ is close to $\mathbf{g}(\mathbf{f}(\mathbf{x}))$. To see the third claim is likely, note that closeness in $\mathbb{F}^{p}$ is the same as closeness in each coordinate. The fourth claim is immediate from the triangle inequality.

For functions defined on $\mathbb{F}^{n}$, there is a notion of polynomial just as there is for functions defined on $\mathbb{R}$.

Definition 6.3.3 Let $\alpha$ be an $n$ dimensional multi-index. This means

$$
\alpha=\left(\alpha_{1}, \cdots, \alpha_{n}\right)
$$

where each $\alpha_{i}$ is a natural number or zero. Also, let

$$
|\alpha| \equiv \sum_{i=1}^{n}\left|\alpha_{i}\right|
$$

The symbol, $\mathbf{x}^{\alpha}$,means

$$
\mathbf{x}^{\alpha} \equiv x_{1}^{\alpha_{1}} x_{2}^{\alpha_{2}} \cdots x_{3}^{\alpha_{n}}
$$

An $n$ dimensional polynomial of degree $m$ is a function of the form

$$
p(\mathbf{x})=\sum_{|\alpha| \leq m} d_{\alpha} \mathbf{x}^{\alpha}
$$

where the $d_{\alpha}$ are complex or real numbers.
The above theorem implies that polynomials are all continuous.

### 6.4 Exercises

1. Let $\mathbf{f}(t)=(t, \sin t)$. Show $f$ is continuous at every point $t$.
2. Suppose $|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})| \leq K|\mathbf{x}-\mathbf{y}|$ where $K$ is a constant. Show that $\mathbf{f}$ is everywhere continuous. Functions satisfying such an inequality are called Lipschitz functions.
3. Suppose $|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})| \leq K|\mathbf{x}-\mathbf{y}|^{\alpha}$ where $K$ is a constant and $\alpha \in(0,1)$. Show that $\mathbf{f}$ is everywhere continuous.
4. Suppose $f: \mathbb{F}^{3} \rightarrow \mathbb{F}$ is given by $f(\mathbf{x})=3 x_{1} x_{2}+2 x_{3}^{2}$. Use Theorem 6.3.2 to verify that $f$ is continuous. Hint: You should first verify that the function, $\pi_{k}: \mathbb{F}^{3} \rightarrow \mathbb{F}$ given by $\pi_{k}(\mathbf{x})=x_{k}$ is a continuous function.
5. Generalize the previous problem to the case where $f: \mathbb{F}^{q} \rightarrow \mathbb{F}$ is a polynomial.
6. State and prove a theorem which involves quotients of functions encountered in the previous problem.

### 6.5 Limits Of A Function

As in the case of scalar valued functions of one variable, a concept closely related to continuity is that of the limit of a function. The notion of limit of a function makes sense at points, $\mathbf{x}$, which are limit points of $D(\mathbf{f})$ and this concept is defined next.

Definition 6.5.1 Let $A \subseteq \mathbb{F}^{m}$ be a set. A point, $\mathbf{x}$, is a limit point of $A$ if $B(\mathbf{x}, r)$ contains infinitely many points of $A$ for every $r>0$.

Definition 6.5.2 Let $\mathbf{f}: D(\mathbf{f}) \subseteq \mathbb{F}^{p} \rightarrow \mathbb{F}^{q}$ be a function and let $\mathbf{x}$ be a limit point of $D(\mathbf{f})$. Then

$$
\lim _{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{f}(\mathbf{y})=\mathbf{L}
$$

if and only if the following condition holds. For all $\varepsilon>0$ there exists $\delta>0$ such that if

$$
0<|\mathbf{y}-\mathbf{x}|<\delta, \text { and } \mathbf{y} \in D(\mathbf{f})
$$

then,

$$
|\mathbf{L}-\mathbf{f}(\mathbf{y})|<\varepsilon .
$$

Theorem 6.5.3 If $\lim _{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{f}(\mathbf{y})=\mathbf{L}$ and $\lim _{y \rightarrow x} \mathbf{f}(\mathbf{y})=\mathbf{L}_{1}$, then $\mathbf{L}=\mathbf{L}_{1}$.
Proof: Let $\varepsilon>0$ be given. There exists $\delta>0$ such that if $0<|\mathbf{y}-\mathbf{x}|<\delta$ and $\mathbf{y} \in D(\mathbf{f})$, then

$$
|\mathbf{f}(\mathbf{y})-\mathbf{L}|<\varepsilon,\left|\mathbf{f}(\mathbf{y})-\mathbf{L}_{1}\right|<\varepsilon .
$$

Pick such a $\mathbf{y}$. There exists one because $\mathbf{x}$ is a limit point of $D(\mathbf{f})$. Then

$$
\left|\mathbf{L}-\mathbf{L}_{1}\right| \leq|\mathbf{L}-\mathbf{f}(\mathbf{y})|+\left|\mathbf{f}(\mathbf{y})-\mathbf{L}_{1}\right|<\varepsilon+\varepsilon=2 \varepsilon .
$$

Since $\varepsilon>0$ was arbitrary, this shows $\mathbf{L}=\mathbf{L}_{1}$.
As in the case of functions of one variable, one can define $\lim _{\mathbf{y} \rightarrow \mathbf{x}} f(\mathbf{x})= \pm \infty$.
Definition 6.5.4 If $f(\mathbf{x}) \in \mathbb{F}, \lim _{\mathbf{y} \rightarrow \mathbf{x}} f(\mathbf{x})=\infty$ if for every number $l$, there exists $\delta>0$ such that whenever $|\mathbf{y}-\mathbf{x}|<\delta$ and $\mathbf{y} \in D(\mathbf{f})$, then $f(\mathbf{x})>l$.

The following theorem is just like the one variable version presented earlier.
Theorem 6.5.5 Suppose $\lim _{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{f}(\mathbf{y})=\mathbf{L}$ and $\lim _{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{g}(\mathbf{y})=\mathbf{K}$ where $\mathbf{K}, \mathbf{L} \in \mathbb{F}^{q}$. Then if $a, b \in \mathbb{F}$,

$$
\begin{gather*}
\lim _{\mathbf{y} \rightarrow \mathbf{x}}(a \mathbf{f}(\mathbf{y})+b \mathbf{g}(\mathbf{y}))=a \mathbf{L}+b \mathbf{K}  \tag{6.5.1}\\
\lim _{y \rightarrow x} \mathbf{f} \cdot \mathbf{g}(y)=\mathbf{L K} \tag{6.5.2}
\end{gather*}
$$

and if $g$ is scalar valued with $\lim _{\mathbf{y} \rightarrow \mathbf{x}} g(\mathbf{y})=K \neq 0$,

$$
\begin{equation*}
\lim _{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{f}(\mathbf{y}) g(\mathbf{y})=\mathbf{L} K \tag{6.5.3}
\end{equation*}
$$

Also, if $\mathbf{h}$ is a continuous function defined near $\mathbf{L}$, then

$$
\begin{equation*}
\lim _{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{h} \circ \mathbf{f}(\mathbf{y})=\mathbf{h}(\mathbf{L}) . \tag{6.5.4}
\end{equation*}
$$

Suppose $\lim _{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{f}(\mathbf{y})=\mathbf{L}$. If $|\mathbf{f}(\mathbf{y})-\mathbf{b}| \leq r$ for all $\mathbf{y}$ sufficiently close to $\mathbf{x}$, then $|\mathbf{L}-\mathbf{b}| \leq r$ also.

Proof: The proof of 6.5 .1 is left for you. It is like a corresponding theorem for continuous functions. Now 6.5 .2is to be verified. Let $\varepsilon>0$ be given. Then by the triangle inequality,

$$
\begin{aligned}
|\mathbf{f} \cdot \mathbf{g}(\mathbf{y})-\mathbf{L} \cdot \mathbf{K}| & \leq|\mathbf{f g}(\mathbf{y})-\mathbf{f}(\mathbf{y}) \cdot \mathbf{K}|+|\mathbf{f}(\mathbf{y}) \cdot \mathbf{K}-\mathbf{L} \cdot \mathbf{K}| \\
& \leq|\mathbf{f}(\mathbf{y})||\mathbf{g}(\mathbf{y})-\mathbf{K}|+|\mathbf{K}||\mathbf{f}(\mathbf{y})-\mathbf{L}| .
\end{aligned}
$$

There exists $\delta_{1}$ such that if $0<|\mathbf{y}-\mathbf{x}|<\delta_{1}$ and $\mathbf{y} \in D(\mathbf{f})$, then

$$
|\mathbf{f}(\mathbf{y})-\mathbf{L}|<1
$$

and so for such $\mathbf{y}$, the triangle inequality implies, $|\mathbf{f}(\mathbf{y})|<1+|\mathbf{L}|$. Therefore, for $0<$ $|\mathbf{y}-\mathbf{x}|<\delta_{1}$,

$$
\begin{equation*}
|\mathbf{f} \cdot \mathbf{g}(\mathbf{y})-\mathbf{L} \cdot \mathbf{K}| \leq(1+|\mathbf{K}|+|\mathbf{L}|)[|\mathbf{g}(\mathbf{y})-\mathbf{K}|+|\mathbf{f}(\mathbf{y})-\mathbf{L}|] . \tag{6.5.5}
\end{equation*}
$$

Now let $0<\boldsymbol{\delta}_{2}$ be such that if $\mathbf{y} \in D(\mathbf{f})$ and $0<|\mathbf{x}-\mathbf{y}|<\boldsymbol{\delta}_{2}$,

$$
|\mathbf{f}(\mathbf{y})-\mathbf{L}|<\frac{\varepsilon}{2(1+|\mathbf{K}|+|\mathbf{L}|)},|\mathbf{g}(\mathbf{y})-\mathbf{K}|<\frac{\varepsilon}{2(1+|\mathbf{K}|+|\mathbf{L}|)}
$$

Then letting $0<\delta \leq \min \left(\delta_{1}, \delta_{2}\right)$, it follows from 6.5.5 that

$$
|\mathbf{f} \cdot \mathbf{g}(\mathbf{y})-\mathbf{L} \cdot \mathbf{K}|<\varepsilon
$$

and this proves 6.5.2.
The proof of 6.5.3 is left to you.
Consider 6.5.4. Since $\mathbf{h}$ is continuous near $\mathbf{L}$, it follows that for $\varepsilon>0$ given, there exists $\eta>0$ such that if $|\mathbf{y}-\mathbf{L}|<\eta$, then

$$
|\mathbf{h}(\mathbf{y})-\mathbf{h}(\mathbf{L})|<\varepsilon
$$

Now since $\lim _{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{f}(\mathbf{y})=\mathbf{L}$, there exists $\delta>0$ such that if $0<|\mathbf{y}-\mathbf{x}|<\boldsymbol{\delta}$, then

$$
|\mathbf{f}(\mathbf{y})-\mathbf{L}|<\eta
$$

Therefore, if $0<|\mathbf{y}-\mathbf{x}|<\boldsymbol{\delta}$,

$$
|\mathbf{h}(\mathbf{f}(\mathbf{y}))-\mathbf{h}(\mathbf{L})|<\varepsilon .
$$

It only remains to verify the last assertion. Assume $|\mathbf{f}(\mathbf{y})-\mathbf{b}| \leq r$ for all $\mathbf{y}$ close enough to $\mathbf{x}$. It is required to show that $|\mathbf{L}-\mathbf{b}| \leq r$. If this is not true, then $|\mathbf{L}-\mathbf{b}|>r$. Consider $B(\mathbf{L},|\mathbf{L}-\mathbf{b}|-r)$. Since $\mathbf{L}$ is the limit of $\mathbf{f}$, it follows $\mathbf{f}(\mathbf{y}) \in B(\mathbf{L},|\mathbf{L}-\mathbf{b}|-r)$ whenever $\mathbf{y} \in D(\mathbf{f})$ is close enough to $\mathbf{x}$. Thus, by the triangle inequality,

$$
|\mathbf{f}(\mathbf{y})-\mathbf{L}|<|\mathbf{L}-\mathbf{b}|-r
$$

and so

$$
\begin{aligned}
r & <|\mathbf{L}-\mathbf{b}|-|\mathbf{f}(\mathbf{y})-\mathbf{L}| \leq||\mathbf{b}-\mathbf{L}|-| \mathbf{f}(\mathbf{y})-\mathbf{L} \| \\
& \leq|\mathbf{b}-\mathbf{f}(\mathbf{y})|
\end{aligned}
$$

a contradiction to the assumption that $|\mathbf{b}-\mathbf{f}(\mathbf{y})| \leq r$.
Theorem 6.5.6 For $\mathbf{f}: D(\mathbf{f}) \rightarrow \mathbb{F}^{q}$ and $\mathbf{x} \in D(\mathbf{f})$ a limit point of $D(\mathbf{f})$, $\mathbf{f}$ is continuous at $\mathbf{x}$ if and only if

$$
\lim _{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{f}(\mathbf{y})=\mathbf{f}(\mathbf{x})
$$

Proof: First suppose $\mathbf{f}$ is continuous at $\mathbf{x}$ a limit point of $D(\mathbf{f})$. Then for every $\varepsilon>0$ there exists $\delta>0$ such that if $|\mathbf{y}-\mathbf{x}|<\delta$ and $\mathbf{y} \in D(\mathbf{f})$, then $|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})|<\varepsilon$. In particular, this holds if $0<|\mathbf{x}-\mathbf{y}|<\delta$ and this is just the definition of the limit. Hence $\mathbf{f}(\mathbf{x})=$ $\lim _{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{f}(\mathbf{y})$.

Next suppose $\mathbf{x}$ is a limit point of $D(\mathbf{f})$ and $\lim _{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{f}(\mathbf{y})=\mathbf{f}(\mathbf{x})$. This means that if $\varepsilon>0$ there exists $\delta>0$ such that for $0<|\mathbf{x}-\mathbf{y}|<\delta$ and $\mathbf{y} \in D(\mathbf{f})$, it follows $|\mathbf{f}(\mathbf{y})-\mathbf{f}(\mathbf{x})|<\varepsilon$. However, if $\mathbf{y}=\mathbf{x}$, then $|\mathbf{f}(\mathbf{y})-\mathbf{f}(\mathbf{x})|=|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{x})|=0$ and so whenever $\mathbf{y} \in D(\mathbf{f})$ and $|\mathbf{x}-\mathbf{y}|<\delta$, it follows $|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})|<\varepsilon$, showing $\mathbf{f}$ is continuous at $\mathbf{x}$.

The following theorem is important.
Theorem 6.5.7 Suppose $\mathbf{f}: D(\mathbf{f}) \rightarrow \mathbb{F}^{q}$. Then for $\mathbf{x}$ a limit point of $D(\mathbf{f})$,

$$
\begin{equation*}
\lim _{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{f}(\mathbf{y})=\mathbf{L} \tag{6.5.6}
\end{equation*}
$$

if and only if

$$
\begin{equation*}
\lim _{\mathbf{y} \rightarrow \mathbf{x}} f_{k}(\mathbf{y})=L_{k} \tag{6.5.7}
\end{equation*}
$$

where $\mathbf{f}(\mathbf{y}) \equiv\left(f_{1}(\mathbf{y}), \cdots, f_{p}(\mathbf{y})\right)$ and $\mathbf{L} \equiv\left(L_{1}, \cdots, L_{p}\right)$.
Proof: Suppose 6.5.6. Then letting $\varepsilon>0$ be given there exists $\delta>0$ such that if $0<|\mathbf{y}-\mathbf{x}|<\delta$, it follows

$$
\left|f_{k}(\mathbf{y})-L_{k}\right| \leq|\mathbf{f}(\mathbf{y})-\mathbf{L}|<\varepsilon
$$

which verifies 6.5.7.
Now suppose 6.5.7 holds. Then letting $\varepsilon>0$ be given, there exists $\delta_{k}$ such that if $0<|\mathbf{y}-\mathbf{x}|<\boldsymbol{\delta}_{k}$, then

$$
\left|f_{k}(\mathbf{y})-L_{k}\right|<\frac{\varepsilon}{\sqrt{p}}
$$

Let $0<\boldsymbol{\delta}<\min \left(\delta_{1}, \cdots, \delta_{p}\right)$. Then if $0<|\mathbf{y}-\mathbf{x}|<\boldsymbol{\delta}$, it follows

$$
\begin{aligned}
|\mathbf{f}(\mathbf{y})-\mathbf{L}| & =\left(\sum_{k=1}^{p}\left|f_{k}(\mathbf{y})-L_{k}\right|^{2}\right)^{1 / 2} \\
& <\left(\sum_{k=1}^{p} \frac{\varepsilon^{2}}{p}\right)^{1 / 2}=\varepsilon
\end{aligned}
$$

This proves the theorem.
This theorem shows it suffices to consider the components of a vector valued function when computing the limit.

Example 6.5.8 Find $\lim _{(x, y) \rightarrow(3,1)}\left(\frac{x^{2}-9}{x-3}, y\right)$.
It is clear that $\lim _{(x, y) \rightarrow(3,1)} \frac{x^{2}-9}{x-3}=6$ and $\lim _{(x, y) \rightarrow(3,1)} y=1$. Therefore, this limit equals $(6,1)$.

Example 6.5.9 Find $\lim _{(x, y) \rightarrow(0,0)} \frac{x y}{x^{2}+y^{2}}$.
First of all observe the domain of the function is $\mathbb{F}^{2} \backslash\{(0,0)\}$, every point in $\mathbb{F}^{2}$ except the origin. Therefore, $(0,0)$ is a limit point of the domain of the function so it might make sense to take a limit. However, just as in the case of a function of one variable, the limit may not exist. In fact, this is the case here. To see this, take points on the line $y=0$. At these points, the value of the function equals 0 . Now consider points on the line $y=x$ where the value of the function equals $1 / 2$. Since arbitrarily close to $(0,0)$ there are points where the function equals $1 / 2$ and points where the function has the value 0 , it follows there can be no limit. Just take $\varepsilon=1 / 10$ for example. You can't be within $1 / 10$ of $1 / 2$ and also within $1 / 10$ of 0 at the same time.

Note it is necessary to rely on the definition of the limit much more than in the case of a function of one variable and it is the case there are no easy ways to do limit problems for functions of more than one variable. It is what it is and you will not deal with these concepts without agony.

### 6.6 Exercises

1. Find the following limits if possible
(a) $\lim _{(x, y) \rightarrow(0,0)} \frac{x^{2}-y^{2}}{x^{2}+y^{2}}$
(b) $\lim _{(x, y) \rightarrow(0,0)} \frac{x\left(x^{2}-y^{2}\right)}{\left(x^{2}+y^{2}\right)}$
(c) $\lim _{(x, y) \rightarrow(0,0)} \frac{\left(x^{2}-y^{4}\right)^{2}}{\left(x^{2}+y^{4}\right)^{2}}$ Hint: Consider along $y=0$ and along $x=y^{2}$.
(d) $\lim _{(x, y) \rightarrow(0,0)} x \sin \left(\frac{1}{x^{2}+y^{2}}\right)$
(e) The limit as $(x, y) \rightarrow(1,2)$ of the expression

$$
\frac{-2 y x^{2}+8 y x+34 y+3 y^{3}-18 y^{2}+6 x^{2}-13 x-20-x y^{2}-x^{3}}{-y^{2}+4 y-5-x^{2}+2 x} .
$$

Hint: It might help to write this in terms of the variables $(s, t)=(x-1, y-2)$.
2. In the definition of limit, why must $\mathbf{x}$ be a limit point of $D(\mathbf{f})$ ? Hint: If $\mathbf{x}$ were not a limit point of $D(\mathbf{f})$, show there exists $\delta>0$ such that $B(\mathbf{x}, \boldsymbol{\delta})$ contains no points of $D(\mathbf{f})$ other than possibly $\mathbf{x}$ itself. Argue that 33.3 is a limit and that so is 22 and 7 and 11. In other words the concept is totally worthless.

### 6.7 The Limit Of A Sequence

As in the case of real numbers, one can consider the limit of a sequence of points in $\mathbb{F}^{p}$.
Definition 6.7.1 A sequence $\left\{\mathbf{a}_{n}\right\}_{n=1}^{\infty}$ converges to $\mathbf{a}$, and write

$$
\lim _{n \rightarrow \infty} \mathbf{a}_{n}=\mathbf{a} \text { or } \mathbf{a}_{n} \rightarrow \mathbf{a}
$$

if and only if for every $\varepsilon>0$ there exists $n_{\varepsilon}$ such that whenever $n \geq n_{\varepsilon}$,

$$
\left|\mathbf{a}_{n}-\mathbf{a}\right|<\varepsilon .
$$

In words the definition says that given any measure of closeness, $\varepsilon$, the terms of the sequence are eventually all this close to $\mathbf{a}$. There is absolutely no difference between this and the definition for sequences of numbers other than here bold face is used to indicate $\mathbf{a}_{n}$ and $\mathbf{a}$ are points in $\mathbb{F}^{p}$.

Theorem 6.7.2 If $\lim _{n \rightarrow \infty} \mathbf{a}_{n}=\mathbf{a}$ and $\lim _{n \rightarrow \infty} \mathbf{a}_{n}=\mathbf{a}_{1}$ then $\mathbf{a}_{1}=\mathbf{a}$.
Proof: Suppose $\mathbf{a}_{1} \neq \mathbf{a}$. Then let $0<\varepsilon<\left|\mathbf{a}_{1}-\mathbf{a}\right| / 2$ in the definition of the limit. It follows there exists $n_{\varepsilon}$ such that if $n \geq n_{\varepsilon}$, then $\left|\mathbf{a}_{n}-\mathbf{a}\right|<\varepsilon$ and $\left|\mathbf{a}_{n}-\mathbf{a}_{1}\right|<\varepsilon$. Therefore, for such $n$,

$$
\begin{aligned}
\left|\mathbf{a}_{1}-\mathbf{a}\right| & \leq\left|\mathbf{a}_{1}-\mathbf{a}_{n}\right|+\left|\mathbf{a}_{n}-\mathbf{a}\right| \\
& <\varepsilon+\varepsilon<\left|\mathbf{a}_{1}-\mathbf{a}\right| / 2+\left|\mathbf{a}_{1}-\mathbf{a}\right| / 2=\left|\mathbf{a}_{1}-\mathbf{a}\right|
\end{aligned}
$$

a contradiction.
As in the case of a vector valued function, it suffices to consider the components. This is the content of the next theorem.

Theorem 6.7.3 Let $\mathbf{a}_{n}=\left(a_{1}^{n}, \cdots, a_{p}^{n}\right) \in \mathbb{F}^{p}$. Then $\lim _{n \rightarrow \infty} \mathbf{a}_{n}=\mathbf{a} \equiv\left(a_{1}, \cdots, a_{p}\right)$ if and only if for each $k=1, \cdots, p$,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} a_{k}^{n}=a_{k} \tag{6.7.8}
\end{equation*}
$$

Proof: First suppose $\lim _{n \rightarrow \infty} \mathbf{a}_{n}=\mathbf{a}$. Then given $\varepsilon>0$ there exists $n_{\varepsilon}$ such that if $n>n_{\varepsilon}$, then

$$
\left|a_{k}^{n}-a_{k}\right| \leq\left|\mathbf{a}_{n}-\mathbf{a}\right|<\varepsilon
$$

which establishes 6.7.8.
Now suppose 6.7.8 holds for each $k$. Then letting $\varepsilon>0$ be given there exist $n_{k}$ such that if $n>n_{k}$,

$$
\left|a_{k}^{n}-a_{k}\right|<\varepsilon / \sqrt{p} .
$$

Therefore, letting $n_{\varepsilon}>\max \left(n_{1}, \cdots, n_{p}\right)$, it follows that for $n>n_{\varepsilon}$,

$$
\left|\mathbf{a}_{n}-\mathbf{a}\right|=\left(\sum_{k=1}^{n}\left|a_{k}^{n}-a_{k}\right|^{2}\right)^{1 / 2}<\left(\sum_{k=1}^{n} \frac{\varepsilon^{2}}{p}\right)^{1 / 2}=\varepsilon
$$

showing that $\lim _{n \rightarrow \infty} \mathbf{a}_{n}=\mathbf{a}$. This proves the theorem.
Example 6.7.4 Let $\mathbf{a}_{n}=\left(\frac{1}{n^{2}+1}, \frac{1}{n} \sin (n), \frac{n^{2}+3}{3 n^{2}+5 n}\right)$.
It suffices to consider the limits of the components according to the following theorem. Thus the limit is $(0,0,1 / 3)$.

Theorem 6.7.5 Suppose $\left\{\mathbf{a}_{n}\right\}$ and $\left\{\mathbf{b}_{n}\right\}$ are sequences and that

$$
\lim _{n \rightarrow \infty} \mathbf{a}_{n}=\mathbf{a} \text { and } \lim _{n \rightarrow \infty} \mathbf{b}_{n}=\mathbf{b}
$$

Also suppose $x$ and $y$ are numbers in $\mathbb{F}$. Then

$$
\begin{align*}
\lim _{n \rightarrow \infty} x \mathbf{a}_{n}+y \mathbf{b}_{n} & =x \mathbf{a}+y \mathbf{b}  \tag{6.7.9}\\
\lim _{n \rightarrow \infty} \mathbf{a}_{n} \cdot \mathbf{b}_{n} & =\mathbf{a} \cdot \mathbf{b} \tag{6.7.10}
\end{align*}
$$

If $b_{n} \in \mathbb{F}$, then

$$
\mathbf{a}_{n} b_{n} \rightarrow \mathbf{a} b
$$

Proof: The first of these claims is left for you to do. To do the second, let $\varepsilon>0$ be given and choose $n_{1}$ such that if $n \geq n_{1}$ then

$$
\left|\mathbf{a}_{n}-\mathbf{a}\right|<1 .
$$

Then for such $n$, the triangle inequality and Cauchy Schwarz inequality imply

$$
\begin{aligned}
\left|\mathbf{a}_{n} \cdot \mathbf{b}_{n}-\mathbf{a} \cdot \mathbf{b}\right| & \leq\left|\mathbf{a}_{n} \cdot \mathbf{b}_{n}-\mathbf{a}_{n} \cdot \mathbf{b}\right|+\left|\mathbf{a}_{n} \cdot \mathbf{b}-\mathbf{a} \cdot \mathbf{b}\right| \\
& \leq\left|\mathbf{a}_{n}\right|\left|\mathbf{b}_{n}-\mathbf{b}\right|+|\mathbf{b}|\left|\mathbf{a}_{n}-\mathbf{a}\right| \\
& \leq(|\mathbf{a}|+1)\left|\mathbf{b}_{n}-\mathbf{b}\right|+|\mathbf{b}|\left|\mathbf{a}_{n}-\mathbf{a}\right|
\end{aligned}
$$

Now let $n_{2}$ be large enough that for $n \geq n_{2}$,

$$
\left|\mathbf{b}_{n}-\mathbf{b}\right|<\frac{\varepsilon}{2(|\mathbf{a}|+1)}, \text { and }\left|\mathbf{a}_{n}-\mathbf{a}\right|<\frac{\varepsilon}{2(|\mathbf{b}|+1)}
$$

Such a number exists because of the definition of limit. Therefore, let

$$
n_{\varepsilon}>\max \left(n_{1}, n_{2}\right)
$$

For $n \geq n_{\varepsilon}$,

$$
\begin{aligned}
\left|\mathbf{a}_{n} \cdot \mathbf{b}_{n}-\mathbf{a} \cdot \mathbf{b}\right| & \leq(|\mathbf{a}|+1)\left|\mathbf{b}_{n}-\mathbf{b}\right|+|\mathbf{b}|\left|\mathbf{a}_{n}-\mathbf{a}\right| \\
& <(|\mathbf{a}|+1) \frac{\varepsilon}{2(|\mathbf{a}|+1)}+|\mathbf{b}| \frac{\varepsilon}{2(|\mathbf{b}|+1)} \leq \varepsilon
\end{aligned}
$$

This proves 6.7.9. The proof of 6.7.10 is entirely similar and is left for you.

### 6.7.1 Sequences And Completeness

Recall the definition of a Cauchy sequence.
Definition 6.7.6 $\left\{\mathbf{a}_{n}\right\}$ is a Cauchy sequence iffor all $\varepsilon>0$, there exists $n_{\varepsilon}$ such that whenever $n, m \geq n_{\varepsilon}$,

$$
\left|\mathbf{a}_{n}-\mathbf{a}_{m}\right|<\varepsilon .
$$

A sequence is Cauchy means the terms are "bunching up to each other" as $m, n$ get large.

Theorem 6.7.7 Let $\left\{\mathbf{a}_{n}\right\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{F}^{p}$. Then there exists a unique $\mathbf{a} \in \mathbb{F}^{p}$ such that $\mathbf{a}_{n} \rightarrow \mathbf{a}$.

Proof: Let $\mathbf{a}_{n}=\left(a_{1}^{n}, \cdots, a_{p}^{n}\right)$. Then

$$
\left|a_{k}^{n}-a_{k}^{m}\right| \leq\left|\mathbf{a}_{n}-\mathbf{a}_{m}\right|
$$

which shows for each $k=1, \cdots, p$, it follows $\left\{a_{k}^{n}\right\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{F}$. This requires that both the real and imaginary parts of $a_{k}^{n}$ are Cauchy sequences in $\mathbb{R}$ which means the real and imaginary parts converge in $\mathbb{R}$. This shows $\left\{a_{k}^{n}\right\}_{n=1}^{\infty}$ must converge to some $a_{k}$. That is $\lim _{n \rightarrow \infty} a_{k}^{n}=a_{k}$. Letting $\mathbf{a}=\left(a_{1}, \cdots, a_{p}\right)$, it follows from Theorem 6.7.3 that

$$
\lim _{n \rightarrow \infty} \mathbf{a}_{n}=\mathbf{a}
$$

This proves the theorem.
Theorem 6.7.8 The set of terms in a Cauchy sequence in $\mathbb{F}^{p}$ is bounded in the sense that for all $n,\left|\mathbf{a}_{n}\right|<M$ for some $M<\infty$.

Proof: Let $\varepsilon=1$ in the definition of a Cauchy sequence and let $n>n_{1}$. Then from the definition,

$$
\left|\mathbf{a}_{n}-\mathbf{a}_{n_{1}}\right|<1
$$

It follows that for all $n>n_{1}$,

$$
\left|\mathbf{a}_{n}\right|<1+\left|\mathbf{a}_{n_{1}}\right| .
$$

Therefore, for all $n$,

$$
\left|\mathbf{a}_{n}\right| \leq 1+\left|\mathbf{a}_{n_{1}}\right|+\sum_{k=1}^{n_{1}}\left|\mathbf{a}_{k}\right|
$$

This proves the theorem.

Theorem 6.7.9 If a sequence $\left\{\mathbf{a}_{n}\right\}$ in $\mathbb{F}^{p}$ converges, then the sequence is a Cauchy sequence.

Proof: Let $\varepsilon>0$ be given and suppose $\mathbf{a}_{n} \rightarrow \mathbf{a}$. Then from the definition of convergence, there exists $n_{\varepsilon}$ such that if $n>n_{\varepsilon}$, it follows that

$$
\left|\mathbf{a}_{n}-\mathbf{a}\right|<\frac{\varepsilon}{2}
$$

Therefore, if $m, n \geq n_{\varepsilon}+1$, it follows that

$$
\left|\mathbf{a}_{n}-\mathbf{a}_{m}\right| \leq\left|\mathbf{a}_{n}-\mathbf{a}\right|+\left|\mathbf{a}-\mathbf{a}_{m}\right|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon
$$

showing that, since $\varepsilon>0$ is arbitrary, $\left\{\mathbf{a}_{n}\right\}$ is a Cauchy sequence.

### 6.7.2 Continuity And The Limit Of A Sequence

Just as in the case of a function of one variable, there is a very useful way of thinking of continuity in terms of limits of sequences found in the following theorem. In words, it says a function is continuous if it takes convergent sequences to convergent sequences whenever possible.

Theorem 6.7.10 A function $\mathbf{f}: D(\mathbf{f}) \rightarrow \mathbb{F}^{q}$ is continuous at $\mathbf{x} \in D(\mathbf{f})$ if and only if, whenever $\mathbf{x}_{n} \rightarrow \mathbf{x}$ with $\mathbf{x}_{n} \in D(\mathbf{f})$, it follows $\mathbf{f}\left(\mathbf{x}_{n}\right) \rightarrow \mathbf{f}(\mathbf{x})$.

Proof: Suppose first that $\mathbf{f}$ is continuous at $\mathbf{x}$ and let $\mathbf{x}_{n} \rightarrow \mathbf{x}$. Let $\varepsilon>0$ be given. By continuity, there exists $\delta>0$ such that if $|\mathbf{y}-\mathbf{x}|<\delta$, then $|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})|<\varepsilon$. However, there exists $n_{\delta}$ such that if $n \geq n_{\delta}$, then $\left|\mathbf{x}_{n}-\mathbf{x}\right|<\delta$ and so for all $n$ this large,

$$
\left|\mathbf{f}(\mathbf{x})-\mathbf{f}\left(\mathbf{x}_{n}\right)\right|<\varepsilon
$$

which shows $\mathbf{f}\left(\mathbf{x}_{n}\right) \rightarrow \mathbf{f}(\mathbf{x})$.
Now suppose the condition about taking convergent sequences to convergent sequences holds at $\mathbf{x}$. Suppose $\mathbf{f}$ fails to be continuous at $\mathbf{x}$. Then there exists $\varepsilon>0$ and $\mathbf{x}_{n} \in D(f)$ such that $\left|\mathbf{x}-\mathbf{x}_{n}\right|<\frac{1}{n}$, yet

$$
\left|\mathbf{f}(\mathbf{x})-\mathbf{f}\left(\mathbf{x}_{n}\right)\right| \geq \varepsilon .
$$

But this is clearly a contradiction because, although $\mathbf{x}_{n} \rightarrow \mathbf{x}, \mathbf{f}\left(\mathbf{x}_{n}\right)$ fails to converge to $\mathbf{f}(\mathbf{x})$. It follows $\mathbf{f}$ must be continuous after all. This proves the theorem.

### 6.8 Properties Of Continuous Functions

Functions of $p$ variables have many of the same properties as functions of one variable. First there is a version of the extreme value theorem generalizing the one dimensional case.

Theorem 6.8.1 Let $C$ be closed and bounded and let $f: C \rightarrow \mathbb{R}$ be continuous. Then $f$ achieves its maximum and its minimum on $C$. This means there exist, $\mathbf{x}_{1}, \mathbf{x}_{2} \in C$ such that for all $\mathbf{x} \in C$,

$$
f\left(\mathbf{x}_{1}\right) \leq f(\mathbf{x}) \leq f\left(\mathbf{x}_{2}\right)
$$

There is also the long technical theorem about sums and products of continuous functions. These theorems are proved in the next section.

Theorem 6.8.2 The following assertions are valid

1. The function, af $+b \mathbf{g}$ is continuous at $\mathbf{x}$ when $\mathbf{f}, \mathbf{g}$ are continuous at $\mathbf{x} \in D(\mathbf{f}) \cap D(\mathbf{g})$ and $a, b \in \mathbb{F}$.
2. If and $f$ and $g$ are each $\mathbb{F}$ valued functions continuous at $\mathbf{x}$, then $f g$ is continuous at $\mathbf{x}$. If, in addition to this, $g(\mathbf{x}) \neq 0$, then $f / g$ is continuous at $\mathbf{x}$.
3. If $\mathbf{f}$ is continuous at $\mathbf{x}, \mathbf{f}(\mathbf{x}) \in D(\mathbf{g}) \subseteq \mathbb{F}^{p}$, and $\mathbf{g}$ is continuous at $\mathbf{f}(\mathbf{x})$, then $\mathbf{g} \circ \mathbf{f}$ is continuous at $\mathbf{x}$.
4. If $\mathbf{f}=\left(f_{1}, \cdots, f_{q}\right): D(\mathbf{f}) \rightarrow \mathbb{F}^{q}$, then $\mathbf{f}$ is continuous if and only if each $f_{k}$ is a continuous $\mathbb{F}$ valued function.
5. The function $f: \mathbb{F}^{p} \rightarrow \mathbb{F}$, given by $f(\mathbf{x})=|\mathbf{x}|$ is continuous.

### 6.9 Exercises

1. $\mathbf{f}: D \subseteq \mathbb{F}^{p} \rightarrow \mathbb{F}^{q}$ is Lipschitz continuous or just Lipschitz for short if there exists a constant, $K$ such that

$$
|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})| \leq K|\mathbf{x}-\mathbf{y}|
$$

for all $\mathbf{x}, \mathbf{y} \in D$. Show every Lipschitz function is uniformly continuous which means that given $\varepsilon>0$ there exists $\delta>0$ independent of $\mathbf{x}$ such that if $|\mathbf{x}-\mathbf{y}|<\delta$, then $|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})|<\varepsilon$.
2. If $\mathbf{f}$ is uniformly continuous, does it follow that $|\mathbf{f}|$ is also uniformly continuous? If $|\mathbf{f}|$ is uniformly continuous does it follow that $\mathbf{f}$ is uniformly continuous? Answer the same questions with "uniformly continuous" replaced with "continuous". Explain why.

### 6.10 Proofs Of Theorems

This section contains the proofs of the theorems which were just stated without proof.
Theorem 6.10.1 The following assertions are valid

1. The function, $a \mathbf{f}+b \mathbf{g}$ is continuous at $\mathbf{x}$ when $\mathbf{f}, \mathbf{g}$ are continuous at $\mathbf{x} \in D(\mathbf{f}) \cap D(\mathbf{g})$ and $a, b \in \mathbb{F}$.
2. If and $f$ and $g$ are each $\mathbb{F}$ valued functions continuous at $\mathbf{x}$, then $f g$ is continuous at $\mathbf{x}$. If, in addition to this, $g(\mathbf{x}) \neq 0$, then $f / g$ is continuous at $\mathbf{x}$.
3. If $\mathbf{f}$ is continuous at $\mathbf{x}, \mathbf{f}(\mathbf{x}) \in D(\mathbf{g}) \subseteq \mathbb{F}^{p}$, and $\mathbf{g}$ is continuous at $\mathbf{f}(\mathbf{x})$, then $\mathbf{g} \circ \mathbf{f}$ is continuous at $\mathbf{x}$.
4. If $\mathbf{f}=\left(f_{1}, \cdots, f_{q}\right): D(\mathbf{f}) \rightarrow \mathbb{F}^{q}$, then $\mathbf{f}$ is continuous if and only if each $f_{k}$ is a continuous $\mathbb{F}$ valued function.
5. The function $f: \mathbb{F}^{p} \rightarrow \mathbb{F}$, given by $f(\mathbf{x})=|\mathbf{x}|$ is continuous.

Proof: Begin with 1.) Let $\varepsilon>0$ be given. By assumption, there exist $\delta_{1}>0$ such that whenever $|\mathbf{x}-\mathbf{y}|<\delta_{1}$, it follows $|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})|<\frac{\varepsilon}{2(|a|+|b|+1)}$ and there exists $\boldsymbol{\delta}_{2}>0$ such that whenever $|\mathbf{x}-\mathbf{y}|<\delta_{2}$, it follows that $|\mathbf{g}(\mathbf{x})-\mathbf{g}(\mathbf{y})|<\frac{\varepsilon}{2(|a|+|b|+1)}$. Then let $0<\delta \leq$ $\min \left(\delta_{1}, \boldsymbol{\delta}_{2}\right)$. If $|\mathbf{x}-\mathbf{y}|<\boldsymbol{\delta}$, then everything happens at once. Therefore, using the triangle inequality

$$
\begin{gathered}
|a \mathbf{f}(\mathbf{x})+b \mathbf{f}(\mathbf{x})-(a \mathbf{g}(\mathbf{y})+b \mathbf{g}(\mathbf{y}))| \\
\leq|a||\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})|+|b||\mathbf{g}(\mathbf{x})-\mathbf{g}(\mathbf{y})| \\
<|a|\left(\frac{\varepsilon}{2(|a|+|b|+1)}\right)+|b|\left(\frac{\varepsilon}{2(|a|+|b|+1)}\right)<\varepsilon
\end{gathered}
$$

Now begin on 2.) There exists $\delta_{1}>0$ such that if $|\mathbf{y}-\mathbf{x}|<\delta_{1}$, then

$$
|f(\mathbf{x})-f(\mathbf{y})|<1
$$

Therefore, for such $\mathbf{y}$,

$$
|f(\mathbf{y})|<1+|f(\mathbf{x})|
$$

It follows that for such $\mathbf{y}$,

$$
\begin{aligned}
\mid f g(\mathbf{x}) & -f g(\mathbf{y})|\leq|f(\mathbf{x}) g(\mathbf{x})-g(\mathbf{x}) f(\mathbf{y})|+|g(\mathbf{x}) f(\mathbf{y})-f(\mathbf{y}) g(\mathbf{y})| \\
& \leq|g(\mathbf{x})||f(\mathbf{x})-f(\mathbf{y})|+|f(\mathbf{y})||g(\mathbf{x})-g(\mathbf{y})| \\
& \leq(1+|g(\mathbf{x})|+|f(\mathbf{y})|)[|g(\mathbf{x})-g(\mathbf{y})|+|f(\mathbf{x})-f(\mathbf{y})|]
\end{aligned}
$$

Now let $\varepsilon>0$ be given. There exists $\boldsymbol{\delta}_{2}$ such that if $|\mathbf{x}-\mathbf{y}|<\boldsymbol{\delta}_{2}$, then

$$
|g(\mathbf{x})-g(\mathbf{y})|<\frac{\varepsilon}{2(1+|g(\mathbf{x})|+|f(\mathbf{y})|)},
$$

and there exists $\boldsymbol{\delta}_{3}$ such that if $|\mathbf{x}-\mathbf{y}|<\boldsymbol{\delta}_{3}$, then

$$
|f(\mathbf{x})-f(\mathbf{y})|<\frac{\varepsilon}{2(1+|g(\mathbf{x})|+|f(\mathbf{y})|)}
$$

Now let $0<\boldsymbol{\delta} \leq \min \left(\delta_{1}, \delta_{2}, \delta_{3}\right)$. Then if $|\mathbf{x}-\mathbf{y}|<\boldsymbol{\delta}$, all the above hold at once and

$$
\begin{gathered}
|f g(\mathbf{x})-f g(\mathbf{y})| \leq \\
(1+|g(\mathbf{x})|+|f(\mathbf{y})|)[|g(\mathbf{x})-g(\mathbf{y})|+|f(\mathbf{x})-f(\mathbf{y})|] \\
<(1+|g(\mathbf{x})|+|f(\mathbf{y})|)\left(\frac{\varepsilon}{2(1+|g(\mathbf{x})|+|f(\mathbf{y})|)}+\frac{\varepsilon}{2(1+|g(\mathbf{x})|+|f(\mathbf{y})|)}\right)=\varepsilon .
\end{gathered}
$$

This proves the first part of 2.) To obtain the second part, let $\delta_{1}$ be as described above and let $\delta_{0}>0$ be such that for $|\mathbf{x}-\mathbf{y}|<\boldsymbol{\delta}_{0}$,

$$
|g(\mathbf{x})-g(\mathbf{y})|<|g(\mathbf{x})| / 2
$$

and so by the triangle inequality,

$$
-|g(\mathbf{x})| / 2 \leq|g(\mathbf{y})|-|g(\mathbf{x})| \leq|g(\mathbf{x})| / 2
$$

which implies $|g(\mathbf{y})| \geq|g(\mathbf{x})| / 2$, and $|g(\mathbf{y})|<3|g(\mathbf{x})| / 2$.
Then if $|\mathbf{x}-\mathbf{y}|<\min \left(\boldsymbol{\delta}_{0}, \boldsymbol{\delta}_{1}\right)$,

$$
\begin{aligned}
& \left|\frac{f(\mathbf{x})}{g(\mathbf{x})}-\frac{f(\mathbf{y})}{g(\mathbf{y})}\right|=\left|\frac{f(\mathbf{x}) g(\mathbf{y})-f(\mathbf{y}) g(\mathbf{x})}{g(\mathbf{x}) g(\mathbf{y})}\right| \\
& \leq \frac{|f(\mathbf{x}) g(\mathbf{y})-f(\mathbf{y}) g(\mathbf{x})|}{\left(\frac{|g(\mathbf{x})|^{2}}{2}\right)} \\
& =\frac{2|f(\mathbf{x}) g(\mathbf{y})-f(\mathbf{y}) g(\mathbf{x})|}{|g(\mathbf{x})|^{2}} \\
& \leq \frac{2}{|g(\mathbf{x})|^{2}}[|f(\mathbf{x}) g(\mathbf{y})-f(\mathbf{y}) g(\mathbf{y})+f(\mathbf{y}) g(\mathbf{y})-f(\mathbf{y}) g(\mathbf{x})|] \\
& \leq \frac{2}{|g(\mathbf{x})|^{2}}[|g(\mathbf{y})||f(\mathbf{x})-f(\mathbf{y})|+|f(\mathbf{y})||g(\mathbf{y})-g(\mathbf{x})|] \\
& \leq \frac{2}{|g(\mathbf{x})|^{2}}\left[\frac{3}{2}|\mathbf{g}(\mathbf{x})||f(\mathbf{x})-f(\mathbf{y})|+(1+|f(\mathbf{x})|)|g(\mathbf{y})-g(\mathbf{x})|\right] \\
& \leq \frac{2}{|g(\mathbf{x})|^{2}}(1+2|f(\mathbf{x})|+2|g(\mathbf{x})|)[|f(\mathbf{x})-f(\mathbf{y})|+|g(\mathbf{y})-g(\mathbf{x})|] \\
& \equiv M[|f(\mathbf{x})-f(\mathbf{y})|+|g(\mathbf{y})-g(\mathbf{x})|]
\end{aligned}
$$

where

$$
M \equiv \frac{2}{|g(\mathbf{x})|^{2}}(1+2|f(\mathbf{x})|+2|g(\mathbf{x})|)
$$

Now let $\delta_{2}$ be such that if $|\mathbf{x}-\mathbf{y}|<\boldsymbol{\delta}_{2}$, then

$$
|f(\mathbf{x})-f(\mathbf{y})|<\frac{\varepsilon}{2} M^{-1}
$$

and let $\delta_{3}$ be such that if $|\mathbf{x}-\mathbf{y}|<\delta_{3}$, then

$$
|g(\mathbf{y})-g(\mathbf{x})|<\frac{\varepsilon}{2} M^{-1} .
$$

Then if $0<\boldsymbol{\delta} \leq \min \left(\boldsymbol{\delta}_{0}, \boldsymbol{\delta}_{1}, \boldsymbol{\delta}_{2}, \boldsymbol{\delta}_{3}\right)$, and $|\mathbf{x}-\mathbf{y}|<\boldsymbol{\delta}$, everything holds and

$$
\left|\frac{f(\mathbf{x})}{g(\mathbf{x})}-\frac{f(\mathbf{y})}{g(\mathbf{y})}\right| \leq M[|f(\mathbf{x})-f(\mathbf{y})|+|g(\mathbf{y})-g(\mathbf{x})|]
$$

$$
<M\left[\frac{\varepsilon}{2} M^{-1}+\frac{\varepsilon}{2} M^{-1}\right]=\varepsilon
$$

This completes the proof of the second part of 2.) Note that in these proofs no effort is made to find some sort of "best" $\delta$. The problem is one which has a yes or a no answer. Either it is or it is not continuous.

Now begin on 3.). If $\mathbf{f}$ is continuous at $\mathbf{x}, \mathbf{f}(\mathbf{x}) \in D(\mathbf{g}) \subseteq \mathbb{F}^{p}$, and $\mathbf{g}$ is continuous at $\mathbf{f}(\mathbf{x})$, then $\mathbf{g} \circ \mathbf{f}$ is continuous at $\mathbf{x}$. Let $\varepsilon>0$ be given. Then there exists $\eta>0$ such that if $|\mathbf{y}-\mathbf{f}(\mathbf{x})|<\eta$ and $\mathbf{y} \in D(\mathbf{g})$, it follows that $|\mathbf{g}(\mathbf{y})-\mathbf{g}(\mathbf{f}(\mathbf{x}))|<\varepsilon$. It follows from continuity of $\mathbf{f}$ at $\mathbf{x}$ that there exists $\delta>0$ such that if $|\mathbf{x}-\mathbf{z}|<\delta$ and $\mathbf{z} \in D(\mathbf{f})$, then $|\mathbf{f}(\mathbf{z})-\mathbf{f}(\mathbf{x})|<\eta$. Then if $|\mathbf{x}-\mathbf{z}|<\delta$ and $\mathbf{z} \in D(\mathbf{g} \circ \mathbf{f}) \subseteq D(\mathbf{f})$, all the above hold and so

$$
|\mathbf{g}(\mathbf{f}(\mathbf{z}))-\mathbf{g}(\mathbf{f}(\mathbf{x}))|<\varepsilon .
$$

This proves part 3.)
Part 4.) says: If $\mathbf{f}=\left(f_{1}, \cdots, f_{q}\right): D(\mathbf{f}) \rightarrow \mathbb{F}^{q}$, then $\mathbf{f}$ is continuous if and only if each $f_{k}$ is a continuous $\mathbb{F}$ valued function. Then

$$
\begin{align*}
& \left|f_{k}(\mathbf{x})-f_{k}(\mathbf{y})\right| \leq|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})| \\
& \equiv\left(\sum_{i=1}^{q}\left|f_{i}(\mathbf{x})-f_{i}(\mathbf{y})\right|^{2}\right)^{1 / 2} \\
& \quad \leq \sum_{i=1}^{q}\left|f_{i}(\mathbf{x})-f_{i}(\mathbf{y})\right| \tag{6.10.11}
\end{align*}
$$

Suppose first that $\mathbf{f}$ is continuous at $\mathbf{x}$. Then there exists $\delta>0$ such that if $|\mathbf{x}-\mathbf{y}|<\delta$, then $|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})|<\varepsilon$. The first part of the above inequality then shows that for each $k=$ $1, \cdots, q,\left|f_{k}(\mathbf{x})-f_{k}(\mathbf{y})\right|<\varepsilon$. This shows the only if part. Now suppose each function, $f_{k}$ is continuous. Then if $\varepsilon>0$ is given, there exists $\boldsymbol{\delta}_{k}>0$ such that whenever $|\mathbf{x}-\mathbf{y}|<\boldsymbol{\delta}_{k}$

$$
\left|f_{k}(\mathbf{x})-f_{k}(\mathbf{y})\right|<\varepsilon / q
$$

Now let $0<\delta \leq \min \left(\delta_{1}, \cdots, \delta_{q}\right)$. For $|\mathbf{x}-\mathbf{y}|<\delta$, the above inequality holds for all $k$ and so the last part of 6.10.11 implies

$$
\begin{aligned}
|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})| & \leq \sum_{i=1}^{q}\left|f_{i}(\mathbf{x})-f_{i}(\mathbf{y})\right| \\
& <\sum_{i=1}^{q} \frac{\varepsilon}{q}=\varepsilon
\end{aligned}
$$

This proves part 4.)
To verify part 5.), let $\varepsilon>0$ be given and let $\delta=\varepsilon$. Then if $|\mathbf{x}-\mathbf{y}|<\delta$, the triangle inequality implies

$$
\begin{aligned}
|f(\mathbf{x})-f(\mathbf{y})| & =\| \mathbf{x}|-|\mathbf{y}|| \\
& \leq|\mathbf{x}-\mathbf{y}|<\delta=\varepsilon
\end{aligned}
$$

This proves part 5.) and completes the proof of the theorem.
Here is a multidimensional version of the nested interval lemma.
The following definition is similar to that given earlier. It defines what is meant by a sequentially compact set in $\mathbb{F}^{p}$.

Definition 6.10.2 $A$ set, $K \subseteq \mathbb{F}^{p}$ is sequentially compact if and only if whenever $\left\{\mathbf{x}_{n}\right\}_{n=1}^{\infty}$ is a sequence of points in $K$, there exists a point, $\mathbf{x} \in K$ and a subsequence, $\left\{\mathbf{x}_{n_{k}}\right\}_{k=1}^{\infty}$ such that $\mathbf{x}_{n_{k}} \rightarrow \mathbf{x}$.

It turns out the sequentially compact sets in $\mathbb{F}^{p}$ are exactly those which are closed and bounded. Only half of this result will be needed in this book and this is proved next. First note that $\mathbb{C}$ can be considered as $\mathbb{R}^{2}$. Therefore, $\mathbb{C}^{p}$ may be considered as $\mathbb{R}^{2 p}$.

Theorem 6.10.3 Let $C \subseteq \mathbb{F}^{p}$ be closed and bounded. Then $C$ is sequentially compact.
Proof: Let $\left\{\mathbf{a}_{n}\right\} \subseteq C$. Then let $\mathbf{a}_{n}=\left(a_{1}^{n}, \cdots, a_{p}^{n}\right)$. It follows the real and imaginary parts of the terms of the sequence, $\left\{a_{j}^{n}\right\}_{n=1}^{\infty}$ are each contained in some sufficiently large closed bounded interval. By Theorem 3.0.3 on Page 37, there is a subsequence of the sequence of real parts of $\left\{a_{j}^{n}\right\}_{n=1}^{\infty}$ which converges. Also there is a further subsequence of the imaginary parts of $\left\{a_{j}^{n}\right\}_{n=1}^{\infty}$ which converges. Thus there is a subsequence, $n_{k}$ with the property that $a_{j}^{n_{k}}$ converges to a point, $a_{j} \in \mathbb{F}$. Taking further subsequences, one obtains the existence of a subsequence, still called $n_{k}$ such that for each $r=1, \cdots, p, a_{r}^{n_{k}}$ converges to a point, $a_{r} \in \mathbb{F}$ as $k \rightarrow \infty$. Therefore, letting $\mathbf{a} \equiv\left(a_{1}, \cdots, a_{p}\right), \lim _{k \rightarrow \infty} \mathbf{a}^{n_{k}}=\mathbf{a}$. Since $C$ is closed, it follows $\mathbf{a} \in C$. This proves the theorem.

Here is a proof of the extreme value theorem.
Theorem 6.10.4 Let $C$ be closed and bounded and let $f: C \rightarrow \mathbb{R}$ be continuous. Then $f$ achieves its maximum and its minimum on $C$. This means there exist, $\mathbf{x}_{1}, \mathbf{x}_{2} \in C$ such that for all $\mathbf{x} \in C$,

$$
f\left(\mathbf{x}_{1}\right) \leq f(\mathbf{x}) \leq f\left(\mathbf{x}_{2}\right)
$$

Proof: Let $M=\sup \{f(\mathbf{x}): \mathbf{x} \in C\}$. Recall this means $+\infty$ if $f$ is not bounded above and it equals the least upper bound of these values of $f$ if $f$ is bounded above. Then there exists a sequence, $\left\{\mathbf{x}_{n}\right\}$ such that $f\left(\mathbf{x}_{n}\right) \rightarrow M$. Since $C$ is sequentially compact, there exists a subsequence, $\mathbf{x}_{n_{k}}$, and a point, $\mathbf{x} \in C$ such that $\mathbf{x}_{n_{k}} \rightarrow \mathbf{x}$. But then since $f$ is continuous at $\mathbf{x}$, it follows from Theorem 6.7.10 on Page 108 that $f(\mathbf{x})=\lim _{k \rightarrow \infty} f\left(\mathbf{x}_{n_{k}}\right)=M$. This proves $f$ achieves its maximum and also shows its maximum is less than $\infty$. Let $\mathbf{x}_{2}=\mathbf{x}$. The case of a minimum is handled similarly.

Recall that a function is uniformly continuous if the following definition holds.
Definition 6.10.5 Let $\mathbf{f}: D(\mathbf{f}) \rightarrow \mathbb{F}^{q}$. Then $\mathbf{f}$ is uniformly continuous iffor every $\varepsilon>0$ there exists $\delta>0$ such that whenever $|\mathbf{x}-\mathbf{y}|<\delta$, it follows $|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})|<\varepsilon$.

Theorem 6.10.6 Let $\mathbf{f}: C \rightarrow \mathbb{F}^{q}$ be continuous where $C$ is a closed and bounded set in $\mathbb{F}^{p}$. Then $\mathbf{f}$ is uniformly continuous on $C$.

Proof: If this is not so, there exists $\varepsilon>0$ and pairs of points, $\mathbf{x}_{n}$ and $\mathbf{y}_{n}$ satisfying $\left|\mathbf{x}_{n}-\mathbf{y}_{n}\right|<1 / n$ but $\left|\mathbf{f}\left(\mathbf{x}_{n}\right)-\mathbf{f}\left(\mathbf{y}_{n}\right)\right| \geq \varepsilon$. Since $C$ is sequentially compact, there exists $\mathbf{x} \in C$ and a subsequence, $\left\{\mathbf{x}_{n_{k}}\right\}$ satisfying $\mathbf{x}_{n_{k}} \rightarrow \mathbf{x}$. But $\left|\mathbf{x}_{n_{k}}-\mathbf{y}_{n_{k}}\right|<1 / k$ and so $\mathbf{y}_{n_{k}} \rightarrow \mathbf{x}$ also. Therefore, from Theorem 6.7.10 on Page 108,

$$
\varepsilon \leq \lim _{k \rightarrow \infty}\left|\mathbf{f}\left(\mathbf{x}_{n_{k}}\right)-\mathbf{f}\left(\mathbf{y}_{n_{k}}\right)\right|=|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{x})|=0
$$

a contradiction. This proves the theorem.

### 6.11 The Space $\mathscr{L}\left(\mathbb{F}^{n}, \mathbb{F}^{m}\right)$

Definition 6.11.1 The symbol, $\mathscr{L}\left(\mathbb{F}^{n}, \mathbb{F}^{m}\right)$ will denote the set of linear transformations mapping $\mathbb{F}^{n}$ to $\mathbb{F}^{m}$. Thus $L \in \mathscr{L}\left(\mathbb{F}^{n}, \mathbb{F}^{m}\right)$ means that for $\alpha, \beta$ scalars and $\mathbf{x}, \mathbf{y}$ vectors in $\mathbb{F}^{n}$,

$$
L(\alpha \mathbf{x}+\beta \mathbf{y})=\alpha L(\mathbf{x})+\beta L(\mathbf{y})
$$

It is convenient to give a norm for the elements of $\mathscr{L}\left(\mathbb{F}^{n}, \mathbb{F}^{m}\right)$. This will allow the consideration of questions such as whether a function having values in this space of linear transformations is continuous.

### 6.11.1 The Operator Norm

How do you measure the distance between linear transformations defined on $\mathbb{F}^{n}$ ? It turns out there are many ways to do this but I will give the most common one here.

Definition 6.11.2 $\mathscr{L}\left(\mathbb{F}^{n}, \mathbb{F}^{m}\right)$ denotes the space of linear transformations mapping $\mathbb{F}^{n}$ to $\mathbb{F}^{m}$. For $A \in \mathscr{L}\left(\mathbb{F}^{n}, \mathbb{F}^{m}\right)$, the operator norm is defined by

$$
\|A\| \equiv \max \left\{|A x|_{\mathbb{F}^{m}}:|x|_{\mathbb{F}^{n}} \leq 1\right\}<\infty
$$

Theorem 6.11.3 Denote by $|\cdot|$ the norm on either $\mathbb{F}^{n}$ or $\mathbb{F}^{m}$. Then $\mathscr{L}\left(\mathbb{F}^{n}, \mathbb{F}^{m}\right)$ with this operator norm is a complete normed linear space of dimension nm with

$$
\|A \mathbf{x}\| \leq\|A\||\mathbf{x}|
$$

Here Completeness means that every Cauchy sequence converges.
Proof: It is necessary to show the norm defined on $\mathscr{L}\left(\mathbb{F}^{n}, \mathbb{F}^{m}\right)$ really is a norm. This means it is necessary to verify

$$
\|A\| \geq 0 \text { and equals zero if and only if } A=0
$$

For $\alpha$ a scalar,

$$
\|\alpha A\|=|\alpha|\|A\|
$$

and for $A, B \in \mathscr{L}\left(\mathbb{F}^{n}, \mathbb{F}^{m}\right)$,

$$
\|A+B\| \leq\|A\|+\|B\|
$$

The first two properties are obvious but you should verify them. It remains to verify the norm is well defined and also to verify the triangle inequality above. First if $|\mathbf{x}| \leq 1$, and $\left(A_{i j}\right)$ is the matrix of the linear transformation with respect to the usual basis vectors, then

$$
\begin{aligned}
\|A\| & =\max \left\{\left(\sum_{i}\left|(A \mathbf{x})_{i}\right|^{2}\right)^{1 / 2}:|\mathbf{x}| \leq 1\right\} \\
& =\max \left\{\left(\sum_{i}\left|\sum_{j} A_{i j} x_{j}\right|^{2}\right)^{1 / 2}:|\mathbf{x}| \leq 1\right\}
\end{aligned}
$$

which is a finite number by the extreme value theorem.
It is clear that a basis for $\mathscr{L}\left(\mathbb{F}^{n}, \mathbb{F}^{m}\right)$ consists of linear transformations whose matrices are of the form $E_{i j}$ where $E_{i j}$ consists of the $m \times n$ matrix having all zeros except for a 1 in the $i j^{t h}$ position. In effect, this considers $\mathscr{L}\left(\mathbb{F}^{n}, \mathbb{F}^{m}\right)$ as $\mathbb{F}^{n m}$. Think of the $m \times n$ matrix as a long vector folded up.

If $\mathbf{x} \neq \mathbf{0}$,

$$
\begin{equation*}
|A \mathbf{x}| \frac{1}{|\mathbf{x}|}=\left|A \frac{\mathbf{x}}{|\mathbf{x}|}\right| \leq \| A| | \tag{6.11.12}
\end{equation*}
$$

It only remains to verify completeness. Suppose then that $\left\{A_{k}\right\}$ is a Cauchy sequence in $\mathscr{L}\left(\mathbb{F}^{n}, \mathbb{F}^{m}\right)$. Then from 6.11.12 $\left\{A_{k} \mathbf{x}\right\}$ is a Cauchy sequence for each $\mathbf{x} \in \mathbb{F}^{n}$. This follows because

$$
\left|A_{k} \mathbf{x}-A_{l} \mathbf{x}\right| \leq\left\|A_{k}-A_{l}|\||\mathbf{x}|\right.
$$

which converges to 0 as $k, l \rightarrow \infty$. Therefore, by completeness of $\mathbb{F}^{m}$, there exists $A \mathbf{x}$, the name of the thing to which the sequence, $\left\{A_{k} \mathbf{x}\right\}$ converges such that

$$
\lim _{k \rightarrow \infty} A_{k} \mathbf{x}=A \mathbf{x}
$$

Then $A$ is linear because

$$
\begin{aligned}
A(a \mathbf{x}+b \mathbf{y}) & \equiv \lim _{k \rightarrow \infty} A_{k}(a \mathbf{x}+b \mathbf{y}) \\
& =\lim _{k \rightarrow \infty}\left(a A_{k} \mathbf{x}+b A_{k} \mathbf{y}\right) \\
& =a \lim _{k \rightarrow \infty} A_{k} \mathbf{x}+b \lim _{k \rightarrow \infty} A_{k} \mathbf{y} \\
& =a A \mathbf{x}+b A \mathbf{y}
\end{aligned}
$$

By the first part of this argument, $\|A\|<\infty$ and so $A \in \mathscr{L}\left(\mathbb{F}^{n}, \mathbb{F}^{m}\right)$. This proves the theorem.
Proposition 6.11.4 Let $A(\mathbf{x}) \in \mathscr{L}\left(\mathbb{F}^{n}, \mathbb{F}^{m}\right)$ for each $\mathbf{x} \in U \subseteq \mathbb{F}^{p}$. Then letting $\left(A_{i j}(\mathbf{x})\right)$ denote the matrix of $A(\mathbf{x})$ with respect to the standard basis, it follows $A_{i j}$ is continuous at $\mathbf{x}$ for each $i, j$ if and only if for all $\varepsilon>0$, there exists a $\delta>0$ such that if $|\mathbf{x}-\mathbf{y}|<\delta$, then $\|A(\mathbf{x})-A(\mathbf{y})\|<\varepsilon$. That is, $A$ is a continuous function having values in $\mathscr{L}\left(\mathbb{F}^{n}, \mathbb{F}^{m}\right)$ at $\mathbf{x}$.

Proof: Suppose first the second condition holds. Then from the material on linear transformations,

$$
\begin{aligned}
\left|A_{i j}(\mathbf{x})-A_{i j}(\mathbf{y})\right| & =\left|\mathbf{e}_{i} \cdot(A(\mathbf{x})-A(\mathbf{y})) \mathbf{e}_{j}\right| \\
& \leq\left|\mathbf{e}_{i}\right|\left|(A(\mathbf{x})-A(\mathbf{y})) \mathbf{e}_{j}\right| \\
& \leq\|A(\mathbf{x})-A(\mathbf{y})\| .
\end{aligned}
$$

Therefore, the second condition implies the first.
Now suppose the first condition holds. That is each $A_{i j}$ is continuous at $\mathbf{x}$. Let $|\mathbf{v}| \leq 1$.

$$
\begin{align*}
|(A(\mathbf{x})-A(\mathbf{y}))(\mathbf{v})| & =\left(\sum_{i}\left|\sum_{j}\left(A_{i j}(\mathbf{x})-A_{i j}(\mathbf{y})\right) v_{j}\right|^{2}\right)^{1 / 2}  \tag{6.11.13}\\
& \leq\left(\sum_{i}\left(\sum_{j}\left|A_{i j}(\mathbf{x})-A_{i j}(\mathbf{y})\right|\left|v_{j}\right|\right)^{2}\right)^{1 / 2} .
\end{align*}
$$

By continuity of each $A_{i j}$, there exists a $\delta>0$ such that for each $i, j$

$$
\left|A_{i j}(\mathbf{x})-A_{i j}(\mathbf{y})\right|<\frac{\varepsilon}{n \sqrt{m}}
$$

whenever $|\mathbf{x}-\mathbf{y}|<\delta$. Then from 6.11.13, if $|\mathbf{x}-\mathbf{y}|<\delta$,

$$
\begin{aligned}
|(A(\mathbf{x})-A(\mathbf{y}))(\mathbf{v})| & <\left(\sum_{i}\left(\sum_{j} \frac{\varepsilon}{n \sqrt{m}}|\mathbf{v}|\right)^{2}\right)^{1 / 2} \\
& \leq\left(\sum_{i}\left(\sum_{j} \frac{\varepsilon}{n \sqrt{m}}\right)^{2}\right)^{1 / 2}=\varepsilon
\end{aligned}
$$

This proves the proposition.

### 6.12 The Frechet Derivative

Let $U$ be an open set in $\mathbb{F}^{n}$, and let $\mathbf{f}: U \rightarrow \mathbb{F}^{m}$ be a function.
Definition 6.12.1 A function $\mathbf{g}$ is $o(\mathbf{v})$ if

$$
\begin{equation*}
\lim _{|\mathbf{v}| \rightarrow 0} \frac{\mathbf{g}(\mathbf{v})}{|\mathbf{v}|}=\mathbf{0} \tag{6.12.14}
\end{equation*}
$$

A function $\mathbf{f}: U \rightarrow \mathbb{F}^{m}$ is differentiable at $\mathbf{x} \in U$ if there exists a linear transformation $L \in \mathscr{L}\left(\mathbb{F}^{n}, \mathbb{F}^{m}\right)$ such that

$$
\mathbf{f}(\mathbf{x}+\mathbf{v})=\mathbf{f}(\mathbf{x})+L \mathbf{v}+o(\mathbf{v})
$$

This linear transformation $L$ is the definition of $D \mathbf{f}(\mathbf{x})$. This derivative is often called the Frechet derivative. .

Usually no harm is occasioned by thinking of this linear transformation as its matrix taken with respect to the usual basis vectors.

The definition 6.12.14 means that the error,

$$
\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})-L \mathbf{v}
$$

converges to $\mathbf{0}$ faster than $|\mathbf{v}|$. Thus the above definition is equivalent to saying

$$
\begin{equation*}
\lim _{|\mathbf{v}| \rightarrow 0} \frac{|\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})-L \mathbf{v}|}{|\mathbf{v}|}=0 \tag{6.12.15}
\end{equation*}
$$

or equivalently,

$$
\begin{equation*}
\lim _{\mathbf{y} \rightarrow \mathbf{x}} \frac{|\mathbf{f}(\mathbf{y})-\mathbf{f}(\mathbf{x})-D \mathbf{f}(\mathbf{x})(\mathbf{y}-\mathbf{x})|}{|\mathbf{y}-\mathbf{x}|}=0 \tag{6.12.16}
\end{equation*}
$$

Now it is clear this is just a generalization of the notion of the derivative of a function of one variable because in this more specialized situation,

$$
\lim _{|v| \rightarrow 0} \frac{\left|f(x+v)-f(x)-f^{\prime}(x) v\right|}{|v|}=0
$$

due to the definition which says

$$
f^{\prime}(x)=\lim _{v \rightarrow 0} \frac{f(x+v)-f(x)}{v}
$$

For functions of $n$ variables, you can't define the derivative as the limit of a difference quotient like you can for a function of one variable because you can't divide by a vector. That is why there is a need for a more general definition.

The term $o(\mathbf{v})$ is notation that is descriptive of the behavior in 6.12.14 and it is only this behavior that is of interest. Thus, if $t$ and $k$ are constants,

$$
o(\mathbf{v})=o(\mathbf{v})+o(\mathbf{v}), o(t \mathbf{v})=o(\mathbf{v}), k o(\mathbf{v})=o(\mathbf{v})
$$

and other similar observations hold. The sloppiness built in to this notation is useful because it ignores details which are not important. It may help to think of $o(\mathbf{v})$ as an adjective describing what is left over after approximating $\mathbf{f}(\mathbf{x}+\mathbf{v})$ by $\mathbf{f}(\mathbf{x})+D \mathbf{f}(\mathbf{x}) \mathbf{v}$.

Theorem 6.12.2 The derivative is well defined.
Proof: First note that for a fixed vector, $\mathbf{v}, o(t \mathbf{v})=o(t)$. Now suppose both $L_{1}$ and $L_{2}$ work in the above definition. Then let $\mathbf{v}$ be any vector and let $t$ be a real scalar which is chosen small enough that $t \mathbf{v}+\mathbf{x} \in U$. Then

$$
\mathbf{f}(\mathbf{x}+t \mathbf{v})=\mathbf{f}(\mathbf{x})+L_{1} t \mathbf{v}+o(t \mathbf{v}), \mathbf{f}(\mathbf{x}+t \mathbf{v})=\mathbf{f}(\mathbf{x})+L_{2} t \mathbf{v}+o(t \mathbf{v}) .
$$

Therefore, subtracting these two yields $\left(L_{2}-L_{1}\right)(t \mathbf{v})=o(t \mathbf{v})=o(t)$. Therefore, dividing by $t$ yields $\left(L_{2}-L_{1}\right)(\mathbf{v})=\frac{o(t)}{t}$. Now let $t \rightarrow 0$ to conclude that $\left(L_{2}-L_{1}\right)(\mathbf{v})=0$. Since this is true for all $\mathbf{v}$, it follows $L_{2}=L_{1}$. This proves the theorem.

Lemma 6.12.3 Let $\mathbf{f}$ be differentiable at $\mathbf{x}$. Then $\mathbf{f}$ is continuous at $\mathbf{x}$ and in fact, there exists $K>0$ such that whenever $|\mathbf{v}|$ is small enough,

$$
|\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})| \leq K|\mathbf{v}|
$$

Proof: From the definition of the derivative, $\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})=D \mathbf{f}(\mathbf{x}) \mathbf{v}+o(\mathbf{v})$. Let $|\mathbf{v}|$ be small enough that $\frac{o(|\mathbf{v}|)}{|\mathbf{v}|}<1$ so that $|o(\mathbf{v})| \leq|\mathbf{v}|$. Then for such $\mathbf{v}$,

$$
\begin{aligned}
|\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})| & \leq|D \mathbf{f}(\mathbf{x}) \mathbf{v}|+|\mathbf{v}| \\
& \leq(|D \mathbf{f}(\mathbf{x})|+1)|\mathbf{v}|
\end{aligned}
$$

This proves the lemma with $K=|D \mathbf{f}(\mathbf{x})|+1$.
Theorem 6.12.4 (The chain rule) Let $U$ and $V$ be open sets, $U \subseteq \mathbb{F}^{n}$ and $V \subseteq \mathbb{F}^{m}$. Suppose $\mathbf{f}: U \rightarrow V$ is differentiable at $\mathbf{x} \in U$ and suppose $\mathbf{g}: V \rightarrow \mathbb{F}^{q}$ is differentiable at $\mathbf{f}(\mathbf{x}) \in V$. Then $\mathbf{g} \circ \mathbf{f}$ is differentiable at $\mathbf{x}$ and

$$
D(\mathbf{g} \circ \mathbf{f})(\mathbf{x})=D \mathbf{g}(\mathbf{f}(\mathbf{x})) D \mathbf{f}(\mathbf{x})
$$

Proof: This follows from a computation. Let $B(\mathbf{x}, r) \subseteq U$ and let $r$ also be small enough that for $|\mathbf{v}| \leq r$, it follows that $\mathbf{f}(\mathbf{x}+\mathbf{v}) \in V$. Such an $r$ exists because $\mathbf{f}$ is continuous at $\mathbf{x}$. For $|\mathbf{v}|<r$, the definition of differentiability of $\mathbf{g}$ and $\mathbf{f}$ implies

$$
\begin{gather*}
\mathbf{g}(\mathbf{f}(\mathbf{x}+\mathbf{v}))-\mathbf{g}(\mathbf{f}(\mathbf{x}))= \\
\\
=\quad D \mathbf{g}(\mathbf{f}(\mathbf{x}))(\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x}))+o(\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})) \\
=  \tag{6.12.17}\\
D \mathbf{g}(\mathbf{f}(\mathbf{x}))[D \mathbf{f}(\mathbf{x}) \mathbf{v}+o(\mathbf{v})]+o(\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x}))) D \mathbf{f}(\mathbf{x}) \mathbf{v}+o(\mathbf{v})+o(\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})) .
\end{gather*}
$$

It remains to show $o(\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x}))=o(\mathbf{v})$.
By Lemma 6.12.3, with $K$ given there, letting $\varepsilon>0$, it follows that for $|\mathbf{v}|$ small enough,

$$
|o(\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x}))| \leq(\varepsilon / K)|\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})| \leq(\varepsilon / K) K|\mathbf{v}|=\varepsilon|\mathbf{v}| .
$$

Since $\varepsilon>0$ is arbitrary, this shows $o(\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x}))=o(\mathbf{v})$ because whenever $|\mathbf{v}|$ is small enough,

$$
\frac{|o(\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x}))|}{|\mathbf{v}|} \leq \varepsilon
$$

By 6.12.17, this shows

$$
\mathbf{g}(\mathbf{f}(\mathbf{x}+\mathbf{v}))-\mathbf{g}(\mathbf{f}(\mathbf{x}))=D \mathbf{g}(\mathbf{f}(\mathbf{x})) D \mathbf{f}(\mathbf{x}) \mathbf{v}+o(\mathbf{v})
$$

which proves the theorem.
The derivative is a linear transformation. What is the matrix of this linear transformation taken with respect to the usual basis vectors? Let $\mathbf{e}_{i}$ denote the vector of $\mathbb{F}^{n}$ which has a one in the $i^{t h}$ entry and zeroes elsewhere. Then the matrix of the linear transformation is
the matrix whose $i^{t h}$ column is $D \mathbf{f}(\mathbf{x}) \mathbf{e}_{i}$. What is this? Let $t \in \mathbb{R}$ such that $|t|$ is sufficiently small.

$$
\begin{aligned}
\mathbf{f}\left(\mathbf{x}+t \mathbf{e}_{i}\right)-\mathbf{f}(\mathbf{x}) & =D \mathbf{f}(\mathbf{x}) t \mathbf{e}_{i}+\mathbf{o}\left(t \mathbf{e}_{i}\right) \\
& =D \mathbf{f}(\mathbf{x}) t \mathbf{e}_{i}+\mathbf{o}(t) .
\end{aligned}
$$

Then dividing by $t$ and taking a limit,

$$
D \mathbf{f}(\mathbf{x}) \mathbf{e}_{i}=\lim _{t \rightarrow 0} \frac{\mathbf{f}\left(\mathbf{x}+t \mathbf{e}_{i}\right)-\mathbf{f}(\mathbf{x})}{t} \equiv \frac{\partial \mathbf{f}}{\partial x_{i}}(\mathbf{x}) .
$$

Thus the matrix of $D \mathbf{f}(\mathbf{x})$ with respect to the usual basis vectors is the matrix of the form

$$
\left(\begin{array}{cccc}
f_{1, x_{1}}(\mathbf{x}) & f_{1, x_{2}}(\mathbf{x}) & \cdots & f_{1, x_{n}}(\mathbf{x}) \\
\vdots & \vdots & & \vdots \\
f_{m, x_{1}}(\mathbf{x}) & f_{m, x_{2}}(\mathbf{x}) & \cdots & f_{m, x_{n}}(\mathbf{x})
\end{array}\right) .
$$

As mentioned before, there is no harm in referring to this matrix as $D \mathbf{f}(\mathbf{x})$ but it may also be referred to as $J \mathbf{f}(\mathbf{x})$.

This is summarized in the following theorem.
Theorem 6.12.5 Let $\mathbf{f}: \mathbb{F}^{n} \rightarrow \mathbb{F}^{m}$ and suppose $\mathbf{f}$ is differentiable at $\mathbf{x}$. Then all the partial derivatives $\frac{\partial f_{i}(\mathbf{x})}{\partial x_{j}}$ exist and if $J \mathbf{f}(\mathbf{x})$ is the matrix of the linear transformation with respect to the standard basis vectors, then the $i j^{\text {th }}$ entry is given by $f_{i, j}$ or $\frac{\partial f_{i}}{\partial x_{j}}(\mathbf{x})$.

What if all the partial derivatives of $\mathbf{f}$ exist? Does it follow that $\mathbf{f}$ is differentiable? Consider the following function.

$$
f(x, y)=\left\{\begin{array}{l}
\frac{x y}{x^{2}+y^{2}} \text { if }(x, y) \neq(0,0) \\
0 \text { if }(x, y)=(0,0)
\end{array} .\right.
$$

Then from the definition of partial derivatives,

$$
\lim _{h \rightarrow 0} \frac{f(h, 0)-f(0,0)}{h}=\lim _{h \rightarrow 0} \frac{0-0}{h}=0
$$

and

$$
\lim _{h \rightarrow 0} \frac{f(0, h)-f(0,0)}{h}=\lim _{h \rightarrow 0} \frac{0-0}{h}=0
$$

However $f$ is not even continuous at $(0,0)$ which may be seen by considering the behavior of the function along the line $y=x$ and along the line $x=0$. By Lemma 6.12.3 this implies $f$ is not differentiable. Therefore, it is necessary to consider the correct definition of the derivative given above if you want to get a notion which generalizes the concept of the derivative of a function of one variable in such a way as to preserve continuity whenever the function is differentiable.

### 6.13 $C^{1}$ Functions

However, there are theorems which can be used to get differentiability of a function based on existence of the partial derivatives.

Definition 6.13.1 When all the partial derivatives exist and are continuous the function is called a $C^{1}$ function.

Because of Proposition 6.11.4 on Page 115 and Theorem 6.12 .5 which identifies the entries of $J \mathbf{f}$ with the partial derivatives, the following definition is equivalent to the above.

Definition 6.13.2 Let $U \subseteq \mathbb{F}^{n}$ be an open set. Then $\mathbf{f}: U \rightarrow \mathbb{F}^{m}$ is $C^{1}(U)$ if $\mathbf{f}$ is differentiable and the mapping

$$
\mathbf{x} \rightarrow D \mathbf{f}(\mathbf{x})
$$

is continuous as a function from $U$ to $\mathscr{L}\left(\mathbb{F}^{n}, \mathbb{F}^{m}\right)$.
The following is an important abstract generalization of the familiar concept of partial derivative.

Definition 6.13.3 Let $\mathbf{g}: U \subseteq \mathbb{F}^{n} \times \mathbb{F}^{m} \rightarrow \mathbb{F}^{q}$, where $U$ is an open set in $\mathbb{F}^{n} \times \mathbb{F}^{m}$. Denote an element of $\mathbb{F}^{n} \times \mathbb{F}^{m}$ by $(\mathbf{x}, \mathbf{y})$ where $\mathbf{x} \in \mathbb{F}^{n}$ and $\mathbf{y} \in \mathbb{F}^{m}$. Then the map $\mathbf{x} \rightarrow \mathbf{g}(\mathbf{x}, \mathbf{y})$ is a function from the open set in $\mathbb{F}^{n}$,

$$
\{\mathbf{x}:(\mathbf{x}, \mathbf{y}) \in U\}
$$

to $\mathbb{F}^{q}$. When this map is differentiable, its derivative is denoted by

$$
D_{1} \mathbf{g}(\mathbf{x}, \mathbf{y}), \text { or sometimes by } D_{\mathbf{x}} \mathbf{g}(\mathbf{x}, \mathbf{y})
$$

Thus,

$$
\mathbf{g}(\mathbf{x}+\mathbf{v}, \mathbf{y})-\mathbf{g}(\mathbf{x}, \mathbf{y})=D_{1} \mathbf{g}(\mathbf{x}, \mathbf{y}) \mathbf{v}+o(\mathbf{v})
$$

A similar definition holds for the symbol $D_{\mathbf{y}} \mathbf{g}$ or $D_{2} \mathbf{g}$. The special case seen in beginning calculus courses is where $\mathbf{g}: U \rightarrow \mathbb{F}^{q}$ and

$$
\mathbf{g}_{x_{i}}(\mathbf{x}) \equiv \frac{\partial \mathbf{g}(\mathbf{x})}{\partial x_{i}} \equiv \lim _{h \rightarrow 0} \frac{\mathbf{g}\left(\mathbf{x}+h \mathbf{e}_{i}\right)-\mathbf{g}(\mathbf{x})}{h}
$$

The following theorem will be very useful in much of what follows. It is a version of the mean value theorem. You might call it the mean value inequality.

Theorem 6.13.4 Suppose $U$ is an open subset of $\mathbb{F}^{n}$ and $\mathbf{f}: U \rightarrow \mathbb{F}^{m}$ has the property that $D \mathbf{f}(\mathbf{x})$ exists for all $\mathbf{x}$ in $U$ and that, $\mathbf{x}+t(\mathbf{y}-\mathbf{x}) \in U$ for all $t \in[0,1]$. (The line segment joining the two points lies in $U$.) Suppose also that for all points on this line segment,

$$
\|D \mathbf{f}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))\| \leq M
$$

Then

$$
|\mathbf{f}(\mathbf{y})-\mathbf{f}(\mathbf{x})| \leq M|\mathbf{y}-\mathbf{x}| .
$$

Proof: Let

$$
\begin{gathered}
S \equiv\{t \in[0,1]: \text { for all } s \in[0, t] \\
|\mathbf{f}(\mathbf{x}+s(\mathbf{y}-\mathbf{x}))-\mathbf{f}(\mathbf{x})| \leq(M+\boldsymbol{\varepsilon}) s|\mathbf{y}-\mathbf{x}|\}
\end{gathered}
$$

Then $0 \in S$ and by continuity of $\mathbf{f}$, it follows that if $t \equiv \sup S$, then $t \in S$ and if $t<1$,

$$
\begin{equation*}
|\mathbf{f}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))-\mathbf{f}(\mathbf{x})|=(M+\boldsymbol{\varepsilon}) t|\mathbf{y}-\mathbf{x}| . \tag{6.13.18}
\end{equation*}
$$

If $t<1$, then there exists a sequence of positive numbers, $\left\{h_{k}\right\}_{k=1}^{\infty}$ converging to 0 such that

$$
\left|\mathbf{f}\left(\mathbf{x}+\left(t+h_{k}\right)(\mathbf{y}-\mathbf{x})\right)-\mathbf{f}(\mathbf{x})\right|>(M+\boldsymbol{\varepsilon})\left(t+h_{k}\right)|\mathbf{y}-\mathbf{x}|
$$

which implies that

$$
\begin{gathered}
\left|\mathbf{f}\left(\mathbf{x}+\left(t+h_{k}\right)(\mathbf{y}-\mathbf{x})\right)-\mathbf{f}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))\right| \\
+|\mathbf{f}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))-\mathbf{f}(\mathbf{x})|>(M+\boldsymbol{\varepsilon})\left(t+h_{k}\right)|\mathbf{y}-\mathbf{x}| .
\end{gathered}
$$

By 6.13 .18 , this inequality implies

$$
\left|\mathbf{f}\left(\mathbf{x}+\left(t+h_{k}\right)(\mathbf{y}-\mathbf{x})\right)-\mathbf{f}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))\right|>(M+\varepsilon) h_{k}|\mathbf{y}-\mathbf{x}|
$$

which yields upon dividing by $h_{k}$ and taking the limit as $h_{k} \rightarrow 0$,

$$
|D \mathbf{f}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))(\mathbf{y}-\mathbf{x})| \geq(M+\boldsymbol{\varepsilon})|\mathbf{y}-\mathbf{x}| .
$$

Now by the definition of the norm of a linear operator,
a contradiction. Therefore, $t=1$ and so

$$
|\mathbf{f}(\mathbf{x}+(\mathbf{y}-\mathbf{x}))-\mathbf{f}(\mathbf{x})| \leq(M+\varepsilon)|\mathbf{y}-\mathbf{x}| .
$$

Since $\varepsilon>0$ is arbitrary, this proves the theorem.
The next theorem proves that if the partial derivatives exist and are continuous, then the function is differentiable.

Theorem 6.13.5 Let $\mathbf{g}: U \subseteq \mathbb{F}^{n} \times \mathbb{F}^{m} \rightarrow \mathbb{F}^{q}$. Then $\mathbf{g}$ is $C^{1}(U)$ if and only if $D_{1} \mathbf{g}$ and $D_{2} \mathbf{g}$ both exist and are continuous on $U$. In this case,

$$
D \mathbf{g}(\mathbf{x}, \mathbf{y})(\mathbf{u}, \mathbf{v})=D_{1} \mathbf{g}(\mathbf{x}, \mathbf{y}) \mathbf{u}+D_{2} \mathbf{g}(\mathbf{x}, \mathbf{y}) \mathbf{v}
$$

Proof: Suppose first that $\mathbf{g} \in C^{1}(U)$. Then if $(\mathbf{x}, \mathbf{y}) \in U$,

$$
\mathbf{g}(\mathbf{x}+\mathbf{u}, \mathbf{y})-\mathbf{g}(\mathbf{x}, \mathbf{y})=D \mathbf{g}(\mathbf{x}, \mathbf{y})(\mathbf{u}, \mathbf{0})+o(\mathbf{u}) .
$$

Therefore, $D_{1} \mathbf{g}(\mathbf{x}, \mathbf{y}) \mathbf{u}=D \mathbf{g}(\mathbf{x}, \mathbf{y})(\mathbf{u}, \mathbf{0})$. Then

$$
\left|\left(D_{1} \mathbf{g}(\mathbf{x}, \mathbf{y})-D_{1} \mathbf{g}\left(\mathbf{x}^{\prime}, \mathbf{y}^{\prime}\right)\right)(\mathbf{u})\right|=
$$

$$
\begin{aligned}
& \left|\left(D \mathbf{g}(\mathbf{x}, \mathbf{y})-D \mathbf{g}\left(\mathbf{x}^{\prime}, \mathbf{y}^{\prime}\right)\right)(\mathbf{u}, \mathbf{0})\right| \leq \\
& \left|\left|D \mathbf{g}(\mathbf{x}, \mathbf{y})-D \mathbf{g}\left(\mathbf{x}^{\prime}, \mathbf{y}^{\prime}\right)\right|\right||(\mathbf{u}, \mathbf{0})| .
\end{aligned}
$$

Therefore,

$$
\left|D_{1} \mathbf{g}(\mathbf{x}, \mathbf{y})-D_{1} \mathbf{g}\left(\mathbf{x}^{\prime}, \mathbf{y}^{\prime}\right)\right| \leq\left\|D \mathbf{g}(\mathbf{x}, \mathbf{y})-D \mathbf{g}\left(\mathbf{x}^{\prime}, \mathbf{y}^{\prime}\right)\right\|
$$

A similar argument applies for $D_{2} \mathbf{g}$ and this proves the continuity of the function, $(\mathbf{x}, \mathbf{y}) \rightarrow$ $D_{i} \mathbf{g}(\mathbf{x}, \mathbf{y})$ for $i=1,2$. The formula follows from

$$
\begin{aligned}
D \mathbf{g}(\mathbf{x}, \mathbf{y})(\mathbf{u}, \mathbf{v}) & =D \mathbf{g}(\mathbf{x}, \mathbf{y})(\mathbf{u}, \mathbf{0})+D \mathbf{g}(\mathbf{x}, \mathbf{y})(\mathbf{0}, \mathbf{v}) \\
& \equiv D_{1} \mathbf{g}(\mathbf{x}, \mathbf{y}) \mathbf{u}+D_{2} \mathbf{g}(\mathbf{x}, \mathbf{y}) \mathbf{v}
\end{aligned}
$$

Now suppose $D_{1} \mathbf{g}(\mathbf{x}, \mathbf{y})$ and $D_{2} \mathbf{g}(\mathbf{x}, \mathbf{y})$ exist and are continuous.

$$
\begin{gather*}
\mathbf{g}(\mathbf{x}+\mathbf{u}, \mathbf{y}+\mathbf{v})-\mathbf{g}(\mathbf{x}, \mathbf{y})=\mathbf{g}(\mathbf{x}+\mathbf{u}, \mathbf{y}+\mathbf{v})-\mathbf{g}(\mathbf{x}, \mathbf{y}+\mathbf{v}) \\
+\mathbf{g}(\mathbf{x}, \mathbf{y}+\mathbf{v})-\mathbf{g}(\mathbf{x}, \mathbf{y}) \\
=\mathbf{g}(\mathbf{x}+\mathbf{u}, \mathbf{y})-\mathbf{g}(\mathbf{x}, \mathbf{y})+\mathbf{g}(\mathbf{x}, \mathbf{y}+\mathbf{v})-\mathbf{g}(\mathbf{x}, \mathbf{y})+ \\
{[\mathbf{g}(\mathbf{x}+\mathbf{u}, \mathbf{y}+\mathbf{v})-\mathbf{g}(\mathbf{x}+\mathbf{u}, \mathbf{y})-(\mathbf{g}(\mathbf{x}, \mathbf{y}+\mathbf{v})-\mathbf{g}(\mathbf{x}, \mathbf{y}))]} \\
=D_{1} \mathbf{g}(\mathbf{x}, \mathbf{y}) \mathbf{u}+D_{2} \mathbf{g}(\mathbf{x}, \mathbf{y}) \mathbf{v}+o(\mathbf{v})+o(\mathbf{u})+ \\
{[\mathbf{g}(\mathbf{x}+\mathbf{u}, \mathbf{y}+\mathbf{v})-\mathbf{g}(\mathbf{x}+\mathbf{u}, \mathbf{y})-(\mathbf{g}(\mathbf{x}, \mathbf{y}+\mathbf{v})-\mathbf{g}(\mathbf{x}, \mathbf{y}))] .} \tag{6.13.19}
\end{gather*}
$$

Let $\mathbf{h}(\mathbf{x}, \mathbf{u}) \equiv \mathbf{g}(\mathbf{x}+\mathbf{u}, \mathbf{y}+\mathbf{v})-\mathbf{g}(\mathbf{x}+\mathbf{u}, \mathbf{y})$. Then the expression in [ ] is of the form,

$$
\mathbf{h}(\mathbf{x}, \mathbf{u})-\mathbf{h}(\mathbf{x}, \mathbf{0}) .
$$

Also

$$
D_{2} \mathbf{h}(\mathbf{x}, \mathbf{u})=D_{1} \mathbf{g}(\mathbf{x}+\mathbf{u}, \mathbf{y}+\mathbf{v})-D_{1} \mathbf{g}(\mathbf{x}+\mathbf{u}, \mathbf{y})
$$

and so, by continuity of $(\mathbf{x}, \mathbf{y}) \rightarrow D_{1} \mathbf{g}(\mathbf{x}, \mathbf{y})$,

$$
\left\|D_{2} \mathbf{h}(\mathbf{x}, \mathbf{u})\right\|<\varepsilon
$$

whenever $\|(\mathbf{u}, \mathbf{v})\|$ is small enough. By Theorem 6.13 .4 on Page 120 , there exists $\delta>0$ such that if $\|(\mathbf{u}, \mathbf{v})\|<\delta$, the norm of the last term in 6.13 .19 satisfies the inequality,

$$
\begin{equation*}
\|\mathbf{g}(\mathbf{x}+\mathbf{u}, \mathbf{y}+\mathbf{v})-\mathbf{g}(\mathbf{x}+\mathbf{u}, \mathbf{y})-(\mathbf{g}(\mathbf{x}, \mathbf{y}+\mathbf{v})-\mathbf{g}(\mathbf{x}, \mathbf{y}))\|<\varepsilon\|\mathbf{u}\| . \tag{6.13.20}
\end{equation*}
$$

Therefore, this term is $o((\mathbf{u}, \mathbf{v}))$. It follows from 6.13.20 and 6.13.19 that

$$
\begin{gathered}
\mathbf{g}(\mathbf{x}+\mathbf{u}, \mathbf{y}+\mathbf{v})= \\
\mathbf{g}(\mathbf{x}, \mathbf{y})+D_{1} \mathbf{g}(\mathbf{x}, \mathbf{y}) \mathbf{u}+D_{2} \mathbf{g}(\mathbf{x}, \mathbf{y}) \mathbf{v}+o(\mathbf{u})+o(\mathbf{v})+o((\mathbf{u}, \mathbf{v})) \\
=\mathbf{g}(\mathbf{x}, \mathbf{y})+D_{1} \mathbf{g}(\mathbf{x}, \mathbf{y}) \mathbf{u}+D_{2} \mathbf{g}(\mathbf{x}, \mathbf{y}) \mathbf{v}+o((\mathbf{u}, \mathbf{v}))
\end{gathered}
$$

Showing that $D \mathbf{g}(\mathbf{x}, \mathbf{y})$ exists and is given by

$$
D \mathbf{g}(\mathbf{x}, \mathbf{y})(\mathbf{u}, \mathbf{v})=D_{1} \mathbf{g}(\mathbf{x}, \mathbf{y}) \mathbf{u}+D_{2} \mathbf{g}(\mathbf{x}, \mathbf{y}) \mathbf{v}
$$

The continuity of $(\mathbf{x}, \mathbf{y}) \rightarrow D \mathbf{g}(\mathbf{x}, \mathbf{y})$ follows from the continuity of $(\mathbf{x}, \mathbf{y}) \rightarrow D_{i} \mathbf{g}(\mathbf{x}, \mathbf{y})$. This proves the theorem.

Not surprisingly, it can be generalized to many more factors.

Definition 6.13.6 Let $\mathbf{g}: U \subseteq \prod_{i=1}^{n} \mathbb{F}^{r_{i}} \rightarrow \mathbb{F}^{q}$, where $U$ is an open set. Then the map $\mathbf{x}_{i} \rightarrow$ $\mathbf{g}(\mathbf{x})$ is a function from the open set in $\mathbb{F}^{r_{i}}$,

$$
\left\{\mathbf{x}: \mathbf{x}=\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{i-1}, \mathbf{x}, \mathbf{x}_{i+1}, \cdots, \mathbf{x}_{n}\right) \in U\right\}
$$

to $\mathbb{F}^{q}$. When this map is differentiable, its derivative is denoted by $D_{i} \mathbf{g}(\mathbf{x})$. To aid in the notation, for $\mathbf{v} \in \mathbb{F}^{r_{i}}$, let $\theta_{i} \mathbf{v} \in \prod_{i=1}^{n} \mathbb{F}^{r_{i}}$ be the vector $(\mathbf{0}, \cdots, \mathbf{v}, \cdots, \mathbf{0})$ where the $\mathbf{v}$ is in the $i^{\text {th }}$ slot and for $\mathbf{v} \in \prod_{i=1}^{n} \mathbb{F}^{r_{i}}$, let $\mathbf{v}_{i}$ denote the entry in the $i^{\text {th }}$ slot of $\mathbf{v}$. Thus, by saying $\mathbf{x}_{i} \rightarrow \mathbf{g}(\mathbf{x})$ is differentiable is meant that for $\mathbf{v} \in \mathbb{F}^{r_{i}}$ sufficiently small,

$$
\mathbf{g}\left(\mathbf{x}+\theta_{i} \mathbf{v}\right)-\mathbf{g}(\mathbf{x})=D_{i} \mathbf{g}(\mathbf{x}) \mathbf{v}+\mathbf{o}(\mathbf{v})
$$

Note $D_{i} \mathbf{g}(\mathbf{x}) \in \mathscr{L}\left(\mathbb{F}^{r_{i}}, \prod_{i=1}^{n} \mathbb{F}^{r_{i}}\right)$.
Here is a generalization of Theorem 6.13.5.
Theorem 6.13.7 Let $\mathbf{g}, U, \prod_{i=1}^{n} \mathbb{F}^{r_{i}}$, be given as in Definition 6.13.6. Then $\mathbf{g}$ is $C^{1}(U)$ if and only if $D_{i} \mathbf{g}$ exists and is continuous on $U$ for each i. In this case,

$$
\begin{equation*}
D \mathbf{g}(\mathbf{x})(\mathbf{v})=\sum_{k} D_{k} \mathbf{g}(\mathbf{x}) \mathbf{v}_{k} \tag{6.13.21}
\end{equation*}
$$

where $\mathbf{v}=\left(\mathbf{v}_{1}, \cdots, \mathbf{v}_{n}\right)$.
Proof: Suppose then that $D_{i} \mathbf{g}$ exists and is continuous for each $i$. Note that $\sum_{j=1}^{k} \theta_{j} \mathbf{v}_{j}=$ $\left(\mathbf{v}_{1}, \cdots, \mathbf{v}_{k}, \mathbf{0}, \cdots, \mathbf{0}\right)$. Thus $\sum_{j=1}^{n} \theta_{j} \mathbf{v}_{j}=\mathbf{v}$ and define $\sum_{j=1}^{0} \theta_{j} \mathbf{v}_{j} \equiv \mathbf{0}$. Therefore,

$$
\begin{equation*}
\mathbf{g}(\mathbf{x}+\mathbf{v})-\mathbf{g}(\mathbf{x})=\sum_{k=1}^{n}\left[\mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k} \theta_{j} \mathbf{v}_{j}\right)-\mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k-1} \theta_{j} \mathbf{v}_{j}\right)\right] \tag{6.13.22}
\end{equation*}
$$

Consider the terms in this sum.

$$
\begin{gather*}
\mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k} \theta_{j} \mathbf{v}_{j}\right)-\mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k-1} \theta_{j} \mathbf{v}_{j}\right)=\mathbf{g}\left(\mathbf{x}+\theta_{k} \mathbf{v}_{k}\right)-\mathbf{g}(\mathbf{x})+  \tag{6.13.23}\\
\left(\mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k} \theta_{j} \mathbf{v}_{j}\right)-\mathbf{g}\left(\mathbf{x}+\theta_{k} \mathbf{v}_{k}\right)\right)-\left(\mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k-1} \theta_{j} \mathbf{v}_{j}\right)-\mathbf{g}(\mathbf{x})\right) \tag{6.13.24}
\end{gather*}
$$

and the expression in 6.13 .24 is of the form $\mathbf{h}\left(\mathbf{v}_{k}\right)-\mathbf{h}(\mathbf{0})$ where for small $\mathbf{w} \in \mathbb{F}^{r_{k}}$,

$$
\mathbf{h}(\mathbf{w}) \equiv \mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k-1} \theta_{j} \mathbf{v}_{j}+\theta_{k} \mathbf{w}\right)-\mathbf{g}\left(\mathbf{x}+\theta_{k} \mathbf{w}\right)
$$

Therefore,

$$
D \mathbf{h}(\mathbf{w})=D_{k} \mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k-1} \theta_{j} \mathbf{v}_{j}+\theta_{k} \mathbf{w}\right)-D_{k} \mathbf{g}\left(\mathbf{x}+\theta_{k} \mathbf{w}\right)
$$

and by continuity, $\|D \mathbf{h}(\mathbf{w})\|<\varepsilon$ provided $|\mathbf{v}|$ is small enough. Therefore, by Theorem 6.13.4, whenever $|\mathbf{v}|$ is small enough, $\left|\mathbf{h}\left(\mathbf{v}_{k}\right)-\mathbf{h}(\mathbf{0})\right| \leq \varepsilon\left|\mathbf{v}_{k}\right| \leq \varepsilon|\mathbf{v}|$ which shows that since $\varepsilon$ is arbitrary, the expression in 6.13.24 is $\mathbf{o}(\mathbf{v})$. Now in 6.13.23

$$
\mathbf{g}\left(\mathbf{x}+\theta_{k} \mathbf{v}_{k}\right)-\mathbf{g}(\mathbf{x})=D_{k} \mathbf{g}(\mathbf{x}) \mathbf{v}_{k}+\mathbf{o}\left(\mathbf{v}_{k}\right)=D_{k} \mathbf{g}(\mathbf{x}) \mathbf{v}_{k}+\mathbf{o}(\mathbf{v}) .
$$

Therefore, referring to 6.13.22,

$$
\mathbf{g}(\mathbf{x}+\mathbf{v})-\mathbf{g}(\mathbf{x})=\sum_{k=1}^{n} D_{k} \mathbf{g}(\mathbf{x}) \mathbf{v}_{k}+\mathbf{o}(\mathbf{v})
$$

which shows $D \mathbf{g}$ exists and equals the formula given in 6.13.21.
Next suppose $\mathbf{g}$ is $C^{1}$. I need to verify that $D_{k} \mathbf{g}(\mathbf{x})$ exists and is continuous. Let $\mathbf{v} \in \mathbb{F}^{r_{k}}$ sufficiently small. Then

$$
\begin{aligned}
\mathbf{g}\left(\mathbf{x}+\theta_{k} \mathbf{v}\right)-\mathbf{g}(\mathbf{x}) & =D \mathbf{g}(\mathbf{x}) \theta_{k} \mathbf{v}+\mathbf{o}\left(\theta_{k} \mathbf{v}\right) \\
& =D \mathbf{g}(\mathbf{x}) \theta_{k} \mathbf{v}+\mathbf{o}(\mathbf{v})
\end{aligned}
$$

since $\left|\theta_{k} \mathbf{v}\right|=|\mathbf{v}|$. Then $D_{k} \mathbf{g}(\mathbf{x})$ exists and equals

$$
D \mathbf{g}(\mathbf{x}) \circ \theta_{k}
$$

Since $\mathbf{x} \rightarrow D \mathbf{g}(\mathbf{x})$ is continuous and $\theta_{k}: \mathbb{F}^{r_{k}} \rightarrow \prod_{i=1}^{n} \mathbb{F}^{r_{i}}$ is also continuous, this proves the theorem

The way this is usually used is in the following corollary, a case of Theorem 6.13.7 obtained by letting $\mathbb{F}^{r_{j}}=\mathbb{F}$ in the above theorem.

Corollary 6.13.8 Let $U$ be an open subset of $\mathbb{F}^{n}$ and let $\mathbf{f}: U \rightarrow \mathbb{F}^{m}$ be $C^{1}$ in the sense that all the partial derivatives of $\mathbf{f}$ exist and are continuous. Then $\mathbf{f}$ is differentiable and

$$
\mathbf{f}(\mathbf{x}+\mathbf{v})=\mathbf{f}(\mathbf{x})+\sum_{k=1}^{n} \frac{\partial \mathbf{f}}{\partial x_{k}}(\mathbf{x}) v_{k}+\mathbf{o}(\mathbf{v}) .
$$

### 6.14 $C^{k}$ Functions

Recall the notation for partial derivatives in the following definition.
Definition 6.14.1 Let $\mathbf{g}: U \rightarrow \mathbb{F}^{n}$. Then

$$
\mathbf{g}_{x_{k}}(\mathbf{x}) \equiv \frac{\partial \mathbf{g}}{\partial x_{k}}(\mathbf{x}) \equiv \lim _{h \rightarrow 0} \frac{\mathbf{g}\left(\mathbf{x}+h \mathbf{e}_{k}\right)-\mathbf{g}(\mathbf{x})}{h}
$$

Higher order partial derivatives are defined in the usual way.

$$
\mathbf{g}_{x_{k} x_{l}}(\mathbf{x}) \equiv \frac{\partial^{2} \mathbf{g}}{\partial x_{l} \partial x_{k}}(\mathbf{x})
$$

and so forth.

To deal with higher order partial derivatives in a systematic way, here is a useful definition.

Definition 6.14.2 $\alpha=\left(\alpha_{1}, \cdots, \alpha_{n}\right)$ for $\alpha_{1} \cdots \alpha_{n}$ positive integers is called a multi-index. For $\alpha$ a multi-index, $|\alpha| \equiv \alpha_{1}+\cdots+\alpha_{n}$ and if $\mathbf{x} \in \mathbb{F}^{n}$,

$$
\mathbf{x}=\left(x_{1}, \cdots, x_{n}\right)
$$

and $\mathbf{f}$ a function, define

$$
\mathbf{x}^{\alpha} \equiv x_{1}^{\alpha_{1}} x_{2}^{\alpha_{2}} \cdots x_{n}^{\alpha_{n}}, D^{\alpha} \mathbf{f}(\mathbf{x}) \equiv \frac{\partial^{|\alpha|} \mathbf{f}(\mathbf{x})}{\partial x_{1}^{\alpha_{1}} \partial x_{2}^{\alpha_{2}} \cdots \partial x_{n}^{\alpha_{n}}}
$$

The following is the definition of what is meant by a $C^{k}$ function.
Definition 6.14.3 Let $U$ be an open subset of $\mathbb{F}^{n}$ and let $\mathbf{f}: U \rightarrow \mathbb{F}^{m}$. Then for $k$ a nonnegative integer, $\mathbf{f}$ is $C^{k}$ if for every $|\alpha| \leq k, D^{\alpha} \mathbf{f}$ exists and is continuous.

### 6.15 Mixed Partial Derivatives

Under certain conditions the mixed partial derivatives will always be equal. This astonishing fact is due to Euler in 1734.

Theorem 6.15.1 Suppose $f: U \subseteq \mathbb{F}^{2} \rightarrow \mathbb{R}$ where $U$ is an open set on which $f_{x}, f_{y}, f_{x y}$ and $f_{y x}$ exist. Then if $f_{x y}$ and $f_{y x}$ are continuous at the point $(x, y) \in U$, it follows

$$
f_{x y}(x, y)=f_{y x}(x, y) .
$$

Proof: Since $U$ is open, there exists $r>0$ such that $B((x, y), r) \subseteq U$. Now let $|t|,|s|<$ $r / 2, t, s$ real numbers and consider

$$
\begin{equation*}
\Delta(s, t) \equiv \frac{1}{s t}\{\overbrace{f(x+t, y+s)-f(x+t, y)}^{h(t)}-\overbrace{(f(x, y+s)-f(x, y))}^{h(0)}\} . \tag{6.15.25}
\end{equation*}
$$

Note that $(x+t, y+s) \in U$ because

$$
\begin{aligned}
|(x+t, y+s)-(x, y)| & =|(t, s)|=\left(t^{2}+s^{2}\right)^{1 / 2} \\
& \leq\left(\frac{r^{2}}{4}+\frac{r^{2}}{4}\right)^{1 / 2}=\frac{r}{\sqrt{2}}<r
\end{aligned}
$$

As implied above, $h(t) \equiv f(x+t, y+s)-f(x+t, y)$. Therefore, by the mean value theorem from calculus and the (one variable) chain rule,

$$
\begin{aligned}
\Delta(s, t) & =\frac{1}{s t}(h(t)-h(0))=\frac{1}{s t} h^{\prime}(\alpha t) t \\
& =\frac{1}{s}\left(f_{x}(x+\alpha t, y+s)-f_{x}(x+\alpha t, y)\right)
\end{aligned}
$$

for some $\alpha \in(0,1)$. Applying the mean value theorem again,

$$
\Delta(s, t)=f_{x y}(x+\alpha t, y+\beta s)
$$

where $\alpha, \beta \in(0,1)$.
If the terms $f(x+t, y)$ and $f(x, y+s)$ are interchanged in 6.15.25, $\Delta(s, t)$ is unchanged and the above argument shows there exist $\gamma, \delta \in(0,1)$ such that

$$
\Delta(s, t)=f_{y x}(x+\gamma t, y+\delta s)
$$

Letting $(s, t) \rightarrow(0,0)$ and using the continuity of $f_{x y}$ and $f_{y x}$ at $(x, y)$,

$$
\lim _{(s, t) \rightarrow(0,0)} \Delta(s, t)=f_{x y}(x, y)=f_{y x}(x, y)
$$

This proves the theorem.
The following is obtained from the above by simply fixing all the variables except for the two of interest.

Corollary 6.15.2 Suppose $U$ is an open subset of $\mathbb{F}^{n}$ and $f: U \rightarrow \mathbb{R}$ has the property that for two indices, $k, l, f_{x_{k}}, f_{x_{l}}, f_{x_{l} x_{k}}$, and $f_{x_{k} x_{l}}$ exist on $U$ and $f_{x_{k} x_{l}}$ and $f_{x_{l} x_{k}}$ are both continuous at $\mathbf{x} \in U$. Then $f_{x_{k} x_{l}}(\mathbf{x})=f_{x_{l} x_{k}}(\mathbf{x})$.

By considering the real and imaginary parts of $f$ in the case where $f$ has values in $\mathbb{F}$ you obtain the following corollary.

Corollary 6.15.3 Suppose $U$ is an open subset of $\mathbb{F}^{n}$ and $f: U \rightarrow \mathbb{F}$ has the property that for two indices, $k, l, f_{x_{k}}, f_{x_{l}}, f_{x_{l} x_{k}}$, and $f_{x_{k} x_{l}}$ exist on $U$ and $f_{x_{k} x_{l}}$ and $f_{x_{l} x_{k}}$ are both continuous at $\mathbf{x} \in U$. Then $f_{x_{k} x_{l}}(\mathbf{x})=f_{x_{l} x_{k}}(\mathbf{x})$.

Finally, by considering the components of $\mathbf{f}$ you get the following generalization.
Corollary 6.15.4 Suppose $U$ is an open subset of $\mathbb{F}^{n}$ and $\mathbf{f}: U \rightarrow \mathbb{F}^{m}$ has the property that for two indices, $k, l, \mathbf{f}_{x_{k}}, \mathbf{f}_{x_{l}}, \mathbf{f}_{x_{l} x_{k}}$, and $\mathbf{f}_{x_{k} x_{l}}$ exist on $U$ and $\mathbf{f}_{x_{k} x_{l}}$ and $\mathbf{f}_{x_{l} x_{k}}$ are both continuous at $\mathbf{x} \in U$. Then $\mathbf{f}_{x_{k} x_{l}}(\mathbf{x})=\mathbf{f}_{x_{l} x_{k}}(\mathbf{x})$.

It is necessary to assume the mixed partial derivatives are continuous in order to assert they are equal. The following is a well known example [6].

Example 6.15.5 Let

$$
f(x, y)=\left\{\begin{array}{l}
\frac{x y\left(x^{2}-y^{2}\right)}{x^{2}+y^{2}} \text { if }(x, y) \neq(0,0) \\
0 \text { if }(x, y)=(0,0)
\end{array}\right.
$$

From the definition of partial derivatives it follows immediately that

$$
f_{x}(0,0)=f_{y}(0,0)=0
$$

Using the standard rules of differentiation, for $(x, y) \neq(0,0)$,

$$
f_{x}=y \frac{x^{4}-y^{4}+4 x^{2} y^{2}}{\left(x^{2}+y^{2}\right)^{2}}, f_{y}=x \frac{x^{4}-y^{4}-4 x^{2} y^{2}}{\left(x^{2}+y^{2}\right)^{2}}
$$

Now

$$
\begin{aligned}
f_{x y}(0,0) & \equiv \lim _{y \rightarrow 0} \frac{f_{x}(0, y)-f_{x}(0,0)}{y} \\
& =\lim _{y \rightarrow 0} \frac{-y^{4}}{\left(y^{2}\right)^{2}}=-1
\end{aligned}
$$

while

$$
\begin{aligned}
f_{y x}(0,0) & \equiv \lim _{x \rightarrow 0} \frac{f_{y}(x, 0)-f_{y}(0,0)}{x} \\
& =\lim _{x \rightarrow 0} \frac{x^{4}}{\left(x^{2}\right)^{2}}=1
\end{aligned}
$$

showing that although the mixed partial derivatives do exist at $(0,0)$, they are not equal there.

### 6.16 Implicit Function Theorem

The implicit function theorem is one of the greatest theorems in mathematics. There are many versions of this theorem. However, I will give a very simple proof valid in finite dimensional spaces.

Theorem 6.16.1 (implicit function theorem) Suppose $U$ is an open set in $\mathbb{R}^{n} \times \mathbb{R}^{m}$. Let $\mathbf{f}: U \rightarrow \mathbb{R}^{n}$ be in $C^{1}(U)$ and suppose

$$
\begin{equation*}
\mathbf{f}\left(\mathbf{x}_{0}, \mathbf{y}_{0}\right)=\mathbf{0}, D_{1} \mathbf{f}\left(\mathbf{x}_{0}, \mathbf{y}_{0}\right)^{-1} \in \mathscr{L}\left(\mathbb{R}^{n}, \mathbb{R}^{n}\right) \tag{6.16.26}
\end{equation*}
$$

Then there exist positive constants, $\delta, \eta$, such that for every $\mathbf{y} \in B\left(\mathbf{y}_{0}, \eta\right)$ there exists $a$ unique $\mathbf{x}(\mathbf{y}) \in B\left(\mathbf{x}_{0}, \delta\right)$ such that

$$
\begin{equation*}
\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})=\mathbf{0} \tag{6.16.27}
\end{equation*}
$$

Furthermore, the mapping, $\mathbf{y} \rightarrow \mathbf{x}(\mathbf{y})$ is in $C^{1}\left(B\left(\mathbf{y}_{0}, \eta\right)\right)$.
Proof: Let

$$
\mathbf{f}(\mathbf{x}, \mathbf{y})=\left(\begin{array}{c}
f_{1}(\mathbf{x}, \mathbf{y}) \\
f_{2}(\mathbf{x}, \mathbf{y}) \\
\vdots \\
f_{n}(\mathbf{x}, \mathbf{y})
\end{array}\right)
$$

Define for $\left(\mathbf{x}^{1}, \cdots, \mathbf{x}^{n}\right) \in{\overline{B\left(\mathbf{x}_{0}, \delta\right)}}^{n}$ and $\mathbf{y} \in B\left(\mathbf{y}_{0}, \eta\right)$ the following matrix.

$$
J\left(\mathbf{x}^{1}, \cdots, \mathbf{x}^{n}, \mathbf{y}\right) \equiv\left(\begin{array}{ccc}
f_{1, x_{1}}\left(\mathbf{x}^{1}, \mathbf{y}\right) & \cdots & f_{1, x_{n}}\left(\mathbf{x}^{1}, \mathbf{y}\right) \\
\vdots & & \vdots \\
f_{n, x_{1}}\left(\mathbf{x}^{n}, \mathbf{y}\right) & \cdots & f_{n, x_{n}}\left(\mathbf{x}^{n}, \mathbf{y}\right)
\end{array}\right)
$$

Then by the assumption of continuity of all the partial derivatives, there exists $\delta_{0}>0$ and $\eta_{0}>0$ such that if $\delta<\delta_{0}$ and $\eta<\eta_{0}$, it follows that for all $\left(\mathbf{x}^{1}, \cdots, \mathbf{x}^{n}\right) \in{\overline{B\left(\mathbf{x}_{0}, \delta\right)}}^{n}$ and $\mathbf{y} \in B\left(\mathbf{y}_{0}, \eta\right)$,

$$
\begin{equation*}
\operatorname{det}\left(J\left(\mathbf{x}^{1}, \cdots, \mathbf{x}^{n}, \mathbf{y}\right)\right)>r>0 \tag{6.16.28}
\end{equation*}
$$

and $\overline{B\left(\mathbf{x}_{0}, \delta_{0}\right)} \times \overline{B\left(\mathbf{y}_{0}, \eta_{0}\right)} \subseteq U$. Pick $\mathbf{y} \in B\left(\mathbf{y}_{0}, \eta\right)$ and suppose there exist $\mathbf{x}, \mathbf{z} \in \overline{B\left(\mathbf{x}_{0}, \delta\right)}$ such that $\mathbf{f}(\mathbf{x}, \mathbf{y})=\mathbf{f}(\mathbf{z}, \mathbf{y})=\mathbf{0}$. Consider $f_{i}$ and let

$$
h(t) \equiv f_{i}(\mathbf{x}+t(\mathbf{z}-\mathbf{x}), \mathbf{y})
$$

Then $h(1)=h(0)$ and so by the mean value theorem, $h^{\prime}\left(t_{i}\right)=0$ for some $t_{i} \in(0,1)$. Therefore, from the chain rule and for this value of $t_{i}$,

$$
\begin{equation*}
h^{\prime}\left(t_{i}\right)=D f_{i}\left(\mathbf{x}+t_{i}(\mathbf{z}-\mathbf{x}), \mathbf{y}\right)(\mathbf{z}-\mathbf{x})=0 \tag{6.16.29}
\end{equation*}
$$

Then denote by $\mathbf{x}^{i}$ the vector, $\mathbf{x}+t_{i}(\mathbf{z}-\mathbf{x})$. It follows from 6.16.29 that

$$
J\left(\mathbf{x}^{1}, \cdots, \mathbf{x}^{n}, \mathbf{y}\right)(\mathbf{z}-\mathbf{x})=\mathbf{0}
$$

and so from $6.16 .28 \mathbf{z}-\mathbf{x}=\mathbf{0}$. Now it will be shown that if $\eta$ is chosen sufficiently small, then for all $\mathbf{y} \in B\left(\mathbf{y}_{0}, \eta\right)$, there exists a unique $\mathbf{x}(\mathbf{y}) \in B\left(\mathbf{x}_{0}, \boldsymbol{\delta}\right)$ such that $\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})=\mathbf{0}$.

Claim: If $\eta$ is small enough, then the function, $h_{\mathbf{y}}(\mathbf{x}) \equiv|\mathbf{f}(\mathbf{x}, \mathbf{y})|^{2}$ achieves its minimum value on $\overline{B\left(\mathbf{x}_{0}, \delta\right)}$ at a point of $B\left(\mathbf{x}_{0}, \boldsymbol{\delta}\right)$.

Proof of claim: Suppose this is not the case. Then there exists a sequence $\eta_{k} \rightarrow 0$ and for some $\mathbf{y}_{k}$ having $\left|\mathbf{y}_{k}-\mathbf{y}_{0}\right|<\eta_{k}$, the minimum of $h_{\mathbf{y}_{k}}$ occurs on a point of the boundary of $\overline{B\left(\mathbf{x}_{0}, \delta\right)}, \mathbf{x}_{k}$ such that $\left|\mathbf{x}_{0}-\mathbf{x}_{k}\right|=\delta$. Now taking a subsequence, still denoted by $k$, it can be assumed that $\mathbf{x}_{k} \rightarrow \mathbf{x}$ with $\left|\mathbf{x}-\mathbf{x}_{0}\right|=\delta$ and $\mathbf{y}_{k} \rightarrow \mathbf{y}_{0}$. Let $\varepsilon>0$. Then for $k$ large enough, $h_{\mathbf{y}_{k}}\left(\mathbf{x}_{0}\right)<\varepsilon$ because $\mathbf{f}\left(\mathbf{x}_{0}, \mathbf{y}_{0}\right)=\mathbf{0}$. Therefore, from the definition of $\mathbf{x}_{k}, h_{\mathbf{y}_{k}}\left(\mathbf{x}_{k}\right)<$ $\varepsilon$. Passing to the limit yields $h_{\mathbf{y}_{0}}(\mathbf{x}) \leq \varepsilon$. Since $\varepsilon>0$ is arbitrary, it follows that $h_{\mathbf{y}_{0}}(\mathbf{x})=0$ which contradicts the first part of the argument in which it was shown that for $\mathbf{y} \in B\left(\mathbf{y}_{0}, \eta\right)$ there is at most one point, $\mathbf{x}$ of $\overline{B\left(\mathbf{x}_{0}, \delta\right)}$ where $\mathbf{f}(\mathbf{x}, \mathbf{y})=\mathbf{0}$. Here two have been obtained, $\mathbf{x}_{0}$ and $\mathbf{x}$. This proves the claim.

Choose $\eta<\eta_{0}$ and also small enough that the above claim holds and let $\mathbf{x}(\mathbf{y})$ denote a point of $B\left(\mathbf{x}_{0}, \delta\right)$ at which the minimum of $h_{\mathbf{y}}$ on $\overline{B\left(\mathbf{x}_{0}, \delta\right)}$ is achieved. Since $\mathbf{x}(\mathbf{y})$ is an interior point, you can consider $h_{\mathbf{y}}(\mathbf{x}(\mathbf{y})+t \mathbf{v})$ for $|t|$ small and conclude this function of $t$ has a zero derivative at $t=0$. Thus

$$
D h_{\mathbf{y}}(\mathbf{x}(\mathbf{y})) \mathbf{v}=0=2 \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})^{T} D_{1} \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) \mathbf{v}
$$

for every vector $\mathbf{v}$. But from 6.16.28 and the fact that $\mathbf{v}$ is arbitrary, it follows $\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})=\mathbf{0}$. This proves the existence of the function $\mathbf{y} \rightarrow \mathbf{x}(\mathbf{y})$ such that $\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})=\mathbf{0}$ for all $\mathbf{y} \in$ $B\left(\mathbf{y}_{0}, \eta\right)$.

It remains to verify this function is a $C^{1}$ function. To do this, let $\mathbf{y}_{1}$ and $\mathbf{y}_{2}$ be points of $B\left(\mathbf{y}_{0}, \eta\right)$. Then as before, consider the $i^{\text {th }}$ component of $\mathbf{f}$ and consider the same argument using the mean value theorem to write

$$
\begin{gathered}
0=f_{i}\left(\mathbf{x}\left(\mathbf{y}_{1}\right), \mathbf{y}_{1}\right)-f_{i}\left(\mathbf{x}\left(\mathbf{y}_{2}\right), \mathbf{y}_{2}\right) \\
=f_{i}\left(\mathbf{x}\left(\mathbf{y}_{1}\right), \mathbf{y}_{1}\right)-f_{i}\left(\mathbf{x}\left(\mathbf{y}_{2}\right), \mathbf{y}_{1}\right)+f_{i}\left(\mathbf{x}\left(\mathbf{y}_{2}\right), \mathbf{y}_{1}\right)-f_{i}\left(\mathbf{x}\left(\mathbf{y}_{2}\right), \mathbf{y}_{2}\right) \\
=D_{1} f_{i}\left(\mathbf{x}^{i}, \mathbf{y}_{1}\right)\left(\mathbf{x}\left(\mathbf{y}_{1}\right)-\mathbf{x}\left(\mathbf{y}_{2}\right)\right)+D_{2} f_{i}\left(\mathbf{x}\left(\mathbf{y}_{2}\right), \mathbf{y}^{i}\right)\left(\mathbf{y}_{1}-\mathbf{y}_{2}\right)
\end{gathered}
$$

Therefore,

$$
\begin{equation*}
J\left(\mathbf{x}^{1}, \cdots, \mathbf{x}^{n}, \mathbf{y}_{1}\right)\left(\mathbf{x}\left(\mathbf{y}_{1}\right)-\mathbf{x}\left(\mathbf{y}_{2}\right)\right)=-M\left(\mathbf{y}_{1}-\mathbf{y}_{2}\right) \tag{6.16.30}
\end{equation*}
$$

where $M$ is the matrix whose $i^{t h}$ row is $D_{2} f_{i}\left(\mathbf{x}\left(\mathbf{y}_{2}\right), \mathbf{y}^{i}\right)$. Then from 6.16 .28 there exists a constant, $C$ independent of the choice of $\mathbf{y} \in B\left(\mathbf{y}_{0}, \eta\right)$ such that

$$
\left\|J\left(\mathbf{x}^{1}, \cdots, \mathbf{x}^{n}, \mathbf{y}\right)^{-1}\right\|<C
$$

whenever $\left(\mathbf{x}^{1}, \cdots, \mathbf{x}^{n}\right) \in{\overline{B\left(\mathbf{x}_{0}, \delta\right)}}^{n}$. By continuity of the partial derivatives of $\mathbf{f}$ it also follows there exists a constant, $C_{1}$ such that $\left\|D_{2} f_{i}(\mathbf{x}, \mathbf{y})\right\|<C_{1}$ whenever, $(\mathbf{x}, \mathbf{y}) \in \overline{B\left(\mathbf{x}_{0}, \boldsymbol{\delta}\right)} \times$ $B\left(\mathbf{y}_{0}, \eta\right)$. Hence $\|M\|$ must also be bounded independent of the choice of $\mathbf{y}_{1}$ and $\mathbf{y}_{2}$ in $B\left(\mathbf{y}_{0}, \eta\right)$. From 6.16.30, it follows there exists a constant, $C$ such that for all $\mathbf{y}_{1}, \mathbf{y}_{2}$ in $B\left(\mathbf{y}_{0}, \eta\right)$,

$$
\begin{equation*}
\left|\mathbf{x}\left(\mathbf{y}_{1}\right)-\mathbf{x}\left(\mathbf{y}_{2}\right)\right| \leq C\left|\mathbf{y}_{1}-\mathbf{y}_{2}\right| \tag{6.16.31}
\end{equation*}
$$

It follows as in the proof of the chain rule that

$$
\begin{equation*}
\mathbf{o}(\mathbf{x}(\mathbf{y}+\mathbf{v})-\mathbf{x}(\mathbf{y}))=\mathbf{o}(\mathbf{v}) . \tag{6.16.32}
\end{equation*}
$$

Now let $\mathbf{y} \in B\left(\mathbf{y}_{0}, \eta\right)$ and let $|\mathbf{v}|$ be sufficiently small that $\mathbf{y}+\mathbf{v} \in B\left(\mathbf{y}_{0}, \eta\right)$. Then

$$
\begin{aligned}
& \mathbf{0}= \mathbf{f}(\mathbf{x}(\mathbf{y}+\mathbf{v}), \mathbf{y}+\mathbf{v})-\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) \\
&= \mathbf{f}(\mathbf{x}(\mathbf{y}+\mathbf{v}), \mathbf{y}+\mathbf{v})-\mathbf{f}(\mathbf{x}(\mathbf{y}+\mathbf{v}), \mathbf{y})+\mathbf{f}(\mathbf{x}(\mathbf{y}+\mathbf{v}), \mathbf{y})-\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) \\
&=D_{2} \mathbf{f}(\mathbf{x}(\mathbf{y}+\mathbf{v}), \mathbf{y}) \mathbf{v}+D_{1} \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})(\mathbf{x}(\mathbf{y}+\mathbf{v})-\mathbf{x}(\mathbf{y}))+\mathbf{o}(|\mathbf{x}(\mathbf{y}+\mathbf{v})-\mathbf{x}(\mathbf{y})|) \\
&= D_{2} \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) \mathbf{v}+D_{1} \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})(\mathbf{x}(\mathbf{y}+\mathbf{v})-\mathbf{x}(\mathbf{y}))+ \\
& \mathbf{o}(|\mathbf{x}(\mathbf{y}+\mathbf{v})-\mathbf{x}(\mathbf{y})|)+\left(D_{2} \mathbf{f}(\mathbf{x}(\mathbf{y}+\mathbf{v}), \mathbf{y}) \mathbf{v}-D_{2} \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) \mathbf{v}\right) \\
&= D_{2} \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) \mathbf{v}+D_{1} \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})(\mathbf{x}(\mathbf{y}+\mathbf{v})-\mathbf{x}(\mathbf{y}))+\mathbf{o}(\mathbf{v}) .
\end{aligned}
$$

Therefore,

$$
\mathbf{x}(\mathbf{y}+\mathbf{v})-\mathbf{x}(\mathbf{y})=-D_{1} \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})^{-1} D_{2} \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) \mathbf{v}+\mathbf{o}(\mathbf{v})
$$

which shows that $D \mathbf{x}(\mathbf{y})=-D_{1} \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})^{-1} D_{2} \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})$ and $\mathbf{y} \rightarrow D \mathbf{x}(\mathbf{y})$ is continuous. This proves the theorem.

In practice, how do you verify the condition, $D_{1} \mathbf{f}\left(\mathbf{x}_{0}, \mathbf{y}_{0}\right)^{-1} \in \mathscr{L}\left(\mathbb{F}^{n}, \mathbb{F}^{n}\right)$ ?

$$
\mathbf{f}(\mathbf{x}, \mathbf{y})=\left(\begin{array}{c}
f_{1}\left(x_{1}, \cdots, x_{n}, y_{1}, \cdots, y_{n}\right) \\
\vdots \\
f_{n}\left(x_{1}, \cdots, x_{n}, y_{1}, \cdots, y_{n}\right)
\end{array}\right)
$$

The matrix of the linear transformation, $D_{1} \mathbf{f}\left(\mathbf{x}_{0}, \mathbf{y}_{0}\right)$ is then

$$
\left(\begin{array}{ccc}
\frac{\partial f_{1}\left(x_{1}, \cdots, x_{n}, y_{1}, \cdots, y_{n}\right)}{\partial x_{1}} & \cdots & \frac{\partial f_{1}\left(x_{1}, \cdots, x_{n}, y_{1}, \cdots, y_{n}\right)}{\partial x_{n}} \\
\vdots & & \vdots \\
\frac{\partial f_{n}\left(x_{1}, \cdots, x_{n}, y_{1}, \cdots, y_{n}\right)}{\partial x_{1}} & \cdots & \frac{\partial f_{n}\left(x_{1}, \cdots, x_{n}, y_{1}, \cdots, y_{n}\right)}{\partial x_{n}}
\end{array}\right)
$$

and from linear algebra, $D_{1} \mathbf{f}\left(\mathbf{x}_{0}, \mathbf{y}_{0}\right)^{-1} \in \mathscr{L}\left(\mathbb{F}^{n}, \mathbb{F}^{n}\right)$ exactly when the above matrix has an inverse. In other words when

$$
\operatorname{det}\left(\begin{array}{ccc}
\frac{\partial f_{1}\left(x_{1}, \cdots, x_{n}, y_{1}, \cdots, y_{n}\right)}{\partial x_{1}} & \cdots & \frac{\partial f_{1}\left(x_{1}, \cdots, x_{n}, y_{1}, \cdots, y_{n}\right)}{\partial x_{n}} \\
\vdots & & \vdots \\
\frac{\partial f_{n}\left(x_{1}, \cdots, x_{n}, y_{1}, \cdots, y_{n}\right)}{\partial x_{1}} & \cdots & \frac{\partial f_{n}\left(x_{1}, \cdots, x_{n}, y_{1}, \cdots, y_{n}\right)}{\partial x_{n}}
\end{array}\right) \neq 0
$$

at $\left(\mathbf{x}_{0}, \mathbf{y}_{0}\right)$. The above determinant is important enough that it is given special notation. Letting $\mathbf{z}=\mathbf{f}(\mathbf{x}, \mathbf{y})$, the above determinant is often written as

$$
\frac{\partial\left(z_{1}, \cdots, z_{n}\right)}{\partial\left(x_{1}, \cdots, x_{n}\right)}
$$

Of course you can replace $\mathbb{R}$ with $\mathbb{F}$ in the above by applying the above to the situation in which each $\mathbb{F}$ is replaced with $\mathbb{R}^{2}$.

Corollary 6.16.2 (implicit function theorem) Suppose $U$ is an open set in $\mathbb{F}^{n} \times \mathbb{F}^{m}$. Let $\mathbf{f}: U \rightarrow \mathbb{F}^{n}$ be in $C^{1}(U)$ and suppose

$$
\begin{equation*}
\mathbf{f}\left(\mathbf{x}_{0}, \mathbf{y}_{0}\right)=\mathbf{0}, D_{1} \mathbf{f}\left(\mathbf{x}_{0}, \mathbf{y}_{0}\right)^{-1} \in \mathscr{L}\left(\mathbb{F}^{n}, \mathbb{F}^{n}\right) \tag{6.16.33}
\end{equation*}
$$

Then there exist positive constants, $\delta, \eta$, such that for every $\mathbf{y} \in B\left(\mathbf{y}_{0}, \eta\right)$ there exists $a$ unique $\mathbf{x}(\mathbf{y}) \in B\left(\mathbf{x}_{0}, \delta\right)$ such that

$$
\begin{equation*}
\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})=\mathbf{0} . \tag{6.16.34}
\end{equation*}
$$

Furthermore, the mapping, $\mathbf{y} \rightarrow \mathbf{x}(\mathbf{y})$ is in $C^{1}\left(B\left(\mathbf{y}_{0}, \eta\right)\right)$.
The next theorem is a very important special case of the implicit function theorem known as the inverse function theorem. Actually one can also obtain the implicit function theorem from the inverse function theorem. It is done this way in [84] and in [4].

Theorem 6.16.3 (inverse function theorem) Let $\mathbf{x}_{0} \in U \subseteq \mathbb{F}^{n}$ and let $\mathbf{f}: U \rightarrow \mathbb{F}^{n}$. Suppose

$$
\begin{equation*}
\mathbf{f} \text { is } C^{1}(U), \text { and } D \mathbf{f}\left(\mathbf{x}_{0}\right)^{-1} \in \mathscr{L}\left(\mathbb{F}^{n}, \mathbb{F}^{n}\right) \tag{6.16.35}
\end{equation*}
$$

Then there exist open sets, $W$, and $V$ such that

$$
\begin{equation*}
\mathbf{x}_{0} \in W \subseteq U \tag{6.16.36}
\end{equation*}
$$

$$
\begin{equation*}
\mathbf{f}: W \rightarrow V \text { is one to one and onto, } \tag{6.16.37}
\end{equation*}
$$

$$
\begin{equation*}
\mathbf{f}^{-1} \text { is } C^{1} \tag{6.16.38}
\end{equation*}
$$

Proof: Apply the implicit function theorem to the function

$$
\mathbf{F}(\mathbf{x}, \mathbf{y}) \equiv \mathbf{f}(\mathbf{x})-\mathbf{y}
$$

where $\mathbf{y}_{0} \equiv \mathbf{f}\left(\mathbf{x}_{0}\right)$. Thus the function $\mathbf{y} \rightarrow \mathbf{x}(\mathbf{y})$ defined in that theorem is $\mathbf{f}^{-1}$. Now let

$$
W \equiv B\left(\mathbf{x}_{0}, \boldsymbol{\delta}\right) \cap \mathbf{f}^{-1}\left(B\left(\mathbf{y}_{0}, \eta\right)\right)
$$

and

$$
V \equiv B\left(\mathbf{y}_{0}, \eta\right)
$$

This proves the theorem.

### 6.16.1 More Continuous Partial Derivatives

Corollary 6.16 .2 will now be improved slightly. If $\mathbf{f}$ is $C^{k}$, it follows that the function which is implicitly defined is also in $C^{k}$, not just $C^{1}$. Since the inverse function theorem comes as a case of the implicit function theorem, this shows that the inverse function also inherits the property of being $C^{k}$.

Theorem 6.16.4 (implicit function theorem) Suppose $U$ is an open set in $\mathbb{F}^{n} \times \mathbb{F}^{m}$. Let $\mathbf{f}: U \rightarrow \mathbb{F}^{n}$ be in $C^{k}(U)$ and suppose

$$
\begin{equation*}
\mathbf{f}\left(\mathbf{x}_{0}, \mathbf{y}_{0}\right)=\mathbf{0}, D_{1} \mathbf{f}\left(\mathbf{x}_{0}, \mathbf{y}_{0}\right)^{-1} \in \mathscr{L}\left(\mathbb{F}^{n}, \mathbb{F}^{n}\right) \tag{6.16.39}
\end{equation*}
$$

Then there exist positive constants, $\delta, \eta$, such that for every $\mathbf{y} \in B\left(\mathbf{y}_{0}, \eta\right)$ there exists $a$ unique $\mathbf{x}(\mathbf{y}) \in B\left(\mathbf{x}_{0}, \delta\right)$ such that

$$
\begin{equation*}
\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})=\mathbf{0} \tag{6.16.40}
\end{equation*}
$$

Furthermore, the mapping, $\mathbf{y} \rightarrow \mathbf{x}(\mathbf{y})$ is in $C^{k}\left(B\left(\mathbf{y}_{0}, \eta\right)\right)$.
Proof: From Corollary 6.16.2 $\mathbf{y} \rightarrow \mathbf{x}(\mathbf{y})$ is $C^{1}$. It remains to show it is $C^{k}$ for $k>1$ assuming that $\mathbf{f}$ is $C^{k}$. From 6.16.40

$$
\frac{\partial \mathbf{x}}{\partial y^{l}}=-D_{1}(\mathbf{x}, \mathbf{y})^{-1} \frac{\partial \mathbf{f}}{\partial y^{l}}
$$

Thus the following formula holds for $q=1$ and $|\alpha|=q$.

$$
\begin{equation*}
D^{\alpha} \mathbf{x}(\mathbf{y})=\sum_{|\beta| \leq q} M_{\beta}(\mathbf{x}, \mathbf{y}) D^{\beta} \mathbf{f}(\mathbf{x}, \mathbf{y}) \tag{6.16.41}
\end{equation*}
$$

where $M_{\beta}$ is a matrix whose entries are differentiable functions of $D^{\gamma}(\mathbf{x})$ for $|\gamma|<q$ and $D^{\tau} \mathbf{f}(\mathbf{x}, \mathbf{y})$ for $|\tau| \leq q$. This follows easily from the description of $D_{1}(\mathbf{x}, \mathbf{y})^{-1}$ in terms of the cofactor matrix and the determinant of $D_{1}(\mathbf{x}, \mathbf{y})$. Suppose 6.16 .41 holds for $|\alpha|=q<k$. Then by induction, this yields $\mathbf{x}$ is $C^{q}$. Then

$$
\frac{\partial D^{\alpha} \mathbf{x}(\mathbf{y})}{\partial y^{p}}=\sum_{|\beta| \leq|\alpha|} \frac{\partial M_{\beta}(\mathbf{x}, \mathbf{y})}{\partial y^{p}} D^{\beta} \mathbf{f}(\mathbf{x}, \mathbf{y})+M_{\beta}(\mathbf{x}, \mathbf{y}) \frac{\partial D^{\beta} \mathbf{f}(\mathbf{x}, \mathbf{y})}{\partial y^{p}}
$$

By the chain rule $\frac{\partial M_{\beta}(\mathbf{x}, \mathbf{y})}{\partial y^{p}}$ is a matrix whose entries are differentiable functions of $D^{\tau} \mathbf{f}(\mathbf{x}, \mathbf{y})$ for $|\tau| \leq q+1$ and $D^{\gamma}(\mathbf{x})$ for $|\gamma|<q+1$. It follows since $y^{p}$ was arbitrary that for any $|\alpha|=q+1$, a formula like 6.16 .41 holds with $q$ being replaced by $q+1$. By induction, $\mathbf{x}$ is
$C^{k}$. This proves the theorem.
As a simple corollary this yields an improved version of the inverse function theorem.
Theorem 6.16.5 (inverse function theorem) Let $\mathbf{x}_{0} \in U \subseteq \mathbb{F}^{n}$ and let $\mathbf{f}: U \rightarrow \mathbb{F}^{n}$. Suppose for $k$ a positive integer,

$$
\begin{equation*}
\mathbf{f} \text { is } C^{k}(U), \text { and } D \mathbf{f}\left(\mathbf{x}_{0}\right)^{-1} \in \mathscr{L}\left(\mathbb{F}^{n}, \mathbb{F}^{n}\right) \tag{6.16.42}
\end{equation*}
$$

Then there exist open sets, $W$, and $V$ such that

$$
\begin{equation*}
\mathbf{x}_{0} \in W \subseteq U \tag{6.16.43}
\end{equation*}
$$

$$
\begin{align*}
& \mathbf{f}: W \rightarrow V \text { is one to one and onto, }  \tag{6.16.44}\\
& \qquad \mathbf{f}^{-1} \text { is } C^{k} \tag{6.16.45}
\end{align*}
$$

### 6.17 The Method Of Lagrange Multipliers

As an application of the implicit function theorem, consider the method of Lagrange multipliers from calculus. Recall the problem is to maximize or minimize a function subject to equality constraints. Let $f: U \rightarrow \mathbb{R}$ be a $C^{1}$ function where $U \subseteq \mathbb{R}^{n}$ and let

$$
\begin{equation*}
g_{i}(\mathbf{x})=0, i=1, \cdots, m \tag{6.17.46}
\end{equation*}
$$

be a collection of equality constraints with $m<n$. Now consider the system of nonlinear equations

$$
\begin{aligned}
f(\mathbf{x}) & =a \\
g_{i}(\mathbf{x}) & =0, i=1, \cdots, m
\end{aligned}
$$

$\mathbf{x}_{0}$ is a local maximum if $f\left(\mathbf{x}_{0}\right) \geq f(\mathbf{x})$ for all $\mathbf{x}$ near $\mathbf{x}_{0}$ which also satisfies the constraints 6.17.46. A local minimum is defined similarly. Let $\mathbf{F}: U \times \mathbb{R} \rightarrow \mathbb{R}^{m+1}$ be defined by

$$
\mathbf{F}(\mathbf{x}, a) \equiv\left(\begin{array}{c}
f(\mathbf{x})-a  \tag{6.17.47}\\
g_{1}(\mathbf{x}) \\
\vdots \\
g_{m}(\mathbf{x})
\end{array}\right)
$$

Now consider the $m+1 \times n$ Jacobian matrix,

$$
\left(\begin{array}{ccc}
f_{x_{1}}\left(\mathbf{x}_{0}\right) & \cdots & f_{x_{n}}\left(\mathbf{x}_{0}\right) \\
g_{1 x_{1}}\left(\mathbf{x}_{0}\right) & \cdots & g_{1 x_{n}}\left(\mathbf{x}_{0}\right) \\
\vdots & & \vdots \\
g_{m x_{1}}\left(\mathbf{x}_{0}\right) & \cdots & g_{m x_{n}}\left(\mathbf{x}_{0}\right)
\end{array}\right)
$$

If this matrix has rank $m+1$ then some $m+1 \times m+1$ submatrix has nonzero determinant. It follows from the implicit function theorem that there exist $m+1$ variables, $x_{i_{1}}, \cdots, x_{i_{m+1}}$ such that the system

$$
\begin{equation*}
\mathbf{F}(\mathbf{x}, a)=\mathbf{0} \tag{6.17.48}
\end{equation*}
$$

specifies these $m+1$ variables as a function of the remaining $n-(m+1)$ variables and $a$ in an open set of $\mathbb{R}^{n-m}$. Thus there is a solution $(\mathbf{x}, a)$ to 6.17 .48 for some $\mathbf{x}$ close to $\mathbf{x}_{0}$ whenever $a$ is in some open interval. Therefore, $\mathbf{x}_{0}$ cannot be either a local minimum or a local maximum. It follows that if $\mathbf{x}_{0}$ is either a local maximum or a local minimum, then the above matrix must have rank less than $m+1$ which requires the rows to be linearly dependent. Thus, there exist $m$ scalars,

$$
\lambda_{1}, \cdots, \lambda_{m}
$$

and a scalar $\mu$, not all zero such that

$$
\mu\left(\begin{array}{c}
f_{x_{1}}\left(\mathbf{x}_{0}\right)  \tag{6.17.49}\\
\vdots \\
f_{x_{n}}\left(\mathbf{x}_{0}\right)
\end{array}\right)=\lambda_{1}\left(\begin{array}{c}
g_{1 x_{1}}\left(\mathbf{x}_{0}\right) \\
\vdots \\
g_{1 x_{n}}\left(\mathbf{x}_{0}\right)
\end{array}\right)+\cdots+\lambda_{m}\left(\begin{array}{c}
g_{m x_{1}}\left(\mathbf{x}_{0}\right) \\
\vdots \\
g_{m x_{n}}\left(\mathbf{x}_{0}\right)
\end{array}\right)
$$

If the column vectors

$$
\left(\begin{array}{c}
g_{1 x_{1}}\left(\mathbf{x}_{0}\right)  \tag{6.17.50}\\
\vdots \\
g_{1 x_{n}}\left(\mathbf{x}_{0}\right)
\end{array}\right), \cdots\left(\begin{array}{c}
g_{m x_{1}}\left(\mathbf{x}_{0}\right) \\
\vdots \\
g_{m x_{n}}\left(\mathbf{x}_{0}\right)
\end{array}\right)
$$

are linearly independent, then, $\mu \neq 0$ and dividing by $\mu$ yields an expression of the form

$$
\left(\begin{array}{c}
f_{x_{1}}\left(\mathbf{x}_{0}\right)  \tag{6.17.51}\\
\vdots \\
f_{x_{n}}\left(\mathbf{x}_{0}\right)
\end{array}\right)=\lambda_{1}\left(\begin{array}{c}
g_{1 x_{1}}\left(\mathbf{x}_{0}\right) \\
\vdots \\
g_{1 x_{n}}\left(\mathbf{x}_{0}\right)
\end{array}\right)+\cdots+\lambda_{m}\left(\begin{array}{c}
g_{m x_{1}}\left(\mathbf{x}_{0}\right) \\
\vdots \\
g_{m x_{n}}\left(\mathbf{x}_{0}\right)
\end{array}\right)
$$

at every point $\mathbf{x}_{0}$ which is either a local maximum or a local minimum. This proves the following theorem.

Theorem 6.17.1 Let $U$ be an open subset of $\mathbb{R}^{n}$ and let $f: U \rightarrow \mathbb{R}$ be a $C^{1}$ function. Then if $\mathbf{x}_{0} \in U$ is either a local maximum or local minimum of $f$ subject to the constraints 6.17.46, then 6.17 .49 must hold for some scalars $\mu, \lambda_{1}, \cdots, \lambda_{m}$ not all equal to zero. If the vectors in 6.17 .50 are linearly independent, it follows that an equation of the form 6.17 .51 holds.

## Chapter 7

## Metric Spaces and Topological Spaces

### 7.1 Metric Space

Definition 7.1.1 A metric space is a set, $X$ and a function $d: X \times X \rightarrow[0, \infty)$ which satisfies the following properties.

$$
\begin{gathered}
d(x, y)=d(y, x) \\
d(x, y) \geq 0 \text { and } d(x, y)=0 \text { if and only if } x=y \\
d(x, y) \leq d(x, z)+d(z, y)
\end{gathered}
$$

You can check that $\mathbb{R}^{n}$ and $\mathbb{C}^{n}$ are metric spaces with $d(\mathbf{x}, \mathbf{y})=|\mathbf{x}-\mathbf{y}|$. However, there are many others. The definitions of open and closed sets are the same for a metric space as they are for $\mathbb{R}^{n}$.

Definition 7.1.2 $A$ set, $U$ in a metric space is open if whenever $x \in U$, there exists $r>0$ such that $B(x, r) \subseteq U$. As before, $B(x, r) \equiv\{y: d(x, y)<r\}$. Closed sets are those whose complements are open. A point $p$ is a limit point of a set, $S$ if for every $r>0, B(p, r)$ contains infinitely many points of $S$. A sequence, $\left\{x_{n}\right\}$ converges to a point $x$ if for every $\varepsilon>0$ there exists $N$ such that if $n \geq N$, then $d\left(x, x_{n}\right)<\varepsilon$. $\left\{x_{n}\right\}$ is a Cauchy sequence iffor every $\varepsilon>0$ there exists $N$ such that if $m, n \geq N$, then $d\left(x_{n}, x_{m}\right)<\varepsilon$.

Lemma 7.1.3 In a metric space, $X$ every ball, $B(x, r)$ is open. A set is closed if and only if it contains all its limit points. If $p$ is a limit point of $S$, then there exists a sequence of distinct points of $S,\left\{x_{n}\right\}$ such that $\lim _{n \rightarrow \infty} x_{n}=p$.

Proof: Let $z \in B(x, r)$. Let $\delta=r-d(x, z)$. Then if $w \in B(z, \delta)$,

$$
d(w, x) \leq d(x, z)+d(z, w)<d(x, z)+r-d(x, z)=r
$$

Therefore, $B(z, \delta) \subseteq B(x, r)$ and this shows $B(x, r)$ is open.
The properties of balls are presented in the following theorem.
Theorem 7.1.4 Suppose $(X, d)$ is a metric space. Then the sets $\{B(x, r): r>0, x \in X\}$ satisfy

$$
\begin{equation*}
\cup\{B(x, r): r>0, x \in X\}=X \tag{7.1.1}
\end{equation*}
$$

If $p \in B\left(x, r_{1}\right) \cap B\left(z, r_{2}\right)$, there exists $r>0$ such that

$$
\begin{equation*}
B(p, r) \subseteq B\left(x, r_{1}\right) \cap B\left(z, r_{2}\right) \tag{7.1.2}
\end{equation*}
$$

Proof: Observe that the union of these balls includes the whole space, $X$ so 7.1.1 is obvious. Consider 7.1.2. Let $p \in B\left(x, r_{1}\right) \cap B\left(z, r_{2}\right)$. Consider

$$
r \equiv \min \left(r_{1}-d(x, p), r_{2}-d(z, p)\right)
$$

and suppose $y \in B(p, r)$. Then

$$
d(y, x) \leq d(y, p)+d(p, x)<r_{1}-d(x, p)+d(x, p)=r_{1}
$$

and so $B(p, r) \subseteq B\left(x, r_{1}\right)$. By similar reasoning, $B(p, r) \subseteq B\left(z, r_{2}\right)$. This proves the theorem.

Let $K$ be a closed set. This means $K^{C} \equiv X \backslash K$ is an open set. Let $p$ be a limit point of $K$. If $p \in K^{C}$, then since $K^{C}$ is open, there exists $B(p, r) \subseteq K^{C}$. But this contradicts $p$ being a limit point because there are no points of $K$ in this ball. Hence all limit points of $K$ must be in $K$.

Suppose next that $K$ contains its limit points. Is $K^{C}$ open? Let $p \in K^{C}$. Then $p$ is not a limit point of $K$. Therefore, there exists $B(p, r)$ which contains at most finitely many points of $K$. Since $p \notin K$, it follows that by making $r$ smaller if necessary, $B(p, r)$ contains no points of $K$. That is $B(p, r) \subseteq K^{C}$ showing $K^{C}$ is open. Therefore, $K$ is closed.

Suppose now that $p$ is a limit point of $S$. Let $x_{1} \in(S \backslash\{p\}) \cap B(p, 1)$. If $x_{1}, \cdots, x_{k}$ have been chosen, let

$$
r_{k+1} \equiv \min \left\{d\left(p, x_{i}\right), i=1, \cdots, k, \frac{1}{k+1}\right\}
$$

Let $x_{k+1} \in(S \backslash\{p\}) \cap B\left(p, r_{k+1}\right)$. This proves the lemma.
Lemma 7.1.5 If $\left\{x_{n}\right\}$ is a Cauchy sequence in a metric space, $X$ and if some subsequence, $\left\{x_{n_{k}}\right\}$ converges to $x$, then $\left\{x_{n}\right\}$ converges to $x$. Also if a sequence converges, then it is $a$ Cauchy sequence.

Proof: Note first that $n_{k} \geq k$ because in a subsequence, the indices, $n_{1}, n_{2}, \cdots$ are strictly increasing. Let $\varepsilon>0$ be given and let $N$ be such that for $k>N, d\left(x, x_{n_{k}}\right)<\varepsilon / 2$ and for $m, n \geq N, d\left(x_{m}, x_{n}\right)<\varepsilon / 2$. Pick $k>n$. Then if $n>N$,

$$
d\left(x_{n}, x\right) \leq d\left(x_{n}, x_{n_{k}}\right)+d\left(x_{n_{k}}, x\right)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon .
$$

Finally, suppose $\lim _{n \rightarrow \infty} x_{n}=x$. Then there exists $N$ such that if $n>N$, then $d\left(x_{n}, x\right)<\varepsilon / 2$. it follows that for $m, n>N$,

$$
d\left(x_{n}, x_{m}\right) \leq d\left(x_{n}, x\right)+d\left(x, x_{m}\right)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon
$$

This proves the lemma.
A useful idea is the idea of distance from a point to a set.

Definition 7.1.6 Let $(X, d)$ be a metric space and let $S$ be a nonempty set in $X$. Then

$$
\operatorname{dist}(x, S) \equiv \inf \{d(x, y): y \in S\}
$$

The following lemma is the fundamental result.
Lemma 7.1.7 The function, $x \rightarrow \operatorname{dist}(x, S)$ is continuous and in fact satisfies

$$
|\operatorname{dist}(x, S)-\operatorname{dist}(y, S)| \leq d(x, y)
$$

Proof: Suppose dist $(x, y)$ is as least as large as dist $(y, S)$. Then pick $z \in S$ such that $d(y, z) \leq \operatorname{dist}(y, S)+\varepsilon$. Then

$$
\begin{aligned}
|\operatorname{dist}(x, S)-\operatorname{dist}(y, S)| & =\operatorname{dist}(x, S)-\operatorname{dist}(y, S) \\
& \leq d(x, z)-(d(y, z)-\varepsilon) \\
& =d(x, z)-d(y, z)+\varepsilon \\
& \leq d(x, y)+d(y, z)-d(y, z)+\varepsilon \\
& =d(x, y)+\varepsilon .
\end{aligned}
$$

Since $\varepsilon>0$ is arbitrary, this proves the lemma.

### 7.2 Open and Closed Sets, Sequences, Limit Points

It is most efficient to discus things in terms of abstract metric spaces to begin with.
Definition 7.2.1 A non empty set $X$ is called a metric space if there is a function $d: X \times$ $X \rightarrow[0, \infty)$ which satisfies the following axioms.

1. $d(x, y)=d(y, x)$
2. $d(x, y) \geq 0$ and equals 0 if and only if $x=y$
3. $d(x, y)+d(y, z) \geq d(x, z)$

This function d is called the metric. We often refer to it as the distance also.
Definition 7.2.2 An open ball, denoted as $B(x, r)$ is defined as follows.

$$
B(x, r) \equiv\{y: d(x, y)<r\}
$$

A set $U$ is said to be open if whenever $x \in U$, it follows that there is $r>0$ such that $B(x, r) \subseteq U$. More generally, a point $x$ is said to be an interior point of $U$ if there exists such a ball. In words, an open set is one for which every point is an interior point.

For example, you could have $X$ be a subset of $\mathbb{R}$ and $d(x, y)=|x-y|$.
Then the first thing to show is the following.
Proposition 7.2.3 An open ball is an open set.
Proof: Suppose $y \in B(x, r)$. We need to verify that $y$ is an interior point of $B(x, r)$. Let $\delta=r-d(x, y)$. Then if $z \in B(y, \boldsymbol{\delta})$, it follows that

$$
d(z, x) \leq d(z, y)+d(y, x)<\delta+d(y, x)=r-d(x, y)+d(y, x)=r
$$

Thus $y \in B(y, \delta) \subseteq B(x, r)$.
Definition 7.2.4 Let $S$ be a nonempty subset of a metric space. Then $p$ is a limit point (accumulation point) of $S$ iffor every $r>0$ there exists a point different than $p$ in $B(p, r) \cap S$. Sometimes people denote the set of limit points as $S^{\prime}$.

The following proposition is fairly obvious from the above definition and will be used whenever convenient. It is equivalent to the above definition and so it can take the place of the above definition if desired.

Proposition 7.2.5 A point $x$ is a limit point of the nonempty set $A$ if and only if every $B(x, r)$ contains infinitely many points of $A$.

Proof: $\Leftarrow$ is obvious. Consider $\Rightarrow$. Let $x$ be a limit point. Let $r_{1}=1$. Then $B\left(x, r_{1}\right)$ contains $a_{1} \neq x$. If $\left\{a_{1}, \cdots, a_{n}\right\}$ have been chosen none equal to $x$ and with no repeats in the list, let $0<r_{n}<\min \left(\frac{1}{n}, \min \left\{d\left(a_{i}, x\right), i=1,2, \cdots n\right\}\right)$. Then let $a_{n+1} \in B\left(x, r_{n}\right)$. Thus every $B(x, r)$ contains $B\left(x, r_{n}\right)$ for all $n$ large enough and hence it contains $a_{k}$ for $k \geq n$ where the $a_{k}$ are distinct, none equal to $x$.

A related idea is the notion of the limit of a sequence. Recall that a sequence is really just a mapping from $\mathbb{N}$ to $X$. We write them as $\left\{x_{n}\right\}$ or $\left\{x_{n}\right\}_{n=1}^{\infty}$ if we want to emphasize the values of $n$. Then the following definition is what it means for a sequence to converge.

Definition 7.2.6 We say that $x=\lim _{n \rightarrow \infty} x_{n}$ when for every $\varepsilon>0$ there exists $N$ such that if $n \geq N$, then

$$
d\left(x, x_{n}\right)<\varepsilon
$$

Often we write $x_{n} \rightarrow x$ for short. This is equivalent to saying

$$
\lim _{n \rightarrow \infty} d\left(x, x_{n}\right)=0
$$

Proposition 7.2.7 The limit is well defined. That is, if $x, x^{\prime}$ are both limits of a sequence, then $x=x^{\prime}$.

Proof: From the definition, there exist $N, N^{\prime}$ such that if $n \geq N$, then $d\left(x, x_{n}\right)<\varepsilon / 2$ and if $n \geq N^{\prime}$, then $d\left(x, x_{n}\right)<\varepsilon / 2$. Then let $M \geq \max \left(N, N^{\prime}\right)$. Let $n>M$. Then

$$
d\left(x, x^{\prime}\right) \leq d\left(x, x_{n}\right)+d\left(x_{n}, x^{\prime}\right)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon
$$

Since $\varepsilon$ is arbitrary, this shows that $x=x^{\prime}$ because $d\left(x, x^{\prime}\right)=0$.
Next there is an important theorem about limit points and convergent sequences.
Theorem 7.2.8 Let $S \neq \emptyset$. Then $p$ is a limit point of $S$ if and only if there exists a sequence of distinct points of $S,\left\{x_{n}\right\}$ none of which equal $p$ such that $\lim _{n \rightarrow \infty} x_{n}=p$.

Proof: $\Longrightarrow$ Suppose $p$ is a limit point. Why does there exist the promissed convergent sequence? Let $x_{1} \in B(p, 1) \cap S$ such that $x_{1} \neq p$. If $x_{1}, \cdots, x_{n}$ have been chosen, let $x_{n+1} \neq p$ be in $B\left(p, \delta_{n+1}\right) \cap S$ where

$$
\delta_{n+1}=\min \left\{\frac{1}{n+1}, d\left(x_{i}, p\right), i=1,2, \cdots, n\right\} .
$$

Then this constructs the necessary convergent sequence.
$\Longleftarrow$ Conversely, if such a sequence $\left\{x_{n}\right\}$ exists, then for every $r>0, B(p, r)$ contains $x_{n} \in S$ for all $n$ large enough. Hence, $p$ is a limit point because none of these $x_{n}$ are equal to $p$.

Definition 7.2.9 $A$ set $H$ is closed means $H^{C}$ is open.
Note that this says that the complement of an open set is closed. If $V$ is open, then the complement of its complement is itself. Thus $\left(V^{C}\right)^{C}=V$ an open set. Hence $V^{C}$ is closed.

Then the following theorem gives the relationship between closed sets and limit points.
Theorem 7.2.10 A set $H$ is closed if and only if it contains all of its limit points.
Proof: $\Longrightarrow$ Let $H$ be closed and let $p$ be a limit point. We need to verify that $p \in H$. If it is not, then since $H$ is closed, its complement is open and so there exists $\delta>0$ such that $B(p, \delta) \cap H=\emptyset$. However, this prevents $p$ from being a limit point.
$\Longleftarrow$ Next suppose $H$ has all of its limit points. Why is $H^{C}$ open? If $p \in H^{C}$ then it is not a limit point and so there exists $\delta>0$ such that $B(p, \delta)$ has no points of $H$. In other words, $H^{C}$ is open. Hence $H$ is closed.

Corollary 7.2.11 $A$ set $H$ is closed if and only if whenever $\left\{h_{n}\right\}$ is a sequence of points of $H$ which converges to a point $x$, it follows that $x \in H$.

Proof: $\Longrightarrow$ Suppose $H$ is closed and $h_{n} \rightarrow x$. If $x \in H$ there is nothing left to show. If $x \notin H$, then from the definition of limit, it is a limit point of $H$ because none of the $h_{n}$ are equal to $x$. Hence $x \in H$ after all.
$\Longleftarrow$ Suppose the limit condition holds, why is $H$ closed? Let $x \in H^{\prime}$ the set of limit points of $H$. By Theorem 7.2.8 there exists a sequence of points of $H,\left\{h_{n}\right\}$ such that $h_{n} \rightarrow x$. Then by assumption, $x \in H$. Thus $H$ contains all of its limit points and so it is closed by Theorem 7.2.10.

Next is the important concept of a subsequence.
Definition 7.2.12 Let $\left\{x_{n}\right\}_{n=1}^{\infty}$ be a sequence. Then if $n_{1}<n_{2}<\cdots$ is a strictly increasing sequence of indices, we say $\left\{x_{n_{k}}\right\}_{k=1}^{\infty}$ is a subsequence of $\left\{x_{n}\right\}_{n=1}^{\infty}$.

The really important thing about subsequences is that they preserve convergence.
Theorem 7.2.13 Let $\left\{x_{n_{k}}\right\}$ be a subsequence of a convergent sequence $\left\{x_{n}\right\}$ where $x_{n} \rightarrow x$. Then

$$
\lim _{k \rightarrow \infty} x_{n_{k}}=x
$$

also.
Proof: Let $\varepsilon>0$ be given. Then there exists $N$ such that

$$
d\left(x_{n}, x\right)<\varepsilon \text { if } n \geq N
$$

It follows that if $k \geq N$, then $n_{k} \geq N$ and so

$$
d\left(x_{n_{k}}, x\right)<\varepsilon \text { if } k \geq N
$$

This is what it means to say $\lim _{k \rightarrow \infty} x_{n_{k}}=x$.

### 7.3 Cauchy Sequences, Completeness

Of course it does not go the other way. For example, you could let $x_{n}=(-1)^{n}$ and it has a convergent subsequence but fails to converge. Here $d(x, y)=|x-y|$ and the metric space is just $\mathbb{R}$.

However, there is a kind of sequence for which it does go the other way. This is called a Cauchy sequence.

Definition 7.3.1 $\left\{x_{n}\right\}$ is called a Cauchy sequence if for every $\varepsilon>0$ there exists $N$ such that if $m, n \geq N$, then

$$
d\left(x_{n}, x_{m}\right)<\varepsilon
$$

Now the major theorem about this is the following.
Theorem 7.3.2 Let $\left\{x_{n}\right\}$ be a Cauchy sequence. Then it converges if and only if any subsequence converges.

Proof: $\Longrightarrow$ This was just done above. $\Longleftarrow$ Suppose now that $\left\{x_{n}\right\}$ is a Cauchy sequence and $\lim _{k \rightarrow \infty} x_{n_{k}}=x$. Then there exists $N_{1}$ such that if $k>N_{1}$, then $d\left(x_{n_{k}}, x\right)<\varepsilon / 2$. From the definition of what it means to be Cauchy, there exists $N_{2}$ such that if $m, n \geq N_{2}$, then $d\left(x_{m}, x_{n}\right)<\varepsilon / 2$. Let $N \geq \max \left(N_{1}, N_{2}\right)$. Then if $k \geq N$, then $n_{k} \geq N$ and so

$$
\begin{equation*}
d\left(x, x_{k}\right) \leq d\left(x, x_{n_{k}}\right)+d\left(x_{n_{k}}, x_{k}\right)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon \tag{7.3.3}
\end{equation*}
$$

It follows from the definition that $\lim _{k \rightarrow \infty} x_{k}=x$.
Definition 7.3.3 A metric space is said to be complete if every Cauchy sequence converges.
There certainly are metric spaces which are not complete. For example, if you consider $\mathbb{Q}$ with $d(x, y) \equiv|x-y|$, this will not be complete because you can get a sequence which is obtained as $x_{n}$ defined as the $n$ decimal place description of $\sqrt{2}$. However, if a sequence converges, then it must be Cauchy.

Lemma 7.3.4 If $x_{n} \rightarrow x$, then $\left\{x_{n}\right\}$ is a Cauchy sequence.
Proof: Let $\varepsilon>0$. Then there exists $n_{\varepsilon}$ such that if $m \geq n_{\varepsilon}$, then $d\left(x, x_{m}\right)<\varepsilon / 2$. If $m, k \geq n_{\mathcal{E}}$, then by the triangle inequality,

$$
d\left(x_{m}, x_{k}\right) \leq d\left(x_{m}, x\right)+d\left(x, x_{k}\right)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon
$$

showing that the convergent sequence is indeed a Cauchy sequence as claimed.
Another nice thing to note is this.
Proposition 7.3.5 If $\left\{x_{n}\right\}$ is a sequence and if $p$ is a limit point of the set $S=\cup_{n=1}^{\infty}\left\{x_{n}\right\}$ then there is a subsequence $\left\{x_{n_{k}}\right\}$ such that $\lim _{k \rightarrow \infty} x_{n_{k}}=x$.

Proof: By Theorem 7.2.8, there exists a sequence of distinct points of $S$ denoted as $\left\{y_{k}\right\}$ such that none of them equal $p$ and $\lim _{k \rightarrow \infty} y_{k}=p$. Thus $B(p, r)$ contains infinitely many different points of the set $D$, this for every $r$. Let $x_{n_{1}} \in B(p, 1)$ where $n_{1}$ is the first index such that $x_{n_{1}} \in B(p, 1)$. Suppose $x_{n_{1}}, \cdots, x_{n_{k}}$ have been chosen, the $n_{i}$ increasing and let $1>\delta_{1}>\delta_{2}>\cdots>\delta_{k}$ where $x_{n_{i}} \in B\left(p, \delta_{i}\right)$. Then let

$$
\delta_{k+1}<\min \left\{\frac{1}{2^{k+1}}, d\left(p, x_{n_{j}}\right), \delta_{j}, j=1,2 \cdots, k\right\}
$$

Let $x_{n_{k+1}} \in B\left(p, \delta_{k+1}\right)$ where $n_{k+1}$ is the first index such that $x_{n_{k+1}}$ is contained $B\left(p, \delta_{k+1}\right)$. Then

$$
\lim _{k \rightarrow \infty} x_{n_{k}}=p
$$

Another useful result is the following.
Lemma 7.3.6 Suppose $x_{n} \rightarrow x$ and $y_{n} \rightarrow y$. Then $d\left(x_{n}, y_{n}\right) \rightarrow d(x, y)$.
Proof: Consider the following.

$$
d(x, y) \leq d\left(x, x_{n}\right)+d\left(x_{n}, y\right) \leq d\left(x, x_{n}\right)+d\left(x_{n}, y_{n}\right)+d\left(y_{n}, y\right)
$$

so

$$
d(x, y)-d\left(x_{n}, y_{n}\right) \leq d\left(x, x_{n}\right)+d\left(y_{n}, y\right)
$$

Similarly

$$
d\left(x_{n}, y_{n}\right)-d(x, y) \leq d\left(x, x_{n}\right)+d\left(y_{n}, y\right)
$$

and so

$$
\left|d\left(x_{n}, y_{n}\right)-d(x, y)\right| \leq d\left(x, x_{n}\right)+d\left(y_{n}, y\right)
$$

and the right side converges to 0 as $n \rightarrow \infty$.

### 7.4 Closure Of A Set

Next is the topic of the closure of a set.
Definition 7.4.1 Let A be a nonempty subset of $(X, d)$ a metric space. Then $\bar{A}$ is defined to be the intersection of all closed sets which contain $A$. Note the whole space, $X$ is one such closed set which contains $A$. The whole space $X$ is closed because its complement is open, its complement being $\emptyset$. It is certainly true that every point of the empty set is an interior point because there are no points of $\emptyset$.

Lemma 7.4.2 Let $A$ be a nonempty set in $(X, d)$. Then $\bar{A}$ is a closed set and

$$
\bar{A}=A \cup A^{\prime}
$$

where $A^{\prime}$ denotes the set of limit points of $A$.

Proof: First of all, denote by $\mathscr{C}$ the set of closed sets which contain $A$. Then

$$
\bar{A}=\cap \mathscr{C}
$$

and this will be closed if its complement is open. However,

$$
\bar{A}^{C}=\cup\left\{H^{C}: H \in \mathscr{C}\right\}
$$

Each $H^{C}$ is open and so the union of all these open sets must also be open. This is because if $x$ is in this union, then it is in at least one of them. Hence it is an interior point of that one. But this implies it is an interior point of the union of them all which is an even larger set. Thus $\bar{A}$ is closed.

The interesting part is the next claim. First note that from the definition, $A \subseteq \bar{A}$ so if $x \in A$, then $x \in \bar{A}$. Now consider $y \in A^{\prime}$ but $y \notin A$. If $y \notin \bar{A}$, a closed set, then there exists $B(y, r) \subseteq \bar{A}^{C}$. Thus $y$ cannot be a limit point of $A$, a contradiction. Therefore,

$$
A \cup A^{\prime} \subseteq \bar{A}
$$

Next suppose $x \in \bar{A}$ and suppose $x \notin A$. Then if $B(x, r)$ contains no points of $A$ different than $x$, since $x$ itself is not in $A$, it would follow that $B(x, r) \cap A=\emptyset$ and so recalling that open balls are open, $B(x, r)^{C}$ is a closed set containing $A$ so from the definition, it also contains $\bar{A}$ which is contrary to the assertion that $x \in \bar{A}$. Hence if $x \notin A$, then $x \in A^{\prime}$ and so

$$
A \cup A^{\prime} \supseteq \bar{A}
$$

### 7.5 Separable Metric Spaces

Definition 7.5.1 A metric space is called separable if there exists a countable dense subset D. This means two things. First, $D$ is countable, and second, that if $x$ is any point and $r>0$, then $B(x, r) \cap D \neq \emptyset$. A metric space is called completely separable if there exists $a$ countable collection of nonempty open sets $\mathscr{B}$ such that every open set is the union of some subset of $\mathscr{B}$. This collection of open sets is called a countable basis.

For those who like to fuss about empty sets, the empty set is open and it is indeed the union of a subset of $\mathscr{B}$ namely the empty subset.

Theorem 7.5.2 A metric space is separable if and only if it is completely separable.
Proof: $\Longleftarrow$ Let $\mathscr{B}$ be the special countable collection of open sets and for each $B \in \mathscr{B}$, let $p_{B}$ be a point of $B$. Then let $\mathscr{P} \equiv\left\{p_{B}: B \in \mathscr{B}\right\}$. If $B(x, r)$ is any ball, then it is the union of sets of $\mathscr{B}$ and so there is a point of $\mathscr{P}$ in it. Since $\mathscr{B}$ is countable, so is $\mathscr{P}$.
$\Longrightarrow$ Let $D$ be the countable dense set and let $\mathscr{B} \equiv\{B(d, r): d \in D, r \in \mathbb{Q} \cap[0, \infty)\}$. Then $\mathscr{B}$ is countable because the Cartesian product of countable sets is countable. It suffices to show that every ball is the union of these sets. Let $B(x, R)$ be a ball. Let $y \in B(y, \delta) \subseteq B(x, R)$. Then there exists $d \in B\left(y, \frac{\delta}{10}\right)$. Let $\varepsilon \in \mathbb{Q}$ and $\frac{\delta}{10}<\varepsilon<\frac{\delta}{5}$. Then $y \in B(d, \varepsilon) \in \mathscr{B}$. Is $B(d, \varepsilon) \subseteq B(x, R)$ ? If so, then the desired result follows because this
would show that every $y \in B(x, R)$ is contained in one of these sets of $\mathscr{B}$ which is contained in $B(x, R)$ showing that $B(x, R)$ is the union of sets of $\mathscr{B}$. Let $z \in B(d, \varepsilon) \subseteq B\left(d, \frac{\delta}{5}\right)$. Then

$$
d(y, z) \leq d(y, d)+d(d, z)<\frac{\delta}{10}+\varepsilon<\frac{\delta}{10}+\frac{\delta}{5}<\delta
$$

Hence $B(d, \varepsilon) \subseteq B(y, \delta) \subseteq B(x, r)$. Therefore, every ball is the union of sets of $\mathscr{B}$ and, since every open set is the union of balls, it follows that every open set is the union of sets of $\mathscr{B}$.

Definition 7.5.3 Let $S$ be a nonempty set. Then a set of open sets $\mathscr{C}$ is called an open cover of $S$ if $\cup \mathscr{C} \supseteq \mathscr{S}$. (It covers up the set $S$. Think lilly pads covering the surface of a pond.)

One of the important properties possessed by separable metric spaces is the Lindeloff property.

Definition 7.5.4 A metric space has the Lindeloff property if whenever $\mathscr{C}$ is an open cover of a set $S$, there exists a countable subset of $\mathscr{C}$ denoted here by $\mathscr{B}$ such that $\mathscr{B}$ is also an open cover of $S$.

Theorem 7.5.5 Every separable metric space has the Lindeloff property.
Proof: Let $\mathscr{C}$ be an open cover of a set $S$. Let $\mathscr{B}$ be a countable basis. Such exists by Theorem 7.5.2. Let $\hat{\mathscr{B}}$ denote those sets of $\mathscr{B}$ which are contained in some set of $\mathscr{C}$. Thus $\hat{\mathscr{B}}$ is a countable open cover of $S$. Now for $B \in \mathscr{B}$, let $U_{B}$ be a set of $\mathscr{C}$ which contains $B$. Letting $\widehat{\mathscr{C}}$ denote these sets $U_{B}$ it follows that $\widehat{\mathscr{C}}$ is countable and is an open cover of $S$.

Definition 7.5.6 A Polish space is a complete separable metric space. These things turn out to be very useful in probability theory and in other areas.

### 7.6 Compactness In Metric Space

Many existence theorems in analysis depend on some set being compact. Therefore, it is important to be able to identify compact sets. The purpose of this section is to describe compact sets in a metric space.

Definition 7.6.1 Let $A$ be a subset of $X$. A is compact if whenever $A$ is contained in the union of a set of open sets, there exists finitely many of these open sets whose union contains A. (Every open cover admits a finite subcover.) A is "sequentially compact" means every sequence has a convergent subsequence converging to an element of $A$.

In a metric space compact is not the same as closed and bounded!
Example 7.6.2 Let $X$ be any infinite set and define $d(x, y)=1$ if $x \neq y$ while $d(x, y)=0$ if $x=y$.

You should verify the details that this is a metric space because it satisfies the axioms of a metric. The set $X$ is closed and bounded because its complement is $\emptyset$ which is clearly open because every point of $\emptyset$ is an interior point. (There are none.) Also $X$ is bounded because $X=B(x, 2)$. However, $X$ is clearly not compact because $\left\{B\left(x, \frac{1}{2}\right): x \in X\right\}$ is a collection of open sets whose union contains $X$ but since they are all disjoint and nonempty, there is no finite subset of these whose union contains $X$. In fact $B\left(x, \frac{1}{2}\right)=\{x\}$.

From this example it is clear something more than closed and bounded is needed. If you are not familiar with the issues just discussed, ignore them and continue.

Definition 7.6.3 In any metric space, a set $E$ is totally bounded if for every $\varepsilon>0$ there exists a finite set of points $\left\{x_{1}, \cdots, x_{n}\right\}$ such that

$$
E \subseteq \cup_{i=1}^{n} B\left(x_{i}, \varepsilon\right)
$$

This finite set of points is called an $\varepsilon$ net.
The following proposition tells which sets in a metric space are compact. First here is an interesting lemma.

Lemma 7.6.4 Let $X$ be a metric space and suppose $D$ is a countable dense subset of $X$. In other words, it is being assumed $X$ is a separable metric space. Consider the open sets of the form $B(d, r)$ where $r$ is a positive rational number and $d \in D$. Denote this countable collection of open sets by $\mathscr{B}$. Then every open set is the union of sets of $\mathscr{B}$. Furthermore, if $\mathscr{C}$ is any collection of open sets, there exists a countable subset, $\left\{U_{n}\right\} \subseteq \mathscr{C}$ such that $\cup_{n} U_{n}=\cup \mathscr{C}$.

Proof: Let $U$ be an open set and let $x \in U$. Let $B(x, \delta) \subseteq U$. Then by density of $D$, there exists $d \in D \cap B(x, \delta / 4)$. Now pick $r \in \mathbb{Q} \cap(\delta / 4,3 \delta / 4)$ and consider $B(d, r)$. Clearly, $B(d, r)$ contains the point $x$ because $r>\delta / 4$. Is $B(d, r) \subseteq B(x, \delta)$ ? if so, this proves the lemma because $x$ was an arbitrary point of $U$. Suppose $z \in B(d, r)$. Then

$$
d(z, x) \leq d(z, d)+d(d, x)<r+\frac{\delta}{4}<\frac{3 \delta}{4}+\frac{\delta}{4}=\delta
$$

Now let $\mathscr{C}$ be any collection of open sets. Each set in this collection is the union of countably many sets of $\mathscr{B}$. Let $\mathscr{B}^{\prime}$ denote the sets of $\mathscr{B}$ which are contained in some set of $\mathscr{C}$. Thus $\cup \mathscr{B}^{\prime}=\cup \mathscr{C}$. Then for each $B \in \mathscr{B}^{\prime}$, pick $U_{B} \in \mathscr{C}$ such that $B \subseteq U_{B}$. Then $\left\{U_{B}: B \in \mathscr{B}^{\prime}\right\}$ is a countable collection of sets of $\mathscr{C}$ whose union equals $\cup \mathscr{C}$. Therefore, this proves the lemma.

Proposition 7.6.5 Let $(X, d)$ be a metric space. Then the following are equivalent.

$$
\begin{equation*}
(X, d) \text { is compact } \tag{7.6.4}
\end{equation*}
$$

( $X, d$ ) is sequentially compact,

$$
\begin{equation*}
(X, d) \text { is complete and totally bounded. } \tag{7.6.5}
\end{equation*}
$$

Proof: Suppose 7.6.4 and let $\left\{x_{k}\right\}$ be a sequence. Suppose $\left\{x_{k}\right\}$ has no convergent subsequence. If this is so, then no value of the sequence is repeated more than finitely many times. Also $\left\{x_{k}\right\}$ has no limit point because if it did, there would exist a subsequence which converges. To see this, suppose $p$ is a limit point of $\left\{x_{k}\right\}$. Then in $B(p, 1)$ there are infinitely many points of $\left\{x_{k}\right\}$. Pick one called $x_{k_{1}}$. Now if $x_{k_{1}}, x_{k_{2}}, \cdots, x_{k_{n}}$ have been picked with $x_{k_{i}} \in B(p, 1 / i)$, consider $B(p, 1 /(n+1))$. There are infinitely many points of $\left\{x_{k}\right\}$ in this ball also. Pick $x_{k_{n+1}}$ such that $k_{n+1}>k_{n}$. Then $\left\{x_{k_{n}}\right\}_{n=1}^{\infty}$ is a subsequence which converges to $p$ and it is assumed this does not happen. Thus $\left\{x_{k}\right\}$ has no limit points. It follows the set

$$
C_{n}=\cup\left\{x_{k}: k \geq n\right\}
$$

is a closed set because it has no limit points and if

$$
U_{n}=C_{n}^{C}
$$

then

$$
X=\cup_{n=1}^{\infty} U_{n}
$$

but there is no finite subcovering, because no value of the sequence is repeated more than finitely many times. This contradicts compactness of $(X, d)$. Note $x_{k}$ is not in $U_{n}$ whenever $k>n$. Thus 7.6.4 implies 7.6.5.

Now suppose 7.6.5 and let $\left\{x_{n}\right\}$ be a Cauchy sequence. Is $\left\{x_{n}\right\}$ convergent? By sequential compactness $x_{n_{k}} \rightarrow x$ for some subsequence. By Lemma 7.1.5 it follows that $\left\{x_{n}\right\}$ also converges to $x$ showing that $(X, d)$ is complete. If $(X, d)$ is not totally bounded, then there exists $\varepsilon>0$ for which there is no $\varepsilon$ net. Hence there exists a sequence $\left\{x_{k}\right\}$ with $d\left(x_{k}, x_{l}\right) \geq \varepsilon$ for all $l \neq k$. By Lemma 7.1.5 again, this contradicts 7.6.5 because no subsequence can be a Cauchy sequence and so no subsequence can converge. This shows 7.6.5 implies 7.6.6.

Now suppose 7.6.6. What about 7.6.5? Let $\left\{p_{n}\right\}$ be a sequence and let $\left\{x_{i}^{n}\right\}_{i=1}^{m_{n}}$ be a $2^{-n}$ net for $n=1,2, \cdots$. Let

$$
B_{n} \equiv B\left(x_{i_{n}}^{n}, 2^{-n}\right)
$$

be such that $B_{n}$ contains $p_{k}$ for infinitely many values of $k$ and $B_{n} \cap B_{n+1} \neq \emptyset$. To do this, suppose $B_{n}$ contains $p_{k}$ for infinitely many values of $k$. Then one of the sets which intersect $B_{n}, B\left(x_{i}^{n+1}, 2^{-(n+1)}\right)$ must contain $p_{k}$ for infinitely many values of $k$ because all these indices of points from $\left\{p_{n}\right\}$ contained in $B_{n}$ must be accounted for in one of finitely many sets, $B\left(x_{i}^{n+1}, 2^{-(n+1)}\right)$. Thus there exists a strictly increasing sequence of integers, $n_{k}$ such that

$$
p_{n_{k}} \in B_{k} .
$$

Then if $k \geq l$,

$$
\begin{aligned}
d\left(p_{n_{k}}, p_{n_{l}}\right) & \leq \sum_{i=l}^{k-1} d\left(p_{n_{i+1}}, p_{n_{i}}\right) \\
& <\sum_{i=l}^{k-1} 2^{-(i-1)}<2^{-(l-2)}
\end{aligned}
$$

Consequently $\left\{p_{n_{k}}\right\}$ is a Cauchy sequence. Hence it converges because the metric space is complete. This proves 7.6.5.

Now suppose 7.6 .5 and 7.6 .6 which have now been shown to be equivalent. Let $D_{n}$ be a $n^{-1}$ net for $n=1,2, \cdots$ and let

$$
D=\cup_{n=1}^{\infty} D_{n}
$$

Thus $D$ is a countable dense subset of $(X, d)$.
Now let $\mathscr{C}$ be any set of open sets such that $\cup \mathscr{C} \supseteq X$. By Lemma 7.6.4, there exists a countable subset of $\mathscr{C}$,

$$
\tilde{\mathscr{C}}=\left\{U_{n}\right\}_{n=1}^{\infty}
$$

such that $\cup \widetilde{\mathscr{C}}=\cup \mathscr{C}$. If $\mathscr{C}$ admits no finite subcover, then neither does $\tilde{\mathscr{C}}$ and there exists $p_{n} \in X \backslash \cup_{k=1}^{n} U_{k}$. Then since $X$ is sequentially compact, there is a subsequence $\left\{p_{n_{k}}\right\}$ such that $\left\{p_{n_{k}}\right\}$ converges. Say

$$
p=\lim _{k \rightarrow \infty} p_{n_{k}} .
$$

All but finitely many points of $\left\{p_{n_{k}}\right\}$ are in $X \backslash \cup_{k=1}^{n} U_{k}$. Therefore $p \in X \backslash \cup_{k=1}^{n} U_{k}$ for each $n$. Hence

$$
p \notin \cup_{k=1}^{\infty} U_{k}
$$

contradicting the construction of $\left\{U_{n}\right\}_{n=1}^{\infty}$ which required that $\cup_{n=1}^{\infty} U_{n} \supseteq X$. Hence $X$ is compact. This proves the proposition.

Consider $\mathbb{R}^{n}$. In this setting totally bounded and bounded are the same. This will yield a proof of the Heine Borel theorem from advanced calculus.

Lemma 7.6.6 A subset of $\mathbb{R}^{n}$ is totally bounded if and only if it is bounded.
Proof: Let $A$ be totally bounded. Is it bounded? Let $\mathbf{x}_{1}, \cdots, \mathbf{x}_{p}$ be a 1 net for $A$. Now consider the ball $B(\mathbf{0}, r+1)$ where $r>\max \left(\left|\mathbf{x}_{i}\right|: i=1, \cdots, p\right)$. If $\mathbf{z} \in A$, then $\mathbf{z} \in B\left(\mathbf{x}_{j}, 1\right)$ for some $j$ and so by the triangle inequality,

$$
|\mathbf{z}-\mathbf{0}| \leq\left|\mathbf{z}-\mathbf{x}_{j}\right|+\left|\mathbf{x}_{j}\right|<1+r
$$

Thus $A \subseteq B(\mathbf{0}, r+1)$ and so $A$ is bounded.
Now suppose $A$ is bounded and suppose $A$ is not totally bounded. Then there exists $\varepsilon>0$ such that there is no $\varepsilon$ net for $A$. Therefore, there exists a sequence of points $\left\{a_{i}\right\}$ with $\left|a_{i}-a_{j}\right| \geq \varepsilon$ if $i \neq j$. Since $A$ is bounded, there exists $r>0$ such that

$$
A \subseteq[-r, r)^{n}
$$

$\left(\mathbf{x} \in[-r, r)^{n}\right.$ means $x_{i} \in[-r, r)$ for each $i$.) Now define $\mathscr{S}$ to be all cubes of the form

$$
\prod_{k=1}^{n}\left[a_{k}, b_{k}\right)
$$

where

$$
a_{k}=-r+i 2^{-p} r, b_{k}=-r+(i+1) 2^{-p} r
$$

for $i \in\left\{0,1, \cdots, 2^{p+1}-1\right\}$. Thus $\mathscr{S}$ is a collection of $\left(2^{p+1}\right)^{n}$ non overlapping cubes whose union equals $[-r, r)^{n}$ and whose diameters are all equal to $2^{-p} r \sqrt{n}$. Now choose $p$ large enough that the diameter of these cubes is less than $\varepsilon$. This yields a contradiction because one of the cubes must contain infinitely many points of $\left\{a_{i}\right\}$. This proves the lemma.

The next theorem is called the Heine Borel theorem and it characterizes the compact sets in $\mathbb{R}^{n}$.

Theorem 7.6.7 A subset of $\mathbb{R}^{n}$ is compact if and only if it is closed and bounded.
Proof: Since a set in $\mathbb{R}^{n}$ is totally bounded if and only if it is bounded, this theorem follows from Proposition 7.6 .5 and the observation that a subset of $\mathbb{R}^{n}$ is closed if and only if it is complete. This proves the theorem.

Proposition 7.6.8 If $K$ is a closed, nonempty subset of a nonempty compact set $H$, then $K$ is compact.

Proof: Let $\mathscr{C}$ be an open cover for $K$. Then $\mathscr{C} \cup\left\{K^{C}\right\}$ is an open cover for $H$. Thus there are finitely many sets from this last collection of open sets, $U_{1}, \cdots, U_{m}$ which covers $H$. Include only those which are in $\mathscr{C}$. These cover $K$ because $K^{C}$ covers no points of $K$.

### 7.7 Some Applications Of Compactness

The following corollary is an important existence theorem which depends on compactness.
Theorem 7.7.1 Let $X$ be a compact metric space and let $f: X \rightarrow \mathbb{R}$ be continuous. Then $\max \{f(x): x \in X\}$ and $\min \{f(x): x \in X\}$ both exist.

Proof: First it is shown $f(X)$ is compact. Suppose $\mathscr{C}$ is a set of open sets whose union contains $f(X)$. Then since $f$ is continuous $f^{-1}(U)$ is open for all $U \in \mathscr{C}$. Therefore, $\left\{f^{-1}(U): U \in \mathscr{C}\right\}$ is a collection of open sets whose union contains $X$. Since $X$ is compact, it follows finitely many of these, $\left\{f^{-1}\left(U_{1}\right), \cdots, f^{-1}\left(U_{p}\right)\right\}$ contains $X$ in their union. Therefore, $f(X) \subseteq \cup_{k=1}^{p} U_{k}$ showing $f(X)$ is compact as claimed.

Now since $f(X)$ is compact, Theorem 7.6.7 implies $f(X)$ is closed and bounded. Therefore, it contains its inf and its sup. Thus $f$ achieves both a maximum and a minimum.

Definition 7.7.2 Let $X, Y$ be metric spaces and $f: X \rightarrow Y$ a function. $f$ is uniformly continuous if for all $\varepsilon>0$ there exists $\delta>0$ such that whenever $x_{1}$ and $x_{2}$ are two points of $X$ satisfying $d\left(x_{1}, x_{2}\right)<\delta$, it follows that $d\left(f\left(x_{1}\right), f\left(x_{2}\right)\right)<\varepsilon$.

A very important theorem is the following.
Theorem 7.7.3 Suppose $f: X \rightarrow Y$ is continuous and $X$ is compact. Then $f$ is uniformly continuous.

Proof: Suppose this is not true and that $f$ is continuous but not uniformly continuous. Then there exists $\varepsilon>0$ such that for all $\delta>0$ there exist points, $p_{\delta}$ and $q_{\delta}$ such that $d\left(p_{\delta}, q_{\delta}\right)<\delta$ and yet $d\left(f\left(p_{\delta}\right), f\left(q_{\delta}\right)\right) \geq \varepsilon$. Let $p_{n}$ and $q_{n}$ be the points which go with $\delta=1 / n$. By Proposition 7.6.5 $\left\{p_{n}\right\}$ has a convergent subsequence, $\left\{p_{n_{k}}\right\}$ converging to a point, $x \in X$. Since $d\left(p_{n}, q_{n}\right)<\frac{1}{n}$, it follows that $q_{n_{k}} \rightarrow x$ also. Therefore,

$$
\varepsilon \leq d\left(f\left(p_{n_{k}}\right), f\left(q_{n_{k}}\right)\right) \leq d\left(f\left(p_{n_{k}}\right), f(x)\right)+d\left(f(x), f\left(q_{n_{k}}\right)\right)
$$

but by continuity of $f$, both $d\left(f\left(p_{n_{k}}\right), f(x)\right)$ and $d\left(f(x), f\left(q_{n_{k}}\right)\right)$ converge to 0 as $k \rightarrow \infty$ contradicting the above inequality. This proves the theorem.

Another important property of compact sets in a metric space concerns the finite intersection property.

Definition 7.7.4 If every finite subset of a collection of sets has nonempty intersection, the collection has the finite intersection property.

Theorem 7.7.5 Suppose $\mathscr{F}$ is a collection of compact sets in a metric space, $X$ which has the finite intersection property. Then there exists a point in their intersection. $(\cap \mathscr{F} \neq \emptyset)$.

Proof: First I show each compact set is closed. Let $K$ be a nonempty compact set and suppose $p \notin K$. Then for each $x \in K$, let $V_{x}=B(x, d(p, x) / 3)$ and $U_{x}=B(p, d(p, x) / 3)$ so that $U_{x}$ and $V_{x}$ have empty intersection. Then since $V$ is compact, there are finitely many $V_{x}$ which cover $K$ say $V_{x_{1}}, \cdots, V_{x_{n}}$. Then let $U=\cap_{i=1}^{n} U_{x_{i}}$. It follows $p \in U$ and $U$ has empty intersection with $K$. In fact $U$ has empty intersection with $\cup_{i=1}^{n} V_{x_{i}}$. Since $U$ is an open set and $p \in K^{C}$ is arbitrary, it follows $K^{C}$ is an open set.

Consider now the claim about the intersection. If this were not so,

$$
\cup\left\{F^{C}: F \in \mathscr{F}\right\}=X
$$

and so, in particular, picking some $F_{0} \in \mathscr{F}$,

$$
\left\{F^{C}: F \in \mathscr{F}\right\}
$$

would be an open cover of $F_{0}$. Since $F_{0}$ is compact, some finite subcover, $F_{1}^{C}, \cdots, F_{m}^{C}$ exists. But then

$$
F_{0} \subseteq \cup_{k=1}^{m} F_{k}^{C}
$$

which means $\cap_{k=0}^{m} F_{k}=\emptyset$, contrary to the finite intersection property. To see this, note that if $x \in F_{0}$, then it must fail to be in some $F_{k}$ and so it is not in $\cap_{k=0}^{m} F_{k}$. Since this is true for every $x$ it follows $\cap_{k=0}^{m} F_{k}=\emptyset$.

Theorem 7.7.6 Let $X_{i}$ be a compact metric space with metric $d_{i}$. Then $\prod_{i=1}^{m} X_{i}$ is also a compact metric space with respect to the metric, $d(\mathbf{x}, \mathbf{y}) \equiv \max _{i}\left(d_{i}\left(x_{i}, y_{i}\right)\right)$.

Proof: This is most easily seen from sequential compactness. Let $\left\{\mathbf{x}^{k}\right\}_{k=1}^{\infty}$ be a sequence of points in $\prod_{i=1}^{m} X_{i}$. Consider the $i^{t h}$ component of $\mathbf{x}^{k}, x_{i}^{k}$. It follows $\left\{x_{i}^{k}\right\}$ is a
sequence of points in $X_{i}$ and so it has a convergent subsequence. Compactness of $X_{1}$ implies there exists a subsequence of $\mathbf{x}^{k}$, denoted by $\left\{\mathbf{x}^{k_{1}}\right\}$ such that

$$
\lim _{k_{1} \rightarrow \infty} x_{1}^{k_{1}} \rightarrow x_{1} \in X_{1}
$$

Now there exists a further subsequence, denoted by $\left\{\mathbf{x}^{k_{2}}\right\}$ such that in addition to this, $x_{2}^{k_{2}} \rightarrow x_{2} \in X_{2}$. After taking $m$ such subsequences, there exists a subsequence, $\left\{\mathbf{x}^{l}\right\}$ such that $\lim _{l \rightarrow \infty} x_{i}^{l}=x_{i} \in X_{i}$ for each $i$. Therefore, letting $\mathbf{x} \equiv\left(x_{1}, \cdots, x_{m}\right), \mathbf{x}^{l} \rightarrow \mathbf{x}$ in $\prod_{i=1}^{m} X_{i}$. This proves the theorem.

### 7.8 Ascoli Arzela Theorem

Definition 7.8.1 Let $(X, d)$ be a complete metric space. Then it is said to be locally compact if $\overline{B(x, r)}$ is compact for each $r>0$.

Thus if you have a locally compact metric space, then if $\left\{a_{n}\right\}$ is a bounded sequence, it must have a convergent subsequence.

Let $K$ be a compact subset of $\mathbb{R}^{n}$ and consider the continuous functions which have values in a locally compact metric space, $(X, d)$ where $d$ denotes the metric on $X$. Denote this space as $C(K, X)$.

Definition 7.8.2 For $f, g \in C(K, X)$, where $K$ is a compact subset of $\mathbb{R}^{n}$ and $X$ is a locally compact complete metric space define

$$
\rho_{K}(f, g) \equiv \sup \{d(f(\mathbf{x}), g(\mathbf{x})): \mathbf{x} \in K\}
$$

Then $\rho_{K}$ provides a distance which makes $C(K, X)$ into a metric space.
The Ascoli Arzela theorem is a major result which tells which subsets of $C(K, X)$ are sequentially compact.

Definition 7.8.3 Let $A \subseteq C(K, X)$ for $K$ a compact subset of $\mathbb{R}^{n}$. Then $A$ is said to be uniformly equicontinuous iffor every $\varepsilon>0$ there exists a $\delta>0$ such that whenever $\mathbf{x}, \mathbf{y} \in K$ with $|\mathbf{x}-\mathbf{y}|<\delta$ and $f \in A$,

$$
d(f(\mathbf{x}), f(\mathbf{y}))<\varepsilon
$$

The set, $A$ is said to be uniformly bounded if for some $M<\infty$, and $a \in X$,

$$
f(\mathbf{x}) \in B(a, M)
$$

for all $f \in A$ and $\mathbf{x} \in K$.
Uniform equicontinuity is like saying that the whole set of functions, $A$, is uniformly continuous on $K$ uniformly for $f \in A$. The version of the Ascoli Arzela theorem I will present here is the following.

Theorem 7.8.4 Suppose $K$ is a nonempty compact subset of $\mathbb{R}^{n}$ and $A \subseteq C(K, X)$ is uniformly bounded and uniformly equicontinuous. Then if $\left\{f_{k}\right\} \subseteq A$, there exists a function, $f \in C(K, X)$ and a subsequence, $f_{k_{l}}$ such that

$$
\lim _{l \rightarrow \infty} \rho_{K}\left(f_{k_{l}}, f\right)=0
$$

To give a proof of this theorem, I will first prove some lemmas.
Lemma 7.8.5 If $K$ is a compact subset of $\mathbb{R}^{n}$, then there exists $D \equiv\left\{\mathbf{x}_{k}\right\}_{k=1}^{\infty} \subseteq K$ such that $D$ is dense in $K$. Also, for every $\varepsilon>0$ there exists a finite set of points, $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{m}\right\} \subseteq K$, called an $\varepsilon$ net such that

$$
\cup_{i=1}^{m} B\left(\mathbf{x}_{i}, \varepsilon\right) \supseteq K
$$

Proof: For $m \in \mathbb{N}$, pick $x_{1}^{m} \in K$. If every point of $K$ is within $1 / m$ of $x_{1}^{m}$, stop. Otherwise, pick

$$
x_{2}^{m} \in K \backslash B\left(x_{1}^{m}, 1 / m\right)
$$

If every point of $K$ contained in $B\left(x_{1}^{m}, 1 / m\right) \cup B\left(x_{2}^{m}, 1 / m\right)$, stop. Otherwise, pick

$$
x_{3}^{m} \in K \backslash\left(B\left(x_{1}^{m}, 1 / m\right) \cup B\left(x_{2}^{m}, 1 / m\right)\right) .
$$

If every point of $K$ is contained in $B\left(x_{1}^{m}, 1 / m\right) \cup B\left(x_{2}^{m}, 1 / m\right) \cup B\left(x_{3}^{m}, 1 / m\right)$, stop. Otherwise, pick

$$
x_{4}^{m} \in K \backslash\left(B\left(x_{1}^{m}, 1 / m\right) \cup B\left(x_{2}^{m}, 1 / m\right) \cup B\left(x_{3}^{m}, 1 / m\right)\right)
$$

Continue this way until the process stops, say at $N(m)$. It must stop because if it didn't, there would be a convergent subsequence due to the compactness of $K$. Ultimately all terms of this convergent subsequence would be closer than $1 / m$, violating the manner in which they are chosen. Then $D=\cup_{m=1}^{\infty} \cup_{k=1}^{N(m)}\left\{x_{k}^{m}\right\}$. This is countable because it is a countable union of countable sets. If $\mathbf{y} \in K$ and $\varepsilon>0$, then for some $m, 2 / m<\varepsilon$ and so $B(\mathbf{y}, \varepsilon)$ must contain some point of $\left\{x_{k}^{m}\right\}$ since otherwise, the process stopped too soon. You could have picked $\mathbf{y}$.

Lemma 7.8.6 Suppose $D$ is defined above and $\left\{g_{m}\right\}$ is a sequence of functions of $A$ having the property that for every $\mathbf{x}_{k} \in D$,

$$
\lim _{m \rightarrow \infty} g_{m}\left(\mathbf{x}_{k}\right) \text { exists }
$$

Then there exists $g \in C(K, X)$ such that

$$
\lim _{m \rightarrow \infty} \rho\left(g_{m}, g\right)=0
$$

Proof: Define $g$ first on $D$.

$$
g\left(\mathbf{x}_{k}\right) \equiv \lim _{m \rightarrow \infty} g_{m}\left(\mathbf{x}_{k}\right)
$$

Next I show that $\left\{g_{m}\right\}$ converges at every point of $K$. Let $\mathbf{x} \in K$ and let $\varepsilon>0$ be given. Choose $\mathbf{x}_{k}$ such that for all $f \in A$,

$$
d\left(f\left(\mathbf{x}_{k}\right), f(\mathbf{x})\right)<\frac{\varepsilon}{3}
$$

I can do this by the equicontinuity. Now if $p, q$ are large enough, say $p, q \geq M$,

$$
d\left(g_{p}\left(\mathbf{x}_{k}\right), g_{q}\left(\mathbf{x}_{k}\right)\right)<\frac{\varepsilon}{3}
$$

Therefore, for $p, q \geq M$,

$$
\begin{aligned}
d\left(g_{p}(\mathbf{x}), g_{q}(\mathbf{x})\right) & \leq d\left(g_{p}(\mathbf{x}), g_{p}\left(\mathbf{x}_{k}\right)\right)+d\left(g_{p}\left(\mathbf{x}_{k}\right), g_{q}\left(\mathbf{x}_{k}\right)\right)+d\left(g_{q}\left(\mathbf{x}_{k}\right), g_{q}(\mathbf{x})\right) \\
& <\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon
\end{aligned}
$$

It follows that $\left\{g_{m}(\mathbf{x})\right\}$ is a Cauchy sequence having values $X$. Therefore, it converges. Let $g(\mathbf{x})$ be the name of the thing it converges to.

Let $\varepsilon>0$ be given and pick $\delta>0$ such that whenever $\mathbf{x}, \mathbf{y} \in K$ and $|\mathbf{x}-\mathbf{y}|<\delta$, it follows $d(f(\mathbf{x}), f(\mathbf{y}))<\frac{\varepsilon}{3}$ for all $f \in A$. Now let $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{m}\right\}$ be a $\delta$ net for $K$ as in Lemma 7.8.5. Since there are only finitely many points in this $\delta$ net, it follows that there exists $N$ such that for all $p, q \geq N$,

$$
d\left(g_{q}\left(\mathbf{x}_{i}\right), g_{p}\left(\mathbf{x}_{i}\right)\right)<\frac{\varepsilon}{3}
$$

for all $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{m}\right\}$. Therefore, for arbitrary $\mathbf{x} \in K$, pick $\mathbf{x}_{i} \in\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{m}\right\}$ such that

$$
\left|\mathbf{x}_{i}-\mathbf{x}\right|<\delta
$$

Then

$$
\begin{aligned}
d\left(g_{q}(\mathbf{x}), g_{p}(\mathbf{x})\right) & \leq d\left(g_{q}(\mathbf{x}), g_{q}\left(\mathbf{x}_{i}\right)\right)+d\left(g_{q}\left(\mathbf{x}_{i}\right), g_{p}\left(\mathbf{x}_{i}\right)\right)+d\left(g_{p}\left(\mathbf{x}_{i}\right), g_{p}(\mathbf{x})\right) \\
& <\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon
\end{aligned}
$$

Since $N$ does not depend on the choice of $\mathbf{x}$, it follows this sequence $\left\{g_{m}\right\}$ is uniformly Cauchy. That is, for every $\varepsilon>0$, there exists $N$ such that if $p, q \geq N$, then

$$
\rho\left(g_{p}, g_{q}\right)<\varepsilon
$$

Next, I need to verify that the function, $g$ is a continuous function. Let $N$ be large enough that whenever $p, q \geq N$, the above holds. Then for all $\mathbf{x} \in K$,

$$
\begin{equation*}
d\left(g(\mathbf{x}), g_{p}(\mathbf{x})\right) \leq \frac{\varepsilon}{3} \tag{7.8.7}
\end{equation*}
$$

whenever $p \geq N$. This follows from observing that for $p, q \geq N$,

$$
d\left(g_{q}(\mathbf{x}), g_{p}(\mathbf{x})\right)<\frac{\varepsilon}{3}
$$

and then taking the limit as $q \rightarrow \infty$ to obtain 7.8.7. In passing to the limit, you can use the following simple claim.

Claim: In a metric space, if $a_{n} \rightarrow a$, then $d\left(a_{n}, b\right) \rightarrow d(a, b)$.
Proof of the claim: You note that by the triangle inequality, $d\left(a_{n}, b\right)-d(a, b) \leq$ $d\left(a_{n}, a\right)$ and $d(a, b)-d\left(a_{n}, b\right) \leq d\left(a_{n}, a\right)$ and so

$$
\left|d\left(a_{n}, b\right)-d(a, b)\right| \leq d\left(a_{n}, a\right)
$$

Now let $p$ satisfy 7.8.7 for all $\mathbf{x}$ whenever $p>N$. Also pick $\delta>0$ such that if $|\mathbf{x}-\mathbf{y}|<$ $\delta$, then

$$
d\left(g_{p}(\mathbf{x}), g_{p}(\mathbf{y})\right)<\frac{\varepsilon}{3}
$$

Then if $|\mathbf{x}-\mathbf{y}|<\boldsymbol{\delta}$,

$$
\begin{aligned}
d(g(\mathbf{x}), g(\mathbf{y})) & \leq d\left(g(\mathbf{x}), g_{p}(\mathbf{x})\right)+d\left(g_{p}(\mathbf{x}), g_{p}(\mathbf{y})\right)+d\left(g_{p}(\mathbf{y}), g(\mathbf{y})\right) \\
& <\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon .
\end{aligned}
$$

Since $\varepsilon$ was arbitrary, this shows that $g$ is continuous.
It only remains to verify that $\rho\left(g, g_{k}\right) \rightarrow 0$. But this follows from 7.8.7.
With these lemmas, it is time to prove Theorem 7.8.4.
Proof of Theorem 7.8.4: Let $D=\left\{\mathbf{x}_{k}\right\}$ be the countable dense set of $K$ gauranteed by Lemma 7.8.5 and let $\{(1,1),(1,2),(1,3),(1,4),(1,5), \cdots\}$ be a subsequence of $\mathbb{N}$ such that

$$
\lim _{k \rightarrow \infty} f_{(1, k)}\left(\mathbf{x}_{1}\right) \text { exists. }
$$

This is where the local compactness of $X$ is being used. Now let

$$
\{(2,1),(2,2),(2,3),(2,4),(2,5), \cdots\}
$$

be a subsequence of $\{(1,1),(1,2),(1,3),(1,4),(1,5), \cdots\}$ which has the property that

$$
\lim _{k \rightarrow \infty} f_{(2, k)}\left(\mathbf{x}_{2}\right) \text { exists. }
$$

Thus it is also the case that

$$
f_{(2, k)}\left(\mathbf{x}_{1}\right) \text { converges to } \lim _{k \rightarrow \infty} f_{(1, k)}\left(\mathbf{x}_{1}\right) .
$$

because every subsequence of a convergent sequence converges to the same thing as the convergent sequence. Continue this way and consider the array

$$
\begin{gathered}
f_{(1,1)}, f_{(1,2)}, f_{(1,3)}, f_{(1,4)}, \cdots \text { converges at } \mathbf{x}_{1} \\
f_{(2,1)}, f_{(2,2)}, f_{(2,3)}, f_{(2,4)} \cdots \text { converges at } \mathbf{x}_{1} \text { and } \mathbf{x}_{2} \\
f_{(3,1)}, f_{(3,2)}, f_{(3,3)}, f_{(3,4)} \cdots \text { converges at } \mathbf{x}_{1}, \mathbf{x}_{2}, \text { and } \mathbf{x}_{3} \\
\vdots
\end{gathered}
$$

Now let $g_{k} \equiv f_{(k, k)}$. Thus $g_{k}$ is ultimately a subsequence of $\left\{f_{(m, k)}\right\}$ whenever $k>m$ and therefore, $\left\{g_{k}\right\}$ converges at each point of $D$. By Lemma 7.8.6 it follows there exists $g \in$ $C(K ; X)$ such that

$$
\lim _{k \rightarrow \infty} \rho\left(g, g_{k}\right)=0
$$

Actually there is an if and only if version of it but the most useful case is what is presented here. The process used to get the subsequence in the proof is called the Cantor diagonalization procedure.

### 7.9 Another General Version

This will use the characterization of compact metric spaces to give a proof of a general version of the Arzella Ascoli theorem. See Naylor and Sell [100] which is where I saw this general formulation.

Definition 7.9.1 Let $\left(X, d_{X}\right)$ be a compact metric space. Let $\left(Y, d_{Y}\right)$ be another complete metric space. Then $C(X, Y)$ will denote the continuous functions which map $X$ to $Y$. Then $\rho$ is a metric on $C(X, Y)$ defined by

$$
\rho(f, g) \equiv \sup _{x \in X} d_{Y}(f(x), g(x)) .
$$

Theorem 7.9.2 $(C(X, Y), \rho)$ is a complete metric space.
Proof: It is first necessary to show that $\rho$ is well defined. In this argument, I will just write $d$ rather than $d_{X}$ or $d_{Y}$. To show this, note that

$$
x \rightarrow d(f(x), g(x))
$$

is a continuous function because $f, g$ are continuous and

$$
|d(f(x), g(x))-d(f(y), g(y))| \leq d(f(x), f(y))+d(g(x), g(y))
$$

This follows from the triangle inequality. Say $d(f(x), g(x)) \geq d(f(y), g(y))$. Otherwise just replace $x$ with $y$ and repeat the argument. Then in this case, it reduces to the claim that

$$
d(f(x), g(x)) \leq d(f(x), f(y))+d(g(x), g(y))+d(f(y), g(y))
$$

However, by the triangle inequality, the right side of the above is at least as large as

$$
d(f(x), f(y))+d(g(x), f(y)) \geq d(f(x), g(x))
$$

It follows that $\rho(f, g)$ is just the maximum of a continuous function defined on a compact set.

Clearly $\rho(f, g)=\rho(g, f)$ and

$$
\begin{aligned}
\rho(f, g)+\rho(g, h) & =\sup _{x \in X} d(f(x), g(x))+\sup _{x \in X} d(g(x), h(x)) \\
& \geq \sup _{x \in X}(d(f(x), g(x))+d(g(x), h(x))) \\
& \geq \sup _{x \in X}(d(f(x), h(x)))=\rho(f, h)
\end{aligned}
$$

so the triangle inequality holds.
It remains to check completeness. Let $\left\{f_{n}\right\}$ be a Cauchy sequence. Then from the definition, $\left\{f_{n}(x)\right\}$ is a Cauchy sequence in $Y$ and so it converges to something called $f(x)$. I have to verify that $x \rightarrow f(x)$ is continuous. Define

$$
\rho^{\prime}\left(f, f_{n}\right) \equiv \lim \sup _{m \rightarrow \infty} \rho\left(f_{m}, f_{n}\right)
$$

Then if $n$ is sufficiently large, $\rho^{\prime}\left(f, f_{n}\right)<\varepsilon / 3$. Also,

$$
\begin{align*}
d\left(f(x), f_{n}(x)\right) & =\lim _{m \rightarrow \infty} d\left(f_{m}(x), f_{n}(x)\right) \\
& \leq \limsup _{m \rightarrow \infty} \rho\left(f_{m}, f_{n}\right)=\rho^{\prime}\left(f, f_{n}\right)<\frac{\varepsilon}{3} \tag{7.9.8}
\end{align*}
$$

Then picking such an $n$,

$$
\begin{aligned}
& d(f(x), f(y)) \leq d\left(f(x), f_{n}(x)\right)+d\left(f_{n}(x), f_{n}(y)\right)+d_{n}\left(f_{n}(y), f(y)\right) \\
& \quad \leq \rho^{\prime}\left(f, f_{n}\right)+d\left(f_{n}(x), f_{n}(y)\right)+\rho^{\prime}\left(f, f_{n}\right)<\frac{2 \varepsilon}{3}+d\left(f_{n}(x), f_{n}(y)\right)
\end{aligned}
$$

which is less than $\varepsilon$ provided $d(x, y)$ is small enough, this by continuity of $f_{n}$. Therefore, $f$ is continuous. By 7.9.8 this shows that, since $x$ is arbitrary, $\rho\left(f, f_{n}\right)<\varepsilon$ whenever $n$ is large enough.

Here is a useful lemma.
Lemma 7.9.3 Let $S$ be a totally bounded subset of $(X, d)$ a metric space. Then $\bar{S}$ is also totally bounded.

Proof: Suppose not. Then there exists a sequence $\left\{p_{n}\right\} \subseteq \bar{S}$ such that $d\left(p_{m}, p_{n}\right) \geq \varepsilon$ for all $m \neq n$. Now let $q_{n} \in B\left(p_{n}, \frac{\varepsilon}{8}\right) \cap S$. Then it follows that

$$
\frac{\varepsilon}{8}+d\left(q_{n}, q_{m}\right)+\frac{\varepsilon}{8} \geq d\left(p_{n}, q_{n}\right)+d\left(q_{n}, q_{m}\right)+d\left(q_{m}, p_{m}\right) \geq d\left(p_{n}, q_{m}\right) \geq \varepsilon
$$

and so $d\left(q_{n}, q_{m}\right)>\frac{\varepsilon}{2}$. This contradicts total boundedness of $S$.
Next, here is an important definition.
Definition 7.9.4 Let $\mathscr{A} \subseteq C(X, Y)$ where $\left(X, d_{X}\right)$ and $\left(Y, d_{Y}\right)$ are metric spaces. Thus $\mathscr{A}$ is a set of continuous functions mapping $X$ to $Y$. Then $\mathscr{A}$ is said to be equicontinuous if for every $\varepsilon>0$ there exists a $\delta>0$ such that if $d_{X}\left(x_{1}, x_{2}\right)<\delta$ then for all $f \in \mathscr{A}$, $d_{Y}\left(f\left(x_{1}\right), f\left(x_{2}\right)\right)<\varepsilon$. (This is uniform continuity which is uniform in $\mathscr{A}$.) $\mathscr{A}$ is said to be pointwise compact if $\{f(x): f \in \mathscr{A}\}$ has compact closure in $Y$.

Here is the Ascoli Arzela theorem.
Theorem 7.9.5 Let $\left(X, d_{X}\right)$ be a compact metric space and let $\left(Y, d_{Y}\right)$ be a complete metric space. Thus $(C(X, Y), \rho)$ is a complete metric space. Let $\mathscr{A} \subseteq C(X, Y)$ be pointwise compact and equicontinuous. Then $\overline{\mathscr{A}}$ is compact. Here the closure is taken in $(C(X, Y), \rho)$. The converse also holds.

Proof: The more useful direction is that the two conditions imply compactness of $\overline{\mathscr{A}}$. I prove this first. Since $\overline{\mathscr{A}}$ is a closed subset of a complete space, it follows that $\overline{\mathscr{A}}$ will be compact if it is totally bounded. In showing this, it follows from Lemma 7.9.3 that it suffices to verify that $\mathscr{A}$ is totally bounded. Suppose this is not so. Then there exists $\varepsilon>0$ and a sequence of points of $\mathscr{A},\left\{f_{n}\right\}$ such that $\rho\left(f_{n}, f_{m}\right) \geq \varepsilon$ whenever $n \neq m$.

By equicontinuity, there exists $\delta>0$ such that if $d(x, y)<\delta$, then $d(f(x), f(y))<\frac{\varepsilon}{8}$ for all $f \in \mathscr{A}$. Let $\left\{x_{i}\right\}_{i=1}^{m}$ be a $\delta / 2$ net for $X$. Since there are only finitely many $x_{i}$, it follows from pointwise compactness that there exists a subsequence, still denoted by $\left\{f_{n}\right\}$ which converges at each $x_{i}$. There exists $x_{m n}$ such that

$$
\begin{align*}
& \rho\left(f_{n}, f_{m}\right)-\frac{\varepsilon}{8}<d\left(f_{n}\left(x_{m n}\right), f_{m}\left(x_{n m}\right)\right) \\
\leq & d\left(f_{n}\left(x_{n m}\right), f_{n}\left(x_{i}\right)\right)+d\left(f_{n}\left(x_{i}\right), f_{m}\left(x_{i}\right)\right)+d\left(f_{m}\left(x_{i}\right), f_{m}\left(x_{n m}\right)\right) \\
< & \frac{\varepsilon}{8}+d\left(f_{n}\left(x_{i}\right), f_{m}\left(x_{i}\right)\right)+\frac{\varepsilon}{8} \tag{7.9.9}
\end{align*}
$$

where here $x_{i}$ is such that $x_{n m} \in B\left(x_{i}, \delta\right)$. From the convergence of $f_{n}$ at each $x_{i}$, there exists $N$ such that if $m, n>N$, then for all $x_{i}$,

$$
d\left(f_{n}\left(x_{i}\right), f_{m}\left(x_{i}\right)\right)<\frac{\varepsilon}{8}
$$

Now 7.9.9 results in the contradiction,

$$
\varepsilon-\frac{\varepsilon}{8}<\frac{\varepsilon}{8}+\frac{\varepsilon}{8}+\frac{\varepsilon}{8}
$$

It follows that $\mathscr{A}$ and hence $\overline{\mathscr{A}}$ is totally bounded. This proves the more important direction.

Next suppose $\overline{\mathscr{A}}$ is compact. Why must $\mathscr{A}$ be pointwise compact and equicontinuous? If it fails to be pointwise compact, then there exists $x \in X$ such that $\{f(x): f \in \mathscr{A}\}$ is not contained in a compact set of $Y$. Thus there exists $\varepsilon>0$ and a sequence of functions in $\mathscr{A}$ $\left\{f_{n}\right\}$ such that $d\left(f_{n}(x), f_{m}(x)\right) \geq \varepsilon$. But this implies $\rho\left(f_{m}, f_{n}\right) \geq \varepsilon$ and so $\overline{\mathscr{A}}$ fails to be totally bounded, a contradiction. Thus $\mathscr{A}$ must be pointwise compact. Now why must it be equicontinuous? If it is not, then for each $n \in \mathbb{N}$ there exists $\varepsilon>0$ and $x_{n}, y_{n} \in X$ such that $d\left(x_{n}, y_{n}\right)<1 / n$ but for some $f_{n} \in \mathscr{A}, d\left(f_{n}\left(x_{n}\right), f_{n}\left(y_{n}\right)\right) \geq \varepsilon$. However, by compactness, there exists a subsequence $\left\{f_{n_{k}}\right\}$ such that $\lim _{k \rightarrow \infty} \rho\left(f_{n_{k}}, f\right)=0$ and also that $x_{n_{k}}, y_{n_{k}} \rightarrow$ $x \in X$. Hence

$$
\begin{aligned}
& \varepsilon \leq d\left(f_{n_{k}}\left(x_{n_{k}}\right), f_{n_{k}}\left(y_{n_{k}}\right)\right) \leq d\left(f_{n_{k}}\left(x_{n_{k}}\right), f\left(x_{n_{k}}\right)\right) \\
&+d\left(f\left(x_{n_{k}}\right), f\left(y_{n_{k}}\right)\right)+d\left(f\left(y_{n_{k}}\right), f_{n_{k}}\left(y_{n_{k}}\right)\right) \\
& \leq \rho\left(f_{n_{k}}, f\right)+d\left(f\left(x_{n_{k}}\right), f\left(y_{n_{k}}\right)\right)+\rho\left(f, f_{n_{k}}\right)
\end{aligned}
$$

and now this is a contradiction because each term on the right converges to 0 . The middle term converges to 0 because $f\left(x_{n_{k}}\right), f\left(y_{n_{k}}\right) \rightarrow f(x)$.

### 7.10 The Tietze Extension Theorem

It turns out that if $H$ is a closed subset of a metric space, $(X, d)$ and if $f: H \rightarrow[a, b]$ is continuous, then there exists $g$ defined on all of $X$ such that $g=f$ on $H$ and $g$ is continuous. This is called the Tietze extension theorem. First it is well to recall continuity in the context of metric space.

Definition 7.10.1 Let $(X, d)$ be a metric space and suppose $f: X \rightarrow Y$ is a function where $(Y, \rho)$ is also a metric space. For example, $Y=\mathbb{R}$. Then $f$ is continuous at $x \in X$ if for every $\varepsilon>0$ there exists $\delta>0$ such that $\rho(f(x), f(z))<\varepsilon$ whenever $d(x, z)<\delta$. As is usual in such definitions, $f$ is said to be continuous if it is continuous at every point of $X$.

The following lemma gives an important example of a continuous real valued function defined on a metric space, $(X, d)$.

Lemma 7.10.2 Let $(X, d)$ be a metric space and let $S \subseteq X$ be a nonempty subset. Define

$$
\operatorname{dist}(x, S) \equiv \inf \{d(x, y): y \in S\}
$$

Then $x \rightarrow \operatorname{dist}(x, S)$ is a continuous function satisfying the inequality,

$$
\begin{equation*}
|\operatorname{dist}(x, S)-\operatorname{dist}(y, S)| \leq d(x, y) \tag{7.10.10}
\end{equation*}
$$

Proof: The continuity of $x \rightarrow \operatorname{dist}(x, S)$ is obvious if the inequality 7.10 .10 is established. So let $x, y \in X$. Without loss of generality, assume $\operatorname{dist}(x, S) \geq \operatorname{dist}(y, S)$ and pick $z \in S$ such that $d(y, z)-\varepsilon<\operatorname{dist}(y, S)$. Then

$$
\begin{aligned}
|\operatorname{dist}(x, S)-\operatorname{dist}(y, S)| & =\operatorname{dist}(x, S)-\operatorname{dist}(y, S) \leq d(x, z)-(d(y, z)-\varepsilon) \\
& \leq d(z, y)+d(x, y)-d(y, z)+\varepsilon=d(x, y)+\varepsilon
\end{aligned}
$$

Since $\varepsilon$ is arbitrary, this proves 7.10.10.
Lemma 7.10.3 Let $H, K$ be two nonempty disjoint closed subsets of a metric space, $(X, d)$. Then there exists a continuous function, $g: X \rightarrow[-1,1]$ such that $g(H)=-1 / 3, g(K)=$ $1 / 3, g(X) \subseteq[-1 / 3,1 / 3]$.

Proof: Let

$$
f(x) \equiv \frac{\operatorname{dist}(x, H)}{\operatorname{dist}(x, H)+\operatorname{dist}(x, K)}
$$

The denominator is never equal to zero because if $\operatorname{dist}(x, H)=0$, then $x \in H$ becasue $H$ is closed. (To see this, pick $h_{k} \in B(x, 1 / k) \cap H$. Then $h_{k} \rightarrow x$ and since $H$ is closed, $x \in H$.) Similarly, if $\operatorname{dist}(x, K)=0$, then $x \in K$ and so the denominator is never zero as claimed. Hence, by Lemma 7.10.2, $f$ is continuous and from its definition, $f=0$ on $H$ and $f=1$ on $K$. Now let $g(x) \equiv \frac{2}{3}\left(f(x)-\frac{1}{2}\right)$. Then $g$ has the desired properties.

Definition 7.10.4 For $f$ a real or complex valued bounded continuous function defined on a metric space, $M$

$$
\|f\|_{M} \equiv \sup \{|f(x)|: x \in M\}
$$

Lemma 7.10.5 Suppose $M$ is a closed set in $X$ where $(X, d)$ is a metric space and suppose $f: M \rightarrow[-1,1]$ is continuous at every point of $M$. Then there exists a function, $g$ which is defined and continuous on all of $X$ such that $\|f-g\|_{M}<\frac{2}{3}$.

Proof: Let $H=f^{-1}([-1,-1 / 3]), K=f^{-1}([1 / 3,1])$. Thus $H$ and $K$ are disjoint closed subsets of $M$. Suppose first $H, K$ are both nonempty. Then by Lemma 7.10.3 there exists $g$ such that $g$ is a continuous function defined on all of $X$ and $g(H)=-1 / 3, g(K)=1 / 3$, and $g(X) \subseteq[-1 / 3,1 / 3]$. It follows $\|f-g\|_{M}<2 / 3$. If $H=\emptyset$, then $f$ has all its values in $[-1 / 3,1]$ and so letting $g \equiv 1 / 3$, the desired condition is obtained. If $K=\emptyset$, let $g \equiv-1 / 3$. This proves the lemma.

Lemma 7.10.6 Suppose $M$ is a closed set in $X$ where $(X, d)$ is a metric space and suppose $f: M \rightarrow[-1,1]$ is continuous at every point of $M$. Then there exists a function, $g$ which is defined and continuous on all of $X$ such that $g=f$ on $M$ and $g$ has its values in $[-1,1]$.

Proof: Let $g_{1}$ be such that $g_{1}(X) \subseteq[-1 / 3,1 / 3]$ and $\left\|f-g_{1}\right\|_{M} \leq \frac{2}{3}$. Suppose $g_{1}, \cdots, g_{m}$ have been chosen such that $g_{j}(X) \subseteq[-1 / 3,1 / 3]$ and

$$
\begin{equation*}
\left\|f-\sum_{i=1}^{m}\left(\frac{2}{3}\right)^{i-1} g_{i}\right\|_{M}<\left(\frac{2}{3}\right)^{m} \tag{7.10.11}
\end{equation*}
$$

Then

$$
\left\|\left(\frac{3}{2}\right)^{m}\left(f-\sum_{i=1}^{m}\left(\frac{2}{3}\right)^{i-1} g_{i}\right)\right\|_{M} \leq 1
$$

and so $\left(\frac{3}{2}\right)^{m}\left(f-\sum_{i=1}^{m}\left(\frac{2}{3}\right)^{i-1} g_{i}\right)$ can play the role of $f$ in the first step of the proof. Therefore, there exists $g_{m+1}$ defined and continuous on all of $X$ such that its values are in $[-1 / 3,1 / 3]$ and

$$
\left\|\left(\frac{3}{2}\right)^{m}\left(f-\sum_{i=1}^{m}\left(\frac{2}{3}\right)^{i-1} g_{i}\right)-g_{m+1}\right\|_{M} \leq \frac{2}{3}
$$

Hence

$$
\left\|\left(f-\sum_{i=1}^{m}\left(\frac{2}{3}\right)^{i-1} g_{i}\right)-\left(\frac{2}{3}\right)^{m} g_{m+1}\right\|_{M} \leq\left(\frac{2}{3}\right)^{m+1}
$$

It follows there exists a sequence, $\left\{g_{i}\right\}$ such that each has its values in $[-1 / 3,1 / 3]$ and for every $m 7.10 .11$ holds. Then let

$$
g(x) \equiv \sum_{i=1}^{\infty}\left(\frac{2}{3}\right)^{i-1} g_{i}(x)
$$

It follows

$$
|g(x)| \leq\left|\sum_{i=1}^{\infty}\left(\frac{2}{3}\right)^{i-1} g_{i}(x)\right| \leq \sum_{i=1}^{m}\left(\frac{2}{3}\right)^{i-1} \frac{1}{3} \leq 1
$$

and since convergence is uniform, $g$ must be continuous. The estimate 7.10 .11 implies $f=g$ on $M$.

The following is the Tietze extension theorem.

Theorem 7.10.7 Let $M$ be a closed nonempty subset of a metric space $(X, d)$ and let $f$ : $M \rightarrow[a, b]$ is continuous at every point of $M$. Then there exists a function, $g$ continuous on all of $X$ which coincides with $f$ on $M$ such that $g(X) \subseteq[a, b]$.

Proof: Let $f_{1}(x)=1+\frac{2}{b-a}(f(x)-b)$. Then $f_{1}$ satisfies the conditions of Lemma 7.10.6 and so there exists $g_{1}: X \rightarrow[-1,1]$ such that $g$ is continuous on $X$ and equals $f_{1}$ on $M$. Let $g(x)=\left(g_{1}(x)-1\right)\left(\frac{b-a}{2}\right)+b$. This works.

### 7.11 Some Simple Fixed Point Theorems

The following is of more interest in the case of normed vector spaces, but there is no harm in stating it in this more general setting. You should verify that the functions described in the following definition are all continuous.

Definition 7.11.1 Let $f: X \rightarrow Y$ where $(X, d)$ and $(Y, \rho)$ are metric spaces. Then $f$ is said to be Lipschitz continuous if for every $x, \hat{x} \in X, \rho(f(x), f(\hat{x})) \leq r d(x, \hat{x})$. The function is called a contraction map if $r<1$.

The big theorem about contraction maps is the following.
Theorem 7.11.2 Let $f:(X, d) \rightarrow(X, d)$ be a contraction map and let $(X, d)$ be a complete metric space. Thus Cauchy sequences converge and also $d(f(x), f(\hat{x})) \leq r d(x, \hat{x})$ where $r<1$. Then $f$ has a unique fixed point. This is a point $x \in X$ such that $f(x)=x$. Also, if $x_{0}$ is any point of $X$, then

$$
d\left(x, x_{0}\right) \leq \frac{d\left(x_{0}, f\left(x_{0}\right)\right)}{1-r}
$$

Also, for each n,

$$
d\left(f^{n}\left(x_{0}\right), x_{0}\right) \leq \frac{d\left(x_{0}, f\left(x_{0}\right)\right)}{1-r}
$$

and $x=\lim _{n \rightarrow \infty} f^{n}\left(x_{0}\right)$.
Proof: Pick $x_{0} \in X$ and consider the sequence of iterates of the map,

$$
x_{0}, f\left(x_{0}\right), f^{2}\left(x_{0}\right), \cdots
$$

We argue that this is a Cauchy sequence. For $m<n$, it follows from the triangle inequality,

$$
d\left(f^{m}\left(x_{0}\right), f^{n}\left(x_{0}\right)\right) \leq \sum_{k=m}^{n-1} d\left(f^{k+1}\left(x_{0}\right), f^{k}\left(x_{0}\right)\right) \leq \sum_{k=m}^{\infty} r^{k} d\left(f\left(x_{0}\right), x_{0}\right)
$$

The reason for this last is as follows.

$$
\begin{gathered}
d\left(f^{2}\left(x_{0}\right), f\left(x_{0}\right)\right) \leq r d\left(f\left(x_{0}\right), x_{0}\right) \\
d\left(f^{3}\left(x_{0}\right), f^{2}\left(x_{0}\right)\right) \leq r d\left(f^{2}\left(x_{0}\right), f\left(x_{0}\right)\right) \leq r^{2} d\left(f\left(x_{0}\right), x_{0}\right)
\end{gathered}
$$

and so forth. Therefore,

$$
d\left(f^{m}\left(x_{0}\right), f^{n}\left(x_{0}\right)\right) \leq d\left(f\left(x_{0}\right), x_{0}\right) \frac{r^{m}}{1-r}
$$

which shows that this is indeed a Cauchy sequence. Therefore, there exists $x$ such that

$$
\lim _{n \rightarrow \infty} f^{n}\left(x_{0}\right)=x
$$

By continuity,

$$
f(x)=f\left(\lim _{n \rightarrow \infty} f^{n}\left(x_{0}\right)\right)=\lim _{n \rightarrow \infty} f^{n+1}\left(x_{0}\right)=x
$$

Also note that this estimate yields

$$
d\left(x_{0}, f^{n}\left(x_{0}\right)\right) \leq \frac{d\left(x_{0}, f\left(x_{0}\right)\right)}{1-r}
$$

Now $d\left(x_{0}, x\right) \leq d\left(x_{0}, f^{n}\left(x_{0}\right)\right)+d\left(f^{n}\left(x_{0}\right), x\right)$ and so

$$
d\left(x_{0}, x\right)-d\left(f^{n}\left(x_{0}\right), x\right) \leq \frac{d\left(x_{0}, f\left(x_{0}\right)\right)}{1-r}
$$

Letting $n \rightarrow \infty$, it follows that

$$
d\left(x_{0}, x\right) \leq \frac{d\left(x_{0}, f\left(x_{0}\right)\right)}{1-r}
$$

It only remains to verify that there is only one fixed point. Suppose then that $x, x^{\prime}$ are two. Then

$$
d\left(x, x^{\prime}\right)=d\left(f(x), f\left(x^{\prime}\right)\right) \leq r d\left(x^{\prime}, x\right)
$$

and so $d\left(x, x^{\prime}\right)=0$ because $r<1$.
The above is the usual formulation of this important theorem, but we actually proved a better result.

Corollary 7.11.3 Let $B$ be a closed subset of the complete metric space $(X, d)$ and let $f: B \rightarrow X$ be a contraction map

$$
d(f(x), f(\hat{x})) \leq r d(x, \hat{x}), r<1
$$

Also suppose there exists $x_{0} \in B$ such that the sequence of iterates $\left\{f^{n}\left(x_{0}\right)\right\}_{n=1}^{\infty}$ remains in $B$. Then $f$ has a unique fixed point in $B$ which is the limit of the sequence of iterates. This is a point $x \in B$ such that $f(x)=x$. In the case that $B=\overline{B\left(x_{0}, \delta\right)}$, the sequence of iterates satisfies the inequality

$$
d\left(f^{n}\left(x_{0}\right), x_{0}\right) \leq \frac{d\left(x_{0}, f\left(x_{0}\right)\right)}{1-r}
$$

and so it will remain in $B$ if

$$
\frac{d\left(x_{0}, f\left(x_{0}\right)\right)}{1-r}<\delta
$$

Proof: By assumption, the sequence of iterates stays in $B$. Then, as in the proof of the preceding theorem, for $m<n$, it follows from the triangle inequality,

$$
\begin{aligned}
d\left(f^{m}\left(x_{0}\right), f^{n}\left(x_{0}\right)\right) & \leq \sum_{k=m}^{n-1} d\left(f^{k+1}\left(x_{0}\right), f^{k}\left(x_{0}\right)\right) \\
& \leq \sum_{k=m}^{\infty} r^{k} d\left(f\left(x_{0}\right), x_{0}\right)=\frac{r^{m}}{1-r} d\left(f\left(x_{0}\right), x_{0}\right)
\end{aligned}
$$

Hence the sequence of iterates is Cauchy and must converge to a point $x$ in $X$. However, $B$ is closed and so it must be the case that $x \in B$. Then as before,

$$
x=\lim _{n \rightarrow \infty} f^{n}\left(x_{0}\right)=\lim _{n \rightarrow \infty} f^{n+1}\left(x_{0}\right)=f\left(\lim _{n \rightarrow \infty} f^{n}\left(x_{0}\right)\right)=f(x)
$$

As to the sequence of iterates remaining in $B$ where $B$ is a ball as described, the inequality above in the case where $m=0$ yields

$$
d\left(x_{0}, f^{n}\left(x_{0}\right)\right) \leq \frac{1}{1-r} d\left(f\left(x_{0}\right), x_{0}\right)
$$

and so, if the right side is less than $\delta$, then the iterates remain in $B$. As to the fixed point being unique, it is as before. If $x, x^{\prime}$ are both fixed points in $B$, then $d\left(x, x^{\prime}\right)=d\left(f(x), f\left(x^{\prime}\right)\right) \leq$ $r d\left(x, x^{\prime}\right)$ and so $x=x^{\prime}$.

Sometimes you have the contraction depending on a parameter $\lambda$. Then there is a principle of uniform contractions.

Corollary 7.11.4 Suppose $f: X \times \Lambda \rightarrow X$ where $\Lambda$ is a metric space and $X$ is a complete metric space. Suppose $f$ satisfies

1. $d(f(x, \lambda), f(y, \lambda)) \leq r d(x, y)$ for each $\lambda \in \Lambda$.
2. $\lambda \rightarrow f(x, \lambda)$ is continuous as a map from $\Lambda$ to $X$.

Then if $x(\lambda)$ is the fixed point, it follows that $\lambda \rightarrow x(\lambda)$ is continuous.
Proof: Pick $x_{0} \in X$ and consider the above sequence of iterates, $\left\{f^{n}(x, \lambda)\right\}$. Let $\rho$ be the metric on $\Lambda$. Then there is a fixed point and if $x(\lambda)$ is this unique fixed point,

$$
d\left(x(\lambda), x_{0}\right) \leq \frac{d\left(f\left(x_{0}, \lambda\right), x_{0}\right)}{1-r}
$$

In particular, you could start with $x_{0}=x(\mu)$ and conclude that

$$
\begin{aligned}
& d(x(\lambda), x(\mu)) \leq \frac{d(f(x(\mu), \lambda), x(\mu))}{1-r} \\
\leq & \frac{d(f(x(\mu), \lambda), f(x(\mu), \mu))}{1-r}+\frac{d(f(x(\mu), \mu), x(\mu))}{1-r} \\
= & \frac{d(f(x(\mu), \lambda), f(x(\mu), \mu))}{1-r}
\end{aligned}
$$

Now by continuity of $\lambda \rightarrow f(x, \lambda)$, it follows that if $\rho(\lambda, \mu)$ is small enough, the above is no larger than

$$
\frac{\varepsilon(1-r)}{1-r}=\varepsilon
$$

Hence, if $\rho(\lambda, \mu)$ is small enough, we have

$$
d(x(\lambda), x(\mu))<\varepsilon .
$$

This is called the uniform contraction principle.
The contraction mapping theorem has an extremely useful generalization. In order to get a unique fixed point, it suffices to have some power of $f$ a contraction map.

Theorem 7.11.5 Let $f:(X, d) \rightarrow(X, d)$ have the property that for some $n \in \mathbb{N}$, $f^{n}$ is a contraction map and let $(X, d)$ be a complete metric space. Then there is a unique fixed point for $f$. As in the earlier theorem the sequence of iterates $\left\{f^{n}\left(x_{0}\right)\right\}_{n=1}^{\infty}$ also converges to the fixed point.

Proof: From Theorem 7.11.2 there is a unique fixed point for $f^{n}$. Thus

$$
f^{n}(x)=x
$$

Then

$$
f^{n}(f(x))=f^{n+1}(x)=f(x)
$$

By uniqueness, $f(x)=x$.
Now consider the sequence of iterates. Suppose it fails to converge to $x$. Then there is $\varepsilon>0$ and a subsequence $n_{k}$ such that

$$
d\left(f^{n_{k}}\left(x_{0}\right), x\right) \geq \varepsilon
$$

Now $n_{k}=p_{k} n+r_{k}$ where $r_{k}$ is one of the numbers $\{0,1,2, \cdots, n-1\}$. It follows that there exists one of these numbers which is repeated infinitely often. Call it $r$ and let the further subsequence continue to be denoted as $n_{k}$. Thus

$$
d\left(f^{p_{k} n+r}\left(x_{0}\right), x\right) \geq \varepsilon
$$

In other words,

$$
d\left(f^{p_{k} n}\left(f^{r}\left(x_{0}\right)\right), x\right) \geq \varepsilon
$$

However, from Theorem 7.11.2, as $k \rightarrow \infty, f^{p_{k} n}\left(f^{r}\left(x_{0}\right)\right) \rightarrow x$ which contradicts the above inequality. Hence the sequence of iterates converges to $x$, as it did for $f$ a contraction map.

Definition 7.11.6 Let $f:(X, d) \rightarrow(Y, \rho)$ be a function. Then it is said to be uniformly continuous on $X$ if for every $\varepsilon>0$ there exists a $\delta>0$ such that whenever $x, \hat{x}$ are two points of $X$ with $d(x, \hat{x})<\delta$, it follows that $\rho(f(x), f(\hat{x}))<\varepsilon$.

Note the difference between this and continuity. With continuity, the $\delta$ could depend on $x$ but here it works for any pair of points in $X$.

Lemma 7.11.7 Suppose $x_{n} \rightarrow x$ and $y_{n} \rightarrow y$. Then $d\left(x_{n}, y_{n}\right) \rightarrow d(x, y)$.
Proof: Consider the following.

$$
d(x, y) \leq d\left(x, x_{n}\right)+d\left(x_{n}, y\right) \leq d\left(x, x_{n}\right)+d\left(x_{n}, y_{n}\right)+d\left(y_{n}, y\right)
$$

so

$$
d(x, y)-d\left(x_{n}, y_{n}\right) \leq d\left(x, x_{n}\right)+d\left(y_{n}, y\right)
$$

Similarly

$$
d\left(x_{n}, y_{n}\right)-d(x, y) \leq d\left(x, x_{n}\right)+d\left(y_{n}, y\right)
$$

and so

$$
\left|d\left(x_{n}, y_{n}\right)-d(x, y)\right| \leq d\left(x, x_{n}\right)+d\left(y_{n}, y\right)
$$

and the right side converges to 0 as $n \rightarrow \infty$.
There is a remarkable result concerning compactness and uniform continuity.
Theorem 7.11.8 Let $f:(X, d) \rightarrow(Y, \rho)$ be a continuous function and let $K$ be a compact subset of $X$. Then the restriction of $f$ to $K$ is uniformly continuous.

Proof: First of all, $K$ is a metric space and $f$ restricted to $K$ is continuous. Now suppose it fails to be uniformly continuous. Then there exists $\varepsilon>0$ and pairs of points $x_{n}, \hat{x}_{n}$ such that $d\left(x_{n}, \hat{x}_{n}\right)<1 / n$ but $\rho\left(f\left(x_{n}\right), f\left(\hat{x}_{n}\right)\right) \geq \varepsilon$. Since $K$ is compact, it is sequentially compact and so there exists a subsequence, still denoted as $\left\{x_{n}\right\}$ such that $x_{n} \rightarrow x \in K$. Then also $\hat{x}_{n} \rightarrow x$ also and so

$$
\rho(f(x), f(x))=\lim _{n \rightarrow \infty} \rho\left(f\left(x_{n}\right), f\left(\hat{x}_{n}\right)\right) \geq \varepsilon
$$

which is a contradiction. Note the use of Lemma 7.11.7 in the equal sign.
Next is to consider the meaning of convergence of sequences of functions. There are two main ways of convergence of interest here, pointwise and uniform convergence.

Definition 7.11.9 Let $f_{n}: X \rightarrow Y$ where $(X, d),(Y, \rho)$ are two metric spaces. Then $\left\{f_{n}\right\}$ is said to converge poinwise to a function $f: X \rightarrow Y$ iffor every $x \in X$,

$$
\lim _{n \rightarrow \infty} f_{n}(x)=f(x)
$$

$\left\{f_{n}\right\}$ is said to converge uniformly iffor all $\varepsilon>0$, there exists $N$ such that if $n \geq N$, then

$$
\sup _{x \in X} \rho\left(f_{n}(x), f(x)\right)<\varepsilon
$$

Here is a well known example illustrating the difference between pointwise and uniform convergence.

Example 7.11.10 Let $f_{n}(x)=x^{n}$ on the metric space $[0,1]$. Then this function converges pointwise to

$$
f(x)=\left\{\begin{array}{l}
0 \text { on }[0,1) \\
1 \text { at } 1
\end{array}\right.
$$

but it does not converge uniformly on this interval to $f$.

Note how the target function $f$ in the above example is not continuous even though each function in the sequence is. The nice thing about uniform convergence is that it takes continuity of the functions in the sequence and imparts it to the target function. It does this for both continuity at a single point and uniform continuity. Thus uniform convergence is a very superior thing.

Theorem 7.11.11 Let $f_{n}: X \rightarrow Y$ where $(X, d),(Y, \rho)$ are two metric spaces and suppose each $f_{n}$ is continuous at $x \in X$ and also that $f_{n}$ converges uniformly to $f$ on $X$. Then $f$ is also continuous at $x$. In addition to this, if each $f_{n}$ is uniformly continuous on $X$, then the same is true for $f$.

Proof: Let $\varepsilon>0$ be given. Then

$$
\rho(f(x), f(\hat{x})) \leq \rho\left(f(x), f_{n}(x)\right)+\rho\left(f_{n}(x), f_{n}(\hat{x})\right)+\rho\left(f_{n}(\hat{x}), f(\hat{x})\right)
$$

By uniform convergence, there exists $N$ such that both $\rho\left(f(x), f_{n}(x)\right)$ and $\rho\left(f_{n}(\hat{x}), f(\hat{x})\right)$ are less than $\varepsilon / 3$ provided $n \geq N$. Thus picking such an $n$,

$$
\rho(f(x), f(\hat{x})) \leq \frac{2 \varepsilon}{3}+\rho\left(f_{n}(x), f_{n}(\hat{x})\right)
$$

Now from the continuity of $f_{n}$, there exists $\delta>0$ such that if $d(x, \hat{x})<\delta$, then

$$
\rho\left(f_{n}(x), f_{n}(\hat{x})\right)<\varepsilon / 3 .
$$

Hence, if $d(x, \hat{x})<\delta$, then

$$
\rho(f(x), f(\hat{x})) \leq \frac{2 \varepsilon}{3}+\rho\left(f_{n}(x), f_{n}(\hat{x})\right)<\frac{2 \varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon
$$

Hence, $f$ is continuous at $x$.
Next consider uniform continuity. It follows from the uniform convergence that if $x, \hat{x}$ are any two points of $X$, then if $n \geq N$, then, picking such an $n$,

$$
\rho(f(x), f(\hat{x})) \leq \frac{2 \varepsilon}{3}+\rho\left(f_{n}(x), f_{n}(\hat{x})\right)
$$

By uniform continuity of $f_{n}$ there exists $\boldsymbol{\delta}$ such that if $d(x, \hat{x})<\boldsymbol{\delta}$, then the term on the right in the above is less than $\varepsilon / 3$. Hence if $d(x, \hat{x})<\delta$, then $\rho(f(x), f(\hat{x}))<\varepsilon$ and so $f$ is uniformly continuous as claimed.

### 7.12 General Topological Spaces

It turns out that metric spaces are not sufficiently general for some applications. This section is a brief introduction to general topology. In making this generalization, the properties of balls which are the conclusion of Theorem 7.1.4 on Page 135 are stated as axioms for a subset of the power set of a given set which will be known as a basis for the topology. More can be found in [83] and the references listed there.

Definition 7.12.1 Let $X$ be a nonempty set and suppose $\mathscr{B} \subseteq \mathscr{P}(X)$. Then $\mathscr{B}$ is a basis for a topology if it satisfies the following axioms.
1.) Whenever $p \in A \cap B$ for $A, B \in \mathscr{B}$, it follows there exists $C \in \mathscr{B}$ such that $p \in C \subseteq$ $A \cap B$.
2.) $\cup \mathscr{B}=X$.

Then a subset, $U$, of $X$ is an open set if for every point, $x \in U$, there exists $B \in \mathscr{B}$ such that $x \in B \subseteq U$. Thus the open sets are exactly those which can be obtained as a union of sets of $\mathscr{B}$. Denote these subsets of $X$ by the symbol $\tau$ and refer to $\tau$ as the topology or the set of open sets.

Note that this is simply the analog of saying a set is open exactly when every point is an interior point.

Proposition 7.12.2 Let $X$ be a set and let $\mathscr{B}$ be a basis for a topology as defined above and let $\tau$ be the set of open sets determined by $\mathscr{B}$. Then

$$
\begin{gather*}
\emptyset \in \tau, X \in \tau  \tag{7.12.12}\\
\text { If } \mathscr{C} \subseteq \tau, \text { then } \cup \mathscr{C} \in \tau  \tag{7.12.13}\\
\text { If } A, B \in \tau, \text { then } A \cap B \in \tau . \tag{7.12.14}
\end{gather*}
$$

Proof: If $p \in \emptyset$ then there exists $B \in \mathscr{B}$ such that $p \in B \subseteq \emptyset$ because there are no points in $\emptyset$. Therefore, $\emptyset \in \tau$. Now if $p \in X$, then by part 2.) of Definition 7.12.1 $p \in B \subseteq X$ for some $B \in \mathscr{B}$ and so $X \in \tau$.

If $\mathscr{C} \subseteq \tau$, and if $p \in \cup \mathscr{C}$, then there exists a set, $B \in \mathscr{C}$ such that $p \in B$. However, $B$ is itself a union of sets from $\mathscr{B}$ and so there exists $C \in \mathscr{B}$ such that $p \in C \subseteq B \subseteq \cup \mathscr{C}$. This verifies 7.12.13.

Finally, if $A, B \in \tau$ and $p \in A \cap B$, then since $A$ and $B$ are themselves unions of sets of $\mathscr{B}$, it follows there exists $A_{1}, B_{1} \in \mathscr{B}$ such that $A_{1} \subseteq A, B_{1} \subseteq B$, and $p \in A_{1} \cap B_{1}$. Therefore, by 1.) of Definition 7.12.1 there exists $C \in \mathscr{B}$ such that $p \in C \subseteq A_{1} \cap B_{1} \subseteq A \cap B$, showing that $A \cap B \in \tau$ as claimed. Of course if $A \cap B=\emptyset$, then $A \cap B \in \tau$. This proves the proposition.

Definition 7.12.3 A set $X$ together with such a collection of its subsets satisfying 7.12.127.12.14 is called a topological space. $\tau$ is called the topology or set of open sets of $X$.

Definition 7.12.4 A topological space is said to be Hausdorff if whenever $p$ and $q$ are distinct points of $X$, there exist disjoint open sets $U, V$ such that $p \in U, q \in V$. In other words points can be separated with open sets.


Definition 7.12.5 A subset of a topological space is said to be closed if its complement is open. Let $p$ be a point of $X$ and let $E \subseteq X$. Then $p$ is said to be a limit point of $E$ if every open set containing $p$ contains a point of $E$ distinct from $p$.

Note that if the topological space is Hausdorff, then this definition is equivalent to requiring that every open set containing $p$ contains infinitely many points from $E$. Why?

Theorem 7.12.6 A subset, $E$, of $X$ is closed if and only if it contains all its limit points.
Proof: Suppose first that $E$ is closed and let $x$ be a limit point of $E$. Is $x \in E$ ? If $x \notin E$, then $E^{C}$ is an open set containing $x$ which contains no points of $E$, a contradiction. Thus $x \in E$.

Now suppose $E$ contains all its limit points. Is the complement of $E$ open? If $x \in E^{C}$, then $x$ is not a limit point of $E$ because $E$ has all its limit points and so there exists an open set, $U$ containing $x$ such that $U$ contains no point of $E$ other than $x$. Since $x \notin E$, it follows that $x \in U \subseteq E^{C}$ which implies $E^{C}$ is an open set because this shows $E^{C}$ is the union of open sets.

Theorem 7.12.7 If $(X, \tau)$ is a Hausdorff space and if $p \in X$, then $\{p\}$ is a closed set.
Proof: If $x \neq p$, there exist open sets $U$ and $V$ such that $x \in U, p \in V$ and $U \cap V=\emptyset$. Therefore, $\{p\}^{C}$ is an open set so $\{p\}$ is closed.

Note that the Hausdorff axiom was stronger than needed in order to draw the conclusion of the last theorem. In fact it would have been enough to assume that if $x \neq y$, then there exists an open set containing $x$ which does not intersect $y$.

Definition 7.12.8 A topological space $(X, \tau)$ is said to be regular if whenever $C$ is a closed set and $p$ is a point not in $C$, there exist disjoint open sets $U$ and $V$ such that $p \in U, C \subseteq V$. Thus a closed set can be separated from a point not in the closed set by two disjoint open sets.


Definition 7.12.9 The topological space, $(X, \tau)$ is said to be normal if whenever $C$ and $K$ are disjoint closed sets, there exist disjoint open sets $U$ and $V$ such that $C \subseteq U, K \subseteq V$. Thus any two disjoint closed sets can be separated with open sets.


Normal

Definition 7.12.10 Let $E$ be a subset of $X . \bar{E}$ is defined to be the smallest closed set containing $E$.

Lemma 7.12.11 The above definition is well defined.
Proof: Let $\mathscr{C}$ denote all the closed sets which contain $E$. Then $\mathscr{C}$ is nonempty because $X \in \mathscr{C}$.

$$
(\cap\{A: A \in \mathscr{C}\})^{C}=\cup\left\{A^{C}: A \in \mathscr{C}\right\},
$$

an open set which shows that $\cap \mathscr{C}$ is a closed set and is the smallest closed set which contains $E$.

Theorem 7.12.12 $\bar{E}=E \cup\{$ limit points of $E\}$.
Proof: Let $x \in \bar{E}$ and suppose that $x \notin E$. If $x$ is not a limit point either, then there exists an open set, $U$,containing $x$ which does not intersect $E$. But then $U^{C}$ is a closed set which contains $E$ which does not contain $x$, contrary to the definition that $\bar{E}$ is the intersection of all closed sets containing $E$. Therefore, $x$ must be a limit point of $E$ after all.

Now $E \subseteq \bar{E}$ so suppose $x$ is a limit point of $E$. Is $x \in \bar{E}$ ? If $H$ is a closed set containing $E$, which does not contain $x$, then $H^{C}$ is an open set containing $x$ which contains no points of $E$ other than $x$ negating the assumption that $x$ is a limit point of $E$.

The following is the definition of continuity in terms of general topological spaces. It is really just a generalization of the $\varepsilon-\delta$ definition of continuity given in calculus.

Definition 7.12.13 Let $(X, \tau)$ and $(Y, \eta)$ be two topological spaces and let $f: X \rightarrow Y . f$ is continuous at $x \in X$ if whenever $V$ is an open set of $Y$ containing $f(x)$, there exists an open set $U \in \tau$ such that $x \in U$ and $f(U) \subseteq V$. $f$ is continuous if $f^{-1}(V) \in \tau$ whenever $V \in \eta$.

You should prove the following.
Proposition 7.12.14 In the situation of Definition $7.12 .13 f$ is continuous if and only if $f$ is continuous at every point of $X$.

Definition 7.12.15 Let $\left(X_{i}, \tau_{i}\right)$ be topological spaces. $\prod_{i=1}^{n} X_{i}$ is the Cartesian product. Define a product topology as follows. Let $\mathscr{B}=\prod_{i=1}^{n} A_{i}$ where $A_{i} \in \tau_{i}$. Then $\mathscr{B}$ is a basis for the product topology.

Theorem 7.12.16 The set $\mathscr{B}$ of Definition 7.12.15 is a basis for a topology.
Proof: Suppose $\mathbf{x} \in \prod_{i=1}^{n} A_{i} \cap \prod_{i=1}^{n} B_{i}$ where $A_{i}$ and $B_{i}$ are open sets. Say

$$
\mathbf{x}=\left(x_{1}, \cdots, x_{n}\right)
$$

Then $x_{i} \in A_{i} \cap B_{i}$ for each $i$. Therefore, $\mathbf{x} \in \prod_{i=1}^{n} A_{i} \cap B_{i} \in \mathscr{B}$ and $\prod_{i=1}^{n} A_{i} \cap B_{i} \subseteq \prod_{i=1}^{n} A_{i}$.
The definition of compactness is also considered for a general topological space. This is given next.

Definition 7.12.17 A subset, $E$, of a topological space $(X, \tau)$ is said to be compact if whenever $\mathscr{C} \subseteq \tau$ and $E \subseteq \cup \mathscr{C}$, there exists a finite subset of $\mathscr{C},\left\{U_{1} \cdots U_{n}\right\}$, such that $E \subseteq \cup_{i=1}^{n} U_{i}$. (Every open covering admits a finite subcovering.) $E$ is precompact if $\bar{E}$ is compact. A topological space is called locally compact if it has a basis $\mathscr{B}$, with the property that $\bar{B}$ is compact for each $B \in \mathscr{B}$.

In general topological spaces there may be no concept of "bounded". Even if there is, closed and bounded is not necessarily the same as compactness. However, in any Hausdorff space every compact set must be a closed set.

Theorem 7.12.18 If $(X, \tau)$ is a Hausdorff space, then every compact subset must also be a closed set.

Proof: Suppose $p \notin K$. For each $x \in X$, there exist open sets, $U_{x}$ and $V_{x}$ such that

$$
x \in U_{x}, p \in V_{x},
$$

and

$$
U_{x} \cap V_{x}=\emptyset .
$$

If $K$ is assumed to be compact, there are finitely many of these sets, $U_{x_{1}}, \cdots, U_{x_{m}}$ which cover $K$. Then let $V \equiv \cap_{i=1}^{m} V_{x_{i}}$. It follows that $V$ is an open set containing $p$ which has empty intersection with each of the $U_{x_{i}}$. Consequently, $V$ contains no points of $K$ and is therefore not a limit point of $K$. This proves the theorem.

A useful construction when dealing with locally compact Hausdorff spaces is the notion of the one point compactification of the space.

Definition 7.12.19 Suppose $(X, \tau)$ is a locally compact Hausdorff space. Then let $\widetilde{X} \equiv$ $X \cup\{\infty\}$ where $\infty$ is just the name of some point which is not in $X$ which is called the point at infinity. A basis for the topology $\widetilde{\tau}$ for $\widetilde{X}$ is

$$
\tau \cup\left\{K^{C} \text { where } K \text { is a compact subset of } X\right\} .
$$

The complement is taken with respect to $\widetilde{X}$ and so the open sets, $K^{C}$ are basic open sets which contain $\infty$.

The reason this is called a compactification is contained in the next lemma.
Lemma 7.12.20 If $(X, \tau)$ is a locally compact Hausdorff space, then $(\widetilde{X}, \tilde{\tau})$ is a compact Hausdorff space. Also if $U$ is an open set of $\tilde{\tau}$, then $U \backslash\{\infty\}$ is an open set of $\tau$.

Proof: Since $(X, \tau)$ is a locally compact Hausdorff space, it follows $(\widetilde{X}, \tilde{\tau})$ is a Hausdorff topological space. The only case which needs checking is the one of $p \in X$ and $\infty$. Since $(X, \tau)$ is locally compact, there exists an open set of $\tau, U$ having compact closure which contains $p$. Then $p \in U$ and $\infty \in \bar{U}^{C}$ and these are disjoint open sets containing the points, $p$ and $\infty$ respectively. Now let $\mathscr{C}$ be an open cover of $\widetilde{X}$ with sets from $\widetilde{\tau}$. Then $\infty$ must be in some set, $U_{\infty}$ from $\mathscr{C}$, which must contain a set of the form $K^{C}$ where $K$ is a
compact subset of $X$. Then there exist sets from $\mathscr{C}, U_{1}, \cdots, U_{r}$ which cover $K$. Therefore, a finite subcover of $\widetilde{X}$ is $U_{1}, \cdots, U_{r}, U_{\infty}$.

To see the last claim, suppose $U$ contains $\infty$ since otherwise there is nothing to show. Notice that if $C$ is a compact set, then $X \backslash C$ is an open set. Therefore, if $x \in U \backslash\{\infty\}$, and if $\widetilde{X} \backslash C$ is a basic open set contained in $U$ containing $\infty$, then if $x$ is in this basic open set of $\widetilde{X}$, it is also in the open set $X \backslash C \subseteq U \backslash\{\infty\}$. If $x$ is not in any basic open set of the form $\widetilde{X} \backslash C$ then $x$ is contained in an open set of $\tau$ which is contained in $U \backslash\{\infty\}$. Thus $U \backslash\{\infty\}$ is indeed open in $\tau$.

Definition 7.12.21 If every finite subset of a collection of sets has nonempty intersection, the collection has the finite intersection property.

Theorem 7.12.22 Let $\mathscr{K}$ be a set whose elements are compact subsets of a Hausdorff topological space, $(X, \tau)$. Suppose $\mathscr{K}$ has the finite intersection property. Then $\emptyset \neq \cap \mathscr{K}$.

Proof: Suppose to the contrary that $\emptyset=\cap \mathscr{K}$. Then consider

$$
\mathscr{C} \equiv\left\{K^{C}: K \in \mathscr{K}\right\}
$$

It follows $\mathscr{C}$ is an open cover of $K_{0}$ where $K_{0}$ is any particular element of $\mathscr{K}$. But then there are finitely many $K \in \mathscr{K}, K_{1}, \cdots, K_{r}$ such that $K_{0} \subseteq \cup_{i=1}^{r} K_{i}^{C}$ implying that $\cap_{i=0}^{r} K_{i}=\emptyset$, contradicting the finite intersection property.

Lemma 7.12.23 Let $(X, \tau)$ be a topological space and let $\mathscr{B}$ be a basis for $\tau$. Then $K$ is compact if and only if every open cover of basic open sets admits a finite subcover.

Proof: Suppose first that $X$ is compact. Then if $\mathscr{C}$ is an open cover consisting of basic open sets, it follows it admits a finite subcover because these are open sets in $\mathscr{C}$.

Next suppose that every basic open cover admits a finite subcover and let $\mathscr{C}$ be an open cover of $X$. Then define $\widetilde{\mathscr{C}}$ to be the collection of basic open sets which are contained in some set of $\mathscr{C}$. It follows $\tilde{\mathscr{C}}$ is a basic open cover of $X$ and so it admits a finite subcover, $\left\{U_{1}, \cdots, U_{p}\right\}$. Now each $U_{i}$ is contained in an open set of $\mathscr{C}$. Let $O_{i}$ be a set of $\mathscr{C}$ which contains $U_{i}$. Then $\left\{O_{1}, \cdots, O_{p}\right\}$ is an open cover of $X$. This proves the lemma.

In fact, much more can be said than Lemma 7.12.23. However, this is all which I will present here.

### 7.13 Connected Sets

Stated informally, connected sets are those which are in one piece. More precisely,
Definition 7.13.1 A set, $S$ in a general topological space is separated if there exist sets, $A, B$ such that

$$
S=A \cup B, A, B \neq \emptyset, \text { and } \bar{A} \cap B=\bar{B} \cap A=\emptyset
$$

In this case, the sets $A$ and $B$ are said to separate $S$. A set is connected if it is not separated.

One of the most important theorems about connected sets is the following.

Theorem 7.13.2 Suppose $U$ and $V$ are connected sets having nonempty intersection. Then $U \cup V$ is also connected.

Proof: Suppose $U \cup V=A \cup B$ where $\bar{A} \cap B=\bar{B} \cap A=\emptyset$. Consider the sets, $A \cap U$ and $B \cap U$. Since

$$
\overline{(A \cap U)} \cap(B \cap U)=(A \cap U) \cap(\overline{B \cap U})=\emptyset,
$$

It follows one of these sets must be empty since otherwise, $U$ would be separated. It follows that $U$ is contained in either $A$ or $B$. Similarly, $V$ must be contained in either $A$ or $B$. Since $U$ and $V$ have nonempty intersection, it follows that both $V$ and $U$ are contained in one of the sets, $A, B$. Therefore, the other must be empty and this shows $U \cup V$ cannot be separated and is therefore, connected.

The intersection of connected sets is not necessarily connected as is shown by the following picture.


Theorem 7.13.3 Let $f: X \rightarrow Y$ be continuous where $X$ and $Y$ are topological spaces and $X$ is connected. Then $f(X)$ is also connected.

Proof: To do this you show $f(X)$ is not separated. Suppose to the contrary that $f(X)=$ $A \cup B$ where $A$ and $B$ separate $f(X)$. Then consider the sets, $f^{-1}(A)$ and $f^{-1}(B)$. If $z$ $\in f^{-1}(B)$, then $f(z) \in B$ and so $f(z)$ is not a limit point of $A$. Therefore, there exists an open set, $U$ containing $f(z)$ such that $U \cap A=\emptyset$. But then, the continuity of $f$ implies that $f^{-1}(U)$ is an open set containing $z$ such that $f^{-1}(U) \cap f^{-1}(A)=\emptyset$. Therefore, $f^{-1}(B)$ contains no limit points of $f^{-1}(A)$. Similar reasoning implies $f^{-1}(A)$ contains no limit points of $f^{-1}(B)$. It follows that $X$ is separated by $f^{-1}(A)$ and $f^{-1}(B)$, contradicting the assumption that $X$ was connected.

An arbitrary set can be written as a union of maximal connected sets called connected components. This is the concept of the next definition.

Definition 7.13.4 Let $S$ be a set and let $p \in S$. Denote by $C_{p}$ the union of all connected subsets of $S$ which contain $p$. This is called the connected component determined by $p$.

Theorem 7.13.5 Let $C_{p}$ be a connected component of a set $S$ in a general topological space. Then $C_{p}$ is a connected set and if $C_{p} \cap C_{q} \neq \emptyset$, then $C_{p}=C_{q}$.

Proof: Let $\mathscr{C}$ denote the connected subsets of $S$ which contain $p$. If $C_{p}=A \cup B$ where

$$
\bar{A} \cap B=\bar{B} \cap A=\emptyset,
$$

then $p$ is in one of $A$ or $B$. Suppose without loss of generality $p \in A$. Then every set of $\mathscr{C}$ must also be contained in $A$ also since otherwise, as in Theorem 7.13.2, the set would be separated. But this implies $B$ is empty. Therefore, $C_{p}$ is connected. From this, and Theorem 7.13.2, the second assertion of the theorem is proved.

This shows the connected components of a set are equivalence classes and partition the set.

A set, $I$ is an interval in $\mathbb{R}$ if and only if whenever $x, y \in I$ then $(x, y) \subseteq I$. The following theorem is about the connected sets in $\mathbb{R}$.

Theorem 7.13.6 A set, $C$ in $\mathbb{R}$ is connected if and only if $C$ is an interval.
Proof: Let $C$ be connected. If $C$ consists of a single point, $p$, there is nothing to prove. The interval is just $[p, p]$. Suppose $p<q$ and $p, q \in C$. You need to show $(p, q) \subseteq C$. If

$$
x \in(p, q) \backslash C
$$

let $C \cap(-\infty, x) \equiv A$, and $C \cap(x, \infty) \equiv B$. Then $C=A \cup B$ and the sets, $A$ and $B$ separate $C$ contrary to the assumption that $C$ is connected.

Conversely, let $I$ be an interval. Suppose $I$ is separated by $A$ and $B$. Pick $x \in A$ and $y \in B$. Suppose without loss of generality that $x<y$. Now define the set,

$$
S \equiv\{t \in[x, y]:[x, t] \subseteq A\}
$$

and let $l$ be the least upper bound of $S$. Then $l \in \bar{A}$ so $l \notin B$ which implies $l \in A$. But if $l \notin \bar{B}$, then for some $\delta>0$,

$$
(l, l+\delta) \cap B=\emptyset
$$

contradicting the definition of $l$ as an upper bound for $S$. Therefore, $l \in \bar{B}$ which implies $l \notin A$ after all, a contradiction. It follows $I$ must be connected.

The following theorem is a very useful description of the open sets in $\mathbb{R}$.
Theorem 7.13.7 Let $U$ be an open set in $\mathbb{R}$. Then there exist countably many disjoint open sets, $\left\{\left(a_{i}, b_{i}\right)\right\}_{i=1}^{\infty}$ such that $U=\cup_{i=1}^{\infty}\left(a_{i}, b_{i}\right)$.

Proof: Let $p \in U$ and let $z \in C_{p}$, the connected component determined by $p$. Since $U$ is open, there exists, $\delta>0$ such that $(z-\delta, z+\delta) \subseteq U$. It follows from Theorem 7.13.2 that

$$
(z-\delta, z+\delta) \subseteq C_{p}
$$

This shows $C_{p}$ is open. By Theorem 7.13.6, this shows $C_{p}$ is an open interval, $(a, b)$ where $a, b \in[-\infty, \infty]$. There are therefore at most countably many of these connected components because each must contain a rational number and the rational numbers are countable. Denote by $\left\{\left(a_{i}, b_{i}\right)\right\}_{i=1}^{\infty}$ the set of these connected components. This proves the theorem.

Definition 7.13.8 A topological space, $E$ is arcwise connected if for any two points, $p, q \in$ $E$, there exists a closed interval, $[a, b]$ and a continuous function, $\gamma:[a, b] \rightarrow E$ such that $\gamma(a)=p$ and $\gamma(b)=q$. $E$ is locally connected if it has a basis of connected open sets. $E$ is locally arcwise connected if it has a basis of arcwise connected open sets.

An example of an arcwise connected topological space would be the any subset of $\mathbb{R}^{n}$ which is the continuous image of an interval. Locally connected is not the same as connected. A well known example is the following.

$$
\begin{equation*}
\left\{\left(x, \sin \frac{1}{x}\right): x \in(0,1]\right\} \cup\{(0, y): y \in[-1,1]\} \tag{7.13.15}
\end{equation*}
$$

You can verify that this set of points considered as a metric space with the metric from $\mathbb{R}^{2}$ is not locally connected or arcwise connected but is connected.

Proposition 7.13.9 If a topological space is arcwise connected, then it is connected.
Proof: Let $X$ be an arcwise connected space and suppose it is separated. Then $X=$ $A \cup B$ where $A, B$ are two separated sets. Pick $p \in A$ and $q \in B$. Since $X$ is given to be arcwise connected, there must exist a continuous function $\gamma:[a, b] \rightarrow X$ such that $\gamma(a)=p$ and $\gamma(b)=q$. But then we would have $\gamma([a, b])=(\gamma([a, b]) \cap A) \cup(\gamma([a, b]) \cap B)$ and the two sets, $\gamma([a, b]) \cap A$ and $\gamma([a, b]) \cap B$ are separated thus showing that $\gamma([a, b])$ is separated and contradicting Theorem 7.13.6 and Theorem 7.13.3. It follows that $X$ must be connected as claimed.

Theorem 7.13.10 Let $U$ be an open subset of a locally arcwise connected topological space, $X$. Then $U$ is arcwise connected if and only if $U$ if connected. Also the connected components of an open set in such a space are open sets, hence arcwise connected.

Proof: By Proposition 7.13.9 it is only necessary to verify that if $U$ is connected and open in the context of this theorem, then $U$ is arcwise connected. Pick $p \in U$. Say $x \in U$ satisfies $\mathscr{P}$ if there exists a continuous function, $\gamma:[a, b] \rightarrow U$ such that $\gamma(a)=p$ and $\gamma(b)=x$.

$$
A \equiv\{x \in U \text { such that } x \text { satisfies } \mathscr{P} .\}
$$

If $x \in A$, there exists, according to the assumption that $X$ is locally arcwise connected, an open set, $V$, containing $x$ and contained in $U$ which is arcwise connected. Thus letting $y \in V$, there exist intervals, $[a, b]$ and $[c, d]$ and continuous functions having values in $U$, $\gamma, \eta$ such that $\gamma(a)=p, \gamma(b)=x, \eta(c)=x$, and $\eta(d)=y$. Then let $\gamma_{1}:[a, b+d-c] \rightarrow U$ be defined as

$$
\gamma_{1}(t) \equiv\left\{\begin{array}{l}
\gamma(t) \text { if } t \in[a, b] \\
\eta(t+c-b) \text { if } t \in[b, b+d-c]
\end{array}\right.
$$

Then it is clear that $\gamma_{1}$ is a continuous function mapping $p$ to $y$ and showing that $V \subseteq A$. Therefore, $A$ is open. $A \neq \emptyset$ because there is an open set, $V$ containing $p$ which is contained in $U$ and is arcwise connected.

Now consider $B \equiv U \backslash A$. This is also open. If $B$ is not open, there exists a point $z \in B$ such that every open set containing $z$ is not contained in $B$. Therefore, letting $V$ be one of
the basic open sets chosen such that $z \in V \subseteq U$, there exist points of $A$ contained in $V$. But then, a repeat of the above argument shows $z \in A$ also. Hence $B$ is open and so if $B \neq \emptyset$, then $U=B \cup A$ and so $U$ is separated by the two sets, $B$ and $A$ contradicting the assumption that $U$ is connected.

It remains to verify the connected components are open. Let $z \in C_{p}$ where $C_{p}$ is the connected component determined by $p$. Then picking $V$ an arcwise connected open set which contains $z$ and is contained in $U, C_{p} \cup V$ is connected and contained in $U$ and so it must also be contained in $C_{p}$. This proves the theorem.

As an application, consider the following corollary.
Corollary 7.13.11 Let $f: \Omega \rightarrow \mathbb{Z}$ be continuous where $\Omega$ is a connected open set. Then $f$ must be a constant.

Proof: Suppose not. Then it achieves two different values, $k$ and $l \neq k$. Then $\Omega=$ $f^{-1}(l) \cup f^{-1}(\{m \in \mathbb{Z}: m \neq l\})$ and these are disjoint nonempty open sets which separate $\Omega$. To see they are open, note

$$
f^{-1}(\{m \in \mathbb{Z}: m \neq l\})=f^{-1}\left(\cup_{m \neq l}\left(m-\frac{1}{6}, m+\frac{1}{6}\right)\right)
$$

which is the inverse image of an open set.

### 7.14 Exercises

1. Let $d(x, y)=|x-y|$ for $x, y \in \mathbb{R}$. Show that this is a metric on $\mathbb{R}$.
2. Now consider $\mathbb{R}^{n}$. Let $\|\mathbf{x}\|_{\infty} \equiv \max \left\{\left|x_{i}\right|, i=1, \cdots, n\right\}$. Define

$$
d(\mathbf{x}, \mathbf{y}) \equiv\|\mathbf{x}-\mathbf{y}\|_{\infty}
$$

Show that this is a metric on $\mathbb{R}^{n}$. In the case of $n=2$, describe the ball $B(\mathbf{0}, r)$. Hint: First show that $\|\mathbf{x}+\mathbf{y}\| \leq\|\mathbf{x}\|+\|\mathbf{y}\|$.
3. Let $C([0, T])$ denote the space of functions which are continuous on $[0, T]$. Define

$$
\|f\| \equiv \sup _{t \in[0, T]}|f(t)|=\max _{t \in[0, T]}|f(t)|
$$

Verify the following. $\|f+g\| \leq\|f\|+\|g\|$. Then use to show that $d(f, g) \equiv\|f-g\|$ is a metric and that with this metric, $(C([0, T]), d)$ is a metric space.
4. Recall that $[a, b]$ is compact. This was done in single variable advanced calculus. That is, every sequence has a convergent subsequence. (We will go over it in here as well.) Also recall that a sequence of numbers $\left\{x_{n}\right\}$ is a Cauchy sequence means that for every $\varepsilon>0$ there exists $N$ such that if $m, n>N$, then $\left|x_{n}-x_{m}\right|<\varepsilon$. First show that every Cauchy sequence is bounded. Next, using the compactness of closed intervals, show that every Cauchy sequence has a convergent subsequence. It is shown later that if this is true, the original Cauchy sequence converges. Thus $\mathbb{R}$ with the usual metric just described is complete because every Cauchy sequence converges.
5. Using the result of the above problem, show that $\left(\mathbb{R}^{n},\|\cdot\|_{\infty}\right)$ is a complete metric space. That is, every Cauchy sequence converges. Here $d(\mathbf{x}, \mathbf{y}) \equiv\|\mathbf{x}-\mathbf{y}\|_{\infty}$.
6. Suppose you had $\left(X_{i}, d_{i}\right)$ is a metric space. Now consider the product space

$$
X \equiv \prod_{i=1}^{n} X_{i}
$$

with $d(\mathbf{x}, \mathbf{y})=\max \left\{d\left(x_{i}, y_{i}\right), i=1 \cdots, n\right\}$. Would this be a metric space? If so, prove that this is the case.
Does triangle inequality hold? Hint: For each $i$,

$$
d_{i}\left(x_{i}, z_{i}\right) \leq d_{i}\left(x_{i}, y_{i}\right)+d_{i}\left(y_{i}, z_{i}\right) \leq d(\mathbf{x}, \mathbf{y})+d(\mathbf{y}, \mathbf{z})
$$

Now take max of the two ends.
7. In the above example, if each $\left(X_{i}, d_{i}\right)$ is complete, explain why $(X, d)$ is also complete.
8. Show that $C([0, T])$ is a complete metric space. That is, show that if $\left\{f_{n}\right\}$ is a Cauchy sequence, then there exists $f \in C([0, T])$ such that

$$
\lim _{n \rightarrow \infty} d\left(f, f_{n}\right)=\lim _{n \rightarrow \infty}\left\|f-f_{n}\right\|=0
$$

Hint: First, you know that $\left\{f_{n}(t)\right\}$ is a Cauchy sequence for each $t$. Why? Now let $f(t)$ be the name of the thing to which $f_{n}(t)$ converges. Recall why the uniform convergence implies $t \rightarrow f(t)$ is continuous. Give the proof. It was done in single variable advanced calculus. Review and write down proof. Also show that $\left\|f-f_{n}\right\| \rightarrow 0$.
9. Let $X$ be a nonempty set of points. Say it has infinitely many points. Define $d(x, y)=$ 1 if $x \neq y$ and $d(x, y)=0$ if $x=y$. Show that this is a metric. Show that in $(X, d)$ every point is open and closed. In fact, show that every set is open and every set is closed. Is this a complete metric space? Explain why. Describe the open balls.
10. Show that the union of any set of open sets is an open set. Show the intersection of any set of closed sets is closed. Let $A$ be a nonempty subset of a metric space $(X, d)$. Then the closure of $A$, written as $\bar{A}$ is defined to be the intersection of all closed sets which contain $A$. Show that $\bar{A}=A \cup A^{\prime}$. That is, to find the closure, you just take the set and include all limit points of the set.
11. Let $A^{\prime}$ denote the set of limit points of $A$, a nonempty subset of a metric space $(X, d)$. Show that $A^{\prime}$ is closed.
12. A theorem was proved which gave three equivalent descriptions of compactness of a metric space. One of them said the following: A metric space is compact if and only if it is complete and totally bounded. Suppose $(X, d)$ is a complete metric space and $K \subseteq X$. Then $(K, d)$ is also clearly a metric space having the same metric as $X$.

Show that $(K, d)$ is compact if and only if it is closed and totally bounded. Note the similarity with the Heine Borel theorem on $\mathbb{R}$. Show that on $\mathbb{R}$, every bounded set is also totally bounded. Thus the earlier Heine Borel theorem for $\mathbb{R}$ is obtained.
13. Suppose $\left(X_{i}, d_{i}\right)$ is a compact metric space. Then the Cartesian product is also a metric space. That is $\left(\prod_{i=1}^{n} X_{i}, d\right)$ is a metric space where $d(\mathbf{x}, \mathbf{y}) \equiv \max \left\{d_{i}\left(x_{i}, y_{i}\right)\right\}$. Show that $\left(\prod_{i=1}^{n} X_{i}, d\right)$ is compact. Recall the Heine Borel theorem for $\mathbb{R}$. Explain why

$$
\prod_{i=1}^{n}\left[a_{i}, b_{i}\right]
$$

is compact in $\mathbb{R}^{n}$ with the distance given by $d(\mathbf{x}, \mathbf{y})=\max \left\{\left|x_{i}-y_{i}\right|\right\}$. Hint: It suffices to show that $\left(\prod_{i=1}^{n} X_{i}, d\right)$ is sequentially compact. Let $\left\{\mathbf{x}^{m}\right\}_{m=1}^{\infty}$ be a sequence. Then $\left\{x_{1}^{m}\right\}_{m=1}^{\infty}$ is a sequence in $X_{i}$. Therefore, it has a subsequence $\left\{x_{1}^{k_{1}}\right\}_{k_{1}=1}^{\infty}$ which converges to a point $x_{1} \in X_{1}$. Now consider $\left\{x_{2}^{k_{1}}\right\}_{k_{1}=1}^{\infty}$ the second components. It has a subsequence denoted as $k_{2}$ such that $\left\{x_{2}^{k_{2}}\right\}_{k_{2}=1}^{\infty_{1}=1}$ converges to a point $x_{2}$ in $X_{2}$. Explain why $\lim _{k_{2} \rightarrow \infty} x_{1}^{k_{2}}=x_{1}$. Continue doing this $n$ times. Explain why $\lim _{k_{n} \rightarrow \infty} x_{l}^{k_{n}}=x_{l} \in X_{l}$ for each $l$. Then explain why this is the same as saying $\lim _{k_{n} \rightarrow \infty} \mathbf{x}^{k_{n}}=\mathbf{x}$ in $\left(\prod_{i=1}^{n} X_{i}, d\right)$.
14. If you have a metric space $(X, d)$ and a compact subset of $(X, d) K$, suppose that $L$ is a closed subset of $K$. Explain why $L$ must also be compact. Hint: Go right to the definition. Take an open covering of $L$ and consider this along with the open set $L^{C}$ to obtain an open covering of $K$. Now use compactness of $K$. Use this to explain why every closed and bounded set in $\mathbb{R}^{n}$ is compact. Here the distance is given by $d(\mathbf{x}, \mathbf{y}) \equiv \max _{1 \leq i \leq n}\left\{\left|x_{i}-y_{i}\right|\right\}$.
15. Show that compactness is a topological property in the following sense. If

$$
(X, d),(Y, \rho)
$$

are both metric spaces and $f: X \rightarrow Y$ has the property that $f$ is one to one, onto, and continuous, and also $f^{-1}$ is one to one onto and continuous, then the two metric spaces are compact or not compact together. That is one is compact if and only if the other is.
16. Consider $\mathbb{R}$ the real numbers. Define a distance in the following way.

$$
\rho(x, y) \equiv|\arctan (x)-\arctan (y)|
$$

Show this is a good enough distance and that the open sets which come from this distance are the same as the open sets which come from the usual distance $d(x, y)=$ $|x-y|$. Explain why this yields that the identity mapping $f(x)=x$ is continuous with continuous inverse as a map from $(\mathbb{R}, d)$ to $(\mathbb{R}, \rho)$. To do this, you show that an open ball taken with respect to one of these is also open with respect to the other. However,
$(\mathbb{R}, \rho)$ is not a complete metric space while $(\mathbb{R}, d)$ is. Thus, unlike compactness. Completeness is not a topological property. Hint: To show the lack of completeness of $(\mathbb{R}, \rho)$, consider $x_{n}=n$. Show it is a Cauchy sequence with respect to $\rho$.
17. A very useful idea in metric space is the following distance function. Let $(X, d)$ be a metric space and $S \subseteq X, S \neq \emptyset$. Then $\operatorname{dist}(x, S) \equiv \inf \{d(x, y): y \in S\}$. Show that this always satisfies

$$
|\operatorname{dist}(x, S)-\operatorname{dist}(z, S)| \leq d(x, z)
$$

This is a really neat result.
18. If $K$ is a compact subset of $(X, d)$ and $y \notin K$, show that there always exists $x \in K$ such that $d(x, y)=\operatorname{dist}(y, K)$. Give an example in $\mathbb{R}$ to show that this is might not be so if $K$ is not compact.
19. You know that if $f: X \rightarrow X$ for $X$ a complete metric space, then if $d(f(x), f(y))<$ $r d(x, y)$ it follows that $f$ has a unique fixed point theorem. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$
f(t)=t+\left(1+e^{t}\right)^{-1}
$$

Show that $|f(t)-f(s)|<|t-s|$, but $f$ has no fixed point.
20. If $(X, d)$ is a metric space, show that there is a bounded metric $\rho$ such that the open sets for $(X, d)$ are the same as those for $(X, \rho)$.
21. Let $(X, d)$ be a metric space where $d$ is a bounded metric. Let $\mathscr{C}$ denote the collection of closed subsets of $X$. For $A, B \in \mathscr{C}$, define

$$
\rho(A, B) \equiv \inf \left\{\delta>0: A_{\delta} \supseteq B \text { and } B_{\delta} \supseteq A\right\}
$$

where for a set $S$,

$$
S_{\delta} \equiv\{x: \operatorname{dist}(x, S) \equiv \inf \{d(x, s): s \in S\} \leq \delta\}
$$

Show $x \rightarrow \operatorname{dist}(x, S)$ is continuous and that therefore, $S_{\delta}$ is a closed set containing $S$. Also show that $\rho$ is a metric on $\mathscr{C}$. This is called the Hausdorff metric.
22. $\uparrow$ Suppose $(X, d)$ is a compact metric space. Show $(\mathscr{C}, \rho)$ is a complete metric space. Hint: Show first that if $W_{n} \downarrow W$ where $W_{n}$ is closed, then $\rho\left(W_{n}, W\right) \rightarrow 0$. Now let $\left\{A_{n}\right\}$ be a Cauchy sequence in $\mathscr{C}$. Then if $\varepsilon>0$ there exists $N$ such that when $m, n \geq N$, then $\rho\left(A_{n}, A_{m}\right)<\varepsilon$. Therefore, for each $n \geq N$,

$$
\left(A_{n}\right)_{\varepsilon} \supseteq \overline{\cup_{k=n}^{\infty} A_{k}}
$$

Let $A \equiv \cap_{n=1}^{\infty} \overline{\cup_{k=n}^{\infty} A_{k}}$. By the first part, there exists $N_{1}>N$ such that for $n \geq N_{1}$,

$$
\rho\left(\overline{\cup_{k=n}^{\infty} A_{k}}, A\right)<\varepsilon, \text { and }\left(A_{n}\right)_{\varepsilon} \supseteq \overline{\cup_{k=n}^{\infty} A_{k}} .
$$

Therefore, for such $n, A_{\varepsilon} \supseteq W_{n} \supseteq A_{n}$ and $\left(W_{n}\right)_{\varepsilon} \supseteq\left(A_{n}\right)_{\varepsilon} \supseteq A$ because

$$
\left(A_{n}\right)_{\varepsilon} \supseteq \overline{\cup_{k=n}^{\infty} A_{k}} \supseteq A .
$$

23. $\uparrow$ Let $X$ be a compact metric space. Show $(\mathscr{C}, \rho)$ is compact. Hint: Let $\mathscr{D}_{n}$ be a $2^{-n}$ net for $X$. Let $\mathscr{K}_{n}$ denote finite unions of sets of the form $\overline{B\left(p, 2^{-n}\right)}$ where $p \in \mathscr{D}_{n}$. Show $\mathscr{K}_{n}$ is a $2^{-(n-1)}$ net for $(\mathscr{C}, \rho)$.

## Chapter 8

## Normed Linear Spaces

The thing which is missing in the above material about metric spaces is any kind of algebra. In most applications, we are interested in adding things and multiplying things by scalars and so forth. This requires the notion of a vector space, also called a linear space. The simplest example is $\mathbb{R}^{n}$ which is described next.

In this chapter, $\mathbb{F}$ will refer to either $\mathbb{R}$ or $\mathbb{C}$. It doesn't make any difference to the arguments which it is and so $\mathbb{F}$ is written to symbolize whichever you wish to think about. In multivariable calculus, the main example is where $\mathbb{F}=\mathbb{R}$. However, it is nice to observe that things work more generally. The big changes take place when you start to consider the derivative. As to notation, when it is desired to emphasize that certain quantities are vectors, bold face will often be used. This is not necessarily done consistently. Sometimes context is considered sufficient.

### 8.1 Algebra in $\mathbb{F}^{n}$, Vector Spaces

There are two algebraic operations done with elements of $\mathbb{F}^{n}$. One is addition and the other is multiplication by numbers, called scalars. In the case of $\mathbb{C}^{n}$ the scalars are complex numbers while in the case of $\mathbb{R}^{n}$ the only allowed scalars are real numbers. Thus, the scalars always come from $\mathbb{F}$ in either case.

Definition 8.1.1 If $\mathbf{x} \in \mathbb{F}^{n}$ and $a \in \mathbb{F}$, also called a scalar, then $a \mathbf{x} \in \mathbb{F}^{n}$ is defined by

$$
\begin{equation*}
a \mathbf{x}=a\left(x_{1}, \cdots, x_{n}\right) \equiv\left(a x_{1}, \cdots, a x_{n}\right) . \tag{8.1.1}
\end{equation*}
$$

This is known as scalar multiplication. If $\mathbf{x}, \mathbf{y} \in \mathbb{F}^{n}$ then $\mathbf{x}+\mathbf{y} \in \mathbb{F}^{n}$ and is defined by

$$
\begin{align*}
\mathbf{x}+\mathbf{y} & =\left(x_{1}, \cdots, x_{n}\right)+\left(y_{1}, \cdots, y_{n}\right) \\
& \equiv\left(x_{1}+y_{1}, \cdots, x_{n}+y_{n}\right) \tag{8.1.2}
\end{align*}
$$

the points in $\mathbb{F}^{n}$ are also referred to as vectors.
With this definition, the algebraic properties satisfy the conclusions of the following theorem. These conclusions are called the vector space axioms. Any time you have a set and a field of scalars satisfying the axioms of the following theorem, it is called a vector space.

Theorem 8.1.2 $\operatorname{For} \mathbf{v}, \mathbf{w} \in \mathbb{F}^{n}$ and $\alpha, \beta$ scalars, (real numbers), the following hold.

$$
\begin{equation*}
\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}, \tag{8.1.3}
\end{equation*}
$$

the commutative law of addition,

$$
\begin{equation*}
(\mathbf{v}+\mathbf{w})+\mathbf{z}=\mathbf{v}+(\mathbf{w}+\mathbf{z}) \tag{8.1.4}
\end{equation*}
$$

the associative law for addition,

$$
\begin{equation*}
\mathbf{v}+\mathbf{0}=\mathbf{v} \tag{8.1.5}
\end{equation*}
$$

the existence of an additive identity,

$$
\begin{equation*}
\mathbf{v}+(-\mathbf{v})=\mathbf{0} \tag{8.1.6}
\end{equation*}
$$

the existence of an additive inverse, Also

$$
\begin{gather*}
\alpha(\mathbf{v}+\mathbf{w})=\alpha \mathbf{v}+\alpha \mathbf{w}  \tag{8.1.7}\\
(\alpha+\beta) \mathbf{v}=\alpha \mathbf{v}+\beta \mathbf{v}  \tag{8.1.8}\\
\alpha(\beta \mathbf{v})=\alpha \beta(\mathbf{v})  \tag{8.1.9}\\
1 \mathbf{v}=\mathbf{v} \tag{8.1.10}
\end{gather*}
$$

In the above $\mathbf{0}=(0, \cdots, 0)$.
You should verify these properties all hold. For example, consider 8.1.7

$$
\begin{aligned}
\alpha(\mathbf{v}+\mathbf{w}) & =\alpha\left(v_{1}+w_{1}, \cdots, v_{n}+w_{n}\right) \\
& =\left(\alpha\left(v_{1}+w_{1}\right), \cdots, \alpha\left(v_{n}+w_{n}\right)\right) \\
& =\left(\alpha v_{1}+\alpha w_{1}, \cdots, \alpha v_{n}+\alpha w_{n}\right) \\
& =\left(\alpha v_{1}, \cdots, \alpha v_{n}\right)+\left(\alpha w_{1}, \cdots, \alpha w_{n}\right) \\
& =\alpha \mathbf{v}+\alpha \mathbf{w} .
\end{aligned}
$$

As usual subtraction is defined as $\mathbf{x}-\mathbf{y} \equiv \mathbf{x}+(-\mathbf{y})$.

### 8.2 Subspaces Spans And Bases

As mentioned above, $\mathbb{F}^{n}$ is an example of a vector space and this is what is studied in linear algebra. The concept of linear combination is fundamental in all of linear algebra. When one considers only algebraic considerations, it makes no difference what field of scalars you are using. It could be $\mathbb{R}, \mathbb{C}, \mathbb{Q}$ or even a field of residue classes. However, go ahead and think $\mathbb{R}$ or $\mathbb{C}$ since the subject of interest here is analysis.

Definition 8.2.1 Let $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{p}\right\}$ be vectors in a vector space, $Y$ having the field of scalars $\mathbb{F}$. A linear combination is any expression of the form

$$
\sum_{i=1}^{p} c_{i} \mathbf{x}_{i}
$$

where the $c_{i}$ are scalars. The set of all linear combinations of these vectors is called $\operatorname{span}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{n}\right)$. A vector $\mathbf{v}$ is said to be in the span of some set $S$ of vectors if $\mathbf{v}$ is a linear combination of vectors of $S$. This means: finite linear combination. If $V \subseteq Y$, then $V$ is called a subspace if whenever $\alpha, \beta$ are scalars and $\mathbf{u}$ and $\mathbf{v}$ are vectors of $V$, it follows $\alpha \mathbf{u}+\beta \mathbf{v} \in V$. That is, it is "closed under the algebraic operations of vector addition and scalar multiplication" and is therefore, a vector space. A linear combination of vectors is said to be trivial if all the scalars in the linear combination equal zero. A set of vectors is said to be linearly independent if the only linear combination of these vectors which
equals the zero vector is the trivial linear combination. Thus $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{n}\right\}$ is called linearly independent if whenever

$$
\sum_{k=1}^{p} c_{k} \mathbf{x}_{k}=\mathbf{0}
$$

it follows that all the scalars, $c_{k}$ equal zero. A set of vectors, $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{p}\right\}$, is called linearly dependent if it is not linearly independent. Thus the set of vectors is linearly dependent if there exist scalars, $c_{i}, i=1, \cdots, n$, not all zero such that $\sum_{k=1}^{p} c_{k} \mathbf{x}_{k}=\mathbf{0}$.

Lemma 8.2.2 A set of vectors $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{p}\right\}$ is linearly independent if and only if none of the vectors can be obtained as a linear combination of the others.

Proof: Suppose first that $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{p}\right\}$ is linearly independent. If

$$
\mathbf{x}_{k}=\sum_{j \neq k} c_{j} \mathbf{x}_{j}
$$

then

$$
\mathbf{0}=1 \mathbf{x}_{k}+\sum_{j \neq k}\left(-c_{j}\right) \mathbf{x}_{j}
$$

a nontrivial linear combination, contrary to assumption. This shows that if the set is linearly independent, then none of the vectors is a linear combination of the others.

Now suppose no vector is a linear combination of the others. Is $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{p}\right\}$ linearly independent? If it is not, there exist scalars, $c_{i}$, not all zero such that

$$
\sum_{i=1}^{p} c_{i} \mathbf{x}_{i}=\mathbf{0}
$$

Say $c_{k} \neq 0$. Then you can solve for $\mathbf{x}_{k}$ as

$$
\mathbf{x}_{k}=\sum_{j \neq k}\left(-c_{j}\right) / c_{k} \mathbf{x}_{j}
$$

contrary to assumption. This proves the lemma.
The following is called the exchange theorem.

## Theorem 8.2.3 If

$$
\operatorname{span}\left(\mathbf{u}_{1}, \cdots, \mathbf{u}_{r}\right) \subseteq \operatorname{span}\left(\mathbf{v}_{1}, \cdots, \mathbf{v}_{s}\right) \equiv V
$$

and $\left\{\mathbf{u}_{1}, \cdots, \mathbf{u}_{r}\right\}$ are linearly independent, then $r \leq s$.
Proof: Suppose $r>s$. Let $F_{p}$ denote the first $p$ vectors in $\left\{\mathbf{u}_{1}, \cdots, \mathbf{u}_{r}\right\}$. In case $p=0, F_{p}$ will denote the empty set. Let $E_{p}$ denote a finite list of vectors of $\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{s}\right\}$ and let $\left|E_{p}\right|$ denote the number of vectors in the list. For $0 \leq p \leq s$, let $E_{p}$ have the property

$$
\operatorname{span}\left(F_{p}, E_{p}\right)=V
$$

and $\left|E_{p}\right|$ is as small as possible for this to happen. I claim $\left|E_{p}\right| \leq s-p$ if $E_{p}$ is nonempty.

Here is why. For $p=0$, it is obvious because there are $s$ vectors from $\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{s}\right\}$ which span $V$, namely those vectors. Of course there might be a smaller list which does so, and so $\left|E_{p}\right| \leq s$. Suppose true for some $p<s$. Then

$$
\mathbf{u}_{p+1} \in \operatorname{span}\left(F_{p}, E_{p}\right)
$$

and so there are constants, $c_{1}, \cdots, c_{p}$ and $d_{1}, \cdots, d_{m}$ where $m \leq s-p$ such that

$$
\mathbf{u}_{p+1}=\sum_{i=1}^{p} c_{i} \mathbf{u}_{i}+\sum_{j=1}^{m} d_{i} \mathbf{z}_{j}
$$

for

$$
\left\{\mathbf{z}_{1}, \cdots, \mathbf{z}_{m}\right\} \subseteq\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{s}\right\} .
$$

Then not all the $d_{i}$ can equal zero because this would violate the linear independence of the $\left\{\mathbf{u}_{1}, \cdots, \mathbf{u}_{r}\right\}$. Therefore, you can solve for one of the $\mathbf{z}_{k}$ as a linear combination of $\left\{\mathbf{u}_{1}, \cdots, \mathbf{u}_{p+1}\right\}$ and the other $\mathbf{z}_{j}$. Thus you can change $F_{p}$ to $F_{p+1}$ and include one fewer vector in $E_{p}$. Thus $\left|E_{p+1}\right| \leq m-1 \leq s-p-1$. This proves the claim.

Therefore, $E_{s}$ is empty and $\operatorname{span}\left(\mathbf{u}_{1}, \cdots, \mathbf{u}_{s}\right)=V$. However, this gives a contradiction because it would require

$$
\mathbf{u}_{s+1} \in \operatorname{span}\left(\mathbf{u}_{1}, \cdots, \mathbf{u}_{s}\right)
$$

which violates the linear independence of these vectors.
Alternate proof: Recall from linear algebra that if you have $A$ an $m \times n$ matrix where $m<n$ so there are more columns than rows, then there exists a nonzero solution $\mathbf{x}$ to the equation $A \mathbf{x}=\mathbf{0}$. Recall why this was. You must have free variables. Then by assumption, you have

$$
\mathbf{u}_{j}=\sum_{i=1}^{s} a_{i j} \mathbf{v}_{i}
$$

If $s<r$, then the matrix $\left(a_{i j}\right)$ has more columns than rows and so there exists a nonzero vector $\mathbf{x} \in \mathbb{F}^{r}$ such that $\sum_{j=1}^{r} a_{i j} x_{j}=0$. Then consider the following.

$$
\sum_{j=1}^{r} x_{j} \mathbf{u}_{j}=\sum_{j=1}^{r} x_{j} \sum_{i=1}^{s} a_{i j} \mathbf{v}_{i}=\sum_{i} \sum_{j} a_{i j} x_{j} \mathbf{v}_{i}=\sum_{i} 0 \mathbf{v}_{j}=\mathbf{0}
$$

and since not all $x_{j}=0$, this contradicts the independence of the vectors $\left\{\mathbf{u}_{1}, \cdots, \mathbf{u}_{r}\right\}$.
Definition 8.2.4 A finite set of vectors, $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{r}\right\}$ is a basis for a vector space $V$ if

$$
\operatorname{span}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{r}\right)=V
$$

and $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{r}\right\}$ is linearly independent. Thus if $\mathbf{v} \in V$ there exist unique scalars, $v_{1}, \cdots, v_{r}$ such that $\mathbf{v}=\sum_{i=1}^{r} v_{i} \mathbf{x}_{i}$. These scalars are called the components of $\mathbf{v}$ with respect to the basis $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{r}\right\}$.

Corollary 8.2.5 Let $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{r}\right\}$ and $\left\{\mathbf{y}_{1}, \cdots, \mathbf{y}_{s}\right\}$ be two bases ${ }^{1}$ of $\mathbb{F}^{n}$. Then $r=s=n$. More generally, if you have two bases for a vector space $V$ then they have the same number of vectors.

Proof: From the exchange theorem, if $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{r}\right\}$ and $\left\{\mathbf{y}_{1}, \cdots, \mathbf{y}_{s}\right\}$ are two bases for $V$, then $r \leq s$ and $s \leq r$. Now note the vectors,

$$
\mathbf{e}_{i}=\overbrace{(0, \cdots, 0,1,0 \cdots, 0)^{T}}^{1 \text { is in the } i^{\text {th }} \text { slot }}
$$

for $i=1,2, \cdots, n$ are a basis for $\mathbb{F}^{n}$. This proves the corollary.
Lemma 8.2.6 Let $\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{r}\right\}$ be a set of vectors. Then $V \equiv \operatorname{span}\left(\mathbf{v}_{1}, \cdots, \mathbf{v}_{r}\right)$ is a subspace.

Proof: Suppose $\alpha, \beta$ are two scalars and let $\sum_{k=1}^{r} c_{k} \mathbf{v}_{k}$ and $\sum_{k=1}^{r} d_{k} \mathbf{v}_{k}$ are two elements of $V$. What about

$$
\alpha \sum_{k=1}^{r} c_{k} \mathbf{v}_{k}+\beta \sum_{k=1}^{r} d_{k} \mathbf{v}_{k} ?
$$

Is it also in $V$ ?

$$
\alpha \sum_{k=1}^{r} c_{k} \mathbf{v}_{k}+\beta \sum_{k=1}^{r} d_{k} \mathbf{v}_{k}=\sum_{k=1}^{r}\left(\alpha c_{k}+\beta d_{k}\right) \mathbf{v}_{k} \in V
$$

so the answer is yes. This proves the lemma.
Definition 8.2.7 Let $V$ be a vector space. Then $\operatorname{dim}(V)$ read as the dimension of $V$ is the number of vectors in a basis.

Of course you should wonder right now whether an arbitrary subspace of a finite dimensional vector space even has a basis. In fact it does and this is in the next theorem. First, here is an interesting lemma.

Lemma 8.2.8 Let $\mathbf{v} \notin \operatorname{span}\left(\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}\right)$ and $\left\{\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}\right\}$ is linearly independent. Then $\left\{\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}, \mathbf{v}\right\}$ is also linearly independent.

Proof: Suppose $\sum_{i=1}^{k} c_{i} \mathbf{u}_{i}+d \mathbf{v}=\mathbf{0}$. It is required to verify that each $c_{i}=0$ and that $d=$ 0 . But if $d \neq 0$, then you can solve for $\mathbf{v}$ as a linear combination of the vectors, $\left\{\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}\right\}$,

$$
\mathbf{v}=-\sum_{i=1}^{k}\left(\frac{c_{i}}{d}\right) \mathbf{u}_{i}
$$

contrary to assumption. Therefore, $d=0$. But then $\sum_{i=1}^{k} c_{i} \mathbf{u}_{i}=0$ and the linear independence of $\left\{\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}\right\}$ implies each $c_{i}=0$ also.

[^6]Theorem 8.2.9 Let $V$ be a nonzero subspace of $Y$ a finite dimensional vector space having dimension $n$. Then $V$ has a basis.

Proof: Let $\mathbf{v}_{1} \in V$ where $\mathbf{v}_{1} \neq 0$. If span $\left\{\mathbf{v}_{1}\right\}=V$, stop. $\left\{\mathbf{v}_{1}\right\}$ is a basis for $V$. Otherwise, there exists $\mathbf{v}_{2} \in V$ which is not in span $\left\{\mathbf{v}_{1}\right\}$. By Lemma 8.2.8 $\left\{\mathbf{v}_{1}, \mathbf{v}_{2}\right\}$ is a linearly independent set of vectors. If span $\left\{\mathbf{v}_{1}, \mathbf{v}_{2}\right\}=V$ stop, $\left\{\mathbf{v}_{1}, \mathbf{v}_{2}\right\}$ is a basis for $V$. If $\operatorname{span}\left\{\mathbf{v}_{1}, \mathbf{v}_{2}\right\} \neq V$, then there exists $\mathbf{v}_{3} \notin \operatorname{span}\left\{\mathbf{v}_{1}, \mathbf{v}_{2}\right\}$ and $\left\{\mathbf{v}_{1}, \mathbf{v}_{2}, \mathbf{v}_{3}\right\}$ is a larger linearly independent set of vectors. Continuing this way, the process must stop before $n+1$ steps because if not, it would be possible to obtain $n+1$ linearly independent vectors contrary to the exchange theorem, Theorem 8.2.3, and the assumed dimension of $Y$.

In words the following corollary states that any linearly independent set of vectors can be enlarged to form a basis.

Corollary 8.2.10 Let $V$ be a subspace of $Y$, a finite dimensional vector space of dimension $n$ and let $\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{r}\right\}$ be a linearly independent set of vectors in $V$. Then either it is a basis for $V$ or there exist vectors, $\mathbf{v}_{r+1}, \cdots, \mathbf{v}_{s}$ such that

$$
\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{r}, \mathbf{v}_{r+1}, \cdots, \mathbf{v}_{s}\right\}
$$

is a basis for $V$.
Proof: This follows immediately from the proof of Theorem 8.2.9. You do exactly the same argument except you start with $\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{r}\right\}$ rather than $\left\{\mathbf{v}_{1}\right\}$.

It is also true that any spanning set of vectors can be restricted to obtain a basis.
Theorem 8.2.11 Let $V$ be a subspace of $Y$, a finite dimensional vector space of dimension $n$ and suppose span $\left(\mathbf{u}_{1} \cdots, \mathbf{u}_{p}\right)=V$ where the $\mathbf{u}_{i}$ are nonzero vectors. Then there exist vectors, $\left\{\mathbf{v}_{1} \cdots, \mathbf{v}_{r}\right\}$ such that $\left\{\mathbf{v}_{1} \cdots, \mathbf{v}_{r}\right\} \subseteq\left\{\mathbf{u}_{1} \cdots, \mathbf{u}_{p}\right\}$ and $\left\{\mathbf{v}_{1} \cdots, \mathbf{v}_{r}\right\}$ is a basis for $V$.

Proof: Let $r$ be the smallest positive integer with the property that for some set,

$$
\begin{gathered}
\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{r}\right\} \subseteq\left\{\mathbf{u}_{1}, \cdots, \mathbf{u}_{p}\right\} \\
\operatorname{span}\left(\mathbf{v}_{1}, \cdots, \mathbf{v}_{r}\right)=V
\end{gathered}
$$

Then $r \leq p$ and it must be the case that $\left\{\mathbf{v}_{1} \cdots, \mathbf{v}_{r}\right\}$ is linearly independent because if it were not so, one of the vectors, say $\mathbf{v}_{k}$ would be a linear combination of the others. But then you could delete this vector from $\left\{\mathbf{v}_{1} \cdots, \mathbf{v}_{r}\right\}$ and the resulting list of $r-1$ vectors would still span $V$ contrary to the definition of $r$.

### 8.3 Inner Product And Normed Linear Spaces

### 8.3.1 The Inner Product In $\mathbb{F}^{n}$

To do calculus, you must understand what you mean by distance. For functions of one variable, the distance was provided by the absolute value of the difference of two numbers. This must be generalized to $\mathbb{F}^{n}$ and to more general situations.

Definition 8.3.1 Let $\mathbf{x}, \mathbf{y} \in \mathbb{F}^{n}$. Thus $\mathbf{x}=\left(x_{1}, \cdots, x_{n}\right)$ where each $x_{k} \in \mathbb{F}$ and a similar formula holding for $\mathbf{y}$. Then the inner product of these two vectors is defined to be

$$
(\mathbf{x}, \mathbf{y}) \equiv \sum_{j} x_{j} \overline{y_{j}} \equiv x_{1} \overline{y_{1}}+\cdots+x_{n} \overline{y_{n}} .
$$

Sometimes it is denoted as $\mathbf{x} \cdot \mathbf{y}$.
Notice how you put the conjugate on the entries of the vector, $\mathbf{y}$. It makes no difference if the vectors happen to be real vectors but with complex vectors you must involve a conjugate. The reason for this is that when you take the inner product of a vector with itself, you want to get the square of the length of the vector, a positive number. Placing the conjugate on the components of $\mathbf{y}$ in the above definition assures this will take place. Thus

$$
(\mathbf{x}, \mathbf{x})=\sum_{j} x_{j} \overline{x_{j}}=\sum_{j}\left|x_{j}\right|^{2} \geq 0 .
$$

If you didn't place a conjugate as in the above definition, things wouldn't work out correctly. For example,

$$
(1+i)^{2}+2^{2}=4+2 i
$$

and this is not a positive number.
The following properties of the inner product follow immediately from the definition and you should verify each of them.

Properties of the inner product:

1. $(\mathbf{u}, \mathbf{v})=\overline{(\mathbf{v}, \mathbf{u})}$
2. If $a, b$ are numbers and $\mathbf{u}, \mathbf{v}, \mathbf{z}$ are vectors then $((a \mathbf{u}+b \mathbf{v}), \mathbf{z})=a(\mathbf{u}, \mathbf{z})+b(\mathbf{v}, \mathbf{z})$.
3. $(\mathbf{u}, \mathbf{u}) \geq 0$ and it equals 0 if and only if $\mathbf{u}=\mathbf{0}$.

Note this implies $(\mathbf{x}, \alpha \mathbf{y})=\bar{\alpha}(\mathbf{x}, \mathbf{y})$ because

$$
(\mathbf{x}, \alpha \mathbf{y})=\overline{(\alpha \mathbf{y}, \mathbf{x})}=\overline{\alpha(\mathbf{y}, \mathbf{x})}=\bar{\alpha}(\mathbf{x}, \mathbf{y})
$$

The norm is defined as follows.
Definition 8.3.2 For $\mathbf{x} \in \mathbb{F}^{n}$,

$$
|\mathbf{x}| \equiv\left(\sum_{k=1}^{n}\left|x_{k}\right|^{2}\right)^{1 / 2}=(\mathbf{x}, \mathbf{x})^{1 / 2}
$$

### 8.3.2 General Inner Product Spaces

Any time you have a vector space which possesses an inner product, something satisfying the properties 1-3 above, it is called an inner product space.

Here is a fundamental inequality called the Cauchy Schwarz inequality which holds in any inner product space. First here is a simple lemma.

Lemma 8.3.3 If $z \in \mathbb{F}$ there exists $\theta \in \mathbb{F}$ such that $\theta z=|z|$ and $|\theta|=1$.
Proof: Let $\theta=1$ if $z=0$ and otherwise, let $\theta=\frac{\bar{z}}{|z|}$. Recall that for $z=x+i y, \bar{z}=x-i y$ and $\bar{z} z=|z|^{2}$. In case $z$ is real, there is no change in the above.

Theorem 8.3.4 (Cauchy Schwarz)Let H be an inner product space. The following inequality holds for $\mathbf{x}$ and $\mathbf{y} \in H$.

$$
\begin{equation*}
|(\mathbf{x}, \mathbf{y})| \leq(\mathbf{x}, \mathbf{x})^{1 / 2}(\mathbf{y}, \mathbf{y})^{1 / 2} \tag{8.3.11}
\end{equation*}
$$

Equality holds in this inequality if and only if one vector is a multiple of the other.
Proof: Let $\theta \in \mathbb{F}$ such that $|\theta|=1$ and

$$
\theta(\mathbf{x}, \mathbf{y})=|(\mathbf{x}, \mathbf{y})|
$$

Consider $p(t) \equiv(\mathbf{x}+\bar{\theta} t \mathbf{y}, \mathbf{x}+t \bar{\theta} \mathbf{y})$ where $t \in \mathbb{R}$. Then from the above list of properties of the inner product,

$$
\begin{align*}
0 & \leq p(t)=(\mathbf{x}, \mathbf{x})+t \boldsymbol{\theta}(\mathbf{x}, \mathbf{y})+t \overline{\boldsymbol{\theta}}(\mathbf{y}, \mathbf{x})+t^{2}(\mathbf{y}, \mathbf{y}) \\
& =(\mathbf{x}, \mathbf{x})+t \boldsymbol{\theta}(\mathbf{x}, \mathbf{y})+t \overline{\boldsymbol{\theta}}(\mathbf{x}, \mathbf{y})+t^{2}(\mathbf{y}, \mathbf{y}) \\
& =(\mathbf{x}, \mathbf{x})+2 t \operatorname{Re}(\boldsymbol{\theta}(\mathbf{x}, \mathbf{y}))+t^{2}(\mathbf{y}, \mathbf{y}) \\
& =(\mathbf{x}, \mathbf{x})+2 t|(\mathbf{x}, \mathbf{y})|+t^{2}(\mathbf{y}, \mathbf{y}) \tag{8.3.12}
\end{align*}
$$

and this must hold for all $t \in \mathbb{R}$. Therefore, if $(\mathbf{y}, \mathbf{y})=0$ it must be the case that $|(\mathbf{x}, \mathbf{y})|=0$ also since otherwise the above inequality would be violated. Therefore, in this case,

$$
|(\mathbf{x}, \mathbf{y})| \leq(\mathbf{x}, \mathbf{x})^{1 / 2}(\mathbf{y}, \mathbf{y})^{1 / 2} .
$$

On the other hand, if $(\mathbf{y}, \mathbf{y}) \neq 0$, then $p(t) \geq 0$ for all $t$ means the graph of $y=p(t)$ is a parabola which opens up and it either has exactly one real zero in the case its vertex touches the $t$ axis or it has no real zeros. From the quadratic formula this happens exactly when

$$
4|(\mathbf{x}, \mathbf{y})|^{2}-4(\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y}) \leq 0
$$

which is equivalent to 8.3.11.
It is clear from a computation that if one vector is a scalar multiple of the other that equality holds in 8.3.11. Conversely, suppose equality does hold. Then this is equivalent to saying $4|(\mathbf{x}, \mathbf{y})|^{2}-4(\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y})=0$ and so from the quadratic formula, there exists one real zero to $p(t)=0$. Call it $t_{0}$. Then

$$
p\left(t_{0}\right) \equiv\left(\mathbf{x}+\bar{\theta} t_{0} \mathbf{y}, \mathbf{x}+t_{0} \bar{\theta} \mathbf{y}\right)=|\mathbf{x}+\bar{\theta} t \mathbf{y}|^{2}=0
$$

and so $\mathbf{x}=-\overline{\boldsymbol{\theta}} t_{0} \mathbf{y}$. This proves the theorem.
Note that in establishing the inequality, I only used part of the above properties of the inner product. It was not necessary to use the one which says that if $(\mathbf{x}, \mathbf{x})=0$ then $\mathbf{x}=\mathbf{0}$.

Now the length of a vector can be defined.

Definition 8.3.5 Let $\mathbf{z} \in H$. Then $|\mathbf{z}| \equiv(\mathbf{z}, \mathbf{z})^{1 / 2}$.
Theorem 8.3.6 For length defined in Definition 8.3.5, the following hold.

$$
\begin{gather*}
|\mathbf{z}| \geq 0 \text { and }|\mathbf{z}|=0 \text { if and only if } \mathbf{z}=\mathbf{0}  \tag{8.3.13}\\
\text { If } \alpha \text { is a scalar, }|\alpha \mathbf{z}|=|\alpha||\mathbf{z}|  \tag{8.3.14}\\
|\mathbf{z}+\mathbf{w}| \leq|\mathbf{z}|+|\mathbf{w}| \tag{8.3.15}
\end{gather*}
$$

Proof: The first two claims are left as exercises. To establish the third,

$$
\begin{aligned}
|\mathbf{z}+\mathbf{w}|^{2} & \equiv(\mathbf{z}+\mathbf{w}, \mathbf{z}+\mathbf{w}) \\
& =(\mathbf{z}, \mathbf{z})+(\mathbf{w}, \mathbf{w})+(\mathbf{w}, \mathbf{z})+(\mathbf{z}, \mathbf{w}) \\
& =|\mathbf{z}|^{2}+|\mathbf{w}|^{2}+2 \operatorname{Re}(\mathbf{w}, \mathbf{z}) \\
& \leq|\mathbf{z}|^{2}+|\mathbf{w}|^{2}+2|(\mathbf{w}, \mathbf{z})| \\
& \leq|\mathbf{z}|^{2}+|\mathbf{w}|^{2}+2|\mathbf{w}||\mathbf{z}|=(|\mathbf{z}|+|\mathbf{w}|)^{2}
\end{aligned}
$$

Note that in an inner product space, you can define

$$
d(\mathbf{x}, \mathbf{y}) \equiv|\mathbf{x}-\mathbf{y}|
$$

and this is a metric for this inner product space. This follows from the above since $d$ satisfies the conditions for a metric,

$$
\begin{gathered}
d(\mathbf{x}, \mathbf{y})=d(\mathbf{y}, \mathbf{x}), d(\mathbf{x}, \mathbf{y}) \geq 0 \text { and equals } 0 \text { if and only if } \mathbf{x}=\mathbf{y} \\
d(\mathbf{x}, \mathbf{y})+d(\mathbf{y}, \mathbf{z})=|\mathbf{x}-\mathbf{y}|+|\mathbf{y}-\mathbf{z}| \geq|\mathbf{x}-\mathbf{y}+\mathbf{y}-\mathbf{z}|=|\mathbf{x}-\mathbf{z}|=d(\mathbf{x}, \mathbf{z}) .
\end{gathered}
$$

It follows that all the theory of metric spaces developed earlier applies to this situation.

### 8.3.3 Normed Vector Spaces

The best sort of a norm is one which comes from an inner product. However, any vector space, $V$ which has a function, $\|\cdot\|$ which maps $V$ to $[0, \infty)$ is called a normed vector space if $\|\cdot\|$ satisfies 8.3.13-8.3.15. That is

$$
\begin{equation*}
\|\mathbf{z}\| \geq 0 \text { and }\|\mathbf{z}\|=0 \text { if and only if } \mathbf{z}=\mathbf{0} \tag{8.3.16}
\end{equation*}
$$

If $\alpha$ is a scalar, $\|\alpha \mathbf{z}\|=|\alpha| \| \mathbf{z}| |$

$$
\begin{equation*}
\|\mathbf{z}+\mathbf{w}\| \leq\|\mathbf{z}\|+\|\mathbf{w}\| \tag{8.3.17}
\end{equation*}
$$

The last inequality above is called the triangle inequality. Another version of this is

$$
\begin{equation*}
\|\mathbf{z}\|-\|\mathbf{w}\| \mid \leq\|\mathbf{z}-\mathbf{w}\| \tag{8.3.19}
\end{equation*}
$$

To see that 8.3.19 holds, note

$$
\|\mathbf{z}\|=\|\mathbf{z}-\mathbf{w}+\mathbf{w}\| \leq\|\mathbf{z}-\mathbf{w}\|+\|\mathbf{w}\|
$$

which implies

$$
\|\mathbf{z}\|-\|\mathbf{w}\| \leq\|\mathbf{z}-\mathbf{w}\|
$$

and now switching $\mathbf{z}$ and $\mathbf{w}$, yields

$$
\|\mathbf{w}\|-\|\mathbf{z}\| \leq\|\mathbf{z}-\mathbf{w}\|
$$

which implies 8.3.19.
Any normed vector space is a metric space, the distance given by

$$
d(\mathbf{x}, \mathbf{y}) \equiv\|\mathbf{x}-\mathbf{y}\|
$$

This satisfies all the axioms of a distance. Therefore, any normed linear space is a metric space with this metric and all the theory of metric spaces applies.

Definition 8.3.7 When $X$ is a normed linear space which is also complete, it is called a Banach space.

### 8.3.4 The $p$ Norms

Examples of norms are the $p$ norms on $\mathbb{C}^{n}$ for $p \neq 2$. These do not come from an inner product but they are norms just the same.

Definition 8.3.8 Let $\mathbf{x} \in \mathbb{C}^{n}$. Then define for $p \geq 1$,

$$
\|\mathbf{x}\|_{p} \equiv\left(\sum_{i=1}^{n}\left|x_{i}\right|^{p}\right)^{1 / p}
$$

The following inequality is called Holder's inequality.
Proposition 8.3.9 For $\mathbf{x}, \mathbf{y} \in \mathbb{C}^{n}$,

$$
\sum_{i=1}^{n}\left|x_{i}\right|\left|y_{i}\right| \leq\left(\sum_{i=1}^{n}\left|x_{i}\right|^{p}\right)^{1 / p}\left(\sum_{i=1}^{n}\left|y_{i}\right|^{p^{\prime}}\right)^{1 / p^{\prime}}
$$

The proof will depend on the following lemma shown later.
Lemma 8.3.10 If $a, b \geq 0$ and $p^{\prime}$ is defined by $\frac{1}{p}+\frac{1}{p^{\prime}}=1$, then

$$
a b \leq \frac{a^{p}}{p}+\frac{b^{p^{\prime}}}{p^{\prime}}
$$

Proof of the Proposition: If $\mathbf{x}$ or $\mathbf{y}$ equals the zero vector there is nothing to prove. Therefore, assume they are both nonzero. Let $A=\left(\sum_{i=1}^{n}\left|x_{i}\right|^{p}\right)^{1 / p}$ and $B=\left(\sum_{i=1}^{n}\left|y_{i}\right|^{p^{\prime}}\right)^{1 / p^{\prime}}$.

Then using Lemma 8.3.10,

$$
\begin{aligned}
\sum_{i=1}^{n} \frac{\left|x_{i}\right|}{A} \frac{\left|y_{i}\right|}{B} & \leq \sum_{i=1}^{n}\left[\frac{1}{p}\left(\frac{\left|x_{i}\right|}{A}\right)^{p}+\frac{1}{p^{\prime}}\left(\frac{\left|y_{i}\right|}{B}\right)^{p^{\prime}}\right] \\
& =\frac{1}{p} \frac{1}{A^{p}} \sum_{i=1}^{n}\left|x_{i}\right|^{p}+\frac{1}{p^{\prime}} \frac{1}{B^{p}} \sum_{i=1}^{n}\left|y_{i}\right|^{p^{\prime}} \\
& =\frac{1}{p}+\frac{1}{p^{\prime}}=1
\end{aligned}
$$

and so

$$
\sum_{i=1}^{n}\left|x_{i}\right|\left|y_{i}\right| \leq A B=\left(\sum_{i=1}^{n}\left|x_{i}\right|^{p}\right)^{1 / p}\left(\sum_{i=1}^{n}\left|y_{i}\right|^{p^{\prime}}\right)^{1 / p^{\prime}}
$$

Theorem 8.3.11 The p norms do indeed satisfy the axioms of a norm.

Proof: It is obvious that $\|\cdot\|_{p}$ does indeed satisfy most of the norm axioms. The only one that is not clear is the triangle inequality. To save notation write $\|\cdot\|$ in place of $\|\cdot\|_{p}$ in what follows. Note also that $\frac{p}{p^{\prime}}=p-1$. Then using the Holder inequality,

$$
\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|^{p} & =\sum_{i=1}^{n}\left|x_{i}+y_{i}\right|^{p} \\
& \leq \sum_{i=1}^{n}\left|x_{i}+y_{i}\right|^{p-1}\left|x_{i}\right|+\sum_{i=1}^{n}\left|x_{i}+y_{i}\right|^{p-1}\left|y_{i}\right| \\
& =\sum_{i=1}^{n}\left|x_{i}+y_{i}\right|^{\frac{p}{p^{\prime}}}\left|x_{i}\right|+\sum_{i=1}^{n}\left|x_{i}+y_{i}\right|^{\frac{p}{p^{\prime}}}\left|y_{i}\right| \\
& \leq\left(\sum_{i=1}^{n}\left|x_{i}+y_{i}\right|^{p}\right)^{1 / p^{\prime}}\left[\left(\sum_{i=1}^{n}\left|x_{i}\right|^{p}\right)^{1 / p}+\left(\sum_{i=1}^{n}\left|y_{i}\right|^{p}\right)^{1 / p}\right] \\
& =\|\mathbf{x}+\mathbf{y}\|^{p / p^{\prime}}\left(\|\mathbf{x}\|_{p}+\|\mathbf{y}\|_{p}\right)
\end{aligned}
$$

so dividing by $\|\mathbf{x}+\mathbf{y}\|^{p / p^{\prime}}$, it follows

$$
\|\mathbf{x}+\mathbf{y}\|^{p}\|\mathbf{x}+\mathbf{y}\|^{-p / p^{\prime}}=\|\mathbf{x}+\mathbf{y}\| \leq\|\mathbf{x}\|_{p}+\|\mathbf{y}\|_{p}
$$

$$
\left(p-\frac{p}{p^{\prime}}=p\left(1-\frac{1}{p^{\prime}}\right)=p \frac{1}{p}=1 .\right) .
$$

It only remains to prove Lemma 8.3.10.
Proof of the lemma: Let $p^{\prime}=q$ to save on notation and consider the following picture:


$$
a b \leq \int_{0}^{a} t^{p-1} d t+\int_{0}^{b} x^{q-1} d x=\frac{a^{p}}{p}+\frac{b^{q}}{q} .
$$

Note equality occurs when $a^{p}=b^{q}$.
Alternate proof of the lemma: First note that if either $a$ or $b$ are zero, then there is nothing to show so we can assume $b, a>0$. Let $b>0$ and let

$$
f(a)=\frac{a^{p}}{p}+\frac{b^{q}}{q}-a b
$$

Then the second derivative of $f$ is positive on $(0, \infty)$ so its graph is convex. Also $f(0)>0$ and $\lim _{a \rightarrow \infty} f(a)=\infty$. Then a short computation shows that there is only one critical point, where $f$ is minimized and this happens when $a$ is such that $a^{p}=b^{q}$. At this point,

$$
f(a)=b^{q}-b^{q / p} b=b^{q}-b^{q-1} b=0
$$

Therefore, $f(a) \geq 0$ for all $a$ and this proves the lemma.
Another example of a very useful norm on $\mathbb{F}^{n}$ is the norm $\|\cdot\|_{\infty}$ defined by

$$
\|\mathbf{x}\|_{\infty} \equiv \max \left\{\left|x_{k}\right|: k=1,2, \cdots, n\right\}
$$

You should verify that this satisfies all the axioms of a norm. Here is the triangle inequality.

$$
\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|_{\infty} & =\max _{k}\left\{\left|x_{k}+y_{k}\right|\right\} \leq \max _{k}\left\{\left|x_{k}\right|+\left|y_{k}\right|\right\} \\
& \leq \max _{k}\left\{\left|x_{k}\right|\right\}+\max _{k}\left\{\left|y_{k}\right|\right\}=\|\mathbf{x}\|_{\infty}+\|\mathbf{y}\|_{\infty}
\end{aligned}
$$

It turns out that in terms of analysis, it makes absolutely no difference which norm you use. This will be explained later. First is a short review of the notion of orthonormal bases which is not needed directly in what follows but is sufficiently important to include.

### 8.3.5 Orthonormal Bases

Not all bases for an inner product space $H$ are created equal. The best bases are orthonormal.

Definition 8.3.12 Suppose $\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{k}\right\}$ is a set of vectors in an inner product space $H$. It is an orthonormal set if

$$
\left(\mathbf{v}_{i}, \mathbf{v}_{j}\right)=\delta_{i j}=\left\{\begin{array}{l}
1 \text { if } i=j \\
0 \text { if } i \neq j
\end{array}\right.
$$

Every orthonormal set of vectors is automatically linearly independent.
Proposition 8.3.13 Suppose $\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{k}\right\}$ is an orthonormal set of vectors. Then it is linearly independent.

Proof: Suppose $\sum_{i=1}^{k} c_{i} \mathbf{v}_{i}=\mathbf{0}$. Then taking inner products with $\mathbf{v}_{j}$,

$$
0=\left(\mathbf{0}, \mathbf{v}_{j}\right)=\sum_{i} c_{i}\left(\mathbf{v}_{i}, \mathbf{v}_{j}\right)=\sum_{i} c_{i} \boldsymbol{\delta}_{i j}=c_{j}
$$

Since $j$ is arbitrary, this shows the set is linearly independent as claimed.
It turns out that if $X$ is any subspace of $H$, then there exists an orthonormal basis for $X$. The process by which this is done is called the Gram Schmidt process.

Lemma 8.3.14 Let $X$ be a subspace of dimension $n$ which is contained in an inner product space $H$. Let a basis for $X$ be $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{n}\right\}$. Then there exists an orthonormal basis for $X,\left\{\mathbf{u}_{1}, \cdots, \mathbf{u}_{n}\right\}$ which has the property that for each $k \leq n, \operatorname{span}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{k}\right)=$ $\operatorname{span}\left(\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}\right)$.

Proof: Let $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{n}\right\}$ be a basis for $X$. Let $\mathbf{u}_{1} \equiv \mathbf{x}_{1} /\left|\mathbf{x}_{1}\right|$. Thus for $k=1$, $\operatorname{span}\left(\mathbf{u}_{1}\right)=$ $\operatorname{span}\left(\mathbf{x}_{1}\right)$ and $\left\{\mathbf{u}_{1}\right\}$ is an orthonormal set. Now suppose for some $k<n, \mathbf{u}_{1}, \cdots, \mathbf{u}_{k}$ have been chosen such that $\left(\mathbf{u}_{j}, \mathbf{u}_{l}\right)=\delta_{j l}$ and $\operatorname{span}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{k}\right)=\operatorname{span}\left(\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}\right)$. Then define

$$
\begin{equation*}
\mathbf{u}_{k+1} \equiv \frac{\mathbf{x}_{k+1}-\sum_{j=1}^{k}\left(\mathbf{x}_{k+1}, \mathbf{u}_{j}\right) \mathbf{u}_{j}}{\left|\mathbf{x}_{k+1}-\sum_{j=1}^{k}\left(\mathbf{x}_{k+1}, \mathbf{u}_{j}\right) \mathbf{u}_{j}\right|}, \tag{8.3.20}
\end{equation*}
$$

where the denominator is not equal to zero because the $\mathbf{x}_{j}$ form a basis and so

$$
\mathbf{x}_{k+1} \notin \operatorname{span}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{k}\right)=\operatorname{span}\left(\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}\right)
$$

Thus by induction,

$$
\mathbf{u}_{k+1} \in \operatorname{span}\left(\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}, \mathbf{x}_{k+1}\right)=\operatorname{span}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{k}, \mathbf{x}_{k+1}\right) .
$$

Also, $\mathbf{x}_{k+1} \in \operatorname{span}\left(\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}, \mathbf{u}_{k+1}\right)$ which is seen easily by solving 8.3.20 for $\mathbf{x}_{k+1}$ and it follows

$$
\operatorname{span}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{k}, \mathbf{x}_{k+1}\right)=\operatorname{span}\left(\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}, \mathbf{u}_{k+1}\right)
$$

If $l \leq k$, then denoting by $C$ the scalar $\left|\mathbf{x}_{k+1}-\sum_{j=1}^{k}\left(\mathbf{x}_{k+1}, \mathbf{u}_{j}\right) \mathbf{u}_{j}\right|^{-1}$,

$$
\begin{aligned}
\left(\mathbf{u}_{k+1}, \mathbf{u}_{l}\right) & =C\left(\left(\mathbf{x}_{k+1}, \mathbf{u}_{l}\right)-\sum_{j=1}^{k}\left(\mathbf{x}_{k+1}, \mathbf{u}_{j}\right)\left(\mathbf{u}_{j}, \mathbf{u}_{l}\right)\right) \\
& =C\left(\left(\mathbf{x}_{k+1}, \mathbf{u}_{l}\right)-\sum_{j=1}^{k}\left(\mathbf{x}_{k+1}, \mathbf{u}_{j}\right) \delta_{l j}\right) \\
& =C\left(\left(\mathbf{x}_{k+1}, \mathbf{u}_{l}\right)-\left(\mathbf{x}_{k+1}, \mathbf{u}_{l}\right)\right)=0
\end{aligned}
$$

The vectors, $\left\{\mathbf{u}_{j}\right\}_{j=1}^{n}$, generated in this way are therefore an orthonormal basis because each vector has unit length.

The process by which these vectors were generated is called the Gram Schmidt process.

### 8.4 Equivalence Of Norms

As mentioned above, it makes absolutely no difference which norm you decide to use. This holds in general finite dimensional normed spaces. First are some simple lemmas featuring one dimensional considerations. In this case, the distance is given by $d(x, y)=|x-y|$ and so the open balls are sets of the form $(x-\boldsymbol{\delta}, x+\boldsymbol{\delta})$.

Also recall the Theorem 3.0.3 which is stated next for convenience.
Lemma 8.4.1 The closed interval $[a, b]$ is sequentially compact.
Corollary 8.4.2 The set $Q \equiv[a, b]+i[c, d] \subseteq \mathbb{C}$ is compact, meaning

$$
\{x+i y: x \in[a, b], y \in[c, d]\}
$$

Proof: Let $\left\{x_{n}+i y_{n}\right\}$ be a sequence in $Q$. Then there is a subsequence such that

$$
\lim _{k \rightarrow \infty} x_{n_{k}}=x \in[a, b] .
$$

There is a further subsequence such that $\lim _{l \rightarrow \infty} y_{n_{k_{l}}}=y \in[c, d]$. Thus, also

$$
\lim _{l \rightarrow \infty} x_{n_{k_{l}}}=x
$$

because subsequences of convergent sequences converge to the same point. Therefore, from the way we measure the distance in $\mathbb{C}$, it follows that $\lim _{l \rightarrow \infty}\left(x_{n_{k_{l}}}+y_{n_{k_{l}}}\right)=x+i y \in$ $Q$.

The next corollary gives the definition of a closed disk and shows that, like a closed interval, a closed disk is compact.

Corollary 8.4.3 In $\mathbb{C}$, let $D(z, r) \equiv\{w \in \mathbb{C}:|z-w| \leq r\}$. Then $D(z, r)$ is compact.
Proof: Note that

$$
D(z, r) \subseteq[\operatorname{Re} z-r, \operatorname{Re} z+r]+i[\operatorname{Im} z-r, \operatorname{Im} z+r]
$$

which was just shown to be compact. Also, if $w_{k} \rightarrow w$ where $w_{k} \in D(z, r)$, then by the triangle inequality,

$$
|z-w|=\lim _{k \rightarrow \infty}\left|z-w_{k}\right| \leq r
$$

and so $D(z, r)$ is a closed subset of a compact set. Hence it is compact by Proposition 7.6.8.
Recall that sequentially compact and compact are the same in any metric space which is the context of the assertions here.

Lemma 8.4.4 Let $K_{i}$ be a nonempty compact set in $\mathbb{F}$. Then $P \equiv \prod_{i=1}^{n} K_{i}$ is compact in $\mathbb{F}^{n}$.
Proof: Let $\left\{\mathbf{x}_{k}\right\}$ be a sequence in $P$. Taking a succession of subsequences as in the proof of Corollary 8.4.2, there exists a subsequence, still denoted as $\left\{\mathbf{x}_{k}\right\}$ such that if $x_{k}^{i}$ is the $i^{\text {th }}$ component of $\mathbf{x}_{k}$, then $\lim _{k \rightarrow \infty} x_{k}^{i}=x^{i} \in K_{i}$. Thus if $\mathbf{x}$ is the vector of $P$ whose $i^{\text {th }}$ component is $x^{i}$,

$$
\lim _{k \rightarrow \infty}\left|\mathbf{x}_{k}-\mathbf{x}\right| \equiv \lim _{k \rightarrow \infty}\left(\sum_{i=1}^{n}\left|x_{k}^{i}-x^{i}\right|^{2}\right)^{1 / 2}=0
$$

It follows that $P$ is sequentially compact, hence compact.
A set $K$ in $\mathbb{F}^{n}$ is said to be bounded if it is contained in some ball $B(\mathbf{0}, r)$.
Theorem 8.4.5 $A$ set $K \subseteq \mathbb{F}^{n}$ is compact if it is closed and bounded. If $f: K \rightarrow \mathbb{R}$, then $f$ achieves its maximum and its minimum on $K$.

Proof: Say $K$ is closed and bounded, being contained in $B(\mathbf{0}, r)$. Then if $\mathbf{x} \in K,\left|x_{i}\right|<r$ where $x_{i}$ is the $i^{t h}$ component. Hence $K \subseteq \prod_{i=1}^{n} D(0, r)$, a compact set by Lemma 8.4.4. By Proposition 7.6.8, since $K$ is a closed subset of a compact set, it is compact. The last claim is just the extreme value theorem, Theorem 7.7.1.

Definition 8.4.6 Let $\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{n}\right\}$ be a basis for $V$ where $(V,\|\cdot\|)$ is a finite dimensional normed vector space with field of scalars equal to either $\mathbb{R}$ or $\mathbb{C}$. Define $\theta: V \rightarrow \mathbb{F}^{n}$ as follows.

$$
\theta\left(\sum_{j=1}^{n} \alpha_{j} v_{j}\right) \equiv \alpha \equiv\left(\alpha_{1}, \cdots, \alpha_{n}\right)^{T}
$$

Thus $\theta$ maps a vector to its coordinates taken with respect to a given basis.
The following fundamental lemma comes from the extreme value theorem for continuous functions defined on a compact set. Let

$$
f(\alpha) \equiv\left\|\sum_{i} \alpha_{i} \mathbf{v}_{i}\right\| \equiv\left\|\theta^{-1} \alpha\right\|
$$

Then it is clear that $f$ is a continuous function defined on $\mathbb{F}^{n}$. This is because $\alpha \rightarrow \sum_{i} \alpha_{i} \mathbf{v}_{i}$ is a continuous map into $V$ and from the triangle inequality $\mathbf{x} \rightarrow\|\mathbf{x}\|$ is continuous as a map from $V$ to $\mathbb{R}$.

Lemma 8.4.7 There exists $\delta>0$ and $\Delta \geq \delta$ such that

$$
\delta=\min \{f(\alpha):|\alpha|=1\}, \Delta=\max \{f(\alpha):|\alpha|=1\}
$$

Also,

$$
\begin{align*}
\delta|\alpha| & \leq\left\|\theta^{-1} \alpha\right\| \leq \Delta|\alpha|  \tag{8.4.21}\\
\delta|\theta \mathbf{v}| & \leq\|\mathbf{v}\| \leq \Delta|\theta \mathbf{v}| \tag{8.4.22}
\end{align*}
$$

Proof: These numbers exist thanks to Theorem 8.4.5. It cannot be that $\delta=0$ because if it were, you would have $|\alpha|=1$ but $\sum_{j=1}^{n} \alpha_{k} \mathbf{v}_{j}=\mathbf{0}$ which is impossible since $\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{n}\right\}$ is linearly independent. The first of the above inequalities follows from

$$
\delta \leq\left\|\theta^{-1} \frac{\alpha}{|\alpha|}\right\|=f\left(\frac{\alpha}{|\alpha|}\right) \leq \Delta
$$

the second follows from observing that $\theta^{-1} \alpha$ is a generic vector $\mathbf{v}$ in $V$.
Note that these inequalities yield the fact that convergence of the coordinates with respect to a given basis is equivalent to convergence of the vectors. More precisely, to say that $\lim _{k \rightarrow \infty} \mathbf{v}^{k}=\mathbf{v}$ is the same as saying that $\lim _{k \rightarrow \infty} \theta \mathbf{v}^{k}=\theta \mathbf{v}$. Indeed,

$$
\delta\left|\theta \mathbf{v}_{n}-\theta \mathbf{v}\right| \leq\left\|\mathbf{v}_{n}-\mathbf{v}\right\| \leq \Delta\left|\theta \mathbf{v}_{n}-\theta \mathbf{v}\right|
$$

Now we can draw several conclusions about $(V,\|\cdot\|)$ for $V$ finite dimensional.
Theorem 8.4.8 Let $(V,\|\cdot\|)$ be a finite dimensional normed linear space. Then the compact sets are exactly those which are closed and bounded. Also $(V,\|\cdot\|)$ is complete. If $K$ is a closed and bounded set in $(V,\|\cdot\|)$ and $f: K \rightarrow \mathbb{R}$, then $f$ achieves its maximum and minimum on $K$.

Proof: First note that the inequalities 8.4 .21 and 8.4 .22 show that both $\theta^{-1}$ and $\theta$ are continuous. Thus these take convergent sequences to convergent sequences.

Let $\left\{\mathbf{w}_{k}\right\}_{k=1}^{\infty}$ be a Cauchy sequence. Then from 8.4.22, $\left\{\theta \mathbf{w}_{k}\right\}_{k=1}^{\infty}$ is a Cauchy sequence. Thanks to Theorem 8.4.5, it converges to some $\beta \in \mathbb{F}^{n}$. It follows that $\lim _{k \rightarrow \infty} \theta^{-1} \theta \mathbf{w}_{k}=$ $\lim _{k \rightarrow \infty} \mathbf{w}_{k}=\theta^{-1} \beta \in V$. This shows completeness.

Next let $K$ be a closed and bounded set. Let $\left\{\mathbf{w}_{k}\right\} \subseteq K$. Then $\left\{\theta \mathbf{w}_{k}\right\} \subseteq \theta K$ which is also a closed and bounded set thanks to the inequalities 8.4.21 and 8.4.22. Thus there is a subsequence still denoted with $k$ such that $\theta \mathbf{w}_{k} \rightarrow \beta \in \mathbb{F}^{n}$. Then as just done, $\mathbf{w}_{k} \rightarrow \theta^{-1} \beta$. Since $K$ is closed, it follows that $\theta^{-1} \beta \in K$.

This has just shown that a closed and bounded set in $V$ is sequentially compact hence compact.

Finally, why are the only compact sets those which are closed and bounded? Let $K$ be compact. If it is not bounded, then there is a sequence of points of $K,\left\{\mathbf{k}^{m}\right\}_{m=1}^{\infty}$ such that $\left\|\mathbf{k}^{m}\right\| \geq\left\|\mathbf{k}^{m-1}\right\|+1$. It follows that it cannot have a convergent subsequence because the points are further apart from each other than $1 / 2$. Indeed,

$$
\left\|\mathbf{k}^{m}-\mathbf{k}^{m+1}\right\| \geq\left\|\mathbf{k}^{m+1}\right\|-\left\|\mathbf{k}^{m}\right\| \geq 1>1 / 2
$$

Hence $K$ is not sequentially compact and consequently it is not compact. It follows that $K$ is bounded. If $K$ is not closed, then there exists a limit point $\mathbf{k}$ which is not in $K$. (Recall that closed means it has all its limit points.) By Theorem 7.2.8, there is a sequence of distinct points having no repeats and none equal to $\mathbf{k}$ denoted as $\left\{\mathbf{k}^{m}\right\}_{m=1}^{\infty}$ such that $\mathbf{k}^{m} \rightarrow \mathbf{k}$. Then this sequence $\left\{\mathbf{k}^{m}\right\}$ fails to have a subsequence which converges to a point of $K$. Hence $K$ is not sequentially compact. Thus, if $K$ is compact then it is closed and bounded.

The last part is the extreme value theorem, Theorem 7.7.1.
Next is the theorem which states that any two norms on a finite dimensional vector space are equivalent.

Theorem 8.4.9 Let $\|\cdot\|,\|\mid \cdot\| \|$ be two norms on $V$ a finite dimensional vector space. Then they are equivalent, which means there are constants $0<a<b$ such that for all $\mathbf{v}$,

$$
a\|\mathbf{v}\| \leq\||\mathbf{v}\|\mid \leq b\| \mathbf{v} \|
$$

Proof: In Lemma 8.4 .7 , let $\delta, \Delta$ go with $\|\cdot\|$ and $\hat{\delta}, \hat{\Delta}$ go with $\||\cdot|\|$. Then using the inequalities of this lemma,

$$
\|\mathbf{v}\| \leq \Delta|\theta \mathbf{v}| \leq \frac{\Delta}{\hat{\delta}}\left|\|\mathbf{v}\|\left\|\leq \frac{\Delta \hat{\Delta}}{\hat{\delta}}|\theta \mathbf{v}| \leq \frac{\Delta}{\delta} \frac{\hat{\Delta}}{\hat{\delta}}\right\| \mathbf{v} \|\right.
$$

and so

$$
\frac{\hat{\delta}}{\Delta}\|\mathbf{v}\| \leq\left\|\left|\mathbf{v}\left\|\left\lvert\, \leq \frac{\hat{\Delta}}{\delta}\right.\right\| \mathbf{v} \|\right.\right.
$$

Thus the norms are equivalent.
It follows right away that the closed and open sets are the same with two different norms. Also, all considerations involving limits are unchanged from one norm to another.

Corollary 8.4.10 Consider the metric spaces $\left(V,\|\cdot\|_{1}\right),\left(V,\|\cdot\|_{2}\right)$ where $V$ has dimension $n$. Then a set is closed or open in one of these if and only if it is respectively closed or open in the other. In other words, the two metric spaces have exactly the same open and closed sets. Also, a set is bounded in one metric space if and only if it is bounded in the other.

Proof: This follows from Theorem 7.6.5, the theorem about the equivalent formulations of continuity. Using this theorem, it follows from Theorem 8.4.9 that the identity map $I(\mathbf{x}) \equiv \mathbf{x}$ is continuous. The reason for this is that the inequality of this theorem implies that if $\left\|\mathbf{v}^{m}-\mathbf{v}\right\|_{1} \rightarrow 0$ then $\left\|I \mathbf{v}^{m}-I \mathbf{v}\right\|_{2}=\left\|I\left(\mathbf{v}^{m}-\mathbf{v}\right)\right\|_{2} \rightarrow 0$ and the same holds on switching 1 and 2 in what was just written.

Therefore, the identity map takes open sets to open sets and closed sets to closed sets. In other words, the two metric spaces have the same open sets and the same closed sets.

Suppose $S$ is bounded in $\left(V,\|\cdot\|_{1}\right)$. This means it is contained in $B(\mathbf{0}, r)_{1}$ where the subscript of 1 indicates the norm is $\|\cdot\|_{1}$. Let $\delta\|\cdot\|_{1} \leq\|\cdot\|_{2} \leq \Delta\|\cdot\|_{1}$ as described above. Then

$$
S \subseteq B(\mathbf{0}, r)_{1} \subseteq B(\mathbf{0}, \Delta r)_{2}
$$

so $S$ is also bounded in $\left(V,\|\cdot\|_{2}\right)$. Similarly, if $S$ is bounded in $\|\cdot\|_{2}$ then it is bounded in $\|\cdot\|_{1}$.

One can show that in the case of $\mathbb{R}$ where it makes sense to consider sup and inf, convergence of Cauchy sequences can be shown to imply the other definition of completeness involving sup, and inf.

### 8.5 Exercises

1. Let $K$ be a nonempty closed and convex set in an inner product space $(X,|\cdot|)$ which is complete. For example, $\mathbb{F}^{n}$ or any other finite dimensional inner product space. Let $y \notin K$ and let $\lambda=\inf \{|y-x|: x \in K\}$. Let $\left\{x_{n}\right\}$ be a minimizing sequence. That is
$\lambda=\lim _{n \rightarrow \infty}\left|y-x_{n}\right|$. Explain why such a minimizing sequence exists. Next explain the following using the parallelogram identity in the above problem as follows.

$$
\begin{gathered}
\left|y-\frac{x_{n}+x_{m}}{2}\right|^{2}=\left|\frac{y}{2}-\frac{x_{n}}{2}+\frac{y}{2}-\frac{x_{m}}{2}\right|^{2} \\
=-\left|\frac{y}{2}-\frac{x_{n}}{2}-\left(\frac{y}{2}-\frac{x_{m}}{2}\right)\right|^{2}+\frac{1}{2}\left|y-x_{n}\right|^{2}+\frac{1}{2}\left|y-x_{m}\right|^{2}
\end{gathered}
$$

Hence

$$
\begin{aligned}
\left|\frac{x_{m}-x_{n}}{2}\right|^{2} & =-\left|y-\frac{x_{n}+x_{m}}{2}\right|^{2}+\frac{1}{2}\left|y-x_{n}\right|^{2}+\frac{1}{2}\left|y-x_{m}\right|^{2} \\
& \leq-\lambda^{2}+\frac{1}{2}\left|y-x_{n}\right|^{2}+\frac{1}{2}\left|y-x_{m}\right|^{2}
\end{aligned}
$$

Next explain why the right hand side converges to 0 as $m, n \rightarrow \infty$. Thus $\left\{x_{n}\right\}$ is a Cauchy sequence and converges to some $x \in X$. Explain why $x \in K$ and $|x-y|=\lambda$. Thus there exists a closest point in $K$ to $y$. Next show that there is only one closest point. Hint: To do this, suppose there are two $x_{1}, x_{2}$ and consider $\frac{x_{1}+x_{2}}{2}$ using the parallelogram law to show that this average works better than either of the two points which is a contradiction unless they are really the same point. This theorem is of enormous significance.
2. Let $K$ be a closed convex nonempty set in a complete inner product space $(H,|\cdot|)$ (Hilbert space) and let $y \in H$. Denote the closest point to $y$ by $P x$. Show that $P x$ is characterized as being the solution to the following variational inequality

$$
\operatorname{Re}(z-P y, y-P y) \leq 0
$$

for all $z \in K$. That is, show that $x=P y$ if and only if $\operatorname{Re}(z-x, y-x) \leq 0$ for all $z \in K$. Hint: Let $x \in K$. Then, due to convexity, a generic thing in $K$ is of the form $x+t(z-x), t \in[0,1]$ for every $z \in K$. Then

$$
|x+t(z-x)-y|^{2}=|x-y|^{2}+t^{2}|z-x|^{2}-t 2 \operatorname{Re}(z-x, y-x)
$$

If $x=P x$, then the minimum value of this on the left occurs when $t=0$. Function defined on $[0,1]$ has its minimum at $t=0$. What does it say about the derivative of this function at $t=0$ ? Next consider the case that for some $x$ the inequality $\operatorname{Re}(z-x, y-x) \leq 0$. Explain why this shows $x=P y$.
3. Using Problem 2 and Problem 1 show the projection map, $P$ onto a closed convex subset is Lipschitz continuous with Lipschitz constant 1. That is $|P x-P y| \leq|x-y|$.

## Chapter 9

## Weierstrass Approximation Theorem

### 9.1 The Bernstein Polynomials

This short chapter is on the important Weierstrass approximation theorem. It is about approximating an arbitrary continuous function uniformly by a polynomial. It will be assumed only that $f$ has values in $\mathbb{C}$ and that all scalars are in $\mathbb{C}$. First here is some notation.

Definition 9.1.1 $\alpha=\left(\alpha_{1}, \cdots, \alpha_{n}\right)$ for $\alpha_{1} \cdots \alpha_{n}$ positive integers is called a multi-index. For $\alpha$ a multi-index, $|\alpha| \equiv \alpha_{1}+\cdots+\alpha_{n}$ and if $\mathbf{x} \in \mathbb{R}^{n}$,

$$
\mathbf{x}=\left(x_{1}, \cdots, x_{n}\right),
$$

and $f$ a function, define

$$
\mathbf{x}^{\alpha} \equiv x_{1}^{\alpha_{1}} x_{2}^{\alpha_{2}} \cdots x_{n}^{\alpha_{n}}
$$

A polynomial in $n$ variables of degree $m$ is a function of the form

$$
p(\mathbf{x})=\sum_{|\alpha| \leq m} a_{\alpha} \mathbf{x}^{\alpha}
$$

Here $\alpha$ is a multi-index as just described. You could have $a_{\alpha}$ have values in a normed linear space.

The following estimate will be the basis for the Weierstrass approximation theorem. It is actually a statement about the variance of a binomial random variable.

Lemma 9.1.2 The following estimate holds for $x \in[0,1]$.

$$
\sum_{k=0}^{m}\binom{m}{k}(k-m x)^{2} x^{k}(1-x)^{m-k} \leq \frac{1}{4} m
$$

Proof: By the Binomial theorem,

$$
\begin{equation*}
\sum_{k=0}^{m}\binom{m}{k}\left(e^{t} x\right)^{k}(1-x)^{m-k}=\left(1-x+e^{t} x\right)^{m} \tag{9.1.1}
\end{equation*}
$$

Differentiating both sides with respect to $t$ and then evaluating at $t=0$ yields

$$
\sum_{k=0}^{m}\binom{m}{k} k x^{k}(1-x)^{m-k}=m x
$$

Now doing two derivatives of 9.1.1 with respect to $t$ yields

$$
\begin{gathered}
\sum_{k=0}^{m}\binom{m}{k} k^{2}\left(e^{t} x\right)^{k}(1-x)^{m-k}=m(m-1)\left(1-x+e^{t} x\right)^{m-2} e^{2 t} x^{2} \\
+m\left(1-x+e^{t} x\right)^{m-1} x e^{t} .
\end{gathered}
$$

Evaluating this at $t=0$,

$$
\sum_{k=0}^{m}\binom{m}{k} k^{2}(x)^{k}(1-x)^{m-k}=m(m-1) x^{2}+m x
$$

Therefore,

$$
\begin{aligned}
\sum_{k=0}^{m}\binom{m}{k}(k-m x)^{2} x^{k}(1-x)^{m-k} & =m(m-1) x^{2}+m x-2 m^{2} x^{2}+m^{2} x^{2} \\
& =m\left(x-x^{2}\right) \leq \frac{1}{4} m
\end{aligned}
$$

This proves the lemma.
Now for $\mathbf{x}=\left(x_{1}, \cdots, x_{n}\right) \in[0,1]^{n}$ consider the polynomial,

$$
\begin{gather*}
\mathbf{p}_{m}(\mathbf{x}) \equiv \sum_{k_{1}=0}^{m} \cdots \sum_{k_{n}=0}^{m}\binom{m}{k_{1}}\binom{m}{k_{2}} \cdots\binom{m}{k_{n}} x_{1}^{k_{1}}\left(1-x_{1}\right)^{m-k_{1}} x_{2}^{k_{2}}\left(1-x_{2}\right)^{m-k_{2}} \\
\cdots x_{n}^{k_{n}}\left(1-x_{n}\right)^{m-k_{n}} \mathbf{f}\left(\frac{k_{1}}{m}, \cdots, \frac{k_{n}}{m}\right) \tag{9.1.2}
\end{gather*}
$$

where $\mathbf{f}$ is a continuous function which takes $[0,1]^{n}$ to a normed linear space. Also define if $I$ is a compact set in $\mathbb{R}^{n}$

$$
\|\mathbf{h}\|_{I} \equiv \sup \{\|\mathbf{h}(\mathbf{x})\|: \mathbf{x} \in I\}
$$

Thus $\mathbf{p}_{m}$ converges uniformly to $\mathbf{f}$ on a set $I$ if

$$
\lim _{m \rightarrow \infty}\left\|\mathbf{p}_{m}-\mathbf{f}\right\|_{I}=0
$$

Also to simplify the notation, let $\mathbf{k}=\left(k_{1}, \cdots, k_{n}\right)$ where each $k_{i} \in[0, m], \frac{\mathbf{k}}{\mathbf{m}} \equiv\left(\frac{k_{1}}{m}, \cdots, \frac{k_{n}}{m}\right)$, and let

$$
\binom{\mathbf{m}}{\mathbf{k}} \equiv\binom{m}{k_{1}}\binom{m}{k_{2}} \cdots\binom{m}{k_{n}} .
$$

Also define

$$
\begin{gathered}
\|\mathbf{k}\|_{\infty} \equiv \max \left\{k_{i}, i=1,2, \cdots, n\right\} \\
\mathbf{x}^{\mathbf{k}}(\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}} \equiv x_{1}^{k_{1}}\left(1-x_{1}\right)^{m-k_{1}} x_{2}^{k_{2}}\left(1-x_{2}\right)^{m-k_{2}} \cdots x_{n}^{k_{n}}\left(1-x_{n}\right)^{m-k_{n}}
\end{gathered}
$$

Thus in terms of this notation,

$$
\mathbf{p}_{m}(\mathbf{x})=\sum_{\|\mathbf{k}\|_{\infty} \leq m}\binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}}(\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}} \mathbf{f}\left(\frac{\mathbf{k}}{\mathbf{m}}\right)
$$

Lemma 9.1.3 For $\mathbf{x} \in[0,1]^{n}$, $\mathbf{f}$ a continuous function defined on $[0,1]^{n}$, and $\mathbf{p}_{m}$ given in 9.1.2, $\mathbf{p}_{m}$ converges uniformly to $\mathbf{f}$ on $[0,1]^{n}$ as $m \rightarrow \infty$.

Proof: The function, $\mathbf{f}$ is uniformly continuous because it is continuous on a compact set. Therefore, there exists $\delta>0$ such that if $\|\mathbf{x}-\mathbf{y}\|<\delta$, then

$$
\|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})\|<\varepsilon
$$

Denote by $G$ the set of $\mathbf{k}$ such that $\left(k_{i}-m x_{i}\right)^{2}<\eta^{2} m^{2}$ for each $i$ where $\eta=\delta / \sqrt{n}$. Note this condition is equivalent to saying that for each $i,\left|\frac{k_{i}}{m}-x_{i}\right|<\eta$. By the binomial theorem,

$$
\sum_{\|\mathbf{k}\|_{\infty} \leq m}\binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}}(\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}}=1
$$

and so for $\mathbf{x} \in[0,1]^{n}$,

$$
\begin{align*}
\| \mathbf{p}_{m}(\mathbf{x})- & \mathbf{f}(\mathbf{x})\left\|\leq \sum_{\|\mathbf{k}\|_{\infty} \leq m}\binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}}(\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}}\right\| \mathbf{f}\left(\frac{\mathbf{k}}{\mathbf{m}}\right)-\mathbf{f}(\mathbf{x}) \| \\
& \leq \sum_{\mathbf{k} \in G}\binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}}(\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}}\left\|\mathbf{f}\left(\frac{\mathbf{k}}{\mathbf{m}}\right)-\mathbf{f}(\mathbf{x})\right\| \\
& +\sum_{\mathbf{k} \in G^{C}}\binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}}(\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}}\left\|\mathbf{f}\left(\frac{\mathbf{k}}{\mathbf{m}}\right)-\mathbf{f}(\mathbf{x})\right\| \tag{9.1.3}
\end{align*}
$$

Now for $\mathbf{k} \in G$ it follows that for each $i$

$$
\begin{equation*}
\left|\frac{k_{i}}{m}-x_{i}\right|<\frac{\delta}{\sqrt{n}} \tag{9.1.4}
\end{equation*}
$$

and so $\left\|\mathbf{f}\left(\frac{\mathbf{k}}{\mathbf{m}}\right)-\mathbf{f}(\mathbf{x})\right\|<\varepsilon$ because the above implies $\left|\frac{\mathbf{k}}{\mathbf{m}}-\mathbf{x}\right|<\delta$. Therefore, the first sum on the right in 9.1.3 is no larger than

$$
\sum_{\mathbf{k} \in G}\binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}}(\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}} \varepsilon \leq \sum_{\|\mathbf{k}\|_{\infty} \leq m}\binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}}(\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}} \varepsilon=\varepsilon
$$

Letting $M \geq \max \left\{\|\mathbf{f}(\mathbf{x})\|: \mathbf{x} \in[0,1]^{n}\right\}$ it follows

$$
\begin{aligned}
& \left\|\mathbf{p}_{m}(\mathbf{x})-\mathbf{f}(\mathbf{x})\right\| \\
\leq & \varepsilon+2 M \sum_{\mathbf{k} \in G^{C}}\binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}}(\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}} \\
\leq & \varepsilon+2 M\left(\frac{1}{\eta^{2} m^{2}}\right)^{n} \sum_{\mathbf{k} \in G^{C}}\binom{\mathbf{m}}{\mathbf{k}} \prod_{j=1}^{n}\left(k_{j}-m x_{j}\right)^{2} \mathbf{x}^{\mathbf{k}}(\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}} \\
\leq & \varepsilon+2 M\left(\frac{1}{\eta^{2} m^{2}}\right)^{n} \sum_{\|\mathbf{k}\|_{\infty} \leq m}\binom{\mathbf{m}}{\mathbf{k}} \prod_{j=1}^{n}\left(k_{j}-m x_{j}\right)^{2} \mathbf{x}^{\mathbf{k}}(\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}}
\end{aligned}
$$

because on $G^{C}$,

$$
\frac{\left(k_{j}-m x_{j}\right)^{2}}{\eta^{2} m^{2}}<1, j=1, \cdots, n
$$

Now by Lemma 9.1.2,

$$
\left\|\mathbf{p}_{m}(\mathbf{x})-\mathbf{f}(\mathbf{x})\right\| \leq \varepsilon+2 M\left(\frac{1}{\eta^{2} m^{2}}\right)^{n}\left(\frac{m}{4}\right)^{n}
$$

Therefore, since the right side does not depend on $\mathbf{x}$, it follows

$$
\lim \sup _{m \rightarrow \infty}\left\|\mathbf{p}_{m}-\mathbf{f}\right\|_{[0,1]^{n}} \leq \varepsilon
$$

and since $\varepsilon$ is arbitrary, this shows $\lim _{m \rightarrow \infty}\left\|\mathbf{p}_{m}-\mathbf{f}\right\|_{[0,1]^{n}}=0$. This proves the lemma.
The following is not surprising.
Lemma 9.1.4 Let $\mathbf{f}$ be a continuous function defined on $[-M, M]^{n}$ having values in a normed linear space. Then there exists a sequence of polynomials, $\left\{\mathbf{p}_{m}\right\}$ converging uniformly to $\mathbf{f}$ on $[-M, M]^{n}$.

Proof: Let $h(t)=-M+2 M t$ so $h:[0,1] \rightarrow[-M, M]$ and let $\mathbf{h}(\mathbf{t}) \equiv\left(h\left(t_{1}\right), \cdots, h\left(t_{n}\right)\right)$. Therefore, $\mathbf{f} \circ \mathbf{h}$ is a continuous function defined on $[0,1]^{n}$. From Lemma 9.1.3 there exists a polynomial, $\mathbf{p}(\mathbf{t})$ such that $\left\|\mathbf{p}_{m}-\mathbf{f} \circ \mathbf{h}\right\|_{[0,1]^{n}}<\frac{1}{m}$. Now for $\mathbf{x} \in[-M, M]^{n}, \mathbf{h}^{-1}(\mathbf{x})=$ $\left(h^{-1}\left(x_{1}\right), \cdots, h^{-1}\left(x_{n}\right)\right)$ and so

$$
\left\|\mathbf{p}_{m} \circ \mathbf{h}^{-1}-\mathbf{f}\right\|_{[-M, M]^{n}}=\left\|\mathbf{p}_{m}-\mathbf{f} \circ \mathbf{h}\right\|_{[0,1]^{n}}<\frac{1}{m}
$$

But $h^{-1}(x)=\frac{x}{2 M}+\frac{1}{2}$ and so $\mathbf{p}_{m}$ is still a polynomial. This proves the lemma.
A similar argument proves the following corollary.
Corollary 9.1.5 Let $\mathbf{f}$ be a continuous function defined on $\prod_{i=1}^{n}\left[a_{i}, b_{i}\right]$ having values in a normed linear space. Then there exists a sequence of polynomials, $\left\{\mathbf{p}_{m}\right\}$ converging uniformly to $\mathbf{f}$ on $\prod_{i=1}^{n}\left[a_{i}, b_{i}\right]$.

Proof: You just let $h_{i}(t)$ map $[0,1]$ one to one and onto $\left[a_{i}, b_{i}\right]$ such that $h_{i}^{-1}(x)$ is a polynomial. Then apply the same argument.

The classical version of the Weierstrass approximation theorem involved showing that a continuous function of one variable defined on a closed and bounded interval is the uniform limit of a sequence of polynomials. This is certainly included as a special case of the above. Now recall the Tietze extension theorem found on Page 158. In the general version about to be presented, the set on which $\mathbf{f}$ is defined is just a compact subset of $\mathbb{R}^{n}$, not the Cartesian product of intervals. For convenience here is the Tietze extension theorem.

Theorem 9.1.6 Let $M$ be a closed nonempty subset of a metric space $(X, d)$ and let $f$ : $M \rightarrow[a, b]$ is continuous at every point of $M$. Then there exists a function, $g$ continuous on all of $X$ which coincides with $f$ on $M$ such that $g(X) \subseteq[a, b]$.

The Weierstrass approximation theorem follows. Here it is assumed the function $\mathbf{f}$ has values in $\mathbb{R}^{p}$. In more general situations, we would need an extension theorem to extend the function off a closed set. There are such theorems, but they have not been presented at this point.

Theorem 9.1.7 Let $K$ be a compact set in $\mathbb{R}^{n}$ and let $\mathbf{f}$ be a continuous function defined on $K$ having values in $\mathbb{R}^{p}$. Then there exists a sequence of polynomials $\left\{\mathbf{p}_{m}\right\}$ converging uniformly to $\mathbf{f}$ on $K$.

Proof: Choose $M$ large enough that $K \subseteq[-M, M]^{n}$ and let $\tilde{f}$ denote a continuous function defined on all of $[-M, M]^{n}$ such that $\tilde{\mathbf{f}}=\mathbf{f}$ on $K$. Such an extension exists by the Tietze extension theorem, Theorem 9.1.6 applied to the components of $\mathbf{f}$. By Lemma 9.1.4 there exists a sequence of polynomials, $\left\{\mathbf{p}_{m}\right\}$ defined on $[-M, M]^{n}$ such that $\left\|\tilde{\mathbf{f}}-\mathbf{p}_{m}\right\|_{[-M, M]^{n}} \rightarrow$ 0 . Therefore, $\left\|\tilde{\mathbf{f}}-\mathbf{p}_{m}\right\|_{K} \rightarrow 0$ also. This proves the theorem.

### 9.2 Stone Weierstrass Theorem

### 9.2.1 The Case Of Compact Sets

There is a profound generalization of the Weierstrass approximation theorem due to Stone.
Definition 9.2.1 $\mathscr{A}$ is an algebra of functions if $\mathscr{A}$ is a vector space and if whenever $f, g \in \mathscr{A}$ then $f g \in \mathscr{A}$.

To begin with assume that the field of scalars is $\mathbb{R}$. This will be generalized later. Theorem 9.1.7 implies the following very special case.

Corollary 9.2.2 The polynomials are dense in $C([a, b])$.
The next result is the key to the profound generalization of the Weierstrass theorem due to Stone in which an interval will be replaced by a compact or locally compact set and polynomials will be replaced with elements of an algebra satisfying certain axioms.

Corollary 9.2.3 On the interval $[-M, M]$, there exist polynomials $p_{n}$ such that

$$
p_{n}(0)=0
$$

and

$$
\lim _{n \rightarrow \infty}\left\|p_{n}-|\cdot|\right\|_{\infty}=0
$$

Proof: By Corollary 9.2 .2 there exists a sequence of polynomials, $\left\{\tilde{p}_{n}\right\}$ such that $\tilde{p}_{n} \rightarrow$ $|\cdot|$ uniformly. Then let $p_{n}(t) \equiv \tilde{p}_{n}(t)-\tilde{p}_{n}(0)$. This proves the corollary.

Definition 9.2.4 An algebra of functions, $\mathscr{A}$ defined on $A$, annihilates no point of $A$ if for all $x \in A$, there exists $g \in \mathscr{A}$ such that $g(x) \neq 0$. The algebra separates points if whenever $x_{1} \neq x_{2}$, then there exists $g \in \mathscr{A}$ such that $g\left(x_{1}\right) \neq g\left(x_{2}\right)$.

The following generalization is known as the Stone Weierstrass approximation theorem.
Theorem 9.2.5 Let $A$ be a compact topological space and let $\mathscr{A} \subseteq C(A ; \mathbb{R})$ be an algebra of functions which separates points and annihilates no point. Then $\mathscr{A}$ is dense in $C(A ; \mathbb{R})$.

Proof: First here is a lemma.

Lemma 9.2.6 Let $c_{1}$ and $c_{2}$ be two real numbers and let $x_{1} \neq x_{2}$ be two points of $A$. Then there exists a function $f_{x_{1} x_{2}}$ such that

$$
f_{x_{1} x_{2}}\left(x_{1}\right)=c_{1}, f_{x_{1} x_{2}}\left(x_{2}\right)=c_{2}
$$

Proof of the lemma: Let $g \in \mathscr{A}$ satisfy

$$
g\left(x_{1}\right) \neq g\left(x_{2}\right) .
$$

Such a $g$ exists because the algebra separates points. Since the algebra annihilates no point, there exist functions $h$ and $k$ such that

$$
h\left(x_{1}\right) \neq 0, k\left(x_{2}\right) \neq 0
$$

Then let

$$
u \equiv g h-g\left(x_{2}\right) h, v \equiv g k-g\left(x_{1}\right) k
$$

It follows that $u\left(x_{1}\right) \neq 0$ and $u\left(x_{2}\right)=0$ while $v\left(x_{2}\right) \neq 0$ and $v\left(x_{1}\right)=0$. Let

$$
f_{x_{1} x_{2}} \equiv \frac{c_{1} u}{u\left(x_{1}\right)}+\frac{c_{2} v}{v\left(x_{2}\right)}
$$

This proves the lemma. Now continue the proof of Theorem 9.2.5.
First note that $\overline{\mathscr{A}}$ satisfies the same axioms as $\mathscr{A}$ but in addition to these axioms, $\overline{\mathscr{A}}$ is closed. The closure of $\mathscr{A}$ is taken with respect to the usual norm on $C(A)$,

$$
\|f\|_{\infty} \equiv \max \{|f(x)|: x \in A\}
$$

Suppose $f \in \overline{\mathscr{A}}$ and suppose $M$ is large enough that

$$
\|f\|_{\infty}<M
$$

Using Corollary 9.2.3, let $p_{n}$ be a sequence of polynomials such that

$$
\left\|p_{n}-|\cdot|\right\|_{\infty} \rightarrow 0, p_{n}(0)=0
$$

It follows that $p_{n} \circ f \in \overline{\mathscr{A}}$ and so $|f| \in \overline{\mathscr{A}}$ whenever $f \in \overline{\mathscr{A}}$. Also note that

$$
\begin{aligned}
& \max (f, g)=\frac{|f-g|+(f+g)}{2} \\
& \min (f, g)=\frac{(f+g)-|f-g|}{2}
\end{aligned}
$$

Therefore, this shows that if $f, g \in \overline{\mathscr{A}}$ then

$$
\max (f, g), \min (f, g) \in \overline{\mathscr{A}}
$$

By induction, if $f_{i}, i=1,2, \cdots, m$ are in $\overline{\mathscr{A}}$ then

$$
\max \left(f_{i}, i=1,2, \cdots, m\right), \min \left(f_{i}, i=1,2, \cdots, m\right) \in \overline{\mathscr{A}}
$$

Now let $h \in C(A ; \mathbb{R})$ and let $x \in A$. Use Lemma 9.2.6 to obtain $f_{x y}$, a function of $\overline{\mathscr{A}}$ which agrees with $h$ at $x$ and $y$. Letting $\varepsilon>0$, there exists an open set $U(y)$ containing $y$ such that

$$
f_{x y}(z)>h(z)-\varepsilon \text { if } z \in U(y)
$$

Since $A$ is compact, let $U\left(y_{1}\right), \cdots, U\left(y_{l}\right)$ cover $A$. Let

$$
f_{x} \equiv \max \left(f_{x y_{1}}, f_{x y_{2}}, \cdots, f_{x y_{l}}\right)
$$

Then $f_{x} \in \overline{\mathscr{A}}$ and

$$
f_{x}(z)>h(z)-\varepsilon
$$

for all $z \in A$ and $f_{x}(x)=h(x)$. This implies that for each $x \in A$ there exists an open set $V(x)$ containing $x$ such that for $z \in V(x)$,

$$
f_{x}(z)<h(z)+\varepsilon
$$

Let $V\left(x_{1}\right), \cdots, V\left(x_{m}\right)$ cover $A$ and let

$$
f \equiv \min \left(f_{x_{1}}, \cdots, f_{x_{m}}\right)
$$

Therefore,

$$
f(z)<h(z)+\varepsilon
$$

for all $z \in A$ and since $f_{x}(z)>h(z)-\varepsilon$ for all $z \in A$, it follows

$$
f(z)>h(z)-\varepsilon
$$

also and so

$$
|f(z)-h(z)|<\varepsilon
$$

for all $z$. Since $\varepsilon$ is arbitrary, this shows $h \in \overline{\mathscr{A}}$ and proves $\overline{\mathscr{A}}=C(A ; \mathbb{R})$. This proves the theorem.

### 9.2.2 The Case Of Locally Compact Sets

Definition 9.2.7 Let $(X, \tau)$ be a locally compact Hausdorff space. $C_{0}(X)$ denotes the space of real or complex valued continuous functions defined on $X$ with the property that if $f \in$ $C_{0}(X)$, then for each $\varepsilon>0$ there exists a compact set $K$ such that $|f(x)|<\varepsilon$ for all $x \notin K$. Define

$$
\|f\|_{\infty}=\sup \{|f(x)|: x \in X\}
$$

Lemma 9.2.8 For $(X, \tau)$ a locally compact Hausdorff space with the above norm, $C_{0}(X)$ is a complete space.

Proof: Let $(\widetilde{X}, \tilde{\tau})$ be the one point compactification described in Lemma 7.12.20.

$$
D \equiv\{f \in C(\widetilde{X}): f(\infty)=0\}
$$

Then $D$ is a closed subspace of $C(\widetilde{X})$. For $f \in C_{0}(X)$,

$$
\widetilde{f}(x) \equiv\left\{\begin{array}{l}
f(x) \text { if } x \in X \\
0 \text { if } x=\infty
\end{array}\right.
$$

and let $\theta: C_{0}(X) \rightarrow D$ be given by $\theta f=\widetilde{f}$. Then $\theta$ is one to one and onto and also satisfies $\|f\|_{\infty}=\|\theta f\|_{\infty}$. Now $D$ is complete because it is a closed subspace of a complete space and so $C_{0}(X)$ with $\|\cdot\|_{\infty}$ is also complete. This proves the lemma.

The above refers to functions which have values in $\mathbb{C}$ but the same proof works for functions which have values in any complete normed linear space.

In the case where the functions in $C_{0}(X)$ all have real values, I will denote the resulting space by $C_{0}(X ; \mathbb{R})$ with similar meanings in other cases.

With this lemma, the generalization of the Stone Weierstrass theorem to locally compact sets is as follows.

Theorem 9.2.9 Let $\mathscr{A}$ be an algebra of functions in $C_{0}(X ; \mathbb{R})$ where $(X, \tau)$ is a locally compact Hausdorff space which separates the points and annihilates no point. Then $\mathscr{A}$ is dense in $C_{0}(X ; \mathbb{R})$.

Proof: Let $(\widetilde{X}, \tilde{\tau})$ be the one point compactification as described in Lemma 7.12.20. Let $\widetilde{\mathscr{A}}$ denote all finite linear combinations of the form

$$
\left\{\sum_{i=1}^{n} c_{i} \widetilde{f}_{i}+c_{0}: f \in \mathscr{A}, c_{i} \in \mathbb{R}\right\}
$$

where for $f \in C_{0}(X ; \mathbb{R})$,

$$
\widetilde{f}(x) \equiv\left\{\begin{array}{l}
f(x) \text { if } x \in X \\
0 \text { if } x=\infty
\end{array}\right.
$$

Then $\widetilde{\mathscr{A}}$ is obviously an algebra of functions in $C(\widetilde{X} ; \mathbb{R})$. It separates points because this is true of $\mathscr{A}$. Similarly, it annihilates no point because of the inclusion of $c_{0}$ an arbitrary element of $\mathbb{R}$ in the definition above. Therefore from Theorem $9.2 .5, \widetilde{\mathscr{A}}$ is dense in $C(\widetilde{X} ; \mathbb{R})$. Letting $f \in C_{0}(X ; \mathbb{R})$, it follows $\widetilde{f} \in C(\widetilde{X} ; \mathbb{R})$ and so there exists a sequence $\left\{h_{n}\right\} \subseteq \widetilde{\mathscr{A}}$ such that $h_{n}$ converges uniformly to $\widetilde{f}$. Now $h_{n}$ is of the form $\sum_{i=1}^{n} c_{i}^{n} \widetilde{f_{i}^{n}}+c_{0}^{n}$ and since $\widetilde{f}(\infty)=0$, you can take each $c_{0}^{n}=0$ and so this has shown the existence of a sequence of functions in $\mathscr{A}$ such that it converges uniformly to $f$. This proves the theorem.

### 9.2.3 The Case Of Complex Valued Functions

What about the general case where $C_{0}(X)$ consists of complex valued functions and the field of scalars is $\mathbb{C}$ rather than $\mathbb{R}$ ? The following is the version of the Stone Weierstrass theorem which applies to this case. You have to assume that for $f \in \mathscr{A}$ it follows $\bar{f} \in \mathscr{A}$. Such an algebra is called self adjoint.

Theorem 9.2.10 Suppose $\mathscr{A}$ is an algebra of functions in $C_{0}(X)$, where $X$ is a locally compact Hausdorff space, which separates the points, annihilates no point, and has the property that if $f \in \mathscr{A}$, then $\bar{f} \in \mathscr{A}$. Then $\mathscr{A}$ is dense in $C_{0}(X)$.

Proof: Let $\operatorname{Re} \mathscr{A} \equiv\{\operatorname{Re} f: f \in \mathscr{A}\}, \operatorname{Im} \mathscr{A} \equiv\{\operatorname{Im} f: f \in \mathscr{A}\}$. First I will show that $\mathscr{A}=\operatorname{Re} \mathscr{A}+i \operatorname{Im} \mathscr{A}=\operatorname{Im} \mathscr{A}+i \operatorname{Re} \mathscr{A}$. Let $f \in \mathscr{A}$. Then

$$
f=\frac{1}{2}(f+\bar{f})+\frac{1}{2}(f-\bar{f})=\operatorname{Re} f+i \operatorname{Im} f \in \operatorname{Re} \mathscr{A}+i \operatorname{Im} \mathscr{A}
$$

and so $\mathscr{A} \subseteq \operatorname{Re} \mathscr{A}+i \operatorname{Im} \mathscr{A}$. Also

$$
f=\frac{1}{2 i}(i f+i \bar{f})-\frac{i}{2}(i f+\overline{(i f)})=\operatorname{Im}(i f)+i \operatorname{Re}(i f) \in \operatorname{Im} \mathscr{A}+i \operatorname{Re} \mathscr{A}
$$

This proves one half of the desired equality. Now suppose $h \in \operatorname{Re} \mathscr{A}+i \operatorname{Im} \mathscr{A}$. Then $h=\operatorname{Re} g_{1}+i \operatorname{Im} g_{2}$ where $g_{i} \in \mathscr{A}$. Then since $\operatorname{Re} g_{1}=\frac{1}{2}\left(g_{1}+\overline{g_{1}}\right)$, it follows $\operatorname{Re} g_{1} \in \mathscr{A}$. Similarly $\operatorname{Im} g_{2} \in \mathscr{A}$. Therefore, $h \in \mathscr{A}$. The case where $h \in \operatorname{Im} \mathscr{A}+i \operatorname{Re} \mathscr{A}$ is similar. This establishes the desired equality.

Now $\operatorname{Re} \mathscr{A}$ and $\operatorname{Im} \mathscr{A}$ are both real algebras. I will show this now. First consider $\operatorname{Im} \mathscr{A}$. It is obvious this is a real vector space. It only remains to verify that the product of two functions in $\operatorname{Im} \mathscr{A}$ is in $\operatorname{Im} \mathscr{A}$. Note that from the first part, $\operatorname{Re} \mathscr{A}, \operatorname{Im} \mathscr{A}$ are both subsets of $\mathscr{A}$ because, for example, if $u \in \operatorname{Im} \mathscr{A}$ then $u+0 \in \operatorname{Im} \mathscr{A}+i \operatorname{Re} \mathscr{A}=\mathscr{A}$. Therefore, if $v, w \in \operatorname{Im} \mathscr{A}$, both $i v$ and $w$ are in $\mathscr{A}$ and so $\operatorname{Im}(i v w)=v w$ and $i v w \in \mathscr{A}$. Similarly, $\operatorname{Re} \mathscr{A}$ is an algebra.

Both $\operatorname{Re} \mathscr{A}$ and $\operatorname{Im} \mathscr{A}$ must separate the points. Here is why: If $x_{1} \neq x_{2}$, then there exists $f \in \mathscr{A}$ such that $f\left(x_{1}\right) \neq f\left(x_{2}\right) . \operatorname{If} \operatorname{Im} f\left(x_{1}\right) \neq \operatorname{Im} f\left(x_{2}\right)$, this shows there is a function in $\operatorname{Im} \mathscr{A}, \operatorname{Im} f$ which separates these two points. If $\operatorname{Im} f$ fails to separate the two points, then $\operatorname{Re} f$ must separate the points and so you could consider $\operatorname{Im}(i f)$ to get a function in $\operatorname{Im} \mathscr{A}$ which separates these points. This shows $\operatorname{Im} \mathscr{A}$ separates the points. Similarly $\operatorname{Re} \mathscr{A}$ separates the points.

Neither $\operatorname{Re} \mathscr{A}$ nor $\operatorname{Im} \mathscr{A}$ annihilate any point. This is easy to see because if $x$ is a point there exists $f \in \mathscr{A}$ such that $f(x) \neq 0$. Thus either $\operatorname{Re} f(x) \neq 0$ or $\operatorname{Im} f(x) \neq 0$. If $\operatorname{Im} f(x) \neq 0$, this shows this point is not annihilated by $\operatorname{Im} \mathscr{A}$. If $\operatorname{Im} f(x)=0$, consider $\operatorname{Im}(i f)(x)=\operatorname{Re} f(x) \neq 0$. Similarly, $\operatorname{Re} \mathscr{A}$ does not annihilate any point.

It follows from Theorem 9.2.9 that $\operatorname{Re} \mathscr{A}$ and $\operatorname{Im} \mathscr{A}$ are dense in the real valued functions of $C_{0}(X)$. Let $f \in C_{0}(X)$. Then there exists $\left\{h_{n}\right\} \subseteq \operatorname{Re} \mathscr{A}$ and $\left\{g_{n}\right\} \subseteq \operatorname{Im} \mathscr{A}$ such that $h_{n} \rightarrow \operatorname{Re} f$ uniformly and $g_{n} \rightarrow \operatorname{Im} f$ uniformly. Therefore, $h_{n}+i g_{n} \in \mathscr{A}$ and it converges to $f$ uniformly. This proves the theorem.

### 9.3 The Holder Spaces

We consider these spaces as spaces of functions defined on an interval $[0,1]$ although one could have $[0, T]$ just as easily. A slightly more general version is in the exercises. They are a very interesting example of spaces which are not separable.

Definition 9.3.1 Let $p>1$. Then $f \in C^{1 / p}([0,1])$ means that $f \in C([0,1])$ and also

$$
\rho_{p}(f) \equiv \sup \left\{\frac{|f(x)-f(y)|}{|x-y|^{1 / p}}: x, y \in X, x \neq y\right\}<\infty
$$

Then the norm is defined as $\|f\|_{C([0,1])}+\rho_{p}(f) \equiv\|f\|_{1 / p}$.
It is an exercise to verify that $C^{1 / p}([0,1])$ is a complete normed linear space.
Let $p>1$. Then $C^{1 / p}([0,1])$ is not separable. Define uncountably many functions, one for each $\varepsilon$ where $\varepsilon$ is a sequence of -1 and 1 . Thus $\varepsilon_{k} \in\{-1,1\}$. Thus $\varepsilon \neq \varepsilon^{\prime}$ if the two sequences differ in at least one slot, one giving 1 and the other equaling -1 . Now define

$$
f_{\varepsilon}(t) \equiv \sum_{k=1}^{\infty} \varepsilon_{k} 2^{-k / p} \sin \left(2^{k} \pi t\right)
$$

Then this is $1 / p$ Holder. Let $s<t$.

$$
\begin{aligned}
\mid f_{\varepsilon}(t)- & f_{\varepsilon}(s)\left|\leq \sum_{k \leq\left|\log _{2}(t-s)\right|}\right| 2^{-k / p} \sin \left(2^{k} \pi t\right)-2^{-k / p} \sin \left(2^{k} \pi s\right) \mid \\
& +\sum_{k>\left|\log _{2}(t-s)\right|}\left|2^{-k / p} \sin \left(2^{k} \pi t\right)-2^{-k / p} \sin \left(2^{k} \pi s\right)\right|
\end{aligned}
$$

If $t=1$ and $s=0$, there is really nothing to show because then the difference equals 0 . There is also nothing to show if $t=s$. From now on, $0<t-s<1$. Let $k_{0}$ be the largest integer which is less than or equal to $\left|\log _{2}(t-s)\right|=-\log _{2}(t-s)$. Note that $-\log (t-s)>0$ because $0<t-s<1$. Then

$$
\begin{aligned}
\left|f_{\varepsilon}(t)-f_{\varepsilon}(s)\right| \leq & \sum_{k \leq k_{0}}\left|2^{-k / p} \sin \left(2^{k} \pi t\right)-2^{-k / p} \sin \left(2^{k} \pi s\right)\right| \\
& +\sum_{k>k_{0}}\left|2^{-k / p} \sin \left(2^{k} \pi t\right)-2^{-k / p} \sin \left(2^{k} \pi s\right)\right| \\
\leq & \sum_{k \leq k_{0}} 2^{-k / p} 2^{k} \pi|t-s|+\sum_{k>k_{0}} 2^{-k / p} 2
\end{aligned}
$$

Now $k_{0} \leq-\log _{2}(t-s)<k_{0}+1$ and so $-k_{0} \geq \log _{2}(t-s) \geq-\left(k_{0}+1\right)$. Hence $2^{-k_{0}} \geq$ $|t-s| \geq 2^{-k_{0}} 2^{-1}$ and so $2^{-k_{0} / p} \geq|t-s|^{1 / p} \geq 2^{-k_{0} / p} 2^{-1 / p}$. Using this in the sums,

$$
\begin{aligned}
& \left|f_{\varepsilon}(t)-f_{\varepsilon}(s)\right| \leq|t-s| C_{p}+\sum_{k>k_{0}} 2^{-k / p} 2^{k_{0} / p} 2^{-k_{0} / p} 2 \\
& \quad \leq|t-s| C_{p}+\sum_{k>k_{0}} 2^{-k / p} 2^{k_{0} / p}\left(2^{1 / p}|t-s|^{1 / p}\right) 2
\end{aligned}
$$

$$
\begin{aligned}
& \leq|t-s| C_{p}+\sum_{k>k_{0}} 2^{-\left(k-k_{0}\right) / p}\left(2^{1 / p}|t-s|^{1 / p}\right) 2 \\
& \leq C_{p}|t-s|+\left(2^{1+1 / p}\right) \sum_{k=1}^{\infty} 2^{-k / p}|t-s|^{1 / p} \\
& =C_{p}|t-s|+D_{p}|t-s|^{1 / p} \leq C_{p}|t-s|^{1 / p}+D_{p}|t-s|^{1 / p}
\end{aligned}
$$

Thus $f_{\varepsilon}$ is indeed $1 / p$ Holder continuous.
Now consider $\varepsilon \neq \varepsilon^{\prime}$. Suppose the first discrepancy in the two sequences occurs with $\varepsilon_{j}$. Thus one is 1 and the other is -1 . Let $t=\frac{i+1}{2^{j+1}}, s=\frac{i}{2^{j+1}}$

$$
\begin{gathered}
\left|f_{\mathcal{E}}(t)-f_{\mathcal{E}}(s)-\left(f_{\varepsilon^{\prime}}(t)-f_{\varepsilon^{\prime}}(s)\right)\right|= \\
\left|\begin{array}{c}
\sum_{k=j}^{\infty} \varepsilon_{k} 2^{-k / p} \sin \left(2^{k} \pi t\right)-\sum_{k=j}^{\infty} \varepsilon_{k} 2^{-k / p} \sin \left(2^{k} \pi s\right) \\
-\left(\sum_{k=j}^{\infty} \varepsilon_{k}^{\prime} 2^{-k / p} \sin \left(2^{k} \pi t\right)-\sum_{k=j}^{\infty} \varepsilon_{k}^{\prime} 2^{-k / p} \sin \left(2^{k} \pi s\right)\right)
\end{array}\right|
\end{gathered}
$$

Now consider what happens for $k>j$. Then $\sin \left(2^{k} \pi \frac{i}{2^{j+1}}\right)=\sin (m \pi)=0$ for some integer $m$. Thus the whole mess reduces to

$$
\begin{aligned}
& \left|\left(\varepsilon_{j}-\varepsilon_{j}^{\prime}\right) 2^{-j / p} \sin \left(\frac{2^{j} \pi(i+1)}{2^{j+1}}\right)-\left(\varepsilon_{j}-\varepsilon_{j}^{\prime}\right) 2^{-j / p} \sin \left(\frac{2^{j} \pi i}{2^{j+1}}\right)\right| \\
= & \left|\left(\varepsilon_{j}-\varepsilon_{j}^{\prime}\right) 2^{-j / p} \sin \left(\frac{\pi(i+1)}{2}\right)-\left(\varepsilon_{j}-\varepsilon_{j}^{\prime}\right) 2^{-j / p} \sin \left(\frac{\pi i}{2}\right)\right| \\
= & 2\left(2^{-j / p}\right)
\end{aligned}
$$

In particular, $|t-s|=\frac{1}{2^{j+1}}$ so $2^{1 / p}|t-s|^{1 / p}=2^{-j / p}$

$$
\left|f_{\varepsilon}(t)-f_{\varepsilon}(s)-\left(f_{\mathcal{\varepsilon}^{\prime}}(t)-f_{\varepsilon^{\prime}}(s)\right)\right|=2\left(2^{1 / p}\right)|t-s|^{1 / p}
$$

which shows that

$$
\sup _{0 \leq s<t \leq 1} \frac{\left|f_{\varepsilon}(t)-f_{\mathcal{\varepsilon}^{\prime}}(t)-\left(f_{\varepsilon}(s)-f_{\mathcal{\varepsilon}^{\prime}}(s)\right)\right|}{|t-s|^{1 / p}} \geq 2^{1 / p}(2)
$$

Thus there exists a set of uncountably many functions in $C^{1 / p}([0, T])$ and for any two of them $f, g$, you get

$$
\|f-g\|_{C^{1 / p}([0,1])}>2
$$

so $C^{1 / p}([0,1])$ is not separable.

### 9.4 Exercises

1. Let $(X, \tau),(Y, \eta)$ be topological spaces and let $A \subseteq X$ be compact. Then if $f: X \rightarrow Y$ is continuous, show that $f(A)$ is also compact.
2. $\uparrow$ In the context of Problem 1, suppose $\mathbb{R}=Y$ where the usual topology is placed on $\mathbb{R}$. Show $f$ achieves its maximum and minimum on $A$.
3. Let $V$ be an open set in $\mathbb{R}^{n}$. Show there is an increasing sequence of compact sets, $K_{m}$, such that $V=\cup_{m=1}^{\infty} K_{m}$. Hint: Let

$$
C_{m} \equiv\left\{\mathbf{x} \in \mathbb{R}^{n}: \operatorname{dist}\left(\mathbf{x}, V^{C}\right) \geq \frac{1}{m}\right\}
$$

where

$$
\operatorname{dist}(\mathbf{x}, S) \equiv \inf \{|\mathbf{y}-\mathbf{x}| \text { such that } \mathbf{y} \in S\}
$$

Consider $K_{m} \equiv C_{m} \cap \overline{\boldsymbol{B}(\mathbf{0}, m)}$.
4. Let $B\left(X ; \mathbb{R}^{n}\right)$ be the space of functions $\mathbf{f}$, mapping $X$ to $\mathbb{R}^{n}$ such that

$$
\sup \{|\mathbf{f}(\mathbf{x})|: \mathbf{x} \in X\}<\infty
$$

Show $B\left(X ; \mathbb{R}^{n}\right)$ is a complete normed linear space if

$$
\|\mathbf{f}\| \equiv \sup \{|\mathbf{f}(\mathbf{x})|: \mathbf{x} \in X\}
$$

5. Let $\alpha \in[0,1]$. Define, for $X$ a compact subset of $\mathbb{R}^{p}$,

$$
C^{\alpha}\left(X ; \mathbb{R}^{n}\right) \equiv\left\{\mathbf{f} \in C\left(X ; \mathbb{R}^{n}\right): \rho_{\alpha}(\mathbf{f})+\|\mathbf{f}\| \equiv\|\mathbf{f}\|_{\alpha}<\infty\right\}
$$

where

$$
\|\mathbf{f}\| \equiv \sup \{|\mathbf{f}(\mathbf{x})|: \mathbf{x} \in X\}
$$

and

$$
\rho_{\alpha}(\mathbf{f}) \equiv \sup \left\{\frac{|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{\alpha}}: \mathbf{x}, \mathbf{y} \in X, \mathbf{x} \neq \mathbf{y}\right\}
$$

Show that $\left(C^{\alpha}\left(X ; \mathbb{R}^{n}\right),\|\cdot\|_{\alpha}\right)$ is a complete normed linear space.
6. Let $\left\{\mathbf{f}_{n}\right\}_{n=1}^{\infty} \subseteq C^{\alpha}\left(X ; \mathbb{R}^{n}\right)$ where $X$ is a compact subset of $\mathbb{R}^{p}$ and suppose

$$
\left\|\mathbf{f}_{n}\right\|_{\alpha} \leq M
$$

for all $n$. Show there exists a subsequence, $n_{k}$, such that $\mathbf{f}_{n_{k}}$ converges in $C\left(X ; \mathbb{R}^{n}\right)$. The given sequence is called precompact when this happens. (This also shows the embedding of $C^{\alpha}\left(X ; \mathbb{R}^{n}\right)$ into $C\left(X ; \mathbb{R}^{n}\right)$ is a compact embedding.) Note that it is likely the case that $C^{\alpha}\left(X ; \mathbb{R}^{n}\right)$ is not separable although it embedds continuously into a nice separable space. In fact, $C^{\alpha}\left([0, T] ; \mathbb{R}^{n}\right)$ can be shown to not be separable. See Definition 9.3.1 and the discussion which follows it.
7. Use the general Stone Weierstrass approximation theorem to prove Theorem 9.1.7.

## Chapter 10

## Brouwer Fixed Point Theorem $\mathbb{R}^{n *}$

This is on the Brouwer fixed point theorem and a discussion of some of the manipulations which are important regarding simplices. This here is an approach based on combinatorics or graph theory. It features the famous Sperner's lemma. It uses very elementary concepts from linear algebra in an essential way. However, it is pretty technical stuff. This elementary proof is harder than those which are based on other approaches like integration theory or degree theory.

### 10.0.1 Simplices and Triangulations

Definition 10.0.1 Define an $n$ simplex, denoted by $\left[\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right]$, to be the convex hull of the $n+1$ points, $\left\{\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right\}$ where $\left\{\mathbf{x}_{i}-\mathbf{x}_{0}\right\}_{i=1}^{n}$ are linearly independent. Thus

$$
\left[\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right] \equiv\left\{\sum_{i=0}^{n} t_{i} \mathbf{x}_{i}: \sum_{i=0}^{n} t_{i}=1, t_{i} \geq 0\right\}
$$

Note that $\left\{\mathbf{x}_{j}-\mathbf{x}_{m}\right\}_{j \neq m}$ are also independent. I will call the $\left\{t_{i}\right\}$ just described the coordinates of a point $\mathbf{x}$.

To see the last claim, suppose $\sum_{j \neq m} c_{j}\left(\mathbf{x}_{j}-\mathbf{x}_{m}\right)=0$. Then you would have

$$
\begin{gathered}
c_{0}\left(\mathbf{x}_{0}-\mathbf{x}_{m}\right)+\sum_{j \neq m, 0} c_{j}\left(\mathbf{x}_{j}-\mathbf{x}_{m}\right)=0 \\
=c_{0}\left(\mathbf{x}_{0}-\mathbf{x}_{m}\right)+\sum_{j \neq m, 0} c_{j}\left(\mathbf{x}_{j}-\mathbf{x}_{0}\right)+\left(\sum_{j \neq m, 0} c_{j}\right)\left(\mathbf{x}_{0}-\mathbf{x}_{m}\right)=0 \\
=\sum_{j \neq m, 0} c_{j}\left(\mathbf{x}_{j}-\mathbf{x}_{0}\right)+\left(\sum_{j \neq m} c_{j}\right)\left(\mathbf{x}_{0}-\mathbf{x}_{m}\right)
\end{gathered}
$$

Then you get $\sum_{j \neq m} c_{j}=0$ and each $c_{j}=0$ for $j \neq m, 0$. Thus $c_{0}=0$ also because the sum is 0 and all other $c_{j}=0$.

Since $\left\{\mathbf{x}_{i}-\mathbf{x}_{0}\right\}_{i=1}^{n}$ is an independent set, the $t_{i}$ used to specify a point in the convex hull are uniquely determined. If two of them are

$$
\sum_{i=0}^{n} t_{i} \mathbf{x}_{i}=\sum_{i=0}^{n} s_{i} \mathbf{x}_{i}
$$

Then

$$
\sum_{i=0}^{n} t_{i}\left(\mathbf{x}_{i}-\mathbf{x}_{0}\right)=\sum_{i=0}^{n} s_{i}\left(\mathbf{x}_{i}-\mathbf{x}_{0}\right)
$$

so $t_{i}=s_{i}$ for $i \geq 1$ by independence. Since the $s_{i}$ and $t_{i}$ sum to 1 , it follows that also $s_{0}=t_{0}$. If $n \leq 2$, the simplex is a triangle, line segment, or point. If $n \leq 3$, it is a tetrahedron, triangle, line segment or point.

Definition 10.0.2 If $S$ is an $n$ simplex. Then it is triangulated if it is the union of smller sub-simplices, the triangulation, such that if $S_{1}, S_{2}$ are two simplices in the triangulation, with

$$
S_{1} \equiv\left[\mathbf{z}_{0}^{1}, \cdots, \mathbf{z}_{m}^{1}\right], S_{2} \equiv\left[\mathbf{z}_{0}^{2}, \cdots, \mathbf{z}_{p}^{2}\right]
$$

then

$$
S_{1} \cap S_{2}=\left[\mathbf{x}_{k_{0}}, \cdots, \mathbf{x}_{k_{r}}\right]
$$

where $\left[\mathbf{x}_{k_{0}}, \cdots, \mathbf{x}_{k_{r}}\right]$ is in the triangulation and

$$
\left\{\mathbf{x}_{k_{0}}, \cdots, \mathbf{x}_{k_{r}}\right\}=\left\{\mathbf{z}_{0}^{1}, \cdots, \mathbf{z}_{m}^{1}\right\} \cap\left\{\mathbf{z}_{0}^{2}, \cdots, \mathbf{z}_{p}^{2}\right\}
$$

or else the two simplices do not intersect.
The following proposition is geometrically fairly clear. It will be used without comment whenever needed in the following argument about triangulations.

Proposition 10.0.3 Say $\left[\mathbf{x}_{1}, \cdots, \mathbf{x}_{r}\right],\left[\hat{\mathbf{x}}_{1}, \cdots, \hat{\mathbf{x}}_{r}\right],\left[\mathbf{z}_{1}, \cdots, \mathbf{z}_{r}\right]$ are all $r-1$ simplices and

$$
\left[\mathbf{x}_{1}, \cdots, \mathbf{x}_{r}\right],\left[\hat{\mathbf{x}}_{1}, \cdots, \hat{\mathbf{x}}_{r}\right] \subseteq\left[\mathbf{z}_{1}, \cdots, \mathbf{z}_{r}\right]
$$

and $\left[\mathbf{z}_{1}, \cdots, \mathbf{z}_{r}, \mathbf{b}\right]$ is an $r+1$ simplex and

$$
\begin{equation*}
\left[\mathbf{y}_{1}, \cdots, \mathbf{y}_{s}\right]=\left[\mathbf{x}_{1}, \cdots, \mathbf{x}_{r}\right] \cap\left[\hat{\mathbf{x}}_{1}, \cdots, \hat{\mathbf{x}}_{r}\right] \tag{10.0.1}
\end{equation*}
$$

where

$$
\begin{equation*}
\left\{\mathbf{y}_{1}, \cdots, \mathbf{y}_{s}\right\}=\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{r}\right\} \cap\left\{\hat{\mathbf{x}}_{1}, \cdots, \hat{\mathbf{x}}_{r}\right\} \tag{10.0.2}
\end{equation*}
$$

Then

$$
\begin{equation*}
\left[\mathbf{x}_{1}, \cdots, \mathbf{x}_{r}, \mathbf{b}\right] \cap\left[\hat{\mathbf{x}}_{1}, \cdots, \hat{\mathbf{x}}_{r}, \mathbf{b}\right]=\left[\mathbf{y}_{1}, \cdots, \mathbf{y}_{s}, \mathbf{b}\right] \tag{10.0.3}
\end{equation*}
$$

Proof: If you have $\sum_{i=1}^{s} t_{i} \mathbf{y}_{i}+t_{s+1} \mathbf{b}$ in the right side, the $t_{i}$ summing to 1 and nonnegative, then it is obviously in both of the two simplices on the left because of 10.0.2. Thus $\left[\mathbf{x}_{1}, \cdots, \mathbf{x}_{r}, \mathbf{b}\right] \cap\left[\hat{\mathbf{x}}_{1}, \cdots, \hat{\mathbf{x}}_{r}, \mathbf{b}\right] \supseteq\left[\mathbf{y}_{1}, \cdots, \mathbf{y}_{s}, \mathbf{b}\right]$.

Now suppose $\mathbf{x}_{k}=\sum_{j=1}^{r} t_{j}^{k} \mathbf{z}_{j}, \hat{\mathbf{x}}_{k}=\sum_{j=1}^{r} \hat{t}_{j}^{k} \mathbf{z}_{j}$, as usual, the scalars adding to 1 and nonnegative.

Consider something in both of the simplices on the left in 10.0.3. Is it in the right? The element on the left is of the form

$$
\sum_{\alpha=1}^{r} s_{\alpha} \mathbf{x}_{\alpha}+s_{r+1} \mathbf{b}=\sum_{\alpha=1}^{r} \hat{s}_{\alpha} \hat{\mathbf{x}}_{\alpha}+\hat{s}_{r+1} \mathbf{b}
$$

where the $s_{\alpha}$, are nonnegative and sum to one, similarly for $\hat{s}_{\alpha}$. Thus

$$
\begin{equation*}
\sum_{j=1}^{r} \sum_{\alpha=1}^{r} s_{\alpha} t_{j}^{\alpha} \mathbf{z}_{j}+s_{r+1} \mathbf{b}=\sum_{\alpha=1}^{r} \sum_{j=1}^{r} \hat{s}_{\alpha} \hat{t}_{j}^{\alpha} \mathbf{z}_{j}+\hat{s}_{r+1} \mathbf{b} \tag{10.0.4}
\end{equation*}
$$

Now observe that

$$
\sum_{j} \sum_{\alpha} s_{\alpha} t_{j}^{\alpha}+s_{r+1}=\sum_{\alpha} \sum_{j} s_{\alpha} t_{j}^{\alpha}+s_{r+1}=\sum_{\alpha} s_{\alpha}+s_{r+1}=1 .
$$

A similar observation holds for the right side of 10.0.4. By uniqueness of the coordinates in an $r+1$ simplex, and assumption that $\left[\mathbf{z}_{1}, \cdots, \mathbf{z}_{r}, \mathbf{b}\right]$ is an $r+1$ simplex, $\hat{s}_{r+1}=s_{r+1}$ and so

$$
\sum_{\alpha=1}^{r} \frac{s_{\alpha}}{1-s_{r+1}} \mathbf{x}_{\alpha}=\sum_{\alpha=1}^{r} \frac{\hat{s}_{\alpha}}{1-s_{r+1}} \hat{\mathbf{x}}_{\alpha}
$$

where $\sum_{\alpha} \frac{s_{\alpha}}{1-s_{r+1}}=\sum_{\alpha} \frac{\hat{s}_{\alpha}}{1-s_{r+1}}=1$, which would say that both sides are a single element of $\left[\mathbf{x}_{1}, \cdots, \mathbf{x}_{r}\right] \cap\left[\hat{\mathbf{x}}_{1}, \cdots, \hat{\mathbf{x}}_{r}\right]=\left[\mathbf{y}_{1}, \cdots, \mathbf{y}_{s}\right]$ and this shows both are equal to something of the form $\sum_{i=1}^{s} t_{i} \mathbf{y}_{i}, \sum_{i} t_{i}=1, t_{i} \geq 0$. Therefore,

$$
\sum_{\alpha=1}^{r} \frac{s_{\alpha}}{1-s_{r+1}} \mathbf{x}_{\alpha}=\sum_{i=1}^{s} t_{i} \mathbf{y}_{i}, \sum_{\alpha=1}^{r} s_{\alpha} \mathbf{x}_{\alpha}=\sum_{i=1}^{s}\left(1-s_{r+1}\right) t_{i} \mathbf{y}_{i}
$$

It follows that

$$
\sum_{\alpha=1}^{r} s_{\alpha} \mathbf{x}_{\alpha}+s_{r+1} \mathbf{b}=\sum_{i=1}^{s}\left(1-s_{r+1}\right) t_{i} \mathbf{y}_{i}+s_{r+1} \mathbf{b} \in\left[\mathbf{y}_{1}, \cdots, \mathbf{y}_{s}, \mathbf{b}\right]
$$

which proves the other inclusion.
Next I will explain why any simplex can be triangulated in such a way that all subsimplices have diameter less than $\varepsilon$.

This is obvious if $n \leq 2$. Supposing it to be true for $n-1$, is it also so for $n$ ? The barycenter $\mathbf{b}$ of a simplex $\left[\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right]$ is $\frac{1}{1+n} \sum_{i} \mathbf{x}_{i}$. This point is not in the convex hull of any of the faces, those simplices of the form $\left[\mathbf{x}_{0}, \cdots, \hat{\mathbf{x}}_{k}, \cdots, \mathbf{x}_{n}\right]$ where the hat indicates $\mathbf{x}_{k}$ has been left out. Thus, placing $\mathbf{b}$ in the $k^{t h}$ position, $\left[\mathbf{x}_{0}, \cdots, \mathbf{b}, \cdots, \mathbf{x}_{n}\right]$ is a $n$ simplex also. First note that $\left[\mathbf{x}_{0}, \cdots, \hat{\mathbf{x}}_{k}, \cdots, \mathbf{x}_{n}\right]$ is an $n-1$ simplex. To be sure $\left[\mathbf{x}_{0}, \cdots, \mathbf{b}, \cdots, \mathbf{x}_{n}\right]$ is an $n$ simplex, we need to check that certain vectors are linearly independent. If

$$
\mathbf{0}=\sum_{j=1}^{k-1} c_{j}\left(\mathbf{x}_{j}-\mathbf{x}_{0}\right)+a_{k}\left(\frac{1}{n+1} \sum_{i=0}^{n} \mathbf{x}_{i}-\mathbf{x}_{0}\right)+\sum_{j=k+1}^{n} d_{j}\left(\mathbf{x}_{j}-\mathbf{x}_{0}\right)
$$

then does it follow that $a_{k}=0=c_{j}=d_{j}$ ?

$$
\begin{gathered}
\mathbf{0}=\sum_{j=1}^{k-1} c_{j}\left(\mathbf{x}_{j}-\mathbf{x}_{0}\right)+a_{k} \frac{1}{n+1}\left(\sum_{i=0}^{n}\left(\mathbf{x}_{i}-\mathbf{x}_{0}\right)\right)+\sum_{j=k+1}^{n} d_{j}\left(\mathbf{x}_{j}-\mathbf{x}_{0}\right) \\
\mathbf{0}=\sum_{j=1}^{k-1}\left(c_{j}+\frac{a_{k}}{n+1}\right)\left(\mathbf{x}_{j}-\mathbf{x}_{0}\right)+a_{k} \frac{1}{n+1}\left(\mathbf{x}_{k}-\mathbf{x}_{0}\right)+\sum_{j=k+1}^{n}\left(d_{j}+\frac{a_{k}}{n+1}\right)\left(\mathbf{x}_{j}-\mathbf{x}_{0}\right)
\end{gathered}
$$

Thus $\frac{a_{k}}{n+1}=0$ and each $c_{j}+\frac{a_{k}}{n+1}=0=d_{j}+\frac{a_{k}}{n+1}$ so each $c_{j}$ and $d_{j}$ are also 0 . Thus, this is also an $n$ simplex.

Actually, a little more is needed. Suppose $\left[\mathbf{y}_{0}, \cdots, \mathbf{y}_{n-1}\right]$ is an $n-1$ simplex such that $\left[\mathbf{y}_{0}, \cdots, \mathbf{y}_{n-1}\right] \subseteq\left[\mathbf{x}_{0}, \cdots, \hat{\mathbf{x}}_{k}, \cdots, \mathbf{x}_{n}\right]$. Why is $\left[\mathbf{y}_{0}, \cdots, \mathbf{y}_{n-1}, \mathbf{b}\right]$ an $n$ simplex? We know the vectors $\left\{\mathbf{y}_{j}-\mathbf{y}_{0}\right\}_{k=1}^{n-1}$ are independent and that $\mathbf{y}_{j}=\sum_{i \neq k} t_{i}^{j} \mathbf{x}_{i}$ where $\sum_{i \neq k} t_{i}^{j}=1$ with each being nonnegative. Suppose

$$
\begin{equation*}
\sum_{j=1}^{n-1} c_{j}\left(\mathbf{y}_{j}-\mathbf{y}_{0}\right)+c_{n}\left(\mathbf{b}-\mathbf{y}_{0}\right)=\mathbf{0} \tag{10.0.5}
\end{equation*}
$$

If $c_{n}=0$, then by assumption, each $c_{j}=0$. The proof goes by assuming $c_{n} \neq 0$ and deriving a contradiction. Assume then that $c_{n} \neq 0$. Then you can divide by it and obtain modified constants, still denoted as $c_{j}$ such that

$$
\mathbf{b}=\frac{1}{n+1} \sum_{i=0}^{n} \mathbf{x}_{i}=\mathbf{y}_{0}+\sum_{j=1}^{n-1} c_{j}\left(\mathbf{y}_{j}-\mathbf{y}_{0}\right)
$$

Thus

$$
\begin{gathered}
\frac{1}{n+1} \sum_{i=0}^{n} \sum_{s \neq k} t_{s}^{0}\left(\mathbf{x}_{i}-\mathbf{x}_{s}\right)=\sum_{j=1}^{n-1} c_{j}\left(\mathbf{y}_{j}-\mathbf{y}_{0}\right)=\sum_{j=1}^{n-1} c_{j}\left(\sum_{s \neq k} t_{s}^{j} \mathbf{x}_{s}-\sum_{s \neq k} t_{s}^{0} \mathbf{x}_{s}\right) \\
=\sum_{j=1}^{n-1} c_{j}\left(\sum_{s \neq k} t_{s}^{j}\left(\mathbf{x}_{s}-\mathbf{x}_{0}\right)-\sum_{s \neq k} t_{s}^{0}\left(\mathbf{x}_{s}-\mathbf{x}_{0}\right)\right)
\end{gathered}
$$

Modify the term on the left and simplify on the right to get

$$
\frac{1}{n+1} \sum_{i=0}^{n} \sum_{s \neq k} t_{s}^{0}\left(\left(\mathbf{x}_{i}-\mathbf{x}_{0}\right)+\left(\mathbf{x}_{0}-\mathbf{x}_{s}\right)\right)=\sum_{j=1}^{n-1} c_{j}\left(\sum_{s \neq k}\left(t_{s}^{j}-t_{s}^{0}\right)\left(\mathbf{x}_{s}-\mathbf{x}_{0}\right)\right)
$$

Thus,

$$
\begin{aligned}
\frac{1}{n+1} \sum_{i=0}^{n}\left(\sum_{s \neq k} t_{s}^{0}\right)\left(\mathbf{x}_{i}-\mathbf{x}_{0}\right)= & \frac{1}{n+1} \sum_{i=0}^{n} \sum_{s \neq k} t_{s}^{0}\left(\mathbf{x}_{s}-\mathbf{x}_{0}\right) \\
& +\sum_{j=1}^{n-1} c_{j}\left(\sum_{s \neq k}\left(t_{s}^{j}-t_{s}^{0}\right)\left(\mathbf{x}_{s}-\mathbf{x}_{0}\right)\right)
\end{aligned}
$$

Then, taking out the $i=k$ term on the left yields

$$
\begin{aligned}
& \frac{1}{n+1}\left(\sum_{s \neq k} t_{s}^{0}\right)\left(\mathbf{x}_{k}-\mathbf{x}_{0}\right)=-\frac{1}{n+1} \sum_{i \neq k}\left(\sum_{s \neq k} t_{s}^{0}\right)\left(\mathbf{x}_{i}-\mathbf{x}_{0}\right) \\
& \frac{1}{n+1} \sum_{i=0}^{n} \sum_{s \neq k} t_{s}^{0}\left(\mathbf{x}_{s}-\mathbf{x}_{0}\right)+\sum_{j=1}^{n-1} c_{j}\left(\sum_{s \neq k}\left(t_{s}^{j}-t_{s}^{0}\right)\left(\mathbf{x}_{s}-\mathbf{x}_{0}\right)\right)
\end{aligned}
$$

That on the right is a linear combination of vectors $\left(\mathbf{x}_{r}-\mathbf{x}_{0}\right)$ for $r \neq k$ so by independence, $\sum_{r \neq k} t_{r}^{0}=0$. However, each $t_{r}^{0} \geq 0$ and these sum to 1 so this is impossible. Hence $c_{n}=0$ after all and so each $c_{j}=0$. Thus $\left[\mathbf{y}_{0}, \cdots, \mathbf{y}_{n-1}, \mathbf{b}\right]$ is an $n$ simplex.

Now in general, if you have an $n$ simplex $\left[\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right]$, its diameter is the maximum of $\left|\mathbf{x}_{k}-\mathbf{x}_{l}\right|$ for all $k \neq l$. Consider $\left|\mathbf{b}-\mathbf{x}_{j}\right|$. It equals

$$
\left|\sum_{i=0}^{n} \frac{1}{n+1}\left(\mathbf{x}_{i}-\mathbf{x}_{j}\right)\right|=\left|\sum_{i \neq j} \frac{1}{n+1}\left(\mathbf{x}_{i}-\mathbf{x}_{j}\right)\right| \leq \frac{n}{n+1} \operatorname{diam}(S)
$$

Next consider the $k^{t h}$ face of $S$

$$
\left[\mathbf{x}_{0}, \cdots, \hat{\mathbf{x}}_{k}, \cdots, \mathbf{x}_{n}\right]
$$

By induction, it has a triangulation into simplices which each have diameter no more than $\frac{n}{n+1} \operatorname{diam}(S)$. Let these $n-1$ simplices be denoted by $\left\{S_{1}^{k}, \cdots, S_{m_{k}}^{k}\right\}$. Then the simplices $\left\{\left[S_{i}^{k}, \mathbf{b}\right]\right\}_{i=1, k=1}^{m_{k}, n+1}$ are a triangulation of $S$ such that $\operatorname{diam}\left(\left[S_{i}^{k}, \mathbf{b}\right]\right) \leq \frac{n}{n+1} \operatorname{diam}(S)$. Do for [ $\left.S_{i}^{k}, \mathbf{b}\right]$ what was just done for $S$ obtaining a triangulation of $S$ as the union of what is obtained such that each simplex has diameter no more than $\left(\frac{n}{n+1}\right)^{2} \operatorname{diam}(S)$. Continuing this way shows the existence of the desired triangulation. You simply do the process $k$ times where $\left(\frac{n}{n+1}\right)^{k} \operatorname{diam}(S)<\varepsilon$.

### 10.0.2 Labeling Vertices

Next is a way to label the vertices. Let $p_{0}, \cdots, p_{n}$ be the first $n+1$ prime numbers. All vertices of a simplex $S=\left[\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right]$ having $\left\{\mathbf{x}_{k}-\mathbf{x}_{0}\right\}_{k=1}^{n}$ independent will be labeled with one of these primes. In particular, the vertex $\mathbf{x}_{k}$ will be labeled as $p_{k}$ if the simplex is $\left[\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right]$. The "value" of a simplex will be the product of its labels. Triangulate this $S$.

Consider a 1 simplex whose vertices are from the vertices of $S$, the original $n$ simplex $\left[\mathbf{x}_{k_{1}}, \mathbf{x}_{k_{2}}\right]$, label $\mathbf{x}_{k_{1}}$ as $p_{k_{1}}$ and $\mathbf{x}_{k_{2}}$ as $p_{k_{2}}$. Then label all other vertices of this triangulation which occur on $\left[\mathbf{x}_{k_{1}}, \mathbf{x}_{k_{2}}\right]$ either $p_{k_{1}}$ or $p_{k_{2}}$. Note that by independence of $\left\{\mathbf{x}_{k}-\mathbf{x}_{r}\right\}_{k \neq r}$, this cannot introduce an inconsistency because the segment cannot contain any other vertex of $S$. Then obviously there will be an odd number of simplices in this triangulation having value $p_{k_{1}} p_{k_{2}}$, that is a $p_{k_{1}}$ at one end and a $p_{k_{2}}$ at the other. Next consider the 2 simplices $\left[\mathbf{x}_{k_{1}}, \mathbf{x}_{k_{2}}, \mathbf{x}_{k_{3}}\right]$ where the $\mathbf{x}_{k_{i}}$ are from $S$. Label all vertices of the triangulation which lie on one of these 2 simplices which have not already been labeled as either $p_{k_{1}}, p_{k_{2}}$, or $p_{k_{2}}$. Continue this way. This labels all vertices of the triangulation of $S$ which have at least one coordinate zero. For the vertices of the triangulation which have all coordinates positive, the interior points of $S$, label these at random from any of $p_{0}, \ldots, p_{n}$. (Essentially, this is the same idea. The "interior" points are the new ones not already labeled.) The idea is to show that there is an odd number of $n$ simplices with value $\prod_{i=0}^{n} p_{i}$ in the triangulation and more generally, for each $m$ simplex $\left[\mathbf{x}_{k_{1}}, \cdots, \mathbf{x}_{k_{m+1}}\right], m \leq n$ with the $\mathbf{x}_{k_{i}}$ an original vertex from $S$, there are an odd number of $m$ simplices of the triangulation contained in $\left[\mathbf{x}_{k_{1}}, \cdots, \mathbf{x}_{k_{m+1}}\right]$, having value $p_{k_{1}} \cdots p_{k_{m+1}}$. It is clear that this is the case for all such 1 simplices. For convenience, call such simplices $\left[\mathbf{x}_{k_{1}}, \cdots, \mathbf{x}_{k_{m+1}}\right] m$ dimensional faces of $S$. An $m$ simplex which is a subspace of this one will have the "correct" value if its value is $p_{k_{1}} \cdots p_{k_{m+1}}$.

Suppose that the labeling has produced an odd number of simplices of the triangulation contained in each $m$ dimensional face of $S$ which have the correct value. Take an $m$ dimensional face $\left[\mathbf{x}_{j_{1}}, \ldots, \mathbf{x}_{j_{k+1}}\right]$. Consider $\hat{S} \equiv\left[\mathbf{x}_{j_{1}}, \ldots \mathbf{x}_{j_{k+1}}, \mathbf{x}_{j_{k+2}}\right]$. Then by induction, there is an odd number of $k$ simplices on the $s^{t h}$ face

$$
\left[\mathbf{x}_{j_{1}}, \ldots, \hat{\mathbf{x}}_{j_{s}}, \cdots, \mathbf{x}_{j_{k+2}}\right]
$$

having value $\prod_{i \neq s} p_{j_{i}}$. In particular, the face $\left[\mathbf{x}_{j_{1}}, \ldots, \mathbf{x}_{j_{k+1}}, \hat{\mathbf{x}}_{j_{k+2}}\right]$ has an odd number of simplices with value $\prod_{i \leq k+1} p_{j_{i}}$.

No simplex in any other face of $\hat{S}$ can have this value by uniqueness of prime factorization. Pick a simplex on the face $\left[\mathbf{x}_{j_{1}}, \ldots, \mathbf{x}_{j_{k+1}}, \hat{\mathbf{x}}_{j_{k+2}}\right]$ which has correct value $\prod_{i \leq k+1} p_{j_{i}}$ and cross this simplex into $\hat{S}$. Continue crossing simplices having value $\prod_{i \leq k+1} p_{j_{i}}$ which have not been crossed till the process ends. It must end because there are an odd number of these simplices having value $\prod_{i \leq k+1} p_{j_{i}}$. If the process leads to the outside of $\hat{S}$, then one can always enter it again because there are an odd number of simplices with value $\prod_{i \leq k+1} p_{j_{i}}$ available and you will have used up an even number. Note that in this process, if you have a simplex with one side labeled $\prod_{i \leq k+1} p_{j_{i}}$, there is either one way in or out of this simplex or two depending on whether the remaining vertex is labeled $p_{j_{k+2}}$. When the process ends, the value of the simplex must be $\prod_{i=1}^{k+2} p_{j_{i}}$ because it will have the additional label $p_{j_{k+2}}$. Otherwise, there would be another route out of this, through the other side labeled $\prod_{i \leq k+1} p_{j_{i}}$. This identifies a simplex in the triangulation with value $\prod_{i=1}^{k+2} p_{j_{i}}$. Then repeat the process with $\prod_{i \leq k+1} p_{j_{i}}$ valued simplices on $\left[\mathbf{x}_{j_{1}}, \ldots, \mathbf{x}_{j_{k+1}}, \hat{\mathbf{x}}_{j_{k+2}}\right]$ which have not been crossed. Repeating the process, entering from the outside, cannot deliver a $\prod_{i=1}^{k+2} p_{j_{i}}$ valued simplex encountered earlier because of what was just noted. There is either one or two ways to cross the simplices. In other words, the process is one to one in selecting a $\prod_{i \leq k+1} p_{j_{i}}$ simplex from crossing such a simplex on the selected face of $\hat{S}$. Continue doing this, crossing a $\prod_{i \leq k+1} p_{j_{i}}$ simplex on the face of $\hat{S}$ which has not been crossed previously. This identifies an odd number of simplices having value $\prod_{i=1}^{k+2} p_{j_{i}}$. These are the ones which are "accessible" from the outside using this process. If there are any which are not accessible from outside, applying the same process starting inside one of these, leads to exactly one other inaccessible simplex with value $\prod_{i=1}^{k+2} p_{j_{i}}$. Hence these inaccessible simplices occur in pairs and so there are an odd number of simplices in the triangulation having value $\prod_{i=1}^{k+2} p_{j_{i}}$. We refer to this procedure of labeling as Sperner's lemma. The system of labeling is well defined thanks to the assumption that $\left\{\mathbf{x}_{k}-\mathbf{x}_{0}\right\}_{k=1}^{n}$ is independent which implies that $\left\{\mathbf{x}_{k}-\mathbf{x}_{i}\right\}_{k \neq i}$ is also linearly independent. Thus there can be no ambiguity in the labeling of vertices on any "face" the convex hull of some of the original vertices of $S$. The following is a description of the system of labeling the vertices.

Lemma 10.0.4 Let $\left[\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right]$ be an $n$ simplex with $\left\{\mathbf{x}_{k}-\mathbf{x}_{0}\right\}_{k=1}^{n}$ independent, and let the first $n+1$ primes be $p_{0}, p_{1}, \cdots, p_{n}$. Label $\mathbf{x}_{k}$ as $p_{k}$ and consider a triangulation of this simplex. Labeling the vertices of this triangulation which occur on $\left[\mathbf{x}_{k_{1}}, \cdots, \mathbf{x}_{k_{s}}\right]$ with any of $p_{k_{1}}, \cdots, p_{k_{s}}$, beginning with all 1 simplices $\left[\mathbf{x}_{k_{1}}, \mathbf{x}_{k_{2}}\right]$ and then 2 simplices and so forth, there are an odd number of simplices $\left[\mathbf{y}_{k_{1}}, \cdots, \mathbf{y}_{k_{s}}\right]$ of the triangulation contained in $\left[\mathbf{x}_{k_{1}}, \cdots, \mathbf{x}_{k_{s}}\right]$ which have value $p_{k_{1}} \cdots p_{k_{s}}$. This for $s=1,2, \cdots, n$.

## A combinatorial method

We now give a brief discussion of the system of labeling for Sperner's lemma from the point of view of counting numbers of faces rather than obtaining them with an algorithm. Let $p_{0}, \cdots, p_{n}$ be the first $n+1$ prime numbers. All vertices of a simplex $S=\left[\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right]$ having $\left\{\mathbf{x}_{k}-\mathbf{x}_{0}\right\}_{k=1}^{n}$ independent will be labeled with one of these primes. In particular, the vertex $\mathbf{x}_{k}$ will be labeled as $p_{k}$. The value of a simplex will be the product of its labels. Triangulate this $S$. Consider a 1 simplex coming from the original simplex $\left[\mathbf{x}_{k_{1}}, \mathbf{x}_{k_{2}}\right]$, label one end as $p_{k_{1}}$ and the other as $p_{k_{2}}$. Then label all other vertices of this triangulation which
occur on $\left[\mathbf{x}_{k_{1}}, \mathbf{x}_{k_{2}}\right]$ either $p_{k_{1}}$ or $p_{k_{2}}$. The assumption of linear independence assures that no other vertex of $S$ can be in $\left[\mathbf{x}_{k_{1}}, \mathbf{x}_{k_{2}}\right]$ so there will be no inconsistency in the labeling. Then obviously there will be an odd number of simplices in this triangulation having value $p_{k_{1}} p_{k_{2}}$, that is a $p_{k_{1}}$ at one end and a $p_{k_{2}}$ at the other. Suppose that the labeling has been done for all vertices of the triangulation which are on $\left[\mathbf{x}_{j_{1}}, \ldots \mathbf{x}_{j_{k+1}}\right]$,

$$
\left\{\mathbf{x}_{j_{1}}, \ldots \mathbf{x}_{j_{k+1}}\right\} \subseteq\left\{\mathbf{x}_{0}, \ldots \mathbf{x}_{n}\right\}
$$

any $k$ simplex for $k \leq n-1$, and there is an odd number of simplices from the triangulation having value equal to $\prod_{i=1}^{k+1} p_{j_{i}}$. Consider $\hat{S} \equiv\left[\mathbf{x}_{j_{1}}, \ldots \mathbf{x}_{j_{k+1}}, \mathbf{x}_{j_{k+2}}\right]$. Then by induction, there is an odd number of $k$ simplices on the $s^{t h}$ face

$$
\left[\mathbf{x}_{j_{1}}, \ldots, \hat{\mathbf{x}}_{j_{s}}, \cdots, \mathbf{x}_{j_{k+1}}\right]
$$

having value $\prod_{i \neq s} p_{j_{i}}$. In particular the face $\left[\mathbf{x}_{j_{1}}, \ldots, \mathbf{x}_{j_{k+1}}, \hat{\mathbf{x}}_{j_{k+2}}\right]$ has an odd number of simplices with value $\prod_{i=1}^{k+1} p_{j_{i}}:=\hat{P}_{k}$. We want to argue that some simplex in the triangulation which is contained in $\hat{S}$ has value $\hat{P}_{k+1}:=\prod_{i=1}^{k+2} p_{j_{i}}$. Let $Q$ be the number of $k+1$ simplices from the triangulation contained in $\hat{S}$ which have two faces with value $\hat{P}_{k}$ (A $k+1$ simplex has either 1 or $2 \hat{P}_{k}$ faces.) and let $R$ be the number of $k+1$ simplices from the triangulation contained in $\hat{S}$ which have exactly one $\hat{P}_{k}$ face. These are the ones we want because they have value $\hat{P}_{k+1}$. Thus the number of faces having value $\hat{P}_{k}$ which is described here is $2 Q+R$. All interior $\hat{P}_{k}$ faces being counted twice by this number. Now we count the total number of $\hat{P}_{k}$ faces another way. There are $P$ of them on the face $\left[\mathbf{x}_{j_{1}}, \ldots, \mathbf{x}_{j_{k+1}}, \hat{\mathbf{x}}_{j_{k+2}}\right]$ and by induction, $P$ is odd. Then there are $O$ of them which are not on this face. These faces got counted twice. Therefore,

$$
2 Q+R=P+2 O
$$

and so, since $P$ is odd, so is $R$. Thus there is an odd number of $\hat{P}_{k+1}$ simplices in $\hat{S}$.
We refer to this procedure of labeling as Sperner's lemma. The system of labeling is well defined thanks to the assumption that $\left\{\mathbf{x}_{k}-\mathbf{x}_{0}\right\}_{k=1}^{n}$ is independent which implies that $\left\{\mathbf{x}_{k}-\mathbf{x}_{i}\right\}_{k \neq i}$ is also linearly independent. Thus there can be no ambiguity in the labeling of vertices on any "face", the convex hull of some of the original vertices of $S$. Sperner's lemma is now a consequence of this discussion.

### 10.1 The Brouwer Fixed Point Theorem

$S \equiv\left[\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right]$ is a simplex in $\mathbb{R}^{n}$. Assume $\left\{\mathbf{x}_{i}-\mathbf{x}_{0}\right\}_{i=1}^{n}$ are linearly independent. Thus a typical point of $S$ is of the form

$$
\sum_{i=0}^{n} t_{i} \mathbf{x}_{i}
$$

where the $t_{i}$ are uniquely determined and the map $\mathbf{x} \rightarrow \mathbf{t}$ is continuous from $S$ to the compact set $\left\{\mathbf{t} \in \mathbb{R}^{n+1}: \sum t_{i}=1, t_{i} \geq 0\right\}$. The map $\mathbf{t} \rightarrow \mathbf{x}$ is one to one and clearly continuous. Since $S$ is compact, it follows that the inverse map is also continuous. This is a general consideration but what follows is a short explanation why this is so in this specific example.

To see this, suppose $\mathbf{x}^{k} \rightarrow \mathbf{x}$ in $S$. Let $\mathbf{x}^{k} \equiv \sum_{i=0}^{n} t_{i}^{k} \mathbf{x}_{i}$ with $\mathbf{x}$ defined similarly with $t_{i}^{k}$ replaced with $t_{i}, \mathbf{x} \equiv \sum_{i=0}^{n} t_{i} \mathbf{x}_{i}$. Then

$$
\mathbf{x}^{k}-\mathbf{x}_{0}=\sum_{i=0}^{n} t_{i}^{k} \mathbf{x}_{i}-\sum_{i=0}^{n} t_{i}^{k} \mathbf{x}_{0}=\sum_{i=1}^{n} t_{i}^{k}\left(\mathbf{x}_{i}-\mathbf{x}_{0}\right)
$$

Thus

$$
\mathbf{x}^{k}-\mathbf{x}_{0}=\sum_{i=1}^{n} t_{i}^{k}\left(\mathbf{x}_{i}-\mathbf{x}_{0}\right), \mathbf{x}-\mathbf{x}_{0}=\sum_{i=1}^{n} t_{i}\left(\mathbf{x}_{i}-\mathbf{x}_{0}\right)
$$

Say $t_{i}^{k}$ fails to converge to $t_{i}$ for all $i \geq 1$. Then there exists a subsequence, still denoted with superscript $k$ such that for each $i=1, \cdots, n$, it follows that $t_{i}^{k} \rightarrow s_{i}$ where $s_{i} \geq 0$ and some $s_{i} \neq t_{i}$. But then, taking a limit, it follows that

$$
\mathbf{x}-\mathbf{x}_{0}=\sum_{i=1}^{n} s_{i}\left(\mathbf{x}_{i}-\mathbf{x}_{0}\right)=\sum_{i=1}^{n} t_{i}\left(\mathbf{x}_{i}-\mathbf{x}_{0}\right)
$$

which contradicts independence of the $\mathbf{x}_{i}-\mathbf{x}_{0}$. It follows that for all $i \geq 1, t_{i}^{k} \rightarrow t_{i}$. Since they all sum to 1 , this implies that also $t_{0}^{k} \rightarrow t_{0}$. Thus the claim about continuity is verified.

Let $\mathbf{f}: S \rightarrow S$ be continuous. When doing $\mathbf{f}$ to a point $\mathbf{x}$, one obtains another point of $S$ denoted as $\sum_{i=0}^{n} s_{i} \mathbf{x}_{i}$. Thus in this argument the scalars $s_{i}$ will be the components after doing $\mathbf{f}$ to a point of $S$ denoted as $\sum_{i=0}^{n} t_{i} \mathbf{x}_{i}$.

Consider a triangulation of $S$ such that all simplices in the triangulation have diameter less than $\varepsilon$. The vertices of the simplices in this triangulation will be labeled from $p_{0}, \cdots, p_{n}$ the first $n+1$ prime numbers. If $\left[\mathbf{y}_{0}, \cdots, \mathbf{y}_{n}\right]$ is one of these simplices in the triangulation, each vertex is of the form $\sum_{l=0}^{n} t_{l} \mathbf{x}_{l}$ where $t_{l} \geq 0$ and $\sum_{l} t_{l}=1$. Let $\mathbf{y}_{i}$ be one of these vertices, $\mathbf{y}_{i}=\sum_{l=0}^{n} t_{l} \mathbf{x}_{l}$. Define $r_{j} \equiv s_{j} / t_{j}$ if $t_{j}>0$ and $\infty$ if $t_{j}=0$. Then $p\left(\mathbf{y}_{i}\right)$ will be the label placed on $\mathbf{y}_{i}$. To determine this label, let $r_{k}$ be the smallest of these ratios. Then the label placed on $\mathbf{y}_{i}$ will be $p_{k}$ where $r_{k}$ is the smallest of all these extended nonnegative real numbers just described. If there is duplication, pick $p_{k}$ where $k$ is smallest.

Note that for the vertices which are on $\left[\mathbf{x}_{i_{1}}, \cdots, \mathbf{x}_{i_{m}}\right]$, these will be labeled from the list $\left\{p_{i_{1}}, \cdots, p_{i_{m}}\right\}$ because $t_{k}=0$ for each of these and so $r_{k}=\infty$ unless $k \in\left\{i_{1}, \cdots, i_{m}\right\}$. In particular, this scheme labels $\mathbf{x}_{i}$ as $p_{i}$.

By the Sperner's lemma procedure described above, there are an odd number of simplices having value $\prod_{i \neq k} p_{i}$ on the $k^{\text {th }}$ face and an odd number of simplices in the triangulation of $S$ for which the product of the labels on their vertices, referred to here as its value, equals $p_{0} p_{1} \cdots p_{n} \equiv P_{n}$. Thus if $\left[\mathbf{y}_{0}, \cdots, \mathbf{y}_{n}\right]$ is one of these simplices, and $p\left(\mathbf{y}_{i}\right)$ is the label for $\mathbf{y}_{i}$,

$$
\prod_{i=0}^{n} p\left(\mathbf{y}_{i}\right)=\prod_{i=0}^{n} p_{i} \equiv P_{n}
$$

What is $r_{k}$, the smallest of those ratios in determining a label? Could it be larger than $1 ? r_{k}$ is certainly finite because at least some $t_{j} \neq 0$ since they sum to 1 . Thus, if $r_{k}>1$, you would have $s_{k}>t_{k}$. The $s_{j}$ sum to 1 and so some $s_{j}<t_{j}$ since otherwise, the sum of the $t_{j}$ equalling 1 would require the sum of the $s_{j}$ to be larger than 1 . Hence $r_{k}$ was not really the smallest after all and so $r_{k} \leq 1$. Hence $s_{k} \leq t_{k}$.

Let $\mathscr{S} \equiv\left\{S_{1}, \cdots, S_{m}\right\}$ denote those simplices whose value is $P_{n}$. In other words, if $\left\{\mathbf{y}_{0}, \cdots, \mathbf{y}_{n}\right\}$ are the vertices of one of these simplices in $\mathscr{S}$, and

$$
\mathbf{y}_{s}=\sum_{i=0}^{n} t_{i}^{s} \mathbf{x}_{i}
$$

$r_{k_{s}} \leq r_{j}$ for all $j \neq k_{s}$ and $\left\{k_{0}, \cdots, k_{n}\right\}=\{0, \cdots, n\}$. Let $\mathbf{b}$ denote the barycenter of $S_{k}=$ $\left[\mathbf{y}_{0}, \cdots, \mathbf{y}_{n}\right]$.

$$
\mathbf{b} \equiv \sum_{i=0}^{n} \frac{1}{n+1} \mathbf{y}_{i}
$$

Do the same system of labeling for each $n$ simplex in a sequence of triangulations where the diameters of the simplices in the $k^{t h}$ triangulation is no more than $2^{-k}$. Thus each of these triangulations has a $n$ simplex having diameter no more than $2^{-k}$ which has value $P_{n}$. Let $\mathbf{b}_{k}$ be the barycenter of one of these $n$ simplices having value $P_{n}$. By compactness, there is a subsequence, still denoted with the index $k$ such that $\mathbf{b}_{k} \rightarrow \mathbf{x}$. This $\mathbf{x}$ is a fixed point.

Consider this last claim. $\mathbf{x}=\sum_{i=0}^{n} t_{i} \mathbf{x}_{i}$ and after applying $\mathbf{f}$, the result is $\sum_{i=0}^{n} s_{i} \mathbf{x}_{i}$. Then $\mathbf{b}_{k}$ is the barycenter of some $\sigma_{k}$ having diameter no more than $2^{-k}$ which has value $P_{n}$. Say $\sigma_{k}$ is a simplex having vertices $\left\{\mathbf{y}_{0}^{k}, \cdots, \mathbf{y}_{n}^{k}\right\}$ and the value of $\left[\mathbf{y}_{0}^{k}, \cdots, \mathbf{y}_{n}^{k}\right]$ is $P_{n}$. Thus also

$$
\lim _{k \rightarrow \infty} \mathbf{y}_{i}^{k}=\mathbf{x}
$$

Re ordering these if necessary, we can assume that the label for $\mathbf{y}_{i}^{k}$ is $p_{i}$ which implies that, as noted above, for each $i=0, \cdots, n$,

$$
\frac{s_{i}}{t_{i}} \leq 1, s_{i} \leq t_{i}
$$

the $i^{\text {th }}$ coordinate of $\mathbf{f}\left(\mathbf{y}_{i}^{k}\right)$ with respect to the original vertices of $S$ decreases and each $i$ is represented for $i=\{0,1, \cdots, n\}$. As noted above,

$$
\mathbf{y}_{i}^{k} \rightarrow \mathbf{x}
$$

and so the $i^{\text {th }}$ coordinate of $\mathbf{y}_{i}^{k}, t_{i}^{k}$ must converge to $t_{i}$. Hence if the $i^{t h}$ coordinate of $\mathbf{f}\left(\mathbf{y}_{i}^{k}\right)$ is denoted by $s_{i}^{k}$,

$$
s_{i}^{k} \leq t_{i}^{k}
$$

By continuity of $\mathbf{f}$, it follows that $s_{i}^{k} \rightarrow s_{i}$. Thus the above inequality is preserved on taking $k \rightarrow \infty$ and so

$$
0 \leq s_{i} \leq t_{i}
$$

this for each $i$. But these $s_{i}$ add to 1 as do the $t_{i}$ and so in fact, $s_{i}=t_{i}$ for each $i$ and so $\mathbf{f}(\mathbf{x})=\mathbf{x}$. This proves the following theorem which is the Brouwer fixed point theorem.

Theorem 10.1.1 Let $S$ be a simplex $\left[\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right]$ such that $\left\{\mathbf{x}_{i}-\mathbf{x}_{0}\right\}_{i=1}^{n}$ are independent. Also let $\mathbf{f}: S \rightarrow S$ be continuous. Then there exists $\mathbf{x} \in S$ such that $\mathbf{f}(\mathbf{x})=\mathbf{x}$.

Corollary 10.1.2 Let $K$ be a closed convex bounded subset of $\mathbb{R}^{n}$. Let $\mathbf{f}: K \rightarrow K$ be continuous. Then there exists $\mathbf{x} \in K$ such that $\mathbf{f}(\mathbf{x})=\mathbf{x}$.

Proof: Let $S$ be a large simplex containing $K$ and let $P$ be the projection map onto $K$. Consider $\mathbf{g}(\mathbf{x}) \equiv \mathbf{f}(P \mathbf{x})$. Then $\mathbf{g}$ satisfies the necessary conditions for Theorem 10.1.1 and so there exists $\mathbf{x} \in S$ such that $\mathbf{g}(\mathbf{x})=\mathbf{x}$. But this says $\mathbf{x} \in K$ and so $\mathbf{g}(\mathbf{x})=\mathbf{f}(\mathbf{x})$.

Definition 10.1.3 $A$ set $B$ has the fixed point property if whenever $f: B \rightarrow B$ for $f$ continuous, it follows that $f$ has a fixed point.

The proof of this corollary is pretty significant. By a homework problem, a closed convex set is a retract of $\mathbb{R}^{n}$. This is what it means when you say there is this continuous projection map which maps onto the closed convex set but does not change any point in the closed convex set. When you have a set $A$ which is a subset of a set $B$ which has the property that continuous functions $\mathbf{f}: B \rightarrow B$ have fixed points, and there is a continuous map $P$ from $B$ to $A$ which leaves points of $A$ unchanged, then it follows that $A$ will have the same "fixed point property". You can probably imagine all sorts of sets which are retracts of closed convex bounded sets. Also, if you have a compact set $B$ which has the fixed point property and $h: B \rightarrow h(B)$ with $h$ one to one and continuous, it will follow that $h^{-1}$ is continuous and that $h(B)$ will also have the fixed point property. This is very easy to show. This will allow further extensions of this theorem. This says that the fixed point property is topological.

### 10.2 Invariance Of Domain

As an application of the inverse function theorem is a simple proof of the important invariance of domain theorem which says that continuous and one to one functions defined on an open set in $\mathbb{R}^{n}$ with values in $\mathbb{R}^{n}$ take open sets to open sets. You know that this is true for functions of one variable because a one to one continuous function must be either strictly increasing or strictly decreasing. This will be used when considering orientations of curves later. However, the $n$ dimensional version isn't at all obvious but is just as important if you want to consider manifolds with boundary for example. The need for this theorem occurs in many other places as well in addition to being extremely interesting for its own sake. The inverse function theorem gives conditions under which a differentiable function maps open sets to open sets. The following lemma, depending on the Brouwer fixed point theorem is the thing which will allow this to be extended to continuous one to one functions. It says roughly that if a continuous function does not move points near $\mathbf{p}$ very far, then the image of a ball centered at $\mathbf{p}$ contains an open set.

Lemma 10.2.1 Let $\mathbf{f}$ be continuous and map $\overline{B(\mathbf{p}, r)} \subseteq \mathbb{R}^{n}$ to $\mathbb{R}^{n}$. Suppose that for all $\mathbf{x} \in \overline{B(\mathbf{p}, r)}$,

$$
|\mathbf{f}(\mathbf{x})-\mathbf{x}|<\varepsilon r
$$

Then it follows that

$$
\mathbf{f}(\overline{B(\mathbf{p}, r)}) \supseteq B(\mathbf{p},(1-\varepsilon) r)
$$

Proof: This is from the Brouwer fixed point theorem, Corollary 10.1.2. Consider for $\mathbf{y} \in B(\mathbf{p},(1-\varepsilon) r)$

$$
\mathbf{h}(\mathbf{x}) \equiv \mathbf{x}-\mathbf{f}(\mathbf{x})+\mathbf{y}
$$

Then $\mathbf{h}$ is continuous and for $\mathbf{x} \in \overline{B(\mathbf{p}, r)}$,

$$
|\mathbf{h}(\mathbf{x})-\mathbf{p}|=|\mathbf{x}-\mathbf{f}(\mathbf{x})+\mathbf{y}-\mathbf{p}|<\varepsilon r+|\mathbf{y}-\mathbf{p}|<\varepsilon r+(1-\varepsilon) r=r
$$

Hence $\mathbf{h}: \overline{B(\mathbf{p}, r)} \rightarrow \overline{B(\mathbf{p}, r)}$ and so it has a fixed point $\mathbf{x}$ by Corollary 10.1.2. Thus

$$
\mathbf{x}-\mathbf{f}(\mathbf{x})+\mathbf{y}=\mathbf{x}
$$

so $\mathbf{f}(\mathbf{x})=\mathbf{y}$.
The notation $\|\mathbf{f}\|_{K}$ means $\sup _{\mathbf{x} \in K}|\mathbf{f}(\mathbf{x})|$. If you have a continuous function $\mathbf{h}$ defined on a compact set $K$, then the Stone Weierstrass theorem implies you can uniformly approximate it with a polynomial $\mathbf{g}$. That is $\|\mathbf{h}-\mathbf{g}\|_{K}$ is small. The following lemma says that you can also have $\mathbf{h}(\mathbf{z})=\mathbf{g}(\mathbf{z})$ and $D \mathbf{g}(\mathbf{z})^{-1}$ exists so that near $\mathbf{z}$, the function $\mathbf{g}$ will map open sets to open sets as claimed by the inverse function theorem.

Lemma 10.2.2 Let $K$ be a compact set in $\mathbb{R}^{n}$ and let $\mathbf{h}: K \rightarrow \mathbb{R}^{n}$ be continuous, $\mathbf{z} \in K$ is fixed. Let $\delta>0$. Then there exists a polynomial $\mathbf{g}$ (each component a polynomial) such that

$$
\|\mathbf{g}-\mathbf{h}\|_{K}<\delta, \mathbf{g}(\mathbf{z})=\mathbf{h}(\mathbf{z}), D \mathbf{g}(\mathbf{z})^{-1} \text { exists }
$$

Proof: By the Weierstrass approximation theorem, Theorem 9.2.5, (apply this theorem to the algebra of real polynomials) there exists a polynomial $\hat{\mathbf{g}}$ such that

$$
\|\hat{\mathbf{g}}-\mathbf{h}\|_{K}<\frac{\delta}{3}
$$

Then define for $\mathbf{y} \in K$

$$
\mathbf{g}(\mathbf{y}) \equiv \hat{\mathbf{g}}(\mathbf{y})+\mathbf{h}(\mathbf{z})-\hat{\mathbf{g}}(\mathbf{z})
$$

Then

$$
\mathbf{g}(\mathbf{z})=\hat{\mathbf{g}}(\mathbf{z})+\mathbf{h}(\mathbf{z})-\hat{\mathbf{g}}(\mathbf{z})=\mathbf{h}(\mathbf{z})
$$

Also

$$
\begin{aligned}
|\mathbf{g}(\mathbf{y})-\mathbf{h}(\mathbf{y})| & \leq|(\hat{\mathbf{g}}(\mathbf{y})+\mathbf{h}(\mathbf{z})-\hat{\mathbf{g}}(\mathbf{z}))-\mathbf{h}(\mathbf{y})| \\
& \leq|\hat{\mathbf{g}}(\mathbf{y})-\mathbf{h}(\mathbf{y})|+|\mathbf{h}(\mathbf{z})-\hat{\mathbf{g}}(\mathbf{z})|<\frac{2 \delta}{3}
\end{aligned}
$$

and so since $\mathbf{y}$ was arbitrary,

$$
\|\mathbf{g}-\mathbf{h}\|_{K} \leq \frac{2 \delta}{3}<\delta
$$

If $D \mathbf{g}(\mathbf{z})^{-1}$ exists, then this is what is wanted. If not, let

$$
0<\eta<\{|\lambda|: \lambda \text { is an eigenvalue of } D \mathbf{g}(\mathbf{z}), \lambda \neq 0\}
$$

Then if $\eta$ is small enough, $\mathbf{g}(\mathbf{y})$ could be replaced with $\mathbf{g}(\mathbf{y})+\eta(\mathbf{y}-\mathbf{z})$ and it will still be the case that $\|\mathbf{g}-\mathbf{h}\|_{K}<\delta$ along with $\mathbf{g}(\mathbf{z})=\mathbf{h}(\mathbf{z})$ but now $D \mathbf{g}(\mathbf{z})$ would have no zero eigenvalues and would therefore be invertible. Simply use the modified $\mathbf{g}$.

The main result is essentially the following lemma which combines the conclusions of the above.

Lemma 10.2.3 Let $\mathbf{f}: \overline{B(\mathbf{p}, r)} \rightarrow \mathbb{R}^{n}$ where the ball is also in $\mathbb{R}^{n}$. Let $\mathbf{f}$ be one to one, $\mathbf{f}$ continuous. Then there exists $\delta>0$ such that

$$
\mathbf{f}(\overline{B(\mathbf{p}, r)}) \supseteq B(\mathbf{f}(\mathbf{p}), \delta) .
$$

In other words, $\mathbf{f}(\mathbf{p})$ is an interior point of $\mathbf{f}(\overline{B(\mathbf{p}, r)})$.
Proof: Since $\mathbf{f}(\overline{B(\mathbf{p}, r)})$ is compact, it follows that $\mathbf{f}^{-1}: \mathbf{f}(\overline{B(\mathbf{p}, r)}) \rightarrow \overline{B(\mathbf{p}, r)}$ is continuous. By Lemma 10.2.2, there exists a polynomial $\mathbf{g}: \mathbf{f}(\overline{B(\mathbf{p}, r)}) \rightarrow \mathbb{R}^{n}$ such that

$$
\left\|\mathbf{g}-\mathbf{f}^{-1}\right\|_{\mathbf{f}(\overline{B(\mathbf{p}, r)})}<\varepsilon r, \varepsilon<1, \quad D \mathbf{g}(\mathbf{f}(\mathbf{p}))^{-1} \text { exists, and } \mathbf{g}(\mathbf{f}(\mathbf{p}))=\mathbf{f}^{-1}(\mathbf{f}(\mathbf{p}))=\mathbf{p}
$$

From the first inequality in the above,

$$
|\mathbf{g}(\mathbf{f}(\mathbf{x}))-\mathbf{x}|=\left|\mathbf{g}(\mathbf{f}(\mathbf{x}))-\mathbf{f}^{-1}(\mathbf{f}(\mathbf{x}))\right| \leq\left\|\mathbf{g}-\mathbf{f}^{-1}\right\|_{\mathbf{f}(\overline{B(\mathbf{p}, r)})}<\varepsilon r
$$

By Lemma 10.2.1,

$$
\mathbf{g} \circ \mathbf{f}(\overline{B(\mathbf{p}, r)}) \supseteq B(\mathbf{p},(1-\varepsilon) r)=B(\mathbf{g}(\mathbf{f}(\mathbf{p})),(1-\varepsilon) r)
$$

Since $D \mathbf{g}(\mathbf{f}(\mathbf{p}))^{-1}$ exists, it follows from the inverse function theorem that $\mathbf{g}^{-1}$ also exists and that $\mathbf{g}, \mathbf{g}^{-1}$ are open maps on small open sets containing $\mathbf{f}(\mathbf{p})$ and $\mathbf{p}$ respectively. Thus there exists $\eta<(1-\varepsilon) r$ such that $\mathbf{g}^{-1}$ is an open map on $B(\mathbf{p}, \eta) \subseteq B(\mathbf{p},(1-\varepsilon) r)$. Thus

$$
\mathbf{g} \circ \mathbf{f}(\overline{B(\mathbf{p}, r)}) \supseteq B(\mathbf{p},(1-\varepsilon) r) \supseteq B(\mathbf{p}, \eta)
$$

So do $\mathbf{g}^{-1^{\star}}$ to both ends. Then you have $\mathbf{g}^{-1}(\mathbf{p})=\mathbf{f}(\mathbf{p})$ is in the open set $\mathbf{g}^{-1}(B(\mathbf{p}, \eta))$. Thus

$$
\mathbf{f}(\overline{B(\mathbf{p}, r)}) \supseteq \mathbf{g}^{-1}(B(\mathbf{p}, \eta)) \supseteq B\left(\mathbf{g}^{-1}(\mathbf{p}), \delta\right)=B(\mathbf{f}(\mathbf{p}), \delta)
$$



With this lemma, the invariance of domain theorem comes right away. This remarkable theorem states that if $\mathbf{f}: U \rightarrow \mathbb{R}^{n}$ for $U$ an open set in $\mathbb{R}^{n}$ and if $\mathbf{f}$ is one to one and continuous, then $\mathbf{f}(U)$ is also an open set in $\mathbb{R}^{n}$.

Theorem 10.2.4 Let $U$ be an open set in $\mathbb{R}^{n}$ and let $\mathbf{f}: U \rightarrow \mathbb{R}^{n}$ be one to one and continuous. Then $\mathbf{f}(U)$ is also an open subset in $\mathbb{R}^{n}$.

Proof: It suffices to show that if $\mathbf{p} \in U$ then $\mathbf{f}(\mathbf{p})$ is an interior point of $\mathbf{f}(U)$. Let $\overline{B(\mathbf{p}, r)} \subseteq U$. By Lemma 10.2.3, $\mathbf{f}(U) \supseteq \mathbf{f}(\overline{B(\mathbf{p}, r)}) \supseteq B(\mathbf{f}(\mathbf{p}), \boldsymbol{\delta})$ so $\mathbf{f}(\mathbf{p})$ is indeed an interior point of $\mathbf{f}(U)$.

The inverse mapping theorem assumed quite a bit about the mapping. In particular it assumed that the mapping had a continuous derivative. The following version of the inverse function theorem seems very interesting because it only needs an invertible derivative at a point.

Corollary 10.2.5 Let $U$ be an open set in $\mathbb{R}^{p}$ and let $\mathbf{f}: U \rightarrow \mathbb{R}^{p}$ be one to one and continuous. Then, $\mathbf{f}^{-1}$ is also continuous on the open set $\mathbf{f}(U)$. If $\mathbf{f}$ is differentiable at $\mathbf{x}_{1} \in U$ and if $D \mathbf{f}\left(\mathbf{x}_{1}\right)^{-1}$ exists for $\mathbf{x}_{1} \in U$, then it follows that $D \mathbf{f}\left(\mathbf{f}\left(\mathbf{x}_{1}\right)\right)=D \mathbf{f}\left(\mathbf{x}_{1}\right)^{-1}$.

Proof: $|\cdot|$ will be a norm on $\mathbb{R}^{p}$, whichever is desired. If you like, let it be the Euclidean norm. $\|\cdot\|$ will be the operator norm. The first part of the conclusion of this corollary is from invariance of domain. From the assumption that $D \mathbf{f}\left(\mathbf{x}_{1}\right)$ and $D \mathbf{f}\left(\mathbf{x}_{1}\right)^{-1}$ exists,

$$
\mathbf{y}-\mathbf{f}\left(\mathbf{x}_{1}\right)=\mathbf{f}\left(\mathbf{f}^{-1}(\mathbf{y})\right)-\mathbf{f}\left(\mathbf{x}_{1}\right)=D \mathbf{f}\left(\mathbf{x}_{1}\right)\left(\mathbf{f}^{-1}(\mathbf{y})-\mathbf{x}_{1}\right)+\mathbf{o}\left(\mathbf{f}^{-1}(\mathbf{y})-\mathbf{x}_{1}\right)
$$

Since $D \mathbf{f}\left(\mathbf{x}_{1}\right)^{-1}$ exists,

$$
D \mathbf{f}\left(\mathbf{x}_{1}\right)^{-1}\left(\mathbf{y}-\mathbf{f}\left(\mathbf{x}_{1}\right)\right)=\mathbf{f}^{-1}(\mathbf{y})-\mathbf{x}_{1}+\mathbf{o}\left(\mathbf{f}^{-1}(\mathbf{y})-\mathbf{x}_{1}\right)
$$

by continuity, if $\left|\mathbf{y}-\mathbf{f}\left(\mathbf{x}_{1}\right)\right|$ is small enough, then $\left|\mathbf{f}^{-1}(\mathbf{y})-\mathbf{x}_{1}\right|$ is small enough that in the above,

$$
\left|\mathbf{o}\left(\mathbf{f}^{-1}(\mathbf{y})-\mathbf{x}_{1}\right)\right|<\frac{1}{2}\left|\mathbf{f}^{-1}(\mathbf{y})-\mathbf{x}_{1}\right|
$$

Hence, if $\left|\mathbf{y}-\mathbf{f}\left(\mathbf{x}_{1}\right)\right|$ is sufficiently small, then from the triangle inequality of the form $|p-q| \geq||p|-|q||$,

$$
\begin{aligned}
&\left\|D \mathbf{f}\left(\mathbf{x}_{1}\right)^{-1}\right\|\left|\left(\mathbf{y}-\mathbf{f}\left(\mathbf{x}_{1}\right)\right)\right| \geq\left|D \mathbf{f}\left(\mathbf{x}_{1}\right)^{-1}\left(\mathbf{y}-\mathbf{f}\left(\mathbf{x}_{1}\right)\right)\right| \\
& \geq\left|\mathbf{f}^{-1}(\mathbf{y})-\mathbf{x}_{1}\right|-\frac{1}{2}\left|\mathbf{f}^{-1}(\mathbf{y})-\mathbf{x}_{1}\right| \\
&=\frac{1}{2}\left|\mathbf{f}^{-1}(\mathbf{y})-\mathbf{x}_{1}\right| \\
&\left|\mathbf{y}-\mathbf{f}\left(\mathbf{x}_{1}\right)\right| \geq\left\|D \mathbf{f}\left(\mathbf{x}_{1}\right)^{-1}\right\|^{-1} \frac{1}{2}\left|\mathbf{f}^{-1}(\mathbf{y})-\mathbf{x}_{1}\right|
\end{aligned}
$$

It follows that for $\left|\mathbf{y}-\mathbf{f}\left(\mathbf{x}_{1}\right)\right|$ small enough,

$$
\left|\frac{\mathbf{o}\left(\mathbf{f}^{-1}(\mathbf{y})-\mathbf{x}_{1}\right)}{\mathbf{y}-\mathbf{f}\left(\mathbf{x}_{1}\right)}\right| \leq\left|\frac{\mathbf{o}\left(\mathbf{f}^{-1}(\mathbf{y})-\mathbf{x}_{1}\right)}{\mathbf{f}^{-1}(\mathbf{y})-\mathbf{x}_{1}}\right| \frac{2}{\left\|D \mathbf{f}\left(\mathbf{x}_{1}\right)^{-1}\right\|^{-1}}
$$

Then, using continuity of the inverse function again, it follows that if $\left|\mathbf{y}-\mathbf{f}\left(\mathbf{x}_{1}\right)\right|$ is possibly still smaller, then $\mathbf{f}^{-1}(\mathbf{y})-\mathbf{x}_{1}$ is sufficiently small that the right side of the above inequality is no larger than $\varepsilon$. Since $\varepsilon$ is arbitrary, it follows

$$
\mathbf{o}\left(\mathbf{f}^{-1}(\mathbf{y})-\mathbf{x}_{1}\right)=\mathbf{o}\left(\mathbf{y}-\mathbf{f}\left(\mathbf{x}_{1}\right)\right)
$$

Now from differentiability of $\mathbf{f}$ at $\mathbf{x}_{1}$,

$$
\begin{aligned}
\mathbf{y}-\mathbf{f}\left(\mathbf{x}_{1}\right) & =\mathbf{f}\left(\mathbf{f}^{-1}(\mathbf{y})\right)-\mathbf{f}\left(\mathbf{x}_{1}\right)=D \mathbf{f}\left(\mathbf{x}_{1}\right)\left(\mathbf{f}^{-1}(\mathbf{y})-\mathbf{x}_{1}\right)+\mathbf{o}\left(\mathbf{f}^{-1}(\mathbf{y})-\mathbf{x}_{1}\right) \\
& =D \mathbf{f}\left(\mathbf{x}_{1}\right)\left(\mathbf{f}^{-1}(\mathbf{y})-\mathbf{x}_{1}\right)+\mathbf{o}\left(\mathbf{y}-\mathbf{f}\left(\mathbf{x}_{1}\right)\right) \\
& =D \mathbf{f}\left(\mathbf{x}_{1}\right)\left(\mathbf{f}^{-1}(\mathbf{y})-\mathbf{f}^{-1}\left(\mathbf{f}\left(\mathbf{x}_{1}\right)\right)\right)+\mathbf{o}\left(\mathbf{y}-\mathbf{f}\left(\mathbf{x}_{1}\right)\right)
\end{aligned}
$$

Therefore,

$$
\mathbf{f}^{-1}(\mathbf{y})-\mathbf{f}^{-1}\left(\mathbf{f}\left(\mathbf{x}_{1}\right)\right)=D \mathbf{f}\left(\mathbf{x}_{1}\right)^{-1}\left(\mathbf{y}-\mathbf{f}\left(\mathbf{x}_{1}\right)\right)+\mathbf{o}\left(\mathbf{y}-\mathbf{f}\left(\mathbf{x}_{1}\right)\right)
$$

From the definition of the derivative, this shows that $D \mathbf{f}^{-1}\left(\mathbf{f}\left(\mathbf{x}_{1}\right)\right)=D \mathbf{f}\left(\mathbf{x}_{1}\right)^{-1}$.

## Part II

## Real And Abstract Analysis

## Chapter 11

## Abstract Measure And Integration

## 11.1 $\sigma$ Algebras

This chapter is on the basics of measure theory and integration. A measure is a real valued mapping from some subset of the power set of a given set which has values in $[0, \infty]$. Many apparently different things can be considered as measures and also there is an integral defined. By discussing this in terms of axioms and in a very abstract setting, many different topics can be considered in terms of one general theory. For example, it will turn out that sums are included as an integral of this sort. So is the usual integral as well as things which are often thought of as being in between sums and integrals.

Let $\Omega$ be a set and let $\mathscr{F}$ be a collection of subsets of $\Omega$ satisfying

$$
\begin{gather*}
\emptyset \in \mathscr{F}, \Omega \in \mathscr{F},  \tag{11.1.1}\\
E \in \mathscr{F} \text { implies } E^{C} \equiv \Omega \backslash E \in \mathscr{F}, \\
\text { If }\left\{E_{n}\right\}_{n=1}^{\infty} \subseteq \mathscr{F}, \text { then } \cup_{n=1}^{\infty} E_{n} \in \mathscr{F} . \tag{11.1.2}
\end{gather*}
$$

Definition 11.1.1 A collection of subsets of a set, $\Omega$, satisfying Formulas 11.1.1-11.1.2 is called a $\sigma$ algebra.

As an example, let $\Omega$ be any set and let $\mathscr{F}=\mathscr{P}(\Omega)$, the set of all subsets of $\Omega$ (power set). This obviously satisfies Formulas 11.1.1-11.1.2.

Lemma 11.1.2 Let $\mathscr{C}$ be a set whose elements are $\sigma$ algebras of subsets of $\Omega$. Then $\cap \mathscr{C}$ is a $\sigma$ algebra also.

Be sure to verify this lemma. It follows immediately from the above definitions but it is important for you to check the details.

Example 11.1.3 Let $\tau$ denote the collection of all open sets in $\mathbb{R}^{n}$ and let $\sigma(\tau) \equiv$ intersection of all $\sigma$ algebras that contain $\tau . \sigma(\tau)$ is called the $\sigma$ algebra of Borel sets. In general, for a collection of sets, $\Sigma, \sigma(\Sigma)$ is the smallest $\sigma$ algebra which contains $\Sigma$.

This is a very important $\sigma$ algebra and it will be referred to frequently as the Borel sets. Attempts to describe a typical Borel set are more trouble than they are worth and it is not easy to do so. Rather, one uses the definition just given in the example. Note, however, that all countable intersections of open sets and countable unions of closed sets are Borel sets. Such sets are called $G_{\delta}$ and $F_{\sigma}$ respectively.

Definition 11.1.4 Let $\mathscr{F}$ be a $\sigma$ algebra of sets of $\Omega$ and let $\mu: \mathscr{F} \rightarrow[0, \infty] . \mu$ is called $a$ measure if

$$
\begin{equation*}
\mu\left(\bigcup_{i=1}^{\infty} E_{i}\right)=\sum_{i=1}^{\infty} \mu\left(E_{i}\right) \tag{11.1.3}
\end{equation*}
$$

whenever the $E_{i}$ are disjoint sets of $\mathscr{F}$. The triple, $(\Omega, \mathscr{F}, \mu)$ is called a measure space and the elements of $\mathscr{F}$ are called the measurable sets. $(\Omega, \mathscr{F}, \mu)$ is a finite measure space when $\mu(\Omega)<\infty$.

Note that the above definition immediately implies that if $E_{i} \in \mathscr{F}$ and the sets $E_{i}$ are not necessarily disjoint,

$$
\mu\left(\bigcup_{i=1}^{\infty} E_{i}\right) \leq \sum_{i=1}^{\infty} \mu\left(E_{i}\right)
$$

To see this, let $F_{1} \equiv E_{1}, F_{2} \equiv E_{2} \backslash E_{1}, \cdots, F_{n} \equiv E_{n} \backslash \cup_{i=1}^{n-1} E_{i}$, then the sets $F_{i}$ are disjoint sets in $\mathscr{F}$ and

$$
\mu\left(\bigcup_{i=1}^{\infty} E_{i}\right)=\mu\left(\bigcup_{i=1}^{\infty} F_{i}\right)=\sum_{i=1}^{\infty} \mu\left(F_{i}\right) \leq \sum_{i=1}^{\infty} \mu\left(E_{i}\right)
$$

because of the fact that each $E_{i} \supseteq F_{i}$ and so

$$
\mu\left(E_{i}\right)=\mu\left(F_{i}\right)+\mu\left(E_{i} \backslash F_{i}\right)
$$

which implies $\mu\left(E_{i}\right) \geq \mu\left(F_{i}\right)$.
The following theorem is the basis for most of what is done in the theory of measure and integration. It is a very simple result which follows directly from the above definition.

Theorem 11.1.5 Let $\left\{E_{m}\right\}_{m=1}^{\infty}$ be measurable sets in a measure space $(\Omega, \mathscr{F}, \mu)$. Then if $\cdots E_{n} \subseteq E_{n+1} \subseteq E_{n+2} \subseteq \cdots$,

$$
\begin{equation*}
\mu\left(\cup_{i=1}^{\infty} E_{i}\right)=\lim _{n \rightarrow \infty} \mu\left(E_{n}\right) \tag{11.1.4}
\end{equation*}
$$

and if $\cdots E_{n} \supseteq E_{n+1} \supseteq E_{n+2} \supseteq \cdots$ and $\mu\left(E_{1}\right)<\infty$, then

$$
\begin{equation*}
\mu\left(\cap_{i=1}^{\infty} E_{i}\right)=\lim _{n \rightarrow \infty} \mu\left(E_{n}\right) \tag{11.1.5}
\end{equation*}
$$

Stated more succinctly, $E_{k} \uparrow E$ implies $\mu\left(E_{k}\right) \uparrow \mu(E)$ and $E_{k} \downarrow E$ with $\mu\left(E_{1}\right)<\infty$ implies $\mu\left(E_{k}\right) \downarrow \mu(E)$.

Proof: First note that $\cap_{i=1}^{\infty} E_{i}=\left(\cup_{i=1}^{\infty} E_{i}^{C}\right)^{C} \in \mathscr{F}$ so $\cap_{i=1}^{\infty} E_{i}$ is measurable. Also note that for $A$ and $B$ sets of $\mathscr{F}, A \backslash B \equiv\left(A^{C} \cup B\right)^{C} \in \mathscr{F}$. To show 11.1.4, note that 11.1.4 is obviously true if $\mu\left(E_{k}\right)=\infty$ for any $k$. Therefore, assume $\mu\left(E_{k}\right)<\infty$ for all $k$. Thus

$$
\mu\left(E_{k+1} \backslash E_{k}\right)+\mu\left(E_{k}\right)=\mu\left(E_{k+1}\right)
$$

and so

$$
\mu\left(E_{k+1} \backslash E_{k}\right)=\mu\left(E_{k+1}\right)-\mu\left(E_{k}\right)
$$

Also,

$$
\bigcup_{k=1}^{\infty} E_{k}=E_{1} \cup \bigcup_{k=1}^{\infty}\left(E_{k+1} \backslash E_{k}\right)
$$

and the sets in the above union are disjoint. Hence by 11.1.3,

$$
\begin{aligned}
\mu\left(\cup_{i=1}^{\infty} E_{i}\right)= & \mu\left(E_{1}\right)+\sum_{k=1}^{\infty} \mu\left(E_{k+1} \backslash E_{k}\right)=\mu\left(E_{1}\right) \\
& +\sum_{k=1}^{\infty} \mu\left(E_{k+1}\right)-\mu\left(E_{k}\right)
\end{aligned}
$$

$$
=\mu\left(E_{1}\right)+\lim _{n \rightarrow \infty} \sum_{k=1}^{n} \mu\left(E_{k+1}\right)-\mu\left(E_{k}\right)=\lim _{n \rightarrow \infty} \mu\left(E_{n+1}\right) .
$$

This shows part 11.1.4.
To verify 11.1.5,

$$
\mu\left(E_{1}\right)=\mu\left(\cap_{i=1}^{\infty} E_{i}\right)+\mu\left(E_{1} \backslash \cap_{i=1}^{\infty} E_{i}\right)
$$

since $\mu\left(E_{1}\right)<\infty$, it follows $\mu\left(\cap_{i=1}^{\infty} E_{i}\right)<\infty$. Also, $E_{1} \backslash \cap_{i=1}^{n} E_{i} \uparrow E_{1} \backslash \cap_{i=1}^{\infty} E_{i}$ and so by 11.1.4,

$$
\begin{gathered}
\mu\left(E_{1}\right)-\mu\left(\cap_{i=1}^{\infty} E_{i}\right)=\mu\left(E_{1} \backslash \cap_{i=1}^{\infty} E_{i}\right)=\lim _{n \rightarrow \infty} \mu\left(E_{1} \backslash \cap_{i=1}^{n} E_{i}\right) \\
=\mu\left(E_{1}\right)-\lim _{n \rightarrow \infty} \mu\left(\cap_{i=1}^{n} E_{i}\right)=\mu\left(E_{1}\right)-\lim _{n \rightarrow \infty} \mu\left(E_{n}\right)
\end{gathered}
$$

Hence, subtracting $\mu\left(E_{1}\right)$ from both sides,

$$
\lim _{n \rightarrow \infty} \mu\left(E_{n}\right)=\mu\left(\cap_{i=1}^{\infty} E_{i}\right)
$$

This proves the theorem.
It is convenient to allow functions to take the value $+\infty$. You should think of $+\infty$, usually referred to as $\infty$ as something out at the right end of the real line and its only importance is the notion of sequences converging to it. $x_{n} \rightarrow \infty$ exactly when for all $l \in \mathbb{R}$, there exists $N$ such that if $n \geq N$, then

$$
x_{n}>l
$$

This is what it means for a sequence to converge to $\infty$. Don't think of $\infty$ as a number. It is just a convenient symbol which allows the consideration of some limit operations more simply. Similar considerations apply to $-\infty$ but this value is not of very great interest. In fact the set of most interest is the complex numbers or some vector space. Therefore, this topic is not considered.

Lemma 11.1.6 Let $f: \Omega \rightarrow(-\infty, \infty]$ where $\mathscr{F}$ is a $\sigma$ algebra of subsets of $\Omega$. Then the following are equivalent.

$$
\begin{gathered}
f^{-1}((d, \infty]) \in \mathscr{F} \text { for all finite } d, \\
f^{-1}((-\infty, d)) \in \mathscr{F} \text { for all finite } d, \\
f^{-1}([d, \infty]) \in \mathscr{F} \text { for all finite } d, \\
f^{-1}((-\infty, d]) \in \mathscr{F} \text { for all finite } d, \\
f^{-1}((a, b)) \in \mathscr{F} \text { for all } a<b,-\infty<a<b<\infty .
\end{gathered}
$$

Proof: First note that the first and the third are equivalent. To see this, observe

$$
f^{-1}([d, \infty])=\cap_{n=1}^{\infty} f^{-1}((d-1 / n, \infty])
$$

and so if the first condition holds, then so does the third.

$$
f^{-1}((d, \infty])=\cup_{n=1}^{\infty} f^{-1}([d+1 / n, \infty])
$$

and so if the third condition holds, so does the first.
Similarly, the second and fourth conditions are equivalent. Now

$$
f^{-1}((-\infty, d])=\left(f^{-1}((d, \infty])\right)^{C}
$$

so the first and fourth conditions are equivalent. Thus the first four conditions are equivalent and if any of them hold, then for $-\infty<a<b<\infty$,

$$
f^{-1}((a, b))=f^{-1}((-\infty, b)) \cap f^{-1}((a, \infty]) \in \mathscr{F} .
$$

Finally, if the last condition holds,

$$
f^{-1}([d, \infty])=\left(\cup_{k=1}^{\infty} f^{-1}((-k+d, d))\right)^{C} \in \mathscr{F}
$$

and so the third condition holds. Therefore, all five conditions are equivalent. This proves the lemma.

This lemma allows for the following definition of a measurable function having values in $(-\infty, \infty]$.

Definition 11.1.7 Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and let $f: \Omega \rightarrow(-\infty, \infty]$. Then $f$ is said to be measurable if any of the equivalent conditions of Lemma 11.1.6 hold. When the $\sigma$ algebra, $\mathscr{F}$ equals the Borel $\sigma$ algebra, $\mathscr{B}$, the function is called Borel measurable. More generally, if $f: \Omega \rightarrow X$ where $X$ is a topological space, $f$ is said to be measurable if $f^{-1}(U) \in \mathscr{F}$ whenever $U$ is open.

Theorem 11.1.8 Let $f_{n}$ and $f$ be functions mapping $\Omega$ to $(-\infty, \infty]$ where $\mathscr{F}$ is a $\sigma$ algebra of measurable sets of $\Omega$. Then if $f_{n}$ is measurable, and $f(\omega)=\lim _{n \rightarrow \infty} f_{n}(\omega)$, it follows that $f$ is also measurable. (Pointwise limits of measurable functions are measurable.)

Proof: First it is shown $f^{-1}((a, b)) \in \mathscr{F}$. Let

$$
V_{m} \equiv\left(a+\frac{1}{m}, b-\frac{1}{m}\right), \bar{V}_{m}=\left[a+\frac{1}{m}, b-\frac{1}{m}\right]
$$

Then for all $m, V_{m} \subseteq(a, b)$ and

$$
(a, b)=\cup_{m=1}^{\infty} V_{m}=\cup_{m=1}^{\infty} \bar{V}_{m}
$$

Note that $V_{m} \neq \emptyset$ for all $m$ large enough. Since $f$ is the pointwise limit of $f_{n}$,

$$
f^{-1}\left(V_{m}\right) \subseteq\left\{\omega: f_{k}(\omega) \in V_{m} \text { for all } k \text { large enough }\right\} \subseteq f^{-1}\left(\bar{V}_{m}\right)
$$

You should note that the expression in the middle is of the form

$$
\cup_{n=1}^{\infty} \cap_{k=n}^{\infty} f_{k}^{-1}\left(V_{m}\right)
$$

Therefore,

$$
f^{-1}((a, b))=\cup_{m=1}^{\infty} f^{-1}\left(V_{m}\right) \subseteq \cup_{m=1}^{\infty} \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} f_{k}^{-1}\left(V_{m}\right)
$$

$$
\subseteq \cup_{m=1}^{\infty} f^{-1}\left(\bar{V}_{m}\right)=f^{-1}((a, b))
$$

It follows $f^{-1}((a, b)) \in \mathscr{F}$ because it equals the expression in the middle which is measurable. This shows $f$ is measurable.

The following theorem considers the case of functions which have values in a metric space. Its proof is similar to the proof of the above.
Theorem 11.1.9 Let $\left\{f_{n}\right\}$ be a sequence of measurable functions mapping $\Omega$ to $(X, d)$ where $(X, d)$ is a metric space and $(\Omega, \mathscr{F})$ is a measure space. Suppose also that $f(\omega)=$ $\lim _{n \rightarrow \infty} f_{n}(\omega)$ for all $\omega$. Then $f$ is also a measurable function.

Proof: It is required to show $f^{-1}(U)$ is measurable for all $U$ open. Let

$$
V_{m} \equiv\left\{x \in U: \operatorname{dist}\left(x, U^{C}\right)>\frac{1}{m}\right\}
$$

Thus

$$
V_{m} \subseteq\left\{x \in U: \operatorname{dist}\left(x, U^{C}\right) \geq \frac{1}{m}\right\}
$$

and $V_{m} \subseteq \overline{V_{m}} \subseteq V_{m+1}$ and $\cup_{m} V_{m}=U$. Then since $V_{m}$ is open,

$$
f^{-1}\left(V_{m}\right)=\cup_{n=1}^{\infty} \cap_{k=n}^{\infty} f_{k}^{-1}\left(V_{m}\right)
$$

and so

$$
\begin{aligned}
f^{-1}(U) & =\cup_{m=1}^{\infty} f^{-1}\left(V_{m}\right) \\
& =\cup_{m=1}^{\infty} \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} f_{k}^{-1}\left(V_{m}\right) \\
& \subseteq \cup_{m=1}^{\infty} f^{-1}\left(\overline{V_{m}}\right)=f^{-1}(U)
\end{aligned}
$$

which shows $f^{-1}(U)$ is measurable. This proves the theorem.
Now here is a simple observation.
Observation 11.1.10 Let $f: \Omega \rightarrow X$ where $X$ is some topological space. Suppose

$$
f(\omega)=\sum_{k=1}^{m} x_{k} \mathscr{X}_{A_{k}}(\omega)
$$

where each $x_{k} \in X$ and the $A_{k}$ are disjoint measurable sets. (Such functions are often referred to as simple functions.) The sum means the function has value $x_{k}$ on set $A_{k}$. Then $f$ is measurable.

Proof: Letting $U$ be open, $f^{-1}(U)=\cup\left\{A_{k}: x_{k} \in U\right\}$, a finite union of measurable sets.

There is also a very interesting theorem due to Kuratowski [82] which is presented next.
To summarize the proof, you get an increasing sequence of $2^{-n}$ nets $C_{n}$ and you obtain a corresponding sequence of simple functions $\left\{s_{n}\right\}$ such that $\left\{s_{n}(\omega)\right\}_{n=1}^{\infty}$ is a Cauchy sequence, and the maximum value of $x \rightarrow \psi(x, \omega)$ for $x \in C_{n}$ equals $\psi\left(s_{n}(\omega), \omega\right)$. Then you let $f(\omega) \equiv \lim _{n \rightarrow \infty} s_{n}(\omega)$. Thus $f(\omega)$ is measurable and

$$
\sup _{x \in E} \psi(x, \omega) \geq \psi(f(\omega), \omega)=\lim _{n \rightarrow \infty} \psi\left(s_{n}(\omega), \omega\right) \geq \sup _{x \in C_{n}} \psi(x, \omega)
$$

Thus, by continuity in the first entry,

$$
\begin{aligned}
\sup _{x \in E} \psi(x, \omega) & \geq \psi(f(\omega), \omega) \geq \lim _{n \rightarrow \infty} \psi\left(s_{n}(\omega), \omega\right) \\
& \geq \sup _{n} \sup _{x \in C_{n}} \psi(x, \omega)=\sup _{x \in \cup_{n} C_{n}} \psi(x, \omega)=\sup _{x \in E} \psi(x, \omega)
\end{aligned}
$$

Theorem 11.1.11 Let $E$ be a compact metric space and let $(\Omega, \mathscr{F})$ be a measure space. Suppose $\psi: E \times \Omega \rightarrow \mathbb{R}$ has the property that $x \rightarrow \psi(x, \omega)$ is continuous and $\omega \rightarrow \psi(x, \omega)$ is measurable. Then there exists a measurable function, $f$ having values in $E$ such that

$$
\psi(f(\omega), \omega)=\sup _{x \in E} \psi(x, \omega) .
$$

Furthermore, $\omega \rightarrow \psi(f(\omega), \omega)$ is measurable.
Proof: Let $C_{1}$ be a $2^{-1}$ net of $E$. Suppose $C_{1}, \cdots, C_{m}$ have been chosen such that $C_{k}$ is a $2^{-k}$ net and $C_{i+1} \supseteq C_{i}$ for all $i$. Then consider $E \backslash \cup\left\{B\left(x, 2^{-(m+1)}\right): x \in C_{m}\right\}$. If this set is empty, let $C_{m+1}=C_{m}$. If it is nonempty, let $\left\{y_{i}\right\}_{i=1}^{r}$ be a $2^{-(m+1)}$ net for this compact set. Then let $C_{m+1}=C_{m} \cup\left\{y_{i}\right\}_{i=1}^{r}$. It follows $\left\{C_{m}\right\}_{m=1}^{\infty}$ satisfies $C_{m}$ is a $2^{-m}$ net and $C_{m} \subseteq C_{m+1}$.

Let $\left\{x_{k}^{1}\right\}_{k=1}^{m(1)}$ equal $C_{1}$. Let

$$
A_{1}^{1} \equiv\left\{\omega: \psi\left(x_{1}^{1}, \omega\right)=\max _{k} \psi\left(x_{k}^{1}, \omega\right)\right\}
$$

For $\omega \in A_{1}^{1}$, define $s_{1}(\omega) \equiv x_{1}^{1}$. Next let

$$
A_{2}^{1} \equiv\left\{\omega \notin A_{1}^{1}: \psi\left(x_{2}^{1}, \omega\right)=\max _{k} \psi\left(x_{k}^{1}, \omega\right)\right\}
$$

and let $s_{1}(\omega) \equiv x_{2}^{1}$ on $A_{2}^{1}$. Continue in this way to obtain a simple function, $s_{1}$ such that

$$
\psi\left(s_{1}(\omega), \omega\right)=\max \left\{\psi(x, \omega): x \in C_{1}\right\}
$$

and $s_{1}$ has values in $C_{1}$.
Suppose $s_{1}(\omega), s_{2}(\omega), \cdots, s_{m}(\omega)$ are simple functions with the property that if $m>1$,

$$
\begin{aligned}
d\left(s_{k}(\omega), s_{k+1}(\omega)\right)< & 2^{-k} \\
\psi\left(s_{k}(\omega), \omega\right)= & \max \left\{\psi(x, \omega): x \in C_{k}\right\} \\
& s_{k} \text { has values in } C_{k}
\end{aligned}
$$

for each $k+1 \leq m$, only the second and third assertions holding if $m=1$. Letting $C_{m}=$ $\left\{x_{k}\right\}_{k=1}^{N}$, it follows $s_{m}(\omega)$ is of the form

$$
\begin{equation*}
s_{m}(\omega)=\sum_{k=1}^{N} x_{k} \mathscr{X}_{A_{k}}(\omega), A_{i} \cap A_{j}=\emptyset \tag{11.1.6}
\end{equation*}
$$

meaning that $s_{m}(\omega)$ has value $x_{k}$ on $A_{k}$. Denote by $\left\{y_{1 i}\right\}_{i=1}^{n_{1}}$ those points of $C_{m+1}$ which are contained in $B\left(x_{1}, 2^{-m}\right)$. Letting $A_{k}$ play the role of $\Omega$ in the first step in which $s_{1}$ was constructed, for each $\omega \in A_{1}$ let $s_{m+1}(\omega)$ be a simple function which has one of the values $y_{1 i}$ and satisfies

$$
\psi\left(s_{m+1}(\omega), \omega\right)=\max _{i \leq n_{1}} \psi\left(y_{1 i}, \omega\right)
$$

for each $\omega \in A_{1}$. Next let $\left\{y_{2 i}\right\}_{i=1}^{n_{2}}$ be those points of $C_{m+1}$ different than $\left\{y_{1 i}\right\}_{i=1}^{n_{1}}$ which are contained in $B\left(x_{2}, 2^{-m}\right)$. Then define $s_{m+1}(\omega)$ on $A_{2}$ to have values taken from $\left\{y_{2 i}\right\}_{i=1}^{n_{2}}$ and

$$
\psi\left(s_{m+1}(\omega), \omega\right)=\max _{i \leq n_{2}} \psi\left(y_{2 i}, \omega\right)
$$

for each $\omega \in A_{2}$. Continuing this way defines $s_{m+1}$ on all of $\Omega$ and it satisfies

$$
\begin{equation*}
d\left(s_{m}(\omega), s_{m+1}(\omega)\right)<2^{-m} \text { for all } \omega \in \Omega \tag{11.1.7}
\end{equation*}
$$

It remains to verify

$$
\begin{equation*}
\psi\left(s_{m+1}(\omega), \omega\right)=\max \left\{\psi(x, \omega): x \in C_{m+1}\right\} \tag{11.1.8}
\end{equation*}
$$

To see this is so, pick $\omega \in \Omega$. Let

$$
\begin{equation*}
\max \left\{\psi(x, \omega): x \in C_{m+1}\right\}=\psi\left(y_{r j}, \omega\right) \tag{11.1.9}
\end{equation*}
$$

where $y_{r j} \in\left\{y_{r i}\right\}_{i=1}^{n_{r}} \subseteq C_{m+1}$ and out of all the balls $B\left(x_{l}, 2^{-m}\right)$, let the first one which contains $y_{r j}$ be $B\left(x_{k}, 2^{-m}\right)$. Then by the construction, $s_{m+1}(\omega)=y_{r j}$ because $\psi\left(y_{r j}, \omega\right)$ is at least as large as $\psi\left(y_{s j}, \omega\right)$ for all the other $y_{s j}$. This and 11.1.9 verifies 11.1.8.

From 11.1.7 it follows $s_{m}(\omega)$ converges uniformly on $\Omega$ to a measurable function, $f(\omega)$. Then from the construction, $\psi(f(\omega), \omega) \geq \psi\left(s_{m}(\omega), \omega\right)$ for all $m$ and $\omega$. Now pick $\omega \in \Omega$ and let $z$ be such that $\psi(z, \omega)=\max _{x \in E} \psi(x, \omega)$. Letting $y_{k} \rightarrow z$ where $y_{k} \in C_{k}$, it follows from continuity of $\psi$ in the first argument that

$$
\begin{aligned}
\max _{x \in E} \psi(x, \omega) & =\psi(z, \omega)=\lim _{k \rightarrow \infty} \psi\left(y_{k}, \omega\right) \\
& \leq \lim _{m \rightarrow \infty} \psi\left(s_{m}(\omega), \omega\right)=\psi(f(\omega), \omega) \leq \max _{x \in E} \psi(x, \omega)
\end{aligned}
$$

To show $\omega \rightarrow \psi(f(\omega), \omega)$ is measurable, note that since $E$ is compact, there exists a countable dense subset, $D$. Then using continuity of $\psi$ in the first argument,

$$
\begin{aligned}
\psi(f(\omega), \omega) & =\sup _{x \in E} \psi(x, \omega) \\
& =\sup _{x \in D} \psi(x, \omega)
\end{aligned}
$$

which equals a measurable function of $\omega$ because $D$ is countable.
Theorem 11.1.12 Let $\mathscr{B}$ consist of open cubes of the form

$$
Q_{\mathbf{x}} \equiv \prod_{i=1}^{n}\left(x_{i}-\delta, x_{i}+\delta\right)
$$

where $\delta$ is a positive rational number and $\mathbf{x} \in \mathbb{Q}^{n}$. Then every open set in $\mathbb{R}^{n}$ can be written as a countable union of open cubes from $\mathscr{B}$. Furthermore, $\mathscr{B}$ is a countable set.

Proof: Let $U$ be an open set and let $\mathbf{y} \in U$. Since $U$ is open, $B(\mathbf{y}, r) \subseteq U$ for some $r>0$ and it can be assumed $r / \sqrt{n} \in \mathbb{Q}$. Let

$$
\mathbf{x} \in B\left(\mathbf{y}, \frac{r}{10 \sqrt{n}}\right) \cap \mathbb{Q}^{n}
$$

and consider the cube, $Q_{\mathbf{x}} \in \mathscr{B}$ defined by

$$
Q_{\mathbf{x}} \equiv \prod_{i=1}^{n}\left(x_{i}-\boldsymbol{\delta}, x_{i}+\boldsymbol{\delta}\right)
$$

where $\delta=r / 4 \sqrt{n}$. The following picture is roughly illustrative of what is taking place.


Then the diameter of $Q_{\mathbf{x}}$ equals

$$
\left(n\left(\frac{r}{2 \sqrt{n}}\right)^{2}\right)^{1 / 2}=\frac{r}{2}
$$

and so, if $\mathbf{z} \in Q_{\mathbf{x}}$, then

$$
\begin{aligned}
|\mathbf{z}-\mathbf{y}| & \leq|\mathbf{z}-\mathbf{x}|+|\mathbf{x}-\mathbf{y}| \\
& <\frac{r}{2}+\frac{r}{2}=r .
\end{aligned}
$$

Consequently, $Q_{\mathbf{x}} \subseteq U$. Now also,

$$
\left(\sum_{i=1}^{n}\left(x_{i}-y_{i}\right)^{2}\right)^{1 / 2}<\frac{r}{10 \sqrt{n}}
$$

and so it follows that for each $i$,

$$
\left|x_{i}-y_{i}\right|<\frac{r}{4 \sqrt{n}}
$$

since otherwise the above inequality would not hold. Therefore, $\mathbf{y} \in Q_{\mathbf{x}} \subseteq U$. Now let $\mathscr{B}_{U}$ denote those sets of $\mathscr{B}$ which are contained in $U$. Then $\cup \mathscr{B}_{U}=U$.

To see $\mathscr{B}$ is countable, note there are countably many choices for $\mathbf{x}$ and countably many choices for $\delta$. This proves the theorem.

Recall that $g: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is continuous means $g^{-1}$ (open set) $=$ an open set. In particular $g^{-1}((a, b))$ must be an open set.

Theorem 11.1.13 Let $f_{i}: \Omega \rightarrow \mathbb{R}$ for $i=1, \cdots, n$ be measurable functions and let $g: \mathbb{R}^{n} \rightarrow$ $\mathbb{R}$ be continuous where $\mathbf{f} \equiv\left(f_{1} \cdots f_{n}\right)^{T}$. Then $g \circ \mathbf{f}$ is a measurable function from $\Omega$ to $\mathbb{R}$.

Proof: First it is shown

$$
(g \circ \mathbf{f})^{-1}((a, b)) \in \mathscr{F} .
$$

Now $(g \circ \mathbf{f})^{-1}((a, b))=\mathbf{f}^{-1}\left(g^{-1}((a, b))\right)$. Since $g$ is continuous, it follows that $g^{-1}((a, b))$ is an open set which is denoted as $U$ for convenience. Now by Theorem 11.1.12 above, it follows there are countably many open cubes, $\left\{Q_{k}\right\}$ such that

$$
U=\cup_{k=1}^{\infty} Q_{k}
$$

where each $Q_{k}$ is a cube of the form

$$
Q_{k}=\prod_{i=1}^{n}\left(x_{i}-\delta, x_{i}+\delta\right)
$$

Now

$$
\mathbf{f}^{-1}\left(\prod_{i=1}^{n}\left(x_{i}-\boldsymbol{\delta}, x_{i}+\boldsymbol{\delta}\right)\right)=\cap_{i=1}^{n} f_{i}^{-1}\left(\left(x_{i}-\boldsymbol{\delta}, x_{i}+\boldsymbol{\delta}\right)\right) \in \mathscr{F}
$$

and so

$$
\begin{aligned}
(g \circ \mathbf{f})^{-1}((a, b)) & =\mathbf{f}^{-1}\left(g^{-1}((a, b))\right)=\mathbf{f}^{-1}(U) \\
& =\mathbf{f}^{-1}\left(\cup_{k=1}^{\infty} Q_{k}\right)=\cup_{k=1}^{\infty} \mathbf{f}^{-1}\left(Q_{k}\right) \in \mathscr{F} .
\end{aligned}
$$

This proves the theorem.
Corollary 11.1.14 Sums, products, and linear combinations of measurable functions are measurable.

Proof: To see the product of two measurable functions is measurable, let $g(x, y)=x y$, a continuous function defined on $\mathbb{R}^{2}$. Thus if you have two measurable functions, $f_{1}$ and $f_{2}$ defined on $\Omega$,

$$
g \circ\left(f_{1}, f_{2}\right)(\omega)=f_{1}(\omega) f_{2}(\omega)
$$

and so $\omega \rightarrow f_{1}(\omega) f_{2}(\omega)$ is measurable. Similarly you can show the sum of two measurable functions is measurable by considering $g(x, y)=x+y$ and you can show a linear combination of two measurable functions is measurable by considering $g(x, y)=a x+b y$. More than two functions can also be considered as well.

The message of this corollary is that starting with measurable real valued functions you can combine them in pretty much any way you want and you end up with a measurable function.

Here is some notation which will be used whenever convenient.

Definition 11.1.15 Let $f: \Omega \rightarrow[-\infty, \infty]$. Define

$$
[\alpha<f] \equiv\{\omega \in \Omega: f(\omega)>\alpha\} \equiv f^{-1}((\alpha, \infty])
$$

with obvious modifications for the symbols $[\alpha \leq f],[\alpha \geq f],[\alpha \geq f \geq \beta]$, etc.
Definition 11.1.16 For a set $E$,

$$
\mathscr{X}_{E}(\omega)=\left\{\begin{array}{l}
1 \text { if } \omega \in E, \\
0 \text { if } \omega \notin E .
\end{array}\right.
$$

This is called the characteristic function of $E$. Sometimes this is called the indicator function which I think is better terminology since the term characteristic function has another meaning. Note that this "indicates" whether a point, $\omega$ is contained in $E$. It is exactly when the function has the value 1 .

Theorem 11.1.17 (Egoroff) Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space,

$$
(\mu(\Omega)<\infty)
$$

and let $f_{n}, f$ be complex valued functions such that $\operatorname{Re} f_{n}, \operatorname{Im} f_{n}$ are all measurable and

$$
\lim _{n \rightarrow \infty} f_{n}(\omega)=f(\omega)
$$

for all $\omega \notin E$ where $\mu(E)=0$. Then for every $\varepsilon>0$, there exists a set,

$$
F \supseteq E, \mu(F)<\varepsilon
$$

such that $f_{n}$ converges uniformly to $f$ on $F^{C}$.
Proof: First suppose $E=\emptyset$ so that convergence is pointwise everywhere. It follows then that $\operatorname{Re} f$ and $\operatorname{Im} f$ are pointwise limits of measurable functions and are therefore measurable. Let $E_{k m}=\left\{\omega \in \Omega:\left|f_{n}(\omega)-f(\omega)\right| \geq 1 / m\right.$ for some $\left.n>k\right\}$. Note that

$$
\left|f_{n}(\omega)-f(\omega)\right|=\sqrt{\left(\operatorname{Re} f_{n}(\omega)-\operatorname{Re} f(\omega)\right)^{2}+\left(\operatorname{Im} f_{n}(\omega)-\operatorname{Im} f(\omega)\right)^{2}}
$$

and so, By Theorem 11.1.13,

$$
\left[\left|f_{n}-f\right| \geq \frac{1}{m}\right]
$$

is measurable. Hence $E_{k m}$ is measurable because

$$
E_{k m}=\cup_{n=k+1}^{\infty}\left[\left|f_{n}-f\right| \geq \frac{1}{m}\right]
$$

For fixed $m, \cap_{k=1}^{\infty} E_{k m}=\emptyset$ because $f_{n}$ converges to $f$. Therefore, if $\omega \in \Omega$ there exists $k$ such that if $n>k,\left|f_{n}(\omega)-f(\omega)\right|<\frac{1}{m}$ which means $\omega \notin E_{k m}$. Note also that

$$
E_{k m} \supseteq E_{(k+1) m}
$$

Since $\mu\left(E_{1 m}\right)<\infty$, Theorem 11.1.5 on Page 224 implies

$$
0=\mu\left(\cap_{k=1}^{\infty} E_{k m}\right)=\lim _{k \rightarrow \infty} \mu\left(E_{k m}\right)
$$

Let $k(m)$ be chosen such that $\mu\left(E_{k(m) m}\right)<\varepsilon 2^{-m}$ and let

$$
F=\bigcup_{m=1}^{\infty} E_{k(m) m}
$$

Then $\mu(F)<\varepsilon$ because

$$
\mu(F) \leq \sum_{m=1}^{\infty} \mu\left(E_{k(m) m}\right)<\sum_{m=1}^{\infty} \varepsilon 2^{-m}=\varepsilon
$$

Now let $\eta>0$ be given and pick $m_{0}$ such that $m_{0}^{-1}<\eta$. If $\omega \in F^{C}$, then

$$
\omega \in \bigcap_{m=1}^{\infty} E_{k(m) m}^{C}
$$

Hence $\omega \in E_{k\left(m_{0}\right) m_{0}}^{C}$ so

$$
\left|f_{n}(\omega)-f(\omega)\right|<1 / m_{0}<\eta
$$

for all $n>k\left(m_{0}\right)$. This holds for all $\omega \in F^{C}$ and so $f_{n}$ converges uniformly to $f$ on $F^{C}$.
Now if $E \neq \emptyset$, consider $\left\{\mathscr{X}_{E^{C}} f_{n}\right\}_{n=1}^{\infty}$. Each $\mathscr{X}_{E^{C}} f_{n}$ has real and imaginary parts measurable and the sequence converges pointwise to $\mathscr{X}_{E} f$ everywhere. Therefore, from the first part, there exists a set of measure less than $\varepsilon, F$ such that on $F^{C},\left\{\mathscr{X}_{E^{C}} f_{n}\right\}$ converges uniformly to $\mathscr{X}_{E^{C}} f$. Therefore, on $(E \cup F)^{C},\left\{f_{n}\right\}$ converges uniformly to $f$. This proves the theorem.

Finally here is a comment about notation.

Definition 11.1.18 Something happens for $\mu$ a.e. $\omega$ said as $\mu$ almost everywhere, if there exists $a$ set $E$ with $\mu(E)=0$ and the thing takes place for all $\omega \notin E$. Thus $f(\omega)=g(\omega)$ a.e. if $f(\omega)=g(\omega)$ for all $\omega \notin E$ where $\mu(E)=0$. A measure space, $(\Omega, \mathscr{F}, \mu)$ is $\sigma$ finite if there exist measurable sets, $\Omega_{n}$ such that $\mu\left(\Omega_{n}\right)<\infty$ and $\Omega=\cup_{n=1}^{\infty} \Omega_{n}$.

### 11.2 Exercises

1. Let $\Omega=\mathbb{N}=\{1,2, \cdots\}$. Let $\mathscr{F}=\mathscr{P}(\mathbb{N})$ and let $\mu(S)=$ number of elements in $S$. Thus $\mu(\{1\})=1=\mu(\{2\}), \mu(\{1,2\})=2$, etc. $\operatorname{Show}(\Omega, \mathscr{F}, \mu)$ is a measure space. It is called counting measure. What functions are measurable in this case?
2. Let $\Omega$ be any uncountable set and let $\mathscr{F}=\left\{A \subseteq \Omega\right.$ : either $A$ or $A^{C}$ is countable $\}$. Let $\mu(A)=1$ if $A$ is uncountable and $\mu(A)=0$ if $A$ is countable. Show $(\Omega, \mathscr{F}, \mu)$ is a measure space. This is a well known bad example.
3. Let $\mathscr{F}$ be a $\sigma$ algebra of subsets of $\Omega$ and suppose $\mathscr{F}$ has infinitely many elements. Show that $\mathscr{F}$ is uncountable. Hint: You might try to show there exists a countable sequence of disjoint sets of $\mathscr{F},\left\{A_{i}\right\}$. It might be easiest to verify this by contradiction if it doesn't exist rather than a direct construction. Once this has been done, you can define a map, $\theta$, from $\mathscr{P}(\mathbb{N})$ into $\mathscr{F}$ which is one to one by $\theta(S)=\cup_{i \in S} A_{i}$. Then argue $\mathscr{P}(\mathbb{N})$ is uncountable and so $\mathscr{F}$ is also uncountable.

## 4. Prove Lemma 11.1.2.

5. $g$ is Borel measurable if whenever $U$ is open, $g^{-1}(U)$ is Borel. Let $\mathbf{f}: \Omega \rightarrow \mathbb{R}^{n}$ and let $g: \mathbb{R}^{n} \rightarrow \mathbb{R}$ and $\mathscr{F}$ is a $\sigma$ algebra of sets of $\Omega$. Suppose $\mathbf{f}$ is measurable and $g$ is Borel measurable. Show $g \circ \mathbf{f}$ is measurable. To say $g$ is Borel measurable means $g^{-1}($ open set $)=($ Borel set $)$ where a Borel set is one of those sets in the smallest $\sigma$ algebra containing the open sets of $\mathbb{R}^{n}$. See Lemma 11.1.2. Hint: You should show, using Theorem 11.1.12 that $\mathbf{f}^{-1}$ (open set) $\in \mathscr{F}$. Now let

$$
\mathscr{S} \equiv\left\{E \subseteq \mathbb{R}^{n}: \mathbf{f}^{-1}(E) \in \mathscr{F}\right\}
$$

By what you just showed, $\mathscr{S}$ contains the open sets. Now verify $\mathscr{S}$ is a $\sigma$ algebra. Argue that from the definition of the Borel sets, it follows $\mathscr{S}$ contains the Borel sets.
6. Let $(\Omega, \mathscr{F})$ be a measure space and suppose $f: \Omega \rightarrow \mathbb{C}$. Then $f$ is said to be mesurable if

$$
f^{-1}(\text { open set }) \in \mathscr{F} .
$$

Show $f$ is measurable if and only if $\operatorname{Re} f$ and $\operatorname{Im} f$ are measurable real-valued functions. Thus it suffices to define a complex valued function to be measurable if the real and imaginary parts are measurable. Hint: Argue that $f^{-1}(((a, b)+i(c, d)))=$ $(\operatorname{Re} f)^{-1}((a, b)) \cap(\operatorname{Im} f)^{-1}((c, d))$. Then use Theorem 11.1.12 to verify that if $\operatorname{Re} f$ and $\operatorname{Im} f$ are measurable, it follows $f$ is. Conversely, argue that $(\operatorname{Re} f)^{-1}((a, b))=$ $f^{-1}((a, b)+i \mathbb{R})$ with a similar formula holding for $\operatorname{Im} f$.
7. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. Define $\bar{\mu}: \mathscr{P}(\Omega) \rightarrow[0, \infty]$ by

$$
\bar{\mu}(A)=\inf \{\mu(B): B \supseteq A, B \in \mathscr{F}\} .
$$

Show $\bar{\mu}$ satisfies

$$
\begin{aligned}
\bar{\mu}(\emptyset) & =0, \text { if } A \subseteq B, \bar{\mu}(A) \leq \bar{\mu}(B) \\
\bar{\mu}\left(\cup_{i=1}^{\infty} A_{i}\right) & \leq \sum_{i=1}^{\infty} \bar{\mu}\left(A_{i}\right), \mu(A)=\bar{\mu}(A) \text { if } A \in \mathscr{F}
\end{aligned}
$$

If $\bar{\mu}$ satisfies these conditions, it is called an outer measure. This shows every measure determines an outer measure on the power set.
8. Let $\left\{E_{i}\right\}$ be a sequence of measurable sets with the property that

$$
\sum_{i=1}^{\infty} \mu\left(E_{i}\right)<\infty .
$$

Let $S=\left\{\omega \in \Omega\right.$ such that $\omega \in E_{i}$ for infinitely many values of $\left.i\right\}$. Show $\mu(S)=0$ and $S$ is measurable. This is part of the Borel Cantelli lemma. Hint: Write $S$ in terms of intersections and unions. Something is in $S$ means that for every $n$ there exists $k>n$ such that it is in $E_{k}$. Remember the tail of a convergent series is small.
9. $\uparrow$ Let $f_{n}, f$ be measurable functions. $f_{n}$ converges in measure if

$$
\lim _{n \rightarrow \infty} \mu\left(x \in \Omega:\left|f(x)-f_{n}(x)\right| \geq \varepsilon\right)=0
$$

for each fixed $\varepsilon>0$. Prove the theorem of F. Riesz. If $f_{n}$ converges to $f$ in measure, then there exists a subsequence $\left\{f_{n_{k}}\right\}$ which converges to $f$ a.e. Hint: Choose $n_{1}$ such that

$$
\mu\left(x:\left|f(x)-f_{n_{1}}(x)\right| \geq 1\right)<1 / 2
$$

Choose $n_{2}>n_{1}$ such that

$$
\mu\left(x:\left|f(x)-f_{n_{2}}(x)\right| \geq 1 / 2\right)<1 / 2^{2}
$$

$n_{3}>n_{2}$ such that

$$
\mu\left(x:\left|f(x)-f_{n_{3}}(x)\right| \geq 1 / 3\right)<1 / 2^{3}
$$

etc. Now consider what it means for $f_{n_{k}}(x)$ to fail to converge to $f(x)$. Then use Problem 8.

### 11.3 The Abstract Lebesgue Integral

### 11.3.1 Preliminary Observations

This section is on the Lebesgue integral and the major convergence theorems which are the reason for studying it. In all that follows $\mu$ will be a measure defined on a $\sigma$ algebra $\mathscr{F}$ of subsets of $\Omega .0 \cdot \infty=0$ is always defined to equal zero. This is a meaningless expression and so it can be defined arbitrarily but a little thought will soon demonstrate that this is the right definition in the context of measure theory. To see this, consider the zero function defined on $\mathbb{R}$. What should the integral of this function equal? Obviously, by an analogy with the Riemann integral, it should equal zero. Formally, it is zero times the length of the set or infinity. This is why this convention will be used.

Lemma 11.3.1 Let $f(a, b) \in[-\infty, \infty]$ for $a \in A$ and $b \in B$ where $A, B$ are sets. Then

$$
\sup _{a \in A} \sup _{b \in B} f(a, b)=\sup _{b \in B} \sup _{a \in A} f(a, b) .
$$

Proof: Note that for all $a, b, f(a, b) \leq \sup _{b \in B} \sup _{a \in A} f(a, b)$ and therefore, for all $a$,

$$
\sup _{b \in B} f(a, b) \leq \sup _{b \in B} \sup _{a \in A} f(a, b)
$$

Therefore,

$$
\sup _{a \in A} \sup _{b \in B} f(a, b) \leq \sup _{b \in B} \sup _{a \in A} f(a, b) .
$$

Repeating the same argument interchanging $a$ and $b$, gives the conclusion of the lemma.

Lemma 11.3.2 If $\left\{A_{n}\right\}$ is an increasing sequence in $[-\infty, \infty]$, then $\sup \left\{A_{n}\right\}=\lim _{n \rightarrow \infty} A_{n}$.
The following lemma is useful also and this is a good place to put it. First $\left\{b_{j}\right\}_{j=1}^{\infty}$ is an enumeration of the $a_{i j}$ if

$$
\cup_{j=1}^{\infty}\left\{b_{j}\right\}=\cup_{i, j}\left\{a_{i j}\right\}
$$

In other words, the countable set, $\left\{a_{i j}\right\}_{i, j=1}^{\infty}$ is listed as $b_{1}, b_{2}, \cdots$.
Lemma 11.3.3 Let $a_{i j} \geq 0$. Then $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i j}=\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{i j}$. Also if $\left\{b_{j}\right\}_{j=1}^{\infty}$ is any enumeration of the $a_{i j}$, then $\sum_{j=1}^{\infty} b_{j}=\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i j}$.

Proof: First note there is no trouble in defining these sums because the $a_{i j}$ are all nonnegative. If a sum diverges, it only diverges to $\infty$ and so $\infty$ is written as the answer.

$$
\begin{align*}
& \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{i j} \geq \sup _{n} \sum_{j=1}^{\infty} \sum_{i=1}^{n} a_{i j}=\sup _{n} \lim _{m \rightarrow \infty} \sum_{j=1}^{m} \sum_{i=1}^{n} a_{i j} \\
= & \sup _{n} \lim _{m \rightarrow \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} a_{i j}=\sup _{n} \sum_{i=1}^{n} \sum_{j=1}^{\infty} a_{i j}=\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i j} \tag{11.3.10}
\end{align*}
$$

Interchanging the $i$ and $j$ in the above argument the first part of the lemma is proved.
Finally, note that for all $p$,

$$
\sum_{j=1}^{p} b_{j} \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i j}
$$

and so $\sum_{j=1}^{\infty} b_{j} \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i j}$. Now let $m, n>1$ be given. Then

$$
\sum_{i=1}^{m} \sum_{j=1}^{n} a_{i j} \leq \sum_{j=1}^{p} b_{j}
$$

where $p$ is chosen large enough that $\left\{b_{1}, \cdots, b_{p}\right\} \supseteq\left\{a_{i j}: i \leq m\right.$ and $\left.j \leq n\right\}$. Therefore, since such a $p$ exists for any choice of $m, n$, it follows that for any $m, n$,

$$
\sum_{i=1}^{m} \sum_{j=1}^{n} a_{i j} \leq \sum_{j=1}^{\infty} b_{j}
$$

Therefore, taking the limit as $n \rightarrow \infty$,

$$
\sum_{i=1}^{m} \sum_{j=1}^{\infty} a_{i j} \leq \sum_{j=1}^{\infty} b_{j}
$$

and finally, taking the limit as $m \rightarrow \infty$,

$$
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i j} \leq \sum_{j=1}^{\infty} b_{j}
$$

proving the lemma.

### 11.3.2 The Lebesgue Integral Nonnegative Functions

The following picture illustrates the idea used to define the Lebesgue integral to be like the area under a curve.


You can see that by following the procedure illustrated in the picture and letting $h$ get smaller, you would expect to obtain better approximations to the area under the curve ${ }^{1}$ although all these approximations would likely be too small. Therefore, define

$$
\int f d \mu \equiv \sup _{h>0} \sum_{i=1}^{\infty} h \mu([i h<f])
$$

Lemma 11.3.4 The following inequality holds.

$$
\sum_{i=1}^{\infty} h \mu([i h<f]) \leq \sum_{i=1}^{\infty} \frac{h}{2} \mu\left(\left[i \frac{h}{2}<f\right]\right) .
$$

Also, it suffices to consider only $h$ smaller than a given positive number in the above definition of the integral.

## Proof:

Let $N \in \mathbb{N}$.

$$
\begin{gathered}
\quad \sum_{i=1}^{2 N} \frac{h}{2} \mu\left(\left[i \frac{h}{2}<f\right]\right)=\sum_{i=1}^{2 N} \frac{h}{2} \mu([i h<2 f]) \\
=\sum_{i=1}^{N} \frac{h}{2} \mu([(2 i-1) h<2 f])+\sum_{i=1}^{N} \frac{h}{2} \mu([(2 i) h<2 f]) \\
=\sum_{i=1}^{N} \frac{h}{2} \mu\left(\left[\frac{(2 i-1)}{2} h<f\right]\right)+\sum_{i=1}^{N} \frac{h}{2} \mu([i h<f]) \\
\geq \sum_{i=1}^{N} \frac{h}{2} \mu([i h<f])+\sum_{i=1}^{N} \frac{h}{2} \mu([i h<f])=\sum_{i=1}^{N} h \mu([i h<f]) .
\end{gathered}
$$

Now letting $N \rightarrow \infty$ yields the claim of the lemma.

[^7]To verify the last claim, suppose $M<\int f d \mu$ and let $\delta>0$ be given. Then there exists $h>0$ such that

$$
M<\sum_{i=1}^{\infty} h \mu([i h<f]) \leq \int f d \mu
$$

By the first part of this lemma,

$$
M<\sum_{i=1}^{\infty} \frac{h}{2} \mu\left(\left[i \frac{h}{2}<f\right]\right) \leq \int f d \mu
$$

and continuing to apply the first part,

$$
M<\sum_{i=1}^{\infty} \frac{h}{2^{n}} \mu\left(\left[i \frac{h}{2^{n}}<f\right]\right) \leq \int f d \mu
$$

Choose $n$ large enough that $h / 2^{n}<\delta$. It follows

$$
M<\sup _{\delta>h>0} \sum_{i=1}^{\infty} h \mu([i h<f]) \leq \int f d \mu
$$

Since $M$ is arbitrary, this proves the last claim.

### 11.3.3 The Lebesgue Integral For Nonnegative Simple Functions

Definition 11.3.5 A function, $s$, is called simple if it is a measurable real valued function and has only finitely many values. These values will never be $\pm \infty$. Thus a simple function is one which may be written in the form

$$
s(\omega)=\sum_{i=1}^{n} c_{i} \mathscr{X}_{E_{i}}(\omega)
$$

where the sets, $E_{i}$ are disjoint and measurable. stakes the value $c_{i}$ at $E_{i}$.
Note that by taking the union of some of the $E_{i}$ in the above definition, you can assume that the numbers, $c_{i}$ are the distinct values of $s$. Simple functions are important because it will turn out to be very easy to take their integrals as shown in the following lemma.

Lemma 11.3.6 Let $s(\omega)=\sum_{i=1}^{p} a_{i} \mathscr{X}_{E_{i}}(\omega)$ be a nonnegative simple function with the $a_{i}$ the distinct non zero values of $s$. Then

$$
\begin{equation*}
\int s d \mu=\sum_{i=1}^{p} a_{i} \mu\left(E_{i}\right) \tag{11.3.11}
\end{equation*}
$$

Also, for any nonnegative measurable function, $f$, if $\lambda \geq 0$, then

$$
\begin{equation*}
\int \lambda f d \mu=\lambda \int f d \mu \tag{11.3.12}
\end{equation*}
$$

Proof: Consider 11.3 .11 first. Without loss of generality, you can assume $0<a_{1}<$ $a_{2}<\cdots<a_{p}$ and that $\mu\left(E_{i}\right)<\infty$. Let $\varepsilon>0$ be given and let

$$
\delta_{1} \sum_{i=1}^{p} \mu\left(E_{i}\right)<\varepsilon
$$

Pick $\delta<\delta_{1}$ such that for $h<\delta$ it is also true that

$$
h<\frac{1}{2} \min \left(a_{1}, a_{2}-a_{1}, a_{3}-a_{2}, \cdots, a_{n}-a_{n-1}\right) .
$$

Then for $0<h<\delta$

$$
\begin{align*}
\sum_{k=1}^{\infty} h \mu([s>k h]) & =\sum_{k=1}^{\infty} h \sum_{i=k}^{\infty} \mu([i h<s \leq(i+1) h]) \\
& =\sum_{i=1}^{\infty} \sum_{k=1}^{i} h \mu([i h<s \leq(i+1) h]) \\
& =\sum_{i=1}^{\infty} i h \mu([i h<s \leq(i+1) h]) \tag{11.3.13}
\end{align*}
$$

Because of the choice of $h$ there exist positive integers, $i_{k}$ such that $i_{1}<i_{2}<\cdots,<i_{p}$ and

$$
\begin{aligned}
i_{1} h & <a_{1} \leq\left(i_{1}+1\right) h<\cdots<i_{2} h<a_{2}< \\
& <\left(i_{2}+1\right) h<\cdots<i_{p} h<a_{p} \leq\left(i_{p}+1\right) h
\end{aligned}
$$

Then in the sum of 11.3 .13 the only terms which are nonzero are those for which $i \in$ $\left\{i_{1}, i_{2} \cdots, i_{p}\right\}$. To see this, you might consider the following picture.


When $i h$ and $(i+1) h$ are both in between two of the $a_{i}$ the set $[i h<s \leq(i+1) h]$ must be empty because the only values of the function are one of the $a_{i}$. At an $i_{k}, i_{k} h$ is smaller than $a_{k}$ while $\left(i_{k}+1\right) h$ is at least as large. Therefore, the set $[i h<s \leq(i+1) h]$ equals $E_{k}$ and so

$$
\mu\left(\left[i_{k} h<s \leq\left(i_{k}+1\right) h\right]\right)=\mu\left(E_{k}\right) .
$$

Therefore,

$$
\sum_{k=1}^{\infty} h \mu([s>k h])=\sum_{k=1}^{p} i_{k} h \mu\left(E_{k}\right) .
$$

It follows that for all $h$ this small,

$$
\begin{aligned}
0 & <\sum_{k=1}^{p} a_{k} \mu\left(E_{k}\right)-\sum_{k=1}^{\infty} h \mu([s>k h]) \\
& =\sum_{k=1}^{p} a_{k} \mu\left(E_{k}\right)-\sum_{k=1}^{p} i_{k} h \mu\left(E_{k}\right) \leq h \sum_{k=1}^{p} \mu\left(E_{k}\right)<\varepsilon .
\end{aligned}
$$

Taking the inf for $h$ this small and using Lemma 11.3.4,

$$
\begin{aligned}
0 & \leq \sum_{k=1}^{p} a_{k} \mu\left(E_{k}\right)-\sup _{\delta>h>0} \sum_{k=1}^{\infty} h \mu([s>k h]) \\
& =\sum_{k=1}^{p} a_{k} \mu\left(E_{k}\right)-\int s d \mu \leq \varepsilon
\end{aligned}
$$

Since $\varepsilon>0$ is arbitrary, this proves the first part.
To verify 11.3.12 Note the formula is obvious if $\lambda=0$ because then $[i h<\lambda f]=\emptyset$ for all $i>0$. Assume $\lambda>0$. Then

$$
\begin{aligned}
\int \lambda f d \mu & \equiv \sup _{h>0} \sum_{i=1}^{\infty} h \mu([i h<\lambda f]) \\
& =\sup _{h>0} \sum_{i=1}^{\infty} h \mu([i h / \lambda<f]) \\
& =\sup _{h>0} \lambda \sum_{i=1}^{\infty}(h / \lambda) \mu([i(h / \lambda)<f]) \\
& =\lambda \int f d \mu
\end{aligned}
$$

This proves the lemma.
Lemma 11.3.7 Let the nonnegative simple function, $s$ be defined as

$$
s(\omega)=\sum_{i=1}^{n} c_{i} \mathscr{X}_{E_{i}}(\omega)
$$

where the $c_{i}$ are not necessarily distinct but the $E_{i}$ are disjoint. It follows that

$$
\int s=\sum_{i=1}^{n} c_{i} \mu\left(E_{i}\right)
$$

Proof: Let the values of $s$ be $\left\{a_{1}, \cdots, a_{m}\right\}$. Therefore, since the $E_{i}$ are disjoint, each $a_{i}$ equal to one of the $c_{j}$. Let $A_{i} \equiv \cup\left\{E_{j}: c_{j}=a_{i}\right\}$. Then from Lemma 11.3.6 it follows that

$$
\begin{aligned}
\int s & =\sum_{i=1}^{m} a_{i} \mu\left(A_{i}\right)=\sum_{i=1}^{m} a_{i} \sum_{\left\{j: c_{j}=a_{i}\right\}} \mu\left(E_{j}\right) \\
& =\sum_{i=1}^{m} \sum_{\left\{j: c_{j}=a_{i}\right\}} c_{j} \mu\left(E_{j}\right)=\sum_{i=1}^{n} c_{i} \mu\left(E_{i}\right) .
\end{aligned}
$$

This proves the lemma.
Note that $\int s$ could equal $+\infty$ if $\mu\left(A_{k}\right)=\infty$ and $a_{k}>0$ for some $k$, but $\int s$ is well defined because $s \geq 0$. Recall that $0 \cdot \infty=0$.

Lemma 11.3.8 If $a, b \geq 0$ and if $s$ and $t$ are nonnegative simple functions, then

$$
\int a s+b t=a \int s+b \int t
$$

Proof: Let

$$
s(\omega)=\sum_{i=1}^{n} \alpha_{i} \mathscr{X}_{A_{i}}(\omega), t(\omega)=\sum_{i=1}^{m} \beta_{j} \mathscr{X}_{B_{j}}(\omega)
$$

where $\alpha_{i}$ are the distinct values of $s$ and the $\beta_{j}$ are the distinct values of $t$. Clearly $a s+b t$ is a nonnegative simple function because it is measurable and has finitely many values. Also,

$$
(a s+b t)(\omega)=\sum_{j=1}^{m} \sum_{i=1}^{n}\left(a \alpha_{i}+b \beta_{j}\right) \mathscr{X}_{A_{i} \cap B_{j}}(\omega)
$$

where the sets $A_{i} \cap B_{j}$ are disjoint. By Lemma 11.3.7,

$$
\begin{aligned}
\int a s+b t & =\sum_{j=1}^{m} \sum_{i=1}^{n}\left(a \alpha_{i}+b \beta_{j}\right) \mu\left(A_{i} \cap B_{j}\right) \\
& =a \sum_{i=1}^{n} \alpha_{i} \mu\left(A_{i}\right)+b \sum_{j=1}^{m} \beta_{j} \mu\left(B_{j}\right) \\
& =a \int s+b \int t
\end{aligned}
$$

This proves the lemma.

### 11.3.4 Simple Functions And Measurable Functions

There is a fundamental theorem about the relationship of simple functions to measurable functions given in the next theorem.

Theorem 11.3.9 Let $f \geq 0$ be measurable. Then there exists a sequence of nonnegative simple functions $\left\{s_{n}\right\}$ satisfying

$$
\begin{gather*}
0 \leq s_{n}(\omega)  \tag{11.3.14}\\
\cdots s_{n}(\omega) \leq s_{n+1}(\omega) \cdots \\
f(\omega)=\lim _{n \rightarrow \infty} s_{n}(\omega) \text { for all } \omega \in \Omega \tag{11.3.15}
\end{gather*}
$$

If $f$ is bounded the convergence is actually uniform.
Proof: Letting $I \equiv\{\omega: f(\omega)=\infty\}$, define

$$
t_{n}(\omega)=\sum_{k=0}^{2^{n}} \frac{k}{n} \mathscr{X}_{[k / n \leq f<(k+1) / n]}(\omega)+n \mathscr{X}_{I}(\omega)
$$

Then $t_{n}(\omega) \leq f(\omega)$ for all $\omega$ and $\lim _{n \rightarrow \infty} t_{n}(\omega)=f(\omega)$ for all $\omega$. This is because $t_{n}(\omega)=n$ for $\omega \in I$ and if $f(\omega) \in\left[0, \frac{2^{n}+1}{n}\right)$, then

$$
\begin{equation*}
0 \leq f(\omega)-t_{n}(\omega) \leq \frac{1}{n} \tag{11.3.16}
\end{equation*}
$$

Thus whenever $\omega \notin I$, the above inequality will hold for all $n$ large enough. Let

$$
s_{1}=t_{1}, s_{2}=\max \left(t_{1}, t_{2}\right), s_{3}=\max \left(t_{1}, t_{2}, t_{3}\right), \cdots
$$

Then the sequence $\left\{s_{n}\right\}$ satisfies 11.3.14-11.3.15.
To verify the last claim, note that in this case the term $n \mathscr{X}_{I}(\omega)$ is not present. Therefore, for all $n$ large enough, 11.3.16 holds for all $\omega$. Thus the convergence is uniform. This proves the theorem.

Although it is not needed here, there is a similar theorem which applies to measurable functions which have values in a separable metric space. In this context, a simple function is one which is of the form

$$
\sum_{k=1}^{m} x_{k} \mathscr{X}_{E_{k}}(\omega)
$$

where the $E_{k}$ are disjoint measurable sets and the $x_{k}$ are in $X$. I am abusing notation somewhat by using a sum. You can't add in a general metric space. The symbol means the function has value $x_{k}$ on the set $E_{k}$.

Theorem 11.3.10 Let $(\Omega, \mathscr{F})$ be a measure space and let $f: \Omega \rightarrow X$ where $(X, d)$ is a separable metric space. Then $f$ is a measurable function if and only if there exists a sequence of simple functions, $\left\{f_{n}\right\}$ such that for each $\omega \in \Omega$ and $n \in \mathbb{N}$,

$$
\begin{equation*}
d\left(f_{n}(\omega), f(\omega)\right) \geq d\left(f_{n+1}(\omega), f(\omega)\right) \tag{11.3.17}
\end{equation*}
$$

and

$$
\begin{equation*}
\lim _{n \rightarrow \infty} d\left(f_{n}(\omega), f(\omega)\right)=0 \tag{11.3.18}
\end{equation*}
$$

Proof: Let $D=\left\{x_{k}\right\}_{k=1}^{\infty}$ be a countable dense subset of $X$. First suppose $f$ is measurable. Then since in a metric space every open set is the countable intersection of closed sets, it follows $f^{-1}($ closed set $) \in \mathscr{F}$. Now let $D_{n}=\left\{x_{k}\right\}_{k=1}^{n}$. Let

$$
A_{1} \equiv\left\{\omega: d\left(x_{1}, f(\omega)\right)=\min _{k \leq n} d\left(x_{k}, f(\omega)\right)\right\}
$$

That is, $A_{1}$ are those $\omega$ such that $f(\omega)$ is approximated best out of $D_{n}$ by $x_{1}$. Why is this a measurable set? It is because $\omega \rightarrow d(x, f(\omega))$ is a real valued measurable function, being the composition of a continuous function, $y \rightarrow d(x, y)$ and a measurable function, $\omega \rightarrow f(\omega)$. Next let

$$
A_{2} \equiv\left\{\omega \notin A_{1}: d\left(x_{2}, f(\omega)\right)=\min _{k \leq n} d\left(x_{k}, f(\omega)\right)\right\}
$$

and continue in this manner obtaining disjoint measurable sets, $\left\{A_{k}\right\}_{k=1}^{n}$ such that for $\omega \in$ $A_{k}$ the best approximation to $f(\omega)$ from $D_{n}$ is $x_{k}$. Then

$$
f_{n}(\omega) \equiv \sum_{k=1}^{n} x_{k} \mathscr{X}_{A_{k}}(\omega)
$$

Note

$$
\min _{k \leq n+1} d\left(x_{k}, f(\omega)\right) \leq \min _{k \leq n} d\left(x_{k}, f(\omega)\right)
$$

and so this verifies 11.3 .17 . It remains to verify 11.3 .18 .
Let $\varepsilon>0$ be given and pick $\omega \in \Omega$. Then there exists $x_{n} \in D$ such that $d\left(x_{n}, f(\omega)\right)<\varepsilon$. It follows from the construction that $d\left(f_{n}(\omega), f(\omega)\right) \leq d\left(x_{n}, f(\omega)\right)<\varepsilon$. This proves the first half.

Now suppose the existence of the sequence of simple functions as described above. Each $f_{n}$ is a measurable function because $f_{n}^{-1}(U)=\cup\left\{A_{k}: x_{k} \in U\right\}$. Therefore, the conclusion that $f$ is measurable follows from Theorem 11.1.9 on Page 227.

In the context of this more general notion of measurable function having values in a metric space, here is a version of Egoroff's theorem.

Theorem 11.3.11 (Egoroff) Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space,

$$
(\mu(\Omega)<\infty)
$$

and let $f_{n}, f$ be $X$ valued measurable functions where $X$ is a separable metric space and for all $\omega \notin E$ where $\mu(E)=0$

$$
f_{n}(\omega) \rightarrow f(\omega)
$$

Then for every $\varepsilon>0$, there exists a set,

$$
F \supseteq E, \mu(F)<\varepsilon,
$$

such that $f_{n}$ converges uniformly to $f$ on $F^{C}$.
Proof: First suppose $E=\emptyset$ so that convergence is pointwise everywhere. Let

$$
E_{k m}=\left\{\omega \in \Omega: d\left(f_{n}(\omega), f(\omega)\right) \geq 1 / m \text { for some } n>k\right\}
$$

Claim: $\left[\omega: d\left(f_{n}(\omega), f(\omega)\right) \geq \frac{1}{m}\right]$ is measurable.
Proof of claim: Let $\left\{x_{k}\right\}_{k=1}^{\infty}$ be a countable dense subset of $X$ and let $r$ denote a positive rational number, $\mathbb{Q}^{+}$. Then

$$
\begin{align*}
& \cup_{k \in \mathbb{N}, r \in \mathbb{Q}^{+}} f_{n}^{-1}\left(B\left(x_{k}, r\right)\right) \cap f^{-1}\left(B\left(x_{k}, \frac{1}{m}-r\right)\right) \\
= & {\left[d\left(f, f_{n}\right)<\frac{1}{m}\right] } \tag{11.3.19}
\end{align*}
$$

Here is why. If $\omega$ is in the set on the left, then $d\left(f_{n}(\omega), x_{k}\right)<r$ and

$$
d\left(f(\omega), x_{k}\right)<\frac{1}{m}-r .
$$

Therefore,

$$
d\left(f(\omega), f_{n}(\omega)\right)<r+\frac{1}{m}-r=\frac{1}{m}
$$

Thus the left side is contained in the right. Now let $\omega$ be in the right side. That is $d\left(f_{n}(\omega), f(\omega)\right)<\frac{1}{m}$. Choose $2 r<\frac{1}{m}-d\left(f_{n}(\omega), f(\omega)\right)$ and pick $x_{k} \in B\left(f_{n}(\omega), r\right)$. Then

$$
\begin{aligned}
d\left(f(\omega), x_{k}\right) & \leq d\left(f(\omega), f_{n}(\omega)\right)+d\left(f_{n}(\omega), x_{k}\right) \\
& <\frac{1}{m}-2 r+r=\frac{1}{m}-r
\end{aligned}
$$

Thus $\omega \in f_{n}^{-1}\left(B\left(x_{k}, r\right)\right) \cap f^{-1}\left(B\left(x_{k}, \frac{1}{m}-r\right)\right)$ and so $\omega$ is in the left side. Thus the two sets are equal. Now the set on the left in 11.3.19 is measurable because it is a countable union of measurable sets. This proves the claim since

$$
\left[\omega: d\left(f_{n}(\omega), f(\omega)\right) \geq \frac{1}{m}\right]
$$

is the complement of this measurable set.
Hence $E_{k m}$ is measurable because

$$
E_{k m}=\cup_{n=k+1}^{\infty}\left[\omega: d\left(f_{n}(\omega), f(\omega)\right) \geq \frac{1}{m}\right]
$$

For fixed $m, \cap_{k=1}^{\infty} E_{k m}=\emptyset$ because $f_{n}(\omega)$ converges to $f(\omega)$. Therefore, if $\omega \in \Omega$ there exists $k$ such that if $n>k,\left|f_{n}(\omega)-f(\omega)\right|<\frac{1}{m}$ which means $\omega \notin E_{k m}$. Note also that

$$
E_{k m} \supseteq E_{(k+1) m}
$$

Since $\mu\left(E_{1 m}\right)<\infty$, Theorem 11.1.5 on Page 224 implies

$$
0=\mu\left(\cap_{k=1}^{\infty} E_{k m}\right)=\lim _{k \rightarrow \infty} \mu\left(E_{k m}\right)
$$

Let $k(m)$ be chosen such that $\mu\left(E_{k(m) m}\right)<\varepsilon 2^{-m}$ and let

$$
F=\bigcup_{m=1}^{\infty} E_{k(m) m}
$$

Then $\mu(F)<\varepsilon$ because

$$
\mu(F) \leq \sum_{m=1}^{\infty} \mu\left(E_{k(m) m}\right)<\sum_{m=1}^{\infty} \varepsilon 2^{-m}=\varepsilon
$$

Now let $\eta>0$ be given and pick $m_{0}$ such that $m_{0}^{-1}<\eta$. If $\omega \in F^{C}$, then

$$
\omega \in \bigcap_{m=1}^{\infty} E_{k(m) m}^{C}
$$

Hence $\omega \in E_{k\left(m_{0}\right) m_{0}}^{C}$ so

$$
d\left(f(\omega), f_{n}(\omega)\right)<1 / m_{0}<\eta
$$

for all $n>k\left(m_{0}\right)$. This holds for all $\omega \in F^{C}$ and so $f_{n}$ converges uniformly to $f$ on $F^{C}$.
Now if $E \neq \emptyset$, consider $\left\{\mathscr{X}_{E^{C}} f_{n}\right\}_{n=1}^{\infty}$. Then $\mathscr{X}_{E^{C}} f_{n}$ is measurable and the sequence converges pointwise to $\mathscr{X}_{E} f$ everywhere. Therefore, from the first part, there exists a set of measure less than $\varepsilon, F$ such that on $F^{C},\left\{\mathscr{X}_{E^{C}} f_{n}\right\}$ converges uniformly to $\mathscr{X}_{E^{C}} f$. Therefore, on $(E \cup F)^{C},\left\{f_{n}\right\}$ converges uniformly to $f$. This proves the theorem.

### 11.3.5 The Monotone Convergence Theorem

The following is called the monotone convergence theorem. This theorem and related convergence theorems are the reason for using the Lebesgue integral.

Theorem 11.3.12 (Monotone Convergence theorem) Let $f$ have values in $[0, \infty]$ and suppose $\left\{f_{n}\right\}$ is a sequence of nonnegative measurable functions having values in $[0, \infty]$ and satisfying

$$
\begin{gathered}
\lim _{n \rightarrow \infty} f_{n}(\omega)=f(\omega) \text { for each } \omega \\
\cdots f_{n}(\omega) \leq f_{n+1}(\omega) \cdots
\end{gathered}
$$

Then $f$ is measurable and

$$
\int f d \mu=\lim _{n \rightarrow \infty} \int f_{n} d \mu
$$

Proof: From Lemmas 11.3.1 and 11.3.2,

$$
\begin{aligned}
\int f d \mu & \equiv \sup _{h>0} \sum_{i=1}^{\infty} h \mu([i h<f]) \\
& =\sup _{h>0} \sup _{k} \sum_{i=1}^{k} h \mu([i h<f]) \\
& =\sup _{h>0} \sup _{k} \sup _{m} \sum_{i=1}^{k} h \mu\left(\left[i h<f_{m}\right]\right) \\
& =\sup _{m} \sup _{h>0} \sum_{i=1}^{\infty} h \mu\left(\left[i h<f_{m}\right]\right) \\
& \equiv \sup _{m} \int f_{m} d \mu \\
& =\lim _{m \rightarrow \infty} \int f_{m} d \mu
\end{aligned}
$$

The third equality follows from the observation that

$$
\lim _{m \rightarrow \infty} \mu\left(\left[i h<f_{m}\right]\right)=\mu([i h<f])
$$

which follows from Theorem 11.1.5 since the sets, $\left[i h<f_{m}\right]$ are increasing in $m$ and their union equals $[i h<f]$. This proves the theorem.

To illustrate what goes wrong without the Lebesgue integral, consider the following example.

Example 11.3.13 Let $\left\{r_{n}\right\}$ denote the rational numbers in $[0,1]$ and let

$$
f_{n}(t) \equiv\left\{\begin{array}{l}
1 \text { if } t \notin\left\{r_{1}, \cdots, r_{n}\right\} \\
0 \text { otherwise }
\end{array}\right.
$$

Then $f_{n}(t) \uparrow f(t)$ where $f$ is the function which is one on the rationals and zero on the irrationals. Each $f_{n}$ is Riemann integrable (why?) but $f$ is not Riemann integrable. Therefore, you can't write $\int f d x=\lim _{n \rightarrow \infty} \int f_{n} d x$.

A meta-mathematical observation related to this type of example is this. If you can choose your functions, you don't need the Lebesgue integral. The Riemann integral is just fine. It is when you can't choose your functions and they come to you as pointwise limits that you really need the superior Lebesgue integral or at least something more general than the Riemann integral. The Riemann integral is entirely adequate for evaluating the seemingly endless lists of boring problems found in calculus books.

### 11.3.6 Other Definitions

To review and summarize the above, if $f \geq 0$ is measurable,

$$
\begin{equation*}
\int f d \mu \equiv \sup _{h>0} \sum_{i=1}^{\infty} h \mu([f>i h]) \tag{11.3.20}
\end{equation*}
$$

another way to get the same thing for $\int f d \mu$ is to take an increasing sequence of nonnegative simple functions, $\left\{s_{n}\right\}$ with $s_{n}(\omega) \rightarrow f(\omega)$ and then by monotone convergence theorem,

$$
\int f d \mu=\lim _{n \rightarrow \infty} \int s_{n}
$$

where if $s_{n}(\omega)=\sum_{j=1}^{m} c_{i} \mathscr{X}_{E_{i}}(\omega)$,

$$
\int s_{n} d \mu=\sum_{i=1}^{m} c_{i} m\left(E_{i}\right)
$$

Similarly this also shows that for such nonnegative measurable function,

$$
\int f d \mu=\sup \left\{\int s: 0 \leq s \leq f, s \text { simple }\right\}
$$

which is the usual way of defining the Lebesgue integral for nonnegative simple functions in most books. I have done it differently because this approach led to an easier proof of the Monotone convergence theorem. Here is an equivalent definition of the integral. The fact it is well defined has been discussed above.

Definition 11.3.14 For s a nonnegative simple function,

$$
s(\omega)=\sum_{k=1}^{n} c_{k} \mathscr{X}_{E_{k}}(\omega), \int s=\sum_{k=1}^{n} c_{k} \mu\left(E_{k}\right) .
$$

For $f$ a nonnegative measurable function,

$$
\int f d \mu=\sup \left\{\int s: 0 \leq s \leq f, \text { s simple }\right\}
$$

### 11.3.7 Fatou's Lemma

Sometimes the limit of a sequence does not exist. There are two more general notions known as limsup and liminf which do always exist in some sense. These notions are dependent on the following lemma.

Lemma 11.3.15 Let $\left\{a_{n}\right\}$ be an increasing/decreasing in $[-\infty, \infty]$. Then $\lim _{n \rightarrow \infty} a_{n}$ exists.
Proof: Suppose first $\left\{a_{n}\right\}$ is increasing. Recall this means $a_{n} \leq a_{n+1}$ for all $n$. If the sequence is bounded above, then it has a least upper bound and so $a_{n} \rightarrow a$ where $a$ is its least upper bound. If the sequence is not bounded above, then for every $l \in \mathbb{R}$, it follows $l$ is not an upper bound and so eventually, $a_{n}>l$. But this is what is meant by $a_{n} \rightarrow \infty$. The situation for decreasing sequences is completely similar.

Now take any sequence, $\left\{a_{n}\right\} \subseteq[-\infty, \infty]$ and consider the sequence $\left\{A_{n}\right\}$ where

$$
A_{n} \equiv \inf \left\{a_{k}: k \geq n\right\}
$$

Then as $n$ increases, the set of numbers whose inf is being taken is getting smaller. Therefore, $A_{n}$ is an increasing sequence and so it must converge. Similarly, defining the sequence $B_{n} \equiv \sup \left\{a_{k}: k \geq n\right\}$, it follows $B_{n}$ is decreasing and so $\left\{B_{n}\right\}$ also must converge. With this preparation, the following definition can be given.

Definition 11.3.16 Let $\left\{a_{n}\right\}$ be a sequence of points in $[-\infty, \infty]$. Then define

$$
\lim \inf _{n \rightarrow \infty} a_{n} \equiv \lim _{n \rightarrow \infty} \inf \left\{a_{k}: k \geq n\right\}
$$

and

$$
\limsup _{n \rightarrow \infty} a_{n} \equiv \lim _{n \rightarrow \infty} \sup \left\{a_{k}: k \geq n\right\}
$$

In the case of functions having values in $[-\infty, \infty]$,

$$
\left(\lim \inf _{n \rightarrow \infty} f_{n}\right)(\omega) \equiv \lim \inf _{n \rightarrow \infty}\left(f_{n}(\omega)\right)
$$

A similar definition applies to $\lim \sup _{n \rightarrow \infty} f_{n}$.
Lemma 11.3.17 Let $\left\{a_{n}\right\}$ be a sequence in $[-\infty, \infty]$. Then $\lim _{n \rightarrow \infty} a_{n}$ exists if and only if

$$
\lim _{n \rightarrow \infty} \inf _{n} a_{n}=\lim \sup _{n \rightarrow \infty} a_{n}
$$

and in this case, the limit equals the common value of these two numbers.
Proof: Suppose first $\lim _{n \rightarrow \infty} a_{n}=a \in \mathbb{R}$. Then, letting $\varepsilon>0$ be given, $a_{n} \in(a-\varepsilon, a+\varepsilon)$ for all $n$ large enough, say $n \geq N$. Therefore, both $\inf \left\{a_{k}: k \geq n\right\}$ and $\sup \left\{a_{k}: k \geq n\right\}$ are contained in $[a-\varepsilon, a+\varepsilon]$ whenever $n \geq N$. It follows $\limsup _{n \rightarrow \infty} a_{n}$ and $\liminf _{n \rightarrow \infty} a_{n}$ are both in $[a-\varepsilon, a+\varepsilon]$, showing

$$
\left|\lim \inf _{n \rightarrow \infty} a_{n}-\lim \sup _{n \rightarrow \infty} a_{n}\right|<2 \varepsilon .
$$

Since $\varepsilon$ is arbitrary, the two must be equal and they both must equal $a$. Next suppose $\lim _{n \rightarrow \infty} a_{n}=\infty$. Then if $l \in \mathbb{R}$, there exists $N$ such that for $n \geq N$,

$$
l \leq a_{n}
$$

and therefore, for such $n$,

$$
l \leq \inf \left\{a_{k}: k \geq n\right\} \leq \sup \left\{a_{k}: k \geq n\right\}
$$

and this shows, since $l$ is arbitrary that

$$
\lim \inf _{n \rightarrow \infty} a_{n}=\lim \sup _{n \rightarrow \infty} a_{n}=\infty
$$

The case for $-\infty$ is similar.
Conversely, suppose $\liminf _{n \rightarrow \infty} a_{n}=\limsup _{n \rightarrow \infty} a_{n}=a$. Suppose first that $a \in \mathbb{R}$. Then, letting $\varepsilon>0$ be given, there exists $N$ such that if $n \geq N$,

$$
\sup \left\{a_{k}: k \geq n\right\}-\inf \left\{a_{k}: k \geq n\right\}<\varepsilon
$$

therefore, if $k, m>N$, and $a_{k}>a_{m}$,

$$
\left|a_{k}-a_{m}\right|=a_{k}-a_{m} \leq \sup \left\{a_{k}: k \geq n\right\}-\inf \left\{a_{k}: k \geq n\right\}<\varepsilon
$$

showing that $\left\{a_{n}\right\}$ is a Cauchy sequence. Therefore, it converges to $a \in \mathbb{R}$, and as in the first part, the liminf and limsup both equal $a$. If $\liminf _{n \rightarrow \infty} a_{n}=\limsup _{n \rightarrow \infty} a_{n}=\infty$, then given $l \in \mathbb{R}$, there exists $N$ such that for $n \geq N$,

$$
\inf _{n>N} a_{n}>l
$$

Therefore, $\lim _{n \rightarrow \infty} a_{n}=\infty$. The case for $-\infty$ is similar. This proves the lemma.
The next theorem, known as Fatou's lemma is another important theorem which justifies the use of the Lebesgue integral.

Theorem 11.3.18 (Fatou's lemma) Let $f_{n}$ be a nonnegative measurable function with values in $[0, \infty]$. Let $g(\omega)=\liminf _{n \rightarrow \infty} f_{n}(\omega)$. Then $g$ is measurable and

$$
\int g d \mu \leq \lim \inf _{n \rightarrow \infty} \int f_{n} d \mu
$$

In other words,

$$
\int\left(\lim \inf _{n \rightarrow \infty} f_{n}\right) d \mu \leq \lim \inf _{n \rightarrow \infty} \int f_{n} d \mu
$$

Proof: Let $g_{n}(\omega)=\inf \left\{f_{k}(\omega): k \geq n\right\}$. Then

$$
g_{n}^{-1}([a, \infty])=\cap_{k=n}^{\infty} f_{k}^{-1}([a, \infty]) \in \mathscr{F} .
$$

Thus $g_{n}$ is measurable by Lemma 11.1.6 on Page 225. Also $g(\omega)=\lim _{n \rightarrow \infty} g_{n}(\omega)$ so $g$ is measurable because it is the pointwise limit of measurable functions. Now the functions
$g_{n}$ form an increasing sequence of nonnegative measurable functions so the monotone convergence theorem applies. This yields

$$
\int g d \mu=\lim _{n \rightarrow \infty} \int g_{n} d \mu \leq \lim \inf _{n \rightarrow \infty} \int f_{n} d \mu
$$

The last inequality holding because

$$
\int g_{n} d \mu \leq \int f_{n} d \mu
$$

(Note that it is not known whether $\lim _{n \rightarrow \infty} \int f_{n} d \mu$ exists.) This proves the Theorem.

### 11.3.8 The Righteous Algebraic Desires Of The Lebesgue Integral

The monotone convergence theorem shows the integral wants to be linear. This is the essential content of the next theorem.

Theorem 11.3.19 Let $f, g$ be nonnegative measurable functions and let $a, b$ be nonnegative numbers. Then

$$
\begin{equation*}
\int(a f+b g) d \mu=a \int f d \mu+b \int g d \mu \tag{11.3.21}
\end{equation*}
$$

Proof: By Theorem 11.3 .9 on Page 241 there exist sequences of nonnegative simple functions, $s_{n} \rightarrow f$ and $t_{n} \rightarrow g$. Then by the monotone convergence theorem and Lemma 11.3.8,

$$
\begin{aligned}
\int(a f+b g) d \mu & =\lim _{n \rightarrow \infty} \int a s_{n}+b t_{n} d \mu \\
& =\lim _{n \rightarrow \infty}\left(a \int s_{n} d \mu+b \int t_{n} d \mu\right) \\
& =a \int f d \mu+b \int g d \mu
\end{aligned}
$$

As long as you are allowing functions to take the value $+\infty$, you cannot consider something like $f+(-g)$ and so you can't very well expect a satisfactory statement about the integral being linear until you restrict yourself to functions which have values in a vector space. This is discussed next.

### 11.4 The Space $L^{1}$

The functions considered here have values in $\mathbb{C}$, a vector space.
Definition 11.4.1 Let $(\Omega, \mathscr{S}, \mu)$ be a measure space and suppose $f: \Omega \rightarrow \mathbb{C}$. Then $f$ is said to be measurable if both $\operatorname{Re} f$ and $\operatorname{Im} f$ are measurable real valued functions.

Definition 11.4.2 A complex simple function will be a function which is of the form

$$
s(\omega)=\sum_{k=1}^{n} c_{k} \mathscr{X}_{E_{k}}(\omega)
$$

where $c_{k} \in \mathbb{C}$ and $\mu\left(E_{k}\right)<\infty$. For $s$ a complex simple function as above, define

$$
I(s) \equiv \sum_{k=1}^{n} c_{k} \mu\left(E_{k}\right)
$$

Lemma 11.4.3 The definition, 11.4.2 is well defined. Furthermore, I is linear on the vector space of complex simple functions. Also the triangle inequality holds,

$$
|I(s)| \leq I(|s|)
$$

Proof: Suppose $\sum_{k=1}^{n} c_{k} \mathscr{X}_{E_{k}}(\omega)=0$. Does it follow that $\sum_{k} c_{k} \mu\left(E_{k}\right)=0$ ? The supposition implies

$$
\begin{equation*}
\sum_{k=1}^{n} \operatorname{Re} c_{k} \mathscr{X}_{E_{k}}(\omega)=0, \sum_{k=1}^{n} \operatorname{Im} c_{k} \mathscr{X}_{E_{k}}(\omega)=0 \tag{11.4.22}
\end{equation*}
$$

Choose $\lambda$ large and positive so that $\lambda+\operatorname{Re} c_{k} \geq 0$. Then adding $\sum_{k} \lambda \mathscr{X}_{E_{k}}$ to both sides of the first equation above,

$$
\sum_{k=1}^{n}\left(\lambda+\operatorname{Re} c_{k}\right) \mathscr{X}_{E_{k}}(\omega)=\sum_{k=1}^{n} \lambda \mathscr{X}_{E_{k}}
$$

and by Lemma 11.3.8 on Page 241, it follows upon taking $\int$ of both sides that

$$
\sum_{k=1}^{n}\left(\lambda+\operatorname{Re} c_{k}\right) \mu\left(E_{k}\right)=\sum_{k=1}^{n} \lambda \mu\left(E_{k}\right)
$$

which implies $\sum_{k=1}^{n} \operatorname{Re} c_{k} \mu\left(E_{k}\right)=0$. Similarly, $\sum_{k=1}^{n} \operatorname{Im} c_{k} \mu\left(E_{k}\right)=0$ so

$$
\sum_{k=1}^{n} c_{k} \mu\left(E_{k}\right)=0
$$

Thus if

$$
\sum_{j} c_{j} \mathscr{X}_{E_{j}}=\sum_{k} d_{k} \mathscr{X}_{F_{k}}
$$

then $\sum_{j} c_{j} \mathscr{X}_{E_{j}}+\sum_{k}\left(-d_{k}\right) \mathscr{X}_{F_{k}}=0$ and so the result just established verifies $\sum_{j} c_{j} \mu\left(E_{j}\right)-$ $\sum_{k} d_{k} \mu\left(F_{k}\right)=0$ which proves $I$ is well defined.

That $I$ is linear is now obvious. It only remains to verify the triangle inequality.
Let $s$ be a simple function,

$$
s=\sum_{j} c_{j} \mathscr{X}_{E_{j}}
$$

Then pick $\theta \in \mathbb{C}$ such that $\theta I(s)=|I(s)|$ and $|\theta|=1$. Then from the triangle inequality for sums of complex numbers,

$$
\begin{aligned}
|I(s)| & =\theta I(s)=I(\theta s)=\sum_{j} \theta c_{j} \mu\left(E_{j}\right) \\
& =\left|\sum_{j} \theta c_{j} \mu\left(E_{j}\right)\right| \leq \sum_{j}\left|\theta c_{j}\right| \mu\left(E_{j}\right)=I(|s|) .
\end{aligned}
$$

This proves the lemma.
With this lemma, the following is the definition of $L^{1}(\Omega)$.

Definition 11.4.4 $f \in L^{1}(\Omega)$ means there exists a sequence of complex simple functions, $\left\{s_{n}\right\}$ such that

$$
\begin{gather*}
s_{n}(\omega) \rightarrow f(\omega) \text { for all } \omega \in \Omega \\
\lim _{m, n \rightarrow \infty} I\left(\left|s_{n}-s_{m}\right|\right)=\lim _{n, m \rightarrow \infty} \int\left|s_{n}-s_{m}\right| d \mu=0 \tag{11.4.23}
\end{gather*}
$$

Then

$$
\begin{equation*}
I(f) \equiv \lim _{n \rightarrow \infty} I\left(s_{n}\right) \tag{11.4.24}
\end{equation*}
$$

Lemma 11.4.5 Definition 11.4.4 is well defined.
Proof: There are several things which need to be verified. First suppose 11.4.23. Then by Lemma 11.4.3

$$
\left|I\left(s_{n}\right)-I\left(s_{m}\right)\right|=\left|I\left(s_{n}-s_{m}\right)\right| \leq I\left(\left|s_{n}-s_{m}\right|\right)
$$

and for $m, n$ large enough this last is given to be small so $\left\{I\left(s_{n}\right)\right\}$ is a Cauchy sequence in $\mathbb{C}$ and so it converges. This verifies the limit in 11.4.24 at least exists. It remains to consider another sequence $\left\{t_{n}\right\}$ having the same properties as $\left\{s_{n}\right\}$ and verifying $I(f)$ determined by this other sequence is the same. By Lemma 11.4.3 and Fatou's lemma, Theorem 11.3.18 on Page 248,

$$
\begin{aligned}
\left|I\left(s_{n}\right)-I\left(t_{n}\right)\right| & \leq I\left(\left|s_{n}-t_{n}\right|\right)=\int\left|s_{n}-t_{n}\right| d \mu \\
& \leq \int\left|s_{n}-f\right|+\left|f-t_{n}\right| d \mu \\
& \leq \liminf _{k \rightarrow \infty} \int\left|s_{n}-s_{k}\right| d \mu+\liminf _{k \rightarrow \infty} \int\left|t_{n}-t_{k}\right| d \mu<\varepsilon
\end{aligned}
$$

whenever $n$ is large enough. Since $\varepsilon$ is arbitrary, this shows the limit from using the $t_{n}$ is the same as the limit from using $s_{n}$. This proves the lemma.

What if $f$ has values in $[0, \infty)$ ? Earlier $\int f d \mu$ was defined for such functions and now $I(f)$ has been defined. Are they the same? If so, $I$ can be regarded as an extension of $\int d \mu$ to a larger class of functions.
Lemma 11.4.6 Suppose $f$ has values in $[0, \infty)$ and $f \in L^{1}(\Omega)$. Then $f$ is measurable and

$$
I(f)=\int f d \mu
$$

Proof: Since $f$ is the pointwise limit of a sequence of complex simple functions, $\left\{s_{n}\right\}$ having the properties described in Definition 11.4.4, it follows $f(\omega)=\lim _{n \rightarrow \infty} \operatorname{Re} s_{n}(\omega)$ and so $f$ is measurable. Also

$$
\int\left|\left(\operatorname{Re} s_{n}\right)^{+}-\left(\operatorname{Re} s_{m}\right)^{+}\right| d \mu \leq \int\left|\operatorname{Re} s_{n}-\operatorname{Re} s_{m}\right| d \mu \leq \int\left|s_{n}-s_{m}\right| d \mu
$$

where $x^{+} \equiv \frac{1}{2}(|x|+x)$, the positive part of the real number, $x$. ${ }^{2}$ Thus there is no loss of generality in assuming $\left\{s_{n}\right\}$ is a sequence of complex simple functions having values in

[^8]$[0, \infty)$. Then since for such complex simple functions, $I(s)=\int s d \mu$,
\[

$$
\begin{aligned}
& \left|I(f)-\int f d \mu\right| \leq\left|I(f)-I\left(s_{n}\right)\right|+\left|\int s_{n} d \mu-\int f d \mu\right| \\
& \quad<\varepsilon+\mid \int_{\left[s_{n}-f \geq 0\right]} s_{n} d \mu-\int_{\left[s_{n}-f \geq 0\right]} f d \mu \\
& \quad+\int_{\left[s_{n}-f<0\right]} s_{n} d \mu-\int_{\left[s_{n}-f<0\right]} f d \mu \mid \\
& \leq \varepsilon+\left|\int_{\left[s_{n}-f \geq 0\right]}\left(s_{n}-f\right) d \mu\right|+\left|\int_{\left[s_{n}-f<0\right]}\left(s_{n}-f\right) d \mu\right| \\
& \leq \varepsilon+\int_{\left[s_{n}-f \geq 0\right]}\left|s_{n}-f\right| d \mu+\int_{\left[s_{n}-f>0\right]}\left|s_{n}-f\right| d \mu \\
& =\varepsilon+\int\left|s_{n}-f\right| d \mu
\end{aligned}
$$
\]

whenever $n$ is large enough. But by Fatou's lemma, Theorem 11.3.18 on Page 248, the last term is no larger than

$$
\lim _{\inf _{k \rightarrow \infty}} \int\left|s_{n}-s_{k}\right| d \mu<\varepsilon
$$

whenever $n$ is large enough. Since $\varepsilon$ is arbitrary, this shows $I(f)=\int f d \mu$ as claimed.
As explained above, $I$ can be regarded as an extension of $\int d \mu$ so from now on, the usual symbol, $\int d \mu$ will be used. It is now easy to verify $\int d \mu$ is linear on $L^{1}(\Omega)$.

Theorem 11.4.7 $\int d \mu$ is linear on $L^{1}(\Omega)$ and $L^{1}(\Omega)$ is a complex vector space. If $f \in$ $L^{1}(\Omega)$, then $\operatorname{Re} f, \operatorname{Im} f$, and $|f|$ are all in $L^{1}(\Omega)$. Furthermore, for $f \in L^{1}(\Omega)$,

$$
\int f d \mu=\int(\operatorname{Re} f)^{+} d \mu-\int(\operatorname{Re} f)^{-} d \mu+i\left(\int(\operatorname{Im} f)^{+} d \mu-\int(\operatorname{Im} f)^{-} d \mu\right)
$$

Also the triangle inequality holds,

$$
\left|\int f d \mu\right| \leq \int|f| d \mu
$$

Proof: First it is necessary to verify that $L^{1}(\Omega)$ is really a vector space because it makes no sense to speak of linear maps without having these maps defined on a vector space. Let $f, g$ be in $L^{1}(\Omega)$ and let $a, b \in \mathbb{C}$. Then let $\left\{s_{n}\right\}$ and $\left\{t_{n}\right\}$ be sequences of complex simple functions associated with $f$ and $g$ respectively as described in Definition 11.4.4. Consider $\left\{a s_{n}+b t_{n}\right\}$, another sequence of complex simple functions. Then $a s_{n}(\omega)+b t_{n}(\omega) \rightarrow$ $a f(\omega)+b g(\omega)$ for each $\omega$. Also, from Lemma 11.4.3

$$
\int\left|a s_{n}+b t_{n}-\left(a s_{m}+b t_{m}\right)\right| d \mu \leq|a| \int\left|s_{n}-s_{m}\right| d \mu+|b| \int\left|t_{n}-t_{m}\right| d \mu
$$

and the sum of the two terms on the right converge to zero as $m, n \rightarrow \infty$. Thus $a f+b g \in$ $L^{1}(\Omega)$. Also

$$
\begin{aligned}
\int(a f+b g) d \mu & =\lim _{n \rightarrow \infty} \int\left(a s_{n}+b t_{n}\right) d \mu \\
& =\lim _{n \rightarrow \infty}\left(a \int s_{n} d \mu+b \int t_{n} d \mu\right) \\
& =a \lim _{n \rightarrow \infty} \int s_{n} d \mu+b \lim _{n \rightarrow \infty} \int t_{n} d \mu \\
& =a \int f d \mu+b \int g d \mu
\end{aligned}
$$

If $\left\{s_{n}\right\}$ is a sequence of complex simple functions described in Definition 11.4.4 corresponding to $f$, then $\left\{\left|s_{n}\right|\right\}$ is a sequence of complex simple functions satisfying the conditions of Definition 11.4.4 corresponding to $|f|$. This is because $\left|s_{n}(\omega)\right| \rightarrow|f(\omega)|$ and

$$
\int\left|\left|s_{n}\right|-\left|s_{m}\right|\right| d \mu \leq \int\left|s_{m}-s_{n}\right| d \mu
$$

with this last expression converging to 0 as $m, n \rightarrow \infty$. Thus $|f| \in L^{1}(\Omega)$. Also, by similar reasoning, $\left\{\operatorname{Re} s_{n}\right\}$ and $\left\{\operatorname{Im} s_{n}\right\}$ correspond to $\operatorname{Re} f$ and $\operatorname{Im} f$ respectively in the manner described by Definition 11.4.4 showing that $\operatorname{Re} f$ and $\operatorname{Im} f$ are in $L^{1}(\Omega)$. Now $(\operatorname{Re} f)^{+}=$ $\frac{1}{2}(|\operatorname{Re} f|+\operatorname{Re} f)$ and $(\operatorname{Re} f)^{-}=\frac{1}{2}(|\operatorname{Re} f|-\operatorname{Re} f)$ so both of these functions are in $L^{1}(\Omega)$. Similar formulas establish that $(\operatorname{Im} f)^{+}$and $(\operatorname{Im} f)^{-}$are in $L^{1}(\Omega)$.

The formula follows from the observation that

$$
f=(\operatorname{Re} f)^{+}-(\operatorname{Re} f)^{-}+i\left((\operatorname{Im} f)^{+}-(\operatorname{Im} f)^{-}\right)
$$

and the fact shown first that $\int d \mu$ is linear.
To verify the triangle inequality, let $\left\{s_{n}\right\}$ be complex simple functions for $f$ as in Definition 11.4.4. Then

$$
\left|\int f d \mu\right|=\lim _{n \rightarrow \infty}\left|\int s_{n} d \mu\right| \leq \lim _{n \rightarrow \infty} \int\left|s_{n}\right| d \mu=\int|f| d \mu .
$$

This proves the theorem.
The following description of $L^{1}(\Omega)$ is the version most often used because it is easy to verify the conditions for it.

Corollary 11.4.8 Let $(\Omega, \mathscr{S}, \mu)$ be a measure space and let $f: \Omega \rightarrow \mathbb{C}$. Then $f \in L^{1}(\Omega)$ if and only if $f$ is measurable and $\int|f| d \mu<\infty$.

Proof: Suppose $f \in L^{1}(\Omega)$. Then from Definition 11.4.4, it follows both real and imaginary parts of $f$ are measurable. Just take real and imaginary parts of $s_{n}$ and observe the real and imaginary parts of $f$ are limits of the real and imaginary parts of $s_{n}$ respectively. Why is $\int|f| d \mu<\infty$ ? It follows from Theorem 11.4.7. Recall why this was so. Let $\left\{s_{n}\right\}$ be a sequence of simple functions attached to $f$ as in the definition of what it means to be $L^{1}$. Then from the definition of $I(s)$ for $s$ simple,

$$
\left|I\left(\left|s_{n}\right|-\left|s_{m}\right|\right)\right| \leq I\left(\left|s_{n}-s_{m}\right|\right)
$$

which converges to 0 . Since $\left\{I\left(\left|s_{n}\right|\right)\right\}$ is a Cauchy sequence, it is bounded by a constant $C$ and also $\left\{\left|s_{n}\right|\right\}$ is a sequence of simple functions of the right sort which converges pointwise to $|f|$ and so by definition,

$$
\int|f| d \mu=I(f)=\lim _{n \rightarrow \infty} I\left(\left|s_{n}\right|\right) \leq C
$$

This shows the only if part.
The more interesting part is the if part. Suppose then that $f$ is measurable and $\int|f| d \mu<$ $\infty$. Suppose first that $f$ has values in $[0, \infty)$. It is necessary to obtain the sequence of complex simple functions. By Theorem 11.3.9, there exists a sequence of nonnegative simple functions, $\left\{s_{n}\right\}$ such that $s_{n}(\omega) \uparrow f(\omega)$. Then by the monotone convergence theorem,

$$
\lim _{n \rightarrow \infty} \int\left(2 f-\left(f-s_{n}\right)\right) d \mu=\int 2 f d \mu
$$

and so

$$
\lim _{n \rightarrow \infty} \int\left(f-s_{n}\right) d \mu=0
$$

Letting $m$ be large enough, it follows $\int\left(f-s_{m}\right) d \mu<\varepsilon$ and so if $n>m$

$$
\int\left|s_{m}-s_{n}\right| d \mu \leq \int\left|f-s_{m}\right| d \mu<\varepsilon
$$

Therefore, $f \in L^{1}(\Omega)$ because $\left\{s_{n}\right\}$ is a suitable sequence.
The general case follows from considering positive and negative parts of real and imaginary parts of $f$. These are each measurable and nonnegative and their integral is finite so each is in $L^{1}(\Omega)$ by what was just shown. Thus

$$
f=\operatorname{Re} f^{+}-\operatorname{Re} f^{-}+i\left(\operatorname{Im} f^{+}-\operatorname{Im} f^{-}\right)
$$

and so $f \in L^{1}(\Omega)$. This proves the corollary.
Theorem 11.4.9 (Dominated Convergence theorem) Let $f_{n} \in L^{1}(\Omega)$ and suppose

$$
f(\omega)=\lim _{n \rightarrow \infty} f_{n}(\omega)
$$

and there exists a measurable function $g$, with values in $[0, \infty],{ }^{3}$ such that

$$
\left|f_{n}(\omega)\right| \leq g(\omega) \text { and } \int g(\omega) d \mu<\infty
$$

Then $f \in L^{1}(\Omega)$ and

$$
\int f d \mu=\lim _{n \rightarrow \infty} \int f_{n} d \mu
$$

[^9]Proof: $f$ is measurable by Theorem 11.1.8. Since $|f| \leq g$, it follows that

$$
f \in L^{1}(\Omega) \text { and }\left|f-f_{n}\right| \leq 2 g
$$

By Fatou's lemma (Theorem 11.3.18),

$$
\begin{aligned}
\int 2 g d \mu & \leq \liminf _{n \rightarrow \infty} \int 2 g-\left|f-f_{n}\right| d \mu \\
& =\int 2 g d \mu-\lim \sup _{n \rightarrow \infty} \int\left|f-f_{n}\right| d \mu
\end{aligned}
$$

Subtracting $\int 2 g d \mu$,

$$
0 \leq-\lim \sup _{n \rightarrow \infty} \int\left|f-f_{n}\right| d \mu
$$

Hence

$$
\begin{aligned}
0 & \geq \lim \sup _{n \rightarrow \infty}\left(\int\left|f-f_{n}\right| d \mu\right) \geq \lim \sup _{n \rightarrow \infty}\left|\int f d \mu-\int f_{n} d \mu\right| \\
& \geq \lim \inf _{n \rightarrow \infty}\left|\int f d \mu-\int f_{n} d \mu\right| \geq 0
\end{aligned}
$$

This proves the theorem by Lemma 11.3.17 on Page 247 because the limsup and liminf are equal.

Corollary 11.4.10 Suppose $f_{n} \in L^{1}(\Omega)$ and $f(\omega)=\lim _{n \rightarrow \infty} f_{n}(\omega)$. Suppose also there exist measurable functions, $g_{n}, g$ with values in $[0, \infty]$ such that

$$
\lim _{n \rightarrow \infty} \int g_{n} d \mu=\int g d \mu
$$

$g_{n}(\omega) \rightarrow g(\omega) \mu$ a.e. and both $\int g_{n} d \mu$ and $\int g d \mu$ are finite. Also suppose $\left|f_{n}(\omega)\right| \leq$ $g_{n}(\omega)$. Then

$$
\lim _{n \rightarrow \infty} \int\left|f-f_{n}\right| d \mu=0
$$

Proof: It is just like the above. This time $g+g_{n}-\left|f-f_{n}\right| \geq 0$ and so by Fatou's lemma,

$$
\begin{gathered}
\int 2 g d \mu-\lim _{n \rightarrow \infty} \int\left|f-f_{n}\right| d \mu= \\
\lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty} \int\left(g_{n}+g\right)-\lim \sup _{n \rightarrow \infty} \int\left|f-f_{n}\right| d \mu \\
=\lim _{n \rightarrow \infty} \int\left(\left(g_{n}+g\right)-\left|f-f_{n}\right|\right) d \mu \geq \int 2 g d \mu
\end{gathered}
$$

and so $-\lim \sup _{n \rightarrow \infty} \int\left|f-f_{n}\right| d \mu \geq 0$.
Definition 11.4.11 Let $E$ be a measurable subset of $\Omega$.

$$
\int_{E} f d \mu \equiv \int f \mathscr{X}_{E} d \mu
$$

If $L^{1}(E)$ is written, the $\sigma$ algebra is defined as

$$
\{E \cap A: A \in \mathscr{F}\}
$$

and the measure is $\mu$ restricted to this smaller $\sigma$ algebra. Clearly, if $f \in L^{1}(\Omega)$, then

$$
f \mathscr{X}_{E} \in L^{1}(E)
$$

and if $f \in L^{1}(E)$, then letting $\tilde{f}$ be the 0 extension of $f$ off of $E$, it follows $\tilde{f} \in L^{1}(\Omega)$.

### 11.5 Vitali Convergence Theorem

The Vitali convergence theorem is a convergence theorem which in the case of a finite measure space is superior to the dominated convergence theorem.

Definition 11.5.1 Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and let $\mathfrak{S} \subseteq L^{1}(\Omega)$. $\mathfrak{S}$ is uniformly integrable iffor every $\varepsilon>0$ there exists $\delta>0$ such that for all $f \in \mathfrak{S}$

$$
\left|\int_{E} f d \mu\right|<\varepsilon \text { whenever } \mu(E)<\delta
$$

Lemma 11.5.2 If $\mathfrak{S}$ is uniformly integrable, then $|\mathfrak{S}| \equiv\{|f|: f \in \mathfrak{S}\}$ is uniformly integrable. Also $\mathfrak{S}$ is uniformly integrable if $\mathfrak{S}$ is finite.

Proof: Let $\varepsilon>0$ be given and suppose $\mathfrak{S}$ is uniformly integrable. First suppose the functions are real valued. Let $\delta$ be such that if $\mu(E)<\delta$, then

$$
\left|\int_{E} f d \mu\right|<\frac{\varepsilon}{2}
$$

for all $f \in \mathfrak{S}$. Let $\mu(E)<\delta$. Then if $f \in \mathfrak{S}$,

$$
\begin{aligned}
\int_{E}|f| d \mu & \leq \int_{E \cap[f \leq 0]}(-f) d \mu+\int_{E \cap[f>0]} f d \mu \\
& =\left|\int_{E \cap[f \leq 0]} f d \mu\right|+\left|\int_{E \cap[f>0]} f d \mu\right| \\
& <\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon
\end{aligned}
$$

In general, if $\mathfrak{S}$ is a uniformly integrable set of complex valued functions, the inequalities,

$$
\left|\int_{E} \operatorname{Re} f d \mu\right| \leq\left|\int_{E} f d \mu\right|,\left|\int_{E} \operatorname{Im} f d \mu\right| \leq\left|\int_{E} f d \mu\right|
$$

imply $\operatorname{Re} \mathfrak{S} \equiv\{\operatorname{Re} f: f \in \mathfrak{S}\}$ and $\operatorname{Im} \mathfrak{S} \equiv\{\operatorname{Im} f: f \in \mathfrak{S}\}$ are also uniformly integrable. Therefore, applying the above result for real valued functions to these sets of functions, it follows $|\mathfrak{S}|$ is uniformly integrable also.

For the last part, is suffices to verify a single function in $L^{1}(\Omega)$ is uniformly integrable. To do so, note that from the dominated convergence theorem,

$$
\lim _{R \rightarrow \infty} \int_{[|f|>R]}|f| d \mu=0
$$

Let $\varepsilon>0$ be given and choose $R$ large enough that $\int_{[|f|>R]}|f| d \mu<\frac{\varepsilon}{2}$. Now let $\mu(E)<\frac{\varepsilon}{2 R}$. Then

$$
\begin{aligned}
\int_{E}|f| d \mu & =\int_{E \cap[|f| \leq R]}|f| d \mu+\int_{E \cap[|f|>R]}|f| d \mu \\
& <R \mu(E)+\frac{\varepsilon}{2}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon
\end{aligned}
$$

This proves the lemma.
The following theorem is Vitali's convergence theorem.
Theorem 11.5.3 Let $\left\{f_{n}\right\}$ be a uniformly integrable set of complex valued functions,

$$
\mu(\Omega)<\infty \text { and } f_{n}(x) \rightarrow f(x)
$$

a.e. where $f$ is a measurable complex valued function. Then $f \in L^{1}(\Omega)$ and

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \int_{\Omega}\left|f_{n}-f\right| d \mu=0 \tag{11.5.25}
\end{equation*}
$$

Proof: First it will be shown that $f \in L^{1}(\Omega)$. By uniform integrability, there exists $\delta>0$ such that if $\mu(E)<\delta$, then

$$
\int_{E}\left|f_{n}\right| d \mu<1
$$

for all $n$. By Egoroff's theorem, there exists a set, $E$ of measure less than $\delta$ such that on $E^{C},\left\{f_{n}\right\}$ converges uniformly. Therefore, for $p$ large enough, and $n>p$,

$$
\int_{E^{C}}\left|f_{p}-f_{n}\right| d \mu<1
$$

which implies

$$
\int_{E^{C}}\left|f_{n}\right| d \mu<1+\int_{\Omega}\left|f_{p}\right| d \mu
$$

Then since there are only finitely many functions, $f_{n}$ with $n \leq p$, there exists a constant, $M_{1}$ such that for all $n$,

$$
\int_{E^{C}}\left|f_{n}\right| d \mu<M_{1}
$$

But also,

$$
\begin{aligned}
\int_{\Omega}\left|f_{m}\right| d \mu & =\int_{E^{C}}\left|f_{m}\right| d \mu+\int_{E}\left|f_{m}\right| \\
& \leq M_{1}+1 \equiv M
\end{aligned}
$$

Therefore, by Fatou's lemma,

$$
\int_{\Omega}|f| d \mu \leq \lim \inf _{n \rightarrow \infty} \int\left|f_{n}\right| d \mu \leq M
$$

showing that $f \in L^{1}$ as hoped.
Now $\mathfrak{S} \cup\{f\}$ is uniformly integrable so there exists $\delta_{1}>0$ such that if $\mu(E)<\delta_{1}$, then $\int_{E}|g| d \mu<\varepsilon / 3$ for all $g \in \mathfrak{S} \cup\{f\}$. By Egoroff's theorem, there exists a set, $F$ with $\mu(F)<\delta_{1}$ such that $f_{n}$ converges uniformly to $f$ on $F^{C}$. Therefore, there exists $N$ such that if $n>N$, then

$$
\int_{F^{C}}\left|f-f_{n}\right| d \mu<\frac{\varepsilon}{3}
$$

It follows that for $n>N$,

$$
\begin{aligned}
\int_{\Omega}\left|f-f_{n}\right| d \mu & \leq \int_{F^{C}}\left|f-f_{n}\right| d \mu+\int_{F}|f| d \mu+\int_{F}\left|f_{n}\right| d \mu \\
& <\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon
\end{aligned}
$$

which verifies 11.5.25.

### 11.6 Measures and Regularity

It is often the case that $\Omega$ has more going on than to simply be a set. In particular, it is often the case that $\Omega$ is some sort of topological space, often a metric space. In this case, it is usually if not always the case that the open sets will be in the $\sigma$ algebra of measurable sets. This leads to the following definition.

Definition 11.6.1 A Polish space is a complete separable metric space. For a Polish space $E$ or more generally a metric space or even a general topological space, $\mathscr{B}(E)$ denotes the Borel sets of $E$. This is defined to be the smallest $\sigma$ algebra which contains the open sets. Thus it contains all open sets and closed sets and compact sets and many others.

Don't ever try to describe a generic Borel set. Always work with the definition that it is the smallest $\sigma$ algebra containing the open sets. Attempts to give an explicit description of a "typical" Borel set tend to lead nowhere because there are so many things which can be done. You can take countable unions and complements and then countable intersections of what you get and then another countable union followed by complements and on and on. You just can't get a good useable description in this way. However, it is easy to see that something like $\left(\cap_{i=1}^{\infty} \cup_{j=i}^{\infty} E_{j}\right)^{C}$ is a Borel set if the $E_{j}$ are. This is useful. This said, you can look at Hewitt and Stromberg in their discussion of why there are more Lebesgue measurable sets than Borel measurable sets to see the kind of technicalities which result by describing Borel sets.

For example, $\mathbb{R}$ is a Polish space as is any separable Banach space. Amazing things can be said about finite measures on the Borel sets of a Polish space. First the case of a finite measure on a metric space will be considered.

Definition 11.6.2 A measure $\mu$ defined on $\mathscr{B}(E)$ will be called inner regular if for all $F \in \mathscr{B}(E)$,

$$
\begin{equation*}
\mu(F)=\sup \{\mu(K): K \subseteq F \text { and } K \text { is closed }\} \tag{11.6.26}
\end{equation*}
$$

A measure, $\mu$ defined on $\mathscr{B}(E)$ will be called outer regular if for all $F \in \mathscr{B}(E)$,

$$
\begin{equation*}
\mu(F)=\inf \{\mu(V): V \supseteq F \text { and } V \text { is open }\} \tag{11.6.27}
\end{equation*}
$$

When a measure is both inner and outer regular, it is called regular. Actually, it is more useful and likely more standard to refer to $\mu$ being inner regular as

$$
\begin{equation*}
\mu(F)=\sup \{\mu(K): K \subseteq F \text { and } K \text { is compact }\} \tag{11.6.28}
\end{equation*}
$$

Thus the word "closed" is replaced with "compact". A complete measure defined on a $\sigma$ algebra $\mathscr{F}$ which includes the Borel sets which is finite on compact sets and also satisfies 11.6.27 and 11.6.28 for each $F \in \mathscr{F}$ is called a Radon measure.

For finite measures, defined on the Borel sets of a metric space, the first definition of regularity is automatic. These are always outer and inner regular provided inner regularity refers to closed sets.

Lemma 11.6.3 Let $\mu$ be a finite measure defined on $\mathscr{B}(X)$ where $X$ is a metric space. Then $\mu$ is regular.

Proof: First note every open set is the countable union of closed sets and every closed set is the countable intersection of open sets. Here is why. Let $V$ be an open set and let

$$
K_{k} \equiv\left\{x \in V: \operatorname{dist}\left(x, V^{C}\right) \geq 1 / k\right\} .
$$

Then clearly the union of the $K_{k}$ equals $V$. Thus

$$
\mu(V)=\sup \{\mu(K): K \subseteq V \text { and } K \text { is closed }\}
$$

If $U$ is open and contains $V$, then $\mu(U) \geq \mu(V)$ and so

$$
\mu(V) \leq \inf \{\mu(U): U \supseteq V, U \text { open }\} \leq \mu(V) \text { since } V \subseteq V
$$

Thus $\mu$ is inner and outer regular on open sets. In what follows, $K$ will be closed and $V$ will be open.

Let $\mathscr{K}$ be the open sets. This is a $\pi$ system. Let

$$
\mathscr{G} \equiv\{E \in \mathscr{B}(X): \mu \text { is inner and outer regular on } E\} \text { so } \mathscr{G} \supseteq \mathscr{K} .
$$

For $E \in \mathscr{G}$, let $V \supseteq E \supseteq K$ such that $\mu(V \backslash K)=\mu(V \backslash E)+\mu(E \backslash K)<\varepsilon$. Thus $K^{C} \supseteq E^{C}$ and so $\mu\left(K^{C} \backslash E^{C}\right)=\mu(E \backslash K)<\varepsilon$. Thus $\mu$ is outer regular on $E^{C}$ because

$$
\mu\left(K^{C}\right)=\mu\left(E^{C}\right)+\mu\left(K^{C} \backslash E^{C}\right)<\mu\left(E^{C}\right)+\varepsilon, K^{C} \supseteq E^{C}
$$

Also, $E^{C} \supseteq V^{C}$ and $\mu\left(E^{C} \backslash V^{C}\right)=\mu(V \backslash E)<\varepsilon$ so $\mu$ is inner regular on $E^{C}$ and so $\mathscr{G}$ is closed for complements. If the sets of $\mathscr{G}\left\{E_{i}\right\}$ are disjoint, let $K_{i} \subseteq E_{i} \subseteq V_{i}$ with $\mu\left(V_{i} \backslash K_{i}\right)<$ $\varepsilon 2^{-i}$. Then for $E \equiv \cup_{i} E_{i}$, and choosing $m$ sufficiently large,

$$
\mu(E)=\sum_{i} \mu\left(E_{i}\right) \leq \sum_{i=1}^{m} \mu\left(E_{i}\right)+\varepsilon \leq \sum_{i=1}^{m} \mu\left(K_{i}\right)+2 \varepsilon=\mu\left(\cup_{i=1}^{m} K_{i}\right)+2 \varepsilon
$$

and so $\mu$ is inner regular on $E \equiv \cup_{i} E_{i}$. It remains to show that $\mu$ is outer regular on $E$. Letting $V \equiv \cup_{i} V_{i}$,

$$
\mu(V \backslash E) \leq \mu\left(\cup_{i}\left(V_{i} \backslash E_{i}\right)\right) \leq \sum_{i} \varepsilon 2^{-i}=\varepsilon
$$

Hence $\mu$ is outer regular on $E$ since $\mu(V)=\mu(E)+\mu(V \backslash E) \leq \mu(E)+\varepsilon$ and $V \supseteq E$.
By Dynkin's lemma, $\mathscr{G}=\sigma(\mathscr{K}) \equiv \mathscr{B}(X)$.
One can say more if the metric space is complete and separable. In fact in this case the above definition of inner regularity can be shown to imply the usual one where the closed sets are replaced with compact sets.

Lemma 11.6.4 Let $\mu$ be a finite measure on a $\sigma$ algebra containing $\mathscr{B}(X)$, the Borel sets of $X$, a separable complete metric space. Then if $C$ is a closed set,

$$
\mu(C)=\sup \{\mu(K): K \subseteq C \text { and } K \text { is compact. }\}
$$

It follows that for a finite measure on $\mathscr{B}(X)$ where $X$ is a Polish space, $\mu$ is inner regular in the sense that for all $F \in \mathscr{B}(X)$,

$$
\mu(F)=\sup \{\mu(K): K \subseteq F \text { and } K \text { is compact }\}
$$

Proof: Let $\left\{a_{k}\right\}$ be a countable dense subset of $C$. Thus $\cup_{k=1}^{\infty} B\left(a_{k}, \frac{1}{n}\right) \supseteq C$. Therefore, there exists $m_{n}$ such that

$$
\mu\left(C \backslash \cup_{k=1}^{m_{n}} \overline{B\left(a_{k}, \frac{1}{n}\right)}\right) \equiv \mu\left(C \backslash C_{n}\right)<\frac{\varepsilon}{2^{n}}
$$

Now let $K=C \cap\left(\cap_{n=1}^{\infty} C_{n}\right)$. Then $K$ is a subset of $C_{n}$ for each $n$ and so for each $\varepsilon>0$ there exists an $\varepsilon$ net for $K$ since $C_{n}$ has a $1 / n$ net, namely $a_{1}, \cdots, a_{m_{n}}$. Since $K$ is closed, it is complete and so it is also compact since it is complete and totally bounded, Theorem 7.6.5. Now

$$
\mu(C \backslash K) \leq \mu\left(\cup_{n=1}^{\infty}\left(C \backslash C_{n}\right)\right)<\sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n}}=\varepsilon
$$

Thus $\mu(C)$ can be approximated by $\mu(K)$ for $K$ a compact subset of $C$. The last claim follows from Lemma 11.6.3.

An important example is the case of a random vector and its distribution measure.
Definition 11.6.5 A measurable function $\mathbf{X}:(\Omega, \mathscr{F}, \mu) \rightarrow Z$ a metric space is called a random variable when $\mu(\Omega)=1$. For such a random variable, one can define a distribution measure $\lambda_{\mathbf{x}}$ on the Borel sets of $Z$ as follows.

$$
\lambda_{\mathbf{X}}(G) \equiv \mu\left(\mathbf{X}^{-1}(G)\right)
$$

This is a well defined measure on the Borel sets of $Z$ because it makes sense for every $G$ open and $\mathscr{G} \equiv\left\{G \subseteq Z: \mathbf{X}^{-1}(G) \in \mathscr{F}\right\}$ is a $\sigma$ algebra which contains the open sets, hence the Borel sets. Such a random variable is also called a random vector when $Z$ is a vector space.

Corollary 11.6.6 Let $\mathbf{X}$ be a random variable with values in a separable complete metric space $Z$. Then $\lambda_{\mathbf{x}}$ is an inner and outer regular measure defined on $\mathscr{B}(Z)$.

What if the measure $\mu$ is defined on a Polish space but is not finite. Sometimes one can still get the assertion that $\mu$ is regular. In every case of interest in this book, the measure will also be $\sigma$ finite.

Definition 11.6.7 Let $(E, \mathscr{B}(E), \mu)$ be a measurable space with the measure $\mu$. Then $\mu$ is said to be $\sigma$ finite if there is a sequence of disjoint Borel sets $\left\{B_{i}\right\}_{i=1}^{\infty}$ such that $\cup_{i=1}^{\infty} B_{i}=E$ and $\mu\left(B_{i}\right)<\infty$. More generally, if $(X, \mathscr{F}, \mu)$ is a measure space, it is $\sigma$ finite if there are $X_{n} \in \mathscr{F}$ with $\cup_{n} X_{n}=X$ and $\mu\left(X_{n}\right)<\infty$.

One such example of a complete metric space and a measure which is finite on compact sets is the following where the closures of balls are compact. Thus, this involves finite dimensional situations essentially. Note that if you have a metric space in which the closures of balls are compact sets, then the metric space must be separable. This is because you can pick a point $\xi$ and consider the closures of balls $\overline{B(\xi, n)}$. Then $\overline{B(\xi, n)}$ is complete and totally bounded so it has a countable dense subset $D_{n}$. Let $D=\cup_{n} D_{n}$.

Corollary 11.6.8 Let $\Omega$ be a complete metric space which is the countable union of compact sets $K_{n}$ and suppose, for $\mu$ a Borel measure, $\mu\left(K_{n}\right)$ is finite. Then $\mu$ must be regular. In particular, if $\Omega$ is a metric space and the closure of each ball is compact, and $\mu$ is finite on balls, then $\mu$ must be regular.

Proof: Let the compact sets be increasing without loss of generality, and let $\mu_{n}(E) \equiv$ $\mu\left(K_{n} \cap E\right)$. Thus $\mu_{n}$ is a finite measure defined on the Borel sets of a Polish space so it is regular. Letting $l<\mu(E)$, there exists $n$ such that $l<\mu_{n}(E) \leq \mu(E)$. By what was shown above in Lemma 11.6.4, there exists $H$ compact, $H \subseteq E$ such that also for a large $n, \mu_{n}(H)>l$. Hence $\mu\left(H \cap K_{n}\right)>l$ and so $\mu$ is inner regular. It remains to verify that $\mu$ is outer regular. If $\mu(E)=\infty$, there is nothing to show. Assume then that $\mu(E)<\infty$. Let $V_{n} \supseteq E$ with $\mu_{n}\left(V_{n} \backslash E\right)<\varepsilon 2^{-n}$ so also $\mu\left(V_{n}\right)<\infty$. We can assume also that $V_{n} \supseteq V_{n+1}$ for all $n$. Thus $\mu\left(\left(V_{n} \backslash E\right) \cap K_{n}\right)<2^{-n} \varepsilon$. Let $G=\cap_{k} V_{k}$. Then $G \subseteq V_{n}$ so $\mu\left((G \backslash E) \cap K_{n}\right)<2^{-n} \varepsilon$. Letting $n \rightarrow \infty, \mu(G \backslash E)=0$ and $G \supseteq E$. Then, since $V_{1}$ has finite measure, $\mu(G \backslash E)=\lim _{n \rightarrow \infty} \mu\left(V_{n} \backslash E\right)$ and so for all $n$ large enough, $\mu\left(V_{n} \backslash E\right)<\varepsilon$ so $\mu(E)+\varepsilon>\mu\left(V_{n}\right)$ and so $\mu$ is outer regular. In the last case, if the closure of each ball is compact, then $\Omega$ is automatically complete because every Cauchy sequence is contained in some ball and so has a convergent subsequence. Since the sequence is Cauchy, it also converges by Theorem 7.3.2 on Page 140 .

### 11.7 Regular Measures in a Metric Space

In this section $X$ will be a metric space in which the closed balls are compact. The extra generality involving a metric space instead of $\mathbb{R}^{p}$ would allow the consideration of manifolds for example. However, $\mathbb{R}^{p}$ is an important case.

Definition 11.7.1 The symbol $C_{c}(V)$ for $V$ an open set will denote the continuous functions having compact support which is contained in $V$. Recall that the support of a continuous function $f$ is defined as the closure of the set on which the function is nonzero. $L: C_{c}(X) \rightarrow$ $\mathbb{C}$ is called a positive linear functional if it is linear, $L(\alpha f+\beta g)=\alpha L f+\beta L g$ and satisfies $L f \leq L g$ if $f \leq g$. Also, recall that a measure $\mu$ is regular on some $\sigma$ algebra $\mathscr{F}$ containing the Borel sets iffor every $F \in \mathscr{F}$,

$$
\begin{aligned}
& \mu(F)=\sup \{\mu(K): K \subseteq F \text { and } K \text { compact }\} \\
& \mu(F)=\inf \{\mu(V): V \supseteq F \text { and } V \text { is open }\}
\end{aligned}
$$

A complete measure, finite on compact sets, which is regular as above, is called a Radon measure. A set is called an $F_{\sigma}$ set if it is the countable union of closed sets and a set is $G_{\delta}$ if it is the countable intersection of open sets. If $K$ is compact and $V$ is open, we say $K \prec \phi \prec V$ if $\phi$ is continuous, has values in $[0,1], \phi(x)=1$ for all $x \in K$, and $\overline{\{x: \phi(x) \neq 0\}}$ called $\operatorname{spt}(\phi)$ is contained in $V$.

Remarkable things happen in the above context. Some are described in the following proposition. First is a lemma.

Lemma 11.7.2 Let $(\Omega, d)$ be a metric space in which closed balls are compact. Then if $K$ is a compact subset of an open set $V$, then there exists $\phi$ such that $K \prec \phi \prec V$.

Proof: Since $K$ is compact, the distance between $K$ and $V^{C}$ is positive, $\delta>0$. Otherwise there would be $x_{n} \in K$ and $y_{n} \in V^{C}$ with $d\left(x_{n}, y_{n}\right)<1 / n$. Taking a subsequence, still denoted with $n$, we can assume $x_{n} \rightarrow x$ and $y_{n} \rightarrow x$ but this would imply $x$ is in both $K$ and $V^{C}$ which is not possible. Now consider $\{B(x, \delta / 2)\}$ for $x \in K$. This is an open cover and the closure of each ball is contained in $V$. Since $K$ is compact, finitely many of these balls cover $K$. Denote their union as $W$. Then $\bar{W}$ is compact because it is the finite union of the closed balls. Hence $K \subseteq W \subseteq \bar{W} \subseteq V$. Now consider

$$
\phi(x) \equiv \frac{\operatorname{dist}\left(x, W^{C}\right)}{\operatorname{dist}(x, K)+\operatorname{dist}\left(x, W^{C}\right)}
$$

the denominator is never zero because $x$ cannot be in both $K$ and $W^{C}$. Thus $\phi$ is continuous because the denominator is never 0 and the functions of $x$ are continuous. Also if $x \in K$, then $\phi(x)=1$ and if $x \notin W$, then $\phi(x)=0$.

Proposition 11.7.3 Suppose $(X, d)$ is a metric space in which the balls are compact and $X$ is a countable union of closed balls. Also suppose $(X, \mathscr{F}, \mu)$ is a complete measure space, $\mathscr{F}$ contains the Borel sets, and that $\mu$ is regular and finite on finite balls. Then

1. For each $E \in \mathscr{F}$, there is an $F_{\sigma}$ set $F$ and $a G_{\delta}$ set $G$ such that $F \subseteq E \subseteq G$ and $\mu(G \backslash F)=0$.
2. Also if $f \geq 0$ is $\mathscr{F}$ measurable, then there exists $g \leq f$ such that $g$ is Borel measurable and $g=f$ a.e. and $h \geq f$ such that $h$ is Borel measurable and $h=f$ a.e.
3. If $E \in \mathscr{F}$ is a bounded set contained in a ball $B\left(x_{0}, r\right)=V$, then there exists $a$ sequence of continuous functions in $C_{c}(V)\left\{h_{n}\right\}$ having values in $[0,1]$ and a set of measure zero $N$ such that for $x \notin N, h_{n}(x) \rightarrow \mathscr{X}_{E}(x)$. Also $\int\left|h_{n}-\mathscr{X}_{E}\right| d \mu \rightarrow 0$. Letting $\tilde{N}$ be a $G_{\delta}$ set of measure zero containing $N, h_{n} \mathscr{X}_{\tilde{N}^{C}} \rightarrow \mathscr{X}_{F}$ where $F \subseteq E$ and $\mu(E \backslash F)=0$.
4. If $f \in L^{1}(X, \mathscr{F}, \mu)$, there exists $g \in C_{c}(X)$, such that $\int_{X}|f-g| d \mu<\varepsilon$. There also exists a sequence of functions in $C_{c}(X)\left\{g_{n}\right\}$ which converges pointwise to $f$.

Proof: 1. Let $R_{n} \equiv B\left(x_{0}, n\right), R_{0}=\emptyset$. If $E$ is Lebesgue measurable, let $E_{n} \equiv E \cap$ $\left(R_{n} \backslash R_{n-1}\right)$. Thus these $E_{n}$ are disjoint and their union is $E$. By outer regularity, there exists open $U_{n} \supseteq E_{n}$ such that $\mu\left(U_{n} \backslash E_{n}\right)<\varepsilon / 2^{n}$. Now if $U \equiv \cup_{n} U_{n}$, it follows that $\mu(U \backslash E) \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n}}=\varepsilon$. Let $V_{n}$ be open, containing $E$ and $\mu\left(V_{n} \backslash E\right)<\frac{1}{2^{n}}, V_{n} \supseteq V_{n+1}$. Let $G \equiv \cap_{n} V_{n}$. This is a $G_{\delta}$ set containing $E$ and $\mu(G \backslash E) \leq \mu\left(V_{n} \backslash E\right)<\frac{1}{2^{n}}$ and so $\mu(G \backslash E)=0$. By inner regularity, there is $F_{n}$ an $F_{\sigma}$ set contained in $E_{n}$ with $\mu\left(E_{n} \backslash F_{n}\right)=0$. Then let $F \equiv \cup_{n} F_{n}$. This is an $F_{\sigma}$ set and $\mu(E \backslash F) \leq \sum_{n} \mu\left(E_{n} \backslash F_{n}\right)=0$. Thus $F \subseteq E \subseteq G$ and $\mu(G \backslash F) \leq \mu(G \backslash E)+\mu(E \backslash F)=0$.
2. If $f$ is measurable and nonnegative, there is an increasing sequence of simple functions $s_{n}$ such that $\lim _{n \rightarrow \infty} s_{n}(x)=f(x)$. Say $\sum_{k=1}^{m_{n}} c_{k}^{n} \mathscr{X}_{E_{k}^{n}}(x)$. Let $m_{p}\left(E_{k}^{n} \backslash F_{k}^{n}\right)=0$ where $F_{k}^{n}$ is an $F_{\sigma}$ set. Replace $E_{k}^{n}$ with $F_{k}^{n}$ and let $\tilde{s}_{n}$ be the resulting simple function. Let $g(x) \equiv \lim _{n \rightarrow \infty} \tilde{S}_{n}(x)$. Then $g$ is Borel measurable and $g \leq f$ and $g=f$ except for a set of measure zero, the union of the sets where $s_{n}$ is not equal to $\tilde{s}_{n}$. As to the other claim, let $h_{n}(x) \equiv \sum_{k=1}^{\infty} \mathscr{X}_{A_{k n}}(x) \frac{k}{2^{n}}$ where $A_{k n}$ is a $G_{\delta}$ set containing $f^{-1}\left(\left(\frac{k-1}{2^{n}}, \frac{k}{2^{n}}\right]\right)$ for which $\mu\left(A_{k n} \backslash f^{-1}\left(\left(\frac{k-1}{2^{n}}, \frac{k}{2^{n}}\right]\right)\right) \equiv \mu\left(D_{k n}\right)=0$. If $N=\cup_{k, n} D_{k n}$, then $N$ is a set of measure zero. On $N^{C}, h_{n}(x) \rightarrow f(x)$. Let $h(x)=\liminf _{n \rightarrow \infty} h_{n}(x)$.
3. Let $K_{n} \subseteq E \subseteq V_{n}$ with $K_{n}$ compact and $V_{n}$ open such that $V_{n} \subseteq B\left(x_{0}, r\right)$ and

$$
\mu\left(V_{n} \backslash K_{n}\right)<2^{-(n+1)}
$$

Then from Lemma 11.7.2, there exists $h_{n}$ with $K_{n} \prec h_{n} \prec V_{n}$. Then $\int\left|h_{n}-\mathscr{X}_{E}\right| d \mu<2^{-n}$ and so

$$
\mu\left(\left|h_{n}-\mathscr{X}_{E}\right|>\left(\frac{2}{3}\right)^{n}\right)<\left(\left(\frac{3}{2}\right)^{n} \int_{\left[\left|h_{n}-\mathscr{X}_{E}\right|>\left(\frac{2}{3}\right)^{n}\right]}\left|h_{n}-\mathscr{X}_{E}\right| d \mu\right) \leq\left(\frac{3}{4}\right)^{n}
$$

Letting $A_{n} \equiv\left[\left|h_{n}-\mathscr{X}_{E}\right|>\left(\frac{2}{3}\right)^{n}\right]$, the set of $x$ which is in infinitely many $A_{n}$ is $N \equiv \cap_{n} \cup_{k \geq n}$ $A_{k}$ and so

$$
\mu\left(\cap_{n} \cup_{k \geq n} A_{k}\right) \leq \mu\left(\cup_{k \geq n} A_{k}\right) \leq \sum_{k=n}^{\infty}\left(\frac{3}{4}\right)^{k}=\left(\frac{3}{4}\right)^{n}\left(\frac{1}{1 / 4}\right)
$$

and since $n$ is arbitrary the set of $x$ in infinitely many $A_{n}$ called $N$ has measure zero. Thus if $x \notin N$, it is in only finitely many of the sets $\left\{\left|h_{n}-\mathscr{X}_{E}\right|>\left(\frac{2}{3}\right)^{n}\right\}$. Thus on $N^{C}$, eventually, for all $k$ large enough, $\left|h_{k}-\mathscr{X}_{E}\right| \leq\left(\frac{2}{3}\right)^{k}$ so $h_{k}(x) \rightarrow \mathscr{X}_{E}(x)$ off $N$. The assertion about convergence of the integrals follows from the dominated convergence theorem and the fact that each $h_{n}$ is nonnegative, bounded by 1 , and is 0 off some ball. In the last claim, it only remains to verify that $h_{n} \mathscr{X}_{\tilde{N}^{c}}$ converges to an indicator function because each $h_{n} \mathscr{X}_{\tilde{N}^{c}}$ is Borel measurable. Thus its limit will also be Borel measurable. However, it converges to 1 on $E \cap \tilde{N}^{C}, 0$ on $E^{C} \cap \tilde{N}^{C}$ and 0 on $\tilde{N}$. Thus $E \cap \tilde{N}^{C}=F$ and $h_{n} \mathscr{X}_{\tilde{N}^{C}}(x) \rightarrow \mathscr{X}_{F}$ where $F \subseteq E$ and $\mu(E \backslash F) \leq \mu(\tilde{N})=0$.
4. It suffices to assume $f \geq 0$ because you can consider the positive and negative parts of the real and imaginary parts of $f$ and reduce to this case. Let $f_{n}(x) \equiv \mathscr{X}_{B\left(x_{0}, n\right)}(x) f(x)$. Then by the dominated convergence theorem, if $n$ is large enough, $\int\left|f-f_{n}\right| d \mu<\varepsilon$. There is a nonnegative simple function $s \leq f_{n}$ such that $\int\left|f_{n}-s\right| d \mu<\varepsilon$. This follows from picking $k$ large enough in an increasing sequence of simple functions $\left\{s_{k}\right\}$ converging to $f_{n}$ and the monotone or dominated convergence theorem. Say $s(x)=\sum_{k=1}^{m} c_{k} \mathscr{X}_{E_{k}}(x)$. Then let $K_{k} \subseteq E_{k} \subseteq V_{k}$ where $K_{k}, V_{k}$ are compact and open respectively and $\sum_{k=1}^{m} c_{k} \mu\left(V_{k} \backslash K_{k}\right)<\varepsilon$. By Lemma 11.7.2, there exists $h_{k}$ with $K_{k} \prec h_{k} \prec V_{k}$. Then

$$
\begin{aligned}
\int\left|\sum_{k=1}^{m} c_{k} \mathscr{X}_{E_{k}}(x)-\sum_{k=1}^{m} c_{k} h_{k}(x)\right| d \mu & \leq \sum_{k} c_{k} \int\left|\mathscr{X}_{E_{k}}(x)-h_{k}(x)\right| d x \\
& <2 \sum_{k} c_{k} \mu\left(V_{k} \backslash K_{k}\right)<2 \varepsilon
\end{aligned}
$$

Let $g \equiv \sum_{k=1}^{m} c_{k} h_{k}(x)$. Thus $\int|s-g| d \mu \leq 2 \varepsilon$. Then

$$
\int|f-g| d \mu \leq \int\left|f-f_{n}\right| d \mu+\int\left|f_{n}-s\right| d \mu+\int|s-g| d \mu<4 \varepsilon
$$

Since $\varepsilon$ is arbitrary, this proves the first part of 4 . For the second part, let $g_{n} \in C_{c}(X)$ such that $\int\left|f-g_{n}\right| d \mu<2^{-n}$. Let $A_{n} \equiv\left\{x:\left|f-g_{n}\right|>\left(\frac{2}{3}\right)^{n}\right\}$. Then

$$
\mu\left(A_{n}\right) \leq\left(\frac{3}{2}\right)^{n} \int_{A_{n}}\left|f-g_{n}\right| d \mu \leq\left(\frac{3}{4}\right)^{n}
$$

Thus, if $N$ is all $x$ in infinitely many $A_{n}$, then by the Borel Cantelli lemma, $\mu(N)=0$ and if $x \notin N$, then $x$ is in only finitely many $A_{n}$ and so for all $n$ large enough, $\left|f(x)-g_{n}(x)\right| \leq$ $\left(\frac{2}{3}\right)^{n}$.

### 11.8 Exercises

1. Let $\Omega=\mathbb{N}=\{1,2, \cdots\}$ and $\mu(S)=$ number of elements in $S$. If

$$
f: \Omega \rightarrow \mathbb{C}
$$

what is meant by $\int f d \mu$ ? Which functions are in $L^{1}(\Omega)$ ? Which functions are measurable?
2. Show that for $f \geq 0$ and measurable, $\int f d \mu \equiv \lim _{h \rightarrow 0+} \sum_{i=1}^{\infty} h \mu([i h<f])$.
3. For the measure space of Problem 1, give an example of a sequence of nonnegative measurable functions $\left\{f_{n}\right\}$ converging pointwise to a function $f$, such that inequality is obtained in Fatou's lemma.
4. Fill in all the details of the proof of Lemma 11.5.2.
5. Let $\sum_{i=1}^{n} c_{i} \mathscr{X}_{E_{i}}(\omega)=s(\omega)$ be a nonnegative simple function for which the $c_{i}$ are the distinct nonzero values. Show with the aid of the monotone convergence theorem that the two definitions of the Lebesgue integral given in the chapter are equivalent.
6. Suppose $(\Omega, \mu)$ is a finite measure space and $\mathfrak{S} \subseteq L^{1}(\Omega)$. Show $\mathfrak{S}$ is uniformly integrable and bounded in $L^{1}(\Omega)$ if there exists an increasing function $h$ which satisfies

$$
\lim _{t \rightarrow \infty} \frac{h(t)}{t}=\infty, \sup \left\{\int_{\Omega} h(|f|) d \mu: f \in \mathfrak{S}\right\}<\infty
$$

$\mathfrak{S}$ is bounded if there is some number, $M$ such that

$$
\int|f| d \mu \leq M
$$

for all $f \in \mathfrak{S}$.
7. Let $\left\{a_{n}\right\},\left\{b_{n}\right\}$ be sequences in $[-\infty, \infty]$ and $a \in \mathbb{R}$. Show

$$
\lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty}\left(a-a_{n}\right)=a-\lim \sup _{n \rightarrow \infty} a_{n} .
$$

This was used in the proof of the Dominated convergence theorem. Also show

$$
\begin{gathered}
\lim \sup _{n \rightarrow \infty}\left(-a_{n}\right)=-\lim \inf _{n \rightarrow \infty}\left(a_{n}\right) \\
\lim \sup _{n \rightarrow \infty}\left(a_{n}+b_{n}\right) \leq \lim \sup _{n \rightarrow \infty} a_{n}+\lim \sup _{n \rightarrow \infty} b_{n}
\end{gathered}
$$

provided no sum is of the form $\infty-\infty$. Also show strict inequality can hold in the inequality. State and prove corresponding statements for liminf.
8. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and suppose $f, g: \Omega \rightarrow(-\infty, \infty]$ are measurable. Prove the sets

$$
\{\omega: f(\omega)<g(\omega)\} \text { and }\{\omega: f(\omega)=g(\omega)\}
$$

are measurable. Hint: The easy way to do this is to write

$$
\{\omega: f(\omega)<g(\omega)\}=\cup_{r \in \mathbb{Q}}[f<r] \cap[g>r] .
$$

Note that $l(x, y)=x-y$ is not continuous on $(-\infty, \infty]$ so the obvious idea doesn't work.
9. Let $\left\{f_{n}\right\}$ be a sequence of real or complex valued measurable functions. Let

$$
S=\left\{\omega:\left\{f_{n}(\omega)\right\} \text { converges }\right\}
$$

Show $S$ is measurable. Hint: You might try to exhibit the set where $f_{n}$ converges in terms of countable unions and intersections using the definition of a Cauchy sequence.
10. Let $(\Omega, \mathscr{S}, \mu)$ be a measure space and let $f$ be a nonnegative measurable function defined on $\Omega$. Also let $\phi:[0, \infty) \rightarrow[0, \infty)$ be strictly increasing and have a continuous derivative and $\phi(0)=0$. Suppose $f$ is bounded and that $0 \leq \phi(f(\omega)) \leq M$ for some number, $M$. Show that

$$
\int_{\Omega} \phi(f) d \mu=\int_{0}^{\infty} \phi^{\prime}(s) \mu([s<f]) d s
$$

where the integral on the right is the ordinary improper Riemann integral. Hint: First note that $s \rightarrow \phi^{\prime}(s) \mu([s<f])$ is Riemann integrable because $\phi^{\prime}$ is continuous and $s \rightarrow \mu([s<f])$ is a nonincreasing function, hence Riemann integrable. From the second description of the Lebesgue integral and the assumption that $\phi(f(\omega)) \leq M$, argue that for $[M / h]$ the greatest integer less than $M / h$,

$$
\begin{aligned}
\int_{\Omega} \phi(f) d \mu & =\sup _{h>0}^{[M / h]} \sum_{i=1}^{[M \mu([i h<\phi(f)])} \\
& =\sup _{h>0}^{[M / h]} \sum_{i=1}^{\left[M \mu\left(\left[\phi^{-1}(i h)<f\right]\right)\right.} \\
& =\sup _{h>0} \sum_{i=1}^{[M / h]} \frac{h \Delta_{i}}{\Delta_{i}} \mu\left(\left[\phi^{-1}(i h)<f\right]\right)
\end{aligned}
$$

where $\Delta_{i}=\left(\phi^{-1}(i h)-\phi^{-1}((i-1) h)\right)$. Now use the mean value theorem to write

$$
\begin{aligned}
\Delta_{i} & =\left(\phi^{-1}\right)^{\prime}\left(t_{i}\right) h \\
& =\frac{1}{\phi^{\prime}\left(\phi^{-1}\left(t_{i}\right)\right)} h
\end{aligned}
$$

for some $t_{i}$ between $(i-1) h$ and $i h$. Therefore, the right side is of the form

$$
\sup _{h} \sum_{i=1}^{[M / h]} \phi^{\prime}\left(\phi^{-1}\left(t_{i}\right)\right) \Delta_{i} \mu\left(\left[\phi^{-1}(i h)<f\right]\right)
$$

where $\phi^{-1}\left(t_{i}\right) \in\left(\phi^{-1}((i-1) h), \phi^{-1}(i h)\right)$. Argue that if $t_{i}$ were replaced with $i h$, this would be a Riemann sum for the Riemann integral

$$
\int_{0}^{\phi^{-1}(M)} \phi^{\prime}(t) \mu([t<f]) d t=\int_{0}^{\infty} \phi^{\prime}(t) \mu([t<f]) d t
$$

11. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and suppose $f_{n}$ converges uniformly to $f$ and that $f_{n}$ is in $L^{1}(\Omega)$. When is

$$
\lim _{n \rightarrow \infty} \int f_{n} d \mu=\int f d \mu ?
$$

12. Suppose $u_{n}(t)$ is a differentiable function for $t \in(a, b)$ and suppose that for $t \in(a, b)$,

$$
\left|u_{n}(t)\right|,\left|u_{n}^{\prime}(t)\right|<K_{n}
$$

where $\sum_{n=1}^{\infty} K_{n}<\infty$. Show

$$
\left(\sum_{n=1}^{\infty} u_{n}(t)\right)^{\prime}=\sum_{n=1}^{\infty} u_{n}^{\prime}(t)
$$

Hint: This is an exercise in the use of the dominated convergence theorem and the mean value theorem.
13. Show that $\left\{\sum_{i=1}^{\infty} 2^{-n} \mu\left(\left[i 2^{-n}<f\right]\right)\right\}$ for $f$ a nonnegative measurable function is an increasing sequence. Could you define

$$
\int f d \mu \equiv \lim _{n \rightarrow \infty} \sum_{i=1}^{\infty} 2^{-n} \mu\left(\left[i 2^{-n}<f\right]\right)
$$

and would it be equivalent to the above definitions of the Lebesgue integral?
14. Suppose $\left\{f_{n}\right\}$ is a sequence of nonnegative measurable functions defined on a measure space, $(\Omega, \mathscr{S}, \mu)$. Show that

$$
\int \sum_{k=1}^{\infty} f_{k} d \mu=\sum_{k=1}^{\infty} \int f_{k} d \mu
$$

Hint: Use the monotone convergence theorem along with the fact the integral is linear.

## Chapter 12

## The Construction Of Measures

### 12.1 Outer Measures

What are some examples of measure spaces? In this chapter, a general procedure is discussed called the method of outer measures. It is due to Caratheodory (1918). This approach shows how to obtain measure spaces starting with an outer measure. This will then be used to construct measures determined by positive linear functionals.

Definition 12.1.1 Let $\Omega$ be a nonempty set and let $\mu: \mathscr{P}(\Omega) \rightarrow[0, \infty]$ satisfy

$$
\begin{gathered}
\mu(\emptyset)=0, \\
\text { If } A \subseteq B, \text { then } \mu(A) \leq \mu(B), \\
\mu\left(\cup_{i=1}^{\infty} E_{i}\right) \leq \sum_{i=1}^{\infty} \mu\left(E_{i}\right) .
\end{gathered}
$$

Such a function is called an outer measure. For $E \subseteq \Omega, E$ is $\mu$ measurable if for all $S \subseteq \Omega$,

$$
\begin{equation*}
\mu(S)=\mu(S \backslash E)+\mu(S \cap E) \tag{12.1.1}
\end{equation*}
$$

To help in remembering 12.1.1, think of a measurable set, $E$, as a process which divides a given set into two pieces, the part in $E$ and the part not in $E$ as in 12.1.1. In the Bible, there are four incidents recorded in which a process of division resulted in more stuff than was originally present. ${ }^{1}$ Measurable sets are exactly those for which no such miracle occurs. You might think of the measurable sets as the nonmiraculous sets. The idea is to show that they form a $\sigma$ algebra on which the outer measure, $\mu$ is a measure.

First here is a definition and a lemma.
Definition 12.1.2 $(\mu\lfloor S)(A) \equiv \mu(S \cap A)$ for all $A \subseteq \Omega$. Thus $\mu\lfloor S$ is the name of a new outer measure, called $\mu$ restricted to $S$.

The next lemma indicates that the property of measurability is not lost by considering this restricted measure.

Lemma 12.1.3 If $A$ is $\mu$ measurable, then $A$ is $\mu\lfloor S$ measurable.
Proof: Suppose $A$ is $\mu$ measurable. It is desired to to show that for all $T \subseteq \Omega$,

$$
(\mu\lfloor S)(T)=(\mu\lfloor S)(T \cap A)+(\mu\lfloor S)(T \backslash A)
$$

[^10]Thus it is desired to show

$$
\begin{equation*}
\mu(S \cap T)=\mu(T \cap A \cap S)+\mu\left(T \cap S \cap A^{C}\right) \tag{12.1.2}
\end{equation*}
$$

But 12.1.2 holds because $A$ is $\mu$ measurable. Apply Definition 12.1.1 to $S \cap T$ instead of $S$.
If $A$ is $\mu\lfloor S$ measurable, it does not follow that $A$ is $\mu$ measurable. Indeed, if you believe in the existence of non measurable sets, you could let $A=S$ for such a $\mu$ non measurable set and verify that $S$ is $\mu\lfloor S$ measurable.

The next theorem is the main result on outer measures. It is a very general result which applies whenever one has an outer measure on the power set of any set. This theorem will be referred to as Caratheodory's procedure in the rest of the book.

Theorem 12.1.4 The collection of $\mu$ measurable sets, $\mathscr{S}$, forms a $\sigma$ algebra and

$$
\begin{equation*}
\text { If } F_{i} \in \mathscr{S}, F_{i} \cap F_{j}=\emptyset, \text { then } \mu\left(\cup_{i=1}^{\infty} F_{i}\right)=\sum_{i=1}^{\infty} \mu\left(F_{i}\right) \tag{12.1.3}
\end{equation*}
$$

If $\cdots F_{n} \subseteq F_{n+1} \subseteq \cdots$, then if $F=\cup_{n=1}^{\infty} F_{n}$ and $F_{n} \in \mathscr{S}$, it follows that

$$
\begin{equation*}
\mu(F)=\lim _{n \rightarrow \infty} \mu\left(F_{n}\right) \tag{12.1.4}
\end{equation*}
$$

If $\cdots F_{n} \supseteq F_{n+1} \supseteq \cdots$, and if $F=\cap_{n=1}^{\infty} F_{n}$ for $F_{n} \in \mathscr{S}$ then if $\mu\left(F_{1}\right)<\infty$,

$$
\begin{equation*}
\mu(F)=\lim _{n \rightarrow \infty} \mu\left(F_{n}\right) \tag{12.1.5}
\end{equation*}
$$

Also, $(\mathscr{S}, \mu)$ is complete. By this it is meant that if $F \in \mathscr{S}$ and if $E \subseteq \Omega$ with $\mu(E \backslash F)+$ $\mu(F \backslash E)=0$, then $E \in \mathscr{S}$.

Proof: First note that $\emptyset$ and $\Omega$ are obviously in $\mathscr{S}$. Now suppose $A, B \in \mathscr{S}$. I will show $A \backslash B \equiv A \cap B^{C}$ is in $\mathscr{S}$. To do so, consider the following picture.


Since $\mu$ is subadditive,

$$
\mu(S) \leq \mu\left(S \cap A \cap B^{C}\right)+\mu(A \cap B \cap S)+\mu\left(S \cap B \cap A^{C}\right)+\mu\left(S \cap A^{C} \cap B^{C}\right)
$$

Now using $A, B \in \mathscr{S}$,

$$
\begin{aligned}
\mu(S) & \leq \mu\left(S \cap A \cap B^{C}\right)+\mu(S \cap A \cap B)+\mu\left(S \cap A^{C} \cap B\right)+\mu\left(S \cap A^{C} \cap B^{C}\right) \\
& =\mu(S \cap A)+\mu\left(S \cap A^{C}\right)=\mu(S)
\end{aligned}
$$

It follows equality holds in the above. Now observe using the picture if you like that

$$
(A \cap B \cap S) \cup\left(S \cap B \cap A^{C}\right) \cup\left(S \cap A^{C} \cap B^{C}\right)=S \backslash(A \backslash B)
$$

and therefore,

$$
\begin{aligned}
\mu(S) & =\mu\left(S \cap A \cap B^{C}\right)+\mu(A \cap B \cap S)+\mu\left(S \cap B \cap A^{C}\right)+\mu\left(S \cap A^{C} \cap B^{C}\right) \\
& \geq \mu(S \cap(A \backslash B))+\mu(S \backslash(A \backslash B))
\end{aligned}
$$

Therefore, since $S$ is arbitrary, this shows $A \backslash B \in \mathscr{S}$.
Since $\Omega \in \mathscr{S}$, this shows that $A \in \mathscr{S}$ if and only if $A^{C} \in \mathscr{S}$. Now if $A, B \in \mathscr{S}, A \cup B=$ $\left(A^{C} \cap B^{C}\right)^{C}=\left(A^{C} \backslash B\right)^{C} \in \mathscr{S}$. By induction, if $A_{1}, \cdots, A_{n} \in \mathscr{S}$, then so is $\cup_{i=1}^{n} A_{i}$. If $A, B \in \mathscr{S}$, with $A \cap B=\emptyset$,

$$
\mu(A \cup B)=\mu((A \cup B) \cap A)+\mu((A \cup B) \backslash A)=\mu(A)+\mu(B)
$$

By induction, if $A_{i} \cap A_{j}=\emptyset$ and $A_{i} \in \mathscr{S}, \mu\left(\cup_{i=1}^{n} A_{i}\right)=\sum_{i=1}^{n} \mu\left(A_{i}\right)$.

Now let $A=\cup_{i=1}^{\infty} A_{i}$ where $A_{i} \cap A_{j}=\emptyset$ for $i \neq j$.

$$
\sum_{i=1}^{\infty} \mu\left(A_{i}\right) \geq \mu(A) \geq \mu\left(\cup_{i=1}^{n} A_{i}\right)=\sum_{i=1}^{n} \mu\left(A_{i}\right) .
$$

Since this holds for all $n$, you can take the limit as $n \rightarrow \infty$ and conclude,

$$
\sum_{i=1}^{\infty} \mu\left(A_{i}\right)=\mu(A)
$$

which establishes 12.1.3. Part 12.1.4 follows from part 12.1 .3 just as in the proof of Theorem 11.1.5 on Page 224. That is, letting $F_{0} \equiv \emptyset$, use part 12.1.3 to write

$$
\begin{aligned}
\mu(F) & =\mu\left(\cup_{k=1}^{\infty}\left(F_{k} \backslash F_{k-1}\right)\right)=\sum_{k=1}^{\infty} \mu\left(F_{k} \backslash F_{k-1}\right) \\
& =\lim _{n \rightarrow \infty} \sum_{k=1}^{n}\left(\mu\left(F_{k}\right)-\mu\left(F_{k-1}\right)\right)=\lim _{n \rightarrow \infty} \mu\left(F_{n}\right)
\end{aligned}
$$

In order to establish 12.1.5, let the $F_{n}$ be as given there. Then from what was just shown,

$$
\mu\left(F_{1} \backslash F_{n}\right)+\mu\left(F_{n}\right)=\mu\left(F_{1}\right)
$$

Then, since $\left(F_{1} \backslash F_{n}\right)$ increases to $\left(F_{1} \backslash F\right)$, 12.1.4 implies

$$
\lim _{n \rightarrow \infty}\left(\mu\left(F_{1} \backslash F_{n}\right)\right)=\lim _{n \rightarrow \infty}\left(\mu\left(F_{1}\right)-\mu\left(F_{n}\right)\right)=\mu\left(F_{1} \backslash F\right)
$$

Now I don't know whether $F \in \mathscr{S}$ and so all that can be said is that

$$
\mu\left(F_{1} \backslash F\right)+\mu(F) \geq \mu\left(F_{1}\right)
$$

but this implies

$$
\mu\left(F_{1} \backslash F\right) \geq \mu\left(F_{1}\right)-\mu(F)
$$

Hence

$$
\lim _{n \rightarrow \infty}\left(\mu\left(F_{1}\right)-\mu\left(F_{n}\right)\right)=\mu\left(F_{1} \backslash F\right) \geq \mu\left(F_{1}\right)-\mu(F)
$$

which implies

$$
\lim _{n \rightarrow \infty} \mu\left(F_{n}\right) \leq \mu(F)
$$

But since $F \subseteq F_{n}$,

$$
\mu(F) \leq \lim _{n \rightarrow \infty} \mu\left(F_{n}\right)
$$

and this establishes 12.1.5. Note that it was assumed $\mu\left(F_{1}\right)<\infty$ because $\mu\left(F_{1}\right)$ was subtracted from both sides.

It remains to show $\mathscr{S}$ is closed under countable unions. Recall that if $A \in \mathscr{S}$, then $A^{C} \in \mathscr{S}$ and $\mathscr{S}$ is closed under finite unions. Let $A_{i} \in \mathscr{S}, A=\cup_{i=1}^{\infty} A_{i}, B_{n}=\cup_{i=1}^{n} A_{i}$. Then

$$
\begin{align*}
\mu(S) & =\mu\left(S \cap B_{n}\right)+\mu\left(S \backslash B_{n}\right)  \tag{12.1.6}\\
& =\left(\mu\lfloor S)\left(B_{n}\right)+\left(\mu\lfloor S)\left(B_{n}^{C}\right)\right.\right.
\end{align*}
$$

By Lemma 12.1.3 $B_{n}$ is $\left(\mu\lfloor S)\right.$ measurable and so is $B_{n}^{C}$. I want to show $\mu(S) \geq \mu(S \backslash A)+$ $\mu(S \cap A)$. If $\mu(S)=\infty$, there is nothing to prove. Assume $\mu(S)<\infty$. Then apply Parts 12.1.5 and 12.1.4 to the outer measure, $\mu\lfloor S$ in 12.1.6 and let $n \rightarrow \infty$. Thus

$$
B_{n} \uparrow A, B_{n}^{C} \downarrow A^{C}
$$

and this yields

$$
\mu(S)=\left(\mu\lfloor S)(A)+\left(\mu\lfloor S)\left(A^{C}\right)=\mu(S \cap A)+\mu(S \backslash A)\right.\right.
$$

Therefore $A \in \mathscr{S}$ and this proves Parts 12.1.3, 12.1.4, and 12.1.5. It remains to prove the last assertion about the measure being complete.

Let $F \in \mathscr{S}$ and let $\mu(E \backslash F)+\mu(F \backslash E)=0$. Consider the following picture.


Then referring to this picture and using $F \in \mathscr{S}$,

$$
\begin{aligned}
\mu(S) & \leq \mu(S \cap E)+\mu(S \backslash E) \\
& \leq \mu(S \cap E \cap F)+\mu((S \cap E) \backslash F)+\mu(S \backslash F)+\mu(F \backslash E) \\
& \leq \mu(S \cap F)+\mu(E \backslash F)+\mu(S \backslash F)+\mu(F \backslash E) \\
& =\mu(S \cap F)+\mu(S \backslash F)=\mu(S)
\end{aligned}
$$

Hence $\mu(S)=\mu(S \cap E)+\mu(S \backslash E)$ and so $E \in \mathscr{S}$. This shows that $(\mathscr{S}, \mu)$ is complete and proves the theorem.

Completeness usually occurs in the following form. $E \subseteq F \in \mathscr{S}$ and $\mu(F)=0$. Then $E \in \mathscr{S}$.

Proposition 12.1.5 Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. Let $\bar{\mu}$ be the outer measure determined by $\mu$. Also denote as $\overline{\mathscr{F}}$, the $\sigma$ algebra of $\bar{\mu}$ measurable sets. Thus $(\Omega, \overline{\mathscr{F}}, \bar{\mu})$ is a complete measure space in which $\overline{\mathscr{F}} \supseteq \mathscr{F}$ and $\bar{\mu}=\mu$ on $\mathscr{F}$. Also, in this situation, if $\bar{\mu}(E)=0$, then $E \in \mathscr{\mathscr { F }}$. No new sets are obtained if $(\Omega, \mathscr{F}, \mu)$ is already complete.

Proof: All that remains to show is the last claim. But this is obvious because if $S$ is a set,

$$
\begin{aligned}
\bar{\mu}(S) & \leq \bar{\mu}(S \cap E)+\bar{\mu}(S \backslash E) \\
& \leq \bar{\mu}(E)+\bar{\mu}(S \backslash E) \\
& =\bar{\mu}(S \backslash E) \leq \bar{\mu}(S)
\end{aligned}
$$

and so all inequalities are equal signs.

Suppose now that $(\Omega, \mathscr{F}, \mu)$ is complete. Let $F \in \overline{\mathscr{F}}$. Then there exists $E \supseteq F$ such that $\mu(E)=\bar{\mu}(F)$. This is obvious if $\bar{\mu}(F)=\infty$. Otherwise, let $E_{n} \supseteq F, \bar{\mu}(F)+\frac{1}{n}>\mu\left(E_{n}\right)$. Just let $E=\cap_{n} E_{n}$. Now $\bar{\mu}(E \backslash F)=0$. Now also, there exists a set of $\mathscr{F}$ called $W$ such that $\mu(W)=0$ and $W \supseteq E \backslash F$. Thus $E \backslash F \subseteq W$, a set of measure zero. Hence by completeness of $(\Omega, \mathscr{F}, \mu)$, it must be the case that $E \backslash F=E \cap F^{C}=G \in \mathscr{F}$. Then taking complements of both sides, $E^{C} \cup F=G^{C} \in \mathscr{F}$. Now take intersections with $E . F \in E \cap G^{C} \in \mathscr{F}$.

In the case of a Hausdorff topological space, the following lemma gives conditions under which the $\sigma$ algebra of $\mu$ measurable sets for an outer measure $\mu$ contains the Borel sets. In words, it assumes the outer measure is inner regular on open sets and outer regular on all sets. Also it assumes you can approximate the measure of an open set with a compact set and the measure of a compact set with an open set.

Lemma 12.1.6 Let $\Omega$ be a Hausdorff space and suppose $\mu$ is an outer measure satisfying $\mu$ is finite on compact sets and the following conditions,

1. $\mu(E)=\inf \{\mu(V), V \supseteq E, V$ open $\}$ for all $E$. (Outer regularity.)
2. For every open set $V, \mu(V)=\sup \{\mu(K): K \subseteq V, K$ compact $\}$ (Inner regularity on open sets.)
3. If $A, B$ are compact disjoint sets, then $\mu(A \cup B)=\mu(A)+\mu(B)$.

Then the following hold.

1. If $\varepsilon>0$ and if $K$ is compact, there exists $V$ open such that $V \supseteq K$ and

$$
\mu(V \backslash K)<\varepsilon
$$

2. If $\varepsilon>0$ and if $V$ is open with $\mu(V)<\infty$, there exists a compact subset $K$ of $V$ such that

$$
\mu(V \backslash K)<\varepsilon
$$

3. Then the $\mu$ measurable sets $\mathscr{S}$ contain the Borel sets and also $\mu$ is inner regular on every open set and for every $E \in \mathscr{S}$ with $\mu(E)<\infty$. Here $\mathscr{S}$ consists of those subsets of $\Omega E$ with the property that for any subset $S$ of $\Omega$,

$$
\mu(S)=\mu(S \cap E)+\mu\left(S \cap E^{C}\right)
$$

Proof: First we establish 1 and 2 and use them to establish the last assertion. Consider 2. Suppose it is not true. Then there exists an open set $V$ having $\mu(V)<\infty$ but for all $K \subseteq V, \mu(V \backslash K) \geq \varepsilon$ for some $\varepsilon>0$. By inner regularity on open sets, there exists $K_{1} \subseteq$ $V, K_{1}$ compact, such that $\mu\left(K_{1}\right) \geq \varepsilon / 2$. Now by assumption, $\mu\left(V \backslash K_{1}\right) \geq \varepsilon$ and so by inner regularity on open sets again, there exists compact $K_{2} \subseteq V \backslash K_{1}$ such that $\mu\left(K_{2}\right) \geq \varepsilon / 2$. Continuing this way, there is a sequence of disjoint compact sets contained in $V\left\{K_{i}\right\}$ such that $\mu\left(K_{i}\right) \geq \varepsilon / 2$.


Now this is an obvious contradiction because by 3,

$$
\mu(V) \geq \mu\left(\cup_{i=1}^{n} K_{i}\right)=\sum_{i=1}^{n} \mu\left(K_{i}\right) \geq n \frac{\varepsilon}{2}
$$

for each $n$, contradicting $\mu(V)<\infty$.
Next consider 1. By outer regularity, there exists an open set $W \supseteq K$ such that $\mu(W)<$ $\mu(K)+1$. By 2 , there exists compact $K_{1} \subseteq W \backslash K$ such that $\mu\left((W \backslash K) \backslash K_{1}\right)<\varepsilon$. Then consider $V \equiv W \backslash K_{1}$. This is an open set containing $K$ and from what was just shown,

$$
\mu\left(\left(W \backslash K_{1}\right) \backslash K\right)=\mu\left((W \backslash K) \backslash K_{1}\right)<\varepsilon .
$$

Now consider the last assertion.
Define

$$
\mathscr{S}_{1}=\{E \in \mathscr{P}(\Omega): E \cap K \in \mathscr{S}\}
$$

for all compact $K$.
First it will be shown the compact sets are in $\mathscr{S}$. From this it will follow the closed sets are in $\mathscr{S}_{1}$. Then you show $\mathscr{S}_{1}=\mathscr{S}$. Thus $\mathscr{S}_{1}=\mathscr{S}$ is a $\sigma$ algebra and so it contains the Borel sets. Finally you show the inner regularity assertion.

Claim 1: Compact sets are in $\mathscr{S}$.
Proof of claim: Let $V$ be an open set with $\mu(V)<\infty$. I will show that for $C$ compact,

$$
\mu(V) \geq \mu(V \backslash C)+\mu(V \cap C)
$$

Here is a diagram to help keep things straight.


By 2, there exists a compact set $K \subseteq V \backslash C$ such that

$$
\mu((V \backslash C) \backslash K)<\varepsilon
$$

and a compact set $H \subseteq V$ such that

$$
\mu(V \backslash H)<\varepsilon
$$

Thus $\mu(V) \leq \mu(V \backslash H)+\mu(H)<\varepsilon+\mu(H)$. Then

$$
\begin{gathered}
\mu(V) \leq \mu(H)+\varepsilon \leq \mu(H \cap C)+\mu(H \backslash C)+\varepsilon \\
\leq \mu(V \cap C)+\mu(V \backslash C)+\varepsilon \leq \mu(H \cap C)+\mu(K)+3 \varepsilon
\end{gathered}
$$

By 3,

$$
=\mu(H \cap C)+\mu(K)+3 \varepsilon=\mu((H \cap C) \cup K)+3 \varepsilon \leq \mu(V)+3 \varepsilon
$$

Since $\varepsilon$ is arbitrary, this shows that

$$
\begin{equation*}
\mu(V)=\mu(V \backslash C)+\mu(V \cap C) \tag{12.1.7}
\end{equation*}
$$

Of course 12.1 .7 is exactly what needs to be shown for arbitrary $S$ in place of $V$. It suffices to consider only $S$ having $\mu(S)<\infty$. If $S \subseteq \Omega$, with $\mu(S)<\infty$, let $V \supseteq S, \mu(S)+\varepsilon>$ $\mu(V)$. Then from what was just shown, if $C$ is compact,

$$
\begin{aligned}
\varepsilon+\mu(S) & >\mu(V)=\mu(V \backslash C)+\mu(V \cap C) \\
& \geq \mu(S \backslash C)+\mu(S \cap C) .
\end{aligned}
$$

Since $\varepsilon$ is arbitrary, this shows the compact sets are in $\mathscr{S}$. This proves the claim.
As discussed above, this verifies the closed sets are in $\mathscr{S}_{1}$ because if $H$ is closed and $C$ is compact, then $H \cap C \in \mathscr{S}$. If $\mathscr{S}_{1}$ is a $\sigma$ algebra, this will show that $\mathscr{S}_{1}$ contains the Borel sets. Thus I first show $\mathscr{S}_{1}$ is a $\sigma$ algebra.

To see that $\mathscr{S}_{1}$ is closed with respect to taking complements, let $E \in \mathscr{S}_{1}$ and $K$ a compact set.

$$
K=\left(E^{C} \cap K\right) \cup(E \cap K)
$$

Then from the fact, just established, that the compact sets are in $\mathscr{S}$,

$$
E^{C} \cap K=K \backslash(E \cap K) \in \mathscr{S} .
$$

$\mathscr{S}_{1}$ is closed under countable unions because if $K$ is a compact set and $E_{n} \in \mathscr{S}_{1}$,

$$
K \cap \cup_{n=1}^{\infty} E_{n}=\cup_{n=1}^{\infty} K \cap E_{n} \in \mathscr{S}
$$

because it is a countable union of sets of $\mathscr{S}$. Thus $\mathscr{S}_{1}$ is a $\sigma$ algebra.
Therefore, if $E \in \mathscr{S}$ and $K$ is a compact set, just shown to be in $\mathscr{S}$, it follows $K \cap E \in \mathscr{S}$ because $\mathscr{S}$ is a $\sigma$ algebra which contains the compact sets and so $\mathscr{S}_{1} \supseteq \mathscr{S}$. It remains to verify $\mathscr{S}_{1} \subseteq \mathscr{S}$. Recall that

$$
\mathscr{S}_{1} \equiv\{E: E \cap K \in \mathscr{S} \text { for all } K \text { compact }\}
$$

Let $E \in \mathscr{S}_{1}$ and let $V$ be an open set with $\mu(V)<\infty$ and choose $K \subseteq V$ such that $\mu(V \backslash K)<\varepsilon$. Then since $E \in \mathscr{S}_{1}$, it follows $E \cap K, E^{C} \cap K \in \mathscr{S}$ and so

$$
\begin{aligned}
\mu(V) & \leq \mu(V \backslash E)+\mu(V \cap E) \leq \overbrace{\mu(K \backslash E)+\mu(K \cap E)}^{\text {The two sets are disjoint and in } \mathscr{S}}+2 \varepsilon \\
& =\mu(K)+2 \varepsilon \leq \mu(V)+3 \varepsilon
\end{aligned}
$$

Since $\varepsilon$ is arbitrary, this shows

$$
\mu(V)=\mu(V \backslash E)+\mu(V \cap E)
$$

which would show $E \in \mathscr{S}$ if $V$ were an arbitrary set.
Now let $S \subseteq \Omega$ be such an arbitrary set. If $\mu(S)=\infty$, then

$$
\mu(S)=\mu(S \cap E)+\mu(S \backslash E)
$$

If $\mu(S)<\infty$, let

$$
V \supseteq S, \mu(S)+\varepsilon \geq \mu(V)
$$

Then

$$
\mu(S)+\varepsilon \geq \mu(V)=\mu(V \backslash E)+\mu(V \cap E) \geq \mu(S \backslash E)+\mu(S \cap E)
$$

Since $\varepsilon$ is arbitrary, this shows that $E \in \mathscr{S}$ and so $\mathscr{S}_{1}=\mathscr{S}$. Thus $\mathscr{S} \supseteq$ Borel sets as claimed.

From $2 \mu$ is inner regular on all open sets. It remains to show that

$$
\mu(F)=\sup \{\mu(K): K \subseteq F\}
$$

for all $F \in \mathscr{S}$ with $\mu(F)<\infty$. It might help to refer to the following crude picture to keep things straight. It also might not help. I am not sure. In the picture, the green marks the boundary of $V$ while red marks $U$ and black marks $F$ and $V^{C} \cap K$. This last set is as shown because $K$ is a comnact subset of $U$ such that $u(U \backslash K)<\varepsilon$.


Let $\mu(F)<\infty$ and let $U$ be an open set, $U \supseteq F, \mu(U)<\infty$. Let $V$ be open, $V \supseteq U \backslash F$, and

$$
\mu(V \backslash(U \backslash F))<\varepsilon
$$

(This can be obtained as follows, because $\mu$ is a measure on $\mathscr{S}$.

$$
\mu(V)=\mu(U \backslash F)+\mu(V \backslash(U \backslash F))
$$

Thus from the outer regularity of $\mu, 1$ above, there exists $V$ such that it contains $U \backslash F$ and

$$
\mu(U \backslash F)+\varepsilon>\mu(V)
$$

and so

$$
\mu(V \backslash(U \backslash F))=\mu(V)-\mu(U \backslash F)<\varepsilon .)
$$

Also,

$$
\begin{aligned}
V \backslash(U \backslash F) & =V \cap\left(U \cap F^{C}\right)^{C} \\
& =V \cap\left[U^{C} \cup F\right] \\
& =(V \cap F) \cup\left(V \cap U^{C}\right) \\
& \supseteq V \cap F
\end{aligned}
$$

and so

$$
\mu(V \cap F) \leq \mu(V \backslash(U \backslash F))<\varepsilon
$$

Since $V \supseteq U \cap F^{C}, V^{C} \subseteq U^{C} \cup F$ so $U \cap V^{C} \subseteq U \cap F=F$. Hence $U \cap V^{C}$ is a subset of $F$. Now let $K \subseteq U, \mu(U \backslash K)<\varepsilon$. Thus $K \cap V^{C}$ is a compact subset of $F$ and

$$
\begin{aligned}
\mu(F) & =\mu(V \cap F)+\mu(F \backslash V) \\
& <\varepsilon+\mu(F \backslash V) \leq \varepsilon+\mu\left(U \cap V^{C}\right) \leq 2 \varepsilon+\mu\left(K \cap V^{C}\right)
\end{aligned}
$$

Since $\varepsilon$ is arbitrary, this proves the second part of the lemma.
Where do outer measures come from? One way to obtain an outer measure is to start with a measure $\mu$, defined on a $\sigma$ algebra of sets, $\mathscr{S}$, and use the following definition of the outer measure induced by the measure.

Definition 12.1.7 Let $\mu$ be a measure defined on a $\sigma$ algebra of sets, $\mathscr{S} \subseteq \mathscr{P}(\Omega)$. Then the outer measure induced by $\mu$, denoted by $\bar{\mu}$ is defined on $\mathscr{P}(\Omega)$ as

$$
\bar{\mu}(E)=\inf \{\mu(F): F \in \mathscr{S} \text { and } F \supseteq E\}
$$

A measure space, $(\mathscr{S}, \Omega, \mu)$ is $\sigma$ finite if there exist measurable sets, $\Omega_{i}$ with $\mu\left(\Omega_{i}\right)<\infty$ and $\Omega=\cup_{i=1}^{\infty} \Omega_{i}$.

You should prove the following lemma.
Lemma 12.1.8 If $(\mathscr{S}, \Omega, \mu)$ is $\sigma$ finite then there exist disjoint measurable sets, $\left\{B_{n}\right\}$ such that $\mu\left(B_{n}\right)<\infty$ and $\cup_{n=1}^{\infty} B_{n}=\Omega$.

The following lemma deals with the outer measure generated by a measure which is $\sigma$ finite. It says that if the given measure is $\sigma$ finite and complete then no new measurable sets are gained by going to the induced outer measure and then considering the measurable sets in the sense of Caratheodory.

Lemma 12.1.9 Let $(\Omega, \mathscr{S}, \mu)$ be any measure space and let $\bar{\mu}: \mathscr{P}(\Omega) \rightarrow[0, \infty]$ be the outer measure induced by $\mu$. Then $\bar{\mu}$ is an outer measure as claimed and if $\overline{\mathscr{S}}$ is the set of $\bar{\mu}$ measurable sets in the sense of Caratheodory, then $\overline{\mathscr{S}} \supseteq \mathscr{S}$ and $\bar{\mu}=\mu$ on $\mathscr{S}$. Furthermore, if $\mu$ is $\sigma$ finite and $(\Omega, \mathscr{S}, \mu)$ is complete, then $\overline{\mathscr{S}}=\mathscr{S}$.

Proof: It is easy to see that $\bar{\mu}$ is an outer measure. Let $E \in \mathscr{S}$. The plan is to show $E \in \overline{\mathscr{S}}$ and $\bar{\mu}(E)=\mu(E)$. To show this, let $S \subseteq \Omega$ and then show

$$
\begin{equation*}
\bar{\mu}(S) \geq \bar{\mu}(S \cap E)+\bar{\mu}(S \backslash E) \tag{12.1.8}
\end{equation*}
$$

This will verify that $E \in \overline{\mathscr{S}}$. If $\bar{\mu}(S)=\infty$, there is nothing to prove, so assume $\bar{\mu}(S)<\infty$. Thus there exists $T \in \mathscr{S}, T \supseteq S$, and

$$
\begin{aligned}
\bar{\mu}(S) & >\mu(T)-\varepsilon=\mu(T \cap E)+\mu(T \backslash E)-\varepsilon \\
& \geq \bar{\mu}(T \cap E)+\bar{\mu}(T \backslash E)-\varepsilon \\
& \geq \bar{\mu}(S \cap E)+\bar{\mu}(S \backslash E)-\varepsilon .
\end{aligned}
$$

Since $\varepsilon$ is arbitrary, this proves 12.1 .8 and verifies $\mathscr{S} \subseteq \overline{\mathscr{S}}$. Now if $E \in \mathscr{S}$ and $V \supseteq E$ with $V \in \mathscr{S}, \mu(E) \leq \mu(V)$. Hence, taking inf, $\mu(E) \leq \bar{\mu}(E)$. But also $\mu(E) \geq \bar{\mu}(E)$ since $E \in \mathscr{S}$ and $E \supseteq E$. Hence

$$
\bar{\mu}(E) \leq \mu(E) \leq \bar{\mu}(E)
$$

Next consider the claim about not getting any new sets from the outer measure in the case the measure space is $\sigma$ finite and complete.

Suppose first $F \in \overline{\mathscr{S}}$ and $\bar{\mu}(F)<\infty$. Then there exists $E \in \mathscr{S}$ such that $E \supseteq F$ and $\mu(E)=\bar{\mu}(F)$. Since $\bar{\mu}(F)<\infty$,

$$
\bar{\mu}(E \backslash F)=\mu(E)-\bar{\mu}(F)=0
$$

Then there exists $D \supseteq E \backslash F$ such that $D \in \mathscr{S}$ and $\mu(D)=\bar{\mu}(E \backslash F)=0$. Then by completeness of $\mathscr{S}$, it follows $E \backslash F \in \mathscr{S}$ and so

$$
E=(E \backslash F) \cup F
$$

Hence $F=E \backslash(E \backslash F) \in \mathscr{S}$. In the general case where $\bar{\mu}(F)$ is not known to be finite, let $\mu\left(B_{n}\right)<\infty$, with $B_{n} \cap B_{m}=\emptyset$ for all $n \neq m$ and $\cup_{n} B_{n}=\Omega$. Apply what was just shown to $F \cap B_{n}$, obtaining each of these is in $\mathscr{S}$. Then $F=\cup_{n} F \cap B_{n} \in \mathscr{S}$. This proves the Lemma.

Usually $\Omega$ is not just a set. It is also a topological space. It is very important to consider how the measure is related to this topology. The following definition tells what it means for a measure to be regular.

Definition 12.1.10 Let $\mu$ be a measure on a $\sigma$ algebra $\mathscr{S}$, of subsets of $\Omega$, where $(\Omega, \tau)$ is a topological space. $\mu$ is a Borel measure if $\mathscr{S}$ contains all Borel sets. $\mu$ is called outer regular if $\mu$ is Borel and for all $E \in \mathscr{S}$,

$$
\mu(E)=\inf \{\mu(V): V \text { is open and } V \supseteq E\} .
$$

$\mu$ is called inner regular if $\mu$ is Borel and

$$
\mu(E)=\sup \{\mu(K): K \subseteq E, \text { and } K \text { is compact }\}
$$

If the measure is both outer and inner regular, it is called regular.

There is an interesting situation in which regularity is obtained automatically. To save on words, let $\mathscr{B}(E)$ denote the $\sigma$ algebra of Borel sets in $E$, a closed subset of $\mathbb{R}^{n}$. It is a very interesting fact that every finite measure on $\mathscr{B}(E)$ must be regular.

Lemma 12.1.11 Let $\mu$ be a finite measure defined on $\mathscr{B}(E)$ where $E$ is a closed subset of $\mathbb{R}^{n}$. Then for every $F \in \mathscr{B}(E)$,

$$
\begin{gathered}
\mu(F)=\sup \{\mu(K): K \subseteq F, K \text { is closed }\} \\
\mu(F)=\inf \{\mu(V): V \supseteq F, V \text { is open }\}
\end{gathered}
$$

Proof: For convenience, I will call a measure which satisfies the above two conditions "almost regular". It would be regular if closed were replaced with compact. First note every open set is the countable union of compact sets and every closed set is the countable intersection of open sets. Here is why. Let $V$ be an open set and let

$$
K_{k} \equiv\left\{x \in V: \operatorname{dist}\left(x, V^{C}\right) \geq 1 / k\right\}
$$

Then clearly the union of the $K_{k}$ equals $V$ and each is closed because $x \rightarrow \operatorname{dist}(x, S)$ is always a continuous function whenever $S$ is any nonempty set. Next, for $K$ closed let

$$
V_{k} \equiv\{x \in E: \operatorname{dist}(x, K)<1 / k\}
$$

Clearly the intersection of the $V_{k}$ equals $K$. Therefore, letting $V$ denote an open set and $K$ a closed set,

$$
\begin{aligned}
& \mu(V)=\sup \{\mu(K): K \subseteq V \text { and } K \text { is closed }\} \\
& \mu(K)=\inf \{\mu(V): V \supseteq K \text { and } V \text { is open }\}
\end{aligned}
$$

Also since $V$ is open and $K$ is closed,

$$
\begin{aligned}
& \mu(V)=\inf \{\mu(U): U \supseteq V \text { and } V \text { is open }\} \\
& \mu(K)=\sup \{\mu(L): L \subseteq K \text { and } L \text { is closed }\}
\end{aligned}
$$

In words, $\mu$ is almost regular on open and closed sets. Let

$$
\mathscr{F} \equiv\{F \in \mathscr{B}(E) \text { such that } \mu \text { is almost regular on } F\}
$$

Then $\mathscr{F}$ contains the open sets. I want to show $\mathscr{F}$ is a $\sigma$ algebra and then it will follow $\mathscr{F}=\mathscr{B}(E)$.

First I will show $\mathscr{F}$ is closed with respect to complements. Let $F \in \mathscr{F}$. Then since $\mu$ is finite and $F$ is inner regular, there exists $K \subseteq F$ such that $\mu(F \backslash K)<\varepsilon$. But $K^{C} \backslash F^{C}=F \backslash K$ and so $\mu\left(K^{C} \backslash F^{C}\right)<\varepsilon$ showing that $F^{C}$ is outer regular. I have just approximated the measure of $F^{C}$ with the measure of $K^{C}$, an open set containing $F^{C}$. A similar argument works to show $F^{C}$ is inner regular. You start with $V \supseteq F$ such that $\mu(V \backslash F)<\varepsilon$, note $F^{C} \backslash V^{C}=V \backslash F$, and then conclude $\mu\left(F^{C} \backslash V^{C}\right)<\varepsilon$, thus approximating $F^{C}$ with the closed subset, $V^{C}$.

Next I will show $\mathscr{F}$ is closed with respect to taking countable unions. Let $\left\{F_{k}\right\}$ be a sequence of sets in $\mathscr{F}$. Then since $F_{k} \in \mathscr{F}$, there exist $\left\{K_{k}\right\}$ such that $K_{k} \subseteq F_{k}$ and $\mu\left(F_{k} \backslash K_{k}\right)<\varepsilon / 2^{k+1}$. First choose $m$ large enough that

$$
\mu\left(\left(\cup_{k=1}^{\infty} F_{k}\right) \backslash\left(\cup_{k=1}^{m} F_{k}\right)\right)<\frac{\varepsilon}{2} .
$$

Then

$$
\mu\left(\left(\cup_{k=1}^{m} F_{k}\right) \backslash\left(\cup_{k=1}^{m} K_{k}\right)\right) \leq \sum_{k=1}^{m} \frac{\varepsilon}{2^{k+1}}<\frac{\varepsilon}{2}
$$

and so

$$
\begin{aligned}
\mu\left(\left(\cup_{k=1}^{\infty} F_{k}\right) \backslash\left(\cup_{k=1}^{m} K_{k}\right)\right) \leq & \mu\left(\left(\cup_{k=1}^{\infty} F_{k}\right) \backslash\left(\cup_{k=1}^{m} F_{k}\right)\right) \\
& +\mu\left(\left(\cup_{k=1}^{m} F_{k}\right) \backslash\left(\cup_{k=1}^{m} K_{k}\right)\right) \\
< & \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon
\end{aligned}
$$

Since $\mu$ is outer regular on $F_{k}$, there exists $V_{k}$ such that $\mu\left(V_{k} \backslash F_{k}\right)<\varepsilon / 2^{k}$. Then

$$
\begin{aligned}
\mu\left(\left(\cup_{k=1}^{\infty} V_{k}\right) \backslash\left(\cup_{k=1}^{\infty} F_{k}\right)\right) & \leq \sum_{k=1}^{\infty} \mu\left(V_{k} \backslash F_{k}\right) \\
& <\sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k}}=\varepsilon
\end{aligned}
$$

and this completes the demonstration that $\mathscr{F}$ is a $\sigma$ algebra. This proves the lemma.
Theorem 12.1.12 Let $\mu$ be a finite measure defined on $\mathscr{B}(E)$ where $E$ is a closed subset of $\mathbb{R}^{n}$. Then $\mu$ is regular.

Proof: From Lemma $12.1 .11 \mu$ is outer regular. Now let $F \in \mathscr{B}(E)$. Then since $\mu$ is finite, there exists $K \subseteq F$ such that $K$ is closed, $K \subseteq F$, and

$$
\mu(F)<\mu(K)+\varepsilon .
$$

Then let $K_{k} \equiv K \cap \overline{B(\mathbf{0}, k)}$. Thus $K_{k}$ is a closed and bounded, hence compact set and $\cup_{k=1}^{\infty} K_{k}=K$. Therefore, for all $k$ large enough,

$$
\begin{aligned}
& \mu(F) \\
< & \mu\left(K_{k}\right)+\varepsilon \\
< & \sup \{\mu(K): K \subseteq F \text { and } K \text { compact }\}+\varepsilon \\
\leq & \mu(F)+\varepsilon
\end{aligned}
$$

Since $\varepsilon$ was arbitrary, it follows

$$
\sup \{\mu(K): K \subseteq F \text { and } K \text { compact }\}=\mu(F)
$$

This proves the theorem.

It will be assumed in what follows that $(\Omega, \tau)$ is a locally compact Hausdorff space. This means it is Hausdorff: If $p, q \in \Omega$ such that $p \neq q$, there exist open sets, $U_{p}$ and $U_{q}$ containing $p$ and $q$ respectively such that $U_{p} \cap U_{q}=\emptyset$ and Locally compact: There exists a basis of open sets for the topology, $\mathscr{B}$ such that for each $U \in \mathscr{B}, \bar{U}$ is compact. Recall $\mathscr{B}$ is a basis for the topology if $\cup \mathscr{B}=\Omega$ and if every open set in $\tau$ is the union of sets of $\mathscr{B}$. Also recall a Hausdorff space is normal if whenever $H$ and $C$ are two closed sets, there exist disjoint open sets, $U_{H}$ and $U_{C}$ containing $H$ and $C$ respectively. A regular space is one which has the property that if $p$ is a point not in $H$, a closed set, then there exist disjoint open sets, $U_{p}$ and $U_{H}$ containing $p$ and $H$ respectively.

### 12.2 Urysohn's lemma

Urysohn's lemma which characterizes normal spaces is a very important result which is useful in general topology and in the construction of measures. Because it is somewhat technical a proof is given for the part which is needed.

Theorem 12.2.1 (Urysohn) Let $(X, \tau)$ be normal and let $H \subseteq U$ where $H$ is closed and $U$ is open. Then there exists $g: X \rightarrow[0,1]$ such that $g$ is continuous, $g(x)=1$ on $H$ and $g(x)=0$ if $x \notin U$.

Proof: Let $D \equiv\left\{r_{n}\right\}_{n=1}^{\infty}$ be the rational numbers in $(0,1]$. Choose $V_{r_{1}}$ an open set such that

$$
H \subseteq V_{r_{1}} \subseteq \bar{V}_{r_{1}} \subseteq U
$$

This can be done by applying the assumption that $X$ is normal to the disjoint closed sets, $H$ and $U^{C}$, to obtain open sets $V$ and $W$ with

$$
H \subseteq V, U^{C} \subseteq W, \text { and } V \cap W=\emptyset
$$

Then

$$
H \subseteq V \subseteq \bar{V}, \bar{V} \cap U^{C}=\emptyset
$$

and so let $V_{r_{1}}=V$.
Suppose $V_{r_{1}}, \cdots, V_{r_{k}}$ have been chosen and list the rational numbers $r_{1}, \cdots, r_{k}$ in order,

$$
r_{l_{1}}<r_{l_{2}}<\cdots<r_{l_{k}} \text { for }\left\{l_{1}, \cdots, l_{k}\right\}=\{1, \cdots, k\}
$$

If $r_{k+1}>r_{l_{k}}$ then letting $p=r_{l_{k}}$, let $V_{r_{k+1}}$ satisfy

$$
\bar{V}_{p} \subseteq V_{r_{k+1}} \subseteq \bar{V}_{r_{k+1}} \subseteq U
$$

If $r_{k+1} \in\left(r_{l_{i}}, r_{l_{i+1}}\right)$, let $p=r_{l_{i}}$ and let $q=r_{l_{i+1}}$. Then let $V_{r_{k+1}}$ satisfy

$$
\bar{V}_{p} \subseteq V_{r_{k+1}} \subseteq \bar{V}_{r_{k+1}} \subseteq V_{q}
$$

If $r_{k+1}<r_{l_{1}}$, let $p=r_{l_{1}}$ and let $V_{r_{k+1}}$ satisfy

$$
H \subseteq V_{r_{k+1}} \subseteq \bar{V}_{r_{k+1}} \subseteq V_{p}
$$

Thus there exist open sets $V_{r}$ for each $r \in \mathbb{Q} \cap(0,1)$ with the property that if $r<s$,

$$
H \subseteq V_{r} \subseteq \bar{V}_{r} \subseteq V_{s} \subseteq \bar{V}_{s} \subseteq U
$$

Now let

$$
f(x)=\min \left(\inf \left\{t \in D: x \in V_{t}\right\}, 1\right), f(x) \equiv 1 \text { if } x \notin \bigcup_{t \in D} V_{t} .
$$

(Recall $D=\mathbb{Q} \cap(0,1]$.) I claim $f$ is continuous.

$$
f^{-1}([0, a))=\cup\left\{V_{t}: t<a, t \in D\right\}
$$

an open set.
Next consider $x \in f^{-1}([0, a])$ so $f(x) \leq a$. If $t>a$, then $x \in V_{t}$ because if not, then

$$
f(x) \equiv \inf \left\{t \in D: x \in V_{t}\right\}>a .
$$

Thus

$$
f^{-1}([0, a]) \subseteq \cap\left\{V_{t}: t>a\right\}=\cap\left\{\bar{V}_{t}: t>a\right\}
$$

which is a closed set. If $x \in \cap\left\{\bar{V}_{t}: t>a\right\}$, then $x \in \cap\left\{V_{t}: t>a\right\}$ and so $f(x) \leq a$.
If $a=1, f^{-1}([0,1])=f^{-1}([0, a])=X$. Therefore,

$$
f^{-1}((a, 1])=X \backslash f^{-1}([0, a])=\text { open set. }
$$

It follows $f$ is continuous. Clearly $f(x)=0$ on $H$. If $x \in U^{C}$, then $x \notin V_{t}$ for any $t \in D$ so $f(x)=1$ on $U^{C}$. Let $g(x)=1-f(x)$. This proves the theorem.

In any metric space there is a much easier proof of the conclusion of Urysohn's lemma which applies.

Lemma 12.2.2 Let $S$ be a nonempty subset of a metric space, $(X, d)$. Define

$$
f(x) \equiv \operatorname{dist}(x, S) \equiv \inf \{d(x, y): y \in S\}
$$

Then $f$ is continuous.
Proof: Consider $\left|f(x)-f\left(x_{1}\right)\right|$ and suppose without loss of generality that $f\left(x_{1}\right) \geq$ $f(x)$. Then choose $y \in S$ such that $f(x)+\varepsilon>d(x, y)$. Then

$$
\begin{aligned}
\left|f\left(x_{1}\right)-f(x)\right| & =f\left(x_{1}\right)-f(x) \leq f\left(x_{1}\right)-d(x, y)+\varepsilon \\
& \leq d\left(x_{1}, y\right)-d(x, y)+\varepsilon \\
& \leq d\left(x, x_{1}\right)+d(x, y)-d(x, y)+\varepsilon \\
& =d\left(x_{1}, x\right)+\varepsilon
\end{aligned}
$$

Since $\varepsilon$ is arbitrary, it follows that $\left|f\left(x_{1}\right)-f(x)\right| \leq d\left(x_{1}, x\right)$ and this proves the lemma.
Theorem 12.2.3 (Urysohn's lemma for metric space) Let H be a closed subset of an open set, $U$ in a metric space, $(X, d)$. Then there exists a continuous function, $g: X \rightarrow[0,1]$ such that $g(x)=1$ for all $x \in H$ and $g(x)=0$ for all $x \notin U$.

Proof: If $x \notin C$, a closed set, then $\operatorname{dist}(x, C)>0$ because if not, there would exist a sequence of points of $C$ converging to $x$ and it would follow that $x \in C$. Therefore, $\operatorname{dist}(x, H)+\operatorname{dist}\left(x, U^{C}\right)>0$ for all $x \in X$. Now define a continuous function, $g$ as

$$
g(x) \equiv \frac{\operatorname{dist}\left(x, U^{C}\right)}{\operatorname{dist}(x, H)+\operatorname{dist}\left(x, U^{C}\right)}
$$

It is easy to see this verifies the conclusions of the theorem and this proves the theorem.
Theorem 12.2.4 Every compact Hausdorff space is normal.
Proof: First it is shown that $X$, is regular. Let $H$ be a closed set and let $p \notin H$. Then for each $h \in H$, there exists an open set $U_{h}$ containing $p$ and an open set $V_{h}$ containing $h$ such that $U_{h} \cap V_{h}=\emptyset$. Since $H$ must be compact, it follows there are finitely many of the sets $V_{h}, V_{h_{1}} \cdots V_{h_{n}}$ such that $H \subseteq \cup_{i=1}^{n} V_{h_{i}}$. Then letting $U=\cap_{i=1}^{n} U_{h_{i}}$ and $V=\cup_{i=1}^{n} V_{h_{i}}$, it follows that $p \in U, H \in V$ and $U \cap V=\emptyset$. Thus $X$ is regular as claimed.

Next let $K$ and $H$ be disjoint nonempty closed sets.Using regularity of $X$, for every $k \in K$, there exists an open set $U_{k}$ containing $k$ and an open set $V_{k}$ containing $H$ such that these two open sets have empty intersection. Thus $H \cap \bar{U}_{k}=\emptyset$. Finitely many of the $U_{k}$, $U_{k_{1}}, \cdots, U_{k_{p}}$ cover $K$ and so $\cup_{i=1}^{p} \bar{U}_{k_{i}}$ is a closed set which has empty intersection with $H$. Therefore, $K \subseteq \cup_{i=1}^{p} U_{k_{i}}$ and $H \subseteq\left(\cup_{i=1}^{p} \bar{U}_{k_{i}}\right)^{C}$. This proves the theorem.

A useful construction when dealing with locally compact Hausdorff spaces is the notion of the one point compactification of the space discussed earler. However, it is reviewed here for the sake of convenience or in case you have not read the earlier treatment.

Definition 12.2.5 Suppose $(X, \tau)$ is a locally compact Hausdorff space. Then let $\widetilde{X} \equiv$ $X \cup\{\infty\}$ where $\infty$ is just the name of some point which is not in $X$ which is called the point at infinity. A basis for the topology $\widetilde{\tau}$ for $\widetilde{X}$ is

$$
\tau \cup\left\{K^{C} \text { where } K \text { is a compact subset of } X\right\}
$$

The complement is taken with respect to $\widetilde{X}$ and so the open sets, $K^{C}$ are basic open sets which contain $\infty$.

The reason this is called a compactification is contained in the next lemma.
Lemma 12.2.6 If $(X, \tau)$ is a locally compact Hausdorff space, then $(\widetilde{X}, \tilde{\tau})$ is a compact Hausdorff space. Also if $U$ is an open set of $\widetilde{\tau}$, then $U \backslash\{\infty\}$ is an open set of $\tau$.

Proof: Since $(X, \tau)$ is a locally compact Hausdorff space, it follows $(\widetilde{X}, \tilde{\tau})$ is a Hausdorff topological space. The only case which needs checking is the one of $p \in X$ and $\infty$. Since $(X, \tau)$ is locally compact, there exists an open set of $\tau, U$ having compact closure which contains $p$. Then $p \in U$ and $\infty \in \bar{U}^{C}$ and these are disjoint open sets containing the points, $p$ and $\infty$ respectively. Now let $\mathscr{C}$ be an open cover of $\widetilde{X}$ with sets from $\tilde{\tau}$. Then $\infty$ must be in some set, $U_{\infty}$ from $\mathscr{C}$, which must contain a set of the form $K^{C}$ where $K$ is a
compact subset of $X$. Then there exist sets from $\mathscr{C}, U_{1}, \cdots, U_{r}$ which cover $K$. Therefore, a finite subcover of $\widetilde{X}$ is $U_{1}, \cdots, U_{r}, U_{\infty}$.

To see the last claim, suppose $U$ contains $\infty$ since otherwise there is nothing to show. Notice that if $C$ is a compact set, then $X \backslash C$ is an open set. Therefore, if $x \in U \backslash\{\infty\}$, and if $\widetilde{X} \backslash C$ is a basic open set contained in $U$ containing $\infty$, then if $x$ is in this basic open set of $\widetilde{X}$, it is also in the open set $X \backslash C \subseteq U \backslash\{\infty\}$. If $x$ is not in any basic open set of the form $\widetilde{X} \backslash C$ then $x$ is contained in an open set of $\tau$ which is contained in $U \backslash\{\infty\}$. Thus $U \backslash\{\infty\}$ is indeed open in $\tau$.

Theorem 12.2.7 Let $X$ be a locally compact Hausdorff space, and let $K$ be a compact subset of the open set $V$. Then there exists a continuous function, $f: X \rightarrow[0,1]$, such that $f$ equals 1 on $K$ and $\overline{\{x: f(x) \neq 0\}} \equiv \operatorname{spt}(f)$ is a compact subset of $V$.

Proof: Let $\widetilde{X}$ be the space just described. Then $K$ and $V$ are respectively closed and open in $\tilde{\tau}$. By Theorem 12.2.4 there exist open sets in $\tilde{\tau}, U$, and $W$ such that $K \subseteq U, \infty \in$ $V^{C} \subseteq W$, and $U \cap W=U \cap(W \backslash\{\infty\})=\emptyset$.


Thus $W \backslash\{\infty\}$ is an open set in the original topological space which contains $V^{C}, U$ is an open set in the original topological space which contains $K$, and $W \backslash\{\infty\}$ and $U$ are disjoint.

Now for each $x \in K$, let $U_{x}$ be a basic open set whose closure is compact and such that

$$
x \in U_{x} \subseteq U .
$$

Thus $\overline{U_{x}}$ must have empty intersection with $V^{C}$ because the open set, $W \backslash\{\infty\}$ contains no points of $U_{x}$. Since $K$ is compact, there are finitely many of these sets, $U_{x_{1}}, U_{x_{2}}, \cdots, U_{x_{n}}$ which cover $K$. Now let $H \equiv \cup_{i=1}^{n} U_{x_{i}}$.

Claim: $\bar{H}=\cup_{i=1}^{n} \overline{U_{x_{i}}}$
Proof of claim: Suppose $p \in \bar{H}$. If $p \notin \cup_{i=1}^{n} \overline{U_{x_{i}}}$ then it follows $p \notin \overline{U_{x_{i}}}$ for each $i$. Therefore, there exists an open set, $R_{i}$ containing $p$ such that $R_{i}$ contains no other points of $U_{x_{i}}$. Therefore, $R \equiv \cap_{i=1}^{n} R_{i}$ is an open set containing $p$ which contains no other points of $\cup_{i=1}^{n} U_{x_{i}}=W$, a contradiction. Therefore, $\bar{H} \subseteq \cup_{i=1}^{n} \overline{U_{x_{i}}}$. On the other hand, if $p \in \overline{U_{x_{i}}}$ then $p$ is obviously in $\bar{H}$ so this proves the claim.

From the claim, $K \subseteq H \subseteq \bar{H} \subseteq V$ and $\bar{H}$ is compact because it is the finite union of compact sets. By Urysohn's lemma, there exists $f_{1}$ continuous on $\bar{H}$ which has values in $[0,1]$ such that $f_{1}$ equals 1 on $K$ and equals 0 off $H$. Let $f$ denote the function which extends
$f_{1}$ to be 0 off $\bar{H}$. Then for $\alpha>0$, the continuity of $f_{1}$ implies there exists $U$ open in the topological space such that

$$
f^{-1}((-\infty, \alpha))=f_{1}^{-1}((-\infty, \alpha)) \cup \bar{H}^{C}=(U \cap \bar{H}) \cup \bar{H}^{C}=U \cup \bar{H}^{C}
$$

an open set. If $\alpha \leq 0$,

$$
f^{-1}((-\infty, \alpha))=\emptyset
$$

an open set. If $\alpha>0$, there exists an open set $U$ such that

$$
f^{-1}((\alpha, \infty))=f_{1}^{-1}((\alpha, \infty))=U \cap \bar{H}=U \cap H
$$

because $U$ must be a subset of $H$ since by definition $f=0$ off $H$. If $\alpha \leq 0$, then

$$
f^{-1}((\alpha, \infty))=X
$$

an open set. Thus $f$ is continuous and $\operatorname{spt}(f) \subseteq \bar{H}$, a compact subset of $V$. This proves the theorem.

In fact, the conclusion of the above theorem could be used to prove that the topological space is locally compact. However, this is not needed here.

In case you would like a more elementary proof which does not use the one point compactification idea, here is such a proof.

Theorem 12.2.8 Let $X$ be a locally compact Hausdorff space, and let $K$ be a compact subset of the open set $V$. Then there exists a continuous function, $f: X \rightarrow[0,1]$, such that $f$ equals 1 on $K$ and $\overline{\{x: f(x) \neq 0\}} \equiv \operatorname{spt}(f)$ is a compact subset of $V$.

Proof: To begin with, here is a claim. This claim is obvious in the case of a metric space but requires some proof in this more general case.

Claim: If $k \in K$ then there exists an open set $U_{k}$ containing $k$ such that $\overline{U_{k}}$ is contained in $V$.

Proof of claim: Since $X$ is locally compact, there exists a basis of open sets whose closures are compact, $\mathscr{U}$. Denote by $\mathscr{C}$ the set of all $U \in \mathscr{U}$ which contain $k$ and let $\mathscr{C}^{\prime}$ denote the set of all closures of these sets of $\mathscr{C}$ intersected with the closed set $V^{C}$. Thus $\mathscr{C}^{\prime}$ is a collection of compact sets. I will argue that there are finitely many of the sets of $\mathscr{C}^{\prime}$ which have empty intersection. If not, then $\mathscr{C}^{\prime}$ has the finite intersection property and so there exists a point $p$ in all of them. Since $X$ is a Hausdorff space, there exist disjoint basic open sets from $\mathscr{U}, A, B$ such that $k \in A$ and $p \in B$. Therefore, $p \notin \bar{A}$ contrary to the above requirement that $p$ be in all such sets. It follows there are sets $A_{1}, \cdots, A_{m}$ in $\mathscr{C}$ such that

$$
V^{C} \cap \overline{A_{1}} \cap \cdots \cap \overline{A_{m}}=\emptyset
$$

Let $U_{k}=A_{1} \cap \cdots \cap A_{m}$. Then $\overline{U_{k}} \subseteq \overline{A_{1}} \cap \cdots \cap \overline{A_{m}}$ and so it has empty intersection with $V^{C}$. Thus it is contained in $V$. Also $\overline{U_{k}}$ is a closed subset of the compact set $\overline{A_{1}}$ so it is compact. This proves the claim.

Now to complete the proof of the theorem, since $K$ is compact, there are finitely many $U_{k}$ of the sort just described which cover $K, U_{k_{1}}, \cdots, U_{k_{r}}$. Let

$$
H=\cup_{i=1}^{r} U_{k_{i}}
$$

so it follows

$$
\bar{H}=\cup_{i=1}^{r} \overline{U_{k_{i}}}
$$

and so $K \subseteq H \subseteq \bar{H} \subseteq V$ and $\bar{H}$ is a compact set. By Urysohn's lemma, there exists $f_{1}$ continuous on $\bar{H}$ which has values in $[0,1]$ such that $f_{1}$ equals 1 on $K$ and equals 0 off $H$. Let $f$ denote the function which extends $f_{1}$ to be 0 off $\bar{H}$. Then for $\alpha>0$, the continuity of $f_{1}$ implies there exists $U$ open in the topological space such that

$$
f^{-1}((-\infty, \alpha))=f_{1}^{-1}((-\infty, \alpha)) \cup \bar{H}^{C}=(U \cap \bar{H}) \cup \bar{H}^{C}=U \cup \bar{H}^{C}
$$

an open set. If $\alpha \leq 0$,

$$
f^{-1}((-\infty, \alpha))=\emptyset
$$

an open set. If $\alpha>0$, there exists an open set $U$ such that

$$
f^{-1}((\alpha, \infty))=f_{1}^{-1}((\alpha, \infty))=U \cap \bar{H}=U \cap H
$$

because $U$ must be a subset of $H$ since by definition $f=0$ off $H$. If $\alpha \leq 0$, then

$$
f^{-1}((\alpha, \infty))=X
$$

an open set. Thus $f$ is continuous and $\operatorname{spt}(f) \subseteq \bar{H}$, a compact subset of $V$. This proves the theorem.

Definition 12.2.9 Define $\operatorname{spt}(f)$ (support of $f$ ) to be the closure of the set $\{x: f(x) \neq 0\}$. If $V$ is an open set, $C_{c}(V)$ will be the set of continuous functions $f$, defined on $\Omega$ having $\operatorname{spt}(f) \subseteq V$. Thus in Theorem 12.2.7 or 12.2.8, $f \in C_{c}(V)$.

Definition 12.2.10 If $K$ is a compact subset of an open set, $V$, then $K \prec \phi \prec V$ if

$$
\phi \in C_{c}(V), \phi(K)=\{1\}, \phi(\Omega) \subseteq[0,1]
$$

where $\Omega$ denotes the whole topological space considered. Also for $\phi \in C_{c}(\Omega), K \prec \phi$ if

$$
\phi(\Omega) \subseteq[0,1] \text { and } \phi(K)=1
$$

and $\phi \prec V$ if

$$
\phi(\Omega) \subseteq[0,1] \text { and } \operatorname{spt}(\phi) \subseteq V
$$

Theorem 12.2.11 (Partition of unity) Let $K$ be a compact subset of a locally compact Hausdorff topological space satisfying Theorem 12.2.7 or 12.2.8 and suppose

$$
K \subseteq V=\cup_{i=1}^{n} V_{i}, V_{i} \text { open. }
$$

Then there exist $\psi_{i} \prec V_{i}$ with

$$
\sum_{i=1}^{n} \psi_{i}(x)=1
$$

for all $x \in K$.

Proof: Let $K_{1}=K \backslash \cup_{i=2}^{n} V_{i}$. Thus $K_{1}$ is compact and $K_{1} \subseteq V_{1}$. Let $K_{1} \subseteq W_{1} \subseteq \bar{W}_{1} \subseteq$ $V_{1}$ with $\bar{W}_{1}$ compact. To obtain $W_{1}$, use Theorem 12.2 .7 or 12.2 .8 to get $f$ such that $K_{1} \prec$ $f \prec V_{1}$ and let $W_{1} \equiv\{x: f(x) \neq 0\}$. Thus $W_{1}, V_{2}, \cdots V_{n}$ covers $K$ and $\bar{W}_{1} \subseteq V_{1}$. Let $K_{2}=$ $K \backslash\left(\cup_{i=3}^{n} V_{i} \cup W_{1}\right)$. Then $K_{2}$ is compact and $K_{2} \subseteq V_{2}$. Let $K_{2} \subseteq W_{2} \subseteq \bar{W}_{2} \subseteq V_{2} \bar{W}_{2}$ compact. Continue this way finally obtaining $W_{1}, \cdots, W_{n}, K \subseteq W_{1} \cup \cdots \cup W_{n}$, and $\bar{W}_{i} \subseteq V_{i} \bar{W}_{i}$ compact. Now let $\bar{W}_{i} \subseteq U_{i} \subseteq \bar{U}_{i} \subseteq V_{i}, \bar{U}_{i}$ compact.


By Theorem 12.2.7 or 12.2.8, let $\bar{U}_{i} \prec \phi_{i} \prec V_{i}, \cup_{i=1}^{n} \bar{W}_{i} \prec \gamma \prec \cup_{i=1}^{n} U_{i}$. Define

$$
\psi_{i}(x)=\left\{\begin{array}{l}
\gamma(x) \phi_{i}(x) / \sum_{j=1}^{n} \phi_{j}(x) \text { if } \sum_{j=1}^{n} \phi_{j}(x) \neq 0 \\
0 \text { if } \sum_{j=1}^{n} \phi_{j}(x)=0
\end{array}\right.
$$

If $x$ is such that $\sum_{j=1}^{n} \phi_{j}(x)=0$, then $x \notin \cup_{i=1}^{n} \bar{U}_{i}$. Consequently $\gamma(y)=0$ for all $y$ near $x$ and so $\psi_{i}(y)=0$ for all $y$ near $x$. Hence $\psi_{i}$ is continuous at such $x$. If $\sum_{j=1}^{n} \phi_{j}(x) \neq 0$, this situation persists near $x$ and so $\psi_{i}$ is continuous at such points. Therefore $\psi_{i}$ is continuous. If $x \in K$, then $\gamma(x)=1$ and so $\sum_{j=1}^{n} \psi_{j}(x)=1$. Clearly $0 \leq \psi_{i}(x) \leq 1$ and $\operatorname{spt}\left(\psi_{j}\right) \subseteq V_{j}$. This proves the theorem.

The following corollary won't be needed immediately but is of considerable interest later.

Corollary 12.2.12 If $H$ is a compact subset of $V_{i}$, there exists a partition of unity such that $\psi_{i}(x)=1$ for all $x \in H$ in addition to the conclusion of Theorem 12.2.11.

Proof: Keep $V_{i}$ the same but replace $V_{j}$ with $\widetilde{V}_{j} \equiv V_{j} \backslash H$. Now in the proof above, applied to this modified collection of open sets, if $j \neq i, \phi_{j}(x)=0$ whenever $x \in H$. Therefore, $\psi_{i}(x)=1$ on $H$.

### 12.3 Positive Linear Functionals

Definition 12.3.1 Let $(\Omega, \tau)$ be a topological space. $L: C_{c}(\Omega) \rightarrow \mathbb{C}$ is called a positive linear functional if $L$ is linear,

$$
L\left(a f_{1}+b f_{2}\right)=a L f_{1}+b L f_{2}
$$

and if $L f \geq 0$ whenever $f \geq 0$.
Theorem 12.3.2 (Riesz representation theorem) Let $(\Omega, \tau)$ be a locally compact Hausdorff space and let $L$ be a positive linear functional on $C_{c}(\Omega)$. Then there exists a $\sigma$ algebra $\mathscr{S}$ containing the Borel sets and a unique measure $\mu$, defined on $\mathscr{S}$, such that

$$
\begin{align*}
& \mu \text { is complete },  \tag{12.3.9}\\
\mu(K)< & \infty \text { for all } K \text { compact } \tag{12.3.10}
\end{align*}
$$

$$
\mu(F)=\sup \{\mu(K): K \subseteq F, K \text { compact }\}
$$

for all $F$ open and for all $F \in \mathscr{S}$ with $\mu(F)<\infty$,

$$
\mu(F)=\inf \{\mu(V): V \supseteq F, V \text { open }\}
$$

for all $F \in \mathscr{S}$, and

$$
\begin{equation*}
\int f d \mu=L f \text { for all } f \in C_{c}(\Omega) \tag{12.3.11}
\end{equation*}
$$

The plan is to define an outer measure and then to show that it, together with the $\sigma$ algebra of sets measurable in the sense of Caratheodory, satisfies the conclusions of the theorem. Always, $K$ will be a compact set and $V$ will be an open set.

Definition 12.3.3 $\mu(V) \equiv \sup \{L f: f \prec V\}$ for $V$ open, $\mu(\emptyset)=0$. $\mu(E) \equiv \inf \{\mu(V): V \supseteq$ $E\}$ for arbitrary sets $E$.

Lemma 12.3.4 $\mu$ is a well-defined outer measure.
Proof: First it is necessary to verify $\mu$ is well defined because there are two descriptions of it on open sets. Suppose then that $\mu_{1}(V) \equiv \inf \{\mu(U): U \supseteq V$ and $U$ is open $\}$. It is required to verify that $\mu_{1}(V)=\mu(V)$ where $\mu$ is given as $\sup \{L f: f \prec V\}$. If $U \supseteq V$, then $\mu(U) \geq \mu(V)$ directly from the definition. Hence from the definition of $\mu_{1}$, it follows $\mu_{1}(V) \geq \mu(V)$. On the other hand, $V \supseteq V$ and so $\mu_{1}(V) \leq \mu(V)$. This verifies $\mu$ is well defined.

It remains to show that $\mu$ is an outer measure. Let $V=\cup_{i=1}^{\infty} V_{i}$ and let $f \prec V$. Then $\operatorname{spt}(f) \subseteq \cup_{i=1}^{n} V_{i}$ for some $n$. Let $\psi_{i} \prec V_{i}, \sum_{i=1}^{n} \psi_{i}=1$ on $\operatorname{spt}(f)$.

$$
L f=\sum_{i=1}^{n} L\left(f \psi_{i}\right) \leq \sum_{i=1}^{n} \mu\left(V_{i}\right) \leq \sum_{i=1}^{\infty} \mu\left(V_{i}\right) .
$$

Hence

$$
\mu(V) \leq \sum_{i=1}^{\infty} \mu\left(V_{i}\right)
$$

since $f \prec V$ is arbitrary. Now let $E=\cup_{i=1}^{\infty} E_{i}$. Is $\mu(E) \leq \sum_{i=1}^{\infty} \mu\left(E_{i}\right)$ ? Without loss of generality, it can be assumed $\mu\left(E_{i}\right)<\infty$ for each $i$ since if not so, there is nothing to prove. Let $V_{i} \supseteq E_{i}$ with $\mu\left(E_{i}\right)+\varepsilon 2^{-i}>\mu\left(V_{i}\right)$.

$$
\mu(E) \leq \mu\left(\cup_{i=1}^{\infty} V_{i}\right) \leq \sum_{i=1}^{\infty} \mu\left(V_{i}\right) \leq \varepsilon+\sum_{i=1}^{\infty} \mu\left(E_{i}\right)
$$

Since $\varepsilon$ was arbitrary, $\mu(E) \leq \sum_{i=1}^{\infty} \mu\left(E_{i}\right)$ which proves the lemma.
Lemma 12.3.5 Let $K$ be compact, $g \geq 0, g \in C_{c}(\Omega)$, and $g=1$ on $K$. Then $\mu(K) \leq L g$. Also $\mu(K)<\infty$ whenever $K$ is compact.

Proof: Let $\alpha \in(0,1)$ and $V_{\alpha}=\{x: g(x)>\alpha\}$ so $V_{\alpha} \supseteq K$ and let $h \prec V_{\alpha}$.


Then $h \leq 1$ on $V_{\alpha}$ while $g \alpha^{-1} \geq 1$ on $V_{\alpha}$ and so $g \alpha^{-1} \geq h$ which implies $L\left(g \alpha^{-1}\right) \geq L h$ and that therefore, since $L$ is linear,

$$
L g \geq \alpha L h
$$

Since $h \prec V_{\alpha}$ is arbitrary, and $K \subseteq V_{\alpha}$,

$$
L g \geq \alpha \mu\left(V_{\alpha}\right) \geq \alpha \mu(K)
$$

Letting $\alpha \uparrow 1$ yields $L g \geq \mu(K)$. This proves the first part of the lemma. The second assertion follows from this and Theorem 12.2.7. If $K$ is given, let

$$
K \prec g \prec \Omega
$$

and so from what was just shown, $\mu(K) \leq L g<\infty$. This proves the lemma.
Lemma 12.3.6 If $A$ and $B$ are disjoint compact subsets of $\Omega$, then $\mu(A \cup B)=\mu(A)+\mu(B)$.
Proof: By Theorem 12.2 .7 or 12.2 .8 , there exists $h \in C_{c}(\Omega)$ such that $A \prec h \prec B^{C}$. Let $U_{1}=h^{-1}\left(\left(\frac{1}{2}, 1\right]\right), V_{1}=h^{-1}\left(\left[0, \frac{1}{2}\right)\right)$. Then $A \subseteq U_{1}, B \subseteq V_{1}$ and $U_{1} \cap V_{1}=\emptyset$.


From Lemma 12.3.5 $\mu(A \cup B)<\infty$ and so there exists an open set, $W$ such that

$$
W \supseteq A \cup B, \mu(A \cup B)+\varepsilon>\mu(W)
$$

Now let $U=U_{1} \cap W$ and $V=V_{1} \cap W$. Then

$$
U \supseteq A, V \supseteq B, U \cap V=\emptyset, \text { and } \mu(A \cup B)+\varepsilon \geq \mu(W) \geq \mu(U \cup V)
$$

Let $A \prec f \prec U, B \prec g \prec V$. Then by Lemma 12.3.5,

$$
\mu(A \cup B)+\varepsilon \geq \mu(U \cup V) \geq L(f+g)=L f+L g \geq \mu(A)+\mu(B)
$$

Since $\varepsilon>0$ is arbitrary, this proves the lemma.
From Lemma 12.3.5 the following lemma is obtained.

Lemma 12.3.7 Let $f \in C_{c}(\Omega), f(\Omega) \subseteq[0,1]$. Then $\mu(\operatorname{spt}(f)) \geq L f$. Also, every open set, $V$ satisfies

$$
\mu(V)=\sup \{\mu(K): K \subseteq V\}
$$

Proof: Let $V \supseteq \operatorname{spt}(f)$ and let $\operatorname{spt}(f) \prec g \prec V$. Then $L f \leq L g \leq \mu(V)$ because $f \leq g$. Since this holds for all $V \supseteq \operatorname{spt}(f), L f \leq \mu(\operatorname{spt}(f))$ by definition of $\mu$.


Finally, let $V$ be open and let $l<\mu(V)$. Then from the definition of $\mu$, there exists $f \prec V$ such that $L(f)>l$. Therefore, $l<\mu(\operatorname{spt}(f)) \leq \mu(V)$ and so this shows the claim about inner regularity of the measure on an open set.

At this point, the conditions of Lemma 12.1.6 have been verified. Thus $\mathscr{S}$ contains the Borel sets and $\mu$ is inner regular on sets of $\mathscr{S}$ having finite measure.

It remains to show $\mu$ satisfies 12.3.11.

Lemma 12.3.8 $\int f d \mu=L f$ for all $f \in C_{c}(\Omega)$.
Proof: Let $f \in C_{c}(\Omega), f$ real-valued, and suppose $f(\Omega) \subseteq[a, b]$. Choose $t_{0}<a$ and let $t_{0}<t_{1}<\cdots<t_{n}=b, t_{i}-t_{i-1}<\varepsilon$. Let

$$
\begin{equation*}
E_{i}=f^{-1}\left(\left(t_{i-1}, t_{i}\right]\right) \cap \operatorname{spt}(f) . \tag{12.3.12}
\end{equation*}
$$

Note that $\cup_{i=1}^{n} E_{i}$ is a closed set, and in fact

$$
\begin{equation*}
\cup_{i=1}^{n} E_{i}=\operatorname{spt}(f) \tag{12.3.13}
\end{equation*}
$$

since $\Omega=\cup_{i=1}^{n} f^{-1}\left(\left(t_{i-1}, t_{i}\right]\right)$. Let $V_{i} \supseteq E_{i}, V_{i}$ is open and let $V_{i}$ satisfy

$$
\begin{equation*}
f(x)<t_{i}+\varepsilon \text { for all } x \in V_{i} \tag{12.3.14}
\end{equation*}
$$

$$
\mu\left(V_{i} \backslash E_{i}\right)<\varepsilon / n
$$

By Theorem 12.2 .11 there exists $h_{i} \in C_{c}(\Omega)$ such that

$$
h_{i} \prec V_{i}, \quad \sum_{i=1}^{n} h_{i}(x)=1 \text { on } \operatorname{spt}(f) .
$$

Now note that for each $i$,

$$
f(x) h_{i}(x) \leq h_{i}(x)\left(t_{i}+\varepsilon\right) .
$$

(If $x \in V_{i}$, this follows from 12.3.14. If $x \notin V_{i}$ both sides equal 0 .) Therefore,

$$
\begin{aligned}
L f & =L\left(\sum_{i=1}^{n} f h_{i}\right) \leq L\left(\sum_{i=1}^{n} h_{i}\left(t_{i}+\varepsilon\right)\right) \\
& =\sum_{i=1}^{n}\left(t_{i}+\varepsilon\right) L\left(h_{i}\right) \\
& =\sum_{i=1}^{n}\left(\left|t_{0}\right|+t_{i}+\varepsilon\right) L\left(h_{i}\right)-\left|t_{0}\right| L\left(\sum_{i=1}^{n} h_{i}\right) .
\end{aligned}
$$

Now note that $\left|t_{0}\right|+t_{i}+\varepsilon \geq 0$ and so from the definition of $\mu$ and Lemma 12.3.5, this is no larger than

$$
\begin{gathered}
\sum_{i=1}^{n}\left(\left|t_{0}\right|+t_{i}+\varepsilon\right) \mu\left(V_{i}\right)-\left|t_{0}\right| \mu(\operatorname{spt}(f)) \\
\leq \sum_{i=1}^{n}\left(\left|t_{0}\right|+t_{i}+\varepsilon\right)\left(\mu\left(E_{i}\right)+\varepsilon / n\right)-\left|t_{0}\right| \mu(\operatorname{spt}(f)) \\
\leq\left|t_{0}\right| \sum_{i=1}^{n} \mu\left(E_{i}\right)+\left|t_{0}\right| \varepsilon+\sum_{i=1}^{n} t_{i} \mu\left(E_{i}\right)+\varepsilon\left(\left|t_{0}\right|+|b|\right) \\
\sum_{i=1}^{n} t_{i} \frac{\varepsilon}{n}+\varepsilon \sum_{i=1}^{n} \mu\left(E_{i}\right)+\varepsilon^{2}-\left|t_{0}\right| \mu(\operatorname{spt}(f))
\end{gathered}
$$

From 12.3.13 and 12.3.12, the first and last terms cancel. Therefore this is no larger than

$$
\begin{gathered}
\quad\left(2\left|t_{0}\right|+|b|+\mu(\operatorname{spt}(f))+\varepsilon\right) \varepsilon \\
+\sum_{i=1}^{n} t_{i-1} \mu\left(E_{i}\right)+\varepsilon \mu(\operatorname{spt}(f))+\sum_{i=1}^{n}\left(\left|t_{0}\right|+|b|\right) \frac{\varepsilon}{n} \\
\leq \int f d \mu+\left(2\left|t_{0}\right|+|b|+2 \mu(\operatorname{spt}(f))+\varepsilon\right) \varepsilon+\left(\left|t_{0}\right|+|b|\right) \varepsilon
\end{gathered}
$$

Since $\varepsilon>0$ is arbitrary,

$$
\begin{equation*}
L f \leq \int f d \mu \tag{12.3.15}
\end{equation*}
$$

for all $f \in C_{c}(\Omega), f$ real. Hence equality holds in 12.3.15 because $L(-f) \leq-\int f d \mu$ so $L(f) \geq \int f d \mu$. Thus $L f=\int f d \mu$ for all $f \in C_{c}(\Omega)$. Just apply the result for real functions to the real and imaginary parts of $f$. This proves the Lemma.

This gives the existence part of the Riesz representation theorem.
It only remains to prove uniqueness. Suppose both $\mu_{1}$ and $\mu_{2}$ are measures on $\mathscr{S}$ satisfying the conclusions of the theorem. Then if $K$ is compact and $V \supseteq K$, let $K \prec f \prec V$. Then

$$
\mu_{1}(K) \leq \int f d \mu_{1}=L f=\int f d \mu_{2} \leq \mu_{2}(V)
$$

Thus $\mu_{1}(K) \leq \mu_{2}(K)$ for all $K$. Similarly, the inequality can be reversed and so it follows the two measures are equal on compact sets. By the assumption of inner regularity on open sets, the two measures are also equal on all open sets. By outer regularity, they are equal on all sets of $\mathscr{S}$. This proves the theorem.

An important example of a locally compact Hausdorff space is any metric space in which the closures of balls are compact. For example, $\mathbb{R}^{n}$ with the usual metric is an example of this. Not surprisingly, more can be said in this important special case.

Theorem 12.3.9 Let $(\Omega, \tau)$ be a metric space in which the closures of the balls are compact and let $L$ be a positive linear functional defined on $C_{c}(\Omega)$. Then there exists a measure representing the positive linear functional which satisfies all the conclusions of Theorem 12.2.7 or 12.2.8 and in addition the property that $\mu$ is regular. The same conclusion follows if $(\Omega, \tau)$ is a compact Hausdorff space.

Theorem 12.3.10 Let $(\Omega, \tau)$ be a metric space in which the closures of the balls are compact and let $L$ be a positive linear functional defined on $C_{c}(\Omega)$. Then there exists a measure representing the positive linear functional which satisfies all the conclusions of Theorem 12.2.7 or 12.2.8 and in addition the property that $\mu$ is regular. The same conclusion follows if $(\Omega, \tau)$ is a compact Hausdorff space.

Proof: Let $\mu$ and $\mathscr{S}$ be as described in Theorem 12.3.2. The outer regularity comes automatically as a conclusion of Theorem 12.3.2. It remains to verify inner regularity. Let $F \in \mathscr{S}$ and let $l<k<\mu(F)$. Now let $z \in \Omega$ and $\Omega_{n}=\overline{B(z, n)}$ for $n \in \mathbb{N}$. Thus $F \cap \Omega_{n} \uparrow F$. It follows that for $n$ large enough,

$$
k<\mu\left(F \cap \Omega_{n}\right) \leq \mu(F)
$$

Since $\mu\left(F \cap \Omega_{n}\right)<\infty$ it follows there exists a compact set, $K$ such that $K \subseteq F \cap \Omega_{n} \subseteq F$ and

$$
l<\mu(K) \leq \mu(F)
$$

This proves inner regularity. In case $(\Omega, \tau)$ is a compact Hausdorff space, the conclusion of inner regularity follows from Theorem 12.3.2. This proves the theorem.

The proof of the above yields the following corollary.
Corollary 12.3.11 Let $(\Omega, \tau)$ be a locally compact Hausdorff space and suppose $\mu$ defined on a $\sigma$ algebra, $\mathscr{S}$ represents the positive linear functional $L$ where $L$ is defined on $C_{c}(\Omega)$ in the sense of Theorem 12.2 .7 or 12.2.8. Suppose also that there exist $\Omega_{n} \in \mathscr{S}$ such that $\Omega=\cup_{n=1}^{\infty} \Omega_{n}$ and $\mu\left(\Omega_{n}\right)<\infty$. Then $\mu$ is regular.

The following is on the uniqueness of the $\sigma$ algebra in some cases.
Definition 12.3.12 Let $(\Omega, \tau)$ be a locally compact Hausdorff space and let $L$ be a positive linear functional defined on $C_{c}(\Omega)$ such that the complete measure defined by the Riesz representation theorem for positive linear functionals is inner regular. Then this is called a Radon measure. Thus a Radon measure is complete, and regular.

Corollary 12.3.13 Let $(\Omega, \tau)$ be a locally compact Hausdorff space which is also $\sigma$ compact meaning

$$
\Omega=\cup_{n=1}^{\infty} \Omega_{n}, \Omega_{n} \text { is compact }
$$

and let $L$ be a positive linear functional defined on $C_{c}(\Omega)$. Then if $\left(\mu_{1}, \mathscr{S}_{1}\right)$, and $\left(\mu_{2}, \mathscr{S}_{2}\right)$ are two Radon measures, together with their $\sigma$ algebras which represent $L$ then the two $\sigma$ algebras are equal and the two measures are equal.

Proof: Suppose $\left(\mu_{1}, \mathscr{S}_{1}\right)$ and $\left(\mu_{2}, \mathscr{S}_{2}\right)$ both work. It will be shown the two measures are equal on every compact set. Let $K$ be compact and let $V$ be an open set containing $K$. Then let $K \prec f \prec V$. Then

$$
\mu_{1}(K)=\int_{K} d \mu_{1} \leq \int f d \mu_{1}=L(f)=\int f d \mu_{2} \leq \mu_{2}(V)
$$

Therefore, taking the infimum over all $V$ containing $K$ implies $\mu_{1}(K) \leq \mu_{2}(K)$. Reversing the argument shows $\mu_{1}(K)=\mu_{2}(K)$. This also implies the two measures are equal on all open sets because they are both inner regular on open sets. It is being assumed the two measures are regular. Now let $F \in \mathscr{S}_{1}$ with $\mu_{1}(F)<\infty$. Then there exist sets, $H, G$ such that $H \subseteq F \subseteq G$ such that $H$ is the countable union of compact sets and $G$ is a countable intersection of open sets such that $\mu_{1}(G)=\mu_{1}(H)$ which implies $\mu_{1}(G \backslash H)=0$. Now $G \backslash$ $H$ can be written as the countable intersection of sets of the form $V_{k} \backslash K_{k}$ where $V_{k}$ is open, $\mu_{1}\left(V_{k}\right)<\infty$ and $K_{k}$ is compact. From what was just shown, $\mu_{2}\left(V_{k} \backslash K_{k}\right)=\mu_{1}\left(V_{k} \backslash K_{k}\right)$ so it follows $\mu_{2}(G \backslash H)=0$ also. Since $\mu_{2}$ is complete, and $G$ and $H$ are in $\mathscr{S}_{2}$, it follows $F \in \mathscr{S}_{2}$ and $\mu_{2}(F)=\mu_{1}(F)$. Now for arbitrary $F$ possibly having $\mu_{1}(F)=\infty$, consider $F \cap \Omega_{n}$. From what was just shown, this set is in $\mathscr{S}_{2}$ and $\mu_{2}\left(F \cap \Omega_{n}\right)=\mu_{1}\left(F \cap \Omega_{n}\right)$. Taking the union of these $F \cap \Omega_{n}$ gives $F \in \mathscr{S}_{2}$ and also $\mu_{1}(F)=\mu_{2}(F)$. This shows $\mathscr{S}_{1} \subseteq \mathscr{S}_{2}$. Similarly, $\mathscr{S}_{2} \subseteq \mathscr{S}_{1}$.

The following lemma is often useful.
Lemma 12.3.14 Let $(\Omega, \mathscr{F}, \mu)$ be a measure space where $\Omega$ is a topological space. Suppose $\mu$ is a Radon measure and $f$ is measurable with respect to $\mathscr{F}$. Then there exists a Borel measurable function, $g$, such that $g=f$ a.e.

Proof: Assume without loss of generality that $f \geq 0$. Then let $s_{n} \uparrow f$ pointwise. Say

$$
s_{n}(\omega)=\sum_{k=1}^{P_{n}} c_{k}^{n} \mathscr{X}_{E_{k}^{n}}(\omega)
$$

where $E_{k}^{n} \in \mathscr{F}$. By the outer regularity of $\mu$, there exists a Borel set, $F_{k}^{n} \supseteq E_{k}^{n}$ such that $\mu\left(F_{k}^{n}\right)=\mu\left(E_{k}^{n}\right)$. In fact $F_{k}^{n}$ can be assumed to be a $G_{\delta}$ set. Let

$$
t_{n}(\omega) \equiv \sum_{k=1}^{P_{n}} c_{k}^{n} \mathscr{X}_{F_{k}^{n}}(\omega)
$$

Then $t_{n}$ is Borel measurable and $t_{n}(\omega)=s_{n}(\omega)$ for all $\omega \notin N_{n}$ where $N_{n} \in \mathscr{F}$ is a set of measure zero. Now let $N \equiv \cup_{n=1}^{\infty} N_{n}$. Then $N$ is a set of measure zero and if $\omega \notin N$, then $t_{n}(\omega) \rightarrow f(\omega)$. Let $N^{\prime} \supseteq N$ where $N^{\prime}$ is a Borel set and $\mu\left(N^{\prime}\right)=0$. Then $t_{n} \mathscr{X}_{\left(N^{\prime}\right)}{ }^{C}$ converges pointwise to a Borel measurable function, $g$, and $g(\omega)=f(\omega)$ for all $\omega \notin N^{\prime}$. Therefore, $g=f$ a.e. and this proves the lemma.

### 12.4 One Dimensional Lebesgue Measure

To obtain one dimensional Lebesgue measure, you use the positive linear functional $L$ given by

$$
L f=\int f(x) d x
$$

whenever $f \in C_{c}(\mathbb{R})$. Lebesgue measure, denoted by $m$ is the measure obtained from the Riesz representation theorem such that

$$
\int f d m=L f=\int f(x) d x
$$

From this it is easy to verify that

$$
\begin{equation*}
m([a, b])=m((a, b))=b-a . \tag{12.4.16}
\end{equation*}
$$

This will be done in general a little later but for now, consider the following picture of functions, $f^{k}$ and $g^{k}$. Note that $f^{k} \leq \mathscr{X}_{(a, b)} \leq \mathscr{X}_{[a, b]} \leq g^{k}$.


Then considering lower sums and upper sums in the inequalities on the ends,

$$
\begin{aligned}
\left(b-a-\frac{2}{k}\right) & \leq \int f^{k} d x=\int f^{k} d m \leq m((a, b)) \leq m([a, b]) \\
& =\int \mathscr{X}_{[a, b]} d m \leq \int g^{k} d m=\int g^{k} d x \leq\left(b-a+\frac{2}{k}\right)
\end{aligned}
$$

From this the claim in 12.4.16 follows.

### 12.5 One Dimensional Lebesgue Stieltjes Measure

This is just a generalization of Lebesgue measure. Instead of the functional,

$$
L f \equiv \int f(x) d x, f \in C_{c}(\mathbb{R})
$$

you use the functional

$$
L f \equiv \int f(x) d F(x) f \in C_{c}(\mathbb{R})
$$

where $F$ is an increasing function defined on $\mathbb{R}$. By Theorem 4.3.4 this functional is easily seen to be well defined. Therefore, by the Riesz representation theorem there exists a
unique Radon measure $\mu$ representing the functional. Thus

$$
\int_{\mathbb{R}} f d \mu=\int f d F
$$

for all $f \in C_{c}(\mathbb{R})$. Now consider what this measure does to intervals. To begin with, consider what it does to the closed interval, $[a, b]$. The following picture may help.


In this picture $\left\{a_{n}\right\}$ increases to $a$ and $b_{n}$ decreases to $b$. Also suppose $a, b$ are points of continuity of $F$. Therefore,

$$
F(b)-F(a) \leq L f_{n}=\int_{\mathbb{R}} f_{n} d \mu \leq F\left(b_{n}\right)-F\left(a_{n}\right)
$$

Passing to the limit and using the dominated convergence theorem, this shows

$$
\mu([a, b])=F(b)-F(a)=F(b+)-F(a-) .
$$

Next suppose $a, b$ are arbitrary, maybe not points of continuity of $F$. Then letting $a_{n}$ and $b_{n}$ be as in the above picture which are points of continuity of $F$,

$$
\begin{aligned}
\mu([a, b]) & =\lim _{n \rightarrow \infty} \mu\left(\left[a_{n}, b_{n}\right]\right)=\lim _{n \rightarrow \infty} F\left(b_{n}\right)-F\left(a_{n}\right) \\
& =F(b+)-F(a-) .
\end{aligned}
$$

In particular $\mu(a)=F(a+)-F(a-)$ and so

$$
\begin{aligned}
\mu((a, b))= & F(b+)-F(a-)-(F(a+)-F(a-)) \\
& -(F(b+)-F(b-)) \\
= & F(b-)-F(a+)
\end{aligned}
$$

This shows what $\mu$ does to intervals. This is stated as the following proposition.
Proposition 12.5.1 Let $\mu$ be the measure representing the functional

$$
L f \equiv \int f d F, f \in C_{c}(\mathbb{R})
$$

for $F$ an increasing function defined on $\mathbb{R}$. Then

$$
\begin{aligned}
\mu([a, b]) & =F(b+)-F(a-) \\
\mu((a, b)) & =F(b-)-F(a+) \\
\mu(a) & =F(a+)-F(a-) .
\end{aligned}
$$

Observation 12.5.2 Note that all the above would work as well if

$$
L f \equiv \int f d F, f \in C_{c}([0, \infty))
$$

where $F$ is continuous at 0 and $v$ is the measure representing this functional. This is because you could just extend $F(x)$ to equal $F(0)$ for $x \leq 0$ and apply the above to the extended $F$. In this case, $v([0, b])=F(b+)-F(0)$.

### 12.6 The Distribution Function

There is an interesting connection between the Lebesgue integral of a nonnegative function with something called the distribution function.
Definition 12.6.1 Let $f \geq 0$ and suppose $f$ is measurable. The distribution function is the function defined by

$$
t \rightarrow \mu([t<f])
$$

Lemma 12.6.2 If $\left\{f_{n}\right\}$ is an increasing sequence of functions converging pointwise to $f$ then

$$
\mu([f>t])=\lim _{n \rightarrow \infty} \mu\left(\left[f_{n}>t\right]\right)
$$

Proof: The sets, $\left[f_{n}>t\right]$ are increasing and their union is $[f>t]$ because if $f(\omega)>t$, then for all $n$ large enough, $f_{n}(\omega)>t$ also. Therefore, the desired conclusion follows from properties of measures.
Lemma 12.6.3 Suppose $s \geq 0$ is a measurable simple function,

$$
s(\omega) \equiv \sum_{k=1}^{n} a_{k} \mathscr{X}_{E_{k}}(\omega)
$$

where the $a_{k}$ are the distinct nonzero values of $s, 0<a_{1}<a_{2}<\cdots<a_{n}$. Suppose $\phi$ is a $C^{1}$ function defined on $[0, \infty)$ which has the property that $\phi(0)=0, \phi^{\prime}(t)>0$ for all $t$. Then

$$
\int_{0}^{\infty} \phi^{\prime}(t) \mu([s>t]) d m=\int \phi(s) d \mu .
$$

Proof: First note that if $\mu\left(E_{k}\right)=\infty$ for any $k$ then both sides equal $\infty$ and so without loss of generality, assume $\mu\left(E_{k}\right)<\infty$ for all $k$. Letting $a_{0} \equiv 0$, the left side equals

$$
\begin{aligned}
\sum_{k=1}^{n} \int_{a_{k-1}}^{a_{k}} \phi^{\prime}(t) \mu([s>t]) d m(t) & =\sum_{k=1}^{n} \int_{a_{k-1}}^{a_{k}} \phi^{\prime}(t) \sum_{i=k}^{n} \mu\left(E_{i}\right) d m \\
& =\sum_{k=1}^{n} \sum_{i=k}^{n} \mu\left(E_{i}\right) \int_{a_{k-1}}^{a_{k}} \phi^{\prime}(t) d m \\
& =\sum_{k=1}^{n} \sum_{i=k}^{n} \mu\left(E_{i}\right)\left(\phi\left(a_{k}\right)-\phi\left(a_{k-1}\right)\right) \\
& =\sum_{i=1}^{n} \mu\left(E_{i}\right) \sum_{k=1}^{i}\left(\phi\left(a_{k}\right)-\phi\left(a_{k-1}\right)\right) \\
& =\sum_{i=1}^{n} \mu\left(E_{i}\right) \phi\left(a_{i}\right)=\int \phi(s) d \mu
\end{aligned}
$$

With this lemma the next theorem which is the main result follows easily.
Theorem 12.6.4 Let $f \geq 0$ be measurable and let $\phi$ be a $C^{1}$ function defined on $[0, \infty)$ which satisfies $\phi^{\prime}(t)>0$ for all $t>0$ and $\phi(0)=0$. Then

$$
\int \phi(f) d \mu=\int_{0}^{\infty} \phi^{\prime}(t) \mu([f>t]) d m
$$

Proof: By Theorem 11.3.9 on Page 241 there exists an increasing sequence of nonnegative simple functions, $\left\{s_{n}\right\}$ which converges pointwise to $f$. By the monotone convergence theorem and Lemma 12.6.2,

$$
\begin{aligned}
\int \phi(f) d \mu & =\lim _{n \rightarrow \infty} \int \phi\left(s_{n}\right) d \mu=\lim _{n \rightarrow \infty} \int_{0}^{\infty} \phi^{\prime}(t) \mu\left(\left[s_{n}>t\right]\right) d m \\
& =\int_{0}^{\infty} \phi^{\prime}(t) \mu([f>t]) d m
\end{aligned}
$$

This theorem can be generalized to a situation in which $\phi$ is only increasing and continuous. In the generalization I will replace the symbol $\phi$ with $F$ to coincide with earlier notation.

Lemma 12.6.5 Suppose $s \geq 0$ is a measurable simple function,

$$
s(\omega) \equiv \sum_{k=1}^{n} a_{k} \mathscr{X}_{E_{k}}(\omega)
$$

where the $a_{k}$ are the distinct nonzero values of $s, a_{1}<a_{2}<\cdots<a_{n}$. Suppose $F$ is an increasing function defined on $[0, \infty), F(0)=0, F$ being continuous at 0 from the right and continuous at every $a_{k}$. Then letting $\mu$ be a measure and $(\Omega, \mathscr{F}, \mu)$ a measure space,

$$
\int_{(0, \infty]} \mu([s>t]) d \nu=\int_{\Omega} F(s) d \mu
$$

where the integral on the left is the Lebesgue integral for the measure $v$ given as the Radon measure representing the functional

$$
\int_{0}^{\infty} g d F
$$

for $g \in C_{c}([0, \infty))$.
Proof: This follows from the following computation and Proposition 12.5.1. Since $F$ is continuous at 0 and the values $a_{k}$,

$$
\begin{gathered}
\int_{0}^{\infty} \mu([s>t]) d v(t)=\sum_{k=1}^{n} \int_{\left(a_{k-1}, a_{k}\right]} \mu([s>t]) d v(t) \\
=\sum_{k=1}^{n} \int_{\left(a_{k-1}, a_{k}\right]} \sum_{j=k}^{n} \mu\left(E_{j}\right) d F(t)=\sum_{j=1}^{n} \mu\left(E_{j}\right) \sum_{k=1}^{j} v\left(\left(a_{k-1}, a_{k}\right]\right) \\
=\sum_{j=1}^{n} \mu\left(E_{j}\right) \sum_{k=1}^{j}\left(F\left(a_{k}\right)-F\left(a_{k-1}\right)\right)=\sum_{j=1}^{n} \mu\left(E_{j}\right) F\left(a_{j}\right) \equiv \int_{\Omega} F(s) d \mu
\end{gathered}
$$

Now here is the generalization to nonnegative measurable $f$.

Theorem 12.6.6 Let $f \geq 0$ be measurable with respect to $\mathscr{F}$ where $(\Omega, \mathscr{F}, \mu)$ a measure space, and let $F$ be an increasing continuous function defined on $[0, \infty)$ and $F(0)=0$. Then

$$
\int_{\Omega} F(f) d \mu=\int_{(0, \infty]} \mu([f>t]) d v(t)
$$

where $v$ is the Radon measure representing

$$
L g=\int_{0}^{\infty} g d F
$$

for $g \in C_{c}([0, \infty))$.
Proof: By Theorem 11.3.9 on Page 241 there exists an increasing sequence of nonnegative simple functions, $\left\{s_{n}\right\}$ which converges pointwise to $f$. By the monotone convergence theorem and Lemma 12.6.5,

$$
\begin{aligned}
\int_{\Omega} F(f) d \mu & =\lim _{n \rightarrow \infty} \int_{\Omega} F\left(s_{n}\right) d \mu=\lim _{n \rightarrow \infty} \int_{(0, \infty]} \mu\left(\left[s_{n}>t\right]\right) d v \\
& =\int_{(0, \infty]} \mu([f>t]) d v
\end{aligned}
$$

Note that the function $t \rightarrow \mu([f>t])$ is a decreasing function. Therefore, one can make sense of an improper Riemann Stieltjes integral

$$
\int_{0}^{\infty} \mu([f>t]) d F(t)
$$

With more work, one can have this equal to the corresponding Lebesgue integral above.

### 12.7 Good Lambda Inequality

There is a very interesting and important inequality called the good lambda inequality (I am not sure if there is a bad lambda inequality.) which follows from the above theory of distribution functions. It involves the inequality

$$
\mu([f>\beta \lambda] \cap[g \leq \delta \lambda]) \leq \phi(\delta) \mu([f>\lambda])
$$

for $\beta>1$, nonnegative functions $f, g$ and is supposed to hold for all small positive $\delta$ and $\phi(\boldsymbol{\delta}) \rightarrow 0$ as $\boldsymbol{\delta} \rightarrow 0$. Note the left side is small when $g$ is large and $f$ is small. The inequality involves dominating an integral involving $f$ with one involving $g$ as described below. As above, $v$ is the measure which comes from the functional $\int_{\mathbb{R}} g d F$ for $g \in C_{c}(\mathbb{R})$.

Theorem 12.7.1 Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space and let $F$ be a continuous increasing function defined on $[0, \infty)$ such that $F(0)=0$. Suppose also that for all $\alpha>1$, there exists a constant $C_{\alpha}$ such that for all $x \in[0, \infty)$,

$$
F(\alpha x) \leq C_{\alpha} F(x)
$$

Also suppose $f, g$ are nonnegative measurable functions and there exists $\beta>1,0<r \leq 1$, such that for all $\lambda>0$ and $1>\delta>0$,

$$
\begin{equation*}
\mu([f>\beta \lambda] \cap[g \leq r \delta \lambda]) \leq \phi(\delta) \mu([f>\lambda]) \tag{12.7.17}
\end{equation*}
$$

where $\lim _{\delta \rightarrow 0+} \phi(\delta)=0$ and $\phi$ is increasing. Under these conditions, there exists a constant $C$ depending only on $\beta, \phi, r$ such that

$$
\int_{\Omega} F(f(\omega)) d \mu(\omega) \leq C \int_{\Omega} F(g(\omega)) d \mu(\omega)
$$

Proof: Let $\beta>1$ be as given above. First suppose $f$ is bounded.

$$
\begin{aligned}
\int_{\Omega} F(f) d \mu & =\int_{\Omega} F\left(\beta \frac{f}{\beta}\right) d \mu \leq C_{\beta} \int_{\Omega} F\left(\frac{f}{\beta}\right) d \mu \\
& =C_{\beta} \int_{0}^{\infty} \mu([f>\beta \lambda]) d \nu
\end{aligned}
$$

Now using the given inequality,

$$
\begin{aligned}
= & C_{\beta} \int_{0}^{\infty} \mu([f>\beta \lambda] \cap[g \leq r \delta \lambda]) d v \\
& +C_{\beta} \int_{0}^{\infty} \mu([f>\beta \lambda] \cap[g>r \delta \lambda]) d v \\
\leq & C_{\beta} \phi(\delta) \int_{0}^{\infty} \mu([f>\lambda]) d v+C_{\beta} \int_{0}^{\infty} \mu([g>r \delta \lambda]) d v \\
\leq & C_{\beta} \phi(\delta) \int_{\Omega} F(f) d \mu+C_{\beta} \int_{\Omega} F\left(\frac{g}{r \delta}\right) d \mu
\end{aligned}
$$

Now choose $\delta$ small enough that $C_{\beta} \phi(\delta)<\frac{1}{2}$ and then subtract the first term on the right in the above from both sides. It follows from the properties of $F$ again that

$$
\frac{1}{2} \int_{\Omega} F(f) d \mu \leq C_{\beta} C_{(r \delta)^{-1}} \int_{\Omega} F(g) d \mu
$$

This establishes the inequality in the case where $f$ is bounded.
In general, let $f_{n}=\min (f, n)$. Then for $n \leq \lambda$, the inequality

$$
\mu([f>\beta \lambda] \cap[g \leq r \delta \lambda]) \leq \phi(\delta) \mu([f>\lambda])
$$

holds with $f$ replaced with $f_{n}$ because both sides equal 0 thanks to $\beta>1$. If $n>\lambda$, then $[f>\lambda]=\left[f_{n}>\lambda\right]$ and so the inequality still holds because in this case,

$$
\begin{aligned}
\mu\left(\left[f_{n}>\beta \lambda\right] \cap[g \leq r \delta \lambda]\right) & \leq \mu([f>\beta \lambda] \cap[g \leq r \delta \lambda]) \\
& \leq \phi(\delta) \mu([f>\lambda])=\phi(\delta) \mu\left(\left[f_{n}>\lambda\right]\right)
\end{aligned}
$$

Therefore, 12.7 .17 is valid with $f$ replaced with $f_{n}$. Now pass to the limit as $n \rightarrow \infty$ and use the monotone convergence theorem.

### 12.8 The Ergodic Theorem

I am putting this theorem here because it seems to fit in well with the material of this chapter.

In this section $(\Omega, \mathscr{F}, \mu)$ will be a finite measure space. This means that $\mu(\Omega)<\infty$. The mapping, $T: \Omega \rightarrow \Omega$ will satisfy the following condition.

$$
\begin{equation*}
T(A), T^{-1}(A) \in \mathscr{F} \text { whenever } A \in \mathscr{F}, T \text { is one to one. } \tag{12.8.18}
\end{equation*}
$$

For example, you could have $T$ a homeomorphism on some topological space $X$ and the $\sigma$ algebra could be the Borel sets.

Lemma 12.8.1 If $T$ satisfies 12.8.18, then $f \circ T$ is measurable whenever $f$ is measurable.
Proof: Let $U$ be an open set. Then

$$
(f \circ T)^{-1}(U)=T^{-1}\left(f^{-1}(U)\right) \in \mathscr{F}
$$

by 12.8.18.
Now suppose that in addition to $12.8 .18, T$ also satisfies

$$
\begin{equation*}
\mu\left(T^{-1} A\right)=\mu(A) \tag{12.8.19}
\end{equation*}
$$

for all $A \in \mathscr{F}$. In words, $T^{-1}$ is measure preserving. Note that also

$$
\mu(T A)=\mu\left(T^{-1} T A\right)=\mu(A)
$$

so also $T$ is measure preserving. Then for $T$ satisfying 12.8 .18 and 12.8.19, we have the following simple lemma.

Lemma 12.8.2 If $T$ satisfies 12.8 .18 and 12.8 .19 then whenever $f$ is nonnegative and measurable,

$$
\begin{equation*}
\int_{\Omega} f(\omega) d \mu=\int_{\Omega} f(T \omega) d \mu \tag{12.8.20}
\end{equation*}
$$

Also 12.8.20 holds whenever $f \in L^{1}(\Omega)$.
Proof: Let $f \geq 0$ and $f$ is measurable. Let $A \in \mathscr{F}$. Then from 12.8.19,

$$
\int_{\Omega} \mathscr{X}_{A}(\omega) d \mu=\mu(A)=\mu\left(T^{-1}(A)\right)=\int_{\Omega} \mathscr{X}_{T^{-1}(A)}(\omega) d \mu=\int_{\Omega} \mathscr{X}_{A}(T(\omega)) d \mu .
$$

It follows that whenever $s$ is a simple function,

$$
\int s(\omega) d \mu=\int s(T \omega) d \mu
$$

If $f \geq 0$ and measurable, Theorem 11.3.9 on Page 241, implies there exists an increasing sequence of simple functions, $\left\{s_{n}\right\}$ converging pointwise to $f$. Then the result follows from monotone convergence theorem. Splitting $f \in L^{1}$ into real and imaginary parts we apply this to the positive and negative parts of these and obtain 12.8 .20 in this case also.

Definition 12.8.3 A measurable function $f$, is said to be invariant if

$$
f(T \omega)=f(\omega)
$$

A set, $A \in \mathscr{F}$ is said to be invariant if $\mathscr{X}_{A}$ is an invariant function. Thus a set is invariant if and only if $T^{-1} A=A .\left(\mathscr{X}_{A}(T \omega)=\mathscr{X}_{T^{-1}(A)}(\omega)\right.$ so to say that $\mathscr{X}_{A}$ is invariant is to say that $T^{-1} A=A$.)

The following theorem, the individual ergodic theorem, is the main result. Define $T^{0}(\omega)=\omega$. Let

$$
S_{n} f(\omega) \equiv \sum_{k=1}^{n} f\left(T^{k-1} \omega\right), S_{0} f(\omega) \equiv 0
$$

Also define the following maximal type function $M_{\infty} f(\omega)$

$$
\begin{equation*}
M_{\infty} f(\omega) \equiv \sup \left\{S_{k} f(\omega): 0 \leq k\right\} \tag{12.8.21}
\end{equation*}
$$

and let

$$
\begin{equation*}
M_{n} f(\omega) \equiv \sup \left\{S_{k} f(\omega): 0 \leq k \leq n\right\} \tag{12.8.22}
\end{equation*}
$$

Then one can prove the following interesting lemma.
Lemma 12.8.4 Let $f \in L^{1}(\mu)$ where $f$ has real values. Then $\int_{\left[M_{\infty} f>0\right]} f d \mu \geq 0$.
Proof: First note that $M_{n} f(\omega) \geq 0$ for all $n$ and $\omega$. This follows easily from the observation that by definition, $S_{0} f(\omega)=0$ and so $M_{n} f(\omega)$ is at least as large. There is certainly something to show here because the integrand is not known to be nonnegative. The integral involves $f$ not $M_{\infty} f$.

Let $T^{*} h \equiv h \circ T$. Thus $T^{*}$ is linear and maps measurable functions to measurable functions by Lemma 12.8.1. It is also clear that if $h \geq 0$, then $T^{*} h \geq 0$ also. Therefore, for large $k \leq n$,

$$
\begin{aligned}
S_{k} f(\omega) & \equiv \sum_{j=1}^{k} f\left(T^{j-1} \omega\right)=f(\omega)+\sum_{j=2}^{k} f\left(T^{j-1} \omega\right) \\
& =f(\omega)+T^{*} \sum_{j=1}^{k-1} f\left(T^{j-1} \omega\right)\left(\text { factored out } T^{*}\right) \\
& =f(\omega)+T^{*} S_{k-1} f(\omega) \leq f(\omega)+T^{*} M_{n} f
\end{aligned}
$$

and so, taking the supremum for $k \leq n$,

$$
M_{n} f(\omega) \leq f(\omega)+T^{*} M_{n} f(\omega)
$$

Now since $M_{n} f \geq 0$,

$$
\begin{aligned}
& \int_{\Omega} M_{n} f(\omega) d \mu=\int_{\left[M_{n} f>0\right]} M_{n} f(\omega) d \mu \\
\leq & \int_{\left[M_{n} f>0\right]} f(\omega) d \mu+\int_{\Omega} T^{*} M_{n} f(\omega) d \mu
\end{aligned}
$$

$$
=\int_{\left[M_{n} f>0\right]} f(\omega) d \mu+\int_{\Omega} M_{n} f(\omega) d \mu
$$

by Lemma 12.8.2. It follows that

$$
\int_{\left[M_{n} f>0\right]} f(\omega) d \mu \geq 0
$$

for each $n$. Also, since $M_{n} f(\omega) \rightarrow M_{\infty} f(\omega)$, the following pointwise convergence holds.

$$
\mathscr{X}_{\left[M_{n} f>0\right]}(\omega) f(\omega) \rightarrow \mathscr{X}_{\left[M_{\infty} f>0\right]}(\omega) f(\omega)
$$

Since $f$ is in $L^{1}$, the dominated convergence theorem implies

$$
\int_{\left[M_{\infty} f>0\right]} f(\omega) d \mu=\lim _{n \rightarrow \infty} \int_{\left[M_{n} f>0\right]} f(\omega) d \mu \geq 0
$$

Theorem 12.8.5 Let $(\Omega, \mathscr{F}, \mu)$ be a probability space and let $T: \Omega \rightarrow \Omega$ satisfy 12.8.18 and 12.8.19, $T^{-1}$ is measure preserving and $T^{-1}$ maps $\mathscr{F}$ to $\mathscr{F}$ and $T$ is one to one. Then if $f \in L^{1}(\Omega)$ having real or complex values and

$$
\begin{equation*}
S_{n} f(\omega) \equiv \sum_{k=1}^{n} f\left(T^{k-1} \omega\right), S_{0} f(\omega) \equiv 0 \tag{12.8.23}
\end{equation*}
$$

it follows there exists a set of measure zero $N$, and an invariant function $g$ such that for all $\omega \notin N$,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{1}{n} S_{n} f(\omega)=g(\omega) \tag{12.8.24}
\end{equation*}
$$

and also

$$
\lim _{n \rightarrow \infty} \frac{1}{n} S_{n} f=g \text { in } L^{1}(\Omega)
$$

Proof: To begin with, we assume $f$ has real values. Now if $A$ is an invariant set, $\mathscr{X}_{A}\left(T^{m} \omega\right)=\mathscr{X}_{A}(\omega)$ and so

$$
\begin{aligned}
S_{n}\left(\mathscr{X}_{A} f\right)(\omega) & \equiv \sum_{k=1}^{n} f\left(T^{k-1} \omega\right) \mathscr{X}_{A}\left(T^{k-1} \omega\right)=\sum_{k=1}^{n} f\left(T^{k-1} \omega\right) \mathscr{X}_{A}(\omega) \\
& =\mathscr{X}_{A}(\omega) \sum_{k=1}^{n} f\left(T^{k-1} \omega\right)=\mathscr{X}_{A}(\omega) S_{n} f(\omega)
\end{aligned}
$$

Therefore, for such an invariant set,

$$
\begin{equation*}
M_{n}\left(\mathscr{X}_{A} f\right)(\omega)=\mathscr{X}_{A}(\omega) M_{n} f(\omega), M_{\infty}\left(\mathscr{X}_{A} f\right)(\omega)=\mathscr{X}_{A}(\omega) M_{\infty} f(\omega) \tag{12.8.25}
\end{equation*}
$$

Let $-\infty<a<b<\infty$ and define

$$
\begin{equation*}
N_{a b} \equiv\left[-\infty<\lim \inf _{n \rightarrow \infty} \frac{1}{n} S_{n} f(\omega)<a<b<\lim \sup _{n \rightarrow \infty} \frac{1}{n} S_{n} f(\omega)<\infty\right] \tag{12.8.26}
\end{equation*}
$$

Observe that from the definition,

$$
\lim \inf _{n \rightarrow \infty} \frac{1}{n} S_{n} f(\omega)=\lim \inf _{n \rightarrow \infty} \frac{1}{n} S_{n} f(T \omega)
$$

and

$$
\lim \sup _{n \rightarrow \infty} \frac{1}{n} S_{n} f(\omega)=\lim \sup _{n \rightarrow \infty} \frac{1}{n} S_{n} f(T \omega)
$$

Thus if $\omega \in N_{a b}$, it follows that $T \omega \in N_{a b}$ and if $T \omega \in N_{a b}$, then so is $\omega$. Thus $N_{a b}$ is an invariant set. Also, if $\omega \in N_{a b}$, then

$$
a-\lim \inf _{n \rightarrow \infty} \frac{1}{n} S_{n} f(\omega)=\lim \sup _{n \rightarrow \infty}\left(a-\frac{1}{n} S_{n} f(\omega)\right)>0
$$

and

$$
\lim \sup _{n \rightarrow \infty}\left(\frac{1}{n} S_{n} f(\omega)-b\right)>0
$$

It follows that

$$
N_{a b} \subseteq\left[M_{\infty}(f-b)>0\right] \cap\left[M_{\infty}(a-f)>0\right]
$$

Consequently, since $N_{a b}$ is invariant, argued above,

$$
\mathscr{X}_{N_{a b}} M_{\infty}(f-b)=M_{\infty}\left(\mathscr{X}_{N_{a b}}(f-b)\right)
$$

and so from Lemma 12.8.4

$$
\begin{gather*}
\int_{N_{a b}}(f(\omega)-b) d \mu=\int_{\left[\mathscr{X}_{N_{a b}} M_{\infty}(f-b)>0\right]} \mathscr{X}_{N_{a b}}(\omega)(f(\omega)-b) d \mu \\
\quad=\int_{\left[M_{\infty}\left(\mathscr{X}_{N_{a b}}(f-b)\right)>0\right]} \mathscr{X}_{N_{a b}}(\omega)(f(\omega)-b) d \mu \geq 0 \tag{12.8.27}
\end{gather*}
$$

and

$$
\begin{gather*}
\int_{N_{a b}}(a-f(\omega)) d \mu=\int_{\left[\mathscr{X}_{\left.N_{a b} M_{\infty}(a-f)>0\right]} \mathscr{X}_{N_{a b}}(\omega)(a-f(\omega)) d \mu\right.}=\int_{\left[M _ { \infty } \left(\mathscr{X}_{\left.\left.N_{a b}(a-f)\right)>0\right]} \mathscr{X}_{N_{a b}}(\omega)(a-f(\omega)) d \mu \geq 0\right.\right.}
\end{gather*}
$$

It follows that

$$
\begin{equation*}
a \mu\left(N_{a b}\right) \geq \int_{N_{a b}} f d \mu \geq b \mu\left(N_{a b}\right) \tag{12.8.29}
\end{equation*}
$$

Since $a<b$, it follows that $\mu\left(N_{a b}\right)=0$.
Now let

$$
N \equiv \cup\left\{N_{a b}: a<b, a, b \in \mathbb{Q}\right\}
$$

It follows that $\mu(N)=0$. Now $T N_{a, b}=N_{a, b}$ and so

$$
T(N)=\cup_{a, b} T\left(N_{a, b}\right)=\cup_{a, b} N_{a, b}=N
$$

Thus, $T^{n} N=N$ for all $n \in \mathbb{N}$. For $\omega \notin N, \lim _{n \rightarrow \infty} \frac{1}{n} S_{n} f(\omega)$ exists. Now let

$$
g(\omega) \equiv\left\{\begin{array}{l}
0 \text { if } \omega \in N \\
\lim _{n \rightarrow \infty} \frac{1}{n} S_{n} f(\omega) \text { if } \omega \notin N
\end{array}\right.
$$

Then it is clear $g$ satisfies the conditions of the theorem because if $\omega \in N$, then $T \omega \in N$ also and so in this case, $g(T \omega)=g(\omega) \equiv 0$. On the other hand, if $\omega \notin N$, then

$$
g(T \omega)=\lim _{n \rightarrow \infty} \frac{1}{n} S_{n} f(T \omega)=\lim _{n \rightarrow \infty} \frac{1}{n} S_{n} f(\omega)=g(\omega) .
$$

Which shows that $g$ is invariant. Also, from Lemma 12.8.2,

$$
\begin{aligned}
\int_{\Omega}|g| d \mu & \leq \lim _{n \rightarrow \infty} \int_{\Omega}\left|\frac{1}{n} S_{n} f\right| d \mu \leq \lim _{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{n} \int_{\Omega}\left|f\left(T^{k-1} \omega\right)\right| d \mu \\
& =\liminf _{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{n} \int_{\Omega}|f(\omega)| d \mu=\|f\|_{L^{1}}
\end{aligned}
$$

so $g \in L^{1}(\Omega, \mu)$.
The last claim about convergence in $L^{1}$ follows from the Vitali convergence theorem if we verify the sequence, $\left\{\frac{1}{n} S_{n} f\right\}_{n=1}^{\infty}$ is uniformly integrable. To see this is the case, we know $f \in L^{1}(\Omega)$ and so if $\varepsilon>0$ is given, there exists $\delta>0$ such that whenever $B \in \mathscr{F}$ and $\mu(B) \leq \delta$, then $\left|\int_{B} f(\omega) d \mu\right|<\varepsilon$. Taking $\mu(A)<\delta$, it follows

$$
\begin{gathered}
\left|\int_{A} \frac{1}{n} S_{n} f(\omega) d \mu\right|=\left|\frac{1}{n} \sum_{k=1}^{n} \int_{A} f\left(T^{k-1} \omega\right) d \mu\right|=\left|\frac{1}{n} \sum_{k=1}^{n} \int_{\Omega} \mathscr{X}_{A}(\omega) f\left(T^{k-1} \omega\right) d \mu\right| \\
=\left|\frac{1}{n} \sum_{k=1}^{n} \int_{\Omega} \mathscr{X}_{A}\left(T^{k-1} T^{-(k-1)} \omega\right) f\left(T^{k-1} \omega\right) d \mu\right| \\
=\left|\frac{1}{n} \sum_{k=1}^{n} \int_{\Omega} \mathscr{X}_{A}\left(T^{-(k-1)} \omega\right) f(\omega) d \mu\right| \\
=\left|\frac{1}{n} \sum_{k=1}^{n} \int_{T^{k-1}(A)} f(\omega) d \mu\right| \leq \frac{1}{n} \sum_{k=1}^{n}\left|\int_{T^{k-1}(A)} f(\omega) d \mu\right|<\frac{1}{n} \sum_{k=1}^{n} \varepsilon=\varepsilon
\end{gathered}
$$

because $\mu\left(T^{(k-1)} A\right)=\mu(A)$ by assumption. This proves the above sequence is uniformly integrable and so, by the Vitali convergence theorem,

$$
\lim _{n \rightarrow \infty} \int_{\Omega}\left|\frac{1}{n} S_{n} f-g\right| d \mu=0
$$

This proves the theorem in the case the function has real values. In the case where $f$ has complex values, apply the above result to the real and imaginary parts of $f$.

Definition 12.8.6 The above mapping $T$ is ergodic if the only invariant sets have measure 0 or 1 .

If the map, $T$ is ergodic, the following corollary holds.
Corollary 12.8.7 In the situation of Theorem 12.8.5, if $T$ is ergodic, then

$$
g(\omega)=\int f(\omega) d \mu
$$

for a.e. $\omega$.
Proof: Let $g$ be the function of Theorem 12.8 .5 and let $R_{1}$ be a rectangle in $\mathbb{R}^{2}=\mathbb{C}$ of the form $[-a, a] \times[-a, a]$ such that $g^{-1}\left(R_{1}\right)$ has measure greater than 0 . This set is invariant because the function, $g$ is invariant and so it must have measure 1. Divide $R_{1}$ into four equal rectangles, $R_{1}^{\prime}, R_{2}^{\prime}, R_{3}^{\prime}, R_{4}^{\prime}$. Then one of these, renamed $R_{2}$ has the property that $g^{-1}\left(R_{2}\right)$ has positive measure. Therefore, since the set is invariant, it must have measure 1. Continue in this way obtaining a sequence of closed rectangles, $\left\{R_{i}\right\}$ such that the diameter of $R_{i}$ converges to zero and $g^{-1}\left(R_{i}\right)$ has measure 1 . Then let $c=\cap_{j=1}^{\infty} R_{j}$. We know $\mu\left(g^{-1}(c)\right)=\lim _{n \rightarrow \infty} \mu\left(g^{-1}\left(R_{i}\right)\right)=1$. It follows that $g(\omega)=c$ for a.e. $\omega$. Now from Theorem 12.8.5,

$$
c=\int c d \mu=\lim _{n \rightarrow \infty} \frac{1}{n} \int S_{n} f d \mu=\int f d \mu
$$

### 12.9 Product Measures

Let $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{T}, v)$ be two complete measure spaces. In this section consider the problem of defining a product measure, $\overline{\mu \times v}$ which is defined on a $\sigma$ algebra of sets of $X \times Y$ such that $(\overline{\mu \times v})(E \times F)=\mu(E) v(F)$ whenever $E \in \mathscr{S}$ and $F \in \mathscr{T}$. I found the following approach to product measures in [47] and they say they got it from [50].

Definition 12.9.1 Let $\mathscr{R}$ denote the set of countable unions of sets of the form $A \times B$, where $A \in \mathscr{S}$ and $B \in \mathscr{T}$ (Sets of the form $A \times B$ are referred to as measurable rectangles) and also let

$$
\begin{equation*}
\rho(A \times B)=\mu(A) v(B) \tag{12.9.30}
\end{equation*}
$$

More generally, define

$$
\begin{equation*}
\rho(E) \equiv \iint \mathscr{X}_{E}(x, y) d \mu d v \tag{12.9.31}
\end{equation*}
$$

whenever $E$ is such that

$$
\begin{equation*}
x \rightarrow \mathscr{X}_{E}(x, y) \text { is } \mu \text { measurable for all } y \tag{12.9.32}
\end{equation*}
$$

and

$$
\begin{equation*}
y \rightarrow \int \mathscr{X}_{E}(x, y) d \mu \text { is } v \text { measurable. } \tag{12.9.33}
\end{equation*}
$$

Note that if $E=A \times B$ as above, then

$$
\begin{aligned}
\iint \mathscr{X}_{E}(x, y) d \mu d v & =\iint \mathscr{X}_{A \times B}(x, y) d \mu d v \\
& =\iint \mathscr{X}_{A}(x) \mathscr{X}_{B}(y) d \mu d v=\mu(A) v(B)=\rho(E)
\end{aligned}
$$

and so there is no contradiction between 12.9.31 and 12.9.30.
The first goal is to show that for $Q \in \mathscr{R}, 12.9 .32$ and 12.9.33 both hold. That is, $x \rightarrow$ $\mathscr{X}_{Q}(x, y)$ is $\mu$ measurable for all $y$ and $y \rightarrow \int \mathscr{X}_{Q}(x, y) d \mu$ is $v$ measurable. This is done so that it is possible to speak of $\rho(Q)$. The following lemma will be the fundamental result which will make this possible. First here is a picture.


Lemma 12.9.2 Given $C \times D$ and $\left\{A_{i} \times B_{i}\right\}_{i=1}^{n}$, there exist finitely many disjoint rectangles, $\left\{C_{i}^{\prime} \times D_{i}^{\prime}\right\}_{i=1}^{p}$ such that none of these sets intersect any of the $A_{i} \times B_{i}$, each set is contained in $C \times D$ and

$$
\left(\cup_{i=1}^{n} A_{i} \times B_{i}\right) \cup\left(\cup_{k=1}^{p} C_{k}^{\prime} \times D_{k}^{\prime}\right)=(C \times D) \cup\left(\cup_{i=1}^{n} A_{i} \times B_{i}\right) .
$$

Proof: From the above picture, you see that

$$
(C \times D) \backslash\left(A_{1} \times B_{1}\right)=C \times\left(D \backslash B_{1}\right) \cup\left(C \backslash A_{1}\right) \times\left(D \cap B_{1}\right)
$$

and these last two sets are disjoint, have empty intersection with $A_{1} \times B_{1}$, and

$$
\left(C \times\left(D \backslash B_{1}\right) \cup\left(C \backslash A_{1}\right) \times\left(D \cap B_{1}\right)\right) \cup\left(\cup_{i=1}^{n} A_{i} \times B_{i}\right)=(C \times D) \cup\left(\cup_{i=1}^{n} A_{i} \times B_{i}\right)
$$

Now suppose disjoint sets, $\left\{\widetilde{C}_{i} \times \widetilde{D}_{i}\right\}_{i=1}^{m}$ have been obtained, each being a subset of $C \times D$ such that

$$
\left(\cup_{i=1}^{n} A_{i} \times B_{i}\right) \cup\left(\cup_{k=1}^{m} \widetilde{C}_{k} \times \widetilde{D}_{k}\right)=\left(\cup_{i=1}^{n} A_{i} \times B_{i}\right) \cup(C \times D)
$$

and for all $k, \widetilde{C}_{k} \times \widetilde{D}_{k}$ has empty intersection with each set of $\left\{A_{i} \times B_{i}\right\}_{i=1}^{p}$. Then using the same procedure, replace each of $\widetilde{C}_{k} \times \widetilde{D}_{k}$ with finitely many disjoint rectangles such that none of these intersect $A_{p+1} \times B_{p+1}$ while preserving the union of all the sets involved. The process stops when you have gotten to $n$. This proves the lemma.

Lemma 12.9.3 If $Q=\cup_{i=1}^{\infty} A_{i} \times B_{i} \in \mathscr{R}$, then there exist disjoint sets, of the form $A_{i}^{\prime} \times B_{i}^{\prime}$ such that $Q=\cup_{i=1}^{\infty} A_{i}^{\prime} \times B_{i}^{\prime}$, each $A_{i}^{\prime} \times B_{i}^{\prime}$ is a subset of some $A_{i} \times B_{i}$, and $A_{i}^{\prime} \in \mathscr{S}$ while
$B_{i}^{\prime} \in \mathscr{T}$. Also, the intersection of finitely many sets of $\mathscr{R}$ is a set of $\mathscr{R}$. For $\rho$ defined in 12.9.31, it follows that 12.9 .32 and 12.9.33 hold for any element of $\mathscr{R}$. Furthermore,

$$
\rho(Q)=\sum_{i} \mu\left(A_{i}^{\prime}\right) v\left(B_{i}^{\prime}\right)=\sum_{i} \rho\left(A_{i}^{\prime} \times B_{i}^{\prime}\right)
$$

Proof: Let $Q$ be given as above. Let $A_{1}^{\prime} \times B_{1}^{\prime}=A_{1} \times B_{1}$. By Lemma 12.9.2, it is possible to replace $A_{2} \times B_{2}$ with finitely many disjoint rectangles, $\left\{A_{i}^{\prime} \times B_{i}^{\prime}\right\}_{i=2}^{m_{2}}$ such that none of these rectangles intersect $A_{1}^{\prime} \times B_{1}^{\prime}$, each is a subset of $A_{2} \times B_{2}$, and

$$
\cup_{i=1}^{\infty} A_{i} \times B_{i}=\left(\cup_{i=1}^{m_{2}} A_{i}^{\prime} \times B_{i}^{\prime}\right) \cup\left(\cup_{k=3}^{\infty} A_{k} \times B_{k}\right)
$$

Now suppose disjoint rectangles, $\left\{A_{i}^{\prime} \times B_{i}^{\prime}\right\}_{i=1}^{m_{p}}$ have been obtained such that each rectangle is a subset of $A_{k} \times B_{k}$ for some $k \leq p$ and

$$
\cup_{i=1}^{\infty} A_{i} \times B_{i}=\left(\cup_{i=1}^{m_{p}} A_{i}^{\prime} \times B_{i}^{\prime}\right) \cup\left(\cup_{k=p+1}^{\infty} A_{k} \times B_{k}\right) .
$$

By Lemma 12.9.2 again, there exist disjoint rectangles $\left\{A_{i}^{\prime} \times B_{i}^{\prime}\right\}_{i=m_{p}+1}^{m_{p+1}}$ such that each is contained in $A_{p+1} \times B_{p+1}$, none have intersection with any of $\left\{A_{i}^{\prime} \times B_{i}^{\prime}\right\}_{i=1}^{m_{p}}$ and

$$
\cup_{i=1}^{\infty} A_{i} \times B_{i}=\left(\cup_{i=1}^{m_{p+1}} A_{i}^{\prime} \times B_{i}^{\prime}\right) \cup\left(\cup_{k=p+2}^{\infty} A_{k} \times B_{k}\right)
$$

Note that no change is made in $\left\{A_{i}^{\prime} \times B_{i}^{\prime}\right\}_{i=1}^{m_{p}}$. Continuing this way proves the existence of the desired sequence of disjoint rectangles, each of which is a subset of at least one of the original rectangles and such that

$$
Q=\cup_{i=1}^{\infty} A_{i}^{\prime} \times B_{i}^{\prime}
$$

It remains to verify $x \rightarrow \mathscr{X}_{Q}(x, y)$ is $\mu$ measurable for all $y$ and

$$
y \rightarrow \int \mathscr{X}_{Q}(x, y) d \mu
$$

is $v$ measurable whenever $Q \in \mathscr{R}$. Let $Q \equiv \cup_{i=1}^{\infty} A_{i} \times B_{i} \in \mathscr{R}$. Then by the first part of this lemma, there exists $\left\{A_{i}^{\prime} \times B_{i}^{\prime}\right\}_{i=1}^{\infty}$ such that the sets are disjoint and $\cup_{i=1}^{\infty} A_{i}^{\prime} \times B_{i}^{\prime}=Q$. Therefore, since the sets are disjoint,

$$
\mathscr{X}_{Q}(x, y)=\sum_{i=1}^{\infty} \mathscr{X}_{A_{i}^{\prime} \times B_{i}^{\prime}}(x, y)=\sum_{i=1}^{\infty} \mathscr{X}_{A_{i}^{\prime}}(x) \mathscr{X}_{B_{i}^{\prime}}(y) .
$$

It follows $x \rightarrow \mathscr{X}_{Q}(x, y)$ is measurable. Now by the monotone convergence theorem,

$$
\begin{aligned}
\int \mathscr{X}_{Q}(x, y) d \mu & =\int \sum_{i=1}^{\infty} \mathscr{X}_{A_{i}^{\prime}}(x) \mathscr{X}_{B_{i}^{\prime}}(y) d \mu \\
& =\sum_{i=1}^{\infty} \mathscr{X}_{B_{i}^{\prime}}(y) \int \mathscr{X}_{A_{i}^{\prime}}(x) d \mu \\
& =\sum_{i=1}^{\infty} \mathscr{X}_{B_{i}^{\prime}}(y) \mu\left(A_{i}^{\prime}\right) .
\end{aligned}
$$

It follows $y \rightarrow \int \mathscr{X}_{Q}(x, y) d \mu$ is measurable and so by the monotone convergence theorem again,

$$
\begin{align*}
\iint \mathscr{X}_{Q}(x, y) d \mu d v & =\int \sum_{i=1}^{\infty} \mathscr{X}_{B_{i}^{\prime}}(y) \mu\left(A_{i}^{\prime}\right) d v \\
& =\sum_{i=1}^{\infty} \int \mathscr{X}_{B_{i}^{\prime}}(y) \mu\left(A_{i}^{\prime}\right) d v \\
& =\sum_{i=1}^{\infty} v\left(B_{i}^{\prime}\right) \mu\left(A_{i}^{\prime}\right) \tag{12.9.34}
\end{align*}
$$

This shows the measurability conditions, 12.9.32 and 12.9 .33 hold for $Q \in \mathscr{R}$ and also establishes the formula for $\rho(Q), 12.9 .34$.

If $\cup_{i} A_{i} \times B_{i}$ and $\cup_{j} C_{j} \times D_{j}$ are two sets of $\mathscr{R}$, then their intersection is

$$
\cup_{i} \cup_{j}\left(A_{i} \cap C_{j}\right) \times\left(B_{i} \cap D_{j}\right)
$$

a countable union of measurable rectangles. Thus finite intersections of sets of $\mathscr{R}$ are in $\mathscr{R}$. This proves the lemma.

Now note that from the definition of $\mathscr{R}$ if you have a sequence of elements of $\mathscr{R}$ then their union is also in $\mathscr{R}$. The next lemma will enable the definition of an outer measure.

Lemma 12.9.4 Suppose $\left\{R_{i}\right\}_{i=1}^{\infty}$ is a sequence of sets of $\mathscr{R}$ then

$$
\rho\left(\cup_{i=1}^{\infty} R_{i}\right) \leq \sum_{i=1}^{\infty} \rho\left(R_{i}\right)
$$

Proof: Let $R_{i}=\cup_{j=1}^{\infty} A_{j}^{i} \times B_{j}^{i}$. Using Lemma 12.9.3, let $\left\{A_{m}^{\prime} \times B_{m}^{\prime}\right\}_{m=1}^{\infty}$ be a sequence of disjoint rectangles each of which is contained in some $A_{j}^{i} \times B_{j}^{i}$ for some $i, j$ such that

$$
\cup_{i=1}^{\infty} R_{i}=\cup_{m=1}^{\infty} A_{m}^{\prime} \times B_{m}^{\prime} .
$$

Now define

$$
S_{i} \equiv\left\{m: A_{m}^{\prime} \times B_{m}^{\prime} \subseteq A_{j}^{i} \times B_{j}^{i} \text { for some } j\right\}
$$

It is not important to consider whether some $m$ might be in more than one $S_{i}$. The important thing to notice is that

$$
\cup_{m \in S_{i}} A_{m}^{\prime} \times B_{m}^{\prime} \subseteq \cup_{j=1}^{\infty} A_{j}^{i} \times B_{j}^{i}=R_{i}
$$

Then by Lemma 12.9.3,

$$
\begin{aligned}
\rho\left(\cup_{i=1}^{\infty} R_{i}\right) & =\sum_{m} \rho\left(A_{m}^{\prime} \times B_{m}^{\prime}\right) \\
& \leq \sum_{i=1}^{\infty} \sum_{m \in S_{i}} \rho\left(A_{m}^{\prime} \times B_{m}^{\prime}\right) \\
& \leq \sum_{i=1}^{\infty} \rho\left(\cup_{m \in S_{i}} A_{m}^{\prime} \times B_{m}^{\prime}\right) \leq \sum_{i=1}^{\infty} \rho\left(R_{i}\right) .
\end{aligned}
$$

This proves the lemma.
So far, there is no measure and no $\sigma$ algebra. However, the next step is to define an outer measure which will lead to a measure on a $\sigma$ algebra of measurable sets from the Caratheodory procedure. When this is done, it will be shown that this measure can be computed using $\rho$ which implies the important Fubini theorem.

Now it is possible to define an outer measure.
Definition 12.9.5 For $S \subseteq X \times Y$, define

$$
\begin{equation*}
(\overline{\mu \times v})(S) \equiv \inf \{\rho(R): S \subseteq R, R \in \mathscr{R}\} \tag{12.9.35}
\end{equation*}
$$

The following proposition is interesting but is not needed in the development which follows. It gives a different description of $(\overline{\mu \times v})$.

Proposition 12.9.6 $(\overline{\mu \times v})(S)=\inf \left\{\sum_{i=1}^{\infty} \mu\left(A_{i}\right) v\left(B_{i}\right): S \subseteq \cup_{i=1}^{\infty} A_{i} \times B_{i}\right\}$
Proof: Let $\lambda(S) \equiv \inf \left\{\sum_{i=1}^{\infty} \mu\left(A_{i}\right) v\left(B_{i}\right): S \subseteq \cup_{i=1}^{\infty} A_{i} \times B_{i}\right\}$. Suppose $S \subseteq \cup_{i=1}^{\infty} A_{i} \times$ $B_{i} \equiv Q \in \mathscr{R}$. Then by Lemma 12.9.3, $Q=\cup_{i} A_{i}^{\prime} \times B_{i}^{\prime}$ where these rectangles are disjoint. Thus by this lemma, $\rho(Q)=\sum_{i=1}^{\infty} \mu\left(A_{i}^{\prime}\right) v\left(B_{i}^{\prime}\right) \geq \lambda(S)$ and so $\lambda(S) \leq(\overline{\mu \times v})(S)$. If $\lambda(S)=\infty$, this shows $\lambda(S)=(\overline{\mu \times v})(S)$. Suppose then that $\lambda(S)<\infty$ and $\lambda(S)+\varepsilon>$ $\sum_{i=1}^{\infty} \mu\left(A_{i}\right) v\left(B_{i}\right)$ where $Q=\cup_{i=1}^{\infty} A_{i} \times B_{i} \supseteq S$. Then by Lemma 12.9.3 again, $\cup_{i=1}^{\infty} A_{i} \times B_{i}=$ $\cup_{i=1}^{\infty} A_{i}^{\prime} \times B_{i}^{\prime}$ where the primed rectangles are disjoint, each is a subset of some $A_{i} \times B_{i}$ and so

$$
\lambda(S)+\varepsilon \geq \sum_{i=1}^{\infty} \mu\left(A_{i}\right) v\left(B_{i}\right) \geq \sum_{i=1}^{\infty} \mu\left(A_{i}^{\prime}\right) v\left(B_{i}^{\prime}\right)=\rho(Q) \geq(\overline{\mu \times v})(S)
$$

Since $\varepsilon$ is arbitrary, this shows $\lambda(S) \geq(\overline{\mu \times v})(S)$ and this proves the proposition.
Lemma 12.9.7 $\overline{\mu \times v}$ is an outer measure on $X \times Y$ and for $R \in \mathscr{R}$

$$
\begin{equation*}
(\overline{\mu \times v})(R)=\rho(R) . \tag{12.9.36}
\end{equation*}
$$

Proof: First consider 12.9.36. Since $R \supseteq R$, it follows $\rho(R) \geq(\overline{\mu \times v})(R)$. On the other hand, if $Q \in \mathscr{R}$ and $Q \supseteq R$, then $\rho(Q) \geq \rho(R)$ and so, taking the infimum on the left yields $(\overline{\mu \times v})(R) \geq \rho(R)$. This shows 12.9.36.

It is necessary to show that if $S \subseteq T$, then

$$
\begin{align*}
(\overline{\mu \times v})(S) & \leq(\overline{\mu \times v})(T)  \tag{12.9.37}\\
(\overline{\mu \times v})\left(\cup_{i=1}^{\infty} S_{i}\right) & \leq \sum_{i=1}^{\infty}(\overline{\mu \times v})\left(S_{i}\right) . \tag{12.9.38}
\end{align*}
$$

To do this, note that 12.9 .37 is obvious. To verify 12.9 .38 , note that it is obvious if $(\overline{\mu \times v})\left(S_{i}\right)=\infty$ for any $i$. Therefore, assume $(\overline{\mu \times v})\left(S_{i}\right)<\infty$. Then letting $\varepsilon>0$ be given, there exist $R_{i} \in \mathscr{R}$ such that

$$
(\overline{\mu \times v})\left(S_{i}\right)+\frac{\varepsilon}{2^{i}}>\rho\left(R_{i}\right), R_{i} \supseteq S_{i}
$$

Then by Lemma 12.9.4, 12.9.36, and the observation that $\cup_{i=1}^{\infty} R_{i} \in \mathscr{R}$,

$$
\begin{aligned}
(\overline{\mu \times v})\left(\cup_{i=1}^{\infty} S_{i}\right) & \leq(\overline{\mu \times v})\left(\cup_{i=1}^{\infty} R_{i}\right) \\
& =\rho\left(\cup_{i=1}^{\infty} R_{i}\right) \leq \sum_{i=1}^{\infty} \rho\left(R_{i}\right) \\
& \leq \sum_{i=1}^{\infty}\left((\overline{\mu \times v})\left(S_{i}\right)+\frac{\varepsilon}{2^{i}}\right) \\
& =\left(\sum_{i=1}^{\infty}(\overline{\mu \times v})\left(S_{i}\right)\right)+\varepsilon .
\end{aligned}
$$

Since $\varepsilon$ is arbitrary, this proves the lemma.
By Caratheodory's procedure, it follows there is a $\sigma$ algebra of subsets of $X \times Y$, denoted here by $\overline{\mathscr{S} \times \mathscr{T}}$ such that $(\overline{\mu \times v})$ is a complete measure on this $\sigma$ algebra. The first thing to note is that every rectangle is in this $\sigma$ algebra.

Lemma 12.9.8 Every rectangle is $(\overline{\mu \times v})$ measurable.
Proof: Let $S \subseteq X \times Y$. The following inequality must be established.

$$
\begin{equation*}
(\overline{\mu \times v})(S) \geq(\overline{\mu \times v})(S \cap(A \times B))+(\overline{\mu \times v})(S \backslash(A \times B)) \tag{12.9.39}
\end{equation*}
$$

The following claim will be used to establish this inequality.
Claim: Let $P, A \times B \in \mathscr{R}$. Then

$$
\rho(P \cap(A \times B))+\rho(P \backslash(A \times B))=\rho(P) .
$$

Proof of the claim: From Lemma 12.9.3, $P=\cup_{i=1}^{\infty} A_{i}^{\prime} \times B_{i}^{\prime}$ where the $A_{i}^{\prime} \times B_{i}^{\prime}$ are disjoint. Therefore,

$$
P \cap(A \times B)=\bigcup_{i=1}^{\infty}\left(A \cap A_{i}^{\prime}\right) \times\left(B \cap B_{i}^{\prime}\right)
$$

while

$$
P \backslash(A \times B)=\bigcup_{i=1}^{\infty}\left(A_{i}^{\prime} \backslash A\right) \times B_{i}^{\prime} \cup \bigcup_{i=1}^{\infty}\left(A \cap A_{i}^{\prime}\right) \times\left(B_{i}^{\prime} \backslash B\right) .
$$

Since all of the sets in the above unions are disjoint,

$$
\begin{gathered}
\rho(P \cap(A \times B))+\rho(P \backslash(A \times B))= \\
\iint \sum_{i=1}^{\infty} \mathscr{X}_{\left(A \cap A_{i}^{\prime}\right)}(x) \mathscr{X}_{B \cap B_{i}^{\prime}}(y) d \mu d v+\iint \sum_{i=1}^{\infty} \mathscr{X}_{\left(A_{i}^{\prime} \backslash A\right)}(x) \mathscr{X}_{B_{i}^{\prime}}(y) d \mu d v \\
+\iint \sum_{i=1}^{\infty} \mathscr{X}_{A \cap A_{i}^{\prime}}(x) \mathscr{X}_{B_{i}^{\prime} \backslash B}(y) d \mu d v
\end{gathered}
$$

$$
\begin{aligned}
& =\sum_{i=1}^{\infty} \mu\left(A \cap A_{i}^{\prime}\right) v\left(B \cap B_{i}^{\prime}\right)+\mu\left(A_{i}^{\prime} \backslash A\right) v\left(B_{i}^{\prime}\right)+\mu\left(A \cap A_{i}^{\prime}\right) v\left(B_{i}^{\prime} \backslash B\right) \\
& =\sum_{i=1}^{\infty} \mu\left(A \cap A_{i}^{\prime}\right) v\left(B_{i}^{\prime}\right)+\mu\left(A_{i}^{\prime} \backslash A\right) v\left(B_{i}^{\prime}\right)=\sum_{i=1}^{\infty} \mu\left(A_{i}^{\prime}\right) v\left(B_{i}^{\prime}\right)=\rho(P) .
\end{aligned}
$$

This proves the claim.
Now continuing to verify 12.9 .39 , without loss of generality, $(\overline{\mu \times v})(S)$ can be assumed finite. Let $P \supseteq S$ for $P \in \mathscr{R}$ and

$$
(\overline{\mu \times v})(S)+\varepsilon>\rho(P) .
$$

Then from the claim,

$$
\begin{aligned}
(\overline{\mu \times v})(S)+\varepsilon & >\rho(P)=\rho(P \cap(A \times B))+\rho(P \backslash(A \times B)) \\
& \geq(\overline{\mu \times v})(S \cap(A \times B))+(\overline{\mu \times v})(S \backslash(A \times B))
\end{aligned}
$$

Since $\varepsilon>0$ this shows $A \times B$ is $\overline{\mu \times v}$ measurable as claimed.
Lemma 12.9.9 Let $\mathscr{R}_{1}$ be defined as the set of all countable intersections of sets of $\mathscr{R}$. Then if $S \subseteq X \times Y$, there exists $R \in \mathscr{R}_{1}$ for which it makes sense to write $\rho(R)$ because 12.9.32 and 12.9.33 hold such that

$$
\begin{equation*}
(\overline{\mu \times v})(S)=\rho(R) \tag{12.9.40}
\end{equation*}
$$

Also, every element of $\mathscr{R}_{1}$ is $\overline{\mu \times v}$ measurable.
Proof: Consider 12.9.40. Let $S \subseteq X \times Y$. If $(\overline{\mu \times v})(S)=\infty$, let $R=X \times Y$ and it follows $\rho(X \times Y)=\infty=(\overline{\mu \times v})(S)$. Assume then that $(\overline{\mu \times v})(S)<\infty$.

Therefore, there exists $P_{n} \in \mathscr{R}$ such that $P_{n} \supseteq S$ and

$$
\begin{equation*}
(\overline{\mu \times v})(S) \leq \rho\left(P_{n}\right)<(\overline{\mu \times v})(S)+1 / n \tag{12.9.41}
\end{equation*}
$$

Let $Q_{n}=\cap_{i=1}^{n} P_{i} \in \mathscr{R}$. Define

$$
P \equiv \cap_{i=1}^{\infty} Q_{i} \supseteq S
$$

Then 12.9.41 holds with $Q_{n}$ in place of $P_{n}$. It is clear that

$$
x \rightarrow \mathscr{X}_{P}(x, y) \text { is } \mu \text { measurable }
$$

because this function is the pointwise limit of functions for which this is so. It remains to consider whether $y \rightarrow \int \mathscr{X}_{P}(x, y) d \mu$ is $v$ measurable. First observe $Q_{n} \supseteq Q_{n+1}, \mathscr{X}_{Q_{i}} \leq$ $\mathscr{X}_{P_{i}}$, and

$$
\begin{equation*}
\rho\left(Q_{1}\right)=\rho\left(P_{1}\right)=\iint \mathscr{X}_{P_{1}}(x, y) d \mu d v<\infty \tag{12.9.42}
\end{equation*}
$$

Therefore, there exists a set of $v$ measure $0, N$, such that if $y \notin N$, then

$$
\int \mathscr{X}_{P_{1}}(x, y) d \mu<\infty .
$$

It follows from the dominated convergence theorem that

$$
\lim _{n \rightarrow \infty} \mathscr{X}_{N^{C}}(y) \int \mathscr{X}_{Q_{n}}(x, y) d \mu=\mathscr{X}_{N^{C}}(y) \int \mathscr{X}_{P}(x, y) d \mu
$$

and so

$$
y \rightarrow \mathscr{X}_{N^{c}}(y) \int \mathscr{X}_{P}(x, y) d \mu
$$

is also measurable. By completeness of $v$,

$$
y \rightarrow \int \mathscr{X}_{P}(x, y) d \mu
$$

must also be $v$ measurable and so it makes sense to write

$$
\iint \mathscr{X}_{P}(x, y) d \mu d v
$$

for every $P \in \mathscr{R}_{1}$. Also, by the dominated convergence theorem,

$$
\begin{aligned}
\iint \mathscr{X}_{P}(x, y) d \mu d v & =\int \mathscr{X}_{N^{C}}(y) \int \mathscr{X}_{P}(x, y) d \mu d v \\
& =\lim _{n \rightarrow \infty} \int \mathscr{X}_{N^{C}}(y) \int \mathscr{X}_{Q_{n}}(x, y) d \mu d v \\
& =\lim _{n \rightarrow \infty} \iint \mathscr{X}_{Q_{n}}(x, y) d \mu d v \\
& =\lim _{n \rightarrow \infty} \rho\left(Q_{n}\right) \in[(\overline{\mu \times v})(S),(\overline{\mu \times v})(S)+1 / n]
\end{aligned}
$$

for all $n$. Therefore,

$$
\rho(P) \equiv \iint \mathscr{X}_{P}(x, y) d \mu d v=(\overline{\mu \times v})(S)
$$

The sets of $\mathscr{R}_{1}$ are $\overline{\mu \times v}$ measurable because these sets are countable intersections of countable unions of rectangles and Lemma 12.9.8 verifies the rectangles are $\overline{\mu \times v}$ measurable. This proves the Lemma.

The following theorem is the main result.
Theorem 12.9.10 Let $E \subseteq X \times Y$ be $\overline{\mu \times v}$ measurable and suppose $(\overline{\mu \times v})(E)<\infty$. Then

$$
x \rightarrow \mathscr{X}_{E}(x, y) \text { is } \mu \text { measurable a.e. } y .
$$

Modifying $\mathscr{X}_{E}$ on a set of measure zero, it is possible to write

$$
\int \mathscr{X}_{E}(x, y) d \mu
$$

The function,

$$
y \rightarrow \int \mathscr{X}_{E}(x, y) d \mu
$$

is $v$ measurable and

$$
(\overline{\mu \times v})(E)=\iint \mathscr{X}_{E}(x, y) d \mu d v
$$

Similarly,

$$
(\overline{\mu \times v})(E)=\iint \mathscr{X}_{E}(x, y) d v d \mu
$$

Proof: By Lemma 12.9.9, there exists $R \in \mathscr{R}_{1}$ such that

$$
\rho(R)=(\overline{\mu \times v})(E), R \supseteq E .
$$

Therefore, since $R$ is $\overline{\mu \times v}$ measurable and $\rho(R)=(\overline{\mu \times v})(R)$, it follows

$$
(\overline{\mu \times v})(R \backslash E)=0
$$

By Lemma 12.9.9 again, there exists $P \supseteq R \backslash E$ with $P \in \mathscr{R}_{1}$ and

$$
\rho(P)=(\overline{\mu \times v})(R \backslash E)=0
$$

Thus

$$
\begin{equation*}
\iint \mathscr{X}_{P}(x, y) d \mu d v=0 \tag{12.9.43}
\end{equation*}
$$

Since $P \in \mathscr{R}_{1}$ Lemma 12.9.9 implies $x \rightarrow \mathscr{X}_{P}(x, y)$ is $\mu$ measurable and it follows from the above there exists a set of $v$ measure zero, $N$ such that if $y \notin N$, then $\int \mathscr{X}_{P}(x, y) d \mu=0$. Therefore, by completeness of $v$,

$$
x \rightarrow \mathscr{X}_{N^{C}}(y) \mathscr{X}_{R \backslash E}(x, y)
$$

is $\mu$ measurable and

$$
\begin{equation*}
\int \mathscr{X}_{N^{C}}(y) \mathscr{X}_{R \backslash E}(x, y) d \mu=0 . \tag{12.9.44}
\end{equation*}
$$

Now also

$$
\begin{equation*}
\mathscr{X}_{N^{C}}(y) \mathscr{X}_{R}(x, y)=\mathscr{X}_{N^{C}}(y) \mathscr{X}_{R \backslash E}(x, y)+\mathscr{X}_{N^{C}}(y) \mathscr{X}_{E}(x, y) \tag{12.9.45}
\end{equation*}
$$

and this shows that

$$
x \rightarrow \mathscr{X}_{N^{C}}(y) \mathscr{X}_{E}(x, y)
$$

is $\mu$ measurable because it is the difference of two functions with this property. Then by 12.9.44 it follows

$$
\int \mathscr{X}_{N^{C}}(y) \mathscr{X}_{E}(x, y) d \mu=\int \mathscr{X}_{N^{C}}(y) \mathscr{X}_{R}(x, y) d \mu
$$

The right side of this equation equals a $v$ measurable function and so the left side which equals it is also a $v$ measurable function. It follows from completeness of $v$ that $y \rightarrow$ $\int \mathscr{X}_{E}(x, y) d \mu$ is $v$ measurable because for $y$ outside of a set of $v$ measure zero, $N$ it equals $\int \mathscr{X}_{R}(x, y) d \mu$. Therefore,

$$
\begin{aligned}
\iint \mathscr{X}_{E}(x, y) d \mu d v & =\iint \mathscr{X}_{N^{C}}(y) \mathscr{X}_{E}(x, y) d \mu d v \\
& =\iint \mathscr{X}_{N^{C}}(y) \mathscr{X}_{R}(x, y) d \mu d v \\
& =\iint \mathscr{X}_{R}(x, y) d \mu d v \\
& =\rho(R)=(\overline{\mu \times v})(E)
\end{aligned}
$$

In all the above there would be no change in writing $d \nu d \mu$ instead of $d \mu d \nu$. The same result would be obtained. This proves the theorem.

Now let $f: X \times Y \rightarrow[0, \infty]$ be $\overline{\mu \times v}$ measurable and

$$
\begin{equation*}
\int f d(\overline{\mu \times v})<\infty \tag{12.9.46}
\end{equation*}
$$

Let $s(x, y) \equiv \sum_{i=1}^{m} c_{i} \mathscr{X}_{E_{i}}(x, y)$ be a nonnegative simple function with $c_{i}$ being the nonzero values of $s$ and suppose

$$
0 \leq s \leq f
$$

Then from the above theorem,

$$
\int s d(\overline{\mu \times v})=\iint s d \mu d v
$$

In which

$$
\int s d \mu=\int \mathscr{X}_{N^{C}}(y) s d \mu
$$

for $N$ a set of $v$ measure zero such that $y \rightarrow \int \mathscr{X}_{N^{C}}(y) s d \mu$ is $v$ measurable. This follows because 12.9.46 implies $(\overline{\mu \times v})\left(E_{i}\right)<\infty$. Now let $s_{n} \uparrow f$ where $s_{n}$ is a nonnegative simple function and

$$
\int s_{n} d(\overline{\mu \times v})=\iint \mathscr{X}_{N_{n}^{C}}(y) s_{n}(x, y) d \mu d v
$$

where

$$
y \rightarrow \int \mathscr{X}_{N_{n}^{C}}(y) s_{n}(x, y) d \mu
$$

is $v$ measurable. Then let $N \equiv \cup_{n=1}^{\infty} N_{n}$. It follows $N$ is a set of $v$ measure zero. Thus

$$
\int s_{n} d(\overline{\mu \times v})=\iint \mathscr{X}_{N^{C}}(y) s_{n}(x, y) d \mu d v
$$

and letting $n \rightarrow \infty$, the monotone convergence theorem implies

$$
\begin{aligned}
\int f d(\overline{\mu \times v}) & =\iint \mathscr{X}_{N^{C}}(y) f(x, y) d \mu d v \\
& =\iint f(x, y) d \mu d v
\end{aligned}
$$

because of completeness of the measures, $\mu$ and $v$. This proves Fubini's theorem.
Theorem 12.9.11 (Fubini) Let $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{T}, v)$ be complete measure spaces and let

$$
(\overline{\mu \times v})(E) \equiv \inf \left\{\iint \mathscr{X}_{R}(x, y) d \mu d v: E \subseteq R \in \mathscr{R}\right\}^{2}
$$

$$
\begin{aligned}
& { }^{2} \text { Recall this is the same as } \\
& \qquad \inf \left\{\sum_{i=1}^{\infty} \mu\left(A_{i}\right) v\left(B_{i}\right): E \subseteq \cup_{i=1}^{\infty} A_{i} \times B_{i}\right\}
\end{aligned}
$$

in which the $A_{i}$ and $B_{i}$ are measurable.
where $A_{i} \in \mathscr{S}$ and $B_{i} \in \mathscr{T}$. Then $\overline{\mu \times v}$ is an outer measure on the subsets of $X \times Y$ and the $\sigma$ algebra of $\overline{\mu \times v}$ measurable sets, $\overline{\mathscr{S} \times \mathscr{T}}$, contains all measurable rectangles. If $f \geq 0$ is a $\overline{\mu \times v}$ measurable function satisfying

$$
\begin{equation*}
\int_{X \times Y} f d(\overline{\mu \times v})<\infty \tag{12.9.47}
\end{equation*}
$$

then

$$
\int_{X \times Y} f d(\overline{\mu \times v})=\int_{Y} \int_{X} f d \mu d v
$$

where the iterated integral on the right makes sense because for $v$ a.e. $y, x \rightarrow f(x, y)$ is $\mu$ measurable and $y \rightarrow \int f(x, y) d \mu$ is $v$ measurable. Similarly,

$$
\int_{X \times Y} f d(\overline{\mu \times v})=\int_{X} \int_{Y} f d v d \mu
$$

In the case where $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{T}, v)$ are both $\sigma$ finite, it is not necessary to assume 12.9.47.

Corollary 12.9.12 (Fubini) Let $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{T}, v)$ be complete measure spaces such that $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{T}, v)$ are both $\sigma$ finite and let

$$
(\overline{\mu \times v})(E) \equiv \inf \left\{\iint \mathscr{X}_{R}(x, y) d \mu d v: E \subseteq R \in \mathscr{R}\right\}
$$

where $A_{i} \in \mathscr{S}$ and $B_{i} \in \mathscr{T}$. Then $\overline{\mu \times v}$ is an outer measure. If $f \geq 0$ is a $\overline{\mu \times v}$ measurable function then

$$
\int_{X \times Y} f d(\overline{\mu \times v})=\int_{Y} \int_{X} f d \mu d v
$$

where the iterated integral on the right makes sense because for $v$ a.e. $y, x \rightarrow f(x, y)$ is $\mu$ measurable and $y \rightarrow \int f(x, y) d \mu$ is $v$ measurable. Similarly,

$$
\int_{X \times Y} f d(\overline{\mu \times v})=\int_{X} \int_{Y} f d v d \mu
$$

Proof: Let $\cup_{n=1}^{\infty} X_{n}=X$ and $\cup_{n=1}^{\infty} Y_{n}=Y$ where $X_{n} \in \mathscr{S}, Y_{n} \in \mathscr{T}, X_{n} \subseteq X_{n+1}, Y_{n} \subseteq Y_{n+1}$ for all $n$ and $\mu\left(X_{n}\right)<\infty, v\left(Y_{n}\right)<\infty$. From Theorem 12.9.11 applied to $X_{n}, Y_{n}$ and $f_{m} \equiv$ $\min (f, m)$,

$$
\int_{X_{n} \times Y_{n}} f_{m} d(\overline{\mu \times v})=\int_{Y_{n}} \int_{X_{n}} f_{m} d \mu d v
$$

Now take $m \rightarrow \infty$ and use the monotone convergence theorem to obtain

$$
\int_{X_{n} \times Y_{n}} f d(\overline{\mu \times v})=\int_{Y_{n}} \int_{X_{n}} f d \mu d v
$$

Then use the monotone convergence theorem again letting $n \rightarrow \infty$ to obtain the desired conclusion. The argument for the other order of integration is similar.

Corollary 12.9.13 If $f \in L^{1}(X \times Y)$, then

$$
\int f d(\overline{\mu \times v})=\iint f(x, y) d \mu d v=\iint f(x, y) d v d \mu
$$

If $\mu$ and $v$ are $\sigma$ finite, then if $f$ is $\overline{\mu \times v}$ measurable having complex values and either $\iint|f| d \mu d \nu<\infty$ or $\iint|f| d v d \mu<\infty$, then $\int|f| d(\overline{\mu \times v})<\infty$ so $f \in L^{1}(X \times Y)$.

Proof: Without loss of generality, it can be assumed that $f$ has real values. Then

$$
f=\frac{|f|+f-(|f|-f)}{2}
$$

and both $f^{+} \equiv \frac{|f|+f}{2}$ and $f^{-} \equiv \frac{|f|-f}{2}$ are nonnegative and are less than $|f|$. Therefore, $\int g d(\overline{\mu \times v})<\infty$ for $g=f^{+}$and $g=f^{-}$so the above theorem applies and

$$
\begin{aligned}
\int f d(\overline{\mu \times v}) & \equiv \int f^{+} d(\overline{\mu \times v})-\int f^{-} d(\overline{\mu \times v}) \\
& =\iint f^{+} d \mu d v-\iint f^{-} d \mu d v \\
& =\iint f d \mu d v
\end{aligned}
$$

It remains to verify the last claim. Suppose $s$ is a simple function,

$$
s(x, y) \equiv \sum_{i=1}^{m} c_{i} \mathscr{X}_{E_{i}} \leq|f|(x, y)
$$

where the $c_{i}$ are the nonzero values of $s$. Then

$$
s \mathscr{X}_{R_{n}} \leq|f| \mathscr{X}_{R_{n}}
$$

where $R_{n} \equiv X_{n} \times Y_{n}$ where $X_{n} \uparrow X$ and $Y_{n} \uparrow Y$ with $\mu\left(X_{n}\right)<\infty$ and $v\left(Y_{n}\right)<\infty$. It follows, since the nonzero values of $s \mathscr{X}_{R_{n}}$ are achieved on sets of finite measure,

$$
\int s \mathscr{X}_{R_{n}} d(\overline{\mu \times v})=\iint s \mathscr{X}_{R_{n}} d \mu d v .
$$

Letting $n \rightarrow \infty$ and applying the monotone convergence theorem, this yields

$$
\begin{equation*}
\int s d(\overline{\mu \times v})=\iint s d \mu d v \tag{12.9.48}
\end{equation*}
$$

Now let $s_{n} \uparrow|f|$ where $s_{n}$ is a nonnegative simple function. From 12.9.48,

$$
\int s_{n} d(\overline{\mu \times v})=\iint s_{n} d \mu d v
$$

Letting $n \rightarrow \infty$ and using the monotone convergence theorem, yields

$$
\int|f| d(\overline{\mu \times v})=\iint|f| d \mu d v<\infty
$$

### 12.10 Alternative Treatment Of Product Measure

### 12.10.1 Monotone Classes And Algebras

Measures are defined on $\sigma$ algebras which are closed under countable unions. It is for this reason that the theory of measure and integration is so useful in dealing with limits of sequences. However, there is a more basic notion which involves only finite unions and differences.

Definition 12.10.1 $\mathscr{A}$ is said to be an algebra of subsets of a set, $Z$ if $Z \in \mathscr{A}, \emptyset \in \mathscr{A}$, and when $E, F \in \mathscr{A}, E \cup F$ and $E \backslash F$ are both in $\mathscr{A}$.

It is important to note that if $\mathscr{A}$ is an algebra, then it is also closed under finite intersections. This is because $E \cap F=\left(E^{C} \cup F^{C}\right)^{C} \in \mathscr{A}$ since $E^{C}=Z \backslash E \in \mathscr{A}$ and $F^{C}=Z \backslash F \in \mathscr{A}$. Note that every $\sigma$ algebra is an algebra but not the other way around.

Something satisfying the above definition is called an algebra because union is like addition, the set difference is like subtraction and intersection is like multiplication. Furthermore, only finitely many operations are done at a time and so there is nothing like a limit involved.

How can you recognize an algebra when you see one? The answer to this question is the purpose of the following lemma.

Lemma 12.10.2 Suppose $\mathscr{R}$ and $\mathscr{E}$ are subsets of $\mathscr{P}(Z)^{3}$ such that $\mathscr{E}$ is defined as the set of all finite disjoint unions of sets of $\mathscr{R}$. Suppose also

$$
\begin{gathered}
\emptyset, Z \in \mathscr{R} \\
A \cap B \in \mathscr{R} \text { whenever } A, B \in \mathscr{R}, \\
A \backslash B \in \mathscr{E} \text { whenever } A, B \in \mathscr{R} .
\end{gathered}
$$

Then $\mathscr{E}$ is an algebra of sets of $Z$.
Proof: Note first that if $A \in \mathscr{R}$, then $A^{C} \in \mathscr{E}$ because $A^{C}=Z \backslash A$.
Now suppose that $E_{1}$ and $E_{2}$ are in $\mathscr{E}$,

$$
E_{1}=\cup_{i=1}^{m} R_{i}, \quad E_{2}=\cup_{j=1}^{n} R_{j}
$$

where the $R_{i}$ are disjoint sets in $\mathscr{R}$ and the $R_{j}$ are disjoint sets in $\mathscr{R}$. Then

$$
E_{1} \cap E_{2}=\cup_{i=1}^{m} \cup_{j=1}^{n} R_{i} \cap R_{j}
$$

which is clearly an element of $\mathscr{E}$ because no two of the sets in the union can intersect and by assumption they are all in $\mathscr{R}$. Thus by induction, finite intersections of sets of $\mathscr{E}$ are in $\mathscr{E}$. Consider the difference of two elements of $\mathscr{E}$ next.

If $E=\cup_{i=1}^{n} R_{i} \in \mathscr{E}$,

$$
E^{C}=\cap_{i=1}^{n} R_{i}^{C}=\text { finite intersection of sets of } \mathscr{E}
$$

[^11]which was just shown to be in $\mathscr{E}$. Now, if $E_{1}, E_{2} \in \mathscr{E}$,
$$
E_{1} \backslash E_{2}=E_{1} \cap E_{2}^{C} \in \mathscr{E}
$$
from what was just shown about finite intersections.
Finally consider finite unions of sets of $\mathscr{E}$. Let $E_{1}$ and $E_{2}$ be sets of $\mathscr{E}$. Then
$$
E_{1} \cup E_{2}=\left(E_{1} \backslash E_{2}\right) \cup E_{2} \in \mathscr{E}
$$
because $E_{1} \backslash E_{2}$ consists of a finite disjoint union of sets of $\mathscr{R}$ and these sets must be disjoint from the sets of $\mathscr{R}$ whose union yields $E_{2}$ because $\left(E_{1} \backslash E_{2}\right) \cap E_{2}=\emptyset$. This proves the lemma.

The following corollary is particularly helpful in verifying the conditions of the above lemma.

Corollary 12.10.3 Let $\left(Z_{1}, \mathscr{R}_{1}, \mathscr{E}_{1}\right)$ and $\left(Z_{2}, \mathscr{R}_{2}, \mathscr{E}_{2}\right)$ be as described in Lemma 14.1.2. Then $\left(Z_{1} \times Z_{2}, \mathscr{R}, \mathscr{E}\right)$ also satisfies the conditions of Lemma 14.1.2 if $\mathscr{R}$ is defined as

$$
\mathscr{R} \equiv\left\{R_{1} \times R_{2}: R_{i} \in \mathscr{R}_{i}\right\}
$$

and

$$
\mathscr{E} \equiv\{\text { finite disjoint unions of sets of } \mathscr{R}\}
$$

Consequently, $\mathscr{E}$ is an algebra of sets.
Proof: It is clear $\emptyset, Z_{1} \times Z_{2} \in \mathscr{R}$. Let $A \times B$ and $C \times D$ be two elements of $\mathscr{R}$.

$$
A \times B \cap C \times D=A \cap C \times B \cap D \in \mathscr{R}
$$

by assumption.

$$
\begin{aligned}
& A \times B \backslash(C \times D)= \\
& A \times \overbrace{(B \backslash D)}^{\in \mathscr{E}_{2}} \cup \overbrace{(A \backslash C)}^{\in \mathscr{E}_{1}} \times \overbrace{(D \cap B)}^{\in \mathscr{R}_{2}} \\
& =(A \times Q) \cup(P \times R)
\end{aligned}
$$

where $Q \in \mathscr{E}_{2}, P \in \mathscr{E}_{1}$, and $R \in \mathscr{R}_{2}$.


Since $A \times Q$ and $P \times R$ do not intersect, it follows the above expression is in $\mathscr{E}$ because each of these terms are. This proves the corollary.

Definition 12.10.4 $\mathscr{M} \subseteq \mathscr{P}(Z)$ is called a monotone class if
a.) $\cdots E_{n} \supseteq E_{n+1} \cdots, E=\cap_{n=1}^{\infty} E_{n}$, and $E_{n} \in \mathscr{M}$, then $E \in \mathscr{M}$.
b.) $\cdots E_{n} \subseteq E_{n+1} \cdots, E=\cup_{n=1}^{\infty} E_{n}$, and $E_{n} \in \mathscr{M}$, then $E \in \mathscr{M}$.
(In simpler notation, $E_{n} \downarrow E$ and $E_{n} \in \mathscr{M}$ implies $E \in \mathscr{M} . E_{n} \uparrow E$ and $E_{n} \in \mathscr{M}$ implies $E \in \mathscr{M}$.)

Theorem 12.10.5 (Monotone Class theorem) Let $\mathscr{A}$ be an algebra of subsets of $Z$ and let $\mathscr{M}$ be a monotone class containing $\mathscr{A}$. Then $\mathscr{M} \supseteq \sigma(\mathscr{A})$, the smallest $\sigma$-algebra containing $\mathscr{A}$.

Proof: Consider all monotone classes which contain $\mathscr{A}$, and take their intersection. The result is still a monotone class which contains $\mathscr{A}$ and is therefore the smallest monotone class containing $\mathscr{A}$. Therefore, assume without loss of generality that $\mathscr{M}$ is the smallest monotone class containing $\mathscr{A}$ because if it is shown the smallest monotone class containing $\mathscr{A}$ contains $\sigma(\mathscr{A})$, then the given monotone class does also. To avoid more notation, let $\mathscr{M}$ denote this smallest monotone class.

The plan is to show $\mathscr{M}$ is a $\sigma$-algebra. It will then follow $\mathscr{M} \supseteq \sigma(\mathscr{A})$ because $\sigma(\mathscr{A})$ is defined as the intersection of all $\sigma$ algebras which contain $\mathscr{A}$. For $A \in \mathscr{A}$, define

$$
\mathscr{M}_{A} \equiv\{B \in \mathscr{M} \text { such that } A \cup B \in \mathscr{M}\}
$$

Clearly $\mathscr{M}_{A}$ is a monotone class containing $\mathscr{A}$. Hence $\mathscr{M}_{A} \supseteq \mathscr{M}$ because $\mathscr{M}$ is the smallest such monotone class. But by construction, $\mathscr{M}_{A} \subseteq \mathscr{M}$. Therefore, $\mathscr{M}=\mathscr{M}_{A}$. This shows that $A \cup B \in \mathscr{M}$ whenever $A \in \mathscr{A}$ and $B \in \mathscr{M}$. Now pick $B \in \mathscr{M}$ and define

$$
\mathscr{M}_{B} \equiv\{D \in \mathscr{M} \text { such that } D \cup B \in \mathscr{M}\} .
$$

It was just shown that $\mathscr{A} \subseteq \mathscr{M}_{B}$. It is clear that $\mathscr{M}_{B}$ is a monotone class. Thus by a similar argument, $\mathscr{M}_{B}=\mathscr{M}$ and it follows that $D \cup B \in \mathscr{M}$ whenever $D \in \mathscr{M}$ and $B \in \mathscr{M}$. This shows $\mathscr{M}$ is closed under finite unions.

Next consider the diference of two sets. Let $A \in \mathscr{A}$

$$
\mathscr{M}_{A} \equiv\{B \in \mathscr{M} \text { such that } B \backslash A \text { and } A \backslash B \in \mathscr{M}\}
$$

Then $\mathscr{M}_{A}$, is a monotone class containing $\mathscr{A}$. As before, $\mathscr{M}=\mathscr{M}_{A}$. Thus $B \backslash A$ and $A \backslash B$ are both in $\mathscr{M}$ whenever $A \in \mathscr{A}$ and $B \in \mathscr{M}$. Now pick $A \in \mathscr{M}$ and consider

$$
\mathscr{M}_{A} \equiv\{B \in \mathscr{M} \text { such that } B \backslash A \text { and } A \backslash B \in \mathscr{M}\}
$$

It was just shown $\mathscr{M}_{A}$ contains $\mathscr{A}$. Now $\mathscr{M}_{A}$ is a monotone class and so $\mathscr{M}_{A}=\mathscr{M}$ as before.

Thus $\mathscr{M}$ is both a monotone class and an algebra. Hence, if $E \in \mathscr{M}$ then $Z \backslash E \in \mathscr{M}$. Next consider the question of whether $\mathscr{M}$ is a $\sigma$-algebra. If $E_{i} \in \mathscr{M}$ and $F_{n}=\cup_{i=1}^{n} E_{i}$, then $F_{n} \in \mathscr{M}$ and $F_{n} \uparrow \cup_{i=1}^{\infty} E_{i}$. Since $\mathscr{M}$ is a monotone class, $\cup_{i=1}^{\infty} E_{i} \in \mathscr{M}$ and so $\mathscr{M}$ is a $\sigma$-algebra. This proves the theorem.

### 12.10.2 Product Measure

Definition 12.10.6 Let $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{F}, \lambda)$ be two measure spaces. A measurable rectangle is a set $A \times B \subseteq X \times Y$ where $A \in \mathscr{S}$ and $B \in \mathscr{F}$. An elementary set will be any subset of $X \times Y$ which is a finite union of disjoint measurable rectangles. $\mathscr{S} \times \mathscr{F}$ will denote the smallest $\sigma$ algebra of sets in $\mathscr{P}(X \times Y)$ containing all elementary sets.

Example 12.10.7 It follows from Lemma 14.1.2 or more easily from Corollary 14.1.3 that the elementary sets form an algebra.

Definition 12.10.8 Let $E \subseteq X \times Y$,

$$
\begin{aligned}
& E_{x}=\{y \in Y:(x, y) \in E\}, \\
& E^{y}=\{x \in X:(x, y) \in E\} .
\end{aligned}
$$

These are called the $x$ and $y$ sections.


Theorem 12.10.9 If $E \in \mathscr{S} \times \mathscr{F}$, then $E_{x} \in \mathscr{F}$ and $E^{y} \in \mathscr{S}$ for all $x \in X$ and $y \in Y$.
Proof: Let

$$
\begin{gathered}
\mathscr{M}=\left\{E \subseteq \mathscr{S} \times \mathscr{F} \text { such that for all } x \in X, E_{x} \in \mathscr{F},\right. \\
\text { and for all } \left.y \in Y, E^{y} \in \mathscr{S} .\right\}
\end{gathered}
$$

Then $\mathscr{M}$ contains all measurable rectangles. If $E_{i} \in \mathscr{M}$,

$$
\left(\cup_{i=1}^{\infty} E_{i}\right)_{x}=\cup_{i=1}^{\infty}\left(E_{i}\right)_{x} \in \mathscr{F} .
$$

Similarly,

$$
\left(\cup_{i=1}^{\infty} E_{i}\right)^{y}=\cup_{i=1}^{\infty} E_{i}^{y} \in \mathscr{S}
$$

It follows $\mathscr{M}$ is closed under countable unions.
If $E \in \mathscr{M}$,

$$
\left(E^{C}\right)_{x}=\left(E_{x}\right)^{C} \in \mathscr{F}
$$

Similarly, $\left(E^{C}\right)^{y} \in \mathscr{S}$. Thus $\mathscr{M}$ is closed under complementation. Therefore $\mathscr{M}$ is a $\sigma$-algebra containing the elementary sets. Hence, $\mathscr{M} \supseteq \mathscr{S} \times \mathscr{F}$ because $\mathscr{S} \times \mathscr{F}$ is the
smallest $\sigma$ algebra containing these elementary sets. But $\mathscr{M} \subseteq \mathscr{S} \times \mathscr{F}$ by definition and so $\mathscr{M}=\mathscr{S} \times \mathscr{F}$. This proves the theorem.

It follows from Lemma 14.1.2 that the elementary sets form an algebra because clearly the intersection of two measurable rectangles is a measurable rectangle and

$$
(A \times B) \backslash\left(A_{0} \times B_{0}\right)=\left(A \backslash A_{0}\right) \times B \cup\left(A \cap A_{0}\right) \times\left(B \backslash B_{0}\right)
$$

an elementary set.
Theorem 12.10.10 If $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{F}, \lambda)$ are both finite measure spaces

$$
(\mu(X), \lambda(Y)<\infty)
$$

then for every $E \in \mathscr{S} \times \mathscr{F}$,
a.) $x \rightarrow \lambda\left(E_{x}\right)$ is $\mu$ measurable, $y \rightarrow \mu\left(E^{y}\right)$ is $\lambda$ measurable
b.) $\int_{X} \lambda\left(E_{x}\right) d \mu=\int_{Y} \mu\left(E^{y}\right) d \lambda$.

Proof: Let

$$
\mathscr{M}=\{E \in \mathscr{S} \times \mathscr{F} \text { such that both } a .) \text { and } b \text {.) hold }\} .
$$

Since $\mu$ and $\lambda$ are both finite, the monotone convergence and dominated convergence theorems imply $\mathscr{M}$ is a monotone class.

Next I will argue $\mathscr{M}$ contains the elementary sets. Let

$$
E=\cup_{i=1}^{n} A_{i} \times B_{i}
$$

where the measurable rectangles, $A_{i} \times B_{i}$ are disjoint. Then

$$
\begin{aligned}
\lambda\left(E_{x}\right) & =\int_{Y} \mathscr{X}_{E}(x, y) d \lambda=\int_{Y} \sum_{i=1}^{n} \mathscr{X}_{A_{i} \times B_{i}}(x, y) d \lambda \\
& =\sum_{i=1}^{n} \int_{Y} \mathscr{X}_{A_{i} \times B_{i}}(x, y) d \lambda=\sum_{i=1}^{n} \mathscr{X}_{A_{i}}(x) \lambda\left(B_{i}\right)
\end{aligned}
$$

which is clearly $\mu$ measurable. Furthermore,

$$
\int_{X} \lambda\left(E_{x}\right) d \mu=\int_{X} \sum_{i=1}^{n} \mathscr{X}_{A_{i}}(x) \lambda\left(B_{i}\right) d \mu=\sum_{i=1}^{n} \mu\left(A_{i}\right) \lambda\left(B_{i}\right) .
$$

Similarly,

$$
\int_{Y} \mu\left(E^{y}\right) d \lambda=\sum_{i=1}^{n} \mu\left(A_{i}\right) \lambda\left(B_{i}\right)
$$

and $y \rightarrow \mu\left(E^{y}\right)$ is $\lambda$ measurable and this shows $\mathscr{M}$ contains the algebra of elementary sets. By the monotone class theorem, $\mathscr{M}=\mathscr{S} \times \mathscr{F}$. This proves the theorem.

One can easily extend this theorem to the case where the measure spaces are $\sigma$ finite.

Theorem 12.10.11 If $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{F}, \lambda)$ are both $\sigma$ finite measure spaces, then for every $E \in \mathscr{S} \times \mathscr{F}$,
a.) $x \rightarrow \lambda\left(E_{x}\right)$ is $\mu$ measurable, $y \rightarrow \mu\left(E^{y}\right)$ is $\lambda$ measurable.
b.) $\int_{X} \lambda\left(E_{x}\right) d \mu=\int_{Y} \mu\left(E^{y}\right) d \lambda$.

Proof: Let $X=\cup_{n=1}^{\infty} X_{n}, Y=\cup_{n=1}^{\infty} Y_{n}$ where,

$$
X_{n} \subseteq X_{n+1}, Y_{n} \subseteq Y_{n+1}, \mu\left(X_{n}\right)<\infty, \lambda\left(Y_{n}\right)<\infty
$$

Let

$$
\mathscr{S}_{n}=\left\{A \cap X_{n}: A \in \mathscr{S}\right\}, \mathscr{F}_{n}=\left\{B \cap Y_{n}: B \in \mathscr{F}\right\} .
$$

Thus $\left(X_{n}, \mathscr{S}_{n}, \mu\right)$ and $\left(Y_{n}, \mathscr{F}_{n}, \lambda\right)$ are both finite measure spaces.
Claim: If $E \in \mathscr{S} \times \mathscr{F}$, then $E \cap\left(X_{n} \times Y_{n}\right) \in \mathscr{S}_{n} \times \mathscr{F}_{n}$.
Proof: Let

$$
\mathscr{M}_{n}=\left\{E \in \mathscr{S} \times \mathscr{F}: E \cap\left(X_{n} \times Y_{n}\right) \in \mathscr{S}_{n} \times \mathscr{F}_{n}\right\} .
$$

Clearly $\mathscr{M}_{n}$ contains the algebra of elementary sets. It is also clear that $\mathscr{M}_{n}$ is a monotone class. Thus $\mathscr{M}_{n}=\mathscr{S} \times \mathscr{F}$.

Now let $E \in \mathscr{S} \times \mathscr{F}$. By Theorem 12.10.10,

$$
\begin{equation*}
\int_{X_{n}} \lambda\left(\left(E \cap\left(X_{n} \times Y_{n}\right)\right)_{x}\right) d \mu=\int_{Y_{n}} \mu\left(\left(E \cap\left(X_{n} \times Y_{n}\right)\right)^{y}\right) d \lambda \tag{12.10.49}
\end{equation*}
$$

where the integrands are measurable. Also

$$
\left(E \cap\left(X_{n} \times Y_{n}\right)\right)_{x}=\emptyset
$$

if $x \notin X_{n}$ and a similar observation holds for the second integrand in 12.10.49 if $y \notin Y_{n}$. Therefore,

$$
\begin{aligned}
\int_{X} \lambda\left(\left(E \cap\left(X_{n} \times Y_{n}\right)\right)_{x}\right) d \mu & =\int_{X_{n}} \lambda\left(\left(E \cap\left(X_{n} \times Y_{n}\right)\right)_{x}\right) d \mu \\
& =\int_{Y_{n}} \mu\left(\left(E \cap\left(X_{n} \times Y_{n}\right)\right)^{y}\right) d \lambda \\
& =\int_{Y} \mu\left(\left(E \cap\left(X_{n} \times Y_{n}\right)\right)^{y}\right) d \lambda .
\end{aligned}
$$

Then letting $n \rightarrow \infty$, the monotone convergence theorem implies b.) and the measurability assertions of a.) are valid because

$$
\begin{aligned}
\lambda\left(E_{x}\right) & =\lim _{n \rightarrow \infty} \lambda\left(\left(E \cap\left(X_{n} \times Y_{n}\right)\right)_{x}\right) \\
\mu\left(E^{y}\right) & =\lim _{n \rightarrow \infty} \mu\left(\left(E \cap\left(X_{n} \times Y_{n}\right)\right)^{y}\right) .
\end{aligned}
$$

This proves the theorem.
This theorem makes it possible to define product measure.
Definition 12.10.12 For $E \in \mathscr{S} \times \mathscr{F}$ and $(X, \mathscr{S}, \mu),(Y, \mathscr{F}, \lambda) \sigma$ finite,

$$
(\mu \times \lambda)(E) \equiv \int_{X} \lambda\left(E_{x}\right) d \mu=\int_{Y} \mu\left(E^{y}\right) d \lambda
$$

This definition is well defined because of Theorem 12.10.11.
Theorem 12.10.13 If $A \in \mathscr{S}, B \in \mathscr{F}$, then $(\mu \times \lambda)(A \times B)=\mu(A) \lambda(B)$, and $\mu \times \lambda$ is a measure on $\mathscr{S} \times \mathscr{F}$ called product measure.

Proof: The first assertion about the measure of a measurable rectangle was established above. Now suppose $\left\{E_{i}\right\}_{i=1}^{\infty}$ is a disjoint collection of sets of $\mathscr{S} \times \mathscr{F}$. Then using the monotone convergence theorem along with the observation that $\left(E_{i}\right)_{x} \cap\left(E_{j}\right)_{x}=\emptyset$,

$$
\begin{aligned}
(\mu \times \lambda)\left(\cup_{i=1}^{\infty} E_{i}\right) & =\int_{X} \lambda\left(\left(\cup_{i=1}^{\infty} E_{i}\right)_{x}\right) d \mu \\
& =\int_{X} \lambda\left(\cup_{i=1}^{\infty}\left(E_{i}\right)_{x}\right) d \mu=\int_{X} \sum_{i=1}^{\infty} \lambda\left(\left(E_{i}\right)_{x}\right) d \mu \\
& =\sum_{i=1}^{\infty} \int_{X} \lambda\left(\left(E_{i}\right)_{x}\right) d \mu \\
& =\sum_{i=1}^{\infty}(\mu \times \lambda)\left(E_{i}\right)
\end{aligned}
$$

This proves the theorem.
The next theorem is one of several theorems due to Fubini and Tonelli. These theorems all have to do with interchanging the order of integration in a multiple integral.

Theorem 12.10.14 Let $f: X \times Y \rightarrow[0, \infty]$ be measurable with respect to $\mathscr{S} \times \mathscr{F}$ and suppose $\mu$ and $\lambda$ are $\sigma$ finite. Then

$$
\begin{equation*}
\int_{X \times Y} f d(\mu \times \lambda)=\int_{X} \int_{Y} f(x, y) d \lambda d \mu=\int_{Y} \int_{X} f(x, y) d \mu d \lambda \tag{12.10.50}
\end{equation*}
$$

and all integrals make sense.
Proof: For $E \in \mathscr{S} \times \mathscr{F}$,

$$
\int_{Y} \mathscr{X}_{E}(x, y) d \lambda=\lambda\left(E_{x}\right), \int_{X} \mathscr{X}_{E}(x, y) d \mu=\mu\left(E^{y}\right)
$$

Thus from Definition 12.10.12, 12.10.50 holds if $f=\mathscr{X}_{E}$. It follows that 12.10 .50 holds for every nonnegative simple function. By Theorem 11.3.9 on Page 241, there exists an increasing sequence, $\left\{f_{n}\right\}$, of simple functions converging pointwise to $f$. Then

$$
\begin{aligned}
\int_{Y} f(x, y) d \lambda & =\lim _{n \rightarrow \infty} \int_{Y} f_{n}(x, y) d \lambda \\
\int_{X} f(x, y) d \mu & =\lim _{n \rightarrow \infty} \int_{X} f_{n}(x, y) d \mu
\end{aligned}
$$

This follows from the monotone convergence theorem. Since

$$
x \rightarrow \int_{Y} f_{n}(x, y) d \lambda
$$

is measurable with respect to $\mathscr{S}$, it follows that $x \rightarrow \int_{Y} f(x, y) d \lambda$ is also measurable with respect to $\mathscr{S}$. A similar conclusion can be drawn about $y \rightarrow \int_{X} f(x, y) d \mu$. Thus the two iterated integrals make sense. Since 12.10 .50 holds for $f_{n}$, another application of the Monotone Convergence theorem shows 12.10 .50 holds for $f$. This proves the theorem.

Corollary 12.10.15 Let $f: X \times Y \rightarrow \mathbb{C}$ be $\mathscr{S} \times \mathscr{F}$ measurable. Suppose either

$$
\int_{X} \int_{Y}|f| d \lambda d \mu \text { or } \int_{Y} \int_{X}|f| d \mu d \lambda<\infty
$$

Then $f \in L^{1}(X \times Y, \mu \times \lambda)$ and

$$
\begin{equation*}
\int_{X \times Y} f d(\mu \times \lambda)=\int_{X} \int_{Y} f d \lambda d \mu=\int_{Y} \int_{X} f d \mu d \lambda \tag{12.10.51}
\end{equation*}
$$

with all integrals making sense.
Proof: Suppose first that $f$ is real valued. Apply Theorem 12.10 .14 to $f^{+}$and $f^{-}$. Then 12.10.51 follows from observing that $f=f^{+}-f^{-}$; and that all integrals are finite. If $f$ is complex valued, consider real and imaginary parts. This proves the corollary.

Suppose $f$ is product measurable. From the above discussion, and breaking $f$ down into a sum of positive and negative parts of real and imaginary parts and then using Theorem 11.3.9 on Page 241 on approximation by simple functions, it follows that whenever $f$ is $\mathscr{S} \times \mathscr{F}$ measurable, $x \rightarrow f(x, y)$ is $\mu$ measurable, $y \rightarrow f(x, y)$ is $\lambda$ measurable.

### 12.11 Completion Of Measures

Suppose $(\Omega, \mathscr{F}, \mu)$ is a measure space. Then it is always possible to enlarge the $\sigma$ algebra and define a new measure $\bar{\mu}$ on this larger $\sigma$ algebra such that $(\Omega, \overline{\mathscr{F}}, \bar{\mu})$ is a complete measure space. Recall this means that if $N \subseteq N^{\prime} \in \overline{\mathscr{F}}$ and $\bar{\mu}\left(N^{\prime}\right)=0$, then $N \in \overline{\mathscr{F}}$. The following theorem is the main result. The new measure space is called the completion of the measure space.

Theorem 12.11.1 Let $(\Omega, \mathscr{F}, \mu)$ be a $\sigma$ finite measure space. Then there exists a unique measure space, $(\Omega, \overline{\mathscr{F}}, \bar{\mu})$ satisfying

1. $(\Omega, \overline{\mathscr{F}}, \bar{\mu})$ is a complete measure space.
2. $\bar{\mu}=\mu$ on $\mathscr{F}$
3. $\overline{\mathscr{F}} \supseteq \mathscr{F}$
4. For every $E \in \overline{\mathscr{F}}$ there exists $G \in \mathscr{F}$ such that $G \supseteq E$ and $\mu(G)=\bar{\mu}(E)$.
5. For every $E \in \overline{\mathscr{F}}$ there exists $F \in \mathscr{F}$ such that $F \subseteq E$ and $\mu(F)=\bar{\mu}(E)$.

Also for every $E \in \overline{\mathscr{F}}$ there exist sets $G, F \in \mathscr{F}$ such that $G \supseteq E \supseteq F$ and

$$
\begin{equation*}
\mu(G \backslash F)=\bar{\mu}(G \backslash F)=0 \tag{12.11.52}
\end{equation*}
$$

Proof: First consider the claim about uniqueness. Suppose $\left(\Omega, \mathscr{F}_{1}, v_{1}\right)$ and $\left(\Omega, \mathscr{F}_{2}, \nu_{2}\right)$ both work and let $E \in \mathscr{F}_{1}$. Also let $\mu\left(\Omega_{n}\right)<\infty, \cdots \Omega_{n} \subseteq \Omega_{n+1} \cdots$, and $\cup_{n=1}^{\infty} \Omega_{n}=\Omega$. Define $E_{n} \equiv E \cap \Omega_{n}$. Then pick $G_{n} \supseteq E_{n} \supseteq F_{n}$ such that $\mu\left(G_{n}\right)=\mu\left(F_{n}\right)=v_{1}\left(E_{n}\right)$. It follows $\mu\left(G_{n} \backslash F_{n}\right)=0$. Then letting $G=\cup_{n} G_{n}, F \equiv \cup_{n} F_{n}$, it follows $G \supseteq E \supseteq F$ and

$$
\begin{aligned}
\mu(G \backslash F) & \leq \mu\left(\cup_{n}\left(G_{n} \backslash F_{n}\right)\right) \\
& \leq \sum_{n} \mu\left(G_{n} \backslash F_{n}\right)=0 .
\end{aligned}
$$

It follows that $v_{2}(G \backslash F)=0$ also. Now $E \backslash F \subseteq G \backslash F$ and since $\left(\Omega, \mathscr{F}_{2}, v_{2}\right)$ is complete, it follows $E \backslash F \in \mathscr{F}_{2}$. Since $F \in \mathscr{F}_{2}$, it follows $E=(E \backslash F) \cup F \in \mathscr{F}_{2}$. Thus $\mathscr{F}_{1} \subseteq \mathscr{F}_{2}$. Similarly $\mathscr{F}_{2} \subseteq \mathscr{F}_{1}$. Now it only remains to verify $v_{1}=v_{2}$. Thus let $E \in \mathscr{F}_{1}=\mathscr{F}_{2}$ and let $G$ and $F$ be as just described. Since $v_{i}=\mu$ on $\mathscr{F}$,

$$
\begin{aligned}
\mu(F) & \leq v_{1}(E) \\
& =v_{1}(E \backslash F)+v_{1}(F) \\
& \leq v_{1}(G \backslash F)+v_{1}(F) \\
& =v_{1}(F)=\mu(F)
\end{aligned}
$$

Similarly $v_{2}(E)=\mu(F)$. This proves uniqueness. The construction has also verified 12.11.52.

Next define an outer measure, $\bar{\mu}$ on $\mathscr{P}(\Omega)$ as follows. For $S \subseteq \Omega$,

$$
\bar{\mu}(S) \equiv \inf \{\mu(E): E \in \mathscr{F}\}
$$

Then it is clear $\bar{\mu}$ is increasing. It only remains to verify $\bar{\mu}$ is subadditive. Then let $S=$ $\cup_{i=1}^{\infty} S_{i}$. If any $\bar{\mu}\left(S_{i}\right)=\infty$, there is nothing to prove so suppose $\bar{\mu}\left(S_{i}\right)<\infty$ for each $i$. Then there exist $E_{i} \in \mathscr{F}$ such that $E_{i} \supseteq S_{i}$ and

$$
\bar{\mu}\left(S_{i}\right)+\varepsilon / 2^{i}>\mu\left(E_{i}\right)
$$

Then

$$
\begin{aligned}
\bar{\mu}(S) & =\bar{\mu}\left(\cup_{i} S_{i}\right) \\
& \leq \mu\left(\cup_{i} E_{i}\right) \leq \sum_{i} \mu\left(E_{i}\right) \\
& \leq \sum_{i}\left(\bar{\mu}\left(S_{i}\right)+\varepsilon / 2^{i}\right)=\sum_{i} \bar{\mu}\left(S_{i}\right)+\varepsilon
\end{aligned}
$$

Since $\varepsilon$ is arbitrary, this verifies $\bar{\mu}$ is subadditive and is an outer measure as claimed.
Denote by $\overline{\mathscr{F}}$ the $\sigma$ algebra of measurable sets in the sense of Caratheodory. Then it follows from the Caratheodory procedure, Theorem 12.1.4, on Page 270 that $(\Omega, \overline{\mathscr{F}}, \bar{\mu})$ is a complete measure space. This verifies 1 .

Now let $E \in \mathscr{F}$. Then from the definition of $\bar{\mu}$, it follows

$$
\bar{\mu}(E) \equiv \inf \{\mu(F): F \in \mathscr{F} \text { and } F \supseteq E\} \leq \mu(E)
$$

If $F \supseteq E$ and $F \in \mathscr{F}$, then $\mu(F) \geq \mu(E)$ and so $\mu(E)$ is a lower bound for all such $\mu(F)$ which shows that

$$
\bar{\mu}(E) \equiv \inf \{\mu(F): F \in \mathscr{F} \text { and } F \supseteq E\} \geq \mu(E)
$$

This verifies 2 .
Next consider 3. Let $E \in \mathscr{F}$ and let $S$ be a set. I must show

$$
\bar{\mu}(S) \geq \bar{\mu}(S \backslash E)+\bar{\mu}(S \cap E)
$$

If $\bar{\mu}(S)=\infty$ there is nothing to show. Therefore, suppose $\bar{\mu}(S)<\infty$. Then from the definition of $\bar{\mu}$ there exists $G \supseteq S$ such that $G \in \mathscr{F}$ and $\mu(G)=\bar{\mu}(S)$. Then from the definition of $\bar{\mu}$,

$$
\begin{aligned}
\bar{\mu}(S) & \leq \bar{\mu}(S \backslash E)+\bar{\mu}(S \cap E) \\
& \leq \mu(G \backslash E)+\mu(G \cap E) \\
& =\mu(G)=\bar{\mu}(S)
\end{aligned}
$$

This verifies 3.
Claim 4 comes by the definition of $\bar{\mu}$ as used above. The only other case is when $\bar{\mu}(S)=\infty$. However, in this case, you can let $G=\Omega$.

It only remains to verify 5 . Let the $\Omega_{n}$ be as described above and let $E \in \overline{\mathscr{F}}$ such that $E \subseteq \Omega_{n}$. By 4 there exists $H \in \mathscr{F}$ such that $H \subseteq \Omega_{n}, H \supseteq \Omega_{n} \backslash E$, and

$$
\begin{equation*}
\mu(H)=\bar{\mu}\left(\Omega_{n} \backslash E\right) \tag{12.11.53}
\end{equation*}
$$

Then let $F \equiv \Omega_{n} \cap H^{C}$. It follows $F \subseteq E$ and

$$
\begin{aligned}
E \backslash F & =E \cap F^{C}=E \cap\left(H \cup \Omega_{n}^{C}\right) \\
& =E \cap H=H \backslash\left(\Omega_{n} \backslash E\right)
\end{aligned}
$$

Hence from 12.11.53

$$
\bar{\mu}(E \backslash F)=\bar{\mu}\left(H \backslash\left(\Omega_{n} \backslash E\right)\right)=0
$$

It follows

$$
\bar{\mu}(E)=\bar{\mu}(F)=\mu(F)
$$

In the case where $E \in \overline{\mathscr{F}}$ is arbitrary, not necessarily contained in some $\Omega_{n}$, it follows from what was just shown that there exists $F_{n} \in \mathscr{F}$ such that $F_{n} \subseteq E \cap \Omega_{n}$ and

$$
\mu\left(F_{n}\right)=\bar{\mu}\left(E \cap \Omega_{n}\right) .
$$

Letting $F \equiv \cup_{n} F_{n}$

$$
\bar{\mu}(E \backslash F) \leq \bar{\mu}\left(\cup_{n}\left(E \cap \Omega_{n} \backslash F_{n}\right)\right) \leq \sum_{n} \bar{\mu}\left(E \cap \Omega_{n} \backslash F_{n}\right)=0
$$

Therefore, $\bar{\mu}(E)=\mu(F)$ and this proves 5. This proves the theorem.
Now here is an interesting theorem about complete measure spaces.

Theorem 12.11.2 Let $(\Omega, \mathscr{F}, \mu)$ be a complete measure space and let $f \leq g \leq h$ be functions having values in $[0, \infty]$. Suppose also that $f(\omega)=h(\omega)$ a.e. $\omega$ and that $f$ and $h$ are measurable. Then $g$ is also measurable. If $(\Omega, \overline{\mathscr{F}}, \bar{\mu})$ is the completion of a $\sigma$ finite measure space $(\Omega, \mathscr{F}, \mu)$ as described above in Theorem 12.11.1 then if $f$ is measurable with respect to $\overline{\mathscr{F}}$ having values in $[0, \infty]$, it follows there exists $g$ measurable with respect to $\mathscr{F}$ , $g \leq f$, and a set $N \in \mathscr{F}$ with $\mu(N)=0$ and $g=f$ on $N^{C}$. There also exists $h$ measurable with respect to $\mathscr{F}$ such that $h \geq f$, and a set of measure zero, $M \in \mathscr{F}$ such that $f=h$ on $M^{C}$.

Proof: Let $\alpha \in \mathbb{R}$.

$$
[f>\alpha] \subseteq[g>\alpha] \subseteq[h>\alpha]
$$

Thus

$$
[g>\alpha]=[f>\alpha] \cup([g>\alpha] \backslash[f>\alpha])
$$

and $[g>\alpha] \backslash[f>\alpha]$ is a measurable set because it is a subset of the set of measure zero,

$$
[h>\alpha] \backslash[f>\alpha]
$$

Now consider the last assertion. By Theorem 11.3.9 on Page 241 there exists an increasing sequence of nonnegative simple functions, $\left\{s_{n}\right\}$ measurable with respect to $\overline{\mathscr{F}}$ which converges pointwise to $f$. Letting

$$
\begin{equation*}
s_{n}(\omega)=\sum_{k=1}^{m_{n}} c_{k}^{n} \mathscr{X}_{E_{k}^{n}}(\omega) \tag{12.11.54}
\end{equation*}
$$

be one of these simple functions, it follows from Theorem 12.11.1 there exist sets, $F_{k}^{n} \in \mathscr{F}$ such that $F_{k}^{n} \subseteq E_{k}^{n}$ and $\mu\left(F_{k}^{n}\right)=\bar{\mu}\left(E_{k}^{n}\right)$. Then let

$$
t_{n}(\omega) \equiv \sum_{k=1}^{m_{n}} c_{k}^{n} \mathscr{X}_{F_{k}^{n}}(\omega)
$$

Thus $t_{n}=s_{n}$ off a set of measure zero, $N_{n} \in \overline{\mathscr{F}}, t_{n} \leq s_{n}$. Let $N^{\prime} \equiv \cup_{n} N_{n}$. Then by Theorem 12.11.1 again, there exists $N \in \mathscr{F}$ such that $N \supseteq N^{\prime}$ and $\mu(N)=0$. Consider the simple functions,

$$
s_{n}^{\prime}(\omega) \equiv t_{n}(\omega) \mathscr{X}_{N^{C}}(\omega)
$$

It is an increasing sequence so let $g(\omega)=\lim _{n \rightarrow \infty} s_{n^{\prime}}(\omega)$. It follows $g$ is mesurable with respect to $\mathscr{F}$ and equals $f$ off $N$.

Finally, to obtain the function, $h \geq f$, in 12.11.54 use Theorem 12.11.1 to obtain the existence of $F_{k}^{n} \in \mathscr{F}$ such that $F_{k}^{n} \supseteq E_{k}^{n}$ and $\mu\left(F_{k}^{n}\right)=\bar{\mu}\left(E_{k}^{n}\right)$. Then let

$$
t_{n}(\omega) \equiv \sum_{k=1}^{m_{n}} c_{k}^{n} \mathscr{X}_{F_{k}^{n}}(\omega)
$$

Thus $t_{n}=s_{n}$ off a set of measure zero, $M_{n} \in \overline{\mathscr{F}}, t_{n} \geq s_{n}$, and $t_{n}$ is measurable with respect to $\mathscr{F}$. Then define

$$
s_{n}^{\prime}=\max _{k \leq n} t_{n}
$$

It follows $s_{n}^{\prime}$ is an increasing sequence of $\mathscr{F}$ measurable nonnegative simple functions. Since each $s_{n}^{\prime} \geq s_{n}$, it follows that if $h(\omega)=\lim _{n \rightarrow \infty} s_{n}^{\prime}(\omega)$, then $h(\omega) \geq f(\omega)$. Also if $h(\omega)>f(\omega)$, then $\omega \in \cup_{n} M_{n} \equiv M^{\prime}$, a set of $\overline{\mathscr{F}}$ having measure zero. By Theorem 12.11.1, there exists $M \supseteq M^{\prime}$ such that $M \in \mathscr{F}$ and $\mu(M)=0$. It follows $h=f$ off $M$. This proves the theorem.

### 12.12 Another Version Of Product Measures

### 12.12.1 General Theory

Given two finite measure spaces, $(X, \mathscr{F}, \mu)$ and $(Y, \mathscr{S}, v)$, there is a way to define a $\sigma$ algebra of subsets of $X \times Y$, denoted by $\mathscr{F} \times \mathscr{S}$ and a measure, denoted by $\mu \times v$ defined on this $\sigma$ algebra such that

$$
\mu \times v(A \times B)=\mu(A) v(B)
$$

whenever $A \in \mathscr{F}$ and $B \in \mathscr{S}$. This is naturally related to the concept of iterated integrals similar to what is used in calculus to evaluate a multiple integral. The approach is based on something called a $\pi$ system, [36].

Definition 12.12.1 Let $(X, \mathscr{F}, \mu)$ and $(Y, \mathscr{S}, v)$ be two measure spaces. A measurable rectangle is a set of the form $A \times B$ where $A \in \mathscr{F}$ and $B \in \mathscr{S}$.

Definition 12.12.2 Let $\Omega$ be a set and let $\mathscr{K}$ be a collection of subsets of $\Omega$. Then $\mathscr{K}$ is called a $\pi$ system if $\emptyset, \Omega \in \mathscr{K}$ and whenever $A, B \in \mathscr{K}$, it follows $A \cap B \in \mathscr{K}$.

Obviously an example of a $\pi$ system is the set of measurable rectangles because

$$
A \times B \cap A^{\prime} \times B^{\prime}=\left(A \cap A^{\prime}\right) \times\left(B \cap B^{\prime}\right)
$$

The following is the fundamental lemma which shows these $\pi$ systems are useful. This lemma is due to Dynkin.

Lemma 12.12.3 Let $\mathscr{K}$ be a $\pi$ system of subsets of $\Omega$, a set. Also let $\mathscr{G}$ be a collection of subsets of $\Omega$ which satisfies the following three properties.

1. $\mathscr{K} \subseteq \mathscr{G}$
2. If $A \in \mathscr{G}$, then $A^{C} \in \mathscr{G}$
3. If $\left\{A_{i}\right\}_{i=1}^{\infty}$ is a sequence of disjoint sets from $\mathscr{G}$ then $\cup_{i=1}^{\infty} A_{i} \in \mathscr{G}$.

Then $\mathscr{G} \supseteq \sigma(\mathscr{K})$, where $\sigma(\mathscr{K})$ is the smallest $\sigma$ algebra which contains $\mathscr{K}$.
Proof: First note that if

$$
\mathscr{H} \equiv\{\mathscr{G}: 1-3 \text { all hold }\}
$$

then $\cap \mathscr{H}$ yields a collection of sets which also satisfies $1-3$. Therefore, I will assume in the argument that $\mathscr{G}$ is the smallest collection satisfying 1-3. Let $A \in \mathscr{K}$ and define

$$
\mathscr{G}_{A} \equiv\{B \in \mathscr{G}: A \cap B \in \mathscr{G}\} .
$$

I want to show $\mathscr{G}_{A}$ satisfies $1-3$ because then it must equal $\mathscr{G}$ since $\mathscr{G}$ is the smallest collection of subsets of $\Omega$ which satisfies $1-3$. This will give the conclusion that for $A \in \mathscr{K}$ and $B \in \mathscr{G}, A \cap B \in \mathscr{G}$. This information will then be used to show that if $A, B \in \mathscr{G}$ then $A \cap B \in \mathscr{G}$. From this it will follow very easily that $\mathscr{G}$ is a $\sigma$ algebra which will imply it contains $\sigma(\mathscr{K})$. Now here are the details of the argument.

Since $\mathscr{K}$ is given to be a $\pi$ system, $\mathscr{K} \subseteq \mathscr{G}_{A}$. Property 3 is obvious because if $\left\{B_{i}\right\}$ is a sequence of disjoint sets in $\mathscr{G}_{A}$, then

$$
A \cap \cup_{i=1}^{\infty} B_{i}=\cup_{i=1}^{\infty} A \cap B_{i} \in \mathscr{G}
$$

because $A \cap B_{i} \in \mathscr{G}$ and the property 3 of $\mathscr{G}$.
It remains to verify Property 2 so let $B \in \mathscr{G}_{A}$. I need to verify that $B^{C} \in \mathscr{G}_{A}$. In other words, I need to show that $A \cap B^{C} \in \mathscr{G}$. However,

$$
A \cap B^{C}=\left(A^{C} \cup(A \cap B)\right)^{C} \in \mathscr{G}
$$

Here is why. Since $B \in \mathscr{G}_{A}, A \cap B \in \mathscr{G}$ and since $A \in \mathscr{K} \subseteq \mathscr{G}$ it follows $A^{C} \in \mathscr{G}$ by assumption 2. It follows from assumption 3 the union of the disjoint sets, $A^{C}$ and $(A \cap B)$ is in $\mathscr{G}$ and then from 2 the complement of their union is in $\mathscr{G}$. Thus $\mathscr{G}_{A}$ satisfies 1-3 and this implies since $\mathscr{G}$ is the smallest such, that $\mathscr{G}_{A} \supseteq \mathscr{G}$. However, $\mathscr{G}_{A}$ is constructed as a subset of $\mathscr{G}$. This proves that for every $B \in \mathscr{G}$ and $A \in \mathscr{K}, A \cap B \in \mathscr{G}$. Now pick $B \in \mathscr{G}$ and consider

$$
\mathscr{G}_{B} \equiv\{A \in \mathscr{G}: A \cap B \in \mathscr{G}\} .
$$

I just proved $\mathscr{K} \subseteq \mathscr{G}_{B}$. The other arguments are identical to show $\mathscr{G}_{B}$ satisfies 1-3 and is therefore equal to $\mathscr{G}$. This shows that whenever $A, B \in \mathscr{G}$ it follows $A \cap B \in \mathscr{G}$.

This implies $\mathscr{G}$ is a $\sigma$ algebra. To show this, all that is left is to verify $\mathscr{G}$ is closed under countable unions because then it follows $\mathscr{G}$ is a $\sigma$ algebra. Let $\left\{A_{i}\right\} \subseteq \mathscr{G}$. Then let $A_{1}^{\prime}=A_{1}$ and

$$
\begin{aligned}
A_{n+1}^{\prime} & \equiv A_{n+1} \backslash\left(\cup_{i=1}^{n} A_{i}\right) \\
& =A_{n+1} \cap\left(\cap_{i=1}^{n} A_{i}^{C}\right) \\
& =\cap_{i=1}^{n}\left(A_{n+1} \cap A_{i}^{C}\right) \in \mathscr{G}
\end{aligned}
$$

because finite intersections of sets of $\mathscr{G}$ are in $\mathscr{G}$. Since the $A_{i}^{\prime}$ are disjoint, it follows

$$
\cup_{i=1}^{\infty} A_{i}=\cup_{i=1}^{\infty} A_{i}^{\prime} \in \mathscr{G}
$$

Therefore, $\mathscr{G} \supseteq \sigma(\mathscr{K})$ and this proves the Lemma.
With this lemma, it is easy to define product measure.
Let $(X, \mathscr{F}, \mu)$ and $(Y, \mathscr{S}, v)$ be two finite measure spaces. Define $\mathscr{K}$ to be the set of measurable rectangles, $A \times B, A \in \mathscr{F}$ and $B \in \mathscr{S}$. Let

$$
\begin{equation*}
\mathscr{G} \equiv\left\{E \subseteq X \times Y: \int_{Y} \int_{X} \mathscr{X}_{E} d \mu d v=\int_{X} \int_{Y} \mathscr{X}_{E} d v d \mu\right\} \tag{12.12.55}
\end{equation*}
$$

where in the above, part of the requirement is for all integrals to make sense.

Then $\mathscr{K} \subseteq \mathscr{G}$. This is obvious.
Next I want to show that if $E \in \mathscr{G}$ then $E^{C} \in \mathscr{G}$. Observe $\mathscr{X}_{E^{C}}=1-\mathscr{X}_{E}$ and so

$$
\begin{aligned}
\int_{Y} \int_{X} \mathscr{X}_{E^{C}} d \mu d v & =\int_{Y} \int_{X}\left(1-\mathscr{X}_{E}\right) d \mu d v \\
& =\int_{X} \int_{Y}\left(1-\mathscr{X}_{E}\right) d v d \mu \\
& =\int_{X} \int_{Y} \mathscr{X}_{E^{C}} d v d \mu
\end{aligned}
$$

which shows that if $E \in \mathscr{G}$, then $E^{C} \in \mathscr{G}$.
Next I want to show $\mathscr{G}$ is closed under countable unions of disjoint sets of $\mathscr{G}$. Let $\left\{A_{i}\right\}$ be a sequence of disjoint sets from $\mathscr{G}$. Then

$$
\begin{align*}
\int_{Y} \int_{X} \mathscr{X}_{\cup_{i=1}^{\infty} A_{i}} d \mu d \nu & =\int_{Y} \int_{X} \sum_{i=1}^{\infty} \mathscr{X}_{A_{i}} d \mu d \nu \\
& =\int_{Y} \sum_{i=1}^{\infty} \int_{X} \mathscr{X}_{A_{i}} d \mu d \nu \\
& =\sum_{i=1}^{\infty} \int_{Y} \int_{X} \mathscr{X}_{A_{i}} d \mu d \nu \\
& =\sum_{i=1}^{\infty} \int_{X} \int_{Y} \mathscr{X}_{A_{i}} d v d \mu \\
& =\int_{X} \sum_{i=1}^{\infty} \int_{Y} \mathscr{X}_{A_{i}} d v d \mu \\
& =\int_{X} \int_{Y} \sum_{i=1}^{\infty} \mathscr{X}_{A_{i}} d v d \mu \\
& =\int_{X} \int_{Y} \mathscr{X}_{U_{i=1}^{\infty} A_{i}} d v d \mu \tag{12.12.56}
\end{align*}
$$

the interchanges between the summation and the integral depending on the monotone convergence theorem. Thus $\mathscr{G}$ is closed with respect to countable disjoint unions.

From Lemma 12.12.3, $\mathscr{G} \supseteq \sigma(\mathscr{K})$. Also the computation in 12.12 .56 implies that on $\sigma(\mathscr{K})$ one can define a measure, denoted by $\mu \times v$ and that for every $E \in \sigma(\mathscr{K})$,

$$
\begin{equation*}
(\mu \times v)(E)=\int_{Y} \int_{X} \mathscr{X}_{E} d \mu d v=\int_{X} \int_{Y} \mathscr{X}_{E} d v d \mu \tag{12.12.57}
\end{equation*}
$$

Now here is Fubini's theorem.
Theorem 12.12.4 Let $f: X \times Y \rightarrow[0, \infty]$ be measurable with respect to the $\sigma$ algebra, $\sigma(\mathscr{K})$ just defined and let $\mu \times v$ be the product measure of 12.12 .57 where $\mu$ and $v$ are finite measures on $(X, \mathscr{F})$ and $(Y, \mathscr{S})$ respectively. Then

$$
\int_{X \times Y} f d(\mu \times v)=\int_{Y} \int_{X} f d \mu d v=\int_{X} \int_{Y} f d v d \mu
$$

Proof: Let $\left\{s_{n}\right\}$ be an increasing sequence of $\sigma(\mathscr{K})$ measurable simple functions which converges pointwise to $f$. The above equation holds for $s_{n}$ in place of $f$ from what was shown above. The final result follows from passing to the limit and using the monotone convergence theorem.

The symbol, $\mathscr{F} \times \mathscr{S}$ denotes $\sigma(\mathscr{K})$.
Of course one can generalize right away to measures which are only $\sigma$ finite.
Theorem 12.12.5 Let $f: X \times Y \rightarrow[0, \infty]$ be measurable with respect to the $\sigma$ algebra, $\sigma(\mathscr{K})$ just defined and let $\mu \times v$ be the product measure of 12.12 .57 where $\mu$ and $v$ are $\sigma$ finite measures on $(X, \mathscr{F})$ and $(Y, \mathscr{S})$ respectively. Then

$$
\int_{X \times Y} f d(\mu \times v)=\int_{Y} \int_{X} f d \mu d v=\int_{X} \int_{Y} f d v d \mu .
$$

Proof: Since the measures are $\sigma$ finite, there exist increasing sequences of sets, $\left\{X_{n}\right\}$ and $\left\{Y_{n}\right\}$ such that $\mu\left(X_{n}\right)<\infty$ and $v\left(Y_{n}\right)<\infty$. Then $\mu$ and $v$ restricted to $X_{n}$ and $Y_{n}$ respectively are finite. Then from Theorem 12.12.4,

$$
\int_{Y_{n}} \int_{X_{n}} f d \mu d v=\int_{X_{n}} \int_{Y_{n}} f d v d \mu
$$

Passing to the limit yields

$$
\int_{Y} \int_{X} f d \mu d v=\int_{X} \int_{Y} f d v d \mu
$$

whenever $f$ is as above. In particular, you could take $f=\mathscr{X}_{E}$ where $E \in \mathscr{F} \times \mathscr{S}$ and define

$$
(\mu \times v)(E) \equiv \int_{Y} \int_{X} \mathscr{X}_{E} d \mu d v=\int_{X} \int_{Y} \mathscr{X}_{E} d v d \mu .
$$

Then just as in the proof of Theorem 12.12.4, the conclusion of this theorem is obtained. This proves the theorem.

It is also useful to note that all the above holds for $\prod_{i=1}^{n} X_{i}$ in place of $X \times Y$. You would simply modify the definition of $\mathscr{G}$ in 12.12.55 including all permutations for the iterated integrals and for $\mathscr{K}$ you would use sets of the form $\prod_{i=1}^{n} A_{i}$ where $A_{i}$ is measurable. Everything goes through exactly as above. Thus the following is obtained.

Theorem 12.12.6 Let $\left\{\left(X_{i}, \mathscr{F}_{i}, \mu_{i}\right)\right\}_{i=1}^{n}$ be $\sigma$ finite measure spaces and let $\prod_{i=1}^{n} \mathscr{F}_{i}$ denote the smallest $\sigma$ algebra which contains the measurable boxes of the form $\prod_{i=1}^{n} A_{i}$ where $A_{i} \in \mathscr{F}_{i}$. Then there exists a measure, $\lambda$ defined on $\prod_{i=1}^{n} \mathscr{F}_{i}$ such that if $f: \prod_{i=1}^{n} X_{i} \rightarrow[0, \infty]$ is $\prod_{i=1}^{n} \mathscr{F}_{i}$ measurable, and $\left(i_{1}, \cdots, i_{n}\right)$ is any permutation of $(1, \cdots, n)$, then

$$
\int f d \lambda=\int_{X_{i_{n}}} \cdots \int_{X_{i_{1}}} f d \mu_{i_{1}} \cdots d \mu_{i_{n}}
$$

### 12.12.2 Completion Of Product Measure Spaces

Using Theorem 12.11.2 it is easy to give a generalization to yield a theorem for the completion of product spaces.

Theorem 12.12.7 Let $\left\{\left(X_{i}, \mathscr{F}_{i}, \mu_{i}\right)\right\}_{i=1}^{n}$ be $\sigma$ finite measure spaces and let $\prod_{i=1}^{n} \mathscr{F}_{i}$ denote the smallest $\sigma$ algebra which contains the measurable boxes of the form $\prod_{i=1}^{n} A_{i}$ where $A_{i} \in \mathscr{F}_{i}$. Then there exists a measure, $\lambda$ defined on $\prod_{i=1}^{n} \mathscr{F}_{i}$ such that if $f: \prod_{i=1}^{n} X_{i} \rightarrow[0, \infty]$ is $\prod_{i=1}^{n} \mathscr{F}_{i}$ measurable, and $\left(i_{1}, \cdots, i_{n}\right)$ is any permutation of $(1, \cdots, n)$, then

$$
\int f d \lambda=\int_{X_{i_{n}}} \cdots \int_{X_{i_{1}}} f d \mu_{i_{1}} \cdots d \mu_{i_{n}}
$$

Let $\left(\prod_{i=1}^{n} X_{i}, \overline{\prod_{i=1}^{n} \mathscr{F}_{i}}, \bar{\lambda}\right)$ denote the completion of this product measure space and let

$$
f: \prod_{i=1}^{n} X_{i} \rightarrow[0, \infty]
$$

be $\overline{\prod_{i=1}^{n} \mathscr{F}_{i}}$ measurable. Then there exists $N \in \prod_{i=1}^{n} \mathscr{F}_{i}$ such that $\lambda(N)=0$ and a nonnegative function, $f_{1}$ measurable with respect to $\prod_{i=1}^{n} \mathscr{F}_{i}$ such that $f_{1}=f$ off $N$ and if $\left(i_{1}, \cdots, i_{n}\right)$ is any permutation of $(1, \cdots, n)$, then

$$
\int f d \bar{\lambda}=\int_{X_{i_{n}}} \cdots \int_{X_{i_{1}}} f_{1} d \mu_{i_{1}} \cdots d \mu_{i_{n}} .
$$

Furthermore, $f_{1}$ may be chosen to satisfy either $f_{1} \leq f$ or $f_{1} \geq f$.
Proof: This follows immediately from Theorem 12.12.6 and Theorem 12.11.2. By the second theorem, there exists a function $f_{1} \geq f$ such that $f_{1}=f$ for all $\left(x_{1}, \cdots, x_{n}\right) \notin N$, a set of $\prod_{i=1}^{n} \mathscr{F}_{i}$ having measure zero. Then by Theorem 12.11.1 and Theorem 12.12.6

$$
\int f d \bar{\lambda}=\int f_{1} d \lambda=\int_{X_{i_{n}}} \cdots \int_{X_{i_{1}}} f_{1} d \mu_{i_{1}} \cdots d \mu_{i_{n}}
$$

Since $f_{1}=f$ off a set of measure zero, I will dispense with the subscript. Also it is customary to write

$$
\lambda=\mu_{1} \times \cdots \times \mu_{n}
$$

and

$$
\bar{\lambda}=\overline{\mu_{1} \times \cdots \times \mu_{n}} .
$$

Thus in more standard notation, one writes

$$
\int f d\left(\overline{\mu_{1} \times \cdots \times \mu_{n}}\right)=\int_{X_{i_{n}}} \cdots \int_{X_{i_{1}}} f d \mu_{i_{1}} \cdots d \mu_{i_{n}}
$$

This theorem is often referred to as Fubini's theorem. The next theorem is also called this.
Corollary 12.12 .8 Suppose $f \in L^{1}\left(\prod_{i=1}^{n} X_{i}, \overline{\prod_{i=1}^{n} \mathscr{F}_{i}}, \overline{\mu_{1} \times \cdots \times \mu_{n}}\right)$ where each $X_{i}$ is a $\sigma$ finite measure space. Then if $\left(i_{1}, \cdots, i_{n}\right)$ is any permutation of $(1, \cdots, n)$, it follows

$$
\int f d\left(\overline{\mu_{1} \times \cdots \times \mu_{n}}\right)=\int_{X_{i_{n}}} \cdots \int_{X_{i_{1}}} f d \mu_{i_{1}} \cdots d \mu_{i_{n}}
$$

Proof: Just apply Theorem 12.12 .7 to the positive and negative parts of the real and imaginary parts of $f$. This proves the theorem.

Here is another easy corollary.
Corollary 12.12.9 Suppose in the situation of Corollary 12.12.8, $f=f_{1}$ off $N$, a set of $\prod_{i=1}^{n} \mathscr{F}_{i}$ having $\mu_{1} \times \cdots \times \mu_{n}$ measure zero and that $f_{1}$ is a complex valued function measurable with respect to $\prod_{i=1}^{n} \mathscr{F}_{i}$. Suppose also that for some permutation of $(1,2, \cdots, n)$ , $\left(j_{1}, \cdots, j_{n}\right)$

$$
\int_{X_{j_{n}}} \cdots \int_{X_{j_{1}}}\left|f_{1}\right| d \mu_{j_{1}} \cdots d \mu_{j_{n}}<\infty .
$$

Then

$$
f \in L^{1}\left(\prod_{i=1}^{n} X_{i}, \overline{\prod_{i=1}^{n} \mathscr{F}_{i}}, \overline{\mu_{1} \times \cdots \times \mu_{n}}\right)
$$

and the conclusion of Corollary 12.12.8 holds.
Proof: Since $\left|f_{1}\right|$ is $\prod_{i=1}^{n} \mathscr{F}_{i}$ measurable, it follows from Theorem 12.12.6 that

$$
\begin{aligned}
\infty & >\int_{X_{j_{n}}} \cdots \int_{X_{j_{1}}}\left|f_{1}\right| d \mu_{j_{1}} \cdots d \mu_{j_{n}} \\
& =\int\left|f_{1}\right| d\left(\mu_{1} \times \cdots \times \mu_{n}\right) \\
& =\int\left|f_{1}\right| d\left(\overline{\mu_{1} \times \cdots \times \mu_{n}}\right) \\
& =\int|f| d\left(\overline{\mu_{1} \times \cdots \times \mu_{n}}\right) .
\end{aligned}
$$

Thus $f \in L^{1}\left(\prod_{i=1}^{n} X_{i}, \overline{\prod_{i=1}^{n} \mathscr{F}_{i}}, \overline{\mu_{1} \times \cdots \times \mu_{n}}\right)$ as claimed and the rest follows from Corollary 12.12 .8 . This proves the corollary.

The following lemma is also useful.
Lemma 12.12.10 Let $(X, \mathscr{F}, \mu)$ and $(Y, \mathscr{S}, v)$ be $\sigma$ finite complete measure spaces and suppose $f \geq 0$ is $\overline{\mathscr{F}} \times \mathscr{S}$ measurable. Then for a.e. $x$,

$$
y \rightarrow f(x, y)
$$

is $\mathscr{S}$ measurable. Similarly for a.e. y,

$$
x \rightarrow f(x, y)
$$

is $\mathscr{F}$ measurable.
Proof: By Theorem 12.11.2, there exist $\mathscr{F} \times \mathscr{S}$ measurable functions, $g$ and $h$ and a set, $N \in \mathscr{F} \times \mathscr{S}$ of $\mu \times \lambda$ measure zero such that $g \leq f \leq h$ and for $(x, y) \notin N$, it follows that $g(x, y)=h(x, y)$. Then

$$
\int_{X} \int_{Y} g d v d \mu=\int_{X} \int_{Y} h d v d \mu
$$

and so for a.e. $x$,

$$
\int_{Y} g d v=\int_{Y} h d v
$$

Then it follows that for these values of $x, g(x, y)=h(x, y)$ and so by Theorem 12.11.2 again and the assumption that $(Y, \mathscr{S}, v)$ is complete, $y \rightarrow f(x, y)$ is $\mathscr{S}$ measurable. The other claim is similar. This proves the lemma.

### 12.13 Disturbing Examples

There are examples which help to define what can be expected of product measures and Fubini type theorems. Three such examples are given in Rudin [113]. Some of the theorems given above are more general than those in this reference but the same examples are still useful for showing that the hypotheses of the above theorems are all necessary.

Example 12.13.1 Let $\left\{a_{n}\right\}$ be an increasing sequence of numbers in $(0,1)$ which converges to 1. Let $g_{n} \in C_{c}\left(a_{n}, a_{n+1}\right)$ such that $\int g_{n} d x=1$. Now for $(x, y) \in[0,1) \times[0,1)$ define

$$
f(x, y) \equiv \sum_{k=1}^{\infty} g_{n}(y)\left(g_{n}(x)-g_{n+1}(x)\right)
$$

Note this is actually a finite sum for each such $(x, y)$. Therefore, this is a continuous function on $[0,1) \times[0,1)$. Now for a fixed $y$,

$$
\int_{0}^{1} f(x, y) d x=\sum_{k=1}^{\infty} g_{n}(y) \int_{0}^{1}\left(g_{n}(x)-g_{n+1}(x)\right) d x=0
$$

showing that $\int_{0}^{1} \int_{0}^{1} f(x, y) d x d y=\int_{0}^{1} 0 d y=0$. Next fix $x$.

$$
\int_{0}^{1} f(x, y) d y=\sum_{k=1}^{\infty}\left(g_{n}(x)-g_{n+1}(x)\right) \int_{0}^{1} g_{n}(y) d y=g_{1}(x) .
$$

Hence $\int_{0}^{1} \int_{0}^{1} f(x, y) d y d x=\int_{0}^{1} g_{1}(x) d x=1$. The iterated integrals are not equal. Note the function, $g$ is not nonnegative even though it is measurable. In addition, neither $\int_{0}^{1} \int_{0}^{1}|f(x, y)| d x d y$ nor $\int_{0}^{1} \int_{0}^{1}|f(x, y)| d y d x$ is finite and so you Corollary 12.9.13 does not apply. The problem here is the function is not nonnegative and is not absolutely integrable.

Example 12.13.2 This time let $\mu=m$, Lebesgue measure on $[0,1]$ and let $v$ be counting measure on $[0,1]$, in this case, the $\sigma$ algebra is $\mathscr{P}([0,1])$. Let l denote the line segment in $[0,1] \times[0,1]$ which goes from $(0,0)$ to $(1,1)$. Thus $l=(x, x)$ where $x \in[0,1]$. Consider the outer measure of $l$ in $\overline{m \times v}$. Let $l \subseteq \cup_{k} A_{k} \times B_{k}$ where $A_{k}$ is Lebesgue measurable and $B_{k}$ is a subset of $[0,1]$. Let $\mathscr{B} \equiv\left\{k \in \mathbb{N}: v\left(B_{k}\right)=\infty\right\}$. If $m\left(\cup_{k \in \mathscr{B}} A_{k}\right)$ has measure zero, then there are uncountably many points of $[0,1]$ outside of $\cup_{k \in \mathscr{B}} A_{k}$. For $p$ one of these points, $(p, p) \in A_{i} \times B_{i}$ and $i \notin \mathscr{B}$. Thus each of these points is in $\cup_{i \notin \mathscr{B}} B_{i}$, a countable set because these $B_{i}$ are each finite. But this is a contradiction because there need to be uncountably many of these points as just indicated. Thus $m\left(A_{k}\right)>0$ for some $k \in \mathscr{B}$ and so $\overline{m \times v}\left(A_{k} \times B_{k}\right)=\infty$. It follows $\overline{m \times v}(l)=\infty$ and so $l$ is $\overline{m \times v}$ measurable. Thus
$\int \mathscr{X}_{l}(x, y) d \overline{m \times v}=\infty$ and so you cannot apply Fubini's theorem, Theorem 12.9.11. Since $v$ is not $\sigma$ finite, you cannot apply the corollary to this theorem either. Thus there is no contradiction to the above theorems in the following observation.

$$
\iint \mathscr{X}_{l}(x, y) d v d m=\int 1 d m=1, \iint \mathscr{X}_{l}(x, y) d m d v=\int 0 d v=0 .
$$

The problem here is that you have neither $\int f d \overline{m \times v}<\infty$ not $\sigma$ finite measure spaces.
The next example is far more exotic. It concerns the case where both iterated integrals make perfect sense but are unequal. In 1877 Cantor conjectured that the cardinality of the real numbers is the next size of infinity after countable infinity. This hypothesis is called the continuum hypothesis and it has never been proved or disproved ${ }^{4}$. Assuming this continuum hypothesis will provide the basis for the following example. It is due to Sierpinski.

Example 12.13.3 Let $X$ be an uncountable set. It follows from the well ordering theorem which says every set can be well ordered which is presented in the appendix that $X$ can be well ordered. Let $\omega \in X$ be the first element of $X$ which is preceded by uncountably many points of $X$. Let $\Omega$ denote $\{x \in X: x<\omega\}$. Then $\Omega$ is uncountable but there is no smaller uncountable set. Thus by the continuum hypothesis, there exists $a$ one to one and onto mapping, $j$ which maps $[0,1]$ onto $\Omega$. Thus, for $x \in[0,1], j(x)$ is preceeded by countably many points. Let $Q \equiv\left\{(x, y) \in[0,1]^{2}: j(x)<j(y)\right\}$ and let $f(x, y)=\mathscr{X}_{Q}(x, y)$. Then

$$
\int_{0}^{1} f(x, y) d y=1, \int_{0}^{1} f(x, y) d x=0
$$

In each case, the integrals make sense. In the first, for fixed $x, f(x, y)=1$ for all but countably many y so the function of $y$ is Borel measurable. In the second where $y$ is fixed, $f(x, y)=0$ for all but countably many $x$. Thus

$$
\int_{0}^{1} \int_{0}^{1} f(x, y) d y d x=1, \int_{0}^{1} \int_{0}^{1} f(x, y) d x d y=0
$$

The problem here must be that $f$ is not $\overline{m \times m}$ measurable.

### 12.14 Exercises

1. Let $\Omega=\mathbb{N}$, the natural numbers and let $d(p, q)=|p-q|$, the usual distance in $\mathbb{R}$. Show that $(\Omega, d)$ the closures of the balls are compact. Now let $\Lambda f \equiv \sum_{k=1}^{\infty} f(k)$ whenever $f \in C_{c}(\Omega)$. Show this is a well defined positive linear functional on the space $C_{c}(\Omega)$. Describe the measure of the Riesz representation theorem which results from this positive linear functional. What if $\Lambda(f)=f(1)$ ? What measure would result from this functional? Which functions are measurable?

[^12]2. Verify that $\bar{\mu}$ defined in Lemma 12.1.9 is an outer measure.
3. Let $F: \mathbb{R} \rightarrow \mathbb{R}$ be increasing and right continuous. Let $\Lambda f \equiv \int f d F$ where the integral is the Riemann Stieltjes integral of $f$. Show the measure $\mu$ from the Riesz representation theorem satisfies
\[

$$
\begin{aligned}
\mu([a, b]) & =F(b)-F(a-), \mu((a, b])=F(b)-F(a), \\
\mu([a, a]) & =F(a)-F(a-) .
\end{aligned}
$$
\]

4. Let $\Omega$ be a metric space with the closed balls compact and suppose $\mu$ is a measure defined on the Borel sets of $\Omega$ which is finite on compact sets. Show there exists a unique Radon measure, $\bar{\mu}$ which equals $\mu$ on the Borel sets.
5. $\uparrow$ Random vectors are measurable functions $\mathbf{X}$, which map a probability space to $\mathbb{R}^{n}$. A probability space is of the form $(\Omega, P, \mathscr{F})$. Thus $\mathbf{X}(\omega) \in \mathbb{R}^{n}$ for each $\omega \in \Omega$ and $P$ is a probability measure defined on the sets of $\mathscr{F}$, a $\sigma$ algebra of subsets of $\Omega$. For $E$ a Borel set in $\mathbb{R}^{n}$, define

$$
\mu(E) \equiv P\left(\mathbf{X}^{-1}(E)\right) \equiv \text { probability that } \mathbf{X} \in E
$$

Show this is a well defined measure on the Borel sets of $\mathbb{R}^{n}$ and use Problem 4 to obtain a Radon measure, $\lambda_{\mathbf{X}}$ defined on a $\sigma$ algebra of sets of $\mathbb{R}^{n}$ including the Borel sets such that for $E$ a Borel set, $\lambda_{\mathbf{x}}(E)=$ Probability that $(\mathbf{X} \in E)$.
6. Suppose $X$ and $Y$ are metric spaces having compact closed balls. Show

$$
\left(X \times Y, d_{X \times Y}\right)
$$

is also a metric space which has the closures of balls compact. Here

$$
d_{X \times Y}\left(\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right)\right) \equiv \max \left(d\left(x_{1}, x_{2}\right), d\left(y_{1}, y_{2}\right)\right)
$$

Let

$$
\mathscr{A} \equiv\{E \times F: E \text { is a Borel set in } X, F \text { is a Borel set in } Y\} .
$$

Show $\sigma(\mathscr{A})$, the smallest $\sigma$ algebra containing $\mathscr{A}$ contains the Borel sets. Hint: Show every open set in a metric space which has closed balls compact can be obtained as a countable union of compact sets. Next show this implies every open set can be obtained as a countable union of open sets of the form $U \times V$ where $U$ is open in $X$ and $V$ is open in $Y$.
7. Suppose $(\Omega, \mathscr{S}, \mu)$ is a measure space which may not be complete. Could you obtain a complete measure space, $\left(\Omega, \overline{\mathscr{S}}, \mu_{1}\right)$ by simply letting $\overline{\mathscr{S}}$ consist of all sets of the form $E$ where there exists $F \in \mathscr{S}$ such that $(F \backslash E) \cup(E \backslash F) \subseteq N$ for some $N \in \mathscr{S}$ which has measure zero and then let $\mu(E)=\mu_{1}(F)$ ?
8. If $\mu$ and $v$ are Radon measures defined on $\mathbb{R}^{n}$ and $\mathbb{R}^{m}$ respectively, show $\overline{\mu \times v}$ is also a radon measure on $\mathbb{R}^{n+m}$. Hint: Show the $\overline{\mu \times v}$ measurable sets include the open sets using the observation that every open set in $\mathbb{R}^{n+m}$ is the countable union of sets
of the form $U \times V$ where $U$ and $V$ are open in $\mathbb{R}^{n}$ and $\mathbb{R}^{m}$ respectively. Next verify outer regularity by considering $A \times B$ for $A, B$ measurable. Argue sets of $\mathscr{R}$ defined above have the property that they can be approximated in measure from above by open sets. Then verify the same is true of sets of $\mathscr{R}_{1}$. Finally conclude using an appropriate lemma that $\overline{\mu \times v}$ is inner regular as well.
9. Let $(\Omega, \mathscr{S}, \mu)$ be a $\sigma$ finite measure space and let $f: \Omega \rightarrow[0, \infty)$ be measurable. Define

$$
A \equiv\{(x, y): y<f(x)\}
$$

Verify that $A$ is $\overline{\mu \times m}$ measurable. Show that

$$
\int f d \mu=\iint \mathscr{X}_{A}(x, y) d \mu d m=\int \mathscr{X}_{A} d \overline{\mu \times m}
$$

## Chapter 13

## Lebesgue Measure

### 13.1 Basic Properties

Definition 13.1.1 Define the following positive linear functional for $f \in C_{c}\left(\mathbb{R}^{n}\right)$.

$$
\Lambda f \equiv \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(\mathbf{x}) d x_{1} \cdots d x_{n}
$$

Then the measure representing this functional is Lebesgue measure.
The following lemma will help in understanding Lebesgue measure.
Lemma 13.1.2 Every open set in $\mathbb{R}^{n}$ is the countable disjoint union of half open boxes of the form

$$
\prod_{i=1}^{n}\left(a_{i}, a_{i}+2^{-k}\right]
$$

where $a_{i}=l 2^{-k}$ for some integers, $l, k$. The sides of these boxes are of equal length. One could also have half open boxes of the form

$$
\prod_{i=1}^{n}\left[a_{i}, a_{i}+2^{-k}\right)
$$

and the conclusion would be unchanged.
Proof: Let

$$
\begin{gathered}
\mathscr{C}_{k}=\left\{\text { All half open boxes } \prod_{i=1}^{n}\left(a_{i}, a_{i}+2^{-k}\right]\right. \text { where } \\
\left.a_{i}=l 2^{-k} \text { for some integer } l .\right\}
\end{gathered}
$$

Thus $\mathscr{C}_{k}$ consists of a countable disjoint collection of boxes whose union is $\mathbb{R}^{n}$. This is sometimes called a tiling of $\mathbb{R}^{n}$. Think of tiles on the floor of a bathroom and you will get the idea. Note that each box has diameter no larger than $2^{-k} \sqrt{n}$. This is because if

$$
\mathbf{x}, \mathbf{y} \in \prod_{i=1}^{n}\left(a_{i}, a_{i}+2^{-k}\right]
$$

then $\left|x_{i}-y_{i}\right| \leq 2^{-k}$. Therefore,

$$
|\mathbf{x}-\mathbf{y}| \leq\left(\sum_{i=1}^{n}\left(2^{-k}\right)^{2}\right)^{1 / 2}=2^{-k} \sqrt{n}
$$

Let $U$ be open and let $\mathscr{B}_{1} \equiv$ all sets of $\mathscr{C}_{1}$ which are contained in $U$. If $\mathscr{B}_{1}, \cdots, \mathscr{B}_{k}$ have been chosen, $\mathscr{B}_{k+1} \equiv$ all sets of $\mathscr{C}_{k+1}$ contained in

$$
U \backslash \cup\left(\cup_{i=1}^{k} \mathscr{B}_{i}\right)
$$

Let $\mathscr{B}_{\infty}=\cup_{i=1}^{\infty} \mathscr{B}_{i}$. In fact $\cup \mathscr{B}_{\infty}=U$. Clearly $\cup \mathscr{B}_{\infty} \subseteq U$ because every box of every $\mathscr{B}_{i}$ is contained in $U$. If $p \in U$, let $k$ be the smallest integer such that $p$ is contained in a box from $\mathscr{C}_{k}$ which is also a subset of $U$. Thus

$$
p \in \cup \mathscr{B}_{k} \subseteq \cup \mathscr{B}_{\infty} .
$$

Hence $\mathscr{B}_{\infty}$ is the desired countable disjoint collection of half open boxes whose union is $U$. The last assertion about the other type of half open rectangle is obvious. This proves the lemma.

Now what does Lebesgue measure do to a rectangle, $\prod_{i=1}^{n}\left(a_{i}, b_{i}\right]$ ?
Lemma 13.1.3 Let $R=\prod_{i=1}^{n}\left[a_{i}, b_{i}\right], R_{0}=\prod_{i=1}^{n}\left(a_{i}, b_{i}\right)$. Then

$$
m_{n}\left(R_{0}\right)=m_{n}(R)=\prod_{i=1}^{n}\left(b_{i}-a_{i}\right)
$$

Proof: Let $k$ be large enough that

$$
a_{i}+1 / k<b_{i}-1 / k
$$

for $i=1, \cdots, n$ and consider functions $g_{i}^{k}$ and $f_{i}^{k}$ having the following graphs.


Let

$$
g^{k}(\mathbf{x})=\prod_{i=1}^{n} g_{i}^{k}\left(x_{i}\right), f^{k}(\mathbf{x})=\prod_{i=1}^{n} f_{i}^{k}\left(x_{i}\right)
$$

Then by elementary calculus along with the definition of $\Lambda$,

$$
\begin{gathered}
\prod_{i=1}^{n}\left(b_{i}-a_{i}+2 / k\right) \geq \Lambda g^{k}=\int g^{k} d m_{n} \geq m_{n}(R) \geq m_{n}\left(R_{0}\right) \\
\geq \int f^{k} d m_{n}=\Lambda f^{k} \geq \prod_{i=1}^{n}\left(b_{i}-a_{i}-2 / k\right)
\end{gathered}
$$

Letting $k \rightarrow \infty$, it follows that

$$
m_{n}(R)=m_{n}\left(R_{0}\right)=\prod_{i=1}^{n}\left(b_{i}-a_{i}\right)
$$

This proves the lemma.

Lemma 13.1.4 Let $U$ be an open or closed set. Then $m_{n}(U)=m_{n}(\mathbf{x}+U)$.
Proof: By Lemma 13.1.2 there is a sequence of disjoint half open rectangles, $\left\{R_{i}\right\}$ such that $\cup_{i} R_{i}=U$. Therefore, $\mathbf{x}+U=\cup_{i}\left(\mathbf{x}+R_{i}\right)$ and the $\mathbf{x}+R_{i}$ are also disjoint rectangles which are identical to the $R_{i}$ but translated. From Lemma 13.1.3, $m_{n}(U)=\sum_{i} m_{n}\left(R_{i}\right)=$ $\sum_{i} m_{n}\left(\mathbf{x}+R_{i}\right)=m_{n}(\mathbf{x}+U)$.

It remains to verify the lemma for a closed set. Let $H$ be a closed bounded set first. Then $H \subseteq B(\mathbf{0}, R)$ for some $R$ large enough. First note that $\mathbf{x}+H$ is a closed set. Thus

$$
\begin{aligned}
m_{n}(B(\mathbf{x}, R)) & =m_{n}(\mathbf{x}+H)+m_{n}((B(\mathbf{0}, R)+\mathbf{x}) \backslash(\mathbf{x}+H)) \\
& =m_{n}(\mathbf{x}+H)+m_{n}((B(\mathbf{0}, R) \backslash H)+\mathbf{x}) \\
& =m_{n}(\mathbf{x}+H)+m_{n}((B(\mathbf{0}, R) \backslash H)) \\
& =m_{n}(B(\mathbf{0}, R))-m_{n}(H)+m_{n}(\mathbf{x}+H) \\
& =m_{n}(B(\mathbf{x}, R))-m_{n}(H)+m_{n}(\mathbf{x}+H)
\end{aligned}
$$

the last equality because of the first part of the lemma which implies $m_{n}(B(\mathbf{x}, R))=$ $m_{n}(B(\mathbf{0}, R))$. Therefore, $m_{n}(\mathbf{x}+H)=m_{n}(H)$ as claimed. If $H$ is not bounded, consider $H_{m} \equiv \overline{B(\mathbf{0}, m)} \cap H$. Then $m_{n}\left(\mathbf{x}+H_{m}\right)=m_{n}\left(H_{m}\right)$. Passing to the limit as $m \rightarrow \infty$ yields the result in general.

Theorem 13.1.5 Lebesgue measure is translation invariant. That is

$$
m_{n}(E)=m_{n}(\mathbf{x}+E)
$$

for all E Lebesgue measurable.
Proof: Suppose $m_{n}(E)<\infty$. By regularity of the measure, there exist sets $G, H$ such that $G$ is a countable intersection of open sets, $H$ is a countable union of compact sets, $m_{n}(G \backslash H)=0$, and $G \supseteq E \supseteq H$. Now $m_{n}(G)=m_{n}(G+\mathbf{x})$ and $m_{n}(H)=m_{n}(H+\mathbf{x})$ which follows from Lemma 13.1.4 applied to the sets which are either intersected to form $G$ or unioned to form $H$. Now

$$
\mathbf{x}+H \subseteq \mathbf{x}+E \subseteq \mathbf{x}+G
$$

and both $\mathbf{x}+H$ and $\mathbf{x}+G$ are measurable because they are either countable unions or countable intersections of measurable sets. Furthermore,

$$
m_{n}(\mathbf{x}+G \backslash \mathbf{x}+H)=m_{n}(\mathbf{x}+G)-m_{n}(\mathbf{x}+H)=m_{n}(G)-m_{n}(H)=0
$$

and so by completeness of the measure, $\mathbf{x}+E$ is measurable. It follows

$$
\begin{aligned}
m_{n}(E) & =m_{n}(H)=m_{n}(\mathbf{x}+H) \leq m_{n}(\mathbf{x}+E) \\
& \leq m_{n}(\mathbf{x}+G)=m_{n}(G)=m_{n}(E)
\end{aligned}
$$

If $m_{n}(E)$ is not necessarily less than $\infty$, consider $E_{m} \equiv B(\mathbf{0}, m) \cap E$. Then $m_{n}\left(E_{m}\right)=$ $m_{n}\left(E_{m}+\mathbf{x}\right)$ by the above. Letting $m \rightarrow \infty$ it follows $m_{n}(E)=m_{n}(E+\mathbf{x})$. This proves the theorem.

Corollary 13.1.6 Let $D$ be an $n \times n$ diagonal matrix and let $U$ be an open set. Then

$$
m_{n}(D U)=|\operatorname{det}(D)| m_{n}(U)
$$

Proof: If any of the diagonal entries of $D$ equals 0 there is nothing to prove because then both sides equal zero. Therefore, it can be assumed none are equal to zero. Suppose these diagonal entries are $k_{1}, \cdots, k_{n}$. From Lemma 13.1.2 there exist half open boxes, $\left\{R_{i}\right\}$ having all sides equal such that $U=\cup_{i} R_{i}$. Suppose one of these is $R_{i}=\prod_{j=1}^{n}\left(a_{j}, b_{j}\right]$, where $b_{j}-a_{j}=l_{i}$. Then $D R_{i}=\prod_{j=1}^{n} I_{j}$ where $I_{j}=\left(k_{j} a_{j}, k_{j} b_{j}\right]$ if $k_{j}>0$ and $I_{j}=\left[k_{j} b_{j}, k_{j} a_{j}\right)$ if $k_{j}<0$. Then the rectangles, $D R_{i}$ are disjoint because $D$ is one to one and their union is $D U$. Also,

$$
m_{n}\left(D R_{i}\right)=\prod_{j=1}^{n}\left|k_{j}\right| l_{i}=|\operatorname{det} D| m_{n}\left(R_{i}\right)
$$

Therefore,

$$
m_{n}(D U)=\sum_{i=1}^{\infty} m_{n}\left(D R_{i}\right)=|\operatorname{det}(D)| \sum_{i=1}^{\infty} m_{n}\left(R_{i}\right)=|\operatorname{det}(D)| m_{n}(U)
$$

and this proves the corollary.
From this the following corollary is obtained.
Corollary 13.1.7 Let $M>0$. Then $m_{n}(B(\mathbf{a}, M r))=M^{n} m_{n}(B(\mathbf{0}, r))$.
Proof: By Lemma 13.1.4 there is no loss of generality in taking $\mathbf{a}=\mathbf{0}$. Let $D$ be the diagonal matrix which has $M$ in every entry of the main diagonal so $|\operatorname{det}(D)|=M^{n}$. Note that $D B(\mathbf{0}, r)=B(\mathbf{0}, M r)$. By Corollary 13.1.6 $m_{n}(B(\mathbf{0}, M r))=m_{n}(D B(\mathbf{0}, r))=$ $M^{n} m_{n}(B(\mathbf{0}, r))$.

There are many norms on $\mathbb{R}^{n}$. Other common examples are

$$
\|\mathbf{x}\|_{\infty} \equiv \max \left\{\left|x_{k}\right|: \mathbf{x}=\left(x_{1}, \cdots, x_{n}\right)\right\}
$$

or

$$
\|\mathbf{x}\|_{p} \equiv\left(\sum_{i=1}^{n}\left|x_{i}\right|^{p}\right)^{1 / p}
$$

With $\|\cdot\|$ any norm for $\mathbb{R}^{n}$ you can define a corresponding ball in terms of this norm.

$$
B(\mathbf{a}, r) \equiv\left\{\mathbf{x} \in \mathbb{R}^{n} \text { such that }\|\mathbf{x}-\mathbf{a}\|<r\right\}
$$

It follows from general considerations involving metric spaces presented earlier that these balls are open sets. Therefore, Corollary 13.1.7 has an obvious generalization.

Corollary 13.1.8 Let $\|\cdot\|$ be a norm on $\mathbb{R}^{n}$. Then for $M>0$,

$$
m_{n}(B(\mathbf{a}, M r))=M^{n} m_{n}(B(\mathbf{0}, r))
$$

where these balls are defined in terms of the norm $\|\cdot\|$.

### 13.2 The Vitali Covering Theorem

The Vitali covering theorem is concerned with the situation in which a set is contained in the union of balls. You can imagine that it might be very hard to get disjoint balls from this collection of balls which would cover the given set. However, it is possible to get disjoint balls from this collection of balls which have the property that if each ball is enlarged appropriately, the resulting enlarged balls do cover the set. When this result is established, it is used to prove another form of this theorem in which the disjoint balls do not cover the set but they only miss a set of measure zero.

Recall the Hausdorff maximal principle, Theorem 2.4.2 on Page 36 which is proved to be equivalent to the axiom of choice in the appendix. For convenience, here it is:

Theorem 13.2.1 (Hausdorff Maximal Principle) Let $\mathscr{F}$ be a nonempty partially ordered set. Then there exists a maximal chain.

I will use this Hausdorff maximal principle to give a very short and elegant proof of the Vitali covering theorem. This follows the treatment in Evans and Gariepy [47] which they got from another book. I am not sure who first did it this way but it is very nice because it is so short. In the following lemma and theorem, the balls will be either open or closed and determined by some norm on $\mathbb{R}^{n}$. When pictures are drawn, I shall draw them as though the norm is the usual norm but the results are unchanged for any norm. Also, I will write (in this section only) $B(\mathbf{a}, r)$ to indicate a set which satisfies

$$
\left\{\mathbf{x} \in \mathbb{R}^{n}:\|\mathbf{x}-\mathbf{a}\|<r\right\} \subseteq B(\mathbf{a}, r) \subseteq\left\{\mathbf{x} \in \mathbb{R}^{n}:\|\mathbf{x}-\mathbf{a}\| \leq r\right\}
$$

and $\widehat{B}(\mathbf{a}, r)$ to indicate the usual ball but with radius 5 times as large,

$$
\left\{\mathbf{x} \in \mathbb{R}^{n}:\|\mathbf{x}-\mathbf{a}\|<5 r\right\}
$$

Lemma 13.2.2 Let $\|\cdot\|$ be a norm on $\mathbb{R}^{n}$ and let $\mathscr{F}$ be a collection of balls determined by this norm. Suppose

$$
\infty>M \equiv \sup \{r: B(\mathbf{p}, r) \in \mathscr{F}\}>0
$$

and $k \in(0, \infty)$. Then there exists $\mathscr{G} \subseteq \mathscr{F}$ such that

$$
\begin{gather*}
\text { if } B(\mathbf{p}, r) \in \mathscr{G} \text { then } r>k,  \tag{13.2.1}\\
\text { if } B_{1}, B_{2} \in \mathscr{G} \text { then } B_{1} \cap B_{2}=\emptyset \tag{13.2.2}
\end{gather*}
$$

$\mathscr{G}$ is maximal with respect to 13.2.1 and 13.2.2.
Note that if there is no ball of $\mathscr{F}$ which has radius larger than $k$ then $\mathscr{G}=\emptyset$.
Proof: Let $\mathscr{H}=\{\mathscr{B} \subseteq \mathscr{F}$ such that 13.2.1 and 13.2.2 hold $\}$. If there are no balls with radius larger than $k$ then $\overline{\mathscr{H}}=\emptyset$ and you let $\mathscr{G}=\emptyset$. In the other case, $\mathscr{H} \neq \emptyset$ because there exists $B(\mathbf{p}, r) \in \mathscr{F}$ with $r>k$. In this case, partially order $\mathscr{H}$ by set inclusion and use the Hausdorff maximal principle (see the appendix on set theory) to let $\mathscr{C}$ be a maximal chain in $\mathscr{H}$. Clearly $\cup \mathscr{C}$ satisfies 13.2 .1 and 13.2.2 because if $B_{1}$ and $B_{2}$ are two balls from $\cup \mathscr{C}$ then since $\mathscr{C}$ is a chain, it follows there is some element of $\mathscr{C}, \mathscr{B}$ such that both $B_{1}$ and $B_{2}$
are elements of $\mathscr{B}$ and $\mathscr{B}$ satisfies 13.2.1 and 13.2.2. If $\cup \mathscr{C}$ is not maximal with respect to these two properties, then $\mathscr{C}$ was not a maximal chain because then there would exist $\mathscr{B} \supsetneq \cup \mathscr{C}$, that is, $\mathscr{B}$ contains $\mathscr{C}$ as a proper subset and $\{\mathscr{C}, \mathscr{B}\}$ would be a strictly larger chain in $\mathscr{H}$. Let $\mathscr{G}=\cup \mathscr{C}$.

Theorem 13.2.3 (Vitali) Let $\mathscr{F}$ be a collection of balls and let

$$
A \equiv \cup\{B: B \in \mathscr{F}\}
$$

Suppose

$$
\infty>M \equiv \sup \{r: B(\mathbf{p}, r) \in \mathscr{F}\}>0
$$

Then there exists $\mathscr{G} \subseteq \mathscr{F}$ such that $\mathscr{G}$ consists of disjoint balls and

$$
A \subseteq \cup\{\widehat{B}: B \in \mathscr{G}\}
$$

Proof: Using Lemma 13.2.2, there exists $\mathscr{G}_{1} \subseteq \mathscr{F} \equiv \mathscr{F}_{0}$ which satisfies

$$
\begin{gather*}
B(\mathbf{p}, r) \in \mathscr{G}_{1} \text { implies } r>\frac{M}{2}  \tag{13.2.3}\\
B_{1}, B_{2} \in \mathscr{G}_{1} \text { implies } B_{1} \cap B_{2}=\emptyset \tag{13.2.4}
\end{gather*}
$$

$\mathscr{G}_{1}$ is maximal with respect to 13.2.3, and 13.2.4.
Suppose $\mathscr{G}_{1}, \cdots, \mathscr{G}_{m}$ have been chosen, $m \geq 1$. Let

$$
\mathscr{F}_{m} \equiv\left\{B \in \mathscr{F}: B \subseteq \mathbb{R}^{n} \backslash \cup\left\{\mathscr{G}_{1} \cup \cdots \cup \mathscr{G}_{m}\right\}\right\}
$$

Using Lemma 13.2.2, there exists $\mathscr{G}_{m+1} \subseteq \mathscr{F}_{m}$ such that

$$
\begin{gather*}
B(\mathbf{p}, r) \in \mathscr{G}_{m+1} \text { implies } r>\frac{M}{2^{m+1}}  \tag{13.2.5}\\
B_{1}, B_{2} \in \mathscr{G}_{m+1} \text { implies } B_{1} \cap B_{2}=\emptyset \tag{13.2.6}
\end{gather*}
$$

$$
\mathscr{G}_{m+1} \text { is a maximal subset of } \mathscr{F}_{m} \text { with respect to 13.2.5 and 13.2.6. }
$$

Note it might be the case that $\mathscr{G}_{m+1}=\emptyset$ which happens if $\mathscr{F}_{m}=\emptyset$. Define

$$
\mathscr{G} \equiv \cup_{k=1}^{\infty} \mathscr{G}_{k}
$$

Thus $\mathscr{G}$ is a collection of disjoint balls in $\mathscr{F}$. I must show $\{\widehat{B}: B \in \mathscr{G}\}$ covers $A$.
Let $\mathbf{x} \in B(\mathbf{p}, r) \in \mathscr{F}$ and let

$$
\frac{M}{2^{m}}<r \leq \frac{M}{2^{m-1}}
$$

Then $B(\mathbf{p}, r)$ must intersect some set, $B\left(\mathbf{p}_{0}, r_{0}\right) \in \mathscr{G}_{1} \cup \cdots \cup \mathscr{G}_{m}$ since otherwise, $\mathscr{G}_{m}$ would fail to be maximal. Then $r_{0}>\frac{M}{2^{m}}$ because all balls in $\mathscr{G}_{1} \cup \cdots \cup \mathscr{G}_{m}$ satisfy this inequality.


Then for $\mathbf{x} \in B(\mathbf{p}, r)$, the following chain of inequalities holds because $r \leq \frac{M}{2^{m-1}}$ and $r_{0}>\frac{M}{2^{m}}$

$$
\begin{aligned}
\left|\mathbf{x}-\mathbf{p}_{0}\right| & \leq|\mathbf{x}-\mathbf{p}|+\left|\mathbf{p}-\mathbf{p}_{0}\right| \leq r+r_{0}+r \\
& \leq \frac{2 M}{2^{m-1}}+r_{0}=\frac{4 M}{2^{m}}+r_{0}<5 r_{0}
\end{aligned}
$$

Thus $B(\mathbf{p}, r) \subseteq \widehat{B}\left(\mathbf{p}_{0}, r_{0}\right)$ and this proves the theorem.

### 13.3 The Vitali Covering Theorem (Elementary Version)

The proof given here is from Basic Analysis [83]. It first considers the case of open balls and then generalizes to balls which may be neither open nor closed or closed.

Lemma 13.3.1 Let $\mathscr{F}$ be a countable collection of balls satisfying

$$
\infty>M \equiv \sup \{r: B(\mathbf{p}, r) \in \mathscr{F}\}>0
$$

and let $k \in(0, \infty)$. Then there exists $\mathscr{G} \subseteq \mathscr{F}$ such that

$$
\begin{gather*}
\text { If } B(\mathbf{p}, r) \in \mathscr{G} \text { then } r>k,  \tag{13.3.7}\\
\text { If } B_{1}, B_{2} \in \mathscr{G} \text { then } B_{1} \cap B_{2}=\emptyset, \tag{13.3.8}
\end{gather*}
$$

$$
\begin{equation*}
\mathscr{G} \text { is maximal with respect to } 13.3 .7 \text { and 13.3.8. } \tag{13.3.9}
\end{equation*}
$$

Proof: If no ball of $\mathscr{F}$ has radius larger than $k$, let $\mathscr{G}=\emptyset$. Assume therefore, that some balls have radius larger than $k$. Let $\mathscr{F} \equiv\left\{B_{i}\right\}_{i=1}^{\infty}$. Now let $B_{n_{1}}$ be the first ball in the list which has radius greater than $k$. If every ball having radius larger than $k$ intersects this one, then stop. The maximal set is just $B_{n_{1}}$. Otherwise, let $B_{n_{2}}$ be the next ball having radius larger than $k$ which is disjoint from $B_{n_{1}}$. Continue this way obtaining $\left\{B_{n_{i}}\right\}_{i=1}^{\infty}$, a finite or infinite sequence of disjoint balls having radius larger than $k$. Then let $\mathscr{G} \equiv\left\{B_{n_{i}}\right\}$. To see that $\mathscr{G}$ is maximal with respect to 13.3 .7 and 13.3.8, suppose $B \in \mathscr{F}, B$ has radius larger than $k$, and $\mathscr{G} \cup\{B\}$ satisfies 13.3.7 and 13.3.8. Then at some point in the process, $B$ would have been chosen because it would be the ball of radius larger than $k$ which has the smallest index. Therefore, $B \in \mathscr{G}$ and this shows $\mathscr{G}$ is maximal with respect to 13.3.7 and 13.3.8.

For the next lemma, for an open ball, $B=B(\mathbf{x}, r)$, denote by $\widetilde{B}$ the open ball, $B(\mathbf{x}, 4 r)$.

Lemma 13.3.2 Let $\mathscr{F}$ be a collection of open balls, and let

$$
A \equiv \cup\{B: B \in \mathscr{F}\}
$$

Suppose

$$
\infty>M \equiv \sup \{r: B(\mathbf{p}, r) \in \mathscr{F}\}>0
$$

Then there exists $\mathscr{G} \subseteq \mathscr{F}$ such that $\mathscr{G}$ consists of disjoint balls and

$$
A \subseteq \cup\{\widetilde{B}: B \in \mathscr{G}\}
$$

Proof: Without loss of generality assume $\mathscr{F}$ is countable. This is because there is a countable subset of $\mathscr{F}, \mathscr{F}^{\prime}$ such that $\cup \mathscr{F}^{\prime}=A$. To see this, consider the set of balls having rational radii and centers having all components rational. This is a countable set of balls and you should verify that every open set is the union of balls of this form. Therefore, you can consider the subset of this set of balls consisting of those which are contained in some open set of $\mathscr{F}, G$ so $\cup G=A$ and use the axiom of choice to define a subset of $\mathscr{F}$ consisting of a single set from $\mathscr{F}$ containing each set of $G$. Then this is $\mathscr{F}^{\prime}$. The union of these sets equals $A$. Then consider $\mathscr{F}^{\prime}$ instead of $\mathscr{F}$. Therefore, assume at the outset $\mathscr{F}$ is countable. By Lemma 13.3.1, there exists $\mathscr{G}_{1} \subseteq \mathscr{F}$ which satisfies 13.3.7, 13.3.8, and 13.3.9 with $k=\frac{2 M}{3}$.

Suppose $\mathscr{G}_{1}, \cdots, \mathscr{G}_{m-1}$ have been chosen for $m \geq 2$. Let

$$
\mathscr{F}_{m}=\{B \in \mathscr{F}: B \subseteq \mathbb{R}^{n} \backslash \overbrace{\cup\left\{\mathscr{G}_{1} \cup \cdots \cup \mathscr{G}_{m-1}\right\}}^{\text {union of the balls in these } \mathscr{G}_{j}}\}
$$

and using Lemma 13.3.1, let $\mathscr{G}_{m}$ be a maximal collection of disjoint balls from $\mathscr{F}_{m}$ with the property that each ball has radius larger than $\left(\frac{2}{3}\right)^{m} M$. Let $\mathscr{G} \equiv \cup_{k=1}^{\infty} \mathscr{G}_{k}$. Let $\mathbf{x} \in B(\mathbf{p}, r) \in$ $\mathscr{F}$. Choose $m$ such that

$$
\left(\frac{2}{3}\right)^{m} M<r \leq\left(\frac{2}{3}\right)^{m-1} M
$$

Then $B(\mathbf{p}, r)$ must have nonempty intersection with some ball from $\mathscr{G}_{1} \cup \cdots \cup \mathscr{G}_{m}$ because if it didn't, then $\mathscr{G}_{m}$ would fail to be maximal. Denote by $B\left(\mathbf{p}_{0}, r_{0}\right)$ a ball in $\mathscr{G}_{1} \cup \cdots \cup \mathscr{G}_{m}$ which has nonempty intersection with $B(\mathbf{p}, r)$. Thus

$$
r_{0}>\left(\frac{2}{3}\right)^{m} M
$$

Consider the picture, in which $\mathbf{w} \in B\left(\mathbf{p}_{0}, r_{0}\right) \cap B(\mathbf{p}, r)$.


Then

$$
\begin{aligned}
\left|\mathbf{x}-\mathbf{p}_{0}\right| & \leq|\mathbf{x}-\mathbf{p}|+|\mathbf{p}-\mathbf{w}|+\overbrace{\left|\mathbf{w}-\mathbf{p}_{0}\right|}^{<r_{0}} \\
& <r+r+r_{0} \leq 2 \overbrace{\left(\frac{2}{3}\right)^{m-1}}^{<\frac{3}{2} r_{0}} M+r_{0} \\
& <2\left(\frac{3}{2}\right) r_{0}+r_{0}=4 r_{0} .
\end{aligned}
$$

This proves the lemma since it shows $B(\mathbf{p}, r) \subseteq B\left(\mathbf{p}_{0}, 4 r_{0}\right)$.
With this Lemma consider a version of the Vitali covering theorem in which the balls do not have to be open. A ball centered at $\mathbf{x}$ of radius $r$ will denote something which contains the open ball, $B(\mathbf{x}, r)$ and is contained in the closed ball, $\overline{B(\mathbf{x}, r)}$. Thus the balls could be open or they could contain some but not all of their boundary points.

Definition 13.3.3 Let $B$ be a ball centered at $\mathbf{x}$ having radius $r$. Denote by $\widehat{B}$ the open ball, $B(\mathbf{x}, 5 r)$.

Theorem 13.3.4 (Vitali) Let $\mathscr{F}$ be a collection of balls, and let

$$
A \equiv \cup\{B: B \in \mathscr{F}\}
$$

Suppose

$$
\infty>M \equiv \sup \{r: B(\mathbf{p}, r) \in \mathscr{F}\}>0
$$

Then there exists $\mathscr{G} \subseteq \mathscr{F}$ such that $\mathscr{G}$ consists of disjoint balls and

$$
A \subseteq \cup\{\widehat{B}: B \in \mathscr{G}\}
$$

Proof: For $B$ one of these balls, say $\overline{B(\mathbf{x}, r)} \supseteq B \supseteq B(\mathbf{x}, r)$, denote by $B_{1}$, the ball $B\left(\mathbf{x}, \frac{5 r}{4}\right)$. Let $\mathscr{F}_{1} \equiv\left\{B_{1}: B \in \mathscr{F}\right\}$ and let $A_{1}$ denote the union of the balls in $\mathscr{F}_{1}$. Apply Lemma 13.3.2 to $\mathscr{F}_{1}$ to obtain

$$
A_{1} \subseteq \cup\left\{\widetilde{B_{1}}: B_{1} \in \mathscr{G}_{1}\right\}
$$

where $\mathscr{G}_{1}$ consists of disjoint balls from $\mathscr{F}_{1}$. Now let $\mathscr{G} \equiv\left\{B \in \mathscr{F}: B_{1} \in \mathscr{G}_{1}\right\}$. Thus $\mathscr{G}$ consists of disjoint balls from $\mathscr{F}$ because they are contained in the disjoint open balls, $\mathscr{G}_{1}$. Then

$$
A \subseteq A_{1} \subseteq \cup\left\{\widetilde{B_{1}}: B_{1} \in \mathscr{G}_{1}\right\}=\cup\{\widehat{B}: B \in \mathscr{G}\}
$$

because for $B_{1}=B\left(\mathbf{x}, \frac{5 r}{4}\right)$, it follows $\widetilde{B_{1}}=B(\mathbf{x}, 5 r)=\widehat{B}$. This proves the theorem.

### 13.4 Vitali Coverings

There is another version of the Vitali covering theorem which is also of great importance. In this one, balls from the original set of balls almost cover the set,leaving out only a set of measure zero. It is like packing a truck with stuff. You keep trying to fill in the holes with smaller and smaller things so as to not waste space. It is remarkable that you can avoid wasting any space at all when you are dealing with balls of any sort provided you can use arbitrarily small balls.

Definition 13.4.1 Let $\mathscr{F}$ be a collection of balls that cover a set $E$, which have the property that if $\mathbf{x} \in E$ and $\varepsilon>0$, then there exists $B \in \mathscr{F}$, diameter of $B<\varepsilon$ and $\mathbf{x} \in B$. Such a collection covers $E$ in the sense of Vitali.

In the following covering theorem, $\overline{m_{n}}$ denotes the outer measure determined by $n$ dimensional Lebesgue measure.

Theorem 13.4.2 Let $E \subseteq \mathbb{R}^{n}$ and suppose $0<\overline{m_{n}}(E)<\infty$ where $\overline{m_{n}}$ is the outer measure determined by $m_{n}, n$ dimensional Lebesgue measure, and let $\mathscr{F}$ be a collection of closed balls of bounded radii such that $\mathscr{F}$ covers $E$ in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathscr{F},\left\{B_{j}\right\}_{j=1}^{\infty}$, such that $\overline{m_{n}}\left(E \backslash \cup_{j=1}^{\infty} B_{j}\right)=0$.

Proof: From the definition of outer measure there exists a Lebesgue measurable set, $E_{1} \supseteq E$ such that $m_{n}\left(E_{1}\right)=\overline{m_{n}}(E)$. Now by outer regularity of Lebesgue measure, there exists $U$, an open set which satisfies

$$
m_{n}\left(E_{1}\right)>\left(1-10^{-n}\right) m_{n}(U), U \supseteq E_{1}
$$



Each point of $E$ is contained in balls of $\mathscr{F}$ of arbitrarily small radii and so there exists a covering of $E$ with balls of $\mathscr{F}$ which are themselves contained in $U$. Therefore, by the Vitali covering theorem, there exist disjoint balls, $\left\{B_{i}\right\}_{i=1}^{\infty} \subseteq \mathscr{F}$ such that

$$
E \subseteq \cup_{j=1}^{\infty} \widehat{B}_{j}, B_{j} \subseteq U
$$

Therefore,

$$
\begin{aligned}
m_{n}\left(E_{1}\right) & =\overline{m_{n}}(E) \leq m_{n}\left(\cup_{j=1}^{\infty} \widehat{B}_{j}\right) \leq \sum_{j} m_{n}\left(\widehat{B}_{j}\right) \\
& =5^{n} \sum_{j} m_{n}\left(B_{j}\right)=5^{n} m_{n}\left(\cup_{j=1}^{\infty} B_{j}\right)
\end{aligned}
$$

Then $E_{1}$ and $\cup_{j=1}^{\infty} B_{j}$ are contained in $U$ and so

$$
\begin{gathered}
m_{n}\left(E_{1}\right)>\left(1-10^{-n}\right) m_{n}(U) \\
\geq\left(1-10^{-n}\right)\left[m_{n}\left(E_{1} \backslash \cup_{j=1}^{\infty} B_{j}\right)+m_{n}\left(\cup_{j=1}^{\infty} B_{j}\right)\right] \\
\geq\left(1-10^{-n}\right)[m_{n}\left(E_{1} \backslash \cup_{j=1}^{\infty} B_{j}\right)+5^{-n} \overbrace{\overline{m_{n}}(E)}^{=m_{n}\left(E_{1}\right)}] .
\end{gathered}
$$

and so

$$
\left(1-\left(1-10^{-n}\right) 5^{-n}\right) m_{n}\left(E_{1}\right) \geq\left(1-10^{-n}\right) m_{n}\left(E_{1} \backslash \cup_{j=1}^{\infty} B_{j}\right)
$$

which implies

$$
m_{n}\left(E_{1} \backslash \cup_{j=1}^{\infty} B_{j}\right) \leq \frac{\left(1-\left(1-10^{-n}\right) 5^{-n}\right)}{\left(1-10^{-n}\right)} m_{n}\left(E_{1}\right)
$$

Now a short computation shows

$$
0<\frac{\left(1-\left(1-10^{-n}\right) 5^{-n}\right)}{\left(1-10^{-n}\right)}<1
$$

Hence, denoting by $\theta_{n}$ a number such that

$$
\begin{gathered}
\frac{\left(1-\left(1-10^{-n}\right) 5^{-n}\right)}{\left(1-10^{-n}\right)}<\theta_{n}<1, \\
\overline{m_{n}}\left(E \backslash \cup_{j=1}^{\infty} B_{j}\right) \leq m_{n}\left(E_{1} \backslash \cup_{j=1}^{\infty} B_{j}\right)<\theta_{n} m_{n}\left(E_{1}\right)=\theta_{n} \overline{m_{n}}(E)
\end{gathered}
$$

Now using Theorem 11.1.5 on Page 224 there exists $N_{1}$ large enough that

$$
\begin{equation*}
\theta_{n} \overline{m_{n}}(E) \geq m_{n}\left(E_{1} \backslash \cup_{j=1}^{N_{1}} B_{j}\right) \geq \overline{m_{n}}\left(E \backslash \cup_{j=1}^{N_{1}} B_{j}\right) \tag{13.4.10}
\end{equation*}
$$

Let $\mathscr{F}_{1}=\left\{B \in \mathscr{F}: B_{j} \cap B=\emptyset, j=1, \cdots, N_{1}\right\}$. If $E \backslash \cup_{j=1}^{N_{1}} B_{j}=\emptyset$, then $\mathscr{F}_{1}=\emptyset$ and

$$
\overline{m_{n}}\left(E \backslash \cup_{j=1}^{N_{1}} B_{j}\right)=0
$$

Therefore, in this case let $B_{k}=\emptyset$ for all $k>N_{1}$. Consider the case where

$$
E \backslash \cup_{j=1}^{N_{1}} B_{j} \neq \emptyset
$$

In this case, since the balls are closed and $\mathscr{F}$ is a Vitali cover, $\mathscr{F} 1 \neq \emptyset$ and covers $E \backslash \cup_{j=1}^{N_{1}} B_{j}$ in the sense of Vitali. Repeat the same argument, letting $E \backslash \cup_{j=1}^{N_{1}} B_{j}$ play the role of $E$. (You pick a different $E_{1}$ whose measure equals the outer measure of $E \backslash \cup_{j=1}^{N_{1}} B_{j}$ and proceed as before.) Then choosing $B_{j}$ for $j=N_{1}+1, \cdots, N_{2}$ as in the above argument,

$$
\theta_{n} \overline{m_{n}}\left(E \backslash \cup_{j=1}^{N_{1}} B_{j}\right) \geq \overline{m_{n}}\left(E \backslash \cup_{j=1}^{N_{2}} B_{j}\right)
$$

and so from 13.4.10,

$$
\theta_{n}^{2} \overline{m_{n}}(E) \geq \overline{m_{n}}\left(E \backslash \cup_{j=1}^{N_{2}} B_{j}\right)
$$

Continuing this way

$$
\theta_{n}^{k} \overline{m_{n}}(E) \geq \overline{m_{n}}\left(E \backslash \cup_{j=1}^{N_{k}} B_{j}\right)
$$

If it is ever the case that $E \backslash \cup_{j=1}^{N_{k}} B_{j}=\emptyset$, then as in the above argument,

$$
\overline{m_{n}}\left(E \backslash \cup_{j=1}^{N_{k}} B_{j}\right)=0
$$

Otherwise, the process continues and

$$
\overline{m_{n}}\left(E \backslash \cup_{j=1}^{\infty} B_{j}\right) \leq \overline{m_{n}}\left(E \backslash \cup_{j=1}^{N_{k}} B_{j}\right) \leq \theta_{n}^{k} \overline{m_{n}}(E)
$$

for every $k \in \mathbb{N}$. Therefore, the conclusion holds in this case also.
There is an obvious corollary which removes the assumption that $0<\overline{m_{n}}(E)$.
Corollary 13.4.3 Let $E \subseteq \mathbb{R}^{n}$ and suppose $\overline{m_{n}}(E)<\infty$ where $\overline{m_{n}}$ is the outer measure determined by $m_{n}$, $n$ dimensional Lebesgue measure, and let $\mathscr{F}$, be a collection of closed balls of bounded radii such that $\mathscr{F}$ covers $E$ in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathscr{F},\left\{B_{j}\right\}_{j=1}^{\infty}$, such that $\overline{m_{n}}\left(E \backslash \cup_{j=1}^{\infty} B_{j}\right)=0$.

Proof: If $0=\overline{m_{n}}(E)$ you simply pick any ball from $\mathscr{F}$ for your collection of disjoint balls.

It is also not hard to remove the assumption that $\overline{m_{n}}(E)<\infty$.
Corollary 13.4.4 Let $E \subseteq \mathbb{R}^{n}$ and let $\mathscr{F}$, be a collection of closed balls of bounded radii such that $\mathscr{F}$ covers $E$ in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathscr{F},\left\{B_{j}\right\}_{j=1}^{\infty}$, such that $\overline{m_{n}}\left(E \backslash \cup_{j=1}^{\infty} B_{j}\right)=0$.

Proof: Let $R_{m} \equiv(-m, m)^{n}$ be the open rectangle having sides of length $2 m$ which is centered at $\mathbf{0}$ and let $R_{0}=\emptyset$. Let $H_{m} \equiv \overline{R_{m}} \backslash R_{m}$. Since both $\overline{R_{m}}$ and $R_{m}$ have the same measure, $(2 m)^{n}$, it follows $m_{n}\left(H_{m}\right)=0$. Now for all $k \in \mathbb{N}, R_{k} \subseteq \overline{R_{k}} \subseteq R_{k+1}$. Consider the disjoint open sets, $U_{k} \equiv R_{k+1} \backslash \overline{R_{k}}$. Thus $\mathbb{R}^{n}=\cup_{k=0}^{\infty} U_{k} \cup N$ where $N$ is a set of measure zero equal to the union of the $H_{k}$. Let $\mathscr{F}_{k}$ denote those balls of $\mathscr{F}$ which are contained in $U_{k}$ and let $E_{k} \equiv U_{k} \cap E$. Then from Theorem 13.4.2, there exists a sequence of disjoint balls, $D_{k} \equiv\left\{B_{i}^{k}\right\}_{i=1}^{\infty}$ of $\mathscr{F}_{k}$ such that $\overline{m_{n}}\left(E_{k} \backslash \cup_{j=1}^{\infty} B_{j}^{k}\right)=0$. Letting $\left\{B_{i}\right\}_{i=1}^{\infty}$ be an enumeration of all the balls of $\cup_{k} D_{k}$, it follows that

$$
\overline{m_{n}}\left(E \backslash \cup_{j=1}^{\infty} B_{j}\right) \leq m_{n}(N)+\sum_{k=1}^{\infty} \overline{m_{n}}\left(E_{k} \backslash \cup_{j=1}^{\infty} B_{j}^{k}\right)=0
$$

Also, you don't have to assume the balls are closed.
Corollary 13.4.5 Let $E \subseteq \mathbb{R}^{n}$ and let $\mathscr{F}$, be a collection of open balls of bounded radii such that $\mathscr{F}$ covers $E$ in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathscr{F},\left\{B_{j}\right\}_{j=1}^{\infty}$, such that $\overline{m_{n}}\left(E \backslash \cup_{j=1}^{\infty} B_{j}\right)=0$.

Proof: Let $\overline{\mathscr{F}}$ be the collection of closures of balls in $\mathscr{F}$. Then $\overline{\mathscr{F}}$ covers $E$ in the sense of Vitali and so from Corollary 13.4.4 there exists a sequence of disjoint closed balls from $\overline{\mathscr{F}}$ satisfying $\overline{m_{n}}\left(E \backslash \cup_{i=1}^{\infty} \overline{B_{i}}\right)=0$. Now boundaries of the balls, $B_{i}$ have measure zero and so $\left\{B_{i}\right\}$ is a sequence of disjoint open balls satisfying $\overline{m_{n}}\left(E \backslash \cup_{i=1}^{\infty} B_{i}\right)=0$. The reason for this is that

$$
\left(E \backslash \cup_{i=1}^{\infty} B_{i}\right) \backslash\left(E \backslash \cup_{i=1}^{\infty} \overline{B_{i}}\right) \subseteq \cup_{i=1}^{\infty} \overline{B_{i}} \backslash \cup_{i=1}^{\infty} B_{i} \subseteq \cup_{i=1}^{\infty} \overline{B_{i}} \backslash B_{i},
$$

a set of measure zero. Therefore,

$$
E \backslash \cup_{i=1}^{\infty} B_{i} \subseteq\left(E \backslash \cup_{i=1}^{\infty} \overline{B_{i}}\right) \cup\left(\cup_{i=1}^{\infty} \overline{B_{i}} \backslash B_{i}\right)
$$

and so

$$
\begin{aligned}
\overline{m_{n}}\left(E \backslash \cup_{i=1}^{\infty} B_{i}\right) & \leq \overline{m_{n}}\left(E \backslash \cup_{i=1}^{\infty} \overline{B_{i}}\right)+m_{n}\left(\cup_{i=1}^{\infty} \overline{B_{i}} \backslash B_{i}\right) \\
& =\overline{m_{n}}\left(E \backslash \cup_{i=1}^{\infty} \overline{B_{i}}\right)=0 .
\end{aligned}
$$

This implies you can fill up an open set with balls which cover the open set in the sense of Vitali.

Corollary 13.4.6 Let $U \subseteq \mathbb{R}^{n}$ be an open set and let $\mathscr{F}$ be a collection of closed or even open balls of bounded radii contained in $U$ such that $\mathscr{F}$ covers $U$ in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathscr{F},\left\{B_{j}\right\}_{j=1}^{\infty}$, such that $\overline{m_{n}}(U \backslash$ $\left.\cup_{j=1}^{\infty} B_{j}\right)=0$.

### 13.5 Change of Variables for Linear Maps

To begin with certain kinds of functions map measurable sets to measurable sets. It will be assumed that $U$ is an open set in $\mathbb{R}^{n}$ and that $\mathbf{h}: U \rightarrow \mathbb{R}^{n}$ satisfies

$$
\begin{equation*}
D \mathbf{h}(\mathbf{x}) \text { exists for all } \mathbf{x} \in U \tag{13.5.11}
\end{equation*}
$$

Lemma 13.5.1 Let $\mathbf{h}$ satisfy 13.5.11. If $T \subseteq U$ and $m_{n}(T)=0$, then $m_{n}(\mathbf{h}(T))=0$.
Proof: Let

$$
T_{k} \equiv\{\mathbf{x} \in T:\|D \mathbf{h}(\mathbf{x})\|<k\}
$$

and let $\varepsilon>0$ be given. Now by outer regularity, there exists an open set, $V$, containing $T_{k}$ which is contained in $U$ such that $m_{n}(V)<\varepsilon$. Let $\mathbf{x} \in T_{k}$. Then by differentiability,

$$
\mathbf{h}(\mathbf{x}+\mathbf{v})=\mathbf{h}(\mathbf{x})+D \mathbf{h}(\mathbf{x}) \mathbf{v}+o(\mathbf{v})
$$

and so there exist arbitrarily small $r_{\mathbf{x}}<1$ such that $B\left(\mathbf{x}, 5 r_{\mathbf{x}}\right) \subseteq V$ and whenever $|\mathbf{v}| \leq$ $r_{\mathbf{x}},|o(\mathbf{v})|<k|\mathbf{v}|$. Thus

$$
\mathbf{h}\left(B\left(\mathbf{x}, r_{\mathbf{x}}\right)\right) \subseteq B\left(\mathbf{h}(\mathbf{x}), 2 k r_{\mathbf{x}}\right)
$$

From the Vitali covering theorem there exists a countable disjoint sequence of these sets, $\left\{B\left(\mathbf{x}_{i}, r_{i}\right)\right\}_{i=1}^{\infty}$ such that $\left\{B\left(\mathbf{x}_{i}, 5 r_{i}\right)\right\}_{i=1}^{\infty}=\left\{\widehat{B}_{i}\right\}_{i=1}^{\infty}$ covers $T_{k}$ Then letting $\overline{m_{n}}$ denote the outer measure determined by $m_{n}$,

$$
\overline{m_{n}}\left(\mathbf{h}\left(T_{k}\right)\right) \leq \overline{m_{n}}\left(\mathbf{h}\left(\cup_{i=1}^{\infty} \widehat{B}_{i}\right)\right)
$$

$$
\begin{aligned}
& \leq \sum_{i=1}^{\infty} \overline{m_{n}}\left(\mathbf{h}\left(\widehat{B}_{i}\right)\right) \leq \sum_{i=1}^{\infty} m_{n}\left(B\left(\mathbf{h}\left(\mathbf{x}_{i}\right), 2 k r_{\mathbf{x}_{i}}\right)\right) \\
& = \\
& \leq \sum_{i=1}^{\infty} m_{n}\left(B\left(\mathbf{x}_{i}, 2 k r_{\mathbf{x}_{i}}\right)\right)=(2 k)^{n} \sum_{i=1}^{\infty} m_{n}\left(B\left(\mathbf{x}_{i}, r_{\mathbf{x}_{i}}\right)\right) \\
& \leq(2 k)^{n} m_{n}(V) \leq(2 k)^{n} \varepsilon .
\end{aligned}
$$

Since $\varepsilon>0$ is arbitrary, this shows $m_{n}\left(\mathbf{h}\left(T_{k}\right)\right)=0$. Now

$$
m_{n}(\mathbf{h}(T))=\lim _{k \rightarrow \infty} m_{n}\left(\mathbf{h}\left(T_{k}\right)\right)=0
$$

This proves the lemma.
Lemma 13.5.2 Let $\mathbf{h}$ satisfy 13.5.11. If $S$ is a Lebesgue measurable subset of $U$, then $\mathbf{h}(S)$ is Lebesgue measurable.

Proof: Let $S_{k}=S \cap B(\mathbf{0}, k), k \in \mathbb{N}$. By inner regularity of Lebesgue measure, there exists a set, $F$, which is the countable union of compact sets and a set $T$ with $m_{n}(T)=0$ such that

$$
F \cup T=S_{k}
$$

Then $\mathbf{h}(F) \subseteq \mathbf{h}\left(S_{k}\right) \subseteq \mathbf{h}(F) \cup \mathbf{h}(T)$. By continuity of $\mathbf{h}, \mathbf{h}(F)$ is a countable union of compact sets and so it is Borel. By Lemma 13.5.1, $m_{n}(\mathbf{h}(T))=0$ and so $\mathbf{h}\left(S_{k}\right)$ is Lebesgue measurable because of completeness of Lebesgue measure. Now $\mathbf{h}(S)=\cup_{k=1}^{\infty} \mathbf{h}\left(S_{k}\right)$ and so it is also true that $\mathbf{h}(S)$ is Lebesgue measurable. This proves the lemma.

In particular, this proves the following corollary.
Corollary 13.5.3 Suppose $A$ is an $n \times n$ matrix. Then if $S$ is a Lebesgue measurable set, it follows AS is also a Lebesgue measurable set.

Lemma 13.5.4 Let $R$ be unitary $\left(R^{*} R=R R^{*}=I\right)$ and let $V$ be $a$ an open or closed set. Then $m_{n}(R V)=m_{n}(V)$.

Proof: First assume $V$ is a bounded open set. By Corollary 13.4.6 there is a disjoint sequence of closed balls, $\left\{B_{i}\right\}$ such that $V=\cup_{i=1}^{\infty} B_{i} \cup N$ where $m_{n}(N)=0$. Denote by $\mathbf{x}_{i}$ the center of $B_{i}$ and let $r_{i}$ be the radius of $B_{i}$. Then by Lemma 13.5.1 $m_{n}(R V)=\sum_{i=1}^{\infty} m_{n}\left(R B_{i}\right)$. Now by invariance of translation of Lebesgue measure, this equals $\sum_{i=1}^{\infty} m_{n}\left(R B_{i}-R \mathbf{x}_{i}\right)=$ $\sum_{i=1}^{\infty} m_{n}\left(R B\left(\mathbf{0}, r_{i}\right)\right)$. Since $R$ is unitary, it preserves all distances and so $R B\left(\mathbf{0}, r_{i}\right)=B\left(\mathbf{0}, r_{i}\right)$ and therefore,

$$
m_{n}(R V)=\sum_{i=1}^{\infty} m_{n}\left(B\left(\mathbf{0}, r_{i}\right)\right)=\sum_{i=1}^{\infty} m_{n}\left(B_{i}\right)=m_{n}(V)
$$

This proves the lemma in the case that $V$ is bounded. Suppose now that $V$ is just an open set. Let $V_{k}=V \cap B(\mathbf{0}, k)$. Then $m_{n}\left(R V_{k}\right)=m_{n}\left(V_{k}\right)$. Letting $k \rightarrow \infty$, this yields the desired conclusion. This proves the lemma in the case that $V$ is open.

Suppose now that $H$ is a closed and bounded set. Let $B(\mathbf{0}, R) \supseteq H$. Then letting $B=$ $B(\mathbf{0}, R)$ for short,

$$
\begin{aligned}
m_{n}(R H) & =m_{n}(R B)-m_{n}(R(B \backslash H)) \\
& =m_{n}(B)-m_{n}(B \backslash H)=m_{n}(H) .
\end{aligned}
$$

In general, let $H_{m}=H \cap \overline{B(\mathbf{0}, m)}$. Then from what was just shown, $m_{n}\left(R H_{m}\right)=m_{n}\left(H_{m}\right)$. Now let $m \rightarrow \infty$ to get the conclusion of the lemma in general. This proves the lemma.

Lemma 13.5.5 Let $E$ be Lebesgue measurable set in $\mathbb{R}^{n}$ and let $R$ be unitary. Then $m_{n}(R E)=m_{n}(E)$.

Proof: First suppose $E$ is bounded. Then there exist sets, $G$ and $H$ such that $H \subseteq E \subseteq G$ and $H$ is the countable union of closed sets while $G$ is the countable intersection of open sets such that $m_{n}(G \backslash H)=0$. By Lemma 13.5.4 applied to these sets whose union or intersection equals $H$ or $G$ respectively, it follows

$$
m_{n}(R G)=m_{n}(G)=m_{n}(H)=m_{n}(R H) .
$$

Therefore,

$$
m_{n}(H)=m_{n}(R H) \leq m_{n}(R E) \leq m_{n}(R G)=m_{n}(G)=m_{n}(E)=m_{n}(H)
$$

In the general case, let $E_{m}=E \cap B(\mathbf{0}, m)$ and apply what was just shown and let $m \rightarrow \infty$.
Lemma 13.5.6 Let $V$ be an open or closed set in $\mathbb{R}^{n}$ and let $A$ be an $n \times n$ matrix. Then $m_{n}(A V)=|\operatorname{det}(A)| m_{n}(V)$.

Proof: Let $R U$ be the right polar decomposition (Theorem 5.9.6 on Page 94) of $A$ and let $V$ be an open set. Then from Lemma 13.5.5,

$$
m_{n}(A V)=m_{n}(R U V)=m_{n}(U V) .
$$

Now $U=Q^{*} D Q$ where $D$ is a diagonal matrix such that $|\operatorname{det}(D)|=|\operatorname{det}(A)|$ and $Q$ is unitary. Therefore,

$$
m_{n}(A V)=m_{n}\left(Q^{*} D Q V\right)=m_{n}(D Q V)
$$

Now $Q V$ is an open set and so by Corollary 13.1.6 on Page 342 and Lemma 13.5.4,

$$
m_{n}(A V)=|\operatorname{det}(D)| m_{n}(Q V)=|\operatorname{det}(D)| m_{n}(V)=|\operatorname{det}(A)| m_{n}(V)
$$

This proves the lemma in case $V$ is open.
Now let $H$ be a closed set which is also bounded. First suppose $\operatorname{det}(A)=0$. Then letting $V$ be an open set containing $H$,

$$
m_{n}(A H) \leq m_{n}(A V)=|\operatorname{det}(A)| m_{n}(V)=0
$$

which shows the desired equation is obvious in the case where $\operatorname{det}(A)=0$. Therefore, assume $A$ is one to one. Since $H$ is bounded, $H \subseteq B(\mathbf{0}, R)$ for some $R>0$. Then letting $B=B(\mathbf{0}, R)$ for short,

$$
\begin{aligned}
m_{n}(A H) & =m_{n}(A B)-m_{n}(A(B \backslash H)) \\
& =|\operatorname{det}(A)| m_{n}(B)-|\operatorname{det}(A)| m_{n}(B \backslash H)=|\operatorname{det}(A)| m_{n}(H)
\end{aligned}
$$

If $H$ is not bounded, apply the result just obtained to $H_{m} \equiv H \cap \overline{\boldsymbol{B}(\mathbf{0}, m)}$ and then let $m \rightarrow \infty$. With this preparation, the main result is the following theorem.

Theorem 13.5.7 Let $E$ be Lebesgue measurable set in $\mathbb{R}^{n}$ and let $A$ be an $n \times n$ matrix. Then $m_{n}(A E)=|\operatorname{det}(A)| m_{n}(E)$.

Proof: First suppose $E$ is bounded. Then there exist sets, $G$ and $H$ such that $H \subseteq E \subseteq G$ and $H$ is the countable union of closed sets while $G$ is the countable intersection of open sets such that $m_{n}(G \backslash H)=0$. By Lemma 13.5.6 applied to these sets whose union or intersection equals $H$ or $G$ respectively, it follows

$$
m_{n}(A G)=|\operatorname{det}(A)| m_{n}(G)=|\operatorname{det}(A)| m_{n}(H)=m_{n}(A H)
$$

Therefore,

$$
\begin{aligned}
|\operatorname{det}(A)| m_{n}(E) & =|\operatorname{det}(A)| m_{n}(H)=m_{n}(A H) \leq m_{n}(A E) \\
& \leq m_{n}(A G)=|\operatorname{det}(A)| m_{n}(G)=|\operatorname{det}(A)| m_{n}(E)
\end{aligned}
$$

In the general case, let $E_{m}=E \cap B(\mathbf{0}, m)$ and apply what was just shown and let $m \rightarrow \infty$.

### 13.6 Change Of Variables For $C^{1}$ Functions

In this section theorems are proved which generalize the above to $C^{1}$ functions. More general versions can be seen in Kuttler [83], Kuttler [84], and Rudin [113]. There is also a very different approach to this theorem given in [83]. The more general version in [83] follows [113] and both are based on the Brouwer fixed point theorem and a very clever lemma presented in Rudin [113]. The proof will be based on a sequence of easy lemmas.

Lemma 13.6.1 Let $U$ and $V$ be bounded open sets in $\mathbb{R}^{n}$ and let $\mathbf{h}, \mathbf{h}^{-1}$ be $C^{1}$ functions such that $\mathbf{h}(U)=V$. Also let $f \in C_{c}(V)$. Then

$$
\int_{V} f(\mathbf{y}) d m_{n}=\int_{U} f(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d m_{n}
$$

Proof: First note $\mathbf{h}^{-1}(\operatorname{spt}(f))$ is a closed subset of the bounded set, $U$ and so it is compact. Thus $\mathbf{x} \rightarrow f(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))|$ is bounded and continuous.

Let $\mathbf{x} \in U$. By the assumption that $\mathbf{h}$ and $\mathbf{h}^{-1}$ are $C^{1}$,

$$
\begin{aligned}
\mathbf{h}(\mathbf{x}+\mathbf{v})-\mathbf{h}(\mathbf{x}) & =D \mathbf{h}(\mathbf{x}) \mathbf{v}+\mathbf{o}(\mathbf{v}) \\
& =D \mathbf{h}(\mathbf{x})\left(\mathbf{v}+D \mathbf{h}^{-1}(\mathbf{h}(\mathbf{x})) \mathbf{o}(\mathbf{v})\right) \\
& =D \mathbf{h}(\mathbf{x})(\mathbf{v}+\mathbf{o}(\mathbf{v}))
\end{aligned}
$$

and so if $r>0$ is small enough then $B(\mathbf{x}, r)$ is contained in $U$ and

$$
\begin{gather*}
\mathbf{h}(B(\mathbf{x}, r))-\mathbf{h}(\mathbf{x})= \\
\mathbf{h}(\mathbf{x}+B(\mathbf{0}, r))-\mathbf{h}(\mathbf{x}) \subseteq D \mathbf{h}(\mathbf{x})(B(\mathbf{0},(1+\varepsilon) r)) \tag{13.6.12}
\end{gather*}
$$

Making $r$ still smaller if necessary, one can also obtain

$$
\begin{equation*}
|f(\mathbf{y})-f(\mathbf{h}(\mathbf{x}))|<\varepsilon \tag{13.6.13}
\end{equation*}
$$

for any $\mathbf{y} \in \mathbf{h}(B(\mathbf{x}, r))$ and also

$$
\begin{equation*}
\left|f\left(\mathbf{h}\left(\mathbf{x}_{1}\right)\right)\right| \operatorname{det}\left(D \mathbf{h}\left(\mathbf{x}_{1}\right)\right)|-f(\mathbf{h}(\mathbf{x}))| \operatorname{det}(D \mathbf{h}(\mathbf{x}))|\mid<\varepsilon \tag{13.6.14}
\end{equation*}
$$

whenever $\mathbf{x}_{1} \in B(\mathbf{x}, r)$. The collection of such balls is a Vitali cover of $U$. By Corollary 13.4.6 there is a sequence of disjoint closed balls $\left\{B_{i}\right\}$ such that $U=\cup_{i=1}^{\infty} B_{i} \cup N$ where $m_{n}(N)=0$. Denote by $\mathbf{x}_{i}$ the center of $B_{i}$ and $r_{i}$ the radius. Then by Lemma 13.5.1, the monotone convergence theorem, and 13.6.12-13.6.14,

$$
\begin{gathered}
\int_{V} f(\mathbf{y}) d m_{n}=\sum_{i=1}^{\infty} \int_{\mathbf{h}\left(B_{i}\right)} f(\mathbf{y}) d m_{n} \\
\leq \varepsilon m_{n}(V)+\sum_{i=1}^{\infty} \int_{\mathbf{h}\left(B_{i}\right)} f\left(\mathbf{h}\left(\mathbf{x}_{i}\right)\right) d m_{n} \\
\leq \varepsilon m_{n}(V)+\sum_{i=1}^{\infty} f\left(\mathbf{h}\left(\mathbf{x}_{i}\right)\right) m_{n}\left(\mathbf{h}\left(B_{i}\right)\right) \\
\leq \varepsilon m_{n}(V)+\sum_{i=1}^{\infty} f\left(\mathbf{h}\left(\mathbf{x}_{i}\right)\right) m_{n}\left(D \mathbf{h}\left(\mathbf{x}_{i}\right)\left(B\left(\mathbf{0},(1+\varepsilon) r_{i}\right)\right)\right) \\
=\varepsilon m_{n}(V)+(1+\varepsilon)^{n} \sum_{i=1}^{\infty} \int_{B_{i}} f\left(\mathbf{h}\left(\mathbf{x}_{i}\right)\right)\left|\operatorname{det}\left(D \mathbf{h}\left(\mathbf{x}_{i}\right)\right)\right| d m_{n} \\
\leq \varepsilon m_{n}(V)+(1+\varepsilon)^{n} \sum_{i=1}^{\infty}\left(\int_{B_{i}} f(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d m_{n}+\varepsilon m_{n}\left(B_{i}\right)\right) \\
\leq \varepsilon m_{n}(V)+(1+\varepsilon)^{n} \sum_{i=1}^{\infty} \int_{B_{i}} f(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d m_{n}+(1+\varepsilon)^{n} \varepsilon m_{n}(U) \\
=\varepsilon m_{n}(V)+(1+\varepsilon)^{n} \int_{U} f(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d m_{n}+(1+\varepsilon)^{n} \varepsilon m_{n}(U)
\end{gathered}
$$

Since $\varepsilon>0$ is arbitrary, this shows

$$
\begin{equation*}
\int_{V} f(\mathbf{y}) d m_{n} \leq \int_{U} f(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d m_{n} \tag{13.6.15}
\end{equation*}
$$

whenever $f \in C_{c}(V)$. Now $\mathbf{x} \rightarrow f(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))|$ is in $C_{c}(U)$ and so using the same argument with $U$ and $V$ switching roles and replacing $\mathbf{h}$ with $\mathbf{h}^{-1}$,

$$
\begin{aligned}
& \int_{U} f(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d m_{n} \\
\leq & \int_{V} f\left(\mathbf{h}\left(\mathbf{h}^{-1}(\mathbf{y})\right)\right)\left|\operatorname{det}\left(D \mathbf{h}\left(\mathbf{h}^{-1}(\mathbf{y})\right)\right)\right|\left|\operatorname{det}\left(D \mathbf{h}^{-1}(\mathbf{y})\right)\right| d m_{n} \\
= & \int_{V} f(\mathbf{y}) d m_{n}
\end{aligned}
$$

by the chain rule. This with 13.6 .15 proves the lemma.
The next task is to relax the assumption that $f$ is continuous.

Corollary 13.6.2 Let $U$ and $V$ be bounded open sets in $\mathbb{R}^{n}$ and let $\mathbf{h}, \mathbf{h}^{-1}$ be $C^{1}$ functions such that $\mathbf{h}(U)=V$ and $|\operatorname{det}(D \mathbf{h}(\mathbf{x}))|$ is bounded. Also let $E \subseteq V$ be measurable. Then

$$
\int_{V} \mathscr{X}_{E}(\mathbf{y}) d m_{n}=\int_{U} \mathscr{X}_{E}(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d m_{n}
$$

Proof: By regularity, there exist compact sets, $K_{k}$ and open sets $G_{k}$ such that

$$
K_{k} \subseteq E \subseteq G_{k}
$$

and $m_{n}\left(G_{k} \backslash K_{k}\right)<2^{-k}$. By Theorem 12.2.7, there exist $f_{k}$ such that $K_{k} \prec f_{k} \prec G_{k}$. Then $f_{k}(\mathbf{y}) \rightarrow \mathscr{X}_{E}(\mathbf{y})$ a.e. because if $\mathbf{y}$ is such that convergence fails, it must be the case that $\mathbf{y}$ is in $G_{k} \backslash K_{k}$ for infinitely many $k$ and $\sum_{k} m_{n}\left(G_{k} \backslash K_{k}\right)<\infty$. This set equals

$$
N=\cap_{m=1}^{\infty} \cup_{k=m}^{\infty} G_{k} \backslash K_{k}
$$

and so for each $m \in \mathbb{N}$

$$
\begin{aligned}
m_{n}(N) & \leq m_{n}\left(\cup_{k=m}^{\infty} G_{k} \backslash K_{k}\right) \\
& \leq \sum_{k=m}^{\infty} m_{n}\left(G_{k} \backslash K_{k}\right)<\sum_{k=m}^{\infty} 2^{-k}=2^{-(m-1)}
\end{aligned}
$$

showing $m_{n}(N)=0$.
Then $f_{k}(\mathbf{h}(\mathbf{x}))$ must converge to $\mathscr{X}_{E}(\mathbf{h}(\mathbf{x}))$ for all $\mathbf{x} \notin \mathbf{h}^{-1}(N)$, a set of measure zero by Lemma 13.5.1. Thus

$$
\int_{V} f_{k}(\mathbf{y}) d m_{n}=\int_{U} f_{k}(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d m_{n}
$$

By the dominated convergence theorem using a dominating function, $\mathscr{X}_{V}$ in the integral on the left and $\mathscr{X}_{U}|\operatorname{det}(D \mathbf{h})|$ on the right, it follows

$$
\int_{V} \mathscr{X}_{E}(\mathbf{y}) d m_{n}=\int_{U} \mathscr{X}_{E}(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d m_{n}
$$

This proves the corollary.
You don't need to assume the open sets are bounded.
Corollary 13.6.3 Let $U$ and $V$ be open sets in $\mathbb{R}^{n}$ and let $\mathbf{h}, \mathbf{h}^{-1}$ be $C^{1}$ functions such that $\mathbf{h}(U)=V$. Also let $E \subseteq V$ be measurable. Then

$$
\int_{V} \mathscr{X}_{E}(\mathbf{y}) d m_{n}=\int_{U} \mathscr{X}_{E}(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d m_{n}
$$

Proof: For each $\mathbf{x} \in U$, there exists $r_{\mathbf{x}}$ such that $\overline{B\left(\mathbf{x}, r_{\mathbf{x}}\right)} \subseteq U$ and $r_{\mathbf{x}}<1$. Then by the mean value inequality Theorem 6.13.4, it follows $\mathbf{h}\left(B\left(\mathbf{x}, r_{\mathbf{x}}\right)\right)$ is also bounded. This is a Vitali cover of $U$ and so by Corollary 13.4.6 there is a sequence of these balls, $\left\{B_{i}\right\}$ such that they are disjoint, $\mathbf{h}\left(B_{i}\right)$ is also bounded and

$$
m_{n}\left(U \backslash \cup_{i} B_{i}\right)=0
$$

It follows from Lemma 13.5.1 that $\mathbf{h}\left(U \backslash \cup_{i} B_{i}\right)$ also has measure zero. Then from Corollary 13.6.2

$$
\begin{aligned}
\int_{V} \mathscr{X}_{E}(\mathbf{y}) d m_{n} & =\sum_{i} \int_{\mathbf{h}\left(B_{i}\right)} \mathscr{X}_{E \cap \mathbf{h}\left(B_{i}\right)}(\mathbf{y}) d m_{n} \\
& =\sum_{i} \int_{B_{i}} \mathscr{X}_{E}(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d m_{n} \\
& =\int_{U} \mathscr{X}_{E}(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d m_{n} .
\end{aligned}
$$

This proves the corollary.
With this corollary, the main theorem follows.
Theorem 13.6.4 Let $U$ and $V$ be open sets in $\mathbb{R}^{n}$ and let $\mathbf{h}, \mathbf{h}^{-1}$ be $C^{1}$ functions such that $\mathbf{h}(U)=V$. Then if $g$ is a nonnegative Lebesgue measurable function,

$$
\begin{equation*}
\int_{V} g(\mathbf{y}) d y=\int_{U} g(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d x \tag{13.6.16}
\end{equation*}
$$

Proof: From Corollary 13.6.3, 13.6.16 holds for any nonnegative simple function in place of $g$. In general, let $\left\{s_{k}\right\}$ be an increasing sequence of simple functions which converges to $g$ pointwise. Then from the monotone convergence theorem

$$
\begin{aligned}
\int_{V} g(\mathbf{y}) d y & =\lim _{k \rightarrow \infty} \int_{V} s_{k} d y=\lim _{k \rightarrow \infty} \int_{U} s_{k}(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d x \\
& =\int_{U} g(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d x
\end{aligned}
$$

This proves the theorem.
This is a pretty good theorem but it isn't too hard to generalize it. In particular, it is not necessary to assume $\mathbf{h}^{-1}$ is $C^{1}$.

Lemma 13.6.5 (Sard) Let $U$ be an open set in $\mathbb{R}^{n}$ and let $\mathbf{h}: U \rightarrow \mathbb{R}^{n}$ be $C^{1}$. Let

$$
Z \equiv\{\mathbf{x} \in U: \operatorname{det} D \mathbf{h}(\mathbf{x})=0\}
$$

Then $m_{n}(\mathbf{h}(Z))=0$.
Proof: Let $Z_{k}$ denote those points $\mathbf{x}$ of $Z$ such that $\|D \mathbf{h}(\mathbf{x})\| \leq k$ and such that $|\mathbf{x}|<k$. Let $\varepsilon>0$ be given. For $\mathbf{x} \in Z_{k}$,

$$
\mathbf{h}(\mathbf{x}+\mathbf{v})=\mathbf{h}(\mathbf{x})+D \mathbf{h}(\mathbf{x}) \mathbf{v}+\mathbf{o}(\mathbf{v})
$$

and so whenever $r$ is small enough,

$$
\mathbf{h}(\mathbf{x}+B(\mathbf{0}, r))=\mathbf{h}(B(\mathbf{x}, r)) \subseteq \mathbf{h}(\mathbf{x})+D \mathbf{h}(\mathbf{x}) B(\mathbf{0}, r)+B(\mathbf{0}, r \varepsilon)
$$

Note $D \mathbf{h}(\mathbf{x}) B(\mathbf{0}, r)$ is contained in an $n-1$ dimensional subspace of $\mathbb{R}^{n}$ due to the fact $D \mathbf{h}(\mathbf{x})$ has rank less than $n$. Now let $Q$ denote an orthogonal transformation preserving all distances,

$$
Q Q^{*}=Q^{*} Q=I
$$

such that

$$
Q D \mathbf{h}(\mathbf{x}) B(\mathbf{0}, r) \subseteq \mathbb{R}^{n-1}
$$

Then

$$
Q \mathbf{h}(B(\mathbf{x}, r)) \subseteq Q \mathbf{h}(\mathbf{x})+Q D \mathbf{h}(\mathbf{x}) B(\mathbf{0}, r)+B(\mathbf{0}, r \varepsilon)
$$

and by translation invariance of Lebesgue measure,

$$
\begin{gathered}
m_{n}(Q \mathbf{h}(B(\mathbf{x}, r))) \leq m_{n}(Q D \mathbf{h}(\mathbf{x}) B(\mathbf{0}, r)+B(\mathbf{0}, r \varepsilon)) \\
\leq(\|Q D \mathbf{h}(\mathbf{x})\|(2 r+2 r \varepsilon))^{n-1} 2 r \varepsilon=C(1+\varepsilon)^{n-1} m_{n}(B(\mathbf{0}, r)) \varepsilon
\end{gathered}
$$

These balls give a Vitali cover of $Z_{k}$ and so there exists a disjoint sequence of them $\left\{B_{i}\right\}$, each contained in $B(\mathbf{0}, k)$ which covers $Z_{k}$ except for a set of measure zero which is mapped by $\mathbf{h}$ to a set of measure zero. Therefore using Theorem 13.5.7,

$$
\begin{gathered}
m_{n}\left(\mathbf{h}\left(Z_{k}\right)\right)=m_{n}\left(\mathbf{h}\left(\cup_{i=1}^{\infty} B_{i}\right)\right) \leq \sum_{i=1}^{\infty} m_{n}\left(\mathbf{h}\left(B_{i}\right)\right) \\
=\sum_{i=1}^{\infty} m_{n}\left(Q \mathbf{h}\left(B_{i}\right)\right) \leq C(1+\varepsilon)^{n-1} \varepsilon \sum_{i=1}^{\infty} m_{n}\left(B_{i}\right) \leq C(1+\varepsilon)^{n-1} \varepsilon m_{n}(B(\mathbf{0}, k))
\end{gathered}
$$

and since $\varepsilon$ is arbitrary, this shows $m_{n}\left(\mathbf{h}\left(Z_{k}\right)\right)=0$. Now

$$
m_{n}(\mathbf{h}(Z))=\lim _{k \rightarrow \infty} m_{n}\left(\mathbf{h}\left(Z_{k}\right)\right)=0
$$

This proves the lemma.
With this important lemma, here is a generalization of Theorem 13.6.4.
Theorem 13.6.6 Let $U$ be an open set and let $\mathbf{h}$ be a $1-1, C^{1}$ function with values in $\mathbb{R}^{n}$. Then if $g$ is a nonnegative Lebesgue measurable function,

$$
\begin{equation*}
\int_{\mathbf{h}(U)} g(\mathbf{y}) d y=\int_{U} g(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d x \tag{13.6.17}
\end{equation*}
$$

Proof: Let $Z=\{\mathbf{x}: \operatorname{det}(D \mathbf{h}(\mathbf{x}))=0\}$. Then by the inverse function theorem, $\mathbf{h}^{-1}$ is $C^{1}$ on $\mathbf{h}(U \backslash Z)$ and $\mathbf{h}(U \backslash Z)$ is an open set. Therefore, from Lemma 13.6.5 and Theorem 13.6.4,

$$
\begin{aligned}
\int_{\mathbf{h}(U)} g(\mathbf{y}) d y & =\int_{\mathbf{h}(U \backslash Z)} g(\mathbf{y}) d y=\int_{U \backslash Z} g(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d x \\
& =\int_{U} g(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d x
\end{aligned}
$$

This proves the theorem.
Of course the next generalization considers the case when $\mathbf{h}$ is not even one to one.

### 13.7 Mappings Which Are Not One To One

Now suppose $\mathbf{h}$ is only $C^{1}$, not necessarily one to one. For

$$
U_{+} \equiv\{\mathbf{x} \in U:|\operatorname{det} D \mathbf{h}(x)|>0\}
$$

and $Z$ the set where $|\operatorname{det} D \mathbf{h}(\mathbf{x})|=0$, Lemma 13.6.5 implies $m_{n}(\mathbf{h}(Z))=0$. For $\mathbf{x} \in U_{+}$, the inverse function theorem implies there exists an open set $B_{\mathbf{x}}$ such that $\mathbf{x} \in B_{\mathbf{x}} \subseteq U_{+}, \mathbf{h}$ is one to one on $B_{\mathbf{x}}$.

Let $\left\{B_{i}\right\}$ be a countable subset of $\left\{B_{\mathbf{x}}\right\}_{\mathbf{x} \in U_{+}}$such that $U_{+}=\cup_{i=1}^{\infty} B_{i}$. Let $E_{1}=B_{1}$. If $E_{1}, \cdots, E_{k}$ have been chosen, $E_{k+1}=B_{k+1} \backslash \cup_{i=1}^{k} E_{i}$. Thus

$$
\cup_{i=1}^{\infty} E_{i}=U_{+}, \mathbf{h} \text { is one to one on } E_{i}, E_{i} \cap E_{j}=\emptyset
$$

and each $E_{i}$ is a Borel set contained in the open set $B_{i}$. Now define

$$
n(\mathbf{y}) \equiv \sum_{i=1}^{\infty} \mathscr{X}_{\mathbf{h}\left(E_{i}\right)}(\mathbf{y})+\mathscr{X}_{\mathbf{h}(Z)}(\mathbf{y})
$$

The set, $\mathbf{h}\left(E_{i}\right), \mathbf{h}(Z)$ are measurable by Lemma 13.5.2. Thus $n(\cdot)$ is measurable.
Lemma 13.7.1 Let $F \subseteq \mathbf{h}(U)$ be measurable. Then

$$
\int_{\mathbf{h}(U)} n(\mathbf{y}) \mathscr{X}_{F}(\mathbf{y}) d y=\int_{U} \mathscr{X}_{F}(\mathbf{h}(\mathbf{x}))|\operatorname{det} D \mathbf{h}(\mathbf{x})| d x .
$$

Proof: Using Lemma 13.6.5 and the Monotone convergence Theorem or Fubini's Theorem,

$$
\begin{aligned}
& \int_{\mathbf{h}(U)} n(\mathbf{y}) \mathscr{X}_{F}(\mathbf{y}) d y= \int_{\mathbf{h}(U)}(\sum_{i=1}^{\infty} \mathscr{X}_{\mathbf{h}\left(E_{i}\right)}(\mathbf{y})+\overbrace{\mathscr{X}_{\mathbf{h}(Z)}(\mathbf{y})}^{m_{n}(\mathbf{h}(Z))=0}) \mathscr{X}_{F}(\mathbf{y}) d y \\
&=\sum_{i=1}^{\infty} \int_{\mathbf{h}(U)} \mathscr{X}_{\mathbf{h}\left(E_{i}\right)}(\mathbf{y}) \mathscr{X}_{F}(\mathbf{y}) d y \\
&=\sum_{i=1}^{\infty} \int_{\mathbf{h}(U) \cap \mathbf{h}\left(E_{i}\right)} \mathscr{X}_{F}(\mathbf{y}) d y \\
&=\sum_{i=1}^{\infty} \int_{\mathbf{h}\left(B_{i}\right) \cap \mathbf{h}\left(E_{i}\right)} \mathscr{X}_{F}(\mathbf{y}) d y \\
&=\sum_{i=1}^{\infty} \int_{\mathbf{h}\left(B_{i}\right)} \mathscr{X}_{\mathbf{h}\left(E_{i}\right)}(\mathbf{y}) \mathscr{X}_{F}(\mathbf{y}) d y \\
&=\sum_{i=1}^{\infty} \int_{B_{i}} \mathscr{X}_{E_{i}}(\mathbf{x}) \mathscr{X}_{F}(\mathbf{h}(\mathbf{x}))|\operatorname{det} D \mathbf{h}(\mathbf{x})| d x \\
&=\sum_{i=1}^{\infty} \int_{U} \mathscr{X}_{E_{i}}(\mathbf{x}) \mathscr{X}_{F}(\mathbf{h}(\mathbf{x}))|\operatorname{det} D \mathbf{h}(\mathbf{x})| d x \\
&= \int_{U} \sum_{i=1}^{\infty} \mathscr{X}_{E_{i}}(\mathbf{x}) \mathscr{X}_{F}(\mathbf{h}(\mathbf{x}))|\operatorname{det} D \mathbf{h}(\mathbf{x})| d x
\end{aligned}
$$

$$
=\int_{U_{+}} \mathscr{X}_{F}(\mathbf{h}(\mathbf{x}))|\operatorname{det} D \mathbf{h}(\mathbf{x})| d x=\int_{U} \mathscr{X}_{F}(\mathbf{h}(\mathbf{x}))|\operatorname{det} D \mathbf{h}(\mathbf{x})| d x .
$$

This proves the lemma.
Definition 13.7.2 For $\mathbf{y} \in \mathbf{h}(U)$, define a function, \#, according to the formula

$$
\#(\mathbf{y}) \equiv \text { number of elements in } \mathbf{h}^{-1}(\mathbf{y})
$$

Observe that

$$
\begin{equation*}
\#(\mathbf{y})=n(\mathbf{y}) \quad \text { a.e. } \tag{13.7.18}
\end{equation*}
$$

because $n(\mathbf{y})=\#(\mathbf{y})$ if $\mathbf{y} \notin \mathbf{h}(Z)$, a set of measure 0 . Therefore, \# is a measurable function.
Theorem 13.7.3 Let $g \geq 0, g$ measurable, and let $\mathbf{h}$ be $C^{1}(U)$. Then

$$
\begin{equation*}
\int_{\mathbf{h}(U)} \#(\mathbf{y}) g(\mathbf{y}) d y=\int_{U} g(\mathbf{h}(\mathbf{x}))|\operatorname{det} D \mathbf{h}(\mathbf{x})| d x \tag{13.7.19}
\end{equation*}
$$

Proof: From 13.7.18 and Lemma 13.7.1, 13.7.19 holds for all $g$, a nonnegative simple function. Approximating an arbitrary measurable nonnegative function, $g$, with an increasing pointwise convergent sequence of simple functions and using the monotone convergence theorem, yields 13.7 .19 for an arbitrary nonnegative measurable function, $g$. This proves the theorem.

### 13.8 Lebesgue Measure And Iterated Integrals

The following is the main result.
Theorem 13.8.1 Let $f \geq 0$ and suppose $f$ is a Lebesgue measurable function defined on $\mathbb{R}^{n}$ and $\int_{\mathbb{R}^{n}} f d m_{n}<\infty$. Then

$$
\int_{\mathbb{R}^{n}} f d m_{n}=\int_{\mathbb{R}^{k}} \int_{\mathbb{R}^{n-k}} f d m_{n-k} d m_{k}
$$

This will be accomplished by Fubini's theorem, Theorem 12.9.11 and the following lemma.

Lemma 13.8.2 $\overline{m_{k} \times m_{n-k}}=m_{n}$ on the $m_{n}$ measurable sets.
Proof: First of all, let $R=\prod_{i=1}^{n}\left(a_{i}, b_{i}\right]$ be a measurable rectangle and let

$$
R_{k}=\prod_{i=1}^{k}\left(a_{i}, b_{i}\right], R_{n-k}=\prod_{i=k+1}^{n}\left(a_{i}, b_{i}\right]
$$

Then by Fubini's theorem,

$$
\begin{aligned}
\int \mathscr{X}_{R} d\left(\overline{m_{k} \times m_{n-k}}\right) & =\int_{\mathbb{R}^{k}} \int_{\mathbb{R}^{n-k}} \mathscr{X}_{R_{k}} \mathscr{X}_{R_{n-k}} d m_{k} d m_{n-k} \\
& =\int_{\mathbb{R}^{k}} \mathscr{X}_{R_{k}} d m_{k} \int_{\mathbb{R}^{n-k}} \mathscr{X}_{R_{n-k}} d m_{n-k} \\
& =\int \mathscr{X}_{R} d m_{n}
\end{aligned}
$$

and so $\overline{m_{k} \times m_{n-k}}$ and $m_{n}$ agree on every half open rectangle. By Lemma 13.1.2 these two measures agree on every open set. Now if $K$ is a compact set, then $K=\cap_{k=1}^{\infty} U_{k}$ where $U_{k}$ is the open set, $K+B\left(\mathbf{0}, \frac{1}{k}\right)$. Another way of saying this is $U_{k} \equiv\left\{\mathbf{x}: \operatorname{dist}(\mathbf{x}, K)<\frac{1}{k}\right\}$ which is obviously open because $\mathbf{x} \rightarrow \operatorname{dist}(\mathbf{x}, K)$ is a continuous function. Since $K$ is the countable intersection of these decreasing open sets, each of which has finite measure with respect to either of the two measures, it follows that $\overline{m_{k} \times m_{n-k}}$ and $m_{n}$ agree on all the compact sets. Now let $E$ be a bounded Lebesgue measurable set. Then there are sets, $H$ and $G$ such that $H$ is a countable union of compact sets, $G$ a countable intersection of open sets, $H \subseteq E \subseteq G$, and $m_{n}(G \backslash H)=0$. Then from what was just shown about compact and open sets, the two measures agree on $G$ and on $H$. Therefore,

$$
\begin{aligned}
m_{n}(H) & =\overline{m_{k} \times m_{n-k}}(H) \leq \overline{m_{k} \times m_{n-k}}(E) \\
& \leq \overline{m_{k} \times m_{n-k}}(G)=m_{n}(E)=m_{n}(H)
\end{aligned}
$$

By completeness of the measure space for $\overline{m_{k} \times m_{n-k}}$, it follows $E$ is $\overline{m_{k} \times m_{n-k}}$ measurable and

$$
\overline{m_{k} \times m_{n-k}}(E)=m_{n}(E) .
$$

This proves the lemma.
You could also show that the two $\sigma$ algebras are the same. However, this is not needed for the lemma or the theorem.

Proof of Theorem 13.8.1: By the lemma and Fubini's theorem, Theorem 12.9.11,

$$
\int_{\mathbb{R}^{n}} f d m_{n}=\int_{\mathbb{R}^{n}} f d\left(\overline{m_{k} \times m_{n-k}}\right)=\int_{\mathbb{R}^{k}} \int_{\mathbb{R}^{n-k}} f d m_{n-k} d m_{k} .
$$

Corollary 13.8.3 Let $f$ be a nonnegative real valued measurable function. Then

$$
\int_{\mathbb{R}^{n}} f d m_{n}=\int_{\mathbb{R}^{k}} \int_{\mathbb{R}^{n-k}} f d m_{n-k} d m_{k}
$$

Proof: Let $S_{p} \equiv\left\{\mathbf{x} \in \mathbb{R}^{n}: 0 \leq f(\mathbf{x}) \leq p\right\} \cap B(\mathbf{0}, p)$. Then $\int_{\mathbb{R}^{n}} f \mathscr{X}_{S_{p}} d m_{n}<\infty$. Therefore, from Theorem 13.8.1,

$$
\int_{\mathbb{R}^{n}} f \mathscr{X}_{S_{p}} d m_{n}=\int_{\mathbb{R}^{k}} \int_{\mathbb{R}^{n-k}} \mathscr{X}_{S_{p}} f d m_{n-k} d m_{k}
$$

Now let $p \rightarrow \infty$ and use the Monotone convergence theorem and the Fubini Theorem 12.9.11 on Page 315.

Not surprisingly, the following corollary follows from this.

Corollary 13.8.4 Let $f \in L^{1}\left(\mathbb{R}^{n}\right)$ where the measure is $m_{n}$. Then

$$
\int_{\mathbb{R}^{n}} f d m_{n}=\int_{\mathbb{R}^{k}} \int_{\mathbb{R}^{n-k}} f d m_{n-k} d m_{k}
$$

Proof: Apply Corollary 13.8.3 to the postive and negative parts of the real and imaginary parts of $f$.

### 13.9 Spherical Coordinates In $p$ Dimensions

Sometimes there is a need to deal with spherical coordinates in more than three dimensions. In this section, this concept is defined and formulas are derived for these coordinate systems. Recall polar coordinates are of the form

$$
\begin{aligned}
& y_{1}=\rho \cos \theta \\
& y_{2}=\rho \sin \theta
\end{aligned}
$$

where $\rho>0$ and $\theta \in \mathbb{R}$. Thus these transformation equations are not one to one but they are one to one on $(0, \infty) \times[0,2 \pi)$. Here I am writing $\rho$ in place of $r$ to emphasize a pattern which is about to emerge. I will consider polar coordinates as spherical coordinates in two dimensions. I will also simply refer to such coordinate systems as polar coordinates regardless of the dimension. This is also the reason I am writing $y_{1}$ and $y_{2}$ instead of the more usual $x$ and $y$. Now consider what happens when you go to three dimensions. The situation is depicted in the following picture.


From this picture, you see that $y_{3}=\rho \cos \phi_{1}$. Also the distance between $\left(y_{1}, y_{2}\right)$ and $(0,0)$ is $\rho \sin \left(\phi_{1}\right)$. Therefore, using polar coordinates to write $\left(y_{1}, y_{2}\right)$ in terms of $\theta$ and this distance,

$$
\begin{aligned}
& y_{1}=\rho \sin \phi_{1} \cos \theta \\
& y_{2}=\rho \sin \phi_{1} \sin \theta \\
& y_{3}=\rho \cos \phi_{1} .
\end{aligned}
$$

where $\phi_{1} \in \mathbb{R}$ and the transformations are one to one if $\phi_{1}$ is restricted to be in $[0, \pi]$. What was done is to replace $\rho$ with $\rho \sin \phi_{1}$ and then to add in $y_{3}=\rho \cos \phi_{1}$. Having done this, there is no reason to stop with three dimensions. Consider the following picture:


From this picture, you see that $y_{4}=\rho \cos \phi_{2}$. Also the distance between $\left(y_{1}, y_{2}, y_{3}\right)$ and $(0,0,0)$ is $\rho \sin \left(\phi_{2}\right)$. Therefore, using polar coordinates to write $\left(y_{1}, y_{2}, y_{3}\right)$ in terms of $\theta, \phi_{1}$, and this distance,

$$
\begin{aligned}
& y_{1}=\rho \sin \phi_{2} \sin \phi_{1} \cos \theta, \\
& y_{2}=\rho \sin \phi_{2} \sin \phi_{1} \sin \theta, \\
& y_{3}=\rho \sin \phi_{2} \cos \phi_{1}, \\
& y_{4}=\rho \cos \phi_{2}
\end{aligned}
$$

where $\phi_{2} \in \mathbb{R}$ and the transformations will be one to one if

$$
\phi_{2}, \phi_{1} \in(0, \pi), \theta \in(0,2 \pi), \rho \in(0, \infty) .
$$

Continuing this way, given spherical coordinates in $\mathbb{R}^{p}$, to get the spherical coordinates in $\mathbb{R}^{p+1}$, you let $y_{p+1}=\rho \cos \phi_{p-1}$ and then replace every occurance of $\rho$ with $\rho \sin \phi_{p-1}$ to obtain $y_{1} \cdots y_{p}$ in terms of $\phi_{1}, \phi_{2}, \cdots, \phi_{p-1}, \theta$, and $\rho$.

It is always the case that $\rho$ measures the distance from the point in $\mathbb{R}^{p}$ to the origin in $\mathbb{R}^{p}, \mathbf{0}$. Each $\phi_{i} \in \mathbb{R}$ and the transformations will be one to one if each $\phi_{i} \in(0, \pi)$, and $\theta \in(0,2 \pi)$. Denote by $\mathbf{h}_{p}(\rho, \vec{\phi}, \theta)$ the above transformation.

It can be shown using math induction and geometric reasoning that these coordinates map $\prod_{i=1}^{p-2}(0, \pi) \times(0,2 \pi) \times(0, \infty)$ one to one onto an open subset of $\mathbb{R}^{p}$ which is everything except for the set of measure zero $\Psi_{p}(N)$ where $N$ results from having some $\phi_{i}$ equal to 0 or $\pi$ or for $\rho=0$ or for $\theta$ equal to either $2 \pi$ or 0 . Each of these are sets of Lebesgue measure zero and so their union is also a set of measure zero. You can see that $\mathbf{h}_{p}\left(\prod_{i=1}^{p-2}(0, \pi) \times(0,2 \pi) \times(0, \infty)\right)$ omits the union of the coordinate axes except for maybe one of them. This is not important to the integral because it is just a set of measure zero.

Theorem 13.9.1 Let $\mathbf{y}=\mathbf{h}_{p}(\vec{\phi}, \theta, \rho)$ be the spherical coordinate transformations in $\mathbb{R}^{p}$. Then letting $A=\prod_{i=1}^{p-2}(0, \pi) \times(0,2 \pi)$, it follows $\mathbf{h}$ maps $A \times(0, \infty)$ one to one onto all of $\mathbb{R}^{p}$ except a set of measure zero given by $\mathbf{h}_{p}(N)$ where $N$ is the set of measure zero

$$
(\bar{A} \times[0, \infty)) \backslash(A \times(0, \infty))
$$

Also $\left|\operatorname{det} D \mathbf{h}_{p}(\vec{\phi}, \theta, \rho)\right|$ will always be of the form

$$
\begin{equation*}
\left|\operatorname{det} D \mathbf{h}_{p}(\vec{\phi}, \theta, \rho)\right|=\rho^{p-1} \Phi(\vec{\phi}, \theta) . \tag{13.9.20}
\end{equation*}
$$

where $\Phi$ is a continuous function of $\vec{\phi}$ and $\theta .{ }^{1}$ Then if $f$ is nonnegative and Lebesgue measurable,

$$
\begin{equation*}
\int_{\mathbb{R}^{p}} f(\mathbf{y}) d m_{p}=\int_{\mathbf{h}_{p}(A)} f(\mathbf{y}) d m_{p}=\int_{A} f\left(\mathbf{h}_{p}(\vec{\phi}, \theta, \rho)\right) \rho^{p-1} \Phi(\vec{\phi}, \theta) d m_{p} \tag{13.9.21}
\end{equation*}
$$

Furthermore whenever $f$ is Borel measurable and nonnegative, one can apply Fubini's theorem and write

$$
\begin{equation*}
\int_{\mathbb{R}^{p}} f(\mathbf{y}) d y=\int_{0}^{\infty} \rho^{p-1} \int_{A} f(\mathbf{h}(\vec{\phi}, \theta, \rho)) \Phi(\vec{\phi}, \theta) d \vec{\phi} d \theta d \rho \tag{13.9.22}
\end{equation*}
$$

where here $d \vec{\phi} d \theta$ denotes $d m_{p-1}$ on $A$. The same formulas hold if $f \in L^{1}\left(\mathbb{R}^{p}\right)$.
Proof: Formula 13.9.20 is obvious from the definition of the spherical coordinates because in the matrix of the derivative, there will be a $\rho$ in $p-1$ columns. The first claim is also clear from the definition and math induction or from the geometry of the above description. It remains to verify 13.9 .21 and 13.9.22. It is clear $\mathbf{h}_{p}$ maps $\bar{A} \times[0, \infty)$ onto $\mathbb{R}^{p}$. Since $\mathbf{h}_{p}$ is differentiable, it maps sets of measure zero to sets of measure zero. Then

$$
\mathbb{R}^{p}=\mathbf{h}_{p}(N \cup A \times(0, \infty))=\mathbf{h}_{p}(N) \cup \mathbf{h}_{p}(A \times(0, \infty)),
$$

the union of a set of measure zero with $\mathbf{h}_{p}(A \times(0, \infty))$. Therefore, from the change of variables formula,

$$
\begin{aligned}
\int_{\mathbb{R}^{p}} f(\mathbf{y}) d m_{p} & =\int_{\mathbf{h}_{p}(A \times(0, \infty))} f(\mathbf{y}) d m_{p} \\
& =\int_{A \times(0, \infty)} f\left(\mathbf{h}_{p}(\vec{\phi}, \theta, \rho)\right) \rho^{p-1} \Phi(\vec{\phi}, \theta) d m_{p}
\end{aligned}
$$

which proves 13.9.21. This formula continues to hold if $f$ is in $L^{1}\left(\mathbb{R}^{p}\right)$. Finally, if $f \geq 0$ or in $L^{1}\left(\mathbb{R}^{n}\right)$ and is Borel measurable, then it is $\mathscr{F}^{p}$ measurable as well. Recall that $\mathscr{F} p$ includes the smallest $\sigma$ algebra which contains products of open intervals. Hence $\mathscr{F} p$ includes the Borel sets $\mathscr{B}\left(\mathbb{R}^{p}\right)$. Thus from the definition of $m_{p}$

$$
\begin{aligned}
& \int_{A \times(0, \infty)} f\left(\mathbf{h}_{p}(\vec{\phi}, \theta, \rho)\right) \rho^{p-1} \Phi(\vec{\phi}, \theta) d m_{p} \\
= & \int_{(0, \infty)} \int_{A} f\left(\mathbf{h}_{p}(\vec{\phi}, \theta, \rho)\right) \rho^{p-1} \Phi(\vec{\phi}, \theta) d m_{p-1} d m \\
= & \int_{(0, \infty)} \rho^{p-1} \int_{A} f\left(\mathbf{h}_{p}(\vec{\phi}, \theta, \rho)\right) \Phi(\vec{\phi}, \theta) d m_{p-1} d m
\end{aligned}
$$

Now the claim about $f \in L^{1}$ follows routinely from considering the positive and negative parts of the real and imaginary parts of $f$ in the usual way.

[^13]Note that the above equals

$$
\int_{\bar{A} \times[0, \infty)} f\left(\mathbf{h}_{p}(\vec{\phi}, \theta, \rho)\right) \rho^{p-1} \Phi(\vec{\phi}, \theta) d m_{p}
$$

and the iterated integral is also equal to

$$
\int_{[0, \infty)} \rho^{p-1} \int_{\vec{A}} f\left(\mathbf{h}_{p}(\vec{\phi}, \theta, \rho)\right) \Phi(\vec{\phi}, \theta) d m_{p-1} d m
$$

because the difference is just a set of measure zero.
Notation 13.9.2 Often this is written differently. Note that from the spherical coordinate formulas, $f(\mathbf{h}(\vec{\phi}, \theta, \rho))=f(\rho \omega)$ where $|\omega|=1$. Letting $S^{p-1}$ denote the unit sphere, $\left\{\omega \in \mathbb{R}^{p}:|\omega|=1\right\}$, the inside integral in the above formula is sometimes written as

$$
\int_{S^{p-1}} f(\rho \omega) d \sigma
$$

where $\sigma$ is a measure on $S^{p-1}$. See [83] for another description of this measure. It isn't an important issue here. Either 13.9.22 or the formula

$$
\int_{0}^{\infty} \rho^{p-1}\left(\int_{S^{p-1}} f(\rho \omega) d \sigma\right) d \rho
$$

will be referred to as polar coordinates and is very useful in establishing estimates. Here $\sigma\left(S^{p-1}\right) \equiv \int_{A} \Phi(\vec{\phi}, \theta) d m_{p-1}$.

Example 13.9.3 For what values of $s$ is the integral $\int_{B(\mathbf{0}, R)}\left(1+|\mathbf{x}|^{2}\right)^{s} d y$ bounded independent of $R$ ? Here $B(\mathbf{0}, R)$ is the ball, $\left\{\mathbf{x} \in \mathbb{R}^{p}:|\mathbf{x}| \leq R\right\}$.

I think you can see immediately that $s$ must be negative but exactly how negative? It turns out it depends on $p$ and using polar coordinates, you can find just exactly what is needed. From the polar coordinates formula above,

$$
\begin{aligned}
\int_{B(\mathbf{0}, R)}\left(1+|\mathbf{x}|^{2}\right)^{s} d y & =\int_{0}^{R} \int_{S^{p-1}}\left(1+\rho^{2}\right)^{s} \rho^{p-1} d \sigma d \rho \\
& =C_{p} \int_{0}^{R}\left(1+\rho^{2}\right)^{s} \rho^{p-1} d \rho
\end{aligned}
$$

Now the very hard problem has been reduced to considering an easy one variable problem of finding when

$$
\int_{0}^{R} \rho^{p-1}\left(1+\rho^{2}\right)^{s} d \rho
$$

is bounded independent of $R$. You need $2 s+(p-1)<-1$ so you need $s<-p / 2$.

### 13.10 The Brouwer Fixed Point Theorem

This seems to be a good place to present a short proof of one of the most important of all fixed point theorems. There are many approaches to this but one of the easiest and shortest I have ever seen is the one in Dunford and Schwartz [45]. This is what is presented here. In Evans [48] there is a different proof which depends on integration theory. A good reference for an introduction to various kinds of fixed point theorems is the book by Smart [118]. This book also gives an entirely different approach to the Brouwer fixed point theorem.

The proof given here is based on the following lemma. Recall that for $A$ an $n \times n$ matrix, $\operatorname{cof}(A)_{i j}$ is the determinant of the matrix which results from deleting the $i^{t h}$ row and the $j^{t h}$ column and multiplying by $(-1)^{i+j}$. The following lemma is proved earlier. See Lemma 16.3.1.

Lemma 13.10.1 Let $\mathbf{g}: U \rightarrow \mathbb{R}^{n}$ be $C^{2}$ where $U$ is an open subset of $\mathbb{R}^{n}$. Then

$$
\sum_{j=1}^{n} \operatorname{cof}(D \mathbf{g})_{i j, j}=0
$$

where here $(D \mathbf{g})_{i j} \equiv g_{i, j} \equiv \frac{\partial g_{i}}{\partial x_{j}}$. Also, $\operatorname{cof}(D \mathbf{g})_{i j}=\frac{\partial \operatorname{det}(D \mathbf{g})}{\partial g_{i, j}}$.
To prove the Brouwer fixed point theorem, first consider a version of it valid for $C^{2}$ mappings. This is the following lemma.

Lemma 13.10.2 Let $B_{r}=\overline{B(\mathbf{0}, r)}$ and suppose $\mathbf{g}$ is a $C^{2}$ function defined on $\mathbb{R}^{n}$ which maps $B_{r}$ to $B_{r}$. Then $\mathbf{g}(\mathbf{x})=\mathbf{x}$ for some $\mathbf{x} \in B_{r}$.

Proof: Suppose not. Then $|\mathbf{g}(\mathbf{x})-\mathbf{x}|$ must be bounded away from zero on $B_{r}$. Let $a(\mathbf{x})$ be the larger of the two roots of the equation,

$$
\begin{equation*}
|\mathbf{x}+z(\mathbf{x}-\mathbf{g}(\mathbf{x}))|^{2}=|\mathbf{x}|^{2}+2(\mathbf{x}, \mathbf{x}-\mathbf{g}(\mathbf{x})) z+z^{2}|\mathbf{x}-\mathbf{g}(\mathbf{x})|^{2}=r^{2} \tag{13.10.23}
\end{equation*}
$$

Thus, from the quadratic formula,

$$
\begin{equation*}
a(\mathbf{x})=\frac{-(\mathbf{x},(\mathbf{x}-\mathbf{g}(\mathbf{x})))+\sqrt{(\mathbf{x},(\mathbf{x}-\mathbf{g}(\mathbf{x})))^{2}+\left(r^{2}-|\mathbf{x}|^{2}\right)|\mathbf{x}-\mathbf{g}(\mathbf{x})|^{2}}}{|\mathbf{x}-\mathbf{g}(\mathbf{x})|^{2}} \tag{13.10.24}
\end{equation*}
$$

That under the square root is positive if $|\mathbf{x}|<r$. What if $|\mathbf{x}|=r$ ? In this case, since $\mathbf{g}(\mathbf{x}) \in B_{r}$, you cannot have $(\mathbf{x},(\mathbf{x}-\mathbf{g}(\mathbf{x})))=0$ because if so, you would have

$$
r^{2}=(\mathbf{x}, \mathbf{g}(\mathbf{x}))=|\mathbf{x}||\mathbf{g}(\mathbf{x})| \cos (\theta)
$$

so $r=|\mathbf{g}(\mathbf{x})| \cos \theta$ where $\theta$ is the angle between the vectors $\mathbf{x}, \mathbf{g}(\mathbf{x})$. The only way this can happen is for $\mathbf{g}(\mathbf{x})=\mathbf{x}$ and this is assumed not to occur. Thus what is under the square root sign is always positive. It follows that $a(\cdot)$ is a $C^{2}$ function because $t \rightarrow \sqrt{t}$ is smooth on $t>0$. When $|\mathbf{x}|=r$, one solution to 13.10 .23 is $z=0$. There is also a solution for negative $z$ based on geometric reasoning. Therefore, these are the two roots to 13.10 .23 , one negative
and one 0 , and so $a(\mathbf{x})=0$ when $|\mathbf{x}|=r$. Thus also, if $|\mathbf{x}|=r$, 13.10.24 implies that $0=-(\mathbf{x},(\mathbf{x}-\mathbf{g}(\mathbf{x})))+|(\mathbf{x},(\mathbf{x}-\mathbf{g}(\mathbf{x})))|$ so $(\mathbf{x},(\mathbf{x}-\mathbf{g}(\mathbf{x}))) \geq 0$.

Now define for $t \in[0,1]$,

$$
\mathbf{f}(t, \mathbf{x}) \equiv \mathbf{x}+t a(\mathbf{x})(\mathbf{x}-\mathbf{g}(\mathbf{x}))
$$

The important properties of $\mathbf{f}(t, \mathbf{x})$ and $a(\mathbf{x})$ are that

$$
\begin{equation*}
a(\mathbf{x})=0 \text { if }|\mathbf{x}|=r \tag{13.10.25}
\end{equation*}
$$

and

$$
\begin{equation*}
|\mathbf{f}(t, \mathbf{x})|=r \text { for all }|\mathbf{x}|=r \tag{13.10.26}
\end{equation*}
$$

Also from 13.10.24, $a$ is a $C^{2}$ function near $B_{r}$ because, as shown above, the expression under $\sqrt{ }$ is always positive and $t \rightarrow \sqrt{t}$ is infinitely differentiable for $t>0$.

Now define

$$
I(t) \equiv \int_{B_{r}} \operatorname{det}\left(D_{2} \mathbf{f}(t, \mathbf{x})\right) d x
$$

Then

$$
\begin{equation*}
I(0)=\int_{B_{r}} d x=m_{n}\left(B_{r}\right)>0 \tag{13.10.27}
\end{equation*}
$$

Using the dominated convergence theorem one can differentiate $I(t)$ as follows.

$$
\begin{aligned}
I^{\prime}(t) & =\int_{B_{r}} \sum_{i j} \frac{\partial \operatorname{det}\left(D_{2} \mathbf{f}(t, \mathbf{x})\right)}{\partial f_{i, j}} \frac{\partial f_{i, j}}{\partial t} d x \\
& =\int_{B_{r}} \sum_{i j} \operatorname{cof}\left(D_{2} \mathbf{f}\right)_{i j} \frac{\partial\left(a(\mathbf{x})\left(x_{i}-g_{i}(\mathbf{x})\right)\right)}{\partial x_{j}} d x .
\end{aligned}
$$

Now from 13.10.25 $a(\mathbf{x})=0$ when $|\mathbf{x}|=r$ and so integration by parts and Lemma 16.3.1 yields

$$
\begin{aligned}
I^{\prime}(t) & =\int_{B_{r}} \sum_{i j} \operatorname{cof}\left(D_{2} \mathbf{f}\right)_{i j} \frac{\partial\left(a(\mathbf{x})\left(x_{i}-g_{i}(\mathbf{x})\right)\right)}{\partial x_{j}} d x \\
& =-\int_{B_{r}} \sum_{i j} \operatorname{cof}\left(D_{2} \mathbf{f}\right)_{i j, j} a(\mathbf{x})\left(x_{i}-g_{i}(\mathbf{x})\right) d x=0 .
\end{aligned}
$$

Therefore, $I(1)=I(0)$. However, from 13.10.23

$$
|\mathbf{x}+a(\mathbf{x}) t(\mathbf{x}-\mathbf{g}(\mathbf{x}))|^{2}=|\mathbf{x}|^{2}+2(\mathbf{x}, \mathbf{x}-\mathbf{g}(\mathbf{x})) a(\mathbf{x}) t+(a(\mathbf{x}) t)^{2}|\mathbf{x}-\mathbf{g}(\mathbf{x})|^{2}=r^{2}
$$

it follows that for $t=1$ in the above expression, you have

$$
|\mathbf{f}(1, \mathbf{x})|^{2}=\sum_{i} f_{i} f_{i}=r^{2}
$$

and so, $\sum_{i} f_{i, j} f_{i}=0$ which implies since $|\mathbf{f}(1, \mathbf{x})|=r$ by 13.10.23, that

$$
\operatorname{det}\left(f_{i, j}\right)=\operatorname{det}\left(D_{2} \mathbf{f}(1, \mathbf{x})\right)=0
$$

and so $I(1)=0$, a contradiction to 13.10 .27 since $I(1)=I(0)$.
The following theorem is the Brouwer fixed point theorem for a ball.

Theorem 13.10.3 Let $B_{r}$ be the above closed ball and let $\mathbf{f}: B_{r} \rightarrow B_{r}$ be continuous. Then there exists $\mathbf{x} \in B_{r}$ such that $\mathbf{f}(\mathbf{x})=\mathbf{x}$.

Proof: Let $\mathbf{f}_{k}(\mathbf{x}) \equiv \frac{\mathbf{f}(\mathbf{x})}{1+k^{-1}}$. Thus

$$
\begin{aligned}
\left\|\mathbf{f}_{k}-\mathbf{f}\right\| & =\max _{\mathbf{x} \in B_{r}}\left\{\left|\frac{\mathbf{f}(\mathbf{x})}{1+(1 / k)}-\mathbf{f}(\mathbf{x})\right|\right\}=\max _{\mathbf{x} \in B_{r}}\left\{\left|\frac{\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{x})(1+(1 / k))}{1+(1 / k)}\right|\right\} \\
& =\max _{\mathbf{x} \in B_{r}}\left\{\left|\frac{\mathbf{f}(\mathbf{x})(1 / k)}{1+(1 / k)}\right|\right\} \leq \frac{r}{1+k}
\end{aligned}
$$

Letting $\|\mathbf{h}\| \equiv \max \left\{|\mathbf{h}(\mathbf{x})|: \mathbf{x} \in B_{r}\right\}$, It follows from the Weierstrass approximation theorem, there exists a function whose components are polynomials $\mathbf{g}_{k}$ such that $\left\|\mathbf{g}_{k}-\mathbf{f}_{k}\right\|<$ $\frac{r}{k+1}$. Then if $\mathbf{x} \in B_{r}$, it follows

$$
\begin{aligned}
\left|\mathbf{g}_{k}(\mathbf{x})\right| & \leq\left|\mathbf{g}_{k}(\mathbf{x})-\mathbf{f}_{k}(\mathbf{x})\right|+\left|\mathbf{f}_{k}(\mathbf{x})\right| \\
& <\frac{r}{1+k}+\frac{k r}{1+k}=r
\end{aligned}
$$

and so $\mathbf{g}_{k}$ maps $B_{r}$ to $B_{r}$. By Lemma 13.10.2 each of these $\mathbf{g}_{k}$ has a fixed point $\mathbf{x}_{k}$ such that $\mathbf{g}_{k}\left(\mathbf{x}_{k}\right)=\mathbf{x}_{k}$. The sequence of points, $\left\{\mathbf{x}_{k}\right\}$ is contained in the compact set, $B_{r}$ and so there exists a convergent subsequence still denoted by $\left\{\mathbf{x}_{k}\right\}$ which converges to a point $\mathbf{x} \in B_{r}$. Then

$$
\begin{aligned}
|\mathbf{f}(\mathbf{x})-\mathbf{x}| & \leq\left|\mathbf{f}(\mathbf{x})-\mathbf{f}_{k}(\mathbf{x})\right|+\left|\mathbf{f}_{k}(\mathbf{x})-\mathbf{f}_{k}\left(\mathbf{x}_{k}\right)\right|+|\mathbf{f}_{k}\left(\mathbf{x}_{k}\right)-\overbrace{\mathbf{g}_{k}\left(\mathbf{x}_{k}\right)}^{=\mathbf{x}_{k}}|+\left|\mathbf{x}_{k}-\mathbf{x}\right| \\
& \leq \frac{r}{1+k}+\left|\mathbf{f}(\mathbf{x})-\mathbf{f}\left(\mathbf{x}_{k}\right)\right|+\frac{r}{1+k}+\left|\mathbf{x}_{k}-\mathbf{x}\right| .
\end{aligned}
$$

Now let $k \rightarrow \infty$ in the right side to conclude $\mathbf{f}(\mathbf{x})=\mathbf{x}$.
It is not surprising that the ball does not need to be centered at $\mathbf{0}$.
Corollary 13.10.4 Let $\mathbf{f}: \overline{B(\mathbf{a}, r)} \rightarrow \overline{B(\mathbf{a}, r)}$ be continuous. Then there exists $\mathbf{x} \in \overline{B(\mathbf{a}, r)}$ such that $\mathbf{f}(\mathbf{x})=\mathbf{x}$.

Proof: Let $\mathbf{g}: B_{r} \rightarrow B_{r}$ be defined by $\mathbf{g}(\mathbf{y}) \equiv \mathbf{f}(\mathbf{y}+\mathbf{a})-\mathbf{a}$. Then $\mathbf{g}$ is a continuous map from $B_{r}$ to $B_{r}$. Therefore, there exists $\mathbf{y} \in B_{r}$ such that $\mathbf{g}(\mathbf{y})=\mathbf{y}$. Therefore, $\mathbf{f}(\mathbf{y}+\mathbf{a})-\mathbf{a}=\mathbf{y}$ and so letting $\mathbf{x}=\mathbf{y}+\mathbf{a}, \mathbf{f}$ also has a fixed point as claimed.

Definition 13.10.5 $A$ set $A$ is a retract of a set $B$ if $A \subseteq B$, and there is a continuous map $\mathbf{h}: B \rightarrow A$ such that $\mathbf{h}(\mathbf{x})=\mathbf{x}$ for all $\mathbf{x} \in A$ and $\mathbf{h}$ is onto. $B$ has the fixed point property means that whenever $\mathbf{g}$ is continuous and $\mathbf{g}: B \rightarrow B$, it follows that $\mathbf{g}$ has a fixed point.

Proposition 13.10.6 Let $A$ be a retract of $B$ and suppose $B$ has the fixed point property. Then so does $A$.

Proof: Suppose $\mathbf{f}: A \rightarrow A$. Let $\mathbf{h}$ be the retract of $B$ onto $A$. Then $\mathbf{f} \circ \mathbf{h}: B \rightarrow B$ is continuous. Thus, it has a fixed point $\mathbf{x} \in B$ so $\mathbf{f}(\mathbf{h}(\mathbf{x}))=\mathbf{x}$. However, $\mathbf{h}(\mathbf{x}) \in A$ and $\mathbf{f}: A \rightarrow A$ so in fact, $\mathbf{x} \in A$. Now $h(\mathbf{x})=\mathbf{x}$ and so $\mathbf{f}(\mathbf{x})=\mathbf{x}$.

Recall that every convex compact subset $K$ of $\mathbb{R}^{p}$ is a retract of all of $\mathbb{R}^{p}$ obtained by using the projection map. In particular, $K$ is a retract of a large closed ball containing $K$ which has the fixed point property. Therefore, $K$ also has the fixed point property. This shows the following which is often called the Brouwer fixed point theorem.

Theorem 13.10.7 Every convex closed and bounded subset of $\mathbb{R}^{p}$ has the fixed point property.

### 13.11 The Brouwer Fixed Point Theorem Another Proof

This proof is also based on Lemma 16.3.1. I found this proof of the Brouwer fixed point theorem or one close to it in Evans [48]. It is even shorter than the proof just presented. I think it might be easier to remember also. It is also based on Lemma 16.3.1 which is stated next for convenience.

Lemma 13.11.1 Let $\mathbf{g}: U \rightarrow \mathbb{R}^{p}$ be $C^{2}$ where $U$ is an open subset of $\mathbb{R}^{p}$. Then

$$
\sum_{j=1}^{p} \operatorname{cof}(D \mathbf{g})_{i j, j}=0
$$

where here $(D \mathbf{g})_{i j} \equiv g_{i, j} \equiv \frac{\partial g_{i}}{\partial x_{j}}$. Also, $\operatorname{cof}(D \mathbf{g})_{i j}=\frac{\partial \operatorname{det}(D \mathbf{g})}{\partial g_{i, j}}$.
Definition 13.11.2 Let $\mathbf{h}$ be a function defined on an open set, $U \subseteq \mathbb{R}^{p}$. Then $\mathbf{h} \in C^{k}(\bar{U})$ if there exists a function $\mathbf{g}$ defined on an open set, $W$ containng $\bar{U}$ such that $\mathbf{g}=\mathbf{h}$ on $U$ and $\mathbf{g}$ is $C^{k}(W)$.
Lemma 13.11.3 There does not exist $\mathbf{h} \in C^{2}(\overline{B(\mathbf{0}, R)})$ such that $\mathbf{h}: \overline{B(\mathbf{0}, R)} \rightarrow \partial B(\mathbf{0}, R)$ which also has the property that $\mathbf{h}(\mathbf{x})=\mathbf{x}$ for all $\mathbf{x} \in \partial B(\mathbf{0}, R)$. Such a function is called a retract.

Proof: Here and below, let $B_{R}$ denote $\overline{B(\mathbf{0}, R)}$. Suppose such an $\mathbf{h}$ exists. Let $\lambda \in[0,1]$ and let $\mathbf{p}_{\lambda}(\mathbf{x}) \equiv \mathbf{x}+\lambda(\mathbf{h}(\mathbf{x})-\mathbf{x})$. This function, $\mathbf{p}_{\lambda}$ is a homotopy of the identity map and the retraction, $\mathbf{h}$. Let

$$
I(\lambda) \equiv \int_{B(\mathbf{0}, R)} \operatorname{det}\left(D \mathbf{p}_{\lambda}(\mathbf{x})\right) d x
$$

Then using the dominated convergence theorem,

$$
\begin{aligned}
I^{\prime}(\lambda) & =\int_{B(\mathbf{0}, R)} \sum_{i . j} \frac{\partial \operatorname{det}\left(D \mathbf{p}_{\lambda}(\mathbf{x})\right)}{\partial p_{\lambda i, j}} \frac{\partial p_{\lambda i j}(\mathbf{x})}{\partial \lambda} d x \\
& =\int_{B(\mathbf{0}, R)} \sum_{i} \sum_{j} \frac{\partial \operatorname{det}\left(D \mathbf{p}_{\lambda}(\mathbf{x})\right)}{\partial p_{\lambda i, j}}\left(h_{i}(\mathbf{x})-x_{i}\right)_{, j} d x \\
& =\int_{B(\mathbf{0}, R)} \sum_{i} \sum_{j} \operatorname{cof}\left(D \mathbf{p}_{\lambda}(\mathbf{x})\right)_{i j}\left(h_{i}(\mathbf{x})-x_{i}\right)_{, j} d x
\end{aligned}
$$

Now by assumption, $h_{i}(\mathbf{x})=x_{i}$ on $\partial B(\mathbf{0}, R)$ and so one can integrate by parts and write

$$
I^{\prime}(\lambda)=-\sum_{i} \int_{B(\mathbf{0}, R)} \sum_{j} \operatorname{cof}\left(D \mathbf{p}_{\lambda}(\mathbf{x})\right)_{i j, j}\left(h_{i}(\mathbf{x})-x_{i}\right) d x=0
$$

Therefore, $I(\lambda)$ equals a constant. However, $|\mathbf{h}(\mathbf{x})|^{2}=R^{2}$ so $\sum_{i=1}^{p} h_{i}(\mathbf{x}) h_{i}(\mathbf{x})=R^{2}$ and so, differentiating with respect to $j$,

$$
2 \sum_{i=1}^{p} h_{i, j}(\mathbf{x}) h_{i}(\mathbf{x})=0 \text { so } D \mathbf{h}(\mathbf{x})^{T} \mathbf{h}(\mathbf{x})=0
$$

and so, since $\mathbf{h}(\mathbf{x}) \neq \mathbf{0}, D \mathbf{h}(\mathbf{x})^{T}$ is not invertible. Hence $\operatorname{det}(D \mathbf{h}(\mathbf{x}))=0$ and so $I(1)=0 \neq$ $I(0)=m_{p}(B(\mathbf{0}, R))$. This is a contradiction.

The following is the Brouwer fixed point theorem for $C^{2}$ maps.
Lemma 13.11.4 If $\mathbf{h} \in C^{2}(\overline{B(\mathbf{0}, R)})$ and $\mathbf{h}: \overline{B(\mathbf{0}, R)} \rightarrow \overline{B(\mathbf{0}, R)}$, then $\mathbf{h}$ has a fixed point, $\mathbf{x}$ such that $\mathbf{h}(\mathbf{x})=\mathbf{x}$.

Proof: Suppose the lemma is not true. Then for all $\mathbf{x},|\mathbf{x}-\mathbf{h}(\mathbf{x})| \neq 0$. Then define

$$
\mathbf{g}(\mathbf{x})=\mathbf{h}(\mathbf{x})+\frac{\mathbf{x}-\mathbf{h}(\mathbf{x})}{|\mathbf{x}-\mathbf{h}(\mathbf{x})|} t(\mathbf{x})
$$

where $t(\mathbf{x})$ is nonnegative and is chosen such that $\mathbf{g}(\mathbf{x}) \in \partial B(\mathbf{0}, R)$.
This mapping is illustrated in the following picture.


If $\mathbf{x} \rightarrow t(\mathbf{x})$ is $C^{2}$ near $\overline{B(\mathbf{0}, R)}$, it will follow $\mathbf{g}$ is a $C^{2}$ retraction onto $\partial B(\mathbf{0}, R)$ contrary to Lemma 13.11.3. Thus $t(\mathbf{x})$ is the nonnegative solution to

$$
\begin{equation*}
|\mathbf{h}(\mathbf{x})|^{2}+2\left(\mathbf{h}(\mathbf{x}), \frac{\mathbf{x}-\mathbf{h}(\mathbf{x})}{|\mathbf{x}-\mathbf{h}(\mathbf{x})|}\right) t+t^{2}=R^{2} \tag{13.11.28}
\end{equation*}
$$

then by the quadratic formula,

$$
t(\mathbf{x})=-\left(\mathbf{h}(\mathbf{x}), \frac{\mathbf{x}-\mathbf{h}(\mathbf{x})}{|\mathbf{x}-\mathbf{h}(\mathbf{x})|}\right)+\sqrt{\left(\mathbf{h}(\mathbf{x}), \frac{\mathbf{x}-\mathbf{h}(\mathbf{x})}{|\mathbf{x}-\mathbf{h}(\mathbf{x})|}\right)^{2}+\left(R^{2}-|\mathbf{h}(\mathbf{x})|^{2}\right)}
$$

Is $\mathbf{x} \rightarrow t(\mathbf{x}) C^{2}$ ? If what is under the radical is positive, then there is no problem because $s \rightarrow \sqrt{s}$ is smooth for $s>0$. In fact, this is the case here. The inside of the radical is positive
if $R>|\mathbf{h}(\mathbf{x})|$. If $|\mathbf{h}(\mathbf{x})|=R$, it is still positive because in this case, the angle between $\mathbf{h}(\mathbf{x})$ and $\mathbf{x}-\mathbf{h}(\mathbf{x})$ cannot be $\pi / 2$. This shows that $\mathbf{x} \rightarrow t(\mathbf{x})$ is the composition of $C^{2}$ functions and is therefore $C^{2}$. Thus this $\mathbf{g}(\mathbf{x})$ is a $C^{2}$ retract and by the above lemma, there isn't one.

Now it is easy to prove the Brouwer fixed point theorem. The following theorem is the Brouwer fixed point theorem for a ball.

Theorem 13.11.5 Let $B_{R}$ be the above closed ball and let $\mathbf{f}: B_{R} \rightarrow B_{R}$ be continuous. Then there exists $\mathbf{x} \in B_{R}$ such that $\mathbf{f}(\mathbf{x})=\mathbf{x}$.

Proof: Let $\mathbf{f}_{k}(\mathbf{x}) \equiv \frac{\mathbf{f}(\mathbf{x})}{1+k^{-1}}$. Thus

$$
\begin{aligned}
\left\|\mathbf{f}_{k}-\mathbf{f}\right\| & =\max _{\mathbf{x} \in B_{R}}\left\{\left|\frac{\mathbf{f}(\mathbf{x})}{1+(1 / k)}-\mathbf{f}(\mathbf{x})\right|\right\}=\max _{\mathbf{x} \in B_{R}}\left\{\left|\frac{\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{x})(1+(1 / k))}{1+(1 / k)}\right|\right\} \\
& =\max _{\mathbf{x} \in B_{R}}\left\{\left|\frac{\mathbf{f}(\mathbf{x})(1 / k)}{1+(1 / k)}\right|\right\} \leq \frac{R}{1+k}
\end{aligned}
$$

Letting $\|\mathbf{h}\| \equiv \max \left\{|\mathbf{h}(\mathbf{x})|: \mathbf{x} \in B_{R}\right\}$, It follows from the Weierstrass approximation theorem, there exists a function whose components are polynomials $\mathbf{g}_{k}$ such that $\left\|\mathbf{g}_{k}-\mathbf{f}_{k}\right\|<$ $\frac{R}{k+1}$. Then if $\mathbf{x} \in B_{R}$, it follows

$$
\begin{aligned}
\left|\mathbf{g}_{k}(\mathbf{x})\right| & \leq\left|\mathbf{g}_{k}(\mathbf{x})-\mathbf{f}_{k}(\mathbf{x})\right|+\left|\mathbf{f}_{k}(\mathbf{x})\right| \\
& <\frac{R}{1+k}+\frac{k R}{1+k}=R
\end{aligned}
$$

and so $\mathbf{g}_{k}$ maps $B_{R}$ to $B_{R}$. By Lemma 13.10.2 each of these $\mathbf{g}_{k}$ has a fixed point $\mathbf{x}_{k}$ such that $\mathbf{g}_{k}\left(\mathbf{x}_{k}\right)=\mathbf{x}_{k}$. The sequence of points, $\left\{\mathbf{x}_{k}\right\}$ is contained in the compact set, $B_{R}$ and so there exists a convergent subsequence still denoted by $\left\{\mathbf{x}_{k}\right\}$ which converges to a point $\mathbf{x} \in B_{R}$. Then

$$
\begin{aligned}
|\mathbf{f}(\mathbf{x})-\mathbf{x}| & \leq\left|\mathbf{f}(\mathbf{x})-\mathbf{f}_{k}(\mathbf{x})\right|+\left|\mathbf{f}_{k}(\mathbf{x})-\mathbf{f}_{k}\left(\mathbf{x}_{k}\right)\right|+|\mathbf{f}_{k}\left(\mathbf{x}_{k}\right)-\overbrace{\mathbf{g}_{k}\left(\mathbf{x}_{k}\right)}^{=\mathbf{x}_{k}}|+\left|\mathbf{x}_{k}-\mathbf{x}\right| \\
& \leq \frac{R}{1+k}+\left|\mathbf{f}(\mathbf{x})-\mathbf{f}\left(\mathbf{x}_{k}\right)\right|+\frac{R}{1+k}+\left|\mathbf{x}_{k}-\mathbf{x}\right| .
\end{aligned}
$$

Now let $k \rightarrow \infty$ in the right side to conclude $\mathbf{f}(\mathbf{x})=\mathbf{x}$.
It is not surprising that the ball does not need to be centered at $\mathbf{0}$.
Corollary 13.11.6 Let $\mathbf{f}: \overline{B(\mathbf{a}, R)} \rightarrow \overline{B(\mathbf{a}, R)}$ be continuous. Then there exists $\mathbf{x} \in \overline{B(\mathbf{a}, R)}$ such that $\mathbf{f}(\mathbf{x})=\mathbf{x}$.

Proof: Let $\mathbf{g}: B_{R} \rightarrow B_{R}$ be defined by $\mathbf{g}(\mathbf{y}) \equiv \mathbf{f}(\mathbf{y}+\mathbf{a})-\mathbf{a}$. Then $\mathbf{g}$ is a continuous map from $B_{R}$ to $B_{R}$. Therefore, there exists $\mathbf{y} \in B_{R}$ such that $\mathbf{g}(\mathbf{y})=\mathbf{y}$. Therefore, $\mathbf{f}(\mathbf{y}+\mathbf{a})-$ $\mathbf{a}=\mathbf{y}$ and so letting $\mathbf{x}=\mathbf{y}+\mathbf{a}, \mathbf{f}$ also has a fixed point as claimed.

Definition 13.11.7 $A$ set $A$ is a retract of a set $B$ if $A \subseteq B$, and there is a continuous map $\mathbf{h}: B \rightarrow A$ such that $\mathbf{h}(\mathbf{x})=\mathbf{x}$ for all $\mathbf{x} \in A$ and $\mathbf{h}$ is onto. $B$ has the fixed point property means that whenever $\mathbf{g}$ is continuous and $\mathbf{g}: B \rightarrow B$, it follows that $\mathbf{g}$ has a fixed point.

Proposition 13.11.8 Let $A$ be a retract of $B$ and suppose $B$ has the fixed point property. Then so does $A$.

Proof: Suppose $\mathbf{f}: A \rightarrow A$. Let $\mathbf{h}$ be the retract of $B$ onto $A$. Then $\mathbf{f} \circ \mathbf{h}: B \rightarrow B$ is continuous. Thus, it has a fixed point $\mathbf{x} \in B$ so $\mathbf{f}(\mathbf{h}(\mathbf{x}))=\mathbf{x}$. However, $\mathbf{h}(\mathbf{x}) \in A$ and $\mathbf{f}: A \rightarrow A$ so in fact, $\mathbf{x} \in A$. Now $h(\mathbf{x})=\mathbf{x}$ and so $\mathbf{f}(\mathbf{x})=\mathbf{x}$.

Recall that every convex compact subset $K$ of $\mathbb{R}^{p}$ is a retract of all of $\mathbb{R}^{p}$ obtained by using the projection map. In particular, $K$ is a retract of a large closed ball containing $K$ which has the fixed point property. Therefore, $K$ also has the fixed point property. This shows the following which is often called the Brouwer fixed point theorem.

Corollary 13.11.9 Every convex closed and bounded subset of $\mathbb{R}^{p}$ has the fixed point property.

### 13.12 Invariance Of Domain

This principal says that if $\mathbf{f}: U \subseteq \mathbb{R}^{n} \rightarrow \mathbf{f}(U) \subseteq \mathbb{R}^{n}$ where $U$ is open and $\mathbf{f}$ is one to one and continuous, then $\mathbf{f}(U)$ is also open. To do this, we first prove the following lemma. I found something like this on the web. I liked it a lot because it shows how the Brouwer fixed point theorem implies the invariance of domain. The other ways I know about involve degree theory or some sort of algebraic topology. I will give two proofs of the following lemma, the first being somewhat more informal than the second.

Lemma 13.12.1 Let $B$ be a closed ball in $\mathbb{R}^{n}$ centered at a which has radius $r$. Let $\mathbf{f}: B \rightarrow$ $\mathbb{R}^{n}$. Then $\mathbf{f}(\mathbf{a})$ is an interior point of $\mathbf{f}(B)$.

Proof: Since $\mathbf{f}(B)$ is compact and $\mathbf{f}$ is one to one, $\mathbf{f}^{-1}$ is continuous on $\mathbf{f}(B)$. Use Tietze extension theorem on components of $\mathbf{f}^{-1}$ or some such thing to obtain $\mathbf{g}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ such that $\mathbf{g}$ is continuous and equals $\mathbf{f}^{-1}$ on $\mathbf{f}(B)$. Then multiply by a suitable truncation function to get $\mathbf{g}$ uniformly continuous on $\mathbb{R}^{n}$.

Suppose $\mathbf{f}(\mathbf{a})$ is not an interior point of $\mathbf{f}(B)$. Then there exists $\mathbf{c}_{k} \rightarrow \mathbf{f}(\mathbf{a})$ but $\mathbf{c}_{k} \notin \mathbf{f}(B)$. In the picture, let $C_{k}$ be a sphere whose radius is

$$
2\left|\mathbf{c}_{k}-\mathbf{f}(\mathbf{a})\right|
$$



Let $\hat{\mathbf{g}}_{k}$ be $C^{1}$ and let it satisfy

$$
\left\|\hat{\mathbf{g}}_{k}-\mathbf{g}\right\|_{\mathbf{f}(B) \cup D} \equiv \max _{\mathbf{y} \in \mathbf{f}(B) \cup D}|\hat{\mathbf{g}}(\mathbf{y})-\mathbf{g}(\mathbf{y})|<\varepsilon_{k}
$$

$\varepsilon_{k}$ is very small, $\varepsilon_{k} \rightarrow 0$. How small will be considered later. Here $D$ is a large closed disk which contains all of the spheres $C_{k}$ considered above. The idea is to have a large compact set which includes everything of interest below.

To get $\hat{\mathbf{g}}_{k}$,you could use the Weierstrass approximation theorem, Theorem 9.2.9. An easier way involving convolution will be presented in the next chapter. Also let $\mathbf{a} \notin \hat{\mathbf{g}}_{k}\left(C_{k}\right)$. This is no problem. $C_{k}$ has measure zero and so $\hat{\mathbf{g}}\left(C_{k}\right)$ also has measure zero thanks to the assumption that $\hat{\mathbf{g}}$ is $C^{1}$ and Lemma 13.5.1. Therefore, you could simply add a small enough nonzero vector to $\hat{\mathbf{g}}$ to preserve the above inequality of $\hat{\mathbf{g}}$ and $\mathbf{g}$ so that $\hat{\mathbf{g}}\left(C_{k}\right)$ no longer contains $\mathbf{a}$. That is, replace $\hat{\mathbf{g}}$ with $\hat{\mathbf{g}}+\mathbf{a}-\mathbf{b}$ where $|\mathbf{a}-\mathbf{b}|$ is very small but $\mathbf{b} \notin$ $\hat{\mathbf{g}}\left(C_{k}\right)$.

There is a set $\Sigma_{k}$ consisting of that part of $\mathbf{f}(B)$ which is outside of the sphere $C_{k}$ in the picture along with the sphere $C_{k}$ itself. By construction, $\hat{\mathbf{g}}_{k}$ misses a on $C_{k}$. As to the other part of $\Sigma_{k}, \mathbf{g}$ misses a on this part, because $\mathbf{f}$ is one to one and so $\mathbf{f}^{-1}$ is also. Now we will squash the part of $\mathbf{f}(B)$ inside $C_{k}$ onto $C_{k}$ while leaving the rest of $\mathbf{f}(B)$ unchanged.

Let $\Phi_{k}$ be defined on $\mathbf{f}(B)$

$$
\Phi_{k}(\mathbf{y}) \equiv \max \left(\frac{2\left|\mathbf{c}_{k}-\mathbf{f}(\mathbf{a})\right|}{\left|\mathbf{y}-\mathbf{c}_{k}\right|}, 1\right)\left(\mathbf{y}-\mathbf{c}_{k}\right)+\mathbf{c}_{k}
$$

This $\Phi_{k}$ squishes the part of $\mathbf{f}(B)$ inside $C_{k}$ to $C_{k}$ and leaves the rest of $\mathbf{f}(B)$ unchanged. Thus

$$
\Phi_{k}: \mathbf{f}(B) \rightarrow \mathbf{f}(B) \cap\left[\mathbf{y}:\left|\mathbf{y}-\mathbf{c}_{k}\right| \geq 2\left|\mathbf{c}_{k}-\mathbf{f}(\mathbf{a})\right|\right] \cup C_{k}
$$

a compact set. Now $\left\|\hat{\mathbf{g}}_{k} \circ \Phi_{k}-\mathbf{g}\right\|_{\mathbf{f}(B)} \rightarrow 0$ and $\mathbf{g}$ misses a on the part of $\mathbf{f}(B)$ outside of $C_{k}$. In the above, we chose $\hat{\mathbf{g}}_{k}$ so close to $\mathbf{g}$ that it also misses a on the part of $\mathbf{f}(B)$ which is outside of $C_{k}$. Then by construction, $\hat{\mathbf{g}}_{k}$ misses $\mathbf{a}$ on $C_{k}$ and so in fact $\hat{\mathbf{g}}_{k} \circ \Phi_{k}$ misses $\mathbf{a}$ on $\mathbf{f}(B)$. Now consider $\mathbf{a}+\mathbf{x}-\hat{\mathbf{g}}_{k}\left(\Phi_{k}(\mathbf{f}(\mathbf{x}))\right)$ for $\mathbf{x} \in B$.

$$
\begin{gathered}
\left|\mathbf{a}+\mathbf{x}-\hat{\mathbf{g}}_{k}\left(\Phi_{k}(\mathbf{f}(\mathbf{x}))\right)-\mathbf{a}\right|=|\mathbf{x}-\hat{\mathbf{g}}(\Phi(\mathbf{f}(\mathbf{x})))| \\
=\left|\mathbf{g}(\mathbf{f}(\mathbf{x}))-\hat{\mathbf{g}}_{k}\left(\Phi_{k}(\mathbf{f}(\mathbf{x}))\right)\right|
\end{gathered}
$$

For $\mathbf{f}(\mathbf{x})$ outside of $C_{k}$, we could have chosen $\hat{\mathbf{g}}_{k}$ such that $\left\|\mathbf{g}-\hat{\mathbf{g}}_{k}\right\|_{\mathbf{f}(B)}<\frac{r}{2}$ and this was indeed done. When $\mathbf{f}(\mathbf{x})$ is inside $C_{k}$, then eventually, for large $k$, both $\mathbf{g}(\mathbf{f}(\mathbf{x})), \hat{\mathbf{g}}_{k}\left(\Phi_{k}(\mathbf{f}(\mathbf{x}))\right)$ are close to $\mathbf{g}(\mathbf{f}(\mathbf{a}))$. To see this,

$$
\begin{aligned}
& \left|\hat{\mathbf{g}}_{k}\left(\Phi_{k}(\mathbf{f}(\mathbf{x}))\right)-\mathbf{g}\left(\Phi_{k}(\mathbf{f}(\mathbf{x}))\right)\right|+\left|\mathbf{g}\left(\Phi_{k}(\mathbf{f}(\mathbf{x}))\right)-\mathbf{g}(\mathbf{f}(\mathbf{x}))\right| \\
\leq & \left\|\hat{\mathbf{g}}_{k}-\mathbf{g}\right\|_{\mathbf{f}(B) \cup D}+\left|\mathbf{g}\left(\Phi_{k}(\mathbf{f}(\mathbf{x}))\right)-\mathbf{g}(\mathbf{f}(\mathbf{x}))\right|
\end{aligned}
$$

the last term being small for large $k$, and so for large $k, \mathbf{x} \rightarrow \mathbf{a}+\mathbf{x}-\hat{\mathbf{g}}_{k}\left(\Phi_{k}(\mathbf{f}(\mathbf{x}))\right)$ maps $B$ to $B$ and so by Brouwer fixed point theorem, it has a fixed point and hence $\hat{\mathbf{g}}_{k}\left(\Phi_{k}(\mathbf{f}(\mathbf{x}))\right)=\mathbf{a}$ contrary to what was argued above. Hence, $\mathbf{f}(\mathbf{a})$ must be an interior point after all.

Now here is the same lemma with the details.

Lemma 13.12.2 Let $B$ be a closed ball in $\mathbb{R}^{n}$ centered at a which has radius $r$. Let $\mathbf{f}: B \rightarrow$ $\mathbb{R}^{n}$. Then $\mathbf{f}(\mathbf{a})$ is an interior point of $\mathbf{f}(B)$.

Proof: Since $\mathbf{f}(B)$ is compact and $\mathbf{f}$ is one to one, $\mathbf{f}^{-1}$ is continuous on $\mathbf{f}(\boldsymbol{B})$. Use Tietze extension theorem on components of $\mathbf{f}^{-1}$ or some such thing to obtain $\mathbf{g}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ such that $\mathbf{g}$ is continuous and equals $\mathbf{f}^{-1}$ on $\mathbf{f}(B)$.

Suppose $\mathbf{f}(\mathbf{a})$ is not an interior point of $\mathbf{f}(B)$. Then for every $\varepsilon>0$ there exists $\mathbf{c}_{\varepsilon} \in$ $B(\mathbf{f}(\mathbf{a}), \varepsilon) \backslash \mathbf{f}(B)$. So fix $\varepsilon$ small and refer to $\mathbf{c}_{\varepsilon}$ as $\mathbf{c}$. $\varepsilon$ will be so small that

$$
|\mathbf{g}(\mathbf{y})-\mathbf{g}(\mathbf{f}(\mathbf{a}))|<\frac{r}{10} \text { for } \mathbf{y} \in B(\mathbf{f}(\mathbf{a}), 4 \varepsilon)
$$

There is $\delta>0$ such that if $|\mathbf{x}-\hat{\mathbf{x}}|<\boldsymbol{\delta}$, then

$$
\begin{equation*}
|\mathbf{f}(\mathbf{x})-\mathbf{f}(\hat{\mathbf{x}})|<\varepsilon \tag{13.12.29}
\end{equation*}
$$

Let $\hat{\mathbf{g}}$ be $C^{1}$ and on $\mathbf{f}(B)$, let it satisfy

$$
\|\hat{\mathbf{g}}-\mathbf{g}\|_{\mathbf{f}(B)} \equiv \max _{\mathbf{y} \in \mathbf{f}(B)}|\hat{\mathbf{g}}(\mathbf{y})-\mathbf{g}(\mathbf{y})|<\min \left(\frac{r}{10}, \frac{\delta}{2}\right)
$$

To get $\hat{\mathbf{g}}_{k}$,you could use the Weierstrass approximation theorem, Theorem 9.2.9. An easier way involving convolution will be presented in the next chapter. Also let $\mathbf{a} \notin \hat{\mathbf{g}}(\partial B(\mathbf{c}, 2 \varepsilon))$. This is no problem. $\partial B(\mathbf{a}, 2 \varepsilon)$ has measure zero and so $\hat{\mathbf{g}}(\partial B(\mathbf{c}, \varepsilon))$ also has measure zero thanks to the assumption that $\hat{\mathbf{g}}$ is $C^{1}$ and Lemma 13.5.1. Therefore, you could simply add a small enough nonzero vector to $\hat{\mathbf{g}}$ to preserve the above inequality of $\hat{\mathbf{g}}$ and $\mathbf{g}$ so that $\hat{\mathbf{g}}(\partial B(\mathbf{c}, 2 \varepsilon))$ no longer contains $\mathbf{a}$. That is, replace $\hat{\mathbf{g}}$ with $\hat{\mathbf{g}}+\mathbf{a}-\mathbf{b}$ where $|\mathbf{a}-\mathbf{b}|$ is very small but $\mathbf{b} \notin \hat{\mathbf{g}}(\partial B(\mathbf{c}, 2 \varepsilon))$. A summary of the rest of the argument is contained in the following picture in which the sphere has radius $2 \varepsilon$.


There is a set $\Sigma$ consisting of that part of $\mathbf{f}(B)$ which is outside of the sphere $C$ in the picture along with the sphere $C$ itself. By construction, $\hat{\mathbf{g}}$ misses a on $C$. As to the other part of $\Sigma$, that $\mathbf{g}$ misses a on this part, follows from the assumption that $\mathbf{f}$ is one to one and so $\mathbf{f}^{-1}$ is also. Then $\hat{\mathbf{g}}$ missing a follows from $\hat{\mathbf{g}}$ being close enough to $\mathbf{g}$. We define a continuous mapping $\Phi$ which maps $\mathbf{f}(B)$ to this set $\Sigma$. This map squishes that part of $\mathbf{f}(B)$ which is inside $C$ onto $C$ and does nothing to the part of $\mathbf{f}(B)$ which is outside of $C$. This is where $\mathbf{c}$ not in $\mathbf{f}(B)$ but close to $\mathbf{f}(B)$ is used. Then we argue that $\hat{\mathbf{g}} \circ \Phi$ is continuous and close to $\mathbf{g}$ and misses $\mathbf{a}$. It will be close to $\mathbf{g}$ and $\hat{\mathbf{g}}$ because of the above assumption that everything inside $C$ is close to $\mathbf{f}(\mathbf{a})$ and $\mathbf{g}$ and $\hat{\mathbf{g}}$ are continuous. This will yield an easy contradiction from a use of the Brouwer fixed point theorem.

Now let $\Phi$ be defined on $\mathbf{f}(B)$

$$
\Phi(\mathbf{y}) \equiv \max \left(\frac{2 \varepsilon}{|\mathbf{y}-\mathbf{c}|}, 1\right)(\mathbf{y}-\mathbf{c})+\mathbf{c}
$$

If $|\mathbf{y}-\mathbf{c}| \geq 2 \varepsilon$, then $\Phi(\mathbf{y})=\mathbf{y}$. If $|\mathbf{y}-\mathbf{c}|<2 \varepsilon$, then $\Phi(\mathbf{y})=\frac{2 \varepsilon}{|\mathbf{y}-\mathbf{c}|}(\mathbf{y}-\mathbf{c})+\mathbf{c}$ and so

$$
|\Phi(\mathbf{y})-\mathbf{c}|=2 \varepsilon \frac{|\mathbf{y}-\mathbf{c}|}{|\mathbf{y}-\mathbf{c}|}=2 \varepsilon
$$

Note that this function is well defined because $\mathbf{c} \notin \mathbf{f}(B)$. Thus

$$
\Phi: \mathbf{f}(B) \rightarrow \mathbf{f}(B) \cap[\mathbf{y}:|\mathbf{y}-\mathbf{c}| \geq 2 \varepsilon] \cup \partial B(\mathbf{c}, 2 \varepsilon) \equiv \Sigma
$$

a compact set. Now the interesting thing about this set $\Sigma$ is this. For $\mathbf{y} \in \Sigma, \hat{\mathbf{g}}(\mathbf{y}) \neq \mathbf{a}$. Why is this? It is because by construction, $\mathbf{a} \notin \hat{\mathbf{g}}(\partial B(\mathbf{c}, 2 \varepsilon))$. What if $\mathbf{y} \in \mathbf{f}(B) \cap[\mathbf{y}:|\mathbf{y}-\mathbf{c}| \geq 2 \varepsilon]$, the other set in $\Sigma$ ? Could $\hat{\mathbf{g}}(\mathbf{y})=\mathbf{a}$ ? If so, then

$$
|\mathbf{g}(\mathbf{y})-\mathbf{a}| \leq|\mathbf{g}(\mathbf{y})-\hat{\mathbf{g}}(\mathbf{y})|+|\hat{\mathbf{g}}(\mathbf{y})-\mathbf{a}| \leq \frac{\delta}{2}
$$

and so by 13.12.29,

$$
|\mathbf{y}-\mathbf{f}(\mathbf{a})|<\varepsilon
$$

But $|\mathbf{y}-\mathbf{c}| \geq 2 \varepsilon$ and so

$$
|\mathbf{y}-\mathbf{f}(\mathbf{a})| \geq|\mathbf{y}-\mathbf{c}|-|\mathbf{c}-\mathbf{f}(\mathbf{a})| \geq 2 \varepsilon-|\mathbf{c}-\mathbf{f}(\mathbf{a})| \geq 2 \varepsilon-\varepsilon=\varepsilon
$$

which contradicts the above inequality.
Therefore, $\hat{\mathbf{g}}(\mathbf{y}) \neq \mathbf{a}$ for any $\mathbf{y} \in \Sigma$. So consider $\hat{\mathbf{g}}(\Phi(\mathbf{y}))$ for $\mathbf{y} \in \mathbf{f}(B)$.

$$
\begin{align*}
& |\hat{\mathbf{g}}(\Phi(\mathbf{y}))-\mathbf{g}(\mathbf{y})| \leq|\hat{\mathbf{g}}(\Phi(\mathbf{y}))-\mathbf{g}(\Phi(\mathbf{y}))|+|\mathbf{g}(\Phi(\mathbf{y}))-\mathbf{g}(\mathbf{y})| \\
& \quad \leq \frac{r}{10}+|\mathbf{g}(\Phi(\mathbf{y}))-\mathbf{g}(\mathbf{f}(\mathbf{a}))-(\mathbf{g}(\mathbf{y})-\mathbf{g}(\mathbf{f}(\mathbf{a})))| \tag{13.12.30}
\end{align*}
$$

This last equals 0 if $|\mathbf{y}-\mathbf{c}| \geq 2 \varepsilon$. On the other hand, if $|\mathbf{y}-\mathbf{c}|<2 \varepsilon$,

$$
\mathbf{y} \in B(\mathbf{f}(\mathbf{a}), 2 \varepsilon), \Phi(\mathbf{y}) \in \partial B(\mathbf{c}, 2 \varepsilon)
$$

so both $\mathbf{y}, \Phi(\mathbf{y})$ are in $B(\mathbf{f}(\mathbf{a}), 4 \varepsilon)$ and so this last term in 13.12.30 is no larger than $|\mathbf{g}(\Phi(\mathbf{y})-\mathbf{g}(\mathbf{f}(\mathbf{a})))|+|\mathbf{g}(\mathbf{y})-\mathbf{g}(\mathbf{f}(\mathbf{a}))|<\frac{r}{10}+\frac{r}{10}$ and so for all $\mathbf{y} \in \mathbf{f}(B)$,

$$
|\hat{\mathbf{g}}(\Phi(\mathbf{y}))-\mathbf{g}(\mathbf{y})| \leq \frac{3 r}{10}
$$

Now note that for $\mathbf{x} \in B$, from what was just shown,

$$
|\hat{\mathbf{g}}(\Phi(\mathbf{f}(\mathbf{x})))-\mathbf{x}|=|\hat{\mathbf{g}}(\Phi(\mathbf{f}(\mathbf{x})))-\mathbf{g}(\mathbf{f}(\mathbf{x}))| \leq \frac{3 r}{10}
$$

It follows that for every $\mathbf{x} \in B, \mathbf{a}+\mathbf{x}-\hat{\mathbf{g}}(\Phi(\mathbf{f}(\mathbf{x}))) \in B$ and so by the Brouwer fixed point theorem, there is a fixed point $\mathbf{x}$ and hence

$$
\mathbf{a}+\mathbf{x}-\hat{\mathbf{g}}(\Phi(\mathbf{f}(\mathbf{x})))=\mathbf{x}
$$

so $\hat{\mathbf{g}}(\Phi(\mathbf{f}(\mathbf{x})))=\mathbf{a}$ contrary to what was just shown that there is no solution to $\hat{\mathbf{g}}(\mathbf{y})=\mathbf{a}$ for $\mathbf{y} \in \Sigma$.

With the lemma, it is easy to prove the invariance of domain theorem which is as follows.

Theorem 13.12.3 Let $U$ be an open set in $\mathbb{R}^{n}$ and let $\mathbf{f}: U \rightarrow \mathbf{f}(U) \subseteq \mathbb{R}^{n}$. Then $\mathbf{f}(U)$ is also an open set in $\mathbb{R}^{n}$.

Proof: For $\mathbf{a} \in U$, let $\mathbf{a} \in B_{\mathbf{a}} \subseteq U$, where $B_{\mathbf{a}}$ is a closed ball centered at $\mathbf{a}$. Then from Lemma 13.12.2, $\mathbf{f}(\mathbf{a}) \in V_{\mathbf{f}(\mathbf{a})}$ an open subset of $\mathbf{f}\left(B_{\mathbf{a}}\right)$. Hence $\mathbf{f}(U)=\cup_{\mathbf{a} \in U} V_{\mathbf{f}(\mathbf{a})}$ which is open.

### 13.13 Besicovitch Covering Theorem

The Besicovitch covering theorem is one of the most amazing and insightful ideas that I have ever encountered. It is simultaneously elegant, elementary and profound. The next section is an attempt to present this wonderful result.

When dealing with probability distribution functions or some other Radon measure, it is necessary to have a better covering theorem than the Vitali covering theorem which works well for Lebesgue measure. However, for a Radon measure, if you enlarge the ball by making the radius larger, you don't know what happens to the measure of the enlarged
ball except that its measure does not get smaller. Thus the thing required is a covering theorem which does not depend on enlarging balls.

This all works in a normed linear space $(X,\|\cdot\|)$ which has dimension $p$. Here is a sequence of balls from $\mathscr{F}$ in the case that the set of centers of these balls is bounded. I will denote by $r\left(B_{k}\right)$ the radius of a ball $B_{k}$.

## A construction of a sequence of balls

Lemma 13.13.1 Let $\mathscr{F}$ be a nonempty set of nonempty balls in $X$ with

$$
\sup \{\operatorname{diam}(B): B \in \mathscr{F}\}=D<\infty
$$

and let A denote the set of centers of these balls. Suppose $A$ is bounded. Define a sequence of balls from $\mathscr{F},\left\{B_{j}\right\}_{j=1}^{J}$ where $J \leq \infty$ such that

$$
\begin{equation*}
r\left(B_{1}\right)>\frac{3}{4} \sup \{r(B): B \in \mathscr{F}\} \tag{13.13.31}
\end{equation*}
$$

and if

$$
\begin{equation*}
A_{m} \equiv A \backslash\left(\cup_{i=1}^{m} B_{i}\right) \neq \emptyset, \tag{13.13.32}
\end{equation*}
$$

then $B_{m+1} \in \mathscr{F}$ is chosen with center in $A_{m}$ such that

$$
\begin{equation*}
r\left(B_{m}\right)>r\left(B_{m+1}\right)>\frac{3}{4} \sup \left\{r: B(\mathbf{a}, r) \in \mathscr{F}, \mathbf{a} \in A_{m}\right\} \tag{13.13.33}
\end{equation*}
$$

Then letting $B_{j}=B\left(\mathbf{a}_{j}, r_{j}\right)$, this sequence satisfies $\left\{B\left(\mathbf{a}_{j}, r_{j} / 3\right)\right\}_{j=1}^{J}$ are disjoint,.

$$
\begin{equation*}
A \subseteq \cup_{i=1}^{J} B_{i} \tag{13.13.34}
\end{equation*}
$$

Proof: First note that $B_{m+1}$ can be chosen as in 13.13.33. This is because the $A_{m}$ are decreasing and so

$$
\begin{aligned}
& \frac{3}{4} \sup \left\{r: B(\mathbf{a}, r) \in \mathscr{F}, \mathbf{a} \in A_{m}\right\} \\
\leq & \frac{3}{4} \sup \left\{r: B(\mathbf{a}, r) \in \mathscr{F}, \mathbf{a} \in A_{m-1}\right\}<r\left(B_{m}\right)
\end{aligned}
$$

Thus the $r\left(B_{k}\right)$ are strictly decreasing and so no $B_{k}$ contains a center of any other $B_{j}$.
If $\mathbf{x} \in B\left(\mathbf{a}_{j}, r_{j} / 3\right) \cap B\left(\mathbf{a}_{i}, r_{i} / 3\right)$ where these balls are two which are chosen by the above scheme such that $j>i$, then from what was just shown

$$
\left\|\mathbf{a}_{j}-\mathbf{a}_{i}\right\| \leq\left\|\mathbf{a}_{j}-\mathbf{x}\right\|+\left\|\mathbf{x}-\mathbf{a}_{i}\right\| \leq \frac{r_{j}}{3}+\frac{r_{i}}{3} \leq\left(\frac{1}{3}+\frac{1}{3}\right) r_{i}=\frac{2}{3} r_{i}<r_{i}
$$

and this contradicts the construction because $\mathbf{a}_{j}$ is not covered by $B\left(\mathbf{a}_{i}, r_{i}\right)$.
Finally consider the claim that $A \subseteq \cup_{i=1}^{J} B_{i}$. Pick $B_{1}$ satisfying 13.13.31. If $B_{1}, \cdots, B_{m}$ have been chosen, and $A_{m}$ is given in 13.13.32, then if $A_{m}=\emptyset$, it follows $A \subseteq \cup_{i=1}^{m} B_{i}$. Set $J=m$.

Now let a be the center of $B_{\mathbf{a}} \in \mathscr{F}$. If $\mathbf{a} \in A_{m}$ for all $m$, (That is a does not get covered by the $B_{i}$.) then $r_{m+1} \geq \frac{3}{4} r\left(B_{\mathbf{a}}\right)$ for all $m$, a contradiction since the balls $B\left(\mathbf{a}_{j}, \frac{r_{j}}{3}\right)$ are disjoint and $A$ is bounded, implying that $r_{j} \rightarrow 0$. Thus a must fail to be in some $A_{m}$ which means it was covered by some ball in the sequence.

The covering theorem is obtained by estimating how many $B_{j}$ can intersect $B_{k}$ for $j<k$. The thing to notice is that from the construction, no $B_{j}$ contains the center of another $B_{i}$. Also, the $r\left(B_{k}\right)$ is a decreasing sequence.

Let $\alpha>1$. There are two cases for an intersection. Either $r\left(B_{j}\right) \geq \alpha r\left(B_{k}\right)$ or $\alpha r\left(B_{k}\right)>$ $r\left(B_{j}\right)>r\left(B_{k}\right)$.

First consider the case where we have a ball $B(\mathbf{a}, r)$ intersected with other balls of radius larger than $\alpha r$ such that none of the balls contains the center of any other. This is illustrated in the following picture with two balls. This has to do with estimating the number of $B_{j}$ for $j \leq k$ where $r\left(B_{j}\right) \geq \alpha r\left(B_{k}\right)$.


Imagine projecting the center of each big ball as in the above picture onto the surface of the given ball, assuming the given ball has radius 1 . By scaling the balls, you could reduce to this case that the given ball has radius 1 . Then from geometric reasoning, there should be a lower bound to the distance between these two projections depending on dimension. Thus there is an estimate on how many large balls can intersect the given ball with no ball containing a center of another one.

## Intersections with relatively big balls

Lemma 13.13.2 Let the balls $B_{\mathbf{a}}, B_{\mathbf{x}}, B_{\mathbf{y}}$ be as shown, having radii $r, r_{x}, r_{y}$ respectively. Suppose the centers of $B_{\mathbf{x}}$ and $B_{\mathbf{y}}$ are not both in any of the balls shown, and suppose $r_{y} \geq r_{x} \geq \alpha r$ where $\alpha$ is a number larger than 1 . Also let $P_{\mathbf{x}} \equiv \mathbf{a}+r \frac{\mathbf{x}-\mathbf{a}}{\|\mathbf{x}-\mathbf{a}\|}$ with $P_{\mathbf{y}}$ being defined similarly. Then it follows that $\left\|P_{\mathbf{x}}-P_{\mathbf{y}}\right\| \geq \frac{\alpha-1}{\alpha+1} r$. There exists a constant $L(p, \alpha)$ depending on $\alpha$ and the dimension, such that if $B_{1}, \cdots, B_{m}$ are all balls such that any pair are in the same situation relative to $B_{\mathbf{a}}$ as $B_{\mathbf{x}}$, and $B_{\mathbf{y}}$, then $m \leq L(p, \alpha)$.

Proof: From the definition,

$$
\begin{gathered}
\left\|P_{\mathbf{x}}-P_{\mathbf{y}}\right\|=r\left\|\frac{\mathbf{x}-\mathbf{a}}{\|\mathbf{x}-\mathbf{a}\|}-\frac{\mathbf{y}-\mathbf{a}}{\|\mathbf{y}-\mathbf{a}\|}\right\| \\
=r\left\|\frac{(\mathbf{x}-\mathbf{a})\|\mathbf{y}-\mathbf{a}\|-(\mathbf{y}-\mathbf{a})\|\mathbf{x}-\mathbf{a}\|}{\|\mathbf{x}-\mathbf{a}\|\|\mathbf{y}-\mathbf{a}\|}\right\| \\
=r\left\|\frac{\|\mathbf{y}-\mathbf{a}\|(\mathbf{x}-\mathbf{y})+(\mathbf{y}-\mathbf{a})(\|\mathbf{y}-\mathbf{a}\|-\|\mathbf{x}-\mathbf{a}\|)}{\|\mathbf{x}-\mathbf{a}\|\|\mathbf{y}-\mathbf{a}\|}\right\|
\end{gathered}
$$

$$
\begin{align*}
& \geq r \frac{\|\mathbf{x}-\mathbf{y}\|}{\|\mathbf{x}-\mathbf{a}\|}-r \frac{\|\mathbf{y}-\mathbf{a}\| \mid\|\mathbf{y}-\mathbf{a}\|-\|\mathbf{x}-\mathbf{a}\|}{\|\mathbf{x}-\mathbf{a}\|\|\mathbf{y}-\mathbf{a}\|} \\
& =r \frac{\|\mathbf{x}-\mathbf{y}\|}{\|\mathbf{x}-\mathbf{a}\|}-\frac{r}{\|\mathbf{x}-\mathbf{a}\|}|\|\mathbf{y}-\mathbf{a}\|-\|\mathbf{x}-\mathbf{a}\|| . \tag{13.13.35}
\end{align*}
$$

There are two cases. First suppose that $\|\mathbf{y}-\mathbf{a}\|-\|\mathbf{x}-\mathbf{a}\| \geq 0$. Then the above

$$
=r \frac{\|\mathbf{x}-\mathbf{y}\|}{\|\mathbf{x}-\mathbf{a}\|}-\frac{r}{\|\mathbf{x}-\mathbf{a}\|}\|\mathbf{y}-\mathbf{a}\|+r
$$

From the assumptions, $\|\mathbf{x}-\mathbf{y}\| \geq r_{y}$ and also $\|\mathbf{y}-\mathbf{a}\| \leq r+r_{y}$. Hence the above

$$
\begin{aligned}
& \geq r \frac{r_{y}}{\|\mathbf{x}-\mathbf{a}\|}-\frac{r}{\|\mathbf{x}-\mathbf{a}\|}\left(r+r_{y}\right)+r=r-r \frac{r}{\|\mathbf{x}-\mathbf{a}\|} \\
& \geq r\left(1-\frac{r}{\|\mathbf{x}-\mathbf{a}\|}\right) \geq r\left(1-\frac{r}{r_{x}}\right) \geq r\left(1-\frac{1}{\alpha}\right) \geq r \frac{\alpha-1}{\alpha+1} .
\end{aligned}
$$

The other case is that $\|\mathbf{y}-\mathbf{a}\|-\|\mathbf{x}-\mathbf{a}\|<0$ in 13.13.35. Then in this case 13.13.35 equals

$$
\begin{aligned}
& =r\left(\frac{\|\mathbf{x}-\mathbf{y}\|}{\|\mathbf{x}-\mathbf{a}\|}-\frac{1}{\|\mathbf{x}-\mathbf{a}\|}(\|\mathbf{x}-\mathbf{a}\|-\|\mathbf{y}-\mathbf{a}\|)\right) \\
& =\frac{r}{\|\mathbf{x}-\mathbf{a}\|}(\|\mathbf{x}-\mathbf{y}\|-(\|\mathbf{x}-\mathbf{a}\|-\|\mathbf{y}-\mathbf{a}\|))
\end{aligned}
$$

Then since $\|\mathbf{x}-\mathbf{a}\| \leq r+r_{x},\|\mathbf{x}-\mathbf{y}\| \geq r_{y},\|\mathbf{y}-\mathbf{a}\| \geq r_{y}$, and remembering that $r_{y} \geq r_{x} \geq \alpha r$,

$$
\begin{aligned}
& \geq \frac{r}{r_{x}+r}\left(r_{y}-\left(r+r_{x}\right)+r_{y}\right) \geq \frac{r}{r_{x}+r}\left(r_{y}-\left(r+r_{y}\right)+r_{y}\right) \\
& \geq \frac{r}{r_{x}+r}\left(r_{y}-r\right) \geq \frac{r}{r_{x}+r}\left(r_{x}-r\right) \geq \frac{r}{r_{x}+\frac{1}{\alpha} r_{x}}\left(r_{x}-\frac{1}{\alpha} r_{x}\right) \\
& =\frac{r}{1+(1 / \alpha)}(1-1 / \alpha)=\frac{\alpha-1}{\alpha+1} r
\end{aligned}
$$

Replacing $r$ with something larger, $\frac{1}{\alpha} r_{x}$ is justified by the observation that $x \rightarrow \frac{\alpha-x}{\alpha+x}$ is decreasing. This proves the estimate between $P_{\mathbf{x}}$ and $P_{\mathbf{y}}$.

Finally, in the case of the balls $B_{i}$ having centers at $\mathbf{x}_{i}$, then as above, let $P_{\mathbf{x}_{i}}=\mathbf{a}+$ $r \frac{\mathbf{x}_{i}-\mathbf{a}}{\left\|\mathbf{x}_{i}-\mathbf{a}\right\|}$. Then $\left(P_{\mathbf{x}_{i}}-\mathbf{a}\right) r^{-1}$ is on the unit sphere having center $\mathbf{0}$. Furthermore,

$$
\left\|\left(P_{\mathbf{x}_{i}}-\mathbf{a}\right) r^{-1}-\left(P_{\mathbf{y}_{i}}-\mathbf{a}\right) r^{-1}\right\|=r^{-1}\left\|P_{\mathbf{x}_{i}}-P_{\mathbf{y}_{i}}\right\| \geq r^{-1} r \frac{\alpha-1}{\alpha+1}=\frac{\alpha-1}{\alpha+1} .
$$

How many points on the unit sphere can be pairwise this far apart? The unit sphere is compact and so there exists a $\frac{1}{4}\left(\frac{\alpha-1}{\alpha+1}\right)$ net having $L(p, \alpha)$ points. Thus $m$ cannot be any larger than $L(p, \alpha)$ because if it were, then by the pigeon hole principal, two of the points $\left(P_{\mathbf{x}_{i}}-\mathbf{a}\right) r^{-1}$ would lie in a single ball $B\left(p, \frac{1}{4}\left(\frac{\alpha-1}{\alpha+1}\right)\right)$ so they could not be $\frac{\alpha-1}{\alpha+1}$ apart.

The above lemma has to do with balls which are relatively large intersecting a given ball. Next is a lemma which has to do with relatively small balls intersecting a given ball. First is another lemma.

Lemma 13.13.3 Let $\Gamma>1$ and $B(\mathbf{a}, \Gamma r)$ be a ball and suppose $\left\{B\left(\mathbf{x}_{i}, r_{i}\right)\right\}_{i=1}^{m}$ are balls contained in $B(\mathbf{a}, \Gamma r)$ such that $r \leq r_{i}$ and none of these balls contains the center of another ball. Then there is a constant $M(p, \Gamma)$ such that $m \leq M(p, \Gamma)$.

Proof: Let $\mathbf{z}_{i}=\mathbf{x}_{i}-\mathbf{a}$. Then $B\left(\mathbf{z}_{i}, r_{i}\right)$ are balls contained in $B(\mathbf{0}, \Gamma r)$ with no ball containing a center of another. Then $B\left(\frac{\mathbf{z}_{i}}{\Gamma r}, \frac{r_{i}}{\Gamma r}\right)$ are balls in $B(\mathbf{0}, 1)$ with no ball containing the center of another. By compactness, there is a $\frac{1}{8 \Gamma}$ net for $\overline{B(\mathbf{0}, 1)},\left\{\mathbf{y}_{i}\right\}_{i=1}^{M(p, \Gamma)}$. Thus the balls $B\left(\mathbf{y}_{i}, \frac{1}{8 \Gamma}\right)$ cover $\overline{B(\mathbf{0}, 1)}$. If $m \geq M(p, \Gamma)$, then by the pigeon hole principle, one of these $B\left(\mathbf{y}_{i}, \frac{1}{8 \Gamma}\right)$ would contain some $\frac{\mathbf{z}_{i}}{\Gamma r}$ and $\frac{\mathbf{z}_{j}}{\Gamma r}$ which requires $\left\|\frac{\mathbf{z}_{i}}{\Gamma r}-\frac{\mathbf{z}_{j}}{\Gamma r}\right\| \leq \frac{1}{4 \Gamma}<\frac{r_{j}}{4 \Gamma r}$ so $\frac{\mathbf{z}_{i}}{\Gamma r} \in B\left(\frac{\mathbf{z}_{j}}{\Gamma r}, \frac{r_{j}}{\Gamma r}\right)$. Thus $m \leq M(p, \gamma, \Gamma)$.

## Intersections with small balls

Lemma 13.13.4 Let $B$ be a ball having radius $r$ and suppose $B$ has nonempty intersection with the balls $B_{1}, \cdots, B_{m}$ having radii $r_{1}, \cdots, r_{m}$ respectively, and as before, no $B_{i}$ contains the center of any other and the centers of the $B_{i}$ are not contained in B. Suppose $\alpha>1$ and $r \leq \min \left(r_{1}, \cdots, r_{m}\right)$, each $r_{i}<\alpha r$. Then there exists a constant $M(p, \alpha)$ such that $m \leq M(p, \alpha)$.

Proof: Let $B=B(\mathbf{a}, r)$. Then each $B_{i}$ is contained in $B(\mathbf{a}, 2 r+\alpha r+\alpha r)$. This is because if $\mathbf{y} \in B_{i} \equiv B\left(\mathbf{x}_{i}, r_{i}\right)$,

$$
\|\mathbf{y}-\mathbf{a}\| \leq\left\|\mathbf{y}-\mathbf{x}_{i}\right\|+\left\|\mathbf{x}_{i}-\mathbf{a}\right\| \leq r_{i}+r+r_{i}<2 r+\alpha r+\alpha r
$$

Thus $B_{i}$ does not contain the center of any other $B_{j}$. Then these balls are contained in $B(\mathbf{a}, r(2 \alpha+2))$, and each radius is at least as large as $r$. By Lemma 13.13.3 there is a constant $M(p, \alpha)$ such that $m \leq M(p, \alpha)$.

Now here is the Besicovitch covering theorem. In the proof, we are considering the sequence of balls described above.

Theorem 13.13.5 There exists a constant $N_{p}$, depending only on $p$ with the following property. If $\mathscr{F}$ is any collection of nonempty balls in $X$ with

$$
\sup \{\operatorname{diam}(B): B \in \mathscr{F}\}<D<\infty
$$

and if $A$ is the set of centers of the balls in $\mathscr{F}$, then there exist subsets of $\mathscr{F}, \mathscr{H}_{1}, \cdots, \mathscr{H}_{N_{p}}$, such that each $\mathscr{H}_{i}$ is a countable collection of disjoint balls from $\mathscr{F}$ (possibly empty) and

$$
A \subseteq \cup_{i=1}^{N_{p}} \cup\left\{B: B \in \mathscr{H}_{i}\right\}
$$

Proof: To begin with, suppose $A$ is bounded. Let $L(p, \alpha)$ be the constant of Lemma 13.13.2 and let $M_{p}=L(p, \alpha)+M(p, \alpha)+1$. Define the following sequence of subsets of $\mathscr{F}, \mathscr{G}_{1}, \mathscr{G}_{2}, \cdots, \mathscr{G}_{M_{p}}$. Referring to the sequence $\left\{B_{k}\right\}$ considered in Lemma 13.13.1, let $B_{1} \in \mathscr{G}_{1}$ and if $B_{1}, \cdots, B_{m}$ have been assigned, each to a $\mathscr{G}_{i}$, place $B_{m+1}$ in the first $\mathscr{G}_{j}$ such that $B_{m+1}$ intersects no set already in $\mathscr{G}_{j}$. The existence of such a $j$ follows from Lemmas 13.13.2 and 13.13.4 and the pigeon hole principle. Here is why. $B_{m+1}$ can intersect at most $L(p, \alpha)$ sets of $\left\{B_{1}, \cdots, B_{m}\right\}$ which have radii at least as large as $\alpha r\left(B_{m+1}\right)$ thanks to

Lemma 13.13.2. It can intersect at most $M(p, \alpha)$ sets of $\left\{B_{1}, \cdots, B_{m}\right\}$ which have radius smaller than $\alpha r\left(B_{m+1}\right)$ thanks to Lemma 13.13.4. Thus each $\mathscr{G}_{j}$ consists of disjoint sets of $\mathscr{F}$ and the set of centers is covered by the union of these $\mathscr{G}_{j}$. This proves the theorem in case the set of centers is bounded.

Now let $R_{1}=B(\mathbf{0}, 5 D)$ and if $R_{m}$ has been chosen, let

$$
R_{m+1}=B(\mathbf{0},(m+1) 5 D) \backslash R_{m}
$$

Thus, if $|k-m| \geq 2$, no ball from $\mathscr{F}$ having nonempty intersection with $R_{m}$ can intersect any ball from $\mathscr{F}$ which has nonempty intersection with $R_{k}$. This is because all these balls have radius less than $D$. Now let $A_{m} \equiv A \cap R_{m}$ and apply the above result for a bounded set of centers to those balls of $\mathscr{F}$ which intersect $R_{m}$ to obtain sets of disjoint balls $\mathscr{G}_{1}\left(R_{m}\right), \mathscr{G}_{2}\left(R_{m}\right), \cdots, \mathscr{G}_{M_{p}}\left(R_{m}\right)$ covering $A_{m}$. Then simply define $\mathscr{G}_{j}^{\prime} \equiv \cup_{k=1}^{\infty} \mathscr{G}_{j}\left(R_{2 k}\right), \mathscr{G}_{j} \equiv$ $\cup_{k=1}^{\infty} \mathscr{G}_{j}\left(R_{2 k-1}\right)$. Let $N_{p}=2 M_{p}$ and

$$
\left\{\mathscr{H}_{1}, \cdots, \mathscr{H}_{N_{p}}\right\} \equiv\left\{\mathscr{G}_{1}^{\prime}, \cdots, \mathscr{G}_{M_{p}}^{\prime}, \mathscr{G}_{1}, \cdots, \mathscr{G}_{M_{p}}\right\}
$$

Note that the balls in $\mathscr{G}_{j}^{\prime}$ are disjoint. This is because those in $\mathscr{G}_{j}\left(R_{2 k}\right)$ are disjoint and if you consider any ball in $\mathscr{G}_{j}\left(R_{2 m}\right)$, it cannot intersect a ball of $\mathscr{G}_{j}\left(R_{2 k}\right)$ for $m \neq k$ because $|2 k-2 m| \geq 2$. Similar considerations apply to the balls of $\mathscr{G}_{j}$.

Of course, you could pick a particular $\alpha$. If you make $\alpha \operatorname{larger,~} L(p, \alpha)$ should get smaller and $M(p, \alpha)$ should get larger. Obviously one could explore this at length to try and get a best choice of $\alpha$.

### 13.14 Vitali Coverings and Radon Measures

There is another covering theorem which may also be referred to as the Besicovitch covering theorem. As before, the balls can be taken with respect to any norm on $\mathbb{R}^{n}$. At first, the balls will be closed but this assumption will be removed.

Definition 13.14.1 A collection of balls in $\mathbb{R}^{p}, \mathscr{F}$ covers a set $E$ in the sense of Vitali if whenever $\mathbf{x} \in E$ and $\varepsilon>0$, there exists a ball $B \in \mathscr{F}$ whose center is $\mathbf{x}$ having diameter less than $\varepsilon$.

I will give a proof of the following theorem.
Theorem 13.14.2 Let $\mu$ be a Radon measure on $\mathbb{R}^{p}$ and let $E$ be a set with $\bar{\mu}(E)<\infty$. Where $\bar{\mu}$ is the outer measure determined by $\mu$. Suppose $\mathscr{F}$ is a collection of closed balls which cover $E$ in the sense of Vitali. Then there exists a sequence of disjoint balls, $\left\{B_{i}\right\} \subseteq \mathscr{F}$ such that

$$
\bar{\mu}\left(E \backslash \cup_{j=1}^{\infty} B_{j}\right)=0
$$

Proof: Let $N_{p}$ be the constant of the Besicovitch covering theorem. Choose $r>0$ such that

$$
(1-r)^{-1}\left(1-\frac{1}{2 N_{p}+2}\right) \equiv \lambda<1
$$

If $\bar{\mu}(E)=0$, there is nothing to prove so assume $\bar{\mu}(E)>0$. Let $U_{1}$ be an open set containing $E$ with $(1-r) \mu\left(U_{1}\right)<\bar{\mu}(E)$ and $2 \bar{\mu}(E)>\mu\left(U_{1}\right)$, and let $\mathscr{F}_{1}$ be those sets of $\mathscr{F}$ which are contained in $U_{1}$ whose centers are in $E$. Thus $\mathscr{F}_{1}$ is also a Vitali cover of $E$. Now by the Besicovitch covering theorem proved earlier, there exist balls, $B$, of $\mathscr{F}_{1}$ such that

$$
E \subseteq \cup_{i=1}^{N_{p}}\left\{B: B \in \mathscr{G}_{i}\right\}
$$

where $\mathscr{G}_{i}$ consists of a collection of disjoint balls of $\mathscr{F}_{1}$. Therefore,

$$
\bar{\mu}(E) \leq \sum_{i=1}^{N_{p}} \sum_{B \in \mathscr{G}_{i}} \mu(B)
$$

and so, for some $i \leq N_{p}$,

$$
\left(N_{p}+1\right) \sum_{B \in \mathscr{G}_{i}} \mu(B)>\bar{\mu}(E)
$$

It follows there exists a finite set of balls of $\mathscr{G}_{i},\left\{B_{1}, \cdots, B_{m_{1}}\right\}$ such that

$$
\begin{equation*}
\left(N_{p}+1\right) \sum_{i=1}^{m_{1}} \mu\left(B_{i}\right)>\bar{\mu}(E) \tag{13.14.36}
\end{equation*}
$$

and so

$$
\left(2 N_{p}+2\right) \sum_{i=1}^{m_{1}} \mu\left(B_{i}\right)>2 \bar{\mu}(E)>\mu\left(U_{1}\right)
$$

Since $2 \bar{\mu}(E) \geq \mu\left(U_{1}\right), 13.14 .36$ implies

$$
\frac{\mu\left(U_{1}\right)}{2 N_{2}+2} \leq \frac{2 \bar{\mu}(E)}{2 N_{2}+2}=\frac{\bar{\mu}(E)}{N_{2}+1}<\sum_{i=1}^{m_{1}} \mu\left(B_{i}\right)
$$

Also $U_{1}$ was chosen such that $(1-r) \mu\left(U_{1}\right)<\bar{\mu}(E)$, and so

$$
\begin{gathered}
\lambda \bar{\mu}(E) \geq \lambda(1-r) \mu\left(U_{1}\right)=\left(1-\frac{1}{2 N_{p}+2}\right) \mu\left(U_{1}\right) \\
\geq \mu\left(U_{1}\right)-\sum_{i=1}^{m_{1}} \mu\left(B_{i}\right)=\mu\left(U_{1}\right)-\mu\left(\cup_{j=1}^{m_{1}} B_{j}\right) \\
=\mu\left(U_{1} \backslash \cup_{j=1}^{m_{1}} B_{j}\right) \geq \bar{\mu}\left(E \backslash \cup_{j=1}^{m_{1}} B_{j}\right)
\end{gathered}
$$

Since the balls are closed, you can consider the sets of $\mathscr{F}$ which have empty intersection with $\cup_{j=1}^{m_{1}} B_{j}$ and this new collection of sets will be a Vitali cover of $E \backslash \cup_{j=1}^{m_{1}} B_{j}$. Letting this collection of balls play the role of $\mathscr{F}$ in the above argument and letting $E \backslash \cup_{j=1}^{m_{1}} B_{j}$ play the role of $E$, repeat the above argument and obtain disjoint sets of $\mathscr{F}$,

$$
\left\{B_{m_{1}+1}, \cdots, B_{m_{2}}\right\}
$$

such that

$$
\lambda \bar{\mu}\left(E \backslash \cup_{j=1}^{m_{1}} B_{j}\right)>\bar{\mu}\left(\left(E \backslash \cup_{j=1}^{m_{1}} B_{j}\right) \backslash \cup_{j=m_{1}+1}^{m_{2}} B_{j}\right)=\bar{\mu}\left(E \backslash \cup_{j=1}^{m_{2}} B_{j}\right),
$$

and so

$$
\lambda^{2} \bar{\mu}(E)>\bar{\mu}\left(E \backslash \cup_{j=1}^{m_{2}} B_{j}\right)
$$

Continuing in this way, yields a sequence of disjoint balls $\left\{B_{i}\right\}$ contained in $\mathscr{F}$ and

$$
\bar{\mu}\left(E \backslash \cup_{j=1}^{\infty} B_{j}\right) \leq \bar{\mu}\left(E \backslash \cup_{j=1}^{m_{k}} B_{j}\right)<\lambda^{k} \bar{\mu}(E)
$$

for all $k$. Therefore, $\bar{\mu}\left(E \backslash \cup_{j=1}^{\infty} B_{j}\right)=0$.
It is not necessary to assume $\bar{\mu}(E)<\infty$.
Corollary 13.14.3 Let $\mu$ be a Radon measure on $\mathbb{R}^{p}$. Letting $\bar{\mu}$ be the outer measure determined by $\mu$, suppose $\mathscr{F}$ is a collection of closed balls which cover $E$ in the sense of Vitali. Then there exists a sequence of disjoint balls, $\left\{B_{i}\right\} \subseteq \mathscr{F}$ such that

$$
\bar{\mu}\left(E \backslash \cup_{j=1}^{\infty} B_{j}\right)=0
$$

Proof: Since $\mu$ is a Radon measure it is finite on compact sets. Therefore, there are at most countably many numbers, $\left\{b_{i}\right\}_{i=1}^{\infty}$ such that $\mu\left(\partial B\left(\mathbf{0}, b_{i}\right)\right)>0$. It follows there exists an increasing sequence of positive numbers, $\left\{r_{i}\right\}_{i=1}^{\infty}$ such that $\lim _{i \rightarrow \infty} r_{i}=\infty$ and $\mu\left(\partial B\left(\mathbf{0}, r_{i}\right)\right)=0$. Now let

$$
\begin{aligned}
D_{1} & \equiv\left\{\mathbf{x}:\|\mathbf{x}\|<r_{1}\right\}, D_{2} \equiv\left\{\mathbf{x}: r_{1}<\|\mathbf{x}\|<r_{2}\right\} \\
\cdots, D_{m} & \equiv\left\{\mathbf{x}: r_{m-1}<\|\mathbf{x}\|<r_{m}\right\}, \cdots
\end{aligned}
$$

Let $\mathscr{F}_{m}$ denote those closed balls of $\mathscr{F}$ which are contained in $D_{m}$. Then letting $E_{m}$ denote $E \cap D_{m}, \mathscr{F}_{m}$ is a Vitali cover of $E_{m}, \bar{\mu}\left(E_{m}\right)<\infty$, and so by Theorem 13.14.2, there exists a countable sequence of balls from $\mathscr{F}_{m}\left\{B_{j}^{m}\right\}_{j=1}^{\infty}$, such that $\bar{\mu}\left(E_{m} \backslash \cup_{j=1}^{\infty} B_{j}^{m}\right)=0$. Then consider the countable collection of balls, $\left\{B_{j}^{m}\right\}_{j, m=1}^{\infty}$.

$$
\begin{aligned}
\bar{\mu}\left(E \backslash \cup_{m=1}^{\infty} \cup_{j=1}^{\infty} B_{j}^{m}\right) & \leq \bar{\mu}\left(\cup_{j=1}^{\infty} \partial B\left(\mathbf{0}, r_{i}\right)\right)+ \\
+\sum_{m=1}^{\infty} \bar{\mu}\left(E_{m} \backslash \cup_{j=1}^{\infty} B_{j}^{m}\right) & =0 \boldsymbol{\square}
\end{aligned}
$$

You don't need to assume the balls are closed. In fact, the balls can be open, closed or anything in between and the same conclusion can be drawn provided you change the definition of a Vitali cover a little.

Corollary 13.14.4 Let $\mu$ be a Radon measure on $\mathbb{R}^{p}$. Letting $\bar{\mu}$ be the outer measure determined by $\mu$, suppose $\mathscr{F}$ is a collection of balls which cover $E$ in the sense that for all $\varepsilon>0$ there are uncountably many balls of $\mathscr{F}$ centered at $\mathbf{x}$ having radius less than $\varepsilon$. Then there exists a sequence of disjoint balls, $\left\{B_{i}\right\} \subseteq \mathscr{F}$ such that

$$
\bar{\mu}\left(E \backslash \cup_{j=1}^{\infty} B_{j}\right)=0
$$

Proof: Let $\mathbf{x} \in E$. Thus $\mathbf{x}$ is the center of arbitrarily small balls from $\mathscr{F}$. Since $\mu$ is a Radon measure, at most countably many radii, $r$ of these balls can have the property that $\mu(\partial B(\mathbf{0}, r))=0$. Let $\mathscr{F}^{\prime}$ denote the closures of the balls of $\mathscr{F}, \overline{B(\mathbf{x}, r)}$ with the property that $\mu(\partial B(\mathbf{x}, r))=0$. Since for each $\mathbf{x} \in E$ there are only countably many exceptions, $\mathscr{F}^{\prime}$ is still a Vitali cover of $E$. Therefore, by Corollary 13.14.3 there is a disjoint sequence of these balls of $\mathscr{F}^{\prime},\left\{\overline{B_{i}}\right\}_{i=1}^{\infty}$ for which

$$
\bar{\mu}\left(E \backslash \cup_{j=1}^{\infty} \overline{B_{j}}\right)=0
$$

However, since their boundaries have $\mu$ measure zero, it follows

$$
\bar{\mu}\left(E \backslash \cup_{j=1}^{\infty} B_{j}\right)=0
$$

## Chapter 14

## Some Extension Theorems

### 14.1 Algebras

First of all, here is the definition of an algebra and theorems which tell how to recognize one when you see it. An algebra is like a $\sigma$ algebra except it is only closed with respect to finite unions.

Definition 14.1.1 $\mathscr{A}$ is said to be an algebra of subsets of a set, $Z$ if $Z \in \mathscr{A}, \emptyset \in \mathscr{A}$, and when $E, F \in \mathscr{A}, E \cup F$ and $E \backslash F$ are both in $\mathscr{A}$.

It is important to note that if $\mathscr{A}$ is an algebra, then it is also closed under finite intersections. This is because $E \cap F=\left(E^{C} \cup F^{C}\right)^{C} \in \mathscr{A}$ since $E^{C}=Z \backslash E \in \mathscr{A}$ and $F^{C}=Z \backslash F \in \mathscr{A}$. Note that every $\sigma$ algebra is an algebra but not the other way around.

Something satisfying the above definition is called an algebra because union is like addition, the set difference is like subtraction and intersection is like multiplication. Furthermore, only finitely many operations are done at a time and so there is nothing like a limit involved.

How can you recognize an algebra when you see one? The answer to this question is the purpose of the following lemma.

Lemma 14.1.2 Suppose $\mathscr{R}$ and $\mathscr{E}$ are subsets of $\mathscr{P}(Z)^{1}$ such that $\mathscr{E}$ is defined as the set of all finite disjoint unions of sets of $\mathscr{R}$. Suppose also

$$
\begin{gathered}
\emptyset, Z \in \mathscr{R} \\
A \cap B \in \mathscr{R} \text { whenever } A, B \in \mathscr{R} \\
A \backslash B \in \mathscr{E} \text { whenever } A, B \in \mathscr{R} .
\end{gathered}
$$

Then $\mathscr{E}$ is an algebra of sets of $Z$.
Proof: Note first that if $A \in \mathscr{R}$, then $A^{C} \in \mathscr{E}$ because $A^{C}=Z \backslash A$.
Now suppose that $E_{1}$ and $E_{2}$ are in $\mathscr{E}$,

$$
E_{1}=\cup_{i=1}^{m} R_{i}, \quad E_{2}=\cup_{j=1}^{n} R_{j}
$$

where the $R_{i}$ are disjoint sets in $\mathscr{R}$ and the $R_{j}$ are disjoint sets in $\mathscr{R}$. Then

$$
E_{1} \cap E_{2}=\cup_{i=1}^{m} \cup_{j=1}^{n} R_{i} \cap R_{j}
$$

which is clearly an element of $\mathscr{E}$ because no two of the sets in the union can intersect and by assumption they are all in $\mathscr{R}$. Thus by induction, finite intersections of sets of $\mathscr{E}$ are in $\mathscr{E}$. Consider the difference of two elements of $\mathscr{E}$ next.

If $E=\cup_{i=1}^{n} R_{i} \in \mathscr{E}$,

$$
E^{C}=\cap_{i=1}^{n} R_{i}^{C}=\text { finite intersection of sets of } \mathscr{E}
$$

[^14]which was just shown to be in $\mathscr{E}$. Now, if $E_{1}, E_{2} \in \mathscr{E}$,
$$
E_{1} \backslash E_{2}=E_{1} \cap E_{2}^{C} \in \mathscr{E}
$$
from what was just shown about finite intersections.
Finally consider finite unions of sets of $\mathscr{E}$. Let $E_{1}$ and $E_{2}$ be sets of $\mathscr{E}$. Then
$$
E_{1} \cup E_{2}=\left(E_{1} \backslash E_{2}\right) \cup E_{2} \in \mathscr{E}
$$
because $E_{1} \backslash E_{2}$ consists of a finite disjoint union of sets of $\mathscr{R}$ and these sets must be disjoint from the sets of $\mathscr{R}$ whose union yields $E_{2}$ because $\left(E_{1} \backslash E_{2}\right) \cap E_{2}=\emptyset$. This proves the lemma.

The following corollary is particularly helpful in verifying the conditions of the above lemma.

Corollary 14.1.3 Let $\left(Z_{1}, \mathscr{R}_{1}, \mathscr{E}_{1}\right)$ and $\left(Z_{2}, \mathscr{R}_{2}, \mathscr{E}_{2}\right)$ be as described in Lemma 14.1.2. Then $\left(Z_{1} \times Z_{2}, \mathscr{R}, \mathscr{E}\right)$ also satisfies the conditions of Lemma 14.1.2 if $\mathscr{R}$ is defined as

$$
\mathscr{R} \equiv\left\{R_{1} \times R_{2}: R_{i} \in \mathscr{R}_{i}\right\}
$$

and

$$
\mathscr{E} \equiv\{\text { finite disjoint unions of sets of } \mathscr{R}\}
$$

Consequently, $\mathscr{E}$ is an algebra of sets.
Proof: It is clear $\emptyset, Z_{1} \times Z_{2} \in \mathscr{R}$. Let $A \times B$ and $C \times D$ be two elements of $\mathscr{R}$.

$$
A \times B \cap C \times D=A \cap C \times B \cap D \in \mathscr{R}
$$

by assumption.

$$
\begin{aligned}
A \times & \overbrace{(B \backslash D)}^{\in \mathscr{E}_{2}} \cup \overbrace{(A \backslash C)}^{\in \mathscr{E}_{1}} \times \overbrace{(D \cap B)}^{\in \mathscr{R}_{2}} \\
& =(A \times Q) \cup(P \times R)
\end{aligned}
$$

where $Q \in \mathscr{E}_{2}, P \in \mathscr{E}_{1}$, and $R \in \mathscr{R}_{2}$.


Since $A \times Q$ and $P \times R$ do not intersect, it follows the above expression is in $\mathscr{E}$ because each of these terms are. This proves the corollary.

### 14.2 Caratheodory Extension Theorem

The Caratheodory extension theorem is a fundamental result which makes possible the consideration of measures on infinite products among other things. The idea is that if a finite measure defined only on an algebra is trying to be a measure, then in fact it can be extended to a measure.

Definition 14.2.1 Let $\mathscr{E}$ be an algebra of sets of $\Omega$ and let $\mu_{0}$ be a finite measure on $\mathscr{E}$. This means $\mu_{0}$ is finitely additive and if $E_{i}, E$ are sets of $\mathscr{E}$ with the $E_{i}$ disjoint and

$$
E=\cup_{i=1}^{\infty} E_{i}
$$

then

$$
\mu_{0}(E)=\sum_{i=1}^{\infty} \mu_{0}\left(E_{i}\right)
$$

while $\mu_{0}(\Omega)<\infty$.
In this definition, $\mu_{0}$ is trying to be a measure and acts like one whenever possible. Under these conditions, $\mu_{0}$ can be extended uniquely to a complete measure, $\mu$, defined on a $\sigma$ algebra of sets containing $\mathscr{E}$ such that $\mu$ agrees with $\mu_{0}$ on $\mathscr{E}$. The following is the main result.

Theorem 14.2.2 Let $\mu_{0}$ be a measure on an algebra of sets, $\mathscr{E}$, which satisfies $\mu_{0}(\Omega)<\infty$. Then there exists a complete measure space $(\Omega, \mathscr{S}, \mu)$ such that

$$
\mu(E)=\mu_{0}(E)
$$

for all $E \in \mathscr{E}$. Also if $v$ is any such measure which agrees with $\mu_{0}$ on $\mathscr{E}$, then $v=\mu$ on $\sigma(\mathscr{E})$, the $\sigma$ algebra generated by $\mathscr{E}$.

Proof: Define an outer measure as follows.

$$
\mu(S) \equiv \inf \left\{\sum_{i=1}^{\infty} \mu_{0}\left(E_{i}\right): S \subseteq \cup_{i=1}^{\infty} E_{i}, E_{i} \in \mathscr{E}\right\}
$$

Claim 1: $\mu$ is an outer measure.
Proof of Claim 1: Let $S \subseteq \cup_{i=1}^{\infty} S_{i}$ and let $S_{i} \subseteq \cup_{j=1}^{\infty} E_{i j}$, where

$$
\mu\left(S_{i}\right)+\frac{\varepsilon}{2^{i}} \geq \sum_{j=1}^{\infty} \mu\left(E_{i j}\right)
$$

Then

$$
\mu(S) \leq \sum_{i} \sum_{j} \mu\left(E_{i j}\right)=\sum_{i}\left(\mu\left(S_{i}\right)+\frac{\varepsilon}{2^{i}}\right)=\sum_{i} \mu\left(S_{i}\right)+\varepsilon .
$$

Since $\varepsilon$ is arbitrary, this shows $\mu$ is an outer measure as claimed.
By the Caratheodory procedure, there exists a unique $\sigma$ algebra, $\mathscr{S}$, consisting of the $\mu$ measurable sets such that

$$
(\Omega, \mathscr{S}, \mu)
$$

is a complete measure space. It remains to show $\mu$ extends $\mu_{0}$.
Claim 2: If $\mathscr{S}$ is the $\sigma$ algebra of $\mu$ measurable sets, $\mathscr{S} \supseteq \mathscr{E}$ and $\mu=\mu_{0}$ on $\mathscr{E}$.
Proof of Claim 2: First observe that if $A \in \mathscr{E}$, then $\mu(A) \leq \mu_{0}(A)$ by definition. Letting

$$
\mu(A)+\varepsilon>\sum_{i=1}^{\infty} \mu_{0}\left(E_{i}\right), \cup_{i=1}^{\infty} E_{i} \supseteq A, E_{i} \in \mathscr{E}
$$

it follows

$$
\mu(A)+\varepsilon>\sum_{i=1}^{\infty} \mu_{0}\left(E_{i} \cap A\right) \geq \mu_{0}(A)
$$

since $A=\cup_{i=1}^{\infty} E_{i} \cap A$. Therefore, $\mu=\mu_{0}$ on $\mathscr{E}$.
Consider the assertion that $\mathscr{E} \subseteq \mathscr{S}$. Let $A \in \mathscr{E}$ and let $S \subseteq \Omega$ be any set. There exist sets $\left\{E_{i}\right\} \subseteq \mathscr{E}$ such that $\cup_{i=1}^{\infty} E_{i} \supseteq S$ but

$$
\mu(S)+\varepsilon>\sum_{i=1}^{\infty} \mu\left(E_{i}\right)
$$

Then

$$
\begin{gathered}
\mu(S) \leq \mu(S \cap A)+\mu(S \backslash A) \\
\leq \mu\left(\cup_{i=1}^{\infty} E_{i} \backslash A\right)+\mu\left(\cup_{i=1}^{\infty}\left(E_{i} \cap A\right)\right) \\
\leq \sum_{i=1}^{\infty} \mu\left(E_{i} \backslash A\right)+\sum_{i=1}^{\infty} \mu\left(E_{i} \cap A\right)=\sum_{i=1}^{\infty} \mu\left(E_{i}\right)<\mu(S)+\varepsilon
\end{gathered}
$$

Since $\varepsilon$ is arbitrary, this shows $A \in \mathscr{S}$.
This has proved the existence part of the theorem. To verify uniqueness, Let

$$
\mathscr{G} \equiv\{E \in \sigma(\mathscr{E}): \mu(E)=v(E)\}
$$

Then $\mathscr{G}$ is given to contain $\mathscr{E}$ and is obviously closed with respect to countable disjoint unions and complements. Therefore by Lemma 12.12.3, $\mathscr{G} \supseteq \sigma(\mathscr{E})$ and this proves the lemma.

The following lemma is also very significant.
Lemma 14.2.3 Let $M$ be a metric space with the closed balls compact and suppose $\mu$ is a measure defined on the Borel sets of $M$ which is finite on compact sets. Then there exists a unique Radon measure, $\bar{\mu}$ which equals $\mu$ on the Borel sets. In particular $\mu$ must be both inner and outer regular on all Borel sets.

Proof: Define a positive linear functional, $\Lambda(f)=\int f d \mu$. Let $\bar{\mu}$ be the Radon measure which comes from the Riesz representation theorem for positive linear functionals. Thus for all $f \in C_{0}(M)$,

$$
\int f d \mu=\int f d \bar{\mu}
$$

If $V$ is an open set, let $\left\{f_{n}\right\}$ be a sequence of continuous functions in $C_{0}(M)$ which is increasing and converges to $\mathscr{X}_{V}$ pointwise. Then applying the monotone convergence theorem,

$$
\int \mathscr{X}_{V} d \mu=\mu(V)=\int \mathscr{X}_{V} d \bar{\mu}=\bar{\mu}(V)
$$

and so the two measures coincide on all open sets. Every compact set is a countable intersection of open sets and so the two measures coincide on all compact sets. Now let $B(a, n)$ be a ball of radius $n$ and let $E$ be a Borel set contained in this ball. Then by regularity of $\bar{\mu}$ there exist sets $F, G$ such that $G$ is a countable intersection of open sets and $F$ is a countable union of compact sets such that $F \subseteq E \subseteq G$ and $\bar{\mu}(G \backslash F)=0$. Now $\mu(G)=\bar{\mu}(G)$ and $\mu(F)=\bar{\mu}(F)$. Thus

$$
\begin{aligned}
\bar{\mu}(G \backslash F)+\bar{\mu}(F) & =\bar{\mu}(G) \\
& =\mu(G)=\mu(G \backslash F)+\mu(F)
\end{aligned}
$$

and so $\mu(G \backslash F)=\bar{\mu}(G \backslash F)$. It follows

$$
\mu(E)=\mu(F)=\bar{\mu}(F)=\bar{\mu}(G)=\bar{\mu}(E)
$$

If $E$ is an arbitrary Borel set, then

$$
\mu(E \cap B(a, n))=\bar{\mu}(E \cap B(a, n))
$$

and letting $n \rightarrow \infty$, this yields $\mu(E)=\bar{\mu}(E)$.

### 14.3 The Tychonoff Theorem

Sometimes it is necessary to consider infinite Cartesian products of topological spaces. When you have finitely many topological spaces in the product and each is compact, it can be shown that the Cartesian product is compact with the product topology. It turns out that the same thing holds for infinite products but you have to be careful how you define the topology. The first thing likely to come to mind by analogy with finite products is not the right way to do it.

First recall the Hausdorff maximal principle.

Theorem 14.3.1 (Hausdorff maximal principle) Let $\mathscr{F}$ be a nonempty partially ordered set. Then there exists a maximal chain.

The main tool in the study of products of compact topological spaces is the Alexander subbasis theorem which is presented next. Recall a set is compact if every basic open cover admits a finite subcover. This was pretty easy to prove. However, there is a much smaller set of open sets called a subbasis which has this property. The proof of this result is much harder.

Definition 14.3.2 $\mathscr{S} \subseteq \tau$ is called a subbasis for the topology $\tau$ if the set $\mathscr{B}$ of finite intersections of sets of $\mathscr{S}$ is a basis for the topology, $\tau$.

Theorem 14.3.3 Let $(X, \tau)$ be a topological space and let $\mathscr{S} \subseteq \tau$ be a subbasis for $\tau$. Then if $H \subseteq X, H$ is compact if and only if every open cover of $H$ consisting entirely of sets of $\mathscr{S}$ admits a finite subcover.

Proof: The only if part is obvious because the subasic sets are themselves open.
If every basic open cover admits a finite subcover then the set in question is compact. Suppose then that $H$ is a subset of $X$ having the property that subbasic open covers admit finite subcovers. Is $H$ compact? Assume this is not so. Then what was just observed about basic covers implies there exists a basic open cover of $H, \mathscr{O}$, which admits no finite subcover. Let $\mathscr{F}$ be defined as

$$
\{\mathscr{O}: \mathscr{O} \text { is a basic open cover of } H \text { which admits no finite subcover }\} .
$$

The assumption is that $\mathscr{F}$ is nonempty. Partially order $\mathscr{F}$ by set inclusion and use the Hausdorff maximal principle to obtain a maximal chain, $\mathscr{C}$, of such open covers and let

$$
\mathscr{D}=\cup \mathscr{C} .
$$

If $\mathscr{D}$ admits a finite subcover, then since $\mathscr{C}$ is a chain and the finite subcover has only finitely many sets, some element of $\mathscr{C}$ would also admit a finite subcover, contrary to the definition of $\mathscr{F}$. Therefore, $\mathscr{D}$ admits no finite subcover. If $\mathscr{D}^{\prime}$ properly contains $\mathscr{D}$ and $\mathscr{D}^{\prime}$ is a basic open cover of $H$, then $\mathscr{D}^{\prime}$ has a finite subcover of $H$ since otherwise, $\mathscr{C}$ would fail to be a maximal chain, being properly contained in $\mathscr{C} \cup\left\{\mathscr{D}^{\prime}\right\}$. Every set of $\mathscr{D}$ is of the form

$$
U=\cap_{i=1}^{m} B_{i}, B_{i} \in \mathscr{S}
$$

because they are all basic open sets. If it is the case that for all $U \in \mathscr{D}$ one of the $B_{i}$ is found in $\mathscr{D}$, then replace each such $U$ with the subbasic set from $\mathscr{D}$ containing it. But then this would be a subbasic open cover of $H$ which by assumption would admit a finite subcover contrary to the properties of $\mathscr{D}$. Therefore, one of the sets of $\mathscr{D}$, denoted by $U$, has the property that

$$
U=\cap_{i=1}^{m} B_{i}, B_{i} \in \mathscr{S}
$$

and no $B_{i}$ is in $\mathscr{D}$. Thus $\mathscr{D} \cup\left\{B_{i}\right\}$ admits a finite subcover, for each of the above $B_{i}$ because it is strictly larger than $\mathscr{D}$. Let this finite subcover corresponding to $B_{i}$ be denoted by

$$
V_{1}^{i}, \cdots, V_{m_{i}}^{i}, B_{i}
$$

Consider

$$
\left\{U, V_{j}^{i}, j=1, \cdots, m_{i}, i=1, \cdots, m\right\}
$$

If $p \in H \backslash \cup\left\{V_{j}^{i}\right\}$, then $p \in B_{i}$ for each $i$ and so $p \in U$. This is therefore a finite subcover of $\mathscr{D}$ contradicting the properties of $\mathscr{D}$. Therefore, $\mathscr{F}$ must be empty and this proves the theorem.

Definition 14.3.4 Let I be a set and suppose for each $i \in I,\left(X_{i}, \tau_{i}\right)$ is a nonempty topological space. The Cartesian product of the $X_{i}$, denoted by $\prod_{i \in I} X_{i}$, consists of the set of all choice functions defined on I which select a single element of each $X_{i}$. Thus $f \in \prod_{i \in I} X_{i}$ means for every $i \in I, f(i) \in X_{i}$. The axiom of choice says $\prod_{i \in I} X_{i}$ is nonempty. Let

$$
P_{j}(A) \equiv \prod_{i \in I} B_{i}
$$

where $B_{i} \equiv X_{i}$ if $i \neq j$ and $B_{j}=A$. A subbasis for a topology on the product space consists of all sets $P_{j}(A)$ where $A \in \tau_{j}$. (These sets have an open set from the topology of $X_{j}$ in the $j^{\text {th }}$ slot and the whole space in the other slots.) Thus a basis consists of finite intersections of these sets. Note that the intersection of two of these basic sets is another basic set and their union yields $\prod_{i \in I} X_{i}$. Therefore, they satisfy the condition needed for a collection of sets to serve as a basis for a topology. This topology is called the product topology and is denoted by $\Pi \tau_{i}$.

Proposition 14.3.5 The product topology is the smallest topology $\tau$ for $X \equiv \prod_{i \in I} X_{i}$ such that each $\pi_{i}$ is continuous. Here $\pi_{i}$ is defined in the following manner. For $\mathbf{x} \in X, \pi_{i}(\mathbf{x}) \equiv$ $x_{i}$. Thus $\pi_{i}$ delivers the $i^{\text {th }}$ entry of $\mathbf{x}$.

Proof: If each $\pi_{i}$ is continuous, then for $A \in \tau_{i}, \pi_{i}^{-1}(A)$ must be in $\tau$. However, $\pi_{i}^{-1}(A)=P_{j}(A)$ having $A$ in the $i^{t h}$ slot and $X_{j}$ in every other. Therefore, $\tau$ must contain the sets $P_{j}(A)$. Since it must be a topology, it must also contain all finite intersections of these sets. Thus the topology $\tau$ must contain the product topology described in the above definition. Is it any larger? No, because if it were, it would not be the smallest topology making the coordinate maps continuous, due to the observation that these coordinate maps are indeed continuous with respect to the product topology.

It is tempting to define a basis for a topology to be sets of the form $\prod_{i \in I} A_{i}$ where $A_{i}$ is open in $X_{i}$. This is not the same thing at all. Note that the basis just described has at most finitely many slots filled with an open set which is not the whole space. The thing just mentioned in which every slot may be filled by a proper open set is called the box topology and there exist people who are interested in it.

The Alexander subbasis theorem is used to prove the Tychonoff theorem which says that if each $X_{i}$ is a compact topological space, then in the product topology, $\prod_{i \in I} X_{i}$ is also compact.

Theorem 14.3.6 If $\left(X_{i}, \tau_{i}\right)$ is compact, then so is $\left(\prod_{i \in I} X_{i}, \tau\right)$ where $\tau$ is the product topology.

Proof: By the Alexander subbasis theorem, the theorem will be proved if every subbasic open cover admits a finite subcover. Therefore, let $\mathscr{O}$ be a subbasic open cover of $X \equiv \prod_{i \in I} X_{i}$. Let

$$
\begin{aligned}
\mathscr{O}_{j} & =\left\{Q \in \mathscr{O}: \pi_{i} Q=X_{i} \text { for } i \neq j\right\} \\
\pi_{j} \mathscr{O}_{j} & \equiv\left\{\pi_{j} Q: Q \in \mathscr{O}_{j}\right\}
\end{aligned}
$$

Thus $\mathscr{O}_{j}$ are those sets of $\mathscr{O}$ which might have a proper open subset of $X_{j}$ in the $j^{t h}$ position. If each $\pi_{j} \mathscr{O}_{j}$ fails to cover $X_{j}$, then there exists

$$
f \in \prod_{j \in I} X_{j} \backslash \cup \pi_{j} \mathscr{O}_{j}
$$

Now $f$ is contained in some open set from $\mathscr{O}$ which must be in some $\mathscr{O}_{j}$. Hence $\pi_{j} f=$ $f(j) \in \cup \pi_{j} \mathscr{O}_{j}$ but this does not happen. Hence for some $j, \pi_{j} \mathscr{O}_{j}$ must cover $X_{j}$.

$$
X_{j}=\cup \pi_{j} \mathscr{O}_{j}
$$

and so by compactness of $X_{j}$, there exist $A_{1}, \cdots, A_{m}$, sets in $\tau_{j}$ such that $X_{j} \subseteq \cup_{k=1}^{m} A_{k}$ and letting $\pi_{j} U_{k}=A_{k}$ for $U_{k} \in \mathscr{O}_{j},\left\{U_{k}\right\}_{k=1}^{m}$ covers $\prod_{i \in I} X_{i}$. By the Alexander subbasis theorem this proves $\prod_{i \in I} X_{i}$ is compact.

### 14.4 Kolmogorov Extension Theorem

Let a subbasis for $[-\infty, \infty]$ be sets of the form $[-\infty, a)$ and $(a, \infty]$. Thus with this subbasis, $[-\infty, \infty]$ is a compact Hausdorff space. Also let $M_{t} \equiv[-\infty, \infty]^{n_{t}}$ where $n_{t}$ is a positive integer and endow this product with the product topology so that $M_{t}$ is also a compact Hausdorff space.
$I$ will denote a totally ordered index set, (Like $\mathbb{R})$ and the interest will be in building a measure on the product space, $\prod_{t \in I} M_{t}$. By the well ordering principle, you can always put an order on any index set so this order is no restriction, but we do not insist on a well order and in fact, index sets of great interest are $\mathbb{R}$ or $[0, \infty)$. Also for $X$ a topological space, $\mathscr{B}(X)$ will denote the Borel sets.

Notation 14.4.1 The symbol $J$ will denote a finite subset of $I, J=\left(t_{1}, \cdots, t_{n}\right)$, the $t_{i}$ taken in order. $\mathbf{E}_{J}$ will denote a set which has a set $E_{t}$ of $\mathscr{B}\left(M_{t}\right)$ in the $t^{t h}$ position for $t \in J$ and for $t \notin J$, the set in the $t^{\text {th }}$ position will be $M_{t} . \mathbf{K}_{J}$ will denote a set which has a compact set in the $t^{\text {th }}$ position for $t \in J$ and for $t \notin J$, the set in the $t^{\text {th }}$ position will be $M_{t}$. Thus $\mathbf{K}_{J}$ is compact in the product topology of $\Omega \equiv \prod_{t \in I} M_{t}$.Also denote by $\mathscr{R}_{J}$ the sets $\mathbf{E}_{J}$ and $\mathscr{R}$ the union of the $\mathscr{R}_{J}$. Let $\mathscr{E}_{J}$ denote finite disjoint unions of sets of $\mathscr{R}_{J}$ and let $\mathscr{E}$ denote finite disjoint unions of sets of $\mathscr{R}$. Thus if $\mathbf{F}$ is a set of $\mathscr{E}$, there exists $J$ such that $\mathbf{F}$ is a finite disjoint union of sets of $\mathscr{R}_{J}$. For $\mathbf{F} \in \Omega$, denote by $\pi_{J}(\mathbf{F})$ the set $\prod_{t \in J} F_{t}$ where $\mathbf{F}=\prod_{t \in I} F_{t}$.

Lemma 14.4.2 The sets, $\mathscr{E}, \mathscr{E}_{J}$ defined above form an algebra of sets of $\prod_{t \in I} M_{t}$.
Proof: First consider $\mathscr{R}_{J}$. If $\mathbf{A}, \mathbf{B} \in \mathscr{R}_{J}$, then $\mathbf{A} \cap \mathbf{B} \in \mathscr{R}_{J}$ also. Is $\mathbf{A} \backslash \mathbf{B}$ a finite disjoint union of sets of $\mathscr{R}_{J}$ ? It suffices to verify that $\pi_{J}(\mathbf{A} \backslash \mathbf{B})$ is a finite disjoint union of $\pi_{J}\left(\mathscr{R}_{J}\right)$. Let $|J|$ denote the number of indices in $J$. If $|J|=1$, then it is obvious that $\pi_{J}(\mathbf{A} \backslash \mathbf{B})$ is a finite disjoint union of sets of $\pi_{J}\left(\mathscr{R}_{J}\right)$. In fact, letting $J=(t)$ and the $t^{t h}$ entry of $\mathbf{A}$ is $A$ and the $t^{\text {th }}$ entry of $\mathbf{B}$ is $B$, then the $t^{t h}$ entry of $\mathbf{A} \backslash \mathbf{B}$ is $A \backslash B$, a Borel set of $M_{t}$, a finite disjoint union of Borel sets of $M_{t}$.

Suppose then that for $\mathbf{A}, \mathbf{B}$ sets of $\mathscr{R}_{J}, \pi_{J}(\mathbf{A} \backslash \mathbf{B})$ is a finite disjoint union of sets of $\pi_{J}\left(\mathscr{R}_{J}\right)$ for $|J| \leq n$, and consider $J=\left(t_{1}, \cdots, t_{n}, t_{n+1}\right)$. Let the $t_{i}^{t h}$ entry of $\mathbf{A}$ and $\mathbf{B}$ be respectively $A_{i}$ and $B_{i}$. It follows that $\pi_{J}(\mathbf{A} \backslash \mathbf{B})$ has the following in the entries for $J$

$$
\left(A_{1} \times A_{2} \times \cdots \times A_{n} \times A_{n+1}\right) \backslash\left(B_{1} \times B_{2} \times \cdots \times B_{n} \times B_{n+1}\right)
$$

Letting $A$ represent $A_{1} \times A_{2} \times \cdots \times A_{n}$ and $B$ represent $B_{1} \times B_{2} \times \cdots \times B_{n}$, this is of the form

$$
A \times\left(A_{n+1} \backslash B_{n+1}\right) \cup(A \backslash B) \times\left(A_{n+1} \cap B_{n+1}\right)
$$

By induction, $(A \backslash B)$ is the finite disjoint union of sets of $\mathscr{R}_{\left(t_{1}, \cdots, t_{n}\right)}$. Therefore, the above is the finite disjoint union of sets of $\mathscr{R}_{J}$. It follows that $\mathscr{E}_{J}$ is an algebra.

Now suppose $\mathbf{A}, \mathbf{B} \in \mathscr{R}$. Then for some finite set $J$, both are in $\mathscr{R}_{J}$. Then from what was just shown,

$$
\mathbf{A} \backslash \mathbf{B} \in \mathscr{E}_{J} \subseteq \mathscr{E}, \mathbf{A} \cap \mathbf{B} \in \mathscr{R} .
$$

By Lemma 12.10.2 on Page 318 this shows $\mathscr{E}$ is an algebra.
With this preparation, here is the Kolmogorov extension theorem. In the statement and proof of the theorem, $F_{i}, G_{i}$, and $E_{i}$ will denote Borel sets. Any list of indices from $I$ will always be assumed to be taken in order. Thus, if $J \subseteq I$ and $J=\left(t_{1}, \cdots, t_{n}\right)$, it will always be assumed $t_{1}<t_{2}<\cdots<t_{n}$.

Theorem 14.4.3 For each finite set

$$
J=\left(t_{1}, \cdots, t_{n}\right) \subseteq I
$$

suppose there exists a Borel probability measure, $v_{J}=v_{t_{1} \cdots t_{n}}$ defined on the Borel sets of $\prod_{t \in J} M_{t}$ such that the following consistency condition holds. If

$$
\left(t_{1}, \cdots, t_{n}\right) \subseteq\left(s_{1}, \cdots, s_{p}\right)
$$

then

$$
\begin{equation*}
v_{t_{1} \cdots t_{n}}\left(F_{t_{1}} \times \cdots \times F_{t_{n}}\right)=v_{s_{1} \cdots s_{p}}\left(G_{s_{1}} \times \cdots \times G_{s_{p}}\right) \tag{14.4.1}
\end{equation*}
$$

where if $s_{i}=t_{j}$, then $G_{s_{i}}=F_{t_{j}}$ and if $s_{i}$ is not equal to any of the indices, $t_{k}$, then $G_{s_{i}}=M_{s_{i}}$. Then for $\mathscr{E}$ defined in Definition 14.4.1, there exists a probability measure, $P$ and a $\sigma$ algebra $\mathscr{F}=\sigma(\mathscr{E})$ such that

$$
\left(\prod_{t \in I} M_{t}, P, \mathscr{F}\right)
$$

is a probability space. Also there exist measurable functions, $X_{s}: \prod_{t \in I} M_{t} \rightarrow M_{s}$ defined as

$$
X_{s} \mathbf{x} \equiv x_{s}
$$

for each $s \in I$ such that for each $\left(t_{1} \cdots t_{n}\right) \subseteq I$,

$$
\begin{gather*}
v_{t_{1} \cdots t_{n}}\left(F_{t_{1}} \times \cdots \times F_{t_{n}}\right)=P\left(\left[X_{t_{1}} \in F_{t_{1}}\right] \cap \cdots \cap\left[X_{t_{n}} \in F_{t_{n}}\right]\right) \\
\quad=P\left(\left(X_{t_{1}}, \cdots, X_{t_{n}}\right) \in \prod_{j=1}^{n} F_{t_{j}}\right)=P\left(\prod_{t \in I} F_{t}\right) \tag{14.4.2}
\end{gather*}
$$

where $F_{t}=M_{t}$ for every $t \notin\left\{t_{1} \cdots t_{n}\right\}$ and $F_{t_{i}}$ is a Borel set. Also if $f$ is a nonnegative function of finitely many variables, $x_{t_{1}}, \cdots, x_{t_{n}}$, measurable with respect to $\mathscr{B}\left(\prod_{j=1}^{n} M_{t_{j}}\right)$, then $f$ is also measurable with respect to $\mathscr{F}$ and

$$
\begin{align*}
& \int_{M_{t_{1} \times \cdots \times M_{t_{n}}}} f\left(x_{t_{1}}, \cdots, x_{t_{n}}\right) d v_{t_{1} \cdots t_{n}} \\
= & \int_{\Pi_{t \in I} M_{t}} f\left(x_{t_{1}}, \cdots, x_{t_{n}}\right) d P \tag{14.4.3}
\end{align*}
$$

Proof: Let $\mathscr{E}$ be the algebra of sets defined in Definition 14.4.1. I want to define a measure on $\mathscr{E}$. For $\mathbf{F} \in \mathscr{E}$, there exists $J$ such that $\mathbf{F}$ is the finite disjoint unions of sets of $\mathscr{R}_{J}$. Define

$$
P_{0}(\mathbf{F}) \equiv v_{J}\left(\pi_{J}(\mathbf{F})\right)
$$

Then $P_{0}$ is well defined because of the consistency condition on the measures $v_{J} . P_{0}$ is clearly finitely additive because the $v_{J}$ are measures and one can pick $J$ as large as desired to include all $t$ where there may be something other than $M_{t}$. Also, from the definition,

$$
P_{0}(\Omega) \equiv P_{0}\left(\prod_{t \in I} M_{t}\right)=v_{t_{1}}\left(M_{t_{1}}\right)=1
$$

Next I will show $P_{0}$ is a finite measure on $\mathscr{E}$. After this it is only a matter of using the Caratheodory extension theorem to get the existence of the desired probability measure $P$.

Claim: Suppose $\mathbf{E}^{n}$ is in $\mathscr{E}$ and suppose $\mathbf{E}^{n} \downarrow \emptyset$. Then $P_{0}\left(\mathbf{E}^{n}\right) \downarrow 0$.
Proof of the claim: If not, there exists a sequence such that although $\mathbf{E}^{n} \downarrow \emptyset, P_{0}\left(\mathbf{E}^{n}\right) \downarrow$ $\varepsilon>0$. Let $\mathbf{E}^{n} \in \mathscr{E}_{J_{n}}$. Thus it is a finite disjoint union of sets of $\mathscr{R}_{J_{n}}$. By regularity of the measures $v_{J}$, there exists a compact set $\mathbf{K}_{J_{n}} \subseteq \mathbf{E}^{n}$ such that

$$
v_{J_{n}}\left(\pi_{J_{n}}\left(\mathbf{K}_{J_{n}}\right)\right)+\frac{\varepsilon}{2^{n+2}}>v_{J_{n}}\left(\pi_{J_{n}}\left(\mathbf{E}^{n}\right)\right)
$$

Thus

$$
\begin{aligned}
P_{0}\left(\mathbf{K}_{J_{n}}\right)+\frac{\varepsilon}{2^{n+2}} & \equiv v_{J_{n}}\left(\pi_{J_{n}}\left(\mathbf{K}_{J_{n}}\right)\right)+\frac{\varepsilon}{2^{n+2}} \\
& >v_{J_{n}}\left(\pi_{J_{n}}\left(\mathbf{E}^{n}\right)\right) \equiv P_{0}\left(\mathbf{E}^{n}\right)
\end{aligned}
$$

The interesting thing about these $\mathbf{K}_{J_{n}}$ is: they have the finite intersection property. Here is why.

$$
\begin{aligned}
\varepsilon & \leq P_{0}\left(\cap_{k=1}^{m} \mathbf{K}_{J_{k}}\right)+P_{0}\left(\mathbf{E}^{m} \backslash \cap_{k=1}^{m} \mathbf{K}_{J_{k}}\right) \\
& \leq P_{0}\left(\cap_{k=1}^{m} \mathbf{K}_{J_{k}}\right)+P_{0}\left(\cup_{k=1}^{m} \mathbf{E}^{k} \backslash \mathbf{K}_{J_{k}}\right) \\
& <P_{0}\left(\cap_{k=1}^{m} \mathbf{K}_{J_{k}}\right)+\sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k+2}}<P_{0}\left(\cap_{k=1}^{m} \mathbf{K}_{J_{k}}\right)+\varepsilon / 2
\end{aligned}
$$

and so $P_{0}\left(\cap_{k=1}^{m} \mathbf{K}_{J_{k}}\right)>\varepsilon / 2$. Now this yields a contradiction, because this finite intersection property implies the intersection of all the $\mathbf{K}_{J_{k}}$ is nonempty, contradicting $\mathbf{E}^{n} \downarrow \emptyset$ since each $\mathbf{K}_{J_{n}}$ is contained in $\mathbf{E}^{n}$.

With the claim, it follows $P_{0}$ is a measure on $\mathscr{E}$. Here is why: If $\mathbf{E}=\cup_{k=1}^{\infty} \mathbf{E}^{k}$ where $\mathbf{E}, \mathbf{E}^{k} \in \mathscr{E}$, then $\left(\mathbf{E} \backslash \cup_{k=1}^{n} \mathbf{E}_{k}\right) \downarrow \emptyset$ and so

$$
P_{0}\left(\cup_{k=1}^{n} \mathbf{E}_{k}\right) \rightarrow P_{0}(\mathbf{E})
$$

Hence if the $\mathbf{E}_{k}$ are disjoint, $P_{0}\left(\cup_{k=1}^{n} \mathbf{E}_{k}\right)=\sum_{k=1}^{n} P_{0}\left(\mathbf{E}_{k}\right) \rightarrow P_{0}(\mathbf{E})$. Thus for disjoint $\mathbf{E}_{k}$ having $\cup_{k} \mathbf{E}_{k}=\mathbf{E} \in \mathscr{E}$,

$$
P_{0}\left(\cup_{k=1}^{\infty} \mathbf{E}_{k}\right)=\sum_{k=1}^{\infty} P_{0}\left(\mathbf{E}_{k}\right)
$$

Now to conclude the proof, apply the Caratheodory extension theorem to obtain $P$ a probability measure which extends $P_{0}$ to a $\sigma$ algebra which contains $\sigma(\mathscr{E})$ the sigma algebra generated by $\mathscr{E}$ with $P=P_{0}$ on $\mathscr{E}$. Thus for $\mathbf{E}_{J} \in \mathscr{E}, P\left(\mathbf{E}_{J}\right)=P_{0}\left(\mathbf{E}_{J}\right)=v_{J}\left(P_{J} \mathbf{E}_{j}\right)$.

Next, let $\left(\prod_{t \in I} M_{t}, \mathscr{F}, P\right)$ be the probability space and for $\mathbf{x} \in \prod_{t \in I} M_{t}$ let $X_{t}(\mathbf{x})=x_{t}$, the $t^{t h}$ entry of $\mathbf{x}$. It follows $X_{t}$ is measurable (also continuous) because if $U$ is open in $M_{t}$, then $X_{t}^{-1}(U)$ has a $U$ in the $t^{t h}$ slot and $M_{s}$ everywhere else for $s \neq t$. Thus inverse images of open sets are measurable. Also, letting $J$ be a finite subset of $I$ and for $J=\left(t_{1}, \cdots, t_{n}\right)$, and $F_{t_{1}}, \cdots, F_{t_{n}}$ Borel sets in $M_{t_{1}} \cdots M_{t_{n}}$ respectively, it follows $\mathbf{F}_{J}$, where $\mathbf{F}_{J}$ has $F_{t_{i}}$ in the $t_{i}^{t h}$ entry, is in $\mathscr{E}$ and therefore,

$$
\begin{gathered}
P\left(\left[X_{t_{1}} \in F_{t_{1}}\right] \cap\left[X_{t_{2}} \in F_{t_{2}}\right] \cap \cdots \cap\left[X_{t_{n}} \in F_{t_{n}}\right]\right)= \\
P\left(\left[\left(X_{t_{1}}, X_{t_{2}}, \cdots, X_{t_{n}}\right) \in F_{t_{1}} \times \cdots \times F_{t_{n}}\right]\right)=P\left(\mathbf{F}_{J}\right)=P_{0}\left(\mathbf{F}_{J}\right) \\
=v_{t_{1} \cdots t_{n}}\left(F_{t_{1}} \times \cdots \times F_{t_{n}}\right)
\end{gathered}
$$

Finally consider the claim about the integrals. Suppose $f\left(x_{t_{1}}, \cdots, x_{t_{n}}\right)=\mathscr{X}_{F}$ where $F$ is a Borel set of $\prod_{t \in J} M_{t}$ where $J=\left(t_{1}, \cdots, t_{n}\right)$. To begin with suppose

$$
\begin{equation*}
F=F_{t_{1}} \times \cdots \times F_{t_{n}} \tag{14.4.4}
\end{equation*}
$$

where each $F_{t_{j}}$ is in $\mathscr{B}\left(M_{t_{j}}\right)$. Then

$$
\begin{gather*}
\int_{M_{t_{1} \times \cdots \times M_{t_{n}}}} \mathscr{X}_{F}\left(x_{t_{1}}, \cdots, x_{t_{n}}\right) d v_{t_{1} \cdots t_{n}}=v_{t_{1} \cdots t_{n}}\left(F_{t_{1}} \times \cdots \times F_{t_{n}}\right) \\
=P\left(\prod_{t \in I} F_{t}\right)=\int_{\Omega} \mathscr{X}_{\prod_{t \in I} F_{t}}(\mathbf{x}) d P \\
=\int_{\Omega} \mathscr{X}_{F}\left(x_{t_{1}}, \cdots, x_{t_{n}}\right) d P \tag{14.4.5}
\end{gather*}
$$

where $F_{t}=M_{t}$ if $t \notin J$. Let $\mathscr{K}$ denote sets, $F$ of the sort in 14.4.4. It is clearly a $\pi$ system. Now let $\mathscr{G}$ denote those sets $F$ in $\mathscr{B}\left(\prod_{t \in J} M_{t}\right)$ such that 14.4 .5 holds. Thus $\mathscr{G} \supseteq \mathscr{K}$. It is clear that $\mathscr{G}$ is closed with respect to countable disjoint unions and complements. Hence $\mathscr{G} \supseteq \sigma(\mathscr{K})$ but $\sigma(\mathscr{K})=\mathscr{B}\left(\prod_{t \in J} M_{t}\right)$ because every open set in $\prod_{t \in J} M_{t}$ is the countable union of rectangles like 14.4.4 in which each $F_{t_{i}}$ is open. Therefore, 14.4.5 holds for every $F \in \mathscr{B}\left(\prod_{t \in J} M_{t}\right)$.

Passing to simple functions and then using the monotone convergence theorem yields the final claim of the theorem.

The next task is to consider the case where $M_{t}=(-\infty, \infty)^{n_{t}}$. To consider this case, here is a lemma which will allow this case to be deduced from the above theorem. In this lemma, $M_{t}^{\prime} \equiv[-\infty, \infty]^{n_{t}}$.

Lemma 14.4.4 Let $J$ be a finite subset of $I$. Then $\mathbf{U}$ is a Borel set in $\prod_{t \in J} M_{t}$ if and only if there exists a Borel set, $\mathbf{U}^{\prime}$ in $\prod_{t \in J} M_{t}^{\prime}$ such that $\mathbf{U}=\mathbf{U}^{\prime} \cap \prod_{t \in J} M_{t}$.

Proof: A subbasis for the topology for $[-\infty, \infty]$ is sets of the form $[-\infty, a)$ and $(a, \infty]$. Also a subbasis for the topology of $[-\infty, \infty]^{n}$ consists of sets of the form $\prod_{i=1}^{n}\left[-\infty, a_{i}\right)$ and $\prod_{i=1}^{n}\left(a_{i}, \infty\right]$. Similarly, a subbasis for the topology of $(-\infty, \infty)^{n}$ consists of sets of the form $\prod_{i=1}^{n}\left(-\infty, a_{i}\right)$ and $\prod_{i=1}^{n}\left(a_{i}, \infty\right)$. Thus the basic open sets of $\prod_{t \in J} M_{t}$ are of the form
$\mathbf{U}^{\prime} \cap \prod_{t \in J} M_{t}$ where $\mathbf{U}^{\prime}$ is a basic open set in $\prod_{t \in J} M_{t}^{\prime}$. It follows the open sets of $\prod_{t \in J} M_{t}$ are of the form $\mathbf{U}^{\prime} \cap \prod_{t \in J} M_{t}$ where $\mathbf{U}^{\prime}$ is open in $\prod_{t \in J} M_{t}^{\prime}$. Let $\mathscr{F}$ denote those Borel sets of $\prod_{t \in J} M_{t}$ which are of the form $\mathbf{U}^{\prime} \cap \prod_{t \in J} M_{t}$ for $\mathbf{U}^{\prime}$ a Borel set in $\prod_{t \in J} M_{t}^{\prime}$. Then as just shown, $\mathscr{F}$ contains the $\pi$ system of open sets in $\prod_{t \in J} M_{t}$. Let $\mathscr{G}$ denote those Borel sets of $\prod_{t \in J} M_{t}$ which are of the desired form. It is clearly closed with respect to complements and countable disjoint unions. Hence $\mathscr{G}$ equals the Borel sets of $\prod_{t \in J} M_{t}$.

Now here is the Kolmogorov extension theorem in the desired form. However, a more general version is given later where $M_{t}$ is just a Polish space (complete separable metric space).

Theorem 14.4.5 (Kolmogorov extension theorem) For each finite set

$$
J=\left(t_{1}, \cdots, t_{n}\right) \subseteq I
$$

suppose there exists a Borel probability measure, $v_{J}=v_{t_{1} \cdots t_{n}}$ defined on the Borel sets of $\prod_{t \in J} M_{t}$ for $M_{t}=\mathbb{R}^{n_{t}}$ for $n_{t}$ an integer, such that the following consistency condition holds. If

$$
\left(t_{1}, \cdots, t_{n}\right) \subseteq\left(s_{1}, \cdots, s_{p}\right)
$$

then

$$
\begin{equation*}
v_{t_{1} \cdots t_{n}}\left(F_{t_{1}} \times \cdots \times F_{t_{n}}\right)=v_{s_{1} \cdots s_{p}}\left(G_{s_{1}} \times \cdots \times G_{s_{p}}\right) \tag{14.4.6}
\end{equation*}
$$

where if $s_{i}=t_{j}$, then $G_{s_{i}}=F_{t_{j}}$ and if $s_{i}$ is not equal to any of the indices, $t_{k}$, then $G_{s_{i}}=M_{s_{i}}$. Then for $\mathscr{E}$ defined as in Definition 14.4.1, adjusted so that $\pm \infty$ never appears as any endpoint of any interval, there exists a probability measure, $P$ and a $\sigma$ algebra $\mathscr{F}=\sigma(\mathscr{E})$ such that

$$
\left(\prod_{t \in I} M_{t}, P, \mathscr{F}\right)
$$

is a probability space. Also there exist measurable functions, $X_{s}: \prod_{t \in I} M_{t} \rightarrow M_{s}$ defined as

$$
X_{S} \mathbf{x} \equiv x_{s}
$$

for each $s \in I$ such that for each $\left(t_{1} \cdots t_{n}\right) \subseteq I$,

$$
\begin{gather*}
v_{t_{1} \cdots t_{n}}\left(F_{t_{1}} \times \cdots \times F_{t_{n}}\right)=P\left(\left[X_{t_{1}} \in F_{t_{1}}\right] \cap \cdots \cap\left[X_{t_{n}} \in F_{t_{n}}\right]\right) \\
\quad=P\left(\left(X_{t_{1}}, \cdots, X_{t_{n}}\right) \in \prod_{j=1}^{n} F_{t_{j}}\right)=P\left(\prod_{t \in I} F_{t}\right) \tag{14.4.7}
\end{gather*}
$$

where $F_{t}=M_{t}$ for every $t \notin\left\{t_{1} \cdots t_{n}\right\}$ and $F_{t_{i}}$ is a Borel set. Also if $f$ is a nonnegative function of finitely many variables, $x_{t_{1}}, \cdots, x_{t_{n}}$, measurable with respect to $\mathscr{B}\left(\prod_{j=1}^{n} M_{t_{j}}\right)$, then $f$ is also measurable with respect to $\mathscr{F}$ and

$$
\begin{align*}
& \int_{M_{t_{1} \times \cdots \times M_{t_{n}}}} f\left(x_{t_{1}}, \cdots, x_{t_{n}}\right) d v_{t_{1} \cdots t_{n}} \\
= & \int_{\prod_{t \in I} M_{t}} f\left(x_{t_{1}}, \cdots, x_{t_{n}}\right) d P \tag{14.4.8}
\end{align*}
$$

Proof: Using Lemma 14.4.4, extend each measure, $v_{J}$ to $M_{t}^{\prime}$, defined by adding in the points $\pm \infty$ at the ends, by letting $v_{J}(\mathbf{E}) \equiv v_{J}\left(\mathbf{E} \cap \prod_{t \in I} M_{t}\right)$ for all $\mathbf{E} \in \mathscr{B}\left(\prod_{t \in I} M_{t}^{\prime}\right)$. Then apply Theorem 14.4 .3 to these extended measures and use the definition of the extensions of each $v_{J}$ to replace each $M_{t}^{\prime}$ with $M_{t}$ everywhere it occurs.

As a special case, you can obtain a version of product measure for possibly infinitely many factors. Suppose in the context of the above theorem that $v_{t}$ is a probability measure defined on the Borel sets of $M_{t} \equiv \mathbb{R}^{n_{t}}$ for $n_{t}$ a positive integer, and let the measures, $v_{t_{1} \cdots t_{n}}$ be defined on the Borel sets of $\prod_{i=1}^{n} M_{t_{i}}$ by

$$
v_{t_{1} \cdots t_{n}}(\mathbf{E}) \equiv \overbrace{\left(v_{t_{1}} \times \cdots \times v_{t_{n}}\right)}^{\text {product measure }}(\mathbf{E}) .
$$

Then these measures satisfy the necessary consistency condition and so the Kolmogorov extension theorem given above can be applied to obtain a measure $P$ defined on a measurable space $\left(\prod_{t \in I} M_{t}, \mathscr{F}\right)$ and measurable functions $X_{s}: \prod_{t \in I} M_{t} \rightarrow M_{s}$ such that for $F_{t_{i}}$ a Borel set in $M_{t_{i}}$,

$$
\begin{gather*}
P\left(\left(X_{t_{1}}, \cdots, X_{t_{n}}\right) \in \prod_{i=1}^{n} F_{t_{i}}\right)=v_{t_{1} \cdots t_{n}}\left(F_{t_{1}} \times \cdots \times F_{t_{n}}\right) \\
=v_{t_{1}}\left(F_{t_{1}}\right) \cdots v_{t_{n}}\left(F_{t_{n}}\right) \tag{14.4.9}
\end{gather*}
$$

In particular, $P\left(X_{t} \in F_{t}\right)=v_{t}\left(F_{t}\right)$. Then $P$ in the resulting probability space,

$$
\left(\prod_{t \in I} M_{t}, \mathscr{F}, P\right)
$$

will be denoted as $\prod_{t \in I} v_{t}$. This proves the following theorem which describes an infinite product measure.

Theorem 14.4.6 Let $M_{t}$ for $t \in I$ be given as in Theorem 14.4.5 and let $v_{t}$ be a Borel probability measure defined on the Borel sets of $M_{t}$. Then there exists a measure $P$ and a $\sigma$ algebra $\mathscr{F}=\sigma(\mathscr{E})$ where $\mathscr{E}$ is given in Definition 14.4.1 such that $\left(\prod_{t} M_{t}, \mathscr{F}, P\right)$ is a probability space satisfying 14.4 .9 whenever each $F_{t_{i}}$ is a Borel set of $M_{t_{i}}$. This probability measure is sometimes denoted as $\prod_{t} v_{t}$.

### 14.5 Exercises

1. Let $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{F}, \lambda)$ be two finite measure spaces. A subset of $X \times Y$ is called a measurable rectangle if it is of the form $A \times B$ where $A \in \mathscr{S}$ and $B \in \mathscr{F}$. A subset of $X \times Y$ is called an elementary set if it is a finite disjoint union of measurable rectangles. Denote this set of functions by $\mathscr{E}$. Show that $\mathscr{E}$ is an algebra of sets.
2. $\uparrow$ For $A \in \sigma(\mathscr{E})$, the smallest $\sigma$ algebra containing $\mathscr{E}$, show that $x \rightarrow \mathscr{X}_{A}(x, y)$ is $\mu$ measurable and that

$$
y \rightarrow \int \mathscr{X}_{A}(x, y) d \mu
$$

is $\lambda$ measurable. Show similar assertions hold for $y \rightarrow \mathscr{X}_{A}(x, y)$ and

$$
x \rightarrow \int \mathscr{X}_{A}(x, y) d \lambda
$$

and that

$$
\begin{equation*}
\iint \mathscr{X}_{A}(x, y) d \mu d \lambda=\iint \mathscr{X}_{A}(x, y) d \lambda d \mu . \tag{14.5.10}
\end{equation*}
$$

Hint: Let $\mathscr{M} \equiv\{A \in \sigma(\mathscr{E}): 14.5 .10$ holds $\}$ along with all relevant measurability assertions. Show $\mathscr{M}$ contains $\mathscr{E}$ and is a monotone class. Then apply the Theorem 12.10.5.
3. $\uparrow$ For $A \in \sigma(\mathscr{E})$ define $(\mu \times \lambda)(A) \equiv \iint \mathscr{X}_{A}(x, y) d \mu d \lambda$. Show that $(\mu \times \lambda)$ is a measur on $\sigma(\mathscr{E})$ and that whenever $f \geq 0$ is measurable with respect to $\sigma(\mathscr{E})$,

$$
\int_{X \times Y} f d(\mu \times \lambda)=\iint f(x, y) d \mu d \lambda=\iint f(x, y) d \lambda d \mu .
$$

This is a common approach to Fubini's theorem.
4. $\uparrow$ Generalize the above version of Fubini's theorem to the case where the measure spaces are only $\sigma$ finite.
5. $\uparrow$ Suppose now that $\mu$ and $\lambda$ are both complete $\sigma$ finite measures. Let $\overline{(\mu \times \lambda)}$ denote the completion of this measure. Let the larger measure space be

$$
(X \times Y, \overline{\sigma(\mathscr{E})}, \overline{(\mu \times \lambda)})
$$

Thus if $E \in \overline{\sigma(\mathscr{E})}$, it follows there exists a set $A \in \sigma(\mathscr{E})$ such that $E \cup N=A$ where $\overline{(\mu \times \lambda)}(N)=0$. Now argue that for $\lambda$ a.e. $y, x \rightarrow \mathscr{X}_{N}(x, y)$ is measurable because it is equal to zero $\mu$ a.e. and $\mu$ is complete. Therefore,

$$
\iint \mathscr{X}_{N}(x, y) d \mu d \lambda
$$

makes sense and equals zero. Use to argue that for $\lambda$ a.e. $y, x \rightarrow \mathscr{X}_{E}(x, y)$ is $\mu$ measurable and equals $\int \mathscr{X}_{A}(x, y) d \mu$. Then by completeness of $\lambda, y \rightarrow \int \mathscr{X}_{E}(x, y) d \mu$ is $\lambda$ measurable and

$$
\iint \mathscr{X}_{A}(x, y) d \mu d \lambda=\iint \mathscr{X}_{E}(x, y) d \mu d \lambda=\overline{(\mu \times \lambda)}(E) .
$$

Similarly

$$
\iint \mathscr{X}_{E}(x, y) d \lambda d \mu=\overline{(\mu \times \lambda)}(E)
$$

Use this to give a generalization of the above Fubini theorem. Prove that if $f$ is measurable with respect to the $\sigma$ algebra, $\overline{\sigma(\mathscr{E})}$ and nonnegative, then

$$
\int_{X \times Y} f d \overline{(\mu \times \lambda)}=\iint f(x, y) d \mu d \lambda=\iint f(x, y) d \lambda d \mu
$$

where the iterated integrals make sense.

## Chapter 15

## The $L^{p}$ Spaces

### 15.1 Basic Inequalities And Properties

One of the main applications of the Lebesgue integral is to the study of various sorts of functions space. These are vector spaces whose elements are functions of various types. One of the most important examples of a function space is the space of measurable functions whose absolute values are $p^{t h}$ power integrable where $p \geq 1$. These spaces, referred to as $L^{p}$ spaces, are very useful in applications. In the chapter $(\Omega, \mathscr{S}, \mu)$ will be a measure space.

Definition 15.1.1 Let $1 \leq p<\infty$. Define

$$
L^{p}(\Omega) \equiv\left\{f: f \text { is measurable and } \int_{\Omega}|f(\omega)|^{p} d \mu<\infty\right\}
$$

In terms of the distribution function,

$$
L^{p}(\Omega)=\left\{f: f \text { is measurable and } \int_{0}^{\infty} p t^{p-1} \mu([|f|>t]) d t<\infty\right\}
$$

For each $p>1$ define $q$ by

$$
\frac{1}{p}+\frac{1}{q}=1
$$

Often one uses $p^{\prime}$ instead of $q$ in this context.
$L^{p}(\Omega)$ is a vector space and has a norm. This is similar to the situation for $\mathbb{R}^{n}$ but the proof requires the following fundamental inequality. .

Theorem 15.1.2 (Holder's inequality) If $f$ and $g$ are measurable functions, then if $p>1$,

$$
\begin{equation*}
\int|f||g| d \mu \leq\left(\int|f|^{p} d \mu\right)^{\frac{1}{p}}\left(\int|g|^{q} d \mu\right)^{\frac{1}{q}} \tag{15.1.1}
\end{equation*}
$$

Proof: First here is a proof of Young's inequality .
Lemma 15.1.3 If $p>1$, and $0 \leq a, b$ then $a b \leq \frac{a^{p}}{p}+\frac{b^{q}}{q}$.
Proof: Consider the following picture:


From this picture, the sum of the area between the $x$ axis and the curve added to the
area between the $t$ axis and the curve is at least as large as $a b$. Using beginning calculus, this is equivalent to the following inequality.

$$
a b \leq \int_{0}^{a} t^{p-1} d t+\int_{0}^{b} x^{q-1} d x=\frac{a^{p}}{p}+\frac{b^{q}}{q} .
$$

The above picture represents the situation which occurs when $p>2$ because the graph of the function is concave up. If $2 \geq p>1$ the graph would be concave down or a straight line. You should verify that the same argument holds in these cases just as well. In fact, the only thing which matters in the above inequality is that the function $x=t^{p-1}$ be strictly increasing.

Note equality occurs when $a^{p}=b^{q}$.
Here is an alternate proof.
Lemma 15.1.4 For $a, b \geq 0$,

$$
a b \leq \frac{a^{p}}{p}+\frac{b^{q}}{q}
$$

and equality occurs when if and only if $a^{p}=b^{q}$.
Proof: If $b=0$, the inequality is obvious. Fix $b>0$ and consider

$$
f(a) \equiv \frac{a^{p}}{p}+\frac{b^{q}}{q}-a b
$$

Then $f^{\prime}(a)=a^{p-1}-b$. This is negative when $a<b^{1 /(p-1)}$ and is positive when $a>$ $b^{1 /(p-1)}$. Therefore, $f$ has a minimum when $a=b^{1 /(p-1)}$. In other words, when $a^{p}=$ $b^{p /(p-1)}=b^{q}$ since $1 / p+1 / q=1$. Thus the minimum value of $f$ is

$$
\frac{b^{q}}{p}+\frac{b^{q}}{q}-b^{1 /(p-1)} b=b^{q}-b^{q}=0
$$

It follows $f \geq 0$ and this yields the desired inequality.
Proof of Holder's inequality: If either $\int|f|^{p} d \mu$ or $\int|g|^{p} d \mu$ equals $\infty$, the inequality 15.1.1 is obviously valid because $\infty \geq$ anything. If either $\int|f|^{p} d \mu$ or $\int|g|^{p} d \mu$ equals 0 , then $f=0$ a.e. or that $g=0$ a.e. and so in this case the left side of the inequality equals 0 and so the inequality is therefore true. Therefore assume both $\int|f|^{p} d \mu$ and $\int|g|^{p} d \mu$ are less than $\infty$ and not equal to 0 . Let

$$
\left(\int|f|^{p} d \mu\right)^{1 / p}=I(f)
$$

and let $\left(\int|g|^{p} d \mu\right)^{1 / q}=I(g)$. Then using the lemma,

$$
\int \frac{|f|}{I(f)} \frac{|g|}{I(g)} d \mu \leq \frac{1}{p} \int \frac{|f|^{p}}{I(f)^{p}} d \mu+\frac{1}{q} \int \frac{|g|^{q}}{I(g)^{q}} d \mu=1
$$

Hence,

$$
\int|f||g| d \mu \leq I(f) I(g)=\left(\int|f|^{p} d \mu\right)^{1 / p}\left(\int|g|^{q} d \mu\right)^{1 / q}
$$

This proves Holder's inequality.
The following lemma will be needed.

Lemma 15.1.5 Suppose $x, y \in \mathbb{C}$. Then

$$
|x+y|^{p} \leq 2^{p-1}\left(|x|^{p}+|y|^{p}\right) .
$$

Proof: The function $f(t)=t^{p}$ is concave up for $t \geq 0$ because $p>1$. Therefore, the secant line joining two points on the graph of this function must lie above the graph of the function. This is illustrated in the following picture.


Now as shown above,

$$
\left(\frac{|x|+|y|}{2}\right)^{p} \leq \frac{|x|^{p}+|y|^{p}}{2}
$$

which implies

$$
|x+y|^{p} \leq(|x|+|y|)^{p} \leq 2^{p-1}\left(|x|^{p}+|y|^{p}\right)
$$

and this proves the lemma.
Note that if $y=\phi(x)$ is any function for which the graph of $\phi$ is concave up, you could get a similar inequality by the same argument.

Corollary 15.1.6 (Minkowski inequality) Let $1 \leq p<\infty$. Then

$$
\begin{equation*}
\left(\int|f+g|^{p} d \mu\right)^{1 / p} \leq\left(\int|f|^{p} d \mu\right)^{1 / p}+\left(\int|g|^{p} d \mu\right)^{1 / p} \tag{15.1.2}
\end{equation*}
$$

Proof: If $p=1$, this is obvious because it is just the triangle inequality. Let $p>1$. Without loss of generality, assume

$$
\left(\int|f|^{p} d \mu\right)^{1 / p}+\left(\int|g|^{p} d \mu\right)^{1 / p}<\infty
$$

and $\left(\int|f+g|^{p} d \mu\right)^{1 / p} \neq 0$ or there is nothing to prove. Therefore, using the above lemma,

$$
\int|f+g|^{p} d \mu \leq 2^{p-1}\left(\int|f|^{p}+|g|^{p} d \mu\right)<\infty
$$

Now $|f(\omega)+g(\omega)|^{p} \leq|f(\omega)+g(\omega)|^{p-1}(|f(\omega)|+|g(\omega)|)$. Also, it follows from the definition of $p$ and $q$ that $p-1=\frac{p}{q}$. Therefore, using this and Holder's inequality,

$$
\int|f+g|^{p} d \mu \leq
$$

$$
\begin{aligned}
& \int|f+g|^{p-1}|f| d \mu+\int|f+g|^{p-1}|g| d \mu \\
= & \int|f+g|^{\frac{p}{q}}|f| d \mu+\int|f+g|^{\frac{p}{q}}|g| d \mu \\
\leq & \left(\int|f+g|^{p} d \mu\right)^{\frac{1}{q}}\left(\int|f|^{p} d \mu\right)^{\frac{1}{p}}+\left(\int|f+g|^{p} d \mu\right)^{\frac{1}{q}}\left(\int|g|^{p} d \mu\right)^{\frac{1}{p}} .
\end{aligned}
$$

Dividing both sides by $\left(\int|f+g|^{p} d \mu\right)^{\frac{1}{q}}$ yields 15.1.2. This proves the corollary.
The following follows immediately from the above.
Corollary 15.1.7 Let $f_{i} \in L^{p}(\Omega)$ for $i=1,2, \cdots, n$. Then

$$
\left(\int\left|\sum_{i=1}^{n} f_{i}\right|^{p} d \mu\right)^{1 / p} \leq \sum_{i=1}^{n}\left(\int\left|f_{i}\right|^{p}\right)^{1 / p}
$$

This shows that if $f, g \in L^{p}$, then $f+g \in L^{p}$. Also, it is clear that if $a$ is a constant and $f \in L^{p}$, then $a f \in L^{p}$ because

$$
\int|a f|^{p} d \mu=|a|^{p} \int|f|^{p} d \mu<\infty
$$

Thus $L^{p}$ is a vector space and
a.) $\left(\int|f|^{p} d \mu\right)^{1 / p} \geq 0,\left(\int|f|^{p} d \mu\right)^{1 / p}=0$ if and only if $f=0$ a.e.
b.) $\left(\int|a f|^{p} d \mu\right)^{1 / p}=|a|\left(\int|f|^{p} d \mu\right)^{1 / p}$ if $a$ is a scalar.
c.) $\left(\int|f+g|^{p} d \mu\right)^{1 / p} \leq\left(\int|f|^{p} d \mu\right)^{1 / p}+\left(\int|g|^{p} d \mu\right)^{1 / p}$.
$f \rightarrow\left(\int|f|^{p} d \mu\right)^{1 / p}$ would define a norm if $\left(\int|f|^{p} d \mu\right)^{1 / p}=0$ implied $f=0$. Unfortunately, this is not so because if $f=0$ a.e. but is nonzero on a set of measure zero, $\left(\int|f|^{p} d \mu\right)^{1 / p}=0$ and this is not allowed. However, all the other properties of a norm are available and so a little thing like a set of measure zero will not prevent the consideration of $L^{p}$ as a normed vector space if two functions in $L^{p}$ which differ only on a set of measure zero are considered the same. That is, an element of $L^{p}$ is really an equivalence class of functions where two functions are equivalent if they are equal a.e. With this convention, here is a definition.

Definition 15.1.8 Let $f \in L^{p}(\Omega)$. Define

$$
\|f\|_{p} \equiv\|f\|_{L^{p}} \equiv\left(\int|f|^{p} d \mu\right)^{1 / p}
$$

Then with this definition and using the convention that elements in $L^{p}$ are considered to be the same if they differ only on a set of measure zero, $\left\|\|_{p}\right.$ is a norm on $L^{p}(\Omega)$ because if $\|f\|_{p}=0$ then $f=0$ a.e. and so $f$ is considered to be the zero function because it differs from 0 only on a set of measure zero.

The following is an important definition.

Definition 15.1.9 A complete normed linear space is called a Banach ${ }^{1}$ space.
$L^{p}$ is a Banach space. This is the next big theorem.
Theorem 15.1.10 The following hold for $L^{p}(\Omega)$
a.) $L^{p}(\Omega)$ is complete.
b.) If $\left\{f_{n}\right\}$ is a Cauchy sequence in $L^{p}(\Omega)$, then there exists $f \in L^{p}(\Omega)$ and a subsequence which converges a.e. to $f \in L^{p}(\Omega)$, and $\left\|f_{n}-f\right\|_{p} \rightarrow 0$.

Proof: Let $\left\{f_{n}\right\}$ be a Cauchy sequence in $L^{p}(\Omega)$. This means that for every $\varepsilon>0$ there exists $N$ such that if $n, m \geq N$, then $\left\|f_{n}-f_{m}\right\|_{p}<\varepsilon$. Now select a subsequence as follows. Let $n_{1}$ be such that $\left\|f_{n}-f_{m}\right\|_{p}<2^{-1}$ whenever $n, m \geq n_{1}$. Let $n_{2}$ be such that $n_{2}>n_{1}$ and $\left\|f_{n}-f_{m}\right\|_{p}<2^{-2}$ whenever $n, m \geq n_{2}$. If $n_{1}, \cdots, n_{k}$ have been chosen, let $n_{k+1}>n_{k}$ and whenever $n, m \geq n_{k+1},\left\|f_{n}-f_{m}\right\|_{p}<2^{-(k+1)}$. The subsequence just mentioned is $\left\{f_{n_{k}}\right\}$. Thus, $\left\|f_{n_{k}}-f_{n_{k+1}}\right\|_{p}<2^{-k}$. Let

$$
g_{k+1}=f_{n_{k+1}}-f_{n_{k}} .
$$

Then by the corollary to Minkowski’s inequality,

$$
\infty>\sum_{k=1}^{\infty}\left\|g_{k+1}\right\|_{p} \geq \sum_{k=1}^{m}\left\|g_{k+1}\right\|_{p} \geq\left\|\sum_{k=1}^{m}\left|g_{k+1}\right|\right\|_{p}
$$

for all $m$. It follows that

$$
\begin{equation*}
\int\left(\sum_{k=1}^{m}\left|g_{k+1}\right|\right)^{p} d \mu \leq\left(\sum_{k=1}^{\infty}\left\|g_{k+1}\right\|_{p}\right)^{p}<\infty \tag{15.1.3}
\end{equation*}
$$

for all $m$ and so the monotone convergence theorem implies that the sum up to $m$ in 15.1.3 can be replaced by a sum up to $\infty$. Thus,

$$
\int\left(\sum_{k=1}^{\infty}\left|g_{k+1}\right|\right)^{p} d \mu<\infty
$$

which requires

$$
\sum_{k=1}^{\infty}\left|g_{k+1}(x)\right|<\infty \text { a.e. } x
$$

[^15]Therefore, $\sum_{k=1}^{\infty} g_{k+1}(x)$ converges for a.e. $x$ because the functions have values in a complete space, $\mathbb{C}$, and this shows the partial sums form a Cauchy sequence. Now let $x$ be such that this sum is finite. Then define

$$
f(x) \equiv f_{n_{1}}(x)+\sum_{k=1}^{\infty} g_{k+1}(x)=\lim _{m \rightarrow \infty} f_{n_{m}}(x)
$$

since $\sum_{k=1}^{m} g_{k+1}(x)=f_{n_{m+1}}(x)-f_{n_{1}}(x)$. Therefore there exists a set, $E$ having measure zero such that

$$
\lim _{k \rightarrow \infty} f_{n_{k}}(x)=f(x)
$$

for all $x \notin E$. Redefine $f_{n_{k}}$ to equal 0 on $E$ and let $f(x)=0$ for $x \in E$. It then follows that $\lim _{k \rightarrow \infty} f_{n_{k}}(x)=f(x)$ for all $x$. By Fatou's lemma, and the Minkowski inequality,

$$
\begin{gather*}
\left\|f-f_{n_{k}}\right\|_{p}=\left(\int\left|f-f_{n_{k}}\right|^{p} d \mu\right)^{1 / p} \leq \\
\lim \inf _{m \rightarrow \infty}\left(\int\left|f_{n_{m}}-f_{n_{k}}\right|^{p} d \mu\right)^{1 / p}=\lim _{m \rightarrow \infty} \inf _{m \rightarrow{ }^{2}}\left\|f_{n_{m}}-f_{n_{k}}\right\|_{p} \leq \\
\lim \inf _{m \rightarrow \infty} \sum_{j=k}^{m-1}\left\|\mid f_{n_{j+1}}-f_{n_{j}}\right\|_{p} \leq \sum_{i=k}^{\infty}\left\|f_{n_{i+1}}-f_{n_{i}}\right\|_{p} \leq 2^{-(k-1)} . \tag{15.1.4}
\end{gather*}
$$

Therefore, $f \in L^{p}(\Omega)$ because

$$
\|f\|_{p} \leq\left\|f-f_{n_{k}}\right\|_{p}+\left\|f_{n_{k}}\right\|_{p}<\infty
$$

and $\lim _{k \rightarrow \infty}\left\|f_{n_{k}}-f\right\|_{p}=0$. This proves b.).
This has shown $f_{n_{k}}$ converges to $f$ in $L^{p}(\Omega)$. It follows the original Cauchy sequence also converges to $f$ in $L^{p}(\Omega)$. This is a general fact that if a subsequence of a Cauchy sequence converges, then so does the original Cauchy sequence. You should give a proof of this. This proves the theorem.

In working with the $L^{p}$ spaces, the following inequality also known as Minkowski's inequality is very useful. It is similar to the Minkowski inequality for sums. To see this, replace the integral, $\int_{X}$ with a finite summation sign and you will see the usual Minkowski inequality or rather the version of it given in Corollary 15.1.7.

To prove this theorem first consider a special case of it in which technical considerations which shed no light on the proof are excluded.

Lemma 15.1.11 Let $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{F}, \lambda)$ be finite complete measure spaces and let $f$ be $\overline{\mu \times \lambda}$ measurable and uniformly bounded. Then the following inequality is valid for $p \geq 1$.

$$
\begin{equation*}
\int_{X}\left(\int_{Y}|f(x, y)|^{p} d \lambda\right)^{\frac{1}{p}} d \mu \geq\left(\int_{Y}\left(\int_{X}|f(x, y)| d \mu\right)^{p} d \lambda\right)^{\frac{1}{p}} \tag{15.1.5}
\end{equation*}
$$

Proof: Since $f$ is bounded and $\mu(X), \lambda(Y)<\infty$,

$$
\left(\int_{Y}\left(\int_{X}|f(x, y)| d \mu\right)^{p} d \lambda\right)^{\frac{1}{p}}<\infty .
$$

Let

$$
J(y)=\int_{X}|f(x, y)| d \mu
$$

Note there is no problem in writing this for a.e. $y$ because $f$ is product measurable. Then by Fubini's theorem,

$$
\begin{aligned}
\int_{Y}\left(\int_{X}|f(x, y)| d \mu\right)^{p} d \lambda & =\int_{Y} J(y)^{p-1} \int_{X}|f(x, y)| d \mu d \lambda \\
& =\int_{X} \int_{Y} J(y)^{p-1}|f(x, y)| d \lambda d \mu
\end{aligned}
$$

Now apply Holder's inequality in the last integral above and recall $p-1=\frac{p}{q}$. This yields

$$
\begin{gather*}
\int_{Y}\left(\int_{X}|f(x, y)| d \mu\right)^{p} d \lambda \\
\leq \int_{X}\left(\int_{Y} J(y)^{p} d \lambda\right)^{\frac{1}{q}}\left(\int_{Y}|f(x, y)|^{p} d \lambda\right)^{\frac{1}{p}} d \mu \\
=\left(\int_{Y} J(y)^{p} d \lambda\right)^{\frac{1}{q}} \int_{X}\left(\int_{Y}|f(x, y)|^{p} d \lambda\right)^{\frac{1}{p}} d \mu \\
=\left(\int_{Y}\left(\int_{X}|f(x, y)| d \mu\right)^{p} d \lambda\right)^{\frac{1}{q}} \int_{X}\left(\int_{Y}|f(x, y)|^{p} d \lambda\right)^{\frac{1}{p}} d \mu \tag{15.1.6}
\end{gather*}
$$

Therefore, dividing both sides by the first factor in the above expression,

$$
\begin{equation*}
\left(\int_{Y}\left(\int_{X}|f(x, y)| d \mu\right)^{p} d \lambda\right)^{\frac{1}{p}} \leq \int_{X}\left(\int_{Y}|f(x, y)|^{p} d \lambda\right)^{\frac{1}{p}} d \mu \tag{15.1.7}
\end{equation*}
$$

Note that 15.1.7 holds even if the first factor of 15.1.6 equals zero. This proves the lemma.
Now consider the case where $f$ is not assumed to be bounded and where the measure spaces are $\sigma$ finite.

Theorem 15.1.12 Let $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{F}, \lambda)$ be $\sigma$-finite measure spaces and let $f$ be product measurable. Then the following inequality is valid for $p \geq 1$.

$$
\begin{equation*}
\int_{X}\left(\int_{Y}|f(x, y)|^{p} d \lambda\right)^{\frac{1}{p}} d \mu \geq\left(\int_{Y}\left(\int_{X}|f(x, y)| d \mu\right)^{p} d \lambda\right)^{\frac{1}{p}} \tag{15.1.8}
\end{equation*}
$$

Proof: Since the two measure spaces are $\sigma$ finite, there exist measurable sets, $X_{m}$ and $Y_{k}$ such that $X_{m} \subseteq X_{m+1}$ for all $m, Y_{k} \subseteq Y_{k+1}$ for all $k$, and $\mu\left(X_{m}\right), \lambda\left(Y_{k}\right)<\infty$. Now define

$$
f_{n}(x, y) \equiv\left\{\begin{array}{l}
f(x, y) \text { if }|f(x, y)| \leq n \\
n \text { if }|f(x, y)|>n
\end{array}\right.
$$

Thus $f_{n}$ is uniformly bounded and product measurable. By the above lemma,

$$
\begin{equation*}
\int_{X_{m}}\left(\int_{Y_{k}}\left|f_{n}(x, y)\right|^{p} d \lambda\right)^{\frac{1}{p}} d \mu \geq\left(\int_{Y_{k}}\left(\int_{X_{m}}\left|f_{n}(x, y)\right| d \mu\right)^{p} d \lambda\right)^{\frac{1}{p}} . \tag{15.1.9}
\end{equation*}
$$

Now observe that $\left|f_{n}(x, y)\right|$ increases in $n$ and the pointwise limit is $|f(x, y)|$. Therefore, using the monotone convergence theorem in 15.1.9 yields the same inequality with $f$ replacing $f_{n}$. Next let $k \rightarrow \infty$ and use the monotone convergence theorem again to replace $Y_{k}$ with $Y$. Finally let $m \rightarrow \infty$ in what is left to obtain 15.1.8. This proves the theorem.

Note that the proof of this theorem depends on two manipulations, the interchange of the order of integration and Holder's inequality. Note that there is nothing to check in the case of double sums. Thus if $a_{i j} \geq 0$, it is always the case that

$$
\left(\sum_{j}\left(\sum_{i} a_{i j}\right)^{p}\right)^{1 / p} \leq \sum_{i}\left(\sum_{j} a_{i j}^{p}\right)^{1 / p}
$$

because the integrals in this case are just sums and $(i, j) \rightarrow a_{i j}$ is measurable.
The $L^{p}$ spaces have many important properties.

### 15.2 Density Considerations

Theorem 15.2.1 Let $p \geq 1$ and let $(\Omega, \mathscr{S}, \mu)$ be a measure space. Then the simple functions are dense in $L^{p}(\Omega)$.

Proof: Recall that a function, $f$, having values in $\mathbb{R}$ can be written in the form $f=$ $f^{+}-f^{-}$where

$$
f^{+}=\max (0, f), f^{-}=\max (0,-f)
$$

Therefore, an arbitrary complex valued function, $f$ is of the form

$$
f=\operatorname{Re} f^{+}-\operatorname{Re} f^{-}+i\left(\operatorname{Im} f^{+}-\operatorname{Im} f^{-}\right) .
$$

If each of these nonnegative functions is approximated by a simple function, it follows $f$ is also approximated by a simple function. Therefore, there is no loss of generality in assuming at the outset that $f \geq 0$.

Since $f$ is measurable, Theorem 11.3.9 implies there is an increasing sequence of simple functions, $\left\{s_{n}\right\}$, converging pointwise to $f(x)$. Now

$$
\left|f(x)-s_{n}(x)\right| \leq|f(x)|
$$

By the Dominated Convergence theorem,

$$
0=\lim _{n \rightarrow \infty} \int\left|f(x)-s_{n}(x)\right|^{p} d \mu
$$

Thus simple functions are dense in $L^{p}$.
Recall that for $\Omega$ a topological space, $C_{c}(\Omega)$ is the space of continuous functions with compact support in $\Omega$. Also recall the following definition.

Definition 15.2.2 Let $(\Omega, \mathscr{S}, \mu)$ be a measure space and suppose $(\Omega, \tau)$ is also a topological space. Then $(\Omega, \mathscr{S}, \mu)$ is called a regular measure space if the $\sigma$ algebra of Borel sets is contained in $\mathscr{S}$ and for all $E \in \mathscr{S}$,

$$
\mu(E)=\inf \{\mu(V): V \supseteq E \text { and } V \text { open }\}
$$

and if $\mu(E)<\infty$,

$$
\mu(E)=\sup \{\mu(K): K \subseteq E \text { and } K \text { is compact }\}
$$

and $\mu(K)<\infty$ for any compact set, $K$.
For example Lebesgue measure is an example of such a measure. More generally these measures are often refered to as Radon measures.

Lemma 15.2.3 Let $\Omega$ be a metric space in which the closed balls are compact and let $K$ be a compact subset of $V$, an open set. Then there exists a continuous function $f: \Omega \rightarrow[0,1]$ such that $f(x)=1$ for all $x \in K$ and $\operatorname{spt}(f)$ is a compact subset of $V$. That is, $K \prec f \prec V$.

Proof: Let $K \subseteq W \subseteq \bar{W} \subseteq V$ and $\bar{W}$ is compact. To obtain this list of inclusions consider a point in $K, x$, and take $B\left(x, r_{x}\right)$ a ball containing $x$ such that $\overline{B\left(x, r_{x}\right)}$ is a compact subset of $V$. Next use the fact that $K$ is compact to obtain the existence of a list, $\left\{B\left(x_{i}, r_{x_{i}} / 2\right)\right\}_{i=1}^{m}$ which covers $K$. Then let

$$
W \equiv \cup_{i=1}^{m} B\left(x_{i}, \frac{r_{x_{i}}}{2}\right) .
$$

It follows since this is a finite union that

$$
\bar{W}=\cup_{i=1}^{m} \overline{B\left(x_{i}, \frac{r_{x_{i}}}{2}\right)}
$$

and so $\bar{W}$, being a finite union of compact sets is itself a compact set. Also, from the construction

$$
\bar{W} \subseteq \cup_{i=1}^{m} B\left(x_{i}, r_{x_{i}}\right)
$$

Define $f$ by

$$
f(x)=\frac{\operatorname{dist}\left(x, W^{C}\right)}{\operatorname{dist}(x, K)+\operatorname{dist}\left(x, W^{C}\right)}
$$

It is clear that $f$ is continuous if the denominator is always nonzero. But this is clear because if $x \in W^{C}$ there must be a ball $B(x, r)$ such that this ball does not intersect $K$. Otherwise, $x$ would be a limit point of $K$ and since $K$ is closed, $x \in K$. However, $x \notin K$ because $K \subseteq W$.

It is not necessary to be in a metric space to do this. You can accomplish the same thing using Urysohn's lemma.

Theorem 15.2.4 Let $(\Omega, \mathscr{S}, \mu)$ be a regular measure space as in Definition 15.2.2 where the conclusion of Lemma 15.2.3 holds. Then $C_{c}(\Omega)$ is dense in $L^{p}(\Omega)$.

Proof: First consider a measurable set, $E$ where $\mu(E)<\infty$. Let $K \subseteq E \subseteq V$ where $\mu(V \backslash K)<\varepsilon$. Now let $K \prec h \prec V$. Then

$$
\int\left|h-\mathscr{X}_{E}\right|^{p} d \mu \leq \int \mathscr{X}_{V \backslash K}^{p} d \mu=\mu(V \backslash K)<\varepsilon
$$

It follows that for each $s$ a simple function in $L^{p}(\Omega)$, there exists $h \in C_{c}(\Omega)$ such that $\|s-h\|_{p}<\varepsilon$. This is because if

$$
s(x)=\sum_{i=1}^{m} c_{i} \mathscr{X}_{E_{i}}(x)
$$

is a simple function in $L^{p}$ where the $c_{i}$ are the distinct nonzero values of $s$ each $\mu\left(E_{i}\right)<\infty$ since otherwise $s \notin L^{p}$ due to the inequality

$$
\int|s|^{p} d \mu \geq\left|c_{i}\right|^{p} \mu\left(E_{i}\right)
$$

By Theorem 15.2.1, simple functions are dense in $L^{p}(\Omega)$, and so this proves the Theorem.

### 15.3 Separability

Theorem 15.3.1 For $p \geq 1$ and $\mu$ a Radon measure, $L^{p}\left(\mathbb{R}^{n}, \mu\right)$ is separable. Recall this means there exists a countable set, $\mathscr{D}$, such that if $f \in L^{p}\left(\mathbb{R}^{n}, \mu\right)$ and $\varepsilon>0$, there exists $g \in \mathscr{D}$ such that $\|f-g\|_{p}<\varepsilon$.

Proof: Let $Q$ be all functions of the form $c \mathscr{X}_{[\mathbf{a}, \mathbf{b})}$ where

$$
[\mathbf{a}, \mathbf{b}) \equiv\left[a_{1}, b_{1}\right) \times\left[a_{2}, b_{2}\right) \times \cdots \times\left[a_{n}, b_{n}\right),
$$

and both $a_{i}, b_{i}$ are rational, while $c$ has rational real and imaginary parts. Let $\mathscr{D}$ be the set of all finite sums of functions in $Q$. Thus, $\mathscr{D}$ is countable. In fact $\mathscr{D}$ is dense in $L^{p}\left(\mathbb{R}^{n}, \mu\right)$. To prove this it is necessary to show that for every $f \in L^{p}\left(\mathbb{R}^{n}, \mu\right)$, there exists an element of $\mathscr{D}, s$ such that $\|s-f\|_{p}<\varepsilon$. If it can be shown that for every $g \in C_{c}\left(\mathbb{R}^{n}\right)$ there exists $h \in \mathscr{D}$ such that $\|g-h\|_{p}<\varepsilon$, then this will suffice because if $f \in L^{p}\left(\mathbb{R}^{n}\right)$ is arbitrary, Theorem 15.2.4 implies there exists $g \in C_{c}\left(\mathbb{R}^{n}\right)$ such that $\|f-g\|_{p} \leq \frac{\varepsilon}{2}$ and then there would exist $h \in C_{c}\left(\mathbb{R}^{n}\right)$ such that $\|h-g\|_{p}<\frac{\varepsilon}{2}$. By the triangle inequality,

$$
\|f-h\|_{p} \leq\|h-g\|_{p}+\|g-f\|_{p}<\varepsilon .
$$

Therefore, assume at the outset that $f \in C_{c}\left(\mathbb{R}^{n}\right)$.
Let $\mathscr{P}_{m}$ consist of all sets of the form $[\mathbf{a}, \mathbf{b}) \equiv \prod_{i=1}^{n}\left[a_{i}, b_{i}\right)$ where $a_{i}=j 2^{-m}$ and $b_{i}=(j+$ 1)2 $2^{-m}$ for $j$ an integer. Thus $\mathscr{P}_{m}$ consists of a tiling of $\mathbb{R}^{n}$ into half open rectangles having diameters $2^{-m} n^{\frac{1}{2}}$. There are countably many of these rectangles; so, let $\mathscr{P}_{m}=\left\{\left[\mathbf{a}_{i}, \mathbf{b}_{i}\right)\right\}_{i=1}^{\infty}$ and $\mathbb{R}^{n}=\cup_{i=1}^{\infty}\left[\mathbf{a}_{i}, \mathbf{b}_{i}\right)$. Let $c_{i}^{m}$ be complex numbers with rational real and imaginary parts satisfying

$$
\left|f\left(\mathbf{a}_{i}\right)-c_{i}^{m}\right|<2^{-m}
$$

$$
\begin{equation*}
\left|c_{i}^{m}\right| \leq\left|f\left(\mathbf{a}_{i}\right)\right| \tag{15.3.10}
\end{equation*}
$$

Let

$$
s_{m}(\mathbf{x})=\sum_{i=1}^{\infty} c_{i}^{m} \mathscr{X}_{\left[\mathbf{a}_{i}, \mathbf{b}_{i}\right)}(\mathbf{x})
$$

Since $f\left(\mathbf{a}_{i}\right)=0$ except for finitely many values of $i$, the above is a finite sum. Then 15.3.10 implies $s_{m} \in \mathscr{D}$. If $s_{m}$ converges uniformly to $f$ then it follows $\left\|s_{m}-f\right\|_{p} \rightarrow 0$ because $\left|s_{m}\right| \leq|f|$ and so

$$
\begin{aligned}
\left\|s_{m}-f\right\|_{p} & =\left(\int\left|s_{m}-f\right|^{p} d \mu\right)^{1 / p} \\
& =\left(\int_{\operatorname{spt}(f)}\left|s_{m}-f\right|^{p} d \mu\right)^{1 / p} \\
& \leq\left[\varepsilon m_{n}(\operatorname{spt}(f))\right]^{1 / p}
\end{aligned}
$$

whenever $m$ is large enough.
Since $f \in C_{c}\left(\mathbb{R}^{n}\right)$ it follows that $f$ is uniformly continuous and so given $\varepsilon>0$ there exists $\delta>0$ such that if $|\mathbf{x}-\mathbf{y}|<\delta,|f(\mathbf{x})-f(\mathbf{y})|<\varepsilon / 2$. Now let $m$ be large enough that every box in $\mathscr{P}_{m}$ has diameter less than $\delta$ and also that $2^{-m}<\varepsilon / 2$. Then if $\left[\mathbf{a}_{i}, \mathbf{b}_{i}\right)$ is one of these boxes of $\mathscr{P}_{m}$, and $\mathbf{x} \in\left[\mathbf{a}_{i}, \mathbf{b}_{i}\right)$,

$$
\left|f(\mathbf{x})-f\left(\mathbf{a}_{i}\right)\right|<\varepsilon / 2
$$

and

$$
\left|f\left(\mathbf{a}_{i}\right)-c_{i}^{m}\right|<2^{-m}<\varepsilon / 2
$$

Therefore, using the triangle inequality, it follows that

$$
\left|f(\mathbf{x})-c_{i}^{m}\right|=\left|s_{m}(\mathbf{x})-f(\mathbf{x})\right|<\varepsilon
$$

and since $\mathbf{x}$ is arbitrary, this establishes uniform convergence. This proves the theorem.
Here is an easier proof if you know the Weierstrass approximation theorem.
Theorem 15.3.2 For $p \geq 1$ and $\mu$ a Radon measure, $L^{p}\left(\mathbb{R}^{n}, \mu\right)$ is separable. Recall this means there exists a countable set, $\mathscr{D}$, such that if $f \in L^{p}\left(\mathbb{R}^{n}, \mu\right)$ and $\varepsilon>0$, there exists $g \in \mathscr{D}$ such that $\|f-g\|_{p}<\varepsilon$.

Proof: Let $\mathscr{P}$ denote the set of all polynomials which have rational coefficients. Then $\mathscr{P}$ is countable. Let $\tau_{k} \in C_{c}\left((-(k+1),(k+1))^{n}\right)$ such that

$$
\overline{[-k, k]^{n}} \prec \tau_{k} \prec(-(k+1),(k+1))^{n} .
$$

Let $\mathscr{D}_{k}$ denote the functions which are of the form, $p \tau_{k}$ where $p \in \mathscr{P}$. Thus $\mathscr{D}_{k}$ is also countable. Let $\mathscr{D} \equiv \cup_{k=1}^{\infty} \mathscr{D}_{k}$. It follows each function in $\mathscr{D}$ is in $C_{c}\left(\mathbb{R}^{n}\right)$ and so it in $L^{p}\left(\mathbb{R}^{n}, \mu\right)$. Let $f \in L^{p}\left(\mathbb{R}^{n}, \mu\right)$. By regularity of $\mu$ there exists $g \in C_{c}\left(\mathbb{R}^{n}\right)$ such that
$\|f-g\|_{L^{p}\left(\mathbb{R}^{n}, \mu\right)}<\frac{\varepsilon}{3}$. Let $k$ be such that $\operatorname{spt}(g) \subseteq(-k, k)^{n}$. Now by the Weierstrass approximation theorem there exists a polynomial $q$ such that

$$
\begin{aligned}
\|g-q\|_{[-(k+1), k+1]^{n}} & \equiv \sup \left\{|g(\mathbf{x})-q(\mathbf{x})|: \mathbf{x} \in[-(k+1),(k+1)]^{n}\right\} \\
& <\frac{\varepsilon}{3 \mu\left((-(k+1), k+1)^{n}\right)}
\end{aligned}
$$

It follows

$$
\begin{aligned}
\left\|g-\tau_{k} q\right\|_{[-(k+1), k+1]^{n}} & =\left\|\tau_{k} g-\tau_{k} q\right\|_{[-(k+1), k+1]^{n}} \\
& <\frac{\varepsilon}{3 \mu\left((-(k+1), k+1)^{n}\right)}
\end{aligned}
$$

Without loss of generality, it can be assumed this polynomial has all rational coefficients. Therefore, $\tau_{k} q \in \mathscr{D}$.

$$
\begin{aligned}
\left\|g-\tau_{k} q\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}^{p} & =\int_{(-(k+1), k+1)^{n}}\left|g(\mathbf{x})-\tau_{k}(\mathbf{x}) q(\mathbf{x})\right|^{p} d \mu \\
& \leq\left(\frac{\varepsilon}{3 \mu\left((-(k+1), k+1)^{n}\right)}\right)^{p} \mu\left((-(k+1), k+1)^{n}\right) \\
& <\left(\frac{\varepsilon}{3}\right)^{p} .
\end{aligned}
$$

It follows

$$
\left\|f-\tau_{k} q\right\|_{L^{p}\left(\mathbb{R}^{n}, \mu\right)} \leq\|f-g\|_{L^{p}\left(\mathbb{R}^{n}, \mu\right)}+\left\|g-\tau_{k} q\right\|_{L^{p}\left(\mathbb{R}^{n}, \mu\right)}<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}<\varepsilon
$$

This proves the theorem.
Corollary 15.3.3 Let $\Omega$ be any $\mu$ measurable subset of $\mathbb{R}^{n}$ and let $\mu$ be a Radon measure. Then $L^{p}(\Omega, \mu)$ is separable. Here the $\sigma$ algebra of measurable sets will consist of all intersections of measurable sets with $\Omega$ and the measure will be $\mu$ restricted to these sets.

Proof: Let $\widetilde{\mathscr{D}}$ be the restrictions of $\mathscr{D}$ to $\Omega$. If $f \in L^{p}(\Omega)$, let $F$ be the zero extension of $f$ to all of $\mathbb{R}^{n}$. Let $\varepsilon>0$ be given. By Theorem 15.3 .1 or 15.3.2 there exists $s \in \mathscr{D}$ such that $\|F-s\|_{p}<\varepsilon$. Thus

$$
\|s-f\|_{L^{p}(\Omega, \mu)} \leq\|s-F\|_{L^{p}\left(\mathbb{R}^{n}, \mu\right)}<\varepsilon
$$

and so the countable set $\widetilde{\mathscr{D}}$ is dense in $L^{p}(\Omega)$.

### 15.4 Continuity Of Translation

Definition 15.4.1 Let $f$ be a function defined on $U \subseteq \mathbb{R}^{n}$ and let $\mathbf{w} \in \mathbb{R}^{n}$. Then $f_{\mathbf{w}}$ will be the function defined on $\mathbf{w}+U$ by

$$
f_{\mathbf{w}}(\mathbf{x})=f(\mathbf{x}-\mathbf{w})
$$

Theorem 15.4.2 (Continuity of translation in $\left.L^{p}\right)$ Let $f \in L^{p}\left(\mathbb{R}^{n}\right)$ with the measure being Lebesgue measure. Then

$$
\lim _{\|\mathbf{w}\| \rightarrow 0}\left\|f_{\mathbf{w}}-f\right\|_{p}=0
$$

Proof: Let $\varepsilon>0$ be given and let $g \in C_{c}\left(\mathbb{R}^{n}\right)$ with $\|g-f\|_{p}<\frac{\varepsilon}{3}$. Since Lebesgue measure is translation invariant $\left(m_{n}(\mathbf{w}+E)=m_{n}(E)\right)$,

$$
\left\|g_{\mathbf{w}}-f_{\mathbf{w}}\right\|_{p}=\|g-f\|_{p}<\frac{\varepsilon}{3}
$$

You can see this from looking at simple functions and passing to the limit or you could use the change of variables formula to verify it.

Therefore

$$
\begin{align*}
\left\|f-f_{\mathbf{w}}\right\|_{p} & \leq\|f-g\|_{p}+\left\|g-g_{\mathbf{w}}\right\|_{p}+\left\|g_{\mathbf{w}}-f_{\mathbf{w}}\right\| \\
& <\frac{2 \varepsilon}{3}+\left\|g-g_{\mathbf{w}}\right\|_{p} \tag{15.4.11}
\end{align*}
$$

But $\lim _{|\mathbf{w}| \rightarrow 0} g_{\mathbf{w}}(\mathbf{x})=g(\mathbf{x})$ uniformly in $\mathbf{x}$ because $g$ is uniformly continuous. Now let $B$ be a large ball containing $\operatorname{spt}(g)$ and let $\delta_{1}$ be small enough that $B(\mathbf{x}, \boldsymbol{\delta}) \subseteq B$ whenever $\mathbf{x} \in \operatorname{spt}(g)$. If $\varepsilon>0$ is given there exists $\delta<\delta_{1}$ such that if $|\mathbf{w}|<\delta$, it follows that $|g(\mathbf{x}-\mathbf{w})-g(\mathbf{x})|<\varepsilon / 3\left(1+m_{n}(B)^{1 / p}\right)$. Therefore,

$$
\begin{aligned}
\left\|g-g_{\mathbf{w}}\right\|_{p} & =\left(\int_{B}|g(\mathbf{x})-g(\mathbf{x}-\mathbf{w})|^{p} d m_{n}\right)^{1 / p} \\
& \leq \varepsilon \frac{m_{n}(B)^{1 / p}}{3\left(1+m_{n}(B)^{1 / p}\right)}<\frac{\varepsilon}{3}
\end{aligned}
$$

Therefore, whenever $|\mathbf{w}|<\delta$, it follows $\left\|g-g_{\mathbf{w}}\right\|_{p}<\frac{\varepsilon}{3}$ and so from 15.4.11 $\left\|f-f_{\mathbf{w}}\right\|_{p}<\varepsilon$. This proves the theorem.

### 15.5 Mollifiers And Density Of Smooth Functions

Definition 15.5.1 Let $U$ be an open subset of $\mathbb{R}^{n} . C_{c}^{\infty}(U)$ is the vector space of all infinitely differentiable functions which equal zero for all $\mathbf{x}$ outside of some compact set contained in $U$. Similarly, $C_{c}^{m}(U)$ is the vector space of all functions which are $m$ times continuously differentiable and whose support is a compact subset of $U$.

Example 15.5.2 Let $U=B(\mathbf{z}, 2 r)$

$$
\psi(\mathbf{x})=\left\{\begin{array}{l}
\exp \left[\left(|\mathbf{x}-\mathbf{z}|^{2}-r^{2}\right)^{-1}\right] \text { if }|\mathbf{x}-\mathbf{z}|<r \\
0 \text { if }|\mathbf{x}-\mathbf{z}| \geq r
\end{array}\right.
$$

Then a little work shows $\psi \in C_{c}^{\infty}(U)$. Note that if $\mathbf{z}=\mathbf{0}$ then $\psi(\mathbf{x})=\psi(-\mathbf{x})$. The following also is easily obtained.

Lemma 15.5.3 Let $U$ be any open set. Then $C_{c}^{\infty}(U) \neq \emptyset$.
Proof: Pick $\mathbf{z} \in U$ and let $r$ be small enough that $B(\mathbf{z}, 2 r) \subseteq U$. Then let

$$
\psi \in C_{c}^{\infty}(B(\mathbf{z}, 2 r)) \subseteq C_{c}^{\infty}(U)
$$

be the function of the above example.
Definition 15.5.4 Let $U=\left\{\mathbf{x} \in \mathbb{R}^{n}:|\mathbf{x}|<1\right\}$. A sequence $\left\{\psi_{m}\right\} \subseteq C_{c}^{\infty}(U)$ is called a mollifier (This is sometimes called an approximate identity if the differentiability is not included.) if

$$
\psi_{m}(\mathbf{x}) \geq 0, \psi_{m}(\mathbf{x})=0, \text { if }|\mathbf{x}| \geq \frac{1}{m}
$$

and $\int \psi_{m}(\mathbf{x})=1$. Sometimes it may be written as $\left\{\psi_{\varepsilon}\right\}$ where $\psi_{\varepsilon}$ satisfies the above conditions except $\psi_{\varepsilon}(\mathbf{x})=0$ if $|\mathbf{x}| \geq \varepsilon$. In other words, $\varepsilon$ takes the place of $1 / m$ and in everything that follows $\varepsilon \rightarrow 0$ instead of $m \rightarrow \infty$.

As before, $\int f(\mathbf{x}, \mathbf{y}) d \mu(\mathbf{y})$ will mean $\mathbf{x}$ is fixed and the function $\mathbf{y} \rightarrow f(\mathbf{x}, \mathbf{y})$ is being integrated. To make the notation more familiar, $d x$ is written instead of $d m_{n}(x)$.

Example 15.5.5 Let

$$
\psi \in C_{c}^{\infty}(B(0,1))(B(0,1)=\{\mathbf{x}:|\mathbf{x}|<1\})
$$

with $\psi(\mathbf{x}) \geq 0$ and $\int \psi d m=1$. Let $\psi_{m}(\mathbf{x})=c_{m} \psi(m \mathbf{x})$ where $c_{m}$ is chosen in such a way that $\int \psi_{m} d m=1$. By the change of variables theorem $c_{m}=m^{n}$.

Definition 15.5.6 A function, $f$, is said to be in $L_{l o c}^{1}\left(\mathbb{R}^{n}, \mu\right)$ if $f$ is $\mu$ measurable and if $|f| \mathscr{X}_{K} \in L^{1}\left(\mathbb{R}^{n}, \mu\right)$ for every compact set, $K$. Here $\mu$ is a Radon measure on $\mathbb{R}^{n}$. Usually $\mu=m_{n}$, Lebesgue measure. When this is so, write $L_{\text {loc }}^{1}\left(\mathbb{R}^{n}\right)$ or $L^{p}\left(\mathbb{R}^{n}\right)$, etc. If $f \in L_{l o c}^{1}\left(\mathbb{R}^{n}, \mu\right)$, and $g \in C_{c}\left(\mathbb{R}^{n}\right)$,

$$
f * g(\mathbf{x}) \equiv \int f(\mathbf{y}) g(\mathbf{x}-\mathbf{y}) d \mu
$$

The following lemma will be useful in what follows. It says that one of these very unregular functions in $L_{l o c}^{1}\left(\mathbb{R}^{n}, \mu\right)$ is smoothed out by convolving with a mollifier.

Lemma 15.5.7 Let $f \in L_{l o c}^{1}\left(\mathbb{R}^{n}, \mu\right)$, and $g \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$. Then $f * g$ is an infinitely differentiable function. Here $\mu$ is a Radon measure on $\mathbb{R}^{n}$.

Proof: Consider the difference quotient for calculating a partial derivative of $f * g$.

$$
\frac{f * g\left(\mathbf{x}+t \mathbf{e}_{j}\right)-f * g(\mathbf{x})}{t}=\int f(\mathbf{y}) \frac{g\left(\mathbf{x}+t \mathbf{e}_{j}-\mathbf{y}\right)-g(\mathbf{x}-\mathbf{y})}{t} d \mu(y)
$$

Using the fact that $g \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$, the quotient,

$$
\frac{g\left(\mathbf{x}+t \mathbf{e}_{j}-\mathbf{y}\right)-g(\mathbf{x}-\mathbf{y})}{t}
$$

is uniformly bounded. To see this easily, use Theorem 6.13.4 on Page 120 to get the existence of a constant, $M$ depending on

$$
\max \left\{\|D g(\mathbf{x})\|: \mathbf{x} \in \mathbb{R}^{n}\right\}
$$

such that

$$
\left|g\left(\mathbf{x}+t \mathbf{e}_{j}-\mathbf{y}\right)-g(\mathbf{x}-\mathbf{y})\right| \leq M|t|
$$

for any choice of $\mathbf{x}$ and $\mathbf{y}$. Therefore, there exists a dominating function for the integrand of the above integral which is of the form $C|f(\mathbf{y})| \mathscr{X}_{K}$ where $K$ is a compact set depending on the support of $g$. It follows the limit of the difference quotient above passes inside the integral as $t \rightarrow 0$ and

$$
\frac{\partial}{\partial x_{j}}(f * g)(\mathbf{x})=\int f(\mathbf{y}) \frac{\partial}{\partial x_{j}} g(\mathbf{x}-\mathbf{y}) d \mu(y)
$$

Now letting $\frac{\partial}{\partial x_{j}} g$ play the role of $g$ in the above argument, partial derivatives of all orders exist. A similar use of the dominated convergence theorem shows all these partial derivatives are also continuous. This proves the lemma.

Theorem 15.5.8 Let $K$ be a compact subset of an open set, $U$. Then there exists a function, $h \in C_{c}^{\infty}(U)$, such that $h(\mathbf{x})=1$ for all $\mathbf{x} \in K$ and $h(\mathbf{x}) \in[0,1]$ for all $\mathbf{x}$.

Proof: Let $r>0$ be small enough that $K+B(\mathbf{0}, 3 r) \subseteq U$. The symbol, $K+B(\mathbf{0}, 3 r)$ means

$$
\{\mathbf{k}+\mathbf{x}: \mathbf{k} \in K \text { and } \mathbf{x} \in B(\mathbf{0}, 3 r)\}
$$

Thus this is simply a way to write

$$
\cup\{B(\mathbf{k}, 3 r): \mathbf{k} \in K\} .
$$

Think of it as fattening up the set, $K$. Let $K_{r}=K+B(0, r)$. A picture of what is happening follows.


Consider $\mathscr{X}_{K_{r}} * \psi_{m}$ where $\psi_{m}$ is a mollifier. Let $m$ be so large that $\frac{1}{m}<r$. Then from the definition of what is meant by a convolution, and using that $\psi_{m}$ has support in $B\left(\mathbf{0}, \frac{1}{m}\right)$, $\mathscr{X}_{K_{r}} * \psi_{m}=1$ on $K$ and that its support is in $K+B(\mathbf{0}, 3 r)$. Now using Lemma 15.5.7, $\mathscr{X}_{K_{r}} * \psi_{m}$ is also infinitely differentiable. Therefore, let $h=\mathscr{X}_{K_{r}} * \psi_{m}$.

The following corollary will be used later.

Corollary 15.5.9 Let $K$ be a compact set in $\mathbb{R}^{n}$ and let $\left\{U_{i}\right\}_{i=1}^{\infty}$ be an open cover of $K$. Then there exist functions, $\psi_{k} \in C_{c}^{\infty}\left(U_{i}\right)$ such that $\psi_{i} \prec U_{i}$ and for all $\mathbf{x} \in K$,

$$
\sum_{i=1}^{\infty} \psi_{i}(\mathbf{x})=1
$$

If $K_{1}$ is a compact subset of $U_{1}$ there exist such functions such that also $\psi_{1}(\mathbf{x})=1$ for all $\mathbf{x} \in K_{1}$.

Proof: This follows from a repeat of the proof of Theorem 12.2.11 on Page 287, replacing the lemma used in that proof with Theorem 15.5.8.

Note that in the last conclusion of above corollary, the set $U_{1}$ could be replaced with $U_{i}$ for any fixed $i$ by simply renumbering.

Theorem 15.5.10 For each $p \geq 1, C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$ is dense in $L^{p}\left(\mathbb{R}^{n}\right)$. Here the measure is Lebesgue measure.

Proof: Let $f \in L^{p}\left(\mathbb{R}^{n}\right)$ and let $\varepsilon>0$ be given. Choose $g \in C_{c}\left(\mathbb{R}^{n}\right)$ such that $\|f-g\|_{p}<$ $\frac{\varepsilon}{2}$. This can be done by using Theorem 15.2.4. Now let

$$
g_{m}(\mathbf{x})=g * \psi_{m}(\mathbf{x}) \equiv \int g(\mathbf{x}-\mathbf{y}) \psi_{m}(\mathbf{y}) d m_{n}(y)=\int g(\mathbf{y}) \psi_{m}(\mathbf{x}-\mathbf{y}) d m_{n}(y)
$$

where $\left\{\psi_{m}\right\}$ is a mollifier. It follows from Lemma 15.5.7 $g_{m} \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$. It vanishes if $\mathbf{x} \notin \operatorname{spt}(g)+B\left(0, \frac{1}{m}\right)$.

$$
\begin{aligned}
\left\|g-g_{m}\right\|_{p} & =\left(\int\left|g(\mathbf{x})-\int g(\mathbf{x}-\mathbf{y}) \psi_{m}(\mathbf{y}) d m_{n}(\mathbf{y})\right|^{p} d m_{n}(\mathbf{x})\right)^{\frac{1}{p}} \\
& \leq\left(\int\left(\int|g(\mathbf{x})-g(\mathbf{x}-\mathbf{y})| \psi_{m}(\mathbf{y}) d m_{n}(\mathbf{y})\right)^{p} d m_{n}(\mathbf{x})\right)^{\frac{1}{p}} \\
& \leq \int\left(\int|g(\mathbf{x})-g(\mathbf{x}-\mathbf{y})|^{p} d m_{n}(\mathbf{x})\right)^{\frac{1}{p}} \psi_{m}(\mathbf{y}) d m_{n}(\mathbf{y}) \\
& =\int_{B\left(0, \frac{1}{m}\right)}\left\|g-g_{\mathbf{y}}\right\|_{p} \psi_{m}(\mathbf{y}) d m_{n}(\mathbf{y})<\frac{\varepsilon}{2}
\end{aligned}
$$

whenever $m$ is large enough thanks to the uniform continuity of $g$. Theorem 15.1.12 was used to obtain the third inequality. There is no measurability problem because the function

$$
(\mathbf{x}, \mathbf{y}) \rightarrow|g(\mathbf{x})-g(\mathbf{x}-\mathbf{y})| \psi_{m}(\mathbf{y})
$$

is continuous. Thus when $m$ is large enough,

$$
\left\|f-g_{m}\right\|_{p} \leq\|f-g\|_{p}+\left\|g-g_{m}\right\|_{p}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon
$$

This proves the theorem.
This is a very remarkable result. Functions in $L^{p}\left(\mathbb{R}^{n}\right)$ don't need to be continuous anywhere and yet every such function is very close in the $L^{p}$ norm to one which is infinitely differentiable having compact support. The same result holds for $L^{p}(U)$ for $U$ an open set. This is the next corollary.

Corollary 15.5.11 Let $U$ be an open set. For each $p \geq 1, C_{c}^{\infty}(U)$ is dense in $L^{p}(U)$. Here the measure is Lebesgue measure.

Proof: Let $f \in L^{p}(U)$ and let $\varepsilon>0$ be given. Choose $g \in C_{c}(U)$ such that $\|f-g\|_{p}<\frac{\varepsilon}{2}$. This is possible because Lebesgue measure restricted to the open set, $U$ is regular. Thus the existence of such a $g$ follows from Theorem 15.2.4. Now let

$$
g_{m}(\mathbf{x})=g * \psi_{m}(\mathbf{x}) \equiv \int g(\mathbf{x}-\mathbf{y}) \psi_{m}(\mathbf{y}) d m_{n}(y)=\int g(\mathbf{y}) \psi_{m}(\mathbf{x}-\mathbf{y}) d m_{n}(y)
$$

where $\left\{\psi_{m}\right\}$ is a mollifier. It follows from Lemma 15.5.7 $g_{m} \in C_{c}^{\infty}(U)$ for all $m$ sufficiently large. It vanishes if $\mathbf{x} \notin \operatorname{spt}(g)+B\left(0, \frac{1}{m}\right)$. Then

$$
\begin{aligned}
\left\|g-g_{m}\right\|_{p} & =\left(\int\left|g(\mathbf{x})-\int g(\mathbf{x}-\mathbf{y}) \psi_{m}(\mathbf{y}) d m_{n}(\mathbf{y})\right|^{p} d m_{n}(\mathbf{x})\right)^{\frac{1}{p}} \\
& \leq\left(\int\left(\int|g(\mathbf{x})-g(\mathbf{x}-\mathbf{y})| \psi_{m}(\mathbf{y}) d m_{n}(\mathbf{y})\right)^{p} d m_{n}(\mathbf{x})\right)^{\frac{1}{p}} \\
& \leq \int\left(\int|g(\mathbf{x})-g(\mathbf{x}-\mathbf{y})|^{p} d m_{n}(\mathbf{x})\right)^{\frac{1}{p}} \psi_{m}(\mathbf{y}) d m_{n}(\mathbf{y}) \\
& =\int_{B\left(0, \frac{1}{m}\right)}\left\|g-g_{\mathbf{y}}\right\|_{p} \psi_{m}(\mathbf{y}) d m_{n}(\mathbf{y})<\frac{\varepsilon}{2}
\end{aligned}
$$

whenever $m$ is large enough thanks to uniform continuity of $g$. Theorem 15.1.12 was used to obtain the third inequality. There is no measurability problem because the function

$$
(\mathbf{x}, \mathbf{y}) \rightarrow|g(\mathbf{x})-g(\mathbf{x}-\mathbf{y})| \psi_{m}(\mathbf{y})
$$

is continuous. Thus when $m$ is large enough,

$$
\left\|f-g_{m}\right\|_{p} \leq\|f-g\|_{p}+\left\|g-g_{m}\right\|_{p}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon
$$

This proves the corollary.
Another thing should probably be mentioned. If you have had a course in complex analysis, you may be wondering whether these infinitely differentiable functions having compact support have anything to do with analytic functions which also have infinitely many derivatives. The answer is no! Recall that if an analytic function has a limit point in the set of zeros then it is identically equal to zero. Thus these functions in $C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$ are not analytic. This is a strictly real analysis phenomenon and has absolutely nothing to do with the theory of functions of a complex variable.

### 15.6 Exercises

1. Let $E$ be a Lebesgue measurable set in $\mathbb{R}$. Suppose $m(E)>0$. Consider the set

$$
E-E=\{x-y: x \in E, y \in E\}
$$

Show that $E-E$ contains an interval. Hint: Let

$$
f(x)=\int \mathscr{X}_{E}(t) \mathscr{X}_{E}(x+t) d t
$$

Note $f$ is continuous at 0 and $f(0)>0$ and use continuity of translation in $L^{p}$.
2. Establish the inequality $\|f g\|_{r} \leq\|f\|_{p}\|g\|_{q}$ whenever $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$.
3. Let $(\Omega, \mathscr{S}, \mu)$ be counting measure on $\mathbb{N}$. Thus $\Omega=\mathbb{N}$ and $\mathscr{S}=\mathscr{P}(\mathbb{N})$ with $\mu(S)=$ number of things in $S$. Let $1 \leq p \leq q$. Show that in this case,

$$
L^{1}(\mathbb{N}) \subseteq L^{p}(\mathbb{N}) \subseteq L^{q}(\mathbb{N})
$$

Hint: This is real easy if you consider what $\int_{\Omega} f d \mu$ equals. How are the norms related?
4. Consider the function, $f(x, y)=\frac{x^{p-1}}{p y}+\frac{y^{q-1}}{q x}$ for $x, y>0$ and $\frac{1}{p}+\frac{1}{q}=1$. Show directly that $f(x, y) \geq 1$ for all such $x, y$ and show this implies $x y \leq \frac{x^{p}}{p}+\frac{y^{q}}{q}$.
5. Give an example of a sequence of functions in $L^{p}(\mathbb{R})$ which converges to zero in $L^{p}$ but does not converge pointwise to 0 . Does this contradict the proof of the theorem that $L^{p}$ is complete?
6. Let $K$ be a bounded subset of $L^{p}\left(\mathbb{R}^{n}\right)$ and suppose that there exists $G$ such that $\bar{G}$ is compact with

$$
\int_{\mathbb{R}^{n} \backslash \bar{G}}|u(\mathbf{x})|^{p} d x<\varepsilon^{p}
$$

and for all $\varepsilon>0$, there exist a $\delta>0$ and such that if $|\mathbf{h}|<\delta$, then

$$
\int|u(\mathbf{x}+\mathbf{h})-u(\mathbf{x})|^{p} d x<\varepsilon^{p}
$$

for all $u \in K$. Show that $K$ is precompact in $L^{p}\left(\mathbb{R}^{n}\right)$. Hint: Let $\phi_{k}$ be a mollifier and consider

$$
K_{k} \equiv\left\{u * \phi_{k}: u \in K\right\}
$$

Verify the conditions of the Ascoli Arzela theorem for these functions defined on $\bar{G}$ and show there is an $\varepsilon$ net for each $\varepsilon>0$. Can you modify this to let an arbitrary open set take the place of $\mathbb{R}^{n}$ ?
7. Let $(\Omega, d)$ be a metric space and suppose also that $(\Omega, \mathscr{S}, \mu)$ is a regular measure space such that $\mu(\Omega)<\infty$ and let $f \in L^{1}(\Omega)$ where $f$ has complex values. Show that for every $\varepsilon>0$, there exists an open set of measure less than $\varepsilon$, denoted here by $V$ and a continuous function, $g$ defined on $\Omega$ such that $f=g$ on $V^{C}$. Thus, aside from a set of small measure, $f$ is continuous. If $|f(\omega)| \leq M$, show that it can be assumed that $|g(\omega)| \leq M$. This is called Lusin's theorem. Hint: Use Theorems 15.2.4 and 15.1.10 to obtain a sequence of functions in $C_{c}(\Omega),\left\{g_{n}\right\}$ which converges pointwise a.e. to $f$ and then use Egoroff's theorem to obtain a small set, $W$ of measure less
than $\varepsilon / 2$ such that convergence is uniform on $W^{C}$. Now let $F$ be a closed subset of $W^{C}$ such that $\mu\left(W^{C} \backslash F\right)<\varepsilon / 2$. Let $V=F^{C}$. Thus $\mu(V)<\varepsilon$ and on $F=V^{C}$, the convergence of $\left\{g_{n}\right\}$ is uniform showing that the restriction of $f$ to $V^{C}$ is continuous. Now use the Tietze extension theorem.
8. Let $\phi_{m} \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right), \phi_{m}(\mathbf{x}) \geq 0$, and $\int_{\mathbb{R}^{n}} \phi_{m}(\mathbf{y}) d y=1$ with

$$
\lim _{m \rightarrow \infty} \sup \left\{|\mathbf{x}|: \mathbf{x} \in \operatorname{spt}\left(\phi_{m}\right)\right\}=0
$$

Show if $f \in L^{p}\left(\mathbb{R}^{n}\right), \lim _{m \rightarrow \infty} f * \phi_{m}=f$ in $L^{p}\left(\mathbb{R}^{n}\right)$.
9. Let $\phi: \mathbb{R} \rightarrow \mathbb{R}$ be convex. This means

$$
\phi(\lambda x+(1-\lambda) y) \leq \lambda \phi(x)+(1-\lambda) \phi(y)
$$

whenever $\lambda \in[0,1]$. Verify that if $x<y<z$, then $\frac{\phi(y)-\phi(x)}{y-x} \leq \frac{\phi(z)-\phi(y)}{z-y}$ and that $\frac{\phi(z)-\phi(x)}{z-x} \leq \frac{\phi(z)-\phi(y)}{z-y}$. Show if $s \in \mathbb{R}$ there exists $\lambda$ such that $\phi(s) \leq \phi(t)+\lambda(s-t)$ for all $t$. Show that if $\phi$ is convex, then $\phi$ is continuous.
10. $\uparrow$ Prove Jensen's inequality. If $\phi: \mathbb{R} \rightarrow \mathbb{R}$ is convex, $\mu(\Omega)=1$, and $f: \Omega \rightarrow \mathbb{R}$ is in $L^{1}(\Omega)$, then $\phi\left(\int_{\Omega} f d u\right) \leq \int_{\Omega} \phi(f) d \mu$. Hint: Let $s=\int_{\Omega} f d \mu$ and use Problem 9.
11. Let $\frac{1}{p}+\frac{1}{p^{\prime}}=1, p>1$, let $f \in L^{p}(\mathbb{R}), g \in L^{p^{\prime}}(\mathbb{R})$. Show $f * g$ is uniformly continuous on $\mathbb{R}$ and $|(f * g)(x)| \leq\|f\|_{L^{p}}\|g\|_{L^{p^{\prime}}}$. Hint: You need to consider why $f * g$ exists and then this follows from the definition of convolution and continuity of translation in $L^{p}$.
12. $B(p, q)=\int_{0}^{1} x^{p-1}(1-x)^{q-1} d x, \Gamma(p)=\int_{0}^{\infty} e^{-t} t^{p-1} d t$ for $p, q>0$. The first of these is called the beta function, while the second is the gamma function. Show a.) $\Gamma(p+$ $1)=p \Gamma(p) ;$ b.) $\Gamma(p) \Gamma(q)=B(p, q) \Gamma(p+q)$.
13. Let $f \in C_{c}(0, \infty)$ and define $F(x)=\frac{1}{x} \int_{0}^{x} f(t) d t$. Show

$$
\|F\|_{L^{p}(0, \infty)} \leq \frac{p}{p-1}\|f\|_{L^{p}(0, \infty)} \text { whenever } p>1
$$

Hint: Argue there is no loss of generality in assuming $f \geq 0$ and then assume this is so. Integrate $\int_{0}^{\infty}|F(x)|^{p} d x$ by parts as follows:

$$
\int_{0}^{\infty} F^{p} d x=\overbrace{\left.x F^{p}\right|_{0} ^{\infty}}^{\text {show }=0}-p \int_{0}^{\infty} x F^{p-1} F^{\prime} d x .
$$

Now show $x F^{\prime}=f-F$ and use this in the last integral. Complete the argument by using Holder's inequality and $p-1=p / q$.
14. $\uparrow$ Now suppose $f \in L^{p}(0, \infty), p>1$, and $f$ not necessarily in $C_{c}(0, \infty)$. Show that $F(x)=\frac{1}{x} \int_{0}^{x} f(t) d t$ still makes sense for each $x>0$. Show the inequality of Problem 13 is still valid. This inequality is called Hardy's inequality. Hint: To show this, use the above inequality along with the density of $C_{c}(0, \infty)$ in $L^{p}(0, \infty)$.
15. Suppose $f, g \geq 0$. When does equality hold in Holder's inequality?
16. Prove Vitali's Convergence theorem: Let $\left\{f_{n}\right\}$ be uniformly integrable and complex valued, $\mu(\Omega)<\infty, f_{n}(x) \rightarrow f(x)$ a.e. where $f$ is measurable. Then $f \in L^{1}$ and $\lim _{n \rightarrow \infty} \int_{\Omega}\left|f_{n}-f\right| d \mu=0$. Hint: Use Egoroff's theorem to show $\left\{f_{n}\right\}$ is a Cauchy sequence in $L^{1}(\Omega)$. This yields a different and easier proof than what was done earlier. See Theorem 11.5.3 on Page 257.
17. $\uparrow$ Show the Vitali Convergence theorem implies the Dominated Convergence theorem for finite measure spaces but there exist examples where the Vitali convergence theorem works and the dominated convergence theorem does not.
18. $\uparrow$ Suppose $\mu(\Omega)<\infty,\left\{f_{n}\right\} \subseteq L^{1}(\Omega)$, and

$$
\int_{\Omega} h\left(\left|f_{n}\right|\right) d \mu<C
$$

for all $n$ where $h$ is a continuous, nonnegative function satisfying

$$
\lim _{t \rightarrow \infty} \frac{h(t)}{t}=\infty
$$

Show $\left\{f_{n}\right\}$ is uniformly integrable. In applications, this often occurs in the form of a bound on $\left\|f_{n}\right\|_{p}$.
19. $\uparrow$ Sometimes, especially in books on probability, a different definition of uniform integrability is used than that presented here. A set of functions, $\mathfrak{S}$, defined on a finite measure space, $(\Omega, \mathscr{S}, \mu)$ is said to be uniformly integrable if for all $\varepsilon>0$ there exists $\alpha>0$ such that for all $f \in \mathfrak{S}$,

$$
\int_{[|f| \geq \alpha]}|f| d \mu \leq \varepsilon
$$

Show that this definition is equivalent to the definition of uniform integrability given earlier in Definition 11.5.1 on Page 256 with the addition of the condition that there is a constant, $C<\infty$ such that

$$
\int|f| d \mu \leq C
$$

for all $f \in \mathfrak{S}$.
20. $f \in L^{\infty}(\Omega, \mu)$ if there exists a set of measure zero, $E$, and a constant $C<\infty$ such that $|f(x)| \leq C$ for all $x \notin E$.

$$
\|f\|_{\infty} \equiv \inf \{C:|f(x)| \leq C \text { a.e. }\} .
$$

Show $\left\|\|_{\infty}\right.$ is a norm on $L^{\infty}(\Omega, \mu)$ provided $f$ and $g$ are identified if $f(x)=g(x)$ a.e. Show $L^{\infty}(\Omega, \mu)$ is complete. Hint: You might want to show that $\left[|f|>\|f\|_{\infty}\right]$ has measure zero so $\|f\|_{\infty}$ is the smallest number at least as large as $|f(x)|$ for a.e. $x$. Thus $\|f\|_{\infty}$ is one of the constants, $C$ in the above.
21. Suppose $f \in L^{\infty} \cap L^{1}$. Show $\lim _{p \rightarrow \infty}\|f\|_{L^{p}}=\|f\|_{\infty}$. Hint:

$$
\begin{gathered}
\left(\left|\mid f \|_{\infty}-\varepsilon\right)^{p} \mu\left(\left[|f|>| | f \|_{\infty}-\varepsilon\right]\right) \leq \int_{\left[|f|>| | f \|_{\infty}-\varepsilon\right]}|f|^{p} d \mu \leq\right. \\
\int|f|^{p} d \mu=\int|f|^{p-1}|f| d \mu \leq\|f\|_{\infty}^{p-1} \int|f| d \mu .
\end{gathered}
$$

Now raise both ends to the $1 / p$ power and take liminf and limsup as $p \rightarrow \infty$. You should get $\|f\|_{\infty}-\varepsilon \leq \liminf \|f\|_{p} \leq \limsup \|f\|_{p} \leq\|f\|_{\infty}$
22. Suppose $\mu(\Omega)<\infty$. Show that if $1 \leq p<q$, then $L^{q}(\Omega) \subseteq L^{p}(\Omega)$. Hint Use Holder's inequality.
23. Show $L^{1}(\mathbb{R}) \nsubseteq L^{2}(\mathbb{R})$ and $L^{2}(\mathbb{R}) \nsubseteq L^{1}(\mathbb{R})$ if Lebesgue measure is used. Hint: Consider $1 / \sqrt{x}$ and $1 / x$.
24. Suppose that $\theta \in[0,1]$ and $r, s, q>0$ with

$$
\frac{1}{q}=\frac{\theta}{r}+\frac{1-\theta}{s}
$$

show that

$$
\left(\int|f|^{q} d \mu\right)^{1 / q} \leq\left(\left(\int|f|^{r} d \mu\right)^{1 / r}\right)^{\theta}\left(\left(\int|f|^{s} d \mu\right)^{1 / s}\right)^{1-\theta}
$$

If $q, r, s \geq 1$ this says that

$$
\|f\|_{q} \leq\|f\|_{r}^{\theta}\|f\|_{s}^{1-\theta}
$$

Using this, show that

$$
\ln \left(\|f\|_{q}\right) \leq \theta \ln \left(\|f\|_{r}\right)+(1-\theta) \ln \left(\|f\|_{s}\right)
$$

## Hint:

$$
\int|f|^{q} d \mu=\int|f|^{q \theta}|f|^{q(1-\theta)} d \mu
$$

Now note that $1=\frac{\theta q}{r}+\frac{q(1-\theta)}{s}$ and use Holder's inequality.
25. Suppose $f$ is a function in $L^{1}(\mathbb{R})$ and $f$ is infinitely differentiable. Is $f^{\prime} \in L^{1}(\mathbb{R})$ ? Hint: What if $\phi \in C_{c}^{\infty}(0,1)$ and $f(x)=\phi\left(2^{n}(x-n)\right)$ for $x \in(n, n+1), f(x)=0$ if $x<0$ ?

## Chapter 16

## Stone's Theorem

This section is devoted to Stone's theorem which says that a metric space is paracompact, defined below. See [98] for this which is where I read it. First is the definition of what is meant by a refinement.

Definition 16.0.1 Let $S$ be a topological space. We say that a collection of sets $\mathfrak{D}$ is a refinement of an open cover $\mathfrak{S}$, if every set of $\mathfrak{D}$ is contained in some set of $\mathfrak{S}$. An open refinement would be one in which all sets are open, with a similar convention holding for the term " closed refinement".

Definition 16.0.2 We say that a collection of sets $\mathfrak{D}$, is locally finite if for all $p \in S$, there exists $V$ an open set containing $p$ such that $V$ has nonempty intersection with only finitely many sets of $\mathfrak{D}$.

Definition 16.0.3 We say $S$ is paracompact if it is Hausdorff and for every open cover $\mathfrak{S}$, there exists an open refinement $\mathfrak{D}$ such that $\mathfrak{D}$ is locally finite and $\mathfrak{D}$ covers $S$.

Theorem 16.0.4 If $\mathfrak{D}$ is locally finite then

$$
\cup\{\bar{D}: D \in \mathfrak{D}\}=\overline{\cup\{D: D \in \mathfrak{D}\}}
$$

Proof: It is clear the left side is a subset of the right. Let $p$ be a limit point of

$$
\cup\{D: D \in \mathfrak{D}\}
$$

and let $p \in V$, an open set intersecting only finitely many sets of $\mathfrak{D}, D_{1} \ldots D_{n}$. If $p$ is not in any of $\overline{D_{i}}$ then $p \in W$ where $W$ is some open set which contains no points of $\cup_{i=1}^{n} D_{i}$. Then $V \cap W$ contains no points of any set of $\mathfrak{D}$ and this contradicts the assumption that $p$ is a limit point of

$$
\cup\{D: D \in \mathfrak{D}\} .
$$

Thus $p \in \overline{D_{i}}$ for some $i$.
We say $\mathfrak{S} \subseteq \mathscr{P}(S)$ is countably locally finite if

$$
\mathfrak{S}=\cup_{n=1}^{\infty} \mathfrak{S}_{n}
$$

and each $\mathfrak{S}_{n}$ is locally finite. The following theorem appeared in the 1950's. It will be used to prove Stone's theorem.

Theorem 16.0.5 Let $S$ be a regular topological space. (If $p \in U$ open, then there exists an open set $V$ such that $p \in \bar{V} \subseteq U$. ) The following are equivalent
1.) Every open covering of $S$ has a refinement that is open, covers $S$ and is countably locally finite.
2.) Every open covering of $S$ has a refinement that is locally finite and covers $S$. (The sets in refinement maybe not open.)
3.) Every open covering of $S$ has a refinement that is closed, locally finite, and covers S. (Sets in refinement are closed.)
4.) Every open covering of $S$ has a refinement that is open, locally finite, and covers $S$. (Sets in refinement are open.)

## Proof:

1.) $\Rightarrow 2$.)

Let $\mathfrak{S}$ be an open cover of $S$ and let $\mathfrak{B}$ be an open countably locally finite refinement

$$
\mathfrak{B}=\cup_{n=1}^{\infty} \mathfrak{B}_{n}
$$

where $\mathfrak{B}_{n}$ is an open refinement of $\mathfrak{S}$ and $\mathfrak{B}_{n}$ is locally finite. For $B \in \mathfrak{B}_{n}$, let

$$
E_{n}(B)=B \backslash \bigcup_{k<n}\left(\cup\left\{B: B \in \mathfrak{B}_{k}\right\}\right)
$$

Thus, in words, $E_{n}(B)$ consists of points in $B$ which are not in any set from any $\mathfrak{B}_{k}$ for $k<n$.

Claim: $\left\{E_{n}(B): n \in \mathbb{N}, B \in \mathfrak{B}_{n}\right\}$ is locally finite.
Proof of the claim: Let $p \in S$. Then $p \in B_{0} \in \mathfrak{B}_{n}$ for some $n$. Let $V$ be open, $p \in V$, and $V$ intersects only finitely many sets of $\mathfrak{B}_{1} \cup \ldots \cup \mathfrak{B}_{n}$. Then consider $B_{0} \cap V$. If $m>n$,

$$
\left(B_{0} \cap V\right) \cap E_{m}(B) \subseteq\left[\bigcup_{k<m}\left(\cup\left\{B: B \in \mathfrak{B}_{k}\right)\right]^{C} \subseteq B_{0}^{C}\right.
$$

In words, $E_{m}(B)$ has nothing in it from any of the $\mathfrak{B}_{k}$ for $k<m$. In particular, it has nothing in it from $B_{0}$. Thus $\left(B_{0} \cap V\right) \cap E_{m}(B)=\emptyset$ for $m>n$. Thus $p \in B_{0} \cap V$ which intersects only finitely many sets of $\mathfrak{S}$, no more than those intersected by $V$. This establishes the claim.

Claim: $\left\{E_{n}(B): n \in \mathbb{N}, B \in \mathfrak{B}_{n}\right\}$ covers $S$.
Proof: Let $p \in S$ and let $n=\min \left\{k \in \mathbb{N}: p \in B\right.$ for some $\left.B \in \mathfrak{B}_{k}\right\}$. Let $p \in B \in \mathfrak{B}_{n}$. Then $p \in E_{n}(B)$.

The two claims show that 1.) $\Rightarrow 2$.).
2.) $\Rightarrow 3$.)

Let $\mathfrak{S}$ be an open cover and let

$$
\mathscr{G} \equiv\{U: U \text { is open and } \bar{U} \subseteq V \in \mathfrak{S} \text { for some } V \in \mathfrak{S}\}
$$

Then since $S$ is regular, $\mathscr{G}$ covers $S$. (If $p \in S$, then $p \in U \subseteq \bar{U} \subseteq V \in \mathfrak{S}$.) By 2.), $\mathscr{G}$ has a locally finite refinement $\mathfrak{C}$, covering $S$. Consider

$$
\{\bar{E}: E \in \mathfrak{C}\}
$$

This collection of closed sets covers $S$ and is locally finite because if $p \in S$, there exists $V, p \in V$, and $V$ has nonempty intersections with only finitely many elements of $\mathfrak{C}$, say $E_{1}, \cdots, E_{n}$. If $\bar{E} \cap V \neq \emptyset$, then $E \cap V \neq \emptyset$ and so $V$ intersects only $\overline{E_{1}}, \cdots, \overline{E_{n}}$. This shows 2 .) $\Rightarrow 3$.).

3 .) $\Rightarrow$ 4.) Here is a table of symbols with a short summary of their meaning.

| Open covering | Locally finite refinement |
| :--- | :--- |
| $\mathfrak{S}$ original covering | $\mathfrak{B}$ by 3. can be closed refinement |
| $\mathfrak{F}$ open intersectors | $\mathfrak{C}$ closed refinement |

Let $\mathfrak{S}$ be an open cover and let $\mathfrak{B}$ be a locally finite refinement which covers $S$. By 3.) we can take $\mathfrak{B}$ to be a closed refinement but this is not important here. Let

$$
\mathfrak{F} \equiv\{U: U \text { is open and } U \text { intersects only finitely many sets of } \mathfrak{B}\} .
$$

Then $\mathfrak{F}$ covers $S$ because $\mathfrak{B}$ is locally finite. If $p \in S$, then there exists an open set $U$ containing $p$ which intersects only finitely many sets of $\mathfrak{B}$. Thus $p \in U \in \mathfrak{F}$. By 3 ., $\mathfrak{F}$ has a locally finite closed refinement $\mathfrak{C}$, which covers $S$. Define for $B \in \mathfrak{B}$

$$
\mathfrak{C}(B) \equiv\{C \in \mathfrak{C}: C \cap B=\emptyset\}
$$

Thus these closed sets $C$ do not intersect $B$ and so $B$ is in their complement. We use $\mathfrak{C}(B)$ to fatten up B. Let

$$
E(B) \equiv(\cup\{C: C \in \mathfrak{C}(B)\})^{C}
$$

In words, $E(B)$ is the complement of the union of all closed sets of $\mathfrak{C}$ which do not intersect $B$. Thus $E(B) \supseteq B$, and has fattened up $B$. Then since $\mathfrak{C}(B)$ is locally finite, $E(B)$ is an open set by Theorem 16.0.4. Now let $F(B)$ be defined such that for $B \in \mathfrak{B}$,

$$
B \subseteq F(B) \in \mathfrak{S}
$$

(by definition $B$ is in some set of $\mathfrak{S}$ ), and let

$$
\mathfrak{L}=\{E(B) \cap F(B): B \in \mathfrak{B}\}
$$

The intersection with $F(B)$ is to ensure that $\mathfrak{L}$ is a refinement of $\mathfrak{S}$. The important thing to notice is that if $C \in \mathfrak{C}$ intersects $E(B)$, then it must also intersect $B$. If not, you could include it in the list of closed sets which do not intersect $B$ and whose complement is $E(B)$. Thus $E(B)$ would be too large.

Claim: $\mathfrak{L}$ covers $S$.
This claim is obvious because if $p \in S$ then $p \in B$ for some $B \in \mathfrak{B}$. Hence

$$
p \in E(B) \cap F(B) \in \mathfrak{L}
$$

Claim: $\mathfrak{L}$ is locally finite and a refinement of $\mathfrak{S}$.
Proof: It is clear $\mathfrak{L}$ is a refinement of $\mathfrak{S}$ because every set of $\mathfrak{L}$ is a subset of a set of $\mathfrak{S}, F(B)$. Let $p \in S$. There exists an open set $W$, such that $p \in W$ and $W$ intersects only $C_{1}, \cdots, C_{n}$, elements of $\mathfrak{C}$. Hence $W \subseteq \cup_{i=1}^{n} C_{i}$ since $\mathfrak{C}$ covers $S$.

But $C_{i}$ is contained in a set $U_{i} \in \mathfrak{F}$ which intersects only finitely many sets of $\mathfrak{B}$. Thus each $C_{i}$ intersects only finitely many $B \in \mathfrak{B}$ and so each $C_{i}$ intersects only finitely many of the sets, $E(B)$. (If it intersects $E(B)$, then it intersects $B$.) Thus $W$ intersects only finitely many of the $E(B)$, hence finitely many of the $E(B) \cap F(B)$. It follows that $\mathfrak{L}$ is locally finite.

It is obvious that 4.) $\Rightarrow 1$.).
The following theorem is Stone's theorem.
Theorem 16.0.6 If $S$ is a metric space then $S$ is paracompact (Every open cover has a locally finite open refinement also an open cover.)

Proof: Let $\mathfrak{S}$ be an open cover. Well order $\mathfrak{S}$. For $B \in \mathfrak{S}$,

$$
B_{n} \equiv\left\{x \in B: \operatorname{dist}\left(x, B^{C}\right)<\frac{1}{2^{n}}\right\}, n=1,2, \cdots .
$$

Thus $B_{n}$ is contained in $B$ but approximates it up to $2^{-n}$. Let

$$
E_{n}(B)=B_{n} \backslash \cup\{D: D \prec B \text { and } D \neq B\}
$$

where $\prec$ denotes the well order. If $B, D \in \mathfrak{S}$, then one is first in the well order. Let $D \prec B$. Then from the construction, $E_{n}(B) \subseteq D^{C}$ and $E_{n}(D)$ is further than $1 / 2^{n}$ from $D^{C}$. Hence, assuming neither set is empty,

$$
\operatorname{dist}\left(E_{n}(B), E_{n}(D)\right) \geq 2^{-n}
$$

for all $B, D \in \mathfrak{S}$. Fatten up $E_{n}(B)$ as follows.

$$
\widetilde{E_{n}(B)} \equiv \cup\left\{B\left(x, 8^{-n}\right): x \in E_{n}(B)\right\}
$$

Thus $\widetilde{E_{n}(B)} \subseteq B$ and

$$
\operatorname{dist}\left(\widetilde{E_{n}(B)}, \widetilde{E_{n}(D)}\right) \geq \frac{1}{2^{n}}-2\left(\frac{1}{8}\right)^{n} \equiv \delta_{n}>0
$$

It follows that the collection of open sets

$$
\left\{\widetilde{E_{n}(B)}: B \in \mathfrak{S}\right\} \equiv \mathfrak{B}_{n}
$$

is locally finite. In fact, $B\left(p, \frac{\delta_{n}}{2}\right)$ cannot intersect more than one of them. In addition to this,

$$
S \subseteq \cup\left\{\widetilde{E_{n}(B)}: n \in \mathbb{N}, B \in \mathfrak{S}\right\}
$$

because if $p \in S$, let $B$ be the first set in $\mathfrak{S}$ to contain $p$. Then $p \in E_{n}(B)$ for $n$ large enough because it will not be in anything deleted. Thus this is an open countably locally finite refinement. Thus 1.) in the above theorem is satisfied.

### 16.1 Partitions Of Unity And Stone's Theorem

First observe that if $S$ is a nonempty set, then $\operatorname{dist}(x, S)$ satisfies $|\operatorname{dist}(x, S)-\operatorname{dist}(y, S)| \leq$ $d(x, y)$. To see this,

$$
|\operatorname{dist}(x, S)-\operatorname{dist}(y, S)| \leq d(x, y)
$$

To see this, say dist $(x, S)$ is the larger of the two. Then there exists $z \in S$ such that

$$
\operatorname{dist}(y, S) \geq d(y, z)-\varepsilon
$$

It follows that

$$
\begin{aligned}
& |\operatorname{dist}(x, S)-\operatorname{dist}(y, S)| \\
= & \operatorname{dist}(x, S)-\operatorname{dist}(y, S)
\end{aligned}
$$

$$
\begin{aligned}
& \leq \operatorname{dist}(x, S)-(d(y, z)-\varepsilon) \\
& \leq d(x, z)-d(y, z)+\varepsilon \\
& \leq d(x, y)+d(y, z)-d(y, z)+\varepsilon=d(x, y)+\varepsilon
\end{aligned}
$$

Since $\varepsilon>0$ is arbitrary, this shows the desired conclusion.
Theorem 16.1.1 Let $S$ be a metric space and let $\mathfrak{S}$ be any open cover of $S$. Then there exists a set $\mathfrak{F}$, an open refinement of $\mathfrak{S}$, and functions $\left\{\phi_{F}: F \in \mathfrak{F}\right\}$ such that

$$
\begin{gathered}
\phi_{F}: S \rightarrow[0,1] \\
\phi_{F} \text { is continuous } \\
\phi_{F}(x) \text { equals } 0 \text { for all but finitely many } F \in \mathfrak{F} \\
\sum\left\{\phi_{F}(x): F \in \mathfrak{F}\right\}=1 \text { for all } x \in S .
\end{gathered}
$$

Each $\phi_{F}$ is locally Lipschitz continuous which means that for each $z$ there is an open set $W$ containing $z$ for which, if $x, y \in W$, then there is a constant $K$ such that

$$
\left|\phi_{F}(x)-\phi_{F}(y)\right| \leq K d(x, y)
$$

Proof: By Stone's theorem, there exists a locally finite refinement $\mathfrak{F}$ covering $S$. For $F \in \mathfrak{F}$

$$
g_{F}(x) \equiv \operatorname{dist}\left(x, F^{C}\right)
$$

Let

$$
\phi_{F}(x) \equiv\left(\sum\left\{g_{F}(x): F \in \mathfrak{F}\right\}\right)^{-1} g_{F}(x)
$$

Now

$$
\sum\left\{g_{F}(x): F \in \mathfrak{F}\right\}
$$

is a continuous function because if $x \in S$, then there exists an open set $W$ with $x \in W$ and $W$ has nonempty intersection with only finitely many sets of $F \in \mathfrak{F}$. Then for $y \in W$,

$$
\sum\left\{g_{F}(y): F \in \mathfrak{F}\right\}=\sum_{i=1}^{n} g_{F_{i}}(y)
$$

Since $\mathfrak{F}$ is a cover of $S$,

$$
\sum\left\{g_{F}(x): F \in \mathfrak{F}\right\} \neq 0
$$

for any $x \in S$. Hence $\phi_{F}$ is continuous. This also shows $\phi_{F}(x)=0$ for all but finitely many $F \in \mathfrak{F}$. It is obvious that

$$
\sum\left\{\phi_{F}(x): F \in \mathfrak{F}\right\}=1
$$

from the definition.
Let $z \in S$. Then there is an open set $W$ containing $z$ such that $W$ has nonempty intersection with only finitely many $F \in \mathscr{F}$. Thus for $y, x \in W$,

$$
\left|\phi_{F_{j}}(x)-\phi_{F_{j}}(y)\right| \leq\left|\frac{g_{F_{j}}(x) \sum_{i=1}^{n} g_{F_{i}}(y)-g_{F_{j}}(y) \sum_{i=1}^{n} g_{F_{i}}(x)}{\sum_{i=1}^{n} g_{F_{i}}(x) \sum_{i=1}^{n} g_{F_{i}}(y)}\right|
$$

If $F$ is not one of these $F_{i}$, then $g_{F}(x)=\phi_{F}(x)=\phi_{F}(y)=g_{F}(y)=0$. Thus there is nothing to show for these. It suffices to consider the ones above. Restricting $W$ if necessary, we can assume that for $x \in W$,

$$
\sum_{F} g_{F}(x)=\sum_{i=1}^{n} g_{F_{i}}(x)>\delta>0, g_{F_{j}}(x)<\Delta<\infty, j \leq n
$$

Then, simplifying the above, and letting $x, y \in W$, for each $j \leq n$,

$$
\begin{aligned}
& \left|\phi_{F_{j}}(x)-\phi_{F_{j}}(y)\right| \leq \frac{1}{\delta^{2}}\left|\begin{array}{c}
g_{F_{j}}(x) \sum_{F} g_{F}(y)-g_{F_{j}}(y) \sum_{F} g_{F}(y) \\
+g_{F_{j}}(y) \sum_{F} g_{F}(y)-g_{F_{j}}(y) \sum_{F} g_{F}(x)
\end{array}\right| \\
& \quad \leq \frac{1}{\delta^{2}} \Delta\left|g_{F_{j}}(x)-g_{F_{j}}(y)\right|+\frac{1}{\delta^{2}} \Delta \sum_{i=1}^{n}\left|g_{F_{i}}(y)-g_{F_{i}}(x)\right| \\
& \quad \leq \frac{\Delta}{\delta^{2}} d(x, y)+\frac{\Delta}{\delta^{2}} n d(x, y)=(n+1) \frac{\Delta}{\delta^{2}} d(x, y)
\end{aligned}
$$

Thus on this set $W$ containing $z$, all $\phi_{F}$ are Lipschitz continuous with Lipschitz constant $(n+1) \frac{\Delta}{\delta^{2}}$.

The functions described above are called a partition of unity subordinate to the open cover $\mathfrak{S}$. A useful observation is contained in the following corollary.

Corollary 16.1.2 Let $S$ be a metric space and let $\mathfrak{S}$ be any open cover of $S$. Then there exists a set $\mathfrak{F}$, an open refinement of $\mathfrak{S}$, and functions $\left\{\phi_{F}: F \in \mathfrak{F}\right\}$ such that

$$
\begin{gathered}
\phi_{F}: S \rightarrow[0,1] \\
\phi_{F} \text { is continuous } \\
\phi_{F}(x) \text { equals } 0 \text { for all but finitely many } F \in \mathfrak{F} \\
\sum\left\{\phi_{F}(x): F \in \mathfrak{F}\right\}=1 \text { for all } x \in S
\end{gathered}
$$

Each $\phi_{F}$ is Lipschitz continuous. If $U \in \mathfrak{S}$ and $H$ is a closed subset of $U$, the partition of unity can be chosen such that each $\phi_{F}=0$ on $H$ except for one which equals 1 on $H$.

Proof: Just change your open cover to consist of $U$ and $V \backslash H$ for each $V \in \mathfrak{S}$. Then every function but one equals 0 on $H$ and so exactly one of them equals 1 on $H$.

### 16.2 An Extension Theorem, Retracts

Lemma 16.2.1 Let $A$ be a closed set in a metric space and let $x_{n} \notin A, x_{n} \rightarrow a_{0} \in A$ and $a_{n} \in A$ such that $d\left(a_{n}, x_{n}\right)<6 \operatorname{dist}\left(x_{n}, A\right)$. Then $a_{n} \rightarrow a_{0}$.

Proof: By assumption,

$$
\begin{aligned}
d\left(a_{n}, a_{0}\right) & \leq d\left(a_{n}, x_{n}\right)+d\left(x_{n}, a_{0}\right)<6 \operatorname{dist}\left(x_{n}, A\right)+d\left(x_{n}, a_{0}\right) \\
& \leq 6 d\left(x_{n}, a_{0}\right)+d\left(x_{n}, a_{0}\right)=7 d\left(x_{n}, a_{0}\right)
\end{aligned}
$$

and this converges to 0 .


Note that there was nothing magic about 6 in the above. Another number would work as well.

In the proof of the following theorem, you get a covering of $A^{C}$ with open balls $B$ such that for each of these balls, there exists $a \in A$ such that for all $x \in B,\|x-a\| \leq 6 \operatorname{dist}(x, A)$. The 6 is not important. Any other constant with this property would work. Then you use Stone's theorem.

A Banach space is a normed vector space which is also a complete metric space where the metric comes from the norm.

$$
d(x, y)=\|x-y\|
$$

Thus you can add things in a Banach space. Much more will be considered about Banach spaces a little later.

Definition 16.2.2 A Banach space is a complete normed linear space. If you have a subset $B$ of a Banach space, then $\operatorname{conv}(B)$ denotes the smallest closed convex set which contains $B$. It can be obtained by taking the intersection of all closed convex sets containing $B$. Recall that a set $C$ is convex if whenever $x, y \in C$, then so is $\lambda x+(1-\lambda) y$ for all $\lambda \in[0,1]$. Note how this makes sense in a vector space but maybe not in a general metric space.

In the following theorem, we have in mind both $X$ and $Y$ are Banach spaces, but this is not needed in the proof. All that is needed is that $X$ is a metric space and $Y$ a normed linear space or possibly something more general in which it makes sense to do addition and scalar multiplication.

Theorem 16.2.3 Let $A$ be a closed subset of a metric space $X$ and let $F: A \rightarrow Y, Y$ a normed linear space. Then there exists an extension of $F$ denoted as $\hat{F}$ such that $\hat{F}$ is defined on all of $X$ and agrees with $F$ on A. It has values in $\operatorname{conv}(F(A))$, the convex hull of $F(A)$.

Proof: For each $c \notin A$, let $B_{c}$ be a ball contained in $A^{C}$ centered at $c$ where distance of $c$ to $A$ is at least $\operatorname{diam}\left(B_{c}\right)$.


So for $x \in B_{c}$ what about dist $(x, A)$ ? How does it compare with dist $(c, A)$ ?

$$
\begin{aligned}
\operatorname{dist}(c, A) & \leq d(c, x)+\operatorname{dist}(x, A) \\
& \leq \frac{1}{2} \operatorname{diam}\left(B_{c}\right)+\operatorname{dist}(x, A) \\
& \leq \frac{1}{2} \operatorname{dist}(c, A)+\operatorname{dist}(x, A)
\end{aligned}
$$

so

$$
\operatorname{dist}(c, A) \leq 2 \operatorname{dist}(x, A)
$$

Now the following is also valid. Letting $x \in B_{c}$ be arbitrary, it follows from the assumption on the diameter that there exists $a_{0} \in A$ such that $d\left(c, a_{0}\right)<2 \operatorname{dist}(c, A)$. Then

$$
\begin{gather*}
d\left(x, a_{0}\right) \leq \sup _{y \in B_{c}} d\left(y, a_{0}\right) \leq \sup _{y \in B_{c}}\left(d(y, c)+d\left(c, a_{0}\right)\right) \leq \frac{\operatorname{diam}\left(B_{c}\right)}{2}+2 \operatorname{dist}(c, A) \\
\leq \frac{\operatorname{dist}(c, A)}{2}+2 \operatorname{dist}(c, A)<3 \operatorname{dist}(c, A) \tag{16.2.1}
\end{gather*}
$$

It follows from 16.2.1,

$$
d\left(x, a_{0}\right) \leq 3 \operatorname{dist}(c, A) \leq 6 \operatorname{dist}(x, A)
$$

Thus for any $x \in B_{c}$, there is an $a_{0} \in A$ such that $d\left(x, a_{0}\right)$ is bounded by a fixed multiple of the distance from $x$ to $A$.

By Stone's theorem, there is a locally finite open refinement $\mathscr{R}$. These are open sets each of which is contained in one of the balls just mentioned such that each of these balls is the union of sets of $\mathscr{R}$. Thus $\mathscr{R}$ is a locally finite cover of $A^{C}$. Since $x \in A^{C}$ is in one of those balls, it was just shown that there exists $a_{R} \in A$ such that for all $x \in R \in \mathscr{R}$ we have $d\left(x, a_{R}\right) \leq 6 \operatorname{dist}(x, A)$. Of course there may be more than one because $R$ might be contained in more than one of those special balls. One $a_{R}$ is chosen for each $R \in \mathscr{R}$.

Now let $\phi_{R}(x) \equiv \operatorname{dist}\left(x, R^{C}\right)$. Then let

$$
\hat{F}(x) \equiv\left\{\begin{array}{l}
F(x) \text { for } x \in A \\
\sum_{R \in \mathscr{R}} F\left(a_{R}\right) \frac{\phi_{R}(x)}{\sum_{\hat{R} \in \mathscr{R}} \phi_{\hat{R}}(x)} \text { for } x \notin A
\end{array}\right.
$$

The sum in the bottom is always finite because the covering is locally finite. Also, this sum is never 0 because $\mathscr{R}$ is a covering. Also $\hat{F}$ has values in $\operatorname{conv}(F(K))$. It only remains to verify that $\hat{F}$ is continuous. It is clearly so on the interior of $A$ thanks to continuity of $F$. It is also clearly continuous on $A^{C}$ because the functions $\phi_{R}$ are continuous. So it suffices to consider $x_{n} \rightarrow a \in \partial A \subseteq A$ where $x_{n} \notin A$ and see whether $F(a)=\lim _{n \rightarrow \infty} \hat{F}\left(x_{n}\right)$.

Suppose this does not happen. Then there is a sequence converging to some $a \in \partial A$ and $\varepsilon>0$ such that

$$
\varepsilon \leq\left\|\hat{F}(a)-\hat{F}\left(x_{n}\right)\right\| \text { all } n
$$

For $x_{n} \in R$, it was shown above that $d\left(x_{n}, a_{R_{n}}\right) \leq 6 \operatorname{dist}\left(x_{n}, A\right)$. By the above Lemma 16.2.1, it follows that $a_{R n} \rightarrow a$ and so $F\left(a_{R n}\right) \rightarrow F(a)$.

$$
\varepsilon \leq\left\|\hat{F}(a)-\hat{F}\left(x_{n}\right)\right\| \leq \sum_{R \in \mathscr{R}}\left\|F\left(a_{R n}\right)-F(a)\right\| \frac{\phi_{R}\left(x_{R n}\right)}{\sum_{\hat{R} \in \mathscr{R}} \phi_{\hat{R}}\left(x_{R n}\right)}
$$

By local finiteness of the cover, each $x_{n}$ involves only finitely many $R$ Thus, in this limit process, there are countably many $R$ involved $\left\{R_{j}\right\}_{j=1}^{\infty}$. Thus one can apply Fatou's lemma.

$$
\begin{aligned}
\varepsilon & \leq \lim \inf _{n \rightarrow \infty}\left\|\hat{F}(a)-\hat{F}\left(x_{n}\right)\right\| \\
& \leq \sum_{j=1}^{\infty} \lim _{n \rightarrow \infty} \inf _{n \rightarrow}\left\|F\left(a_{R_{j} n}\right)-F(a)\right\| \frac{\phi_{R_{j}}\left(x_{R_{j} n}\right)}{\sum_{j=1}^{\infty} \phi_{\hat{R}_{j}}\left(x_{R_{j} n}\right)} \\
& \leq \sum_{j=1}^{\infty} \lim _{n \rightarrow \infty} \inf _{n}\left\|F\left(a_{R_{j} n}\right)-F(a)\right\|=0 ■
\end{aligned}
$$

The last step is needed because you lose local finiteness as you approach $\partial A$. Note that the only thing needed was that $X$ is a metric space. The addition takes place in $Y$ so it needs to be a vector space. Did it need to be complete? No, this was not used. Nor was completeness of $X$ used. The main interest here is in Banach spaces, but the result is more general than that.

It also appears that $\hat{F}$ is locally Lipschitz on $A^{C}$.
Definition 16.2.4 Let $S$ be a subset of $X$, a Banach space. Then it is a retract if there exists a continuous function $R: X \rightarrow S$ such that $R s=s$ for all $s \in S$. This $R$ is a retraction. More generally, $S \subseteq T$ is called a retract of $T$ if there is a continuous $R: T \rightarrow S$ such that $R s=s$ for all $s \in S$.

Theorem 16.2.5 Let $K$ be closed and convex subset of $X$ a Banach space. Then $K$ is $a$ retract.

Proof: By Theorem 16.2.3, there is a continuous function $\hat{I}$ extending $I$ to all of $X$. Then also $\hat{I}$ has values in $\operatorname{conv}(I K)=\operatorname{conv}(K)=K$. Hence $\hat{I}$ is a continuous function which does what is needed. It maps everything into $K$ and keeps the points of $K$ unchanged.

Sometimes people call the set a retraction also or the function which does the job a retraction. This seems like strange thing to call it because a retraction is the act of repudiating something you said earlier. Nevertheless, I will call it that. Note that if $S$ is a retract of the whole metric space $X$, then it must be a retract of every set which contains $S$.

### 16.3 Something Which Is Not A Retract

The next lemma is a fundamental result which will be used to develop the Brouwer degree. It will also be used to give a short proof of the Brouwer fixed point theorem in the exercises. This major fixed point theorem is probably the most fundamental theorem in nonlinear analysis. The proof outlined in the exercises is from [48].

Lemma 16.3.1 Let $\mathbf{g}: U \rightarrow \mathbb{R}^{n}$ be $C^{2}$ where $U$ is an open subset of $\mathbb{R}^{n}$. Then

$$
\sum_{j=1}^{n} \operatorname{cof}(D \mathbf{g})_{i j, j}=0
$$

where here $(D \mathbf{g})_{i j} \equiv g_{i, j} \equiv \frac{\partial g_{i}}{\partial x_{j}}$. Also, $\operatorname{cof}(D \mathbf{g})_{i j}=\frac{\partial \operatorname{det}(D \mathbf{g})}{\partial g_{i, j}}$.

Proof: From the cofactor expansion theorem,

$$
\operatorname{det}(D \mathbf{g})=\sum_{i=1}^{n} g_{i, j} \operatorname{cof}(D \mathbf{g})_{i j}
$$

and so

$$
\begin{equation*}
\frac{\partial \operatorname{det}(D \mathbf{g})}{\partial g_{i, j}}=\operatorname{cof}(D \mathbf{g})_{i j} \tag{16.3.2}
\end{equation*}
$$

which shows the last claim of the lemma. Also

$$
\begin{equation*}
\delta_{k j} \operatorname{det}(D \mathbf{g})=\sum_{i} g_{i, k}(\operatorname{cof}(D \mathbf{g}))_{i j} \tag{16.3.3}
\end{equation*}
$$

because if $k \neq j$ this is just the cofactor expansion of the determinant of a matrix in which the $k^{t h}$ and $j^{t h}$ columns are equal. Differentiate 16.3 .3 with respect to $x_{j}$ and sum on $j$. This yields

$$
\sum_{r, s, j} \delta_{k j} \frac{\partial(\operatorname{det} D \mathbf{g})}{\partial g_{r, s}} g_{r, s j}=\sum_{i j} g_{i, k j}(\operatorname{cof}(D \mathbf{g}))_{i j}+\sum_{i j} g_{i, k} \operatorname{cof}(D \mathbf{g})_{i j, j}
$$

Hence, using $\delta_{k j}=0$ if $j \neq k$ and 16.3.2,

$$
\sum_{r s}(\operatorname{cof}(D \mathbf{g}))_{r s} g_{r, s k}=\sum_{r s} g_{r, k s}(\operatorname{cof}(D \mathbf{g}))_{r s}+\sum_{i j} g_{i, k} \operatorname{cof}(D \mathbf{g})_{i j, j}
$$

Subtracting the first sum on the right from both sides and using the equality of mixed partials,

$$
\sum_{i} g_{i, k}\left(\sum_{j}(\operatorname{cof}(D \mathbf{g}))_{i j, j}\right)=0
$$

If $\operatorname{det}\left(g_{i, k}\right) \neq 0$ so that $\left(g_{i, k}\right)$ is invertible, this shows $\sum_{j}(\operatorname{cof}(D \mathbf{g}))_{i j, j}=0$. If $\operatorname{det}(D \mathbf{g})=0$, let

$$
g_{k}=g+\varepsilon_{k} I
$$

where $\varepsilon_{k} \rightarrow 0$ and $\operatorname{det}\left(D \mathbf{g}+\varepsilon_{k} I\right) \equiv \operatorname{det}\left(D \mathbf{g}_{k}\right) \neq 0$. Then

$$
\sum_{j}(\operatorname{cof}(D \mathbf{g}))_{i j, j}=\lim _{k \rightarrow \infty} \sum_{j}\left(\operatorname{cof}\left(D \mathbf{g}_{k}\right)\right)_{i j, j}=0
$$

Definition 16.3.2 Let $\mathbf{h}$ be a function defined on an open set, $U \subseteq \mathbb{R}^{n}$. Then $\mathbf{h} \in C^{k}(\bar{U})$ if there exists a function $\mathbf{g}$ defined on an open set, $W$ containng $\bar{U}$ such that $\mathbf{g}=\mathbf{h}$ on $\bar{U}$ and $\mathbf{g}$ is $C^{k}(W)$.

Lemma 16.3.3 There does not exist $\mathbf{h} \in C^{2}(\overline{B(\mathbf{0}, R)})$ such that $\mathbf{h}: \overline{B(\mathbf{0}, R)} \rightarrow \partial B(\mathbf{0}, R)$ which also has the property that $\mathbf{h}(\mathbf{x})=\mathbf{x}$ for all $\mathbf{x} \in \partial B(\mathbf{0}, R)$. That is, there is no retraction of $\overline{B(\mathbf{0}, R)}$ to $\partial B(\mathbf{0}, R)$.

Proof: Suppose such an $\mathbf{h}$ exists. Let $\lambda \in[0,1]$ and let $\mathbf{p}_{\lambda}(\mathbf{x}) \equiv \mathbf{x}+\lambda(\mathbf{h}(\mathbf{x})-\mathbf{x})$. This function, $\mathbf{p}_{\lambda}$ is a homotopy of the identity map and the retraction, $\mathbf{h}$. Let

$$
I(\lambda) \equiv \int_{\overline{B(\mathbf{0}, R)}} \operatorname{det}\left(D \mathbf{p}_{\lambda}(\mathbf{x})\right) d x
$$

Then using the dominated convergence theorem,

$$
\begin{aligned}
I^{\prime}(\lambda) & =\int \frac{\overline{B(\mathbf{0}, R)}}{} \sum_{i . j} \frac{\partial \operatorname{det}\left(D \mathbf{p}_{\lambda}(\mathbf{x})\right)}{\partial p_{\lambda i, j}} \frac{\partial p_{\lambda i j}(\mathbf{x})}{\partial \lambda} \\
& =\int_{\overline{B(\mathbf{0}, R)}} \sum_{i} \sum_{j} \frac{\partial \operatorname{det}\left(D \mathbf{p}_{\lambda}(\mathbf{x})\right)}{\partial p_{\lambda i, j}}\left(h_{i}(\mathbf{x})-x_{i}\right)_{, j} d x \\
& =\int_{\overline{B(\mathbf{0}, R)}} \sum_{i} \sum_{j} \operatorname{cof}\left(D \mathbf{p}_{\lambda}(\mathbf{x})\right)_{i j}\left(h_{i}(\mathbf{x})-x_{i}\right)_{, j} d x
\end{aligned}
$$

Now by assumption, $h_{i}(\mathbf{x})=x_{i}$ on $\partial \overline{B(\mathbf{0}, R)}$ and so one can integrate by parts and write

$$
I^{\prime}(\lambda)=-\sum_{i} \int_{\overline{B(\mathbf{0}, R)}} \sum_{j} \operatorname{cof}\left(D \mathbf{p}_{\lambda}(\mathbf{x})\right)_{i j, j}\left(h_{i}(\mathbf{x})-x_{i}\right) d x=0
$$

Therefore, $I(\lambda)$ equals a constant. However,

$$
I(0)=m_{n}(\overline{B(\mathbf{0}, R)})>0
$$

but

$$
I(1)=\int_{B(\mathbf{0}, R)} \operatorname{det}(D \mathbf{h}(\mathbf{x})) d m_{n}=\int_{\partial B(\mathbf{0}, R)} \#(\mathbf{y}) d m_{n}=0
$$

because from polar coordinates or other elementary reasoning, $m_{n}(\partial B(\mathbf{0}, 1))=0$.
The last formula uses the change of variables formula for functions which are not one to one. In this formula, $\#(\mathbf{y})$ equals the number of $\mathbf{x}$ such that $\mathbf{h}(\mathbf{x})=\mathbf{y}$. To see this is so in case you have not seen this, note that $\mathbf{h}$ is $C^{1}$ and so the inverse function theorem from advanced calculus applies. Thus

$$
\begin{aligned}
\int_{\overline{B(\mathbf{0}, R)}} \operatorname{det}(D \mathbf{h}(\mathbf{x})) d m_{n}= & \int_{[\operatorname{det}(D \mathbf{h}(\mathbf{x}))>0]} \operatorname{det}(D \mathbf{h}(\mathbf{x})) d m_{n} \\
& +\int_{[\operatorname{det}(D \mathbf{h}(\mathbf{x}))<0]} \operatorname{det}(D \mathbf{h}(\mathbf{x})) d m_{n}
\end{aligned}
$$

Thus $\mathbf{h}$ is locally one to one on the two open sets $[\operatorname{det}(D \mathbf{h}(\mathbf{x}))>0],[\operatorname{det}(D \mathbf{h}(\mathbf{x}))<0]$. Now use inverse function theorem and change of variables for one to one $\mathbf{h}$ to verify that both of these integrals equal 0 . You cover $[\operatorname{det}(D \mathbf{h}(\mathbf{x}))>0$ ] with countably many balls on which $\mathbf{h}$ is one to one and then use change of variables for each of these integrals over $[\operatorname{det}(D \mathbf{h}(\mathbf{x}))>0]$ intersected with this ball.

The following is the Brouwer fixed point theorem for $C^{2}$ maps.
Lemma 16.3.4 If $\mathbf{h} \in C^{2}(\overline{B(\mathbf{0}, R)})$ and $\mathbf{h}: \overline{B(\mathbf{0}, R)} \rightarrow \overline{B(\mathbf{0}, R)}$, then $\mathbf{h}$ has a fixed point, $\mathbf{x}$ such that $\mathbf{h}(\mathbf{x})=\mathbf{x}$.

Proof: Suppose the lemma is not true. Then for all $\mathbf{x},|\mathbf{x}-\mathbf{h}(\mathbf{x})| \neq 0$. Then define

$$
\mathbf{g}(\mathbf{x})=\mathbf{h}(\mathbf{x})+\frac{\mathbf{x}-\mathbf{h}(\mathbf{x})}{|\mathbf{x}-\mathbf{h}(\mathbf{x})|} t(\mathbf{x})
$$

where $t(\mathbf{x})$ is nonnegative and is chosen such that $\mathbf{g}(\mathbf{x}) \in \partial B(\mathbf{0}, R)$. This mapping is illustrated in the following picture.


If $\mathbf{x} \rightarrow t(\mathbf{x})$ is $C^{2}$ near $\overline{B(\mathbf{0}, R)}$, it will follow $\mathbf{g}$ is a $C^{2}$ retraction onto $\partial B(\mathbf{0}, R)$ contrary to Lemma 16.3.3. Thus $t(\mathbf{x})$ is the nonnegative solution to

$$
\begin{equation*}
H(\mathbf{x}, t) \equiv|\mathbf{h}(\mathbf{x})|^{2}+2\left(\mathbf{h}(\mathbf{x}), \frac{\mathbf{x}-\mathbf{h}(\mathbf{x})}{|\mathbf{x}-\mathbf{h}(\mathbf{x})|}\right) t+t^{2}=R^{2} \tag{16.3.4}
\end{equation*}
$$

Then

$$
H_{t}(\mathbf{x}, t)=2\left(\mathbf{h}(\mathbf{x}), \frac{\mathbf{x}-\mathbf{h}(\mathbf{x})}{|\mathbf{x}-\mathbf{h}(\mathbf{x})|}\right)+2 t
$$

If this is nonzero for all $\mathbf{x}$ near $\overline{B(\mathbf{0}, R)}$, it follows from the implicit function theorem that $t$ is a $C^{2}$ function of $\mathbf{x}$. Then from 16.3.4

$$
\begin{aligned}
2 t= & -2\left(\mathbf{h}(\mathbf{x}), \frac{\mathbf{x}-\mathbf{h}(\mathbf{x})}{|\mathbf{x}-\mathbf{h}(\mathbf{x})|}\right) \\
& \pm \sqrt{4\left(\mathbf{h}(\mathbf{x}), \frac{\mathbf{x}-\mathbf{h}(\mathbf{x})}{|\mathbf{x}-\mathbf{h}(\mathbf{x})|}\right)^{2}-4\left(|\mathbf{h}(\mathbf{x})|^{2}-R^{2}\right)}
\end{aligned}
$$

and so

$$
\begin{aligned}
H_{t}(\mathbf{x}, t) & =2 t+2\left(\mathbf{h}(\mathbf{x}), \frac{\mathbf{x}-\mathbf{h}(\mathbf{x})}{|\mathbf{x}-\mathbf{h}(\mathbf{x})|}\right) \\
& = \pm \sqrt{4\left(R^{2}-|\mathbf{h}(\mathbf{x})|^{2}\right)+4\left(\mathbf{h}(\mathbf{x}), \frac{\mathbf{x}-\mathbf{h}(\mathbf{x})}{|\mathbf{x}-\mathbf{h}(\mathbf{x})|}\right)^{2}}
\end{aligned}
$$

If $|\mathbf{h}(\mathbf{x})|<R$, this is nonzero. If $|\mathbf{h}(\mathbf{x})|=R$, then it is still nonzero unless

$$
(\mathbf{h}(\mathbf{x}), \mathbf{x}-\mathbf{h}(\mathbf{x}))=0
$$

But this cannot happen because the angle between $\mathbf{h}(\mathbf{x})$ and $\mathbf{x}-\mathbf{h}(\mathbf{x})$ cannot be $\pi / 2$. Alternatively, if the above equals zero, you would need

$$
(\mathbf{h}(\mathbf{x}), \mathbf{x})=|\mathbf{h}(\mathbf{x})|^{2}=R^{2}
$$

which cannot happen unless $\mathbf{x}=\mathbf{h}(\mathbf{x})$ which is assumed not to happen. Therefore, $\mathbf{x} \rightarrow t(\mathbf{x})$ is $C^{2}$ near $\overline{B(\mathbf{0}, R)}$ and so $\mathbf{g}(\mathbf{x})$ given above contradicts Lemma 16.3.3.

Then the Brouwer fixed point theorem is as follows.
Theorem 16.3.5 Let $\mathbf{f}: \overline{B(\mathbf{0}, R)} \rightarrow \overline{B(\mathbf{0}, R)}$ be continuous, this being a ball in $\mathbb{R}^{p}$. Then it has a fixed point $\mathbf{x} \in \overline{B(\mathbf{0}, R)}$ such that $\mathbf{f}(\mathbf{x})=\mathbf{x}$.

Proof: You can extend $\mathbf{f}$ to assume it is defined on all of $\mathbb{R}^{p}, \mathbf{f}\left(\mathbb{R}^{p}\right) \subseteq \overline{B(\mathbf{0}, R)}$, the convex hull of $\overline{B(\mathbf{0}, R)}$. Then letting $\left\{\psi_{n}\right\}$ be a mollifier, let $\mathbf{f}_{n} \equiv \mathbf{f} * \psi_{n}$. Thus

$$
\left|\mathbf{f}_{n}(\mathbf{x})\right|=\left|\int_{\mathbb{R}^{p}} \mathbf{f}(\mathbf{t}) \psi_{n}(\mathbf{x}-\mathbf{t}) d t\right| \leq \int_{\mathbb{R}^{p}}|\mathbf{f}(\mathbf{t})| \psi_{n}(\mathbf{x}-\mathbf{t}) d t \leq R \int_{\mathbb{R}^{p}} \psi_{n}(\mathbf{x}-\mathbf{t}) d t=R
$$

and so the restriction of $\mathbf{f}_{n}$ to $\overline{B(\mathbf{0}, R)}$ is $C^{2}(\overline{B(\mathbf{0}, R)})$. Therefore, there exists $\mathbf{x}_{n} \in \overline{B(\mathbf{0}, R)}$ such that $\mathbf{f}_{n}\left(\mathbf{x}_{n}\right)=\mathbf{x}_{n}$. The functions $\mathbf{f}_{n}$ converge uniformly to $\mathbf{f}$ on $\overline{B(\mathbf{0}, R)}$.

$$
\begin{aligned}
\left|\mathbf{f}(\mathbf{x})-\mathbf{f}_{n}(\mathbf{x})\right| & =\left|\int_{B\left(\mathbf{0}, \frac{1}{n}\right)}(\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{x}-\mathbf{t})) \psi_{n}(\mathbf{t}) d t\right| \\
& \leq \int_{B\left(\mathbf{0}, \frac{1}{n}\right)}|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{x}-\mathbf{t})| \psi_{n}(\mathbf{t}) d t<\varepsilon
\end{aligned}
$$

provided $n$ is large enough, this for every $\mathbf{x} \in \overline{B(\mathbf{0}, R)}$, this by uniform continuity of $\mathbf{f}$ on $\overline{B(\mathbf{0}, R+1)}$. There exists a subsequence, still called $\left\{\mathbf{x}_{n}\right\}$ which converges to $\mathbf{x} \in \overline{B(\mathbf{0}, R)}$. Then using the uniform convergence of $\mathbf{f}_{n}$ to $\mathbf{f}$,

$$
\mathbf{f}(\mathbf{x})=\lim _{n \rightarrow \infty} \mathbf{f}\left(\mathbf{x}_{n}\right)=\lim _{n \rightarrow \infty} \mathbf{f}_{n}\left(\mathbf{x}_{n}\right)=\lim _{n \rightarrow \infty} \mathbf{x}_{n}=\mathbf{x}
$$

Definition 16.3.6 A nonempty topological space $A$ is said to have the fixed point property if every continuous mapping $f: A \rightarrow A$ has a fixed point.

### 16.4 Exercises

1. Suppose you have a Banach space $X$ and a set $A \subseteq X$. Suppose $A$ is a retract of $B$ where $B$ has the fixed point property. By this is meant that $A \subseteq B$ and there is a continuous function $f: B \rightarrow A$ such that $f$ equals the identity on $A$. Show that it follows that then $A$ also has the fixed point property.
2. Show that the fixed point property is a topological property. That is, if you have $A, B$ two topological spaces and there is a continuous one to one onto mapping $f: A \rightarrow B$ which has continuous inverse, then the two topological spaces either both have the fixed point property or neither one does.
3. The Brouwer fixed point theorem says that every closed ball in $\mathbb{R}^{n}$ centered at 0 has the fixed point property. Show that it follows that every bounded convex closed set in $\mathbb{R}^{n}$ has the fixed point property. Hint: You know that the closed convex set is a retract of $\mathbb{R}^{n}$. Now if it is also a bounded set, then you could enclose it in $B(\mathbf{0}, r)$ for some large enough $r$.
4. Convex closed sets in $\mathbb{R}^{n}$ are retracts. Are there other examples of retracts not considered by Theorem 16.2.3?
5. In $\mathbb{R}^{2}$, consider an annulus, $\{\mathbf{x}: 1 \leq|\mathbf{x}| \leq 2\}$. Show that this set does not have the fixed point property. Could it be a retract of $\mathbb{R}^{2}$ ?
6. Does $\left\{\mathbf{x} \in \mathbb{R}^{n}:|\mathbf{x}|=1\right\}$ have the fixed point property?
7. Suppose you have a closed subset $H$ of $X$ a metric space and suppose also that $\mathscr{C}$ is an open cover of $H$. Show there is another open cover $\hat{\mathscr{C}}$ such that the closure of each open set in $\hat{\mathscr{C}}$ is contained in some set of $\mathscr{C}$. Hint: You might want to use the fact that metric space is normal.
8. If $H$ is a closed nonempty subset of $\mathbb{R}^{n}$ and $\mathscr{C}$ is an open cover of $H$, show that there is a refined open cover such that each of the new open sets are bounded. In the partition of unity result obtained above, applied to $H$ show that the functions in the partition of unity can be assumed to be infinitely differentiable with compact support.
9. Check that the conclusion of Theorem 16.2.3 applies for $X$ just a metric space. Then apply it to give another proof of the Tietze extension theorem.
10. Suppose you have that $h_{k}: B \rightarrow B$ for $B$ a compact set and each $h_{k}$ has a fixed point. Suppose also that $h_{k}$ converges to $h$ uniformly on $B$. Then $h$ also has a fixed point. Verify this.
11. The Brouwer fixed point theorem is a finite dimensional creature. Consider a separable Hilbert space $H$ with a complete orthonormal basis $\left\{e_{k}\right\}_{k=1}^{\infty}$. Then define the following map. For $x=\sum_{i=1}^{\infty} x_{i} e_{i}$, define $L\left(\sum_{i=1}^{\infty} x_{i} e_{i}\right) \equiv \sum_{i=1}^{\infty} x_{i} e_{i+1}$. Now let $f(x) \equiv \frac{1}{2}\left(1-\|x\|_{H}\right) e_{1}+L x$. Verify that $f: \overline{B(0,1)} \rightarrow \overline{B(0,1)}$ is continuous and yet it has no fixed point. This example is in [55].

## Chapter 17

## Banach Spaces

### 17.1 Theorems Based On Baire Category

### 17.1.1 Baire Category Theorem

Some examples of Banach spaces that have been discussed up to now are $\mathbb{R}^{n}, \mathbb{C}^{n}$, and $L^{p}(\Omega)$. Theorems about general Banach spaces are proved in this chapter. The main theorems to be presented here are the uniform boundedness theorem, the open mapping theorem, the closed graph theorem, and the Hahn Banach Theorem. The first three of these theorems come from the Baire category theorem which is about to be presented. They are topological in nature. The Hahn Banach theorem has nothing to do with topology. Banach spaces are all normed linear spaces and as such, they are all metric spaces because a normed linear space may be considered as a metric space with $d(x, y) \equiv\|x-y\|$. You can check that this satisfies all the axioms of a metric. As usual, if every Cauchy sequence converges, the metric space is called complete.

Definition 17.1.1 A complete normed linear space is called a Banach space.

The following remarkable result is called the Baire category theorem. To get an idea of its meaning, imagine you draw a line in the plane. The complement of this line is an open set and is dense because every point, even those on the line, are limit points of this open set. Now draw another line. The complement of the two lines is still open and dense. Keep drawing lines and looking at the complements of the union of these lines. You always have an open set which is dense. Now what if there were countably many lines? The Baire category theorem implies the complement of the union of these lines is dense. In particular it is nonempty. Thus you cannot write the plane as a countable union of lines. This is a rather rough description of this very important theorem. The precise statement and proof follow.

Theorem 17.1.2 Let $(X, d)$ be a complete metric space and let $\left\{U_{n}\right\}_{n=1}^{\infty}$ be a sequence of open subsets of $X$ satisfying $\overline{U_{n}}=X$ ( $U_{n}$ is dense). Then $D \equiv \cap_{n=1}^{\infty} U_{n}$ is a dense subset of $X$.

Proof: Let $p \in X$ and let $r_{0}>0$. I need to show $D \cap B\left(p, r_{0}\right) \neq \emptyset$. Since $U_{1}$ is dense, there exists $p_{1} \in U_{1} \cap B\left(p, r_{0}\right)$, an open set. Let $p_{1} \in B\left(p_{1}, r_{1}\right) \subseteq \overline{B\left(p_{1}, r_{1}\right)} \subseteq U_{1} \cap B\left(p, r_{0}\right)$ and $r_{1}<2^{-1}$. This is possible because $U_{1} \cap B\left(p, r_{0}\right)$ is an open set and so there exists $r_{1}$ such that $B\left(p_{1}, 2 r_{1}\right) \subseteq U_{1} \cap B\left(p, r_{0}\right)$. But

$$
B\left(p_{1}, r_{1}\right) \subseteq \overline{B\left(p_{1}, r_{1}\right)} \subseteq B\left(p_{1}, 2 r_{1}\right)
$$

because $\overline{B\left(p_{1}, r_{1}\right)}=\left\{x \in X: d(x, p) \leq r_{1}\right\}$. (Why?)


There exists $p_{2} \in U_{2} \cap B\left(p_{1}, r_{1}\right)$ because $U_{2}$ is dense. Let

$$
p_{2} \in B\left(p_{2}, r_{2}\right) \subseteq \overline{B\left(p_{2}, r_{2}\right)} \subseteq U_{2} \cap B\left(p_{1}, r_{1}\right) \subseteq U_{1} \cap U_{2} \cap B\left(p, r_{0}\right)
$$

and let $r_{2}<2^{-2}$. Continue in this way. Thus

$$
\begin{gathered}
r_{n}<2^{-n}, \\
\overline{B\left(p_{n}, r_{n}\right)} \subseteq U_{1} \cap U_{2} \cap \ldots \cap U_{n} \cap B\left(p, r_{0}\right), \\
\overline{B\left(p_{n}, r_{n}\right)} \subseteq B\left(p_{n-1}, r_{n-1}\right) .
\end{gathered}
$$

The sequence, $\left\{p_{n}\right\}$ is a Cauchy sequence because all terms of $\left\{p_{k}\right\}$ for $k \geq n$ are contained in $B\left(p_{n}, r_{n}\right)$, a set whose diameter is no larger than $2^{-n}$. Since $X$ is complete, there exists $p_{\infty}$ such that

$$
\lim _{n \rightarrow \infty} p_{n}=p_{\infty}
$$

Since all but finitely many terms of $\left\{p_{n}\right\}$ are in $\overline{B\left(p_{m}, r_{m}\right)}$, it follows that $p_{\infty} \in \overline{B\left(p_{m}, r_{m}\right)}$ for each $m$. Therefore,

$$
p_{\infty} \in \cap_{m=1}^{\infty} \overline{B\left(p_{m}, r_{m}\right)} \subseteq \cap_{i=1}^{\infty} U_{i} \cap B\left(p, r_{0}\right)
$$

This proves the theorem.
The following corollary is also called the Baire category theorem.
Corollary 17.1.3 Let $X$ be a complete metric space and suppose $X=\cup_{i=1}^{\infty} F_{i}$ where each $F_{i}$ is a closed set. Then for some $i$, interior $F_{i} \neq \emptyset$.

Proof: If all $F_{i}$ has empty interior, then $F_{i}^{C}$ would be a dense open set. Therefore, from Theorem 17.1.2, it would follow that

$$
\emptyset=\left(\cup_{i=1}^{\infty} F_{i}\right)^{C}=\cap_{i=1}^{\infty} F_{i}^{C} \neq \emptyset
$$

The set $D$ of Theorem 17.1.2 is called a $G_{\delta}$ set because it is the countable intersection of open sets. Thus $D$ is a dense $G_{\delta}$ set.

Recall that a norm satisfies:
a.) $\|x\| \geq 0,\|x\|=0$ if and only if $x=0$.
b.) $\|x+y\| \leq\|x\|+\|y\|$.
c.) $\|c x\|=|c|\|x\|$ if $c$ is a scalar and $x \in X$.

From the definition of continuity, it follows easily that a function is continuous if

$$
\lim _{n \rightarrow \infty} x_{n}=x
$$

implies

$$
\lim _{n \rightarrow \infty} f\left(x_{n}\right)=f(x)
$$

Theorem 17.1.4 Let $X$ and $Y$ be two normed linear spaces and let $L: X \rightarrow Y$ be linear $(L(a x+b y)=a L(x)+b L(y)$ for $a, b$ scalars and $x, y \in X)$. The following are equivalent
a.) $L$ is continuous at 0
b.) $L$ is continuous
c.) There exists $K>0$ such that $\|L x\|_{Y} \leq K\|x\|_{X}$ for all $x \in X$ ( $L$ is bounded).

Proof: a.) $\Rightarrow \mathrm{b}$.) Let $x_{n} \rightarrow x$. It is necessary to show that $L x_{n} \rightarrow L x$. But $\left(x_{n}-x\right) \rightarrow 0$ and so from continuity at 0 , it follows

$$
L\left(x_{n}-x\right)=L x_{n}-L x \rightarrow 0
$$

so $L x_{n} \rightarrow L x$. This shows a.) implies b.).
b. $) \Rightarrow$ c.) Since $L$ is continuous, $L$ is continuous at 0 . Hence $\|L x\|_{Y}<1$ whenever $\|x\|_{X} \leq \delta$ for some $\delta$. Therefore, suppressing the subscript on the \| \|,

$$
\left\|L\left(\frac{\delta x}{\|x\|}\right)\right\| \leq 1
$$

Hence

$$
\|L x\| \leq \frac{1}{\delta}\|x\|
$$

c.) $\Rightarrow$ a.) follows from the inequality given in c .).

Definition 17.1.5 Let $L: X \rightarrow Y$ be linear and continuous where $X$ and $Y$ are normed linear spaces. Denote the set of all such continuous linear maps by $\mathscr{L}(X, Y)$ and define

$$
\begin{equation*}
\|L\|=\sup \{\|L x\|:\|x\| \leq 1\} \tag{17.1.1}
\end{equation*}
$$

This is called the operator norm.
Note that from Theorem 17.1.4 $\|L\|$ is well defined because of part c .) of that Theorem.
The next lemma follows immediately from the definition of the norm and the assumption that $L$ is linear.

Lemma 17.1.6 With $\|L\|$ defined in 17.1.1, $\mathscr{L}(X, Y)$ is a normed linear space. Also $\|L x\| \leq\|L\|\|x\|$.

Proof: Let $x \neq 0$ then $x /\|x\|$ has norm equal to 1 and so

$$
\left\|L\left(\frac{x}{\|x\|}\right)\right\| \leq\|L\|
$$

Therefore, multiplying both sides by $\|x\|,\|L x\| \leq\|L\|\|x\|$. This is obviously a linear space. It remains to verify the operator norm really is a norm. First of all, if $\|L\|=0$, then $L x=0$ for all $\|x\| \leq 1$. It follows that for any $x \neq 0,0=L\left(\frac{x}{\|x\|}\right)$ and so $L x=0$. Therefore, $L=0$. Also, if $c$ is a scalar,

$$
\|c L\|=\sup _{\|x\| \leq 1}\|c L(x)\|=|c| \sup _{\|x\| \leq 1}\|L x\|=|c|\|L\| .
$$

It remains to verify the triangle inequality. Let $L, M \in \mathscr{L}(X, Y)$.

$$
\begin{aligned}
\|L+M\| & \equiv \sup _{\|x\| \leq 1}\|(L+M)(x)\| \leq \sup _{\|x\| \leq 1}(\|L x\|+\|M x\|) \\
& \leq \sup _{\|x\| \leq 1}\|L x\|+\sup _{\|x\| \leq 1}\|M x\|=\|L\|+\|M\| .
\end{aligned}
$$

This shows the operator norm is really a norm as hoped. This proves the lemma.
For example, consider the space of linear transformations defined on $\mathbb{R}^{n}$ having values in $\mathbb{R}^{m}$. The fact the transformation is linear automatically imparts continuity to it. You should give a proof of this fact. Recall that every such linear transformation can be realized in terms of matrix multiplication.

Thus, in finite dimensions the algebraic condition that an operator is linear is sufficient to imply the topological condition that the operator is continuous. The situation is not so simple in infinite dimensional spaces such as $C\left(X ; \mathbb{R}^{n}\right)$. This explains the imposition of the topological condition of continuity as a criterion for membership in $\mathscr{L}(X, Y)$ in addition to the algebraic condition of linearity.

Theorem 17.1.7 If $Y$ is a Banach space, then $\mathscr{L}(X, Y)$ is also a Banach space.
Proof: Let $\left\{L_{n}\right\}$ be a Cauchy sequence in $\mathscr{L}(X, Y)$ and let $x \in X$.

$$
\left\|L_{n} x-L_{m} x\right\| \leq\|x\|\left\|L_{n}-L_{m}\right\|
$$

Thus $\left\{L_{n} x\right\}$ is a Cauchy sequence. Let

$$
L x=\lim _{n \rightarrow \infty} L_{n} x
$$

Then, clearly, $L$ is linear because if $x_{1}, x_{2}$ are in $X$, and $a, b$ are scalars, then

$$
\begin{aligned}
L\left(a x_{1}+b x_{2}\right) & =\lim _{n \rightarrow \infty} L_{n}\left(a x_{1}+b x_{2}\right) \\
& =\lim _{n \rightarrow \infty}\left(a L_{n} x_{1}+b L_{n} x_{2}\right) \\
& =a L x_{1}+b L x_{2}
\end{aligned}
$$

Also $L$ is continuous. To see this, note that $\left\{\left\|L_{n}\right\|\right\}$ is a Cauchy sequence of real numbers because $\left\|\mid L_{n}\right\|-\left\|L_{m}\right\|\|\leq\| L_{n}-L_{m} \|$. Hence there exists $K>\sup \left\{\left\|L_{n}\right\|: n \in \mathbb{N}\right\}$. Thus, if $x \in X$,

$$
\|L x\|=\lim _{n \rightarrow \infty}\left\|L_{n} x\right\| \leq K\|x\|
$$

This proves the theorem.

### 17.1.2 Uniform Boundedness Theorem

The next big result is sometimes called the Uniform Boundedness theorem, or the BanachSteinhaus theorem. This is a very surprising theorem which implies that for a collection of bounded linear operators, if they are bounded pointwise, then they are also bounded
uniformly. As an example of a situation in which pointwise bounded does not imply uniformly bounded, consider the functions $f_{\alpha}(x) \equiv \mathscr{X}_{(\alpha, 1)}(x) x^{-1}$ for $\alpha \in(0,1)$. Clearly each function is bounded and the collection of functions is bounded at each point of $(0,1)$, but there is no bound for all these functions taken together. One problem is that $(0,1)$ is not a Banach space. Therefore, the functions cannot be linear.

Theorem 17.1.8 Let $X$ be a Banach space and let $Y$ be a normed linear space. Let $\left\{L_{\alpha}\right\}_{\alpha \in \Lambda}$ be a collection of elements of $\mathscr{L}(X, Y)$. Then one of the following happens.
a.) $\sup \left\{\left\|L_{\alpha}\right\|: \alpha \in \Lambda\right\}<\infty$
b.) There exists a dense $G_{\delta}$ set, $D$, such that for all $x \in D$,

$$
\sup \left\{\left\|L_{\alpha} x\right\| \alpha \in \Lambda\right\}=\infty
$$

Proof: For each $n \in \mathbb{N}$, define

$$
U_{n}=\left\{x \in X: \sup \left\{\left\|L_{\alpha} x\right\|: \alpha \in \Lambda\right\}>n\right\} .
$$

Then $U_{n}$ is an open set because if $x \in U_{n}$, then there exists $\alpha \in \Lambda$ such that

$$
\left\|L_{\alpha} x\right\|>n
$$

But then, since $L_{\alpha}$ is continuous, this situation persists for all $y$ sufficiently close to $x$, say for all $y \in B(x, \delta)$. Then $B(x, \delta) \subseteq U_{n}$ which shows $U_{n}$ is open.

Case b.) is obtained from Theorem 17.1.2 if each $U_{n}$ is dense.
The other case is that for some $n, U_{n}$ is not dense. If this occurs, there exists $x_{0}$ and $r>0$ such that for all $x \in B\left(x_{0}, r\right),\left\|L_{\alpha} x\right\| \leq n$ for all $\alpha$. Now if $y \in B(0, r), x_{0}+y \in B\left(x_{0}, r\right)$. Consequently, for all such $y,\left\|L_{\alpha}\left(x_{0}+y\right)\right\| \leq n$. This implies that for all $\alpha \in \Lambda$ and $\|y\|<r$,

$$
\left\|L_{\alpha} y\right\| \leq n+\left\|L_{\alpha}\left(x_{0}\right)\right\| \leq 2 n
$$

Therefore, if $\|y\| \leq 1,\left\|\frac{r}{2} y\right\|<r$ and so for all $\alpha$,

$$
\left\|L_{\alpha}\left(\frac{r}{2} y\right)\right\| \leq 2 n
$$

Now multiplying by $r / 2$ it follows that whenever $\|y\| \leq 1,\left\|L_{\alpha}(y)\right\| \leq 4 n / r$. Hence case a.) holds.

### 17.1.3 Open Mapping Theorem

Another remarkable theorem which depends on the Baire category theorem is the open mapping theorem. Unlike Theorem 17.1.8 it requires both $X$ and $Y$ to be Banach spaces.

Theorem 17.1.9 Let $X$ and $Y$ be Banach spaces, let $L \in \mathscr{L}(X, Y)$, and suppose $L$ is onto. Then L maps open sets onto open sets.

To aid in the proof, here is a lemma.

Lemma 17.1.10 Let $a$ and $b$ be positive constants and suppose

$$
B(0, a) \subseteq \overline{L(B(0, b))}
$$

Then

$$
\overline{L(B(0, b))} \subseteq L(B(0,2 b))
$$

Proof of Lemma 17.1.10: Let $y \in \overline{L(B(0, b))}$. There exists $x_{1} \in B(0, b)$ such that $\| y-$ $L x_{1} \|<\frac{a}{2}$. Now this implies

$$
2 y-2 L x_{1} \in B(0, a) \subseteq \overline{L(B(0, b))}
$$

Thus $2 y-2 L x_{1} \in \overline{L(B(0, b))}$ just like $y$ was. Therefore, there exists $x_{2} \in B(0, b)$ such that $\left\|2 y-2 L x_{1}-L x_{2}\right\|<a / 2$. Hence $\left\|4 y-4 L x_{1}-2 L x_{2}\right\|<a$, and there exists $x_{3} \in$ $B(0, b)$ such that $\left\|4 y-4 L x_{1}-2 L x_{2}-L x_{3}\right\|<a / 2$. Continuing in this way, there exist $x_{1}, x_{2}, x_{3}, x_{4}, \ldots$ in $B(0, b)$ such that

$$
\left\|2^{n} y-\sum_{i=1}^{n} 2^{n-(i-1)} L\left(x_{i}\right)\right\|<a
$$

which implies

$$
\begin{equation*}
\left\|y-\sum_{i=1}^{n} 2^{-(i-1)} L\left(x_{i}\right)\right\|=\left\|y-L\left(\sum_{i=1}^{n} 2^{-(i-1)}\left(x_{i}\right)\right)\right\|<2^{-n} a \tag{17.1.2}
\end{equation*}
$$

Now consider the partial sums of the series, $\sum_{i=1}^{\infty} 2^{-(i-1)} x_{i}$.

$$
\left\|\sum_{i=m}^{n} 2^{-(i-1)} x_{i}\right\| \leq b \sum_{i=m}^{\infty} 2^{-(i-1)}=b 2^{-m+2}
$$

Therefore, these partial sums form a Cauchy sequence and so since $X$ is complete, there exists $x=\sum_{i=1}^{\infty} 2^{-(i-1)} x_{i}$. Letting $n \rightarrow \infty$ in 17.1.2 yields $\|y-L x\|=0$. Now

$$
\begin{gathered}
\|x\|=\lim _{n \rightarrow \infty}\left\|\sum_{i=1}^{n} 2^{-(i-1)} x_{i}\right\| \\
\leq \lim _{n \rightarrow \infty} \sum_{i=1}^{n} 2^{-(i-1)}\left\|x_{i}\right\|<\lim _{n \rightarrow \infty} \sum_{i=1}^{n} 2^{-(i-1)} b=2 b .
\end{gathered}
$$

This proves the lemma.
Proof of Theorem 17.1.9: $Y=\cup_{n=1}^{\infty} \overline{L(B(0, n))}$. By Corollary 17.1.3, the set, $\overline{L\left(B\left(0, n_{0}\right)\right)}$ has nonempty interior for some $n_{0}$. Thus $B(y, r) \subseteq \overline{L\left(B\left(0, n_{0}\right)\right)}$ for some $y$ and some $r>0$. Since $L$ is linear $B(-y, r) \subseteq \overline{L\left(B\left(0, n_{0}\right)\right)}$ also. Here is why. If $z \in B(-y, r)$, then
 and $-x_{n} \in B\left(0, n_{0}\right)$ also. Therefore $z \in \overline{L\left(B\left(0, n_{0}\right)\right)}$. Then it follows that

$$
\begin{aligned}
B(0, r) & \subseteq B(y, r)+B(-y, r) \\
& \equiv\left\{y_{1}+y_{2}: y_{1} \in B(y, r) \text { and } y_{2} \in B(-y, r)\right\} \\
& \subseteq \frac{L\left(B\left(0,2 n_{0}\right)\right)}{}
\end{aligned}
$$

The reason for the last inclusion is that from the above, if $y_{1} \in B(y, r)$ and $y_{2} \in B(-y, r)$, there exists $x_{n}, z_{n} \in B\left(0, n_{0}\right)$ such that

$$
L x_{n} \rightarrow y_{1}, L z_{n} \rightarrow y_{2} .
$$

Therefore,

$$
\left\|x_{n}+z_{n}\right\| \leq 2 n_{0}
$$

and so $\left(y_{1}+y_{2}\right) \in \overline{L\left(B\left(0,2 n_{0}\right)\right)}$.
By Lemma 17.1.10, $\overline{L\left(B\left(0,2 n_{0}\right)\right)} \subseteq L\left(B\left(0,4 n_{0}\right)\right)$ which shows

$$
B(0, r) \subseteq L\left(B\left(0,4 n_{0}\right)\right)
$$

Letting $a=r\left(4 n_{0}\right)^{-1}$, it follows, since $L$ is linear, that $B(0, a) \subseteq L(B(0,1))$. It follows since $L$ is linear,

$$
\begin{equation*}
L(B(0, r)) \supseteq B(0, a r) . \tag{17.1.3}
\end{equation*}
$$

Now let $U$ be open in $X$ and let $x+B(0, r)=B(x, r) \subseteq U$. Using 17.1.3,

$$
\begin{aligned}
L(U) & \supseteq L(x+B(0, r)) \\
=L x+L(B(0, r)) & \supseteq L x+B(0, a r)=B(L x, a r) .
\end{aligned}
$$

Hence

$$
L x \in B(L x, a r) \subseteq L(U)
$$

which shows that every point, $L x \in L U$, is an interior point of $L U$ and so $L U$ is open. This proves the theorem.

This theorem is surprising because it implies that if $|\cdot|$ and $\|\cdot\|$ are two norms with respect to which a vector space $X$ is a Banach space such that $|\cdot| \leq K\|\cdot\|$, then there exists a constant $k$, such that $\|\cdot\| \leq k|\cdot|$. This can be useful because sometimes it is not clear how to compute $k$ when all that is needed is its existence. To see the open mapping theorem implies this, consider the identity map id $x=x$. Then id : $(X,\|\cdot\|) \rightarrow(X,|\cdot|)$ is continuous and onto. Hence id is an open map which implies $\mathrm{id}^{-1}$ is continuous. Theorem 17.1.4 gives the existence of the constant $k$.

### 17.1.4 Closed Graph Theorem

Definition 17.1.11 Let $f: D \rightarrow E$. The set of all ordered pairs of the form $\{(x, f(x)): x \in$ $D\}$ is called the graph of $f$.

Definition 17.1.12 If $X$ and $Y$ are normed linear spaces, make $X \times Y$ into a normed linear space by using the norm $\|(x, y)\|=\max (\|x\|,\|y\|)$ along with component-wise addition and scalar multiplication. Thus $a(x, y)+b(z, w) \equiv(a x+b z, a y+b w)$.

There are other ways to give a norm for $X \times Y$. For example, you could define $\|(x, y)\|=$ $\|x\|+\|y\|$

Lemma 17.1.13 The norm defined in Definition 17.1.12 on $X \times Y$ along with the definition of addition and scalar multiplication given there make $X \times Y$ into a normed linear space.

Proof: The only axiom for a norm which is not obvious is the triangle inequality. Therefore, consider

$$
\begin{aligned}
&\left\|\left(x_{1}, y_{1}\right)+\left(x_{2}, y_{2}\right)\right\|=\left\|\left(x_{1}+x_{2}, y_{1}+y_{2}\right)\right\| \\
&=\max \left(\left\|x_{1}+x_{2}\right\|,\left\|y_{1}+y_{2}\right\|\right) \\
& \leq \max \left(\left\|x_{1}\right\|+\left\|x_{2}\right\|,\left\|y_{1}\right\|+\left\|y_{2}\right\|\right) \\
& \leq \max \left(\left\|x_{1}\right\|,\left\|y_{1}\right\|\right)+\max \left(\left\|x_{2}\right\|,\left\|y_{2}\right\|\right) \\
&=\left\|\left(x_{1}, y_{1}\right)\right\|+\left\|\left(x_{2}, y_{2}\right)\right\| .
\end{aligned}
$$

It is obvious $X \times Y$ is a vector space from the above definition. This proves the lemma.
Lemma 17.1.14 If $X$ and $Y$ are Banach spaces, then $X \times Y$ with the norm and vector space operations defined in Definition 17.1.12 is also a Banach space.

Proof: The only thing left to check is that the space is complete. But this follows from the simple observation that $\left\{\left(x_{n}, y_{n}\right)\right\}$ is a Cauchy sequence in $X \times Y$ if and only if $\left\{x_{n}\right\}$ and $\left\{y_{n}\right\}$ are Cauchy sequences in $X$ and $Y$ respectively. Thus if $\left\{\left(x_{n}, y_{n}\right)\right\}$ is a Cauchy sequence in $X \times Y$, it follows there exist $x$ and $y$ such that $x_{n} \rightarrow x$ and $y_{n} \rightarrow y$. But then from the definition of the norm, $\left(x_{n}, y_{n}\right) \rightarrow(x, y)$.

Lemma 17.1.15 Every closed subspace of a Banach space is a Banach space.
Proof: If $F \subseteq X$ where $X$ is a Banach space and $\left\{x_{n}\right\}$ is a Cauchy sequence in $F$, then since $X$ is complete, there exists a unique $x \in X$ such that $x_{n} \rightarrow x$. However this means $x \in \bar{F}=F$ since $F$ is closed.

Definition 17.1.16 Let $X$ and $Y$ be Banach spaces and let $D \subseteq X$ be a subspace. A linear map $L: D \rightarrow Y$ is said to be closed if its graph is a closed subspace of $X \times Y$. Equivalently, $L$ is closed if $x_{n} \rightarrow x$ and $L x_{n} \rightarrow y$ implies $x \in D$ and $y=L x$.

Note the distinction between closed and continuous. If the operator is closed the assertion that $y=L x$ only follows if it is known that the sequence $\left\{L x_{n}\right\}$ converges. In the case of a continuous operator, the convergence of $\left\{L x_{n}\right\}$ follows from the assumption that $x_{n} \rightarrow x$. It is not always the case that a mapping which is closed is necessarily continuous. Consider the function $f(x)=\tan (x)$ if $x$ is not an odd multiple of $\frac{\pi}{2}$ and $f(x) \equiv 0$ at every odd multiple of $\frac{\pi}{2}$. Then the graph is closed and the function is defined on $\mathbb{R}$ but it clearly fails to be continuous. Of course this function is not linear. You could also consider the map,

$$
\frac{d}{d x}:\left\{y \in C^{1}([0,1]): y(0)=0\right\} \equiv D \rightarrow C([0,1])
$$

where the norm is the uniform norm on $C([0,1]),\|y\|_{\infty}$. If $y \in D$, then

$$
y(x)=\int_{0}^{x} y^{\prime}(t) d t
$$

Therefore, if $\frac{d y_{n}}{d x} \rightarrow f \in C([0,1])$ and if $y_{n} \rightarrow y$ in $C([0,1])$ it follows that

$$
\begin{aligned}
y_{n}(x) & =\int_{0}^{x} \frac{d y_{n}(t)}{d x} d t \\
\downarrow & \downarrow \\
y(x) & =\int_{0}^{x} f(t) d t
\end{aligned}
$$

and so by the fundamental theorem of calculus $f(x)=y^{\prime}(x)$ and so the mapping is closed. It is obviously not continuous because it takes $y(x)$ and $y(x)+\frac{1}{n} \sin (n x)$ to two functions which are far from each other even though these two functions are very close in $C([0,1])$. Furthermore, it is not defined on the whole space, $C([0,1])$.

The next theorem, the closed graph theorem, gives conditions under which closed implies continuous.

Theorem 17.1.17 Let $X$ and $Y$ be Banach spaces and suppose $L: X \rightarrow Y$ is closed and linear. Then $L$ is continuous.

Proof: Let $G$ be the graph of $L . G=\{(x, L x): x \in X\}$. By Lemma 17.1.15 it follows that $G$ is a Banach space. Define $P: G \rightarrow X$ by $P(x, L x)=x$. $P$ maps the Banach space $G$ onto the Banach space $X$ and is continuous and linear. By the open mapping theorem, $P$ maps open sets onto open sets. Since $P$ is also one to one, this says that $P^{-1}$ is continuous. Thus $\left\|P^{-1} x\right\| \leq K\|x\|$. Hence

$$
\|L x\| \leq \max (\|x\|,\|L x\|) \leq K\|x\|
$$

By Theorem 17.1.4 on Page 437, this shows $L$ is continuous and proves the theorem.
The following corollary is quite useful. It shows how to obtain a new norm on the domain of a closed operator such that the domain with this new norm becomes a Banach space.

Corollary 17.1.18 Let $L: D \subseteq X \rightarrow Y$ where $X, Y$ are a Banach spaces, and $L$ is a closed operator. Then define a new norm on $D$ by

$$
\|x\|_{D} \equiv\|x\|_{X}+\|L x\|_{Y} .
$$

Then $D$ with this new norm is a Banach space.
Proof: If $\left\{x_{n}\right\}$ is a Cauchy sequence in $D$ with this new norm, it follows both $\left\{x_{n}\right\}$ and $\left\{L x_{n}\right\}$ are Cauchy sequences and therefore, they converge. Since $L$ is closed, $x_{n} \rightarrow x$ and $L x_{n} \rightarrow L x$ for some $x \in D$. Thus $\left\|x_{n}-x\right\|_{D} \rightarrow 0$.

### 17.2 Hahn Banach Theorem

The closed graph, open mapping, and uniform boundedness theorems are the three major topological theorems in functional analysis. The other major theorem is the Hahn-Banach theorem which has nothing to do with topology. Before presenting this theorem, here are some preliminaries about partially ordered sets.

### 17.2.1 Partially Ordered Sets

Definition 17.2.1 Let $\mathscr{F}$ be a nonempty set. $\mathscr{F}$ is called a partially ordered set if there is a relation, denoted here by $\leq$, such that

$$
\begin{gathered}
x \leq x \text { for all } x \in \mathscr{F} \\
\text { If } x \leq y \text { and } y \leq z \text { then } x \leq z
\end{gathered}
$$

$\mathscr{C} \subseteq \mathscr{F}$ is said to be a chain if every two elements of $\mathscr{C}$ are related. This means that if $x, y \in \mathscr{C}$, then either $x \leq y$ or $y \leq x$. Sometimes a chain is called a totally ordered set. $\mathscr{C}$ is said to be a maximal chain if whenever $\mathscr{D}$ is a chain containing $\mathscr{C}, \mathscr{D}=\mathscr{C}$.

The most common example of a partially ordered set is the power set of a given set with $\subseteq$ being the relation. It is also helpful to visualize partially ordered sets as trees. Two points on the tree are related if they are on the same branch of the tree and one is higher than the other. Thus two points on different branches would not be related although they might both be larger than some point on the trunk. You might think of many other things which are best considered as partially ordered sets. Think of food for example. You might find it difficult to determine which of two favorite pies you like better although you may be able to say very easily that you would prefer either pie to a dish of lard topped with whipped cream and mustard. The following theorem is equivalent to the axiom of choice. For a discussion of this, see the appendix on the subject.

Theorem 17.2.2 (Hausdorff Maximal Principle) Let $\mathscr{F}$ be a nonempty partially ordered set. Then there exists a maximal chain.

### 17.2.2 Gauge Functions And Hahn Banach Theorem

Definition 17.2.3 Let $X$ be a real vector space $\rho: X \rightarrow \mathbb{R}$ is called a gauge function if

$$
\begin{align*}
& \rho(x+y) \leq \rho(x)+\rho(y) \\
& \rho(a x)=a \rho(x) \text { if } a \geq 0 . \tag{17.2.4}
\end{align*}
$$

Suppose $M$ is a subspace of $X$ and $z \notin M$. Suppose also that $f$ is a linear real-valued function having the property that $f(x) \leq \rho(x)$ for all $x \in M$. Consider the problem of extending $f$ to $M \oplus \mathbb{R} z$ such that if $F$ is the extended function, $F(y) \leq \rho(y)$ for all $y \in$ $M \oplus \mathbb{R} z$ and $F$ is linear. Since $F$ is to be linear, it suffices to determine how to define $F(z)$. Letting $a>0$, it is required to define $F(z)$ such that the following hold for all $x, y \in M$.

$$
\begin{align*}
& \overbrace{F(x)}^{f(x)}+a F(z)=F(x+a z) \leq \rho(x+a z), \\
& \overbrace{F(y)}^{f(y)}-a F(z)=F(y-a z) \leq \rho(y-a z) . \tag{17.2.5}
\end{align*}
$$

Now if these inequalities hold for all $y / a$, they hold for all $y$ because $M$ is given to be a subspace. Therefore, multiplying by $a^{-1} 17.2 .4$ implies that what is needed is to choose $F(z)$ such that for all $x, y \in M$,

$$
f(x)+F(z) \leq \rho(x+z), f(y)-\rho(y-z) \leq F(z)
$$

and that if $F(z)$ can be chosen in this way, this will satisfy 17.2 .5 for all $x, y$ and the problem of extending $f$ will be solved. Hence it is necessary to choose $F(z)$ such that for all $x, y \in M$

$$
\begin{equation*}
f(y)-\rho(y-z) \leq F(z) \leq \rho(x+z)-f(x) \tag{17.2.6}
\end{equation*}
$$

Is there any such number between $f(y)-\rho(y-z)$ and $\rho(x+z)-f(x)$ for every pair $x, y \in$ $M$ ? This is where $f(x) \leq \rho(x)$ on $M$ and that $f$ is linear is used. For $x, y \in M$,

$$
\begin{gathered}
\rho(x+z)-f(x)-[f(y)-\rho(y-z)] \\
=\rho(x+z)+\rho(y-z)-(f(x)+f(y)) \\
\geq \rho(x+y)-f(x+y) \geq 0 .
\end{gathered}
$$

Therefore there exists a number between

$$
\sup \{f(y)-\rho(y-z): y \in M\}
$$

and

$$
\inf \{\rho(x+z)-f(x): x \in M\}
$$

Choose $F(z)$ to satisfy 17.2 .6 . This has proved the following lemma.
Lemma 17.2.4 Let $M$ be a subspace of $X$, a real linear space, and let $\rho$ be a gauge function on $X$. Suppose $f: M \rightarrow \mathbb{R}$ is linear, $z \notin M$, and $f(x) \leq \rho(x)$ for all $x \in M$. Then $f$ can be extended to $M \oplus \mathbb{R} z$ such that, if $F$ is the extended function, $F$ is linear and $F(x) \leq \rho(x)$ for all $x \in M \oplus \mathbb{R} z$.

With this lemma, the Hahn Banach theorem can be proved.
Theorem 17.2.5 (Hahn Banach theorem) Let $X$ be a real vector space, let $M$ be a subspace of $X$, let $f: M \rightarrow \mathbb{R}$ be linear, let $\rho$ be a gauge function on $X$, and suppose $f(x) \leq \rho(x)$ for all $x \in M$. Then there exists a linear function, $F: X \rightarrow \mathbb{R}$, such that
a.) $F(x)=f(x)$ for all $x \in M$
b.) $F(x) \leq \rho(x)$ for all $x \in X$.

Proof: Let $\mathscr{F}=\{(V, g): V \supseteq M, V$ is a subspace of $X, g: V \rightarrow \mathbb{R}$ is linear, $g(x)=f(x)$ for all $x \in M$, and $g(x) \leq \rho(x)$ for $x \in V\}$. Then $(M, f) \in \mathscr{F}$ so $\mathscr{F} \neq \emptyset$. Define a partial order by the following rule.

$$
(V, g) \leq(W, h)
$$

means

$$
V \subseteq W \text { and } h(x)=g(x) \text { if } x \in V
$$

By Theorem 17.2.2, there exists a maximal chain, $\mathscr{C} \subseteq \mathscr{F}$. Let $Y=\cup\{V:(V, g) \in \mathscr{C}\}$ and let $h: Y \rightarrow \mathbb{R}$ be defined by $h(x)=g(x)$ where $x \in V$ and $(V, g) \in \mathscr{C}$. This is well defined because if $x \in V_{1}$ and $V_{2}$ where $\left(V_{1}, g_{1}\right)$ and $\left(V_{2}, g_{2}\right)$ are both in the chain, then since $\mathscr{C}$ is a chain, the two element related. Therefore, $g_{1}(x)=g_{2}(x)$. Also $h$ is linear because if $a x+b y \in Y$, then $x \in V_{1}$ and $y \in V_{2}$ where $\left(V_{1}, g_{1}\right)$ and $\left(V_{2}, g_{2}\right)$ are elements of $\mathscr{C}$. Therefore, letting $V$ denote the larger of the two $V_{i}$, and $g$ be the function that goes with $V$, it follows $a x+b y \in V$ where $(V, g) \in \mathscr{C}$. Therefore,

$$
\begin{aligned}
h(a x+b y) & =g(a x+b y) \\
& =a g(x)+b g(y) \\
& =a h(x)+b h(y)
\end{aligned}
$$

Also, $h(x)=g(x) \leq \rho(x)$ for any $x \in Y$ because for such $x, x \in V$ where $(V, g) \in \mathscr{C}$.
Is $Y=X$ ? If not, there exists $z \in X \backslash Y$ and there exists an extension of $h$ to $Y \oplus \mathbb{R} z$ using Lemma 17.2.4. Letting $\bar{h}$ denote this extended function, contradicts the maximality of $\mathscr{C}$. Indeed, $\mathscr{C} \cup\{(Y \oplus \mathbb{R} z, \bar{h})\}$ would be a longer chain. This proves the Hahn Banach theorem.

This is the original version of the theorem. There is also a version of this theorem for complex vector spaces which is based on a trick.

### 17.2.3 The Complex Version Of The Hahn Banach Theorem

Corollary 17.2.6 (Hahn Banach) Let $M$ be a subspace of a complex normed linear space, $X$, and suppose $f: M \rightarrow \mathbb{C}$ is linear and satisfies $|f(x)| \leq K| | x| |$ for all $x \in M$. Then there exists a linear function, $F$, defined on all of $X$ such that $F(x)=f(x)$ for all $x \in M$ and $|F(x)| \leq K| | x \|$ for all $x$.

Proof: First note $f(x)=\operatorname{Re} f(x)+i \operatorname{Im} f(x)$ and so

$$
\operatorname{Re} f(i x)+i \operatorname{Im} f(i x)=f(i x)=i f(x)=i \operatorname{Re} f(x)-\operatorname{Im} f(x)
$$

Therefore, $\operatorname{Im} f(x)=-\operatorname{Re} f(i x)$, and

$$
f(x)=\operatorname{Re} f(x)-i \operatorname{Re} f(i x)
$$

This is important because it shows it is only necessary to consider $\operatorname{Re} f$ in understanding $f$. Now it happens that $\operatorname{Re} f$ is linear with respect to real scalars so the above version of the Hahn Banach theorem applies. This is shown next.

If $c$ is a real scalar

$$
\operatorname{Re} f(c x)-i \operatorname{Re} f(i c x)=c f(x)=c \operatorname{Re} f(x)-i c \operatorname{Re} f(i x)
$$

Thus $\operatorname{Re} f(c x)=c \operatorname{Re} f(x)$. Also,

$$
\begin{aligned}
\operatorname{Re} f(x+y)-i \operatorname{Re} f(i(x+y)) & =f(x+y) \\
& =f(x)+f(y)
\end{aligned}
$$

$$
=\operatorname{Re} f(x)-i \operatorname{Re} f(i x)+\operatorname{Re} f(y)-i \operatorname{Re} f(i y)
$$

Equating real parts, $\operatorname{Re} f(x+y)=\operatorname{Re} f(x)+\operatorname{Re} f(y)$. Thus $\operatorname{Re} f$ is linear with respect to real scalars as hoped.

Consider $X$ as a real vector space and let $\rho(x) \equiv K\|x\|$. Then for all $x \in M$,

$$
|\operatorname{Re} f(x)| \leq|f(x)| \leq K \| x| |=\rho(x)
$$

From Theorem 17.2.5, $\operatorname{Re} f$ may be extended to a function, $h$ which satisfies

$$
\begin{aligned}
h(a x+b y) & =a h(x)+b h(y) \text { if } a, b \in \mathbb{R} \\
h(x) & \leq K\|x\| \text { for all } x \in X .
\end{aligned}
$$

Actually, $|h(x)| \leq K\|x\|$. The reason for this is that $h(-x)=-h(x) \leq K\|-x\|=K\|x\|$ and therefore, $h(x) \geq-K\|x\|$. Let

$$
F(x) \equiv h(x)-i h(i x) .
$$

By arguments similar to the above, $F$ is linear.

$$
\begin{aligned}
F(i x) & =h(i x)-i h(-x) \\
& =\operatorname{ih}(x)+h(i x) \\
& =i(h(x)-i h(i x))=i F(x)
\end{aligned}
$$

If $c$ is a real scalar,

$$
\begin{aligned}
F(c x) & =h(c x)-i h(i c x) \\
& =\operatorname{ch}(x)-\operatorname{cih}(i x)=c F(x)
\end{aligned}
$$

Now

$$
\begin{aligned}
F(x+y) & =h(x+y)-i h(i(x+y)) \\
& =h(x)+h(y)-i h(i x)-i h(i y) \\
& =F(x)+F(y)
\end{aligned}
$$

Thus

$$
\begin{aligned}
F((a+i b) x) & =F(a x)+F(i b x) \\
& =a F(x)+i b F(x) \\
& =(a+i b) F(x) .
\end{aligned}
$$

This shows $F$ is linear as claimed.
Now $w F(x)=|F(x)|$ for some $|w|=1$. Therefore

$$
\begin{aligned}
|F(x)| & =w F(x)=h(w x)-\overbrace{i h(i w x)}^{\text {must equal zero }}=h(w x) \\
& =|h(w x)| \leq K\|w x\|=K\|x\| .
\end{aligned}
$$

This proves the corollary.

### 17.2.4 The Dual Space And Adjoint Operators

Definition 17.2.7 Let $X$ be a Banach space. Denote by $X^{\prime}$ the space of continuous linear functions which map $X$ to the field of scalars. Thus $X^{\prime}=\mathscr{L}(X, \mathbb{F})$. By Theorem 17.1.7 on Page 438, $X^{\prime}$ is a Banach space. Remember with the norm defined on $\mathscr{L}(X, \mathbb{F})$,

$$
\|f\|=\sup \{|f(x)|:\|x\| \leq 1\}
$$

$X^{\prime}$ is called the dual space.
Definition 17.2.8 Let $X$ and $Y$ be Banach spaces and suppose $L \in \mathscr{L}(X, Y)$. Then define the adjoint map in $\mathscr{L}\left(Y^{\prime}, X^{\prime}\right)$, denoted by $L^{*}$, by

$$
L^{*} y^{*}(x) \equiv y^{*}(L x)
$$

for all $y^{*} \in Y^{\prime}$.
The following diagram is a good one to help remember this definition.


This is a generalization of the adjoint of a linear transformation on an inner product space. Recall

$$
(A x, y)=\left(x, A^{*} y\right)
$$

What is being done here is to generalize this algebraic concept to arbitrary Banach spaces. There are some issues which need to be discussed relative to the above definition. First of all, it must be shown that $L^{*} y^{*} \in X^{\prime}$. Also, it will be useful to have the following lemma which is a useful application of the Hahn Banach theorem.

Lemma 17.2.9 Let $X$ be a normed linear space and let $x \in X \backslash V$ where $V$ is a closed subspace of $X$. Then there exists $x^{*} \in X^{\prime}$ such that $x^{*}(x)=\|x\|, x^{*}(V)=\{0\}$, and

$$
\left\|x^{*}\right\| \leq \frac{1}{\operatorname{dist}(x, V)}
$$

In the case that $V=\{0\},\left\|x^{*}\right\|=1$.
Proof: Let $f: \mathbb{F} x+V \rightarrow \mathbb{F}$ be defined by $f(\alpha x+v)=\alpha\|x\|$. First it is necessary to show $f$ is well defined and continuous. If $\alpha_{1} x+v_{1}=\alpha_{2} x+v_{2}$ then if $\alpha_{1} \neq \alpha_{2}$, then $x \in V$ which is assumed not to happen so $f$ is well defined. It remains to show $f$ is continuous. Suppose then that $\alpha_{n} x+v_{n} \rightarrow 0$. It is necessary to show $\alpha_{n} \rightarrow 0$. If this does not happen, then there exists a subsequence, still denoted by $\alpha_{n}$ such that $\left|\alpha_{n}\right| \geq \delta>0$. Then $x+\left(1 / \alpha_{n}\right) v_{n} \rightarrow 0$ contradicting the assumption that $x \notin V$ and $V$ is a closed subspace. Hence $f$ is continuous on $\mathbb{F} x+V$. Being a little more careful,

$$
\|f\|=\sup _{\|\alpha x+\nu\| \leq 1}|f(\alpha x+v)|=\sup _{|\alpha|\|x+(v / \alpha)\| \leq 1}|\alpha|\|x\|=\frac{1}{\operatorname{dist}(x, V)}\|x\|
$$

By the Hahn Banach theorem, there exists $x^{*} \in X^{\prime}$ such that $x^{*}=f$ on $\mathbb{F} x+V$. Thus $x^{*}(x)=$ $\|x\|$ and also

$$
\left\|x^{*}\right\| \leq\|f\|=\frac{1}{\operatorname{dist}(x, V)}
$$

In case $V=\{0\}$, the result follows from the above or alternatively,

$$
\|f\| \equiv \sup _{\|\alpha x\| \leq 1}|f(\alpha x)|=\sup _{|\alpha| \leq 1 /\|x\|}|\alpha|\|x\|=1
$$

and so, in this case, $\left\|x^{*}\right\| \leq\|f\|=1$. Since $x^{*}(x)=\|x\|$ it follows

$$
\left\|x^{*}\right\| \geq\left|x^{*}\left(\frac{x}{\|x\|}\right)\right|=\frac{\|x\|}{\|x\|}=1
$$

Thus $\left\|x^{*}\right\|=1$ and this proves the lemma.
Theorem 17.2.10 Let $L \in \mathscr{L}(X, Y)$ where $X$ and $Y$ are Banach spaces. Then
a.) $L^{*} \in \mathscr{L}\left(Y^{\prime}, X^{\prime}\right)$ as claimed and $\left\|L^{*}\right\|=\|L\|$.
b.) If $L$ maps one to one onto a closed subspace of $Y$, then $L^{*}$ is onto.
c.) If $L$ maps onto a dense subset of $Y$, then $L^{*}$ is one to one.

Proof: It is routine to verify $L^{*} y^{*}$ and $L^{*}$ are both linear. This follows immediately from the definition. As usual, the interesting thing concerns continuity.

$$
\left\|L^{*} y^{*}\right\|=\sup _{\|x\| \leq 1}\left|L^{*} y^{*}(x)\right|=\sup _{\|x\| \leq 1}\left|y^{*}(L x)\right| \leq\left\|y^{*}\right\|\|L\| .
$$

Thus $L^{*}$ is continuous as claimed and $\left\|L^{*}\right\| \leq\|L\|$.
By Lemma 17.2.9, there exists $y_{x}^{*} \in Y^{\prime}$ such that $\left\|y_{x}^{*}\right\|=1$ and $y_{x}^{*}(L x)=\|L x\|$.Therefore,

$$
\begin{aligned}
\left\|L^{*}\right\| & =\sup _{\left\|y^{*}\right\| \leq 1}\left\|L^{*} y^{*}\right\|=\sup _{\left\|y^{*}\right\| \leq 1\|x\| \leq 1} \sup \left|L^{*} y^{*}(x)\right| \\
& =\sup _{\left\|y^{*}\right\| \leq 1\|x\| \mid \leq 1} \sup \left|y^{*}(L x)\right|=\sup _{\|x\| \leq 1\left\|y^{*}\right\| \leq 1} \sup \left|y^{*}(L x)\right| \\
& \geq \sup _{\|x\| \leq 1}\left|y_{x}^{*}(L x)\right|=\sup _{\|x\| \leq 1}\|L x\|=\|L\|
\end{aligned}
$$

showing that $\left\|L^{*}\right\| \geq\|L\|$ and this shows part a.).
If $L$ is one to one and onto a closed subset of $Y$, then $L(X)$ being a closed subspace of a Banach space, is itself a Banach space and so the open mapping theorem implies $L^{-1}: L(X) \rightarrow X$ is continuous. Hence

$$
\|x\|=\left\|L^{-1} L x\right\| \leq\left\|L^{-1}\right\|\|L x\|
$$

Now let $x^{*} \in X^{\prime}$ be given. Define $f \in \mathscr{L}(L(X), \mathbb{C})$ by $f(L x)=x^{*}(x)$. The function, $f$ is well defined because if $L x_{1}=L x_{2}$, then since $L$ is one to one, it follows $x_{1}=x_{2}$ and so
$f\left(L\left(x_{1}\right)\right)=x^{*}\left(x_{1}\right)=x^{*}\left(x_{2}\right)=f\left(L\left(x_{1}\right)\right)$. Also, $f$ is linear because

$$
\begin{aligned}
f\left(a L\left(x_{1}\right)+b L\left(x_{2}\right)\right) & =f\left(L\left(a x_{1}+b x_{2}\right)\right) \\
& \equiv x^{*}\left(a x_{1}+b x_{2}\right) \\
& =a x^{*}\left(x_{1}\right)+b x^{*}\left(x_{2}\right) \\
& =a f\left(L\left(x_{1}\right)\right)+b f\left(L\left(x_{2}\right)\right) .
\end{aligned}
$$

In addition to this,

$$
|f(L x)|=\left|x^{*}(x)\right| \leq\left\|x^{*}\right\|\|x\| \leq\left\|x^{*}\right\|\left\|L^{-1}\right\|\|L x\|
$$

and so the norm of $f$ on $L(X)$ is no larger than $\left\|x^{*}\right\|\left\|L^{-1}\right\|$. By the Hahn Banach theorem, there exists an extension of $f$ to an element $y^{*} \in Y^{\prime}$ such that $\left\|y^{*}\right\| \leq\left\|x^{*}\right\|\left\|L^{-1}\right\|$. Then

$$
L^{*} y^{*}(x)=y^{*}(L x)=f(L x)=x^{*}(x)
$$

so $L^{*} y^{*}=x^{*}$ because this holds for all $x$. Since $x^{*}$ was arbitrary, this shows $L^{*}$ is onto and proves b.).

Consider the last assertion. Suppose $L^{*} y^{*}=0$. Is $y^{*}=0$ ? In other words is $y^{*}(y)=0$ for all $y \in Y$ ? Pick $y \in Y$. Since $L(X)$ is dense in $Y$, there exists a sequence, $\left\{L x_{n}\right\}$ such that $L x_{n} \rightarrow y$. But then by continuity of $y^{*}, y^{*}(y)=\lim _{n \rightarrow \infty} y^{*}\left(L x_{n}\right)=\lim _{n \rightarrow \infty} L^{*} y^{*}\left(x_{n}\right)=0$. Since $y^{*}(y)=0$ for all $y$, this implies $y^{*}=0$ and so $L^{*}$ is one to one.

Corollary 17.2.11 Suppose $X$ and $Y$ are Banach spaces, $L \in \mathscr{L}(X, Y)$, and $L$ is one to one and onto. Then $L^{*}$ is also one to one and onto.

There exists a natural mapping, called the James map from a normed linear space, $X$, to the dual of the dual space which is described in the following definition.

Definition 17.2.12 Define $J: X \rightarrow X^{\prime \prime}$ by $J(x)\left(x^{*}\right)=x^{*}(x)$.
Theorem 17.2.13 The map, J, has the following properties.
a.) $J$ is one to one and linear.
b.) $\|J x\|=\|x\|$ and $\|J\|=1$.
c.) $J(X)$ is a closed subspace of $X^{\prime \prime}$ if $X$ is complete.

Also if $x^{*} \in X^{\prime}$,

$$
\left\|x^{*}\right\|=\sup \left\{\left|x^{* *}\left(x^{*}\right)\right|:\left\|x^{* *}\right\| \leq 1, x^{* *} \in X^{\prime \prime}\right\} .
$$

Proof:

$$
\begin{aligned}
J(a x+b y)\left(x^{*}\right) & \equiv x^{*}(a x+b y) \\
& =a x^{*}(x)+b x^{*}(y) \\
& =(a J(x)+b J(y))\left(x^{*}\right)
\end{aligned}
$$

Since this holds for all $x^{*} \in X^{\prime}$, it follows that

$$
J(a x+b y)=a J(x)+b J(y)
$$

and so $J$ is linear. If $J x=0$, then by Lemma 17.2.9 there exists $x^{*}$ such that $x^{*}(x)=\|x\|$ and $\left\|x^{*}\right\|=1$. Then

$$
0=J(x)\left(x^{*}\right)=x^{*}(x)=\|x\| .
$$

This shows a.).
To show b.), let $x \in X$ and use Lemma 17.2.9 to obtain $x^{*} \in X^{\prime}$ such that $x^{*}(x)=\|x\|$ with $\left\|x^{*}\right\|=1$. Then

$$
\begin{aligned}
\|x\| & \geq \sup \left\{\left|y^{*}(x)\right|:\left\|y^{*}\right\| \leq 1\right\} \\
& =\sup \left\{\left|J(x)\left(y^{*}\right)\right|:\left\|y^{*}\right\| \leq 1\right\}=\|J x\| \\
& \geq\left|J(x)\left(x^{*}\right)\right|=\left|x^{*}(x)\right|=\|x\|
\end{aligned}
$$

Therefore, $\|J x\|=\|x\|$ as claimed. Therefore,

$$
\|J\|=\sup \{\|J x\|:\|x\| \leq 1\}=\sup \{\|x\|:\|x\| \leq 1\}=1
$$

This shows b.).
To verify c.), use b.). If $J x_{n} \rightarrow y^{* *} \in X^{\prime \prime}$ then by b.), $x_{n}$ is a Cauchy sequence converging to some $x \in X$ because

$$
\left\|x_{n}-x_{m}\right\|=\left\|J x_{n}-J x_{m}\right\|
$$

and $\left\{J x_{n}\right\}$ is a Cauchy sequence. Then $J x=\lim _{n \rightarrow \infty} J x_{n}=y^{* *}$.
Finally, to show the assertion about the norm of $x^{*}$, use what was just shown applied to the James map from $X^{\prime}$ to $X^{\prime \prime \prime}$ still referred to as $J$.

$$
\begin{gathered}
\left\|x^{*}\right\|=\sup \left\{\left|x^{*}(x)\right|:\|x\| \leq 1\right\}=\sup \left\{\left|J(x)\left(x^{*}\right)\right|:\|J x\| \leq 1\right\} \\
\leq \sup \left\{\left|x^{* *}\left(x^{*}\right)\right|: \mid\left\|x^{* *}\right\| \leq 1\right\}=\sup \left\{\left|J\left(x^{*}\right)\left(x^{* *}\right)\right|:\left\|x^{* *}\right\| \leq 1\right\} \\
\equiv\left\|J x^{*}\right\|=\left\|x^{*}\right\| .
\end{gathered}
$$

This proves the theorem.
Definition 17.2.14 When J maps $X$ onto $X^{\prime \prime}, X$ is called reflexive.
It happens the $L^{p}$ spaces are reflexive whenever $p>1$. This is shown later.

### 17.3 Uniform Convexity Of $L^{p}$

These terms refer roughly to how round the unit ball is. Here is the definition.
Definition 17.3.1 A Banach space is uniformly convex if whenever $\left\|x_{n}\right\|,\left\|y_{n}\right\| \leq 1$ and $\left\|x_{n}+y_{n}\right\| \rightarrow 2$, it follows that $\left\|x_{n}-y_{n}\right\| \rightarrow 0$.

You can show that uniform convexity implies strict convexity. There are various other things which can also be shown. See the exercises for some of these. In this section, it will be shown that the $L^{p}$ spaces are examples of uniformly convex spaces. This involves some inequalities known as Clarkson's inequalities. Before presenting these, here are the backwards Holder inequality and the backwards Minkowski inequality.

Lemma 17.3.2 Let $0<p<1$ and let $f, g$ be measurable functions. Also

$$
\int_{\Omega}|g|^{p /(p-1)} d \mu<\infty, \int_{\Omega}|f|^{p} d \mu<\infty
$$

Then the following backwards Holder inequality holds.

$$
\int_{\Omega}|f g| d \mu \geq\left(\int_{\Omega}|f|^{p} d \mu\right)^{1 / p}\left(\int_{\Omega}|g|^{p /(p-1)} d \mu\right)^{(p-1) / p}
$$

Proof: If $\int|f g| d \mu=\infty$, there is nothing to prove. Hence assume this is finite. Then

$$
\int|f|^{p} d \mu=\int|g|^{-p}|f g|^{p} d \mu
$$

This makes sense because, due to the hypothesis on $g$ it must be the case that $g$ equals 0 only on a set of measure zero, since $p /(p-1)<0$. Then

$$
\begin{aligned}
\int|f|^{p} d \mu & \leq\left(\int|f g| d \mu\right)^{p}\left(\int\left(\frac{1}{|g|^{p}}\right)^{1 /(1-p)} d \mu\right)^{1-p} \\
& =\left(\int|f g| d \mu\right)^{p}\left(\int|g|^{p / p-1} d \mu\right)^{1-p}
\end{aligned}
$$

Now divide and then take the $p^{t h}$ root.
Here is the backwards Minkowski inequality.
Corollary 17.3.3 Let $0<p<1$ and suppose $\int|h|^{p} d \mu<\infty$ for $h=f, g$. Then

$$
\left(\int(|f|+|g|)^{p} d \mu\right)^{1 / p} \geq\left(\int|f|^{p} d \mu\right)^{1 / p}+\left(\int|g|^{p} d \mu\right)^{1 / p}
$$

Proof: If $\int(|f|+|g|)^{p} d \mu=0$ then there is nothing to prove so assume this is not zero.

$$
\int(|f|+|g|)^{p} d \mu=\int(|f|+|g|)^{p-1}(|f|+|g|) d \mu
$$

$(|f|+|g|)^{p} \leq|f|^{p}+|g|^{p}$ and so

$$
\int\left((|f|+|g|)^{p-1}\right)^{p / p-1} d \mu<\infty
$$

Hence the backward Holder inequality applies and it follows that

$$
\begin{aligned}
& \int(|f|+|g|)^{p} d \mu=\int(|f|+|g|)^{p-1}|f| d \mu+\int(|f|+|g|)^{p-1}|g| d \mu \\
\geq & \left(\int\left((|f|+|g|)^{p-1}\right)^{p / p-1}\right)^{(p-1) / p}\left[\left(\int|f|^{p} d \mu\right)^{1 / p}+\left(\int|g|^{p} d \mu\right)^{1 / p}\right] \\
= & \left(\int(|f|+|g|)^{p}\right)^{(p-1) / p}\left[\left(\int|f|^{p} d \mu\right)^{1 / p}+\left(\int|g|^{p} d \mu\right)^{1 / p}\right]
\end{aligned}
$$

and so, dividing gives the desired inequality.
Consider the easy Clarkson inequalities.

Lemma 17.3.4 For any $p \geq 2$ the following inequality holds for any $t \in[0,1]$,

$$
\left|\frac{1+t}{2}\right|^{p}+\left|\frac{1-t}{2}\right|^{p} \leq \frac{1}{2}\left(|t|^{p}+1\right)
$$

Proof: It is clear that, since $p \geq 2$, the inequality holds for $t=0$ and $t=1$.Thus it suffices to consider only $t \in(0,1)$. Let $x=1 / t$. Then, dividing by $1 / t^{p}$, the inequality holds if and only if

$$
\left(\frac{x+1}{2}\right)^{p}+\left(\frac{x-1}{2}\right)^{p} \leq \frac{1}{2}\left(1+x^{p}\right)
$$

for all $x \geq 1$. Let

$$
f(x)=\frac{1}{2}\left(1+x^{p}\right)-\left(\left(\frac{x+1}{2}\right)^{p}+\left(\frac{x-1}{2}\right)^{p}\right)
$$

Then $f(1)=0$ and

$$
f^{\prime}(x)=\frac{p}{2} x^{p-1}-\left(\frac{p}{2}\left(\frac{x+1}{2}\right)^{p-1}+\frac{p}{2}\left(\frac{x-1}{2}\right)^{p-1}\right)
$$

Since $p-1 \geq 1$, by convexity of $f(x)=x^{p-1}$,

$$
f^{\prime}(x) \geq \frac{p}{2} x^{p-1}-p\left(\frac{\frac{x+1}{2}+\frac{x-1}{2}}{2}\right)^{p-1}=\frac{p}{2} x^{p-1}-p\left(\frac{x}{2}\right)^{p-1} \geq 0
$$

Hence $f(x) \geq 0$ for all $x \geq 1$.
Corollary 17.3.5 If $z, w \in \mathbb{C}$ and $p \geq 2$, then

$$
\begin{equation*}
\left|\frac{z+w}{2}\right|^{p}+\left|\frac{z-w}{2}\right|^{p} \leq \frac{1}{2}\left(|z|^{p}+|w|^{p}\right) \tag{17.3.7}
\end{equation*}
$$

Proof: One of $|w|,|z|$ is larger. Say $|z| \geq|w|$. Then dividing both sides of the proposed inequality by $|z|^{p}$ it suffices to verify that for all complex $t$ having $|t| \leq 1$,

$$
\left|\frac{1+t}{2}\right|^{p}+\left|\frac{1-t}{2}\right|^{p} \leq \frac{1}{2}\left(|t|^{p}+1\right)
$$

Say $t=r e^{i \theta}$ where $r \leq 1$.Then consider the expression

$$
\left|\frac{1+r e^{i \theta}}{2}\right|^{p}+\left|\frac{1-r e^{i \theta}}{2}\right|^{p}
$$

It is $2^{-p}$ times

$$
\begin{aligned}
& \left((1+r \cos \theta)^{2}+r^{2} \sin ^{2}(\theta)\right)^{p / 2}+\left((1-r \cos \theta)^{2}+r^{2} \sin ^{2}(\theta)\right)^{p / 2} \\
= & \left(1+r^{2}+2 r \cos \theta\right)^{p / 2}+\left(1+r^{2}-2 r \cos \theta\right)^{p / 2}
\end{aligned}
$$

a continuous periodic function for $\theta \in \mathbb{R}$ which achieves its maximum value when $\theta=0$. This follows from the first derivative test from calculus. Therefore, for $|t| \leq 1$,

$$
\left|\frac{1+t}{2}\right|^{p}+\left|\frac{1-t}{2}\right|^{p} \leq\left|\frac{1+|t|}{2}\right|^{p}+\left|\frac{1-|t|}{2}\right|^{p} \leq \frac{1}{2}\left(1+|t|^{p}\right)
$$

by the above lemma.
With this corollary, here is the easy Clarkson inequality.
Theorem 17.3.6 Let $p \geq 2$. Then

$$
\left\|\frac{f+g}{2}\right\|_{L^{p}}^{p}+\left\|\frac{f-g}{2}\right\|_{L^{p}}^{p} \leq \frac{1}{2}\left(\|f\|_{L^{p}}^{p}+\|g\|_{L^{p}}^{p}\right)
$$

Proof: This follows right away from the above corollary.

$$
\int_{\Omega}\left|\frac{f+g}{2}\right|^{p} d \mu+\int_{\Omega}\left|\frac{f-g}{2}\right|^{p} d \mu \leq \frac{1}{2} \int_{\Omega}\left(|f|^{p}+|g|^{p}\right) d \mu
$$

Now it remains to consider the hard Clarkson inequalities. These pertain to $p<2$. First is the following elementary inequality.

Lemma 17.3.7 For $1<p<2$, the following inequality holds for all $t \in[0,1]$.

$$
\left|\frac{1+t}{2}\right|^{q}+\left|\frac{1-t}{2}\right|^{q} \leq\left(\frac{1}{2}+\frac{1}{2}|t|^{p}\right)^{q / p}
$$

where here $1 / p+1 / q=1$ so $q>2$.
Proof: First note that if $t=0$ or 1 , the inequality holds. Next observe that the map $s \rightarrow \frac{1-s}{1+s}$ maps $(0,1)$ onto $(0,1)$. Replace $t$ with $(1-s) /(1+s)$. Then you get

$$
\left|\frac{1}{s+1}\right|^{q}+\left|\frac{s}{s+1}\right|^{q} \leq\left(\frac{1}{2}+\frac{1}{2}\left|\frac{1-s}{s+1}\right|^{p}\right)^{q / p}
$$

Multiplying both sides by $(1+s)^{q}$, this is equivalent to showing that for all $s \in(0,1)$,

$$
\begin{aligned}
1+s^{q} & \leq\left((1+s)^{p}\right)^{q / p}\left(\frac{1}{2}+\frac{1}{2}\left|\frac{1-s}{s+1}\right|^{p}\right)^{q / p} \\
& =\left(\frac{1}{2}\right)^{q / p}\left((1+s)^{p}+(1-s)^{p}\right)^{q / p}
\end{aligned}
$$

This is the same as establishing

$$
\begin{equation*}
\frac{1}{2}\left((1+s)^{p}+(1-s)^{p}\right)-\left(1+s^{q}\right)^{p-1} \geq 0 \tag{17.3.8}
\end{equation*}
$$

where $p-1=p / q$ due to the definition of $q$ above.

$$
\binom{p}{l} \equiv \frac{p(p-1) \cdots(p-k+1)}{l!}, l \geq 1
$$

and $\binom{p}{0} \equiv 1$. What is the sign of $\binom{p}{l}$ ? Recall that $1<p<2$ so the sign is positive if $l=0, l=1, l=2$. What about $l=3$ ? $\binom{p}{3}=\frac{p(p-1)(p-2)}{3!}$ so this is negative. Then $\binom{p}{4}$ is positive. Thus these alternate between positive and negative with $\binom{p}{2 k}>0$ for all $k$. What about $\binom{p-1}{k}$ ? When $k=0$ it is positive. When $k=1$ it is also positive. When $k=2$ it equals $\frac{(p-1)(p-2)}{2!}<0$. Then when $k=3,\binom{p-1}{3}>0$. Thus $\binom{p-1}{k}$ is positive when $k$ is odd and is negative when $k$ is even.

Now return to 17.3.8. The left side equals

$$
\frac{1}{2}\left(\sum_{k=0}^{\infty}\binom{p}{k} s^{k}+\sum_{k=0}^{\infty}\binom{p}{k}(-s)^{k}\right)-\sum_{k=0}^{\infty}\binom{p-1}{k} s^{q k}
$$

The first term equals 0 . Then this reduces to

$$
\sum_{k=1}^{\infty}\binom{p}{2 k} s^{2 k}-\binom{p-1}{2 k} s^{q 2 k}-\binom{p-1}{2 k-1} s^{q(2 k-1)}
$$

From the above observation about the binomial coefficients, the above is larger than

$$
\sum_{k=1}^{\infty}\binom{p}{2 k} s^{2 k}-\binom{p-1}{2 k-1} s^{q(2 k-1)}
$$

It remains to show the $k^{t h}$ term in the above sum is nonnegative. Now $q(2 k-1)>2 k$ for all $k \geq 1$ because $q>2$. Then since $0<s<1$

$$
\binom{p}{2 k} s^{2 k}-\binom{p-1}{2 k-1} s^{q(2 k-1)} \geq s^{2 k}\left(\binom{p}{2 k}-\binom{p-1}{2 k-1}\right)
$$

However, this is nonnegative because it equals

$$
\begin{aligned}
& s^{2 k}(\frac{p(p-1) \cdots(p-2 k+1)}{(2 k)!}-\frac{\overbrace{(p-1)(p-2) \cdots(p-2 k+1)}}{>0}) \\
\geq & s^{2 k}\left(\frac{p(p-1)!}{(2 k)!}\right) \\
= & s^{2 k} \frac{(p-1)(p-2) \cdots(p-2 k+1)}{(2 k)!}(p-1)>0 .
\end{aligned}
$$

As before, this leads to the following corollary.

Corollary 17.3.8 Let $z, w \in \mathbb{C}$. Then for $p \in(1,2)$,

$$
\left|\frac{z+w}{2}\right|^{q}+\left|\frac{z-w}{2}\right|^{q} \leq\left(\frac{1}{2}|z|^{p}+\frac{1}{2}|w|^{p}\right)^{q / p}
$$

Proof: One of $|w|,|z|$ is larger. Say $|w| \geq|z|$. Then dividing by $|w|^{q}$, for $t=z / w$, showing the above inequality is equivalent to showing that for all $t \in \mathbb{C},|t| \leq 1$,

$$
\left|\frac{t+1}{2}\right|^{q}+\left|\frac{1-t}{2}\right|^{q} \leq\left(\frac{1}{2}|t|^{p}+\frac{1}{2}\right)^{q / p}
$$

Now $q>2$ and so by the same argument given in proving Corollary 17.3.5, for $t=r e^{i \theta}$, the left side of the above inequality is maximized when $\theta=0$. Hence, from Lemma 17.3.7,

$$
\begin{aligned}
\left|\frac{t+1}{2}\right|^{q}+ & \left|\frac{1-t}{2}\right|^{q} \leq\left|\frac{|t|+1}{2}\right|^{q}+\left|\frac{1-|t|}{2}\right|^{q} \\
& \leq\left(\frac{1}{2}|t|^{p}+\frac{1}{2}\right)^{q / p} .
\end{aligned}
$$

From this the hard Clarkson inequality follows. The two Clarkson inequalities are summarized in the following theorem.

Theorem 17.3.9 Let $2 \leq p$. Then

$$
\left\|\frac{f+g}{2}\right\|_{L^{p}}^{p}+\left\|\frac{f-g}{2}\right\|_{L^{p}}^{p} \leq \frac{1}{2}\left(\|f\|_{L^{p}}^{p}+\|g\|_{L^{p}}^{p}\right)
$$

Let $1<p<2$. Then for $1 / p+1 / q=1$,

$$
\left\|\frac{f+g}{2}\right\|_{L^{p}}^{q}+\left\|\frac{f-g}{2}\right\|_{L^{p}}^{q} \leq\left(\frac{1}{2}\|f\|_{L^{p}}^{p}+\frac{1}{2}\|g\|_{L^{p}}^{p}\right)^{q / p}
$$

Proof: The first was established above.

$$
\begin{gathered}
\left\|\frac{f+g}{2}\right\|_{L^{p}}^{q}+\left\|\frac{f-g}{2}\right\|_{L^{p}}^{q} \leq \\
\left(\int_{\Omega}\left|\frac{f+g}{2}\right|^{p} d \mu\right)^{q / p}+\left(\int_{\Omega}\left|\frac{f-g}{2}\right|^{p} d \mu\right)^{q / p} \\
=\left(\int_{\Omega}\left(\left|\frac{f+g}{2}\right|^{q}\right)^{p / q} d \mu\right)^{q / p}+\left(\int_{\Omega}\left(\left|\frac{f-g}{2}\right|^{q}\right)^{p / q} d \mu\right)^{q / p}
\end{gathered}
$$

Now $p / q<1$ and so the backwards Minkowski inequality applies. Thus

$$
\leq\left(\int_{\Omega}\left(\left|\frac{f+g}{2}\right|^{q}+\left|\frac{f-g}{2}\right|^{q}\right)^{p / q} d \mu\right)^{q / p}
$$

From Corollary 17.3.8,

$$
\begin{aligned}
& \leq\left(\int_{\Omega}\left(\left(\frac{1}{2}|f|^{p}+\frac{1}{2}|g|^{p}\right)^{q / p}\right)^{p / q} d \mu\right)^{q / p} \\
& =\left(\int_{\Omega}\left(\frac{1}{2}|f|^{p}+\frac{1}{2}|g|^{p}\right) d \mu\right)^{q / p}=\left(\frac{1}{2}\|f\|_{L^{p}}^{p}+\frac{1}{2}\|g\|_{L^{p}}^{p}\right)^{q / p}
\end{aligned}
$$

Now with these Clarkson inequalities, it is not hard to show that all the $L^{p}$ spaces are uniformly convex.

Theorem 17.3.10 The $L^{p}$ spaces are uniformly convex.
Proof: First suppose $p \geq 2$. Suppose $\left\|f_{n}\right\|_{L^{p}},\left\|g_{n}\right\|_{L^{p}} \leq 1$ and $\left\|\frac{f_{n}+g_{n}}{2}\right\|_{L^{p}} \rightarrow 1$. Then from the first Clarkson inequality,

$$
\left\|\frac{f_{n}+g_{n}}{2}\right\|_{L^{p}}^{p}+\left\|\frac{f_{n}-g_{n}}{2}\right\|_{L^{p}}^{p} \leq \frac{1}{2}\left(\left\|f_{n}\right\|_{L^{p}}^{p}+\left\|g_{n}\right\|_{L^{p}}^{p}\right) \leq 1
$$

and so $\left\|f_{n}-g_{n}\right\|_{L^{p}} \rightarrow 0$.
Next suppose $1<p<2$ and $\left\|\frac{f_{n}+g_{n}}{2}\right\|_{L^{p}} \rightarrow 1$. Then from the second Clarkson inequality

$$
\left\|\frac{f_{n}+g_{n}}{2}\right\|_{L^{p}}^{q}+\left\|\frac{f_{n}-g_{n}}{2}\right\|_{L^{p}}^{q} \leq\left(\frac{1}{2}\left\|f_{n}\right\|_{L^{p}}^{p}+\frac{1}{2}\left\|g_{n}\right\|_{L^{p}}^{p}\right)^{q / p} \leq 1
$$

which shows that $\left\|f_{n}-g_{n}\right\|_{L^{p}} \rightarrow 0$.

### 17.4 Closed Subspaces

Theorem 17.4.1 Let $X$ be a Banach space and let $V=\operatorname{span}\left(x_{1}, \cdots, x_{n}\right)$. Then $V$ is a closed subspace of $X$.

Proof: Without loss of generality, it can be assumed $\left\{x_{1}, \cdots, x_{n}\right\}$ is linearly independent. Otherwise, delete those vectors which are in the span of the others till a linearly independent set is obtained. Let

$$
\begin{equation*}
x=\lim _{p \rightarrow \infty} \sum_{k=1}^{n} c_{k}^{p} x_{k} \in \bar{V} \tag{17.4.9}
\end{equation*}
$$

First suppose $\mathbf{c}^{p} \equiv\left(c_{1}^{p}, \cdots, c_{n}^{p}\right)$ is not bounded in $\mathbb{F}^{n}$. Then $\mathbf{d}^{p} \equiv \mathbf{c}^{p} /\left|\mathbf{c}^{p}\right|_{\mathbb{F}^{n}}$ is a unit vector in $\mathbb{F}^{n}$ and so there exists a subsequence, still denoted by $\mathbf{d}^{p}$ which converges to $\mathbf{d}$ where $|\mathbf{d}|=1$. Then

$$
\mathbf{0}=\lim _{p \rightarrow \infty} \frac{x}{\left\|\mathbf{c}^{p}\right\|}=\lim _{p \rightarrow \infty} \sum_{k=1}^{n} d_{k}^{p} x_{k}=\sum_{k=1}^{n} d_{k} x_{k}
$$

where $\sum_{k}\left|d_{k}\right|^{2}=1$ in contradiction to the linear independence of the $\left\{x_{1}, \cdots, x_{n}\right\}$. Hence it must be the case that $\mathbf{c}^{p}$ is bounded in $\mathbb{F}^{n}$. Then taking a subsequence, still denoted as $p$, it can be assumed $\mathbf{c}^{p} \rightarrow \mathbf{c}$ and then in 17.4.9 it follows

$$
x=\sum_{k=1}^{n} c_{k} x_{k} \in \operatorname{span}\left(x_{1}, \cdots, x_{n}\right)
$$

Proposition 17.4.2 Let $E$ be a separable Banach space. Then there exists an increasing sequence of subspaces, $\left\{F_{n}\right\}$ such that $\operatorname{dim}\left(F_{n+1}\right)-\operatorname{dim}\left(F_{n}\right) \leq 1$ and equals 1 for all $n$ if the dimension of $E$ is infinite. Also $\cup_{n=1}^{\infty} F_{n}$ is dense in $E$. In the case where $E$ is infinite dimensional, $F_{n}=\operatorname{span}\left(e_{1}, \cdots, e_{n}\right)$ where for each $n$

$$
\begin{equation*}
\operatorname{dist}\left(e_{n+1}, F_{n}\right) \geq \frac{1}{2} \tag{17.4.10}
\end{equation*}
$$

and defining,

$$
\begin{gather*}
G_{k} \equiv \operatorname{span}\left(\left\{e_{j}: j \neq k\right\}\right) \\
\operatorname{dist}\left(e_{k}, G_{k}\right) \geq \frac{1}{4} \tag{17.4.11}
\end{gather*}
$$

Proof: Since $E$ is separable, so is $\partial B(0,1)$, the boundary of the unit ball. Let $\left\{w_{k}\right\}_{k=1}^{\infty}$ be a countable dense subset of $\partial B(0,1)$.

Let $e_{1}=w_{1}$. Let $F_{1}=\mathbb{F} e_{1}$. Suppose $F_{n}$ has been obtained and equals span $\left(e_{1}, \cdots, e_{n}\right)$ where $\left\{e_{1}, \cdots, e_{n}\right\}$ is independent, $\left\|e_{k}\right\|=1$, and

$$
\operatorname{dist}\left(e_{n}, \operatorname{span}\left(e_{1}, \cdots, e_{n-1}\right)\right) \geq \frac{1}{2}
$$

For each $n, F_{n}$ is closed by Theorem 17.4.1.
If $F_{n}$ contains $\left\{w_{k}\right\}_{k=1}^{\infty}$, let $F_{m}=F_{n}$ for all $m>n$. Otherwise, pick $w \in\left\{w_{k}\right\}$ to be the point of $\left\{w_{k}\right\}_{k=1}^{\infty}$ having the smallest subscript which is not contained in $F_{n}$. Then $w$ is at a positive distance, $\lambda$ from $F_{n}$ because $F_{n}$ is closed. Therefore, there exists $y \in F_{n}$ such that $\lambda \leq\|y-w\| \leq 2 \lambda$. Let $e_{n+1}=\frac{w-y}{\|w-y\|}$. It follows

$$
w=\|w-y\| e_{n+1}+y \in \operatorname{span}\left(e_{1}, \cdots, e_{n+1}\right) \equiv F_{n+1}
$$

Then if $x \in \operatorname{span}\left(e_{1}, \cdots, e_{n}\right)$,

$$
\begin{aligned}
\left\|e_{n+1}-x\right\| & =\left\|\frac{w-y}{\|w-y\|}-x\right\| \\
& =\left\|\frac{w-y}{\|w-y\|}-\frac{\|w-y\| x}{\|w-y\|}\right\| \\
& \geq \frac{1}{2 \lambda}\|w-y-\| w-y\|x\| \\
& \geq \frac{\lambda}{2 \lambda}=\frac{1}{2}
\end{aligned}
$$

This has shown the existence of an increasing sequence of subspaces, $\left\{F_{n}\right\}$ as described above. It remains to show the union of these subspaces is dense. First note that the union of these subspaces must contain the $\left\{w_{k}\right\}_{k=1}^{\infty}$ because if $w_{m}$ is missing, then it would contradict the construction at the $m^{\text {th }}$ step. That one should have been chosen. However, $\left\{w_{k}\right\}_{k=1}^{\infty}$ is dense in $\partial B(0,1)$. If $x \in E$ and $x \neq 0$, then $\frac{x}{\|x\|} \in \partial B(0,1)$ then there exists

$$
w_{m} \in\left\{w_{k}\right\}_{k=1}^{\infty} \subseteq \cup_{n=1}^{\infty} F_{n}
$$

such that $\left\|w_{m}-\frac{x}{\|x\|}\right\|<\frac{\varepsilon}{\|x\|}$. But then

$$
\left\|\|x\| w_{m}-x\right\|<\varepsilon
$$

and so $\|x\| w_{m}$ is a point of $\cup_{n=1}^{\infty} F_{n}$ which is within $\varepsilon$ of $x$. This proves $\cup_{n=1}^{\infty} F_{n}$ is dense as desired. 17.4.10 follows from the construction. It remains to verify 17.4.11.

Let $y \in G_{k}$. Thus for some $n$,

$$
y=\sum_{j=1}^{k-1} c_{j} e_{j}+\sum_{j=k+1}^{n} c_{j} e_{j}
$$

and I need to show $\left\|y-e_{k}\right\| \geq 1 / 4$. Without loss of generality, $c_{n} \neq 0$ and $n>k$. Suppose 17.4.11 does not hold for some such $y$ so that

$$
\begin{equation*}
\left\|e_{k}-\left(\sum_{j=1}^{k-1} c_{j} e_{j}+\sum_{j=k+1}^{n} c_{j} e_{j}\right)\right\|<\frac{1}{4} . \tag{17.4.12}
\end{equation*}
$$

Then from the construction,

$$
\begin{aligned}
\frac{1}{4} & >\left|c_{n}\right|\left\|e_{k}-\left(\sum_{j=1}^{k-1}\left(c_{j} / c_{n}\right) e_{j}+\sum_{j=k+1}^{n-1}\left(c_{j} / c_{n}\right) e_{j}+e_{n}\right)\right\| \\
& \geq\left|c_{n}\right| \frac{1}{2}
\end{aligned}
$$

and so $\left|c_{n}\right|<1 / 2$. Consider the left side of 17.4.12. By the construction

$$
\begin{gathered}
\left\|c_{n}\left(e_{k}-e_{n}\right)+\left(1-c_{n}\right) e_{k}-\left(\sum_{j=1}^{k-1} c_{j} e_{j}+\sum_{j=k+1}^{n-1} c_{j} e_{j}\right)\right\| \\
\geq\left|1-c_{n}\right|-\left|c_{n}\right|\left\|\left(e_{k}-e_{n}\right)-\left(\sum_{j=1}^{k-1}\left(c_{j} / c_{n}\right) e_{j}+\sum_{j=k+1}^{n-1}\left(c_{j} / c_{n}\right) e_{j}\right)\right\| \\
\geq\left|1-c_{n}\right|-\left|c_{n}\right| \frac{1}{2} \geq 1-\frac{3}{2}\left|c_{n}\right|>1-\frac{3}{2} \frac{1}{2}=\frac{1}{4}
\end{gathered}
$$

a contradiction. This proves the desired estimate.

### 17.5 Weak And Weak * Topologies

### 17.5.1 Basic Definitions

Let $X$ be a Banach space and let $X^{\prime}$ be its dual space. ${ }^{1}$ For $A^{\prime}$ a finite subset of $X^{\prime}$, denote by $\rho_{A^{\prime}}$ the function defined on $X$

$$
\begin{equation*}
\rho_{A^{\prime}}(x) \equiv \max _{x^{*} \in A^{\prime}}\left|x^{*}(x)\right| \tag{17.5.13}
\end{equation*}
$$

and also let $B_{A^{\prime}}(x, r)$ be defined by

$$
\begin{equation*}
B_{A^{\prime}}(x, r) \equiv\left\{y \in X: \rho_{A^{\prime}}(y-x)<r\right\} \tag{17.5.14}
\end{equation*}
$$

Then certain things are obvious. First of all, if $a \in \mathbb{F}$ and $x, y \in X$,

$$
\begin{aligned}
\rho_{A^{\prime}}(x+y) & \leq \rho_{A^{\prime}}(x)+\rho_{A^{\prime}}(y) \\
\rho_{A^{\prime}}(a x) & =|a| \rho_{A^{\prime}}(x)
\end{aligned}
$$

Similarly, letting $A$ be a finite subset of $X$, denote by $\rho_{A}$ the function defined on $X^{\prime}$

$$
\begin{equation*}
\rho_{A}\left(x^{*}\right) \equiv \max _{x \in A}\left|x^{*}(x)\right| \tag{17.5.15}
\end{equation*}
$$

and let $B_{A}\left(x^{*}, r\right)$ be defined by

$$
\begin{equation*}
B_{A}\left(x^{*}, r\right) \equiv\left\{y^{*} \in X^{\prime}: \rho_{A}\left(y^{*}-x^{*}\right)<r\right\} . \tag{17.5.16}
\end{equation*}
$$

It is also clear that

$$
\begin{aligned}
\rho_{A}\left(x^{*}+y^{*}\right) & \leq \rho\left(x^{*}\right)+\rho_{A}\left(y^{*}\right) \\
\rho_{A}\left(a x^{*}\right) & =|a| \rho_{A}\left(x^{*}\right)
\end{aligned}
$$

Lemma 17.5.1 The sets, $B_{A^{\prime}}(x, r)$ where $A^{\prime}$ is a finite subset of $X^{\prime}$ and $x \in X$ form a basis for a topology on $X$ known as the weak topology. The sets $B_{A}\left(x^{*}, r\right)$ where $A$ is a finite subset of $X$ and $x^{*} \in X^{\prime}$ form a basis for a topology on $X^{\prime}$ known as the weak $*$ topology.

Proof: The two assertions are very similar. I will verify the one for the weak topology. The union of these sets, $B_{A^{\prime}}(x, r)$ for $x \in X$ and $r>0$ is all of $X$. Now suppose $z$ is contained in the intersection of two of these sets. Say

$$
z \in B_{A^{\prime}}(x, r) \cap B_{A_{1}^{\prime}}\left(x_{1}, r_{1}\right)
$$

Then let $C^{\prime}=A^{\prime} \cup A_{1}^{\prime}$ and let

$$
0<\delta \leq \min \left(r-\rho_{A^{\prime}}(z-x), r_{1}-\rho_{A_{1}^{\prime}}\left(z-x_{1}\right)\right)
$$

[^16]Consider $y \in B_{C^{\prime}}(z, \boldsymbol{\delta})$. Then

$$
r-\rho_{A^{\prime}}(z-x) \geq \delta>\rho_{C^{\prime}}(y-z) \geq \rho_{A^{\prime}}(y-z)
$$

and so

$$
r>\rho_{A^{\prime}}(y-z)+\rho_{A^{\prime}}(z-x) \geq \rho_{A^{\prime}}(y-x)
$$

which shows $y \in B_{A^{\prime}}(x, r)$. Similar reasoning shows $y \in B_{A_{1}^{\prime}}\left(x_{1}, r_{1}\right)$ and so

$$
B_{C^{\prime}}(z, \boldsymbol{\delta}) \subseteq B_{A^{\prime}}(x, r) \cap B_{A_{1}^{\prime}}\left(x_{1}, r_{1}\right)
$$

Therefore, the weak topology consists of the union of all sets of the form $B_{A}(x, r)$.

### 17.5.2 Banach Alaoglu Theorem

Why does anyone care about these topologies? The short answer is that in the weak * topology, closed unit ball in $X^{\prime}$ is compact. This is not true in the normal topology. This wonderful result is the Banach Alaoglu theorem. First recall the notion of the product topology, and the Tychonoff theorem, Theorem 14.3.6 on Page 391 which are stated here for convenience.

Definition 17.5.2 Let I be a set and suppose for each $i \in I,\left(X_{i}, \tau_{i}\right)$ is a nonempty topological space. The Cartesian product of the $X_{i}$, denoted by $\prod_{i \in I} X_{i}$, consists of the set of all choice functions defined on I which select a single element of each $X_{i}$. Thus $f \in \prod_{i \in I} X_{i}$ means for every $i \in I, f(i) \in X_{i}$. The axiom of choice says $\prod_{i \in I} X_{i}$ is nonempty. Let

$$
P_{j}(A)=\prod_{i \in I} B_{i}
$$

where $B_{i}=X_{i}$ if $i \neq j$ and $B_{j}=A$. A subbasis for a topology on the product space consists of all sets $P_{j}(A)$ where $A \in \tau_{j}$. (These sets have an open set from the topology of $X_{j}$ in the $j^{\text {th }}$ slot and the whole space in the other slots.) Thus a basis consists of finite intersections of these sets. Note that the intersection of two of these basic sets is another basic set and their union yields $\prod_{i \in I} X_{i}$. Therefore, they satisfy the condition needed for a collection of sets to serve as a basis for a topology. This topology is called the product topology and is denoted by $\Pi \tau_{i}$.

Theorem 17.5.3 If $\left(X_{i}, \tau_{i}\right)$ is compact, then so is $\left(\prod_{i \in I} X_{i}, \Pi \tau_{i}\right)$.
The Banach Alaoglu theorem is as follows.
Theorem 17.5.4 Let $B^{\prime}$ be the closed unit ball in $X^{\prime}$. Then $B^{\prime}$ is compact in the weak * topology.

Proof: By the Tychonoff theorem, Theorem 17.5.3

$$
P \equiv \prod_{x \in X} \overline{B(0,\|x\|)}
$$

is compact in the product topology where the topology on $\overline{B(0,\|x\|)}$ is the usual topology of $\mathbb{F}$. Recall $P$ is the set of functions which map a point, $x \in X$ to a point in $\overline{B(0,\|x\|)}$. Therefore, $B^{\prime} \subseteq P$. Also the basic open sets in the weak $*$ topology on $B^{\prime}$ are obtained as the intersection of basic open sets in the product topology of $P$ to $B^{\prime}$ and so it suffices to show $B^{\prime}$ is a closed subset of $P$. Suppose then that $f \in P \backslash B^{\prime}$. Since $|f(x)| \leq\|x\|$ for each $x$, it follows $f$ cannot be linear. There are two ways this can happen. One way is that for some $x, y$

$$
f(x+y) \neq f(x)+f(y)
$$

for some $x, y \in X$. However, if $g$ is close enough to $f$ at the three points, $x+y, x$, and $y$, the above inequality will hold for $g$ in place of $f$. In other words there is a basic open set containing $f$, such that for all $g$ in this basic open set, $g \notin B^{\prime}$. A similar consideration applies in case $f(\lambda x) \neq \lambda f(x)$ for some scalar $\lambda$ and $x$. Since $P \backslash B^{\prime}$ is open, it follows $B^{\prime}$ is a closed subset of $P$ and is therefore, compact.

Sometimes one can consider the weak $*$ topology in terms of a metric space.
Theorem 17.5.5 If $K \subseteq X^{\prime}$ is compact in the weak $*$ topology and $X$ is separable in the weak topology then there exists a metric, $d$, on $K$ such that if $\tau_{d}$ is the topology on $K$ induced by $d$ and if $\tau$ is the topology on $K$ induced by the weak * topology of $X^{\prime}$, then $\tau=\tau_{d}$. Thus one can consider $K$ with the weak $*$ topology as a metric space.

Proof: Let $D=\left\{x_{n}\right\}$ be the dense countable subset in $X$. The metric is

$$
d(f, g) \equiv \sum_{n=1}^{\infty} 2^{-n} \frac{\rho_{x_{n}}(f-g)}{1+\rho_{x_{n}}(f-g)}
$$

where $\rho_{x_{n}}(f)=\left|f\left(x_{n}\right)\right|$. Clearly $d(f, g)=d(g, f) \geq 0$. If $d(f, g)=0$, then this requires $f\left(x_{n}\right)=g\left(x_{n}\right)$ for all $x_{n} \in D$. Is it the case that $f=g ? B_{\{f, g\}}(x, r)$ contains some $x_{n} \in D$. Hence

$$
\max \left\{\left|f\left(x_{n}\right)-f(x)\right|,\left|g\left(x_{n}\right)-g(x)\right|\right\}<r
$$

and $f\left(x_{n}\right)=g\left(x_{n}\right)$. It follows that $|f(x)-g(x)|<2 r$. Since $r$ is arbitrary, this implies $f(x)=g(x)$. It is routine to verify the triangle inequality from the easy to establish inequality,

$$
\frac{x}{1+x}+\frac{y}{1+y} \geq \frac{x+y}{1+x+y}
$$

valid whenever $x, y \geq 0$. Therefore this is a metric.
Thus there are two topological spaces, $(K, \tau)$ and $(K, d)$, the first being $K$ with the weak * topology and the second being $K$ with this metric. It is clear that if $i$ is the identity map, $i:(K, \tau) \rightarrow(K, d)$, then $i$ is continuous. Therefore, sets which are open in $(K, d)$ are open in $(K, \tau)$. Letting $\tau_{d}$ denote those sets which are open with respect to the metric, $\tau_{d} \subseteq \tau$.

Now suppose $U \in \tau$. Is $U$ in $\tau_{d}$ ? Since $K$ is compact with respect to $\tau$, it follows from the above that $K$ is compact with respect to $\tau_{d} \subseteq \tau$. Hence $K \backslash U$ is compact with respect to $\tau_{d}$ and so it is closed with respect to $\tau_{d}$. Thus $U$ is open with respect to $\tau_{d}$.

The fact that this set with the weak $*$ topology can be considered a metric space is very significant because if a point is a limit point in a metric space, one can extract a convergent sequence.

Note that if a Banach space is separable, then it is weakly separable.

Corollary 17.5.6 If $X$ is weakly separable and $K \subseteq X^{\prime}$ is compact in the weak $*$ topology, then $K$ is sequentially compact. That is, if $\left\{f_{n}\right\}_{n=1}^{\infty} \subseteq K$, then there exists a subsequence $f_{n_{k}}$ and $f \in K$ such that for all $x \in X$,

$$
\lim _{k \rightarrow \infty} f_{n_{k}}(x)=f(x)
$$

Proof: By Theorem 17.5.5, $K$ is a metric space for the metric described there and it is compact. Therefore by the characterization of compact metric spaces, Proposition 7.6.5 on Page $144, K$ is sequentially compact. This proves the corollary.

### 17.5.3 Eberlein Smulian Theorem

Next consider the weak topology. The most interesting results have to do with a reflexive Banach space. The following lemma ties together the weak and weak $*$ topologies in the case of a reflexive Banach space.

Lemma 17.5.7 Let $J: X \rightarrow X^{\prime \prime}$ be the James map

$$
J x(f) \equiv f(x)
$$

and let $X$ be reflexive so that $J$ is onto. Then $J$ is a homeomorphism of ( $X$, weak topology) and ( $X^{\prime \prime}$, weak * topology).This means $J$ is one to one, onto, and both $J$ and $J^{-1}$ are continuous.

Proof: Let $f \in X^{\prime}$ and let

$$
B_{f}(x, r) \equiv\{y:|f(x)-f(y)|<r\} .
$$

Thus $B_{f}(x, r)$ is a subbasic set for the weak topology on $X$. I claim that

$$
J B_{f}(x, r)=B_{f}(J x, r)
$$

where $B_{f}(J x, r)$ is a subbasic set for the weak $*$ topology. If $y \in B_{f}(x, r)$, then $\|J y-J x\|=$ $\|x-y\|<r$ and so $J B_{f}(x, r) \subseteq B_{f}(J x, r)$. Now if $x^{* *} \in B_{f}(J x, r)$, then since $J$ is reflexive, there exists $y \in X$ such that $J y=x^{* *}$ and so

$$
\|y-x\|=\|J y-J x\|<r
$$

showing that $J B_{f}(x, r)=B_{f}(J x, r)$. A typical subbasic set in the weak $*$ topology is of the form $B_{f}(J x, r)$. Thus $J$ maps the subbasic sets of the weak topology to the subbasic sets of the weak $*$ topology. Therefore, $J$ is a homeomorphism as claimed.

The following is an easy corollary.
Corollary 17.5.8 If $X$ is a reflexive Banach space, then the closed unit ball is weakly compact.

Proof: Let $B$ be the closed unit ball. Then $B=J^{-1}\left(B^{* *}\right)$ where $B^{* *}$ is the unit ball in $X^{\prime \prime}$ which is compact in the weak $*$ topology. Therefore $B$ is weakly compact because $J^{-1}$ is continuous.

Corollary 17.5.9 Let $X$ be a reflexive Banach space. If $K \subseteq X$ is compact in the weak topology and $X^{\prime}$ is separable in the weak $*$ topology, then there exists a metric $d$, on $K$ such that if $\tau_{d}$ is the topology on $K$ induced by $d$ and if $\tau$ is the topology on $K$ induced by the weak topology of $X$, then $\tau=\tau_{d}$. Thus one can consider $K$ with the weak topology as $a$ metric space.

Proof: This follows from Theorem 17.5.5 and Lemma 17.5.7. Lemma 17.5.7 implies $J(K)$ is compact in $X^{\prime \prime}$. Then since $X^{\prime}$ is separable in the weak $*$ topology, $X$ is separable in the weak topology and so there is a metric, $d^{\prime \prime}$ on $J(K)$ which delivers the weak $*$ topology on $J(K)$. Let $d(x, y) \equiv d^{\prime \prime}(J x, J y)$. Then

$$
\left(K, \tau_{d}\right) \xrightarrow{J}\left(J(K), \tau_{d^{\prime \prime}}\right) \xrightarrow{i d}\left(J(K), \tau_{\text {weak } *}\right) \xrightarrow{J^{-1}}\left(K, \tau_{\text {weak }}\right)
$$

and all the maps are homeomorphisms.
Here is a useful lemma.
Lemma 17.5.10 Let $Y$ be a closed subspace of a Banach space $X$ and let $y \in X \backslash Y$. Then there exists $x^{*} \in X^{\prime}$ such that $x^{*}(Y)=0$ but $x^{*}(y) \neq 0$.

Proof: Define $f(x+\alpha y) \equiv\|y\| \alpha$. Thus $f$ is linear on $Y \oplus \mathbb{F} y$. I claim that $f$ is also continuous on this subspace of $X$. If not, then there exists $x_{n}+\alpha_{n} y \rightarrow 0$ but $\left|f\left(x_{n}+\alpha_{n} y\right)\right| \geq$ $\varepsilon>0$ for all $n$. First suppose $\left|\alpha_{n}\right|$ is bounded. Then, taking a further subsequence, we can assume $\alpha_{n} \rightarrow \alpha$. It follows then that $\left\{x_{n}\right\}$ must also converge to some $x \in Y$ since $Y$ is closed. Therefore, in this case, $x+\alpha y=0$ and so $\alpha=0$ since otherwise, $y \in Y$. In the other case when $\alpha_{n}$ is unbounded, you have $\left(x_{n} / \alpha_{n}+y\right) \rightarrow 0$ and so it would require that $y \in \bar{Y}$ which cannot happen because $Y$ is closed. Hence $f$ is continuous as claimed. It follows that for some $k$,

$$
|f(x+\alpha y)| \leq k\|x+\alpha y\|
$$

Now apply the Hahn Banach theorem to extend $f$ to $x^{*} \in X^{\prime}$.
Next is the Eberlein Smulian theorem which states that a Banach space is reflexive if and only if the closed unit ball is weakly sequentially compact. Actually, only half the theorem is proved here, the more useful only if part. The book by Yoshida [127] has the complete theorem discussed. First here is an interesting lemma for its own sake.

Lemma 17.5.11 A closed subspace of a reflexive Banach space is reflexive.
Proof: Let $Y$ be the closed subspace of the reflexive space, $X$. Consider the following diagram


This diagram follows from Theorem 17.2.10 on Page 449, the theorem on adjoints. Now let $y^{* *} \in Y^{\prime \prime}$. Then $i^{* *} y^{* *}=J_{X}(y)$ because $X$ is reflexive. I want to show that $y \in Y$. If it
is not in $Y$ then since $Y$ is closed, there exists $x^{*} \in X^{\prime}$ such that $x^{*}(y) \neq 0$ but $x^{*}(Y)=0$. Then $i^{*} x^{*}=0$. Hence

$$
0=y^{* *}\left(i^{*} x^{*}\right)=i^{* *} y^{* *}\left(x^{*}\right)=J(y)\left(x^{*}\right)=x^{*}(y) \neq 0
$$

a contradiction. Hence $y \in Y$. Letting $J_{Y}$ denote the James map from $Y$ to $Y^{\prime \prime}$ and $x^{*} \in X^{\prime}$,

$$
\begin{aligned}
y^{* *}\left(i^{*} x^{*}\right) & =i^{* *} y^{* *}\left(x^{*}\right)=J_{X}(y)\left(x^{*}\right) \\
& =x^{*}(y)=x^{*}(i y)=i^{*} x^{*}(y)=J_{Y}(y)\left(i^{*} x^{*}\right)
\end{aligned}
$$

Since $i^{*}$ is onto, this shows $y^{* *}=J_{Y}(y)$.
Theorem 17.5.12 (Eberlein Smulian) The closed unit ball in a reflexive Banach space $X$, is weakly sequentially compact. By this is meant that if $\left\{x_{n}\right\}$ is contained in the closed unit ball, there exists a subsequence, $\left\{x_{n_{k}}\right\}$ and $x \in X$ such that for all $x^{*} \in X^{\prime}$,

$$
x^{*}\left(x_{n_{k}}\right) \rightarrow x^{*}(x) .
$$

Proof: Let $\left\{x_{n}\right\} \subseteq B \equiv \overline{B(0,1)}$. Let $Y$ be the closure of the linear span of $\left\{x_{n}\right\}$. Thus $Y$ is a separable. It is reflexive because it is a closed subspace of a reflexive space so the above lemma applies. By the Banach Alaoglu theorem, the closed unit ball $B^{*}$ in $Y^{\prime}$ is weak * compact. Also by Theorem 17.5.5, $B^{*}$ is a metric space with a suitable metric.

$$
\begin{array}{rlc}
B^{* *} Y^{\prime \prime} & \stackrel{i^{* *} 1-1}{\rightarrow} & X^{\prime \prime} \\
\text { weakly separable } B^{*} Y^{\prime} & \stackrel{i^{*} \text { onto }}{\leftarrow} & X^{\prime} \\
\text { separable } B Y & \xrightarrow{i} & X
\end{array}
$$

Thus $B^{*}$ is complete and totally bounded with respect to this metric and it follows that $B^{*}$ with the weak $*$ topology is separable. This implies $Y^{\prime}$ is also separable in the weak * topology. To see this, let $\left\{y_{n}^{*}\right\} \equiv D$ be a weak $*$ dense set in $B^{*}$ and let $y^{*} \in Y^{\prime}$. Let $p$ be a large enough positive rational number that $y^{*} / p \in B^{*}$. Then if $A$ is any finite set from $Y$, there exists $y_{n}^{*} \in D$ such that $\rho_{A}\left(y^{*} / p-y_{n}^{*}\right)<\frac{\varepsilon}{p}$. It follows $p y_{n}^{*} \in B_{A}\left(y^{*}, \varepsilon\right)$ showing that rational multiples of $D$ are weak $*$ dense in $Y^{\prime}$. Since $Y$ is reflexive, the weak and weak * topologies on $Y^{\prime}$ coincide and so $Y^{\prime}$ is weakly separable. Since $Y^{\prime}$ is weakly separable, Corollary 17.5 .6 implies $B^{* *}$, the closed unit ball in $Y^{\prime \prime}$ is weak $*$ sequentially compact. Then by Lemma 17.5.7 $B$, the unit ball in $Y$, is weakly sequentially compact. It follows there exists a subsequence $x_{n_{k}}$, of the sequence $\left\{x_{n}\right\}$ and a point $x \in Y$, such that for all $f \in Y^{\prime}$,

$$
f\left(x_{n_{k}}\right) \rightarrow f(x) .
$$

Now if $x^{*} \in X^{\prime}$, and $i$ is the inclusion map of $Y$ into $X$,

$$
x^{*}\left(x_{n_{k}}\right)=i^{*} x^{*}\left(x_{n_{k}}\right) \rightarrow i^{*} x^{*}(x)=x^{*}(x) .
$$

which shows $x_{n_{k}}$ converges weakly and this shows the unit ball in $X$ is weakly sequentially compact.

Corollary 17.5.13 Let $\left\{x_{n}\right\}$ be any bounded sequence in a reflexive Banach space $X$. Then there exists $x \in X$ and a subsequence, $\left\{x_{n_{k}}\right\}$ such that for all $x^{*} \in X^{\prime}$,

$$
\lim _{k \rightarrow \infty} x^{*}\left(x_{n_{k}}\right)=x^{*}(x)
$$

Proof: If a subsequence, $x_{n_{k}}$ has $\left\|x_{n_{k}}\right\| \rightarrow 0$, then the conclusion follows. Simply let $x=0$. Suppose then that $\left\|x_{n}\right\|$ is bounded away from 0 . That is, $\left\|x_{n}\right\| \in[\delta, C]$. Take a subsequence such that $\left\|x_{n_{k}}\right\| \rightarrow a$. Then consider $x_{n_{k}} /\left\|x_{n_{k}}\right\|$. By the Eberlein Smulian theorem, this subsequence has a further subsequence, $x_{n_{k_{j}}} /\left\|x_{n_{k_{j}}}\right\|$ which converges weakly to $x \in B$ where $B$ is the closed unit ball. It follows from routine considerations that $x_{n_{k_{j}}} \rightarrow a x$ weakly. This proves the corollary.

### 17.6 Operators With Closed Range

When is $T(X)$ a closed subset of $Y$ for $T \in \mathscr{L}(X, Y)$ ? One way this happens is when $T=I-C$ for $C$ compact.

Definition 17.6.1 Let $C \in \mathscr{L}(X, Y)$ where $X, Y$ are two Banach spaces. Then $C$ is called a compact operator if $C$ (bounded set $)=($ precompact set $)$.

Lemma 17.6.2 Suppose $C \in \mathscr{L}(X, X)$ is compact. Then $(I-C)(X)$ is closed.
Proof: Let $(I-C) x_{n} \rightarrow y$. Let $z_{n} \in \operatorname{ker}(I-C)$ such that

$$
\begin{aligned}
\operatorname{dist}\left(x_{n}, \operatorname{ker}(I-C)\right) & \leq\left\|x_{n}-z_{n}\right\| \\
& \leq\left(1+\frac{1}{n}\right) \operatorname{dist}\left(x_{n}, \operatorname{ker}(I-C)\right)
\end{aligned}
$$

Case 1: $\left\|x_{n}-z_{n}\right\| \rightarrow \infty$.
In this case, you get $(I-C)\left(x_{n}-z_{n}\right) \rightarrow y$ and so there is a subsequence such that $C\left(\frac{x_{n}-z_{n}}{\left\|x_{n}-z_{n}\right\|}\right)$ converges. Also $\frac{x_{n}-z_{n}}{\left\|x_{n}-z_{n}\right\|}$ converges to the same thing. Let it be called $w$. Thus

$$
\begin{gathered}
\frac{x_{n}-z_{n}}{\left\|x_{n}-z_{n}\right\|} \rightarrow w, C \frac{x_{n}-z_{n}}{\left\|x_{n}-z_{n}\right\|} \rightarrow C w \\
C\left(\frac{x_{n}-z_{n}}{\left\|x_{n}-z_{n}\right\|}\right) \rightarrow \quad \rightarrow \text { so } C w=w, w \in \operatorname{ker}(I-C) \\
\left\|\frac{x_{n}-z_{n}}{\left\|x_{n}-z_{n}\right\|}-w\right\|=\frac{1}{\left\|x_{n}-z_{n}\right\|}\|\left(x_{n}-z_{n}\right)-\overbrace{w\left\|x_{n}-z_{n}\right\|}^{\in \operatorname{ker}(I-C)}\| \\
\geq \frac{1}{\left\|x_{n}-z_{n}\right\|} \operatorname{dist}\left(x_{n}, \operatorname{ker}(I-C)\right) \\
\geq \frac{1}{\left(\left(1+\frac{1}{n}\right) \operatorname{dist}\left(x_{n}, \operatorname{ker}(I-C)\right)\right)} \operatorname{dist}\left(x_{n}, \operatorname{ker}(I-C)\right)
\end{gathered}
$$

Now passing to a limit,

$$
0 \geq \lim _{n \rightarrow \infty} \frac{1}{1+1 / n}=1
$$

so Case 1 cannot occur.
Case 2: A subsequence of $\left\|x_{n}-z_{n}\right\|$ is bounded.
Let $n$ denote the subscript for the subsequence. Then there is a further subsequence still denoted with $n$ such that $C\left(x_{n}-z_{n}\right)$ converges. Then also $\left(x_{n}-z_{n}\right)$ converges because $(I-C)\left(x_{n}\right)=(I-C)\left(x_{n}-z_{n}\right)$ is given to converge. Let $\left(x_{n}-z_{n}\right) \rightarrow x$. Then

$$
y=\lim _{n \rightarrow \infty}(I-C) x_{n}=\lim _{n \rightarrow \infty}(I-C)\left(x_{n}-z_{n}\right)=(I-C) x
$$

and so $y \in(I-C)(X)$ showing that $(I-C)(X)$ is closed.
Here is a useful lemma.
Lemma 17.6.3 Suppose $W$ and $V$ are closed subspaces of a Banach space $X$ and $V \varsubsetneqq W$ $(V$ is a proper subset of $W$.) while $(\lambda I-L)(W) \subseteq V, \lambda \neq 0$. Then there exists $w \in W \backslash V$ such that $\|w\|=1$ and

$$
\operatorname{dist}(L w, L V) \geq 1 / 2
$$

Proof: Let $w_{0} \in W \backslash V$. Then let $v \in V$ be such that $\left\|\lambda w_{0}-v\right\| \leq 2 \operatorname{dist}\left(\lambda w_{0}, V\right)$. Then let

$$
w=\frac{\lambda w_{0}-v}{\left\|\lambda w_{0}-v\right\|}
$$

It follows that $\|w\|=1$ and is in $W \backslash V$. Now let $x \in V$. Then

$$
\begin{gathered}
L x-L w=\lambda(x-w)+\overbrace{(L-\lambda I)(x-w)}^{\text {in } V} \\
=\lambda x+(L-\lambda I)(x-w)-\lambda w \\
=\frac{1}{\left\|\lambda w_{0}-v\right\|}\left(\lambda x\left\|\lambda w_{0}-v\right\|+(L-\lambda I)(x-w)\left\|\lambda w_{0}-v\right\|-\lambda\left\|\lambda w_{0}-v\right\| w\right) \\
=\frac{1}{\left\|\lambda w_{0}-v\right\|}\left(\lambda x\left\|\lambda w_{0}-v\right\|+(L-\lambda I)(x-w)\left\|\lambda w_{0}-v\right\|-\lambda\left(\lambda w_{0}-v\right)\right) \\
=\frac{1}{\left\|\lambda w_{0}-v\right\|}(\overbrace{\lambda x\left\|w_{0}-v\right\|+(L-\lambda I)(x-w)\left\|\lambda w_{0}-v\right\|+\lambda v}^{\in V}-\lambda w_{0})
\end{gathered}
$$

Thus

$$
\begin{gathered}
\|L x-L w\| \geq \frac{1}{\left\|\lambda w_{0}-v\right\|}\|\lambda x\| \lambda w_{0}-v\|+(L-\lambda I)(x-w)\| \lambda w_{0}-v\left\|+\lambda v-\lambda w_{0}\right\| \\
\geq \frac{1}{2 \operatorname{dist}\left(\lambda w_{0}, V\right)} \operatorname{dist}\left(\lambda w_{0}, V\right)=\frac{1}{2} \square
\end{gathered}
$$

Here is another fairly elementary lemma a little like the above.

Lemma 17.6.4 Let $Y$ be an infinite dimensional Banach space. Then there exists a sequence $\left\{x_{n}\right\}$ in the unit sphere $S,\left\|x_{n}\right\|=1$, such that $\left\|x_{n}-x_{m}\right\| \geq \frac{1}{2}$ whenever $n \neq m$.

Proof: Pick $x_{1} \in S$. Now the span of $x_{1}$ is not everything and so there exists $u_{2} \notin$ $\operatorname{span}\left(x_{1}\right)$. Let $w_{2}$ be a point of $\operatorname{span}\left(x_{1}\right)$ such that $\left\|u_{2}-w_{2}\right\| \leq 2 \operatorname{dist}\left(u_{2}, \operatorname{span}\left(x_{1}\right)\right)$. Then $x_{2}=\frac{u_{2}-w_{2}}{\left\|u_{2}-w_{2}\right\|}$. Then

$$
\left\|x_{1}-x_{2}\right\|=\left\|\frac{\left\|u_{2}-w_{2}\right\| x_{1}-\left(u_{2}-w_{2}\right)}{\left\|u_{2}-w_{2}\right\|}\right\| \geq \frac{\operatorname{dist}\left(u_{2}, \operatorname{span}\left(x_{1}\right)\right)}{2 \operatorname{dist}\left(u_{2}, \operatorname{span}\left(x_{1}\right)\right)}=\frac{1}{2}
$$

Now repeat the argument with span $\left(x_{1}, x_{2}\right)$ in place of $\operatorname{span}\left(x_{1}\right)$ and continue to get the desired sequence.

Lemma 17.6.5 Let $L$ be a compact linear map. Then the eigenspace of $L$ is finite dimensional for each eigenvalue $\lambda \neq 0$.

Proof: Consider $(L-\lambda I)^{-1}(0) \cap S$ where $S$ is the unit sphere. The eigenspace is just $(L-\lambda I)^{-1}(0)$. Let $Y$ be this inverse image. If $Y$ is infinite dimensional, then the above Lemma 17.6.4 applies. There exists $\left\{x_{n}\right\} \subseteq(L-\lambda I)^{-1}(0) \cap S$ where $\left\|x_{n}-x_{m}\right\| \geq 1 / 2$ for all $n \neq m$. Then there is a subsequence, still denoted with subscript $n$ such that $\left\{L x_{n}\right\}$ is a Cauchy sequence. Thus $L x_{n}=\lambda x_{n}$ and so, since $\lambda \neq 0$, it follows that $\left\{x_{n}\right\}$ is also a Cauchy sequence and converges to some $x$. But this is impossible because of the construction of the $\left\{x_{n}\right\}$ which prevents there being any Cauchy sequence. Thus $Y$ must be finite dimensional.

This lemma is useful in proving the following major spectral theorem about the eigenvalues of a compact operator. I found this theorem in Deimling [38].

Theorem 17.6.6 Let $L \in \mathscr{L}(X, X)$ with $L$ compact. Let $\Lambda$ be the eigenvalues of $L$. That is $\lambda \in \Lambda$ means there exists $x \neq 0$ such that $L x=\lambda x$. It is assumed the field of scalars is $\mathbb{R}$ or $\mathbb{C}$. Let $R_{\lambda} \equiv L-\lambda I$. Then the following hold.

1. If $\mu \in \Lambda$ then $|\mu| \leq\|L\|, \Lambda$ is at most countable and has no limit points other than possibly 0.
2. $R_{\lambda}$ is a homeomorphism onto $X$ whenever $\lambda \notin \Lambda \cup\{0\}$.
3. For all $\lambda \in \Lambda \backslash\{0\}$, there exists a smallest $k=k(\lambda)$,
(a) $R_{\lambda}^{k} X \oplus N\left(R_{\lambda}^{k}\right)=X$ where $N\left(R_{\lambda}^{k}\right)$ is the vectors $x$ such that $R_{\lambda}^{k} x=0 . R_{\lambda}^{k} X$ is closed, $\operatorname{dim}\left(N\left(R_{\lambda}^{k}\right)\right)<\infty$.
(b) $R_{\lambda}^{k} X$ and $N\left(R_{\lambda}^{k}\right)$ are invariant under $L$ and $\left.R_{\lambda}\right|_{R_{k}^{k} X}$ is a homeomorphism onto $R_{\lambda}^{k} X$.
(c) $N\left(R_{\mu}^{k}\right) \subseteq R_{\lambda}^{k} X$ for all $\lambda, \mu \in \Lambda \backslash\{0\}$ where $\lambda \neq \mu$.

Proof: Consider $\lambda \neq 0$. The $N\left(R_{\lambda}^{k}\right)$ are increasing in $k$ and $R_{\lambda}\left(N\left(R_{\lambda}^{k+1}\right)\right) \subseteq N\left(R_{\lambda}^{k}\right)$. This follows from the definition. (It isn't necessary to assume in most of this that $\lambda \in \Lambda$, just a nonzero number will do.) Now

$$
R_{\lambda}=-\lambda\left(I-\frac{1}{\lambda} L\right)
$$

If these things are strictly increasing for infinitely many $k$, then by Lemma 17.6.3, there is an infinite sequence $x_{k}, x_{k} \in N\left(R_{\lambda}^{k+1}\right) \backslash N\left(R_{\lambda}^{k}\right)$, $\operatorname{dist}\left(L x_{k}, L N\left(R_{\lambda}^{k}\right)\right) \geq 1 / 2$. Hence

$$
\left\|L x_{k}-L x_{k-1}\right\| \geq 1 / 2
$$

and this can't happen because $L$ is compact so $\left\{L x_{k}\right\}$ has a Cauchy subsequence. Therefore there exists a smallest $k$ such that

$$
N\left(R_{\lambda}^{k}\right)=N\left(R_{\lambda}^{m}\right), m \geq k
$$

On the other hand, $\left\{R_{\lambda}^{k} X\right\}$ are decreasing in $k$. By similar reasoning using Lemma 17.6.3 and the observation that $R_{\lambda}\left(R_{\lambda}^{k} X\right) \supseteq R_{\lambda}^{k+1} X$ (in fact they are equal) it follows that the $\left\{R_{\lambda}^{k} X\right\}$ are also eventually constant, say for $m \geq l$.

Now if you have $y \in N\left(R_{\lambda}^{k}\right) \cap R_{\lambda}^{k} X$, then $y=R_{\lambda}^{k} w$ and also $R_{\lambda}^{k} y=0$. Hence $R_{\lambda}^{2 k} w=0$ and so, $w \in N\left(R_{\lambda}^{2 k}\right)=N\left(R_{\lambda}^{k}\right)$ which implies $R_{\lambda}^{k} w=0$ and so $y=0$. It follows $N\left(R_{\lambda}^{k}\right) \cap$ $R_{\lambda}^{k} X=\{0\}$.

Now suppose $l>k$. Then there exists $y \in R_{\lambda}^{l-1} X \backslash R_{\lambda}^{l} X$ and so $R_{\lambda} y \in R_{\lambda}^{l} X=R_{\lambda}^{l+1} X=$ $R_{\lambda} R_{\lambda}^{l} X$. So $R_{\lambda} y=R_{\lambda} z$ for some $z \in R_{\lambda}^{l} X$. Thus $y-z \neq 0$ because $y \notin R_{\lambda}^{l} X$ but $z$ is. However, $R_{\lambda}(y-z)=0$ and so

$$
(y-z) \in N\left(R_{\lambda}\right) \cap R_{\lambda}^{k} X \subseteq N\left(R_{\lambda}^{k}\right) \cap R_{\lambda}^{k} X
$$

which cannot happen from the above which showed that $N\left(R_{\lambda}^{k}\right) \cap R_{\lambda}^{k} X=\{0\}$. Thus $l \leq k$.
Next suppose $l<k$. Then you would have $R_{\lambda}^{l} X=R_{\lambda}^{k} X$ and $N\left(R_{\lambda}^{k}\right) \supsetneqq N\left(R_{\lambda}^{l}\right)$. Thus there exists $y \in N\left(R_{\lambda}^{k}\right)$ but not in $N\left(R_{\lambda}^{l}\right)$. Hence $R_{\lambda}^{k} y=0$ but $R_{\lambda}^{l} y \neq 0$. However, $R_{\lambda}^{l} y$ is in $R_{\lambda}^{k} X$ from the definition of $l$ and so there is $u$ such that $R_{\lambda}^{l} y=R_{\lambda}^{k} u$. Thus

$$
0=R_{\lambda}^{k} y=R_{\lambda}^{k-l+l} y=R_{\lambda}^{k-l} R_{\lambda}^{l} y=R_{\lambda}^{k-l} R_{\lambda}^{k} u=R_{\lambda}^{2 k-l} u
$$

Now it follows that $u \in N\left(R_{\lambda}^{2 k-l}\right)=N\left(R_{\lambda}^{k}\right)$. This is a contradiction because it says that $R_{\lambda}^{k} u=0$ but right above the displayed equation, we had $R_{\lambda}^{l} y=R_{\lambda}^{k} u$ and $R_{\lambda}^{l} y \neq 0$. Thus, with the above paragraph, $k=l$.

What about the claim that $R_{\lambda}$ restricted to $R_{\lambda}^{k} X$ is a homeomorphism? It maps $R_{\lambda}^{k} X$ to $R_{\lambda}^{k+1} X=R_{\lambda}^{k} X$. Also, if $R_{\lambda}(y)=0$ for $y \in R_{\lambda}^{k} X$, then $R_{\lambda}^{k} y=0$ also and so $y \in R_{\lambda}^{k} X \cap$ $N\left(R_{\lambda}^{k}\right)$. It was shown above that this implies $y=0$. Thus $R_{\lambda}$ appears to be one to one. By assumption, it is continuous. Also from Lemma 17.6.2,

$$
R_{\lambda}^{k} X \text { is closed. }
$$

This follows from the observation that

$$
\begin{equation*}
R_{\lambda}^{k}=(L-\lambda I)^{k}=\sum_{j=0}^{k}\binom{k}{j} L^{j}(-\lambda I)^{k-j}=(-\lambda)^{k} I+\sum_{j=1}^{k}\binom{k}{j} L^{j}(-\lambda I)^{k-j} \tag{17.6.17}
\end{equation*}
$$

which is a multiple of $I-C$ where $C$ is a compact map. Then by the open mapping theorem, it follows that $R_{\lambda}$ is a homeomorphism onto $R_{\lambda}^{k+1} X=R_{\lambda}^{k} X$.

What about $R_{\lambda}^{k} X \oplus N\left(R_{\lambda}^{k}\right)=X$ ? It only remains to verify that $R_{\lambda}^{k} X+N\left(R_{\lambda}^{k}\right)=X$ because the only vector in the intersection was shown to be 0 . Thus if you have $x+y=0$ where $x$ is in one of these and $y$ in the other, then $x=-y$ so each is in both and hence both are 0 . Pick $x \in X$. Then $R_{\lambda}^{k} x \in R_{\lambda}^{k}\left(R_{\lambda}^{k} X\right)=R_{\lambda}^{k} X$. Therefore, $R_{\lambda}^{k} x=R_{\lambda}^{k}\left(R_{\lambda}^{k} y\right)$ for some $y$ and so $R_{\lambda}^{k}\left(x-R_{\lambda}^{k} y\right)=0$. Hence

$$
x-R_{\lambda}^{k} y \in N\left(R_{\lambda}^{k}\right)
$$

showing that $x \in R_{\lambda}^{k} X+N\left(R_{\lambda}^{k}\right)$.
It is obvious that $R_{\lambda}^{k} X$ and $N\left(R_{\lambda}^{k}\right)$ are invariant under $L$. If $\lambda_{0} \notin \Lambda \backslash\{0\}$, then $L-\lambda_{0} I$ is one to one and so the compactness of $L$ and Lemma 17.6.2 implies that $\left(L-\lambda_{0} I\right) X$ is closed. Hence the open mapping theorem implies $L-\lambda_{0} I$ is a homeomorphism onto $\left(L-\lambda_{0} I\right) X$. Is this last all of $X$ ? There is nothing in the above argument which involved an essential assumption that $\lambda \in \Lambda$. Hence, repeating this argument, you see that $\left(L-\lambda_{0} I\right) X \oplus$ $N\left(L-\lambda_{0} I\right)=X$ but $N\left(L-\lambda_{0} I\right)=0$. Hence $\left(L-\lambda_{0} I\right) X=X$ and so indeed $\left(L-\lambda_{0} I\right)$ is a homeomorphism.

For $\mu \in \Lambda, L x=\mu x$ and so $|\mu|\|x\| \leq\|L\|\|x\|$ so $|\mu| \leq\|L\|$. Why is $\Lambda$ at most countable and has only one possible limit point at 0 ? It was shown that $R_{\lambda}$ is a homeomorphism when restricted to $R_{\lambda}^{k} X$. It follows that for $x \in R_{\lambda}^{k} X,\left\|R_{\lambda} x\right\|>\delta\|x\|$ for some $\delta>0$, this for every such $x \in R_{\lambda}^{k} X$. Now consider $\mu$ close to $\lambda$ and consider $R_{\mu}$. Then for $x \in R_{\lambda}^{k} X,\left\|R_{\mu} x\right\|=$ $\left\|\left(R_{\lambda}+(\lambda-\mu)\right) x\right\| \geq \delta\|x\|-|\lambda-\mu|\|x\|>\frac{\delta}{2}\|x\|$ provided $|\lambda-\mu|<\delta / 2$. Thus for $\mu$ close enough to $\lambda, R_{\mu}$ is one to one on $R_{\lambda}^{k} X$. But also $R_{\mu}$ is one to one on $N\left(R_{\lambda}^{k}\right)$. Lets see why this is so. Suppose $(L-\mu I) x=0$ for $x \in N\left(R_{\lambda}^{k}\right)$. Then

$$
\begin{aligned}
0 & =(L-\mu I+(\mu-\lambda) I)^{k} x \\
& =(\mu-\lambda)^{k} x+\sum_{j=1}^{k}\binom{k}{j}(L-\mu I)^{j}(\mu-\lambda)^{k-j} x
\end{aligned}
$$

and the second term involving the sum yields 0 . Since $R_{\lambda}^{k} X \oplus N\left(R_{\lambda}^{k}\right)=X$, this shows that $(L-\mu I)$ is one to one for $\mu$ near $\lambda$. It follows that for $\mu$ near $\lambda, \mu \notin \Lambda$. Thus the only possible limit point is 0 . Note that there is no restriction on the size of $\mu$ for $(L-\mu I)$ to be one to one on $N\left(R_{\lambda}^{k}\right)$.

Why is $\operatorname{dim}\left(N\left(R_{\lambda}^{k}\right)\right)<\infty$ for each $\lambda \neq 0$. This follows from 17.6.17. $R_{\lambda}^{k}$ is a multiple of $I-C$ for $C$ a compact operator. Hence this is finite dimensional by Lemma 17.6.5.

What about $N\left(R_{\mu}^{k}\right) \subseteq R_{\lambda}^{k} X$ for $\mu$ an eigenvalue different than $\lambda$ ? Say $R_{\mu}^{k} x=0$. Then, does it follow that $x \in R_{\lambda}^{k} X$ ? From what was just shown

$$
x=y+z, y \in R_{\lambda}^{k} X, z \in N\left(R_{\lambda}^{k}\right)
$$

Then

$$
0=R_{\mu}^{p} x=R_{\mu}^{p} y+R_{\mu}^{p} z
$$

Here $p=k(\mu)$. This is where it is important that $\mu \in \Lambda$. However, $N\left(R_{\lambda}^{k}\right)$ and $R_{\lambda}^{k} X$ are invariant under $R_{\mu}^{p}$ since it is clear that $R_{\lambda}$ and $R_{\mu}$ commute. Thus $R_{\mu}^{p} y=-R_{\mu}^{p} z$ and $R_{\mu}^{p} y \in R_{\lambda}^{k} X,-R_{\mu}^{p} z \in N\left(R_{\lambda}^{k}\right)$ and these are equal. Hence they are both 0 . Now it was just shown that $R_{\mu}$ is one to one on $N\left(R_{\lambda}^{k}\right)$ and so $z=0$. Hence $x=y \in R_{\lambda}^{k} X$.

Note that in the last step, we can't conclude that $y=0$ because we only know that $R_{\mu}$ is one to one on $R_{\lambda}^{k} X$ if $\mu$ is sufficiently close to $\lambda$. The above is about compact mappings from a single space to itself. However, there are also mappings which have closed range which map from one space to another. The Fredholm operators have this property that their image is closed. These are discussed next.

Suppose $T \in \mathscr{L}(X, Y)$. Then $T X$ is a subspace of $Y$ and so it has a Hamel basis $\mathscr{B}$. Extending $\mathscr{B}$ to a Hamel basis for $Y$ yields $\mathscr{C}$. Then $Y=\operatorname{span}(\mathscr{B}) \oplus \operatorname{span}(\mathscr{C} \backslash \mathscr{B})$. Thus $Y=T X \oplus E$. For more on this, see [55].

Definition 17.6.7 Let $T \in \mathscr{L}(X, Y)$. Then this is a Fredholm operator means

1. $\operatorname{dim}(\operatorname{ker}(T))<\infty$
2. $\operatorname{dim}(E)<\infty$ where $Y=T X \oplus E$

Proposition 17.6.8 Let $T \in \mathscr{L}(X, Y)$. Then $T X$ is closed if and only if there exists $\delta>0$ such that

$$
\|T x\| \geq \delta \operatorname{dist}(x, \operatorname{ker}(T))
$$

Proof: First suppose $T X$ is closed. Let $\hat{T}: X / \operatorname{ker}(T) \rightarrow Y$ be defined as $\hat{T}([x]) \equiv T x$. Then by Theorem 18.7.2, $\hat{T}$ is one to one and continuous and $X / \operatorname{ker}(T)$ is a Banach space, $\|\hat{T}\| \leq\|T\|$. Also $\hat{T}$ has the same range as $T$. Thus $T X$ is the same as $\hat{T}(X / \operatorname{ker}(T))$ and $\hat{T} \in \mathscr{L}(X / \operatorname{ker}(T), Y)$. By the open mapping theorem, $\hat{T}$ is continuous and has continuous inverse. Recall

$$
\|[x]\| \equiv \inf \{\|x+z\|: z \in \operatorname{ker} T\}=\operatorname{dist}(x, \operatorname{ker}(T))
$$

Then

$$
\operatorname{dist}(x, \operatorname{ker}(T))=\|[x]\|=\left\|\hat{T}^{-1} \hat{T}[x]\right\| \leq\left\|\hat{T}^{-1}\right\|\|\hat{T}[x]\|=\left\|\hat{T}^{-1}\right\|\|T x\|
$$

and so,

$$
\|T x\| \geq \delta \operatorname{dist}(x, \operatorname{ker}(T))
$$

where $\delta=1 /\left\|\hat{T}^{-1}\right\|$.
Next suppose the inequality holds. Why will $T X$ be closed? Say $\left\{T x_{n}\right\}$ is a sequence in $T X$ converging to $y$. Then by the inequality,

$$
\left\|T x_{n}-T x_{m}\right\| \geq \delta \operatorname{dist}\left(x_{n}-x_{m}, \operatorname{ker}(T)\right)=\delta\left\|\left[x_{n}\right]-\left[x_{m}\right]\right\|_{X / \operatorname{ker}(T)}
$$

showing that $\left\{\left[x_{n}\right]\right\}$ is a Cauchy sequence in $X / \operatorname{ker}(T)$. Therefore, since this is a Banach space, there exists $[x]$ such that $\left[x_{n}\right] \rightarrow[x]$ in $X / \operatorname{ker}(T)$ and so $\hat{T}\left(\left[x_{n}\right]\right) \rightarrow \hat{T}([x])$ in $Y$. But this is the same as saying that $T\left(x_{n}\right) \rightarrow T(x)$. It follows that $y=T x$ and so $T X$ is indeed closed.

Theorem 17.6.9 If $T$ is a Fredholm operator, then $T X$ is closed in $Y$.
Proof: Recall that $Y=T X \oplus E$ where $E$ is a closed subspace of $Y$. In fact, $E$ is finite dimensional, but it is only needed that $E$ is closed. Let $T_{0} \in \mathscr{L}(X \times E, T X \oplus E)$ be given by

$$
T_{0}(x, e) \equiv T x+e
$$

Let the norm on $X \times E$ be

$$
\|(x, e)\|_{X \times E} \equiv \max \left\{\|x\|_{X},\|e\|_{E}\right\}
$$

Thus $T_{0}(x, e)=0$ implies both $T x=0$ and $e=0$. Thus $\operatorname{ker}\left(T_{0}\right)=\operatorname{ker}(T) \times\{0\}$. Also, $T_{0}(X \times E)$ is closed in $Y$ because in fact it is all of $Y, T X \oplus E$. By Proposition 17.6.8, there exists $\delta>0$ such that

$$
\begin{aligned}
\left\|T_{0}(x, e)\right\|_{Y} & \geq \delta \operatorname{dist}\left((x, e), \operatorname{ker}\left(T_{0}\right)\right) \\
& =\delta \operatorname{dist}((x, e), \operatorname{ker}(T) \times\{0\}) \geq \delta \operatorname{dist}(x, \operatorname{ker}(T))
\end{aligned}
$$

Then

$$
\|T x\|_{Y} \equiv\left\|T_{0}(x, 0)\right\|_{Y} \geq \delta \operatorname{dist}(x, \operatorname{ker}(T))
$$

and by Proposition 17.6.8, TX is closed.
Actually, the above proves the following corollary.
Corollary 17.6.10 If $T X \oplus E$ is closed in $Y$ and $E$ is a closed subspace of $Y$, then $T X$ is closed. Here $T \in \mathscr{L}(X, Y)$.

Note that it appears that $\operatorname{dim}(\operatorname{ker}(T))<\infty$ was not really needed.
Let $\mathscr{B}$ be a Hamel basis for $T X$ and consider $\mathscr{A} \equiv\{x: T x \in \mathscr{B}\}$. Then this is a linearly independent set of vectors in $X$. Suppose now that $\operatorname{ker}(T)=\operatorname{span}\left(z_{1}, \cdots, z_{n}\right)$ where $\left\{z_{1}, \cdots, z_{n}\right\}$ is linearly independent so here the assumption that $\operatorname{ker}(T)$ has finite dimensions is being used. Then if $x \in X, T x \in T X$ and so there are finitely many vectors $x_{i} \in \mathscr{A}$ such that

$$
T x=\sum_{i} c_{i} T x_{i} .
$$

Hence

$$
T\left(x-\sum_{i} c_{i} x_{i}\right)=0
$$

so

$$
x-\sum_{i} c_{i} x_{i}=\sum_{j=1}^{n} a_{j} z_{j}
$$

Hence $X=\operatorname{span}(\mathscr{A})+\operatorname{ker}(T)$. In fact, $\left\{\mathscr{A},\left\{z_{1}, \cdots, z_{n}\right\}\right\}$ is linearly independent as is easily seen and so this is a basis for $X$. Hence

$$
X=\operatorname{span}(\mathscr{A}) \oplus \operatorname{ker}(T) \equiv X_{1} \oplus \operatorname{ker}(T)
$$

Is $X_{1}$ closed? Define $S: T X \rightarrow X_{1}$ as follows: $S y=x \in X_{1}$ such that $T x=y$. Since $T$ is one to one on $X_{1}$, there is only one such $x$. Is $S$ continuous? Yes, this is so by the open mapping theorem. It is just the inverse of a continuous one to one linear onto map. Now this reduces to the situation discussed above in Corollary 17.6.10. You have $S \in \mathscr{L}\left(T X, X_{1}\right)$ and $S(T X) \oplus \operatorname{ker}(T)$ is all of $X$ and so it is closed in $X$. Therefore, $S(T X)=X_{1}$ is closed. This, along with the above proves the following.

Theorem 17.6.11 Let $T \in \mathscr{L}(X, Y)$ be a Fredholm operator and suppose $\operatorname{ker}(T)$ is finite dimensional and that $Y=T X \oplus E$ where $E$ is a finite dimensional subspace or more generally closed. Then $T X$ is closed and also for $X=X_{1} \oplus \operatorname{ker}(T)$, it follows that $X_{1}$ is closed.

### 17.7 Exercises

1. Is $\mathbb{N}$ a $G_{\delta}$ set? What about $\mathbb{Q}$ ? What about a countable dense subset of a complete metric space?
2. $\uparrow$ Let $f: \mathbb{R} \rightarrow \mathbb{C}$ be a function. Define the oscillation of a function in $B(x, r)$ by $\omega_{r} f(x)=\sup \{|f(z)-f(y)|: y, z \in B(x, r)\}$. Define the oscillation of the function at the point, $x$ by $\omega f(x)=\lim _{r \rightarrow 0} \omega_{r} f(x)$. Show $f$ is continuous at $x$ if and only if $\omega f(x)=0$. Then show the set of points where $f$ is continuous is a $G_{\delta}$ set (try $\left.U_{n}=\left\{x: \omega f(x)<\frac{1}{n}\right\}\right)$. Does there exist a function continuous at only the rational numbers? Does there exist a function continuous at every irrational and discontinuous elsewhere? Hint: Suppose $D$ is any countable set, $D=\left\{d_{i}\right\}_{i=1}^{\infty}$, and define the function, $f_{n}(x)$ to equal zero for every $x \notin\left\{d_{1}, \cdots, d_{n}\right\}$ and $2^{-n}$ for $x$ in this finite set. Then consider $g(x) \equiv \sum_{n=1}^{\infty} f_{n}(x)$. Show that this series converges uniformly.
3. Let $f \in C([0,1])$ and suppose $f^{\prime}(x)$ exists. Show there exists a constant, $K$, such that $|f(x)-f(y)| \leq K|x-y|$ for all $y \in[0,1]$. Let $U_{n}=\{f \in C([0,1])$ such that for each $x \in[0,1]$ there exists $y \in[0,1]$ such that $|f(x)-f(y)|>n|x-y|\}$. Show that $U_{n}$ is open and dense in $C([0,1])$ where for $f \in C([0,1])$,

$$
\|f\| \equiv \sup \{|f(x)|: x \in[0,1]\}
$$

Show that $\cap_{n} U_{n}$ is a dense $G_{\delta}$ set of nowhere differentiable continuous functions. Thus every continuous function is uniformly close to one which is nowhere differentiable.
4. $\uparrow$ Suppose $f(x)=\sum_{k=1}^{\infty} u_{k}(x)$ where the convergence is uniform and each $u_{k}$ is a polynomial. Is it reasonable to conclude that $f^{\prime}(x)=\sum_{k=1}^{\infty} u_{k}^{\prime}(x)$ ? The answer is no. Use the Weierstrass approximation theorem do show this.
5. Let $X$ be a normed linear space. We say $A \subseteq X$ is "weakly bounded" if for each $x^{*} \in X^{\prime}, \sup \left\{\left|x^{*}(x)\right|: x \in A\right\}<\infty$, while $A$ is bounded if $\sup \{\|x\|: x \in A\}<\infty$. Show $A$ is weakly bounded if and only if it is bounded.
6. Let $X$ and $Y$ be two Banach spaces. Define the norm

$$
\|\|(x, y) \mid\| \equiv\| x\left\|_{X}+\right\| y \|_{Y} .
$$

Show this is a norm on $X \times Y$ which is equivalent to the norm given in the chapter for $X \times Y$. Can you do the same for the norm defined for $p>1$ by

$$
|(x, y)| \equiv\left(\|x\|_{X}^{p}+\|y\|_{Y}^{p}\right)^{1 / p_{?}}
$$

7. Let $f$ be a $2 \pi$ periodic locally integrable function on $\mathbb{R}$. The Fourier series for $f$ is given by

$$
\sum_{k=-\infty}^{\infty} a_{k} e^{i k x} \equiv \lim _{n \rightarrow \infty} \sum_{k=-n}^{n} a_{k} e^{i k x} \equiv \lim _{n \rightarrow \infty} S_{n} f(x)
$$

where

$$
a_{k}=\frac{1}{2 \pi} \int_{-\pi}^{\pi} e^{-i k x} f(x) d x
$$

Show

$$
S_{n} f(x)=\int_{-\pi}^{\pi} D_{n}(x-y) f(y) d y
$$

where

$$
D_{n}(t)=\frac{\sin \left(\left(n+\frac{1}{2}\right) t\right)}{2 \pi \sin \left(\frac{t}{2}\right)}
$$

Verify that $\int_{-\pi}^{\pi} D_{n}(t) d t=1$. Also show that if $g \in L^{1}(\mathbb{R})$, then

$$
\lim _{a \rightarrow \infty} \int_{\mathbb{R}} g(x) \sin (a x) d x=0
$$

This last is called the Riemann Lebesgue lemma. Hint: For the last part, assume first that $g \in C_{c}^{\infty}(\mathbb{R})$ and integrate by parts. Then exploit density of the set of functions in $L^{1}(\mathbb{R})$.
8. $\uparrow$ It turns out that the Fourier series sometimes converges to the function pointwise. Suppose $f$ is $2 \pi$ periodic and Holder continuous. That is $|f(x)-f(y)| \leq K|x-y|^{\theta}$ where $\theta \in(0,1]$. Show that if $f$ is like this, then the Fourier series converges to $f$ at every point. Next modify your argument to show that if at every point, $x$, $|f(x+)-f(y)| \leq K|x-y|^{\theta}$ for $y$ close enough to $x$ and larger than $x$ and

$$
|f(x-)-f(y)| \leq K|x-y|^{\theta}
$$

for every $y$ close enough to $x$ and smaller than $x$, then $S_{n} f(x) \rightarrow \frac{f(x+)+f(x-)}{2}$, the midpoint of the jump of the function. Hint: Use Problem 7.
9. $\uparrow$ Let $Y=\{f$ such that $f$ is continuous, defined on $\mathbb{R}$, and $2 \pi$ periodic $\}$. Define $\|f\|_{Y}=\sup \{|f(x)|: x \in[-\pi, \pi]\}$. Show that $\left(Y,\| \|_{Y}\right)$ is a Banach space. Let $x \in \mathbb{R}$ and define $L_{n}(f)=S_{n} f(x)$. Show $L_{n} \in Y^{\prime}$ but $\lim _{n \rightarrow \infty}\left\|L_{n}\right\|=\infty$. Show that for each $x \in \mathbb{R}$, there exists a dense $G_{\delta}$ subset of $Y$ such that for $f$ in this set, $\left|S_{n} f(x)\right|$ is unbounded. Finally, show there is a dense $G_{\delta}$ subset of $Y$ having the property that $\left|S_{n} f(x)\right|$ is unbounded on the rational numbers. Hint: To do the first part, let $f(y)$ approximate $\operatorname{sgn}\left(D_{n}(x-y)\right)$. Here $\operatorname{sgn} r=1$ if $r>0,-1$ if $r<0$ and 0 if $r=0$. This rules out one possibility of the uniform boundedness principle. After this, show the countable intersection of dense $G_{\delta}$ sets must also be a dense $G_{\delta}$ set.
10. Let $\alpha \in(0,1]$. Define, for $X$ a compact subset of $\mathbb{R}^{p}$,

$$
C^{\alpha}\left(X ; \mathbb{R}^{n}\right) \equiv\left\{\mathbf{f} \in C\left(X ; \mathbb{R}^{n}\right): \rho_{\alpha}(\mathbf{f})+\|\mathbf{f}\| \equiv\|\mathbf{f}\|_{\alpha}<\infty\right\}
$$

where

$$
||\mathbf{f}|| \equiv \sup \{|\mathbf{f}(\mathbf{x})|: \mathbf{x} \in X\}
$$

and

$$
\rho_{\alpha}(\mathbf{f}) \equiv \sup \left\{\frac{|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{\alpha}}: \mathbf{x}, \mathbf{y} \in X, \mathbf{x} \neq \mathbf{y}\right\}
$$

Show that $\left(C^{\alpha}\left(X ; \mathbb{R}^{n}\right),\|\cdot\| \|_{\alpha}\right)$ is a complete normed linear space. This is called a Holder space. What would this space consist of if $\alpha>1$ ?
11. $\uparrow$ Now recall Problem 10 about the Holder spaces. Let $X$ be the Holder functions which are periodic of period $2 \pi$. Define $L_{n} f(x)=S_{n} f(x)$ where $L_{n}: X \rightarrow Y$ for $Y$ given in Problem 9. Show $\left\|L_{n}\right\|$ is bounded independent of $n$. Conclude that $L_{n} f \rightarrow f$ in $Y$ for all $f \in X$. In other words, for the Holder continuous and $2 \pi$ periodic functions, the Fourier series converges to the function uniformly. Hint: $L_{n} f(x)$ is given by

$$
L_{n} f(x)=\int_{-\pi}^{\pi} D_{n}(y) f(x-y) d y
$$

where $f(x-y)=f(x)+g(x, y)$ where $|g(x, y)| \leq C|y|^{\alpha}$. Use the fact the Dirichlet kernel integrates to one to write

$$
\begin{array}{r}
\left|\int_{-\pi}^{\pi} D_{n}(y) f(x-y) d y\right| \leq \overbrace{\left|\int_{-\pi}^{\pi} D_{n}(y) f(x) d y\right|}^{=|f(x)|} \\
+C\left|\int_{-\pi}^{\pi} \sin \left(\left(n+\frac{1}{2}\right) y\right)(g(x, y) / \sin (y / 2)) d y\right|
\end{array}
$$

Show the functions, $y \rightarrow g(x, y) / \sin (y / 2)$ are bounded in $L^{1}$ independent of $x$ and get a uniform bound on $\left\|L_{n}\right\|$. Now use a similar argument to show $\left\{L_{n} f\right\}$ is equicontinuous in addition to being uniformly bounded. If $L_{n} f$ fails to converge to $f$ uniformly, then there exists $\varepsilon>0$ and a subsequence, $n_{k}$ such that $\left\|L_{n_{k}} f-f\right\|_{\infty} \geq \varepsilon$ where this is the norm in $Y$ or equivalently the sup norm on $[-\pi, \pi]$. By the Arzela Ascoli theorem, there is a further subsequence, $L_{n_{k_{l}}} f$ which converges uniformly on $[-\pi, \pi]$. But by Problem $8 L_{n} f(x) \rightarrow f(x)$.
12. Let $X$ be a normed linear space and let $M$ be a convex open set containing 0 . Define

$$
\rho(x)=\inf \left\{t>0: \frac{x}{t} \in M\right\} .
$$

Show $\rho$ is a gauge function defined on $X$. This particular example is called a Minkowski functional. It is of fundamental importance in the study of locally convex topological vector spaces. A set, $M$, is convex if $\lambda x+(1-\lambda) y \in M$ whenever $\lambda \in[0,1]$ and $x, y \in M$.
13. $\uparrow$ The Hahn Banach theorem can be used to establish separation theorems. Let $M$ be an open convex set containing 0 . Let $x \notin M$. Show there exists $x^{*} \in X^{\prime}$ such that $\operatorname{Re} x^{*}(x) \geq 1>\operatorname{Re} x^{*}(y)$ for all $y \in M$. Hint: If $y \in M, \rho(y)<1$. Show this. If $x \notin M, \rho(x) \geq 1$. Try $f(\alpha x)=\alpha \rho(x)$ for $\alpha \in \mathbb{R}$. Then extend $f$ to the whole space using the Hahn Banach theorem and call the result $F$, show $F$ is continuous, then fix it so $F$ is the real part of $x^{*} \in X^{\prime}$.
14. A Banach space is said to be strictly convex if whenever $\|x\|=\|y\|$ and $x \neq y$, then

$$
\left\|\frac{x+y}{2}\right\|<\|x\|
$$

$F: X \rightarrow X^{\prime}$ is said to be a duality map if it satisfies the following: a.) $\|F(x)\|=$ $\|x\|$. b.) $F(x)(x)=\|x\|^{2}$. Show that if $X^{\prime}$ is strictly convex, then such a duality map exists. The duality map is an attempt to duplicate some of the features of the Riesz map in Hilbert space which is discussed in the chapter on Hilbert space. Hint: For an arbitrary Banach space, let

$$
F(x) \equiv\left\{x^{*}:\left\|x^{*}\right\| \leq\|x\| \text { and } x^{*}(x)=\|x\|^{2}\right\}
$$

Show $F(x) \neq \emptyset$ by using the Hahn Banach theorem on $f(\alpha x)=\alpha\|x\|^{2}$. Next show $F(x)$ is closed and convex. Finally show that you can replace the inequality in the definition of $F(x)$ with an equal sign. Now use strict convexity to show there is only one element in $F(x)$.
15. Prove the following theorem which is an improved version of the open mapping theorem, [42]. Let $X$ and $Y$ be Banach spaces and let $A \in \mathscr{L}(X, Y)$. Then the following are equivalent.

$$
A X=Y
$$

$A$ is an open map.
There exists a constant $M$ such that for every $y \in Y$, there exists $x \in X$ with $y=A x$ and

$$
\|x\| \leq M\|y\|
$$

Note this gives the equivalence between $A$ being onto and $A$ being an open map. The open mapping theorem says that if $A$ is onto then it is open.
16. Suppose $D \subseteq X$ and $D$ is dense in $X$. Suppose $L: D \rightarrow Y$ is linear and $\|L x\| \leq K\|x\|$ for all $x \in D$. Show there is a unique extension of $L, \widetilde{L}$, defined on all of $X$ with $\|\widetilde{L} x\| \leq K| | x \|$ and $\widetilde{L}$ is linear. You do not get uniqueness when you use the Hahn Banach theorem. Therefore, in the situation of this problem, it is better to use this result.
17. $\uparrow$ A Banach space is uniformly convex if whenever $\left\|x_{n}\right\|,\left\|y_{n}\right\| \leq 1$ and $\left\|x_{n}+y_{n}\right\| \rightarrow$ 2 , it follows that $\left\|x_{n}-y_{n}\right\| \rightarrow 0$. Show uniform convexity implies strict convexity (See Problem 14). Hint: Suppose it is not strictly convex. Then there exist $\|x\|$
and $\|y\|$ both equal to 1 and $\left\|\frac{x_{n}+y_{n}}{2}\right\|=1$ consider $x_{n} \equiv x$ and $y_{n} \equiv y$, and use the conditions for uniform convexity to get a contradiction. It can be shown that $L^{p}$ is uniformly convex whenever $\infty>p>1$. See Hewitt and Stromberg [64] or Ray [109].
18. Show that a closed subspace of a reflexive Banach space is reflexive. Hint: The proof of this is an exercise in the use of the Hahn Banach theorem. Let $Y$ be the closed subspace of the reflexive space $X$ and let $y^{* *} \in Y^{\prime \prime}$. Then $i^{* *} y^{* *} \in X^{\prime \prime}$ and so $i^{* *} y^{* *}=J x$ for some $x \in X$ because $X$ is reflexive. Now argue that $x \in Y$ as follows. If $x \notin Y$, then there exists $x^{*}$ such that $x^{*}(Y)=0$ but $x^{*}(x) \neq 0$. Thus, $i^{*} x^{*}=0$. Use this to get a contradiction. When you know that $x=y \in Y$, the Hahn Banach theorem implies $i^{*}$ is onto $Y^{\prime}$ and for all $x^{*} \in X^{\prime}$,

$$
y^{* *}\left(i^{*} x^{*}\right)=i^{* *} y^{* *}\left(x^{*}\right)=J x\left(x^{*}\right)=x^{*}(x)=x^{*}(i y)=i^{*} x^{*}(y) .
$$

19. We say that $x_{n}$ converges weakly to $x$ if for every $x^{*} \in X^{\prime}, x^{*}\left(x_{n}\right) \rightarrow x^{*}(x) . x_{n} \rightharpoonup x$ denotes weak convergence. Show that if $\left\|x_{n}-x\right\| \rightarrow 0$, then $x_{n} \rightharpoonup x$.
20. $\uparrow$ Show that if $X$ is uniformly convex, then if $x_{n} \rightharpoonup x$ and $\left\|x_{n}\right\| \rightarrow\|x\|$, it follows $\left\|x_{n}-x\right\| \rightarrow 0$. Hint: Use Lemma 17.2.9 to obtain $f \in X^{\prime}$ with $\|f\|=1$ and $f(x)=$ $\|x\|$. See Problem 17 for the definition of uniform convexity. Now by the weak convergence, you can argue that if $x \neq 0, f\left(x_{n} /\left\|x_{n}\right\|\right) \rightarrow f(x /\|x\|)$. You also might try to show this in the special case where $\left\|x_{n}\right\|=\|x\|=1$.
21. Suppose $L \in \mathscr{L}(X, Y)$ and $M \in \mathscr{L}(Y, Z)$. Show $M L \in \mathscr{L}(X, Z)$ and that $(M L)^{*}=$ $L^{*} M^{*}$.
22. Let $X$ and $Y$ be Banach spaces and suppose $f \in \mathscr{L}(X, Y)$ is compact. Recall this means that if $B$ is a bounded set in $X$, then $f(B)$ has compact closure in $Y$. Show that $f^{*}$ is also a compact map. Hint: Take a bounded subset of $Y^{\prime}, S$. You need to show $f^{*}(S)$ is totally bounded. You might consider using the Ascoli Arzela theorem on the functions of $S$ applied to $f(B)$ where $B$ is the closed unit ball in $X$.

## Chapter 18

## Topological Vector Spaces

### 18.1 Fundamental Considerations

The right context to consider certain topics like separation theorems is in locally convex topological vector spaces, a generalization of normed linear spaces. Let $X$ be a vector space and let $\Psi$ be a collection of functions defined on $X$ such that if $\rho \in \Psi$,

$$
\begin{gathered}
\rho(x+y) \leq \rho(x)+\rho(y) \\
\rho(a x)=|a| \rho(x) \text { if } a \in \mathbb{F} \\
\rho(x) \geq 0
\end{gathered}
$$

where $\mathbb{F}$ denotes the field of scalars, either $\mathbb{R}$ or $\mathbb{C}$, assumed to be $\mathbb{C}$ unless otherwise specified. These functions are called seminorms because it is not necessarily true that $x=0$ when $\rho(x)=0$. A basis for a topology, $\mathscr{B}$, is defined as follows.

Definition 18.1.1 For $A$ a finite subset of $\Psi$ and $r>0$,

$$
B_{A}(x, r) \equiv\{y \in X: \rho(x-y)<r \text { for all } \rho \in A\}
$$

Then

$$
\mathscr{B} \equiv\left\{B_{A}(x, r): x \in X, r>0, \text { and } A \subseteq \Psi, A \text { finite }\right\} .
$$

That this really is a basis is the content of the next theorem.
Theorem 18.1.2 $\mathscr{B}$ is the basis for a topology.
Proof: I need to show that if $B_{A}\left(x, r_{1}\right)$ and $B_{B}\left(y, r_{2}\right)$ are two elements of $\mathscr{B}$ and if $z \in$ $B_{A}\left(x, r_{1}\right) \cap B_{B}\left(y, r_{2}\right)$, then there exists $U \in \mathscr{B}$ such that

$$
z \in U \subseteq B_{A}\left(x, r_{1}\right) \cap B_{B}\left(y, r_{2}\right)
$$

Let

$$
\begin{aligned}
r= & \min \left(\min \left\{\left(r_{1}-\rho(z-x)\right): \rho \in A\right\},\right. \\
& \left.\min \left\{\left(r_{2}-\rho(z-y)\right): \rho \in B\right\}\right)
\end{aligned}
$$

and consider $B_{A \cup B}(z, r)$. If $w$ belongs to this set, then for $\rho \in A$,

$$
\rho(w-z)<r_{1}-\rho(z-x) .
$$

Hence

$$
\rho(w-x) \leq \rho(w-z)+\rho(z-x)<r_{1}
$$

for each $\rho \in A$ and so $B_{A \cup B}(z, r) \subseteq B_{A}\left(x, r_{1}\right)$. Similarly, $B_{A \cup B}(z, r) \subseteq B_{B}\left(y, r_{2}\right)$. This proves the theorem.

Let $\tau$ be the topology consisting of unions of all subsets of $\mathscr{B}$. Then $(X, \tau)$ is a locally convex topological vector space.

Theorem 18.1.3 The vector space operations of addition and scalar multiplication are continuous. More precisely,

$$
+: X \times X \rightarrow X, \cdot: \mathbb{F} \times X \rightarrow X
$$

are continuous.
Proof: It suffices to show $+^{-1}(B)$ is open in $X \times X$ and $\cdot^{-1}(B)$ is open in $\mathbb{F} \times X$ if $B$ is of the form

$$
B=\{y \in X: \rho(y-x)<r\}
$$

because finite intersections of such sets form the basis $\mathscr{B}$. (This collection of sets is a subbasis.) Suppose $u+v \in B$ where $B$ is described above. Then

$$
\rho(u+v-x)<\lambda r
$$

for some $\lambda<1$. Consider

$$
B_{\rho}(u, \delta) \times B_{\rho}(v, \boldsymbol{\delta})
$$

If $\left(u_{1}, v_{1}\right)$ is in this set, then

$$
\begin{aligned}
\rho\left(u_{1}+v_{1}-x\right) & \leq \rho(u+v-x)+\rho\left(u_{1}-u\right)+\rho\left(v_{1}-v\right) \\
& <\lambda r+2 \delta .
\end{aligned}
$$

Let $\delta$ be positive but small enough that

$$
2 \delta+\lambda r<r
$$

Thus this choice of $\delta$ shows that $+^{-1}(B)$ is open and this shows + is continuous.
Now suppose $\alpha z \in B$. Then

$$
\rho(\alpha z-x)<\lambda r<r
$$

for some $\lambda \in(0,1)$. Let $\delta>0$ be small enough that $\delta<1$ and also

$$
\lambda r+\delta(\rho(z)+1)+\delta|\alpha|<r
$$

Then consider $(\beta, w) \in B(\alpha, \delta) \times B_{\rho}(z, \delta)$.

$$
\begin{aligned}
\rho(\beta w-x)-\rho(\alpha z-x) & \leq \rho(\beta w-\alpha z) \\
& \leq|\beta-\alpha| \rho(w)+\rho(w-z)|\alpha| \\
& \leq|\beta-\alpha|(\rho(z)+1)+\rho(w-z)|\alpha| \\
& <\delta(\rho(z)+1)+\delta|\alpha|
\end{aligned}
$$

Hence

$$
\rho(\beta w-x)<\lambda r+\delta(\rho(z)+1)+\delta|\alpha|<r
$$

and so

$$
B(\alpha, \delta) \times B_{\rho}(z, \delta) \subseteq \cdot^{-1}(B)
$$

This proves the theorem.

Theorem 18.1.4 Let $x$ be given and let $f_{x}(y)=x+y$. Then $f_{x}$ is $1-1$, onto, and continuous. If $\alpha \neq 0$ and $g_{\alpha}(x)=\alpha x$, then $g_{\alpha}$ is also $1-1$ onto and continuous.

Proof: The assertions about $1-1$ and onto are obvious. It remains to show $f_{x}$ and $g_{\alpha}$ are continuous. Let $B=B_{\rho}(z, r)$ and consider $f_{x}^{-1}(B)$. Then it is easy to see that

$$
f_{x}^{-1}(B)=B_{\rho}(z-x, r)
$$

and so $f_{x}$ is continuous. To see that $g_{\alpha}$ is continuous, note that

$$
g_{\alpha}^{-1}(B)=B_{\rho}\left(\frac{z}{\alpha}, \frac{r}{|\alpha|}\right) .
$$

This proves the theorem.
As in the case of a normed linear space, the vector space of continuous linear functionals, is denoted by $X^{\prime}$.

Definition 18.1.5 Define, for A a finite subset of $\Psi$,

$$
\rho_{A}(x)=\max \{\rho(x): \rho \in A\}
$$

The following theorem is the equivalent to the earlier theorems concerning continuous linear functionals on normed linear spaces.

Theorem 18.1.6 The following are equivalent for $f$, a linear function mapping $X$ to $\mathbb{F}$.

$$
\begin{equation*}
f \text { is continuous at } 0 . \tag{18.1.1}
\end{equation*}
$$

For some $A \subseteq \Psi, A$ finite,

$$
\begin{equation*}
|f(x)| \leq C \rho_{A}(x) \tag{18.1.2}
\end{equation*}
$$

for all $x \in X$ where the constant may depend on $A$ but is independent of $x$.

$$
\begin{equation*}
f \text { is continuous at } x \tag{18.1.3}
\end{equation*}
$$

for all $x$.
Proof: Clearly 18.1.3 implies 18.1.1. Suppose 18.1.1. Then

$$
0=f(0) \in B(0,1) \subseteq \mathbb{F}
$$

Since $f$ is continuous at $0,0 \in f^{-1}(B(0,1))$ and there exists an open set $V \in \tau$ such that

$$
0 \in V \subseteq f^{-1}(B(0,1))
$$

Then $0 \in B_{A}(0, r) \subseteq V$ for some $r$ and some $A \subseteq \Psi, A$ finite. Hence

$$
|f(y)|<1 \text { if } \rho_{A}(y)<r .
$$

Since $f$ is linear

$$
|f(x)| \leq \frac{2}{r} \rho_{A}(x)
$$

To see this, note that if $x \neq 0$, then

$$
\frac{r x}{2 \rho_{A}(x)} \in B_{A}(0, r)
$$

and so

$$
\frac{|f(r x)|}{2 \rho_{A}(x)} \leq 1
$$

which shows that 18.1.1 implies 18.1.2.
Now suppose 18.1.2 and suppose $f(x) \in V$, an open set in $\mathbb{F}$. Then

$$
f(x) \in B(f(x), r) \subseteq V
$$

for some $r>0$. Suppose $\rho_{A}(x-y)<r\left(C_{A}+1\right)^{-1}$. Then

$$
|f(x)-f(y)|=|f(x-y)| \leq C_{A} \rho_{A}(y-x)<r .
$$

Hence

$$
f\left(B_{A}\left(x, r\left(C_{A}+1\right)^{-1}\right)\right) \subseteq B(f(x), r) \subseteq V
$$

Thus $f$ is continuous at $x$. This proves the theorem.
What are some examples of locally convex topological vector spaces? It is obvious that any normed linear space is such an example. More generally, here is a theorem which shows how to make any vector space into a locally convex topological vector space.

Theorem 18.1.7 Let $X$ be a vector space and let $Y$ be a vector space of linear functionals defined on $X$. For each $y \in Y$, define

$$
\rho_{y}(x) \equiv|y(x)| .
$$

Then the collection of seminorms $\left\{\rho_{y}\right\}_{y \in Y}$ defined on $X$ makes $X$ into a locally convex topological vector space and $Y=X^{\prime}$.

Proof: Clearly $\left\{\rho_{y}\right\}_{y \in Y}$ is a collection of seminorms defined on $X$; so, $X$ supplied with the topology induced by this collection of seminorms is a locally convex topological vector space. Is $Y=X^{\prime}$ ?

Let $y \in Y$, let $U \subseteq \mathbb{F}$ be open and let $x \in y^{-1}(U)$. Then $B(y(x), r) \subseteq U$ for some $r>0$. Letting $A=\{y\}$, it is easy to see from the definition that $B_{A}(x, r) \subseteq y^{-1}(U)$ and so $y^{-1}(U)$ is an open set as desired. Thus, $Y \subseteq X^{\prime}$.

Now suppose $z \in X^{\prime}$. Then by 18.1 .2 , there exists a finite subset of $Y, A \equiv\left\{y_{1}, \cdots, y_{n}\right\}$, such that

$$
|z(x)| \leq C \rho_{A}(x)
$$

Let

$$
\pi(x) \equiv\left(y_{1}(x), \cdots, y_{n}(x)\right)
$$

and let $f$ be a linear map from $\pi(X)$ to $\mathbb{F}$ defined by

$$
f(\pi x) \equiv z(x)
$$

(This is well defined because if $\pi(x)=\pi\left(x_{1}\right)$, then $y_{i}(x)=y_{i}\left(x_{1}\right)$ for $i=1, \cdots, n$ and so

$$
\rho_{A}\left(x-x_{1}\right)=0 .
$$

Thus,

$$
\left.\left|z\left(x_{1}\right)-z(x)\right|=\left|z\left(x_{1}-x\right)\right| \leq C \rho_{A}\left(x-x_{1}\right)=0 .\right)
$$

Extend $f$ to all of $\mathbb{F}^{n}$ and denote the resulting linear map by $F$. Then there exists a vector

$$
\alpha=\left(\alpha_{1}, \cdots, \alpha_{n}\right) \in \mathbb{F}^{n}
$$

with $\alpha_{i}=F\left(\mathbf{e}_{i}\right)$ such that

$$
F(\beta)=\alpha \cdot \beta
$$

Hence for each $x \in X$,

$$
z(x)=f(\pi x)=F(\pi x)=\sum_{i=1}^{n} \alpha_{i} y_{i}(x)
$$

and so

$$
z=\sum_{i=1}^{n} \alpha_{i} y_{i} \in Y
$$

This proves the theorem.

### 18.2 Separation Theorems

It will always be assumed that $X$ is a locally convex topological vector space. A set, $K$, is said to be convex if whenever $x, y \in K$,

$$
\lambda x+(1-\lambda) y \in K
$$

for all $\lambda \in[0,1]$.
Definition 18.2.1 Let $U$ be an open convex set containing 0 and define

$$
m(x) \equiv \inf \{t>0: x / t \in U\}
$$

This is called a Minkowski functional.
Proposition 18.2.2 Let $X$ be a locally convex topological vector space. Then $m$ is defined on $X$ and satisfies

$$
\begin{gather*}
m(x+y) \leq m(x)+m(y)  \tag{18.2.4}\\
m(\lambda x)=\lambda m(x) \text { if } \lambda>0 . \tag{18.2.5}
\end{gather*}
$$

Thus, $m$ is a gauge function on $X$.

Proof: Let $x \in X$ be arbitrary. There exists $A \subseteq \Psi$ such that

$$
0 \in B_{A}(0, r) \subseteq U
$$

Then

$$
\frac{r x}{2 \rho_{A}(x)} \in B_{A}(0, r) \subseteq U
$$

which implies

$$
\begin{equation*}
\frac{2 \rho_{A}(x)}{r} \geq m(x) \tag{18.2.6}
\end{equation*}
$$

Thus $m(x)$ is defined on $X$.
Let $x / t \in U, y / s \in U$. Then since $U$ is convex,

$$
\frac{x+y}{t+s}=\left(\frac{t}{t+s}\right)\left(\frac{x}{t}\right)+\left(\frac{s}{t+s}\right)\left(\frac{y}{s}\right) \in U .
$$

It follows that

$$
m(x+y) \leq t+s
$$

Choosing $s, t$ such that $t-\varepsilon<m(x)$ and $s-\varepsilon<m(y)$,

$$
m(x+y) \leq m(x)+m(y)+2 \varepsilon .
$$

Since $\varepsilon$ is arbitrary, this shows 18.2.4. It remains to show 18.2.5. Let $x / t \in U$. Then if $\lambda>0$,

$$
\frac{\lambda x}{\lambda t} \in U
$$

and so $m(\lambda x) \leq \lambda t$. Thus $m(\lambda x) \leq \lambda m(x)$ for all $\lambda>0$. Hence

$$
m(x)=m\left(\lambda^{-1} \lambda x\right) \leq \lambda^{-1} m(\lambda x) \leq \lambda^{-1} \lambda m(x)=m(x)
$$

and so

$$
\lambda m(x)=m(\lambda x) .
$$

This proves the proposition.
Lemma 18.2.3 Let $U$ be an open convex set containing 0 and let $q \notin U$. Then there exists $f \in X^{\prime}$ such that

$$
\operatorname{Re} f(q)>\operatorname{Re} f(x)
$$

for all $x \in U$.
Proof: Let $m$ be the Minkowski functional just defined and let

$$
F(c q)=c m(q)
$$

for $c \in \mathbb{R}$. If $c>0$ then

$$
F(c q)=m(c q)
$$

while if $c \leq 0$,

$$
F(c q)=c m(q) \leq 0 \leq m(c q) .
$$

By the Hahn Banach theorem, $F$ has an extension, $g$, defined on all of $X$ satisfying

$$
g(x+y)=g(x)+g(y), g(c x)=c g(x)
$$

for all $c \in \mathbb{R}$, and

$$
g(x) \leq m(x)
$$

Thus, $g(-x) \leq m(-x)$ and so

$$
-m(-x) \leq g(x) \leq m(x)
$$

It follows as in 18.2.6 that for some $A \subseteq \Psi, A$ finite, and $r>0$,

$$
\begin{gathered}
|g(x)| \leq m(x)+m(-x) \\
\leq \frac{2}{r} \rho_{A}(x)+\frac{2}{r} \rho_{A}(-x)=\frac{4}{r} \rho_{A}(x)
\end{gathered}
$$

because

$$
\rho_{A}(-x)=|-1| \rho_{A}(x)=\rho_{A}(x)
$$

Hence $g$ is continuous by Theorem 18.1.6. Now define

$$
f(x) \equiv g(x)-i g(i x)
$$

Thus $f$ is linear and continuous so $f \in X^{\prime}$ and $\operatorname{Re} f(x)=g(x)$. But for $x \in U$, Theorem 18.1.3 implies that $x / t \in U$ for some $t<1$ and so $m(x)<1$. Since $U$ is convex and $0 \in U$, it follows $q / t \notin U$ if $t<1$ because if it were,

$$
q=t\left(\frac{q}{t}\right)+(1-t) 0 \in U
$$

Therefore, $m(q) \geq 1$ and for $x \in U$,

$$
\operatorname{Re} f(x)=g(x) \leq m(x)<1 \leq m(q)=g(q)=\operatorname{Re} f(q)
$$

and this proves the lemma.
Corollary 18.2.4 Let $U$ be an open nonempty convex set and let $q \notin U$. Then there exists $f \in X^{\prime}$ such that

$$
f(q)>f(x)
$$

for all $x \in U$.
Proof: Let $u_{0} \in U$ and consider $\hat{U} \equiv U-u_{0}$. Then $0 \in \hat{U}$ and $q-u_{0} \notin \hat{U}$. By separation theorems, Lemma 18.2.3 there exists $f \in X^{\prime}$ such that

$$
f\left(q-u_{0}\right)>f\left(x-u_{0}\right)
$$

for all $x \in U$. Thus $f(q)>f(x)$ for all $x \in U$.

Theorem 18.2.5 Let $K$ be closed and convex in a locally convex topological vector space and let $p \notin K$. Then there exists a real number $c$, and $f \in X^{\prime}$ such that

$$
\operatorname{Re} f(p)>c>\operatorname{Re} f(k)
$$

for all $k \in K$.
Proof: Since $K$ is closed, and $p \notin K$, there exists a finite subset of $\Psi, A$, and a positive $r>0$ such that

$$
K \cap B_{A}(p, 2 r)=\emptyset .
$$

Pick $k_{0} \in K$ and let

$$
U=K+B_{A}(0, r)-k_{0}, q=p-k_{0}
$$

It follows that $U$ is an open convex set containing 0 and $q \notin U$. Therefore, by Lemma 18.2.3, there exists $f \in X^{\prime}$ such that

$$
\begin{equation*}
\operatorname{Re} f\left(p-k_{0}\right)=\operatorname{Re} f(q)>\operatorname{Re} f\left(k+e-k_{0}\right) \tag{18.2.7}
\end{equation*}
$$

for all $k \in K$ and $e \in B_{A}(0, r)$. If $\operatorname{Re} f(e)=0$ for all $e \in B_{A}(0, r)$, then $\operatorname{Re} f=0$ and 18.2.7 could not hold. Therefore, $\operatorname{Re} f(e)>0$ for some $e \in B_{A}(0, r)$ and so,

$$
\operatorname{Re} f(p)>\operatorname{Re} f(k)+\operatorname{Re} f(e)
$$

for all $k \in K$. Let $c_{1} \equiv \sup \{\operatorname{Re} f(k): k \in K\}$. Then for all $k \in K$,

$$
\operatorname{Re} f(p) \geq c_{1}+\operatorname{Re} f(e)>c_{1}+\frac{\operatorname{Re} f(e)}{2}>\operatorname{Re} f(k)
$$

Let $c=c_{1}+\frac{\operatorname{Re} f(e)}{2}$.


Corollary 18.2.6 In the situation of the above theorem, there exist real numbers $c, d$ such that $\operatorname{Re} f(p)>d>c>\operatorname{Re} f(k)$ for all $k \in K$.

Proof: From the theorem, there exists $\hat{c}$ such that $\operatorname{Re} f(p)>\hat{c}>\operatorname{Re} f(k)$ for all $k \in$ $K$. Thus $\operatorname{Re} f(p)>\hat{c} \geq \sup _{k \in K} \operatorname{Re} f(k)$. Now choose $d, c$ such that $f(p)>d>c>\hat{c} \geq$ $\sup _{k \in K} \operatorname{Re} f(k)>f(k)$.

Note that if the field of scalars comes from $\mathbb{R}$ rather than $\mathbb{C}$ there is no essential change to the above conclusions. Just eliminate all references to the real part.

### 18.2.1 Convex Functionals

As an important application, this theorem gives the basis for proving something about lower semicontinuity of functionals.

Definition 18.2.7 Let $X$ be a Banach space and let $\phi: X \rightarrow(0, \infty]$ be convex and lower semicontinuous. This means whenever $x \in X$ and $\lim _{n \rightarrow \infty} x_{n}=x$,

$$
\phi(x) \leq \lim \inf _{n \rightarrow \infty} \phi\left(x_{n}\right)
$$

Also assume $\phi$ is not identically equal to $\infty$.
Lemma 18.2.8 Let $X, Y$ be two Banach spaces. Then letting

$$
\|(x, y)\| \equiv \max \left(\|x\|_{X},\|y\|_{Y}\right)
$$

it follows $X \times Y$ is a Banach space and $\phi \in(X \times Y)^{\prime}$ if and only if there exist $x^{*} \in X^{\prime}$ and $y^{*} \in Y^{\prime}$ such that

$$
\phi((x, y))=x^{*}(x)+y^{*}(y) .
$$

The topology coming from this norm is called the strong topology.
Proof: Most of these conclusions are obvious. In particular it is clear $X \times Y$ is a Banach space with the given norm. Let $\phi \in(X \times Y)^{\prime}$. Also let $\pi_{X}(x, y) \equiv(x, 0)$ and $\pi_{Y}(x, y) \equiv$ $(0, y)$. Then each of $\pi_{X}$ and $\pi_{Y}$ is continuous and

$$
\begin{aligned}
\phi((x, y)) & =\phi\left(\pi_{X}+\pi_{Y}\right)((x, y)) \\
& =\phi((x, 0))+\phi((0, y))
\end{aligned}
$$

Thus $\phi \circ \pi_{X}$ and $\phi \circ \pi_{Y}$ are both continuous and their sum equals $\phi$. Let $x^{*}(x) \equiv \phi \circ \pi_{X}(x, 0)$ and let $y^{*} \equiv \phi \circ \pi_{Y}(x, 0)$. Then it is clear both $x^{*}$ and $y^{*}$ are continuous and linear defined on $X$ and $Y$ respectively. Also, if $\left(x^{*}, y^{*}\right) \in X^{\prime} \times Y^{\prime}$, then if $\phi((x, y)) \equiv x^{*}(x)+y^{*}(y)$, it follows $\phi \in(X \times Y)^{\prime}$. This proves the lemma.

Lemma 18.2.9 Let $\phi$ be a functional as described in Definition 18.2.7. Then $\phi$ is lower semicontinuous if and only if the epigraph of $\phi$ is closed in $X \times \mathbb{R}$ with the strong topology. Here the epigraph is defined as

$$
\operatorname{epi}(\phi) \equiv\{(x, y): y \geq \phi(x)\}
$$

In this case the functional is called strongly lower semicontinuous.
Proof: First suppose epi $(\phi)$ is closed and suppose $x_{n} \rightarrow x$. Let $l<\phi(x)$. Then $(x, l) \notin$ epi $(\phi)$ and so there exists $\delta>0$ such that if $|x-y|<\delta$ and $|\alpha-l|<\delta$, then $\alpha<\phi(y)$. This implies that if $|x-y|<\delta$ and $\alpha<l+\delta$, then the above holds. Therefore, $\left(x_{n}, \phi\left(x_{n}\right)\right)$, being in epi $(\phi)$ cannot satisfy both conditions,

$$
\left|x_{n}-x\right|<\delta, \phi\left(x_{n}\right)<l+\delta
$$

However, for all $n$ large enough, the first condition is satisfied. Consequently, for all $n$ large enough, $\phi\left(x_{n}\right) \geq l+\delta \geq l$. Thus

$$
\lim _{n \rightarrow \infty} \phi\left(x_{n}\right) \geq l
$$

and since $l<\phi(x)$ is arbitrary, it follows

$$
\lim \inf _{n \rightarrow \infty} \phi\left(x_{n}\right) \geq \phi(x)
$$

Next suppose the condition about the liminf. If epi $(\phi)$ is not closed, then there exists $(x, l) \notin \mathrm{epi}(\phi)$ which is a limit point of points of epi $(\phi)$, Thus there exists $\left(x_{n}, l_{n}\right) \in \operatorname{epi}(\phi)$ such that $\left(x_{n}, l_{n}\right) \rightarrow(x, l)$ and so

$$
l=\lim \inf _{n \rightarrow \infty} l_{n} \geq \lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty} \phi\left(x_{n}\right) \geq \phi(x)
$$

contradicting $(x, l) \notin$ epi $(\phi)$. This proves the lemma.
Definition 18.2.10 Let $\phi$ be convex and defined on $X$, a Banach space. Then $\phi$ is said to be weakly lower semicontinuous if $\mathrm{epi}(\phi)$ is closed in $X \times \mathbb{R}$ where a basis for the topology of $X \times \mathbb{R}$ consists of sets of the form $U \times(a, b)$ for $U$ a weakly open set in $X$.

Theorem 18.2.11 Let $\phi$ be a lower semicontinuous convex functional as described in Definition 18.2 .7 and let $X$ be a real Banach space. Then $\phi$ is also weakly lower semicontinuous.

Proof: By Lemma 18.2.9 epi $(\phi)$ is closed in $X \times \mathbb{R}$ with the strong topology as well as being convex. Letting $(z, l) \notin \operatorname{epi}(\phi)$, it follows from Theorem 18.2.5 and Lemma 18.2.8 there exists $\left(x^{*}, \alpha\right) \in X^{\prime} \times \mathbb{R}$ such that for some $c$

$$
x^{*}(z)+\alpha l>c>x^{*}(x)+\alpha \beta
$$

whenever $\beta \geq \phi(x)$. Consider $B_{\left\{\left(x^{*}, \alpha\right)\right\}}((z, l), r)$ where $r$ is chosen so small that if $(y, \gamma) \in$ $B_{\left\{\left(x^{*}, \alpha\right)\right\}}((z, l), r)$, then

$$
x^{*}(y)+\alpha \gamma>c .
$$

This shows that the complement of epi $(\phi)$ is weakly open and this proves the theorem.
Corollary 18.2.12 Let $\phi$ be a lower semicontinuous convex functional as described in Definition 18.2 .7 and let $X$ be a real Banach space. Then if $x_{n}$ converges weakly to $x$, it follows that

$$
\phi(x) \leq \lim \inf _{n \rightarrow \infty} \phi\left(x_{n}\right)
$$

Proof: Let $l<\phi(x)$ so that $(x, l) \notin \operatorname{epi}(\phi)$. Then by Theorem 18.2.11 there exists $B \times(-\infty, l+\delta)$ such that $B$ is a weakly open set in $X$ containing $x$ and

$$
B \times(-\infty, l+\delta) \subseteq \operatorname{epi}(\phi)^{C}
$$

Thus $\left(x_{n}, \phi\left(x_{n}\right)\right) \notin B \times(-\infty, l+\delta)$ for all $n$. However, $x_{n} \in B$ for all $n$ large enough. Therefore, for those values of $n$, it must be the case that $\phi\left(x_{n}\right) \notin(-\infty, l+\delta)$ and so

$$
\lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty} \phi\left(x_{n}\right) \geq l+\delta \geq l
$$

which shows, since $l<\phi(x)$ is arbitrary that

$$
\lim _{n \rightarrow \infty} \phi\left(x_{n}\right) \geq \phi(x)
$$

This proves the corollary.
The following is a convenient fact which follows from the above.
Proposition 18.2.13 Let $A$ be a linear operator which maps a real normed linear space $\left(X,\|\cdot\|_{X}\right)$ to a real normed linear space $\left(Y,\|\cdot\|_{Y}\right)$. Then $x_{n} \rightarrow x$ strongly implies $A x_{n} \rightarrow A x$ if and only if whenever $x_{n} \rightarrow x$ weakly, it follows that $A x_{n} \rightarrow A x$ weakly.

Proof: $\Rightarrow$ Define $\phi(x) \equiv f(A x)$ where $f \in Y^{\prime}$. Then $\phi$ is convex and continuous. Therefore, if $x_{n} \rightarrow x$ weakly, then

$$
\phi(x)=f(A x) \leq \lim \inf _{n \rightarrow \infty} f\left(A x_{n}\right)=\lim \inf _{n \rightarrow \infty} \phi\left(x_{n}\right)
$$

Then substituting $-A$ for $A$,

$$
-f(A x) \leq \lim \inf _{n \rightarrow \infty} f\left(-A x_{n}\right), f(A x) \geq \lim \sup _{n \rightarrow \infty} f\left(A x_{n}\right)
$$

which shows that for each $f \in Y^{\prime}$,

$$
\lim \sup _{n \rightarrow \infty} f\left(A x_{n}\right) \leq f(A x) \leq \lim _{n \rightarrow \infty} f\left(A x_{n}\right)
$$

and so the second condition holds.
$\Leftarrow$ By the second condition, $x \rightarrow f(A x)$ satisfies the condition that if $x_{n} \rightarrow x$ weakly, then

$$
f(A x)=\lim _{n \rightarrow \infty} f\left(A x_{n}\right)
$$

If $A$ is not bounded, then there exists $x_{n},\left\|x_{n}\right\| \leq 1$ but $\left\|A x_{n}\right\| \geq n$. It follows that $x_{n} / \sqrt{n} \rightarrow 0$ and so $A\left(\frac{x_{n}}{\sqrt{n}}\right) \rightarrow 0$ weakly. Therefore, $A\left(\frac{x_{n}}{\sqrt{n}}\right)$ is bounded contrary to the construction which says that $\left\|A\left(\frac{x_{n}}{\sqrt{n}}\right)\right\| \geq \sqrt{n}$. Since $A$ is bounded, it must be continuous.

### 18.2.2 More Separation Theorems

There are other separation theorems which can be proved in a similar way. The next theorem considers the separation of an open convex set from a convex set.
Theorem 18.2.14 Let $A$ and $B$ be disjoint, convex and nonempty sets with $B$ open. Then there exists $f \in X^{\prime}$ such that

$$
\operatorname{Re} f(a)<\operatorname{Re} f(b)
$$

for all $a \in A$ and $b \in B$.

Proof: Let $b_{0} \in B, a_{0} \in A$. Then the set

$$
B-A+a_{0}-b_{0}
$$

is open, convex, contains 0 , and does not contain $a_{0}-b_{0}$. By Lemma 18.2.3 there exists $f \in X^{\prime}$ such that

$$
\operatorname{Re} f\left(a_{0}-b_{0}\right)>\operatorname{Re} f\left(b-a+a_{0}-b_{0}\right)
$$

for all $a \in A$ and $b \in B$. Therefore, for all $a \in A, b \in B$,

$$
\operatorname{Re} f(b)>\operatorname{Re} f(a)
$$

Before giving another separation theorem, here is a lemma.
Lemma 18.2.15 If $B$ is convex, then $\operatorname{int}(B) \equiv$ union of all open sets contained in $B$ is convex. Also, if $\operatorname{int}(B) \neq \emptyset$, then $B \subseteq \overline{\operatorname{int}(B)}$.

Proof: Suppose $x, y \in \operatorname{int}(B)$. Then there exists $r>0$ and a finite set $A \subseteq \Psi$ such that

$$
B_{A}(x, r), B_{A}(y, r) \subseteq B
$$

Let

$$
V \equiv \cup_{\lambda \in[0,1]} \lambda B_{A}(x, r)+(1-\lambda) B_{A}(y, r) .
$$

Then $V$ is open, $V \subseteq B$, and if $\lambda \in[0,1]$, then

$$
\lambda x+(1-\lambda) y \in V \subseteq B
$$

Therefore, $\operatorname{int}(B)$ is convex as claimed.
Now let $y \in B$ and $x \in \operatorname{int}(B)$. Let

$$
x \in B_{A}(x, r) \subseteq \operatorname{int}(B)
$$

and let $x_{\lambda} \equiv(1-\lambda) x+\lambda y$. Define the open cone,

$$
C \equiv \cup_{\lambda \in[0,1)} B_{A}\left(x_{\lambda},(1-\lambda) r\right)
$$

Thus $C$ is represented in the following picture.


I claim $C \subseteq B$ as suggested in the picture. To see this, let

$$
z \in B_{A}\left(x_{\lambda},(1-\lambda) r\right), \lambda \in(0,1)
$$

Then

$$
\rho_{A}\left(z-x_{\lambda}\right)<(1-\lambda) r
$$

and so

$$
\rho_{A}\left(\frac{z}{1-\lambda}-x-\frac{\lambda y}{1-\lambda}\right)<r
$$

Therefore,

$$
\frac{z}{1-\lambda}-\frac{\lambda y}{1-\lambda} \in B_{A}(x, r) \subseteq B .
$$

It follows

$$
(1-\lambda)\left(\frac{z}{1-\lambda}-\frac{\lambda y}{1-\lambda}\right)+\lambda y=z \in B
$$

and so $C \subseteq B$ as claimed. Now this shows $x_{\lambda} \in \operatorname{int}(B)$ and $\lim _{\lambda \rightarrow 1} x_{\lambda}=y$. Thus, $y \in \overline{\operatorname{int}(B)}$ and this proves the lemma.

Note this also shows that $\bar{B}=\overline{\operatorname{int}(B)}$.
Corollary 18.2.16 Let $A, B$ be convex, nonempty sets. Suppose $\operatorname{int}(B) \neq \emptyset$ and $A \cap \operatorname{int}(B)=$ $\emptyset$. Then there exists $f \in X^{\prime}, f \neq 0$, such that for all $a \in A$ and $b \in B$,

$$
\operatorname{Re} f(b) \geq \operatorname{Re} f(a)
$$

Proof: By Theorem 18.2.14, there exists $f \in X^{\prime}$ such that for all $b \in \operatorname{int}(B)$, and $a \in A$,

$$
\operatorname{Re} f(b)>\operatorname{Re} f(a)
$$

Thus, in particular, $f \neq 0$. By Lemma 18.2.15, if $b \in B$ and $a \in A$,

$$
\operatorname{Re} f(b) \geq \operatorname{Re} f(a)
$$

This proves the theorem.
Lemma 18.2.17 If $X$ is a topological Hausdorff space then compact implies closed.
Proof: Let $K$ be compact and suppose $K^{C}$ is not open. Then there exists $p \in K^{C}$ such that

$$
V_{p} \cap K \neq \emptyset
$$

for all open sets $V_{p}$ containing $p$. Let

$$
\mathscr{C}=\left\{\left(\bar{V}_{p}\right)^{C}: V_{p} \text { is an open set containing } p\right\}
$$

Then $\mathscr{C}$ is an open cover of $K$ because if $q \in K$, there exist disjoint open sets $V_{p}$ and $V_{q}$ containing $p$ and $q$ respectively. Thus $q \in\left(\bar{V}_{p}\right)^{C}$. This is an example of an open cover of $K$ which has no finite subcover, contradicting the assumption that $K$ is compact. This proves the lemma.

Lemma 18.2.18 If $X$ is a locally convex topological vector space, and if every point is a closed set, then the seminorms and $X^{\prime}$ separate the points. This means if $x \neq y$, then for some $\rho \in \Psi$,

$$
\rho(x-y) \neq 0
$$

and for some $f \in X^{\prime}$,

$$
f(x) \neq f(y) .
$$

In this case, X is a Hausdorff space.
Proof: Let $x \neq y$. Then by Theorem 18.2.5, there exists $f \in X^{\prime}$ such that $f(x) \neq f(y)$. Thus $X^{\prime}$ separates the points. Since $f \in X^{\prime}$, Theorem 18.1.6 implies

$$
|f(z)| \leq C \rho_{A}(z)
$$

for some $A$ a finite subset of $\Psi$. Thus

$$
0<|f(x-y)| \leq C \rho_{A}(x-y)
$$

and so $\rho(x-y) \neq 0$ for some $\rho \in A \subseteq \Psi$. Now to show $X$ is Hausdorff, let

$$
0<r<\rho(x-y) 2^{-1}
$$

Then the two disjoint open sets containing $x$ and $y$ respectively are

$$
B_{\rho}(x, r) \text { and } B_{\rho}(y, r) .
$$

This proves the lemma.

### 18.3 The Weak And Weak* Topologies

The weak and weak $*$ topologies are examples which make the underlying vector space into a topological vector space. This section gives a description of these topologies. Unless otherwise specified, $X$ is a locally convex topological vector space. For $G$ a finite subset of $X^{\prime}$ define $\delta_{G}: X \rightarrow[0, \infty)$ by

$$
\delta_{G}(x)=\max \{|f(x)|: f \in G\} .
$$

Lemma 18.3.1 The functions $\delta_{G}$ for $G$ a finite subset of $X^{\prime}$ are seminorms and the sets

$$
B_{G}(x, r) \equiv\left\{y \in X: \delta_{G}(x-y)<r\right\}
$$

form a basis for a topology on $X$. Furthermore, $X$ with this topology is a locally convex topological vector space. If each point in $X$ is a closed set, then the same is true of $X$ with respect to this new topology.

Proof: It is obvious that the functions $\delta_{G}$ are seminorms and therefore the proof that the sets $B_{G}(x, r)$ form a basis for a topology is the same as in Theorem 18.1.2. To see every point is a closed set in this new topology, assuming this is true for $X$ with the original
topology, use Lemma 18.2 .18 to assert $X^{\prime}$ separates the points. Let $x \in X$ and let $y \neq x$. There exists $f \in X^{\prime}$ such that $f(x) \neq f(y)$. Let $G=\{f\}$ and consider

$$
B_{G}(y,|f(x-y)| / 2)
$$

Then this open set does not contain $x$. Thus $\{x\}^{C}$ is open and so $\{x\}$ is closed. This proves the Lemma.

This topology for $X$ is called the weak topology for $X$. For $F$ a finite subset of $X$, define $\gamma_{F}: X^{\prime} \rightarrow[0, \infty)$ by

$$
\gamma_{F}(f)=\max \{|f(x)|: x \in F\}
$$

Lemma 18.3.2 The functions $\gamma_{F}$ for $F$ a finite subset of $X$ are seminorms and the sets

$$
B_{F}(f, r) \equiv\left\{g \in X^{\prime}: \gamma_{F}(f-g)<r\right\}
$$

form a basis for a topology on $X^{\prime}$. Furthermore, $X^{\prime}$ with this topology is a locally convex topological vector space having the property that every point is a closed set.

Proof: The proof is similar to that of Lemma 18.3 .1 but there is a difference in the part where every point is shown to be a closed set. Let $f \in X^{\prime}$ and let $g \neq f$. Thus there exists $x \in X$ such that $f(x) \neq g(x)$. Let $F=\{x\}$. Then

$$
B_{F}(g,|(f-g)(x)| / 2)
$$

contains $g$ but not $f$. Thus $\{f\}^{C}$ is open and so $\{f\}$ is closed.
Note that it was not necessary to assume points in $X$ are closed sets to get this.
The topology for $X^{\prime}$ just described is called the weak $*$ topology. In terms of Theorem 18.1.7 the weak topology is obtained by letting $Y=X^{\prime}$ in that theorem while the weak $*$ topology is obtained by letting $Y=X$ with the understanding that $X$ is a vector space of linear functionals on $X^{\prime}$ defined by

$$
x\left(x^{*}\right) \equiv x^{*}(x)
$$

By Theorem 18.2.5, there is a useful result which follows immediately.
Theorem 18.3.3 Let $K$ be closed and convex in a Banach space $X$. Then it is also weakly closed. Furthermore, if $p \notin K$, there exists $f \in X^{\prime}$ such that

$$
\begin{equation*}
\operatorname{Re} f(p)>c>\operatorname{Re} f(k) \tag{18.3.8}
\end{equation*}
$$

for all $k \in K$. If $K^{*}$ is closed and convex in the dual of a Banach space, $X^{\prime}$, then it is also weak $*$ closed.

Proof: By Theorem 18.2.5 there exists $f \in X^{\prime}$ such that 18.3 .8 holds. Therefore, letting $A=\{f\}$, it follows that for $r$ small enough, $B_{A}(p, r) \cap K=\emptyset$. Thus $K$ is weakly closed. This establishes the first part.

For the second part, the seminorms for the weak $*$ toplogy are determined from $X$ and the continuous linear functionals are of the form $x^{*} \rightarrow x^{*}(x)$ where $x \in X$. Thus if $p^{*} \notin K^{*}$, it follows from Theorem 18.2.5 there exists $x \in X$ such that

$$
\operatorname{Re} p^{*}(x)>c>\operatorname{Re} k^{*}(x)
$$

for all $k^{*} \in K^{*}$. Therefore, letting $A=\{x\}, B_{A}\left(p^{*}, r\right) \cap K^{*}=\emptyset$ whenever $r$ is small enough and this shows $K^{*}$ is weak $*$ closed.

### 18.4 Mean Ergodic Theorem

The following theorem is called the mean ergodic theorem.
Theorem 18.4.1 Let $(\Omega, \mathscr{S}, \mu)$ be a finite measure space and let $T: \Omega \rightarrow \Omega$ satisfy

$$
T^{-1}(E) \in \mathscr{S}, T(E) \in \mathscr{S}
$$

for all $E \in \mathscr{S}$. Also suppose for all positive integers, $n$, that

$$
\mu\left(T^{-n}(E)\right) \leq K \mu(E)
$$

For $f \in L^{p}(\Omega)$, and $p>1$, let

$$
\begin{equation*}
T^{*} f \equiv f \circ T \tag{18.4.9}
\end{equation*}
$$

Then $T^{*} \in \mathscr{L}\left(L^{p}(\Omega), L^{p}(\Omega)\right)$, the continuous linear mappings form $L^{p}(\Omega)$ to itself with

$$
\begin{equation*}
\left\|T^{* n}\right\| \leq K^{1 / p} \tag{18.4.10}
\end{equation*}
$$

Defining $A_{n} \in \mathscr{L}\left(L^{p}(\Omega), L^{p}(\Omega)\right)$ by

$$
A_{n} \equiv \frac{1}{n} \sum_{k=0}^{n-1} T^{* k}
$$

there exists $A \in \mathscr{L}\left(L^{p}(\Omega), L^{p}(\Omega)\right)$ such that for all $f \in L^{p}(\Omega)$,

$$
\begin{equation*}
A_{n} f \rightarrow A f \text { weakly } \tag{18.4.11}
\end{equation*}
$$

and $A$ is a projection, $A^{2}=A$, onto the space of all $f \in L^{p}(\Omega)$ such that $T^{*} f=f$. (The invariant functions.) The norm of $A$ satisfies

$$
\begin{equation*}
\|A\| \leq K^{1 / p} \tag{18.4.12}
\end{equation*}
$$

Proof: To begin with, it follows from simple considerations that

$$
\int\left|\mathscr{X}_{A}\left(T^{n}(\omega)\right)\right|^{p} d \mu=\int\left|\mathscr{X}_{T^{-n}(A)}(\omega)\right|^{p} d \mu=\mu\left(T^{-n}(A)\right) \leq K \mu(A)
$$

Hence

$$
\left\|T^{* n}\left(\mathscr{X}_{A}\right)\right\| \leq K^{1 / p} \mu(A)^{1 / p}=K^{1 / p}\left\|\mathscr{X}_{A}\right\|_{L^{p}}
$$

Next suppose you have a simple function $s(\omega)=\sum_{k=1}^{n} \mathscr{X}_{A_{i}}(\omega) c_{i}$ where we assume the $A_{i}$ are disjoint. From the above,

$$
\begin{aligned}
\int\left|\sum_{k=1}^{n} \mathscr{X}_{A_{i}}\left(T^{m}(\omega)\right) c_{i}\right|^{p} d \mu & =\int \sum_{k=1}^{n} \mathscr{X}_{A_{i}}\left(T^{m}(\omega)\right)^{p}\left|c_{i}\right|^{p} d \mu \\
& \leq \sum_{k=1}^{n} K \mu\left(A_{i}\right)\left|c_{i}\right|^{p}=K \int|s|^{p} d \mu
\end{aligned}
$$

and so

$$
\left\|T^{* m} s\right\| \leq K^{1 / p}\|s\|
$$

and so the density of the simple functions implies that $\left\|T^{* m}\right\| \leq K^{1 / p}$.
Next let

$$
M \equiv\left\{g \in L^{p}(\Omega):\left\|A_{n} g\right\|_{p} \rightarrow 0\right\}
$$

It follows from 18.4.10 that $M$ is a closed subspace of $L^{p}(\Omega)$ containing $\left(I-T^{*}\right)\left(L^{p}(\Omega)\right)$. This is shown next.

Claim 1: $M$ is a closed subspace which contains $\left(I-T^{* m}\right)\left(L^{p}(\Omega)\right)$.
First it is shown that this is true if $m=1$ and then it will be observed that the same argument would work for any positive integer $m$.

$$
\begin{aligned}
A_{n}\left(f-T^{*} f\right) & \equiv \frac{1}{n} \sum_{k=0}^{n-1} T^{* k} f-T^{* k+1} f \\
& =\frac{1}{n} \sum_{k=0}^{n-1} T^{* k} f-\frac{1}{n} \sum_{k=1}^{n} T^{* k} f=\frac{1}{n}\left(f-T^{* n} f\right)
\end{aligned}
$$

Hence

$$
\left\|A_{n}\left(f-T^{*} f\right)\right\|_{p} \leq \frac{1}{n}\left(\|f\|_{p}+\left\|T^{* n} f\right\|_{p}\right) \leq \frac{1}{n}\left(\|f\|_{p}+K^{1 / p}\|f\|_{p}\right)
$$

and this clearly converges to 0 . In fact, the same argument shows that $M$ contains the set $\left(I-T^{* m}\right)\left(L^{p}(\Omega)\right)$ for any $m$. Now suppose $g_{n} \in M$ and $g_{n} \rightarrow g$. Does it follow that $g \in M$ also? Note that $T^{* m}$ is clearly linear. Thus

$$
\left\|T^{* m} g\right\| \leq\left\|T^{* m} g-T^{* m} g_{n}\right\|+\left\|T^{* m} g_{n}\right\| \leq K^{1 / p}\left\|g-g_{n}\right\|+\left\|T^{* m} g_{n}\right\|
$$

Now pick $n$ large enough that $\left\|g_{n}-g\right\|<\varepsilon /\left(2 K^{1 / p}\right)$ so that

$$
\left\|T^{* m} g\right\| \leq \frac{\varepsilon}{2}+\left\|T^{* m} g_{n}\right\|
$$

Then for all $m$ large enough, the right side of the above is less than $\varepsilon$ and this shows that $g \in M$. Note that $M$ is also a subspace and so it is a closed subspace.

Claim 2: If $A_{n_{k}} f \rightarrow g$ weakly and $A_{m_{k}} f \rightarrow h$ weakly, then $g=h$.
It is first shown that if $\xi \in L^{p^{\prime}}(\Omega)$ and $\int \xi g d \mu=0$ for all $g \in M$, then $\int \xi(g-h) d \mu=$ 0.

If $\xi \in L^{p^{\prime}}(\Omega)$ is such that $\int \xi g d \mu=0$ for all $g \in M$, then since $M \supseteq\left(I-T^{* n}\right)\left(L^{p}(\Omega)\right)$, it follows that for all $k \in L^{p}(\Omega)$,

$$
\int \xi k d \mu=\int\left(\xi T^{* n} k+\xi\left(I-T^{* n}\right) k\right) d \mu=\int \xi T^{* n} k d \mu
$$

and so from the definition of $A_{n}$ as an average, for such $\xi$,

$$
\begin{equation*}
\int \xi k d \mu=\int \xi A_{n} k d \mu \tag{18.4.13}
\end{equation*}
$$

Since $A_{n_{k}} f \rightarrow g$ weakly and $A_{m_{k}} f \rightarrow h$ weakly. Then 18.4.13 shows that

$$
\begin{equation*}
\int \xi g d \mu=\lim _{k \rightarrow \infty} \int \xi A_{n_{k}} f d \mu=\int \xi f d \mu=\lim _{k \rightarrow \infty} \int \xi A_{m_{k}} f d \mu=\int \xi h d \mu . \tag{18.4.14}
\end{equation*}
$$

Thus for these special $\xi$, it follows that

$$
\begin{equation*}
\int \xi(g-h) d \mu=0 . \tag{18.4.15}
\end{equation*}
$$

Next observe that for each fixed $n$, if $n_{k} \rightarrow \infty$,

$$
\begin{equation*}
\lim _{k \rightarrow \infty}\left\|T^{* n} A_{n_{k}} f-A_{n_{k}} f\right\|=0 \tag{18.4.16}
\end{equation*}
$$

this follows like the arguments given above in Claim 1. Note that if $L \in \mathscr{L}(X, X)$ and $x_{n} \rightarrow x$ weakly in $X$, then for $\phi \in X^{\prime}$

$$
\left\langle\phi, L x_{n}\right\rangle=\left\langle L^{*} \phi, x_{n}\right\rangle \rightarrow\left\langle L^{*} \phi, x\right\rangle=\langle\phi, L x\rangle
$$

and so $L x_{n} \rightarrow L x$ weakly. Therefore, this simple observation along with the above strong convergence 18.4.16 implies

$$
T^{* n} g=\text { weak } \lim _{k \rightarrow \infty} T^{* n} A_{n_{k}} f=\text { weak } \lim _{k \rightarrow \infty} A_{n_{k}} f=g
$$

Similarly $T^{* n} h=h$ where $A_{m_{k}} f \rightarrow h$ weakly. It follows that $A_{n}(g-h)=g-h$ so if $g \neq h$, then $g-h \notin M$ because

$$
A_{n}(g-h) \rightarrow g-h \neq 0
$$

It follows that since $M$ is a closed subspace, there exists $\xi \in L^{p^{\prime}}(\Omega)$ such that

$$
\int \xi(g-h) d \mu \neq 0
$$

but $\int \xi k d \mu=0$ for all $k \in M$, contradicting 18.4.15. This verifies Claim 2.
Now

$$
\begin{align*}
\left\|A_{n} f\right\|_{p} & =\left(\int\left|\frac{1}{n} \sum_{k=0}^{n-1} f\left(T^{k} \omega\right)\right|^{p} d \mu\right)^{1 / p} \leq \frac{1}{n} \sum_{k=0}^{n-1}\left(\int\left|f\left(T^{k} \omega\right)\right|^{p} d \mu\right)^{1 / p} \\
& =\frac{1}{n} \sum_{k=0}^{n-1}\left\|T^{* k} f\right\|_{p} \leq \frac{1}{n} \sum_{k=0}^{n-1} K^{1 / p}\|f\|_{p}=K^{1 / p}\|f\|_{p} \tag{18.4.17}
\end{align*}
$$

Hence, by the Eberlein Smulian theorem, Theorem 17.5.12, in case $p>1$, there is a subsequence for which $A_{n} f$ converges weakly in $L^{p}(\Omega)$. From the above, it follows that the original sequence must converge. That is, $A_{n} f$ converges weakly for each $f \in L^{p}(\Omega)$. Let $A f$ denote this weak limit. Then it is clear that $A$ is linear because this is true for each $A_{n}$. What of the claim about the estimate? From weak lower semicontinuity of the norm, Corollary 18.2.12,

$$
\|A f\|_{p} \leq \lim \inf _{n \rightarrow \infty}\left\|A_{n} f\right\| \leq K^{1 / p}\|f\|_{p}
$$

### 18.5 The Tychonoff And Schauder Fixed Point Theorems

First we give a proof of the Schauder fixed point theorem which is an infinite dimensional generalization of the Brouwer fixed point theorem. This is a theorem which lives in Banach space. After this, we give a generalization to locally convex topological vector spaces where the theorem is sometimes called Tychonoff's theorem. First here is an interesting example [55].

Exercise 18.5.1 Let $B$ be the closed unit ball in a separable Hilbert space $H$ which is infinite dimensional. Then there exists continuous $f: B \rightarrow B$ which has no fixed point.

Let $\left\{e_{k}\right\}_{k=1}^{\infty}$ be a complete orthonormal set in $H$. Let $L \in \mathscr{L}(H, H)$ be defined as follows. $L e_{k}=e_{k+1}$ and then extend linearly. Then in particular,

$$
L\left(\sum_{i} x_{i} e_{i}\right)=\sum_{i} x_{i} e_{i+1}
$$

Then it is clear that $L$ preserves norms and so it is linear and continuous. Note how this would not work at all if the Hilbert space were finite dimensional. Then define $f(x)=$ $\frac{1}{2}\left(1-\|x\|_{H}\right) e_{1}+L x$. Then if $\|x\| \leq 1$,

$$
\|f(x)\|=\frac{1}{2}(1-\|x\|)^{2}+\|L x\|^{2}=\frac{1}{2}(1-\|x\|)^{2}+\|x\|^{2}=\frac{1}{2}\|x\|^{2}+\frac{1}{2} \leq 1
$$

and so $f: B \rightarrow B$ yet has no fixed point because if it did, you would need to have

$$
x=\frac{1}{2}\left(1-\|x\|_{H}\right) e_{1}+L x
$$

and so

$$
\begin{aligned}
&\|x\|^{2}= \frac{1}{4}(1-\|x\|)^{2}+\|L x\|^{2}=\frac{1}{4}(1-\|x\|)^{2}+\|L x\|^{2} \\
&= \frac{1}{4}+\frac{5}{4}\|x\|^{2}-\frac{1}{2}\|x\| \\
& \frac{1}{2}\|x\|=\frac{1}{4}+\frac{1}{4}\|x\|^{2}
\end{aligned}
$$

this requires $\|x\|=1$. But then you would need to have $x=L x$ which is not so because if $x$ is in the closure of the span of $\left\{e_{i}\right\}_{i=m}^{\infty}$, such that the first nonzero Fourier coefficient is the $m^{t h}$, then $L x$ is in the closure of the span of $\left\{e_{i}\right\}_{i=m+1}^{\infty}$.

This shows you need something other than continuity if you want to get a fixed point. This also shows that there is a retraction of $B$ onto $\partial B$ in any infinite dimensional separable Hilbert space. You get it the usual way. Take the line from $x$ to $f(x)$ and the retraction will be the function which gives the point on $\partial B$ which is obtained by extending this line till it hits the boundary of $B$. Thus for Hilbert spaces, those which have $\partial B$ a retraction of $B$
are exactly those which are infinite dimensional. The above reference claims this retraction property holds for any infinite dimensional normed linear space. I think it is fairly clear to see from the above example that this is not a surprising assertion. Recall that one of the proofs of the Brouwer fixed point theorem used the non existence of such a retraction, obtained using integration theory, to prove the theorem.

We let $K$ be a closed convex subset of $X$ a Banach space and let

$$
f \text { be continuous, } f: K \rightarrow K \text {, and } \overline{f(K)} \text { is compact. }
$$

Lemma 18.5.2 For each $r>0$ there exists a finite set of points

$$
\left\{y_{1}, \cdots, y_{n}\right\} \subseteq \overline{f(K)}
$$

and continuous functions $\psi_{i}$ defined on $\overline{f(K)}$ such that for $x \in \overline{f(K)}$,

$$
\begin{gather*}
\sum_{i=1}^{n} \psi_{i}(x)=1  \tag{18.5.18}\\
\psi_{i}(x)=0 \text { if } x \notin B\left(y_{i}, r\right), \psi_{i}(x)>0 \text { if } x \in B\left(y_{i}, r\right)
\end{gather*}
$$

If

$$
\begin{equation*}
f_{r}(x) \equiv \sum_{i=1}^{n} y_{i} \psi_{i}(f(x)) \tag{18.5.19}
\end{equation*}
$$

then whenever $x \in K$,

$$
\left\|f(x)-f_{r}(x)\right\| \leq r
$$

Proof: Using the compactness of $\overline{f(K)}$, there exists

$$
\left\{y_{1}, \cdots, y_{n}\right\} \subseteq \overline{f(K)} \subseteq K
$$

such that

$$
\left\{B\left(y_{i}, r\right)\right\}_{i=1}^{n}
$$

covers $\overline{f(K)}$. Let

$$
\phi_{i}(y) \equiv\left(r-\left\|y-y_{i}\right\|\right)^{+}
$$

Thus $\phi_{i}(y)>0$ if $y \in B\left(y_{i}, r\right)$ and $\phi_{i}(y)=0$ if $y \notin B\left(y_{i}, r\right)$. For $x \in \overline{f(K)}$, let

$$
\psi_{i}(x) \equiv \phi_{i}(x)\left(\sum_{j=1}^{n} \phi_{j}(x)\right)^{-1}
$$

Then 18.5.18 is satisfied. Indeed the denominator is not zero because $x$ is in one of the $B\left(y_{i}, r\right)$. Thus it is obvious that the sum of these equals 1 on $K$. Now let $f_{r}$ be given by 18.5.19 for $x \in K$. For such $x$,

$$
f(x)-f_{r}(x)=\sum_{i=1}^{n}\left(f(x)-y_{i}\right) \psi_{i}(f(x))
$$

Thus

$$
\begin{gathered}
f(x)-f_{r}(x)=\sum_{\left\{i: f(x) \in B\left(y_{i}, r\right)\right\}}\left(f(x)-y_{i}\right) \psi_{i}(f(x)) \\
+\sum_{\left\{i: f(x) \notin B\left(y_{i}, r\right)\right\}}\left(f(x)-y_{i}\right) \psi_{i}(f(x)) \\
=\sum_{\left\{i: f(x)-y_{i} \in B(0, r)\right\}}\left(f(x)-y_{i}\right) \psi_{i}(f(x))= \\
\sum_{\left\{i: f(x)-y_{i} \in B(0, r)\right\}}\left(f(x)-y_{i}\right) \psi_{i}(f(x))+\sum_{\left\{i: f(x) \notin B\left(y_{i}, r\right)\right\}} 0 \psi_{i}(f(x)) \in B(0, r)
\end{gathered}
$$

because $0 \in B(0, r), B(0, r)$ is convex, and 18.5.18. It is just a convex combination of things in $B(0, r)$.

Note that we could have had the $y_{i}$ in $f(K)$ in addition to being in $\overline{f(K)}$. This would make it possible to eliminate the assumption that $K$ is closed later on. All you really need is that $K$ is convex.

We think of $f_{r}$ as an approximation to $f$. In fact it is uniformly within $r$ of $f$ on $K$. The next lemma shows that this $f_{r}$ has a fixed point. This is the main result and comes from the Brouwer fixed point theorem in $\mathbb{R}^{n}$. It is an approximate fixed point.

Lemma 18.5.3 For each $r>0$, there exists $x_{r} \in$ convex hull of $\overline{f(K)} \subseteq K$ such that

$$
f_{r}\left(x_{r}\right)=x_{r},\left\|f_{r}(x)-f(x)\right\|<r \text { for all } x
$$

Proof: If $f_{r}\left(x_{r}\right)=x_{r}$ and

$$
x_{r}=\sum_{i=1}^{n} a_{i} y_{i}
$$

for $\sum_{i=1}^{n} a_{i}=1$ and the $y_{i}$ described in the above lemma, we need

$$
f_{r}\left(x_{r}\right)=\sum_{i=1}^{n} y_{i} \psi_{i}\left(f\left(x_{r}\right)\right)=\sum_{j=1}^{n} y_{j} \psi_{j}\left(f\left(\sum_{i=1}^{n} a_{i} y_{i}\right)\right)=\sum_{j=1}^{n} a_{j} y_{j}=x_{r}
$$

Also, if this is satisfied, then we have the desired fixed point.
This will be satisfied if for each $j=1, \cdots, n$,

$$
\begin{equation*}
a_{j}=\psi_{j}\left(f\left(\sum_{i=1}^{n} a_{i} y_{i}\right)\right) \tag{18.5.20}
\end{equation*}
$$

so, let

$$
\Sigma_{n-1} \equiv\left\{\mathbf{a} \in \mathbb{R}^{n}: \sum_{i=1}^{n} a_{i}=1, a_{i} \geq 0\right\}
$$

and let $h: \Sigma_{n-1} \rightarrow \Sigma_{n-1}$ be given by

$$
h(\mathbf{a})_{j} \equiv \psi_{j}\left(f\left(\sum_{i=1}^{n} a_{i} y_{i}\right)\right)
$$

Since $h$ is a continuous function of $\mathbf{a}$, the Brouwer fixed point theorem applies and there exists a fixed point for $h$ which is a solution to 18.5.20.

The following is the Schauder fixed point theorem.
Theorem 18.5.4 Let $K$ be a closed and convex subset of $X$, a normed linear space. Let $f: K \rightarrow K$ be continuous and suppose $\overline{f(K)}$ is compact. Then $f$ has a fixed point.

Proof: Recall that $f\left(x_{r}\right)-f_{r}\left(x_{r}\right) \in B(0, r)$ and $f_{r}\left(x_{r}\right)=x_{r}$ with $x_{r} \in$ convex hull of $\overline{f(K)} \subseteq K$.

There is a subsequence, still denoted with subscript $r$ such that $f\left(x_{r}\right) \rightarrow x \in \overline{f(K)}$. Note that the fact that $K$ is convex is what makes $f$ defined at $x_{r} . x_{r}$ is in the convex hull of $\overline{f(K)} \subseteq K$. This is where we use $K$ convex. Then since $f_{r}$ is uniformly close to $f$, it follows that $f_{r}\left(x_{r}\right)=x_{r} \rightarrow x$ also. Thus $x_{r}$ converges strongly to $x$. Therefore,

$$
f(x)=\lim _{r \rightarrow 0} f\left(x_{r}\right)=\lim _{r \rightarrow 0} f_{r}\left(x_{r}\right)=\lim _{r \rightarrow 0} x_{r}=x
$$

We usually have in mind the mapping defined on a Banach space. However, the completeness was never used. Thus the result holds in a normed linear space.

There is a nice corollary of this major theorem which is called the Schaefer fixed point theorem or the Leray Schauder alterative principle [55].

Theorem 18.5.5 Let $f: X \rightarrow X$ be a compact map. Then either

1. There is a fixed point for $t f$ for all $t \in[0,1]$ or
2. For every $r>0$, there exists a solution to $x=t f(x)$ for some $t \in(0,1)$ such that $\|x\|>r$. (The solutions to $x=t f(x)$ for $t \in(0,1)$ are unbounded.)

Proof: Suppose there is $t_{0} \in[0,1]$ such that $t_{0} f$ has no fixed point. Then $t_{0} \neq 0 . t_{0} f$ obviously has a fixed point if $t_{0}=0$. Thus $t_{0} \in(0,1]$. Then let $r_{M}$ be the radial retraction onto $\overline{B(0, M)}$. By Schauder's theorem there exists $x \in \overline{B(0, M)}$ such that $t_{0} r_{M} f(x)=x$. Then if $\|f(x)\| \leq M, r_{M}$ has no effect and so $t_{0} f(x)=x$ which is assumed not to take place. Hence $\|f(x)\|>M$ and so $\left\|r_{M} f(x)\right\|=M$ so $\|x\|=t_{0} M$. Also $t_{0} r_{M} f(x)=t_{0} M \frac{f(x)}{\|f(x)\|}=$ $x$ and so $x=\hat{t} f(x), \hat{t}=t_{0} \frac{M}{\|f(x)\|}<1$. Since $M$ is arbitrary, it follows that the solutions to $x=t f(x)$ for $t \in(0,1)$ are unbounded. It was just shown that there is a solution to $x=\hat{t} f(x), \hat{t}<1$ such that $\|x\|=t_{0} M$ where $M$ is arbitrary. Thus the second of the two alternatives holds.

Proof: Suppose that alternative 2 does not hold and yet alternative 1 also fails to hold. Then there is $M_{0}$ such that if you have any solution to $x=t f(x)$ for $t \in(0,1)$, then $\|x\| \leq$ $M_{0}$. Nevertheless, there is $t \in(0,1]$ for which there is no fixed point for $t f$. (obviously there is a fixed point for $t=0$.) However, there is $x$ such that for $M>M_{0}$,

$$
x=t\left(r_{M} f(x)\right)=t \frac{M f(x)}{\|f(x)\|}=\hat{t} f(x), \hat{t}=\frac{M}{\|f(x)\|} t<t
$$

We must have $\|f(x)\|>M$ and $r_{M} f(x)=\frac{M f(x)}{\|f(x)\|}$ since if $\|f(x)\| \leq M, r_{M} f(x)=f(x)$ and there would be a fixed point for this $t$. Thus, letting $M$ get larger and larger, there are
$t_{M} \in(0,1)$ and $x_{M},\left\|x_{M}\right\| \leq M_{0}$ such that $x_{M}=t_{M} f\left(x_{M}\right),\left\|f\left(x_{M}\right)\right\|>M$. However, $f$ is given to be a compact map so it takes $\overline{B\left(0, M_{0}\right)}$ to a compact set but this shows that $f$ must take this set to an unbounded set which is a contradiction. This results from assuming there is $t$ such that $t f$ fails to have a fixed point for some $t \in[0,1]$. Thus alternative 1 must hold.

Next this is considered in the more general setting of locally convex topological vector space. This is the Tychonoff fixed point theorem. In this theorem, $X$ will be a locally convex topological vector space in which every point is a closed set. Let $\mathscr{B}$ be the basis described earlier and let $\mathscr{B}_{0}$ consist of all sets of $\mathscr{B}$ which are of the form $B_{A}(0, r)$ where $A$ is a finite subset of $\Psi$ as described earlier. Note that for $U \in \mathscr{B}_{0}, U=-U$ and $U$ is convex. Also, if $U \in \mathscr{B}_{0}$, there exists $V \in \mathscr{B}_{0}$ such that

$$
V+V \subseteq U
$$

where

$$
V+V \equiv\left\{v_{1}+v_{2}: v_{i} \in V\right\}
$$

To see this, note

$$
B_{A}(0, r / 2)+B_{A}(0, r / 2) \subseteq B_{A}(0, r)
$$

We let $K$ be a closed convex subset of $X$ and let
$f$ be continuous, $f: K \rightarrow K$, and $\overline{f(K)}$ is compact.
Lemma 18.5.6 For each $U \in \mathscr{B}_{0}$, there exists a finite set of points

$$
\left\{y_{1} \cdots y_{n}\right\} \subseteq \overline{f(K)}
$$

and continuous functions $\psi_{i}$ defined on $\overline{f(K)}$ such that for $x \in \overline{f(K)}$,

$$
\begin{gather*}
\sum_{i=1}^{n} \psi_{i}(x)=1  \tag{18.5.21}\\
\psi_{i}(x)=0 \text { if } x \notin y_{i}+U, \psi_{i}(x)>0 \text { if } x \in y_{i}+U
\end{gather*}
$$

If

$$
\begin{equation*}
f_{U}(x) \equiv \sum_{i=1}^{n} y_{i} \psi_{i}(f(x)) \tag{18.5.22}
\end{equation*}
$$

then whenever $x \in K$,

$$
f(x)-f_{U}(x) \in U
$$

Proof: Let $U=B_{A}(0, r)$. Using the compactness of $\overline{f(K)}$, there exists

$$
\left\{y_{1} \cdots y_{n}\right\} \subseteq \overline{f(K)}
$$

such that

$$
\left\{y_{i}+U\right\}_{i=1}^{n}
$$

covers $\overline{f(K)}$. Let

$$
\phi_{i}(y) \equiv\left(r-\rho_{A}\left(y-y_{i}\right)\right)^{+} .
$$

Thus $\phi_{i}(y)>0$ if $y \in y_{i}+U$ and $\phi_{i}(y)=0$ if $y \notin y_{i}+U$. For $x \in \overline{f(K)}$, let

$$
\psi_{i}(x) \equiv \phi_{i}(x)\left(\sum_{j=1}^{n} \phi_{j}(x)\right)^{-1}
$$

Then 18.5 .21 is satisfied. Now let $f_{U}$ be given by 18.5.22 for $x \in K$. For such $x$,

$$
\begin{aligned}
& f(x)- f_{U}(x)=\sum_{\left\{i: f(x)-y_{i} \in U\right\}}\left(f(x)-y_{i}\right) \psi_{i}(f(x)) \\
&+\sum_{\left\{i: f(x)-y_{i} \notin U\right\}}\left(f(x)-y_{i}\right) \psi_{i}(f(x)) \\
&=\sum_{\left\{i: f(x)-y_{i} \in U\right\}}\left(f(x)-y_{i}\right) \psi_{i}(f(x))= \\
& \sum_{\left\{i: f(x)-y_{i} \in U\right\}}\left(f(x)-y_{i}\right) \psi_{i}(f(x))+\sum_{\left\{i: f(x)-y_{i} \notin U\right\}} 0 \psi_{i}(f(x)) \in U
\end{aligned}
$$

because $0 \in U, U$ is convex, and 18.5.21.
We think of $f_{U}$ as an approximation to $f$.
Lemma 18.5.7 For each $U \in \mathscr{B}_{0}$, there exists $x_{U} \in$ convex hull of $\overline{f(K)} \subseteq K$ such that

$$
f_{U}\left(x_{U}\right)=x_{U}
$$

Proof: If $f_{U}\left(x_{U}\right)=x_{U}$ and

$$
x_{U}=\sum_{i=1}^{n} a_{i} y_{i}
$$

for $\sum_{i=1}^{n} a_{i}=1$, we need

$$
\sum_{j=1}^{n} y_{j} \psi_{j}\left(f\left(\sum_{i=1}^{n} a_{i} y_{i}\right)\right)=\sum_{j=1}^{n} a_{j} y_{j}
$$

Also, if this is satisfied, then we have the desired fixed point. This will be satisfied if for each $j=1, \cdots, n$,

$$
\begin{equation*}
a_{j}=\psi_{j}\left(f\left(\sum_{i=1}^{n} a_{i} y_{i}\right)\right) \tag{18.5.23}
\end{equation*}
$$

so, let

$$
\Sigma_{n-1} \equiv\left\{\mathbf{a} \in \mathbb{R}^{n}: \sum_{i=1}^{n} a_{i}=1, a_{i} \geq 0\right\}
$$

and let $h: \Sigma_{n-1} \rightarrow \Sigma_{n-1}$ be given by

$$
h(\mathbf{a})_{j} \equiv \psi_{j}\left(f\left(\sum_{i=1}^{n} a_{i} y_{i}\right)\right) .
$$

Since $h$ is continuous, the Brouwer fixed point theorem applies and we see there exists a fixed point for $h$ which is a solution to 18.5.23.

Theorem 18.5.8 Let $K$ be a closed and convex subset of $X$, a locally convex topological vector space in which every point is closed. Let $f: K \rightarrow K$ be continuous and suppose $\overline{f(K)}$ is compact. Then $f$ has a fixed point.

Proof: First consider the following claim which will yield a candidate for the fixed point. Recall that $f\left(x_{U}\right)-f_{U}\left(x_{U}\right) \in U$ and $f_{U}\left(x_{U}\right)=x_{U}$ with $x_{U} \in$ convex hull of $\overline{f(K)} \subseteq$ $K$.

Claim: There exists $x \in \overline{f(K)}$ with the property that if $V \in \mathscr{B}_{0}$, there exists $U \subseteq V$, $U \in \mathscr{B}_{0}$, such that

$$
f\left(x_{U}\right) \in x+V
$$

Proof of the claim: If no such $x$ exists, then for each $x \in \overline{f(K)}$, there exists $V_{x} \in \mathscr{B}_{0}$ such that whenever $U \subseteq V_{x}$, with $U \in \mathscr{B}_{0}$,

$$
f\left(x_{U}\right) \notin x+V_{x} .
$$

Since $\overline{f(K)}$ is compact, there exist $x_{1}, \cdots, x_{n} \in \overline{f(K)}$ such that

$$
\left\{x_{i}+V_{x_{i}}\right\}_{i=1}^{n}
$$

cover $\overline{f(K)}$. Let

$$
U \in \mathscr{B}_{0}, U \subseteq \cap_{i=1}^{n} V_{x_{i}}
$$

and consider $x_{U}$.

$$
f\left(x_{U}\right) \in x_{i}+V_{x_{i}}
$$

for some $i$ because these sets cover $\overline{f(K)}$ and $f\left(x_{U}\right)$ is something in $\overline{f(K)}$. But $U \subseteq V_{x_{i}}$, a contradiction. This shows the claim.

Now I show $x$ is the desired fixed point. Let $W \in \mathscr{B}_{0}$ and let $V \in \mathscr{B}_{0}$ with

$$
V+V+V \subseteq W
$$

Since $f$ is continuous at $x$, there exists $V_{0} \in \mathscr{B}_{0}$ such that

$$
V_{0}+V_{0} \subseteq V
$$

and if

$$
y-x \in V_{0}+V_{0}
$$

then

$$
f(x)-f(y) \in V
$$

Using the claim, let $U \in \mathscr{B}_{0}, U \subseteq V_{0}$, such that

$$
f\left(x_{U}\right) \in x+V_{0}
$$

Then

$$
\begin{gathered}
x-x_{U}=x-f\left(x_{U}\right)+f\left(x_{U}\right)-f_{U}\left(x_{U}\right) \in V_{0}+U \\
\subseteq V_{0}+V_{0} \subseteq V
\end{gathered}
$$

and so

$$
\begin{aligned}
f(x)-x & =f(x)-f\left(x_{U}\right)+f\left(x_{U}\right)-f_{U}\left(x_{U}\right)+f_{U}\left(x_{U}\right)-x \\
& =f(x)-f\left(x_{U}\right)+f\left(x_{U}\right)-f_{U}\left(x_{U}\right)+x_{U}-x \\
& \subseteq V+U+V \subseteq W
\end{aligned}
$$

Since $W \in \mathscr{B}_{0}$ is arbitrary, it follows from Lemma 18.2.18 that $f(x)-x=0$.
As an example of the usefulness of this fixed point theorem, consider the following application to the theory of ordinary differential equations. In the context of this theorem, $X=C\left([0, T] ; \mathbb{R}^{n}\right)$, a Banach space with norm given by

$$
\|\mathbf{x}\| \equiv \max \{|\mathbf{x}(t)|: t \in[0, T]\}
$$

Theorem 18.5.9 Let $\mathbf{f}:[0, T] \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be continuous and suppose there exists $L>0$ such that for all $\lambda \in(0,1)$, if

$$
\begin{equation*}
\mathbf{x}^{\prime}=\lambda \mathbf{f}(t, \mathbf{x}), \mathbf{x}(0)=\mathbf{x}_{0} \tag{18.5.24}
\end{equation*}
$$

for all $t \in[0, T]$, then $\|\mathbf{x}\|<L$. Then there exists a solution to

$$
\begin{equation*}
\mathbf{x}^{\prime}=\mathbf{f}(t, \mathbf{x}), \mathbf{x}(0)=\mathbf{x}_{0} \tag{18.5.25}
\end{equation*}
$$

for $t \in[0, T]$.
Proof: Let

$$
N \mathbf{x}(t) \equiv \int_{0}^{t} \mathbf{f}(s, \mathbf{x}(s)) d s
$$

Thus a solution to the initial value problem exists if there exists a solution to

$$
\mathbf{x}_{0}+N(\mathbf{x})=\mathbf{x}
$$

Let

$$
m \equiv \max \{|\mathbf{f}(t, \mathbf{x})|:(t, \mathbf{x}) \in[0, T] \times \overline{B(0, L)}\}, M \equiv\left|\mathbf{x}_{0}\right|+m T
$$

and let

$$
K \equiv\left\{\mathbf{x} \in C\left(0, T ; \mathbb{R}^{n}\right) \text { such that } \mathbf{x}(0)=\mathbf{x}_{0} \text { and }\|\mathbf{x}\| \leq M\right\}
$$

Now define

$$
A \mathbf{x} \equiv\left\{\begin{array}{l}
\mathbf{x}_{0}+N \mathbf{x} \text { if }\|N \mathbf{x}\| \leq M-\left|\mathbf{x}_{0}\right| \\
\mathbf{x}_{0}+\frac{\left(M-\left|\mathbf{x}_{0}\right| \mid N \mathbf{x}\right.}{\|N \mathbf{x}\|} \text { if }\|N \mathbf{x}\|>M-\left|\mathbf{x}_{0}\right|
\end{array}\right.
$$

Then $A$ is continuous and maps $X$ to $K$ because

$$
\|A \mathbf{x}\| \leq\left|\mathbf{x}_{0}\right|+\|N \mathbf{x}\| \leq M \text { if }\|N \mathbf{x}\| \leq M-\left|\mathbf{x}_{0}\right|
$$

and otherwise,

$$
\|A \mathbf{x}\| \leq\left|\mathbf{x}_{0}\right|+\left\|\frac{\left(M-\left|\mathbf{x}_{0}\right|\right) N \mathbf{x}}{\| N \mathbf{x}| |}\right\| \leq\left|\mathbf{x}_{0}\right|+M-\left|\mathbf{x}_{0}\right|=M
$$

Also $A(K)$ is equicontinuous because

$$
\mathbf{x}_{0}+N \mathbf{x}(t)-\left(\mathbf{x}_{0}+N \mathbf{x}\left(t_{1}\right)\right)=\int_{t_{1}}^{t} \mathbf{f}(s, \mathbf{x}(s)) d s
$$

and the integrand is bounded. Thus $\overline{A(K)}$ is a compact set in $X$ by the Ascoli Arzela theorem. By the Schauder fixed point theorem, $A$ has a fixed point, $\mathbf{x} \in K$.

If $\|N(\mathbf{x})\|>M-\left|\mathbf{x}_{0}\right|$, then

$$
\mathbf{x}_{0}+\lambda N(\mathbf{x})=\mathbf{x}
$$

where

$$
\lambda=\frac{\left(M-\left|\mathbf{x}_{0}\right|\right)}{\|N \mathbf{x}\|}<1
$$

and so 18.5 .24 holds. Therefore, by the assumed estimate on the solutions to 18.5 .24 , it follows that

$$
\|\mathbf{x}\|<L
$$

and so $\|N \mathbf{x}\| \leq m T=M-\left|\mathbf{x}_{0}\right|$, a contradiction. Therefore, it must be the case that

$$
\|N(\mathbf{x})\| \leq M-\left|\mathbf{x}_{0}\right|
$$

which implies that

$$
\mathbf{x}_{0}+N(\mathbf{x})=\mathbf{x}
$$

Since this is equivalent to 18.5 .25 , this proves the theorem.
Here is a neater proof which uses the Leray Schauder alternative, also called the Schaefer fixed point theorem presented above.

Theorem 18.5.10 Let $\mathbf{f}:[0, T] \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be continuous and suppose there exists $L>0$ such that for all $\lambda \in(0,1)$, if

$$
\begin{equation*}
\mathbf{x}^{\prime}=\lambda \mathbf{f}(t, \mathbf{x}), \mathbf{x}(0)=\mathbf{x}_{0} \tag{18.5.26}
\end{equation*}
$$

for all $t \in[0, T]$, then $\|\mathbf{x}\|<L$. Then there exists a solution to

$$
\begin{equation*}
\mathbf{x}^{\prime}=\mathbf{f}(t, \mathbf{x}), \mathbf{x}(0)=\mathbf{x}_{0} \tag{18.5.27}
\end{equation*}
$$

for $t \in[0, T]$.

Proof: Let $F: X \rightarrow X$ where $X$ described above.

$$
F \mathbf{y}(t) \equiv \int_{0}^{t} \mathbf{f}\left(s, \mathbf{y}(s)+\mathbf{x}_{0}\right) d s
$$

Let $B$ be a bounded set in $X$. Then $\left|\mathbf{f}\left(s, \mathbf{y}(s)+\mathbf{x}_{0}\right)\right|$ is bounded for $s \in[0, T]$ if $\mathbf{y} \in B$. Say $\left|\mathbf{f}\left(s, \mathbf{y}(s)+\mathbf{x}_{0}\right)\right| \leq C_{B}$. Hence $F(B)$ is bounded in $X$. Also, for $\mathbf{y} \in B, s<t$,

$$
|F \mathbf{y}(t)-F \mathbf{y}(s)| \leq\left|\int_{s}^{t} \mathbf{f}\left(s, \mathbf{y}(s)+\mathbf{x}_{0}\right) d s\right| \leq C_{B}|t-s|
$$

and so $F(B)$ is pre-compact by the Ascoli Arzela theorem. By the Schaefer fixed point theorem, there are two alternatives. Either there are unbounded solutions $\mathbf{y}$ to

$$
\lambda F(\mathbf{y})=\mathbf{y}
$$

for various $\lambda \in(0,1)$ or for all $\lambda \in[0,1]$, there is a fixed point for $\lambda F$. In the first case, there would be unbounded $\mathbf{y}_{\lambda}$ solving

$$
\mathbf{y}_{\lambda}(t)=\lambda \int_{0}^{t} \mathbf{f}\left(s, \mathbf{y}_{\lambda}(s)+\mathbf{x}_{0}\right) d s
$$

Then let $\mathbf{x}_{\lambda}(s) \equiv \mathbf{y}_{\lambda}(s)+\mathbf{x}_{0}$ and you get $\left\|\mathbf{x}_{\lambda}\right\|$ also unbounded for various $\lambda \in(0,1)$. The above implies

$$
\mathbf{x}_{\lambda}(t)-\mathbf{x}_{0}=\lambda \int_{0}^{t} \mathbf{f}\left(s, \mathbf{x}_{\lambda}(s)\right) d s
$$

so $\mathbf{x}_{\lambda}^{\prime}=\lambda \mathbf{f}\left(t, \mathbf{x}_{\lambda}\right), \mathbf{x}_{\lambda}(0)=\mathbf{x}_{0}$ and these would be unbounded for $\lambda \in(0,1)$ contrary to the assumption that there exists an estimate for these valid for all $\lambda \in(0,1)$. Hence the first alternative must hold and hence there is $\mathbf{y} \in X$ such that

$$
F \mathbf{y}=\mathbf{y}
$$

Then letting $\mathbf{x}(s) \equiv \mathbf{y}(s)+\mathbf{x}_{0}$, it follows that

$$
\mathbf{x}(t)-\mathbf{x}_{0}=\int_{0}^{t} \mathbf{f}(s, \mathbf{x}(s)) d s
$$

and so $\mathbf{x}$ is a solution to the differential equation on $[0, T]$.
Note that existence for solutions to 18.5 .24 is not assumed, only estimates of possible solutions. These estimates are called a-priori estimates. Also note this is a global existence theorem, not a local one for a solution defined on only a small interval.

### 18.6 A Variational Principle of Ekeland

Definition 18.6.1 A function $\phi: X \rightarrow(-\infty, \infty]$ is called proper if it is not constantly equal to $\infty$. Here $X$ is assumed to be a complete metric space. The function $\phi$ is lower semicontinuous if

$$
x_{n} \rightarrow x \text { implies } \phi(x) \leq \lim \inf _{n \rightarrow \infty} \phi\left(x_{n}\right)
$$

It is bounded below if there is some constant $C$ such that $C \leq \phi(x)$ for all $x$.

The variational principle of Ekeland is the following theorem [55]. You start with an approximate minimizer $x_{0}$. It says there is $y_{\lambda}$ fairly close to $x_{0}$ such that if you subtract a "cone" from the value of $\phi$ at $y_{\lambda}$, then the resulting function is less than $\phi(x)$ for all $x \neq y_{\lambda}$. This cone is like a supporting plane for a convex function but pertains to functions which are certainly not convex.


Theorem 18.6.2 Let $X$ be a complete metric space and let $\phi: X \rightarrow(-\infty, \infty]$ be proper, lower semicontinuous and bounded below. Let $x_{0}$ be such that

$$
\phi\left(x_{0}\right) \leq \inf _{x \in X} \phi(x)+\varepsilon
$$

Then for every $\lambda>0$ there exists a $y_{\lambda}$ such that

1. $\phi\left(y_{\lambda}\right) \leq \phi\left(x_{0}\right)$
2. $d\left(y_{\lambda}, x_{0}\right) \leq \lambda$
3. $\phi\left(y_{\lambda}\right)-\frac{\varepsilon}{\lambda} d\left(x, y_{\lambda}\right)<\phi(x)$ for all $x \neq y_{\lambda}$

To motivate the proof, see the following picture which illustrates the first two steps. The $S_{i}$ will be sets in $X$ but are denoted symbolically by labeling them in $X \times(-\infty, \infty]$.


Then the end result of this iteration would be a picture like the following.


Thus you would have $\phi\left(y_{\lambda}\right)-\frac{\varepsilon}{\lambda} d\left(y_{\lambda}, x\right) \leq \phi(x)$ for all $x$ which is seen to be what is wanted.

Proof: Let $x_{1}=x_{0}$ and define

$$
S_{1} \equiv\left\{z \in X: \phi(z) \leq \phi\left(x_{1}\right)-\frac{\varepsilon}{\lambda} d\left(z, x_{1}\right)\right\}
$$

Then $S_{1}$ contains $x_{1}$ so it is nonempty. It is also clear that $S_{1}$ is a closed set. This follows from the lower semicontinuity of $\phi$. Let $x_{2}$ be a point of $S_{1}$, possibly different than $x_{1}$ and let

$$
S_{2} \equiv\left\{z \in X: \phi(z) \leq \phi\left(x_{2}\right)-\frac{\varepsilon}{\lambda} d\left(z, x_{2}\right)\right\}
$$

Continue in this way. Now let there be a sequence of points $\left\{x_{k}\right\}$ such that $x_{k} \in S_{k-1}$ and define $S_{k}$ by

$$
S_{k} \equiv\left\{z \in X: \phi(z) \leq \phi\left(x_{k}\right)-\frac{\varepsilon}{\lambda} d\left(z, x_{k}\right)\right\}
$$

where $x_{k}$ is some point of $S_{k-1}$. Then $x_{k}$ is a point of $S_{k}$. Will this yield a nested sequence of nonempty closed sets? Yes, it appears that it would because if $z \in S_{k}$ then

$$
\begin{aligned}
\phi(z) & \leq \phi\left(S_{k-1}\right)-\frac{\varepsilon}{\lambda} d\left(z, x_{k}\right) \leq\left(\phi\left(x_{k-1}\right)-\frac{\varepsilon}{\lambda} d\left(x_{k-1}, x_{k}\right)\right)-\frac{\varepsilon}{\lambda} d\left(z, x_{k}\right) \\
& \leq \phi\left(x_{k-1}\right)-\frac{\varepsilon}{\lambda} d\left(z, x_{k-1}\right)
\end{aligned}
$$

showing that $z$ has what it takes to be in $S_{k-1}$. Thus we would obtain a sequence of nested, nonempty, closed sets according to this scheme.

Now here is how to choose the $x_{k} \in S_{k-1}$. Let

$$
\phi\left(x_{k}\right)<\inf _{x \in S_{k-1}} \phi(x)+\frac{1}{2^{k}}
$$

Then for $z \in S_{n+1} \subseteq S_{n}$,

$$
\phi(z) \leq \phi\left(x_{n+1}\right)-\frac{\varepsilon}{\lambda} d\left(z, x_{n+1}\right)
$$

and so

$$
\begin{aligned}
\frac{\varepsilon}{\lambda} d\left(z, x_{n+1}\right) & \leq \phi\left(x_{n+1}\right)-\phi(z) \leq \inf _{x \in S_{n}} \phi(x)+\frac{1}{2^{n+1}}-\phi(z) \\
& \leq \phi(z)+\frac{1}{2^{n+1}}-\phi(z)=\frac{1}{2^{n+1}}
\end{aligned}
$$

Thus every $z \in S_{n+1}$ is within $\frac{1}{2^{n+1}}$ of the single point $x_{n+1}$ and so the diameter of $S_{n}$ converges to 0 as $n \rightarrow \infty$. By completeness of $X$, there exists a unique $y_{\lambda} \in \cap_{n} S_{n}$. Then it follows in particular that for $x_{0}=x_{1}$ as above,

$$
\phi\left(y_{\lambda}\right) \leq \phi\left(x_{0}\right)-\frac{\varepsilon}{\lambda} d\left(y_{\lambda}, x_{0}\right) \leq \phi\left(x_{0}\right)
$$

which verifies the first of the above conclusions.

As to the second, $\phi\left(x_{0}\right) \leq \inf _{x \in X} \phi(x)+\varepsilon$ and so, for any $x$,

$$
\phi\left(y_{\lambda}\right) \leq \phi\left(x_{0}\right)-\frac{\varepsilon}{\lambda} d\left(y_{\lambda}, x_{0}\right) \leq \phi(x)+\varepsilon-\frac{\varepsilon}{\lambda} d\left(y_{\lambda}, x_{0}\right)
$$

this being true for $x=y_{\lambda}$. Hence $\frac{\varepsilon}{\lambda} d\left(y_{\lambda}, x_{0}\right) \leq \varepsilon$ and so $d\left(y_{\lambda}, x_{0}\right) \leq \lambda$.
Finally consider the third condition. If it does not hold, then there exists $z \neq y_{\lambda}$ such that

$$
\phi\left(y_{\lambda}\right) \geq \phi(z)+\frac{\varepsilon}{\lambda} d\left(z, y_{\lambda}\right)
$$

so that

$$
\phi(z) \leq \phi\left(y_{\lambda}\right)-\frac{\varepsilon}{\lambda} d\left(z, y_{\lambda}\right)
$$

But then, by the definition of $y_{\lambda}$ as being in all the $S_{n}$,

$$
\phi\left(y_{\lambda}\right) \leq \phi\left(x_{n}\right)-\frac{\varepsilon}{\lambda} d\left(x_{n}, y_{\lambda}\right)
$$

and so

$$
\begin{aligned}
\phi(z) & \leq \phi\left(x_{n}\right)-\frac{\varepsilon}{\lambda}\left(d\left(x_{n}, y_{\lambda}\right)+d\left(z, y_{\lambda}\right)\right) \\
& \leq \phi\left(x_{n}\right)-\frac{\varepsilon}{\lambda} d\left(x_{n}, z\right)
\end{aligned}
$$

Since $n$ is arbitrary, this shows that $z \in \cap_{n} S_{n}$ but there is only one element of this intersection and it is $y_{\lambda}$ so $z$ must equal $y_{\lambda}$, a contradiction.

Note how if you make $\lambda$ very small, you could pick $\varepsilon$ very small such that the cone looks pretty flat.

### 18.6.1 Cariste Fixed Point Theorem

As mentioned in [55], the above result can be used to prove a fixed point theorem called the Cariste fixed point theorem.

Theorem 18.6.3 Let $\phi$ be lower semicontinuous, proper, and bounded below on a complete metric space $X$ and let $F: X \rightarrow \mathscr{P}(X)$ be set valued such that $F(x) \neq \emptyset$ for all $x$. Also suppose that for each $x \in X$, there exists $y \in F(x)$ such that

$$
\phi(y) \leq \phi(x)-d(x, y)
$$

Then there exists $x_{0}$ such that $x_{0} \in F\left(x_{0}\right)$.
Proof: In the above Ekeland variational principle, let $\varepsilon=1=\lambda$. Then there exists $x_{0}$ such that for all $y \neq x_{0}$

$$
\begin{equation*}
\phi\left(x_{0}\right)-d\left(y, x_{0}\right)<\phi(y), \text { so } \phi\left(x_{0}\right)<\phi(y)+d\left(y, x_{0}\right) \tag{18.6.28}
\end{equation*}
$$

for all $y \neq x_{0}$.


Suppose $x_{0} \notin F\left(x_{0}\right)$. From the assumption, there is $y \in F\left(x_{0}\right)$ (so $\left.y \neq x_{0}\right)$ such that

$$
\phi(y) \leq \phi\left(x_{0}\right)-d\left(x_{0}, y\right)
$$

Since $y \neq x_{0}$, it follows

$$
\phi(y)+d\left(x_{0}, y\right) \leq \phi\left(x_{0}\right)<\phi(y)+d\left(y, x_{0}\right)
$$

a contradiction. Hence $x_{0} \in F\left(x_{0}\right)$ after all.
It is a funny theorem. It is easy to prove, but you look at it and wonder what it says. If $F$ is single valued, you would need to have a function $\phi$ such that for each $x$,

$$
\phi(F(x)) \leq \phi(x)-d(x, y)
$$

and if you have such a $\phi$ then you can assert there is a fixed point for $F$. Suppose $F$ is single valued and $d(F x, F y) \leq r d(x, y), 0<r<1$. Of course $F$ has a fixed point using easier techniques. However, this also follows from this result. Let

$$
\phi(x)=\frac{1}{1-r} d(x, F(x))
$$

Then is it true that for each $x$, there exists $y \in F(x)$ such that the inequality holds for all $x$ ? Is

$$
\frac{1}{1-r} d(F(x), F(F(x))) \leq \frac{1}{1-r} d(x, F(x))-d(x, F(x))
$$

Yes, this is certainly so because the right side reduces to $\frac{r}{1-r} d(x, F(x))$. Thus this fixed point theorem implies the usual Banach fixed point theorem.

The Ekeland variational principle says that when $\phi$ is lower semicontinuous proper and bounded below, there exists $y$ such that

$$
\phi(y)-d(x, y)<\phi(x) \text { for all } x \neq y
$$

In fact this can be proved from the Cariste fixed point theorem. Suppose the EVP does not hold. This would mean that for all $y$ there exists $x \neq y$ such that

$$
\phi(y)-d(x, y) \geq \phi(x)
$$

Thus, for all $x$ there exists $y \neq x$ such that

$$
\phi(x)-d(x, y) \geq \phi(y)
$$

The inequality is preserved if $x=y$. Then let

$$
F(x) \equiv\{y \neq x: \phi(x)-d(x, y) \geq \phi(y)\} \neq \emptyset
$$

by assumption. This is the hypothesis for the Cariste fixed point theorem. Hence there exists $x_{0} \in F\left(x_{0}\right)=\left\{y \neq x_{0}: \phi\left(x_{0}\right)-d\left(x_{0}, y\right) \geq \phi(y)\right\}$ but this cannot happen because you can't have $x_{0} \neq x_{0}$. Thus the Ekeland variational principle must hold after all.

### 18.6.2 A Density Result

There are several applications of the Ekeland variational principle. For more of them, see [55]. One of these is to show that there is a point where $\phi^{\prime}$ is small assuming $\phi$ is bounded below, lower semicontinuous, and Gateaux differentiable. Here

$$
\left\langle\phi^{\prime}(x), v\right\rangle \equiv \lim _{h \rightarrow 0} \frac{\phi(x+h v)-\phi(x)}{h}, \phi^{\prime}(x) \in X^{\prime}
$$

It is sort of an approximate critical point at a point which causes $\phi$ to be near the infimum.
Theorem 18.6.4 Let $X$ be a Banach space and $\phi: X \rightarrow \mathbb{R}$ be Gateaux differentiable, bounded from below, and lower semicontinuous. Then for every $\varepsilon>0$ there exists $x \in X$ such that

$$
\phi\left(x_{\varepsilon}\right) \leq \inf _{x \in X} \phi(x)+\varepsilon \text { and }\left\|\phi^{\prime}\left(x_{\varepsilon}\right)\right\|_{X^{\prime}} \leq \varepsilon
$$

Proof: From the Ekeland variational principle with $\lambda=1$, there exists $x_{\varepsilon}$ such that

$$
\phi\left(x_{\varepsilon}\right) \leq \phi\left(x_{0}\right) \leq \inf _{x \in X} \phi(x)+\varepsilon
$$

and for all $x$,

$$
\phi\left(x_{\varepsilon}\right)<\phi(x)+\varepsilon\left\|x-x_{\varepsilon}\right\|
$$

Then letting $x=x_{\varepsilon}+h v$ where $\|v\|=1$,

$$
\phi\left(x_{\varepsilon}+h v\right)-\phi\left(x_{\varepsilon}\right)>-\varepsilon|h|
$$

Let $h<0$. Then divide by it

$$
\frac{\phi\left(x_{\varepsilon}+h v\right)-\phi\left(x_{\varepsilon}\right)}{h}<\varepsilon
$$

Passing to a limit as $h \rightarrow 0$ yields

$$
\left\langle\phi^{\prime}\left(x_{\varepsilon}\right), v\right\rangle \leq \varepsilon
$$

Now $v$ was arbitrary with norm 1 and so

$$
\sup _{\|v\|=1}\left\langle\phi^{\prime}\left(x_{\varepsilon}\right), v\right\rangle=\left\|\phi^{\prime}\left(x_{\varepsilon}\right)\right\| \leq \varepsilon
$$

There is another very interesting application of the Ekeland variational principle [55].

Theorem 18.6.5 Let $X$ be a Banach space and $\phi: X \rightarrow \mathbb{R}$ be Gateaux differentiable, bounded from below, and lower semicontinuous. Also suppose there exists $a, c>0$ such that

$$
a\|x\|-c \leq \phi(x) \text { for all } x \in X
$$

Then $\left\{\phi^{\prime}(x): x \in X\right\}$ is dense in the ball of $X^{\prime}$ centered at 0 with radius $a$. Here $\phi^{\prime}(x) \in X^{\prime}$ and is determined by

$$
\left\langle\phi^{\prime}(x), v\right\rangle \equiv \lim _{h \rightarrow 0} \frac{\phi(x+h v)-\phi(x)}{h}
$$

Proof: Let $x^{*} \in X^{\prime},\left\|x^{*}\right\| \leq a$. Let

$$
\psi(x)=\phi(x)-\left\langle x^{*}, x\right\rangle
$$

This is lower semicontinuous. It is also bounded from below because

$$
\psi(x) \geq \phi(x)-a\|x\| \geq(a\|x\|-c)-a\|x\|=-c
$$

It is also clearly Gateaux differentiable and lower semicontinuous because the piece added in is actually continuous. It is clear that the Gateaux derivative is just $\phi^{\prime}(x)-x^{*}$. By Theorem 18.6.4, there exists $x_{\varepsilon}$ such that

$$
\left\|\phi^{\prime}\left(x_{\varepsilon}\right)-x^{*}\right\| \leq \varepsilon
$$

Thus this theorem says that if $\phi(x) \geq a\|x\|-c$ where $\phi$ has the nice properties of the theorem it follows that $\phi^{\prime}(x)$ is dense in $B(0, a)$ in the dual space $X^{\prime}$. It follows that if for every $a$, there exists $c$ such that

$$
\phi(x) \geq a\|x\|-c \text { for all } x \in X
$$

then $\left\{\phi^{\prime}(x): x \in X\right\}$ is dense in $X^{\prime}$. This proves the following lemma.
Lemma 18.6.6 Let $X$ be a Banach space and $\phi: X \rightarrow \mathbb{R}$ be Gateaux differentiable, bounded from below, and lower semicontinuous. Suppose for all $a>0$ there exists a $c>0$ such that

$$
\phi(x) \geq a\|x\|-c \text { for all } x
$$

Then $\left\{\phi^{\prime}(x): x \in X\right\}$ is dense in $X^{\prime}$.
If the above holds, then

$$
\frac{\phi(x)}{\|x\|} \geq a-\frac{c}{\|x\|}
$$

and so, since $a$ is arbitrary, it must be the case that

$$
\begin{equation*}
\lim _{\|x\| \rightarrow \infty} \frac{\phi(x)}{\|x\|}=\infty \tag{18.6.29}
\end{equation*}
$$

In fact, this is sufficient. If not, there would exist $a>0$ such that $\phi\left(x_{n}\right)<a\left\|x_{n}\right\|-n$. Let $-L$ be a lower bound for $\phi(x)$. Then

$$
-L+n \leq a\left\|x_{n}\right\|
$$

and so $\left\|x_{n}\right\| \rightarrow \infty$. Now it follows that

$$
\begin{equation*}
a \geq \frac{\phi\left(x_{n}\right)}{\left\|x_{n}\right\|}+\frac{n}{\left\|x_{n}\right\|} \geq \frac{\phi\left(x_{n}\right)}{\left\|x_{n}\right\|} \tag{18.6.30}
\end{equation*}
$$

which is a contradiction to 18.6 .29 . This proves the following interesting density theorem.
Theorem 18.6.7 Let $X$ be a Banach space and $\phi: X \rightarrow \mathbb{R}$ be Gateaux differentiable, bounded from below, and lower semicontinuous. Also suppose the coercivity condition

$$
\lim _{\|x\| \rightarrow \infty} \frac{\phi(x)}{\|x\|}=\infty
$$

Then $\left\{\phi^{\prime}(x): x \in X\right\}$ is dense in $X^{\prime}$. Here $\phi^{\prime}(x) \in X^{\prime}$ and is determined by

$$
\left\langle\phi^{\prime}(x), v\right\rangle \equiv \lim _{h \rightarrow 0} \frac{\phi(x+h v)-\phi(x)}{h}
$$

### 18.7 Quotient Spaces

A useful idea is that of a quotient space. It is a way to create another Banach space from a given Banach space and a closed subspace. It generalizes similar concepts which are routine in linear algebra.

Definition 18.7.1 Let $X$ be a Banach space and let $V$ be a closed subspace of $X$. Then $X / V$ denotes the set of equivalence classes determined by the equivalence relation which says $x \sim y$ means $x-y \in V$. An individual equivalence class will be denoted by any of the following symbols. $x+V,[x]$, or $[x]_{V}$. Vector space operations are defined as follows:

$$
(x+V)+y+V \equiv x+y+V
$$

or in other symbols,

$$
[x]+[y] \equiv[x+y]
$$

and for $\alpha \in \mathbb{F}$,

$$
\alpha[x] \equiv[\alpha x]
$$

Also a norm is defined by

$$
\|[x]\| \equiv \inf \{\|x+v\|: v \in V\}
$$

It is left as an exercise to verify the above algebraic operations are well defined. With the above definition, here is the major theorem about quotient spaces.

Theorem 18.7.2 Let $X$ be a Banach space and let $V$ be a closed subspace of $X$. Then with the above definitions of vector space operations, $X / V$ is a Banach space. In the case where $V=\operatorname{ker}(A)$ for $A \in \mathscr{L}(X, Y)$ for $Y$ another Banach space, define $\widehat{A}: X / V \rightarrow A(X) \subseteq Y$ by $\widehat{A}([x]) \equiv A x$. Then $\widehat{A}$ is continuous and $1-1$. In fact, $\|\widehat{A}\| \leq\|A\|$.

Proof: First of all, consider the claim that the given norm really is a norm. First note that $\|x+V\| \geq 0$ and $\|x+V\|=0$ only if $x \in V$ because $V$ is closed. Therefore, $x+V=0+V$. Next,

$$
\begin{aligned}
\|\alpha[x]\| & \equiv\|[\alpha x]\| \equiv \inf \{\|\alpha x+v\|: v \in V\} \\
& =\inf \{\|\alpha x+\alpha v\|: v \in V\} \\
& =|\alpha| \inf \{\|x+v\|: v \in V\}=|\alpha|\|[x]\| .
\end{aligned}
$$

Consider the triangle inequality.

$$
\begin{gathered}
\|[x+y]\|=\inf \{\|x+y+v\|: v \in V\} \\
\leq\left\|x+v_{1}\right\|+\left\|y+v_{2}\right\|
\end{gathered}
$$

for any choice of $v_{1}$ and $v_{2}$. Therefore, taking the infimum of both sides over $v_{2}$ yields

$$
\|[x+y]\| \leq\left\|x+v_{1}\right\|+\|[y]\|
$$

and then taking the infimum over all $v_{1}$ yields

$$
\|[x+y]\| \leq\|[x]\|+\|[y]\|
$$

Next consider the claim that $X / V$ is a Banach space. Letting $\left\{\left[x_{n}\right]\right\}$ be a Cauchy sequence in $X / V$, I will show a subsequence of this converges to a point in $X / V$. This is done by defining a suitable sequence in $X$ and then using completeness of $X$. By choosing a subsequence, it can be assumed that $\left\|\left[x_{n}\right]-\left[x_{n+1}\right]\right\|<2^{-n}$. Let $z_{1} \equiv x_{1}$. Then choose $v_{2} \in V$ such that $\left\|x_{2}+v_{2}-z_{1}\right\|<2^{-1}$. Let $z_{2}=x_{2}+v_{2}$. Suppose $\left\{z_{1}, \cdots, z_{n}\right\}$ have been chosen, each having the property that $\left[z_{k}\right]=\left[x_{k}\right]$ and such that $\left\|z_{k}-z_{k+1}\right\|<2^{-k}$. Then let $v_{n+1}$ be chosen such that $\left\|x_{n+1}+v_{n+1}-z_{n}\right\|<2^{-n}$ and let $z_{n+1} \equiv x_{n+1}+v_{n+1}$. Thus $\left\{z_{n}\right\}$ is a Cauchy sequence in $X$ and so it converges to $x \in X$. Then

$$
\left\|[x]-\left[x_{n}\right]\right\| \leq\left\|x-\left(x_{n}+v_{n}\right)\right\|=\left\|x-z_{n}\right\|
$$

and so $\lim _{n \rightarrow \infty}\left[x_{n}\right]=[x]$.
Next consider the claim about $\widehat{A}$. This is well defined and linear because if $[x]=\left[x_{1}\right]$, then $x-x_{1} \in \operatorname{ker}(A)$ and so $A x=A x_{1}$. Thus $\widehat{A}([x])=\widehat{A}\left(\left[x_{1}\right]\right)$. It is linear because

$$
\begin{aligned}
\widehat{A}(\alpha[x]+\beta[y]) & =\widehat{A}([\alpha x+\beta y])=A(\alpha x+\beta y) \\
& =\alpha A x+\beta A y=\alpha \widehat{A}([x])+\beta \widehat{A}([y])
\end{aligned}
$$

Next consider the claim that $\widehat{A}$ is continuous. Letting $v \in V$,

$$
\|\widehat{A}([x])\| \equiv\|A x\|=\|A(x+v)\| \leq\|A\|\|x+v\|
$$

and so, taking the infimum over all $v \in V$,

$$
\|\widehat{A}([x])\| \leq\|A\|\|[x]\|
$$

and this shows $\|\widehat{A}\| \leq\|A\|$.
Now with this theorem, here is an interesting application.

Theorem 18.7.3 Let $X_{1}$ and $X_{2}$ be Banach spaces which are either reflexive or dual spaces for a separable Banach space and let $A_{i} \in \mathscr{L}\left(X_{i}, Y\right)$ for $Y$ a reflexive Banach space. The following are equivalent.

For some $k>0$

$$
\begin{gather*}
A_{1}(\overline{B(0,1)}) \subseteq A_{2}(\overline{B(0, k)})  \tag{18.7.31}\\
\left\|A_{1}^{*} y^{*}\right\| \leq k\left\|A_{2}^{*} y^{*}\right\| \tag{18.7.32}
\end{gather*}
$$

for all $y \in Y^{*}$. If either of the above hold, then

$$
\begin{equation*}
A_{1} X_{1} \subseteq A_{2} X_{2} \tag{18.7.33}
\end{equation*}
$$

Proof: Suppose 18.7.31 first. I show this implies 18.7.33. There are two cases. First suppose $A_{2}$ is one to one. Then in this case, $A_{2}^{-1} A_{1}(\overline{B(0,1)}) \subseteq \overline{B(0, k)}$. Therefore, if $x \in X_{1}$,

$$
A_{2}^{-1} A_{1}(x /\|x\|)=y \in \overline{B(0, k)}
$$

and so

$$
A_{1}(x)=\|x\| A_{2}(y)=A_{2}(\|x\| y) \in A_{2}\left(X_{2}\right)
$$

Next suppose $A_{2}$ is not one to one. In this case, letting $\widehat{A}_{2}$ be the continuous linear map given by

$$
\widehat{A}_{2}([x]) \equiv A_{2} x
$$

it follows $\widehat{A}_{2}$ is $1-1$ on $X_{2} / \operatorname{ker}\left(A_{2}\right)$. Now note that if $\|x\| \leq k$, then it is also the case that $\|[x]\| \leq k$ and so

$$
A_{2}(\overline{B(0, k)}) \subseteq \widehat{A}_{2}\left(\overline{B_{2}(0, k)}\right)
$$

where in the second set, $\overline{B_{2}(0, k)}$ is the unit ball in $X / \operatorname{ker}\left(A_{2}\right)$. It follows from 18.7.31

$$
A_{1}(\overline{B(0,1)}) \subseteq \widehat{A}_{2}\left(\overline{B_{2}(0, k)}\right)
$$

and so $\widehat{A}_{2}^{-1} A_{1}(\overline{B(0,1)}) \subseteq \overline{B_{2}(0, k)}$ which implies

$$
A_{1}(x /\|x\|)=\widehat{A}_{2}[y] \in \widehat{A}_{2}\left(\overline{B_{2}(0, k)}\right)
$$

Therefore, letting $\left[y_{1}\right]=[y]$ be such that $\left\|y_{1}\right\|<2 k$, it follows

$$
A_{1}(x /\|x\|)=\widehat{A}_{2}\left[y_{1}\right]=A_{2}\left(y_{1}\right)
$$

and so

$$
A_{1}(x)=A_{2}\left(\|x\| y_{1}\right) \in A_{2}\left(X_{2}\right)
$$

Next I show the equivalence of 18.7.32 and 18.7.31. First I want to show $A_{i}(\overline{B(0, r)})$ is closed. Suppose then that for $A=A_{1}$ or $A_{2}, A\left(x_{n}\right) \rightarrow y$ where $x_{n} \in \overline{B(0, r)}$. In the case the $X_{i}$ are reflexive, it follows from the Eberlein Smulian theorem there exists a subsequence,
still denoted as $\left\{x_{n}\right\}$ which converges weakly to $x \in \overline{B(0, r)}$. Then $A x_{n} \rightarrow y$ and $x_{n} \rightarrow x$ weakly. Thus $(x, y)$ is in the weak closure of the graph of $A$,

$$
\left\{(x, A x): x \in X_{i}\right\}
$$

This set is strongly closed and convex and hence it is weakly closed by Theorem 18.3.3 so $y=A x$ and this shows $A(\overline{B(0, r)})$ is closed. In the other case where $X_{i}$ is the dual space of a separable Banach space, it follows from Corollary 17.5.6 there exists a subsequence still denoted as $\left\{x_{n}\right\}$ such that $x_{n} \rightarrow x$ weak $*$ and similarly, $(x, y)$ is in the weak $*$ closure of the graph of $A$ which shows again by Theorem 18.3.3 that $(x, y)$ is in the graph of $A$, showing again that $A(\overline{B(0, r)})$ is closed.

Suppose 18.7.31. Then letting $y^{*} \in Y^{\prime}$,

$$
\begin{aligned}
\left\|A_{1}^{*} y^{*}\right\| & =\sup _{\left\|x_{1}\right\|_{X_{1}} \leq 1}\left|y^{*}\left(A_{1} x_{1}\right)\right| \\
& \leq \sup _{\left\|x_{2}\right\|_{X_{2}} \leq k}\left|y^{*}\left(A_{2} x_{2}\right)\right|=k\left\|A_{2}^{*} y^{*}\right\|
\end{aligned}
$$

which shows 18.7.32.
Now suppose 18.7.32. Then if 18.7.31 does not hold, it follows from the first part which gives $A_{i}(\overline{B(0, r)})$ a closed set, there exists

$$
A_{1} x_{0} \in A_{1}(\overline{B(0,1)}) \backslash A_{2}(\overline{B(0, k)})
$$

Now $A_{2}(\overline{B(0, k)})$ is closed and convex, hence weakly closed, and so by Theorem 18.2.5 there exists $y_{0}^{*} \in Y^{\prime}$ such that

$$
\operatorname{Re} y_{0}^{*}\left(A_{2}(\overline{B(0, k)})\right)<c<\operatorname{Re} y_{0}^{*}\left(A_{1} x_{0}\right)
$$

and so

$$
\begin{aligned}
\left\|A_{1}^{*} y_{0}^{*}\right\| & =\sup _{\left\|x_{1}\right\|_{X_{1}} \leq 1}\left|y_{0}^{*}\left(A x_{1}\right)\right| \geq \operatorname{Re} y_{0}^{*}\left(A_{1} x_{0}\right) \\
& >c>\operatorname{Re} y_{0}^{*}\left(A_{2}\left(x_{2}\right)\right)=\operatorname{Re} A_{2}^{*} y_{0}^{*}\left(x_{2}\right)
\end{aligned}
$$

whenever $x_{2} \in \overline{B(0, k)}$ and so, taking the supremum of all such $x_{2}$,

$$
\left\|A_{1}^{*} y_{0}^{*}\right\|>c>k\left\|A_{2}^{*} y_{0}^{*}\right\|
$$

contradicting 18.7.32.

## Chapter 19

## Hilbert Spaces

In this chapter, Hilbert spaces, which have been alluded to earlier are given a complete discussion. These spaces, as noted earlier are just complete inner product spaces.

### 19.1 Basic Theory

Definition 19.1.1 Let $X$ be a vector space. An inner product is a mapping from $X \times X$ to $\mathbb{C}$ if $X$ is complex and from $X \times X$ to $\mathbb{R}$ if $X$ is real, denoted by $(x, y)$ which satisfies the following.

$$
\begin{gather*}
(x, x) \geq 0,(x, x)=0 \text { if and only if } x=0,  \tag{19.1.1}\\
(x, y)=\overline{(y, x) .} \tag{19.1.2}
\end{gather*}
$$

For $a, b \in \mathbb{C}$ and $x, y, z \in X$,

$$
\begin{equation*}
(a x+b y, z)=a(x, z)+b(y, z) \tag{19.1.3}
\end{equation*}
$$

Note that 19.1.2 and 19.1.3 imply $(x, a y+b z)=\bar{a}(x, y)+\bar{b}(x, z)$. Such a vector space is called an inner product space.

The Cauchy Schwarz inequality is fundamental for the study of inner product spaces.
Theorem 19.1.2 (Cauchy Schwarz) In any inner product space

$$
|(x, y)| \leq\|x\|\|y\| .
$$

Proof: Let $\omega \in \mathbb{C},|\omega|=1$, and $\bar{\omega}(x, y)=|(x, y)|=\operatorname{Re}(x, y \omega)$. Let

$$
F(t)=(x+t y \omega, x+t \omega y)
$$

If $y=0$ there is nothing to prove because

$$
(x, 0)=(x, 0+0)=(x, 0)+(x, 0)
$$

and so $(x, 0)=0$. Thus, it can be assumed $y \neq 0$. Then from the axioms of the inner product,

$$
F(t)=\|x\|^{2}+2 t \operatorname{Re}(x, \omega y)+t^{2}\|y\|^{2} \geq 0 .
$$

This yields

$$
\|x\|^{2}+2 t|(x, y)|+t^{2}\|y\|^{2} \geq 0
$$

Since this inequality holds for all $t \in \mathbb{R}$, it follows from the quadratic formula that

$$
4|(x, y)|^{2}-4\|x\|^{2}\|y\|^{2} \leq 0
$$

This yields the conclusion and proves the theorem.
Proposition 19.1.3 For an inner product space, $\|x\| \equiv(x, x)^{1 / 2}$ does specify a norm.

Proof: All the axioms are obvious except the triangle inequality. To verify this,

$$
\begin{aligned}
\|x+y\|^{2} & \equiv(x+y, x+y) \equiv\|x\|^{2}+\|y\|^{2}+2 \operatorname{Re}(x, y) \\
& \leq\|x\|^{2}+\|y\|^{2}+2|(x, y)| \\
& \leq\|x\|^{2}+\|y\|^{2}+2\|x\|\|y\|=(\|x\|+\|y\|)^{2}
\end{aligned}
$$

The following lemma is called the parallelogram identity.

Lemma 19.1.4 In an inner product space,

$$
\|x+y\|^{2}+\|x-y\|^{2}=2\|x\|^{2}+2\|y\|^{2}
$$

The proof, a straightforward application of the inner product axioms, is left to the reader.

Lemma 19.1.5 For $x \in H$, an inner product space,

$$
\begin{equation*}
\|x\|=\sup _{\|y\| \leq 1}|(x, y)| \tag{19.1.4}
\end{equation*}
$$

Proof: By the Cauchy Schwarz inequality, if $x \neq 0$,

$$
\|x\| \geq \sup _{\|y\| \leq 1}|(x, y)| \geq\left(x, \frac{x}{\|x\|}\right)=\|x\| .
$$

It is obvious that 19.1.4 holds in the case that $x=0$.
Definition 19.1.6 A Hilbert space is an inner product space which is complete. Thus a Hilbert space is a Banach space in which the norm comes from an inner product as described above.

In Hilbert space, one can define a projection map onto closed convex nonempty sets.
Definition 19.1.7 $A$ set, $K$, is convex if whenever $\lambda \in[0,1]$ and $x, y \in K, \lambda x+(1-\lambda) y \in K$.
Theorem 19.1.8 Let $K$ be a closed convex nonempty subset of a Hilbert space, $H$, and let $x \in H$. Then there exists a unique point $P x \in K$ such that $\|P x-x\| \leq\|y-x\|$ for all $y \in K$.

Proof: Consider uniqueness. Suppose that $z_{1}$ and $z_{2}$ are two elements of $K$ such that for $i=1,2$,

$$
\begin{equation*}
\left\|z_{i}-x\right\| \leq\|y-x\| \tag{19.1.5}
\end{equation*}
$$

for all $y \in K$. Also, note that since $K$ is convex,

$$
\frac{z_{1}+z_{2}}{2} \in K
$$

Therefore, by the parallelogram identity,

$$
\begin{aligned}
\left\|z_{1}-x\right\|^{2} & \leq\left\|\frac{z_{1}+z_{2}}{2}-x\right\|^{2}=\left\|\frac{z_{1}-x}{2}+\frac{z_{2}-x}{2}\right\|^{2} \\
& =2\left(\left\|\frac{z_{1}-x}{2}\right\|^{2}+\left\|\frac{z_{2}-x}{2}\right\|^{2}\right)-\left\|\frac{z_{1}-z_{2}}{2}\right\|^{2} \\
& =\frac{1}{2}\left\|z_{1}-x\right\|^{2}+\frac{1}{2}\left\|z_{2}-x\right\|^{2}-\left\|\frac{z_{1}-z_{2}}{2}\right\|^{2} \\
& \leq\left\|z_{1}-x\right\|^{2}-\left\|\frac{z_{1}-z_{2}}{2}\right\|^{2}
\end{aligned}
$$

where the last inequality holds because of 19.1.5 letting $z_{i}=z_{2}$ and $y=z_{1}$. Hence $z_{1}=z_{2}$ and this shows uniqueness.

Now let $\lambda=\inf \{\|x-y\|: y \in K\}$ and let $y_{n}$ be a minimizing sequence. This means $\left\{y_{n}\right\} \subseteq K$ satisfies $\lim _{n \rightarrow \infty}\left\|x-y_{n}\right\|=\lambda$. Now the following follows from properties of the norm.

$$
\left\|y_{n}-x+y_{m}-x\right\|^{2}=4\left(\left\|\frac{y_{n}+y_{m}}{2}-x\right\|^{2}\right)
$$

Then by the parallelogram identity, and convexity of $K, \frac{y_{n}+y_{m}}{2} \in K$, and so

$$
\begin{aligned}
\left\|\left(y_{n}-x\right)-\left(y_{m}-x\right)\right\|^{2} & =2\left(\left\|y_{n}-x\right\|^{2}+\left\|y_{m}-x\right\|^{2}\right)-\overbrace{\left(\left\|\frac{y_{n}+y_{m}}{2}-x\right\|^{2}\right)}^{=\left\|y_{n}-x+y_{m}-x\right\|^{2}} \\
& \leq 2\left(\left\|y_{n}-x\right\|^{2}+\left\|y_{m}-x\right\|^{2}\right)-4 \lambda^{2}
\end{aligned}
$$

Since $\left\|x-y_{n}\right\| \rightarrow \lambda$, this shows $\left\{y_{n}-x\right\}$ is a Cauchy sequence. Thus also $\left\{y_{n}\right\}$ is a Cauchy sequence. Since $H$ is complete, $y_{n} \rightarrow y$ for some $y \in H$ which must be in $K$ because $K$ is closed. Therefore

$$
\|x-y\|=\lim _{n \rightarrow \infty}\left\|x-y_{n}\right\|=\lambda
$$

Let $P x=y$.
Corollary 19.1.9 Let $K$ be a closed, convex, nonempty subset of a Hilbert space, $H$, and let $x \in H$. Then for $z \in K, z=P x$ if and only if

$$
\begin{equation*}
\operatorname{Re}(x-z, y-z) \leq 0 \tag{19.1.6}
\end{equation*}
$$

for all $y \in K$.
Before proving this, consider what it says in the case where the Hilbert space is $\mathbb{R}^{n}$.


Condition 19.1.6 says the angle, $\theta$, shown in the diagram is always obtuse. Remember from calculus, the sign of $\mathbf{x} \cdot \mathbf{y}$ is the same as the sign of the cosine of the included angle
between $\mathbf{x}$ and $\mathbf{y}$. Thus, in finite dimensions, the conclusion of this corollary says that $z=P x$ exactly when the indicated angle is obtuse. Surely the picture suggests this is reasonable.

The inequality 19.1 .6 is an example of a variational inequality and this corollary characterizes the projection of $x$ onto $K$ as the solution of this variational inequality.

Proof of Corollary: Let $z \in K$ and let $y \in K$ also. Since $K$ is convex, it follows that if $t \in[0,1]$,

$$
z+t(y-z)=(1-t) z+t y \in K
$$

Furthermore, every point of $K$ can be written in this way. (Let $t=1$ and $y \in K$.) Therefore, $z=P x$ if and only if for all $y \in K$ and $t \in[0,1]$,

$$
\|x-(z+t(y-z))\|^{2}=\|(x-z)-t(y-z)\|^{2} \geq\|x-z\|^{2}
$$

for all $t \in[0,1]$ and $y \in K$ if and only if for all $t \in[0,1]$ and $y \in K$

$$
\|x-z\|^{2}+t^{2}\|y-z\|^{2}-2 t \operatorname{Re}(x-z, y-z) \geq\|x-z\|^{2}
$$

If and only if for all $t \in[0,1]$,

$$
\begin{equation*}
t^{2}\|y-z\|^{2}-2 t \operatorname{Re}(x-z, y-z) \geq 0 \tag{19.1.7}
\end{equation*}
$$

Now this is equivalent to 19.1.7 holding for all $t \in(0,1)$. Therefore, dividing by $t \in(0,1)$, 19.1.7 is equivalent to

$$
t\|y-z\|^{2}-2 \operatorname{Re}(x-z, y-z) \geq 0
$$

for all $t \in(0,1)$ which is equivalent to 19.1.6. This proves the corollary.
Corollary 19.1.10 Let $K$ be a nonempty convex closed subset of a Hilbert space, H. Then the projection map, $P$ is continuous. In fact,

$$
|P x-P y| \leq|x-y|
$$

Proof: Let $x, x^{\prime} \in H$. Then by Corollary 19.1.9,

$$
\operatorname{Re}\left(x^{\prime}-P x^{\prime}, P x-P x^{\prime}\right) \leq 0, \operatorname{Re}\left(x-P x, P x^{\prime}-P x\right) \leq 0
$$

Hence

$$
\begin{aligned}
0 & \leq \operatorname{Re}\left(x-P x, P x-P x^{\prime}\right)-\operatorname{Re}\left(x^{\prime}-P x^{\prime}, P x-P x^{\prime}\right) \\
& =\operatorname{Re}\left(x-x^{\prime}, P x-P x^{\prime}\right)-\left|P x-P x^{\prime}\right|^{2}
\end{aligned}
$$

and so

$$
\left|P x-P x^{\prime}\right|^{2} \leq\left|x-x^{\prime}\right|\left|P x-P x^{\prime}\right|
$$

This proves the corollary.
The next corollary is a more general form for the Brouwer fixed point theorem.
Corollary 19.1.11 Let $\mathbf{f}: K \rightarrow K$ where $K$ is a convex compact subset of $\mathbb{R}^{n}$. Then $\mathbf{f}$ has a fixed point.

Proof: Let $K \subseteq \overline{B(\mathbf{0}, R)}$ and let $P$ be the projection map onto $K$. Then consider the map $\mathbf{f} \circ P$ which maps $\overline{B(\mathbf{0}, R)}$ to $\overline{B(\mathbf{0}, R)}$ and is continuous. By the Brouwer fixed point theorem for balls, this map has a fixed point. Thus there exists $\mathbf{x}$ such that

$$
\mathbf{f} \circ P(\mathbf{x})=\mathbf{x}
$$

Now the equation also requires $\mathbf{x} \in K$ and so $P(\mathbf{x})=\mathbf{x}$. Hence $\mathbf{f}(\mathbf{x})=\mathbf{x}$.
Definition 19.1.12 Let $H$ be a vector space and let $U$ and $V$ be subspaces. $U \oplus V=H$ if every element of $H$ can be written as a sum of an element of $U$ and an element of $V$ in a unique way.

The case where the closed convex set is a closed subspace is of special importance and in this case the above corollary implies the following.

Corollary 19.1.13 Let $K$ be a closed subspace of a Hilbert space, $H$, and let $x \in H$. Then for $z \in K, z=P x$ if and only if

$$
\begin{equation*}
(x-z, y)=0 \tag{19.1.8}
\end{equation*}
$$

for all $y \in K$. Furthermore, $H=K \oplus K^{\perp}$ where

$$
K^{\perp} \equiv\{x \in H:(x, k)=0 \text { for all } k \in K\}
$$

and

$$
\begin{equation*}
\|x\|^{2}=\|x-P x\|^{2}+\|P x\|^{2} . \tag{19.1.9}
\end{equation*}
$$

Proof: Since $K$ is a subspace, the condition 19.1.6 implies $\operatorname{Re}(x-z, y) \leq 0$ for all $y \in K$. Replacing $y$ with $-y$, it follows $\operatorname{Re}(x-z,-y) \leq 0$ which implies $\operatorname{Re}(x-z, y) \geq 0$ for all $y$. Therefore, $\operatorname{Re}(x-z, y)=0$ for all $y \in K$. Now let $|\alpha|=1$ and $\alpha(x-z, y)=|(x-z, y)|$. Since $K$ is a subspace, it follows $\bar{\alpha} y \in K$ for all $y \in K$. Therefore,

$$
0=\operatorname{Re}(x-z, \bar{\alpha} y)=(x-z, \bar{\alpha} y)=\alpha(x-z, y)=|(x-z, y)| .
$$

This shows that $z=P x$, if and only if 19.1.8.
For $x \in H, x=x-P x+P x$ and from what was just shown, $x-P x \in K^{\perp}$ and $P x \in K$. This shows that $K^{\perp}+K=H$. Is there only one way to write a given element of $H$ as a sum of a vector in $K$ with a vector in $K^{\perp}$ ? Suppose $y+z=y_{1}+z_{1}$ where $z, z_{1} \in K^{\perp}$ and $y, y_{1} \in K$. Then $\left(y-y_{1}\right)=\left(z_{1}-z\right)$ and so from what was just shown, $\left(y-y_{1}, y-y_{1}\right)=$ $\left(y-y_{1}, z_{1}-z\right)=0$ which shows $y_{1}=y$ and consequently $z_{1}=z$. Finally, letting $z=P x$,

$$
\begin{aligned}
\|x\|^{2} & =(x-z+z, x-z+z)=\|x-z\|^{2}+(x-z, z)+(z, x-z)+\|z\|^{2} \\
& =\|x-z\|^{2}+\|z\|^{2}
\end{aligned}
$$

This proves the corollary.
The following theorem is called the Riesz representation theorem for the dual of a Hilbert space. If $z \in H$ then define an element $f \in H^{\prime}$ by the rule $(x, z) \equiv f(x)$. It follows from the Cauchy Schwarz inequality and the properties of the inner product that $f \in H^{\prime}$. The Riesz representation theorem says that all elements of $H^{\prime}$ are of this form.

Theorem 19.1.14 Let $H$ be a Hilbert space and let $f \in H^{\prime}$. Then there exists a unique $z \in H$ such that

$$
\begin{equation*}
f(x)=(x, z) \tag{19.1.10}
\end{equation*}
$$

for all $x \in H$.
Proof: Letting $y, w \in H$ the assumption that $f$ is linear implies

$$
f(y f(w)-f(y) w)=f(w) f(y)-f(y) f(w)=0
$$

which shows that $y f(w)-f(y) w \in f^{-1}(0)$, which is a closed subspace of $H$ since $f$ is continuous. If $f^{-1}(0)=H$, then $f$ is the zero map and $z=0$ is the unique element of $H$ which satisfies 19.1.10. If $f^{-1}(0) \neq H$, pick $u \notin f^{-1}(0)$ and let $w \equiv u-P u \neq 0$. Thus Corollary 19.1.13 implies $(y, w)=0$ for all $y \in f^{-1}(0)$. In particular, let $y=x f(w)-f(x) w$ where $x \in H$ is arbitrary. Therefore,

$$
0=(f(w) x-f(x) w, w)=f(w)(x, w)-f(x)\|w\|^{2} .
$$

Thus, solving for $f(x)$ and using the properties of the inner product,

$$
f(x)=\left(x, \frac{\overline{f(w)} w}{\|w\|^{2}}\right)
$$

Let $z=\overline{f(w)} w /\|w\|^{2}$. This proves the existence of $z$. If $f(x)=\left(x, z_{i}\right) i=1,2$, for all $x \in H$, then for all $x \in H$, then $\left(x, z_{1}-z_{2}\right)=0$ which implies, upon taking $x=z_{1}-z_{2}$ that $z_{1}=z_{2}$. This proves the theorem.

If $R: H \rightarrow H^{\prime}$ is defined by $R x(y) \equiv(y, x)$, the Riesz representation theorem above states this map is onto. This map is called the Riesz map. It is routine to show $R$ is linear and $|R x|=|x|$.

### 19.2 The Hilbert Space $L(U)$

Let $L \in \mathscr{L}(U, H)$. Then one can consider the image of $L, L(U)$ as a Hilbert space. This is another interesting application of Theorem 19.1.8. First here is a definition which involves abominable and atrociously misleading notation which nevertheless seems to be well accepted.
Definition 19.2.1 Let $L \in \mathscr{L}(U, H)$, the bounded linear maps from $U$ to $H$ for $U, H$ Hilbert spaces. For $y \in L(U)$, let $L^{-1} y$ denote the unique vector in

$$
\{x: L x=y\} \equiv M_{y}
$$

which is closest in $U$ to 0 .


Note this is a good definition because $\{x: L x=y\}$ is closed thanks to the continuity of $L$ and it is obviously convex. Thus Theorem 19.1.8 applies. With this definition define an inner product on $L(U)$ as follows. For $y, z \in L(U)$,

$$
(y, z)_{L(U)} \equiv\left(L^{-1} y, L^{-1} z\right)_{U}
$$

The notation is abominable because $L^{-1}(y)$ is the normal notation for $M_{y}$.
In terms of linear algebra, this $L^{-1}$ is the Moore Penrose inverse. There you obtain the least squares solution $x$ to $L x=y$ which has smallest norm. Here there is an actual solution and among those solutions you get the one which has least norm. Of course a real honest solution is also a least squares solution so this is the Moore Penrose inverse restricted to $L(U)$.

First I want to understand $L^{-1}$ better. It is actually fairly easy to understand in terms of geometry. Here is a picture of $L^{-1}(y)$ for $y \in L(U)$.


As indicated in the picture, here is a lemma which gives a description of the situation.
Lemma 19.2.2 In the context of the above definition, $L^{-1}(y)$ is characterized by

$$
\begin{aligned}
\left(L^{-1}(y), x\right)_{U} & =0 \text { for all } x \in \operatorname{ker}(L) \\
L\left(L^{-1}(y)\right) & =y, \quad\left(L^{-1}(y) \in M_{y}\right)
\end{aligned}
$$

In addition to this, $L^{-1}$ is linear and the above definition does define an inner product.
Proof: The point $L^{-1}(y)$ is well defined as noted above. I claim it is characterized by the following for $y \in L(U)$

$$
\begin{aligned}
\left(L^{-1}(y), x\right)_{U} & =0 \text { for all } x \in \operatorname{ker}(L) \\
L\left(L^{-1}(y)\right) & =y, \quad\left(L^{-1}(y) \in M_{y}\right)
\end{aligned}
$$

Let $w \in M_{y}$ and suppose

$$
(v, x)_{U}=0, L(v)=y
$$

Then from the above characterization,

$$
\|w\|^{2}=\|\overbrace{w-v}^{\in \operatorname{ker}(L)}+v\|^{2}=\|w-v\|^{2}+\|v\|^{2}
$$

which shows that $w=L^{-1}(y)$ if and only if $w=v$ just described. From this characterization, it is clear that $L^{-1}$ is linear. Then it is also obvious that

$$
(y, z)_{L(U)}=\left(L^{-1} y, L^{-1} z\right)_{U}
$$

also specifies an inner product. The algebraic axioms are all obvious because $L^{-1}$ is linear. If $(y, y)_{L(U)}=0$, then $\left|L^{-1} y\right|_{U}^{2}=0$ and so $L^{-1} y=0$ which requires $y=L\left(L^{-1} y\right)=0$.

With the above definition, here is the main result.
Theorem 19.2.3 Let $U, H$ be Hilbert spaces and let $L \in \mathscr{L}(U, H)$. Then Definition 19.2.1 makes $L(U)$ into a Hilbert space. Also $L: U \rightarrow L(U)$ is continuous and $L^{-1}: L(U) \rightarrow U$ is continuous. Also,

$$
\begin{equation*}
\|L\|_{\mathscr{L}(U, H)}\|L x\|_{L(U)} \geq\|L x\|_{H} \tag{19.2.11}
\end{equation*}
$$

If $U$ is separable, so is $L(U)$. Also $\left(L^{-1}(y), x\right)=0$ for all $x \in \operatorname{ker}(L)$, and $L^{-1}: L(U) \rightarrow U$ is linear. Also, in case that $L$ is one to one, both $L$ and $L^{-1}$ preserve norms.

Proof: First consider the claim that $L: U \rightarrow L(U)$ is continuous and $L^{-1}: L(U) \rightarrow U$ is also continuous. Why is $L$ continuous? Say $u_{n} \rightarrow 0$ in $U$. Then

$$
\left\|L u_{n}\right\|_{L(U)} \equiv\left\|L^{-1}\left(L\left(u_{n}\right)\right)\right\|_{U}
$$

Now $\left\|L^{-1}\left(L\left(u_{n}\right)\right)\right\|_{U} \leq\left\|u_{n}\right\|_{U}$ and so it converges to 0 . (Recall that $L^{-1}\left(L u_{n}\right)$ is the smallest vector in $U$ which maps to $L u_{n}$. Since $u_{n}$ is mapped by $L$ to $L u_{n}$, it follows that $\left.\left\|L^{-1}\left(L\left(u_{n}\right)\right)\right\|_{U} \leq\left\|u_{n}\right\|_{U}.\right)$ Hence $L$ is continuous.

Next, why is $L^{-1}$ continuous? Let $\left\|y_{n}\right\|_{L(U)} \rightarrow 0$. This requires $\left\|L^{-1}\left(y_{n}\right)\right\|_{U} \rightarrow 0$ by definition of the norm in $L(U)$. Thus $L^{-1}$ is continuous.

Why is $L(U)$ a Hilbert space? Let $\left\{y_{n}\right\}$ be a Cauchy sequence in $L(U)$. Then from what was just observed, it follows that $L^{-1}\left(y_{n}\right)$ is a Cauchy sequence in $U$. Hence $L^{-1}\left(y_{n}\right) \rightarrow$ $x \in U$. It follows that $y_{n}=L\left(L^{-1}\left(y_{n}\right)\right) \rightarrow L x$ in $L(U)$. This is in the norm of $L(U)$. It was just shown that $L$ is continuous as a map from $U$ to $L(U)$. This shows that $L(U)$ is a Hilbert space. It was already shown that it is an inner product space and this has shown that it is complete.

If $x \in U$, then $\|L x\|_{H} \leq\|L\|_{\mathscr{L}(U, H)}\|x\|_{U}$. It follows that

$$
\begin{aligned}
\|L(x)\|_{H} & =\left\|L\left(L^{-1}(L(x))\right)\right\|_{H} \leq\|L\|_{\mathscr{L}(U, H)}\left\|L^{-1}(L(x))\right\|_{U} \\
& =\|L\|_{\mathscr{L}(U, H)}\|L(x)\|_{L(U)}
\end{aligned}
$$

This verifies 19.2.11.

If $U$ is separable, then letting $D$ be a countable dense subset, it follows from the continuity of the operators $L, L^{-1}$ discussed above that $L(D)$ is separable in. To see this, note that

$$
\begin{aligned}
\left\|L x_{n}-L x\right\|_{L(U)} & =\left\|L\left(L^{-1}\left(L x_{n}-L x\right)\right)\right\| \\
& \leq\|L\|_{\mathscr{L}(U, H)}\left\|L^{-1}\left(L\left(x_{n}-x\right)\right)\right\|_{U} \\
& \leq\|L\|_{\mathscr{L}(U, H)}\left\|x_{n}-x\right\|_{U}
\end{aligned}
$$

As before, $L^{-1}\left(L\left(x_{n}-x\right)\right)$ is the smallest vector which maps onto $L\left(x_{n}-x\right)$ and so its norm is no larger than $\left\|x_{n}-x\right\|_{U}$.

Consider the last claim. If $L$ is one to one, then for $y \in L(U)$, there is only one vector which maps to $y$. Therefore,

$$
L^{-1}(L(x))=x
$$

Hence for $y \in L(U)$,

$$
\|y\|_{L(U)} \equiv\left\|L^{-1}(y)\right\|_{U}
$$

Also,

$$
\|L u\|_{L(U)} \equiv\left\|L^{-1}(L(u))\right\|_{U} \equiv\|u\|_{U}
$$

Now here is another argument for various continuity claims.

$$
\|L x\|_{L(U)} \equiv\left\|L^{-1}(L x)\right\|_{U} \leq\|x\|_{U}
$$

because $L^{-1}(L x)$ is the smallest thing in $U$ which maps to $L x$ and $x$ is something which maps to $L x$ so it follows that the inequality holds. Hence $L \in \mathscr{L}(U, L(U))$ and in fact, $\|L\|_{\mathscr{L}(U, L(U))}=1$. Next, letting $y \in L(U)$,

$$
\left\|L^{-1} y\right\|_{U} \equiv\|y\|_{L(U)}
$$

and so $\left\|L^{-1}\right\|_{\mathscr{L}(L(U), U)}=1$ and this shows that $L \in \mathscr{L}(U, L(U))$ while $L^{-1} \in \mathscr{L}(L(U), U)$ and both have norm equal to 1 .

Now

$$
\|L x\|_{H}=\left\|L\left(L^{-1}(L x)\right)\right\|_{H} \leq\|L\|_{\mathscr{L}(U, H)}\left\|L^{-1}(L x)\right\|_{U} \equiv\|L\|_{\mathscr{L}(U, H)}\|L x\|_{L(U)}
$$

Now here are some other very interesting results. I am following [108].
Lemma 19.2.4 Let $L \in \mathscr{L}(U, H)$. Then $L(\overline{B(0, r)})$ is closed and convex.
Proof: It is clear this is convex since $L$ is linear. Why is it closed? $\overline{B(0, r)}$ is compact in the weak topology by the Banach Alaoglu theorem, Theorem 17.5.4 on Page 461. Furthermore, $L$ is continuous with respect to the weak topologies on $U$ and $H$. Here is why this is so. Suppose $u_{n} \rightarrow u$ weakly in $U$. Then if $h \in H$,

$$
\left(L u_{n}, h\right)=\left(u_{n}, L^{*} h\right) \rightarrow\left(u, L^{*} h\right)=(L u, h)
$$

which shows $L u_{n} \rightarrow L u$ weakly. Therefore, $L(\overline{B(0, r)})$ is weakly compact because it is the continuous image of a compact set. Therefore, it must also be weakly closed because the weak topology is a Hausdorff space. (See Lemma 18.3.2 on Page 493, and so you can apply the separation theorem, Theorem 18.2.5 on Page 486 to obtain a separating functional. Thus if $x \neq y$, there exists $f \in H^{\prime}$ such that $\operatorname{Re} f(y)>c>\operatorname{Re} f(x)$ and so taking

$$
\begin{gathered}
2 r<\min (c-\operatorname{Re} f(x), \operatorname{Re} f(y)-c), \\
B_{f}(x, r) \cap B_{f}(y, r)=\emptyset
\end{gathered}
$$

where

$$
B_{f}(x, r) \equiv\{y \in H:|f(x-y)|<r\}
$$

is an example of a basic open set in the weak topology.)
Now suppose $p \notin L(\overline{B(0, r)})$. Since the set is weakly closed and convex, it follows by Theorem 18.2.5 and the Riesz representation theorem for Hilbert space that there exists $z \in H$ such that

$$
\operatorname{Re}(p, z)>c>\operatorname{Re}(L x, z)
$$

for all $x \in \overline{B(0, r)}$. Therefore, $p$ cannot be a strong limit point because if it were, there would exist $x_{n} \in \overline{B(0, r)}$ such that $L x_{n} \rightarrow p$ which would require $\operatorname{Re}\left(L x_{n}, z\right) \rightarrow \operatorname{Re}(p, z)$ which is prevented by the above inequality. This proves the lemma.

Now here is a very interesting result about showing that $T_{1}\left(U_{1}\right)=T_{2}\left(U_{2}\right)$ where $U_{i}$ is a Hilbert space and $T_{i} \in \mathscr{L}\left(U_{i}, H\right)$. The situation is as indicated in the diagram.


The question is whether $T_{1} U_{1}=T_{2} U_{2}$.
Theorem 19.2.5 Let $U_{i}, i=1,2$ and $H$ be Hilbert spaces and let $T_{i} \in \mathscr{L}\left(U_{i}, H\right)$. If there exists $c \geq 0$ such that for all $x \in H$

$$
\begin{equation*}
\left\|T_{1}^{*} x\right\|_{1} \leq c\left\|T_{2}^{*} x\right\|_{2} \tag{19.2.12}
\end{equation*}
$$

then

$$
\begin{equation*}
T_{1}(\overline{B(0,1)}) \subseteq T_{2}(\overline{B(0, c)}) \tag{19.2.13}
\end{equation*}
$$

and so $T_{1}\left(U_{1}\right) \subseteq T_{2}\left(U_{2}\right)$. If $\left\|T_{1}^{*} x\right\|_{1}=\left\|T^{*} x\right\|_{2}$ for all $x \in H$, then $T_{1}\left(U_{1}\right)=T_{2}\left(U_{2}\right)$ and in addition to this,

$$
\begin{equation*}
\left\|T_{1}^{-1} x\right\|_{1}=\left\|T_{2}^{-1} x\right\|_{2} \tag{19.2.14}
\end{equation*}
$$

for all $x \in T_{1}\left(U_{1}\right)=T_{2}\left(U_{2}\right)$. In this theorem, $T_{i}^{-1}$ refers to Definition 19.2.1.
Proof: Consider the first claim. If it is not so, then there exists $u_{0},\left\|u_{0}\right\|_{1} \leq 1$ but

$$
T_{1}\left(u_{0}\right) \notin T_{2}(\overline{B(0, c)})
$$

the latter set being a closed convex nonempty set thanks to Lemma 19.2.4. Then by the separation theorem, Theorem 18.2 .5 there exists $z \in H$ such that

$$
\operatorname{Re}\left(T_{1}\left(u_{0}\right), z\right)_{H}>1>\operatorname{Re}\left(T_{2}(v), z\right)_{H}
$$

for all $\|v\|_{2} \leq c$. Therefore, replacing $v$ with $v \theta$ where $\theta$ is a suitable complex number having modulus 1 , it follows

$$
\begin{equation*}
\left\|T_{1}^{*} z\right\|>1>\left|\left(v, T_{2}^{*} z\right)_{U_{2}}\right| \tag{19.2.15}
\end{equation*}
$$

for all $\|v\|_{2} \leq c$. If $c=0,19.2 .15$ gives a contradiction immediately because of 19.2.12. Assume then that $c>0$. From 19.2.15, if $\|v\|_{2} \leq 1$, then

$$
\left|\left(v, T_{2}^{*} z\right)_{U_{2}}\right|<\frac{1}{c}<\frac{1}{c}\left\|T_{1}^{*} z\right\|
$$

Then from 19.2.15,

$$
\left\|T_{2}^{*} z\right\|_{U_{2}}=\sup _{\|v\| \leq 1}\left|\left(v, T_{2}^{*} z\right)_{U_{2}}\right| \leq \frac{1}{c}<\frac{1}{c}\left\|T_{1}^{*} z\right\|
$$

which contradicts 19.2.12. Therefore, it is clear that $T_{1}\left(U_{1}\right) \subseteq T_{2}\left(U_{2}\right)$.
Now consider the second claim. The first part shows $T_{1}\left(U_{1}\right)=T_{2}\left(U_{2}\right)$. Denote by $u_{i} \in U_{i}$, the point $T_{i}^{-1} x$. Without loss of generality, it can be assumed $x \neq 0$ because if $x=0$, then the definition of $T_{i}^{-1}$ gives $T_{i}^{-1}(x)=0$. Thus for $x \neq 0$ neither $u_{i}$ can equal 0 . I need to verify that $\left\|u_{1}\right\|_{1}=\left\|u_{2}\right\|_{2}$. Suppose then that this is not so. Say $\left\|u_{1}\right\|_{1}>\left\|u_{2}\right\|_{2}>0$.

$$
\frac{x}{\left\|u_{2}\right\|_{2}}=T_{2}\left(\frac{u_{2}}{\left\|u_{2}\right\|_{2}}\right) \in T_{2}(\overline{B(0,1)})
$$

But from the first part of the theorem this equals $T_{1}(\overline{B(0,1)})$ and so there exists $u_{1}^{\prime} \in$ $\overline{B(0,1)}$ such that

$$
\frac{x}{\left\|u_{2}\right\|_{2}}=T_{1} u_{1}^{\prime}
$$

Hence

$$
T_{1}\left(u_{1}^{\prime}-\frac{u_{1}}{\left\|u_{2}\right\|_{2}}\right)=\frac{x}{\left\|u_{2}\right\|_{2}}-\frac{x}{\left\|u_{2}\right\|_{2}}=0
$$

From Theorem 19.2.3 this implies

$$
\begin{aligned}
0 & =\left(u_{1}, u_{1}^{\prime}-\frac{u_{1}}{\left\|u_{2}\right\|_{2}}\right) \leq\left\|u_{1}\right\|_{1}\left\|u_{1}^{\prime}\right\|_{1}-\left\|u_{1}\right\|_{1} \frac{\left\|u_{1}\right\|_{1}}{\left\|u_{2}\right\|_{2}} \\
& =\left\|u_{1}\right\|_{1}\left(\left\|u_{1}^{\prime}\right\|_{1}-\frac{\left\|u_{1}\right\|_{1}}{\left\|u_{2}\right\|_{2}}\right) \leq\left\|u_{1}\right\|_{1}\left(1-\frac{\left\|u_{1}\right\|_{1}}{\left\|u_{2}\right\|_{2}}\right)
\end{aligned}
$$

which is a contradiction because it was assumed $\frac{\left\|u_{1}\right\|_{1}}{\left\|u_{2}\right\|_{2}}>1$. This proves the theorem.

### 19.3 Approximations In Hilbert Space

The Gram Schmidt process applies in any Hilbert space.
Theorem 19.3.1 Let $\left\{x_{1}, \cdots, x_{n}\right\}$ be a basis for $M$ a subspace of $H$ a Hilbert space. Then there exists an orthonormal basis for $M,\left\{u_{1}, \cdots, u_{n}\right\}$ which has the property that for each $k \leq n, \operatorname{span}\left(x_{1}, \cdots, x_{k}\right)=\operatorname{span}\left(u_{1}, \cdots, u_{k}\right)$. Also if $\left\{x_{1}, \cdots, x_{n}\right\} \subseteq H$, then

$$
\operatorname{span}\left(x_{1}, \cdots, x_{n}\right)
$$

is a closed subspace.
Proof: Let $\left\{x_{1}, \cdots, x_{n}\right\}$ be a basis for $M$. Let $u_{1} \equiv x_{1} /\left|x_{1}\right|$. Thus for $k=1, \operatorname{span}\left(u_{1}\right)=$ $\operatorname{span}\left(x_{1}\right)$ and $\left\{u_{1}\right\}$ is an orthonormal set. Now suppose for some $k<n, u_{1}, \cdots, u_{k}$ have been chosen such that $\left(u_{j} \cdot u_{l}\right)=\delta_{j l}$ and span $\left(x_{1}, \cdots, x_{k}\right)=\operatorname{span}\left(u_{1}, \cdots, u_{k}\right)$. Then define

$$
\begin{equation*}
u_{k+1} \equiv \frac{x_{k+1}-\sum_{j=1}^{k}\left(x_{k+1} \cdot u_{j}\right) u_{j}}{\left|x_{k+1}-\sum_{j=1}^{k}\left(x_{k+1} \cdot u_{j}\right) u_{j}\right|}, \tag{19.3.16}
\end{equation*}
$$

where the denominator is not equal to zero because the $x_{j}$ form a basis and so

$$
x_{k+1} \notin \operatorname{span}\left(x_{1}, \cdots, x_{k}\right)=\operatorname{span}\left(u_{1}, \cdots, u_{k}\right)
$$

Thus by induction,

$$
u_{k+1} \in \operatorname{span}\left(u_{1}, \cdots, u_{k}, x_{k+1}\right)=\operatorname{span}\left(x_{1}, \cdots, x_{k}, x_{k+1}\right)
$$

Also, $x_{k+1} \in \operatorname{span}\left(u_{1}, \cdots, u_{k}, u_{k+1}\right)$ which is seen easily by solving 19.3 .16 for $x_{k+1}$ and it follows

$$
\operatorname{span}\left(x_{1}, \cdots, x_{k}, x_{k+1}\right)=\operatorname{span}\left(u_{1}, \cdots, u_{k}, u_{k+1}\right)
$$

If $l \leq k$,

$$
\begin{aligned}
\left(u_{k+1} \cdot u_{l}\right) & =C\left(\left(x_{k+1} \cdot u_{l}\right)-\sum_{j=1}^{k}\left(x_{k+1} \cdot u_{j}\right)\left(u_{j} \cdot u_{l}\right)\right) \\
& =C\left(\left(x_{k+1} \cdot u_{l}\right)-\sum_{j=1}^{k}\left(x_{k+1} \cdot u_{j}\right) \delta_{l j}\right) \\
& =C\left(\left(x_{k+1} \cdot u_{l}\right)-\left(x_{k+1} \cdot u_{l}\right)\right)=0
\end{aligned}
$$

The vectors, $\left\{u_{j}\right\}_{j=1}^{n}$, generated in this way are therefore an orthonormal basis because each vector has unit length.

Consider the second claim about finite dimensional subspaces. Without loss of generality, assume $\left\{x_{1}, \cdots, x_{n}\right\}$ is linearly independent. If it is not, delete vectors until a linearly independent set is obtained. Then by the first part, $\operatorname{span}\left(x_{1}, \cdots, x_{n}\right)=\operatorname{span}\left(u_{1}, \cdots, u_{n}\right) \equiv M$ where the $u_{i}$ are an orthonormal set of vectors. Suppose $\left\{y_{k}\right\} \subseteq M$ and $y_{k} \rightarrow y \in H$. Is $y \in M$ ? Let

$$
y_{k} \equiv \sum_{j=1}^{n} c_{j}^{k} u_{j}
$$

Then let $\mathbf{c}^{k} \equiv\left(c_{1}^{k}, \cdots, c_{n}^{k}\right)^{T}$. Then

$$
\begin{aligned}
\left|\mathbf{c}^{k}-\mathbf{c}^{l}\right|^{2} & \equiv \sum_{j=1}^{n}\left|c_{j}^{k}-c_{j}^{l}\right|^{2}=\left(\sum_{j=1}^{n}\left(c_{j}^{k}-c_{j}^{l}\right) u_{j}, \sum_{j=1}^{n}\left(c_{j}^{k}-c_{j}^{l}\right) u_{j}\right) \\
& =\left\|y_{k}-y_{l}\right\|^{2}
\end{aligned}
$$

which shows $\left\{\mathbf{c}^{k}\right\}$ is a Cauchy sequence in $\mathbb{F}^{n}$ and so it converges to $\mathbf{c} \in \mathbb{F}^{n}$. Thus

$$
y=\lim _{k \rightarrow \infty} y_{k}=\lim _{k \rightarrow \infty} \sum_{j=1}^{n} c_{j}^{k} u_{j}=\sum_{j=1}^{n} c_{j} u_{j} \in M
$$

This completes the proof.
Theorem 19.3.2 Let $M$ be the span of $\left\{u_{1}, \cdots, u_{n}\right\}$ in a Hilbert space, $H$ and let $y \in H$. Then Py is given by

$$
\begin{equation*}
P y=\sum_{k=1}^{n}\left(y, u_{k}\right) u_{k} \tag{19.3.17}
\end{equation*}
$$

and the distance is given by

$$
\begin{equation*}
\sqrt{|y|^{2}-\sum_{k=1}^{n}\left|\left(y, u_{k}\right)\right|^{2}} \tag{19.3.18}
\end{equation*}
$$

## Proof:

$$
\begin{aligned}
\left(y-\sum_{k=1}^{n}\left(y, u_{k}\right) u_{k}, u_{p}\right) & =\left(y, u_{p}\right)-\sum_{k=1}^{n}\left(y, u_{k}\right)\left(u_{k}, u_{p}\right) \\
& =\left(y, u_{p}\right)-\left(y, u_{p}\right)=0
\end{aligned}
$$

It follows that

$$
\left(y-\sum_{k=1}^{n}\left(y, u_{k}\right) u_{k}, u\right)=0
$$

for all $u \in M$ and so by Corollary 19.1.13 this verifies 19.3.17.
The square of the distance, $d$ is given by

$$
\begin{aligned}
d^{2} & =\left(y-\sum_{k=1}^{n}\left(y, u_{k}\right) u_{k}, y-\sum_{k=1}^{n}\left(y, u_{k}\right) u_{k}\right) \\
& =|y|^{2}-2 \sum_{k=1}^{n}\left|\left(y, u_{k}\right)\right|^{2}+\sum_{k=1}^{n}\left|\left(y, u_{k}\right)\right|^{2}
\end{aligned}
$$

and this shows 19.3.18.
What if the subspace is the span of vectors which are not orthonormal? There is a very interesting formula for the distance between a point of a Hilbert space and a finite dimensional subspace spanned by an arbitrary basis.

Definition 19.3.3 Let $\left\{x_{1}, \cdots, x_{n}\right\} \subseteq H$, a Hilbert space. Define

$$
\mathscr{G}\left(x_{1}, \cdots, x_{n}\right) \equiv\left(\begin{array}{ccc}
\left(x_{1}, x_{1}\right) & \cdots & \left(x_{1}, x_{n}\right)  \tag{19.3.19}\\
\vdots & & \vdots \\
\left(x_{n}, x_{1}\right) & \cdots & \left(x_{n}, x_{n}\right)
\end{array}\right)
$$

Thus the $i j^{\text {th }}$ entry of this matrix is $\left(x_{i}, x_{j}\right)$. This is sometimes called the Gram matrix. Also define $G\left(x_{1}, \cdots, x_{n}\right)$ as the determinant of this matrix, also called the Gram determinant.

$$
G\left(x_{1}, \cdots, x_{n}\right) \equiv\left|\begin{array}{ccc}
\left(x_{1}, x_{1}\right) & \cdots & \left(x_{1}, x_{n}\right)  \tag{19.3.20}\\
\vdots & & \vdots \\
\left(x_{n}, x_{1}\right) & \cdots & \left(x_{n}, x_{n}\right)
\end{array}\right|
$$

The theorem is the following.
Theorem 19.3.4 Let $M=\operatorname{span}\left(x_{1}, \cdots, x_{n}\right) \subseteq H$, a Real Hilbert space where $\left\{x_{1}, \cdots, x_{n}\right\}$ is a basis and let $y \in H$. Then letting $d$ be the distance from $y$ to $M$,

$$
\begin{equation*}
d^{2}=\frac{G\left(x_{1}, \cdots, x_{n}, y\right)}{G\left(x_{1}, \cdots, x_{n}\right)} \tag{19.3.21}
\end{equation*}
$$

Proof: By Theorem 19.3.1 $M$ is a closed subspace of $H$. Let $\sum_{k=1}^{n} \alpha_{k} x_{k}$ be the element of $M$ which is closest to $y$. Then by Corollary 19.1.13,

$$
\left(y-\sum_{k=1}^{n} \alpha_{k} x_{k}, x_{p}\right)=0
$$

for each $p=1,2, \cdots, n$. This yields the system of equations,

$$
\begin{equation*}
\left(y, x_{p}\right)=\sum_{k=1}^{n}\left(x_{p}, x_{k}\right) \alpha_{k}, p=1,2, \cdots, n \tag{19.3.22}
\end{equation*}
$$

Also by Corollary 19.1.13,

$$
\|y\|^{2}=\overbrace{\left\|y-\sum_{k=1}^{n} \alpha_{k} x_{k}\right\|^{2}}^{d^{2}}+\left\|\sum_{k=1}^{n} \alpha_{k} x_{k}\right\|^{2}
$$

and so, using 19.3.22,

$$
\begin{align*}
\|y\|^{2} & =d^{2}+\sum_{j}\left(\sum_{k} \alpha_{k}\left(x_{k}, x_{j}\right)\right) \alpha_{j} \\
& =d^{2}+\sum_{j}\left(y, x_{j}\right) \alpha_{j}  \tag{19.3.23}\\
& \equiv d^{2}+\mathbf{y}_{x}^{T} \alpha \tag{19.3.24}
\end{align*}
$$

in which

$$
\mathbf{y}_{x}^{T} \equiv\left(\left(y, x_{1}\right), \cdots,\left(y, x_{n}\right)\right), \alpha^{T} \equiv\left(\alpha_{1}, \cdots, \alpha_{n}\right)
$$

Then 19.3.22 and 19.3.23 imply the following system

$$
\left(\begin{array}{cc}
\mathscr{G}\left(x_{1}, \cdots, x_{n}\right) & \mathbf{0} \\
\mathbf{y}_{x}^{T} & 1
\end{array}\right)\binom{\alpha}{d^{2}}=\binom{\mathbf{y}_{x}}{\|y\|^{2}}
$$

By Cramer's rule,

$$
\begin{aligned}
d^{2} & =\frac{\operatorname{det}\left(\begin{array}{cc}
\mathscr{G}\left(x_{1}, \cdots, x_{n}\right) & \mathbf{y}_{x} \\
\mathbf{y}_{x}^{T} & \|y\|^{2}
\end{array}\right)}{\operatorname{det}\left(\begin{array}{cc}
\mathscr{G}\left(x_{1}, \cdots, x_{n}\right) & \mathbf{0} \\
\mathbf{y}_{x}^{T} & 1
\end{array}\right)} \\
& =\frac{\operatorname{det}\left(\begin{array}{cc}
\mathscr{G}\left(x_{1}, \cdots, x_{n}\right) & \mathbf{y}_{x} \\
\mathbf{y}_{x}^{T} & \|y\|^{2}
\end{array}\right)}{\operatorname{det}\left(\mathscr{G}_{1}\left(x_{1}, \cdots, x_{n}\right)\right)} \\
& =\frac{\operatorname{det}\left(\mathscr{G}\left(x_{1}, \cdots, x_{n}, y\right)\right)}{\operatorname{det}\left(\mathscr{G}\left(x_{1}, \cdots, x_{n}\right)\right)}=\frac{G\left(x_{1}, \cdots, x_{n}, y\right)}{G\left(x_{1}, \cdots, x_{n}\right)}
\end{aligned}
$$

and this proves the theorem.

### 19.4 The Müntz Theorem

Recall the polynomials are dense in $C([0,1])$. This is a consequence of the Weierstrass approximation theorem. Now consider finite linear combinations of the functions, $t^{p_{k}}$ where $\left\{p_{0}, p_{1}, p_{2}, \cdots\right\}$ is a sequence of nonnegative real numbers, $p_{0} \equiv 0$. The Müntz theorem says this set, $S$ of finite linear combinations is dense in $C([0,1])$ exactly when $\sum_{k=1}^{\infty} \frac{1}{p_{k}}=\infty$. There are two versions of this theorem, one for density of $S$ in $L^{2}(0,1)$ and one for $C([0,1])$. The presentation follows Cheney [33].

Recall the Cauchy identity presented earlier, Theorem 5.5 .5 on Page 79 which is stated here for convenience.

Theorem 19.4.1 The following identity holds.

$$
\prod_{i, j}\left(a_{i}+b_{j}\right)\left|\begin{array}{ccc}
\frac{1}{a_{1}+b_{1}} & \cdots & \frac{1}{a_{1}+b_{n}}  \tag{19.4.25}\\
\vdots & & \vdots \\
\frac{1}{a_{n}+b_{1}} & \cdots & \frac{1}{a_{n}+b_{n}}
\end{array}\right|=\prod_{j<i}\left(a_{i}-a_{j}\right)\left(b_{i}-b_{j}\right)
$$

Lemma 19.4.2 Let $m, p_{1}, \cdots, p_{n}$ be distinct real numbers larger than $-1 / 2$. Thus the functions, $f_{m}(x) \equiv x^{m}, f_{p_{j}}(x) \equiv x^{p_{j}}$ are all in $L^{2}(0,1)$. Let

$$
M=\operatorname{span}\left(f_{p_{1}}, \cdots, f_{p_{n}}\right)
$$

Then the $L^{2}$ distance, $d$ between $f_{m}$ and $M$ is

$$
d=\frac{1}{\sqrt{2 m+1}} \prod_{j=1}^{n} \frac{\left|m-p_{j}\right|}{m+p_{j}+1}
$$

Proof: By Theorem 19.3.4

$$
\begin{aligned}
d^{2} & =\frac{G\left(f_{p_{1}}, \cdots, f_{p_{n}}, f_{m}\right)}{G\left(f_{p_{1}}, \cdots, f_{p_{n}}\right)} \\
\left(f_{p_{i}}, f_{p_{j}}\right) & =\int_{0}^{1} x^{p_{i}} x^{p_{j}} d x=\frac{1}{1+p_{i}+p_{j}}
\end{aligned}
$$

Therefore,

$$
d^{2}=\frac{\left|\begin{array}{ccccc}
\frac{1}{1+p_{1}+p_{1}} & \frac{1}{1+p_{1}+p_{2}} & \cdots & \frac{1}{1+p_{1}+p_{n}} & \frac{1}{1+m+p_{1}} \\
\frac{1}{1+p_{2}+p_{1}} & \frac{1}{1+p_{2}+p_{2}} & \cdots & \frac{1}{1+p_{2}+p_{n}} & \frac{1}{1+m+p_{2}} \\
\vdots & \vdots & & \vdots & \vdots \\
\frac{1}{1+p_{n}+p_{1}} & \frac{1}{1+p_{n}+p_{2}} & \cdots & \frac{1}{1+p_{n}+p_{n}} & \frac{1}{1+p_{n}+m} \\
\frac{1}{1+m+p_{1}} & \frac{1}{1+m+p_{2}} & \cdots & \frac{1}{1+m+p_{n}} & \frac{1}{1+m+m}
\end{array}\right|}{\left|\begin{array}{cccc|}
\frac{1}{1+p_{1}+p_{1}} & \frac{1}{1+p_{1}+p_{2}} & \cdots & \frac{1}{1+p_{1}+p_{n}} \\
\frac{1}{1+p_{2}+p_{1}} & \frac{1}{1+p_{2}+p_{2}} & \cdots & \frac{1}{1+p_{2}+p_{n}} \\
\vdots & \vdots & & \vdots \\
\frac{1}{1+p_{n}+p_{1}} & \frac{1}{1+p_{n}+p_{2}} & \cdots & \frac{1}{1+p_{n}+p_{n}}
\end{array}\right|}
$$

Now from the Cauchy identity, letting $a_{i}=p_{i}+\frac{1}{2}$ and $b_{j}=\frac{1}{2}+p_{j}$ with $p_{n+1}=m$, the numerator of the above equals

$$
\begin{gathered}
\frac{\prod_{j<i \leq n+1}\left(p_{i}-p_{j}\right)\left(p_{i}-p_{j}\right)}{\prod_{i, j \leq n+1}\left(p_{i}+p_{j}+1\right)} \\
=\frac{\prod_{k=1}^{n}\left(m-p_{k}\right)^{2} \prod_{j<i \leq n}\left(p_{i}-p_{j}\right)^{2}}{\prod_{i=1}^{n}\left(m+p_{i}+1\right) \prod_{j=1}^{n}\left(m+p_{j}+1\right) \prod_{i, j \leq n}\left(p_{i}+p_{j}+1\right)(2 m+1)} \\
=\frac{\prod_{k=1}^{n}\left(m-p_{k}\right)^{2} \prod_{j<i \leq n}\left(p_{i}-p_{j}\right)^{2}}{\prod_{i=1}^{n}\left(m+p_{i}+1\right)^{2} \prod_{i, j \leq n}\left(p_{i}+p_{j}+1\right)(2 m+1)}
\end{gathered}
$$

while the denominator equals

$$
\frac{\prod_{j<i \leq n}\left(p_{i}-p_{j}\right)^{2}}{\prod_{i, j \leq n}\left(p_{i}+p_{j}+1\right)}
$$

Therefore,

$$
\begin{aligned}
d^{2} & =\frac{\left(\frac{\Pi_{k=1}^{n}\left(m-p_{k}\right)^{2} \Pi_{j<i \leq n}\left(p_{i}-p_{j}\right)^{2}}{\prod_{i=1}^{n}\left(m+p_{i}+1\right)^{2} \Pi_{i, j \leq n}\left(p_{i}+p_{j}+1\right)(2 m+1)}\right)}{\left(\frac{\Pi_{j<i \leq n}\left(p_{i}-p_{j}\right)^{2}}{\prod_{i, j \leq n}\left(p_{i}+p_{j}+1\right)}\right)} \\
& =\frac{\prod_{k=1}^{n}\left(m-p_{k}\right)^{2}}{\prod_{i=1}^{n}\left(m+p_{i}+1\right)^{2}(2 m+1)}
\end{aligned}
$$

which shows

$$
d=\frac{1}{\sqrt{2 m+1}} \prod_{k=1}^{n} \frac{\left|m-p_{k}\right|}{m+p_{k}+1} .
$$

and this proves the lemma.
The following lemma relates an infinite sum to a product. First consider the graph of $\ln (1-x)$ for $x \in\left[0, \frac{1}{2}\right]$. Here is a rough sketch with two lines, $y=-x$ which lies above the graph of $\ln (1-x)$ and $y=-2 x$ which lies below.


Lemma 19.4.3 Let $a_{n} \neq 1, a_{n}>0$, and $\lim _{n \rightarrow \infty} a_{n}=0$. Then

$$
\prod_{k=1}^{\infty}\left(1-a_{n}\right) \equiv \lim _{n \rightarrow \infty} \prod_{k=1}^{n}\left(1-a_{n}\right)=0
$$

if and only if

$$
\sum_{n=1}^{\infty} a_{n}=+\infty
$$

Proof:Without loss of generality, you can assume $a_{n}<1 / 2$ because the two conditions are determined by the values of $a_{n}$ for $n$ large. By the above sketch the following is obtained.

$$
\ln \prod_{k=1}^{n}\left(1-a_{k}\right)=\sum_{k=1}^{n} \ln \left(1-a_{k}\right) \in\left[-2 \sum_{k=1}^{n} a_{k},-\sum_{k=1}^{n} a_{k}\right] .
$$

Therefore,

$$
e^{-2 \sum_{k=1}^{n} a_{k}} \leq \prod_{k=1}^{n}\left(1-a_{k}\right) \leq e^{-\sum_{k=1}^{n} a_{k}}
$$

The conclusion follows.
The following is Müntz's first theorem.
Theorem 19.4.4 Let $\left\{p_{n}\right\}$ be a sequence of real numbers larger than $-1 / 2$ such that $\lim _{n \rightarrow \infty} p_{n}=\infty$. Let $S$ denote the set of finite linear combinations of the functions

$$
\left\{x^{p_{1}}, x^{p_{2}}, \cdots\right\}
$$

Then $S$ is dense in $L^{2}(0,1)$ if and only if

$$
\sum_{i=1}^{\infty} \frac{1}{p_{i}}=\infty
$$

Proof: The polynomials are dense in $L^{2}(0,1)$ and so $S$ is dense in $L^{2}(0,1)$ if and only if for every $\varepsilon>0$ there exists a function $f$ from $S$ such that for each integer $m \geq$ $0,\left(\int_{0}^{1}\left|f(x)-x^{m}\right|^{2} d x\right)^{1 / 2}<\varepsilon$. This happens if and only if for all $n$ large enough, the distance in $L^{2}(0,1)$ between the function, $x \rightarrow x^{m}$ and $\operatorname{span}\left(x^{p_{1}}, x^{p_{2}}, \cdots, x^{p_{n}}\right)$ is less than $\varepsilon$. However, from Lemma 19.4.2 this distance equals

$$
\begin{aligned}
& \frac{1}{\sqrt{2 m+1}} \prod_{k=1}^{n} \frac{\left|m-p_{k}\right|}{m+p_{k}+1} \\
= & \frac{1}{\sqrt{2 m+1}} \prod_{k=1}^{n} 1-\left(1-\frac{\left|m-p_{k}\right|}{m+p_{k}+1}\right)
\end{aligned}
$$

Thus $S$ is dense if and only if

$$
\prod_{k=1}^{\infty}\left(1-\left(1-\frac{\left|m-p_{k}\right|}{m+p_{k}+1}\right)\right)=0
$$

which, by Lemma 19.4.3, happens if and only if

$$
\sum_{k=1}^{\infty}\left(1-\frac{\left|m-p_{k}\right|}{m+p_{k}+1}\right)=+\infty
$$

But this sum equals

$$
\sum_{k=1}^{\infty}\left(\frac{m+p_{k}+1-\left|m-p_{k}\right|}{m+p_{k}+1}\right)
$$

which has the same convergence properties as $\sum \frac{1}{p_{k}}$ by the limit comparison test. This proves the theorem.

The following is Müntz's second theorem.
Theorem 19.4.5 Let $S$ be finite linear combinations of $\left\{1, x^{p_{1}}, x^{p_{2}}, \cdots\right\}$ where $p_{j} \geq 1$ and $\lim _{n \rightarrow \infty} p_{n}=\infty$. Then $S$ is dense in $C([0,1])$ if and only if $\sum_{k=1}^{\infty} \frac{1}{p_{k}}=\infty$.

Proof: If $S$ is dense in $C([0,1])$ then $S$ must also be dense in $L^{2}(0,1)$ and so by Theorem 19.4.4 $\sum_{k=1}^{\infty} \frac{1}{p_{k}}=\infty$.

Suppose then that $\sum_{k=1}^{\infty} \frac{1}{p_{k}}=\infty$ so that by Theorem 19.4.4, $S$ is dense in $L^{2}(0,1)$. The theorem will be proved if it is shown that for all $m$ a nonnegative integer,

$$
\max \left\{\left|x^{m}-f(x)\right|: x \in[0,1]\right\}<\varepsilon
$$

for some $f \in S$. This is true if $m=0$ because $1 \in S$. Suppose then that $m>0$. Let $S^{\prime}$ denote finite linear combinations of the functions

$$
\left\{x^{p_{1}-1}, x^{p_{2}-1}, \cdots\right\}
$$

These functions are also dense in $L^{2}(0,1)$ because $\sum \frac{1}{p_{k}-1}=\infty$ by the limit comparison test. Then by Theorem 19.4.4 there exists $f \in S^{\prime}$ such that

$$
\left(\int_{0}^{1}\left|f(x)-m x^{m-1}\right|^{2} d x\right)^{1 / 2}<\varepsilon
$$

Thus $F(x) \equiv \int_{0}^{x} f(t) d t \in S$ and

$$
\begin{aligned}
\left|F(x)-x^{m}\right| & =\left|\int_{0}^{x}\left(f(t)-m t^{m-1}\right) d t\right| \\
& \leq \int_{0}^{x}\left|f(t)-m t^{m-1}\right| d t \\
& \leq\left(\int_{0}^{1}\left|f(t)-m t^{m-1}\right|^{2} d t\right)^{1 / 2}\left(\int_{0}^{1} d x\right)^{1 / 2} \\
& <\varepsilon
\end{aligned}
$$

and this proves the theorem.

### 19.5 Orthonormal Sets

The concept of an orthonormal set of vectors is a generalization of the notion of the standard basis vectors of $\mathbb{R}^{n}$ or $\mathbb{C}^{n}$.

Definition 19.5.1 Let $H$ be a Hilbert space. $S \subseteq H$ is called an orthonormal set if $\|x\|=1$ for all $x \in S$ and $(x, y)=0$ if $x, y \in S$ and $x \neq y$. For any set, $D$,

$$
D^{\perp} \equiv\{x \in H:(x, d)=0 \text { for all } d \in D\} .
$$

If $S$ is a set, $\operatorname{span}(S)$ is the set of all finite linear combinations of vectors from $S$.
You should verify that $D^{\perp}$ is always a closed subspace of $H$.
Theorem 19.5.2 In any separable Hilbert space, $H$, there exists a countable orthonormal set, $S=\left\{x_{i}\right\}$ such that the span of these vectors is dense in $H$. Furthermore, if $\operatorname{span}(S)$ is dense, then for $x \in H$,

$$
\begin{equation*}
x=\sum_{i=1}^{\infty}\left(x, x_{i}\right) x_{i} \equiv \lim _{n \rightarrow \infty} \sum_{i=1}^{n}\left(x, x_{i}\right) x_{i} \tag{19.5.26}
\end{equation*}
$$

Proof: Let $\mathscr{F}$ denote the collection of all orthonormal subsets of $H$. $\mathscr{F}$ is nonempty because $\{x\} \in \mathscr{F}$ where $\|x\|=1$. The set, $\mathscr{F}$ is a partially ordered set with the order given by set inclusion. By the Hausdorff maximal theorem, there exists a maximal chain, $\mathfrak{C}$ in $\mathscr{F}$. Then let $S \equiv \cup \mathfrak{C}$. It follows $S$ must be a maximal orthonormal set of vectors. Why? It remains to verify that $S$ is countable span $(S)$ is dense, and the condition, 19.5.26 holds. To see $S$ is countable note that if $x, y \in S$, then

$$
\|x-y\|^{2}=\|x\|^{2}+\|y\|^{2}-2 \operatorname{Re}(x, y)=\|x\|^{2}+\|y\|^{2}=2 .
$$

Therefore, the open sets, $B\left(x, \frac{1}{2}\right)$ for $x \in S$ are disjoint and cover $S$. Since $H$ is assumed to be separable, there exists a point from a countable dense set in each of these disjoint balls showing there can only be countably many of the balls and that consequently, $S$ is countable as claimed.

It remains to verify 19.5 .26 and that span $(S)$ is dense. If span $(S)$ is not dense, then $\overline{\operatorname{span}(S)}$ is a closed proper subspace of $H$ and letting $y \notin \overline{\operatorname{span}(S)}$,

$$
z \equiv \frac{y-P y}{\|y-P y\|} \in \operatorname{span}(S)^{\perp}
$$

But then $S \cup\{z\}$ would be a larger orthonormal set of vectors contradicting the maximality of $S$.

It remains to verify 19.5 .26 . Let $S=\left\{x_{i}\right\}_{i=1}^{\infty}$ and consider the problem of choosing the constants, $c_{k}$ in such a way as to minimize the expression

$$
\begin{gathered}
\left\|x-\sum_{k=1}^{n} c_{k} x_{k}\right\|^{2}= \\
\|x\|^{2}+\sum_{k=1}^{n}\left|c_{k}\right|^{2}-\sum_{k=1}^{n} \overline{c_{k}}\left(x, x_{k}\right)-\sum_{k=1}^{n} c_{k} \overline{\left(x, x_{k}\right)} .
\end{gathered}
$$

This equals

$$
\|x\|^{2}+\sum_{k=1}^{n}\left|c_{k}-\left(x, x_{k}\right)\right|^{2}-\sum_{k=1}^{n}\left|\left(x, x_{k}\right)\right|^{2}
$$

and therefore, this minimum is achieved when $c_{k}=\left(x, x_{k}\right)$ and equals

$$
\|x\|^{2}-\sum_{k=1}^{n}\left|\left(x, x_{k}\right)\right|^{2}
$$

Now since span $(S)$ is dense, there exists $n$ large enough that for some choice of constants, $c_{k}$,

$$
\left\|x-\sum_{k=1}^{n} c_{k} x_{k}\right\|^{2}<\varepsilon
$$

However, from what was just shown,

$$
\left\|x-\sum_{i=1}^{n}\left(x, x_{i}\right) x_{i}\right\|^{2} \leq\left\|x-\sum_{k=1}^{n} c_{k} x_{k}\right\|^{2}<\varepsilon
$$

showing that $\lim _{n \rightarrow \infty} \sum_{i=1}^{n}\left(x, x_{i}\right) x_{i}=x$ as claimed. This proves the theorem.
The proof of this theorem contains the following corollary.
Corollary 19.5.3 Let $S$ be any orthonormal set of vectors and let

$$
\left\{x_{1}, \cdots, x_{n}\right\} \subseteq S
$$

Then if $x \in H$

$$
\left\|x-\sum_{k=1}^{n} c_{k} x_{k}\right\|^{2} \geq\left\|x-\sum_{i=1}^{n}\left(x, x_{i}\right) x_{i}\right\|^{2}
$$

for all choices of constants, $c_{k}$. In addition to this, Bessel's inequality

$$
\|x\|^{2} \geq \sum_{k=1}^{n}\left|\left(x, x_{k}\right)\right|^{2}
$$

If $S$ is countable and $\operatorname{span}(S)$ is dense, then letting $\left\{x_{i}\right\}_{i=1}^{\infty}=S, 19.5 .26$ follows.

### 19.6 Fourier Series, An Example

In this section consider the Hilbert space, $L^{2}(0,2 \pi)$ with the inner product,

$$
(f, g) \equiv \int_{0}^{2 \pi} f \bar{g} d m
$$

This is a Hilbert space because of the theorem which states the $L^{p}$ spaces are complete, Theorem 15.1.10 on Page 403. An example of an orthonormal set of functions in $L^{2}(0,2 \pi)$ is

$$
\phi_{n}(x) \equiv \frac{1}{\sqrt{2 \pi}} e^{i n x}
$$

for $n$ an integer. Is it true that the span of these functions is dense in $L^{2}(0,2 \pi)$ ?
Theorem 19.6.1 Let $S=\left\{\phi_{n}\right\}_{n \in \mathbb{Z}}$. Then $\operatorname{span}(S)$ is dense in $L^{2}(0,2 \pi)$.
Proof: By regularity of Lebesgue measure, and Theorem 15.2 .4 that $C_{c}(0,2 \pi)$ is dense in $L^{2}(0,2 \pi)$. Therefore, it suffices to show that for $g \in C_{c}(0,2 \pi)$, then for every $\varepsilon>0$ there exists $h \in \operatorname{span}(S)$ such that $\|g-h\|_{L^{2}(0,2 \pi)}<\varepsilon$.

Let $T$ denote the points of $\mathbb{C}$ which are of the form $e^{i t}$ for $t \in \mathbb{R}$. Let $\mathscr{A}$ denote the algebra of functions consisting of polynomials in $z$ and $1 / z$ for $z \in T$. Thus a typical such function would be one of the form

$$
\sum_{k=-m}^{m} c_{k} z^{k}
$$

for $m$ chosen large enough. This algebra separates the points of $T$ because it contains the function, $p(z)=z$. It annihilates no point of $t$ because it contains the constant function 1. Furthermore, it has the property that for $f \in \mathscr{A}, \bar{f} \in \mathscr{A}$. By the Stone Weierstrass approximation theorem, Theorem 9.2 .10 on Page 203, $\mathscr{A}$ is dense in $C(T)$. Now for $g \in$ $C_{c}(0,2 \pi)$, extend $g$ to all of $\mathbb{R}$ to be $2 \pi$ periodic. Then letting $G\left(e^{i t}\right) \equiv g(t)$, it follows $G$ is well defined and continuous on $T$. Therefore, there exists $H \in \mathscr{A}$ such that for all $t \in \mathbb{R}$,

$$
\left|H\left(e^{i t}\right)-G\left(e^{i t}\right)\right|<\varepsilon^{2} / 2 \pi
$$

Thus $H\left(e^{i t}\right)$ is of the form

$$
H\left(e^{i t}\right)=\sum_{k=-m}^{m} c_{k}\left(e^{i t}\right)^{k}=\sum_{k=-m}^{m} c_{k} e^{i k t} \in \operatorname{span}(S)
$$

Let $h(t)=\sum_{k=-m}^{m} c_{k} e^{i k t}$. Then

$$
\begin{aligned}
\left(\int_{0}^{2 \pi}|g-h|^{2} d x\right)^{1 / 2} & \leq\left(\int_{0}^{2 \pi} \max \{|g(t)-h(t)|: t \in[0,2 \pi]\} d x\right)^{1 / 2} \\
& =\left(\int_{0}^{2 \pi} \max \left\{\left|G\left(e^{i t}\right)-H\left(e^{i t}\right)\right|: t \in[0,2 \pi]\right\} d x\right)^{1 / 2} \\
& <\left(\int_{0}^{2 \pi} \frac{\varepsilon^{2}}{2 \pi}\right)^{1 / 2}=\varepsilon
\end{aligned}
$$

This proves the theorem.
Corollary 19.6.2 For $f \in L^{2}(0,2 \pi)$,

$$
\lim _{m \rightarrow \infty}\left\|f-\sum_{k=-m}^{m}\left(f, \phi_{k}\right) \phi_{k}\right\|_{L^{2}(0,2 \pi)}
$$

Proof: This follows from Theorem 19.5.2 on Page 535.

### 19.7 Compact Operators

### 19.7.1 Compact Operators In Hilbert Space

Definition 19.7.1 Let $A \in \mathscr{L}(H, H)$ where $H$ is a Hilbert space. Then

$$
|(A x, y)| \leq\|A\|\|x\|\|y\|
$$

and so the map, $x \rightarrow(A x, y)$ is continuous and linear. By the Riesz representation theorem, there exists a unique element of $H$, denoted by $A^{*}$ y such that

$$
(A x, y)=\left(x, A^{*} y\right) .
$$

It is clear $y \rightarrow A^{*} y$ is linear and continuous. $A^{*}$ is called the adjoint of $A$. $A$ is a self adjoint operator if $A=A^{*}$. Thus for a self adjoint operator, $(A x, y)=(x, A y)$ for all $x, y \in H$. $A$ is a compact operator if whenever $\left\{x_{k}\right\}$ is a bounded sequence, there exists a convergent subsequence of $\left\{A x_{k}\right\}$. Equivalently, A maps bounded sets to sets whose closures are compact.

The big result is called the Hilbert Schmidt theorem. It is a generalization to arbitrary Hilbert spaces of standard finite dimensional results having to do with diagonalizing a symmetric matrix. There is another statement and proof of this theorem around Page 663.

Theorem 19.7.2 Let A be a compact self adjoint operator defined on a Hilbert space, $H$. Then there exists a countable set of eigenvalues, $\left\{\lambda_{i}\right\}$ and an orthonormal set of eigenvectors, $u_{i}$ satisfying

$$
\begin{equation*}
\lambda_{i} \text { is real, }\left|\lambda_{n}\right| \geq\left|\lambda_{n+1}\right|, A u_{i}=\lambda_{i} u_{i} \tag{19.7.27}
\end{equation*}
$$

and either

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \lambda_{n}=0 \tag{19.7.28}
\end{equation*}
$$

or for some $n$,

$$
\begin{equation*}
\operatorname{span}\left(u_{1}, \cdots, u_{n}\right)=H \tag{19.7.29}
\end{equation*}
$$

In any case,

$$
\begin{equation*}
\operatorname{span}\left(\left\{u_{i}\right\}_{i=1}^{\infty}\right) \text { is dense in } A(H) \tag{19.7.30}
\end{equation*}
$$

and for all $x \in H$,

$$
\begin{equation*}
A x=\sum_{k=1}^{\infty} \lambda_{k}\left(x, u_{k}\right) u_{k} \tag{19.7.31}
\end{equation*}
$$

where the sum might be finite. This sequence of eigenvectors and eigenvalues also satisfies

$$
\begin{equation*}
\left|\lambda_{n}\right|=\left\|A_{n}\right\| \tag{19.7.32}
\end{equation*}
$$

and

$$
\begin{equation*}
A_{n}: H_{n} \rightarrow H_{n} \tag{19.7.33}
\end{equation*}
$$

where $H \equiv H_{1}$ and $H_{n} \equiv\left\{u_{1}, \cdots, u_{n-1}\right\}^{\perp}$ and $A_{n}$ is the restriction of $A$ to $H_{n}$.
Proof: If $\|A\|=0$ then pick $u \in H$ with $\|u\|=1$ and let $\lambda_{1}=0$. Since $A(H)=0$ it follows the span of $u$ is dense in $A(H)$ and this proves the theorem in this uninteresting case.

Assume from now on $A \neq 0$. Let $A_{1}=A$ and let $\lambda_{1}$ be real and $\lambda_{1}^{2} \equiv\|A\|^{2}$. From the definition of $\|A\|$ there exists $x_{n},\left\|x_{n}\right\|=1$, and $\left\|A x_{n}\right\| \rightarrow\|A\|=\left|\lambda_{1}\right|$. Now it is clear that $A^{2}$ is also a compact self adjoint operator. Consider

$$
\left(\left(\lambda_{1}^{2}-A^{2}\right) x_{n}, x_{n}\right)=\lambda_{1}^{2}\left(x_{n}, x_{n}\right)-\left(A^{2} x_{n}, x_{n}\right)=\lambda_{1}^{2}-\left\|A x_{n}\right\|^{2} \rightarrow 0
$$

Since $A$ is compact, there exists a subsequence of $\left\{x_{n}\right\}$ still denoted by $\left\{x_{n}\right\}$ such that $A x_{n}$ converges to some element of $H$. Thus since $\lambda_{1}^{2}-A^{2}$ satisfies

$$
\left(\left(\lambda_{1}^{2}-A^{2}\right) y, y\right) \geq 0
$$

in addition to being self adjoint, it follows $x, y \rightarrow\left(\left(\lambda_{1}^{2}-A^{2}\right) x, y\right)$ satisfies all the axioms for an inner product except for the one which says that $(z, z)=0$ only if $z=0$. Therefore, the Cauchy Schwarz inequality may be used to write

$$
\begin{aligned}
\left|\left(\left(\lambda_{1}^{2}-A^{2}\right) x_{n}, y\right)\right| & \leq\left(\left(\lambda_{1}^{2}-A^{2}\right) y, y\right)^{1 / 2}\left(\left(\lambda_{1}^{2}-A^{2}\right) x_{n}, x_{n}\right)^{1 / 2} \\
& \leq e_{n}\|y\|
\end{aligned}
$$

where $e_{n} \rightarrow 0$ as $n \rightarrow \infty$. Therefore, taking the sup over all $\|y\| \leq 1$,

$$
\lim _{n \rightarrow \infty}\left\|\left(\lambda_{1}^{2}-A^{2}\right) x_{n}\right\|=0
$$

Since $A^{2} x_{n}$ converges, it follows, since $\lambda_{1} \neq 0$ that $\left\{x_{n}\right\}$ is a Cauchy sequence converging to $x$ with $\|x\|=1$. Therefore, $A^{2} x_{n} \rightarrow A^{2} x$ and so

$$
\left\|\left(\lambda_{1}^{2}-A^{2}\right) x\right\|=0
$$

Now

$$
\left(\lambda_{1} I-A\right)\left(\lambda_{1} I+A\right) x=\left(\lambda_{1} I+A\right)\left(\lambda_{1} I-A\right) x=0 .
$$

If $\left(\lambda_{1} I-A\right) x=0$, let $u_{1} \equiv x$. If $\left(\lambda_{1} I-A\right) x=y \neq 0$, let $u_{1} \equiv \frac{y}{\|y\|}$.
Suppose $\left\{u_{1}, \cdots, u_{n}\right\}$ is such that $A u_{k}=\lambda_{k} u_{k}$ and $\left|\lambda_{k}\right| \geq\left|\lambda_{k+1}\right|,\left|\lambda_{k}\right|=\left\|A_{k}\right\|$ and $A_{k}:$ $H_{k} \rightarrow H_{k}$ for $k \leq n$, for

$$
H_{k} \equiv\left\{u_{1}, \cdots, u_{k-1}\right\}^{\perp}, H_{0} \equiv\{0\} \text { so } H_{0}^{\perp}=H_{1}=H
$$

From the above, this results in case $n=1$.
If

$$
\operatorname{span}\left(u_{1}, \cdots, u_{n}\right)=H
$$

this yields the conclusion of the theorem 19.7.29. Therefore, assume the span of these vectors is always a proper subspace of $H$.

It is shown next that $A_{n+1}: H_{n+1} \rightarrow H_{n+1}$. Let

$$
y \in H_{n+1} \equiv\left\{u_{1}, \cdots, u_{n}\right\}^{\perp}
$$

Then for $k \leq n$

$$
\left(A y, u_{k}\right)=\left(y, A u_{k}\right)=\lambda_{k}\left(y, u_{k}\right)=0
$$

showing $A_{n+1}: H_{n+1} \rightarrow H_{n+1}$ as claimed.
Say $\lambda_{k}>0$ for $k \leq n-1$. There are two cases. Either $\lambda_{n}=0$ or it is not. In the case where $\lambda_{n}=0$ it follows $A_{n}=0$ since $\left\|A_{n}\right\|=0$. Every element of $H$ is the sum of one in span $\left(u_{1}, \cdots, u_{n}\right)$ and one in span $\left(u_{1}, \cdots, u_{n}\right)^{\perp}$. (note $\operatorname{span}\left(u_{1}, \cdots, u_{n}\right)$ is a closed subspace.) Thus, if $x \in H, x=y+z$ where $y \in \operatorname{span}\left(u_{1}, \cdots, u_{n}\right)$ and $z \in \operatorname{span}\left(u_{1}, \cdots, u_{n}\right)^{\perp}$ and $A z=0$. Say $y=\sum_{j=1}^{n} c_{j} u_{j}$. Then

$$
A x=A y=\sum_{j=1}^{n} c_{j} A u_{j}=\sum_{j=1}^{n} c_{j} \lambda_{j} u_{j} \in \operatorname{span}\left(u_{1}, \cdots, u_{n-1}\right) .
$$

Thus, if $\lambda_{n}=0,19.7 .30$ holds since $x \in H$ was arbitrary. Hence, from the above sum,

$$
\lambda_{k}\left(x, u_{k}\right)=\left(x, A u_{k}\right)=\left(A x, u_{k}\right)=c_{k} \lambda_{k}
$$

and so it suffices to let $c_{k}=\left(x, u_{k}\right)$ yielding the formula in 19.7.30 for any $x \in H$ with $\lambda_{n}=\lambda_{n+1}=\cdots$ all zero.

Now consider the case where $\lambda_{n} \neq 0$. In this case repeat the above argument used to find $u_{n+1}$ and $\lambda_{n+1}$ for the operator, $A_{n+1}$. This yields $u_{n+1} \in H_{n+1} \equiv\left\{u_{1}, \cdots, u_{n}\right\}^{\perp}$ such that

$$
\left\|u_{n+1}\right\|=1,\left\|A u_{n+1}\right\|=\left|\lambda_{n+1}\right|=\left\|A_{n+1}\right\| \leq\left\|A_{n}\right\|=\left|\lambda_{n}\right|
$$

and if it is ever the case that $\lambda_{n}=0$, it follows from the above argument that the conclusion of the theorem is obtained.

I claim $\lim _{n \rightarrow \infty} \lambda_{n}=0$. If this were not so, then for some $\varepsilon>0,0<\varepsilon=\lim _{n \rightarrow \infty}\left|\lambda_{n}\right|$ but then

$$
\begin{aligned}
\left\|A u_{n}-A u_{m}\right\|^{2} & =\left\|\lambda_{n} u_{n}-\lambda_{m} u_{m}\right\|^{2} \\
& =\left|\lambda_{n}\right|^{2}+\left|\lambda_{m}\right|^{2} \geq 2 \varepsilon^{2}
\end{aligned}
$$

and so there would not exist a convergent subsequence of $\left\{A u_{k}\right\}_{k=1}^{\infty}$ contrary to the assumption that $A$ is compact. This verifies the claim that $\lim _{n \rightarrow \infty} \lambda_{n}=0$.

It remains to verify that span $\left(\left\{u_{i}\right\}\right)$ is dense in $A(H)$. If $w \in \operatorname{span}\left(\left\{u_{i}\right\}\right)^{\perp}$ then $w \in H_{n}$ for all $n$ and so for all $n$,

$$
\|A w\| \leq\left\|A_{n}\right\|\|w\| \leq\left|\lambda_{n}\right|\|w\| .
$$

Therefore, $A w=0$. Now every vector from $H$ can be written as a sum of one from

$$
\operatorname{span}\left(\left\{u_{i}\right\}\right)^{\perp}=\overline{\operatorname{span}\left(\left\{u_{i}\right\}\right)}{ }^{\perp}
$$

and one from $\overline{\operatorname{span}\left(\left\{u_{i}\right\}\right)}$. Therefore, if $x \in H, x=y+w$ where $y \in \overline{\operatorname{span}\left(\left\{u_{i}\right\}\right)}$ and $w \in$ $\overline{\operatorname{span}\left(\left\{u_{i}\right\}\right)}{ }^{\perp}$ and $A w=0$. Also, since $y \in \overline{\operatorname{span}\left(\left\{u_{i}\right\}\right)}$, there exist constants, $c_{k}$ and $n$ such that

$$
\left\|y-\sum_{k=1}^{n} c_{k} u_{k}\right\|<\varepsilon,\left(w, u_{k}\right)=0 \text { for all } u_{k} .
$$

Therefore, from Corollary 19.5.3,

$$
\left\|y-\sum_{k=1}^{n}\left(y, u_{k}\right) u_{k}\right\|=\left\|y-\sum_{k=1}^{n}\left(x, u_{k}\right) u_{k}\right\|<\varepsilon
$$

Therefore,

$$
\|A\| \varepsilon>\left\|A\left(y-\sum_{k=1}^{n}\left(x, u_{k}\right) u_{k}\right)\right\|=\left\|A x-\sum_{k=1}^{n}\left(x, u_{k}\right) \lambda_{k} u_{k}\right\| .
$$

Since $\varepsilon$ is arbitrary, this shows span $\left(\left\{u_{i}\right\}\right)$ is dense in $A(H)$ and also implies 19.7.31.
Define $v \otimes u \in \mathscr{L}(H, H)$ by

$$
v \otimes u(x)=(x, u) v,
$$

then 19.7.31 is of the form

$$
A=\sum_{k=1}^{\infty} \lambda_{k} u_{k} \otimes u_{k}
$$

This is the content of the following corollary.
Corollary 19.7.3 The main conclusion of the above theorem can be written as

$$
A=\sum_{k=1}^{\infty} \lambda_{k} u_{k} \otimes u_{k}
$$

where the convergence of the partial sums takes place in the operator norm.

Proof: Using 19.7.31

$$
\begin{aligned}
& \left|\left(\left(A-\sum_{k=1}^{n} \lambda_{k} u_{k} \otimes u_{k}\right) x, y\right)\right|=\left|\left(A x-\sum_{k=1}^{n} \lambda_{k}\left(x, u_{k}\right) u_{k}, y\right)\right| \\
= & \left|\left(\sum_{k=n}^{\infty} \lambda_{k}\left(x, u_{k}\right) u_{k}, y\right)\right|=\left|\sum_{k=n}^{\infty} \lambda_{k}\left(x, u_{k}\right)\left(u_{k}, y\right)\right| \\
\leq & \left|\lambda_{n}\right|\left(\sum_{k=n}^{\infty}\left|\left(x, u_{k}\right)\right|^{2}\right)^{1 / 2}\left(\sum_{k=n}^{\infty}\left|\left(y, u_{k}\right)\right|^{2}\right)^{1 / 2} \leq\left|\lambda_{n}\right|\|x\|\|y\|
\end{aligned}
$$

It follows

$$
\left\|\left(A-\sum_{k=1}^{n} \lambda_{k} u_{k} \otimes u_{k}\right)(x)\right\| \leq\left|\lambda_{n}\right|\|x\|
$$

Lemma 19.7.4 If $V_{\lambda}$ is the eigenspace for $\lambda \neq 0$ and $B: V_{\lambda} \rightarrow V_{\lambda}$ is a compact self adjoint operator, then $V_{\lambda}$ must be finite dimensional.

Proof: This follows from the above theorem because it gives a sequence of eigenvalues on restrictions of $B$ to subspaces with $\lambda_{k} \downarrow 0$. Hence, eventually $\lambda_{n}=0$ because there is no other eigenvalue in $V_{\lambda}$ than $\lambda$. Hence there can be no eigenvector on $V_{\lambda}$ for $\lambda=0$ and $\operatorname{span}\left(u_{1}, \cdots, u_{n}\right)=V_{\lambda}=A\left(V_{\lambda}\right)$ for some $n$.

Corollary 19.7.5 Let A be a compact self adjoint operator defined on a separable Hilbert space, $H$. Then there exists a countable set of eigenvalues, $\left\{\lambda_{i}\right\}$ and an orthonormal set of eigenvectors, $u_{i}$ satisfying

$$
\begin{equation*}
A v_{i}=\lambda_{i} v_{i},\left\|u_{i}\right\|=1 \tag{19.7.34}
\end{equation*}
$$

$$
\begin{equation*}
\operatorname{span}\left(\left\{v_{i}\right\}_{i=1}^{\infty}\right) \text { is dense in } H . \tag{19.7.35}
\end{equation*}
$$

Furthermore, if $\lambda_{i} \neq 0$, the space, $V_{\lambda_{i}} \equiv\left\{x \in H: A x=\lambda_{i} x\right\}$ is finite dimensional.
Proof: Let $B$ be the restriction of $A$ to $V_{\lambda_{i}}$. Thus $B$ is a compact self adjoint operator which maps $V_{\lambda}$ to $V_{\lambda}$ and has only one eigenvalue $\lambda_{i}$ on $V_{\lambda_{i}}$. By Lemma 19.7.4, $V_{\lambda}$ is finite dimensional. As to the density of some span $\left(\left\{v_{i}\right\}_{i=1}^{\infty}\right)$ in $H$, in the proof of the above theorem, let $W \equiv \overline{\operatorname{span}\left(\left\{u_{i}\right\}\right)}{ }^{\perp}$. By Theorem 19.5.2, there is a maximal orthonormal set of vectors, $\left\{w_{i}\right\}_{i=1}^{\infty}$ whose span is dense in $W$. There are only countably many of these since the space $H$ is separable. As shown in the proof of the above theorem, $A w=0$ for all $w \in W$. Let $\left\{v_{i}\right\}_{i=1}^{\infty}=\left\{u_{i}\right\}_{i=1}^{\infty} \cup\left\{w_{i}\right\}_{i=1}^{\infty}$.

Note the last claim of this corollary about $V_{\lambda}$ being finite dimensional if $\lambda \neq 0$ holds independent of the separability of $H$.

Suppose $\lambda \notin\left\{\lambda_{k}\right\}_{k=1}^{\infty}$, the eigenvalues of $A$, and $\lambda \neq 0$. Then the above formula for $A$, 19.7.31, yields an interesting formula for $(A-\lambda I)^{-1}$. Note first that since $\lim _{n \rightarrow \infty} \lambda_{n}=0$, it follows that $\lambda_{n}^{2} /\left(\lambda_{n}-\lambda\right)^{2}$ must be bounded, say by a positive constant, $M$.

Corollary 19.7.6 Let A be a compact self adjoint operator and let $\lambda \notin\left\{\lambda_{n}\right\}_{n=1}^{\infty}$ and $\lambda \neq 0$ where the $\lambda_{n}$ are the eigenvalues of $A .(A x=\lambda x, x \neq 0)$ Then

$$
\begin{equation*}
(A-\lambda I)^{-1} x=-\frac{1}{\lambda} x+\frac{1}{\lambda} \sum_{k=1}^{\infty} \frac{\lambda_{k}}{\lambda_{k}-\lambda}\left(x, u_{k}\right) u_{k} \tag{19.7.36}
\end{equation*}
$$

Proof: Let $m<n$. Then since the $\left\{u_{k}\right\}$ form an orthonormal set,

$$
\begin{align*}
\left|\sum_{k=m}^{n} \frac{\lambda_{k}}{\lambda_{k}-\lambda}\left(x, u_{k}\right) u_{k}\right| & \leq\left(\sum_{k=m}^{n}\left(\frac{\lambda_{k}}{\lambda_{k}-\lambda}\right)^{2}\left|\left(x, u_{k}\right)\right|^{2}\right)^{1 / 2}  \tag{19.7.37}\\
& \leq M\left(\sum_{k=m}^{n}\left|\left(x, u_{k}\right)\right|^{2}\right)^{1 / 2}
\end{align*}
$$

But from Bessel's inequality, $\sum_{k=1}^{\infty}\left|\left(x, u_{k}\right)\right|^{2} \leq\|x\|^{2}$ and so for $m$ large enough, the first term in 19.7.37 is smaller than $\varepsilon$. This shows the infinite series in 19.7.36 converges. It is now routine to verify that the formula in 19.7.36 is the inverse.

### 19.8 Sturm Liouville Problems

A Sturm Liouville problem involves the differential equation,

$$
\begin{equation*}
\left(p(x) y^{\prime}\right)^{\prime}+(\lambda q(x)+r(x)) y=0, x \in[a, b], p(x) \geq 0 \tag{19.8.38}
\end{equation*}
$$

where we assume that $q(x) \geq 0$ for $x \in[a, b]$ and is positive except for finitely many points. Also, assume it is continuous. Probably, you could generalize this to assume less about $q$ if this is of interest. There will also be boundary conditions at $a, b$. These are typically of the form

$$
\begin{align*}
& C_{1} y(a)+C_{2} y^{\prime}(a)=0 \\
& C_{3} y(b)+C_{4} y^{\prime}(b)=0 \tag{19.8.39}
\end{align*}
$$

where

$$
\begin{equation*}
C_{1}^{2}+C_{2}^{2}>0, \text { and } C_{3}^{2}+C_{4}^{2}>0 \tag{19.8.40}
\end{equation*}
$$

Also we assume here that $a$ and $b$ are finite numbers. In the example, the constants $C_{i}$ are given and $\lambda$ is called the eigenvalue while a solution of the differential equation and given boundary conditions corresponding to $\lambda$ is called an eigenfunction.

There is a simple but important identity related to solutions of the above differential equation. Suppose $\lambda_{i}$ and $y_{i}$ for $i=1,2$ are two solutions of 19.8.38. Thus from the equation, we obtain the following two equations.

$$
\begin{aligned}
& \left(p(x) y_{1}^{\prime}\right)^{\prime} y_{2}+\left(\lambda_{1} q(x)+r(x)\right) y_{1} y_{2}=0, \\
& \left(p(x) y_{2}^{\prime}\right)^{\prime} y_{1}+\left(\lambda_{2} q(x)+r(x)\right) y_{1} y_{2}=0 .
\end{aligned}
$$

Subtracting the second from the first yields

$$
\begin{equation*}
\left(p(x) y_{1}^{\prime}\right)^{\prime} y_{2}-\left(p(x) y_{2}^{\prime}\right)^{\prime} y_{1}+\left(\lambda_{1}-\lambda_{2}\right) q(x) y_{1} y_{2}=0 \tag{19.8.41}
\end{equation*}
$$

Now we note that

$$
\left(p(x) y_{1}^{\prime}\right)^{\prime} y_{2}-\left(p(x) y_{2}^{\prime}\right)^{\prime} y_{1}=\frac{d}{d x}\left(\left(p(x) y_{1}^{\prime}\right) y_{2}-\left(p(x) y_{2}^{\prime}\right) y_{1}\right)
$$

and so integrating 19.8.41 from $a$ to $b$, we obtain

$$
\begin{equation*}
\left.\left(\left(p(x) y_{1}^{\prime}\right) y_{2}-\left(p(x) y_{2}^{\prime}\right) y_{1}\right)\right|_{a} ^{b}+\left(\lambda_{1}-\lambda_{2}\right) \int_{a}^{b} q(x) y_{1}(x) y_{2}(x) d x=0 \tag{19.8.42}
\end{equation*}
$$

We have been purposely vague about the nature of the boundary conditions because of a desire to not lose generality. However, we will always assume the boundary conditions are such that whenever $y_{1}$ and $y_{2}$ are two eigenfunctions, it follows that

$$
\begin{equation*}
\left.\left(\left(p(x) y_{1}^{\prime}\right) y_{2}-\left(p(x) y_{2}^{\prime}\right) y_{1}\right)\right|_{a} ^{b}=0 \tag{19.8.43}
\end{equation*}
$$

In the case where the boundary conditions are given by 19.8.39, and 19.8.40, we obtain 19.8.43. To see why this is so, consider the top limit. This yields

$$
p(b)\left[y_{1}^{\prime}(b) y_{2}(b)-y_{2}^{\prime}(b) y_{1}(b)\right]
$$

However we know from the boundary conditions that

$$
\begin{aligned}
& C_{3} y_{1}(b)+C_{4} y_{1}^{\prime}(b)=0 \\
& C_{3} y_{2}(b)+C_{4} y_{2}^{\prime}(b)=0
\end{aligned}
$$

and that from 19.8.40 that not both $C_{3}$ and $C_{4}$ equal zero. Therefore the determinant of the matrix of coefficients must equal zero. But this implies

$$
\left[y_{1}^{\prime}(b) y_{2}(b)-y_{2}^{\prime}(b) y_{1}(b)\right]=0
$$

which yields the top limit is equal to zero. A similar argument holds for the lower limit. Note that $y_{1}, y_{2}$ satisfy different differential equations because of different eigenvalues.

From now on the boundary condition will be conditions $L, \hat{L}$,

$$
L\left(y(a), y^{\prime}(a)\right)=0, \hat{L}\left(y(b), y^{\prime}(b)\right)=0
$$

which imply that if $y_{i}$ correspond to two different eigenvalues,

$$
\begin{equation*}
\left.\left(\left(p(x) y_{1}^{\prime}\right) y_{2}-\left(p(x) y_{2}^{\prime}\right) y_{1}\right)\right|_{a} ^{b}=0 \tag{*}
\end{equation*}
$$

and if $\alpha$ is a constant, if $L\left(y(a), y^{\prime}(a)\right)=0, \hat{L}\left(y(b), y^{\prime}(b)\right)=0$, then also

$$
L\left(\alpha y(a), \alpha y^{\prime}(a)\right)=0, \hat{L}\left(\alpha y(b), \alpha y^{\prime}(b)\right)=0
$$

For example, maybe one wants to say that $y$ is bounded at $a, b$.
With the identity 19.8.42 here is a result on orthogonality of the eigenfunctions.

Proposition 19.8.1 Suppose $y_{i}$ solves the boundary conditions and the differential equation for $\lambda=\lambda_{i}$ where $\lambda_{1} \neq \lambda_{2}$. Then we have the orthogonality relation

$$
\begin{equation*}
\int_{a}^{b} q(x) y_{1}(x) y_{2}(x) d x=0 \tag{19.8.44}
\end{equation*}
$$

In addition to this, if $u, v$ are two solutions to the differential equation corresponding to a single $\lambda, 19.8 .38$, not necessarily the boundary conditions, (same differential equation) then there exists a constant, $C$ such that

$$
\begin{equation*}
W(u, v)(x) p(x)=C \tag{19.8.45}
\end{equation*}
$$

for all $x \in[a, b]$. In this formula, $W(u, v)$ denotes the Wronskian given by

$$
\operatorname{det}\left(\begin{array}{cc}
u(x) & v(x)  \tag{19.8.46}\\
u^{\prime}(x) & v^{\prime}(x)
\end{array}\right)
$$

Proof: The orthogonality relation, 19.8.44 follows from the fundamental assumption, 19.8.43 and 19.8.42.

It remains to verify 19.8.45. We have from 19.8.41,

$$
\begin{aligned}
0 & =(\lambda-\lambda) q(x) u v+\left(p(x) u^{\prime}\right)^{\prime} v-\left(p(x) v^{\prime}\right)^{\prime} u \\
& =\frac{d}{d x}\left(p(x) u^{\prime} v-p(x) v^{\prime} u\right)=\frac{d}{d x}(p(x) W(v, u)(x))
\end{aligned}
$$

and so $p(x) W(u, v)(x)=-p(x) W(v, u)(x)=C$ as claimed.
Now consider the differential equation,

$$
\begin{equation*}
\left(p(x) y^{\prime}\right)^{\prime}+r(x) y=0 \tag{19.8.47}
\end{equation*}
$$

This is obtained from the one of interest by letting $\lambda=0$.
Criterion 19.8.2 Suppose we are able to find functions, $u$ and $v$ such that they solve the differential equation, 19.8 .47 and $u$ solves the boundary condition at $x=a$ while $v$ solves the boundary condition at $x=b$. Assume both are in $L^{2}(a, b)$ and $W(u, v) \neq 0$. It follows that both are in $L^{2}(a, b, q)$, the $L^{2}$ functions with respect to the measure $q(x) d x$. Thus

$$
(f, g)_{L^{2}(a, b, q)} \equiv \int_{a}^{b} f(x) g(x) q(x) d x
$$

If $p(x)>0$ on $[a, b]$ it is typically clear from the fundamental existence and uniqueness theorems for ordinary differential equations that such functions $u$ and $v$ exist. (See any good differential equations book or Problem 10 on Page 750.)

However, such functions might exist even if $p$ vanishes at the end points.
Lemma 19.8.3 Assume Criterion 19.8.2. A function y is a solution to the boundary conditions along with the equation,

$$
\begin{equation*}
\left(p(x) y^{\prime}\right)^{\prime}+r(x) y=g \tag{19.8.48}
\end{equation*}
$$

if

$$
\begin{equation*}
y(x)=\int_{a}^{b} G(t, x) g(t) d t \tag{19.8.49}
\end{equation*}
$$

where

$$
G(t, x)=\left\{\begin{array}{l}
c^{-1}(v(x) u(t)) \text { if } t<x  \tag{19.8.50}\\
c^{-1}(v(t) u(x)) \text { if } t>x
\end{array} .\right.
$$

where $c$ is the constant of Proposition 19.8.1 which satisfies $p(x) W(u, v)(x)=c$.
Proof: Why does $y$ solve the equation 19.8 .48 along with the boundary conditions?

$$
y(x)=\frac{1}{c} \int_{a}^{x} g(t) u(t) v(x) d t+\frac{1}{c} \int_{x}^{b} g(t) v(t) u(x) d t
$$

Differentiate

$$
\begin{array}{r}
y^{\prime}(x)=\frac{1}{c} g(x) u(x) v(x)+\frac{1}{c} \int_{a}^{x} g(t) u(t) v^{\prime}(x) d t \\
-\frac{1}{c} g(x) v(x) u(x)+\frac{1}{c} \int_{x}^{b} g(t) v(t) u^{\prime}(x) d t \\
=\frac{1}{c} \int_{a}^{x} g(t) u(t) v^{\prime}(x) d t+\frac{1}{c} \int_{x}^{b} g(t) v(t) u^{\prime}(x) d t
\end{array}
$$

Then

$$
p(x) y^{\prime}(x)=\frac{1}{c} \int_{a}^{x} g(t) u(t) p(x) v^{\prime}(x) d t+\frac{1}{c} \int_{x}^{b} g(t) v(t) p(x) u^{\prime}(x) d t
$$

Then $\left(p(x) y^{\prime}(x)\right)^{\prime}=$

$$
\begin{gathered}
\frac{1}{c} g(x) p(x) u(x) v^{\prime}(x)-\frac{1}{c} g(x) p(x) v(x) u^{\prime}(x) \\
+\frac{1}{c} \int_{a}^{x} g(t) u(t)\left(p(x) v^{\prime}(x)\right)^{\prime} d t+\frac{1}{c} \int_{x}^{b} g(t) v(t)\left(p(x) u^{\prime}(x)\right)^{\prime} d t
\end{gathered}
$$

From the definition of $c$, this equals

$$
\begin{gathered}
=g(x)+\frac{1}{c} \int_{a}^{x} g(t) u(t)\left(p(x) v^{\prime}(x)\right)^{\prime} d t+\frac{1}{c} \int_{x}^{b} g(t) v(t)\left(p(x) u^{\prime}(x)\right)^{\prime} d t \\
=g(x)+\frac{1}{c} \int_{a}^{x} g(t) u(t)(-r(x) v(x)) d t+\frac{1}{c} \int_{x}^{b} g(t) v(t)(-r(x) u(x)) d t \\
=g(x)-r(x)\left(\frac{1}{c} \int_{a}^{x} g(t) u(t) v(x) d t+\frac{1}{c} \int_{x}^{b} g(t) v(t) u(x) d t\right) \\
=g(x)-r(x) y(x)
\end{gathered}
$$

Thus

$$
\left(p(x) y^{\prime}(x)\right)^{\prime}+r(x) y(x)=g(x)
$$

so $y$ satisfies the equation. As to the boundary conditions, by assumption,

$$
\hat{L}\left(y(b), y^{\prime}(b)\right)=\hat{L}\left(v(b) \frac{1}{c} \int_{a}^{b} g(t) u(t) d t, v^{\prime}(b) \frac{1}{c} \int_{a}^{b} g(t) u(t) d t\right)=0
$$

because $v$ satisfies the boundary condition at $b$. The other boundary condition is exactly similar.

Now in the case of Criterion 19.8.2, $y$ is a solution to the Sturm Liouville eigenvalue problem, if and only if $y$ solves the boundary conditions and the equation,

$$
\left(p(x) y^{\prime}\right)^{\prime}+r(x) y(x)=-\lambda q(x) y(x) .
$$

This happens if

$$
\begin{gather*}
y(x)=\frac{-\lambda}{c} \int_{a}^{x} q(t) y(t) u(t) v(x) d t \\
+\frac{-\lambda}{c} \int_{x}^{b} q(t) y(t) v(t) u(x) d t \tag{19.8.51}
\end{gather*}
$$

Letting $\mu=\frac{1}{\lambda}$, this is of the form

$$
\begin{equation*}
\mu y(x)=\int_{a}^{b} G(t, x) q(t) y(t) d t \tag{19.8.52}
\end{equation*}
$$

where

$$
G(t, x)=\left\{\begin{array}{l}
-c^{-1}(v(x) u(t)) \text { if } t<x  \tag{19.8.53}\\
-c^{-1}(v(t) u(x)) \text { if } t>x
\end{array} .\right.
$$

Could $\mu=0$ ? If this happened, then from Lemma 19.8.3, we would have that $y=0$ is a solution of 19.8 .48 where the right side is $-q(t) y(t)$ which would imply that $q(t) y(t)=0$ (since the left side is 0 ) for all $t$ which implies $y(t)=0$ for all $t$ thanks to assumptions on $q(t)$. Thus we are not interested in this case. It follows from 19.8.53 that $G:[a, b] \times[a, b] \rightarrow$ $\mathbb{R}$ is continuous and symmetric, $G(t, x)=G(x, t)$.

$$
\begin{aligned}
G(x, t) & \equiv\left\{\begin{array}{l}
-c^{-1}(v(t) u(x)) \text { if } x<t \\
-c^{-1}(v(x) u(t)) \text { if } x>t
\end{array}\right. \\
& =G(t, x)
\end{aligned}
$$

Also we see that for $f \in C([a, b])$, and

$$
w(x) \equiv \int_{a}^{b} G(t, x) q(t) f(t) d t
$$

Lemma 19.8.3 implies $w$ is a solution to the boundary conditions and the equation

$$
\begin{equation*}
\left(p(x) y^{\prime}\right)^{\prime}+r(x) y=-q(x) f(x) \tag{19.8.54}
\end{equation*}
$$

Theorem 19.8.4 Suppose $u, v$ are given in Criterion 19.8.2. Then there exists a sequence of functions, $\left\{y_{n}\right\}_{n=1}^{\infty}$ and real numbers, $\lambda_{n}$ such that

$$
\begin{equation*}
\left(p(x) y_{n}^{\prime}\right)^{\prime}+\left(\lambda_{n} q(x)+r(x)\right) y_{n}=0, x \in[a, b], \tag{19.8.55}
\end{equation*}
$$

$$
\begin{align*}
& L\left(y(a), y^{\prime}(a)\right)=0 \\
& \hat{L}\left(y(b), y^{\prime}(b)\right)=0 \tag{19.8.56}
\end{align*}
$$

and

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left|\lambda_{n}\right|=\infty \tag{19.8.57}
\end{equation*}
$$

such that for all $f \in C([a, b])$, whenever $w$ satisfies 19.8 .54 and the boundary conditions,

$$
\begin{equation*}
w(x)=\sum_{n=1}^{\infty} \frac{1}{\lambda_{n}}\left(f, y_{n}\right) y_{n} . \tag{19.8.58}
\end{equation*}
$$

Also the functions, $\left\{y_{n}\right\}$ form a dense set in $L^{2}(a, b, q)$ which satisfy the orthogonality condition, 19.8.44.

Proof: Let $A y(x) \equiv \int_{a}^{b} G(t, x) q(t) y(t) d t$ where $G$ is defined above in 19.8.53. Then from symmetry and Fubini's theorem,

$$
\begin{aligned}
&(A y, z)_{L^{2}(a, b, q)}= \\
& \int_{a}^{b} \int_{a}^{b} G(t, x) y(t) z(x) q(x) q(t) d t d x=\int_{a}^{b} \int_{a}^{b} G(x, t) y(x) z(t) q(t) q(x) d x d t \\
&=\int_{a}^{b} \int_{a}^{b} G(t, x) z(t) q(t) y(x) q(x) d x d t \\
&=(A z, y)_{L^{2}(a, b, q)}
\end{aligned}
$$

This shows that $A$ is self adjoint. For $y \in L^{2}(a, b, q)$,

$$
A y(x)=\int_{a}^{x}\left(-c^{-1}(v(t) u(x))\right) y(t) q(t) d t+\int_{x}^{b}\left(-c^{-1}(v(x) u(t))\right) q(t) y(t) d t
$$

If you have $y_{n} \rightarrow y$ weakly in $L^{2}(a, b, q)$, then it is clear that $A y_{n}(x) \rightarrow A y(x)$ for each $x$, this from the above formula. Consider now $\left\|A y_{n}-A y\right\|_{L^{2}(a, b, q)}$. Look at the first term in the above. Is it true that the following converges to 0 ?

$$
\begin{equation*}
\int_{a}^{b}\left|\int_{a}^{x}\left(-c^{-1}(v(t) u(x))\right)\left(y_{n}(t)-y(t)\right) q(t) d t\right|^{2} q(x) d x \tag{**}
\end{equation*}
$$

We know that the integrand converges to 0 for each $x$. Is there a dominating function? If so, then the dominated convergence theorem gives the result.

$$
\begin{aligned}
& \quad\left|\int_{a}^{x}\left(-c^{-1}(v(t) u(x))\right) q(t)\left(y_{n}(t)-y(t)\right) d t\right| \\
& \leq|u(x)| C(c) \int_{a}^{x}\left|v(t)\left(y_{n}(t)-y(t)\right)\right| d t \\
& \quad \leq|u(x)| C\|v\|_{L^{2}(a, b, q)} \mid\left\|y_{n}-y\right\|_{L^{2}(a, b, q)}
\end{aligned}
$$

The last factor is uniformly bounded due to the weak convergence of $y_{n}$ to $y$. Therefore, there is a constant $C$ such that the integrand is bounded by $|u(x)|^{2} C$. Hence the dominated convergence theorem applies and we can conclude that $* *$ converges to 0 . Thus $A$ is a compact, self adjoint operator on $L^{2}(a, b, q)$

Therefore, by Theorem 19.7.2, there exist functions $y_{n}$ and real constants, $\mu_{n}$ such that $\left\|y_{n}\right\|_{L^{2}}=1$ and $A y_{n}=\mu_{n} y_{n}$ and

$$
\begin{equation*}
\left|\mu_{n}\right| \geq\left|\mu_{n+1}\right|, A u_{i}=\mu_{i} u_{i} \tag{19.8.59}
\end{equation*}
$$

and either

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mu_{n}=0 \tag{19.8.60}
\end{equation*}
$$

or for some $n$,

$$
\begin{equation*}
\operatorname{span}\left(y_{1}, \cdots, y_{n}\right)=H \equiv L^{2}(a, b, q) \tag{19.8.61}
\end{equation*}
$$

Of course, $H$ is not finite dimensional and so the second will not hold. Also from Theorem 19.7.2,

$$
\begin{equation*}
\operatorname{span}\left(\left\{y_{i}\right\}_{i=1}^{\infty}\right) \text { is dense in } A(H) \tag{19.8.62}
\end{equation*}
$$

and so for all $f \in C([a, b])$,

$$
\begin{equation*}
A f=\sum_{k=1}^{\infty} \mu_{k}\left(f, y_{k}\right) y_{k} \tag{19.8.63}
\end{equation*}
$$

Thus for $w$ a solution of 19.8 .54 and suitable boundary conditions as above which cause $*$,

$$
w \equiv A f=\sum_{k=1}^{\infty} \frac{1}{\lambda_{k}}\left(f, y_{k}\right) y_{k} .
$$

The last claim follows from Corollary 19.7 .5 and the observation above that $\mu$ is never equal to zero.

Note that if $q(x) \neq 0$ we can say that for a given $g \in C([a, b])$, one can define $f$ by $g(x)=-q(x) f(x)$ and so if $w$ is a solution to the boundary conditions and the equation

$$
\left(p(x) w^{\prime}(x)\right)^{\prime}+r(x) w(x)=g(x)=-q(x) f(x)
$$

one obtains the formula

$$
\begin{aligned}
w(x) & =\sum_{k=1}^{\infty} \frac{1}{\lambda_{k}}\left(f, y_{k}\right) y_{k} \\
& =\sum_{k=1}^{\infty} \frac{1}{\lambda_{k}}\left(\frac{-g}{q}, y_{k}\right) y_{k}
\end{aligned}
$$

More can be said about convergence of these series based on the eigenfunctions of a Sturm Liouville problem. In particular, it can be shown that for reasonable functions the pointwise convergence properties are like those of Fourier series and that the series converges to the midpoint of the jump. This is partly done for the Legendre polynomials in [28]. For more on these topics see the old book by Ince, written in Egypt in the 1920's, [72], [73] or the 1955 book on differential equations by Coddington and Levinson [31].

As an example, consider the following eigenvalue problem

$$
\begin{equation*}
x^{2} y^{\prime \prime}+x y^{\prime}+\left(\lambda x^{2}-n^{2}\right) y=0, C_{1} y(L)+C_{2} y^{\prime}(L)=0, x \in[0, L] \tag{*}
\end{equation*}
$$

not both $C_{i}$ equal zero. Then you can write the equation in "self adjoint" form as

$$
\left(x y^{\prime}\right)^{\prime}+\left(\lambda x-\frac{n^{2}}{x}\right) y=0
$$

Multiply by $y$ and integrate from 0 to $L$. Then the boundary terms cancel and you get

$$
\int_{0}^{L}\left(\lambda x-\frac{n^{2}}{x}\right) y^{2} d x=0
$$

and so you must have $\lambda>0$.
Now it follows that corresponding to different values of $\lambda$ the eigenfunctions are orthogonal with respect to $x$. So what are the values of $\lambda$ and how can we describe the corresponding eigenfunctions?

Let $y$ be an eigenfunction. Let $z(\sqrt{\lambda} x)=y(x)$. Then

$$
\begin{gathered}
0=x^{2} y^{\prime \prime}+x y^{\prime}+\left(\lambda x^{2}-n^{2}\right) y \\
=\lambda x^{2} z^{\prime \prime}(\sqrt{\lambda} x)+\sqrt{\lambda} x z^{\prime}(\sqrt{\lambda} x)+\left(\lambda x^{2}-n^{2}\right) z(\sqrt{\lambda} x)
\end{gathered}
$$

Now replace $\sqrt{\lambda} x$ with $u$. Then

$$
u^{2} z^{\prime \prime}(u)+u z^{\prime}(u)+\left(u^{2}-n^{2}\right) z(u)=0
$$

Then we need

$$
z(\sqrt{\lambda} L)=0
$$

and $z$ is bounded near 0 . This happens if and only if $z(u)=J_{n}(u)$ because the other solution to the Bessel equation is unbounded near 0 . Then $J_{n}(\sqrt{\lambda} L)=0$ and so for some $\alpha$ a zero of $J_{n}$,

$$
\sqrt{\lambda} L=\alpha, \lambda=\frac{\alpha^{2}}{L^{2}}
$$

Thus the eigenvalues are

$$
\frac{\alpha^{2}}{L^{2}}, \alpha \text { a zero of } J_{n}(x)
$$

and the eigenfunctions are

$$
x \rightarrow J_{n}\left(\frac{\alpha}{L} x\right)
$$

Then Theorem 19.8.4 implies that if you have any $f \in L^{2}(a, b, x)$, you can obtain it as an expansion in terms of the functions $x \rightarrow J_{n}\left(\frac{\alpha_{k}}{L} x\right)$ where $\alpha_{k}$ are the zeros of the Bessel function. Note that this theorem and what was shown above also shows that there are countably many zeros of $J_{n}$ also.

### 19.8.1 Nuclear Operators

Definition 19.8.5 A self adjoint operator $A \in \mathscr{L}(H, H)$ for $H$ a separable Hilbert space is called a nuclear operator if for some complete orthonormal set, $\left\{e_{k}\right\}$,

$$
\sum_{k=1}^{\infty}\left|\left(A e_{k}, e_{k}\right)\right|<\infty
$$

To begin with here is an interesting lemma.
Lemma 19.8.6 Suppose $\left\{A_{n}\right\}$ is a sequence of compact operators in $\mathscr{L}(X, Y)$ for two Banach spaces, $X$ and $Y$ and suppose $A \in \mathscr{L}(X, Y)$ and

$$
\lim _{n \rightarrow \infty}\left\|A-A_{n}\right\|=0
$$

Then $A$ is also compact.
Proof: Let $B$ be a bounded set in $X$ such that $\|b\| \leq C$ for all $b \in B$. I need to verify $A B$ is totally bounded. Suppose then it is not. Then there exists $\varepsilon>0$ and a sequence, $\left\{A b_{i}\right\}$ where $b_{i} \in B$ and

$$
\left\|A b_{i}-A b_{j}\right\| \geq \varepsilon
$$

whenever $i \neq j$. Then let $n$ be large enough that

$$
\left\|A-A_{n}\right\| \leq \frac{\varepsilon}{4 C}
$$

Then

$$
\begin{aligned}
\left\|A_{n} b_{i}-A_{n} b_{j}\right\| & =\left\|A b_{i}-A b_{j}+\left(A_{n}-A\right) b_{i}-\left(A_{n}-A\right) b_{j}\right\| \\
& \geq\left\|A b_{i}-A b_{j}\right\|-\left\|\left(A_{n}-A\right) b_{i}\right\|-\left\|\left(A_{n}-A\right) b_{j}\right\| \\
& \geq\left\|A b_{i}-A b_{j}\right\|-\frac{\varepsilon}{4 C} C-\frac{\varepsilon}{4 C} C \geq \frac{\varepsilon}{2}
\end{aligned}
$$

a contradiction to $A_{n}$ being compact. This proves the lemma.
Then one can prove the following lemma. In this lemma, $A \geq 0$ will mean $(A x, x) \geq 0$.
Lemma 19.8.7 Let $A \geq 0$ be a nuclear operator defined on a separable Hilbert space, $H$. Then $A$ is compact and also, whenever $\left\{e_{k}\right\}$ is a complete orthonormal set,

$$
A=\sum_{j=1}^{\infty} \sum_{i=1}^{\infty}\left(A e_{i}, e_{j}\right) e_{i} \otimes e_{j}
$$

Proof: First consider the formula. Since $A$ is given to be continuous,

$$
A x=A\left(\sum_{j=1}^{\infty}\left(x, e_{j}\right) e_{j}\right)=\sum_{j=1}^{\infty}\left(x, e_{j}\right) A e_{j}
$$

the series converging because

$$
x=\sum_{j=1}^{\infty}\left(x, e_{j}\right) e_{j}
$$

Then also since $A$ is self adjoint,

$$
\begin{aligned}
\sum_{j=1}^{\infty} \sum_{i=1}^{\infty}\left(A e_{i}, e_{j}\right) e_{i} \otimes e_{j}(x) & \equiv \sum_{j=1}^{\infty} \sum_{i=1}^{\infty}\left(A e_{i}, e_{j}\right)\left(x, e_{j}\right) e_{i} \\
& =\sum_{j=1}^{\infty}\left(x, e_{j}\right) \sum_{i=1}^{\infty}\left(A e_{i}, e_{j}\right) e_{i} \\
& =\sum_{j=1}^{\infty}\left(x, e_{j}\right) \sum_{i=1}^{\infty}\left(A e_{j}, e_{i}\right) e_{i} \\
& =\sum_{j=1}^{\infty}\left(x, e_{j}\right) A e_{j}
\end{aligned}
$$

Next consider the claim that $A$ is compact. Let $C_{A} \equiv\left(\sum_{j=1}^{\infty}\left|\left(A e_{j}, e_{j}\right)\right|\right)^{1 / 2}$. Let $A_{n}$ be defined by

$$
A_{n} \equiv \sum_{j=1}^{\infty} \sum_{i=1}^{n}\left(A e_{i}, e_{j}\right)\left(e_{i} \otimes e_{j}\right)
$$

Then $A_{n}$ has values in $\operatorname{span}\left(e_{1}, \cdots, e_{n}\right)$ and so it must be a compact operator because bounded sets in a finite dimensional space must be precompact. Then

$$
\begin{aligned}
&\left|\left(A x-A_{n} x, y\right)\right|=\left|\sum_{j=1}^{\infty} \sum_{i=n+1}^{\infty}\left(A e_{i} e_{j}\right)\left(y, e_{j}\right)\left(e_{i}, x\right)\right| \\
&=\left|\sum_{j=1}^{\infty}\left(y, e_{j}\right) \sum_{i=n+1}^{\infty}\left(A e_{i} e_{j}\right)\left(e_{i}, x\right)\right| \\
& \leq\left|\sum_{j=1}^{\infty}\right|\left(y, e_{j}\right)\left|\left(A e_{j}, e_{j}\right)^{1 / 2} \sum_{i=n+1}^{\infty}\left(A e_{i} e_{i}\right)^{1 / 2}\right|\left(e_{i}, x\right)| | \\
& \leq\left(\sum_{j=1}^{\infty}\left|\left(y, e_{j}\right)\right|^{2}\right)^{1 / 2}\left(\sum_{j=1}^{\infty}\left|\left(A e_{j}, e_{j}\right)\right|\right)^{1 / 2} \\
& \cdot\left(\sum_{i=n+1}^{\infty}\left|\left(x, e_{i}\right)\right|^{2}\right)^{1 / 2}\left(\sum_{i=n+1}^{\infty}\left|\left(A e_{i} e_{i}\right)\right|\right)^{1 / 2} \\
& \leq|y||x| C_{A}\left(\sum_{i=n+1}^{\infty}\left|\left(A e_{i}, e_{i}\right)\right|\right)^{1 / 2}
\end{aligned}
$$

and this shows that if $n$ is sufficiently large,

$$
\left|\left(\left(A-A_{n}\right) x, y\right)\right| \leq \varepsilon|x||y|
$$

Therefore,

$$
\lim _{n \rightarrow \infty}\left\|A-A_{n}\right\|=0
$$

and so $A$ is the limit in operator norm of finite rank bounded linear operators, each of which is compact. Therefore, $A$ is also compact.

Definition 19.8.8 The trace of a nuclear operator $A \in \mathscr{L}(H, H)$ such that $A \geq 0$ is defined to equal

$$
\sum_{k=1}^{\infty}\left(A e_{k}, e_{k}\right)
$$

where $\left\{e_{k}\right\}$ is an orthonormal basis for the Hilbert space, $H$.
Theorem 19.8.9 Definition 19.8 .8 is well defined and equals $\sum_{j=1}^{\infty} \lambda_{j}$ where the $\lambda_{j}$ are the eigenvalues of $A$.

Proof: Suppose $\left\{u_{k}\right\}$ is some other orthonormal basis. Then

$$
e_{k}=\sum_{j=1}^{\infty} u_{j}\left(e_{k}, u_{j}\right)
$$

By Lemma 19.8.7 $A$ is compact and so

$$
A=\sum_{k=1}^{\infty} \lambda_{k} u_{k} \otimes u_{k}
$$

where the $u_{k}$ are the orthonormal eigenvectors of $A$ which form a complete orthonormal set. Then

$$
\begin{aligned}
\sum_{k=1}^{\infty}\left(A e_{k}, e_{k}\right) & =\sum_{k=1}^{\infty}\left(A\left(\sum_{j=1}^{\infty} u_{j}\left(e_{k}, u_{j}\right)\right), \sum_{j=1}^{\infty} u_{j}\left(e_{k}, u_{j}\right)\right) \\
& =\sum_{k=1}^{\infty} \sum_{i j}\left(A u_{j}, u_{i}\right)\left(e_{k}, u_{j}\right)\left(u_{i}, e_{k}\right) \\
& =\sum_{k=1}^{\infty} \sum_{j=1}^{\infty}\left(A u_{j}, u_{j}\right)\left|\left(e_{k}, u_{j}\right)\right|^{2} \\
& =\sum_{j=1}^{\infty}\left(A u_{j}, u_{j}\right) \sum_{k=1}^{\infty}\left|\left(e_{k}, u_{j}\right)\right|^{2}=\sum_{j=1}^{\infty}\left(A u_{j}, u_{j}\right)\left|u_{j}\right|^{2} \\
& =\sum_{j=1}^{\infty}\left(A u_{j}, u_{j}\right)=\sum_{j=1}^{\infty} \lambda_{j}
\end{aligned}
$$

and this proves the theorem.
This is just like it is for a matrix. Recall the trace of a matrix is the sum of the eigenvalues.

It is also easy to see that in any separable Hilbert space, there exist nuclear operators. Let $\sum_{k=1}^{\infty}\left|\lambda_{k}\right|<\infty$. Then let $\left\{e_{k}\right\}$ be a complete orthonormal set of vectors. Let

$$
A \equiv \sum_{k=1}^{\infty} \lambda_{k} e_{k} \otimes e_{k}
$$

It is not too hard to verify this works.
Much more can be said about nuclear operators.

### 19.8.2 Hilbert Schmidt Operators

Definition 19.8.10 Let $H$ and $G$ be two separable Hilbert spaces and let $T$ map $H$ to $G$ be linear. Then $T$ is called a Hilbert Schmidt operator if there exists some orthonormal basis for $H,\left\{e_{j}\right\}$ such that

$$
\sum_{j}\left\|T e_{j}\right\|^{2}<\infty
$$

The collection of all such linear maps will be denoted by $\mathscr{L}_{2}(H, G)$.

Theorem 19.8.11 $\mathscr{L}_{2}(H, G) \subseteq \mathscr{L}(H, G)$ and $\mathscr{L}_{2}(H, G)$ is a separable Hilbert space with norm given by

$$
\|T\|_{\mathscr{L}_{2}} \equiv\left(\sum_{k}\left\|T e_{k}\right\|^{2}\right)^{1 / 2}
$$

where $\left\{e_{k}\right\}$ is some orthonormal basis for $H$. Also $\mathscr{L}_{2}(H, G) \subseteq \mathscr{L}(H, G)$ and

$$
\begin{equation*}
\|T\| \leq\|T\|_{\mathscr{L}_{2}} \tag{19.8.64}
\end{equation*}
$$

All Hilbert Schmidt opearators are compact. Also for $X \in H$ and $Y \in G, X \otimes Y \in \mathscr{L}_{2}(H, G)$ and

$$
\begin{equation*}
\|X \otimes Y\|_{\mathscr{L}_{2}}=\|X\|_{H}\|Y\|_{G} \tag{19.8.65}
\end{equation*}
$$

Proof: First I want to show $\mathscr{L}_{2}(H, G) \subseteq \mathscr{L}(H, G)$ and $\|T\| \leq\|T\|_{\mathscr{L}_{2}}$. Pick an orthonormal basis for $H,\left\{e_{k}\right\}$ and an orthonormal basis for $G,\left\{f_{k}\right\}$. Then letting

$$
\begin{gathered}
x=\sum_{k=1}^{n} x_{k} e_{k} \\
T x=T\left(\sum_{k=1}^{n} x_{k} e_{k}\right)=\sum_{k=1}^{n} x_{k} T\left(e_{k}\right)
\end{gathered}
$$

where $x_{k} \equiv\left(x, e_{k}\right)$. Therefore using Minkowski's inequality,

$$
\begin{aligned}
\|T x\| & =\left(\sum_{k=1}^{\infty}\left|\left(T x, f_{k}\right)\right|^{2}\right)^{1 / 2}=\left(\sum_{k=1}^{\infty}\left|\left(\sum_{j=1}^{n} x_{j} T e_{j}, f_{k}\right)\right|^{2}\right)^{1 / 2} \\
& =\left(\sum_{k=1}^{\infty}\left|\sum_{j=1}^{n}\left(x_{j} T e_{j}, f_{k}\right)\right|^{2}\right)^{1 / 2} \leq \sum_{j=1}^{n}\left(\sum_{k=1}^{\infty}\left|\left(x_{j} T e_{j}, f_{k}\right)\right|^{2}\right)^{1 / 2} \\
& \leq \sum_{j=1}^{n}\left|x_{j}\right|\left(\sum_{k=1}^{\infty}\left|\left(T e_{j}, f_{k}\right)\right|^{2}\right)^{1 / 2} \\
& =\sum_{j=1}^{n}\left|x_{j}\right|\left\|T e_{j}\right\| \leq\left(\sum_{j=1}^{n}\left|x_{j}\right|^{2}\right)^{1 / 2}\|T\|_{\mathscr{L}_{2}}=\|x\|\|T\|_{\mathscr{L}_{2}}
\end{aligned}
$$

Therefore, since finite sums of the form $\sum_{k=1}^{n} x_{k} e_{k}$ are dense in $H$, it follows $T \in \mathscr{L}(H, G)$ and $\|T\| \leq\|T\|_{\mathscr{L}_{2}}$

Next consider the norm. I need to verify the norm does not depend on the choice of orthonormal basis. Let $\left\{f_{k}\right\}$ be an orthonormal basis for $G$. Then for $\left\{e_{k}\right\}$ an orthonormal basis for $H$,

$$
\begin{aligned}
\sum_{k}\left\|T e_{k}\right\|^{2} & =\sum_{k} \sum_{j}\left|\left(T e_{k}, f_{j}\right)\right|^{2}=\sum_{k} \sum_{j}\left|\left(e_{k}, T^{*} f_{j}\right)\right|^{2} \\
& =\sum_{j} \sum_{k}\left|\left(e_{k}, T^{*} f_{j}\right)\right|^{2}=\sum_{j}\left\|T^{*} f_{j}\right\|^{2}
\end{aligned}
$$

The above computation makes sense because it was just shown that $T$ is continuous. The same result would be obtained for any other orthonormal basis $\left\{e_{j}^{\prime}\right\}$ and this shows the norm is at least well defined. It is clear that this does indeed satisfy the axioms of a norm. and this proves the above claims.

It only remains to verify $\mathscr{L}_{2}(H, G)$ is a separable Hilbert space. It is clear that it is an inner product space because you only have to pick an orthonormal basis, $\left\{e_{k}\right\}$ and define the inner product as

$$
(S, T) \equiv \sum_{k}\left(S e_{k}, T e_{k}\right)
$$

This satisfies the axioms of an inner product and delivers the well defined norm so it is a well defined inner product. Indeed, we get it from

$$
(S, T) \equiv \frac{1}{4}\left(\|S+T\|_{\mathscr{L}_{2}}^{2}-\|S-T\|_{\mathscr{L}_{2}}^{2}\right)
$$

and the norm is well defined giving the same thing for any choice of the orthonormal basis so the same is true of the inner product.

Consider completeness. Suppose then that $\left\{T_{n}\right\}$ is a Cauchy sequence in $\mathscr{L}_{2}(H, G)$. Then from 19.8.64 $\left\{T_{n}\right\}$ is a Cauchy sequence in $\mathscr{L}(H, G)$ and so there exists a unique
$T$ such that $\lim _{n \rightarrow \infty}\left\|T_{n}-T\right\|=0$. Then it only remains to verify $T \in \mathscr{L}_{2}(H, G)$. But by Fatou's lemma,

$$
\sum_{k}\left\|T e_{k}\right\|^{2} \leq \lim \inf _{n \rightarrow \infty} \sum_{k}\left\|T_{n} e_{k}\right\|^{2}=\lim \inf _{n \rightarrow \infty}\left\|T_{n}\right\|_{\mathscr{L}_{2}}^{2}<\infty
$$

All that remains is to verify $\mathscr{L}_{2}(H, G)$ is separable and these Hilbert Schmidt operators are compact. I will show an orthonormal basis for $\mathscr{L}_{2}(H, G)$ is $\left\{f_{j} \otimes e_{k}\right\}$ where $\left\{f_{k}\right\}$ is an orthonormal basis for $G$ and $\left\{e_{k}\right\}$ is an orthonormal basis for $H$. Here, for $f \in G$ and $e \in H, f \otimes e(x) \equiv(x, e) f$.

I need to show $f_{j} \otimes e_{k} \in \mathscr{L}_{2}(H, G)$ and that it is an orthonormal basis for $\mathscr{L}_{2}(H, G)$ as claimed.

$$
\sum_{k}\left\|f_{j} \otimes e_{i}\left(e_{k}\right)\right\|^{2}=\sum_{k}\left\|f_{j} \delta_{i k}\right\|^{2}=\left\|f_{j}\right\|^{2}=1<\infty
$$

so each of these operators is in $\mathscr{L}_{2}(H, G)$. Next I show they are orthonormal.

$$
\begin{aligned}
\left(f_{j} \otimes e_{k}, f_{s} \otimes e_{r}\right) & =\sum_{p}\left(f_{j} \otimes e_{k}\left(e_{p}\right), f_{s} \otimes e_{r}\left(e_{p}\right)\right) \\
& =\sum_{p} \delta_{r p} \delta_{k p}\left(f_{j}, f_{s}\right)=\sum_{p} \delta_{r p} \delta_{k p} \delta_{j s}
\end{aligned}
$$

If $j=s$ and $k=r$ this reduces to 1 . Otherwise, this gives 0 . Thus these operators are orthonormal.

Now let $T \in \mathscr{L}_{2}(H, G)$. Consider

$$
T_{n} \equiv \sum_{i=1}^{n} \sum_{j=1}^{n}\left(T e_{i}, f_{j}\right) f_{j} \otimes e_{i}
$$

Then

$$
T_{n} e_{k}=\sum_{i=1}^{n} \sum_{j=1}^{n}\left(T e_{i}, f_{j}\right)\left(e_{k}, e_{i}\right) f_{j}=\sum_{j=1}^{n}\left(T e_{k}, f_{j}\right) f_{j}
$$

It follows $\left\|T_{n} e_{k}\right\| \leq\left\|T e_{k}\right\|$ and $\lim _{n \rightarrow \infty} T_{n} e_{k}=T e_{k}$. Therefore, from the dominated convergence theorem,

$$
\lim _{n \rightarrow \infty}\left\|T-T_{n}\right\|_{\mathscr{L}_{2}}^{2} \equiv \lim _{n \rightarrow \infty} \sum_{k}\left\|\left(T-T_{n}\right) e_{k}\right\|^{2}=0
$$

Therefore, the linear combinations of the $f_{j} \otimes e_{i}$ are dense in $\mathscr{L}_{2}(H, G)$ and this proves completeness of the orthonomal basis.

This also shows $\mathscr{L}_{2}(H, G)$ is separable. From 19.8 .64 it also shows that every $T \in$ $\mathscr{L}_{2}(H, G)$ is the limit in the operator norm of a sequence of compact operators. This follows because each of the $f_{j} \otimes e_{i}$ is easily seen to be a compact operator because if $B \subseteq H$ is bounded, then $\left(f_{j} \otimes e_{i}\right)(B)$ is a bounded subset of a one dimensional vector space so it is pre-compact. Thus $T_{n}$ is compact, being a finite sum of these. By Lemma 19.8.6, so is $T$.

Finally, consider 19.8.65.

$$
\begin{gathered}
\|X \otimes Y\|_{\mathscr{L}_{2}}^{2} \equiv \sum_{k}\left|X \otimes Y\left(f_{k}\right)\right|_{H}^{2} \equiv \sum_{k}\left|X\left(f_{k}, Y\right)\right|_{H}^{2} \\
=\|X\|_{H}^{2} \sum_{k}\left|\left(f_{k}, Y\right)\right|^{2}=\|X\|_{H}^{2}\|Y\|_{G}^{2} \square
\end{gathered}
$$

### 19.9 Compact Operators in Banach Space

In general for $A \in \mathscr{L}(X, Y)$ the following definition holds.

Definition 19.9.1 Let $A \in \mathscr{L}(X, Y)$. Then $A$ is compact if whenever $B \subseteq X$ is a bounded set, $A B$ is precompact. Equivalently, if $\left\{x_{n}\right\}$ is a bounded sequence in $X$, then $\left\{A x_{n}\right\}$ has a subsequence which converges in $Y$.

An important result is the following theorem about the adjoint of a compact operator.
Theorem 19.9.2 Let $A \in \mathscr{L}(X, Y)$ be compact. Then the adjoint operator, $A^{*} \in \mathscr{L}\left(Y^{\prime}, X^{\prime}\right)$ is also compact.

Proof: Let $\left\{y_{n}^{*}\right\}$ be a bounded sequence in $Y^{\prime}$. Let $B$ be the closure of the unit ball in $X$. Then $A B$ is precompact. Then it is clear that the functions $\left\{y_{n}^{*}\right\}$ are equicontinuous and uniformly bounded on the compact set, $\overline{A(B)}$. By the Ascoli Arzela theorem, there is a subsequence $\left\{y_{n_{k}}^{*}\right\}$ which converges uniformly to a continuous function, $f$ on $\overline{A(B)}$. Now define $g$ on $A X$ by

$$
g(A x)=\|x\| f\left(A\left(\frac{x}{\|x\|}\right)\right), g(A 0)=0
$$

Thus for $x_{1}, x_{2} \neq 0$, and $a, b$ scalars,

$$
\begin{aligned}
g\left(a A x_{1}+b A x_{2}\right) & \equiv\left\|a x_{1}+b x_{2}\right\| f\left(\frac{A\left(a x_{1}+b x_{2}\right)}{\left\|a x_{1}+b x_{2}\right\|}\right) \\
& \equiv \lim _{k \rightarrow \infty}\left\|a x_{1}+b x_{2}\right\| y_{n_{k}}^{*}\left(\frac{A\left(a x_{1}+b x_{2}\right)}{\left\|a x_{1}+b x_{2}\right\|}\right) \\
& =\lim _{k \rightarrow \infty} a y_{n_{k}}^{*}\left(A x_{1}\right)+b y_{n_{k}}^{*}\left(A x_{2}\right) \\
& =a \lim _{k \rightarrow \infty}\left\|x_{1}\right\| y_{n_{k}}^{*}\left(\frac{A x_{1}}{\left\|x_{1}\right\|}\right)+b \lim _{k \rightarrow \infty}\left\|x_{2}\right\| y_{n_{k}}^{*}\left(\frac{A x_{2}}{\left\|x_{2}\right\|}\right) \\
& =a\left\|x_{1}\right\| f\left(\frac{A x_{1}}{\left\|x_{1}\right\|}\right)+b\left\|x_{2}\right\| f\left(\frac{A x_{2}}{\left\|x_{2}\right\|}\right) \\
& \equiv a g\left(A x_{1}\right)+b g\left(A x_{2}\right)
\end{aligned}
$$

showing that $g$ is linear on $A X$. Also

$$
|g(A x)|=\lim _{k \rightarrow \infty}\left|\|x\| y_{n_{k}}^{*}\left(A\left(\frac{x}{\|x\|}\right)\right)\right| \leq C\|x\|\left\|A\left(\frac{x}{\|x\|}\right)\right\|=C\|A x\|
$$

and so by the Hahn Banach theorem, there exists $y^{*}$ extending $g$ to all of $Y$ having the same operator norm.

$$
y^{*}(A x)=\lim _{k \rightarrow \infty}\|x\| y_{n_{k}}^{*}\left(A\left(\frac{x}{\|x\|}\right)\right)=\lim _{k \rightarrow \infty} y_{n_{k}}^{*}(A x)
$$

Thus $A^{*} y_{n_{k}}^{*}(x) \rightarrow A^{*} y^{*}(x)$ for every $x$. In addition to this, for $x \in B$,

$$
\begin{aligned}
\left\|A^{*} y^{*}(x)-A^{*} y_{n_{k}}^{*}(x)\right\| & =\left\|y^{*}(A x)-y_{n_{k}}^{*}(A x)\right\| \\
& =\left\|g(A x)-y_{n_{k}}^{*}(A x)\right\| \\
& =\| \| x\left\|f\left(A\left(\frac{x}{\|x\|}\right)\right)-\right\| x\left\|y_{n_{k}}^{*}\left(\frac{A x}{\|x\|}\right)\right\| \\
& \leq\left\|f\left(A\left(\frac{x}{\|x\|}\right)\right)-y_{n_{k}}^{*}\left(\frac{A x}{\|x\|}\right)\right\|
\end{aligned}
$$

and this is uniformly small for large $k$ due to the uniform convergence of $y_{n_{k}}^{*}$ to $f$ on $\overline{A(B)}$. Therefore, $\left\|A^{*} y^{*}-A^{*} y_{n_{k}}^{*}\right\| \rightarrow 0$.

### 19.10 The Fredholm Alternative

Recall that if $A$ is an $n \times n$ matrix and if the only solution to the system, $A \mathbf{x}=0$ is $\mathbf{x}=0$ then for any $\mathbf{y} \in \mathbb{R}^{n}$ it follows that there exists a unique solution to the system $A \mathbf{x}=\mathbf{y}$. This holds because the first condition implies $A$ is one to one and therefore, $A^{-1}$ exists. Of course things are much harder in a general Banach space. Here is a simple example for a Hilbert space.

Example 19.10.1 Let $L^{2}(\mathbb{N} ; \mu)=H$ where $\mu$ is counting measure. Thus an element of $H$ is a sequence, $\mathbf{a}=\left\{a_{i}\right\}_{i=1}^{\infty}$ having the property that

$$
\|\mathbf{a}\|_{H} \equiv\left(\sum_{k=1}^{\infty}\left|a_{k}\right|^{2}\right)^{1 / 2}<\infty
$$

Define $A: H \rightarrow H$ by

$$
A \mathbf{a} \equiv \mathbf{b} \equiv\left\{0, a_{1}, a_{2}, \cdots\right\}
$$

Thus A slides the sequence to the right and puts a zero in the first slot. Clearly $A$ is one to one and linear but it cannot be onto because it fails to yield $\mathbf{e}_{1} \equiv\{1,0,0, \cdots\}$.

Notwithstanding the above example, there are theorems which are like the linear algebra theorem mentioned above which hold in an arbitrary Banach spaces in the case where the operator is compact. To begin with here is an interesting lemma.

Lemma 19.10.2 Suppose $A \in \mathscr{L}(X, X)$ is compact for $X$ a Banach space. Then $(I-A)(X)$ is a closed subspace of $X$.

Proof: Suppose $(I-A) x_{n} \rightarrow y$. Let

$$
\alpha_{n} \equiv \operatorname{dist}\left(x_{n}, \operatorname{ker}(I-A)\right)
$$

and let $z_{n} \in \operatorname{ker}(I-A)$ be such that

$$
\alpha_{n} \leq\left\|x_{n}-z_{n}\right\| \leq\left(1+\frac{1}{n}\right) \alpha_{n}
$$

Thus $(I-A)\left(x_{n}-z_{n}\right) \rightarrow y$ because $(I-A) z_{n}=0$.
Case 1: $\left\{x_{n}-z_{n}\right\}$ has a bounded subsequence.
If this is so, the compactness of $A$ implies there exists a subsequence, still denoted by $n$ such that $\left\{A\left(x_{n}-z_{n}\right)\right\}_{n=1}^{\infty}$ is a Cauchy sequence. Since $(I-A)\left(x_{n}-z_{n}\right) \rightarrow y$, this implies $\left\{\left(x_{n}-z_{n}\right)\right\}$ is also a Cauchy sequence converging to a point, $x \in X$. Then, taking the limit as $n \rightarrow \infty,(I-A) x=y$ and so $y \in(I-A)(X)$.

Case 2: $\lim _{n \rightarrow \infty}\left\|x_{n}-z_{n}\right\|=\infty$. I will show this case cannot occur.
In this case, let $w_{n} \equiv \frac{x_{n}-z_{n}}{\left\|x_{n}-z_{n}\right\|}$. Thus $(I-A) w_{n} \rightarrow 0$ and $w_{n}$ is bounded. Therefore, there exists a subsequence, still denoted by $n$ such that $\left\{A w_{n}\right\}$ is a Cauchy sequence. Now it follows

$$
A w_{n}-A w_{m}+e_{n}-e_{m}=w_{n}-w_{m}
$$

where $e_{k} \rightarrow 0$ as $k \rightarrow \infty$. This implies $\left\{w_{n}\right\}$ is a Cauchy sequence which must converge to some $w_{\infty} \in X$. Therefore, $(I-A) w_{\infty}=0$ and so $w_{\infty} \in \operatorname{ker}(I-A)$. However, this is impossible because of the following argument. If $z \in \operatorname{ker}(I-A)$,

$$
\begin{aligned}
\left\|w_{n}-z\right\| & =\frac{1}{\left\|x_{n}-z_{n}\right\|}\left\|x_{n}-z_{n}-\right\| x_{n}-z_{n}\|z\| \\
& \geq \frac{1}{\left\|x_{n}-z_{n}\right\|} \alpha_{n} \geq \frac{\alpha_{n}}{\left(1+\frac{1}{n}\right) \alpha_{n}}=\frac{n}{n+1} .
\end{aligned}
$$

Taking the limit, $\left\|w_{\infty}-z\right\| \geq 1$. Since $z \in \operatorname{ker}(I-A)$ is arbitrary, this shows

$$
\operatorname{dist}\left(w_{\infty}, \operatorname{ker}(I-A)\right) \geq 1
$$

Since Case 2 does not occur, this proves the lemma.
Theorem 19.10.3 Let $A \in \mathscr{L}(X, X)$ be a compact operator and let $f \in X$. Then there exists a solution, x, to

$$
\begin{equation*}
x-A x=f \tag{19.10.66}
\end{equation*}
$$

if and only if

$$
\begin{equation*}
x^{*}(f)=0 \tag{19.10.67}
\end{equation*}
$$

for all $x^{*} \in \operatorname{ker}\left(I-A^{*}\right)$.
Proof: Suppose $x$ is a solution to 19.10 .66 and let $x^{*} \in \operatorname{ker}\left(I-A^{*}\right)$. Then

$$
x^{*}(f)=x^{*}((I-A)(x))=\left(\left(I-A^{*}\right) x^{*}\right)(x)=0 .
$$

Next suppose $x^{*}(f)=0$ for all $x^{*} \in \operatorname{ker}\left(I-A^{*}\right)$. I will show there exists $x$ solving 19.10.66. By Lemma 19.10.2, $(I-A)(X)$ is a closed subspace of $X$. Is $f \in(I-A)(X)$ ? If not, then by the Hahn Banach theorem, there exists $x^{*} \in X^{\prime}$ such that $x^{*}(f) \neq 0$ but $x^{*}((I-A)(x))=0$ for all $x \in X$. However last statement says nothing more nor less than $\left(I-A^{*}\right) x^{*}=0$. This is a contradiction because for such $x^{*}$, it is given that $x^{*}(f)=0$. This proves the theorem.

The following corollary is called the Fredholm alternative.

Corollary 19.10.4 Let $A \in \mathscr{L}(X, X)$ be a compact operator. Then there exists a solution to the equation

$$
\begin{equation*}
x-A x=f \tag{19.10.68}
\end{equation*}
$$

for all $f \in X$ if and only if $\left(I-A^{*}\right)$ is one to one on $X^{\prime}$.
Proof: Suppose $\left(I-A^{*}\right)$ is one to one first. Then if $x^{*}-A^{*} x^{*}=0$ it follows $x^{*}=0$ and so for any $f \in X, x^{*}(f)=0$ for all $x^{*} \in \operatorname{ker}\left(I-A^{*}\right)$. By 19.10.3 there exists a solution to $(I-A) x=f$.

Now suppose there exists a solution, $x$, to $(I-A) x=f$ for every $f \in X$. If $\left(I-A^{*}\right) x^{*}=$ 0 , then for every $x \in X$,

$$
\left(I-A^{*}\right) x^{*}(x)=x^{*}((I-A)(x))=0
$$

Since $(I-A)$ is onto, this shows $x^{*}=0$ and so $\left(I-A^{*}\right)$ is one to one as claimed. This proves the corollary.

The following is just an easier version of the above.
Corollary 19.10.5 In the case where $X$ is a Hilbert space, the conclusions of Corollary 19.10.4, Theorem 19.10.3, and Lemma 19.10.2 remain true if $H^{\prime}$ is replaced by $H$ and the adjoint is understood in the usual manner for Hilbert space. That is

$$
(A x, y)_{H}=\left(x, A^{*} y\right)_{H}
$$

### 19.11 Square Roots

In this section, $H$ will be a Hilbert space, real or complex, and $T$ will denote an operator which satisfies the following definition. A useful theorem about the existence of square roots of certain operators is presented. This proof is very elementary. I found it in [80].

Definition 19.11.1 Let $T \in \mathscr{L}(H, H)$ satisfy $T=T^{*}$ (Hermitian) and for all $x \in H$,

$$
\begin{equation*}
(T x, x) \geq 0 \tag{19.11.69}
\end{equation*}
$$

Such an operator is referred to as positive and self adjoint. It is probably better to refer to such an operator as "nonnegative" since the possibility that $T x=0$ for some $x \neq 0$ is not being excluded. Instead of "self adjoint" you can also use the term, Hermitian. To save on notation, write $T \geq 0$ to mean $T$ is positive, satisfying 19.11.69.

With the above definition here is a fundamental result about positive self adjoint operators.

Proposition 19.11.2 Let $S, T$ be positive and self adjoint such that $S T=T S$. Then $S T$ is also positive and self adjoint.

Proof: It is obvious that $S T$ is self adjoint. The only problem is to show that $S T$ is positive. To show this, first suppose $S \leq I$. The idea is to write

$$
S=S_{n+1}+\sum_{k=0}^{n} S_{k}^{2}
$$

where $S_{0}=S$ and the operators $S_{k}$ are self adjoint. This is a useful idea because it is then obvious that the sum is positive. If we want such a representation as above for each $n$, then it follows that $S_{0} \equiv S$ and

$$
\begin{equation*}
S=S_{n}+\sum_{k=0}^{n-1} S_{k}^{2} \tag{19.11.70}
\end{equation*}
$$

so, subtracting these yields $0=S_{n+1}-S_{n}+S_{n}^{2}$ and so

$$
S_{n+1}=S_{n}-S_{n}^{2}
$$

Thus it is obvious that the $S_{k}$ are all self adjoint and this shows how to define the $S_{k}$ recursively. Say 19.11.70 holds and $S_{n+1}$ is defined above. Then

$$
S=S_{n}+\sum_{k=0}^{n-1} S_{k}^{2}=S_{n+1}+S_{n}^{2}+\sum_{k=0}^{n-1} S_{k}^{2}=S_{n+1}+\sum_{k=0}^{n} S_{k}^{2}
$$

If we start with $S$ self adjoint, then we end up with each of the $S_{n}$ also being self adjoint. Now the assumption that $I \geq S$ is used. Also, the $S_{n}$ are polynomials in $S$.

Claim: $I \geq S_{n} \geq 0$.
Proof of the claim: This is true if $n=0$ by assumption. Assume true for $n$. Then from the definition,

$$
S_{n+1}=S_{n}-S_{n}^{2}=\left(I-S_{n}\right) S_{n}\left(S_{n}+\left(I-S_{n}\right)\right)=S_{n}^{2}\left(I-S_{n}\right)+\left(I-S_{n}\right)^{2} S_{n}
$$

and it is obvious from the definition that the sum of positive operators is positive. Therefore, it suffices to show the two terms in the above are both positive. It is clear from the definition that each $S_{n}$ is Hermitian (self adjoint) because they are just polynomials in $S$. Also each must commute with $T$ for the same reason. Therefore,

$$
\left(S_{n}^{2}\left(I-S_{n}\right) x, x\right)=\left(\left(I-S_{n}\right) S_{n} x, S_{n} x\right) \geq 0
$$

and also

$$
\left(\left(I-S_{n}\right)^{2} S_{n} x, x\right)=\left(S_{n}\left(I-S_{n}\right) x,\left(I-S_{n}\right) x\right) \geq 0
$$

This proves the claim.
Now each $S_{k}$ commutes with $T$ because this is true of $S_{0}$ and succeding $S_{k}$ are polynomials in terms of $S_{0}$. Therefore,

$$
\begin{align*}
(S T x, x) & =\left(\left(S_{n+1}+\sum_{k=0}^{n} S_{k}^{2}\right) T x, x\right)=\left(S_{n+1} T x, x\right)+\sum_{k=0}^{n}\left(S_{k}^{2} T x, x\right) \\
& =\left(T x, S_{n+1} x\right)+\sum_{k=0}^{n}\left(T S_{k} x, S_{k} x\right) \tag{19.11.71}
\end{align*}
$$

Consider $S_{n+1} x$. From the claim,

$$
(S x, x)=\left(S_{n+1} x, x\right)+\sum_{k=0}^{n}\left|S_{k} x\right|^{2} \geq \sum_{k=0}^{n}\left|S_{k} x\right|^{2}
$$

and so $\lim _{n \rightarrow \infty} S_{n} x=0$. Hence from 19.11.71,

$$
\lim \inf _{n \rightarrow \infty}(S T x, x)=(S T x, x)=\lim \inf _{n \rightarrow \infty} \sum_{k=0}^{n}\left(T S_{k} x, S_{k} x\right) \geq 0
$$

All this was based on the assumption that $S \leq I$. The next task is to remove this assumption. Let $S T=T S$ where $T$ and $S$ are positive self adjoint operators. Then consider $S /\|S\|$. This is still a positive self adjoint operator and it commutes with $T$ just like $S$ does. Therefore, from the first part,

$$
0 \leq\left(\frac{S}{\|S\|} T x, x\right)=\frac{1}{\|S\|}(S T x, x)
$$

The proposition is like the familiar statement about real numbers which says that when you multiply two nonnegative real numbers the result is a nonnegative real number. The next lemma is a generalization of the familiar fact that if you have an increasing sequence of real numbers which is bounded above, then the sequence converges.

Lemma 19.11.3 Let $\left\{T_{n}\right\}$ be a sequence of self adjoint operators on a Hilbert space, $H$ and let $T_{n} \leq T_{n+1}$ for all $n$. Also suppose there exists $K$, a self adjoint operator such that for all $n, T_{n} \leq K$. Suppose also that each operator commutes with all the others and that $K$ commutes with all the $T_{n}$. Then there exists a self adjoint continuous operator, $T$ such that for all $x \in H, T_{n} x \rightarrow T x, T \leq K$, and $T$ commutes with all the $T_{n}$ and with $K$.

Proof: Consider $K-T_{n} \equiv S_{n}$. Then the $\left\{S_{n}\right\}$ are decreasing, that is, $\left\{\left(S_{n} x, x\right)\right\}$ is a decreasing sequence and from the hypotheses, $S_{n} \geq 0$ so the above sequence is bounded below by 0 . Therefore, $\lim _{n \rightarrow \infty}\left(S_{n} x, x\right)$ exists. By Proposition 19.11.2, if $n>m$,

$$
S_{m}^{2}-S_{n} S_{m}=S_{m}\left(S_{m}-S_{n}\right) \geq 0
$$

and similarly from the above proposition,

$$
S_{n} S_{m}-S_{n}^{2}=S_{n}\left(S_{m}-S_{n}\right) \geq 0
$$

Therefore, since $S_{n}$ is self adjoint,

$$
\begin{gathered}
\left|T_{n} x-T_{m} x\right|^{2}=\left|S_{n} x-S_{m} x\right|^{2}=\left(\left(S_{n}-S_{m}\right)^{2} x, x\right) \\
=\left(\left(S_{n}^{2}-2 S_{n} S_{m}+S_{m}^{2}\right) x, x\right)=\left(\left(S_{m}^{2}-S_{m} S_{n}\right) x, x\right)+\left(\left(S_{n}^{2}-S_{n} S_{m}\right) x, x\right) \\
\leq\left(\left(S_{m}^{2}-S_{m} S_{n}\right) x, x\right) \leq\left(\left(S_{m}^{2}-S_{n}^{2}\right) x, x\right) \\
=\left(\left(S_{m}-S_{n}\right)\left(S_{m}+S_{n}\right) x, x\right) \leq 2\left(\left(S_{m}-S_{n}\right) K x, x\right) \\
\leq 2\left(\left(S_{m}-S_{n}\right) K x, K x\right)^{1 / 2}\left(\left(S_{m}-S_{n}\right) x, x\right)^{1 / 2}
\end{gathered}
$$

The last step follows from an application of the Cauchy Schwarz inequality along with the fact $S_{m}-S_{n} \geq 0$. The last expression converges to 0 because $\lim _{n \rightarrow \infty}\left(S_{n} x, x\right)$ exists for each
$x$. It follows $\left\{T_{n} x\right\}$ is a Cauchy sequence. Let $T x$ be the thing to which it converges. $T$ is obviously linear and $(T x, x)=\lim _{n \rightarrow \infty}\left(T_{n} x, x\right) \leq(K x, x)$. Also

$$
(K T x, y)=\lim _{n \rightarrow \infty}\left(K T_{n} x, y\right)=\lim _{n \rightarrow \infty}\left(T_{n} K x, y\right)=(T K x, y)
$$

and so $T K=K T$. Similarly, $T$ commutes with all $T_{n}$.
In order to show $T$ is continuous, apply the uniform boundedness principle, Theorem 17.1.8. The convergence of $\left\{T_{n} x\right\}$ implies there exists a uniform bound on the norms, $\left\|T_{n}\right\|$ and so $\left|\left(T_{n} x, y\right)\right| \leq C|x||y|$. Now take the limit as $n \rightarrow \infty$ to conclude $|(T x, y)| \leq C|x||y|$ which shows $\|T\| \leq C$.

With this preparation, here is the theorem about square roots.
Theorem 19.11.4 Let $T \in \mathscr{L}(H, H)$ be a positive self adjoint linear operator. Then there exists a unique square root, $A$ with the following properties. $A^{2}=T, A$ is positive and self adjoint, A commutes with every operator which commutes with $T$.

Proof: First suppose $T \leq I$. Then define

$$
A_{0} \equiv 0, A_{n+1}=A_{n}+\frac{1}{2}\left(T-A_{n}^{2}\right)
$$

From this it follows that every $A_{n}$ is a polynomial in $T$. Therefore, $A_{n}$ commutes with $T$ and with every operator which commutes with $T$.

Claim 1: $A_{n} \leq I$.
Proof of Claim 1: This is true if $n=0$. Suppose it is true for $n$. Then by the assumption that $T \leq I$,

$$
\begin{aligned}
I-A_{n+1} & =I-A_{n}+\frac{1}{2}\left(A_{n}^{2}-T\right) \geq I-A_{n}+\frac{1}{2}\left(A_{n}^{2}-I\right) \\
& =I-A_{n}-\frac{1}{2}\left(I-A_{n}\right)\left(I+A_{n}\right)=\left(I-A_{n}\right)\left(I-\frac{1}{2}\left(I+A_{n}\right)\right) \\
& =\left(I-A_{n}\right)\left(I-A_{n}\right) \frac{1}{2} \geq 0
\end{aligned}
$$

Claim 2: $A_{n} \leq A_{n+1}$
Proof of Claim 2: From the definition of $A_{n}$, this is true if $n=0$ because

$$
A_{1}=T \geq 0=A_{0} .
$$

Suppose true for $n$. Then from Claim 1,

$$
\begin{aligned}
A_{n+2}-A_{n+1} & =A_{n+1}+\frac{1}{2}\left(T-A_{n+1}^{2}\right)-\left[A_{n}+\frac{1}{2}\left(T-A_{n}^{2}\right)\right] \\
& =A_{n+1}-A_{n}+\frac{1}{2}\left(A_{n}^{2}-A_{n+1}^{2}\right) \\
& =\left(A_{n+1}-A_{n}\right)\left(I-\frac{1}{2}\left(A_{n}+A_{n+1}\right)\right) \\
& \geq\left(A_{n+1}-A_{n}\right)\left(I-\frac{1}{2}(2 I)\right)=0
\end{aligned}
$$

Claim 3: $A_{n} \geq 0$
Proof of Claim 3: This is true if $n=0$. Suppose it is true for $n$.

$$
\begin{aligned}
\left(A_{n+1} x, x\right) & =\left(A_{n} x, x\right)+\frac{1}{2}(T x, x)-\frac{1}{2}\left(A_{n}^{2} x, x\right) \\
& \geq\left(A_{n} x, x\right)+\frac{1}{2}(T x, x)-\frac{1}{2}\left(A_{n} x, x\right) \geq 0
\end{aligned}
$$

because $A_{n}-A_{n}^{2}=A_{n}\left(I-A_{n}\right) \geq 0$ by Proposition 19.11.2.
Now $\left\{A_{n}\right\}$ is a sequence of positive self adjoint operators which are bounded above by $I$ such that each of these operators commutes with every operator which commutes with $T$. By Lemma 19.11.3, there exists a bounded linear operator $A$ such that for all $x$, $A_{n} x \rightarrow A x$. Then $A$ commutes with every operator which commutes with $T$ because each $A_{n}$ has this property. Also $A$ is a positive operator because each $A_{n}$ is. From passing to the limit in the definition of $A_{n}$,

$$
A x=A x+\frac{1}{2}\left(T x-A^{2} x\right)
$$

and so $T x=A^{2} x$. This proves the theorem in the case that $T \leq I$.
In the general case, consider $T /\|T\|$. Then

$$
\left(\frac{T}{\|T\|} x, x\right)=\frac{1}{\|T\|}(T x, x) \leq|x|^{2}=(I x, x)
$$

and so $T /\|T\| \leq I$. Therefore, it has a square root, $B$. Let $A=\sqrt{\|T\|} B$. Then $A$ has all the right properties and $A^{2}=\|T\| B^{2}=\|T\|(T /\|T\|)=T$. This proves the existence part of the theorem.

Next suppose both $A$ and $B$ are square roots of $T$ having all the properties stated in the theorem. Then $A B=B A$ because both $A$ and $B$ commute with every operator which commutes with $T$.

$$
\begin{equation*}
(A(A-B) x,(A-B) x),(B(A-B) x,(A-B) x) \geq 0 \tag{19.11.72}
\end{equation*}
$$

Therefore, on adding these,

$$
\begin{aligned}
& \left(\left(A^{2}-A B+B A-B^{2}\right) x,(A-B) x\right)=\left(\left(A^{2}-B^{2}\right) x,(A-B) x\right) \\
= & ((T-T) x,(A-B) x)=0
\end{aligned}
$$

It follows both expressions in 19.11.72 equal 0 since both are nonnegative and when they are added the result is 0 . Now applying the existence part of the theorem to $A$, there exists a positive square root of $A$ which is self adjoint. Thus

$$
(\sqrt{A}(A-B) x, \sqrt{A}(A-B) x)=0
$$

so $\sqrt{A}(A-B) x=0$ which implies $A(A-B) x=0$. Similarly, $B(A-B) x=0$. Subtracting these and taking the inner product with $x$,

$$
0=((A(A-B)-B(A-B)) x, x)=\left((A-B)^{2} x, x\right)=|(A-B) x|^{2}
$$

and so $A x=B x$ which shows $A=B$ since $x$ was arbitrary.

### 19.12 Ordinary Differential Equations in Banach Space

Here we consider the initial value problem for functions which have values in a Banach space. Let $X$ be a Banach space.
Definition 19.12.1 Define $B C([a, b] ; X)$ as the bounded continuous functions $f$ which have values in the Banach space $X$. For $f \in B C([a, b] ; X)$, $\gamma$ a real number. Then

$$
\begin{equation*}
\|f\|_{\gamma} \equiv \sup _{t \in[a, b]}\left\|f(t) e^{\gamma(t-a)}\right\| \tag{19.12.73}
\end{equation*}
$$

Then this is a norm. The usual norm is given by

$$
\|f\| \equiv \sup _{t \in[a, b]}\|f(t)\|
$$

Lemma 19.12.2 $\|\cdot\|_{\gamma}$ is a norm for $B C([a, b] ; X)$ and $B C([a, b] ; X)$ is a complete normed linear space. Also, a sequence is Cauchy in $\|\cdot\|_{\gamma}$ if and only if it is Cauchy in $\|\cdot\|$.

Proof: First consider the claim about $\|\cdot\|_{\gamma}$ being a norm. To simplify notation, let $T=[a, b]$. It is clear that $\|f\|_{\gamma}=0$ if and only if $f=0$ and $\|f\|_{\gamma} \geq 0$. Also,

$$
\|\alpha f\|_{\gamma} \equiv \sup _{t \in T}\left\|\alpha f(t) e^{\gamma(t-a)}\right\|=|\alpha| \sup _{t \in T}\left\|f(t) e^{\gamma(t-a)}\right\|=|\alpha|\|f\|_{\gamma}
$$

so it does what is should for scalar multiplication. Next consider the triangle inequality.

$$
\begin{aligned}
\|f+g\|_{\gamma} & =\sup _{t \in T}\left\|(f(t)+g(t)) e^{\gamma(t-a)}\right\| \leq \sup _{t \in T}\left(\left|f(t) e^{\gamma(t-a)}\right|+\left|g(t) e^{\gamma(t-a)}\right|\right) \\
& \leq \sup _{t \in T}\left|f(t) e^{\gamma(t-a)}\right|+\sup _{t \in T}\left|g(t) e^{\gamma(t-a)}\right|=\|f\|_{\gamma}+\|g\|_{\gamma}
\end{aligned}
$$

The rest follows from the next inequalities.

$$
\begin{aligned}
\|f\| & \equiv \sup _{t \in T}\|f(t)\|=\sup _{t \in T}\left\|f(t) e^{\gamma(t-a)} e^{-\gamma(t-a)}\right\| \leq e^{|\gamma(b-a)|}\|f\|_{\gamma} \\
& \equiv e^{|\gamma(b-a)|} \sup _{t \in T}\left\|f(t) e^{\gamma(t-a)}\right\| \leq\left(e^{|\gamma|(b-a)}\right)^{2} \sup _{t \in T}\|f(t)\|=\left(e^{|\gamma|(b-a)}\right)^{2}\|f\|
\end{aligned}
$$

Now consider the ordinary initial value problem

$$
\begin{equation*}
x^{\prime}(t)+F(t, x(t))=f(t), x(a)=x_{0}, t \in[a, b] \tag{19.12.74}
\end{equation*}
$$

where here $F:[a, b] \times X \rightarrow X$ is continuous and satisfies the Lipschitz condition

$$
\begin{equation*}
\|F(t, x)-F(t, y)\| \leq K\|x-y\|, F:[a, b] \times X \rightarrow X \text { is continuous } \tag{19.12.75}
\end{equation*}
$$

Thanks to the fundamental theorem of calculus, there exists a solution to 19.12 .74 if and only if it is a solution to the integral equation

$$
\begin{equation*}
x(t)=x_{0}-\int_{a}^{t} F(s, x(s)) d s \tag{19.12.76}
\end{equation*}
$$

Then we have the following theorem.

Theorem 19.12.3 Let 19.12.75 hold. Then there exists a unique solution to 19.12 .74 in $B C([a, b] ; X)$.

Proof: Use the norm of 19.12 .73 where $\gamma \neq 0$ is described later. Let $T: B C([a, b] ; X) \rightarrow$ $B C([a, b] ; X)$ be defined by

$$
T x(t) \equiv x_{0}-\int_{a}^{t} F(s, x(s)) d s
$$

Then

$$
\begin{aligned}
& \|T x(t)-T y(t)\|_{X}=\left\|\int_{a}^{t} F(s, x(s)) d s-\int_{a}^{t} F(s, y(s)) d s\right\| \\
\leq & K \int_{a}^{t}\|x(s)-y(s)\| d s=K \int_{a}^{t}\left\|(x(s)-y(s)) e^{\gamma(s-a)} e^{-\gamma(s-a)}\right\| d s \\
\leq & K \int_{a}^{t} e^{-\gamma(s-a)} d s\|x-y\|_{\gamma}=K\left(\frac{e^{-\gamma(t-a)}}{-\gamma}+\frac{1}{\gamma}\right)\|x-y\|_{\gamma}
\end{aligned}
$$

Therefore,

$$
\begin{gathered}
e^{\gamma(t-a)}\|T x(t)-T y(t)\|_{X} \leq K\left(\frac{e^{\gamma(t-a)}}{\gamma}-\frac{1}{\gamma}\right)\|x-y\|_{\gamma} \\
\|T x-T y\|_{\gamma} \leq \sup _{t \in[a, b]} K\left(\frac{e^{\gamma(t-a)}}{\gamma}-\frac{1}{\gamma}\right)\|x-y\|_{\gamma}
\end{gathered}
$$

Letting $\gamma=-m^{2}$, this reduces to

$$
\|T x-T y\|_{-m^{2}} \leq \frac{K}{m^{2}}\|x-y\|_{-m^{2}}
$$

and so if $K / m^{2}<1 / 2$, this shows the solution to the integral equation is the unique fixed point of a contraction mapping defined on $B C([a, b] ; X)$. This shows existence and uniqueness of the initial value problem 19.12.74.

Definition 19.12.4 Let $S:[0, \infty) \rightarrow \mathscr{L}(X, X)$ be continuous and satisfy

1. $S(t+s)=S(t) S(s)$ called the semigroup identity.
2. $S(0)=I$
3. $\lim _{h \rightarrow 0+} \frac{S(h) x-x}{h}=A x$ for $A$ a densely defined closed linear operator whenever $x \in$ $D(A) \subseteq X$.

Then $S$ is called a continuous semigroup and $A$ is said to generate $S$.

Then we have the following corollary of Theorem 19.12.3. First note the following. For $t \geq 0$ and $h \geq 0$, if $x \in D(A)$, the semigroup identity implies

$$
\lim _{h \rightarrow 0} \frac{S(t+h) x-S(t) x}{h}=\lim _{h \rightarrow 0} S(t) \frac{S(h) x-x}{h}=S(t) \lim _{h \rightarrow 0} \frac{S(h) x-x}{h} \equiv S(t) A x
$$

As shown above, $\mathscr{L}(X, X)$ is a perfectly good Banach space with the operator norm whenever $X$ is a Banach space.

Corollary 19.12.5 Let $X$ be a Banach space and let $A \in \mathscr{L}(X, X)$. Let $S(t)$ be the solution in $\mathscr{L}(X, X)$ to

$$
\begin{equation*}
S^{\prime}(t)=A S(t), S(0)=I \tag{19.12.77}
\end{equation*}
$$

Then $t \rightarrow S(t)$ is a continuous semigroup whose generator is $A$. In this case $A$ is actually defined on all of $X$, not just on a dense subset. Furthermore, in this case where $A \in \mathscr{L}(X, X)$, $S(t) A=A S(t)$. If $T(t)$ is any semigroup having $A$ as a generator, then $T(t)=S(t)$. Also you can express $S(t)$ as a power series,

$$
\begin{equation*}
S(t)=\sum_{n=0}^{\infty} \frac{(A t)^{n}}{n!} \tag{19.12.78}
\end{equation*}
$$

Proof: The solution to the initial value problem 19.12.77 exists on $[0, b]$ for all $b$ so it exists on all of $\mathbb{R}$ thanks to the uniqueness on every finite interval. First consider the semigroup property. Let $\Psi(t) \equiv S(t+s), \Phi(t) \equiv S(t) S(s)$. Then

$$
\begin{gathered}
\Psi^{\prime}(t)=S^{\prime}(t+s)=A S(t+s)=A \Psi(t), \Psi(0)=S(s) \\
\Phi^{\prime}(t)=S^{\prime}(t) S(s)=A S(t) S(s)=A \Phi(t), \Phi(0)=S(s)
\end{gathered}
$$

By uniqueness, $\Phi(t)=\Psi(t)$ for all $t \geq 0$. Thus $S(t) S(s)=S(t+s)=S(s) S(t)$. Now from this, for $t>0$

$$
S(t) A=S(t) \lim _{h \rightarrow 0} \frac{S(h)-I}{h}=\lim _{h \rightarrow 0} S(t) \frac{S(h)-I}{h}=\lim _{h \rightarrow 0} \frac{S(h)-I}{h} S(t)=A S(t)
$$

As to $A$ being the generator of $S(t)$, letting $x \in X$, then from the differential equation solved,

$$
\lim _{h \rightarrow 0+} \frac{S(h) x-x}{h}=\lim _{h \rightarrow 0+} \frac{1}{h} \int_{0}^{h} A S(t) x d t=A S(0) x=A x .
$$

If $T(t)$ is a semigroup generated by $A$ then for $t>0$,

$$
T^{\prime}(t) \equiv \lim _{h \rightarrow 0} \frac{T(t+h)-T(t)}{h}=\lim _{h \rightarrow 0} \frac{T(h)-I}{h} T(t)=A T(t)
$$

and $T(0)=I$. However, uniqueness applies because $T$ and $S$ both satisfy the same initial value problem and this yields $T(t)=S(t)$.

To show the power series equals $S(t)$ it suffices to show it satisfies the initial value problem. Using the mean value theorem,

$$
\sum_{n=0}^{\infty} \frac{A^{n}\left((t+h)^{n}-t^{n}\right)}{n!}=\sum_{n=1}^{\infty} \frac{A^{n}\left(t+\theta_{n}(h)\right)^{n-1}}{(n-1)!}
$$

where $\theta_{n}(h) \in(0, h)$. Then taking a limit as $h \rightarrow 0$ and using the dominated convergence theorem, the limit of the difference quotient is

$$
\sum_{n=1}^{\infty} \frac{A^{n} t^{n-1}}{(n-1)!}=A \sum_{n=1}^{\infty} \frac{A^{n-1} t^{n-1}}{(n-1)!}=A \sum_{n=0}^{\infty} \frac{(A t)^{n}}{n!}
$$

Thus $\sum_{n=0}^{\infty}{\frac{(A t)^{n}}{n!}}^{n}$ satisfies the differential equation. It clearly satisfies the initial condition. Hence it equals $S(t)$.

Note that as a consequence of the above argument showing that $T$ and $S$ are the same, it follows that $T(t) A=A T(t)$ so one obtains that if the generator is a bounded linear operator, then the semigroup commutes with this operator.

When dealing with differential equations, one of the best tools is Gronwall's inequality. This is presented next.

Theorem 19.12.6 Suppose $u$ is nonnegative, continuous, and real valued and that

$$
u(t) \leq C+\int_{0}^{t} k u(s) d s, k \geq 0
$$

Then $u(t) \leq C e^{k t}$.
Proof: Let $w(t) \equiv \int_{0}^{t} k u(s) d s$. Then

$$
w^{\prime}(t)=k u(t) \leq k C+k w(t)
$$

and so $w^{\prime}(t)-k w(t) \leq k C$ which implies $\frac{d}{d t}\left(e^{-k t} w(t)\right) \leq k C e^{-k t}$. Therefore,

$$
e^{-k t} w(t) \leq C k \int_{0}^{t} e^{-k s} d s=C k\left(\frac{1}{k}-\frac{1}{k} e^{-k t}\right)
$$

so $w(t) \leq C\left(e^{k t}-1\right)$. From the original inequality, $u(t) \leq C+w(t) \leq C+C e^{k t}-C=C e^{k t}$.

### 19.13 Fractional Powers of Operators

Let $A \in \mathscr{L}(X, X)$ for $X$ a Hilbert space, $A=A^{*}$. We want to define $A^{\alpha}$ for $\alpha \in(0,1)$ in such a way that things work as they should provided that $(A x, x) \geq 0$.

If $A \in(0, \infty)$ we can get $A^{-\alpha}$ for $\alpha \in(0,1)$ as

$$
A^{-\alpha} \equiv \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} e^{-A t} t^{a-1} d t
$$

Indeed, you change the variable as follows letting $u=A t$,

$$
\begin{aligned}
\int_{0}^{\infty} e^{-A t} t^{a-1} d t & =\int_{0}^{\infty} e^{-u}\left(\frac{u}{A}\right)^{a-1} \frac{1}{A} d u \\
& =\int_{0}^{\infty} e^{-u} u^{\alpha-1} A^{1-\alpha} \frac{1}{A} d u=A^{-\alpha} \Gamma(\alpha)
\end{aligned}
$$

Next we need to define $e^{-A t}$ for $A \in \mathscr{L}(X, X)$.

Definition 19.13.1 By definition, $e^{-A t} x_{0}$ will be $x(t)$ where $x(t)$ is the solution to the initial value problem

$$
x^{\prime}+A x=0, x(0)=x_{0}
$$

Such a solution exists and is unique by standard contraction mapping arguments as in Theorem 19.12.3. Equivalently, one could consider for $\Phi(t) \equiv e^{-A t}$ the solution in $\mathscr{L}(X, X)$ of

$$
\Phi^{\prime}(t)+A \Phi(t)=0, \Phi(0)=I
$$

Now the case of interest here is that $A=A^{*}$ and $(A x, x) \geq \delta|x|^{2}$. We need an estimate for $\left\|e^{-A t}\right\|$.

Lemma 19.13.2 Suppose $A=A^{*}$ and $(A x, x) \geq \varepsilon|x|^{2}$. Then

$$
\left\|e^{-A t}\right\| \leq e^{-\varepsilon t}
$$

Proof: Let $\hat{x}(t)=x(t) e^{\varepsilon t}$. Then the equation for $e^{-A t} x_{0} \equiv x(t)$ becomes

$$
\hat{x}^{\prime}(t)-\varepsilon \hat{x}(t)+A \hat{x}(t)=0, \hat{x}(0)=x_{0}
$$

Then multiplying by $\hat{x}(t)$ and integrating gives

$$
\frac{1}{2}|\hat{x}(t)|^{2}-\varepsilon \int_{0}^{t}|\hat{x}(s)|^{2} d s-\frac{1}{2}\left|x_{0}\right|^{2}+\int_{0}^{t}(A \hat{x}, \hat{x}) d s=0
$$

and so, from the assumed estimate,

$$
\frac{1}{2}|\hat{x}(t)|^{2}-\varepsilon \int_{0}^{t}|\hat{x}(s)|^{2} d s-\frac{1}{2}\left|x_{0}\right|^{2}+\int_{0}^{t} \varepsilon|\hat{x}(s)|^{2} d s \leq 0
$$

and so $|\hat{x}(t)| \leq\left|x_{0}\right|$. Hence, $|x(t)|=\left|e^{-A t} x_{0}\right| \leq\left|x_{0}\right| e^{-\varepsilon t}$. Since $x_{0}$ was arbitrary, it follows that $\left\|e^{-A t}\right\| \leq e^{-\varepsilon t}$.

With this estimate, we can define $A^{-\alpha}$ for $\alpha \in(0,1)$ if $A=A^{*}$ and $(A x, x) \geq \varepsilon|x|^{2}$.
Definition 19.13.3 Let $A \in \mathscr{L}(X, X), A=A^{*}$ and $(A x, x) \geq \varepsilon|x|^{2}$. Then for $\alpha \in(0,1)$,

$$
A^{-\alpha} \equiv \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} e^{-A t} t^{a-1} d t
$$

The integral is well defined thanks to the estimate of the above lemma which gives $\left\|e^{-A t}\right\| \leq$ $e^{-\varepsilon t}$. You can let the integral be a standard improper Riemann integral since everything in sight is continuous. For such A, define

$$
A^{\alpha} \equiv A A^{-(1-\alpha)}
$$

Note the meaning of the integral. It takes place in $\mathscr{L}(X, X)$ and Riemann sums approximating this integral also take place in $\mathscr{L}(X, X)$. Thus

$$
\int_{0}^{\infty} e^{-A t} t^{a-1} d t\left(x_{0}\right)=\int_{0}^{\infty} t^{a-1} e^{-A t}\left(x_{0}\right) d t
$$

by approximating the improper integral with Riemann sums and using the obvious fact that such an identification holds for each sum in the approximation. Similar reasoning shows that an inner product can be taken inside the integral

$$
\left(\int_{0}^{\infty} e^{-A t} t^{a-1} d t\left(x_{0}\right), y_{0}\right)=\int_{0}^{\infty} e^{-A t}\left(x_{0}, y_{0}\right) t^{a-1} d t
$$

With this observation, the following lemma is fairly easy.
Lemma 19.13.4 For all $x_{0}, \int_{0}^{\infty} e^{-A t} t^{a-1} d t\left(x_{0}\right)=\int_{0}^{\infty} t^{a-1} e^{-A t}\left(x_{0}\right) d t$. Also $A^{-\alpha}$ is Hermitian whenever $A$ is. (This is the case considered here.) Also we have the semigroup property $e^{-A(t+s)}=e^{-A t} e^{-A s}$ for $t, s \geq 0$. In addition to this, if $C A=A C$ then $e^{-A t} C=C e^{-A t}$. In words, $e^{-A t}$ commutes with every $C \in \mathscr{L}(X, X)$ which commutes with A. Also, $e^{-A t}$ is Hermitian whenever $A$ is and so is $A^{-\alpha}$. (This is what is being considered here.)

Proof: The semigroup property follows right away from uniqueness considerations for the ordinary initial value problem. Indeed, letting $s, t \geq 0$, fix $s$ and consider the following for $\Phi(t) \equiv e^{-A t}$. Thus $\Phi(0)=I$ and $\Phi(t) x_{0}$ is the solution to $x^{\prime}+A x=0, x(0)=x_{0}$. Then define

$$
t \rightarrow \Phi(t+s)-\Phi(t) \Phi(s) \equiv Y(t)
$$

Then taking the time derivative, you get the following ordinary differential equation in $\mathscr{L}(X, X)$

$$
\begin{aligned}
Y^{\prime}(t) & =\Phi^{\prime}(t+s)-\Phi^{\prime}(t) \Phi(s)=-A \Phi(t+s)+A \Phi(t) \Phi(s) \\
& =-A(\Phi(t+s)-\Phi(t) \Phi(s))=-A Y(t)
\end{aligned}
$$

also, letting $t=0, Y(0)=\Phi(s)-\Phi(0) \Phi(s)=0$. Thus, by uniqueness of solutions to ordinary differential equations, $Y(t)=0$ for all $t \geq 0$ which shows the semigroup property. See Theorem 19.12.75. Actually, this identity holds in this case for all $s, t \in \mathbb{R}$ but this is not needed and the argument given here generalizes well to situations where one can only consider $t \in[0, \infty)$. Note how this shows that it is also the case that $\Phi(t) \Phi(s)=\Phi(s) \Phi(t)$.

Now consider the claim about commuting with operators which commute with $A$. Let $C A=A C$ for $C \in \mathscr{L}(X, X)$ and let $y(t)$ be given by

$$
y(t) \equiv\left(C e^{-A t} x_{0}-e^{-A t} C x_{0}\right)
$$

Then $y(0)=C x_{0}-C x_{0}=0$ and

$$
\begin{aligned}
y^{\prime}(t) & =C\left(-A\left(e^{-A t} x_{0}\right)\right)+\left(A e^{-A t} C x_{0}\right) \\
& =-A C\left(e^{-A t} x_{0}\right)+A\left(e^{-A t} C x_{0}\right)
\end{aligned}
$$

and so

$$
y^{\prime}(t)+A\left(C e^{-A t} x_{0}-e^{-A t} C x_{0}\right)=y^{\prime}+A y=0, y(0)=0
$$

Therefore, by uniqueness to the ODE one obtains $y(t)=0$ which shows that $C$ commutes with $e^{-A t}$.

Finally consider the claim about $e^{-A t}$ being Hermitian. For $\Phi(t) \equiv e^{-A t}$

$$
\begin{aligned}
\Phi^{\prime}(t)+A \Phi(t) & =0, \Phi(0)=I \\
\Phi^{* \prime}(t)+\Phi^{*}(t) A & =0, \Phi^{*}(0)=I
\end{aligned}
$$

and so, from what was just shown about commuting,

$$
\begin{aligned}
\Phi^{* \prime}(t)+\Phi^{*}(t) A & =0, \Phi^{*}(0)=I \\
\Phi^{\prime}(t)+\Phi(t) A & =0, \Phi(0)=I
\end{aligned}
$$

Thus $\Phi(t)$ and $\Phi^{*}(t)$ satisfy the same initial value problem and so they are the same Thanks to Theorem 19.12.3.

Next it follows that $A^{-\alpha}$ is Hermitian because

$$
\begin{aligned}
\left(A^{-\alpha} x_{0}, y_{0}\right) & \equiv\left(\int_{0}^{\infty} t^{\alpha-1} e^{-A t} x_{0} d t, y_{0}\right)=\int_{0}^{\infty}\left(t^{\alpha-1} e^{-A t} x_{0}, y_{0}\right) d t \\
& =\int_{0}^{\infty}\left(x_{0}, t^{\alpha-1} e^{-A t} y_{0}\right) d t=\left(x_{0}, \int_{0}^{\infty} t^{\alpha-1} e^{-A t}\left(y_{0}\right) d t\right) \\
& =\left(x_{0}, A^{-\alpha} y_{0}\right)
\end{aligned}
$$

Note that Lemma 19.13 .2 shows that $A A^{-\alpha}=A^{-\alpha} A$. Also $A^{-\alpha} A^{-\beta}=A^{-\beta} A^{-\alpha}$ and in fact, $A^{-\alpha}$ commutes with every operator which commutes with $A$. Next is a technical lemma which will prove useful.

Lemma 19.13.5 For $\alpha, \beta>0, \Gamma(\alpha) \Gamma(\beta)=\Gamma(\alpha+\beta) \int_{0}^{1}(1-v)^{\alpha-1} v^{\beta-1} d v$
Proof:

$$
\begin{gathered}
\Gamma(\alpha) \Gamma(\beta) \equiv \int_{0}^{\infty} \int_{0}^{\infty} e^{-(t+s)} t^{\alpha-1} s^{\beta-1} d t d s=\int_{0}^{\infty} \int_{s}^{\infty} e^{-u}(u-s)^{\alpha-1} s^{\beta-1} d u d s \\
=\int_{0}^{\infty} e^{-u} \int_{0}^{u}(u-s)^{\alpha-1} s^{\beta-1} d s d u=\int_{0}^{\infty} e^{-u} \int_{0}^{u}(u-s)^{\alpha-1} s^{\beta-1} d s d u \\
=\int_{0}^{\infty} e^{-u} \int_{0}^{1}(u-u v)^{\alpha-1}(u v)^{\beta-1} u d v d u \\
=\int_{0}^{\infty} e^{-u} \int_{0}^{1} u^{\alpha-1} u^{\beta}(1-v)^{\alpha-1} v^{\beta-1} d v d u \\
=\int_{0}^{1}(1-v)^{\alpha-1} v^{\beta-1} d v \int_{0}^{\infty} u^{\alpha+\beta-1} e^{-u} d u \\
=\Gamma(\alpha+\beta) \int_{0}^{1}(1-v)^{\alpha-1} v^{\beta-1} d v \square
\end{gathered}
$$

Now consider whether $A^{-\alpha}$ acts like it should. In particular, is $A^{-\alpha} A^{-(1-\alpha)}=A^{-1}$ ?

Lemma 19.13.6 For $\alpha \in(0,1), A^{-\alpha} A^{-(1-\alpha)}=A^{-1}$. More generally, if

$$
\alpha+\beta<1, A^{-\alpha} A^{-\beta}=A^{-(\alpha+\beta)}
$$

Proof: The product is the following where $\beta=1-\alpha$

$$
\frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} e^{-A t} t^{a-1} d t \frac{1}{\Gamma(\beta)} \int_{0}^{\infty} e^{-A s} s^{\beta-1} d s
$$

Then this equals

$$
\begin{aligned}
& \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_{0}^{\infty} \int_{0}^{\infty} e^{-A(t+s)} t^{\alpha-1} s^{\beta-1} d t d s \\
= & \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_{0}^{\infty} \int_{s}^{\infty} e^{-A u}(u-s)^{\alpha-1} s^{\beta-1} d u d s \\
= & \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_{0}^{\infty} e^{-A u} \int_{0}^{u}(u-s)^{\alpha-1} s^{\beta-1} d s d u \\
= & \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_{0}^{\infty} u^{\alpha+\beta-1} e^{-A u} \int_{0}^{1}(1-v)^{\alpha-1} v^{\beta-1} d v d u \\
= & \int_{0}^{1}(1-v)^{\alpha-1} v^{\beta-1} d v \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_{0}^{\infty} u^{\alpha+\beta-1} e^{-A u} d u
\end{aligned}
$$

From the above lemma, this equals

$$
\frac{1}{\Gamma(\alpha+\beta)} \int_{0}^{\infty} u^{\alpha+\beta-1} e^{-A u} d u \equiv A^{-(\alpha+\beta)}
$$

Note how this shows that these powers of $A$ all commute with each other. If $\alpha+\beta=1$, this becomes

$$
\int_{0}^{\infty} e^{-A u} d u
$$

Is this the usual inverse or is it something else called $A^{-1}$ but not being the real inverse? We show it is the usual inverse. To see this, consider

$$
A \int_{0}^{\infty} e^{-A u} d u\left(x_{0}\right)=\int_{0}^{\infty} A e^{-A u} x_{0} d u=\int_{0}^{\infty}-x^{\prime}(u) d u
$$

where $x^{\prime}(t)+A x(t)=0, x(0)=x_{0}$ and $|x(t)| \leq e^{-\varepsilon t}\left|x_{0}\right|$. The manipulation is routine from approximating with Riemann sums. Then the right side equals $x_{0}$. Thus $A \int_{0}^{\infty} e^{-A u} d u=I$. Similarly,

$$
\int_{0}^{\infty} e^{-A u} d u A\left(x_{0}\right)=\int_{0}^{\infty} e^{-A u} A x_{0} d u=\int_{0}^{\infty} A e^{-A u} x_{0} d u=x_{0}
$$

and so this shows the desired result.
Now it follows that if $A^{\alpha} \equiv A A^{-(1-\alpha)}$ as defined above, then $A^{1-\alpha}=A A^{-(1-(1-\alpha))}=$ $A A^{-\alpha}$ and so,

$$
A^{\alpha} A^{1-\alpha} \equiv A A^{-(1-\alpha)} A A^{-\alpha}=A^{2} A^{-1}=A
$$

Also, if $\alpha+\beta \leq 1$,

$$
\begin{aligned}
A^{\alpha} A^{\beta}= & A A^{-(1-\alpha)} A A^{-(1-\beta)}=A^{2} A^{-(1-\alpha)} A^{-(1-\beta)} \\
A^{\alpha+\beta} & \equiv A A^{-(1-(\alpha+\beta))}=A^{2} A^{-1} A^{-(1-(\alpha+\beta))} \\
& =A^{2} A^{-\beta} A^{-(1-\beta)} A^{-(1-(\alpha+\beta))} \\
& =A^{2} A^{-(1-\alpha)} A^{-(1-\beta)}=A^{\alpha} A^{\beta}
\end{aligned}
$$

This shows the following.
Lemma 19.13.7 If $\alpha, \beta \in(0,1), \alpha+\beta \leq 1$, then $A^{\alpha} A^{\beta}=A^{\alpha+\beta}$. Also $A^{\alpha}$ commutes with every operator in $\mathscr{L}(X, X)$ which commutes with $A$.

Proof: The last assertion follows right away from the fact noted above that $A^{-(1-\alpha)}$ commutes with all operators which commute with $A$ and that so does $A$. Thus if $C$ is such a commuting operator,

$$
C A^{\alpha}=C A A^{-(1-\alpha)}=A C A^{-(1-\alpha)}=A A^{-(1-\alpha)} C=A^{\alpha} C .
$$

The next task is to remove the assumption that $(A x, x) \geq \varepsilon|x|^{2}$ and replace it with $(A x, x) \geq 0$.

Observation 19.13.8 First note that if $\Phi(t)=e^{-(\varepsilon I+A) t}$, and if $x_{0}$ is given, then if $\Phi(t) x_{0}=$ $y(t)$,

$$
y^{\prime}(t)+(\varepsilon I+A) y(t)=0, y(0)=x_{0}
$$

Then taking inner product of both sides with $y(t)$ and integrating,

$$
\frac{|y(t)|^{2}}{2}-\frac{\left|x_{0}\right|^{2}}{2} \leq 0
$$

and so $\left|\Phi(t) x_{0}\right|=|y(t)| \leq\left|x_{0}\right|$ and so $\|\Phi(t)\| \leq 1$. This will be used in the following lemma.

Lemma 19.13.9 Let $(A x, x) \geq 0$. Then for $\alpha \in(0,1)$,

$$
\lim _{\varepsilon \rightarrow 0+}(\varepsilon I+A)(\varepsilon I+A)^{-\alpha}
$$

exists in $\mathscr{L}(X, X)$.
Proof: Let $\delta>\varepsilon$ and both $\varepsilon$ and $\delta$ are small.

$$
\begin{aligned}
& \left((\varepsilon I+A)(\varepsilon I+A)^{-\alpha}-(\delta I+A)(\delta I+A)^{-\alpha}\right) \Gamma(\alpha)= \\
& \int_{0}^{\infty}\left((\varepsilon I+A) e^{-(\varepsilon I+A) t}-(\delta I+A) e^{-(\delta I+A) t}\right) t^{\alpha-1} d t=
\end{aligned}
$$

$$
\begin{gathered}
=\int_{0}^{\infty}\left((\varepsilon I+A) e^{-(\varepsilon I+A) t}-(\varepsilon I+A) e^{-(\delta I+A) t}\right) t^{\alpha-1} d t \\
+\int_{0}^{\infty}\left((\varepsilon I+A) e^{-(\delta I+A) t}-(\delta I+A) e^{-(\delta I+A) t}\right) t^{\alpha-1} d t \\
=P_{\delta, \varepsilon}+Q_{\delta, \varepsilon} .
\end{gathered}
$$

Then $P_{\delta, \varepsilon}=$

$$
\begin{gathered}
\int_{0}^{\infty}(\varepsilon I+A)\left(e^{-(\varepsilon I+A) t}-e^{-(\delta I+A) t}\right) t^{\alpha-1} d t \\
\left\|P_{\delta, \varepsilon}\right\| \leq \int_{0}^{\infty}\|(\varepsilon I+A)\|\left\|\left(e^{-(\varepsilon I+A) t}-e^{-(\delta I+A) t}\right)\right\| t^{\alpha-1} d t
\end{gathered}
$$

We need to estimate the difference of those semigroups. Call the first, $e^{-(\varepsilon I+A) t} \equiv y$ and the second $x$. Then by definition,

$$
\begin{aligned}
y^{\prime}+(\varepsilon I+A) y & =0, y(0)=x_{0} \\
x^{\prime}+(\delta I+A) x & =0, x(0)=x_{0}
\end{aligned}
$$

Then $\hat{y}(t) \equiv e^{\varepsilon t} y(t), \hat{x}(t) \equiv e^{\delta t} x(t)$

$$
\begin{aligned}
\hat{y}^{\prime}-\varepsilon \hat{y}(t)+(\varepsilon I+A) \hat{y} & =0, \hat{y}(0)=x_{0} \\
\hat{x}^{\prime}-\delta \hat{x}(t)+(\delta I+A) \hat{x} & =0, \hat{x}(0)=x_{0}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\hat{y}^{\prime}+A \hat{y} & =0, \hat{y}(0)=x_{0} \\
\hat{x}^{\prime}+A \hat{x} & =0, \hat{x}(0)=x_{0}
\end{aligned}
$$

By uniqueness, $\hat{x}=\hat{y}$. Thus

$$
\left|e^{\varepsilon t} y(t)-e^{\delta t} x(t)\right|=0
$$

So

$$
e^{\delta t}\left|e^{(\varepsilon-\delta) t} y(t)-x(t)\right|=0
$$

So, since $x_{0}$ was arbitrary,

$$
e^{(\varepsilon-\delta) t} e^{-(\varepsilon I+A) t}-e^{-(\delta I+A) t}=0
$$

Then

$$
\begin{gathered}
\left\|P_{\delta, \varepsilon}\right\| \leq C(\|A\|) \int_{0}^{\infty}\left\|\left(e^{-(\varepsilon I+A) t}-e^{(\varepsilon-\delta) t} e^{-(\varepsilon I+A) t}\right)\right\| t^{\alpha-1} d t \\
\leq C(\|A\|) \int_{0}^{\infty}\left|1-e^{(\varepsilon-\delta) t}\right| t^{\alpha-1} d t=C(\|A\|) \int_{0}^{\infty}\left(1-e^{-(\delta-\varepsilon) t}\right) t^{\alpha-1} d t
\end{gathered}
$$

Now we need an estimate. Suppose $1>\lambda>0$. For $t \geq 0$, is

$$
1-e^{-\lambda t} \leq \lambda e^{-\lambda t} ?
$$

Let $f(t)=\lambda e^{-\lambda t}-e^{-\lambda t}+1$. Then

$$
f^{\prime}(t)=-\lambda^{2} e^{-\lambda t}+\lambda e^{-\lambda t}>0, f(0)=\lambda-1+1>0
$$

Thus

$$
\left\|P_{\delta, \varepsilon}\right\| \leq C(\|A\|) \int_{0}^{\infty}(\delta-\varepsilon) e^{-(\delta-\varepsilon) t} t^{\alpha-1} d t
$$

Now change the variables letting $u=(\delta-\varepsilon) t$. Then

$$
\left\|P_{\delta, \varepsilon}\right\|=C(\|A\|) \int_{0}^{\infty} e^{-u}\left(\frac{u}{\delta-\varepsilon}\right)^{\alpha-1} d u=C(\|A\|) \Gamma(\alpha)|\delta-\varepsilon|^{1-\alpha}
$$

Thus $\lim _{\varepsilon, \delta \rightarrow 0}\left\|P_{\delta \varepsilon}\right\|=0$. Consider $Q_{\delta, \varepsilon}$

$$
\begin{aligned}
\left\|Q_{\delta, \varepsilon}\right\| & \leq \int_{0}^{\infty}\left\|e^{-(\delta I+A) t}\right\|\|((\varepsilon I+A)-(\delta I+A))\| t^{\alpha-1} d t \\
& \leq \int_{0}^{\infty} e^{-\delta t}|\delta-\varepsilon| t^{\alpha-1} d t \leq \int_{0}^{\infty} e^{-(\delta-\varepsilon) t}|\varepsilon-\delta| t^{\alpha-1} d t
\end{aligned}
$$

Now let $u=(\delta-\varepsilon) t, d u=(\delta-\varepsilon) d t$. Then the last integral on the right equals

$$
|\varepsilon-\delta| \int_{0}^{\infty} e^{-u} u^{\alpha-1} \frac{d u}{|\delta-\varepsilon|} \frac{1}{|\delta-\varepsilon|^{\alpha-1}}=\Gamma(\alpha)|\delta-\varepsilon|^{1-\alpha}
$$

so also $\lim _{\varepsilon, \delta \rightarrow 0}\left\|Q_{\delta, \varepsilon}\right\|=0$.
Definition 19.13.10 For $\alpha \in(0,1)$, and $(A x, x) \geq 0$ with $A=A^{*}$, we define

$$
A^{\alpha} \equiv \lim _{\varepsilon \rightarrow 0}(\varepsilon I+A)(\varepsilon I+A)^{-(1-\alpha)}
$$

Theorem 19.13.11 In the situation of the definition, if $\alpha+\beta \leq 1$, for $\alpha, \beta \in(0,1)$,

$$
A^{\alpha} A^{\beta}=A^{\alpha+\beta}
$$

and in particular,

$$
A^{\alpha} A^{1-\alpha}=A
$$

Also, $A^{\alpha}$ commutes with every operator which commutes with A. For A a Hermitian operator as here, it follows that $A^{\alpha}$ is also Hermitian.

Proof: $A^{\alpha+\beta} \equiv \lim _{\varepsilon \rightarrow 0}(\varepsilon I+A)(\varepsilon I+A)^{-(1-(\alpha+\beta))}$. Then since $(\varepsilon I+A)$ commutes with $e^{-(\varepsilon I+A) t}$, it follows that this equals

$$
\begin{gathered}
\lim _{\varepsilon \rightarrow 0}(\varepsilon I+A)(\varepsilon I+A)^{-(1-(\alpha+\beta))}= \\
\lim _{\varepsilon \rightarrow 0}(\varepsilon I+A)^{2}(\varepsilon I+A)^{-(1-\alpha)}(\varepsilon I+A)^{-(1-\beta)}
\end{gathered}
$$

$$
=\lim _{\varepsilon \rightarrow 0}(\varepsilon I+A)(\varepsilon I+A)^{-(1-\alpha)}(\varepsilon I+A)(\varepsilon I+A)^{-(1-\beta)}=A^{\alpha} A^{\beta}
$$

There is nothing to show if $\alpha+\beta=1$. In this case, the limit reduces to $\lim _{\varepsilon \rightarrow 0}(\varepsilon I+A)=A$.
Consider the last claim. Let $C$ be such a commuting operator. and let $A_{\varepsilon} \equiv \varepsilon I+A$. Then

$$
C A^{\alpha}=\lim _{\varepsilon \rightarrow 0} C A_{\varepsilon}^{\alpha}=\lim _{\varepsilon \rightarrow 0} A_{\varepsilon}^{\alpha} C=A^{\alpha} C
$$

Note that $A^{\alpha} \equiv \lim _{\varepsilon \rightarrow 0}(\varepsilon I+A)(\varepsilon I+A)^{-(1-\alpha)} \equiv \lim _{\varepsilon \rightarrow 0} A_{\varepsilon}^{\alpha}$. Finally, consider the claim about $A^{\alpha}$ being Hermitian when $A$ is.

$$
\left(x, A^{\alpha} y\right)=\lim _{\varepsilon \rightarrow 0}\left(x, A_{\varepsilon}^{\alpha} y\right)=\lim _{\varepsilon \rightarrow 0}\left(A_{\varepsilon}^{\alpha} x, y\right)=\left(A^{\alpha} x, y\right)
$$

For more on this kind of thing including generalizations to operators defined on Banach space, see [79].

### 19.14 General Theory of Continuous Semigroups

Much more on semigroups is available in Yosida [127]. This is just an introduction to the subject.
Definition 19.14.1 A strongly continuous semigroup defined on $X$, a Banach space is a function $S:[0, \infty) \rightarrow X$ which satisfies the following for all $x_{0} \in X$.

$$
\begin{aligned}
S(t) & \in \mathscr{L}(X, X), S(t+s)=S(t) S(s) \\
t & \rightarrow S(t) x_{0} \text { is continuous, } \lim _{t \rightarrow 0+} S(t) x_{0}=x_{0}
\end{aligned}
$$

Sometimes such a semigroup is said to be $C_{0}$. It is said to have the linear operator $A$ as its generator if

$$
D(A) \equiv\left\{x: \lim _{h \rightarrow 0} \frac{S(h) x-x}{h} \text { exists }\right\}
$$

and for $x \in D(A), A$ is defined by

$$
\lim _{h \rightarrow 0} \frac{S(h) x-x}{h} \equiv A x
$$

The assertion that $t \rightarrow S(t) x_{0}$ is continuous and that $S(t) \in \mathscr{L}(X, X)$ is not sufficient to say there is a bound on $\|S(t)\|$ for all $t \geq 0$. Also the assertion that for each $x_{0}, \lim _{t \rightarrow 0+} S(t) x_{0}=x_{0}$ is not the same as saying that $S(t) \rightarrow I$ in $\mathscr{L}(X, X)$. It is a much weaker assertion. The next theorem gives information on the growth of $\|S(t)\|$. It turns out it has exponential growth.

Lemma 19.14.2 Let $M \equiv \sup \{\|S(t)\|: t \in[0, T]\}$. Then $M<\infty$.
Proof: If this is not true, then there exists $t_{n} \in[0, T]$ such that $\left\|S\left(t_{n}\right)\right\| \geq n$. That is the operators $S\left(t_{n}\right)$ are not uniformly bounded. By the uniform boundedness principle, Theorem 17.1.8, there exists $x \in X$ such that $\left\|S\left(t_{n}\right) x\right\|$ is not bounded. However, this is impossible because it is given that $t \rightarrow S(t) x$ is continuous on $[0, T]$ and so $t \rightarrow\|S(t) x\|$ must achieve its maximum on this compact set.

Now here is the main result for growth of $\|S(t)\|$.

Theorem 19.14.3 For $M$ described in Lemma 19.14.2, there exists $\alpha$ such that

$$
\|S(t)\| \leq M e^{\alpha t}, t \geq 0
$$

In fact, $\alpha$ can be chosen such that $M^{1 / T}=e^{\alpha}$.
Proof: Let $t$ be arbitrary. Then $t=m T+r(t)$ where $0 \leq r(t)<T$. Then by the semigroup property

$$
\|S(t)\|=\|S(m T+r(t))\|=\left\|S(r(t)) S(T)^{m}\right\| \leq M^{m+1}
$$

Now $m T \leq t \leq m T+r(t) \leq(m+1) T$ and so $m \leq \frac{t}{T} \leq m+1$. Therefore,

$$
\|S(t)\| \leq M^{(t / T)+1}=M\left(M^{1 / T}\right)^{t}
$$

Let $M^{1 / T} \equiv e^{\alpha}$ and then $\|S(t)\| \leq M e^{\alpha t}$
Definition 19.14.4 Let $S(t)$ be a continuous semigroup as described above. It is called a contraction semigroup if for all $t \geq 0$

$$
\|S(t)\| \leq 1
$$

It is called a bounded semigroup if there exists $M$ such that for all $t \geq 0$,

$$
\|S(t)\| \leq M
$$

Note that for $S(t)$ an arbitrary continuous semigroup satisfying $\|S(t)\| \leq M e^{\alpha t}$, It follows that the semigroup, $T(t)=e^{-\alpha t} S(t)$ is a bounded semigroup which satisfies $\|T(t)\| \leq$ $M$.

The next proposition has to do with taking a Laplace transform of a semigroup.
Proposition 19.14.5 Given a continuous semigroup $S(t)$, its generator A exists and is a closed densely defined operator. Furthermore, for

$$
\|S(t)\| \leq M e^{\alpha t}
$$

and $\lambda>\alpha, \lambda I-A$ is one to one and onto from $D(A)$ to $X$. Also $(\lambda I-A)^{-1}$ maps $X$ onto $D(A)$ and is in $\mathscr{L}(X, X)$. Also for these values of $\lambda>\alpha$,

$$
(\lambda I-A)^{-1} x=\int_{0}^{\infty} e^{-\lambda t} S(t) x d t
$$

For $\lambda>\alpha$, the following estimate holds.

$$
\begin{equation*}
\left\|(\lambda I-A)^{-1}\right\| \leq \frac{M}{|\lambda-\alpha|} \tag{19.14.79}
\end{equation*}
$$

Proof: First note $D(A) \neq \emptyset$. In fact $0 \in D(A)$. It follows from Theorem 19.14.3 that for all $\lambda$ larger than $\alpha$, one can define a Laplace transform, $R(\lambda) x \equiv \int_{0}^{\infty} e^{-\lambda t} S(t) x d t \in X$. Here the integral is the ordinary improper Riemann integral. I claim each of these $R(\lambda) x$ for $\lambda$ large is in $D(A)$.

$$
\frac{S(h) \int_{0}^{\infty} e^{-\lambda t} S(t) x d t-\int_{0}^{\infty} e^{-\lambda t} S(t) x d t}{h}
$$

Using the semigroup property and changing the variables in the first of the above integrals, this equals

$$
\begin{gathered}
=\frac{1}{h}\left(e^{\lambda h} \int_{h}^{\infty} e^{-\lambda t} S(t) x d t-\int_{0}^{\infty} e^{-\lambda t} S(t) x d t\right) \\
=\frac{1}{h}\left(\left(e^{\lambda h}-1\right) \int_{0}^{\infty} e^{-\lambda t} S(t) x d t-e^{\lambda h} \int_{0}^{h} e^{-\lambda t} S(t) x d t\right)
\end{gathered}
$$

Then it follows that the limit as $h \rightarrow 0$ exists and equals

$$
\begin{equation*}
\lambda R(\lambda) x-x=\lim _{h \rightarrow 0+} \frac{S(h) R(\lambda) x-R(\lambda) x}{h} \equiv A(R(\lambda) x) \tag{19.14.80}
\end{equation*}
$$

and $R(\lambda) x \in D(A)$ as claimed. Hence

$$
\begin{equation*}
x=(\lambda I-A) R(\lambda) x . \tag{19.14.81}
\end{equation*}
$$

Since $x$ is arbitrary, this shows that for $\lambda>\alpha, \lambda I-A$ is onto. Also, if $x \in D(A)$, you could approximate with Riemann sums and pass to a limit and obtain

$$
\frac{1}{h}(R(\lambda) S(h) x-R(\lambda) x)=\int_{0}^{\infty} e^{-\lambda t} S(t) \frac{S(h) x-x}{h} d t
$$

Then, passing to a limit as $h \rightarrow 0$ using the dominated convergence theorem, one obtains

$$
\lim _{h \rightarrow 0} \frac{1}{h}(R(\lambda) S(h) x-R(\lambda) x)=\int_{0}^{\infty} S(t) e^{-\lambda t} A x d t \equiv R(\lambda) A x
$$

Also, $S(h)$ commutes with $R(\lambda)$. This follows in the usual way by approximating with Riemann sums and taking a limit. Thus for $x \in D(A)$

$$
\begin{aligned}
\lambda R(\lambda) x-x & =\lim _{h \rightarrow 0+} \frac{S(h) R(\lambda) x-R(\lambda) x}{h} \\
& =\lim _{h \rightarrow 0+} \frac{R(\lambda) S(h) x-R(\lambda) x}{h}=R(\lambda) A x
\end{aligned}
$$

and so, for $x \in D(A)$,

$$
\begin{equation*}
x=\lambda R(\lambda) x-R(\lambda) A x=R(\lambda)(\lambda I-A) x \tag{19.14.82}
\end{equation*}
$$

which shows that for $\lambda>\alpha,(\lambda I-A)$ is one to one on $D(A)$. Hence from 19.14.80 and 19.14.82, $(\lambda I-A)$ is an algebraic isomorphism from $D(A)$ onto $X$. Also $R(\lambda)=$
$(\lambda I-A)^{-1}$ on $X$. The estimate 19.14.79 follows easily from the definition of $R(\lambda)$ whenever $\lambda>\alpha$ as follows.

$$
\begin{gathered}
\|R(\lambda) x\|=\left\|(\lambda I-A)^{-1} x\right\|=\left\|\int_{0}^{\infty} e^{-\lambda t} S(t) x d t\right\| \\
\leq \int_{0}^{\infty} e^{-\lambda t} M e^{\alpha t} d t\|x\| \leq \frac{M}{|\lambda-\alpha|}\|x\|
\end{gathered}
$$

Why is $D(A)$ dense? I will show that $\|\lambda R(\lambda) x-x\| \rightarrow 0$ as $\lambda \rightarrow \infty$, and it was shown above that $R(\lambda) x$ and therefore $\lambda R(\lambda) x \in D(A)$ so this will show that $D(A)$ is dense in $X$. For $\lambda>\alpha$ where $\|S(t)\| \leq M e^{\alpha t}$,

$$
\begin{aligned}
&\|\lambda R(\lambda) x-x\|=\left\|\int_{0}^{\infty} \lambda e^{-\lambda t} S(t) x d t-\int_{0}^{\infty} \lambda e^{-\lambda t} x d t\right\| \\
& \leq \int_{0}^{\infty}\left\|\lambda e^{-\lambda t}(S(t) x-x)\right\| d t \\
&= \int_{0}^{h}\left\|\lambda e^{-\lambda t}(S(t) x-x)\right\| d t+\int_{h}^{\infty}\left\|\lambda e^{-\lambda t}(S(t) x-x)\right\| d t \\
& \leq \int_{0}^{h}\left\|\lambda e^{-\lambda t}(S(t) x-x)\right\| d t+\int_{h}^{\infty} \lambda e^{-(\lambda-\alpha) t} d t(M+1)\|x\|
\end{aligned}
$$

Now since $S(t) x-x \rightarrow 0$, it follows that for $h$ sufficiently small

$$
\begin{aligned}
& \leq \frac{\varepsilon}{2} \int_{0}^{h} \lambda e^{-\lambda t} d t+\frac{\lambda}{\lambda-\alpha} e^{-(\lambda-\alpha) h}(M+1)\|x\| \\
& \leq \frac{\varepsilon}{2}+\frac{\lambda}{\lambda-\alpha} e^{-(\lambda-\alpha) h}(M+1)\|x\|<\varepsilon
\end{aligned}
$$

whenever $\lambda$ is large enough. Thus $D(A)$ is dense as claimed.
Why is $A$ a closed operator? Suppose $x_{n} \rightarrow x$ where $x_{n} \in D(A)$ and that $A x_{n} \rightarrow \xi$. I need to show that this implies that $x \in D(A)$ and that $A x=\xi$. Thus $x_{n} \rightarrow x$ and for $\lambda>\alpha$, $(\lambda I-A) x_{n} \rightarrow \lambda x-\xi$. However, 19.14.79 shows that $(\lambda I-A)^{-1}$ is continuous and so

$$
x_{n} \rightarrow(\lambda I-A)^{-1}(\lambda x-\xi)=x
$$

It follows that $x \in D(A)$. Then doing $(\lambda I-A)$ to both sides of the equation, $\lambda x-\xi=$ $\lambda x-A x$ and so $A x=\xi$ showing that $A$ is a closed operator as claimed.

Definition 19.14.6 The linear mapping for $\lambda>\alpha$ where $\|S(t)\| \leq M e^{\alpha t}$ given by

$$
(\lambda I-A)^{-1}=R(\lambda)
$$

is called the resolvent.
The following corollary is also very interesting.

Corollary 19.14.7 Let $S(t)$ be a continuous semigroup and let $A$ be its generator. Then for $0<a<b$ and $x \in D(A)$

$$
S(b) x-S(a) x=\int_{a}^{b} S(t) A x d t
$$

and also for $t>0$ you can take the derivative from the left,

$$
\lim _{h \rightarrow 0+} \frac{S(t) x-S(t-h) x}{h}=S(t) A x
$$

Proof:Letting $y^{*} \in X^{\prime}$,

$$
y^{*}\left(\int_{a}^{b} S(t) A x d t\right)=\int_{a}^{b} y^{*}\left(S(t) \lim _{h \rightarrow 0} \frac{S(h) x-x}{h}\right) d t
$$

The difference quotients are bounded because they converge to $A x$. Therefore, from the dominated convergence theorem,

$$
\begin{aligned}
y^{*}\left(\int_{a}^{b} S(t) A x d t\right) & =\lim _{h \rightarrow 0} \int_{a}^{b} y^{*}\left(S(t) \frac{S(h) x-x}{h}\right) d t \\
& =\lim _{h \rightarrow 0} y^{*}\left(\int_{a}^{b} S(t) \frac{S(h) x-x}{h} d t\right) \\
& =\lim _{h \rightarrow 0} y^{*}\left(\frac{1}{h} \int_{a+h}^{b+h} S(t) x d t-\frac{1}{h} \int_{a}^{b} S(t) x d t\right) \\
& =\lim _{h \rightarrow 0} y^{*}\left(\frac{1}{h} \int_{b}^{b+h} S(t) x d t-\frac{1}{h} \int_{a}^{a+h} S(t) x d t\right) \\
& =y^{*}(S(b) x-S(a) x)
\end{aligned}
$$

Since $y^{*}$ is arbitrary, this proves the first part. Now from what was just shown, if $t>0$ and $h$ is small enough,

$$
\frac{S(t) x-S(t-h) x}{h}=\frac{1}{h} \int_{t-h}^{t} S(s) A x d s
$$

which converges to $S(t) A x$ as $h \rightarrow 0+$. This proves the corollary.
Given a closed densely defined operator, when is it the generator of a continuous semigroup? This is answered in the following theorem which is called the Hille Yosida theorem. It concerns the case of a bounded semigroup. However, if you have an arbitrary continuous semigroup, $S(t)$, then it was shown above that $S(t) e^{-\alpha t}$ is bounded for suitable $\alpha$ so the case discussed below is obtained.

Theorem 19.14.8 Suppose $A$ is a densely defined linear operator which has the property that for all $\lambda>0$,

$$
(\lambda I-A)^{-1} \in \mathscr{L}(X, X)
$$

which means that $\lambda I-A: D(A) \rightarrow X$ is one to one and onto with continuous inverse. Suppose also that for all $n \in \mathbb{N}$,

$$
\begin{equation*}
\left\|\left((\lambda I-A)^{-1}\right)^{n}\right\| \leq \frac{M}{\lambda^{n}} \tag{19.14.83}
\end{equation*}
$$

Then there exists a continuous semigroup $S(t)$ which has $A$ as its generator and satisfies $\|S(t)\| \leq M$ and $A$ is closed. In fact letting

$$
S_{\lambda}(t) \equiv \exp \left(-\lambda+\lambda^{2}(\lambda I-A)^{-1}\right)
$$

it follows $\lim _{\lambda \rightarrow \infty} S_{\lambda}(t) x=S(t) x$ uniformly on finite intervals. Conversely, if $A$ is the generator of $S(t)$, a bounded continuous semigroup having $\|S(t)\| \leq M$, then $(\lambda I-A)^{-1} \in$ $\mathscr{L}(X, X)$ for all $\lambda>0$ and 19.14.83 holds.

Proof: The condition 19.14.83 implies, that

$$
\left\|(\lambda I-A)^{-1}\right\| \leq \frac{M}{\lambda}
$$

Consider, for $\lambda>0$, the operator which is defined on $D(A)$,

$$
\lambda(\lambda I-A)^{-1} A
$$

On $D(A)$, this equals

$$
\begin{equation*}
-\lambda I+\lambda^{2}(\lambda I-A)^{-1} \tag{19.14.84}
\end{equation*}
$$

because

$$
\begin{aligned}
(\lambda I-A) \lambda(\lambda I-A)^{-1} A & =\lambda A \\
(\lambda I-A)\left(-\lambda I+\lambda^{2}(\lambda I-A)^{-1}\right) & =-\lambda(\lambda I-A)+\lambda^{2}=\lambda A
\end{aligned}
$$

and, by assumption, $(\lambda I-A)$ is one to one so these are the same. However, the second one in 19.14.84, $-\lambda I+\lambda^{2}(\lambda I-A)^{-1}$ makes sense on all of $X$. Also

$$
\begin{gathered}
\left(-\lambda I+\lambda^{2}(\lambda I-A)^{-1}\right)(\lambda I-A)=-\lambda(\lambda I-A)+\lambda^{2} I=\lambda A \\
\lambda A(\lambda I-A)^{-1}(\lambda I-A)=\lambda A
\end{gathered}
$$

so, since $(\lambda I-A)$ is onto, it follows that on $X$,

$$
-\lambda I+\lambda^{2}(\lambda I-A)^{-1}=A \lambda(\lambda I-A)^{-1} \equiv A_{\lambda}
$$

Denote this as $A_{\lambda}$ to save notation. Thus on $D(A)$,

$$
\lambda A(\lambda I-A)^{-1}=\lambda(\lambda I-A)^{-1} A=A_{\lambda}
$$

although the $\lambda(\lambda I-A)^{-1} A$ only makes sense on $D(A)$. Note that formally

$$
\lim _{\lambda \rightarrow \infty} \lambda(\lambda I-A)^{-1} A=A
$$

This is summarized next.

Lemma 19.14.9 There is a bounded linear operator given for $\lambda>0$ by

$$
-\lambda I+\lambda^{2}(\lambda I-A)^{-1}=\lambda A(\lambda I-A)^{-1} \equiv A_{\lambda}
$$

On $D(A), A_{\lambda}=\lambda(\lambda I-A)^{-1} A$.
For $x \in D(A)$,

$$
\begin{align*}
& \left\|\lambda(\lambda I-A)^{-1} x-x\right\| \\
= & \left\|(\lambda I-A)^{-1}(\lambda x-(\lambda I-A) x)\right\| \\
= & \left\|(\lambda I-A)^{-1} A x\right\| \leq \frac{M}{\lambda}\|A x\| \tag{19.14.85}
\end{align*}
$$

which converges to 0 as $\lambda \rightarrow \infty$.
Now $L_{\lambda} x \rightarrow x$ on a dense subset of $X, L_{\lambda} \equiv \lambda(\lambda I-A)^{-1}$. Also, from the hypothesis, $\left\|L_{\lambda}\right\| \leq M$. Say $x$ is arbitrary. Then does $L_{\lambda} x \rightarrow x$ ? Let $\hat{x} \in D(A)$ and $\|x-\hat{x}\|<\varepsilon$. Then

$$
\begin{aligned}
\left\|L_{\lambda} x-x\right\| & \leq\left\|L_{\lambda} x-L_{\lambda} \hat{x}\right\|+\left\|L_{\lambda} \hat{x}-\hat{x}\right\|+\|\hat{x}-x\| \\
& <M \varepsilon+\varepsilon+\varepsilon
\end{aligned}
$$

whenever $\lambda$ is large enough and so for all $x \in X$,

$$
\lim _{\lambda \rightarrow \infty} \lambda(\lambda I-A)^{-1} x=x
$$

In particular, this holds whenever $x$ is replaced with $A x$ for some $x \in D(A)$. Thus if $x \in$ $D(A)$,

$$
\begin{equation*}
\lim _{\lambda \rightarrow \infty}\left\|A_{\lambda} x-A x\right\|=\lim _{\lambda \rightarrow \infty}\left\|\lambda(\lambda I-A)^{-1} A x-A x\right\|=0 \tag{19.14.86}
\end{equation*}
$$

This is summarized in the following lemma.
Lemma 19.14.10 For all $x \in D(A), \lim _{\lambda \rightarrow \infty}\left\|A_{\lambda} x-A x\right\|=0$.
Now from Corollary 19.12.5, there exists an approximate continuous semigroup $S_{\lambda}(t)$ generated by $A_{\lambda}$ which is the solution to

$$
\begin{equation*}
S_{\lambda}^{\prime}(t)=A_{\lambda} S_{\lambda}(t), S_{\lambda}(0)=I \tag{19.14.87}
\end{equation*}
$$

In terms of power series,

$$
\begin{equation*}
S_{\lambda}(t) \equiv e^{-\lambda t} \sum_{k=0}^{\infty} \frac{t^{k}\left(\lambda^{2}(\lambda I-A)^{-1}\right)^{k}}{k!}=e^{t\left(-\lambda I+\lambda^{2}(\lambda I-A)^{-1}\right)} \tag{19.14.88}
\end{equation*}
$$

Thus, by assumption and triangle inequality,

$$
\begin{equation*}
\left\|S_{\lambda}(t)\right\| \leq e^{-\lambda t} \sum_{k=0}^{\infty} \frac{t^{k}}{k!} \lambda^{2 k} \frac{M}{\lambda^{k}}=e^{-\lambda t} M e^{\lambda t}=M \tag{19.14.89}
\end{equation*}
$$

Note that

$$
\frac{t^{k}\left(\lambda^{2}(\lambda I-A)^{-1}\right)^{k}}{k!} \leq \frac{t^{k}}{k!} \lambda^{2 k} \frac{M}{\lambda^{k}}
$$

so one obtains absolute convergence in 19.14.88.
Next is an easy observation about operators commuting.
Lemma 19.14.11 For $\lambda, \mu>0,(\lambda I-A)^{-1}$ and $(\mu I-A)^{-1}$ commute.
Proof: Suppose

$$
\begin{align*}
& y=(\mu I-A)^{-1}(\lambda I-A)^{-1} x  \tag{19.14.90}\\
& z=(\lambda I-A)^{-1}(\mu I-A)^{-1} x \tag{19.14.91}
\end{align*}
$$

I need to show $y=z$. First note $z, y \in D(A)$. Then also $(\mu I-A) y \in D(A)$ and $(\lambda I-A) z \in$ $D(A)$ and so the following manipulation makes sense.

$$
\begin{aligned}
x & =(\lambda I-A)(\mu I-A) y=(\mu I-A)(\lambda I-A) y \\
x & =(\mu I-A)(\lambda I-A) z
\end{aligned}
$$

so $(\mu I-A)(\lambda I-A) y=(\mu I-A)(\lambda I-A) z$ and since $(\mu I-A),(\lambda I-A)$ are both one to one, this shows $z=y$.

It follows from the description of $S_{\lambda}(t)$ in terms of a power series that $S_{\lambda}(t)$ and $S_{\mu}(s)$ commute and also $A_{\lambda}$ commutes with $S_{\mu}(t)$ for any $t$. One could also exploit uniqueness and the theory of ordinary differential equations to verify this. I will use this fact in what follows whenever needed.

I want to show that for each $x \in D(A)$,

$$
\lim _{\lambda \rightarrow \infty} S_{\lambda}(t) x \equiv S(t) x
$$

where $S(t)$ is the desired semigroup. Let $x \in D(A)$. Then

$$
\begin{aligned}
& S_{\mu}(t) x-S_{\lambda}(t) x=\int_{0}^{t} \frac{d}{d r}\left(S_{\lambda}(t-r) S_{\mu}(r)\right) x d r \\
= & \int_{0}^{t}\left(-S_{\lambda}^{\prime}(t-r) S_{\mu}(r)+S_{\lambda}(t-r) S_{\mu}^{\prime}(r)\right) x d r \\
= & \int_{0}^{t}\left(S_{\lambda}(t-r) S_{\mu}(r) A_{\lambda}-S_{\mu}(r) S_{\lambda}(t-r) A_{\mu}\right) x d r \\
= & \int_{0}^{t} S_{\lambda}(t-r) S_{\mu}(r)\left(A_{\mu} x-A_{\lambda} x\right) d r
\end{aligned}
$$

It follows that

$$
\left\|S_{\mu}(t) x-S_{\lambda}(t) x\right\| \leq \int_{0}^{t}\left\|S_{\lambda}(t-r) S_{\mu}(r)\left(A_{\mu} x-A_{\lambda} x\right)\right\| d r
$$

$$
\leq M^{2} t\left\|A_{\mu} x-A_{\lambda} x\right\| \leq M^{2} t\left(\left\|A_{\mu} x-A x\right\|+\left\|A x-A_{\lambda} x\right\|\right)
$$

Now by Lemma 19.14.10, the right side converges uniformly to 0 in $t \in[0, T]$ an arbitrary finite interval. Denote that to which it converges $S(t) x$. Therefore, $t \rightarrow S(t) x$ is continuous for each $x \in D(A)$ and also from 19.14.89,

$$
\|S(t) x\|=\lim _{\lambda \rightarrow \infty}\left\|S_{\lambda}(t) x\right\| \leq M\|x\|
$$

so that $S(t)$ can be extended to a continuous linear map, still called $S(t)$ defined on all of $X$ which also satisfies $\|S(t)\| \leq M$ since $D(A)$ is dense in $X$.

If $x$ is arbitrary, let $y \in D(A)$ be close to $x$, close enough that $2 M\|x-y\|<\varepsilon$. Then

$$
\begin{aligned}
\left\|S(t) x-S_{\lambda}(t) x\right\| \leq & \|S(t) x-S(t) y\|+\left\|S(t) y-S_{\lambda}(t) y\right\| \\
& +\left\|S_{\lambda}(t) y-S_{\lambda}(t) x\right\|
\end{aligned}
$$

if $\lambda$ is large enough, and so $\lim _{\lambda \rightarrow \infty} S_{\lambda}(t) x=S(t) x$ for all $x$, uniformly on finite intervals. Thus $t \rightarrow S(t) x$ is continuous for any $x \in X$.

It remains to verify $A$ generates $S(t)$ and for all $x, \lim _{t \rightarrow 0+} S(t) x-x=0$. From the above,

$$
\begin{equation*}
S_{\lambda}(t) x=x+\int_{0}^{t} S_{\lambda}(s) A_{\lambda} x d s \tag{19.14.92}
\end{equation*}
$$

and so

$$
\lim _{t \rightarrow 0+}\left\|S_{\lambda}(t) x-x\right\|=0
$$

By the uniform convergence just shown, there exists $\lambda$ large enough that for all $t \in[0, \delta]$,

$$
\left\|S(t) x-S_{\lambda}(t) x\right\|<\varepsilon
$$

Then

$$
\begin{aligned}
{\lim \sup _{t \rightarrow 0+}\|S(t) x-x\|}^{\leq}{\lim \sup _{t \rightarrow 0+}\left(\left\|S(t) x-S_{\lambda}(t) x\right\|+\left\|S_{\lambda}(t) x-x\right\|\right)} & \leq{\lim \sup _{t \rightarrow 0+}}\left(\varepsilon+\left\|S_{\lambda}(t) x-x\right\|\right) \leq \varepsilon
\end{aligned}
$$

It follows $\lim _{t \rightarrow 0+} S(t) x=x$ because $\varepsilon$ is arbitrary.
Next, $\lim _{\lambda \rightarrow \infty} A_{\lambda} x=A x$ for all $x \in D(A)$ by Lemma 19.14.10. Therefore, passing to the limit in 19.14.92 yields from the uniform convergence

$$
S(t) x=x+\int_{0}^{t} S(s) A x d s
$$

and by continuity of $s \rightarrow S(s) A x$, it follows

$$
\lim _{h \rightarrow 0+} \frac{S(h) x-x}{h}=\lim _{h \rightarrow 0} \frac{1}{h} \int_{0}^{h} S(s) A x d s=A x
$$

Thus letting $B$ denote the generator of $S(t), D(A) \subseteq D(B)$ and $A=B$ on $D(A)$. It only remains to verify $D(A)=D(B)$.

To do this, let $\lambda>0$ and consider the following where $y \in X$ is arbitrary.

$$
(\lambda I-B)^{-1} y=(\lambda I-B)^{-1}\left((\lambda I-A)(\lambda I-A)^{-1} y\right)
$$

Now $(\lambda I-A)^{-1} y \in D(A) \subseteq D(B)$ and $A=B$ on $D(A)$ and so

$$
(\lambda I-A)(\lambda I-A)^{-1} y=(\lambda I-B)(\lambda I-A)^{-1} y
$$

which implies,

$$
\begin{gathered}
(\lambda I-B)^{-1} y= \\
(\lambda I-B)^{-1}\left((\lambda I-B)(\lambda I-A)^{-1} y\right)=(\lambda I-A)^{-1} y
\end{gathered}
$$

Recall from Proposition 19.14.5, an arbitrary element of $D(B)$ is of the form $(\lambda I-B)^{-1} y$ and this has shown every such vector is in $D(A)$, in fact it equals $(\lambda I-A)^{-1} y$. Hence $D(B) \subseteq D(A)$ which shows $A$ generates $S(t)$ and this proves the first half of the theorem.

Next suppose $A$ is the generator of a semigroup $S(t)$ having $\|S(t)\| \leq M$. Then by Proposition 19.14.5 for all $\lambda>0,(\lambda I-A)$ is onto and

$$
(\lambda I-A)^{-1}=\int_{0}^{\infty} e^{-\lambda t} S(t) d t
$$

thus

$$
\begin{aligned}
& \left\|\left((\lambda I-A)^{-1}\right)^{n}\right\| \\
= & \left\|\int_{0}^{\infty} \cdots \int_{0}^{\infty} e^{-\lambda\left(t_{1}+\cdots+t_{n}\right)} S\left(t_{1}+\cdots+t_{n}\right) d t_{1} \cdots d t_{n}\right\| \\
\leq & \int_{0}^{\infty} \cdots \int_{0}^{\infty} e^{-\lambda\left(t_{1}+\cdots+t_{n}\right)} M d t_{1} \cdots d t_{n}=\frac{M}{\lambda^{n}} .
\end{aligned}
$$

### 19.14.1 An Evolution Equation

When $\Lambda$ generates a continuous semigroup, one can consider a very interesting theorem about evolution equations of the form

$$
y^{\prime}-\Lambda y=g(t)
$$

provided $t \rightarrow g(t)$ is $C^{1}$.
Theorem 19.14.12 Let $\Lambda$ be the generator of $S(t)$, a continuous semigroup on $X$, a Banach space and let $t \rightarrow g(t)$ be in $C^{1}(0, \infty ; X)$. Then there exists a unique solution to the initial value problem

$$
y^{\prime}-\Lambda y=g, y(0)=y_{0} \in D(\Lambda)
$$

and it is given by

$$
\begin{equation*}
y(t)=S(t) y_{0}+\int_{0}^{t} S(t-s) g(s) d s \tag{19.14.93}
\end{equation*}
$$

This solution is continuous having continuous derivative and has values in $D(\Lambda)$.

Proof: First I show the following claim.
Claim: $\int_{0}^{t} S(t-s) g(s) d s \in D(\Lambda)$ and

$$
\Lambda\left(\int_{0}^{t} S(t-s) g(s) d s\right)=S(t) g(0)-g(t)+\int_{0}^{t} S(t-s) g^{\prime}(s) d s
$$

## Proof of the claim:

$$
\begin{aligned}
& \frac{1}{h}\left(S(h) \int_{0}^{t} S(t-s) g(s) d s-\int_{0}^{t} S(t-s) g(s) d s\right) \\
= & \frac{1}{h}\left(\int_{0}^{t} S(t-s+h) g(s) d s-\int_{0}^{t} S(t-s) g(s) d s\right) \\
= & \frac{1}{h}\left(\int_{-h}^{t-h} S(t-s) g(s+h) d s-\int_{0}^{t} S(t-s) g(s) d s\right) \\
= & \frac{1}{h} \int_{-h}^{0} S(t-s) g(s+h) d s+\int_{0}^{t-h} S(t-s) \frac{g(s+h)-g(s)}{h} \\
& -\frac{1}{h} \int_{t-h}^{t} S(t-s) g(s) d s
\end{aligned}
$$

Using the estimate in Theorem 19.14.3 on Page 577 and the dominated convergence theorem, the limit as $h \rightarrow 0$ of the above equals

$$
S(t) g(0)-g(t)+\int_{0}^{t} S(t-s) g^{\prime}(s) d s
$$

which proves the claim.
Since $y_{0} \in D(\Lambda)$,

$$
\begin{align*}
S(t) \Lambda y_{0} & =S(t) \lim _{h \rightarrow 0} \frac{S(h) y_{0}-y_{0}}{h} \\
& =\lim _{h \rightarrow 0} \frac{S(t+h)-S(t)}{h} y_{0} \\
& =\lim _{h \rightarrow 0} \frac{S(h) S(t) y_{0}-S(t) y_{0}}{h} \tag{19.14.94}
\end{align*}
$$

Since this limit exists, the last limit in the above exists and equals

$$
\begin{equation*}
\Lambda S(t) y_{0} \tag{19.14.95}
\end{equation*}
$$

and so $S(t) y_{0} \in D(\Lambda)$. Now consider 19.14.93.

$$
\begin{gathered}
\frac{y(t+h)-y(t)}{h}=\frac{S(t+h)-S(t)}{h} y_{0}+ \\
\frac{1}{h}\left(\int_{0}^{t+h} S(t-s+h) g(s) d s-\int_{0}^{t} S(t-s) g(s) d s\right)
\end{gathered}
$$

$$
\begin{aligned}
= & \frac{S(t+h)-S(t)}{h} y_{0}+\frac{1}{h} \int_{t}^{t+h} S(t-s+h) g(s) d s \\
& +\frac{1}{h}\left(S(h) \int_{0}^{t} S(t-s) g(s) d s-\int_{0}^{t} S(t-s) g(s) d s\right)
\end{aligned}
$$

From the claim and $19.14 .94,19.14 .95$ the limit of the right side is

$$
\begin{aligned}
& \Lambda S(t) y_{0}+g(t)+\Lambda\left(\int_{0}^{t} S(t-s) g(s) d s\right) \\
= & \Lambda\left(S(t) y_{0}+\int_{0}^{t} S(t-s) g(s) d s\right)+g(t)
\end{aligned}
$$

Hence

$$
y^{\prime}(t)=\Lambda y(t)+g(t)
$$

and from the formula, $y^{\prime}$ is continuous since by the claim and 19.14.95 it also equals

$$
S(t) \Lambda y_{0}+g(t)+S(t) g(0)-g(t)+\int_{0}^{t} S(t-s) g^{\prime}(s) d s
$$

which is continuous. The claim and 19.14.95 also shows $y(t) \in D(\Lambda)$. This proves the existence part of the lemma.

It remains to prove the uniqueness part. It suffices to show that if

$$
y^{\prime}-\Lambda y=0, y(0)=0
$$

and $y$ is $C^{1}$ having values in $D(\Lambda)$, then $y=0$. Suppose then that $y$ is this way. Letting $0<s<t$,

$$
\begin{gathered}
\frac{d}{d s}(S(t-s) y(s)) \\
\equiv \lim _{h \rightarrow 0} S(t-s-h) \frac{y(s+h)-y(s)}{h} \\
-\frac{S(t-s) y(s)-S(t-s-h) y(s)}{h}
\end{gathered}
$$

provided the limit exists. Since $y^{\prime}$ exists and $y(s) \in D(\Lambda)$, this equals

$$
S(t-s) y^{\prime}(s)-S(t-s) \Lambda y(s)=0
$$

Let $y^{*} \in X^{\prime}$. This has shown that on the open interval $(0, t)$ the function $s \rightarrow y^{*}(S(t-s) y(s))$ has a derivative equal to 0 . Also from continuity of $S$ and $y$, this function is continuous on $[0, t]$. Therefore, it is constant on $[0, t]$ by the mean value theorem. At $s=0$, this function equals 0 . Therefore, it equals 0 on $[0, t]$. Thus for fixed $s>0$ and letting $t>s, y^{*}(S(t-s) y(s))=0$. Now let $t$ decrease toward $s$. Then $y^{*}(y(s))=0$ and since $y^{*}$ was arbitrary, it follows $y(s)=0$. This proves uniqueness.

### 19.14.2 Adjoints, Hilbert Space

In Hilbert space, there are some special things which are true.
Definition 19.14.13 Let A be a densely defined closed operator on $H$ a real Hilbert space.
Then $A^{*}$ is defined as follows.

$$
D\left(A^{*}\right) \equiv\{y \in H:|(A x, y)| \leq C|x|\}
$$

Then since $D(A)$ is dense, there exists a unique element of $H$ denoted by $A^{*} y$ such that

$$
(A x, y)=\left(x, A^{*} y\right)
$$

for all $x \in D(A)$.
Lemma 19.14.14 Let $A$ be closed and densely defined on $D(H) \subseteq H$, a Hilbert space. Then $A^{*}$ is also closed and densely defined. Also $\left(A^{*}\right)^{*}=A$. In addition to this, if

$$
(\lambda I-A)^{-1} \in \mathscr{L}(H, H)
$$

then $\left(\lambda I-A^{*}\right)^{-1} \in \mathscr{L}(H, H)$ and

$$
\left(\left((\lambda I-A)^{-1}\right)^{n}\right)^{*}=\left(\left(\lambda I-A^{*}\right)^{-1}\right)^{n}
$$

Proof: Denote by $[x, y]$ an ordered pair in $H \times H$. Define $\tau: H \times H \rightarrow H \times H$ by

$$
\tau[x, y] \equiv[-y, x]
$$

Then the definition of adjoint implies that for $\mathscr{G}(B)$ equal to the graph of $B$,

$$
\begin{equation*}
\mathscr{G}\left(A^{*}\right)=(\tau \mathscr{G}(A))^{\perp} \tag{19.14.96}
\end{equation*}
$$

In this notation the inner product on $H \times H$ with respect to which $\perp$ is defined is given by

$$
([x, y],[a, b]) \equiv(x, a)+(y, b)
$$

Here is why this is so. For $\left[x, A^{*} x\right] \in \mathscr{G}\left(A^{*}\right)$ it follows that for all $y \in D(A)$

$$
\left(\left[x, A^{*} x\right],[-A y, y]\right)=-(A y, x)+\left(y, A^{*} x\right)=0
$$

and so $\left[x, A^{*} x\right] \in\left(\tau_{\mathscr{G}}(A)\right)^{\perp}$ which shows

$$
\mathscr{G}\left(A^{*}\right) \subseteq(\tau \mathscr{G}(A))^{\perp}
$$

To obtain the other inclusion, let $[a, b] \in(\tau \mathscr{G}(A))^{\perp}$. This means that for all $x \in D(A)$,

$$
([a, b],[-A x, x])=0
$$

In other words, for all $x \in D(A)$,

$$
(A x, a)=(x, b)
$$

and so $|(A x, a)| \leq C|x|$ for all $x \in D(A)$ which shows $a \in D\left(A^{*}\right)$ and

$$
\left(x, A^{*} a\right)=(x, b)
$$

for all $x \in D(A)$. Therefore, since $D(A)$ is dense, it follows $b=A^{*} a$ and so $[a, b] \in \mathscr{G}\left(A^{*}\right)$. This shows the other inclusion.

Note that if $V$ is any subspace of the Hilbert space $H \times H$,

$$
\left(V^{\perp}\right)^{\perp}=\bar{V}
$$

and $S^{\perp}$ is always a closed subspace. Also $\tau$ and $\perp$ commute. The reason for this is that $[x, y] \in(\tau V)^{\perp}$ means that

$$
(x,-b)+(y, a)=0
$$

for all $[a, b] \in V$ and $[x, y] \in \tau\left(V^{\perp}\right)$ means $[-y, x] \in V^{\perp}$ so for all $[a, b] \in V$,

$$
(-y, a)+(x, b)=0
$$

which says the same thing. It is also clear that $\tau \circ \tau$ has the effect of multiplication by -1 .
It follows from the above description of the graph of $A^{*}$ that even if $\mathscr{G}(A)$ were not closed it would still be the case that $\mathscr{G}\left(A^{*}\right)$ is closed.

Why is $D\left(A^{*}\right)$ dense? Suppose $z \in D\left(A^{*}\right)^{\perp}$. Then for all $y \in D\left(A^{*}\right)$ so that $[y, A y] \in$ $\mathscr{G}\left(A^{*}\right)$, it follows $[z, 0] \in \mathscr{G}\left(A^{*}\right)^{\perp}=\left((\tau \mathscr{G}(A))^{\perp}\right)^{\perp}=\tau \mathscr{G}(A)$ but this implies

$$
[0, z] \in-\mathscr{G}(A)
$$

and so $z=-A 0=0$. Thus $D\left(A^{*}\right)$ must be dense since there is no nonzero vector in $D\left(A^{*}\right)^{\perp}$.
Since $A$ is a closed operator, meaning $\mathscr{G}(A)$ is closed in $H \times H$, it follows from the above formula that

$$
\begin{aligned}
\mathscr{G}\left(\left(A^{*}\right)^{*}\right) & =\left(\tau\left((\tau \mathscr{G}(A))^{\perp}\right)\right)^{\perp}=\left(\tau(\tau \mathscr{G}(A))^{\perp}\right)^{\perp} \\
& =\left((-\mathscr{G}(A))^{\perp}\right)^{\perp}=\left(\mathscr{G}(A)^{\perp}\right)^{\perp}=\mathscr{G}(A)
\end{aligned}
$$

and so $\left(A^{*}\right)^{*}=A$.
Now consider the final claim. First let $y \in D\left(A^{*}\right)=D\left(\lambda I-A^{*}\right)$. Then letting $x \in H$ be arbitrary,

$$
\begin{gathered}
\left(x,\left((\lambda I-A)(\lambda I-A)^{-1}\right)^{*} y\right) \\
\left((\lambda I-A)(\lambda I-A)^{-1} x, y\right)=\left(x,\left((\lambda I-A)^{-1}\right)^{*}\left(\lambda I-A^{*}\right) y\right)
\end{gathered}
$$

Thus

$$
\begin{equation*}
\left((\lambda I-A)(\lambda I-A)^{-1}\right)^{*}=I=\left((\lambda I-A)^{-1}\right)^{*}\left(\lambda I-A^{*}\right) \tag{19.14.97}
\end{equation*}
$$

on $D\left(A^{*}\right)$. Next let $x \in D(A)=D(\lambda I-A)$ and $y \in H$ arbitrary.

$$
(x, y)=\left((\lambda I-A)^{-1}(\lambda I-A) x, y\right)=\left((\lambda I-A) x,\left((\lambda I-A)^{-1}\right)^{*} y\right)
$$

Now it follows $\left|\left((\lambda I-A) x,\left((\lambda I-A)^{-1}\right)^{*} y\right)\right| \leq|y||x|$ for any $x \in D(A)$ and so

$$
\left((\lambda I-A)^{-1}\right)^{*} y \in D\left(A^{*}\right)
$$

Hence

$$
(x, y)=\left(x,\left(\lambda I-A^{*}\right)\left((\lambda I-A)^{-1}\right)^{*} y\right) .
$$

Since $x \in D(A)$ is arbitrary and $D(A)$ is dense, it follows

$$
\begin{equation*}
\left(\lambda I-A^{*}\right)\left((\lambda I-A)^{-1}\right)^{*}=I \tag{19.14.98}
\end{equation*}
$$

From 19.14.97 and 19.14.98 it follows

$$
\left(\lambda I-A^{*}\right)^{-1}=\left((\lambda I-A)^{-1}\right)^{*}
$$

and $\left(\lambda I-A^{*}\right)$ is one to one and onto with continuous inverse. Finally, from the above,

$$
\left(\left(\lambda I-A^{*}\right)^{-1}\right)^{n}=\left(\left((\lambda I-A)^{-1}\right)^{*}\right)^{n}=\left(\left((\lambda I-A)^{-1}\right)^{n}\right)^{*}
$$

This proves the lemma.
With this preparation, here is an interesting result about the adjoint of the generator of a continuous bounded semigroup. I found this in Balakrishnan [12].

Theorem 19.14.15 Suppose $A$ is a densely defined closed operator which generates a continuous semigroup, $S(t)$. Then $A^{*}$ is also a closed densely defined operator which generates $S^{*}(t)$ and $S^{*}(t)$ is also a continuous semigroup.

Proof: First suppose $S(t)$ is also a bounded semigroup, $\|S(t)\| \leq M$. From Lemma 19.14.14 $A^{*}$ is closed and densely defined. It follows from the Hille Yosida theorem, Theorem 19.14.8 that

$$
\left|\left((\lambda I-A)^{-1}\right)^{n}\right| \leq \frac{M}{\lambda^{n}}
$$

From Lemma 19.14.14 and the fact the adjoint of a bounded linear operator preserves the norm,

$$
\begin{aligned}
\frac{M}{\lambda^{n}} & \geq\left|\left(\left((\lambda I-A)^{-1}\right)^{n}\right)^{*}\right|=\left|\left(\left((\lambda I-A)^{-1}\right)^{*}\right)^{n}\right| \\
& =\left|\left(\left(\lambda I-A^{*}\right)^{-1}\right)^{n}\right|
\end{aligned}
$$

and so by Theorem 19.14 .8 again it follows $A^{*}$ generates a continuous semigroup, $T(t)$ which satisfies $\|T(t)\| \leq M$. I need to identify $T(t)$ with $S^{*}(t)$. However, from the proof of Theorem 19.14.8 and Lemma 19.14.14, it follows that for $x \in D\left(A^{*}\right)$ and a suitable sequence $\left\{\lambda_{n}\right\}$,

$$
(T(t) x, y)=\left(\lim _{n \rightarrow \infty} e^{-\lambda_{n} t} \sum_{k=0}^{\infty} \frac{t^{k}\left(\lambda_{n}^{2}\left(\lambda_{n} I-A^{*}\right)^{-1}\right)^{k}}{k!} x, y\right)
$$

$$
\begin{aligned}
& =\lim _{n \rightarrow \infty}\left(e^{-\lambda_{n} t} \sum_{k=0}^{\infty} \frac{t^{k}\left(\left(\lambda_{n}^{2}\left(\lambda_{n} I-A\right)^{-1}\right)^{k}\right)^{*}}{k!} x, y\right) \\
& =\lim _{n \rightarrow \infty}\left(x, e^{-\lambda_{n} t}\left(\sum_{k=0}^{\infty} \frac{t^{k}\left(\left(\lambda_{n}^{2}\left(\lambda_{n} I-A\right)^{-1}\right)^{k}\right)}{k!}\right) y\right) \\
& =(x, S(t) y)=\left(S^{*}(t) x, y\right) .
\end{aligned}
$$

Therefore, since $y$ is arbitrary, $S^{*}(t)=T(t)$ on $x \in D\left(A^{*}\right)$ a dense set and this shows the two are equal. This proves the proposition in the case where $S(t)$ is also bounded.

Next only assume $S(t)$ is a continuous semigroup. Then by Proposition 19.14.5 there exists $\alpha>0$ such that

$$
\|S(t)\| \leq M e^{\alpha t}
$$

Then consider the operator $-\alpha I+A$ and the bounded semigroup $e^{-\alpha t} S(t)$. For $x \in D(A)$

$$
\begin{aligned}
\lim _{h \rightarrow 0+} \frac{e^{-\alpha h} S(h) x-x}{h} & =\lim _{h \rightarrow 0+}\left(e^{-\alpha h} \frac{S(h) x-x}{h}+\frac{e^{-\alpha h}-1}{h} x\right) \\
& =-\alpha x+A x
\end{aligned}
$$

Thus $-\alpha I+A$ generates $e^{-\alpha t} S(t)$ and it follows from the first part that $-\alpha I+A^{*}$ generates $e^{-\alpha t} S^{*}(t)$. Thus

$$
\begin{aligned}
-\alpha x+A^{*} x & =\lim _{h \rightarrow 0+} \frac{e^{-\alpha h} S^{*}(h) x-x}{h} \\
& =\lim _{h \rightarrow 0+}\left(e^{-\alpha h} \frac{S^{*}(h) x-x}{h}+\frac{e^{-\alpha h}-1}{h} x\right) \\
& =-\alpha x+\lim _{h \rightarrow 0+} \frac{S^{*}(h) x-x}{h}
\end{aligned}
$$

showing that $A^{*}$ generates $S^{*}(t)$. It follows from Proposition 19.14 .5 that $A^{*}$ is closed and densely defined. It is obvious $S^{*}(t)$ is a semigroup. Why is it continuous? This also follows from the first part of the argument which establishes that

$$
e^{-\alpha t} S^{*}(t)
$$

is continuous. This proves the theorem.

### 19.14.3 Adjoints, Reflexive Banach Space

Here the adjoint of a generator of a semigroup is considered. I will show that the adjoint of the generator generates the adjoint of the semigroup in a reflexive Banach space. This is about as far as you can go although a general but less satisfactory result is given in Yosida [127].

Definition 19.14.16 Let A be a densely defined closed operator on $H$ a real Banach space. Then $A^{*}$ is defined as follows.

$$
D\left(A^{*}\right) \equiv\left\{y^{*} \in H^{\prime}:\left|y^{*}(A x)\right| \leq C\|x\| \text { for all } x \in D(A)\right\}
$$

Then since $D(A)$ is dense, there exists a unique element of $H^{\prime}$ denoted by $A^{*} y$ such that

$$
A^{*}\left(y^{*}\right)(x)=y^{*}(A x)
$$

for all $x \in D(A)$.
Lemma 19.14.17 Let $A$ be closed and densely defined on $D(A) \subseteq H$, a Banach space. Then $A^{*}$ is also closed and densely defined. Also $\left(A^{*}\right)^{*}=A$. In addition to this, if $(\lambda I-A)^{-1} \in$ $\mathscr{L}(H, H)$, then $\left(\lambda I-A^{*}\right)^{-1} \in \mathscr{L}\left(H^{\prime}, H^{\prime}\right)$ and

$$
\left(\left((\lambda I-A)^{-1}\right)^{n}\right)^{*}=\left(\left(\lambda I-A^{*}\right)^{-1}\right)^{n}
$$

Proof: Denote by $[x, y]$ an ordered pair in $H \times H$. Define $\tau: H \times H \rightarrow H \times H$ by

$$
\tau[x, y] \equiv[-y, x]
$$

A similar notation will apply to $H^{\prime} \times H^{\prime}$. Then the definition of adjoint implies that for $\mathscr{G}(B)$ equal to the graph of $B$,

$$
\begin{equation*}
\mathscr{G}\left(A^{*}\right)=(\tau \mathscr{G}(A))^{\perp} \tag{19.14.99}
\end{equation*}
$$

For $S \subseteq H \times H$, define $S^{\perp}$ by

$$
\left\{\left[a^{*}, b^{*}\right] \in H^{\prime} \times H^{\prime}: a^{*}(x)+b^{*}(y)=0 \text { for all }[x, y] \in S\right\}
$$

If $S \subseteq H^{\prime} \times H^{\prime}$ a similar definition holds.

$$
\left\{[x, y] \in H \times H: a^{*}(x)+b^{*}(y)=0 \text { for all }\left[a^{*}, b^{*}\right] \in S\right\}
$$

Here is why 19.14 .99 is so. For $\left[x^{*}, A^{*} x^{*}\right] \in \mathscr{G}\left(A^{*}\right)$ it follows that for all $y \in D(A)$

$$
x^{*}(A y)=A^{*} x^{*}(y)
$$

and so for all $[y, A y] \in \mathscr{G}(A)$,

$$
-x^{*}(A y)+A^{*} x^{*}(y)=0
$$

which is what it means to say $\left[x^{*}, A^{*} x^{*}\right] \in\left(\tau_{\mathscr{G}}(A)\right)^{\perp}$. This shows

$$
\mathscr{G}\left(A^{*}\right) \subseteq(\tau \mathscr{G}(A))^{\perp}
$$

To obtain the other inclusion, let $\left[a^{*}, b^{*}\right] \in(\tau \mathscr{G}(A))^{\perp}$. This means that for all $[x, A x] \in$ $\mathscr{G}(A)$,

$$
-a^{*}(A x)+b^{*}(x)=0
$$

In other words, for all $x \in D(A)$,

$$
\left|a^{*}(A x)\right| \leq\left\|b^{*}\right\|\|x\|
$$

which means by definition, $a^{*} \in D\left(A^{*}\right)$ and $A^{*} a^{*}=b^{*}$. Thus $\left[a^{*}, b^{*}\right] \in \mathscr{G}\left(A^{*}\right)$.This shows the other inclusion.

Note that if $V$ is any subspace of $H \times H$,

$$
\left(V^{\perp}\right)^{\perp}=\bar{V}
$$

and $S^{\perp}$ is always a closed subspace. Also $\tau$ and $\perp$ commute. The reason for this is that $\left[x^{*}, y^{*}\right] \in(\tau V)^{\perp}$ means that

$$
-x^{*}(b)+y^{*}(a)=0
$$

for all $[a, b] \in V$ and $\left[x^{*}, y^{*}\right] \in \tau\left(V^{\perp}\right)$ means $\left[-y^{*}, x^{*}\right] \in-\left(V^{\perp}\right)=V^{\perp}$ so for all $[a, b] \in V$,

$$
-y^{*}(a)+x^{*}(b)=0
$$

which says the same thing. It is also clear that $\tau \circ \tau$ has the effect of multiplication by -1 . If $V \subseteq H^{\prime} \times H^{\prime}$, the argument for commuting $\perp$ and $\tau$ is similar.

It follows from the above description of the graph of $A^{*}$ that even if $\mathscr{G}(A)$ were not closed it would still be the case that $\mathscr{G}\left(A^{*}\right)$ is closed.

Why is $D\left(A^{*}\right)$ dense? If it is not dense, then by a typical application of the Hahn Banach theorem, there exists $y^{* *} \in H^{\prime \prime}$ such that $y^{* *}\left(D\left(A^{*}\right)\right)=0$ but $y^{* *} \neq 0$. Since $H$ is reflexive, there exists $y \in H$ such that $x^{*}(y)=0$ for all $x^{*} \in D\left(A^{*}\right)$. Thus

$$
[y, 0] \in \mathscr{G}\left(A^{*}\right)^{\perp}=\left((\tau \mathscr{G}(A))^{\perp}\right)^{\perp}=\tau \mathscr{G}(A)
$$

and so $[0, y] \in \mathscr{G}(A)$ which means $y=A 0=0$, a contradiction. Thus $D\left(A^{*}\right)$ is indeed dense. Note this is where it was important to assume the space is reflexive. If you consider $C([0,1])$ it is not dense in $L^{\infty}([0,1])$ but if $f \in L^{1}([0,1])$ satisfies $\int_{0}^{1} f g d m=0$ for all $g \in C([0,1])$, then $f=0$. Hence there is no nonzero $f \in C([0,1])^{\perp}$.

Since $A$ is a closed operator, meaning $\mathscr{G}(A)$ is closed in $H \times H$, it follows from the above formula that

$$
\begin{aligned}
\mathscr{G}\left(\left(A^{*}\right)^{*}\right) & =\left(\tau\left((\tau \mathscr{G}(A))^{\perp}\right)\right)^{\perp}=\left(\tau(\tau \mathscr{G}(A))^{\perp}\right)^{\perp} \\
& =\left((-\mathscr{G}(A))^{\perp}\right)^{\perp}=\left(\mathscr{G}(A)^{\perp}\right)^{\perp}=\mathscr{G}(A)
\end{aligned}
$$

and so $\left(A^{*}\right)^{*}=A$.
Now consider the final claim. First let $y^{*} \in D\left(A^{*}\right)=D\left(\lambda I-A^{*}\right)$. Then letting $x \in H$ be arbitrary,

$$
\begin{aligned}
y^{*}(x) & =\left((\lambda I-A)(\lambda I-A)^{-1}\right)^{*} y^{*}(x) \\
& =y^{*}\left((\lambda I-A)(\lambda I-A)^{-1} x\right)
\end{aligned}
$$

Since $y^{*} \in D\left(A^{*}\right)$ and $(\lambda I-A)^{-1} x \in D(A)$, this equals

$$
(\lambda I-A)^{*} y^{*}\left((\lambda I-A)^{-1} x\right)
$$

Now by definition, this equals

$$
\left((\lambda I-A)^{-1}\right)^{*}(\lambda I-A)^{*} y^{*}(x)
$$

It follows that for $y^{*} \in D\left(A^{*}\right)$,

$$
\begin{align*}
& \left((\lambda I-A)^{-1}\right)^{*}(\lambda I-A)^{*} y^{*} \\
= & \left((\lambda I-A)^{-1}\right)^{*}\left(\lambda I-A^{*}\right) y^{*}=y^{*} \tag{19.14.100}
\end{align*}
$$

Next let $y^{*} \in H^{\prime}$ be arbitrary and $x \in D(A)$

$$
\begin{aligned}
y^{*}(x) & =y^{*}\left((\lambda I-A)^{-1}(\lambda I-A) x\right) \\
& =\left((\lambda I-A)^{-1}\right)^{*} y^{*}((\lambda I-A) x) \\
& =(\lambda I-A)^{*}\left((\lambda I-A)^{-1}\right)^{*} y^{*}(x)
\end{aligned}
$$

In going from the second to the third line, the first line shows $\left((\lambda I-A)^{-1}\right)^{*} y^{*} \in D\left(A^{*}\right)$ and so the third line follows. Since $D(A)$ is dense, it follows

$$
\begin{equation*}
\left(\lambda I-A^{*}\right)\left((\lambda I-A)^{-1}\right)^{*}=I \tag{19.14.101}
\end{equation*}
$$

Then 19.14 .100 and 19.14 .101 show $\lambda I-A^{*}$ is one to one and onto from $D\left(A^{*}\right)$ to $H^{\prime}$ and

$$
\left(\lambda I-A^{*}\right)^{-1}=\left((\lambda I-A)^{-1}\right)^{*}
$$

Finally, from the above,

$$
\left(\left(\lambda I-A^{*}\right)^{-1}\right)^{n}=\left(\left((\lambda I-A)^{-1}\right)^{*}\right)^{n}=\left(\left((\lambda I-A)^{-1}\right)^{n}\right)^{*}
$$

This proves the lemma.
With this preparation, here is an interesting result about the adjoint of the generator of a continuous bounded semigroup.

Theorem 19.14.18 Suppose $A$ is a densely defined closed operator which generates a continuous semigroup, $S(t)$. Then $A^{*}$ is also a closed densely defined operator which generates $S^{*}(t)$ and $S^{*}(t)$ is also a continuous semigroup.

Proof: First suppose $S(t)$ is also a bounded semigroup, $\|S(t)\| \leq M$. From Lemma 19.14.17 $A^{*}$ is closed and densely defined. It follows from the Hille Yosida theorem, Theorem 19.14.8 that

$$
\left\|\left((\lambda I-A)^{-1}\right)^{n}\right\| \leq \frac{M}{\lambda^{n}}
$$

From Lemma 19.14.17 and the fact the adjoint of a bounded linear operator preserves the norm,

$$
\begin{aligned}
\frac{M}{\lambda^{n}} & \geq\left\|\left(\left((\lambda I-A)^{-1}\right)^{n}\right)^{*}\right\|=\left\|\left(\left((\lambda I-A)^{-1}\right)^{*}\right)^{n}\right\| \\
& =\left\|\left(\left(\lambda I-A^{*}\right)^{-1}\right)^{n}\right\|
\end{aligned}
$$

and so by Theorem 19.14.8 again it follows $A^{*}$ generates a continuous semigroup, $T(t)$ which satisfies $\|T(t)\| \leq M$. I need to identify $T(t)$ with $S^{*}(t)$. However, from the proof of Theorem 19.14.8 and Lemma 19.14.17, it follows that for $x^{*} \in D\left(A^{*}\right)$ and a suitable sequence $\left\{\lambda_{n}\right\}$,

$$
\begin{aligned}
& T(t) x^{*}(y)=\lim _{n \rightarrow \infty} e^{-\lambda_{n} t} \sum_{k=0}^{\infty} \frac{t^{k}\left(\lambda_{n}^{2}\left(\lambda_{n} I-A^{*}\right)^{-1}\right)^{k}}{k!} x^{*}(y) \\
& =\lim _{n \rightarrow \infty} e^{-\lambda_{n} t} \sum_{k=0}^{\infty} \frac{t^{k}\left(\left(\lambda_{n}^{2}\left(\lambda_{n} I-A\right)^{-1}\right)^{k}\right)^{*}}{k!} x^{*}(y) \\
& =\lim _{n \rightarrow \infty} x^{*}\left(e^{-\lambda_{n} t}\left(\sum_{k=0}^{\infty} \frac{t^{k}\left(\left(\lambda_{n}^{2}\left(\lambda_{n} I-A\right)^{-1}\right)^{k}\right)}{k!} y\right)\right) \\
& =x^{*}(S(t) y)=S^{*}(t) x^{*}(y) .
\end{aligned}
$$

Therefore, since $y$ is arbitrary, $S^{*}(t)=T(t)$ on $x \in D\left(A^{*}\right)$ a dense set and this shows the two are equal. In particular, $S^{*}(t)$ is a semigroup because $T(t)$ is. This proves the proposition in the case where $S(t)$ is also bounded.

Next only assume $S(t)$ is a continuous semigroup. Then by Proposition 19.14.5 there exists $\alpha>0$ such that

$$
\|S(t)\| \leq M e^{\alpha t}
$$

Then consider the operator $-\alpha I+A$ and the bounded semigroup $e^{-\alpha t} S(t)$. For $x \in D(A)$

$$
\begin{aligned}
\lim _{h \rightarrow 0+} \frac{e^{-\alpha h} S(h) x-x}{h} & =\lim _{h \rightarrow 0+}\left(e^{-\alpha h} \frac{S(h) x-x}{h}+\frac{e^{-\alpha h}-1}{h} x\right) \\
& =-\alpha x+A x
\end{aligned}
$$

Thus $-\alpha I+A$ generates $e^{-\alpha t} S(t)$ and it follows from the first part that $-\alpha I+A^{*}$ generates the semigroup $e^{-\alpha t} S^{*}(t)$. Thus

$$
\begin{aligned}
-\alpha x+A^{*} x & =\lim _{h \rightarrow 0+} \frac{e^{-\alpha h} S^{*}(h) x-x}{h} \\
& =\lim _{h \rightarrow 0+}\left(e^{-\alpha h} \frac{S^{*}(h) x-x}{h}+\frac{e^{-\alpha h}-1}{h} x\right) \\
& =-\alpha x+\lim _{h \rightarrow 0+} \frac{S^{*}(h) x-x}{h}
\end{aligned}
$$

showing that $A^{*}$ generates $S^{*}(t)$. It follows from Proposition 19.14 .5 that $A^{*}$ is closed and densely defined. It is obvious $S^{*}(t)$ is a semigroup. Why is it continuous? This also follows from the first part of the argument which establishes that

$$
t \rightarrow e^{-\alpha t} S^{*}(t) x
$$

is continuous. This proves the theorem.

## Chapter 20

## Representation Theorems

### 20.1 Radon Nikodym Theorem

This chapter is on various representation theorems. The first theorem, the Radon Nikodym Theorem, is a representation theorem for one measure in terms of another. The approach given here is due to Von Neumann and depends on the Riesz representation theorem for Hilbert space, Theorem 19.1.14 on Page 522.

Definition 20.1.1 Let $\mu$ and $\lambda$ be two measures defined on a $\sigma$-algebra, $\mathscr{S}$, of subsets of a set, $\Omega$. $\lambda$ is absolutely continuous with respect to $\mu$, written as $\lambda \ll \mu$, if $\lambda(E)=0$ whenever $\mu(E)=0$.

It is not hard to think of examples which should be like this. For example, suppose one measure is volume and the other is mass. If the volume of something is zero, it is reasonable to expect the mass of it should also be equal to zero. In this case, there is a function called the density which is integrated over volume to obtain mass. The Radon Nikodym theorem is an abstract version of this notion. Essentially, it gives the existence of the density function.

Theorem 20.1.2 (Radon Nikodym) Let $\lambda$ and $\mu$ be finite measures defined on a $\sigma$-algebra, $\mathscr{S}$, of subsets of $\Omega$. Suppose $\lambda \ll \mu$. Then there exists a unique $f \in L^{1}(\Omega, \mu)$ such that $f(x) \geq 0$ and

$$
\lambda(E)=\int_{E} f d \mu
$$

If it is not necessarily the case that $\lambda \ll \mu$, there are two measures, $\lambda_{\perp}$ and $\lambda_{\|}$such that $\lambda=\lambda_{\perp}+\lambda_{\|}, \lambda_{\|} \ll \mu$ and there exists a set of $\mu$ measure zero, $N$ such that for all $E$ measurable, $\lambda_{\perp}(E)=\lambda(E \cap N)=\lambda_{\perp}(E \cap N)$. In this case the two measures, $\lambda_{\perp}$ and $\lambda_{\|}$ are unique and the representation of $\lambda=\lambda_{\perp}+\lambda_{\|}$is called the Lebesgue decomposition of $\lambda$. The measure $\lambda_{\|}$is the absolutely continuous part of $\lambda$ and $\lambda_{\perp}$ is called the singular part of $\lambda$.

Proof: Let $\Lambda: L^{2}(\Omega, \mu+\lambda) \rightarrow \mathbb{C}$ be defined by

$$
\Lambda g=\int_{\Omega} g d \lambda
$$

By Holder's inequality,

$$
|\Lambda g| \leq\left(\int_{\Omega} 1^{2} d \lambda\right)^{1 / 2}\left(\int_{\Omega}|g|^{2} d(\lambda+\mu)\right)^{1 / 2}=\lambda(\Omega)^{1 / 2}\|g\|_{2}
$$

where $\|g\|_{2}$ is the $L^{2}$ norm of $g$ taken with respect to $\mu+\lambda$. Therefore, since $\Lambda$ is bounded, it follows from Theorem 17.1.4 on Page 437 that $\Lambda \in\left(L^{2}(\Omega, \mu+\lambda)\right)^{\prime}$, the dual space $L^{2}(\Omega, \mu+\lambda)$. By the Riesz representation theorem in Hilbert space, Theorem 19.1.14, there exists a unique $h \in L^{2}(\Omega, \mu+\lambda)$ with

$$
\begin{equation*}
\Lambda g=\int_{\Omega} g d \lambda=\int_{\Omega} h g d(\mu+\lambda) \tag{20.1.1}
\end{equation*}
$$

The plan is to show $h$ is real and nonnegative at least a.e. Therefore, consider the set where $\operatorname{Im} h$ is positive.

$$
E=\{x \in \Omega: \operatorname{Im} h(x)>0\},
$$

Now let $g=\mathscr{X}_{E}$ and use 20.1.1 to get

$$
\begin{equation*}
\lambda(E)=\int_{E}(\operatorname{Re} h+i \operatorname{Im} h) d(\mu+\lambda) \tag{20.1.2}
\end{equation*}
$$

Since the left side of 20.1.2 is real, this shows

$$
\begin{aligned}
0 & =\int_{E}(\operatorname{Im} h) d(\mu+\lambda) \\
& \geq \int_{E_{n}}(\operatorname{Im} h) d(\mu+\lambda) \\
& \geq \frac{1}{n}(\mu+\lambda)\left(E_{n}\right)
\end{aligned}
$$

where

$$
E_{n} \equiv\left\{x: \operatorname{Im} h(x) \geq \frac{1}{n}\right\}
$$

Thus $(\mu+\lambda)\left(E_{n}\right)=0$ and since $E=\cup_{n=1}^{\infty} E_{n}$, it follows $(\mu+\lambda)(E)=0$. A similar argument shows that for

$$
E=\{x \in \Omega: \operatorname{Im} h(x)<0\},
$$

$(\mu+\lambda)(E)=0$. Thus there is no loss of generality in assuming $h$ is real-valued.
The next task is to show $h$ is nonnegative. This is done in the same manner as above. Define the set where it is negative and then show this set has measure zero.

Let $E \equiv\{x: h(x)<0\}$ and let $E_{n} \equiv\left\{x: h(x)<-\frac{1}{n}\right\}$. Then let $g=\mathscr{X}_{E_{n}}$. Since $E=$ $\cup_{n} E_{n}$, it follows that if $(\mu+\lambda)(E)>0$ then this is also true for $(\mu+\lambda)\left(E_{n}\right)$ for all $n$ large enough. Then from 20.1.2

$$
\lambda\left(E_{n}\right)=\int_{E_{n}} h d(\mu+\lambda) \leq-(1 / n)(\mu+\lambda)\left(E_{n}\right)<0
$$

a contradiction. Thus it can be assumed $h \geq 0$.
At this point the argument splits into two cases.
Case Where $\lambda \ll \mu$. In this case, $h<1$.
Let $E=[h \geq 1]$ and let $g=\mathscr{X}_{E}$. Then

$$
\lambda(E)=\int_{E} h d(\mu+\lambda) \geq \mu(E)+\lambda(E)
$$

Therefore $\mu(E)=0$. Since $\lambda \ll \mu$, it follows that $\lambda(E)=0$ also. Thus it can be assumed

$$
0 \leq h(x)<1
$$

for all $x$.

From 20.1.1, whenever $g \in L^{2}(\Omega, \mu+\lambda)$,

$$
\begin{equation*}
\int_{\Omega} g(1-h) d \lambda=\int_{\Omega} h g d \mu \tag{20.1.3}
\end{equation*}
$$

Now let $E$ be a measurable set and define

$$
g(x) \equiv \sum_{i=0}^{n} h^{i}(x) \mathscr{X}_{E}(x)
$$

in 20.1.3. This yields

$$
\begin{equation*}
\int_{E}\left(1-h^{n+1}(x)\right) d \lambda=\int_{E} \sum_{i=1}^{n+1} h^{i}(x) d \mu . \tag{20.1.4}
\end{equation*}
$$

Let $f(x)=\sum_{i=1}^{\infty} h^{i}(x)$ and use the Monotone Convergence theorem in 20.1.4 to let $n \rightarrow \infty$ and conclude

$$
\lambda(E)=\int_{E} f d \mu
$$

$f \in L^{1}(\Omega, \mu)$ because $\lambda$ is finite.
The function, $f$ is unique $\mu$ a.e. because, if $g$ is another function which also serves to represent $\lambda$, consider for each $n \in \mathbb{N}$ the set,

$$
E_{n} \equiv\left[f-g>\frac{1}{n}\right]
$$

and conclude that

$$
0=\int_{E_{n}}(f-g) d \mu \geq \frac{1}{n} \mu\left(E_{n}\right)
$$

Therefore, $\mu\left(E_{n}\right)=0$. It follows that

$$
\mu([f-g>0]) \leq \sum_{n=1}^{\infty} \mu\left(E_{n}\right)=0
$$

Similarly, the set where $g$ is larger than $f$ has measure zero. This proves the theorem.
Case where it is not necessarily true that $\lambda \ll \mu$.
In this case, let $N=[h \geq 1]$ and let $g=\mathscr{X}_{N}$. Then

$$
\lambda(N)=\int_{N} h d(\mu+\lambda) \geq \mu(N)+\lambda(N)
$$

and so $\mu(N)=0$. Now define a measure, $\lambda_{\perp}$ by

$$
\lambda_{\perp}(E) \equiv \lambda(E \cap N)
$$

so $\lambda_{\perp}(E \cap N)=\lambda(E \cap N \cap N) \equiv \lambda_{\perp}(E)$ and let $\lambda_{\|} \equiv \lambda-\lambda_{\perp}$. Therefore,

$$
\mu(E)=\mu\left(E \cap N^{C}\right)
$$

Also,

$$
\lambda_{\|}(E)=\lambda(E)-\lambda_{\perp}(E) \equiv \lambda(E)-\lambda(E \cap N)=\lambda\left(E \cap N^{C}\right)
$$

Suppose $\lambda_{\|}(E)>0$. Therefore, since $h<1$ on $N^{C}$

$$
\begin{aligned}
\lambda_{\|}(E) & =\lambda\left(E \cap N^{C}\right)=\int_{E \cap N^{C}} h d(\mu+\lambda) \\
& <\mu\left(E \cap N^{C}\right)+\lambda\left(E \cap N^{C}\right)=\mu(E)+\lambda_{\|}(E)
\end{aligned}
$$

which is a contradiction unless $\mu(E)>0$. Therefore, $\lambda_{\|} \ll \mu$ because if $\mu(E)=0$, the above inequality cannot hold.

It only remains to verify the two measures $\lambda_{\perp}$ and $\lambda_{\|}$are unique. Suppose then that $v_{1}$ and $v_{2}$ play the roles of $\lambda_{\perp}$ and $\lambda_{\|}$respectively. Let $N_{1}$ play the role of $N$ in the definition of $v_{1}$ and let $f_{1}$ play the role of $f$ for $\nu_{2}$. I will show that $f=f_{1} \mu$ a.e. Let $E_{k} \equiv\left[f_{1}-f>1 / k\right]$ for $k \in \mathbb{N}$. Then on observing that $\lambda_{\perp}-v_{1}=v_{2}-\lambda_{\|}$

$$
\begin{aligned}
0 & =\left(\lambda_{\perp}-v_{1}\right)\left(E_{k} \cap\left(N_{1} \cup N\right)^{C}\right)=\int_{E_{k} \cap\left(N_{1} \cup N\right)^{C}}\left(g_{1}-g\right) d \mu \\
& \geq \frac{1}{k} \mu\left(E_{k} \cap\left(N_{1} \cup N\right)^{C}\right)=\frac{1}{k} \mu\left(E_{k}\right)
\end{aligned}
$$

and so $\mu\left(E_{k}\right)=0$. Therefore, $\mu\left(\left[f_{1}-f>0\right]\right)=0$ because $\left[f_{1}-f>0\right]=\cup_{k=1}^{\infty} E_{k}$. It follows $f_{1} \leq f \mu$ a.e. Similarly, $f_{1} \geq f \mu$ a.e. Therefore, $v_{2}=\lambda_{\|}$and so $\lambda_{\perp}=v_{1}$ also.

The $f$ in the theorem for the absolutely continuous case is sometimes denoted by $\frac{d \lambda}{d \mu}$ and is called the Radon Nikodym derivative.

The next corollary is a useful generalization to $\sigma$ finite measure spaces.
Corollary 20.1.3 Suppose $\lambda \ll \mu$ and there exist sets $S_{n} \in \mathscr{S}$ with

$$
S_{n} \cap S_{m}=\emptyset, \cup_{n=1}^{\infty} S_{n}=\Omega
$$

and $\lambda\left(S_{n}\right), \mu\left(S_{n}\right)<\infty$. Then there exists $f \geq 0$, where $f$ is $\mu$ measurable, and

$$
\lambda(E)=\int_{E} f d \mu
$$

for all $E \in \mathscr{S}$. The function $f$ is $\mu+\lambda$ a.e. unique.
Proof: Define the $\sigma$ algebra of subsets of $S_{n}$,

$$
\mathscr{S}_{n} \equiv\left\{E \cap S_{n}: E \in \mathscr{S}\right\}
$$

Then both $\lambda$, and $\mu$ are finite measures on $\mathscr{S}_{n}$, and $\lambda \ll \mu$. Thus, by Theorem 20.1.2, there exists a nonnegative $\mathscr{S}_{n}$ measurable function $f_{n}$, with $\lambda(E)=\int_{E} f_{n} d \mu$ for all $E \in \mathscr{S}_{n}$. Define $f(x)=f_{n}(x)$ for $x \in S_{n}$. Since the $S_{n}$ are disjoint and their union is all of $\Omega$, this defines $f$ on all of $\Omega$. The function, $f$ is measurable because

$$
f^{-1}((a, \infty])=\cup_{n=1}^{\infty} f_{n}^{-1}((a, \infty]) \in \mathscr{S} .
$$

Also, for $E \in \mathscr{S}$,

$$
\begin{aligned}
\lambda(E) & =\sum_{n=1}^{\infty} \lambda\left(E \cap S_{n}\right)=\sum_{n=1}^{\infty} \int \mathscr{X}_{E \cap S_{n}}(x) f_{n}(x) d \mu \\
& =\sum_{n=1}^{\infty} \int \mathscr{X}_{E \cap S_{n}}(x) f(x) d \mu
\end{aligned}
$$

By the monotone convergence theorem

$$
\begin{aligned}
\sum_{n=1}^{\infty} \int \mathscr{X}_{E \cap S_{n}}(x) f(x) d \mu & =\lim _{N \rightarrow \infty} \sum_{n=1}^{N} \int \mathscr{X}_{E \cap S_{n}}(x) f(x) d \mu \\
& =\lim _{N \rightarrow \infty} \int \sum_{n=1}^{N} \mathscr{X}_{E \cap S_{n}}(x) f(x) d \mu \\
& =\int \sum_{n=1}^{\infty} \mathscr{X}_{E \cap S_{n}}(x) f(x) d \mu=\int_{E} f d \mu
\end{aligned}
$$

This proves the existence part of the corollary.
To see $f$ is unique, suppose $f_{1}$ and $f_{2}$ both work and consider for $n \in \mathbb{N}$

$$
E_{k} \equiv\left[f_{1}-f_{2}>\frac{1}{k}\right]
$$

Then

$$
0=\lambda\left(E_{k} \cap S_{n}\right)-\lambda\left(E_{k} \cap S_{n}\right)=\int_{E_{k} \cap S_{n}} f_{1}(x)-f_{2}(x) d \mu
$$

Hence $\mu\left(E_{k} \cap S_{n}\right)=0$ for all $n$ so

$$
\mu\left(E_{k}\right)=\lim _{n \rightarrow \infty} \mu\left(E \cap S_{n}\right)=0
$$

Hence $\mu\left(\left[f_{1}-f_{2}>0\right]\right) \leq \sum_{k=1}^{\infty} \mu\left(E_{k}\right)=0$. Therefore, $\lambda\left(\left[f_{1}-f_{2}>0\right]\right)=0$ also. Similarly

$$
(\mu+\lambda)\left(\left[f_{1}-f_{2}<0\right]\right)=0
$$

This version of the Radon Nikodym theorem will suffice for most applications, but more general versions are available. To see one of these, one can read the treatment in Hewitt and Stromberg [64]. This involves the notion of decomposable measure spaces, a generalization of $\sigma$ finite.

Not surprisingly, there is a simple generalization of the Lebesgue decomposition part of Theorem 20.1.2.

Corollary 20.1.4 Let $(\Omega, \mathscr{S})$ be a set with a $\sigma$ algebra of sets. Suppose $\lambda$ and $\mu$ are two measures defined on the sets of $\mathscr{S}$ and suppose there exists a sequence of disjoint sets of $\mathscr{S},\left\{\Omega_{i}\right\}_{i=1}^{\infty}$ such that $\lambda\left(\Omega_{i}\right), \mu\left(\Omega_{i}\right)<\infty$. Then there is a set of $\mu$ measure zero, $N$ and measures $\lambda_{\perp}$ and $\lambda_{\|}$such that

$$
\lambda_{\perp}+\lambda_{\|}=\lambda, \lambda_{\|} \ll \mu, \lambda_{\perp}(E)=\lambda(E \cap N)=\lambda_{\perp}(E \cap N)
$$

Proof: Let $\mathscr{S}_{i} \equiv\left\{E \cap \Omega_{i}: E \in \mathscr{S}\right\}$ and for $E \in \mathscr{S}_{i}$, let $\lambda^{i}(E)=\lambda(E)$ and $\mu^{i}(E)=$ $\mu(E)$. Then by Theorem 20.1.2 there exist unique measures $\lambda_{\perp}^{i}$ and $\lambda_{\|}^{i}$ such that $\lambda^{i}=$ $\lambda_{\perp}^{i}+\lambda_{\|}^{i}$, a set of $\mu^{i}$ measure zero, $N_{i} \in \mathscr{S}_{i}$ such that for all $E \in \mathscr{S}_{i}, \lambda_{\perp}^{i}(E)=\lambda^{i}\left(E \cap N_{i}\right)$ and $\lambda_{\|}^{i} \ll \mu^{i}$. Define for $E \in \mathscr{S}$

$$
\lambda_{\perp}(E) \equiv \sum_{i} \lambda_{\perp}^{i}\left(E \cap \Omega_{i}\right), \lambda_{\|}(E) \equiv \sum_{i} \lambda_{\|}^{i}\left(E \cap \Omega_{i}\right), N \equiv \cup_{i} N_{i}
$$

First observe that $\lambda_{\perp}$ and $\lambda_{\|}$are measures.

$$
\begin{aligned}
\lambda_{\perp}\left(\cup_{j=1}^{\infty} E_{j}\right) & \equiv \sum_{i} \lambda_{\perp}^{i}\left(\cup_{j=1}^{\infty} E_{j} \cap \Omega_{i}\right)=\sum_{i} \sum_{j} \lambda_{\perp}^{i}\left(E_{j} \cap \Omega_{i}\right) \\
& =\sum_{j} \sum_{i} \lambda_{\perp}^{i}\left(E_{j} \cap \Omega_{i}\right)=\sum_{j} \sum_{i} \lambda\left(E_{j} \cap \Omega_{i} \cap N_{i}\right) \\
& =\sum_{j} \sum_{i} \lambda_{\perp}^{i}\left(E_{j} \cap \Omega_{i}\right)=\sum_{j} \lambda_{\perp}\left(E_{j}\right) .
\end{aligned}
$$

The argument for $\lambda_{\|}$is similar. Now

$$
\mu(N)=\sum_{i} \mu\left(N \cap \Omega_{i}\right)=\sum_{i} \mu^{i}\left(N_{i}\right)=0
$$

and

$$
\begin{aligned}
\lambda_{\perp}(E) & \equiv \sum_{i} \lambda_{\perp}^{i}\left(E \cap \Omega_{i}\right)=\sum_{i} \lambda^{i}\left(E \cap \Omega_{i} \cap N_{i}\right) \\
& =\sum_{i} \lambda\left(E \cap \Omega_{i} \cap N\right)=\lambda(E \cap N)
\end{aligned}
$$

Also if $\mu(E)=0$, then $\mu^{i}\left(E \cap \Omega_{i}\right)=0$ and so $\lambda_{\|}^{i}\left(E \cap \Omega_{i}\right)=0$. Therefore,

$$
\lambda_{\|}(E)=\sum_{i} \lambda_{\|}^{i}\left(E \cap \Omega_{i}\right)=0
$$

The decomposition is unique because of the uniqueness of the $\lambda_{\|}^{i}$ and $\lambda_{\perp}^{i}$ and the observation that some other decomposition must coincide with the given one on the $\Omega_{i}$.

### 20.2 Vector Measures

The next topic will use the Radon Nikodym theorem. It is the topic of vector and complex measures. The main interest is in complex measures although a vector measure can have values in any topological vector space. Whole books have been written on this subject. See for example the book by Diestal and Uhl [41] titled Vector measures.
Definition 20.2.1 Let $(V,\|\cdot\|)$ be a normed linear space and let $(\Omega, \mathscr{S})$ be a measure space. A function $\mu: \mathscr{S} \rightarrow V$ is a vector measure if $\mu$ is countably additive. That is, if $\left\{E_{i}\right\}_{i=1}^{\infty}$ is a sequence of disjoint sets of $\mathscr{S}$,

$$
\mu\left(\cup_{i=1}^{\infty} E_{i}\right)=\sum_{i=1}^{\infty} \mu\left(E_{i}\right)
$$

Note that it makes sense to take finite sums because it is given that $\mu$ has values in a vector space in which vectors can be summed. In the above, $\mu\left(E_{i}\right)$ is a vector. It might be a point in $\mathbb{R}^{n}$ or in any other vector space. In many of the most important applications, it is a vector in some sort of function space which may be infinite dimensional. The infinite sum has the usual meaning. That is

$$
\sum_{i=1}^{\infty} \mu\left(E_{i}\right)=\lim _{n \rightarrow \infty} \sum_{i=1}^{n} \mu\left(E_{i}\right)
$$

where the limit takes place relative to the norm on $V$.
Definition 20.2.2 Let $(\Omega, \mathscr{S})$ be a measure space and let $\mu$ be a vector measure defined on $\mathscr{S}$. A subset, $\pi(E)$, of $\mathscr{S}$ is called a partition of $E$ if $\pi(E)$ consists of finitely many disjoint sets of $\mathscr{S}$ and $\cup \pi(E)=E$. Let

$$
|\mu|(E)=\sup \left\{\sum_{F \in \pi(E)}\|\mu(F)\|: \pi(E) \text { is a partition of } E\right\} .
$$

$|\mu|$ is called the total variation of $\mu$.
The next theorem may seem a little surprising. It states that, if finite, the total variation is a nonnegative measure.

Theorem 20.2.3 If $|\mu|(\Omega)<\infty$, then $|\mu|$ is a measure on $\mathscr{S}$. Even if $|\mu|(\Omega)=\infty$,

$$
|\mu|\left(\cup_{i=1}^{\infty} E_{i}\right) \leq \sum_{i=1}^{\infty}|\mu|\left(E_{i}\right)
$$

That is $|\mu|$ is subadditive and $|\mu|(A) \leq|\mu|(B)$ whenever $A, B \in \mathscr{S}$ with $A \subseteq B$.
Proof: Consider the last claim. Let $a<|\mu|(A)$ and let $\pi(A)$ be a partition of $A$ such that

$$
a<\sum_{F \in \pi(A)}\|\mu(F)\|
$$

Then $\pi(A) \cup\{B \backslash A\}$ is a partition of $B$ and

$$
|\mu|(B) \geq \sum_{F \in \pi(A)}\|\mu(F)\|+\|\mu(B \backslash A)\|>a .
$$

Since this is true for all such $a$, it follows $|\mu|(B) \geq|\mu|(A)$ as claimed.
Let $\left\{E_{j}\right\}_{j=1}^{\infty}$ be a sequence of disjoint sets of $\mathscr{S}$ and let $E_{\infty}=\cup_{j=1}^{\infty} E_{j}$. Then letting $a<|\mu|\left(E_{\infty}\right)$, it follows from the definition of total variation there exists a partition of $E_{\infty}$, $\pi\left(E_{\infty}\right)=\left\{A_{1}, \cdots, A_{n}\right\}$ such that

$$
a<\sum_{i=1}^{n}\left\|\mu\left(A_{i}\right)\right\|
$$

Also,

$$
A_{i}=\cup_{j=1}^{\infty} A_{i} \cap E_{j}
$$

and so by the triangle inequality, $\left\|\mu\left(A_{i}\right)\right\| \leq \sum_{j=1}^{\infty}\left\|\mu\left(A_{i} \cap E_{j}\right)\right\|$. Therefore, by the above, and either Fubini's theorem or Lemma 11.3.3 on Page 236

$$
\begin{aligned}
a & <\sum_{i=1}^{n} \overbrace{\sum_{j=1}^{\infty}\left\|\mu\left(A_{i} \cap E_{j}\right)\right\|}^{\geq\left\|\mu\left(A_{i}\right)\right\|} \\
& =\sum_{j=1}^{\infty} \sum_{i=1}^{n}\left\|\mu\left(A_{i} \cap E_{j}\right)\right\| \\
& \leq \sum_{j=1}^{\infty}|\mu|\left(E_{j}\right)
\end{aligned}
$$

because $\left\{A_{i} \cap E_{j}\right\}_{i=1}^{n}$ is a partition of $E_{j}$.
Since $a$ is arbitrary, this shows

$$
|\mu|\left(\cup_{j=1}^{\infty} E_{j}\right) \leq \sum_{j=1}^{\infty}|\mu|\left(E_{j}\right)
$$

If the sets, $E_{j}$ are not disjoint, let $F_{1}=E_{1}$ and if $F_{n}$ has been chosen, let $F_{n+1} \equiv E_{n+1} \backslash$ $\cup_{i=1}^{n} E_{i}$. Thus the sets, $F_{i}$ are disjoint and $\cup_{i=1}^{\infty} F_{i}=\cup_{i=1}^{\infty} E_{i}$. Therefore,

$$
|\mu|\left(\cup_{j=1}^{\infty} E_{j}\right)=|\mu|\left(\cup_{j=1}^{\infty} F_{j}\right) \leq \sum_{j=1}^{\infty}|\mu|\left(F_{j}\right) \leq \sum_{j=1}^{\infty}|\mu|\left(E_{j}\right)
$$

and proves $|\mu|$ is always subadditive as claimed regardless of whether $|\mu|(\Omega)<\infty$.
Now suppose $|\mu|(\Omega)<\infty$ and let $E_{1}$ and $E_{2}$ be sets of $\mathscr{S}$ such that $E_{1} \cap E_{2}=\emptyset$ and let $\left\{A_{1}^{i} \cdots A_{n_{i}}^{i}\right\}=\pi\left(E_{i}\right)$, a partition of $E_{i}$ which is chosen such that

$$
|\mu|\left(E_{i}\right)-\varepsilon<\sum_{j=1}^{n_{i}}\left\|\mu\left(A_{j}^{i}\right)\right\| i=1,2
$$

Such a partition exists because of the definition of the total variation. Consider the sets which are contained in either of $\pi\left(E_{1}\right)$ or $\pi\left(E_{2}\right)$, it follows this collection of sets is a partition of $E_{1} \cup E_{2}$ denoted by $\pi\left(E_{1} \cup E_{2}\right)$. Then by the above inequality and the definition of total variation,

$$
|\mu|\left(E_{1} \cup E_{2}\right) \geq \sum_{F \in \pi\left(E_{1} \cup E_{2}\right)} \| \mu(F)| |>|\mu|\left(E_{1}\right)+|\mu|\left(E_{2}\right)-2 \varepsilon,
$$

which shows that since $\varepsilon>0$ was arbitrary,

$$
\begin{equation*}
|\mu|\left(E_{1} \cup E_{2}\right) \geq|\mu|\left(E_{1}\right)+|\mu|\left(E_{2}\right) \tag{20.2.5}
\end{equation*}
$$

Then 20.2.5 implies that whenever the $E_{i}$ are disjoint, $|\mu|\left(\cup_{j=1}^{n} E_{j}\right) \geq \sum_{j=1}^{n}|\mu|\left(E_{j}\right)$. Therefore,

$$
\sum_{j=1}^{\infty}|\mu|\left(E_{j}\right) \geq|\mu|\left(\cup_{j=1}^{\infty} E_{j}\right) \geq|\mu|\left(\cup_{j=1}^{n} E_{j}\right) \geq \sum_{j=1}^{n}|\mu|\left(E_{j}\right) .
$$

Since $n$ is arbitrary,

$$
|\mu|\left(\cup_{j=1}^{\infty} E_{j}\right)=\sum_{j=1}^{\infty}|\mu|\left(E_{j}\right)
$$

which shows that $|\mu|$ is a measure as claimed.
The following corollary is interesting. It concerns the case that $\mu$ is only finitely additive.

Corollary 20.2.4 Suppose $(\Omega, \mathscr{F})$ is a set with a $\sigma$ algebra of subsets $\mathscr{F}$ and suppose $\mu: \mathscr{F} \rightarrow \mathbb{C}$ is only finitely additive. That is, $\mu\left(\cup_{i=1}^{n} E_{i}\right)=\sum_{i=1}^{n} \mu\left(E_{i}\right)$ whenever the $E_{i}$ are disjoint. Then $|\mu|$, defined in the same way as above, is also finitely additive provided $|\mu|$ is finite.

Proof: Say $E \cap F=\emptyset$ for $E, F \in \mathscr{F}$. Let $\pi(E), \pi(F)$ suitable partitions for which the following holds.

$$
|\mu|(E \cup F) \geq \sum_{A \in \pi(E)}|\mu(A)|+\sum_{B \in \pi(F)}|\mu(B)| \geq|\mu|(E)+|\mu|(F)-2 \varepsilon .
$$

Similar considerations apply to any finite union.
Now let $E=\cup_{i=1}^{n} E_{i}$ where the $E_{i}$ are disjoint. Then letting $\pi(E)$ be a partition of $E$,

$$
|\mu|(E)-\varepsilon \leq \sum_{F \in \pi(E)}|\mu(F)|
$$

it follows that

$$
\begin{aligned}
|\mu|(E) & \leq \varepsilon+\sum_{F \in \pi(E)}|\mu(F)|=\varepsilon+\sum_{F \in \pi(E)}\left|\sum_{i=1}^{n} \mu\left(F \cap E_{i}\right)\right| \\
& \leq \varepsilon+\sum_{i=1}^{n} \sum_{F \in \pi(E)}\left|\mu\left(F \cap E_{i}\right)\right| \leq \varepsilon+\sum_{i=1}^{n}|\mu|\left(E_{i}\right)
\end{aligned}
$$

which shows $|\mu|$ is finitely additive.
In the case that $\mu$ is a complex measure, it is always the case that $|\mu|(\Omega)<\infty$.
Theorem 20.2.5 Suppose $\mu$ is a complex measure on $(\Omega, \mathscr{S})$ where $\mathscr{S}$ is a $\sigma$ algebra of subsets of $\Omega$. That is, whenever, $\left\{E_{i}\right\}$ is a sequence of disjoint sets of $\mathscr{S}$,

$$
\mu\left(\cup_{i=1}^{\infty} E_{i}\right)=\sum_{i=1}^{\infty} \mu\left(E_{i}\right) .
$$

Then $|\mu|(\Omega)<\infty$.

Proof: First here is a claim.
Claim: Suppose $|\mu|(E)=\infty$. Then there are disjoint subsets of $E, A$ and $B$ such that $E=A \cup B,|\mu(A)|,|\mu(B)|>1$ and $|\mu|(B)=\infty$.

Proof of the claim: From the definition of $|\mu|$, there exists a partition of $E, \pi(E)$ such that

$$
\begin{equation*}
\sum_{F \in \pi(E)}|\mu(F)|>20(1+|\mu(E)|) . \tag{20.2.6}
\end{equation*}
$$

Here 20 is just a nice sized number. No effort is made to be delicate in this argument. Also note that $\mu(E) \in \mathbb{C}$ because it is given that $\mu$ is a complex measure. Consider the following picture consisting of two lines in the complex plane having slopes 1 and -1 which intersect at the origin, dividing the complex plane into four closed sets, $R_{1}, R_{2}, R_{3}$, and $R_{4}$ as shown.


Let $\pi_{i}$ consist of those sets, $A$ of $\pi(E)$ for which $\mu(A) \in R_{i}$. Thus, some sets, $A$ of $\pi(E)$ could be in two of the $\pi_{i}$ if $\mu(A)$ is on one of the intersecting lines. This is not important. The thing which is important is that if $\mu(A) \in R_{1}$ or $R_{3}$, then $\frac{\sqrt{2}}{2}|\mu(A)| \leq|\operatorname{Re}(\mu(A))|$ and if $\mu(A) \in R_{2}$ or $R_{4}$ then $\frac{\sqrt{2}}{2}|\mu(A)| \leq|\operatorname{Im}(\mu(A))|$ and $\operatorname{Re}(z)$ has the same sign for $z$ in $R_{1}$ and $R_{3}$ while $\operatorname{Im}(z)$ has the same sign for $z$ in $R_{2}$ or $R_{4}$. Then by 20.2.6, it follows that for some $i$,

$$
\begin{equation*}
\sum_{F \in \pi_{i}}|\mu(F)|>5(1+|\mu(E)|) . \tag{20.2.7}
\end{equation*}
$$

Suppose $i$ equals 1 or 3 . A similar argument using the imaginary part applies if $i$ equals 2 or 4. Then,

$$
\begin{aligned}
\left|\sum_{F \in \pi_{i}} \mu(F)\right| & \geq\left|\sum_{F \in \pi_{i}} \operatorname{Re}(\mu(F))\right|=\sum_{F \in \pi_{i}}|\operatorname{Re}(\mu(F))| \\
& \geq \frac{\sqrt{2}}{2} \sum_{F \in \pi_{i}}|\mu(F)|>5 \frac{\sqrt{2}}{2}(1+|\mu(E)|)
\end{aligned}
$$

Now letting $C$ be the union of the sets in $\pi_{i}$,

$$
\begin{equation*}
|\mu(C)|=\left|\sum_{F \in \pi_{i}} \mu(F)\right|>\frac{5}{2}(1+|\mu(E)|)>1 \tag{20.2.8}
\end{equation*}
$$

Define $D \equiv E \backslash C$.


Then $\mu(C)+\mu(E \backslash C)=\mu(E)$ and so

$$
\begin{aligned}
\frac{5}{2}(1+|\mu(E)|) & <|\mu(C)|=|\mu(E)-\mu(E \backslash C)| \\
& =|\mu(E)-\mu(D)| \leq|\mu(E)|+|\mu(D)|
\end{aligned}
$$

and so

$$
1<\frac{5}{2}+\frac{3}{2}|\mu(E)|<|\mu(D)| .
$$

Now since $|\mu|(E)=\infty$, it follows from Theorem 20.2.5 that $\infty=|\mu|(E) \leq|\mu|(C)+|\mu|(D)$ and so either $|\mu|(C)=\infty$ or $|\mu|(D)=\infty$. If $|\mu|(C)=\infty$, let $B=C$ and $A=D$. Otherwise, let $B=D$ and $A=C$. This proves the claim.

Now suppose $|\mu|(\Omega)=\infty$. Then from the claim, there exist $A_{1}$ and $B_{1}$ such that

$$
|\mu|\left(B_{1}\right)=\infty,\left|\mu\left(B_{1}\right)\right|,\left|\mu\left(A_{1}\right)\right|>1
$$

and $A_{1} \cup B_{1}=\Omega$. Let $B_{1} \equiv \Omega \backslash A$ play the same role as $\Omega$ and obtain $A_{2}, B_{2} \subseteq B_{1}$ such that $|\mu|\left(B_{2}\right)=\infty,\left|\mu\left(B_{2}\right)\right|,\left|\mu\left(A_{2}\right)\right|>1$, and $A_{2} \cup B_{2}=B_{1}$. Continue in this way to obtain a sequence of disjoint sets, $\left\{A_{i}\right\}$ such that $\left|\mu\left(A_{i}\right)\right|>1$. Then since $\mu$ is a measure,

$$
\mu\left(\cup_{i=1}^{\infty} A_{i}\right)=\sum_{i=1}^{\infty} \mu\left(A_{i}\right)
$$

but this is impossible because $\lim _{i \rightarrow \infty} \mu\left(A_{i}\right) \neq 0$. This proves the theorem.
Theorem 20.2.6 Let $(\Omega, \mathscr{S})$ be a measure space and let $\lambda: \mathscr{S} \rightarrow \mathbb{C}$ be a complex vector measure. Thus $|\lambda|(\Omega)<\infty$. Let $\mu: \mathscr{S} \rightarrow[0, \mu(\Omega)]$ be a finite measure such that $\lambda \ll \mu$. Then there exists a unique $f \in L^{1}(\Omega)$ such that for all $E \in \mathscr{S}$,

$$
\int_{E} f d \mu=\lambda(E)
$$

Proof: It is clear that $\operatorname{Re} \lambda$ and $\operatorname{Im} \lambda$ are real-valued vector measures on $\mathscr{S}$. Since $|\lambda|(\Omega)<\infty$, it follows easily that $|\operatorname{Re} \lambda|(\Omega)$ and $|\operatorname{Im} \lambda|(\Omega)<\infty$. This is clear because

$$
|\lambda(E)| \geq|\operatorname{Re} \lambda(E)|,|\operatorname{Im} \lambda(E)|
$$

Therefore, each of

$$
\frac{|\operatorname{Re} \lambda|+\operatorname{Re} \lambda}{2}, \frac{|\operatorname{Re} \lambda|-\operatorname{Re}(\lambda)}{2}, \frac{|\operatorname{Im} \lambda|+\operatorname{Im} \lambda}{2}, \text { and } \frac{|\operatorname{Im} \lambda|-\operatorname{Im}(\lambda)}{2}
$$

are finite measures on $\mathscr{S}$. It is also clear that each of these finite measures are absolutely continuous with respect to $\mu$ and so there exist unique nonnegative functions in $L^{1}(\Omega), f_{1}, f_{2}, g_{1}, g_{2}$ such that for all $E \in \mathscr{S}$,

$$
\begin{aligned}
\frac{1}{2}(|\operatorname{Re} \lambda|+\operatorname{Re} \lambda)(E) & =\int_{E} f_{1} d \mu \\
\frac{1}{2}(|\operatorname{Re} \lambda|-\operatorname{Re} \lambda)(E) & =\int_{E} f_{2} d \mu \\
\frac{1}{2}(|\operatorname{Im} \lambda|+\operatorname{Im} \lambda)(E) & =\int_{E} g_{1} d \mu \\
\frac{1}{2}(|\operatorname{Im} \lambda|-\operatorname{Im} \lambda)(E) & =\int_{E} g_{2} d \mu
\end{aligned}
$$

Now let $f=f_{1}-f_{2}+i\left(g_{1}-g_{2}\right)$.
The following corollary is about representing a vector measure in terms of its total variation. It is like representing a complex number in the form $r e^{i \theta}$. The proof requires the following lemma.

Lemma 20.2.7 Suppose $(\Omega, \mathscr{S}, \mu)$ is a measure space and $f$ is a function in $L^{1}(\Omega, \mu)$ with the property that

$$
\left|\int_{E} f d \mu\right| \leq \mu(E)
$$

for all $E \in \mathscr{S}$. Then $|f| \leq 1$ a.e.
Proof of the lemma: Consider the following picture.

where $B(p, r) \cap B(0,1)=\emptyset$. Let $E=f^{-1}(B(p, r))$. In fact $\mu(E)=0$. If $\mu(E) \neq 0$ then

$$
\begin{aligned}
\left|\frac{1}{\mu(E)} \int_{E} f d \mu-p\right| & =\left|\frac{1}{\mu(E)} \int_{E}(f-p) d \mu\right| \\
& \leq \frac{1}{\mu(E)} \int_{E}|f-p| d \mu<r
\end{aligned}
$$

because on $E,|f(x)-p|<r$. Hence

$$
\left|\frac{1}{\mu(E)} \int_{E} f d \mu\right|>1
$$

because it is closer to $p$ than $r$. (Refer to the picture.) However, this contradicts the assumption of the lemma. It follows $\mu(E)=0$. Since the set of complex numbers, $z$ such that $|z|>1$ is an open set, it equals the union of countably many balls, $\left\{B_{i}\right\}_{i=1}^{\infty}$. Therefore,

$$
\begin{aligned}
\mu\left(f^{-1}(\{z \in \mathbb{C}:|z|>1\})\right. & =\mu\left(\cup_{k=1}^{\infty} f^{-1}\left(B_{k}\right)\right) \\
& \leq \sum_{k=1}^{\infty} \mu\left(f^{-1}\left(B_{k}\right)\right)=0
\end{aligned}
$$

Thus $|f(x)| \leq 1$ a.e. as claimed. This proves the lemma.
Corollary 20.2.8 Let $\lambda$ be a complex vector measure with $|\lambda|(\Omega)<\infty^{1}$ Then there exists a unique $f \in L^{1}(\Omega)$ such that $\lambda(E)=\int_{E} f d|\lambda|$. Furthermore, $|f|=1$ for $|\lambda|$ a.e. This is called the polar decomposition of $\lambda$.

Proof: First note that $\lambda \ll|\lambda|$ and so such an $L^{1}$ function exists and is unique. It is required to show $|f|=1$ a.e. If $|\lambda|(E) \neq 0$,

$$
\left|\frac{\lambda(E)}{|\lambda|(E)}\right|=\left|\frac{1}{|\lambda|(E)} \int_{E} f d\right| \lambda| | \leq 1
$$

Therefore by Lemma 20.2.7, $|f| \leq 1,|\lambda|$ a.e. Now let

$$
E_{n}=\left[|f| \leq 1-\frac{1}{n}\right]
$$

Let $\left\{F_{1}, \cdots, F_{m}\right\}$ be a partition of $E_{n}$. Then

$$
\begin{aligned}
\sum_{i=1}^{m}\left|\lambda\left(F_{i}\right)\right| & =\sum_{i=1}^{m}\left|\int_{F_{i}} f d\right| \lambda| | \leq \sum_{i=1}^{m} \int_{F_{i}}|f| d|\lambda| \\
& \leq \sum_{i=1}^{m} \int_{F_{i}}\left(1-\frac{1}{n}\right) d|\lambda|=\sum_{i=1}^{m}\left(1-\frac{1}{n}\right)|\lambda|\left(F_{i}\right) \\
& =|\lambda|\left(E_{n}\right)\left(1-\frac{1}{n}\right)
\end{aligned}
$$

Then taking the supremum over all partitions,

$$
|\lambda|\left(E_{n}\right) \leq\left(1-\frac{1}{n}\right)|\lambda|\left(E_{n}\right)
$$

which shows $|\lambda|\left(E_{n}\right)=0$. Hence $|\lambda|([|f|<1])=0$ because $[|f|<1]=\cup_{n=1}^{\infty} E_{n}$. This proves Corollary 20.2.8.

Corollary 20.2.9 Let $\lambda$ be a complex vector measure such that $\lambda \ll \mu$ where $\mu$ is $\sigma$ finite. Then there exists a unique $g \in L^{1}(\Omega, \mu)$ such that $\lambda(E)=\int_{E} g d \mu$.

[^17]Proof: By Corollary 20.2.8 and Theorem 20.2.5 which says that $|\lambda|$ is finite, there exists a unique $f$ such that $|f|=1|\lambda|$ a.e. and

$$
\lambda(E)=\int_{E} f d|\lambda|
$$

Now $|\lambda| \ll \mu$ and so it follows from Corollary 20.1.3 there exists a unique nonnegative measurable function $h$ such that for all $E$ measurable,

$$
|\lambda|(E)=\int_{E} h d \mu
$$

where since $|\lambda|$ is finite, $h \in L^{1}(\Omega, \mu)$. It follows from approximating $f$ with simple functions and using the above formula that

$$
\lambda(E)=\int_{E} f h d \mu
$$

Then let $g=L^{1}(\Omega, \mu)$. This proves the corollary.
Corollary 20.2.10 Suppose $(\Omega, \mathscr{S})$ is a measure space and $\mu$ is a finite nonnegative measure on $\mathscr{S}$. Then for $h \in L^{1}(\mu)$, define a complex measure, $\lambda$ by

$$
\lambda(E) \equiv \int_{E} h d \mu
$$

Then

$$
|\lambda|(E)=\int_{E}|h| d \mu
$$

Furthermore, $|h|=\bar{g} h$ where $g d|\lambda|$ is the polar decomposition of $\lambda$,

$$
\lambda(E)=\int_{E} g d|\lambda|
$$

Proof: From Corollary 20.2.8 there exists $g$ such that $|g|=1,|\lambda|$ a.e. and for all $E \in \mathscr{S}$

$$
\lambda(E)=\int_{E} g d|\lambda|=\int_{E} h d \mu
$$

Let $s_{n}$ be a sequence of simple functions converging pointwise to $\bar{g}$. Then from the above,

$$
\int_{E} g s_{n} d|\lambda|=\int_{E} s_{n} h d \mu
$$

Passing to the limit using the dominated convergence theorem,

$$
\int_{E} d|\lambda|=\int_{E} \bar{g} h d \mu
$$

It follows $\bar{g} h \geq 0$ a.e. and $|\bar{g}|=1$. Therefore, $|h|=|\bar{g} h|=\bar{g} h$. It follows from the above, that

$$
|\lambda|(E)=\int_{E} d|\lambda|=\int_{E} \bar{g} h d \mu=\int_{E} d|\lambda|=\int_{E}|h| d \mu
$$

and this proves the corollary.

### 20.3 Representation Theorems For The Dual Space Of $L^{p}$

Recall the concept of the dual space of a Banach space in the Chapter on Banach space starting on Page 435. The next topic deals with the dual space of $L^{p}$ for $p \geq 1$ in the case where the measure space is $\sigma$ finite or finite. In what follows $q=\infty$ if $p=1$ and otherwise, $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 20.3.1 (Riesz representation theorem) Let $p>1$ and let $(\Omega, \mathscr{S}, \mu)$ be a finite measure space. If $\Lambda \in\left(L^{p}(\Omega)\right)^{\prime}$, then there exists a unique $h \in L^{q}(\Omega)\left(\frac{1}{p}+\frac{1}{q}=1\right)$ such that

$$
\Lambda f=\int_{\Omega} h f d \mu
$$

This function satisfies $\|h\|_{q}=\|\Lambda\|$ where $\|\Lambda\|$ is the operator norm of $\Lambda$.
Proof: (Uniqueness) If $h_{1}$ and $h_{2}$ both represent $\Lambda$, consider

$$
f=\left|h_{1}-h_{2}\right|^{q-2}\left(\overline{h_{1}}-\overline{h_{2}}\right),
$$

where $\bar{h}$ denotes complex conjugation. By Holder's inequality, it is easy to see that $f \in$ $L^{p}(\Omega)$. Thus

$$
\begin{gathered}
0=\Lambda f-\Lambda f= \\
\int h_{1}\left|h_{1}-h_{2}\right|^{q-2}\left(\overline{h_{1}}-\overline{h_{2}}\right)-h_{2}\left|h_{1}-h_{2}\right|^{q-2}\left(\overline{h_{1}}-\overline{h_{2}}\right) d \mu \\
=\int\left|h_{1}-h_{2}\right|^{q} d \mu
\end{gathered}
$$

Therefore $h_{1}=h_{2}$ and this proves uniqueness.
Now let $\lambda(E)=\Lambda\left(\mathscr{X}_{E}\right)$. Since this is a finite measure space $\mathscr{X}_{E}$ is an element of $L^{p}(\Omega)$ and so it makes sense to write $\Lambda\left(\mathscr{X}_{E}\right)$. In fact $\lambda$ is a complex measure having finite total variation. Let $A_{1}, \cdots, A_{n}$ be a partition of $\Omega$.

$$
\left|\Lambda \mathscr{X}_{A_{i}}\right|=w_{i}\left(\Lambda \mathscr{X}_{A_{i}}\right)=\Lambda\left(w_{i} \mathscr{X}_{A_{i}}\right)
$$

for some $w_{i} \in \mathbb{C},\left|w_{i}\right|=1$. Thus

$$
\begin{gathered}
\sum_{i=1}^{n}\left|\lambda\left(A_{i}\right)\right|=\sum_{i=1}^{n}\left|\Lambda\left(\mathscr{X}_{A_{i}}\right)\right|=\Lambda\left(\sum_{i=1}^{n} w_{i} \mathscr{X}_{A_{i}}\right) \\
\leq\|\Lambda\| \left\lvert\,\left(\int\left|\sum_{i=1}^{n} w_{i} \mathscr{X}_{A_{i}}\right|^{p} d \mu\right)^{\frac{1}{p}}=\|\Lambda\|\left(\int_{\Omega} d \mu\right)^{\frac{1}{p}}=\|\Lambda\| \mu(\Omega)^{\frac{1}{p}} .\right.
\end{gathered}
$$

This is because if $x \in \Omega, x$ is contained in exactly one of the $A_{i}$ and so the absolute value of the sum in the first integral above is equal to 1 . Therefore $|\lambda|(\Omega)<\infty$ because this was an arbitrary partition. Also, if $\left\{E_{i}\right\}_{i=1}^{\infty}$ is a sequence of disjoint sets of $\mathscr{S}$, let

$$
F_{n}=\cup_{i=1}^{n} E_{i}, F=\cup_{i=1}^{\infty} E_{i}
$$

Then by the Dominated Convergence theorem,

$$
\left\|\mathscr{X}_{F_{n}}-\mathscr{X}_{F}\right\|_{p} \rightarrow 0 .
$$

Therefore, by continuity of $\Lambda$,

$$
\lambda(F)=\Lambda\left(\mathscr{X}_{F}\right)=\lim _{n \rightarrow \infty} \Lambda\left(\mathscr{X}_{F_{n}}\right)=\lim _{n \rightarrow \infty} \sum_{k=1}^{n} \Lambda\left(\mathscr{X}_{E_{k}}\right)=\sum_{k=1}^{\infty} \lambda\left(E_{k}\right) .
$$

This shows $\lambda$ is a complex measure with $|\lambda|$ finite.
It is also clear from the definition of $\lambda$ that $\lambda \ll \mu$. Therefore, by the Radon Nikodym theorem, there exists $h \in L^{1}(\Omega)$ with

$$
\lambda(E)=\int_{E} h d \mu=\Lambda\left(\mathscr{X}_{E}\right)
$$

Actually $h \in L^{q}$ and satisfies the other conditions above. Let $s=\sum_{i=1}^{m} c_{i} \mathscr{X}_{E_{i}}$ be a simple function. Then since $\Lambda$ is linear,

$$
\begin{equation*}
\Lambda(s)=\sum_{i=1}^{m} c_{i} \Lambda\left(\mathscr{X}_{E_{i}}\right)=\sum_{i=1}^{m} c_{i} \int_{E_{i}} h d \mu=\int h s d \mu . \tag{20.3.9}
\end{equation*}
$$

Claim: If $f$ is uniformly bounded and measurable, then

$$
\Lambda(f)=\int h f d \mu
$$

Proof of claim: Since $f$ is bounded and measurable, there exists a sequence of simple functions, $\left\{s_{n}\right\}$ which converges to $f$ pointwise and in $L^{p}(\Omega)$. This follows from Theorem 11.3.9 on Page 241 upon breaking $f$ up into positive and negative parts of real and complex parts. In fact this theorem gives uniform convergence. Then

$$
\Lambda(f)=\lim _{n \rightarrow \infty} \Lambda\left(s_{n}\right)=\lim _{n \rightarrow \infty} \int h s_{n} d \mu=\int h f d \mu
$$

the first equality holding because of continuity of $\Lambda$, the second following from 20.3.9 and the third holding by the dominated convergence theorem.

This is a very nice formula but it still has not been shown that $h \in L^{q}(\Omega)$.
Let $E_{n}=\{x:|h(x)| \leq n\}$. Thus $\left|h \mathscr{X}_{E_{n}}\right| \leq n$. Then

$$
\left|h \mathscr{X}_{E_{n}}\right|^{q-2}\left(\bar{h} \mathscr{X}_{E_{n}}\right) \in L^{p}(\Omega) .
$$

By the claim, it follows that

$$
\begin{gathered}
\left\|h \mathscr{X}_{E_{n}}\right\|_{q}^{q}=\int h\left|h \mathscr{X}_{E_{n}}\right|^{q-2}\left(\bar{h} \mathscr{X}_{E_{n}}\right) d \mu=\Lambda\left(\left|h \mathscr{X}_{E_{n}}\right|^{q-2}\left(\bar{h} \mathscr{X}_{E_{n}}\right)\right) \\
\leq\|\Lambda\|\left\|\left|h \mathscr{X}_{E_{n}}\right|^{q-2}\left(\bar{h} \mathscr{X}_{E_{n}}\right)\right\|_{p}=\|\Lambda\|\left\|h \mathscr{X}_{E_{n}}\right\|_{q}^{\frac{q}{p}}
\end{gathered}
$$

the last equality holding because $q-1=q / p$ and so

$$
\begin{aligned}
\left(\int\left|\left|h \mathscr{X}_{E_{n}}\right|^{q-2}\left(\bar{h} \mathscr{X}_{E_{n}}\right)\right|^{p} d \mu\right)^{1 / p} & =\left(\int\left(\left|h \mathscr{X}_{E_{n}}\right|^{q / p}\right)^{p} d \mu\right)^{1 / p} \\
& =\|\left. h \mathscr{X}_{E_{n}}\right|_{q} ^{q}
\end{aligned}
$$

Therefore, since $q-\frac{q}{p}=1$, it follows that

$$
\left\|h \mathscr{X}_{E_{n}}\right\|_{q} \leq\|\Lambda\| .
$$

Letting $n \rightarrow \infty$, the Monotone Convergence theorem implies

$$
\begin{equation*}
\|h\|_{q} \leq\|\Lambda\| \tag{20.3.10}
\end{equation*}
$$

Now that $h$ has been shown to be in $L^{q}(\Omega)$, it follows from 20.3.9 and the density of the simple functions, Theorem 15.2.1 on Page 406, that

$$
\Lambda f=\int h f d \mu
$$

for all $f \in L^{p}(\Omega)$.
It only remains to verify the last claim.

$$
\|\Lambda\|=\sup \left\{\int h f:\|f\|_{p} \leq 1\right\} \leq\|h\|_{q} \leq\|\Lambda\|
$$

by 20.3.10, and Holder's inequality. This proves the theorem.
To represent elements of the dual space of $L^{1}(\Omega)$, another Banach space is needed.
Definition 20.3.2 Let $(\Omega, \mathscr{S}, \mu)$ be a measure space. $L^{\infty}(\Omega)$ is the vector space of measurable functions such that for some $M>0,|f(x)| \leq M$ for all $x$ outside of some set of measure zero $\left(|f(x)| \leq M\right.$ a.e.). Define $f=g$ when $f(x)=g(x)$ a.e. and $\|f\|_{\infty} \equiv \inf \{M$ : $|f(x)| \leq M$ a.e. $\}.$

Theorem 20.3.3 $L^{\infty}(\Omega)$ is a Banach space.
Proof: It is clear that $L^{\infty}(\Omega)$ is a vector space. Is $\left\|\|_{\infty}\right.$ a norm?
Claim: If $f \in L^{\infty}(\Omega)$, then $|f(x)| \leq\|f\|_{\infty}$ a.e.
Proof of the claim: $\left\{x:|f(x)| \geq\|f\|_{\infty}+n^{-1}\right\} \equiv E_{n}$ is a set of measure zero according to the definition of $\|f\|_{\infty}$. Furthermore, $\left\{x:|f(x)|>| | f \|_{\infty}\right\}=\cup_{n} E_{n}$ and so it is also a set of measure zero. This verifies the claim.

Now if $\|f\|_{\infty}=0$ it follows that $f(x)=0$ a.e. Also if $f, g \in L^{\infty}(\Omega)$,

$$
|f(x)+g(x)| \leq|f(x)|+|g(x)| \leq\|f\|_{\infty}+\|g\|_{\infty}
$$

a.e. and so $\|f\|_{\infty}+\|g\|_{\infty}$ serves as one of the constants, $M$ in the definition of $\|f+g\|_{\infty}$. Therefore,

$$
\|f+g\|_{\infty} \leq\|f\|_{\infty}+\|g\|_{\infty}
$$

Next let $c$ be a number. Then $|c f(x)|=|c||f(x)| \leq|c|\|f\|_{\infty}$ and so $\|c f\|_{\infty} \leq|c|\|f\|_{\infty}$. Therefore since $c$ is arbitrary, $\|f\|_{\infty}=\|c(1 / c) f\|_{\infty} \leq\left|\frac{1}{c}\right|\|c f\|_{\infty}$ which implies $|c|\|f\|_{\infty} \leq$ $\|c f\|_{\infty}$. Thus $\left\|\|_{\infty}\right.$ is a norm as claimed.

To verify completeness, let $\left\{f_{n}\right\}$ be a Cauchy sequence in $L^{\infty}(\Omega)$ and use the above claim to get the existence of a set of measure zero, $E_{n m}$ such that for all $x \notin E_{n m}$,

$$
\left|f_{n}(x)-f_{m}(x)\right| \leq\left\|f_{n}-f_{m}\right\|_{\infty}
$$

Let $E=\cup_{n, m} E_{n m}$. Thus $\mu(E)=0$ and for each $x \notin E,\left\{f_{n}(x)\right\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{C}$. Let

$$
f(x)=\left\{\begin{array}{c}
0 \text { if } x \in E \\
\lim _{n \rightarrow \infty} f_{n}(x) \text { if } x \notin E
\end{array}=\lim _{n \rightarrow \infty} \mathscr{X}_{E^{C}}(x) f_{n}(x) .\right.
$$

Then $f$ is clearly measurable because it is the limit of measurable functions. If

$$
F_{n}=\left\{x:\left|f_{n}(x)\right|>\left\|f_{n}\right\|_{\infty}\right\}
$$

and $F=\cup_{n=1}^{\infty} F_{n}$, it follows $\mu(F)=0$ and that for $x \notin F \cup E$,

$$
|f(x)| \leq \lim \inf _{n \rightarrow \infty}\left|f_{n}(x)\right| \leq \lim \inf _{n \rightarrow \infty}\left\|f_{n}\right\|_{\infty}<\infty
$$

because $\left\{\left|\mid f_{n} \|_{\infty}\right\}\right.$ is a Cauchy sequence. $\left(\left\|\left\|f_{n}\right\|_{\infty}-\right\| f_{m}\left\|_{\infty} \mid \leq\right\| f_{n}-f_{m} \|_{\infty}\right.$ by the triangle inequality.) Thus $f \in L^{\infty}(\Omega)$. Let $n$ be large enough that whenever $m>n$,

$$
\left\|f_{m}-f_{n}\right\|_{\infty}<\varepsilon
$$

Then, if $x \notin E$,

$$
\begin{aligned}
\left|f(x)-f_{n}(x)\right| & =\lim _{m \rightarrow \infty}\left|f_{m}(x)-f_{n}(x)\right| \\
& \leq \lim _{m \rightarrow \infty} \inf \left\|f_{m}-f_{n}\right\|_{\infty}<\varepsilon
\end{aligned}
$$

Hence $\left\|f-f_{n}\right\|_{\infty}<\varepsilon$ for all $n$ large enough. This proves the theorem.
The next theorem is the Riesz representation theorem for $\left(L^{1}(\Omega)\right)^{\prime}$.
Theorem 20.3.4 (Riesz representation theorem) Let $(\Omega, \mathscr{S}, \mu)$ be a finite measure space. If $\Lambda \in\left(L^{1}(\Omega)\right)^{\prime}$, then there exists a unique $h \in L^{\infty}(\Omega)$ such that

$$
\Lambda(f)=\int_{\Omega} h f d \mu
$$

for all $f \in L^{1}(\Omega)$. If $h$ is the function in $L^{\infty}(\Omega)$ representing $\Lambda \in\left(L^{1}(\Omega)\right)^{\prime}$, then $\|h\|_{\infty}=$ $|\mid \Lambda \|$.

Proof: Just as in the proof of Theorem 20.3.1, there exists a unique $h \in L^{1}(\Omega)$ such that for all simple functions, $s$,

$$
\begin{equation*}
\Lambda(s)=\int h s d \mu \tag{20.3.11}
\end{equation*}
$$

To show $h \in L^{\infty}(\Omega)$, let $\varepsilon>0$ be given and let

$$
E=\{x:|h(x)| \geq\|\Lambda\|+\varepsilon\} .
$$

Let $|k|=1$ and $h k=|h|$. Since the measure space is finite, $k \in L^{1}(\Omega)$. As in Theorem 20.3.1 let $\left\{s_{n}\right\}$ be a sequence of simple functions converging to $k$ in $L^{1}(\Omega)$, and pointwise. It follows from the construction in Theorem 11.3.9 on Page 241 that it can be assumed $\left|s_{n}\right| \leq 1$. Therefore

$$
\Lambda\left(k \mathscr{X}_{E}\right)=\lim _{n \rightarrow \infty} \Lambda\left(s_{n} \mathscr{X}_{E}\right)=\lim _{n \rightarrow \infty} \int_{E} h s_{n} d \mu=\int_{E} h k d \mu
$$

where the last equality holds by the Dominated Convergence theorem. Therefore,

$$
\begin{aligned}
\|\Lambda\| \mu(E) & \geq\left|\Lambda\left(k \mathscr{X}_{E}\right)\right|=\left|\int_{\Omega} h k \mathscr{X}_{E} d \mu\right|=\int_{E}|h| d \mu \\
& \geq(\|\Lambda\|+\varepsilon) \mu(E)
\end{aligned}
$$

It follows that $\mu(E)=0$. Since $\varepsilon>0$ was arbitrary, $\|\Lambda\| \geq\|h\|_{\infty}$. It was shown that $h \in L^{\infty}(\Omega)$, the density of the simple functions in $L^{1}(\Omega)$ and 20.3 .11 imply

$$
\begin{equation*}
\Lambda f=\int_{\Omega} h f d \mu,\|\Lambda\| \geq\|h\|_{\infty} \tag{20.3.12}
\end{equation*}
$$

This proves the existence part of the theorem. To verify uniqueness, suppose $h_{1}$ and $h_{2}$ both represent $\Lambda$ and let $f \in L^{1}(\Omega)$ be such that $|f| \leq 1$ and $f\left(h_{1}-h_{2}\right)=\left|h_{1}-h_{2}\right|$. Then

$$
0=\Lambda f-\Lambda f=\int\left(h_{1}-h_{2}\right) f d \mu=\int\left|h_{1}-h_{2}\right| d \mu
$$

Thus $h_{1}=h_{2}$. Finally,

$$
\|\Lambda\|=\sup \left\{\left|\int h f d \mu\right|:\|f\|_{1} \leq 1\right\} \leq\|h\|_{\infty} \leq\|\Lambda\|
$$

by 20.3.12.
Next these results are extended to the $\sigma$ finite case.
Lemma 20.3.5 Let $(\Omega, \mathscr{S}, \mu)$ be a measure space and suppose there exists a measurable function, $r$ such that $r(x)>0$ for all $x$, there exists $M$ such that $|r(x)|<M$ for all $x$, and $\int r d \mu<\infty$. Then for

$$
\Lambda \in\left(L^{p}(\Omega, \mu)\right)^{\prime}, p \geq 1
$$

there exists a unique $h \in L^{p^{\prime}}(\Omega, \mu), L^{\infty}(\Omega, \mu)$ if $p=1$ such that

$$
\Lambda f=\int h f d \mu
$$

Also $\|h\|=\|\Lambda\| .\left(\|h\|=\|h\|_{p^{\prime}}\right.$ if $p>1,\|h\|_{\infty}$ if $\left.p=1\right)$. Here

$$
\frac{1}{p}+\frac{1}{p^{\prime}}=1
$$

Proof: Define a new measure $\tilde{\mu}$, according to the rule

$$
\begin{equation*}
\widetilde{\mu}(E) \equiv \int_{E} r d \mu \tag{20.3.13}
\end{equation*}
$$

Thus $\widetilde{\mu}$ is a finite measure on $\mathscr{S}$. Now define a mapping, $\eta: L^{p}(\Omega, \mu) \rightarrow L^{p}(\Omega, \widetilde{\mu})$ by

$$
\eta f=r^{-\frac{1}{p}} f
$$

Then

$$
\|\eta f\|_{L^{p}(\widetilde{\mu})}^{p}=\int\left|r^{-\frac{1}{p}} f\right|^{p} r d \mu=\|f\|_{L^{p}(\mu)}^{p}
$$

and so $\eta$ is one to one and in fact preserves norms. I claim that also $\eta$ is onto. To see this, let $g \in L^{p}(\Omega, \widetilde{\mu})$ and consider the function, $r^{\frac{1}{p}} g$. Then

$$
\int\left|r^{\frac{1}{p}} g\right|^{p} d \mu=\int|g|^{p} r d \mu=\int|g|^{p} d \widetilde{\mu}<\infty
$$

Thus $r^{\frac{1}{p}} g \in L^{p}(\Omega, \mu)$ and $\eta\left(r^{\frac{1}{p}} g\right)=g$ showing that $\eta$ is onto as claimed. Thus $\eta$ is one to one, onto, and preserves norms. Consider the diagram below which is descriptive of the situation in which $\eta^{*}$ must be one to one and onto.

| $h, L^{p^{\prime}}(\widetilde{\mu})$ | $L^{p}(\widetilde{\mu})^{\prime}, \widetilde{\Lambda}$ | $\eta^{*}$ <br>  <br> $L^{p}(\widetilde{\mu})$ | $L^{p}(\mu)^{\prime}, \Lambda$ |
| :---: | :---: | :---: | :---: |
|  | $\leftarrow$ | $L^{p}(\mu)$ |  |

Then for $\Lambda \in L^{p}(\mu)^{\prime}$, there exists a unique $\tilde{\Lambda} \in L^{p}(\widetilde{\mu})^{\prime}$ such that $\eta^{*} \widetilde{\Lambda}=\Lambda,\|\widetilde{\Lambda}\|=\|\Lambda\|$. By the Riesz representation theorem for finite measure spaces, there exists a unique $h \in L^{p^{\prime}}(\widetilde{\mu})$ which represents $\widetilde{\Lambda}$ in the manner described in the Riesz representation theorem. Thus $\|h\|_{L^{p^{\prime}}(\widetilde{\mu})}=\|\widetilde{\Lambda}\|=\|\Lambda\|$ and for all $f \in L^{p}(\mu)$,

$$
\begin{aligned}
\Lambda(f) & =\eta^{*} \widetilde{\Lambda}(f) \equiv \widetilde{\Lambda}(\eta f)=\int h(\eta f) d \widetilde{\mu}=\int r h\left(f^{-\frac{1}{p}} f\right) d \mu \\
& =\int r^{\frac{1}{p^{\prime}}} h f d \mu
\end{aligned}
$$

Now

$$
\int\left|r^{\frac{1}{p^{\prime}}} h\right|^{p^{\prime}} d \mu=\int|h|^{p^{\prime}} r d \mu=\|\left. h\right|_{L^{p^{\prime}}(\widetilde{\mu})} ^{p^{\prime}}<\infty .
$$

Thus $\left\|r^{\frac{1}{p^{\prime}}} h\right\|_{L^{p^{\prime}}(\mu)}=\|h\|_{L^{p^{\prime}}(\widetilde{\mu})}=\|\widetilde{\Lambda}\|=\|\Lambda\|$ and represents $\Lambda$ in the appropriate way. If $p=1$, then $1 / p^{\prime} \equiv 0$. This proves the Lemma.

A situation in which the conditions of the lemma are satisfied is the case where the measure space is $\sigma$ finite. In fact, you should show this is the only case in which the conditions of the above lemma hold.

Theorem 20.3.6 (Riesz representation theorem) Let $(\Omega, \mathscr{S}, \mu)$ be $\sigma$ finite and let

$$
\Lambda \in\left(L^{p}(\Omega, \mu)\right)^{\prime}, p \geq 1
$$

Then there exists a unique $h \in L^{q}(\Omega, \mu), L^{\infty}(\Omega, \mu)$ if $p=1$ such that

$$
\Lambda f=\int h f d \mu
$$

Also $\|h\|=\|\Lambda\| .\left(\|h\|=\|h\|_{q}\right.$ if $p>1,\|h\|_{\infty}$ if $\left.p=1\right)$. Here

$$
\frac{1}{p}+\frac{1}{q}=1
$$

Proof: Let $\left\{\Omega_{n}\right\}$ be a sequence of disjoint elements of $\mathscr{S}$ having the property that

$$
0<\mu\left(\Omega_{n}\right)<\infty, \cup_{n=1}^{\infty} \Omega_{n}=\Omega
$$

Define

$$
r(x)=\sum_{n=1}^{\infty} \frac{1}{n^{2}} \mathscr{X}_{\Omega_{n}}(x) \mu\left(\Omega_{n}\right)^{-1}, \widetilde{\mu}(E)=\int_{E} r d \mu
$$

Thus

$$
\int_{\Omega} r d \mu=\widetilde{\mu}(\Omega)=\sum_{n=1}^{\infty} \frac{1}{n^{2}}<\infty
$$

so $\widetilde{\mu}$ is a finite measure. The above lemma gives the existence part of the conclusion of the theorem. Uniqueness is done as before.

With the Riesz representation theorem, it is easy to show that

$$
L^{p}(\Omega), p>1
$$

is a reflexive Banach space. Recall Definition 17.2.14 on Page 451 for the definition.
Theorem 20.3.7 For $(\Omega, \mathscr{S}, \mu)$ a $\sigma$ finite measure space and $p>1, L^{p}(\Omega)$ is reflexive.
Proof: Let $\delta_{r}:\left(L^{r}(\Omega)\right)^{\prime} \rightarrow L^{r^{\prime}}(\Omega)$ be defined for $\frac{1}{r}+\frac{1}{r^{\prime}}=1$ by

$$
\int\left(\delta_{r} \Lambda\right) g d \mu=\Lambda g
$$

for all $g \in L^{r}(\Omega)$. From Theorem 20.3.6 $\delta_{r}$ is one to one, onto, continuous and linear. By the open map theorem, $\delta_{r}^{-1}$ is also one to one, onto, and continuous ( $\delta_{r} \Lambda$ equals the representor of $\Lambda$ ). Thus $\delta_{r}^{*}$ is also one to one, onto, and continuous by Corollary 17.2.11. Now observe that $J=\delta_{p}^{*} \circ \delta_{q}^{-1}$. To see this, let $z^{*} \in\left(L^{q}\right)^{\prime}, y^{*} \in\left(L^{p}\right)^{\prime}$,

$$
\begin{aligned}
\delta_{p}^{*} \circ \delta_{q}^{-1}\left(\boldsymbol{\delta}_{q} z^{*}\right)\left(y^{*}\right) & =\left(\delta_{p}^{*} z^{*}\right)\left(y^{*}\right) \\
& =z^{*}\left(\boldsymbol{\delta}_{p} y^{*}\right) \\
& =\int\left(\boldsymbol{\delta}_{q} z^{*}\right)\left(\boldsymbol{\delta}_{p} y^{*}\right) d \mu
\end{aligned}
$$

$$
\begin{aligned}
J\left(\boldsymbol{\delta}_{q} z^{*}\right)\left(y^{*}\right) & =y^{*}\left(\boldsymbol{\delta}_{q} z^{*}\right) \\
& =\int\left(\boldsymbol{\delta}_{p} y^{*}\right)\left(\boldsymbol{\delta}_{q} z^{*}\right) d \mu
\end{aligned}
$$

Therefore $\delta_{p}^{*} \circ \delta_{q}^{-1}=J$ on $\delta_{q}\left(L^{q}\right)^{\prime}=L^{p}$. But the two $\delta$ maps are onto and so $J$ is also onto.

### 20.4 The Dual Space Of $L^{\infty}(\Omega)$

What about the dual space of $L^{\infty}(\Omega)$ ? This will involve the following Lemma. Also recall the notion of total variation defined in Definition 20.2.2.

Lemma 20.4.1 Let $(\Omega, \mathscr{F})$ be a measure space. Denote by $B V(\Omega)$ the space of finitely additive complex measures $v$ such that $|v|(\Omega)<\infty$. Then defining $\|v\| \equiv|v|(\Omega)$, it follows that $B V(\Omega)$ is a Banach space.

Proof: It is obvious that $B V(\Omega)$ is a vector space with the obvious conventions involving scalar multiplication. Why is $\|\cdot\|$ a norm? All the axioms are obvious except for the triangle inequality. However, this is not too hard either.

$$
\begin{aligned}
\|\mu+v\| & \equiv|\mu+v|(\Omega)=\sup _{\pi(\Omega)}\left\{\sum_{A \in \pi(\Omega)}|\mu(A)+v(A)|\right\} \\
& \leq \sup _{\pi(\Omega)}\left\{\sum_{A \in \pi(\Omega)}|\mu(A)|\right\}+\sup _{\pi(\Omega)}\left\{\sum_{A \in \pi(\Omega)}|v(A)|\right\} \\
& \equiv|\mu|(\Omega)+|v|(\Omega)=\|v\|+\|\mu\| .
\end{aligned}
$$

Suppose now that $\left\{v_{n}\right\}$ is a Cauchy sequence. For each $E \in \mathscr{F}$,

$$
\left|v_{n}(E)-v_{m}(E)\right| \leq\left\|v_{n}-v_{m}\right\|
$$

and so the sequence of complex numbers $v_{n}(E)$ converges. That to which it converges is called $v(E)$. Then it is obvious that $v(E)$ is finitely additive. Why is $|v|$ finite? Since $\|\cdot\|$ is a norm, it follows that there exists a constant $C$ such that for all $n$,

$$
\left|v_{n}\right|(\Omega)<C
$$

Let $\pi(\Omega)$ be any partition. Then

$$
\sum_{A \in \pi(\Omega)}|v(A)|=\lim _{n \rightarrow \infty} \sum_{A \in \pi(\Omega)}\left|v_{n}(A)\right| \leq C .
$$

Hence $v \in B V(\Omega)$. Let $\varepsilon>0$ be given and let $N$ be such that if $n, m>N$, then

$$
\left\|v_{n}-v_{m}\right\|<\varepsilon / 2
$$

Pick any such $n$. Then choose $\pi(\Omega)$ such that

$$
\begin{gathered}
\left|v-v_{n}\right|(\Omega)-\varepsilon / 2<\sum_{A \in \pi(\Omega)}\left|v(A)-v_{n}(A)\right| \\
=\lim _{m \rightarrow \infty} \sum_{A \in \pi(\Omega)}\left|v_{m}(A)-v_{n}(A)\right|<\lim _{m \rightarrow \infty} \inf _{m}\left|v_{n}-v_{m}\right|(\Omega) \leq \varepsilon / 2
\end{gathered}
$$

It follows that

$$
\lim _{n \rightarrow \infty}\left\|v-v_{n}\right\|=0
$$

Corollary 20.4.2 Suppose $(\Omega, \mathscr{F})$ is a measure space as above and suppose $\mu$ is a measure defined on $\mathscr{F}$. Denote by $B V(\Omega ; \mu)$ those finitely additive measures of $B V(\Omega) v$ such that $v \ll \mu$ in the usual sense that if $\mu(E)=0$, then $v(E)=0$. Then $B V(\Omega ; \mu)$ is a closed subspace of $B V(\Omega)$.

Proof: It is clear that it is a subspace. Is it closed? Suppose $v_{n} \rightarrow v$ and each $v_{n}$ is in $B V(\Omega ; \mu)$. Then if $\mu(E)=0$, it follows that $v_{n}(E)=0$ and so $v(E)=0$ also, being the limit of 0 .

Definition 20.4.3 For s a simple function $s(\omega)=\sum_{k=1}^{n} c_{k} \mathscr{X}_{E_{k}}(\omega)$ and $v \in B V(\Omega)$, define an "integral" with respect to $v$ as follows.

$$
\int s d v \equiv \sum_{k=1}^{n} c_{k} v\left(E_{k}\right)
$$

For $f$ function which is in $L^{\infty}(\Omega ; \mu)$, define $\int f d v$ as follows. Applying Theorem 11.3.9, to the positive and negative parts of real and imaginary parts of $f$, there exists a sequence of simple functions $\left\{s_{n}\right\}$ which converges uniformly to $f$ off a set of $\mu$ measure zero. Then

$$
\int f d v \equiv \lim _{n \rightarrow \infty} \int s_{n} d v
$$

Lemma 20.4.4 The above definition of the integral with respect to a finitely additive measure in $B V(\Omega ; \mu)$ is well defined.

Proof: First consider the claim about the integral being well defined on the simple functions. This is clearly true if it is required that the $c_{k}$ are disjoint and the $E_{k}$ also disjoint having union equal to $\Omega$. Thus define the integral of a simple function in this manner. First write the simple function as

$$
\sum_{k=1}^{n} c_{k} \mathscr{X}_{E_{k}}
$$

where the $c_{k}$ are the values of the simple function. Then use the above formula to define the integral. Next suppose the $E_{k}$ are disjoint but the $c_{k}$ are not necessarily distinct. Let the distinct values of the $c_{k}$ be $a_{1}, \cdots, a_{m}$

$$
\begin{aligned}
\sum_{k} c_{k} \mathscr{X}_{E_{k}} & =\sum_{j} a_{j}\left(\sum_{i: c_{i}=a_{j}} \mathscr{X}_{E_{i}}\right)=\sum_{j} a_{j} v\left(\bigcup_{i: c_{i}=a_{j}} E_{i}\right) \\
& =\sum_{j} a_{j} \sum_{i: c_{i}=a_{j}} v\left(E_{i}\right)=\sum_{k} c_{k} v\left(E_{k}\right)
\end{aligned}
$$

and so the same formula for the integral of a simple function is obtained in this case also. Now consider two simple functions

$$
s=\sum_{k=1}^{n} a_{k} \mathscr{X}_{E_{k}}, t=\sum_{j=1}^{m} b_{j} \mathscr{X}_{F_{j}}
$$

where the $a_{k}$ and $b_{j}$ are the distinct values of the simple functions. Then from what was just shown,

$$
\begin{aligned}
\int(\alpha s+\beta t) d v & =\int\left(\sum_{k=1}^{n} \sum_{j=1}^{m} \alpha a_{k} \mathscr{X}_{E_{k} \cap F_{j}}+\sum_{j=1}^{m} \sum_{k=1}^{n} \beta b_{j} \mathscr{X}_{E_{k} \cap F_{j}}\right) d v \\
& =\int\left(\sum_{j, k} \alpha a_{k} \mathscr{X}_{E_{k} \cap F_{j}}+\beta b_{j} \mathscr{X}_{E_{k} \cap F_{j}}\right) d v \\
& =\sum_{j, k}\left(\alpha a_{k}+\beta b_{j}\right) v\left(E_{k} \cap F_{j}\right) \\
= & \sum_{k=1}^{n} \sum_{j=1}^{m} \alpha a_{k} v\left(E_{k} \cap F_{j}\right)+\sum_{j=1}^{m} \sum_{k=1}^{n} \beta b_{j} v\left(E_{k} \cap F_{j}\right) \\
= & \sum_{k=1}^{n} \alpha a_{k} v\left(E_{k}\right)+\sum_{j=1}^{m} \beta b_{j} v\left(F_{j}\right) \\
= & \alpha \int s d v+\beta \int t d v
\end{aligned}
$$

Thus the integral is linear on simple functions so, in particular, the formula given in the above definition is well defined regardless.

So what about the definition for $f \in L^{\infty}(\Omega ; \mu)$ ? Since $f \in L^{\infty}$, there is a set of $\mu$ measure zero $N$ such that on $N^{C}$ there exists a sequence of simple functions which converges uniformly to $f$ on $N^{C}$. Consider $s_{n}$ and $s_{m}$. As in the above, they can be written as

$$
\sum_{k=1}^{p} c_{k}^{n} \mathscr{X}_{E_{k}}, \sum_{k=1}^{p} c_{k}^{m} \mathscr{X}_{E_{k}}
$$

respectively, where the $E_{k}$ are disjoint having union equal to $\Omega$. Then by uniform convergence, if $m, n$ are sufficiently large, $\left|c_{k}^{n}-c_{k}^{m}\right|<\varepsilon$ or else the corresponding $E_{k}$ is contained in $N^{C}$ a set of $v$ measure 0 thanks to $v \ll \mu$. Hence

$$
\begin{aligned}
\left|\int s_{n} d v-\int s_{m} d v\right| & =\left|\sum_{k=1}^{p}\left(c_{k}^{n}-c_{k}^{m}\right) v\left(E_{k}\right)\right| \\
& \leq \sum_{k=1}^{p}\left|c_{k}^{n}-c_{k}^{m}\right|\left|v\left(E_{k}\right)\right| \leq \boldsymbol{\varepsilon}\|v\|
\end{aligned}
$$

and so the integrals of these simple functions converge. Similar reasoning shows that the definition is not dependent on the choice of approximating sequence.

Note also that for $s$ simple,

$$
\left|\int s d v\right| \leq\|s\|_{L^{\infty}}|v|(\Omega)=\|s\|_{L^{\infty}}\|v\|
$$

Next the dual space of $L^{\infty}(\Omega ; \mu)$ will be identified with $B V(\Omega ; \mu)$. First here is a simple observation. Let $v \in B V(\Omega ; \mu)$. Then define the following for $f \in L^{\infty}(\Omega ; \mu)$.

$$
T_{v}(f) \equiv \int f d v
$$

Lemma 20.4.5 For $T_{v}$ just defined,

$$
\left|T_{v} f\right| \leq\|f\|_{L^{\infty}}\|v\|
$$

Proof: As noted above, the conclusion true if $f$ is simple. Now if $f$ is in $L^{\infty}$, then it is the uniform limit of simple functions off a set of $\mu$ measure zero. Therefore, by the definition of the $T_{V}$,

$$
\left|T_{v} f\right|=\lim _{n \rightarrow \infty}\left|T_{v} s_{n}\right| \leq \lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty}\left\|s_{n}\right\|_{L^{\infty}}\|v\|=\|f\|_{L^{\infty}}\|v\| .
$$

Thus each $T_{v}$ is in $\left(L^{\infty}(\Omega ; \mu)\right)^{\prime}$
Here is the representation theorem, due to Kantorovitch, for the dual of $L^{\infty}(\Omega ; \mu)$.
Theorem 20.4.6 Let $\theta: B V(\Omega ; \mu) \rightarrow\left(L^{\infty}(\Omega ; \mu)\right)^{\prime}$ be given by $\theta(v) \equiv T_{v}$. Then $\theta$ is one to one, onto and preserves norms.

Proof: It was shown in the above lemma that $\theta$ maps into $\left(L^{\infty}(\Omega ; \mu)\right)^{\prime}$. It is obvious that $\theta$ is linear. Why does it preserve norms? From the above lemma,

$$
\|\theta v\| \equiv \sup _{\|f\|_{\infty} \leq 1}\left|T_{v} f\right| \leq\|v\|
$$

It remains to turn the inequality around. Let $\pi(\Omega)$ be a partition. Then

$$
\sum_{A \in \pi(\Omega)}|v(A)|=\sum_{A \in \pi(\Omega)} \operatorname{sgn}(v(A)) v(A) \equiv \int f d v
$$

where $\operatorname{sgn}(v(A))$ is defined to be a complex number of modulus 1 such that

$$
\operatorname{sgn}(v(A)) v(A)=|v(A)|
$$

and

$$
f(\omega)=\sum_{A \in \pi(\Omega)} \operatorname{sgn}(v(A)) \mathscr{X}_{A}(\omega)
$$

Therefore, choosing $\pi(\Omega)$ suitably, since $\|f\|_{\infty} \leq 1$,

$$
\begin{aligned}
\|v\|-\varepsilon & =|v|(\Omega)-\varepsilon \leq \sum_{A \in \pi(\Omega)}|v(A)|=T_{v}(f) \\
& =\left|T_{v}(f)\right|=|\theta(v)(f)| \leq\|\theta(v)\| \leq\|v\|
\end{aligned}
$$

Thus $\theta$ preserves norms. Hence it is one to one also. Why is $\theta$ onto?
Let $\Lambda \in\left(L^{\infty}(\Omega ; \mu)\right)^{\prime}$. Then define

$$
\begin{equation*}
v(E) \equiv \Lambda\left(\mathscr{X}_{E}\right) \tag{20.4.14}
\end{equation*}
$$

This is obviously finitely additive because $\Lambda$ is linear. Also, if $\mu(E)=0$, then $\mathscr{X}_{E}=0$ in $L^{\infty}$ and so $\Lambda\left(\mathscr{X}_{E}\right)=0$. If $\pi(\Omega)$ is any partition of $\Omega$, then

$$
\begin{aligned}
\sum_{A \in \pi(\Omega)}|v(A)| & =\sum_{A \in \pi(\Omega)}\left|\Lambda\left(\mathscr{X}_{A}\right)\right|=\sum_{A \in \pi(\Omega)} \operatorname{sgn}\left(\Lambda\left(\mathscr{X}_{A}\right)\right) \Lambda\left(\mathscr{X}_{A}\right) \\
& =\Lambda\left(\sum_{A \in \pi(\Omega)} \operatorname{sgn}\left(\Lambda\left(\mathscr{X}_{A}\right)\right) \mathscr{X}_{A}\right) \leq\|\Lambda\|
\end{aligned}
$$

and so $\|v\| \leq\|\Lambda\|$ showing that $v \in B V(\Omega ; \mu)$. Also from 20.4.14, if $s=\sum_{k=1}^{n} c_{k} \mathscr{X}_{E_{k}}$ is a simple function,

$$
\int s d v=\sum_{k=1}^{n} c_{k} v\left(E_{k}\right)=\sum_{k=1}^{n} c_{k} \Lambda\left(\mathscr{X}_{E_{k}}\right)=\Lambda\left(\sum_{k=1}^{n} c_{k} \mathscr{X}_{E_{k}}\right)=\Lambda(s)
$$

Then letting $f \in L^{\infty}(\Omega ; \mu)$, there exists a sequence of simple functions converging to $f$ uniformly off a set of $\mu$ measure zero and so passing to a limit in the above with $s$ replaced with $s_{n}$ it follows that

$$
\Lambda(f)=\int f d v
$$

and so $\theta$ is onto.

### 20.5 Non $\sigma$ Finite Case

It turns out that for $p>1$, you don't have to assume the measure space is $\sigma$ finite. The Riesz representation theorem holds always. The proof involves the notion of uniform convexity. First recall Clarkson's inequalities. These fundamental inequalities were used to verify that $L^{p}(\Omega)$ is uniformly convex. More precisely, the unit ball in $L^{p}(\Omega)$ is uniformly convex.

Lemma 20.5.1 Let $2 \leq p$. Then

$$
\left\|\frac{f+g}{2}\right\|_{L^{p}}^{p}+\left\|\frac{f-g}{2}\right\|_{L^{p}}^{p} \leq \frac{1}{2}\left(\|f\|_{L^{p}}^{p}+\|g\|_{L^{p}}^{p}\right)
$$

Let $1<p<2$. then for $1 / p+1 / q=1$,

$$
\left\|\frac{f+g}{2}\right\|_{L^{p}}^{q}+\left\|\frac{f-g}{2}\right\|_{L^{p}}^{q} \leq\left(\frac{1}{2}\|f\|_{L^{p}}^{p}+\frac{1}{2}\|g\|_{L^{p}}^{p}\right)^{q / p}
$$

Recall the following definition of uniform convexity.
Definition 20.5.2 A Banach space, $X$, is said to be uniformly convex if whenever $\left\|x_{n}\right\| \leq 1$ and $\left\|\frac{x_{n}+x_{m}}{2}\right\| \rightarrow 1$ as $n, m \rightarrow \infty$, then $\left\{x_{n}\right\}$ is a Cauchy sequence and $x_{n} \rightarrow x$ where $\|x\|=1$.

Observe that Clarkson's inequalities imply $L^{p}$ is uniformly convex for all $p>1$. Uniformly convex spaces have a very nice property which is described in the following lemma. Roughly, this property is that any element of the dual space achieves its norm at some point of the closed unit ball.

Lemma 20.5.3 Let $X$ be uniformly convex and let $\phi \in X^{\prime}$. Then there exists $x \in X$ such that

$$
\|x\|=1, \phi(x)=\|\phi\| .
$$

Proof: Let $\left\|\left\|\widetilde{x}_{n}\right\| \leq 1\right.$ and $\left.\left|\phi\left(\widetilde{x}_{n}\right)\right| \rightarrow\right\| \phi \|$. Let $x_{n}=w_{n} \widetilde{x}_{n}$ where $\left|w_{n}\right|=1$ and

$$
w_{n} \phi \widetilde{x}_{n}=\left|\phi \widetilde{x}_{n}\right| .
$$

Thus $\phi\left(x_{n}\right)=\left|\phi\left(x_{n}\right)\right|=\left|\phi\left(\widetilde{x}_{n}\right)\right| \rightarrow \| \phi| |$.

$$
\phi\left(x_{n}\right) \rightarrow\|\phi\|,\left\|x_{n}\right\| \leq 1 .
$$

We can assume, without loss of generality, that

$$
\phi\left(x_{n}\right)=\left|\phi\left(x_{n}\right)\right| \geq \frac{\|\phi\|}{2}
$$

and $\phi \neq 0$.
Claim $\left\|\frac{x_{n}+x_{m}}{2}\right\| \rightarrow 1$ as $n, m \rightarrow \infty$.
Proof of Claim: Let $n, m$ be large enough that $\phi\left(x_{n}\right), \phi\left(x_{m}\right) \geq\|\phi\|-\frac{\varepsilon}{2}$ where $0<\varepsilon$. Then $\left\|x_{n}+x_{m}\right\| \neq 0$ because if it equals 0 , then $x_{n}=-x_{m}$ so $-\phi\left(x_{n}\right)=\phi\left(x_{m}\right)$ but both $\phi\left(x_{n}\right)$ and $\phi\left(x_{m}\right)$ are positive. Therefore consider $\frac{x_{n}+x_{m} \|}{\left\|x_{n}+x_{m}\right\|}$, a vector of norm 1. Thus,

$$
\|\phi\| \geq\left|\phi\left(\frac{\left(x_{n}+x_{m}\right)}{\left\|x_{n}+x_{m}\right\|}\right)\right| \geq \frac{2\|\phi\|-\varepsilon}{\left\|x_{n}+x_{m}\right\|} .
$$

Hence

$$
\left\|\left\|x_{n}+x_{m}\right\|\right\| \phi\|\geq 2\| \phi \|-\varepsilon .
$$

Since $\varepsilon>0$ is arbitrary, $\lim _{n, m \rightarrow \infty}\left\|x_{n}+x_{m}\right\|=2$. This proves the claim.
By uniform convexity, $\left\{x_{n}\right\}$ is Cauchy and $x_{n} \rightarrow x,\|x\|=1$. Thus

$$
\phi(x)=\lim _{n \rightarrow \infty} \phi\left(x_{n}\right)=\|\phi\|
$$

The proof of the Riesz representation theorem will be based on the following lemma which says that if you can show a directional derivative exists, then it can be used to represent a functional in terms of this directional derivative. It is very interesting for its own sake.

Lemma 20.5.4 (McShane) Let $X$ be a complex normed linear space and let $\phi \in X^{\prime}$. Suppose there exists $x \in X,\|x\|=1$ with $\phi(x)=\|\phi\| \neq 0$. Let $y \in X$ and let $\psi_{y}(t)=\| x+$ ty $\|$ for $t \in \mathbb{R}$. Suppose $\psi_{y}^{\prime}(0)$ exists for each $y \in X$. Then for all $y \in X$,

$$
\psi_{y}^{\prime}(0)+i \psi_{-i y}^{\prime}(0)=\|\phi\|^{-1} \phi(y)
$$

Proof: Suppose first that $\|\phi\|=1$. Then by assumption, there is $x$ such that $\|x\|=1$ and $\phi(x)=1=\|\phi\|$. Then $\phi(y-\phi(y) x)=0$ and so

$$
\phi(x+t(y-\phi(y) x))=\phi(x)=1=\|\phi\| .
$$

Therefore, $\|x+t(y-\phi(y) x)\| \geq 1$ since otherwise $\|x+t(y-\phi(y) x)\|=r<1$ and so

$$
\phi\left((x+t(y-\phi(y) x)) \frac{1}{r}\right)=\frac{1}{r} \phi(x)=\frac{1}{r}
$$

which would imply that $\|\phi\|>1$.
Also for small $t,|\phi(y) t|<1$, and so

$$
\begin{gathered}
1 \leq\|x+t(y-\phi(y) x)\|=\|(1-\phi(y) t) x+t y\| \\
\leq|1-\phi(y) t|\left\|x+\frac{t}{1-\phi(y) t} y\right\| .
\end{gathered}
$$

Divide both sides by $|1-\phi(y) t|$. Using the standard formula for the sum of a geometric series,

$$
1+t \phi(y)+o(t)=\frac{1}{1-t \phi(y)}
$$

Therefore,

$$
\begin{equation*}
\frac{1}{|1-\phi(y) t|}=|1+\phi(y) t+o(t)| \leq\left\|x+\frac{t}{1-\phi(y) t} y\right\|=\|x+t y+o(t)\| \tag{20.5.15}
\end{equation*}
$$

where $\lim _{t \rightarrow 0} o(t)\left(t^{-1}\right)=0$. Thus,

$$
|1+\phi(y) t| \leq\|x+t y\|+o(t)
$$

Now $|1+t \phi(y)|-1 \geq 1+t \operatorname{Re} \phi(y)-1=t \operatorname{Re} \phi(y)$.
Thus for $t>0$,

$$
\operatorname{Re} \phi(y) \leq \frac{|1+t \phi(y)|-1}{t} \stackrel{\|x\|=1}{\leq} \frac{\|x+t y\|-\|x\|}{t}+\frac{o(t)}{t}
$$

and for $t<0$,

$$
\operatorname{Re} \phi(y) \geq \frac{|1+t \phi(y)|-1}{t} \geq \frac{\|x+t y\|-\|x\|}{t}+\frac{o(t)}{t}
$$

By assumption, letting $t \rightarrow 0+$ and $t \rightarrow 0-$,

$$
\operatorname{Re} \phi(y)=\lim _{t \rightarrow 0} \frac{\|x+t y\|-\|x\|}{t}=\psi_{y}^{\prime}(0)
$$

Now

$$
\phi(y)=\operatorname{Re} \phi(y)+i \operatorname{Im} \phi(y)
$$

so

$$
\phi(-i y)=-i(\phi(y))=-i \operatorname{Re} \phi(y)+\operatorname{Im} \phi(y)
$$

and

$$
\phi(-i y)=\operatorname{Re} \phi(-i y)+i \operatorname{Im} \phi(-i y)
$$

Hence

$$
\operatorname{Re} \phi(-i y)=\operatorname{Im} \phi(y) .
$$

Consequently,

$$
\begin{aligned}
\phi(y)=\operatorname{Re} \phi(y) & +i \operatorname{Im} \phi(y)=\operatorname{Re} \phi(y)+i \operatorname{Re} \phi(-i y) \\
& =\psi_{y}^{\prime}(0)+i \psi_{-i y}^{\prime}(0)
\end{aligned}
$$

This proves the lemma when $\|\phi\|=1$. For arbitrary $\phi \neq 0$, let $\phi(x)=\|\phi\|,\|x\|=1$. Then from above, if $\phi_{1}(y) \equiv\|\phi\|^{-1} \phi(y),\left\|\phi_{1}\right\|=1$ and so from what was just shown,

$$
\phi_{1}(y)=\frac{\phi(y)}{\|\phi\|}=\psi_{y}^{\prime}(0)+i \psi_{-i y}(0)
$$

Now here are some short observations. For $t \in \mathbb{R}, p>1$, and $x, y \in \mathbb{C}, x \neq 0$

$$
\begin{align*}
\lim _{t \rightarrow 0} \frac{|x+t y|^{p}-|x|^{p}}{t} & =p|x|^{p-2}(\operatorname{Re} x \operatorname{Re} y+\operatorname{Im} x \operatorname{Im} y) \\
& =p|x|^{p-2} \operatorname{Re}(\bar{x} y) \tag{20.5.16}
\end{align*}
$$

Also from convexity of $f(r)=r^{p}$, for $|t|<1$,

$$
\begin{aligned}
& |x+t y|^{p}-|x|^{p} \leq||x|+|t|| y| |^{p}-|x|^{p} \\
& =\left[(1+|t|)\left(\frac{|x|+|t||y|}{1+|t|}\right)\right]^{p}-|x|^{p} \\
& \leq(1+|t|)^{p} \frac{|x|^{p}}{1+|t|}+\frac{|t||y|^{p}}{1+|t|}-|x|^{p} \\
& \leq(1+|t|)^{p-1}\left(|x|^{p}+|t||y|^{p}\right)-|x|^{p} \\
& \leq\left((1+|t|)^{p-1}-1\right)|x|^{p}+2^{p-1}|t||y|^{p}
\end{aligned}
$$

Now for $f(t) \equiv(1+t)^{p-1}, f^{\prime}(t)$ is uniformly bounded, depending on $p$, for $t \in[0,1]$. Hence the above is dominated by an expression of the form

$$
\begin{equation*}
C_{p}\left(|x|^{p}+|y|^{p}\right)|t| \tag{20.5.17}
\end{equation*}
$$

The above lemma and uniform convexity of $L^{p}$ can be used to prove a general version of the Riesz representation theorem next. Let $p>1$ and let $\eta: L^{q} \rightarrow\left(L^{p}\right)^{\prime}$ be defined by

$$
\begin{equation*}
\eta(g)(f)=\int_{\Omega} g f d \mu \tag{20.5.18}
\end{equation*}
$$

Theorem 20.5.5 (Riesz representation theorem $p>1$ ) The map $\eta$ is 1-1, onto, continuous, and

$$
\|\eta g\|=\|g\|,\|\eta\|=1
$$

Proof: Obviously $\eta$ is linear. Suppose $\eta g=0$. Then $0=\int g f d \mu$ for all $f \in L^{p}$. Let $f=|g|^{q-2} \bar{g}$. Then $f \in L^{p}$ and so $0=\int|g|^{q} d \mu$. Hence $g=0$ and $\eta$ is one to one. That $\eta g \in\left(L^{p}\right)^{\prime}$ is obvious from the Holder inequality. In fact,

$$
|\eta(g)(f)| \leq\|g\|_{q}\|f\|_{p}
$$

and so $\|\eta(g)\| \leq\|g\|_{q}$. To see that equality holds, let

$$
f=|g|^{q-2} \bar{g}\|g\|_{q}^{1-q}
$$

Then $\|f\|_{p}=1$ and

$$
\eta(g)(f)=\int_{\Omega}|g|^{q} d \mu\|g\|_{q}^{1-q}=\|g\|_{q} .
$$

Thus $\|\eta\|=1$.
It remains to show $\eta$ is onto. Let $\phi \in\left(L^{p}\right)^{\prime}$. Is $\phi=\eta g$ for some $g \in L^{q}$ ? Without loss of generality, assume $\phi \neq 0$. By uniform convexity of $L^{p}$, Lemma 20.5.3, there exists $g$ such that

$$
\phi g=\|\phi\|, g \in L^{p},\|g\|=1
$$

For $f \in L^{p}$, define $\phi_{f}(t) \equiv \int_{\Omega}|g+t f|^{p} d \mu$. Thus

$$
\psi_{f}(t) \equiv\|g+t f\|_{p} \equiv \phi_{f}(t)^{\frac{1}{p}}
$$

Does $\phi_{f}^{\prime}(0)$ exist? Let $[g=0]$ denote the set $\{x: g(x)=0\}$.

$$
\frac{\phi_{f}(t)-\phi_{f}(0)}{t}=\int \frac{\left(|g+t f|^{p}-|g|^{p}\right)}{t} d \mu
$$

From 20.5.17, the integrand is bounded by $C_{p}\left(|f|^{p}+|g|^{p}\right)$. Therefore, using 20.5.16, the dominated convergence theorem applies and it follows $\phi_{f}^{\prime}(0)=$

$$
\begin{gathered}
\lim _{t \rightarrow 0} \frac{\phi_{f}(t)-\phi_{f}(0)}{t}=\lim _{t \rightarrow 0}\left[\int_{[g=0]}|t|^{p-1}|f|^{p} d \mu+\int_{[g \neq 0]} \frac{\left(|g+t f|^{p}-|g|^{p}\right)}{t} d \mu\right] \\
=p \int_{[g \neq 0]}|g|^{p-2} \operatorname{Re}(\bar{g} f) d \mu=p \int|g|^{p-2} \operatorname{Re}(\bar{g} f) d \mu
\end{gathered}
$$

Hence

$$
\psi_{f}^{\prime}(0)=\|g\|^{\frac{-p}{q}} \int|g(x)|^{p-2} \operatorname{Re}(g(x) \bar{f}(x)) d \mu
$$

Note $\frac{1}{p}-1=-\frac{1}{q}$. Therefore,

$$
\psi_{-i f}^{\prime}(0)=\|g\|^{\frac{-p}{q}} \int|g(x)|^{p-2} \operatorname{Re}(i g(x) \bar{f}(x)) d \mu
$$

But $\operatorname{Re}(i g \bar{f})=\operatorname{Im}(-g \bar{f})$ and so by the McShane lemma,

$$
\begin{aligned}
\phi(f) & =\|\phi\|\|g\|^{\frac{-p}{q}} \int|g(x)|^{p-2}[\operatorname{Re}(g(x) \bar{f}(x))+i \operatorname{Re}(i g(x) \bar{f}(x))] d \mu \\
& =\|\phi\|\|g\|^{\frac{-p}{q}} \int|g(x)|^{p-2}[\operatorname{Re}(g(x) \bar{f}(x))+i \operatorname{Im}(-g(x) \bar{f}(x))] d \mu \\
& =\|\phi\|\|g\|^{\frac{-p}{q}} \int|g(x)|^{p-2} \bar{g}(x) f(x) d \mu
\end{aligned}
$$

This shows that

$$
\phi=\eta\left(\|\phi\|\|g\|^{\frac{-p}{q}}|g|^{p-2} \bar{g}\right)
$$

and verifies $\eta$ is onto.

### 20.6 The Dual Space Of $C_{0}(X)$

Consider the dual space of $C_{0}(X)$ where $X$ is a locally compact Hausdorff space. It will turn out to be a space of measures. To show this, the following lemma will be convenient. Recall this space is defined as follows.

Definition 20.6.1 $f \in C_{0}(X)$ means that for every $\varepsilon>0$ there exists a compact set $K$ such that $|f(x)|<\varepsilon$ whenever $x \notin K$. Recall the norm on this space is

$$
\|f\|_{\infty} \equiv\|f\| \equiv \sup \{|f(x)|: x \in X\}
$$

Lemma 20.6.2 Suppose $\lambda$ is a mapping which has nonnegative values which is defined on the nonnegative functions in $C_{0}(X)$ such that

$$
\begin{equation*}
\lambda(a f+b g)=a \lambda(f)+b \lambda(g) \tag{20.6.19}
\end{equation*}
$$

whenever $a, b \geq 0$ and $f, g \geq 0$. Then there exists a unique extension of $\lambda$ to all of $C_{0}(X)$, $\Lambda$ such that whenever $f, g \in C_{0}(X)$ and $a, b \in \mathbb{C}$, it follows

$$
\Lambda(a f+b g)=a \Lambda(f)+b \Lambda(g)
$$

If

$$
|\lambda(f)| \leq C| | f \|_{\infty}
$$

then

$$
|\Lambda f| \leq C| | f \|_{\infty}
$$

Proof: Let $C_{0}(X ; \mathbb{R})$ be the real-valued functions in $C_{0}(X)$ and define

$$
\Lambda_{R}(f)=\lambda f^{+}-\lambda f^{-}
$$

for $f \in C_{0}(X ; \mathbb{R})$. Use the identity

$$
\left(f_{1}+f_{2}\right)^{+}+f_{1}^{-}+f_{2}^{-}=f_{1}^{+}+f_{2}^{+}+\left(f_{1}+f_{2}\right)^{-}
$$

and 20.6.19 to write

$$
\lambda\left(f_{1}+f_{2}\right)^{+}-\lambda\left(f_{1}+f_{2}\right)^{-}=\lambda f_{1}^{+}-\lambda f_{1}^{-}+\lambda f_{2}^{+}-\lambda f_{2}^{-}
$$

it follows that $\Lambda_{R}\left(f_{1}+f_{2}\right)=\Lambda_{R}\left(f_{1}\right)+\Lambda_{R}\left(f_{2}\right)$. To show that $\Lambda_{R}$ is linear, it is necessary to verify that $\Lambda_{R}(c f)=c \Lambda_{R}(f)$ for all $c \in \mathbb{R}$. But

$$
(c f)^{ \pm}=c f^{ \pm}
$$

if $c \geq 0$ while

$$
(c f)^{+}=-c(f)^{-}
$$

if $c<0$ and

$$
(c f)^{-}=(-c) f^{+}
$$

if $c<0$. Thus, if $c<0$,

$$
\begin{gathered}
\Lambda_{R}(c f)=\lambda(c f)^{+}-\lambda(c f)^{-}=\lambda\left((-c) f^{-}\right)-\lambda\left((-c) f^{+}\right) \\
=-c \lambda\left(f^{-}\right)+c \lambda\left(f^{+}\right)=c\left(\lambda\left(f^{+}\right)-\lambda\left(f^{-}\right)\right)=c \Lambda_{R}(f)
\end{gathered}
$$

A similar formula holds more easily if $c \geq 0$. Now let

$$
\Lambda f=\Lambda_{R}(\operatorname{Re} f)+i \Lambda_{R}(\operatorname{Im} f)
$$

for arbitrary $f \in C_{0}(X)$. This is linear as desired.
Here is why. It is obvious that $\Lambda(f+g)=\Lambda(f)+\Lambda(g)$ from the fact that taking the real and imaginary parts are linear operations. The only thing to check is whether you can factor out a complex scalar.

$$
\begin{gathered}
\Lambda((a+i b) f)=\Lambda(a f)+\Lambda(i b f) \\
\equiv \Lambda_{R}(a \operatorname{Re} f)+i \Lambda_{R}(a \operatorname{Im} f)+\Lambda_{R}(-b \operatorname{Im} f)+i \Lambda_{R}(b \operatorname{Re} f)
\end{gathered}
$$

because $i b f=i b \operatorname{Re} f-b \operatorname{Im} f$ and so $\operatorname{Re}(i b f)=-b \operatorname{Im} f$ and $\operatorname{Im}(i b f)=b \operatorname{Re} f$. Therefore, the above equals

$$
\begin{aligned}
& =(a+i b) \Lambda_{R}(\operatorname{Re} f)+i(a+i b) \Lambda_{R}(\operatorname{Im} f) \\
& =\quad(a+i b)\left(\Lambda_{R}(\operatorname{Re} f)+i \Lambda_{R}(\operatorname{Im} f)\right)=(a+i b) \Lambda f
\end{aligned}
$$

The extension is obviously unique because all the above is required in order for $\Lambda$ to be linear.

It remains to verify the claim about continuity of $\Lambda$. From the definition of $\lambda$, if $0 \leq$ $g \leq f$, then

$$
\begin{gathered}
\lambda(f)=\lambda(f-g+g)=\lambda(f-g)+\lambda(g) \geq \lambda(g) \\
\left|\Lambda_{R} f\right| \equiv\left|\lambda f^{+}-\lambda f^{-}\right| \leq \max \left(\lambda f^{+}, \lambda f^{-}\right) \leq \lambda(|f|) \leq C| | f \|_{\infty}
\end{gathered}
$$

Then letting $\omega \Lambda f=|\Lambda f|,|\omega|=1$, and using the above,

$$
\begin{aligned}
|\Lambda f| & =\omega \Lambda f=\Lambda(\omega f) \equiv \Lambda_{R}(\operatorname{Re}(\omega f))=\left|\Lambda_{R}(\operatorname{Re}(\omega f))\right| \\
& \leq C\|\operatorname{Re}(\omega f)\| \leq C\|f\|_{\infty}
\end{aligned}
$$

This proves the lemma.
Let $L \in C_{0}(X)^{\prime}$. Also denote by $C_{0}^{+}(X)$ the set of nonnegative continuous functions defined on $X$. Define for $f \in C_{0}^{+}(X)$

$$
\lambda(f)=\sup \{|L g|:|g| \leq f\} .
$$

Note that $\lambda(f)<\infty$ because $|L g| \leq\|L|\||g\|\leq\| L|\|| \mid f\|$ for $|g| \leq f$. Then the following lemma is important.

Lemma 20.6.3 If $c \geq 0, \lambda(c f)=c \lambda(f), f_{1} \leq f_{2}$ implies $\lambda f_{1} \leq \lambda f_{2}$, and

$$
\lambda\left(f_{1}+f_{2}\right)=\lambda\left(f_{1}\right)+\lambda\left(f_{2}\right)
$$

Also

$$
0 \leq \lambda(f) \leq\|L\|\|f\|_{\infty}
$$

Proof: The first two assertions are easy to see so consider the third.
For $f_{j} \in C_{0}^{+}(X)$, there exists $g_{i} \in C_{0}(X)$ such that $\left|g_{i}\right| \leq f_{i}$ and

$$
\begin{aligned}
\lambda\left(f_{1}\right)+\lambda\left(f_{2}\right) & \leq\left|L\left(g_{1}\right)\right|+\left|L\left(g_{2}\right)\right|+2 \varepsilon \\
& =L\left(\omega_{1} g_{1}\right)+L\left(\omega_{2} g_{2}\right)+2 \varepsilon \\
& =L\left(\omega_{1} g_{1}+\omega_{2} g_{2}\right)+2 \varepsilon \\
& =\left|L\left(\omega_{1} g_{1}+\omega_{2} g_{2}\right)\right|+2 \varepsilon
\end{aligned}
$$

where $\left|g_{i}\right| \leq f_{i}$ and $\left|\omega_{i}\right|=1$ and $\omega_{i} L\left(g_{i}\right)=\left|L\left(g_{i}\right)\right|$. Now

$$
\left|\omega_{1} g_{1}+\omega_{2} g_{2}\right| \leq\left|g_{1}\right|+\left|g_{2}\right| \leq f_{1}+f_{2}
$$

and so the above shows

$$
\lambda\left(f_{1}\right)+\lambda\left(f_{2}\right) \leq \lambda\left(f_{1}+f_{2}\right)+2 \varepsilon
$$

Since $\varepsilon$ is arbitrary, $\lambda\left(f_{1}\right)+\lambda\left(f_{2}\right) \leq \lambda\left(f_{1}+f_{2}\right)$. It remains to verify the other inequality.
Now let $|g| \leq f_{1}+f_{2},|L g| \geq \lambda\left(f_{1}+f_{2}\right)-\varepsilon$. Let

$$
h_{i}(x)=\left\{\begin{array}{l}
\frac{f_{i}(x) g(x)}{f_{1}(x)+f_{2}(x)} \text { if } f_{1}(x)+f_{2}(x)>0 \\
0 \text { if } f_{1}(x)+f_{2}(x)=0
\end{array}\right.
$$

Then $h_{i}$ is continuous and $h_{1}(x)+h_{2}(x)=g(x),\left|h_{i}\right| \leq f_{i}$. The reason it is continuous at a point where $f_{1}(x)+f_{2}(x)=0$ is that at every point $y$ where $f_{1}(y)+f_{2}(y)>0$, the top description of the function gives

$$
\left|\frac{f_{i}(y) g(y)}{f_{1}(y)+f_{2}(y)}\right| \leq|g(y)|
$$

Therefore,

$$
\begin{aligned}
-\varepsilon+\lambda\left(f_{1}+f_{2}\right) & \leq|L g| \leq\left|L h_{1}+L h_{2}\right| \leq\left|L h_{1}\right|+\left|L h_{2}\right| \\
& \leq \lambda\left(f_{1}\right)+\lambda\left(f_{2}\right) .
\end{aligned}
$$

Since $\varepsilon>0$ is arbitrary, this shows

$$
\lambda\left(f_{1}+f_{2}\right) \leq \lambda\left(f_{1}\right)+\lambda\left(f_{2}\right) \leq \lambda\left(f_{1}+f_{2}\right)
$$

The last assertion follows from

$$
\lambda(f)=\sup \{|L g|:|g| \leq f\} \leq \sup _{\|g\|_{\infty} \leq\|f\|_{\infty}}\|L\|\|g\|_{\infty} \leq\|L\|\|f\|_{\infty}
$$

which proves the lemma.
Let $\Lambda$ be defined in Lemma 20.6.2. Then $\Lambda$ is linear by this lemma and also satisfies

$$
\begin{equation*}
|\Lambda f| \leq\|L\|\|f\|_{\infty} \tag{20.6.20}
\end{equation*}
$$

Also, if $f \geq 0$,

$$
\Lambda f=\Lambda_{R} f=\lambda(f) \geq 0
$$

Therefore, $\Lambda$ is a positive linear functional on $C_{0}(X)$. In particular, it is a positive linear functional on $C_{c}(X)$. By Theorem 12.3.2 on Page 288, there exists a unique measure $\mu$ such that

$$
\Lambda f=\int_{X} f d \mu
$$

for all $f \in C_{c}(X)$. This measure is inner regular on all open sets and on all measurable sets having finite measure. In fact, it is actually a finite measure.

Lemma 20.6.4 Let $L \in C_{0}(X)^{\prime}$ as above. Then letting $\mu$ be the Radon measure just described, it follows $\mu$ is finite and

$$
\mu(X)=\|\Lambda\|=\|L\|
$$

Proof: First of all, why is $\|\Lambda\|=\|L\|$ ? From 20.6 .20 it follows $\|\Lambda\| \leq\|L\|$. But also

$$
|L g| \leq \lambda(|g|)=\Lambda(|g|) \leq\|\Lambda\|\|g\|_{\infty}
$$

and so by definition of the operator norm, $\|L\| \leq\|\Lambda\|$.
Now $X$ is an open set and so

$$
\mu(X)=\sup \{\mu(K): K \subseteq X\}
$$

and so letting $K \prec f \prec X$ for one of these $K$, it also follows

$$
\mu(X)=\sup \{\Lambda f: f \prec X\}
$$

However, for such $f \prec X$,

$$
0 \leq \Lambda f=\Lambda_{R} f \leq\|L\|\|f\|_{\infty}=\|L\|
$$

and so

$$
\mu(X) \leq\|L\|
$$

Now since $C_{c}(X)$ is dense in $C_{0}(X)$, there exists $f \in C_{c}(X)$ such that $\|f\| \leq 1$ and

$$
|\Lambda f|+\varepsilon>\|\Lambda\|=\|L\|
$$

Then also $f \prec X$ and so

$$
\|L\|-\varepsilon<|\Lambda f|=\Lambda f \leq \mu(X)
$$

Since $\varepsilon$ is arbitrary, this shows $\|L\|=\mu(X)$. This proves the lemma.
What follows is the Riesz representation theorem for $C_{0}(X)^{\prime}$.
Theorem 20.6.5 Let $L \in\left(C_{0}(X)\right)^{\prime}$ for $X$ a locally compact Hausdorf space. Then there exists a finite Radon measure $\mu$ and a function $\sigma \in L^{\infty}(X, \mu)$ such that for all $f \in C_{0}(X)$,

$$
L(f)=\int_{X} f \sigma d \mu
$$

Furthermore,

$$
\mu(X)=\|L\|,|\sigma|=1 \text { a.e. }
$$

and if

$$
v(E) \equiv \int_{E} \sigma d \mu
$$

then $\mu=|v|$
Proof: From the above there exists a unique Radon measure $\mu$ such that for all $f \in$ $C_{c}(X)$,

$$
\Lambda f=\int_{X} f d \mu
$$

Then for $f \in C_{c}(X)$,

$$
|L f| \leq \Lambda(|f|)=\int_{X}|f| d \mu=\|f\|_{L^{1}(\mu)}
$$

Since $\mu$ is both inner and outer regular thanks to it being finite, $C_{c}(X)$ is dense in $L^{1}(X, \mu)$. (See Theorem 15.2.4 for more than is needed.) Therefore $L$ extends uniquely to an element of $\left(L^{1}(X, \mu)\right)^{\prime}, \widetilde{L}$. By the Riesz representation theorem for $L^{1}$ for finite measure spaces, there exists a unique $\sigma \in L^{\infty}(X, \mu)$ such that for all $f \in L^{1}(X, \mu)$,

$$
\widetilde{L} f=\int_{X} f \sigma d \mu
$$

In particular, for all $f \in C_{0}(X)$,

$$
L f=\int_{X} f \sigma d \mu
$$

and it follows from Lemma 20.6.4, $\mu(X)=\|L\|$.
It remains to verify $|\sigma|=1$ a.e. For any $f \geq 0$,

$$
\Lambda f \equiv \int_{X} f d \mu \geq|L f|=\left|\int_{X} f \sigma d \mu\right|
$$

Now if $E$ is measurable, the regularity of $\mu$ implies there exists a sequence of bounded functions $f_{n} \in C_{c}(X)$ such that $f_{n}(x) \rightarrow \mathscr{X}_{E}(x)$ a.e. Then using the dominated convergence theorem in the above,

$$
\int_{E} d \mu=\lim _{n \rightarrow \infty} \int_{X} f_{n} d \mu \geq \lim _{n \rightarrow \infty}\left|\int_{X} f_{n} \sigma d \mu\right|=\left|\int_{E} \sigma d \mu\right|
$$

and so if $\mu(E)>0$,

$$
1 \geq\left|\frac{1}{\mu(E)} \int_{E} \sigma d \mu\right|
$$

which shows from Lemma 20.2.7 that $|\sigma| \leq 1$ a.e. But also, choosing $f_{1}$ appropriately, $\left\|f_{1}\right\|_{\infty} \leq 1$, and letting $\omega L f_{1}=\left|L f_{1}\right|$,

$$
\begin{aligned}
\mu(X) & =\|L\|=\sup _{\|f\|_{\infty} \leq 1}|L f| \leq\left|L f_{1}\right|+\varepsilon \\
& \leq \int_{X} f_{1} \omega \sigma d \mu+\varepsilon=\int_{X} \operatorname{Re}\left(f_{1} \omega \sigma\right) d \mu+\varepsilon \\
& \leq \int_{X}|\sigma| d \mu+\varepsilon
\end{aligned}
$$

and since $\varepsilon$ is arbitrary,

$$
\mu(X) \leq \int_{X}|\sigma| d \mu
$$

which requires $|\sigma|=1$ a.e. since it was shown to be no larger than 1 and if it is smaller than 1 on a set of positive measure, then the above could not hold.

It only remains to verify $\mu=|v|$. By Corollary 20.2.10,

$$
|v|(E)=\int_{E}|\sigma| d \mu=\int_{E} 1 d \mu=\mu(E)
$$

and so $\mu=|v|$. This proves the Theorem.
Sometimes people write

$$
\int_{X} f d v \equiv \int_{X} f \sigma d|v|
$$

where $\sigma d|v|$ is the polar decomposition of the complex measure $v$. Then with this convention, the above representation is

$$
L(f)=\int_{X} f d v,|v|(X)=\|L\|
$$

### 20.7 The Dual Space Of $C_{0}(X)$, Another Approach

It is possible to obtain the above theorem by a slick trick after first proving it for the special case where $X$ is a compact Hausdorff space. For $X$ a locally compact Hausdorff space, $\widetilde{\widetilde{X}}$ denotes the one point compactification of $X$. Thus, $\widetilde{X}=X \cup\{\infty\}$ and the topology of $\widetilde{X}$ consists of the usual topology of $X$ along with all complements of compact sets which are defined as the open sets containing $\infty$. Also $C_{0}(X)$ will denote the space of continuous functions, $f$, defined on $X$ such that in the topology of $\widetilde{X}, \lim _{x \rightarrow \infty} f(x)=0$. For this space of functions, $\left||f|_{0} \equiv \sup \{|f(x)|: x \in X\}\right.$ is a norm which makes this into a Banach space. Then the generalization is the following corollary.

Corollary 20.7.1 Let $L \in\left(C_{0}(X)\right)^{\prime}$ where $X$ is a locally compact Hausdorff space. Then there exists $\sigma \in L^{\infty}(X, \mu)$ for $\mu$ a finite Radon measure such that for all $f \in C_{0}(X)$,

$$
L(f)=\int_{X} f \sigma d \mu
$$

Proof: Let

$$
\widetilde{D} \equiv\{f \in C(\widetilde{X}): f(\infty)=0\}
$$

Thus $\widetilde{D}$ is a closed subspace of the Banach space $C(\widetilde{X})$. Let $\theta: C_{0}(X) \rightarrow \widetilde{D}$ be defined by

$$
\theta f(x)=\left\{\begin{array}{l}
f(x) \text { if } x \in X \\
0 \text { if } x=\infty
\end{array}\right.
$$

Then $\theta$ is an isometry of $C_{0}(X)$ and $\widetilde{D} .(\|\theta u\|=\|u\|$.$) The following diagram is obtained.$

$$
\begin{array}{lllll}
C_{0}(X)^{\prime} & \stackrel{\theta^{*}}{\leftarrow} & (\widetilde{D})^{\prime} & \stackrel{i^{*}}{\leftarrow} & C(\widetilde{X})^{\prime} \\
C_{0}(X) & \underset{\theta}{\rightarrow} & \widetilde{D} & \rightarrow & C(\widetilde{X})
\end{array}
$$

By the Hahn Banach theorem, there exists $L_{1} \in C(\widetilde{X})^{\prime}$ such that $\theta^{*} i^{*} L_{1}=L$. Now apply Theorem 20.6 .5 to get the existence of a finite Radon measure, $\mu_{1}$, on $\widetilde{X}$ and a function $\sigma \in L^{\infty}\left(\widetilde{X}, \mu_{1}\right)$, such that

$$
L_{1} g=\int_{\widetilde{X}} g \sigma d \mu_{1}
$$

Letting the $\sigma$ algebra of $\mu_{1}$ measurable sets be denoted by $\mathscr{S}_{1}$, define

$$
\mathscr{S} \equiv\left\{E \backslash\{\infty\}: E \in \mathscr{S}_{1}\right\}
$$

and let $\mu$ be the restriction of $\mu_{1}$ to $\mathscr{S}$. If $f \in C_{0}(X)$,

$$
L f=\theta^{*} i^{*} L_{1} f \equiv L_{1} i \theta f=L_{1} \theta f=\int_{\widetilde{X}} \theta f \sigma d \mu_{1}=\int_{X} f \sigma d \mu
$$

This proves the corollary.

### 20.8 More Attractive Formulations

In this section, Corollary 20.7 .1 will be refined and placed in an arguably more attractive form. The measures involved will always be complex Borel measures defined on a $\sigma$ algebra of subsets of $X$, a locally compact Hausdorff space.

Definition 20.8.1 Let $\lambda$ be a complex measure. Then $\int f d \lambda \equiv \int f h d|\lambda|$ where $h d|\lambda|$ is the polar decomposition of $\lambda$ described above. The complex measure, $\lambda$ is called regular if $|\lambda|$ is regular.

The following lemma says that the difference of regular complex measures is also regular.

Lemma 20.8.2 Suppose $\lambda_{i}, i=1,2$ is a complex Borel measure with total variation finite ${ }^{2}$ defined on $X$, a locally compact Hausdorf space. Then $\lambda_{1}-\lambda_{2}$ is also a regular measure on the Borel sets.

Proof: Let $E$ be a Borel set. That way it is in the $\sigma$ algebras associated with both $\lambda_{i}$. Then by regularity of $\lambda_{i}$, there exist $K$ and $V$ compact and open respectively such that $K \subseteq E \subseteq V$ and $\left|\lambda_{i}\right|(V \backslash K)<\varepsilon / 2$. Therefore,

$$
\begin{aligned}
\sum_{A \in \pi(V \backslash K)}\left|\left(\lambda_{1}-\lambda_{2}\right)(A)\right| & =\sum_{A \in \pi(V \backslash K)}\left|\lambda_{1}(A)-\lambda_{2}(A)\right| \\
& \leq \sum_{A \in \pi(V \backslash K)}\left|\lambda_{1}(A)\right|+\left|\lambda_{2}(A)\right| \\
& \leq\left|\lambda_{1}\right|(V \backslash K)+\left|\lambda_{2}\right|(V \backslash K)<\varepsilon .
\end{aligned}
$$

Therefore, $\left|\lambda_{1}-\lambda_{2}\right|(V \backslash K) \leq \varepsilon$ and this shows $\lambda_{1}-\lambda_{2}$ is regular as claimed.
Theorem 20.8.3 Let $L \in C_{0}(X)^{\prime}$ Then there exists a unique complex measure, $\lambda$ with $|\lambda|$ regular and Borel, such that for all $f \in C_{0}(X)$,

$$
L(f)=\int_{X} f d \lambda
$$

Furthermore, $\|L\|=|\lambda|(X)$.
Proof: By Corollary 20.7.1 there exists $\sigma \in L^{\infty}(X, \mu)$ where $\mu$ is a Radon measure such that for all $f \in C_{0}(X)$,

$$
L(f)=\int_{X} f \sigma d \mu
$$

Let a complex Borel measure, $\lambda$ be given by

$$
\lambda(E) \equiv \int_{E} \sigma d \mu
$$

This is a well defined complex measure because $\mu$ is a finite measure. By Corollary 20.2.10

$$
\begin{equation*}
|\lambda|(E)=\int_{E}|\sigma| d \mu \tag{20.8.21}
\end{equation*}
$$

and $\sigma=g|\sigma|$ where $g d|\lambda|$ is the polar decomposition for $\lambda$. Therefore, for $f \in C_{0}(X)$,

$$
\begin{equation*}
L(f)=\int_{X} f \sigma d \mu=\int_{X} f g|\sigma| d \mu=\int_{X} f g d|\lambda| \equiv \int_{X} f d \lambda \tag{20.8.22}
\end{equation*}
$$

From 20.8.21 and the regularity of $\mu$, it follows that $|\lambda|$ is also regular.

[^18]What of the claim about $\|L\|$ ? By the regularity of $|\lambda|$, it follows that $C_{0}(X)$ (In fact, $\left.C_{c}(X)\right)$ is dense in $L^{1}(X,|\lambda|)$. Since $|\lambda|$ is finite, $g \in L^{1}(X,|\lambda|)$. Therefore, there exists a sequence of functions in $C_{0}(X),\left\{f_{n}\right\}$ such that $f_{n} \rightarrow \bar{g}$ in $L^{1}(X,|\lambda|)$. Therefore, there exists a subsequence, still denoted by $\left\{f_{n}\right\}$ such that $f_{n}(x) \rightarrow \bar{g}(x)|\lambda|$ a.e. also. But since $|\bar{g}(x)|=1$ a.e. it follows that $h_{n}(x) \equiv \frac{f_{n}(x)}{\left|f_{n}(x)\right|+\frac{1}{n}}$ also converges pointwise $|\lambda|$ a.e. Then from the dominated convergence theorem and 20.8.22

$$
\|L\| \geq \lim _{n \rightarrow \infty} \int_{X} h_{n} g d|\lambda|=|\lambda|(X)
$$

Also, if $\|f\|_{C_{0}(X)} \leq 1$, then

$$
|L(f)|=\left|\int_{X} f g d\right| \lambda| | \leq \int_{X}|f| d|\lambda| \leq|\lambda|(X)\|f\|_{C_{0}(X)}
$$

and so $\|L\| \leq|\lambda|(X)$. This proves everything but uniqueness.
Suppose $\lambda$ and $\lambda_{1}$ both work. Then for all $f \in C_{0}(X)$,

$$
0=\int_{X} f d\left(\lambda-\lambda_{1}\right)=\int_{X} f h d\left|\lambda-\lambda_{1}\right|
$$

where $h d\left|\lambda-\lambda_{1}\right|$ is the polar decomposition for $\lambda-\lambda_{1}$. By Lemma 20.8.2 $\lambda-\lambda_{1}$ is regular and so, as above, there exists $\left\{f_{n}\right\}$ such that $\left|f_{n}\right| \leq 1$ and $f_{n} \rightarrow \bar{h}$ pointwise. Therefore, $\int_{X} d\left|\lambda-\lambda_{1}\right|=0$ so $\lambda=\lambda_{1}$. This proves the theorem.

### 20.9 Sequential Compactness In $L^{1}$

Lemma 20.9.1 Let $\mathscr{C} \equiv\left\{E_{i}\right\}_{i=1}^{\infty}$ be a countable collection of sets and let $\Omega_{1} \equiv \cup_{i=1}^{\infty} E_{i}$. Then there exists an algebra of sets, $\mathscr{A}$, such that $\mathscr{A} \supseteq \mathscr{C}$ and $\mathscr{A}$ is countable.

Proof: Let $\mathscr{C}_{1}$ denote all finite unions of sets of $\mathscr{C}$ and also include $\Omega_{1}$ and $\emptyset$. Thus $\mathscr{C}_{1}$ is countable. Next let $\mathscr{B}_{1}$ denote all sets of the form $\Omega_{1} \backslash A$ such that $A \in \mathscr{C}_{1}$. Next let $\mathscr{C}_{2}$ denote all finite unions of sets of $\mathscr{B}_{1} \cup \mathscr{C}_{1}$. Then let $\mathscr{B}_{2}$ denote all sets of the form $\Omega_{1} \backslash A$ such that $A \in \mathscr{C}_{2}$ and let $\mathscr{C}_{3}=\mathscr{B}_{2} \cup \mathscr{C}_{2}$. Continuing this way yields an increasing sequence, $\left\{\mathscr{C}_{n}\right\}$ each of which is countable. Let

$$
\mathscr{A} \equiv \cup_{i=1}^{\infty} \mathscr{C}_{i}
$$

Then $\mathscr{A}$ is countable. Also $\mathscr{A}$ is an algebra. Here is why. Suppose $A, B \in \mathscr{A}$. Then there exists $n$ such that both $A, B \in \mathscr{C}_{n-1}$. It follows $A \cup B \in \mathscr{C}_{n} \subseteq \mathscr{A}$ from the construction. It only remains to show that $A \backslash B \in \mathscr{A}$. Taking complements with respect to $\Omega_{1}$, it follows from the construction that $A^{C}, B^{C}$ are both in $\mathscr{B}_{n-1} \subseteq \mathscr{C}_{n}$. Thus,

$$
A^{C} \cup B \in \mathscr{C}_{n}
$$

and so

$$
A \backslash B=\left(A^{C} \cup B\right)^{C} \in \mathscr{B}_{n} \subseteq \mathscr{C}_{n+1} \subseteq \mathscr{A}
$$

This shows $\mathscr{A}$ is an algebra of sets of $\Omega_{1}$ which is also countable and contains $\mathscr{C}$.

Lemma 20.9.2 Let $\left\{f_{n}\right\}$ be a sequence of functions in $L^{1}(\Omega, \mathscr{S}, \mu)$. Then there exists a $\sigma$ finite set of $\mathscr{S}, \Omega_{1}$, and a $\sigma$ algebra of subsets of $\Omega_{1}, \mathscr{S}_{1}$, such that $\mathscr{S}_{1} \subseteq \mathscr{S}, f_{n}=0$ off $\Omega_{1}, f_{n} \in L^{1}\left(\Omega_{1}, \mathscr{S}_{1}, \mu\right)$, and $\mathscr{S}_{1}=\sigma(\mathscr{A})$, the $\sigma$ algebra generated by $\mathscr{A}$, for some $\mathscr{A}$ a countable algebra.

Proof: Let $\mathscr{E}_{n}$ denote the sets which are of the form

$$
\left\{f_{n}^{-1}(B(z, r)): z \in \mathbb{Q}+i \mathbb{Q}, r>0, r \in \mathbb{Q}, \text { and } 0 \notin \overline{B(z, r)}\right\}
$$

Since each $\mathscr{E}_{n}$ is countable, so is

$$
\mathscr{E} \equiv \cup_{n=1}^{\infty} \mathscr{E}_{n}
$$

Now let $\Omega_{1} \equiv \cup \mathscr{E}$. I claim $\Omega_{1}$ is $\sigma$ finite. To see this, let

$$
W_{n}=\left\{\omega \in \Omega:\left|f_{k}(\omega)\right|>\frac{1}{n} \text { for some } k=1,2, \cdots, n\right\}
$$

Thus if $\omega \in W_{n}$, it follows that for some $r \in \mathbb{Q}, z \in \mathbb{Q}+i \mathbb{Q}$ sufficiently close to $f_{k}(\omega)$

$$
\omega \in f_{k}^{-1}(B(z, r)) \in \mathscr{E}_{k}
$$

and so $\omega \in \cup_{k=1}^{n} \mathscr{E}_{k}$ and consequently, $W_{n} \in \cup_{k=1}^{n} \mathscr{E}_{k}$. Also

$$
\mu\left(W_{n}\right) \frac{1}{n} \leq \int_{W_{n}} \sum_{k=1}^{n}\left|f_{k}(\omega)\right| d \mu<\infty .
$$

Now if $\omega \in \Omega_{1}$, then for some $k, \omega$ is contained in a set of $\mathscr{E}_{k}$. Therefore, for that $k$,

$$
f_{k}(\omega) \in B(z, r)
$$

where $r$ is a positive rational number and $z \in \mathbb{Q}+i \mathbb{Q}$ and $\overline{B(z, r)}$ does not contain 0 . Therefore, $f_{k}(\omega)$ is at a positive distance from 0 and so for large enough $n, \omega \in W_{n}$. Take $n$ so large that $1 / n$ is less than the distance from $\overline{B(z, r)}$ to 0 and also larger than $k$.

By Lemma 20.9.1 there exists a countable algebra of sets $\mathscr{A}$ which contains $\mathscr{E}$. Let $\mathscr{S}_{1} \equiv \sigma(\mathscr{A})$. It remains to show $f_{n}(\omega)=0$ off $\Omega_{1}$ for all $n$. Let $\omega \notin \Omega_{1}$. Then $\omega$ is not contained in any set of $\mathscr{E}_{k}$ and so $f_{k}(\omega)$ cannot be nonzero. Hence $f_{k}(\omega)=0$. This proves the lemma.

The following Theorem is the main result on sequential compactness in $L^{1}(\Omega, \mathscr{S}, \mu)$.
Theorem 20.9.3 Let $K \subseteq L^{1}(\Omega, \mathscr{S}, \mu)$ be such that for some $C>0$ and all $f \in K$,

$$
\begin{equation*}
\|f\|_{L^{1}} \leq C \tag{20.9.23}
\end{equation*}
$$

and $K$ also satisfies the property that if $\left\{E_{n}\right\}$ is a decreasing sequence of measurable sets such that $\cap_{n=1}^{\infty} E_{n}=\emptyset$, then for all $\varepsilon>0$ there exists $n_{\varepsilon}$ such that if $n \geq n_{\varepsilon}$, then

$$
\begin{equation*}
\left|\int_{E_{n}} f d \mu\right|<\varepsilon \tag{20.9.24}
\end{equation*}
$$

for all $f \in K$. Then every sequence of functions of $K$ has an $L^{1}(\Omega, \mu)$ weakly convergent subsequence.

Proof: Take $\left\{f_{n}\right\}$ a sequence in $K$ and let $\mathscr{A}, \mathscr{S}_{1}, \Omega_{1}$ be as in Lemma 20.9.2. Thus $\mathscr{A}$ is a countable algebra and by assumption, for each $E \in \mathscr{A}$,

$$
\left\{\int_{E} f_{n} d \mu\right\}
$$

is a bounded sequence and so there exists a convergent subsequence. Therefore, from a Cantor diagonalization argument, there exists a subsequence, denoted by $\left\{g_{n}\right\}$ such that

$$
\left\{\int_{E} g_{n} d \mu\right\}
$$

converges for every $E \in \mathscr{A}$.
Let

$$
\mathscr{M} \equiv\left\{E \in \mathscr{S}_{1}=\sigma(\mathscr{A}) \text { such that } \lim _{n \rightarrow \infty} \int_{E} g_{n} d \mu \text { exists }\right\}
$$

Then it has been shown that $\mathscr{A} \subseteq \mathscr{M}$. Suppose $E_{k} \uparrow E$ where $E_{k} \in \mathscr{M}$. Then letting $\varepsilon>0$ be given, the assumption shows that for $k$ large enough,

$$
\left|\int_{E \backslash E_{k}} g_{n} d \mu\right|<\varepsilon
$$

for all $g_{n}$. Therefore, picking such a $k$,

$$
\left|\int_{E} g_{n} d \mu-\int_{E} g_{m} d \mu\right| \leq 2 \varepsilon+\left|\int_{E_{k}} g_{n} d \mu-\int_{E_{k}} g_{m} d \mu\right|<3 \varepsilon
$$

provided $m, n$ are large enough. Therefore, $\left\{\int_{E} g_{n} d \mu\right\}$ is a Cauchy sequence and so it converges.

In the case that $E_{k} \downarrow E$ use the assumption to conclude there exists a $k$ large enough that

$$
\left|\int_{E_{k} \backslash E} g_{n} d \mu\right|<\varepsilon
$$

for all $g_{n}$. Then

$$
\begin{aligned}
\left|\int_{E} g_{n} d \mu-\int_{E} g_{m} d \mu\right|= & \left|\int_{E_{k}} g_{n} d \mu-\int_{E_{k}} g_{m} d \mu\right| \\
& +\left|\int_{E_{k} \backslash E} g_{n} d \mu\right|+\left|\int_{E_{k} \backslash E} g_{m} d \mu\right| \\
\leq & \left|\int_{E_{k}} g_{n} d \mu-\int_{E_{k}} g_{m} d \mu\right|+2 \varepsilon<3 \varepsilon
\end{aligned}
$$

provided $m, n$ large enough. Again $\left\{\int_{E} g_{n} d \mu\right\}$ is a Cauchy sequence. This shows $\mathscr{M}$ is a monotone class and so by the monotone class theorem, Theorem 12.10.5 on Page 320 it follows $\mathscr{M}=\mathscr{S}_{1} \equiv \sigma(\mathscr{A})$.

Therefore, picking $E \in \mathscr{S}_{1}$, you can define a complex measure,

$$
\lambda(E) \equiv \lim _{n \rightarrow \infty} \int_{E} g_{n} d \mu
$$

Then $\lambda \ll \mu$ and so by Corollary 20.2.9 on Page 609 and the fact shown above that $\Omega_{1}$ is $\sigma$ finite there exists a unique $\mathscr{S}_{1}$ measurable $g \in L^{1}\left(\Omega_{1}, \mu\right)$ such that

$$
\lambda(E)=\int_{E} g d \mu \equiv \lim _{n \rightarrow \infty} \int_{E} g_{n} d \mu
$$

Extend $g$ to equal 0 outside $\Omega_{1}$.
It remains to show $\left\{g_{n}\right\}$ converges weakly. It has just been shown that for every $s$ a simple function measurable with respect to $\mathscr{S}_{1}$

$$
\int_{\Omega} g_{n} s d \mu=\int_{\Omega_{1}} g_{n} s d \mu \rightarrow \int_{\Omega_{1}} g s d \mu=\int_{\Omega} g s d \mu
$$

Now let $f \in L^{\infty}\left(\Omega_{1}, \mathscr{S}_{1}, \mu\right)$ and pick a uniformly bounded representative of this function. Then by Theorem 11.3.9 on Page 241 there exists a sequence of simple functions converging uniformly to $f$ and so

$$
\left\{\int_{\Omega} g_{n} f d \mu\right\}
$$

converges because

$$
\begin{aligned}
\left|\int_{\Omega} g_{n} f d \mu-\int_{\Omega} g f d \mu\right| \leq & \left|\int_{\Omega} g_{n} s d \mu-\int_{\Omega} g s d \mu\right| \\
& +\int_{\Omega}\left|g_{n}\right| \varepsilon d \mu+\int_{\Omega}|g| \varepsilon d \mu \\
\leq & C \varepsilon+\left|\left|g \|_{1} \varepsilon+\left|\int_{\Omega} g_{n} s d \mu-\int_{\Omega} g s d \mu\right|\right.\right.
\end{aligned}
$$

for suitable simple $s$ satisfying $\sup _{\omega \in \Omega_{1}}|s(\omega)-f(\omega)|<\varepsilon$ and the last term converges to 0 as $n \rightarrow \infty$.
$\left(L^{1}(\Omega, \mathscr{S}, \mu)\right)^{\prime}$ is a space I don't know much about due to a possible lack of $\sigma$ finiteness of $\Omega$. However, it does follow that for $i$ the inclusion map of $L^{1}\left(\Omega_{1}, \mathscr{S}_{1}, \mu\right)$ into $L^{1}(\Omega, \mathscr{S}, \mu)$ which merely extends the function as 0 off $\Omega_{1}$ and $f \in\left(L^{1}(\Omega, \mathscr{S}, \mu)\right)^{\prime}$, there exists $h \in L^{\infty}\left(\Omega_{1}\right)$ such that for all $g \in L^{1}\left(\Omega_{1}, \mathscr{S}_{1}, \mu\right)$

$$
i^{*} f(g)=\int_{\Omega_{1}} h g d \mu
$$

This is because $i^{*} f \in\left(L^{1}\left(\Omega_{1}, \mathscr{S}_{1}, \mu\right)\right)^{\prime}$ and $\Omega_{1}$ is $\sigma$ finite and so the Riesz representation theorem applies to get a unique such $h \in L^{\infty}\left(\Omega_{1}\right)$. Then since all the $g_{n}$ equal 0 off $\Omega_{1}$,

$$
f\left(g_{n}\right)=i^{*} f\left(g_{n}\right)=\int_{\Omega_{1}} h g_{n} d \mu
$$

for a unique $h \in L^{\infty}\left(\Omega_{1}, \mathscr{S}_{1}, \mu\right)$ due to the Riesz representation theorem which holds here because $\Omega_{1}$ was shown to be $\sigma$ finite. Therefore,

$$
\lim _{n \rightarrow \infty} f\left(g_{n}\right)=\lim _{n \rightarrow \infty} \int_{\Omega_{1}} h g_{n} d \mu=\int_{\Omega_{1}} h g d \mu=i^{*} f(g)=f(g)
$$

This proves the theorem.
For more on this theorem see [45]. I have only discussed the sufficiency of the conditions to give sequential compactness. They also discuss the necessity of these conditions.

There is another nice condition which implies the above results which is seen in books on probability. It is the concept of equi integrability.

Definition 20.9.4 Let $(\Omega, \mathscr{S}, \mu)$ be a measure space in which $\mu(\Omega)<\infty$. Then

$$
K \subseteq L^{1}(\Omega, \mathscr{S}, \mu)
$$

is said to be equi integrable if

$$
\lim _{\lambda \rightarrow \infty} \sup _{f \in K} \int_{[|f| \geq \lambda]}|f| d \mu=0
$$

Lemma 20.9.5 Let $K$ be an equi integrable set. Then there exists $C>0$ such that for all $f \in K$,

$$
\begin{equation*}
\|f\|_{L^{1}} \leq C \tag{20.9.25}
\end{equation*}
$$

and $K$ also satisfies the property that if $\left\{E_{n}\right\}$ is a decreasing sequence of measurable sets such that $\cap_{n=1}^{\infty} E_{n}=\emptyset$, then for all $\varepsilon>0$ there exists $n_{\varepsilon}$ such that if $n \geq n_{\varepsilon}$, then

$$
\begin{equation*}
\left|\int_{E_{n}} f d \mu\right|<\varepsilon \tag{20.9.26}
\end{equation*}
$$

for all $f \in K$.
Proof: Choose $\lambda_{0}$ such that

$$
\sup _{f \in K} \int_{\left[|f| \geq \lambda_{0}\right]}|f| d \mu \leq 1
$$

Then for $f \in K$,

$$
\begin{aligned}
\int_{\Omega}|f| d \mu & =\int_{\left[|f| \geq \lambda_{0}\right]}|f| d \mu+\int_{\left[|f|<\lambda_{0}\right]}|f| d \mu \\
& \leq 1+\lambda_{0} \mu(\Omega) \equiv C
\end{aligned}
$$

and this proves 20.9.25.
Next suppose $\left\{E_{n}\right\}$ is a decreasing sequence which has empty intersection and let $\varepsilon>0$ and choose $\lambda_{\varepsilon}$ such that

$$
\sup _{f \in K} \int_{\left[|f| \geq \lambda_{\varepsilon}\right]}|f| d \mu \leq \varepsilon / 2 .
$$

Then since $\mu$ is finite, there exists $n_{\varepsilon}$ such that if $n \geq n_{\varepsilon}$, then $\mu\left(E_{n}\right) \leq \varepsilon / 2\left(1+\lambda_{\varepsilon}\right)$. Then letting $f \in K$,

$$
\begin{aligned}
\int_{E_{n}}|f| d \mu & =\int_{E_{n} \cap\left[|f| \geq \lambda_{\varepsilon}\right]}|f| d \mu+\int_{E_{n} \cap\left[|f|<\lambda_{\varepsilon}\right]}|f| d \mu \\
& \leq \varepsilon / 2+\int_{E_{n}} \lambda_{\varepsilon} d \mu<\varepsilon / 2+\varepsilon / 2=\varepsilon
\end{aligned}
$$

This proves 20.9.26
Corollary 20.9.6 Let $(\Omega, \mathscr{S}, \mu)$ be a measure space in which $\mu(\Omega)<\infty$ and let $K \subseteq$ $L^{1}(\Omega, \mathscr{S}, \mu)$ be equi integrable. Then every sequence from $K$ has a weakly convergent subsequence.

Proof: From Lemma 20.9.5 the hypotheses of Theorem 20.9.3 are satisfied.
It is also convenient to consider the following proposition.
Proposition 20.9.7 Let $(\Omega, \mathscr{S}, \mu)$ be a measure space in which $\mu(\Omega)<\infty$. Then $K \subseteq$ $L^{1}(\Omega, \mathscr{S}, \mu)$ is equi integrable if and only if $K$ is uniformly integrable and there exists a constant, $M$ such that for all $f \in K,\|f\|_{L^{1}} \leq M$.

Proof: First suppose $K$ is equi integrable. Then pick $\lambda$ such that for all $f \in K$,

$$
\int_{[|f| \geq \lambda]}|f| d \mu<1
$$

Then for $f \in K$

$$
\begin{aligned}
\int_{\Omega}|f| d \mu & =\int_{[|f| \geq \lambda]}|f| d \mu+\int_{[|f|<\lambda]}|f| d \mu \\
& \leq 1+\lambda \mu(\Omega) \equiv M
\end{aligned}
$$

Also, if $\varepsilon>0$, pick $\lambda$ so large that for all $f \in K$

$$
\int_{[|f| \geq \lambda]}|f| d \mu<\frac{\varepsilon}{2}
$$

Then letting $A \in \mathscr{S}$,

$$
\begin{aligned}
\int_{A}|f| d \mu & =\int_{A \cap[|f| \geq \lambda]}|f| d \mu+\int_{A \cap[|f|<\lambda]}|f| d \mu \\
& <\frac{\varepsilon}{2}+\lambda P(A)
\end{aligned}
$$

and so if $P(A)$ is sufficiently small, this is less than $\varepsilon$. Thus $K$ is uniformly integrable.
Now suppose $\|f\|_{1} \leq M$ for all $f \in K$ and $K$ is uniformly integrable. Then

$$
\int_{[|f| \geq \lambda]}|f| d \mu \geq \lambda P([|f| \geq \lambda])
$$

and so

$$
P([|f| \geq \lambda]) \leq \frac{1}{\lambda} \int_{[|f| \geq \lambda]}|f| d \mu \leq \frac{1}{\lambda} \int_{\Omega}|f| d \mu \leq \frac{M}{\lambda}
$$

and so, by the assumption of uniform integrability,

$$
\int_{[|f| \geq \lambda]}|f| d \mu<\varepsilon
$$

for all $f \in K$ provided $\lambda$ is large enough. This proves the proposition.

### 20.10 Exercises

1. Suppose $\mu$ is a vector measure having values in $\mathbb{R}^{n}$ or $\mathbb{C}^{n}$. Can you show that $|\mu|$ must be finite? Hint: You might define for each $\mathbf{e}_{i}$, one of the standard basis vectors, the real or complex measure, $\mu_{\mathbf{e}_{i}}$ given by $\mu_{\mathbf{e}_{i}}(E) \equiv \mathbf{e}_{i} \cdot \mu(E)$. Why would this approach not yield anything for an infinite dimensional normed linear space in place of $\mathbb{R}^{n}$ ?
2. The Riesz representation theorem of the $L^{p}$ spaces can be used to prove a very interesting inequality. Let $r, p, q \in(1, \infty)$ satisfy

$$
\frac{1}{r}=\frac{1}{p}+\frac{1}{q}-1
$$

Then

$$
\frac{1}{q}=1+\frac{1}{r}-\frac{1}{p}>\frac{1}{r}
$$

and so $r>q$. Let $\theta \in(0,1)$ be chosen so that $\theta r=q$. Then also we have

$$
\frac{1}{r}=(\overbrace{1-\frac{1}{p^{\prime}}}^{1 / p+1 / p^{\prime}=1})+\frac{1}{q}-1=\frac{1}{q}-\frac{1}{p^{\prime}}
$$

and so

$$
\frac{\theta}{q}=\frac{1}{q}-\frac{1}{p^{\prime}}
$$

which implies $p^{\prime}(1-\theta)=q$. Now let $f \in L^{p}\left(\mathbb{R}^{n}\right), g \in L^{q}\left(\mathbb{R}^{n}\right), f, g \geq 0$. Justify the steps in the following argument using what was just shown that $\theta r=q$ and $p^{\prime}(1-\theta)=q$. Let

$$
\begin{gathered}
h \in L^{r^{\prime}}\left(\mathbb{R}^{n}\right) \cdot\left(\frac{1}{r}+\frac{1}{r^{\prime}}=1\right) \\
\int f * g(\mathbf{x})|h(\mathbf{x})| d x=\iint f(\mathbf{y}) g(\mathbf{x}-\mathbf{y})|h(\mathbf{x})| d x d y \\
\leq \iint|f(\mathbf{y})||g(\mathbf{x}-\mathbf{y})|^{\theta}|g(\mathbf{x}-\mathbf{y})|^{1-\theta}|h(\mathbf{x})| d y d x
\end{gathered}
$$

$$
\begin{gather*}
\leq \int\left(\int\left(|g(\mathbf{x}-\mathbf{y})|^{1-\theta}|h(\mathbf{x})|\right)^{r^{\prime}} d x\right)^{1 / r^{\prime}} \cdot \\
\left(\int\left(|f(\mathbf{y})||g(\mathbf{x}-\mathbf{y})|^{\theta}\right)^{r} d x\right)^{1 / r} d y \\
\leq\left[\int\left(\int\left(|g(\mathbf{x}-\mathbf{y})|^{1-\theta}|h(\mathbf{x})|\right)^{r^{\prime}} d x\right)^{p^{\prime} / r^{\prime}} d y\right]^{1 / p^{\prime}} \cdot \\
\leq\left[\int\left(\int\left(|g(\mathbf{x}-\mathbf{y})|^{1-\theta}|h(\mathbf{x})|\right)^{p^{\prime}} d y\right)^{r^{\prime} / p^{\prime}} d x\right]^{1 / r^{\prime}} \cdot \\
\left.\left.\leq\left[\left.\int(\mathbf{y})| | g(\mathbf{x}-\mathbf{y})\right|^{\theta}\right)^{r} d x\right)^{p / r} d y\right]^{1 / p} \\
=\left[\int|h(\mathbf{x})|^{r^{\prime}}\left(\int|g(\mathbf{x}-\mathbf{y})|^{(1-\theta) p^{\prime}} d y\right)^{r^{\prime} / p^{\prime}} d x\right]^{1 / r^{\prime}}\|g\|_{q}^{q / r}\|f\|_{p} \\
=\left[\left.\int(\mathbf{y})\right|^{p}\left(\int|g(\mathbf{x}-\mathbf{y})|^{\theta r} d x\right)^{1 / r} d y\right]^{1 / p} \\
=\|g\|_{q}^{q / r}\|g\|_{q}^{q / p^{\prime}}\|f\|_{p}\|h\|_{r^{\prime}=\|g\|_{q}\|f\|_{p}\|h\|_{r^{\prime}} .} \tag{20.10.27}
\end{gather*}
$$

Young's inequality says that

$$
\begin{equation*}
\|f * g\|_{r} \leq\|g\|_{q}\|f\|_{p} \tag{20.10.28}
\end{equation*}
$$

Therefore $\|f * g\|_{r} \leq\|g\|_{q}\|f\|_{p}$. How does this inequality follow from the above computation? Does 20.10 .27 continue to hold if $r, p, q$ are only assumed to be in $[1, \infty]$ ? Explain. Does 20.10.28 hold even if $r, p$, and $q$ are only assumed to lie in $[1, \infty]$ ?
3. Show that in a reflexive Banach space, weak and weak $*$ convergence are the same.
4. Suppose $(\Omega, \mu, \mathscr{S})$ is a finite measure space and that $\left\{f_{n}\right\}$ is a sequence of functions which converge weakly to 0 in $L^{p}(\Omega)$. Suppose also that $f_{n}(x) \rightarrow 0$ a.e. Show that then $f_{n} \rightarrow 0$ in $L^{p-\varepsilon}(\Omega)$ for every $p>\varepsilon>0$.
5. Give an example of a sequence of functions in $L^{\infty}(-\pi, \pi)$ which converges weak $*$ to zero but which does not converge pointwise a.e. to zero.

## Chapter 21

## The Bochner Integral

### 21.1 Strong and Weak Measurability

In this chapter $(\Omega, \mathscr{S}, \mu)$ will be a $\sigma$ finite measure space and $X$ will be a Banach space which contains the values of either a function or a measure. The Banach space will be either a real or a complex Banach space but the field of scalars does not matter and so it is denoted by $\mathbb{F}$ with the understanding that $\mathbb{F}=\mathbb{C}$ unless otherwise stated. The theory presented here includes the case where $X=\mathbb{R}^{n}$ or $\mathbb{C}^{n}$ but it does not include the situation where $f$ could have values in something like $[0, \infty]$ which is not a vector space. To begin with here is a definition.

Definition 21.1. 1 A function, $x: \Omega \rightarrow X$, for $X$ a Banach space, is a simple function if it is of the form

$$
x(s)=\sum_{i=1}^{n} a_{i} \mathscr{X}_{B_{i}}(s)
$$

where $B_{i} \in \mathscr{S}$ and $\mu\left(B_{i}\right)<\infty$ for each $i$. A function $x$ from $\Omega$ to $X$ is said to be strongly measurable if there exists a sequence of simple functions $\left\{x_{n}\right\}$ converging pointwise to $x$. The function $x$ is said to be weakly measurable if, for each $f \in X^{\prime}, f \circ x$ is a scalar valued measurable function.

The approximating simple functions can be modified so that the norm of each is no more than $2\|x(s)\|$. This is a useful observation.

Lemma 21.1.2 Let $x$ be strongly measurable. Then $\|x\|$ is a real valued measurable function. There exists a sequence of simple functions $\left\{y_{n}\right\}$ which converges to $f(s)$ pointwise and also $\left\|y_{n}(s)\right\| \leq 2\|x(s)\|$ for all $s$.

Proof: Consider the first claim. Letting $x_{n}$ be a sequence of simple functions converging to $x$ pointwise, it follows that $\left\|x_{n}\right\|$ is a real valued measurable function. Since $\|x\|$ is a pointwise limit, so is $\|x\|$ a real valued measurable function.

Let $x_{n}(s)$ be simple functions converging to $x(s)$ pointwise as above. Let

$$
x_{n}(s) \equiv \sum_{k=1}^{m_{n}} a_{k}^{n} \mathscr{X}_{E_{k}^{n}}(s)
$$

Then

$$
y_{n}(s) \equiv\left\{\begin{array}{c}
x_{n}(s) \text { if }\left\|x_{n}(s)\right\|<2\|x(s)\| \\
0 \text { if }\left\|x_{n}(s)\right\| \geq 2\|x(s)\|
\end{array}\right.
$$

Thus, for $\left[\left\|a_{k}^{n}\right\| \leq 2\|x\|\right] \equiv\left\{s:\left\|a_{k}^{n}\right\| \leq 2\|x(s)\|\right\}$,

$$
y_{n}(s)=\sum_{k=1}^{m_{n}} a_{k}^{n} \mathscr{X}_{E_{k}^{n} \cap\left[\left\|a_{k}^{n}\right\| \leq 2\|x\|\right]}(s)
$$

It follows $y_{n}$ is a simple function. If $\|x(s)\|=0$, then $y_{n}(s)=0$ and so $y_{n}(s) \rightarrow x(s)$. If $\|x(s)\|>0$, then eventually, $y_{n}(s)=x_{n}(s)$ and so in this case, $y_{n}(s) \rightarrow x(s)$.

Earlier, a function was measurable if inverse images of open sets were measurable. Something similar holds here. The difference is that another condition needs to hold about the values being separable. First is a somewhat obvious lemma.

Lemma 21.1.3 Suppose $S$ is a nonempty subset of a metric space $(X, d)$ and $S \subseteq T$ where $T$ is separable. Then there exists a countable dense subset of $S$.

Proof: Let $D$ be the countable dense subset of $T$. Now consider the countable set $\mathscr{B}$ of balls having center at a point of $D$ and radius a positive rational number such that also, each ball in $\mathscr{B}$ has nonempty intersection with $S$. Let $\mathscr{D}$ consist of a point from $S \cap B$ whenever $B \in \mathscr{B}$. Let $s \in S$ and consider $B(s, \varepsilon)$. Let $r$ be rational with $r<\varepsilon$. Now $B\left(s, \frac{r}{10}\right)$ contains a point $d \in D$. Thus $B\left(d, \frac{r}{10}\right) \in \mathscr{B}$ and in fact, $s \in B\left(d, \frac{r}{10}\right)$. Let $\hat{d} \in \mathscr{D}$. Thus $d(s, \hat{d})<\frac{r}{5}<r<\varepsilon$ so $\hat{d} \in B(s, \varepsilon)$ and this shows that $\mathscr{D}$ is a countable dense subset of $S$ as claimed.

Theorem 21.1.4 $x$ is strongly measurable if and only if $x^{-1}(U)$ is measurable for all $U$ open in $X$ and $x(\Omega)$ is separable. Thus, if $X$ is separable, $x$ is strongly measurable if and only if $x^{-1}(U)$ is measurable for all $U$ open.

Proof: Suppose first $x^{-1}(U)$ is measurable for all $U$ open in $X$ and $x(\Omega)$ is separable. Let $\left\{a_{n}\right\}_{n=1}^{\infty}$ be the dense subset of $x(\Omega)$. It follows $x^{-1}(B)$ is measurable for all $B$ Borel because

$$
\left\{B: x^{-1}(B) \text { is measurable }\right\}
$$

is a $\sigma$ algebra containing the open sets. Let

$$
U_{k}^{n} \equiv\left\{z \in X:\left\|z-a_{k}\right\| \leq \min \left\{\left\{\left\|z-a_{l}\right\|\right\}_{l=1}^{n}\right\} .\right.
$$

In words, $U_{k}^{m}$ is the set of points of $X$ which are as close to $a_{k}$ as they are to any of the $a_{l}$ for $l \leq n$.

$$
B_{k}^{n} \equiv x^{-1}\left(U_{k}^{n}\right), D_{k}^{n} \equiv B_{k}^{n} \backslash\left(\cup_{i=1}^{k-1} B_{i}^{n}\right), D_{1}^{n} \equiv B_{1}^{n}
$$

and $x_{n}(s) \equiv \sum_{k=1}^{n} a_{k} \mathscr{X}_{D_{k}^{n}}(s)$.Thus $x_{n}(s)$ is a closest approximation to $x(s)$ from $\left\{a_{k}\right\}_{k=1}^{n}$ and so $x_{n}(s) \rightarrow x(s)$ because $\left\{a_{n}\right\}_{n=1}^{\infty}$ is dense in $x(\Omega)$. Furthermore, $x_{n}$ is measurable because each $D_{k}^{n}$ is measurable.

Since $(\Omega, \mathscr{S}, \mu)$ is $\sigma$ finite, there exists $\Omega_{n} \uparrow \Omega$ with $\mu\left(\Omega_{n}\right)<\infty$. Let

$$
y_{n}(s) \equiv \mathscr{X}_{\Omega_{n}}(s) x_{n}(s)
$$

Then $y_{n}(s) \rightarrow x(s)$ for each $s$ because for any $s, s \in \Omega_{n}$ if $n$ is large enough. Also $y_{n}$ is a simple function because it equals 0 off a set of finite measure.

Now suppose that $x$ is strongly measurable. Then some sequence of simple functions, $\left\{x_{n}\right\}$, converges pointwise to $x$. Then $x_{n}^{-1}(W)$ is measurable for every open set $W$ because it is just a finite union of measurable sets. Thus, $x_{n}^{-1}(W)$ is measurable for every Borel set $W$. This follows by considering $\left\{W: x_{n}^{-1}(W)\right.$ is measurable $\}$ and observing this is a $\sigma$ algebra which contains the open sets. Since $X$ is a metric space, it follows that if $U$ is an open set in $X$, there exists a sequence of open sets, $\left\{V_{n}\right\}$ which satisfies

$$
\bar{V}_{n} \subseteq U, \bar{V}_{n} \subseteq V_{n+1}, U=\cup_{n=1}^{\infty} V_{n}
$$

Then

$$
x^{-1}\left(V_{m}\right) \subseteq \bigcup_{n<\infty} \bigcap_{k \geq n} x_{k}^{-1}\left(V_{m}\right) \subseteq x^{-1}\left(\bar{V}_{m}\right)
$$

This implies

$$
\begin{gathered}
x^{-1}(U)=\bigcup_{m<\infty} x^{-1}\left(V_{m}\right) \\
\subseteq \bigcup_{m<\infty} \bigcup_{n<\infty} \bigcap_{k \geq n} x_{k}^{-1}\left(V_{m}\right) \subseteq \bigcup_{m<\infty} x^{-1}\left(\bar{V}_{m}\right) \subseteq x^{-1}(U)
\end{gathered}
$$

Since

$$
x^{-1}(U)=\bigcup_{m<\infty} \bigcup_{n<\infty} \bigcap_{k \geq n} x_{k}^{-1}\left(V_{m}\right),
$$

it follows that $x^{-1}(U)$ is measurable for every open $U$. It remains to show $x(\Omega)$ is separable. Let

$$
D \equiv \text { all values of the simple functions } x_{n}
$$

Then $x(\Omega) \subseteq \bar{D}$, which has a countable dense subset. By Lemma 21.1.3, $x(\Omega)$ is separable.
The next lemma is interesting for its own sake. Roughly it says that if a Banach space is separable, then the unit ball in the dual space is weak $*$ separable. This will be used to prove Pettis's theorem, one of the major theorems in this subject which relates weak measurability to strong measurability. First here is a standard application which comes from earlier material on the Hahn Banach theorem.

Lemma 21.1.5 Let $x \in X$ a normed linear space. Then there exists $f \in X^{\prime}$ such that $\|f\|=1$ and $f(x)=\|x\|$.

Proof: Consider the one dimensional subspace

$$
M \equiv\left\{\alpha \frac{x}{\|x\|}: \alpha \in \mathbb{F}\right\}
$$

and define a continuous linear functional on $M$ by $g\left(\alpha \frac{x}{\|x\|}\right) \equiv \alpha$. Then the norm of $\|g\| \equiv$ $\sup _{|\alpha| \leq 1}|\alpha|=1$. Extend $g$ to all of $X$ using the Hahn Banach theorem calling the extended function $f$. Then $\|f\|=1$ and $f(x)=f\left(\|x\| \frac{x}{\|x\|}\right)=\|x\|$.

Lemma 21.1.6 If $X$ is a separable Banach space with $B^{\prime}$ the closed unit ball in $X^{\prime}$, then there exists a sequence $\left\{f_{n}\right\}_{n=1}^{\infty} \equiv D^{\prime} \subseteq B^{\prime}$ with the property that for every $x \in X$,

$$
\|x\|=\sup _{f \in D^{\prime}}|f(x)|
$$

If $H$ is a dense subset of $X^{\prime}$ then $D^{\prime}$ may be chosen to be contained in $H$.

Proof: Let $\left\{a_{k}\right\}_{k=1}^{\infty}$ be a countable dense set in $X$ and consider the mapping

$$
\phi_{n}: B^{\prime} \rightarrow \mathbb{F}^{n}
$$

given by

$$
\phi_{n}(f) \equiv\left(f\left(a_{1}\right), \cdots, f\left(a_{n}\right)\right)
$$

Then $\phi_{n}\left(B^{\prime}\right)$ is contained in a compact subset of $\mathbb{F}^{n}$ because $\left|f\left(a_{k}\right)\right| \leq\left\|a_{k}\right\|$. Therefore, there exists a countable dense subset of $\phi_{n}\left(B^{\prime}\right),\left\{\phi_{n}\left(f_{k}\right)\right\}_{k=1}^{\infty}$. Then pick $h_{j}^{k} \in H \cap B^{\prime}$ such that $\lim _{j \rightarrow \infty}\left\|f_{k}-h_{j}^{k}\right\|=0$. Then $\left\{\phi_{n}\left(h_{j}^{k}\right), k, j\right\}$ must also be dense in $\phi_{n}\left(B^{\prime}\right)$. Let $D_{n}^{\prime}=$ $\left\{h_{j}^{k}, k, j\right\}$. Thus $D_{n}^{\prime}$ is a countable collection of $f \in B^{\prime}$ which can be used to approximate each $\left\|a_{k}\right\|, k \leq n$. Indeed, if $x$ is arbitrary, there exists $f_{x} \in B^{\prime}$ with $f_{x}(x)=\|x\|$. Thus $\left\|a_{k}\right\|$ is contained in $\phi_{n}\left(B^{\prime}\right)$. Define

$$
D^{\prime} \equiv \cup_{n=1}^{\infty} D_{n}^{\prime}
$$

From the construction, $D^{\prime}$ is countable and can be used to approximate each $\left\|a_{m}\right\|$. That is,

$$
\left\|a_{m}\right\|=\sup \left\{\left|f\left(a_{m}\right)\right|: f \in D^{\prime}\right\}
$$

Then, for $x$ arbitrary, $|f(x)| \leq\|x\|$ and so

$$
\begin{aligned}
\|x\| & \leq\left\|x-a_{m}\right\|+\left\|a_{m}\right\|=\left\|x-a_{m}\right\|+\sup \left\{\left|f\left(a_{m}\right)\right|: f \in D^{\prime}\right\} \\
& \leq \sup \left\{\left|f\left(a_{m}-x\right)+f(x)\right|: f \in D^{\prime}\right\}+\left\|x-a_{m}\right\| \\
& \leq \sup \left\{|f(x)|: f \in D^{\prime}\right\}+2\left\|x-a_{m}\right\| \leq\|x\|+2\left\|x-a_{m}\right\|
\end{aligned}
$$

Since $a_{m}$ is arbitrary and the $\left\{a_{m}\right\}_{m=1}^{\infty}$ are dense, this establishes the claim of the lemma.
Note that the proof would work the same if $H$ were only given to be weak $*$ dense.
The next theorem is one of the most important results in the subject. It is due to Pettis and appeared in 1938 [107].

Theorem 21.1.7 If $x$ has values in a separable Banach space $X$, then $x$ is weakly measurable if and only if $x$ is strongly measurable.

Proof: $\Rightarrow$ It is necessary to show $x^{-1}(U)$ is measurable whenever $U$ is open. Since every open set is a countable union of balls, it suffices to show $x^{-1}(B(a, r))$ is measurable for any ball, $B(a, r)$. Since every open ball is the countable union of closed balls, it suffices to verify $x^{-1}(\overline{B(a, r)})$ is measurable. For $D^{\prime}$ described in Lemma 21.1.6,

$$
\begin{aligned}
x^{-1}(\overline{B(a, r)}) & =\{s:\|x(s)-a\| \leq r\}=\left\{s: \sup _{f \in D^{\prime}}|f(x(s)-a)| \leq r\right\} \\
& =\cap_{f \in D^{\prime}}\{s:|f(x(s)-a)| \leq r\}=\cap_{f \in D^{\prime}}\{s:|f(x(s))-f(a)| \leq r\} \\
& =\cap_{f \in D^{\prime}}(f \circ x)^{-1} \overline{B(f(a), r)}
\end{aligned}
$$

which equals a countable union of measurable sets because it is assumed that $f \circ x$ is measurable for all $f \in X^{\prime}$.
$\Leftarrow$ Next suppose $x$ is strongly measurable. Then there exists a sequence of simple functions $x_{n}$ which converges to $x$ pointwise. Hence for all $f \in X^{\prime}, f \circ x_{n}$ is measurable and $f \circ x_{n} \rightarrow f \circ x$ pointwise. Thus $x$ is weakly measurable.

The same method of proof yields the following interesting corollary.
Corollary 21.1.8 Let $X$ be a separable Banach space and let $\mathscr{B}(X)$ denote the $\sigma$ algebra of Borel sets. Let $H$ be a dense subset of $X^{\prime}$. Then $\mathscr{B}(X)=\sigma(H) \equiv \mathscr{F}$, the smallest $\sigma$ algebra of subsets of $X$ which has the property that every function, $x^{*} \in H$ is measurable.

Proof: First I need to show $\mathscr{F}$ contains open balls because then $\mathscr{F}$ will contain the open sets and hence the Borel sets. As noted above, it suffices to show $\mathscr{F}$ contains closed balls. Let $D^{\prime}$ be those functionals in $B^{\prime}$ defined in Lemma 21.1.6 contained in $H$. Then

$$
\begin{aligned}
\{x:\|x-a\| \leq r\} & =\left\{x: \sup _{x^{*} \in D^{\prime}}\left|x^{*}(x-a)\right| \leq r\right\} \\
& =\cap_{x^{*} \in D^{\prime}}\left\{x:\left|x^{*}(x-a)\right| \leq r\right\} \\
& =\cap_{x^{*} \in D^{\prime}}\left\{x:\left|x^{*}(x)-x^{*}(a)\right| \leq r\right\} \\
& =\cap_{x^{*} \in D^{\prime}} x^{*-1}\left(\overline{B\left(x^{*}(a), r\right)}\right) \in \sigma(H)
\end{aligned}
$$

which is measurable because this is a countable intersection of measurable sets. Thus $\mathscr{F}$ contains open sets so $\sigma(H) \equiv \mathscr{F} \supseteq \mathscr{B}(X)$.

To show the other direction for the inclusion, note that each $x^{*}$ is $\mathscr{B}(X)$ measurable because $x^{*-1}($ open set $)=$ open set. Therefore, $\mathscr{B}(X) \supseteq \sigma(H)$.

It is important to verify the limit of strongly measurable functions is itself strongly measurable. This happens under very general conditions.

Lemma 21.1.9 Let $X$ be a metric space and suppose $V$ is an open set in $V$. Then there exists open sets $V_{m}$ such that

$$
\begin{equation*}
\cdots V_{m} \subseteq \bar{V}_{m} \subseteq V_{m+1} \subseteq \cdots, V=\bigcup_{m=1}^{\infty} V_{m} \tag{21.1.1}
\end{equation*}
$$

Proof: Recall that if $S$ is a nonempty set, $x \rightarrow \operatorname{dist}(x, S)$ is a continuous map from $X$ to $\mathbb{R}$. First assume $V \neq X$. Let

$$
V_{m} \equiv\left\{x \in V: \operatorname{dist}\left(x, V^{C}\right)>\frac{1}{m}\right\}
$$

Then for large enough $m$, this set is nonempty and contained in $V$. Furthermore, if $x \in V$ then it is at a positive distance to the closed set $V^{C}$ so eventually, $x \in V_{m}$. Now

$$
V_{m} \subseteq \overline{V_{m}} \subseteq\left\{x \in V: \operatorname{dist}\left(x, V^{C}\right) \geq \frac{1}{m}\right\} \subseteq V
$$

Indeed, if $p$ is a limit point of $V_{m}$, then there are $x_{n} \in V_{m}$ with $x_{n} \rightarrow p$. Thus dist $\left(x_{n}, V^{C}\right) \rightarrow$ $\operatorname{dist}\left(p, V^{C}\right)$ and so $p$ is in the set on the right. In case $X=V$, let $V_{m} \equiv B(\xi, m)$. Then $V_{m} \subseteq \overline{V_{m}} \subseteq\left\{x \in V: \operatorname{dist}\left(x, V^{C}\right) \geq \frac{1}{m}\right\}$ and the union of these $V_{m}$ equals $V$.

What of limits of measurable functions? The next theorem says that the usual theorem about limits of measurable functions being measurable holds.

Theorem 21.1.10 Let $x_{n}$ and $x$ be functions mapping $\Omega$ to $X$ where $\mathscr{F}$ is a $\sigma$ algebra of measurable sets of $\Omega$ and $X$ is a Banach space. Thus $X$ satisfies 21.1.1. Then if $x_{n}$ is strongly measurable, and $x(s)=\lim _{n \rightarrow \infty} x_{n}(s)$, it follows that $x$ is also strongly measurable. (Pointwise limits of measurable functions are measurable.)

Proof: Let $\left\{V_{m}\right\}$ be the sequence of 21.1.1. Since $x$ is the pointwise limit of $x_{n}$,

$$
x^{-1}\left(V_{m}\right) \subseteq\left\{s: x_{k}(s) \in V_{m} \text { for all } k \text { large enough }\right\} \subseteq x^{-1}\left(\overline{V_{m}}\right)
$$

Therefore,

$$
\begin{gathered}
x^{-1}(V)=\cup_{m=1}^{\infty} x^{-1}\left(V_{m}\right) \subseteq \cup_{m=1}^{\infty} \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} x_{k}^{-1}\left(V_{m}\right) \\
\subseteq \cup_{m=1}^{\infty} x^{-1}\left(\overline{V_{m}}\right)=x^{-1}(V) .
\end{gathered}
$$

It follows $x^{-1}(V) \in \mathscr{F}$ because it equals the expression in the middle which is measurable. Note that this shows the characterization of measurability in terms of inverse images of open sets being measureable sets. Thus the theorem is proved in the case of separable Banach spaces. However, Lemma 21.1.3 can be applied to conclude that this holds in general because each $x_{n}$ is separably valued given they are each strongly measurable and $x(\Omega) \subseteq \bar{D}$ where $D=\cup_{n} D_{n}$ for $D_{n}$ a countable dense subset of $x_{n}(\Omega)$.

Note that the same conclusion in terms of inverse images being measurable would hold for any metric space.

Corollary 21.1.11 $x$ is strongly measurable if and only if $x(\Omega)$ is separable and $x$ is weakly measurable.

Proof: Strong measurability clearly implies weak measurability. If $x_{n}(s) \rightarrow x(s)$ where $x_{n}$ is simple, then $f\left(x_{n}(s)\right) \rightarrow f(x(s))$ for all $f \in X^{\prime}$. Hence $f \circ x$ is measurable by Theorem 21.1.10 because it is the limit of a sequence of measurable functions. Let $D$ denote the set of all values of $x_{n}$. Then $\bar{D}$ is a separable set containing $x(\Omega)$. Thus $\bar{D}$ is a separable metric space. Therefore $x(\Omega)$ is separable also by the last part of the proof of Theorem 21.1.4.

Now suppose $D$ is a countable dense subset of $x(\Omega)$ and $x$ is weakly measurable. Let $Z$ be the subset consisting of all finite linear combinations of $D$ with the scalars coming from the set of rational points of $\mathbb{F}$. Thus, $Z$ is countable. Letting $Y=\bar{Z}, Y$ is a separable Banach space containing $x(\Omega)$. If $f \in Y^{\prime}, f$ can be extended to an element of $X^{\prime}$ by the Hahn Banach theorem. Therefore, $x$ is a weakly measurable $Y$ valued function. Now use Theorem 21.1.7 to conclude $x$ is strongly measurable.

Weakly measurable as defined above means $s \rightarrow x^{*}(x(s))$ is measurable for every $x^{*} \in$ $X^{\prime}$. The next lemma ties this weak measurability to the usual version of measurability in which a function is measurable when inverse images of open sets are measurable.

Lemma 21.1.12 Let $X$ be a Banach space and let $x:(\Omega, \mathscr{F}) \rightarrow K \subseteq X$ where $K$ is weakly compact and $X^{\prime}$ is separable. Then $x$ is weakly measurable if and only if $x^{-1}(U) \in \mathscr{F}$ whenever $U$ is a weakly open set.

Proof: By Corollary 17.5 .9 on Page 464, there exists a metric $d$, such that the metric space topology with respect to $d$ coincides with the weak topology on $K$. Since $K$ is compact, it follows that $K$ is also separable. Hence it is completely separable and so there exists
a countable basis of open sets $\mathscr{B}$ for the weak topology on $K$. It follows that if $U$ is any weakly open set, covered by basic sets of the form $B_{A}(x, r)$ where $A$ is a finite subset of $X^{\prime}$, there exists a countable collection of these sets of the form $B_{A}(x, r)$ which covers $U$.

Suppose now that $x$ is weakly measurable. To show $x^{-1}(U) \in \mathscr{F}$ whenever $U$ is weakly open, it suffices to verify $x^{-1}\left(B_{A}(z, r)\right) \in \mathscr{F}$ for any set, $B_{A}(z, r)$. Let $A=\left\{x_{1}^{*}, \cdots, x_{m}^{*}\right\}$. Then

$$
\begin{aligned}
x^{-1}\left(B_{A}(z, r)\right) & =\left\{s \in \Omega: \rho_{A}(x(s)-z)<r\right\} \\
& \equiv\left\{s \in \Omega: \max _{x^{*} \in A}\left|x^{*}(x(s)-z)\right|<r\right\} \\
& =\cup_{i=1}^{m}\left\{s \in \Omega:\left|x_{i}^{*}(x(s)-z)\right|<r\right\} \\
& =\cup_{i=1}^{m}\left\{s \in \Omega:\left|x_{i}^{*}(x(s))-x_{i}^{*}(z)\right|<r\right\}
\end{aligned}
$$

which is measurable because each $x_{i}^{*} \circ x$ is given to be measurable.
Next suppose $x^{-1}(U) \in \mathscr{F}$ whenever $U$ is weakly open. Then in particular this holds when $U=B_{x^{*}}(z, r)$ for arbitrary $x^{*}$. Hence

$$
\left\{s \in \Omega: x(s) \in B_{x^{*}}(z, r)\right\} \in \mathscr{F} .
$$

But this says the same as

$$
\left\{s \in \Omega:\left|x^{*}(x(s))-x^{*}(z)\right|<r\right\} \in \mathscr{F}
$$

Since $x^{*}(z)$ can be a completely arbitrary element of $\mathbb{F}$, it follows $x^{*} \circ x$ is an $\mathbb{F}$ valued measurable function. In other words, $x$ is weakly measurable according to the former definition.

One can also define weak $*$ measurability and prove a theorem just like the Pettis theorem above. The next lemma is the analogue of Lemma 21.1.6.

Lemma 21.1.13 Let $B$ be the closed unit ball in $X$. If $X^{\prime}$ is separable, there exists a sequence $\left\{x_{m}\right\}_{m=1}^{\infty} \equiv D \subseteq B$ with the property that for all $y^{*} \in X^{\prime}$,

$$
\left\|y^{*}\right\|=\sup _{x \in D}\left|y^{*}(x)\right| .
$$

Proof: Let $\left\{x_{k}^{*}\right\}_{k=1}^{\infty}$ be the dense subset of $X^{\prime}$. Define $\phi_{n}: B \rightarrow \mathbb{F}^{n}$ by

$$
\phi_{n}(x) \equiv\left(x_{1}^{*}(x), \cdots, x_{n}^{*}(x)\right) .
$$

Then $\left|x_{k}^{*}(x)\right| \leq\left\|x_{k}^{*}\right\|$ and so $\phi_{n}(B)$ is contained in a compact subset of $\mathbb{F}^{n}$. Therefore, there exists a countable set, $D_{n} \subseteq B$ such that $\phi_{n}\left(D_{n}\right)$ is dense in $\phi_{n}(B)$. That is, $\left\{\left(x_{1}^{*}(x), \cdots, x_{n}^{*}(x)\right): x \in D_{n}\right\}$ is dense in $\phi_{n}(B)$.

$$
D \equiv \cup_{n=1}^{\infty} D_{n}
$$

It remains to verify this works. Let $y^{*} \in X^{\prime}$. I want to show that $\left\|y^{*}\right\|=\sup _{x \in D}\left|y^{*}(x)\right|$. There exists $y,\|y\| \leq 1$, such that

$$
\left|y^{*}(y)\right|>\left\|y^{*}\right\|-\varepsilon
$$

By density, there exists one of the $x_{k}^{*}$ from the countable dense subset of $X^{\prime}$ such that also

$$
\left\|x_{k}^{*}-y^{*}\right\|<\varepsilon, \text { so }\left|x_{k}^{*}(y)\right|>\left\|y^{*}\right\|-2 \varepsilon
$$

Now $x_{k}^{*}(y) \in \phi_{k}(B)$ and so there exists $x \in D_{k} \subseteq D \subseteq B$ such that also

$$
\left|x_{k}^{*}(x)\right|>\left\|y^{*}\right\|-2 \varepsilon
$$

Then since $\left\|x_{k}^{*}-y^{*}\right\|<\varepsilon$, this implies

$$
\left\|y^{*}\right\| \geq\left|y^{*}(x)\right|=\left|\left(y^{*}-x_{k}^{*}\right)(x)+x_{k}^{*}(x)\right| \geq\left|x_{k}^{*}(x)\right|-\varepsilon>\left\|y^{*}\right\|-3 \varepsilon
$$

It follows that

$$
\left\|y^{*}\right\|-3 \varepsilon \leq \sup _{x \in D}\left|y^{*}(x)\right| \leq\left\|y^{*}\right\|
$$

This proves the lemma because $\varepsilon$ is arbitrary.
The next theorem is another version of the Pettis theorem. First here is a definition.
Definition 21.1.14 A function y having values in $X^{\prime}$ is weak $*$ measurable, when for each $x \in X, y(\cdot)(x)$ is a measurable scalar valued function.

Theorem 21.1.15 If $X^{\prime}$ is separable and $y: \Omega \rightarrow X^{\prime}$ is weak $*$ measurable meaning $s \rightarrow$ $y(s)(x)$ is a $\mathbb{F}$ valued measurable function, then $y$ is strongly measurable.

Proof: It is necessary to show $y^{-1}\left(B\left(a^{*}, r\right)\right)$ is measurable for $a^{*} \in X^{\prime}$. This will suffice because the separability of $X^{\prime}$ implies every open set is the countable union of such balls of the form $B\left(a^{*}, r\right)$. It also suffices to verify inverse images of closed balls are measurable because every open ball is the countable union of closed balls. From Lemma 21.1.13,

$$
\begin{aligned}
y^{-1}\left(\overline{B\left(a^{*}, r\right)}\right) & =\left\{s:\left\|y(s)-a^{*}\right\| \leq r\right\} \\
& =\left\{s: \sup _{x \in D}\left|\left(y(s)-a^{*}\right)(x)\right| \leq r\right\} \\
& =\left\{s: \sup _{x \in D}\left|y(s)(x)-a^{*}(x)\right| \leq r\right\} \\
& =\cap_{x \in D} y(\cdot)(x)^{-1}\left(\overline{B\left(a^{*}(x), r\right)}\right)
\end{aligned}
$$

which is a countable intersection of measurable sets by hypothesis.
The following are interesting consequences of the theory developed so far and are of interest independent of the theory of integration of vector valued functions.

Theorem 21.1.16 If $X^{\prime}$ is separable, then so is $X$.
Proof: Let $D=\left\{x_{m}\right\} \subseteq B$, the unit ball of $X$, be the sequence promised by Lemma 21.1.13. Let $V$ be all finite linear combinations of elements of $\left\{x_{m}\right\}$ with rational scalars. Thus $\bar{V}$ is a separable subspace of $X$. The claim is that $\bar{V}=X$. If not, there exists

$$
x_{0} \in X \backslash \bar{V}
$$

But by the Hahn Banach theorem there exists $x_{0}^{*} \in X^{\prime}$ satisfying $x_{0}^{*}\left(x_{0}\right) \neq 0$, but $x_{0}^{*}(v)=0$ for every $v \in \bar{V}$. Hence

$$
\left\|x_{0}^{*}\right\|=\sup _{x \in D}\left|x_{0}^{*}(x)\right|=0
$$

a contradiction.
Corollary 21.1.17 If $X$ is reflexive, then $X$ is separable if and only if $X^{\prime}$ is separable.
Proof: From the above theorem, if $X^{\prime}$ is separable, then so is $X$. Now suppose $X$ is separable with a dense subset equal to $D$. Then since $X$ is reflexive, $J(D)$ is dense in $X^{\prime \prime}$ where $J$ is the James map satisfying $J x\left(x^{*}\right) \equiv x^{*}(x)$. Then since $X^{\prime \prime}$ is separable, it follows from the above theorem that $X^{\prime}$ is also separable.

Note how this shows that $L^{1}\left(\mathbb{R}^{p}, m_{p}\right)$ is not reflexive because this is a separable space, but $L^{\infty}\left(\mathbb{R}^{p}, m_{p}\right)$ is clearly not. For example, you could consider $\mathscr{X}_{[0, r]}$ for $r$ a positive irrational number. There are uncountably many of these functions in $L^{\infty}([0,1])$ and

$$
\left\|\mathscr{X}_{[0, r]}-\mathscr{X}_{[0, \hat{r}]}\right\|_{\infty}=1
$$

### 21.2 The Bochner Integral

### 21.2.1 Definition and Basic Properties

Definition 21.2.1 Let $a_{k} \in X$, a Banach space and let a simple function $s \rightarrow x(s)$ be

$$
\begin{equation*}
x(s)=\sum_{k=1}^{n} a_{k} \mathscr{X}_{E_{k}}(s) \tag{21.2.2}
\end{equation*}
$$

where for each $k, E_{k}$ is measurable and $\mu\left(E_{k}\right)<\infty$. Then define

$$
\int_{\Omega} x(s) d \mu \equiv \sum_{k=1}^{n} a_{k} \mu\left(E_{k}\right)
$$

Proposition 21.2.2 Definition 21.2.1 is well defined, the integral is linear on simple functions and

$$
\left\|\int_{\Omega} x(s) d \mu\right\| \leq \int_{\Omega}\|x(s)\| d \mu
$$

whenever $x$ is a simple function.
Proof: It suffices to verify that if $\sum_{k=1}^{n} a_{k} \mathscr{X}_{E_{k}}(s)=0$, then $\sum_{k=1}^{n} a_{k} \mu\left(E_{k}\right)=0$. Let $f \in$ $X^{\prime}$. Then

$$
f\left(\sum_{k=1}^{n} a_{k} \mathscr{X}_{E_{k}}(s)\right)=\sum_{k=1}^{n} f\left(a_{k}\right) \mathscr{X}_{E_{k}}(s)=0
$$

and, therefore,

$$
0=\int_{\Omega}\left(\sum_{k=1}^{n} f\left(a_{k}\right) \mathscr{X}_{E_{k}}(s)\right) d \mu=\sum_{k=1}^{n} f\left(a_{k}\right) \mu\left(E_{k}\right)=f\left(\sum_{k=1}^{n} a_{k} \mu\left(E_{k}\right)\right)
$$

Since $f \in X^{\prime}$ is arbitrary, and $X^{\prime}$ separates the points of $X$, it follows that $\sum_{k=1}^{n} a_{k} \mu\left(E_{k}\right)=0$ as hoped. It is now obvious that the integral is linear on simple functions.

As to the triangle inequality, say $x(s)=\sum_{k=1}^{n} a_{k} \mathscr{X}_{E_{k}}(s)$. Then from the triangle inequality,

$$
\left\|\int_{\Omega} x(s) d \mu\right\|=\left\|\sum_{k=1}^{n} a_{k} \mu\left(E_{k}\right)\right\| \leq \sum_{k=1}^{n}\left\|a_{k}\right\| \mu\left(E_{k}\right)=\int_{\Omega}\|x(s)\| d \mu
$$

Definition 21.2.3 A strongly measurable function $x$ is Bochner integrable if there exists a sequence of simple functions $x_{n}$ converging to $x$ pointwise and satisfying

$$
\begin{equation*}
\int_{\Omega}\left\|x_{n}(s)-x_{m}(s)\right\| d \mu \rightarrow 0 \text { as } m, n \rightarrow \infty \tag{21.2.3}
\end{equation*}
$$

If $x$ is Bochner integrable, define

$$
\begin{equation*}
\int_{\Omega} x(s) d \mu \equiv \lim _{n \rightarrow \infty} \int_{\Omega} x_{n}(s) d \mu \tag{21.2.4}
\end{equation*}
$$

First it is important to show that this integral is well defined. When this is done, an easier to use condition will be developed. Note that by Lemma 21.1.2, if $x$ is strongly measurable, $\|x\|$ is a measurable real valued function. Thus, it makes sense to consider $\int_{\Omega}\|x\| d \mu$ and also $\int_{\Omega}\left\|x-x_{n}\right\| d \mu$.

Theorem 21.2.4 The definition of Bochner integrability is well defined. Also, a strongly measurable function $x$ is Bochner integrable if and only if $\int_{\Omega}\|x\| d \mu<\infty$. In this case that the function is Bochner integrable, an approximating sequence $\left\{y_{n}\right\}$ exists such that $\left\|y_{n}(s)\right\| \leq 2\|x(s)\|$ for all $s$ and

$$
\lim _{n \rightarrow \infty} \int_{\Omega}\left\|y_{n}(s)-x(s)\right\| d \mu=0
$$

Proof: $\Rightarrow$ First consider the claim about the integral being well defined. Let $\left\{x_{n}\right\}$ be a sequence of simple functions converging pointwise to $x$ and satisfying the conditions given above for $x$ to be Bochner integrable. Then

$$
\left|\int_{\Omega}\left\|x_{n}(s)\right\| d \mu-\int_{\Omega}\left\|x_{m}(s)\right\| d \mu\right| \leq \int_{\Omega}\left\|x_{n}-x_{m}\right\| d \mu
$$

which is given to converge to 0 as $n, m \rightarrow \infty$ which shows that $\left\{\int_{\Omega}\left\|x_{n}(s)\right\| d \mu\right\}_{n=1}^{\infty}$ is a Cauchy sequence. Hence it is bounded and so, by Fatou's lemma,

$$
\int_{\Omega}\|x(s)\| d \mu \leq \lim \inf _{n \rightarrow \infty} \int_{\Omega}\left\|x_{n}(s)\right\| d \mu<\infty
$$

The limit in 21.2.4 exists because

$$
\left\|\int_{\Omega} x_{n} d \mu-\int_{\Omega} x_{m} d \mu\right\|=\left\|\int_{\Omega}\left(x_{n}-x_{m}\right) d \mu\right\| \leq \int_{\Omega}\left\|x_{n}-x_{m}\right\| d \mu
$$

and the last term is no more than $\varepsilon$ whenever $n, m$ are large enough. From Fatou's lemma, if $n$ is large enough,

$$
\int_{\Omega}\left\|x_{n}-x\right\| d \mu<\varepsilon
$$

Now if you have another sequence $\left\{\hat{x}_{n}\right\}$ satisfying the condition 21.2.3 along with pointwise convergence to $x$,

$$
\begin{aligned}
\left\|\int_{\Omega} x_{n} d \mu-\int_{\Omega} \hat{x}_{n} d \mu\right\| & =\left\|\int_{\Omega}\left(x_{n}-\hat{x}_{n}\right) d \mu\right\| \leq \int_{\Omega}\left\|x_{n}-\hat{x}_{n}\right\| d \mu \\
& \leq \int_{\Omega}\left\|x_{n}-x\right\| d \mu+\int_{\Omega}\left\|x-\hat{x}_{n}\right\| d \mu<2 \varepsilon
\end{aligned}
$$

if $n$ is large enough. Hence convergence of the integrals of the simple functions takes place and these integrals converge to the same thing. Thus the definition is well defined and $\int_{\Omega}\|x\| d \mu<\infty$.
$\Leftarrow$ Next suppose $\int_{\Omega}\|x\| d \mu<\infty$ for $x$ strongly measurable. By Lemma 21.1.2, there is a sequence of simple functions $\left\{y_{n}\right\}$ with $\left\|y_{n}(s)\right\| \leq 2\|x(s)\|$ and $y_{n}(s) \rightarrow x(s)$ for each $s$. Then by the dominated convergence theorem for scalar valued functions,

$$
\lim _{n \rightarrow \infty} \int_{\Omega}\left\|y_{n}-x\right\| d \mu=0
$$

Thus,

$$
\int_{\Omega}\left\|y_{n}-y_{m}\right\| d \mu \leq \int_{\Omega}\left\|y_{n}-x\right\| d \mu+\int_{\Omega}\left\|x-y_{m}\right\| d \mu<\varepsilon
$$

if $m, n$ are large enough so $\left\{y_{n}\right\}$ is a suitable approximating sequence for $x$.
This is a very nice theorem. It says that all you have to do is verify measurability and absolute integrability just like the case of scalar valued functions. Other things which are totally similar are that the integral is linear, the triangle inequality holds, and you can take a continuous linear functional inside the integral. These things are considered in the following theorem.

Theorem 21.2.5 The Bochner integral is well defined and if $x$ is Bochner integrable and $f \in X^{\prime}$,

$$
\begin{equation*}
f\left(\int_{\Omega} x(s) d \mu\right)=\int_{\Omega} f(x(s)) d \mu \tag{21.2.5}
\end{equation*}
$$

and the triangle inequality is valid,

$$
\begin{equation*}
\left\|\int_{\Omega} x(s) d \mu\right\| \leq \int_{\Omega}\|x(s)\| d \mu \tag{21.2.6}
\end{equation*}
$$

Also, the Bochner integral is linear. That is, if $a, b$ are scalars and $x, y$ are two Bochner integrable functions, then

$$
\begin{equation*}
\int_{\Omega}(a x(s)+b y(s)) d \mu=a \int_{\Omega} x(s) d \mu+b \int_{\Omega} y(s) d \mu \tag{21.2.7}
\end{equation*}
$$

Proof: Theorem 21.2.4 shows $\int_{\Omega}\|x(s)\| d \mu<\infty$ and that the definition of the integral is well defined.

It remains to verify the triangle inequality on Bochner integral functions and the claim about passing a continuous linear functional inside the integral. First of all, consider the triangle inequality. From Lemma 21.1.2, there is a sequence of simple functions $\left\{y_{n}\right\}$ satisfying 21.2.3 and converging to $x$ pointwise such that also $\left\|y_{n}(s)\right\| \leq 2\|x(s)\|$. Thus,

$$
\left\|\int_{\Omega} x(s) d \mu\right\| \equiv \lim _{n \rightarrow \infty}\left\|\int_{\Omega} y_{n}(s) d \mu\right\| \leq \lim _{n \rightarrow \infty} \int_{\Omega}\left\|y_{n}(s)\right\| d \mu=\int_{\Omega}\|x(s)\| d \mu
$$

the last step coming from the dominated convergence theorem since $\left\|y_{n}(s)\right\| \leq 2\|x(s)\|$ and $\left\|y_{n}(s)\right\| \rightarrow\|x(s)\|$ for each $s$. This shows the triangle inequality.

From Definition 21.2.1 and Theorem 21.2.4 and $\left\{y_{n}\right\}$ being the approximating sequence described there,

$$
f\left(\int_{\Omega} y_{n} d \mu\right)=\int_{\Omega} f\left(y_{n}\right) d \mu
$$

Thus,

$$
f\left(\int_{\Omega} x d \mu\right)=\lim _{n \rightarrow \infty} f\left(\int_{\Omega} y_{n} d \mu\right)=\lim _{n \rightarrow \infty} \int_{\Omega} f\left(y_{n}\right) d \mu=\int_{\Omega} f(x) d \mu
$$

the last equation holding from the dominated convergence theorem $\left(\left|f\left(y_{n}\right)\right| \leq\|f\|\left\|y_{n}\right\| \leq\right.$ $2\|f\|\|x\|)$. This shows 21.2.5.

It remains to verify 21.2.7. Let $f \in X^{\prime}$. Then from 21.2.5

$$
\begin{aligned}
f\left(\int_{\Omega}(a x(s)+b y(s)) d \mu\right) & =\int_{\Omega}(a f(x(s))+b f(y(s))) d \mu \\
& =a \int_{\Omega} f(x(s)) d \mu+b \int_{\Omega} f(y(s)) d \mu \\
& =f\left(a \int_{\Omega} x(s) d \mu+b \int_{\Omega} y(s) d \mu\right)
\end{aligned}
$$

Since $X^{\prime}$ separates the points of $X$, it follows

$$
\int_{\Omega}(a x(s)+b y(s)) d \mu=a \int_{\Omega} x(s) d \mu+b \int_{\Omega} y(s) d \mu
$$

and this proves 21.2.7.
A similar result is the following corollary.
Corollary 21.2.6 Let an $X$ valued function $x$ be Bochner integrable and let $L \in \mathscr{L}(X, Y)$ where $Y$ is another Banach space. Then $L x$ is a $Y$ valued Bochner integrable function and

$$
L\left(\int_{\Omega} x(s) d \mu\right)=\int_{\Omega} L x(s) d \mu
$$

Proof: From Theorem 21.2.4 there is a sequence of simple functions $\left\{y_{n}\right\}$ having the properties listed in that theorem. Then consider $\left\{L y_{n}\right\}$ which converges pointwise to $L x$. Since $L$ is continuous and linear,

$$
\int_{\Omega}\left\|L y_{n}-L x\right\|_{Y} d \mu \leq\|L\| \int_{\Omega}\left\|y_{n}-x\right\|_{X} d \mu
$$

which converges to 0 . This implies

$$
\lim _{m, n \rightarrow \infty} \int_{\Omega}\left\|L y_{n}-L y_{m}\right\| d \mu=0
$$

and so by definition $L x$ is Bochner integrable. Also

$$
\begin{gathered}
\int_{\Omega} x(s) d \mu=\lim _{n \rightarrow \infty} \int_{\Omega} y_{n}(s) d \mu \\
\int_{\Omega} L x(s) d \mu=\lim _{n \rightarrow \infty} \int_{\Omega} L y_{n}(s) d \mu=\lim _{n \rightarrow \infty} L \int_{\Omega} y_{n}(s) d \mu \\
\left\|L\left(\int_{\Omega} x(s) d \mu\right)-\int_{\Omega} L x(s) d \mu\right\|_{Y} \\
\leq\left\|L\left(\int_{\Omega} x(s) d \mu\right)-L \int_{\Omega} y_{n}(s) d \mu\right\|_{Y} \\
+\left\|\int_{\Omega} L y_{n}(s) d \mu-\int_{\Omega} L x(s) d \mu\right\|_{Y}<\varepsilon / 2+\varepsilon / 2=\varepsilon
\end{gathered}
$$

whenever $n$ large enough.

### 21.2.2 Taking a Closed Operator Out of the Integral

Now let $X$ and $Y$ be separable Banach spaces and suppose $A: D(A) \subseteq X \rightarrow Y$ be a closed operator. Recall this means that the graph of $A$,

$$
G(A) \equiv\{(x, A x): x \in D(A)\}
$$

is a closed subset of $X \times Y$ with respect to the product topology obtained from the norm

$$
\|(x, y)\|=\max (\|x\|,\|y\|)
$$

Thus also $G(A)$ is a separable Banach space with the above norm. You can also consider $D(A)$ as a separable Banach space having the graph norm

$$
\begin{equation*}
\|x\|_{D(A)} \equiv \max (\|x\|,\|A x\|) \tag{21.2.8}
\end{equation*}
$$

which is isometric to $G(A)$ with the mapping, $\theta x \equiv(x, A x)$. Recall why this is. It is clear that $\theta$ is one to one and onto $G(A)$. Is it continuous? If $x_{n} \rightarrow x$ in $D(A)$, this means that $x_{n} \rightarrow x$ in $X$ and $A x_{n} \rightarrow y$. Then, since $A$ is closed, it follows that $y=A x$ so $\left(x_{n}, A x_{n}\right) \rightarrow$ $(x, A x)$ in $G(A)$. Hence $\theta$ is indeed continuous and onto. Similar reasoning shows that $D(A)$ with this norm is complete. Hence it is a Banach space. Thus $\theta^{-1}$ is also continuous. The following lemma is a fundamental result which was proved earlier in the discussion on the Eberlein Smulian theorem in which this was an essential fact to allow the case of a reflexive Banach space which maybe was not separable. See Lemma 17.5.11 for the proof.

Lemma 21.2.7 A closed subspace of a reflexive Banach space is reflexive.

Then, with this lemma, one has the following corollary.
Corollary 21.2.8 Suppose $Y$ is a reflexive Banach space and $X$ is a Banach space such that there exists a continuous one to one mapping, $g: X \rightarrow Y$ such that $g(X)$ is a closed subset of $Y$. Then $X$ is reflexive.

Proof: By the open mapping theorem, $g(X)$ and $X$ are homeomorphic since $g^{-1}$ must also be continuous. Therefore, since $g(X)$ is reflexive because it is a closed subspace of a reflexive space, it follows $X$ is also reflexive.

Lemma 21.2.9 Suppose $V$ is a reflexive Banach space and that $V$ is a dense subset of $W$, another Banach space in the topology of $W$. Then $i^{*} W^{\prime}$ is a dense subset of $V^{\prime}$ where here $i$ is the inclusion map of $V$ into $W$.

Proof: First note that $i^{*}$ is one to one. If $i^{*} w^{*}=0$ for $w^{*} \in W^{\prime}$, then this means that for all $v \in V$,

$$
i^{*} w(v)=w^{*}(v)=0
$$

and since $V$ is dense in $W$, this shows $w^{*}=0$.
Consider the following diagram

$$
\begin{array}{lll}
V^{\prime \prime} & \stackrel{i^{* *}}{\rightarrow} & W^{\prime \prime} \\
V^{\prime} & \stackrel{i^{*}}{\leftarrow} & W^{\prime} \\
V & \xrightarrow{i} & W
\end{array}
$$

in which $i$ is the inclusion map. Next suppose $i^{*} W^{\prime}$ is not dense in $V^{\prime}$. Then, using the Hahn Banach theorem, there exists $v^{* *} \in V^{\prime \prime}$ such that $v^{* *} \neq 0$ but $v^{* *}\left(i^{*} W^{\prime}\right)=0$. It follows from $V$ being reflexive, that $v^{* *}=J v_{0}$ where $J$ is the James map from $V$ to $V^{\prime \prime}$ for some $v_{0} \in V$. Thus for every $w^{*} \in W^{\prime}$,

$$
\begin{aligned}
0 & =v^{* *}\left(i^{*} w^{*}\right) \equiv i^{* *} v^{* *}\left(w^{*}\right) \\
& =i^{* *} J v_{0}\left(w^{*}\right)=J v_{0}\left(i^{*} w^{*}\right) \\
& \equiv i^{*} w^{*}\left(v_{0}\right)=w^{*}\left(v_{0}\right)
\end{aligned}
$$

and since $W^{\prime}$ separates the points of $W$, it follows $v_{0}=0$ which contradicts $v^{* *} \neq 0$.
Note that in the proof, only $V$ reflexive was used.
This lemma implies an easy corollary.
Corollary 21.2.10 Let $E$ and $F$ be reflexive Banach spaces and let $A$ be a closed operator $A: D(A) \subseteq E \rightarrow F$. Suppose also that $D(A)$ is dense in $E$. Then making $D(A)$ into a Banach space by using the above graph norm given in 21.2.8, it follows that $D(A)$ is a Banach space and $i^{*} E^{\prime}$ is a dense subspace of $D(A)^{\prime}$.

Proof: First note that $E \times F$ is a reflexive Banach space and $\mathscr{G}(A)$ is a closed subspace of $E \times F$ so it is also a reflexive Banach space. Now $D(A)$ is isometric to $\mathscr{G}(A)$ and so it follows $D(A)$ is a dense subspace of $E$ which is reflexive. Therefore, from Lemma 21.2.9 the conclusion follows.

With this preparation, here is another interesting theorem. This one is about taking outside the integral a closed linear operator as opposed to a continuous linear operator.

Theorem 21.2.11 Let $X, Y$ be separable Banach spaces and let $A: D(A) \subseteq X \rightarrow Y$ be a closed operator where $D(A)$ is a dense separable subset of $X$ with respect to the graph norm on $D(A)$ described above ${ }^{1}$. Suppose also that $i^{*} X^{\prime}$ is a dense subspace of $D(A)^{\prime}$ where $D(A)$ is a Banach space having the graph norm described in 21.2.8. Suppose that $(\Omega, \mathscr{F}, \mu)$ is a $\sigma$ finite measure space and $x: \Omega \rightarrow X$ is strongly measurable and it happens that $x(s) \in D(A)$ for all $s \in \Omega$. Then $x$ is strongly measurable as a mapping into $D(A)$. Also Ax is strongly measurable as a map into $Y$ and if

$$
\begin{equation*}
\int_{\Omega}\|x(s)\| d \mu, \int_{\Omega}\|A x(s)\| d \mu<\infty \tag{21.2.9}
\end{equation*}
$$

then

$$
\begin{equation*}
\int_{\Omega} x(s) d \mu \in D(A) \tag{21.2.10}
\end{equation*}
$$

and

$$
\begin{equation*}
A \int_{\Omega} x(s) d \mu=\int_{\Omega} A x(s) d \mu \tag{21.2.11}
\end{equation*}
$$

Proof: First of all, consider the assertion that $x$ is strongly measurable into $D(A)$. Letting $f \in D(A)^{\prime}$ be given, there exists a sequence, $\left\{g_{n}\right\} \subseteq i^{*} X^{\prime}$ such that $g_{n} \rightarrow f$ in $D(A)^{\prime}$. Therefore, $s \rightarrow g_{n}(x(s))$ is measurable by assumption and $g_{n}(x(s)) \rightarrow f(x(s))$, which shows that $s \rightarrow f(x(s))$ is measurable. By the Pettis theorem, it follows that $s \rightarrow x(s)$ is strongly measurable as a map into $D(A)$.

It follows from Theorem 21.2.4 there exists a sequence of simple functions, $\left\{x_{n}\right\}$ of the form

$$
x_{n}(s)=\sum_{k=1}^{m_{n}} a_{k}^{n} \mathscr{X}_{E_{k}^{n}}(s), x_{n}(s) \in D(A)
$$

which converges strongly and pointwise to $x(s)$ in $D(A)$. Thus

$$
x_{n}(s) \rightarrow x(s), A x_{n}(s) \rightarrow A x(s)
$$

which shows $s \rightarrow A x(s)$ is stongly measurable in $Y$ as claimed.
It remains to verify the assertions about the integral. 21.2.9 implies $x$ is Bochner integrable as a function having values in $D(A)$ with the norm on $D(A)$ described above. Therefore, by Theorem 21.2.4 there exists a sequence of simple functions $\left\{y_{n}\right\}$ having values in $D(A)$,

$$
\lim _{m, n \rightarrow \infty} \int_{\Omega}\left\|y_{n}-y_{m}\right\|_{D(A)} d \mu=0
$$

$y_{n}(s)$ converging pointwise to $x(s)$,

$$
\left\|y_{n}(s)\right\|_{D(A)} \leq 2\|x(s)\|_{D(A)}
$$

and

$$
\lim _{n \rightarrow \infty} \int_{\Omega}\left\|x(s)-y_{n}(s)\right\|_{D(A)} d s=0
$$

[^19]Therefore,

$$
\int_{\Omega} y_{n}(s) d \mu \in D(A), \int_{\Omega} y_{n}(s) d \mu \rightarrow \int_{\Omega} x(s) d \mu \text { in } X
$$

and since $y_{n}$ is a simple function and $A$ is linear,

$$
A \int_{\Omega} y_{n}(s) d \mu=\int_{\Omega} A y_{n}(s) d \mu \rightarrow \int_{\Omega} A x(s) d \mu \text { in } Y
$$

It follows, since $A$ is a closed operator, that $\int_{\Omega} x(s) d \mu \in D(A)$ and

$$
A \int_{\Omega} x(s) d \mu=\int_{\Omega} A x(s) d \mu
$$

Here is another version of this theorem which has different hypotheses.
Theorem 21.2.12 Let $X$ and $Y$ be separable Banach spaces and let $A: D(A) \subseteq X \rightarrow Y$ be a closed operator. Also let $(\Omega, \mathscr{F}, \mu)$ be a $\sigma$ finite measure space and let $x: \Omega \rightarrow X$ be Bochner integrable such that $x(s) \in D(A)$ for all s. Also suppose Ax is Bochner integrable. Then

$$
\int A x d \mu=A \int x d \mu
$$

and $\int x d \mu \in D(A)$.
Proof: Consider the graph of $A$,

$$
G(A) \equiv\{(x, A x): x \in D(A)\} \subseteq X \times Y
$$

Then since $A$ is closed, $G(A)$ is a closed separable Banach space with the norm $\|(x, y)\| \equiv$ $\max (\|x\|,\|y\|)$. Therefore, for $g^{*} \in G(A)^{\prime}$, one can apply the Hahn Banach theorem and obtain $\left(x^{*}, y^{*}\right) \in(X \times Y)^{\prime}$ such that $g^{*}(x, A x)=\left(x^{*}(x), y^{*}(A x)\right)$. Now it follows from the assumptions that $s \rightarrow\left(x^{*}(x(s)), y^{*}(A x(s))\right)$ is measurable with values in $G(A)$. It is also separably valued because this is true of $G(A)$. By the Pettis theorem, $s \rightarrow(x(s), A(x(s)))$ must be strongly measurable. Also $\int\|x(s)\|+\|A(x(s))\| d \mu<\infty$ by assumption and so there exists a sequence of simple functions having values in $G(A),\left\{\left(x_{n}(s), A x_{n}(s)\right)\right\}$ which converges to $(x(s), A(s))$ pointwise such that $\int\left\|\left(x_{n}, A x_{n}\right)-(x, A x)\right\| d \mu \rightarrow 0$ in $G(A)$. Now for simple functions is it routine to verify that

$$
\int\left(x_{n}, A x_{n}\right) d \mu=\left(\int x_{n} d \mu, \int A x_{n} d \mu\right)=\left(\int x_{n} d \mu, A \int x_{n} d \mu\right)
$$

Also

$$
\begin{aligned}
\left\|\int x_{n} d \mu-\int x d \mu\right\| & \leq \int\left\|x_{n}-x\right\| d \mu \\
& \leq \int\left\|\left(x_{n}, A x_{n}\right)-(x, A x)\right\| d \mu
\end{aligned}
$$

which converges to 0 . Also

$$
\begin{aligned}
\left\|\int A x_{n} d \mu-\int A x d \mu\right\| & =\left\|A \int x_{n} d \mu-\int A x d \mu\right\| \\
& \leq \int\left\|A x_{n}-A x\right\| d \mu \\
& \leq \int\left\|\left(x_{n}, A x_{n}\right)-(x, A x)\right\| d \mu
\end{aligned}
$$

and this converges to 0 . Therefore, $\int x_{n} d \mu \rightarrow \int x d \mu$ and $A \int x_{n} d \mu \rightarrow \int A x d \mu$. Since each $\int x_{n} d \mu \in D(A)$, and $A$ is closed, this implies $\int x d \mu \in D(A)$ and $A \int x d \mu=\int A x d \mu$.

### 21.3 Operator Valued Functions

Consider the case where $A(s) \in \mathscr{L}(X, Y)$ for $X$ and $Y$ separable Banach spaces. With the operator norm $\mathscr{L}(X, Y)$ is a Banach space and so if $A$ is strongly measurable, the Bochner integral can be defined as before. However, it is also possible to define the Bochner integral of such operator valued functions for more general situations. In this section, $(\Omega, \mathscr{F}, \mu)$ will be a $\sigma$ finite measure space as usual.

Lemma 21.3.1 Let $x \in X$ and suppose $A$ is strongly measurable. Then

$$
s \rightarrow A(s) x
$$

is strongly measurable as a map into $Y$.
Proof: Since $A$ is assumed to be strongly measurable, it is the pointwise limit of simple functions of the form

$$
A_{n}(s) \equiv \sum_{k=1}^{m_{n}} A_{k}^{n} \mathscr{X}_{E_{k}^{n}}(s)
$$

where $A_{k}^{n}$ is in $\mathscr{L}(X, Y)$. It follows $A_{n}(s) x \rightarrow A(s) x$ for each $s$ and so, since $s \rightarrow A_{n}(s) x$ is a simple $Y$ valued function, $s \rightarrow A(s) x$ must be strongly measurable.

Definition 21.3.2 Suppose $A(s) \in \mathscr{L}(X, Y)$ for each $s \in \Omega$ where $X, Y$ are separable Banach spaces. Suppose also that for each $x \in X$,

$$
\begin{equation*}
s \rightarrow A(s) x \text { is strongly measurable } \tag{21.3.12}
\end{equation*}
$$

and there exists $C$ such that for each $x \in X$,

$$
\begin{equation*}
\int_{\Omega}\|A(s) x\| d \mu<C\|x\| \tag{21.3.13}
\end{equation*}
$$

Then $\int_{\Omega} A(s) d \mu \in \mathscr{L}(X, Y)$ is defined by the following formula.

$$
\begin{equation*}
\left(\int_{\Omega} A(s) d \mu\right)(x) \equiv \int_{\Omega} A(s) x d \mu \tag{21.3.14}
\end{equation*}
$$

Lemma 21.3.3 The above definition is well defined. Furthermore, if 21.3.12 holds then $s \rightarrow\|A(s)\|$ is measurable and if 21.3.13 holds, then

$$
\left\|\int_{\Omega} A(s) d \mu\right\| \leq \int_{\Omega}\|A(s)\| d \mu
$$

Proof: It is clear that in case $s \rightarrow A(s) x$ is measurable for all $x \in X$ there exists a unique $\Psi \in \mathscr{L}(X, Y)$ such that

$$
\Psi(x)=\int_{\Omega} A(s) x d \mu
$$

This is because $x \rightarrow \int_{\Omega} A(s) x d \mu$ is linear and continuous. It is continuous because

$$
\left\|\int_{\Omega} A(s) x d \mu\right\| \leq \int_{\Omega}\|A(s) x\| d \mu \leq \int_{\Omega}\|A(s)\| d \mu\|x\|
$$

Thus $\Psi=\int_{\Omega} A(s) d \mu$ and the definition is well defined.
Now consider the assertion about $s \rightarrow\|A(s)\|$. Let $D^{\prime} \subseteq B^{\prime}$ the closed unit ball in $Y^{\prime}$ be such that $D^{\prime}$ is countable and

$$
\|y\|=\sup _{y^{*} \in D^{\prime}}\left|y^{*}(y)\right|
$$

This is from Lemma 21.1.6. Recall $X$ is separable. Also let $D$ be a countable dense subset of $B$, the unit ball of $X$. Then

$$
\begin{aligned}
\{s:\|A(s)\|>\alpha\} & =\left\{s: \sup _{x \in D}\|A(s) x\|>\alpha\right\} \\
& =\cup_{x \in D}\{s:\|A(s) x\|>\alpha\} \\
& =\cup_{x \in D}\left(\cup_{y^{*} \in D^{\prime}}\left\{\left|y^{*}(A(s) x)\right|>\alpha\right\}\right)
\end{aligned}
$$

and this is measurable because $s \rightarrow A(s) x$ is strongly, hence weakly measurable.
Now suppose 21.3.13 holds. Then for all $x$,

$$
\int_{\Omega}\|A(s) x\| d \mu<C\|x\|
$$

It follows that for $\|x\| \leq 1$,

$$
\left\|\left(\int_{\Omega} A(s) d \mu\right)(x)\right\|=\left\|\int_{\Omega} A(s) x d \mu\right\| \leq \int_{\Omega}\|A(s) x\| d \mu \leq \int_{\Omega}\|A(s)\| d \mu
$$

and so

$$
\left\|\int_{\Omega} A(s) d \mu\right\| \leq \int_{\Omega}\|A(s)\| d \mu
$$

Now it is interesting to consider the case where $A(s) \in \mathscr{L}(H, H)$ where $s \rightarrow A(s) x$ is strongly measurable and $A(s)$ is compact and self adjoint. Recall the Kuratowski measurable selection theorem, Theorem 11.1.11 on Page 228 listed here for convenience.

Theorem 21.3.4 Let $E$ be a compact metric space and let $(\Omega, \mathscr{F})$ be a measure space. Suppose $\psi: E \times \Omega \rightarrow \mathbb{R}$ has the property that $x \rightarrow \psi(x, \omega)$ is continuous and $\omega \rightarrow \psi(x, \omega)$ is measurable. Then there exists a measurable function, $f$ having values in $E$ such that

$$
\psi(f(\omega), \omega)=\sup _{x \in E} \psi(x, \omega) .
$$

Furthermore, $\omega \rightarrow \psi(f(\omega), \omega)$ is measurable.

### 21.3.1 Review of Hilbert Schmidt Theorem

This section is a review of earlier material and is presented a little differently. I think it does not hurt to repeat some things relative to Hilbert space. I will give a proof of the Hilbert Schmidt theorem which will generalize to a result about measurable operators. It will be a little different then the earlier proof. Recall the following.

Definition 21.3.5 Define $v \otimes u \in \mathscr{L}(H, H)$ by

$$
v \otimes u(x)=(x, u) v .
$$

$A \in \mathscr{L}(H, H)$ is a compact operator if whenever $\left\{x_{k}\right\}$ is a bounded sequence, there exists a convergent subsequence of $\left\{A x_{k}\right\}$. Equivalently, A maps bounded sets to sets whose closures are compact or to use other terminology, A maps bounded sets to sets which are precompact.

Next is a convenient description of compact operators on a Hilbert space.
Lemma 21.3.6 Let $H$ be a Hilbert space and suppose $A \in \mathscr{L}(H, H)$ is a compact operator. Then

1. A is a compact operator if and only if whenever if $x_{n} \rightarrow x$ weakly in $H$, it follows that $A x_{n} \rightarrow A x$ strongly in $H$.
2. For $u, v \in H, v \otimes u: H \rightarrow H$ is a compact operator.
3. Let $B$ be the closed unit ball in $H$. If $A$ is self adjoint and compact, then if $x_{n} \rightarrow x$ weakly on $B$, it follows that $\left(A x_{n}, x_{n}\right) \rightarrow(A x, x)$ so $x \rightarrow|(A x, x)|$ achieves its maximum value on $B$.
4. The function, $v \otimes u$ is compact and the operator $u \otimes u$ is self adjoint.

Proof: Consider $\Rightarrow$ of 1 . Suppose then that $x_{n} \rightarrow x$ weakly. Since $\left\{x_{n}\right\}$ is weakly bounded, it follows from the uniform boundedness principle that $\left\{\left\|x_{n}\right\|\right\}$ is bounded. Let $x_{n} \in \hat{B}$ for $\hat{B}$ some closed ball. If $A x_{n}$ fails to converge to $A x$, then there is $\varepsilon>0$ and a subsequence still denoted as $\left\{x_{n}\right\}$ such that $x_{n} \rightarrow x$ weakly but $\left\|A x_{n}-A x\right\| \geq \varepsilon>0$. Then $A(\hat{B})$ is precompact because $A$ is compact so there is a further subsequence, still denoted by $\left\{x_{n}\right\}$ such that $A x_{n}$ converges to some $y \in H$. Therefore,

$$
\begin{aligned}
(y, w) & =\lim _{n \rightarrow \infty}\left(A x_{n}, w\right)=\lim _{n \rightarrow \infty}\left(x_{n}, A^{*} w\right) \\
& =\left(x, A^{*} w\right)=(A x, w)
\end{aligned}
$$

which shows $A x=y$ since $w$ is arbitrary. However, this is a contradiction to $\left\|A x_{n}-A x\right\| \geq$ $\varepsilon>0$.

Consider $\Leftarrow$ of 1 . Why is $A$ compact if it satisfies the property that it takes weakly convergent sequences to strongly convergent ones? If $A$ is not compact, then there exists $\hat{B}$ a bounded set such that $A(\hat{B})$ is not precompact. Thus, there exists a sequence $\left\{A x_{n}\right\}_{n=1}^{\infty} \subseteq$ $A(\hat{B})$ which has no convergent subsequence where $x_{n} \in \hat{B}$ the bounded set. However, there is a subsequence $\left\{x_{n}\right\} \in \hat{B}$ which converges weakly to some $x \in H$ because of weak compactness. Hence $A x_{n} \rightarrow A x$ by assumption and so this is a contradiction to there being no convergent subsequence of $\left\{A x_{n}\right\}_{n=1}^{\infty}$.

Next consider 2. Letting $\left\{x_{n}\right\}$ be a bounded sequence,

$$
v \otimes u\left(x_{n}\right)=\left(x_{n}, u\right) v .
$$

There exists a weakly convergent subsequence of $\left\{x_{n}\right\}$ say $\left\{x_{n_{k}}\right\}$ converging weakly to $x \in H$. Therefore,

$$
\left\|v \otimes u\left(x_{n_{k}}\right)-v \otimes u(x)\right\|=\left\|\left(x_{n_{k}}, u\right)-(x, u)\right\|\|v\|
$$

which converges to 0 . Thus $v \otimes u$ is compact as claimed. It takes bounded sets to precompact sets.

Next consider 3. To verify the assertion about $x \rightarrow(A x, x)$, let $x_{n} \rightarrow x$ weakly. Since $A$ is compact, $A x_{n} \rightarrow A x$ by part 1. Then, since $A$ is self adjoint,

$$
\begin{aligned}
& \left|\left(A x_{n}, x_{n}\right)-(A x, x)\right| \\
\leq & \left|\left(A x_{n}, x_{n}\right)-\left(A x, x_{n}\right)\right|+\left|\left(A x, x_{n}\right)-(A x, x)\right| \\
\leq & \left|\left(A x_{n}, x_{n}\right)-\left(A x, x_{n}\right)\right|+\left|\left(A x_{n}, x\right)-(A x, x)\right| \\
\leq & \left\|A x_{n}-A x\right\|\left\|x_{n}\right\|+\left\|A x_{n}-A x\right\|\|x\| \leq 2\left\|A x_{n}-A x\right\|
\end{aligned}
$$

which converges to 0 . Now let $\left\{x_{n}\right\}$ be a maximizing sequence for $|(A x, x)|$ for $x \in B$ and let $\lambda \equiv \sup \{|(A x, x)|: x \in B\}$. There is a subsequence still denoted as $\left\{x_{n}\right\}$ which converges weakly to some $x \in B$ by weak compactness. Hence $|(A x, x)|=\lim _{n \rightarrow \infty}\left|\left(A x_{n}, x_{n}\right)\right|=\lambda$.

Next consider 4. It only remains to verify that $u \otimes u$ is self adjoint. This follows from the definition.

$$
\begin{aligned}
((u \otimes u) x, y) & \equiv(u(x, u), y)=(x, u)(u, y) \\
(x,(u \otimes u) y) & \equiv(x, u(y, u))=(u, y)(x, u)
\end{aligned}
$$

the same thing.
Observation 21.3.7 Note that if $A$ is any self adjoint operator,

$$
\overline{(A x, x)}=(x, A x)=(A x, x)
$$

so $(A x, x)$ is real valued.
From Lemma 21.3.6, the maximum of $|(A x, x)|$ exists on the closed unit ball $B$.

Lemma 21.3.8 Let $A \in \mathscr{L}(H, H)$ and suppose it is self adjoint and compact. Let $B$ denote the closed unit ball in $H$. Let $e \in B$ be such that

$$
|(A e, e)|=\max _{x \in B}|(A x, x)|
$$

Then letting $\lambda=(A e, e)$, it follows $A e=\lambda e$. You can always assume $\|e\|=1$.
Proof: From the above observation, $(A x, x)$ is always real and since $A$ is compact, $|(A x, x)|$ achieves a maximum at $e$. It remains to verify $e$ is an eigenvector. If $|(A e, e)|=0$ for all $e \in B$, then $A$ is a self adjoint nonnegative $((A x, x) \geq 0)$ operator and so by Cauchy Schwarz inequality,

$$
(A e, x) \leq(A x, x)^{1 / 2}(A e, e)^{1 / 2}=0
$$

and so $A e=0$ for all $e$. Assume then that $A$ is not 0 . You can always make $|(A e, e)|$ at least as large by replacing $e$ with $e /\|e\|$. Thus, there is no loss of generality in letting $\|e\|=1$ in every case.

Suppose $\lambda=(A e, e) \geq 0$ where $|(A e, e)|=\max _{x \in B}|(A x, x)|$. Thus

$$
((\lambda I-A) e, e)=\lambda\|e\|^{2}-\lambda=0
$$

Then it is easy to verify that $\lambda I-A$ is a nonnegative $(((\lambda I-A) x, x) \geq 0$ for all $x$.) and self adjoint operator. To see this, note that

$$
((\lambda I-A) x, x)=\|x\|^{2}\left((\lambda I-A) \frac{x}{\|x\|}, \frac{x}{\|x\|}\right)=\|x\|^{2} \lambda-\|x\|^{2}\left(A \frac{x}{\|x\|}, \frac{x}{\|x\|}\right) \geq 0
$$

Therefore, the Cauchy Schwarz inequality can be applied to write

$$
((\lambda I-A) e, x) \leq((\lambda I-A) e, e)^{1 / 2}((\lambda I-A) x, x)^{1 / 2}=0
$$

Since this is true for all $x$ it follows $A e=\lambda e$. Just pick $x=(\lambda I-A) e$.
Next suppose $\max _{x \in B}|(A x, x)|=-(A e, e)$. Let $-\lambda=(-A e, e)$ and the previous result can be applied to $-A$ and $-\lambda$. Thus $-\lambda e=-A e$ and so $A e=\lambda e$.

With these lemmas here is a major theorem, the Hilbert Schmidt theorem. I think this proof is a little slicker than the more standard proof given earlier.

Theorem 21.3.9 Let $A \in \mathscr{L}(H, H)$ be a compact self adjoint operator on a Hilbert space. Then there exist real numbers $\left\{\lambda_{k}\right\}_{k=1}^{\infty}$ and vectors $\left\{e_{k}\right\}_{k=1}^{\infty}$ such that

$$
\begin{gather*}
\left\|e_{k}\right\|=1, \\
\left(e_{k}, e_{j}\right)_{H}=0 \text { if } k \neq j, \\
A e_{k}=\lambda_{k} e_{k} \\
\left|\lambda_{n}\right| \geq\left|\lambda_{n+1}\right| \text { for all } n, \\
\lim _{n \rightarrow \infty} \lambda_{n}=0 \\
\lim _{n \rightarrow \infty}\left\|A-\sum_{k=1}^{n} \lambda_{k}\left(e_{k} \otimes e_{k}\right)\right\|_{\mathscr{L}(H, H)}=0 \tag{21.3.15}
\end{gather*}
$$

Proof: This is done by considering a sequence of compact self adjoint operators, $A, A_{1}, A_{2}, \cdots$. Here is how these are defined. Using Lemma 21.3.8 let $e_{1}, \lambda_{1}$ be given by that lemma such that

$$
\left|\left(A e_{1}, e_{1}\right)\right|=\max _{x \in B}|(A x, x)|, \lambda_{1}=\left(A e_{1}, e_{1}\right) \Rightarrow A e_{1}=\lambda_{1} e_{1}
$$

Then by that lemma, $A e_{1}=\lambda_{1} e_{1}$ and $\left\|e_{1}\right\|=1$. Now define $A_{1}=A-\lambda_{1} e_{1} \otimes e_{1}$. This is compact and self adjoint by Lemma 21.3.6. Thus, one could repeat the argument.

If $A_{n}$ has been obtained, use Lemma 21.3.8 to obtain $e_{n+1}$ and $\lambda_{n+1}$ such that

$$
\left|\left(A_{n} e_{n+1}, e_{n+1}\right)\right|=\max _{x \in B}\left|\left(A_{n} x, x\right)\right|, \lambda_{n+1}=\left(A_{n} e_{n+1}, e_{n+1}\right)
$$

By that lemma again, $A_{n} e_{n+1}=\lambda_{n+1} e_{n+1}$ and $\left\|e_{n+1}\right\|=1$. Then

$$
A_{n+1} \equiv A_{n}-\lambda_{n+1} e_{n+1} \otimes e_{n+1}
$$

Thus iterating this,

$$
\begin{equation*}
A_{n}=A-\sum_{k=1}^{n} \lambda_{k} e_{k} \otimes e_{k} \tag{21.3.16}
\end{equation*}
$$

Assume for $j, k \leq n,\left(e_{k}, e_{j}\right)=\delta_{j k}$. Then the new vector $e_{n+1}$ will be orthogonal to the earlier ones. This is the next claim.

Claim 1: If $k<n+1$ then $\left(e_{n+1}, e_{k}\right)=0$. Also $A e_{k}=\lambda_{k} e_{k}$ for all $k$ and from the construction, $A_{n} e_{n+1}=\lambda_{n+1} e_{n+1}$.

Proof of claim: From the above,

$$
\lambda_{n+1} e_{n+1}=A_{n} e_{n+1}=A e_{n+1}-\sum_{k=1}^{n} \lambda_{k}\left(e_{n+1}, e_{k}\right) e_{k}
$$

From the above and induction hypothesis that $\left(e_{k}, e_{j}\right)=\delta_{j k}$ for $j, k \leq n$,

$$
\begin{aligned}
\lambda_{n+1}\left(e_{n+1}, e_{j}\right) & =\left(A e_{n+1}, e_{j}\right)-\sum_{k=1}^{n} \lambda_{k}\left(e_{n+1}, e_{k}\right)\left(e_{k}, e_{j}\right) \\
& =\left(e_{n+1}, A e_{j}\right)-\sum_{k=1}^{n} \lambda_{k}\left(e_{n+1}, e_{k}\right)\left(e_{k}, e_{j}\right) \\
& =\lambda_{j}\left(e_{n+1}, e_{j}\right)-\lambda_{j}\left(e_{n+1}, e_{j}\right)=0
\end{aligned}
$$

To verify the second part of this claim,

$$
\lambda_{n+1} e_{n+1}=A_{n} e_{n+1}=A e_{n+1}-\sum_{k=1}^{n} \lambda_{k} e_{k}\left(e_{n+1}, e_{k}\right)=A e_{n+1}
$$

This proves the claim.
Claim 2: $\left|\lambda_{n}\right| \geq\left|\lambda_{n+1}\right|$.

Proof of claim: From 21.3.16 and the definition of $A_{n}$ and $e_{k} \otimes e_{k}$,

$$
\begin{aligned}
\left(A_{n-1} e_{n+1}, e_{n+1}\right) & =\left(\left(A-\sum_{k=1}^{n-1} \lambda_{k} e_{k} \otimes e_{k}\right) e_{n+1}, e_{n+1}\right) \\
& =\left(A e_{n+1}, e_{n+1}\right)=\left(A_{n} e_{n+1}, e_{n+1}\right)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\lambda_{n+1} & =\left(A_{n} e_{n+1}, e_{n+1}\right) \\
& =\left(A_{n-1} e_{n+1}, e_{n+1}\right)-\lambda_{n}\left|\left(e_{n}, e_{n+1}\right)\right|^{2} \\
& =\left(A_{n-1} e_{n+1}, e_{n+1}\right)
\end{aligned}
$$

By the previous claim. Therefore,

$$
\left|\lambda_{n+1}\right|=\left|\left(A_{n-1} e_{n+1}, e_{n+1}\right)\right| \leq\left|\left(A_{n-1} e_{n}, e_{n}\right)\right|=\left|\lambda_{n}\right|
$$

by the definition of $\left|\lambda_{n}\right|$. ( $e_{n}$ makes $\left|\left(A_{n-1} x, x\right)\right|$ as large as possible.)
Claim 3: $\lim _{n \rightarrow \infty} \lambda_{n}=0$.
Proof of claim: If for some $n, \lambda_{n}=0$, then $\lambda_{k}=0$ for all $k>n$ by claim 2. Thus, for some n,

$$
A=\sum_{k=1}^{n} \lambda_{k} e_{k} \otimes e_{k}
$$

Assume then that $\lambda_{k} \neq 0$ for any $k$. Then if $\lim _{k \rightarrow \infty}\left|\lambda_{k}\right|=\varepsilon>0$, one contradicts, $\left\|e_{k}\right\|=1$ for all $k$ because

$$
\begin{aligned}
\left\|A e_{n}-A e_{m}\right\|^{2} & =\left\|\lambda_{n} e_{n}-\lambda_{m} e_{m}\right\|^{2} \\
& =\lambda_{n}^{2}+\lambda_{m}^{2} \geq 2 \varepsilon^{2}
\end{aligned}
$$

which shows there is no Cauchy subsequence of $\left\{A e_{n}\right\}_{n=1}^{\infty}$, which contradicts the compactness of $A$. This proves the claim.

Claim 4: $\left\|A_{n}\right\| \rightarrow 0$
Proof of claim: Let $x, y \in B$

$$
\begin{aligned}
\left|\lambda_{n+1}\right| & \geq\left|\left(A_{n} \frac{x+y}{2}, \frac{x+y}{2}\right)\right| \\
& =\left|\frac{1}{4}\left(A_{n} x, x\right)+\frac{1}{4}\left(A_{n} y, y\right)+\frac{1}{2}\left(A_{n} x, y\right)\right| \\
& \geq \frac{1}{2}\left|\left(A_{n} x, y\right)\right|-\frac{1}{4}\left|\left(A_{n} x, x\right)+\left(A_{n} y, y\right)\right| \\
& \geq \frac{1}{2}\left|\left(A_{n} x, y\right)\right|-\frac{1}{4}\left(\left|\left(A_{n} x, x\right)\right|+\left|\left(A_{n} y, y\right)\right|\right) \\
& \geq \frac{1}{2}\left|\left(A_{n} x, y\right)\right|-\frac{1}{2}\left|\lambda_{n+1}\right|
\end{aligned}
$$

and so

$$
3\left|\lambda_{n+1}\right| \geq\left|\left(A_{n} x, y\right)\right|
$$

It follows $\left\|A_{n}\right\| \leq 3\left|\lambda_{n+1}\right|$. By 21.3 .16 this proves 21.3 .15 and completes the proof.

### 21.3.2 Measurable Compact Operators

Here the operators will be of the form $A(s)$ where $s \in \Omega$ and $s \rightarrow A(s) x$ is strongly measurable and $A(s)$ is a compact operator in $\mathscr{L}(H, H)$.

Theorem 21.3.10 Let $A(s) \in \mathscr{L}(H, H)$ be a compact self adjoint operator and $H$ is a separable Hilbert space such that $s \rightarrow A(s) x$ is strongly measurable. Then there exist real numbers $\left\{\lambda_{k}(s)\right\}_{k=1}^{\infty}$ and vectors $\left\{e_{k}(s)\right\}_{k=1}^{\infty}$ such that

$$
\begin{gathered}
\left\|e_{k}(s)\right\|=1 \\
\left(e_{k}(s), e_{j}(s)\right)_{H}=0 \text { if } k \neq j \\
A(s) e_{k}(s)=\lambda_{k}(s) e_{k}(s) \\
\left|\lambda_{n}(s)\right| \geq\left|\lambda_{n+1}(s)\right| \text { for all } n \\
\lim _{n \rightarrow \infty} \lambda_{n}(s)=0 \\
\lim _{n \rightarrow \infty}\left\|A(s)-\sum_{k=1}^{n} \lambda_{k}(s)\left(e_{k}(s) \otimes e_{k}(s)\right)\right\|_{\mathscr{L}(H, H)}=0
\end{gathered}
$$

The function $s \rightarrow \lambda_{j}(s)$ is measurable and $s \rightarrow e_{j}(s)$ is strongly measurable.
Proof: It is simply a repeat of the above proof of the Hilbert Schmidt theorem except at every step when the $e_{k}$ and $\lambda_{k}$ are defined, you use the Kuratowski measurable selection theorem, Theorem 21.3.4 on Page 661 to obtain $\lambda_{k}(s)$ is measurable and that $s \rightarrow e_{k}(s)$ is also measurable. This follows because the closed unit ball in a separable Hilbert space is a compact metric space.

When you consider $\max _{x \in B}\left|\left(A_{n}(s) x, x\right)\right|$, let $\psi(x, s)=\left|\left(A_{n}(s) x, x\right)\right|$. Then $\psi$ is continuous in $x$ by Lemma 21.3.6 on Page 661 and it is measurable in $s$ by assumption. Therefore, by the Kuratowski theorem, $e_{k}(s)$ is measurable in the sense that inverse images of weakly open sets in $B$ are measurable. However, by Lemma 21.1.12 on Page 648 this is the same as weakly measurable. Since $H$ is separable, this implies $s \rightarrow e_{k}(s)$ is also strongly measurable. The measurability of $\lambda_{k}$ and $e_{k}$ is the only new thing here and so this completes the proof.

### 21.4 Fubini's Theorem for Bochner Integrals

Now suppose $\left(\Omega_{1}, \mathscr{F}, \mu\right)$ and $\left(\Omega_{2}, \mathscr{S}, \lambda\right)$ are two $\sigma$ finite measure spaces. Recall the notion of product measure. There was a $\sigma$ algebra, denoted by $\mathscr{F} \times \mathscr{S}$ which is the smallest $\sigma$ algebra containing the elementary sets, (finite disjoint unions of measurable rectangles) and a measure, denoted by $\mu \times \lambda$ defined on this $\sigma$ algebra such that for $E \in \mathscr{F} \times \mathscr{S}$,

$$
s_{1} \rightarrow \lambda\left(E_{s_{1}}\right),\left(E_{s_{1}} \equiv\left\{s_{2}:\left(s_{1}, s_{2}\right) \in E\right\}\right)
$$

is $\mu$ measurable and

$$
s_{2} \rightarrow \mu\left(E_{s_{2}}\right),\left(E_{s_{2}} \equiv\left\{s_{1}:\left(s_{1}, s_{2}\right) \in E\right\}\right)
$$

is $\lambda$ measurable. In terms of nonnegative functions which are $\mathscr{F} \times \mathscr{S}$ measurable,

$$
\begin{aligned}
& s_{1} \rightarrow f\left(s_{1}, s_{2}\right) \text { is } \mu \text { measurable } \\
& s_{2} \rightarrow f\left(s_{1}, s_{2}\right) \text { is } \lambda \text { measurable } \\
& s_{1} \rightarrow \int_{\Omega_{2}} f\left(s_{1}, s_{2}\right) d \lambda \text { is } \mu \text { measurable } \\
& s_{2} \rightarrow \int_{\Omega_{1}} f\left(s_{1}, s_{2}\right) d \mu \text { is } \lambda \text { measurable }
\end{aligned}
$$

and the conclusion of Fubini's theorem holds.

$$
\begin{aligned}
\int_{\Omega_{1} \times \Omega_{2}} f d(\mu \times \lambda) & =\int_{\Omega_{1}} \int_{\Omega_{2}} f\left(s_{1}, s_{2}\right) d \lambda d \mu \\
& =\int_{\Omega_{2}} \int_{\Omega_{1}} f\left(s_{1}, s_{2}\right) d \mu d \lambda
\end{aligned}
$$

The following theorem is the version of Fubini's theorem valid for Bochner integrable functions.

Theorem 21.4.1 Let $f: \Omega_{1} \times \Omega_{2} \rightarrow X$ be strongly measurable with respect to $\mu \times \lambda$ and suppose

$$
\begin{equation*}
\int_{\Omega_{1} \times \Omega_{2}}\left\|f\left(s_{1}, s_{2}\right)\right\| d(\mu \times \lambda)<\infty . \tag{21.4.17}
\end{equation*}
$$

Then there exist a set of $\mu$ measure zero, $N$ and a set of $\lambda$ measure zero, $M$ such that the following formula holds with all integrals making sense.

$$
\begin{aligned}
\int_{\Omega_{1} \times \Omega_{2}} f\left(s_{1}, s_{2}\right) d(\mu \times \lambda) & =\int_{\Omega_{1}} \int_{\Omega_{2}} f\left(s_{1}, s_{2}\right) \mathscr{X}_{N}\left(s_{1}\right) d \lambda d \mu \\
& =\int_{\Omega_{2}} \int_{\Omega_{1}} f\left(s_{1}, s_{2}\right) \mathscr{X}_{M}\left(s_{2}\right) d \mu d \lambda
\end{aligned}
$$

Proof: First note that from 21.4.17 and the usual Fubini theorem for nonnegative valued functions,

$$
\int_{\Omega_{1} \times \Omega_{2}}\left\|f\left(s_{1}, s_{2}\right)\right\| d(\mu \times \lambda)=\int_{\Omega_{1}} \int_{\Omega_{2}}\left\|f\left(s_{1}, s_{2}\right)\right\| d \lambda d \mu
$$

and so

$$
\begin{equation*}
\int_{\Omega_{2}}\left\|f\left(s_{1}, s_{2}\right)\right\| d \lambda<\infty \tag{21.4.18}
\end{equation*}
$$

for $\mu$ a.e. $s_{1}$. Say for all $s_{1} \notin N$ where $\mu(N)=0$.
Let $\phi \in X^{\prime}$. Then $\phi \circ f$ is $\mathscr{F} \times \mathscr{S}$ measurable and

$$
\begin{aligned}
& \int_{\Omega_{1} \times \Omega_{2}}\left|\phi \circ f\left(s_{1}, s_{2}\right)\right| d(\mu \times \lambda) \\
\leq & \int_{\Omega_{1} \times \Omega_{2}}\|\phi\|\left\|f\left(s_{1}, s_{2}\right)\right\| d(\mu \times \lambda)<\infty
\end{aligned}
$$

and so from the usual Fubini theorem for complex valued functions,

$$
\begin{equation*}
\int_{\Omega_{1} \times \Omega_{2}} \phi \circ f\left(s_{1}, s_{2}\right) d(\mu \times \lambda)=\int_{\Omega_{1}} \int_{\Omega_{2}} \phi \circ f\left(s_{1}, s_{2}\right) d \lambda d \mu . \tag{21.4.19}
\end{equation*}
$$

Now also if you fix $s_{2}$, it follows from the definition of strongly measurable and the properties of product measure mentioned above that

$$
s_{1} \rightarrow f\left(s_{1}, s_{2}\right)
$$

is strongly measurable. Also, by 21.4.18

$$
\int_{\Omega_{2}}\left\|f\left(s_{1}, s_{2}\right)\right\| d \lambda<\infty
$$

for $s_{1} \notin N$. Therefore, by Theorem 21.2.4 $s_{2} \rightarrow f\left(s_{1}, s_{2}\right) \mathscr{X}_{N^{C}}\left(s_{1}\right)$ is Bochner integrable. By 21.4.19 and 21.2.5

$$
\begin{align*}
& \int_{\Omega_{1} \times \Omega_{2}} \phi \circ f\left(s_{1}, s_{2}\right) d(\mu \times \lambda) \\
= & \int_{\Omega_{1}} \int_{\Omega_{2}} \phi \circ f\left(s_{1}, s_{2}\right) d \lambda d \mu \\
= & \int_{\Omega_{1}} \int_{\Omega_{2}} \phi\left(f\left(s_{1}, s_{2}\right) \mathscr{X}_{N^{C}}\left(s_{1}\right)\right) d \lambda d \mu \\
= & \int_{\Omega_{1}} \phi\left(\int_{\Omega_{2}} f\left(s_{1}, s_{2}\right) \mathscr{X}_{N^{C}}\left(s_{1}\right) d \lambda\right) d \mu . \tag{21.4.20}
\end{align*}
$$

Each iterated integral makes sense and

$$
\begin{align*}
s_{1} & \rightarrow \int_{\Omega_{2}} \phi\left(f\left(s_{1}, s_{2}\right) \mathscr{X}_{N^{C}}\left(s_{1}\right)\right) d \lambda \\
& =\phi\left(\int_{\Omega_{2}} f\left(s_{1}, s_{2}\right) \mathscr{X}_{N^{C}}\left(s_{1}\right) d \lambda\right) \tag{21.4.21}
\end{align*}
$$

is $\mu$ measurable because

$$
\begin{aligned}
\left(s_{1}, s_{2}\right) & \rightarrow \phi\left(f\left(s_{1}, s_{2}\right) \mathscr{X}_{N^{C}}\left(s_{1}\right)\right) \\
& =\phi\left(f\left(s_{1}, s_{2}\right)\right) \mathscr{X}_{N^{C}}\left(s_{1}\right)
\end{aligned}
$$

is product measurable. Now consider the function,

$$
\begin{equation*}
s_{1} \rightarrow \int_{\Omega_{2}} f\left(s_{1}, s_{2}\right) \mathscr{X}_{N^{C}}\left(s_{1}\right) d \lambda \tag{21.4.22}
\end{equation*}
$$

I want to show this is also Bochner integrable with respect to $\mu$ so I can factor out $\phi$ once again. It's measurability follows from the Pettis theorem and the above observation 21.4.21.

Also,

$$
\begin{aligned}
& \int_{\Omega_{1}}\left\|\int_{\Omega_{2}} f\left(s_{1}, s_{2}\right) \mathscr{X}_{N^{C}}\left(s_{1}\right) d \lambda\right\| d \mu \\
\leq & \int_{\Omega_{1}} \int_{\Omega_{2}}\left\|f\left(s_{1}, s_{2}\right)\right\| d \lambda d \mu \\
= & \int_{\Omega_{1} \times \Omega_{2}}\left\|f\left(s_{1}, s_{2}\right)\right\| d(\mu \times \lambda)<\infty .
\end{aligned}
$$

Therefore, the function in 21.4.22 is indeed Bochner integrable and so in 21.4.20 the $\phi$ can be taken outside the last integral. Thus,

$$
\begin{aligned}
& \phi\left(\int_{\Omega_{1} \times \Omega_{2}} f\left(s_{1}, s_{2}\right) d(\mu \times \lambda)\right) \\
= & \int_{\Omega_{1} \times \Omega_{2}} \phi \circ f\left(s_{1}, s_{2}\right) d(\mu \times \lambda) \\
= & \int_{\Omega_{1}} \int_{\Omega_{2}} \phi \circ f\left(s_{1}, s_{2}\right) d \lambda d \mu \\
= & \int_{\Omega_{1}} \phi\left(\int_{\Omega_{2}} f\left(s_{1}, s_{2}\right) \mathscr{X}_{N^{C}}\left(s_{1}\right) d \lambda\right) d \mu \\
= & \phi\left(\int_{\Omega_{1}} \int_{\Omega_{2}} f\left(s_{1}, s_{2}\right) \mathscr{X}_{N^{C}}\left(s_{1}\right) d \lambda d \mu\right) .
\end{aligned}
$$

Since $X^{\prime}$ separates the points,

$$
\int_{\Omega_{1} \times \Omega_{2}} f\left(s_{1}, s_{2}\right) d(\mu \times \lambda)=\int_{\Omega_{1}} \int_{\Omega_{2}} f\left(s_{1}, s_{2}\right) \mathscr{X}_{N^{C}}\left(s_{1}\right) d \lambda d \mu .
$$

The other formula follows from similar reasoning.

### 21.5 The Spaces $L^{p}(\Omega ; X)$

Recall that $x$ is Bochner when it is strongly measurable and $\int_{\Omega}\|x(s)\| d \mu<\infty$. It is natural to generalize to $\int_{\Omega}\|x(s)\|^{p} d \mu<\infty$.
Definition 21.5.1 $x \in L^{p}(\Omega ; X)$ for $p \in[1, \infty)$ if $x$ is strongly measurable and

$$
\int_{\Omega}\|x(s)\|^{p} d \mu<\infty
$$

Also

$$
\begin{equation*}
\|x\|_{L^{p}(\Omega ; X)} \equiv\|x\|_{p} \equiv\left(\int_{\Omega}\|x(s)\|^{p} d \mu\right)^{1 / p} \tag{21.5.23}
\end{equation*}
$$

As in the case of scalar valued functions, two functions in $L^{p}(\Omega ; X)$ are considered equal if they are equal a.e. With this convention, and using the same arguments found in the presentation of scalar valued functions it is clear that $L^{p}(\Omega ; X)$ is a normed linear space with the norm given by 21.5.23. In fact, $L^{p}(\Omega ; X)$ is a Banach space. This is the main contribution of the next theorem.

Lemma 21.5.2 If $x_{n}$ is a Cauchy sequence in $L^{p}(\Omega ; X)$ satisfying

$$
\sum_{n=1}^{\infty}\left\|x_{n+1}-x_{n}\right\|_{p}<\infty
$$

then there exists $x \in L^{p}(\Omega ; X)$ such that $x_{n}(s) \rightarrow x(s)$ a.e. and

$$
\left\|x-x_{n}\right\|_{p} \rightarrow 0
$$

Proof: Let $g_{N}(s) \equiv \sum_{n=1}^{N}\left\|x_{n+1}(s)-x_{n}(s)\right\|_{X}$. Then by the triangle inequality,

$$
\begin{aligned}
\left(\int_{\Omega} g_{N}(s)^{p} d \mu\right)^{1 / p} & \leq \sum_{n=1}^{N}\left(\int_{\Omega}\left\|x_{n+1}(s)-x_{n}(s)\right\|^{p} d \mu\right)^{1 / p} \\
& \leq \sum_{n=1}^{\infty}\left\|x_{n+1}-x_{n}\right\|_{p}<\infty
\end{aligned}
$$

Let

$$
g(s)=\lim _{N \rightarrow \infty} g_{N}(s)=\sum_{n=1}^{\infty}\left\|x_{n+1}(s)-x_{n}(s)\right\|_{X}
$$

By the monotone convergence theorem,

$$
\left(\int_{\Omega} g(s)^{p} d \mu\right)^{1 / p}=\lim _{N \rightarrow \infty}\left(\int_{\Omega} g_{N}(s)^{p} d \mu\right)^{1 / p}<\infty
$$

Therefore, there exists a measurable set of measure 0 called $E$, such that for $s \notin E$, it follows that $g(s)<\infty$. Hence, for $s \notin E, \lim _{N \rightarrow \infty} x_{N+1}(s)$ exists because

$$
x_{N+1}(s)=x_{N+1}(s)-x_{1}(s)+x_{1}(s)=\sum_{n=1}^{N}\left(x_{n+1}(s)-x_{n}(s)\right)+x_{1}(s) .
$$

Thus, if $N>M$, and $s$ is a point where $g(s)<\infty$,

$$
\begin{aligned}
\left\|x_{N+1}(s)-x_{M+1}(s)\right\|_{X} & \leq \sum_{n=M+1}^{N}\left\|x_{n+1}(s)-x_{n}(s)\right\|_{X} \\
& \leq \sum_{n=M+1}^{\infty}\left\|x_{n+1}(s)-x_{n}(s)\right\|_{X}
\end{aligned}
$$

which shows that $\left\{x_{N+1}(s)\right\}_{N=1}^{\infty}$ is a Cauchy sequence for each $s \notin E$. Now let

$$
x(s) \equiv\left\{\begin{array}{l}
\lim _{N \rightarrow \infty} x_{N}(s) \text { if } s \notin E \\
0 \text { if } s \in E
\end{array}\right.
$$

Theorem 21.1.10 shows that $x$ is strongly measurable. By Fatou's lemma,

$$
\int_{\Omega}\left\|x(s)-x_{N}(s)\right\|^{p} d \mu \leq \lim _{M \rightarrow \infty} \inf _{\Omega}\left\|x_{M}(s)-x_{N}(s)\right\|^{p} d \mu .
$$

But if $N$ and $M$ are large enough with $M>N$,

$$
\left(\int_{\Omega}\left\|x_{M}(s)-x_{N}(s)\right\|^{p} d \mu\right)^{1 / p} \leq \sum_{n=N}^{M}\left\|x_{n+1}-x_{n}\right\|_{p} \leq \sum_{n=N}^{\infty}\left\|x_{n+1}-x_{n}\right\|_{p}<\varepsilon
$$

and this shows, since $\varepsilon$ is arbitrary, that

$$
\lim _{N \rightarrow \infty} \int_{\Omega}\left\|x(s)-x_{N}(s)\right\|^{p} d \mu=0
$$

It remains to show $x \in L^{p}(\Omega ; X)$. This follows from the above and the triangle inequality. Thus, for $N$ large enough,

$$
\begin{gathered}
\left(\int_{\Omega}\|x(s)\|^{p} d \mu\right)^{1 / p} \leq\left(\int_{\Omega}\left\|x_{N}(s)\right\|^{p} d \mu\right)^{1 / p} \\
+\left(\int_{\Omega}\left\|x(s)-x_{N}(s)\right\|^{p} d \mu\right)^{1 / p} \leq\left(\int_{\Omega}\left\|x_{N}(s)\right\|^{p} d \mu\right)^{1 / p}+\varepsilon<\infty
\end{gathered}
$$

Theorem 21.5.3 $L^{p}(\Omega ; X)$ is complete. Also every Cauchy sequence has a subsequence which converges pointwise.

Proof: If $\left\{x_{n}\right\}$ is Cauchy in $L^{p}(\Omega ; X)$, extract a subsequence $\left\{x_{n_{k}}\right\}$ satisfying

$$
\left\|x_{n_{k+1}}-x_{n_{k}}\right\|_{p} \leq 2^{-k}
$$

and apply Lemma 21.5.2. The pointwise convergence of this subsequence was established in the proof of this lemma. This proves the theorem because if a subsequence of a Cauchy sequence converges, then the Cauchy sequence must also converge.

Observation 21.5.4 If the measure space is Lebesgue measure then you have continuity of translation in $L^{p}\left(\mathbb{R}^{n} ; X\right)$ in the usual way. More generally, for $\mu$ a Radon measure on $\Omega$ a locally compact Hausdorff space, $C_{c}(\Omega ; X)$ is dense in $L^{p}(\Omega ; X)$. Here $C_{c}(\Omega ; X)$ is the space of continuous $X$ valued functions which have compact support in $\Omega$. The proof of this little observation follows immediately from approximating with simple functions and then applying the appropriate considerations to the simple functions.

Clearly Fatou's lemma and the monotone convergence theorem make no sense for functions with values in a Banach space but the dominated convergence theorem holds in this setting.

Theorem 21.5.5 If $x$ is strongly measurable and $x_{n}(s) \rightarrow x(s)$ a.e. (for $s$ off a set of measure zero) with

$$
\left\|x_{n}(s)\right\| \leq g(s) \text { a.e. }
$$

where $\int_{\Omega} g d \mu<\infty$, then $x$ is Bochner integrable and

$$
\int_{\Omega} x(s) d \mu=\lim _{n \rightarrow \infty} \int_{\Omega} x_{n}(s) d \mu .
$$

Proof: The measurability of $x$ follows from Theorem 21.1.10 if convergence happens for each $s$. Otherwise, $x$ is measurable by assumption. Then $\left\|x_{n}(s)-x(s)\right\| \leq 2 g(s)$ a.e. so, from Fatou's lemma,

$$
\begin{aligned}
\int_{\Omega} 2 g(s) d \mu & \leq \liminf _{n \rightarrow \infty} \int_{\Omega}\left(2 g(s)-\left\|x_{n}(s)-x(s)\right\|\right) d \mu \\
& =\int_{\Omega} 2 g(s) d \mu-\lim \sup _{n \rightarrow \infty} \int_{\Omega}\left\|x_{n}(s)-x(s)\right\| d \mu
\end{aligned}
$$

and so,

$$
\lim \sup _{n \rightarrow \infty} \int_{\Omega}\left\|x_{n}(s)-x(s)\right\| d \mu \leq 0
$$

Also, from Fatou's lemma again,

$$
\int_{\Omega}\|x(s)\| d \mu \leq \lim \inf _{n \rightarrow \infty} \int_{\Omega}\left\|x_{n}(s)\right\| d \mu<\int_{\Omega} g(s) d \mu<\infty
$$

so $x \in L^{1}$. Then by the triangle inequality,

$$
\lim \sup _{n \rightarrow \infty}\left\|\int_{\Omega} x(s) d \mu-\int_{\Omega} x_{n}(s) d \mu\right\| \leq \lim \sup _{n \rightarrow \infty} \int_{\Omega}\left\|x_{n}(s)-x(s)\right\| d \mu=0
$$

One can also give a version of the Vitali convergence theorem.
Definition 21.5.6 Let $\mathscr{A} \subseteq L^{1}(\Omega ; X)$. Then $\mathscr{A}$ is said to be uniformly integrable if for every $\varepsilon>0$ there exists $\delta>0$ such that whenever $\mu(E)<\delta$, it follows

$$
\int_{E}\|f\|_{X} d \mu<\varepsilon
$$

for all $f \in \mathscr{A}$. It is bounded if

$$
\sup _{f \in \mathscr{A}} \int_{\Omega}\|f\|_{X} d \mu<\infty .
$$

Theorem 21.5.7 Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space and let $X$ be a separable Banach space. Let $\left\{f_{n}\right\} \subseteq L^{1}(\Omega ; X)$ be uniformly integrable and bounded such that $f_{n}(\omega) \rightarrow f(\omega)$ for each $\omega \in \Omega$. Then $f \in L^{1}(\Omega ; X)$ and

$$
\lim _{n \rightarrow \infty} \int_{\Omega}\left\|f_{n}-f\right\|_{X} d \mu=0
$$

Proof: Let $\varepsilon>0$ be given. Then by uniform integrability there exists $\delta>0$ such that if $\mu(E)<\delta$ then

$$
\int_{E}\left\|f_{n}\right\| d \mu<\varepsilon / 3
$$

By Fatou's lemma the same inequality holds for $f$. Also Fatou's lemma shows $f \in L^{1}(\Omega ; X)$, $f$ being measurable because of Theorem 11.1.9.

By Egoroff's theorem, Theorem 11.3.11, there exists a set of measure less than $\delta, E$ such that the convergence of $\left\{f_{n}\right\}$ to $f$ is uniform off $E$. Therefore,

$$
\begin{aligned}
\int_{\Omega}\left\|f-f_{n}\right\| d \mu & \leq \int_{E}\left(\|f\|_{X}+\left\|f_{n}\right\|_{X}\right) d \mu+\int_{E^{C}}\left\|f-f_{n}\right\|_{X} d \mu \\
& <\frac{2 \varepsilon}{3}+\int_{E^{C}} \frac{\varepsilon}{(\mu(\Omega)+1) 3} d \mu<\varepsilon
\end{aligned}
$$

if $n$ is large enough.
Note that a convenient way to achieve uniform integrability is to say $\left\{f_{n}\right\}$ is bounded in $L^{p}(\Omega ; X)$ for some $p>1$. This follows from Holder's inequality.

$$
\int_{E}\left\|f_{n}\right\| d \mu \leq\left(\int_{E} d \mu\right)^{1 / p^{\prime}}\left(\int_{\Omega}\left\|f_{n}\right\|^{p} d \mu\right)^{1 / p} \leq C \mu(E)^{1 / p^{\prime}}
$$

The following theorem is interesting.
Theorem 21.5.8 Let $1 \leq p<\infty$ and let $p<r \leq \infty$. Then $L^{r}([0, T], X)$ is a Borel subset of $L^{p}([0, T] ; X)$. Letting $C([0, T] ; X)$ denote the functions having values in $X$ which are continuous, $C([0, T] ; X)$ is also a Borel subset of $L^{p}([0, T] ; X)$. Here the measure is ordinary one dimensional Lebesgue measure on $[0, T]$.

Proof: First consider the claim about $L^{r}([0, T] ; X)$. Let

$$
B_{M} \equiv\left\{x \in L^{p}([0, T] ; X):\|x\|_{L^{r}([0, T] ; X)} \leq M\right\}
$$

Then $B_{M}$ is a closed subset of $L^{p}([0, T] ; X)$. Here is why. If $\left\{x_{n}\right\}$ is a sequence of elements of $B_{M}$ and $x_{n} \rightarrow x$ in $L^{p}([0, T] ; X)$, then passing to a subsequence, still denoted by $x_{n}$, it can be assumed $x_{n}(s) \rightarrow x(s)$ a.e. Hence Fatou's lemma can be applied to conclude

$$
\int_{0}^{T}\|x(s)\|^{r} d s \leq \lim _{n \rightarrow \infty} \inf _{0} \int_{0}^{T}\left\|x_{n}(s)\right\|^{r} d s \leq M^{r}<\infty
$$

Now $\cup_{M=1}^{\infty} B_{M}=L^{r}([0, T] ; X)$. Note this did not depend on the measure space used. It would have been equally valid on any measure space.

Consider now $C([0, T] ; X)$. The norm on this space is the usual norm, $\|\cdot\|_{\infty}$. The argument above shows $\|\cdot\|_{\infty}$ is a Borel measurable function on $L^{p}([0, T] ; X)$. This is because $B_{M} \equiv\left\{x \in L^{p}([0, T] ; X):\|x\|_{\infty} \leq M\right\}$ is a closed, hence Borel subset of $L^{p}([0, T] ; X)$. Now let $\theta \in \mathscr{L}\left(L^{p}([0, T] ; X), L^{p}(\mathbb{R} ; X)\right)$ such that $\theta(x(t))=x(t)$ for all $t \in[0, T]$ and also $\theta \in \mathscr{L}(C([0, T] ; X), B C(\mathbb{R} ; X))$ where $B C(\mathbb{R} ; X)$ denotes the bounded continuous functions with a norm given by $\|x\| \equiv \sup _{t \in \mathbb{R}}\|x(t)\|$, and $\theta x$ has compact support.

For example, you could define

$$
\widetilde{x}(t) \equiv\left\{\begin{array}{l}
x(t) \text { if } t \in[0, T] \\
x(2 T-t) \text { if } t \in[T, 2 T] \\
x(-t) \text { if } t \in[-T, 0] \\
0 \text { if } t \notin[-T, 2 T]
\end{array}\right.
$$

and let $\Phi \in C_{c}^{\infty}(-T, 2 T)$ such that $\Phi(t)=1$ for $t \in[0, T]$. Then you could let

$$
\theta x(t) \equiv \Phi(t) \widetilde{x}(t)
$$

Then let $\left\{\phi_{n}\right\}$ be a mollifier and define

$$
\psi_{n} x(t) \equiv \phi_{n} * \theta x(t)
$$

It follows $\psi_{n} x$ is uniformly continuous because

$$
\begin{aligned}
& \left\|\psi_{n} x(t)-\psi_{n} x\left(t^{\prime}\right)\right\|_{X} \\
\leq & \int_{\mathbb{R}}\left|\phi_{n}\left(t^{\prime}-s\right)-\phi_{n}(t-s)\right|\|\theta x(s)\|_{X} d s \\
\leq & C\|x\|_{p}\left(\int_{\mathbb{R}}\left|\phi_{n}\left(t^{\prime}-s\right)-\phi_{n}(t-s)\right|^{p^{\prime}} d s\right)^{1 / p^{\prime}}
\end{aligned}
$$

Also for $x \in C([0, T] ; X)$, it follows from usual mollifier arguments that

$$
\left\|\psi_{n} x-x\right\|_{L^{\infty}([0, T] ; X)} \rightarrow 0
$$

Here is why. For $t \in[0, T]$,

$$
\begin{aligned}
\left\|\psi_{n} x(t)-x(t)\right\|_{X} & \leq \int_{\mathbb{R}} \phi_{n}(s)\|\theta x(t-s)-\theta x(t)\| d s \\
& \leq C_{\theta} \int_{-1 / n}^{1 / n} \phi_{n}(s) d s \varepsilon=C_{\theta} \varepsilon
\end{aligned}
$$

provided $n$ is large enough due to the compact support and consequent uniform continuity of $\theta x$.

If $\left\|\psi_{n} x-x\right\|_{L^{\infty}([0, T] ; X)} \rightarrow 0$, then $\left\{\psi_{n} x\right\}$ must be a Cauchy sequence in $C([0, T] ; X)$ and this requires that $x$ equals a continuous function a.e. Thus $C([0, T] ; X)$ consists exactly of those functions, $x$ of $L^{p}([0, T] ; X)$ such that $\left\|\psi_{n} x-x\right\|_{\infty} \rightarrow 0$. It follows

$$
\begin{gather*}
C([0, T] ; X)= \\
\cap_{n=1}^{\infty} \cup_{m=1}^{\infty} \cap_{k=m}^{\infty}\left\{x \in L^{p}([0, T] ; X):\left\|\psi_{k} x-x\right\|_{\infty} \leq \frac{1}{n}\right\} \tag{21.5.24}
\end{gather*}
$$

It only remains to show

$$
S \equiv\left\{x \in L^{p}([0, T] ; X):\left\|\psi_{k} x-x\right\|_{\infty} \leq \alpha\right\}
$$

is a Borel set. Suppose then that $x_{n} \in S$ and $x_{n} \rightarrow x$ in $L^{p}([0, T] ; X)$. Then there exists a subsequence, still denoted by $n$ such that $x_{n} \rightarrow x$ pointwise a.e. as well as in $L^{p}$. There exists a set of measure 0 such that for all $n$, and $t$ not in this set,

$$
\begin{aligned}
\left\|\psi_{k} x_{n}(t)-x_{n}(t)\right\| & \equiv\left\|\int_{-1 / k}^{1 / k} \phi_{k}(s)\left(\theta x_{n}(t-s)\right) d s-x_{n}(t)\right\| \leq \alpha \\
x_{n}(t) & \rightarrow x(t)
\end{aligned}
$$

Then

$$
\begin{aligned}
& \left\|\psi_{k} x_{n}(t)-x_{n}(t)-\left(\psi_{k} x(t)-x(t)\right)\right\| \\
\leq & \left\|x_{n}(t)-x(t)\right\|_{X}+\left\|\int_{-1 / k}^{1 / k} \phi_{k}(s)\left(\theta x_{n}(t-s)-\theta x(t-s)\right) d s\right\| \\
\leq & \left\|x_{n}(t)-x(t)\right\|_{X}+C_{k, \theta}\left\|x_{n}-x\right\|_{L^{p}(0, T ; X)}
\end{aligned}
$$

which converges to 0 as $n \rightarrow \infty$. It follows that for a.e. $t$,

$$
\left\|\psi_{k} x(t)-x(t)\right\| \leq \alpha
$$

Thus $S$ is closed and so the set in 21.5.24 is a Borel set.
As in the scalar case, the following lemma holds in this more general context.
Lemma 21.5.9 Let $(\Omega, \mu)$ be a regular measure space where $\Omega$ is a locally compact Hausdorff space. Then $C_{c}(\Omega ; X)$ the space of continuous functions having compact support and values in $X$ is dense in $L^{p}(0, T ; X)$ for all $p \in[0, \infty)$. For any $\sigma$ finite measure space, the simple functions are dense in $L^{p}(0, T ; X)$.

Proof: First is it shown the simple functions are dense in $L^{p}(0, T ; X)$. Let $f$ be a function in $L^{p}(0, T ; X)$ and let $\left\{x_{n}\right\}$ denote a sequence of simple functions which converge to $f$ pointwise which also have the property that

$$
\left\|x_{n}(s)\right\| \leq 2\|f(s)\|
$$

Then

$$
\int_{\Omega}\left\|x_{n}(s)-f(s)\right\|^{p} d \mu \rightarrow 0
$$

from the dominated convergence theorem. Therefore, the simple functions are indeed dense in $L^{p}(0, T ; X)$.

Next suppose $(\Omega, \mu)$ is a regular measure space. If $x(s) \equiv \sum_{i} a_{i} \mathscr{X}_{E_{i}}(s)$ is a simple function, then by regularity, there exist compact sets, $K_{i}$ and open sets, $V_{i}$ such that $K_{i} \subseteq$ $E_{i} \subseteq V_{i}$ and $\mu\left(V_{i} \backslash K_{i}\right)^{1 / p}<\varepsilon / \sum_{i}\left\|a_{i}\right\|$. Let $K_{i} \prec h_{i} \prec V_{i}$. Then consider

$$
\sum_{i} a_{i} h_{i} \in C_{c}(\Omega)
$$

By the triangle inequality,

$$
\begin{aligned}
& \left(\int_{\Omega}\left\|\sum_{i} a_{i} h_{i}(s)-a_{i} \mathscr{X}_{E_{i}}(s)\right\|^{p} d \mu\right)^{1 / p} \leq \sum_{i}\left(\int_{\Omega}\left\|a_{i}\left(h_{i}(s)-\mathscr{X}_{E_{i}}(s)\right)\right\|^{p} d \mu\right)^{1 / p} \\
\leq & \sum_{i}\left(\int_{\Omega}\left\|a_{i}\right\|^{p}\left|h_{i}(s)-\mathscr{X}_{E_{i}}(s)\right|^{p} d \mu\right)^{1 / p} \leq \sum_{i}\left\|a_{i}\right\|\left(\int_{V_{i} \backslash K_{i}} d \mu\right)^{1 / p} \\
\leq & \sum_{i}\left\|a_{i}\right\| \mu\left(V_{i} \backslash K_{i}\right)^{1 / p}<\varepsilon
\end{aligned}
$$

Since $\varepsilon$ is arbitrary, this and the first part of the lemma shows $C_{c}(\Omega ; X)$ is dense in $L^{p}(\Omega ; X)$.

### 21.6 Measurable Representatives

In this section consider the special case where $X=L^{1}(B, v)$ where $(B, \mathscr{F}, v)$ is a $\sigma$ finite measure space and $x \in L^{1}(\Omega ; X)$. Thus for each $s \in \Omega, x(s) \in L^{1}(B, v)$. In general, the map

$$
(s, t) \rightarrow x(s)(t)
$$

will not be product measurable, but one can obtain a measurable representative. This is important because it allows the use of Fubini's theorem on the measurable representative.

By Theorem 21.2.4, there exists a sequence of simple functions, $\left\{x_{n}\right\}$, of the form

$$
\begin{equation*}
x_{n}(s)=\sum_{k=1}^{m} a_{k} \mathscr{X}_{E_{k}}(s) \tag{21.6.25}
\end{equation*}
$$

where $a_{k} \in L^{1}(B, v)$ which satisfy the conditions of Definition 21.2.3 and

$$
\begin{equation*}
\left\|x_{n}-x_{m}\right\|_{L^{1}\left(\Omega, L^{1}(B)\right)} \rightarrow 0 \text { as } m, n \rightarrow \infty \tag{21.6.26}
\end{equation*}
$$

For such a simple function, you can assume the $E_{k}$ are disjoint and then

$$
\begin{aligned}
\left\|x_{n}\right\|_{L^{1}\left(\Omega, L^{1}(B)\right)} & =\sum_{k=1}^{m}\left\|a_{k}\right\|_{L^{1}(B)} \mu\left(E_{k}\right)=\sum_{k=1}^{m} \int_{B}\left|a_{k}\right| d v \mu\left(E_{k}\right) \\
& =\int_{\Omega} \int_{B}\left|a_{k}(t)\right| d v(t) \mathscr{X}_{E_{k}}(s) d \mu(s) \\
& =\int_{\Omega} \int_{B}\left|x_{n}\right| d v d \mu
\end{aligned}
$$

Also, each $x_{n}$ is product measurable. Thus from 21.6.26,

$$
\left\|x_{n}-x_{m}\right\|_{L^{1}\left(\Omega, L^{1}(B)\right)}=\int_{\Omega} \int_{B}\left|x_{n}-x_{m}\right| d \nu d \mu
$$

which shows that $\left\{x_{n}\right\}$ is a Cauchy sequence in $L^{1}(\Omega \times B, \mu \times \lambda)$. Then there exists $y \in$ $L^{1}(\Omega \times B, \mu \times \lambda)$ and a subsequence still called $\left\{x_{n}\right\}$ such that

$$
\lim _{n \rightarrow \infty} \int_{\Omega} \int_{B}\left|x_{n}-y\right| d v d \mu=\lim _{n \rightarrow \infty} \int_{\Omega}\left\|x_{n}-y\right\|_{L^{1}(B)} d \mu=\left\|x_{n}-y\right\|_{L^{1}\left(\Omega, L^{1}(B)\right)}=0
$$

Now consider 21.6.26. Since $\lim _{m \rightarrow \infty} x_{m}(s)=x(s)$ in $L^{1}(B)$, it follows from Fatou's lemma that

$$
\left\|x_{n}-x\right\|_{L^{1}\left(\Omega, L^{1}(B)\right)} \leq \lim \inf _{m \rightarrow \infty}\left\|x_{n}-x_{m}\right\|_{L^{1}\left(\Omega, L^{1}(B)\right)}<\varepsilon
$$

for all $n$ large enough. Hence

$$
\lim _{n \rightarrow \infty}\left\|x_{n}-x\right\|_{L^{1}\left(\Omega, L^{1}(B)\right)}=0
$$

and so

$$
x(s)=y(s) \text { in } L^{1}(B) \mu \text { a.e. } s
$$

In particular, for a.e. $s$, it follows that

$$
x(s)(t)=y(s, t) \text { for a.e. } t
$$

Now $\int_{\Omega} x(s) d \mu \in X=L^{1}(B, v)$ so it makes sense to ask for $\left(\int_{\Omega} x(s) d \mu\right)(t)$, at least $\mu$ a.e. $t$. To find what this is, note

$$
\left\|\int_{\Omega} x_{n}(s) d \mu-\int_{\Omega} x(s) d \mu\right\|_{X} \leq \int_{\Omega}\left\|x_{n}(s)-x(s)\right\|_{X} d \mu .
$$

Therefore, since the right side converges to 0 ,

$$
\begin{gathered}
\lim _{n \rightarrow \infty}\left\|\int_{\Omega} x_{n}(s) d \mu-\int_{\Omega} x(s) d \mu\right\|_{X}= \\
\lim _{n \rightarrow \infty} \int_{B}\left|\left(\int_{\Omega} x_{n}(s) d \mu\right)(t)-\left(\int_{\Omega} x(s) d \mu\right)(t)\right| d v=0 .
\end{gathered}
$$

But

$$
\left(\int_{\Omega} x_{n}(s) d \mu\right)(t)=\int_{\Omega} x_{n}(s, t) d \mu \text { a.e. } t
$$

Therefore

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \int_{B}\left|\int_{\Omega} x_{n}(s, t) d \mu-\left(\int_{\Omega} x(s) d \mu\right)(t)\right| d v=0 \tag{21.6.27}
\end{equation*}
$$

Also, since $x_{n} \rightarrow y$ in $L^{1}(\Omega \times B)$,

$$
\begin{align*}
& 0=\lim _{n \rightarrow \infty} \int_{B} \int_{\Omega}\left|x_{n}(s, t)-y(s, t)\right| d \mu d v \geq \\
& \lim _{n \rightarrow \infty} \int_{B}\left|\int_{\Omega} x_{n}(s, t) d \mu-\int_{\Omega} y(s, t) d \mu\right| d v \tag{21.6.28}
\end{align*}
$$

From 21.6.27 and 21.6.28

$$
\int_{\Omega} y(s, t) d \mu=\left(\int_{\Omega} x(s) d \mu\right)(t) \text { a.e. } t
$$

Theorem 21.6.1 Let $X=L^{1}(B)$ where $(B, \mathscr{F}, v)$ is a $\sigma$ finite measure space and let

$$
x \in L^{1}(\Omega ; X)
$$

Then there exists a measurable representative, $y \in L^{1}(\Omega \times B)$, such that

$$
x(s)=y(s, \cdot) \text { a.e. } s \text { in } \Omega, \text { the equation in } L^{1}(B)
$$

and

$$
\int_{\Omega} y(s, t) d \mu=\left(\int_{\Omega} x(s) d \mu\right)(t) \text { a.e. } t .
$$

### 21.7 Vector Measures

There is also a concept of vector measures.
Definition 21.7.1 Let $(\Omega, \mathscr{S})$ be a set and a $\sigma$ algebra of subsets of $\Omega$. A mapping

$$
F: \mathscr{S} \rightarrow X
$$

is said to be a vector measure if

$$
F\left(\cup_{i=1}^{\infty} E_{i}\right)=\sum_{i=1}^{\infty} F\left(E_{i}\right)
$$

whenever $\left\{E_{i}\right\}_{i=1}^{\infty}$ is a sequence of disjoint elements of $\mathscr{S}$. For $F$ a vector measure,

$$
|F|(A) \equiv \sup \left\{\sum_{F \in \pi(A)}\|\mu(F)\|: \pi(A) \text { is a partition of } A\right\} .
$$

This is the same definition that was given in the case where $F$ would have values in $\mathbb{C}$, the only difference being the fact that now $F$ has values in a general Banach space $X$ as the vector space of values of the vector measure. Recall that a partition of $A$ is a finite set, $\left\{F_{1}, \cdots, F_{m}\right\} \subseteq \mathscr{S}$ such that $\cup_{i=1}^{m} F_{i}=A$. The same theorem about $|F|$ proved in the case of complex valued measures holds in this context with the same proof. For completeness, it is included here.

Theorem 21.7.2 If $|F|(\Omega)<\infty$, then $|F|$ is a measure on $\mathscr{S}$.
Proof: Let $E_{1}$ and $E_{2}$ be sets of $\mathscr{S}$ such that $E_{1} \cap E_{2}=\emptyset$ and let $\left\{A_{1}^{i}, \cdots, A_{n_{i}}^{i}\right\}=\pi\left(E_{i}\right)$, a partition of $E_{i}$ which is chosen such that

$$
|F|\left(E_{i}\right)-\varepsilon<\sum_{j=1}^{n_{i}}\left\|F\left(A_{j}^{i}\right)\right\| i=1,2
$$

Consider the sets which are contained in either of $\pi\left(E_{1}\right)$ or $\pi\left(E_{2}\right)$, it follows this collection of sets is a partition of $E_{1} \cup E_{2}$ which is denoted here by $\pi\left(E_{1} \cup E_{2}\right)$. Then by the above inequality and the definition of total variation,

$$
|F|\left(E_{1} \cup E_{2}\right) \geq \sum_{F \in \pi\left(E_{1} \cup E_{2}\right)}\|F(F)\|>|F|\left(E_{1}\right)+|F|\left(E_{2}\right)-2 \varepsilon
$$

which shows that since $\varepsilon>0$ was arbitrary,

$$
\begin{equation*}
|F|\left(E_{1} \cup E_{2}\right) \geq|F|\left(E_{1}\right)+|F|\left(E_{2}\right) . \tag{21.7.29}
\end{equation*}
$$

Let $\left\{E_{j}\right\}_{j=1}^{\infty}$ be a sequence of disjoint sets of $\mathscr{S}$ and let $E_{\infty}=\cup_{j=1}^{\infty} E_{j}$. Then by the definition of total variation there exists a partition of $E_{\infty}, \pi\left(E_{\infty}\right)=\left\{A_{1}, \cdots, A_{n}\right\}$ such that

$$
|F|\left(E_{\infty}\right)-\varepsilon<\sum_{i=1}^{n}\left\|F\left(A_{i}\right)\right\|
$$

Also,

$$
A_{i}=\cup_{j=1}^{\infty} A_{i} \cap E_{j}, \text { so } F\left(A_{j}\right)=\sum_{j=1}^{\infty} F\left(A_{i} \cap E_{j}\right)
$$

and so by the triangle inequality, $\left\|F\left(A_{i}\right)\right\| \leq \sum_{j=1}^{\infty}\left\|F\left(A_{i} \cap E_{j}\right)\right\|$. Therefore, by the above,

$$
|F|\left(E_{\infty}\right)-\varepsilon<\sum_{i=1}^{n} \overbrace{\sum_{j=1}^{\infty}\left\|F\left(A_{i} \cap E_{j}\right)\right\|}^{\geq\left\|F\left(A_{i}\right)\right\|}=\sum_{j=1}^{\infty} \sum_{i=1}^{n}\left\|F\left(A_{i} \cap E_{j}\right)\right\| \leq \sum_{j=1}^{\infty}|F|\left(E_{j}\right)
$$

because $\left\{A_{i} \cap E_{j}\right\}_{i=1}^{n}$ is a partition of $E_{j}$.
Since $\varepsilon>0$ is arbitrary, this shows

$$
|F|\left(\cup_{j=1}^{\infty} E_{j}\right) \leq \sum_{j=1}^{\infty}|F|\left(E_{j}\right)
$$

Also, 21.7.29 implies that whenever the $E_{i}$ are disjoint, $|F|\left(\cup_{j=1}^{n} E_{j}\right) \geq \sum_{j=1}^{n}|F|\left(E_{j}\right)$. Therefore,

$$
\sum_{j=1}^{\infty}|F|\left(E_{j}\right) \geq|F|\left(\cup_{j=1}^{\infty} E_{j}\right) \geq|F|\left(\cup_{j=1}^{n} E_{j}\right) \geq \sum_{j=1}^{n}|F|\left(E_{j}\right)
$$

Since $n$ is arbitrary,

$$
|F|\left(\cup_{j=1}^{\infty} E_{j}\right)=\sum_{j=1}^{\infty}|F|\left(E_{j}\right)
$$

which shows that $|F|$ is a measure as claimed.
Definition 21.7.3 A Banach space is said to have the Radon Nikodym property if whenever

$$
\begin{gathered}
(\Omega, \mathscr{S}, \mu) \text { is a finite measure space } \\
F: \mathscr{S} \rightarrow X \text { is a vector measure with }|F|(\Omega)<\infty \\
F \ll \mu
\end{gathered}
$$

then one may conclude there exists $g \in L^{1}(\Omega ; X)$ such that

$$
F(E)=\int_{E} g(s) d \mu
$$

for all $E \in \mathscr{S}$.
Some Banach spaces have the Radon Nikodym property and some don't. No attempt is made to give a complete answer to the question of which Banach spaces have this property, but the next theorem gives examples of many spaces which do. This next lemma was used earlier. I am presenting it again.

Lemma 21.7.4 Suppose $v$ is a complex measure defined on $\mathscr{S}$ a $\sigma$ algebra where $(\Omega, \mathscr{S})$ is a measurable space, and let $\mu$ be a measure on $\mathscr{S}$ with $|v(E)| \leq r \mu(E)$ and suppose there is $h \in L^{1}(\Omega, \mu)$ such that for all $E \in \mathscr{S}$,

$$
v(E)=\int_{E} h d \mu,
$$

Then $|h| \leq r$ a.e.
Proof: Let $B(p, \delta) \subseteq \mathbb{C} \backslash \overline{B(0, r)}$ and let $E \equiv h^{-1}(B(p, \delta))$. If $\mu(E)>0$. Then

$$
\left|\frac{1}{\mu(E)} \int_{E} h d \mu-p\right| \leq \frac{1}{\mu(E)} \int_{E}|h(\omega)-p| d \mu<\delta
$$

Thus, $\left|\frac{v(E)}{\mu(E)}-p\right|<\delta$ and so $|v(E)-p \mu(E)|<\delta \mu(E)$ which implies

$$
|v(E)| \geq(|p|-\delta) \mu(E)>r \mu(E) \geq|v(E)|
$$

which contradicts the assumption. Hence $h^{-1}(B(p, \delta))$ is a set of $\mu$ measure zero for all such balls contained in $\mathbb{C} \backslash \overline{B(0, r)}$ and so, since countably many of these balls cover $\mathbb{C} \backslash \overline{B(0, r)}$, it follows that $\mu\left(h^{-1}(\mathbb{C} \backslash \overline{B(0, r)})\right)=0$ and so $|h(\omega)| \leq r$ for a.e. $\omega$.
Theorem 21.7.5 Suppose $X^{\prime}$ is a separable dual space. Then $X^{\prime}$ has the Radon Nikodym property.

Proof: By Theorem 21.1.16, $X$ is separable. Let $D$ be a countable dense subset of $X$. Let $F \ll \mu, \mu$ a finite measure and $F$ a vector measure and let $|F|(\Omega)<\infty$. Pick $x \in X$ and consider the map

$$
E \rightarrow F(E)(x)
$$

for $E \in \mathscr{S}$. This defines a complex measure which is absolutely continuous with respect to $|F|$. Therefore, by the earlier Radon Nikodym theorem, there exists $f_{x} \in L^{1}(\Omega,|F|)$ such that

$$
\begin{equation*}
F(E)(x)=\int_{E} f_{x}(s) d|F| \tag{21.7.30}
\end{equation*}
$$

Also, by definition $\|F(E)\| \leq|F|(E)$ so $|F(E)(x)| \leq|F|(E)\|x\|$. By Lemma 21.7.4, $\left|f_{x}(s)\right| \leq\|x\|$ for $|F|$ a.e. $s$. Let $\tilde{D}$ consist of all finite linear combinations of the form $\sum_{i=1}^{m} a_{i} x_{i}$ where $a_{i}$ is a rational point of $\mathbb{F}$ and $x_{i} \in D$. For each of these countably many vectors, there is an exceptional set of measure zero off which $\left|f_{x}(s)\right| \leq\|x\|$. Let $N$ be the union of all of them and define $f_{x}(s) \equiv 0$ if $s \notin N$. Then since $F(E)$ is in $X^{\prime}$, it is linear and so for $\sum_{i=1}^{m} a_{i} x_{i} \in \tilde{D}$,

$$
\int_{E} f_{\sum_{i=1}^{m} a_{i} x_{i}}(s) d|F|=F(E)\left(\sum_{i=1}^{m} a_{i} x_{i}\right)=\sum_{i=1}^{m} a_{i} F(E)\left(x_{i}\right)=\int_{E} \sum_{i=1}^{m} a_{i} f_{x_{i}}(s) d|F|
$$

and so by uniqueness in the Radon Nikodym theorem,

$$
f_{\sum_{i=1}^{m} a_{i} x_{i}}(s)=\sum_{i=1}^{m} a_{i} f_{x_{i}}(s)|F| \text { a.e. }
$$

and so, we can regard this as holding for all $s \notin N$. Also, if $x \in \tilde{D},\left|f_{x}(s)\right| \leq\|x\|$. Now for $x, y \in \tilde{D}$,

$$
\left|f_{x}(s)-f_{y}(s)\right|=\left|f_{x-y}(s)\right| \leq\|x-y\|
$$

and so, by density of $\tilde{D}$, we can define

$$
h_{x}(s) \equiv \lim _{n \rightarrow \infty} f_{x_{n}}(s) \text { where } x_{n} \rightarrow x, x_{n} \in \tilde{D}
$$

For $s \in N$, all functions equal 0 . Thus for all $x,\left|h_{x}(s)\right| \leq\|x\|$. The dominated convergence theorem and continuity of $F(E)$ implies that for $x_{n} \rightarrow x$, with $x_{n} \in \tilde{D}$,

$$
\begin{equation*}
\int_{E} h_{x}(s) d|F|=\lim _{n \rightarrow \infty} \int_{E} f_{x_{n}}(s) d|F|=\lim _{n \rightarrow \infty} F(E)\left(x_{n}\right)=F(E)(x) . \tag{21.7.31}
\end{equation*}
$$

It follows from the density of $\tilde{D}$ that for all $x, y \in X, s \notin N$, and $a, b \in \mathbb{F}$, let $x_{n} \rightarrow x, y_{n} \rightarrow$ $y, a_{n} \rightarrow a, b_{n} \rightarrow b$, with $x_{n}, y_{n} \in \tilde{D}$ and $a_{n}, b_{n} \in \mathbb{Q}$ or $\mathbb{Q}+i \mathbb{Q}$ in case $\mathbb{F}=\mathbb{C}$. Then

$$
\begin{equation*}
h_{a x+b y}(s)=\lim _{n \rightarrow \infty} f_{a_{n} x_{n}+b_{n} y_{n}}(s)=\lim _{n \rightarrow \infty} a_{n} f_{x_{n}}(s)+b_{n} f_{y_{n}}(s) \equiv a h_{x}(s)+b h_{y}(s) \tag{21.7.32}
\end{equation*}
$$

Let $\theta(s)$ be given by $\theta(s)(x)=h_{x}(s)$ if $s \notin N$ and let $\theta(s)=0$ if $s \in N$. By 21.7.32 it follows that $\theta(s) \in X^{\prime}$ for each $s$. Also

$$
\theta(s)(x)=h_{x}(s) \in L^{1}(\Omega)
$$

so $\theta(\cdot)$ is weak $*$ measurable. Since $X^{\prime}$ is separable, Theorem 21.1.15 implies that $\theta$ is strongly measurable. Furthermore, by 21.7.32,

$$
\|\theta(s)\| \equiv \sup _{\|x\| \leq 1}|\theta(s)(x)| \leq \sup _{\|x\| \leq 1}\left|h_{x}(s)\right| \leq 1
$$

Therefore, $\int_{\Omega}\|\theta(s)\| d|F|<\infty$ so $\theta \in L^{1}\left(\Omega ; X^{\prime}\right)$. Thus, if $E \in \mathscr{S}$,

$$
\begin{equation*}
\int_{E} h_{x}(s) d|F|=\int_{E} \theta(s)(x) d|F|=\left(\int_{E} \theta(s) d|F|\right)(x) . \tag{21.7.33}
\end{equation*}
$$

From 21.7.31 and 21.7.33, $\left(\int_{E} \theta(s) d|F|\right)(x)=F(E)(x)$ for all $x \in X$ and therefore,

$$
\int_{E} \theta(s) d|F|=F(E)
$$

Finally, since $F \ll \mu,|F| \ll \mu$ also and so there exists $k \in L^{1}(\Omega)$ such that

$$
|F|(E)=\int_{E} k(s) d \mu
$$

for all $E \in \mathscr{S}$, by the scalar Radon Nikodym Theorem. It follows

$$
F(E)=\int_{E} \theta(s) d|F|=\int_{E} \theta(s) k(s) d \mu .
$$

Letting $g(s)=\theta(s) k(s)$, this has proved the theorem.
Since each reflexive Banach spaces is a dual space, the following corollary holds.
Corollary 21.7.6 Any separable reflexive Banach space has the Radon Nikodym property.
It is not necessary to assume separability in the above corollary. For the proof of a more general result, consult Vector Measures by Diestal and Uhl, [41].

### 21.8 The Riesz Representation Theorem

The Riesz representation theorem for the spaces $L^{p}(\Omega ; X)$ holds under certain conditions. The proof follows the proofs given earlier for scalar valued functions.

Definition 21.8.1 If $X$ and $Y$ are two Banach spaces, $X$ is isometric to $Y$ if there exists $\theta \in \mathscr{L}(X, Y)$ such that

$$
\|\theta x\|_{Y}=\|x\|_{X} .
$$

This will be written as $X \cong Y$. The map $\theta$ is called an isometry.
The next theorem says that $L^{p^{\prime}}\left(\Omega ; X^{\prime}\right)$ is always isometric to a subspace of $\left(L^{p}(\Omega ; X)\right)^{\prime}$ for any Banach space, $X$.

Theorem 21.8.2 Let $X$ be any Banach space and let $(\Omega, \mathscr{S}, \mu)$ be a finite measure space. Let $p \geq 1$ and let $1 / p+1 / p^{\prime}=1$. (If $p=1, p^{\prime} \equiv \infty$.) Then $L^{p^{\prime}}\left(\Omega ; X^{\prime}\right)$ is isometric to a subspace of $\left(L^{p}(\Omega ; X)\right)^{\prime}$. Also, for $g \in L^{p^{\prime}}\left(\Omega ; X^{\prime}\right)$,

$$
\sup _{\|f\|_{p} \leq 1}\left|\int_{\Omega} g(s)(f(s)) d \mu\right|=\|g\|_{p^{\prime}}
$$

Proof: First observe that for $f \in L^{p}(\Omega ; X)$ and $g \in L^{p^{\prime}}\left(\Omega ; X^{\prime}\right)$,

$$
s \rightarrow g(s)(f(s))
$$

is a function in $L^{1}(\Omega)$. (To obtain measurability, write $f$ as a limit of simple functions. Holder's inequality then yields the function is in $L^{1}(\Omega)$.) Define

$$
\theta: L^{p^{\prime}}\left(\Omega ; X^{\prime}\right) \rightarrow\left(L^{p}(\Omega ; X)\right)^{\prime}
$$

by

$$
\theta g(f) \equiv \int_{\Omega} g(s)(f(s)) d \mu
$$

Holder's inequality implies

$$
\begin{equation*}
\|\theta g\| \leq\|g\|_{p^{\prime}} \tag{21.8.34}
\end{equation*}
$$

and it is also clear that $\theta$ is linear. Next it is required to show $\|\theta g\|=\|g\|$.
This will first be verified for simple functions. Let

$$
g(s)=\sum_{i=1}^{m} c_{i}^{*} \mathscr{X}_{E_{i}}(s)
$$

where $c_{i}^{*} \in X^{\prime}$, the $E_{i}$ are disjoint and $\cup_{i=1}^{m} E_{i}=\Omega$. Then $\|g\| \in L^{p^{\prime}}(\Omega ; \mathbb{R}),\|g(s)\|=$ $\sum_{i=1}^{m}\left\|c_{i}^{*}\right\| \mathscr{X}_{E_{i}}(s)$.

Let $h(s) \equiv\|g(s)\|^{p^{\prime}-1} /\|g\|_{p^{\prime}}^{p^{\prime}-1}$. Then

$$
\begin{equation*}
\int_{\Omega}\|g(s)\|_{X^{\prime}} h(s) d \mu=\int_{\Omega} \frac{\|g(s)\|_{X^{\prime}}^{p^{\prime}}}{\|g\|_{p^{\prime}}^{p^{\prime}-1}} d \mu=\|g\|_{L^{p^{\prime}}\left(\Omega ; X^{\prime}\right)} \tag{21.8.35}
\end{equation*}
$$

Also $h \in L^{p}(\Omega ; \mathbb{R})$ and

$$
\int_{\Omega}|h(s)|^{p} d \mu=\int_{\Omega} \frac{\|g(s)\|^{p^{\prime}}}{\|g\|_{p^{\prime}}^{p^{\prime}}}=\frac{\|g\|_{p^{\prime}}}{\|g\|_{p^{\prime}}}=1
$$

so $\|h\|_{p}=1$. Since the measure space is finite, $h \in L^{1}(\Omega ; \mathbb{R})$.
Now let $d_{i}$ be chosen such that

$$
c_{i}^{*}\left(d_{i}\right) \geq\left\|c_{i}^{*}\right\|_{X^{\prime}}-\varepsilon /\|h\|_{L^{1}(\Omega)}
$$

and $\left\|d_{i}\right\|_{X}=1$. Let

$$
f(s) \equiv \sum_{i=1}^{m} d_{i} h(s) \mathscr{X}_{E_{i}}(s)
$$

Thus $f \in L^{p}(\Omega ; X)$ and $\|f\|_{L^{p}(\Omega ; X)}=1$. This follows from

$$
\|f\|_{p}^{p}=\int_{\Omega} \sum_{i=1}^{m}\left\|d_{i}\right\|_{X}^{p}|h(s)|^{p} \mathscr{X}_{E_{i}}(s) d \mu=\sum_{i=1}^{m}\left(\int_{E_{i}}|h(s)|^{p} d \mu\right)=1
$$

Also

$$
\begin{gathered}
\|\theta g\| \geq|\theta g(f)|=\left|\int_{\Omega} g(s)(f(s)) d \mu\right| \geq \\
\left|\int_{\Omega} \sum_{i=1}^{m}\left(\left\|c_{i}^{*}\right\|_{X^{\prime}}-\varepsilon /\|h\|_{L^{1}(\Omega)}\right) h(s) \mathscr{X}_{E_{i}}(s) d \mu\right|
\end{gathered}
$$

Then from 21.8.35

$$
\geq\left|\int_{\Omega}\|g(s)\|_{X^{\prime}} h(s) d \mu\right|-\varepsilon\left|\int_{\Omega} h(s) /\|h\|_{L^{1}(\Omega)} d \mu\right|=\|g\|_{L^{p^{\prime}}\left(\Omega ; X^{\prime}\right)}-\varepsilon
$$

Since $\varepsilon$ was arbitrary, $\|\theta g\| \geq\|g\|$ and from 21.8.34 this shows equality holds whenever $g$ is a simple function.

In general, let $g \in L^{p^{\prime}}\left(\Omega ; X^{\prime}\right)$ and let $g_{n}$ be a sequence of simple functions converging to $g$ in $L^{p^{\prime}}\left(\Omega ; X^{\prime}\right)$. Such a sequence exists by Lemma 21.1.2. Let $g_{n}(s) \rightarrow g(s),\left\|g_{n}(s)\right\| \leq$ $2\|g(s)\|$. Then each $g_{n}$ is in $L^{p^{\prime}}\left(\Omega ; X^{\prime}\right)$ and by the dominated convergence theorem they converge to $g$ in $L^{p^{\prime}}\left(\Omega ; X^{\prime}\right)$. Then for $\|\cdot\|$ the norm in $\left(L^{p}(\Omega ; X)\right)^{\prime}$,

$$
\|\theta g\|=\lim _{n \rightarrow \infty}\left\|\theta g_{n}\right\|=\lim _{n \rightarrow \infty}\left\|g_{n}\right\|=\|g\|
$$

This proves the theorem and shows $\theta$ is the desired isometry.
Theorem 21.8.3 If $X$ is a Banach space and $X^{\prime}$ has the Radon Nikodym property, then if $(\Omega, \mathscr{S}, \mu)$ is a finite measure space,

$$
\left(L^{p}(\Omega ; X)\right)^{\prime} \cong L^{p^{\prime}}\left(\Omega ; X^{\prime}\right)
$$

and in fact the mapping $\theta$ of Theorem 21.8.2 is onto.

Proof: Let $l \in\left(L^{p}(\Omega ; X)\right)^{\prime}$ and define $F(E) \in X^{\prime}$ by

$$
F(E)(x) \equiv l\left(\mathscr{X}_{E}(\cdot) x\right) .
$$

Lemma 21.8.4 $F$ defined above is a vector measure with values in $X^{\prime}$ and $|F|(\Omega)<\infty$.
Proof of the lemma: Clearly $F(E)$ is linear. Also

$$
\begin{gathered}
\|F(E)\|=\sup _{\|x\| \leq 1}\|F(E)(x)\| \\
\leq\|l\| \sup _{\|x\| \leq 1}\left\|\mathscr{X}_{E}(\cdot) x\right\|_{L^{p}(\Omega ; X)} \leq\|l\| \mu(E)^{1 / p}
\end{gathered}
$$

Let $\left\{E_{i}\right\}_{i=1}^{\infty}$ be a sequence of disjoint elements of $\mathscr{S}$ and let $E=\cup_{n<\infty} E_{n}$.

$$
\begin{align*}
\left|F(E)(x)-\sum_{k=1}^{n} F\left(E_{k}\right)(x)\right| & =\left|l\left(\mathscr{X}_{E}(\cdot) x\right)-\sum_{i=1}^{n} l\left(\mathscr{X}_{E_{i}}(\cdot) x\right)\right|  \tag{21.8.36}\\
& \leq\|l\|\left\|\mathscr{X}_{E}(\cdot) x-\sum_{i=1}^{n} \mathscr{X}_{E_{i}}(\cdot) x\right\|_{L^{p}(\Omega ; X)} \\
& \leq\|l\| \mu\left(\bigcup_{k>n} E_{k}\right)^{1 / p}\|x\| .
\end{align*}
$$

Since $\mu(\Omega)<\infty, \lim _{n \rightarrow \infty} \mu\left(\bigcup_{k>n} E_{k}\right)^{1 / p}=0$ and so inequality 21.8.36 shows that

$$
\lim _{n \rightarrow \infty}\left\|F(E)-\sum_{k=1}^{n} F\left(E_{k}\right)\right\|_{X^{\prime}}=0
$$

To show $|F|(\Omega)<\infty$, let $\varepsilon>0$ be given, let $\left\{H_{1}, \cdots, H_{n}\right\}$ be a partition of $\Omega$, and let $\left\|x_{i}\right\| \leq 1$ be chosen in such a way that

$$
F\left(H_{i}\right)\left(x_{i}\right)>\left\|F\left(H_{i}\right)\right\|-\varepsilon / n
$$

Thus

$$
\begin{gathered}
-\varepsilon+\sum_{i=1}^{n}\left\|F\left(H_{i}\right)\right\|<\sum_{i=1}^{n} l\left(\mathscr{X}_{H_{i}}(\cdot) x_{i}\right) \leq\|l\|\left\|\sum_{i=1}^{n} \mathscr{X}_{H_{i}}(\cdot) x_{i}\right\|_{L^{p}(\Omega ; X)} \\
\leq\|l\|\left(\int_{\Omega} \sum_{i=1}^{n} \mathscr{X}_{H_{i}}(s) d \mu\right)^{1 / p}=\|l\| \mu(\Omega)^{1 / p}
\end{gathered}
$$

Since $\varepsilon>0$ was arbitrary, $\sum_{i=1}^{n}\left\|F\left(H_{i}\right)\right\|<\|l\| \mu(\Omega)^{1 / p}$. Since the partition was arbitrary, this shows $|F|(\Omega) \leq\|l\| \mu(\Omega)^{1 / p}$ and this proves the lemma.

Continuing with the proof of Theorem 21.8.3, note that $F \ll \mu$. Since $X^{\prime}$ has the Radon Nikodym property, there exists $g \in L^{1}\left(\Omega ; X^{\prime}\right)$ such that

$$
F(E)=\int_{E} g(s) d \mu
$$

Also, from the definition of $F(E)$,

$$
\begin{align*}
& l\left(\sum_{i=1}^{n} x_{i} \mathscr{X}_{E_{i}}(\cdot)\right)=\sum_{i=1}^{n} l\left(\mathscr{X}_{E_{i}}(\cdot) x_{i}\right) \\
= & \sum_{i=1}^{n} F\left(E_{i}\right)\left(x_{i}\right)=\sum_{i=1}^{n} \int_{E_{i}} g(s)\left(x_{i}\right) d \mu . \tag{21.8.37}
\end{align*}
$$

It follows from 21.8.37 that whenever $h$ is a simple function,

$$
\begin{equation*}
l(h)=\int_{\Omega} g(s)(h(s)) d \mu \tag{21.8.38}
\end{equation*}
$$

Let $G_{n} \equiv\left\{s:\|g(s)\|_{X^{\prime}} \leq n\right\}$ and let $j: L^{p}\left(G_{n} ; X\right) \rightarrow L^{p}(\Omega ; X)$ be given by

$$
j h(s)=\left\{\begin{array}{l}
h(s) \text { if } s \in G_{n}, \\
0 \text { if } s \notin G_{n} .
\end{array}\right.
$$

Letting $h$ be a simple function in $L^{p}\left(G_{n} ; X\right)$,

$$
\begin{equation*}
j^{*} l(h)=l(j h)=\int_{G_{n}} g(s)(h(s)) d \mu . \tag{21.8.39}
\end{equation*}
$$

Since the simple functions are dense in $L^{p}\left(G_{n} ; X\right)$, and $g \in L^{p^{\prime}}\left(G_{n} ; X^{\prime}\right)$, it follows 21.8.39 holds for all $h \in L^{p}\left(G_{n} ; X\right)$. By Theorem 21.8.2,

$$
\|g\|_{L^{p^{\prime}}\left(G_{n} ; X^{\prime}\right)}=\left\|j^{*} l\right\|_{\left(L^{p}\left(G_{n} ; X\right)\right)^{\prime}} \leq\|l\|_{\left(L^{p}(\Omega ; X)\right)^{\prime}} .
$$

By the monotone convergence theorem,

$$
\|g\|_{L^{p^{\prime}}\left(\Omega ; X^{\prime}\right)}=\lim _{n \rightarrow \infty}\|g\|_{L^{p^{\prime}}\left(G_{n} ; X^{\prime}\right)} \leq\|l\|_{\left(L^{p}(\Omega ; X)\right)^{\prime}} .
$$

Therefore $g \in L^{p^{\prime}}\left(\Omega ; X^{\prime}\right)$ and since simple functions are dense in $L^{p}(\Omega ; X), 21.8 .38$ holds for all $h \in L^{p}(\Omega ; X)$. Thus $l=\theta g$ and the theorem is proved because, by Theorem 21.8.2, $\|l\|=\|g\|$ and the mapping $\theta$ is onto because $l$ was arbitrary.

As in the scalar case, everything generalizes to the case of $\sigma$ finite measure spaces. The proof is almost identical.

Lemma 21.8.5 Let $(\Omega, \mathscr{S}, \mu)$ be a $\sigma$ finite measure space and let $X$ be a Banach space such that $X^{\prime}$ has the Radon Nikodym property. Then there exists a measurable function, $r$ such that $r(x)>0$ for all $x$, such that $|r(x)|<M$ for all $x$, and $\int r d \mu<\infty$. For

$$
\Lambda \in\left(L^{p}(\Omega ; X)\right)^{\prime}, p \geq 1
$$

there exists a unique $h \in L^{p^{\prime}}\left(\Omega ; X^{\prime}\right), L^{\infty}\left(\Omega ; X^{\prime}\right)$ if $p=1$ such that

$$
\Lambda f=\int h(f) d \mu
$$

Also $\|h\|=\|\Lambda\| .\left(\|h\|=\|h\|_{p^{\prime}}\right.$ if $p>1,\|h\|_{\infty}$ if $\left.p=1\right)$. Here

$$
\frac{1}{p}+\frac{1}{p^{\prime}}=1
$$

Proof: First suppose $r$ exists as described. Also, to save on notation and to emphasize the similarity with the scalar case, denote the norm in the various spaces by $|\cdot|$. Define a new measure $\widetilde{\mu}$, according to the rule

$$
\begin{equation*}
\widetilde{\mu}(E) \equiv \int_{E} r d \mu \tag{21.8.40}
\end{equation*}
$$

Thus $\widetilde{\mu}$ is a finite measure on $\mathscr{S}$. Now define a mapping, $\eta: L^{p}(\Omega ; X, \mu) \rightarrow L^{p}(\Omega ; X, \widetilde{\mu})$ by $\eta f=r^{-\frac{1}{p}} f$. Then

$$
\|\eta f\|_{L^{p}(\widetilde{\mu})}^{p}=\int\left|r^{-\frac{1}{p}} f\right|^{p} r d \mu=\|f\|_{L^{p}(\mu)}^{p}
$$

and so $\eta$ is one to one and in fact preserves norms. I claim that also $\eta$ is onto. To see this, let $g \in L^{p}(\Omega ; X, \widetilde{\mu})$ and consider the function, $r^{\frac{1}{p}} g$. Then

$$
\int\left|r^{\frac{1}{p}} g\right|^{p} d \mu=\int|g|^{p} r d \mu=\int|g|^{p} d \widetilde{\mu}<\infty
$$

Thus $r^{\frac{1}{p}} g \in L^{p}(\Omega ; X, \mu)$ and $\eta\left(r^{\frac{1}{p}} g\right)=g$ showing that $\eta$ is onto as claimed. Thus $\eta$ is one to one, onto, and preserves norms. Consider the diagram below which is descriptive of the situation in which $\eta^{*}$ must be one to one and onto.

$$
\begin{array}{cccc}
h, L^{p^{\prime}}(\widetilde{\mu}) & L^{p}(\widetilde{\mu})^{\prime}, \widetilde{\Lambda} & \stackrel{\eta^{*}}{\rightarrow} & L^{p}(\mu)^{\prime}, \Lambda \\
& \eta & \\
L^{p}(\widetilde{\mu}) & \leftarrow & L^{p}(\mu)
\end{array}
$$

Then for $\Lambda \in L^{p}(\mu)^{\prime}$, there exists a unique $\widetilde{\Lambda} \in L^{p}(\widetilde{\mu})^{\prime}$ such that $\eta^{*} \widetilde{\Lambda}=\Lambda,\|\widetilde{\Lambda}\|=\|\Lambda\|$. By the Riesz representation theorem for finite measure spaces, there exists a unique $h \in$ $L^{p^{\prime}}(\widetilde{\mu}) \equiv L^{p^{\prime}}\left(\Omega ; X^{\prime}, \widetilde{\mu}\right)$ which represents $\widetilde{\Lambda}$ in the manner described in the Riesz representation theorem. Thus $\|h\|_{L^{p^{\prime}}(\widetilde{\mu})}=\|\widetilde{\Lambda}\|=\|\Lambda\|$ and for all $f \in L^{p}(\mu)$,

$$
\begin{aligned}
\Lambda(f) & =\eta^{*} \widetilde{\Lambda}(f) \equiv \widetilde{\Lambda}(\eta f)=\int h(\eta f) d \widetilde{\mu}=\int r h\left(r^{-\frac{1}{p}} f\right) d \mu \\
& =\int r^{\frac{1}{p}} h f d \mu
\end{aligned}
$$

Now

$$
\int\left|r^{\frac{1}{p^{\prime}}} h\right|^{p^{\prime}} d \mu=\int|h|^{p^{\prime}} r d \mu=\|h\|_{L^{p^{\prime}}(\widetilde{\mu})}^{p^{\prime}}<\infty
$$

Thus $\left\|r^{\frac{1}{p^{\prime}}}\right\|_{L_{p^{\prime}(\mu)}}=\|h\|_{L^{p^{\prime}}(\widetilde{\mu})}=\|\widetilde{\Lambda}\|=\|\Lambda\|$ and represents $\Lambda$ in the appropriate way. If $p=1$, then $1 / p^{\prime} \equiv 0$. Now consider the existence of $r$. Since the measure space is $\sigma$ finite, there exist $\left\{\Omega_{n}\right\}$ disjoint, each having positive measure and their union equals $\Omega$. Then define

$$
r(\omega) \equiv \sum_{n=1}^{\infty} \frac{1}{n^{2}} \mu\left(\Omega_{n}\right)^{-1} \mathscr{X}_{\Omega_{n}}(\omega)
$$

This proves the Lemma.

Theorem 21.8.6 (Riesz representation theorem) Let $(\Omega, \mathscr{S}, \mu)$ be $\sigma$ finite and let $X^{\prime}$ have the Radon Nikodym property. Then for

$$
\Lambda \in\left(L^{p}(\Omega ; X, \mu)\right)^{\prime}, p \geq 1
$$

there exists a unique $h \in L^{q}\left(\Omega, X^{\prime}, \mu\right), L^{\infty}\left(\Omega, X^{\prime}, \mu\right)$ if $p=1$ such that

$$
\Lambda f=\int h(f) d \mu
$$

Also $\|h\|=\|\Lambda\| .\left(\|h\|=\|h\|_{q}\right.$ if $p>1,\|h\|_{\infty}$ if $\left.p=1\right)$. Here

$$
\frac{1}{p}+\frac{1}{q}=1
$$

Proof: The above lemma gives the existence part of the conclusion of the theorem. Uniqueness is done as before.

Corollary 21.8.7 If $X^{\prime}$ is separable, then for $(\Omega, \mathscr{S}, \mu)$ a $\sigma$ finite measure space,

$$
\left(L^{p}(\Omega ; X)\right)^{\prime} \cong L^{p^{\prime}}\left(\Omega ; X^{\prime}\right)
$$

Corollary 21.8.8 If $X$ is separable and reflexive, then for $(\Omega, \mathscr{S}, \mu)$ a $\sigma$ finite measure space,

$$
\left(L^{p}(\Omega ; X)\right)^{\prime} \cong L^{p^{\prime}}\left(\Omega ; X^{\prime}\right)
$$

Corollary 21.8.9 If $X$ is separable and reflexive and $(\Omega, \mathscr{S}, \mu)$ is a $\sigma$ finite measure space, then if $p \in(1, \infty)$, then $L^{p}(\Omega ; X)$ is reflexive.

Proof: This is just like the scalar valued case.

### 21.8.1 An Example of Polish Space

Here is an interesting example. Obviously $L^{\infty}(0, T, H)$ is not separable with the normed topology. However, bounded sets turn out to be metric spaces which are complete and separable. This is the next lemma. Recall that a Polish space is a complete separable metric space. In this example, $H$ is a separable real Hilbert space or more generally a separable real Banach space.

Lemma 21.8.10 Let $B=\overline{B(\mathbf{0}, L)}$ be a closed ball in $L^{\infty}(0, T, H)$. Then $B$ is a Polish space with respect to the weak $*$ topology. The closure is taken with respect to the usual topology.

Proof: Let $\left\{\mathbf{z}_{k}\right\}_{k=1}^{\infty}=X$ be a dense countable subspace in $L^{1}(0, T, H)$. You start with a dense countable set and then consider all finite linear combinations having coefficients in $\mathbb{Q}$. Then the metric on $B$ is

$$
d(\mathbf{f}, \mathbf{g}) \equiv \sum_{k=1}^{\infty} 2^{-k} \frac{\left|\left\langle\mathbf{f}-\mathbf{g}, \mathbf{z}_{k}\right\rangle_{L^{\infty}, L^{1}}\right|}{1+\left|\left\langle\mathbf{f}-\mathbf{g}, \mathbf{z}_{k}\right\rangle_{L^{\infty}, L^{1}}\right|}
$$

Is $B$ complete? Suppose you have a Cauchy sequence $\left\{\mathbf{f}_{n}\right\}$. This happens if and only if $\left\{\left\langle\mathbf{f}_{n}, \mathbf{z}_{k}\right\rangle\right\}_{n=1}^{\infty}$ is a Cauchy sequence for each $k$. Therefore, there exists

$$
\xi\left(\mathbf{z}_{k}\right)=\lim _{n \rightarrow \infty}\left\langle\mathbf{f}_{n}, \mathbf{z}_{k}\right\rangle .
$$

Then for $a, b \in \mathbb{Q}$, and $\mathbf{z}, \mathbf{w} \in X$

$$
\xi(a \mathbf{z}+b \mathbf{w})=\lim _{n \rightarrow \infty}\left\langle\mathbf{f}_{n}, a \mathbf{z}+b \mathbf{w}\right\rangle=\lim _{n \rightarrow \infty} a\left\langle\mathbf{f}_{n}, \mathbf{z}\right\rangle+b\left\langle\mathbf{f}_{n}, \mathbf{w}\right\rangle=a \xi(\mathbf{z})+b \xi(\mathbf{w})
$$

showing that $\xi$ is linear on $X$ a dense subspace of $L^{1}(0, T, H)$. Is $\xi$ bounded on this dense subspace with bound $L$ ? For $\mathbf{z} \in X$,

$$
|\xi(\mathbf{z})| \equiv \lim _{n \rightarrow \infty}\left|\left\langle\mathbf{f}_{n}, \mathbf{z}\right\rangle\right| \leq \lim \sup _{n \rightarrow \infty}\left\|\mathbf{f}_{n}\right\|_{L^{\infty}}\|\mathbf{z}\|_{L^{1}} \leq L\|\mathbf{z}\|_{L^{1}}
$$

Hence $\xi$ is also bounded on this dense subset of $L^{1}(0, T, H)$. Therefore, there is a unique bounded linear extension of $\xi$ to all of $L^{1}(0, T, H)$ still denoted as $\xi$ such that its norm in $L^{1}(0, T, H)^{\prime}$ is no larger than $L$. It follows from the Riesz representation theorem that there exists a unique $\mathbf{f} \in L^{\infty}(0, T, H)$ such that for all $\mathbf{w} \in L^{1}(0, T, H), \boldsymbol{\xi}(\mathbf{w})=\langle\mathbf{f}, \mathbf{w}\rangle$ and $\|\mathbf{f}\| \leq L$. This $\mathbf{f}$ is the limit of the Cauchy sequence $\left\{\mathbf{f}_{n}\right\}$ in $B$. Thus $B$ is complete.

Is $B$ separable? Let $\mathbf{f} \in B$. Let $\varepsilon>0$ be given. Choose $M$ such that

$$
\sum_{k=M+1}^{\infty} 2^{-k}<\frac{\varepsilon}{4}
$$

Then the finite set $\left\{\mathbf{z}_{1}, \cdots, \mathbf{z}_{M}\right\}$ is uniformly integrable. There exists $\delta>0$ such that if $m(S)<\delta$, then

$$
\int_{S}\left|\mathbf{z}_{k}\right|_{H} d m<\left(\frac{\varepsilon}{4\left(1+\|\mathbf{f}\|_{L^{\infty}}\right)}\right)
$$

Then there is a sequence of simple functions $\left\{\mathbf{s}_{n}\right\}$ which converge uniformly to $\mathbf{f}$ off a set of measure zero, $N,\left\|\mathbf{s}_{n}\right\|_{L^{\infty}} \leq\|\mathbf{f}\|_{L^{\infty}}$. By regularity of the measure, there exists a continuous function with compact support $\mathbf{h}_{n}$ such that $\mathbf{s}_{n}=\mathbf{h}_{n}$ off a set of measure no more than $\delta / 4^{n}$ and also $\left\|\mathbf{h}_{n}\right\|_{L^{\infty}} \leq\|\mathbf{f}\|_{L^{\infty}}$. Then off a set of measure no more than $\frac{1}{3} \delta, \mathbf{h}_{n}(r) \rightarrow \mathbf{f}(r)$. Now by Eggorov's theorem and outer regularity, one can enlarge this exceptional set to obtain an open set $S$ of measure no more than $\delta / 2$ such that the convergence is uniform off this exceptional set. Thus $\mathbf{f}$ equals the uniform limit of continuous functions on $S^{C}$. Define

$$
\mathbf{h}(r) \equiv\left\{\begin{array}{l}
\lim _{n \rightarrow \infty} \mathbf{h}_{n}(r)=\mathbf{f}(r) \text { on } S^{C} \\
\mathbf{0} \text { on } S \backslash N \\
\mathbf{0} \text { on } N
\end{array}\right.
$$

Then $\|\mathbf{h}\|_{L^{\infty}} \leq\|\mathbf{f}\|_{L^{\infty}}$. Now consider

$$
\overline{\mathbf{h}} * \psi_{m}(r)
$$

where $\psi_{r}$ is approximate identity.

$$
\psi_{m}(t)=\frac{1}{2} m \mathscr{X}_{[-1 / m, 1 / m]}(t), \overline{\mathbf{h}} * \psi_{m}(t)=\frac{1}{2} m \int_{-1 / m}^{1 / m} \overline{\mathbf{h}}(t-s) d s=\frac{1}{2} m \int_{t-1 / m}^{t+1 / m} \overline{\mathbf{h}}(s) d s
$$

where we define $\bar{h}$ to be the 0 extension of $\bar{h}$ off $[0, T]$. This is a continuous function of $t$. Also a.e.t is a Lebesgue point and so for a.e.t,

$$
\begin{gathered}
\left|\frac{1}{2} m \int_{t-1 / m}^{t+1 / m} \overline{\mathbf{h}}(s) d s-\overline{\mathbf{h}}(t)\right| \rightarrow 0 \\
\left|\overline{\mathbf{h}} * \psi_{m}(r)\right| \equiv\left|\int_{\mathbb{R}} \overline{\mathbf{h}}(r-s) \psi_{m}(s) d s\right| \leq\|\mathbf{h}\|_{L^{\infty}} \leq\|\mathbf{f}\|_{L^{\infty}}
\end{gathered}
$$

Thus this continuous function is in $L^{\infty}(0, T, H)$. Letting $\mathbf{z}=\mathbf{z}_{k} \in L^{1}(0, T, H)$ be one of those defined above,

$$
\begin{gather*}
\left|\int_{0}^{T}\left\langle\overline{\mathbf{h}} * \psi_{m}(t)-\mathbf{f}(t), \mathbf{z}(t)\right\rangle d t\right| \leq \int_{0}^{T}\left|\left\langle\overline{\mathbf{h}} * \psi_{m}(t)-\mathbf{h}(t), \mathbf{z}(t)\right\rangle\right| d t \\
+\int_{0}^{T}|\langle\mathbf{h}(t)-\mathbf{f}(t), \mathbf{z}(t)\rangle| d t \tag{21.8.41}
\end{gather*}
$$

for a.e. $t, \overline{\mathbf{h}} * \psi_{m}(t)-\mathbf{h}(t) \rightarrow 0$ and the integrand in the first integral is bounded by the expression $2\|\mathbf{f}\|_{L^{\infty}}|\mathbf{z}(t)|_{H}$ so by the dominated convergence theorem, as $m \rightarrow \infty$, the first integral converges to 0 . As to the second, it is dominated by

$$
\int_{S}|\langle\mathbf{h}(t)-\mathbf{f}(t), \mathbf{z}(t)\rangle| d t \leq 2\|\mathbf{f}\|_{L^{\infty}} \int_{S}|\mathbf{z}(t)| d t<\frac{2\|\mathbf{f}\|_{L^{\infty}} \varepsilon}{4\left(1+\|\mathbf{f}\|_{L^{\infty}}\right)} \leq \frac{\varepsilon}{2}
$$

Therefore, choosing $m$ large enough so that the first integral on the right in 21.8.41 is less than $\frac{\varepsilon}{4}$ for each $\mathbf{z}_{k}$ for $k \leq M$, then for each of these,

$$
\begin{aligned}
d\left(\mathbf{f}, \overline{\mathbf{h}} * \psi_{m}\right) & \leq \frac{\varepsilon}{4}+\sum_{k=1}^{M} 2^{-k} \frac{(\varepsilon / 4)+(\varepsilon / 2)}{1+((\varepsilon / 4)+(\varepsilon / 2))}=\frac{\varepsilon}{4}+\sum_{k=1}^{M} 2^{-k} \frac{3}{4} \frac{\varepsilon}{\frac{3}{4} \varepsilon+1} \\
& \leq \frac{\varepsilon}{4}+\frac{3 \varepsilon}{4} \sum_{k=1}^{M} 2^{-k}<\frac{\varepsilon}{4}+\frac{3 \varepsilon}{4}=\varepsilon
\end{aligned}
$$

which appears to show that $C([0, T], H)$ is weak $*$ dense in $L^{\infty}(0, T, H)$. However, this last space is obviously separable in terms of the norm topology. Let $D$ be a countable dense subset of $C([0, T], H)$. For $\mathbf{f} \in L^{\infty}(0, T, H)$ let $\mathbf{g} \in C([0, T], H)$ such that $d(\mathbf{f}, \mathbf{g})<\frac{\varepsilon}{4}$. Then let $\mathbf{h} \in D$ be so close to $\mathbf{g}$ in $C([0, T], H)$ that

$$
\sum_{k=1}^{M} 2^{-k} \frac{\left|\left\langle\mathbf{h}-\mathbf{g}, \mathbf{z}_{k}\right\rangle_{L^{\infty}, L^{1}}\right|}{1+\left|\left\langle\mathbf{h}-\mathbf{g}, \mathbf{z}_{k}\right\rangle_{L^{\infty}, L^{1}}\right|}<\frac{\varepsilon}{2}
$$

Then

$$
d(\mathbf{f}, \mathbf{h}) \leq d(\mathbf{f}, \mathbf{g})+d(\mathbf{g}, \mathbf{h})<\frac{\varepsilon}{4}+\frac{\varepsilon}{2}+\frac{\varepsilon}{4}=\varepsilon
$$

It appears that $D$ is dense in $B$ in the weak $*$ topology.

### 21.9 Pointwise Behavior, Weakly Convergent Sequences

There is an interesting little result which relates to weak limits in $L^{2}(\Gamma, E)$ for $E$ a Banach space. I am not sure where to put this thing but think that this would be a good place for it. It obviously generalizes to $L^{p}$ spaces.

Proposition 21.9.1 Let $E$ be a Banach space and let $\left\{u_{n}\right\}$ be a sequence in $L^{2}(\Gamma, E)$ and let $G(x)$ be a weakly compact set in $E$, and $u_{n}(x) \in G(x)$ a.e. for each $n$. Let $\lim \sup \left\{u_{n}(x)\right\}$ denote the set of all weak limits of subsequences of $\left\{u_{n}(x)\right\}$ and let $H(x)$ be the closure of the convex hull of $\lim \sup \left\{u_{n}(x)\right\}$. Then if $u_{n} \rightarrow u$ weakly in $L^{2}(\Gamma, E)$, then $u(x) \in H(x)$ for a.e. $x$.

Proof: Let $H=\left\{w \in L^{2}(\Gamma, E): w(x) \in H(x)\right.$ a.e. $\}$. Then $H$ is convex. If you have $w_{i} \in H$, then since each $H(x)$ is convex, it follows that $\lambda w_{1}(x)+(1-\lambda) w_{2}(x) \in H$ for a.e. $x$ and $\lambda \in[0,1]$. Is $H$ closed? Suppose you have $w_{n} \in H$ and $w_{n} \rightarrow w$ in $L^{2}(\Gamma, E)$. Then there is a subsequence such that pointwise convergence happens a.e. and so since $H$ is closed, you have $w(x) \in H$ for a.e. $x$. Hence $H$ is also weakly closed in $L^{2}(\Gamma, H)$. Thus if $u$ is the weak limit of $\left\{u_{n}\right\}$ in $L^{2}(\Gamma, E)$, it must be the case that $u(x) \in H(x)$ a.e.

As a case of this which might be pretty interesting, suppose $G(x)$ is not just weakly compact but also convex. Then $H(x)=G(x)$ and you can say that $u(x) \in H(x)$ a.e. whenever it is a weak limit in $L^{2}(\Gamma, E)$ of functions $u_{n}$ for which $u_{n}(x) \in G(x)$.

### 21.10 Some Embedding Theorems

The next lemma is a very useful little result which involves embeddings of Banach spaces.

Lemma 21.10.1 Suppose $V \subseteq W$ and the injection map is compact, hence continuous. Suppose also that $W \subseteq U$ with continuous injection. Then for any $\varepsilon>0$ there exists $C_{\varepsilon}$ such that for all $v \in V$,

$$
\|v\|_{W} \leq \varepsilon\|v\|_{V}+C_{\varepsilon}\|v\|_{U}
$$

Proof: Suppose not. Then there exists $\varepsilon>0$ for which things don't work out. Thus there exists $v_{n} \in V$ such that

$$
\left\|v_{n}\right\|_{W}>\varepsilon\left\|v_{n}\right\|_{V}+n\left\|v_{n}\right\|_{U}
$$

Dividing by $\left\|v_{n}\right\|_{V}$, it can also be assumed that $\left\|v_{n}\right\|_{V}=1$. Thus

$$
\left\|v_{n}\right\|_{W}>\varepsilon+n\left\|v_{n}\right\|_{U}
$$

and so $\left\|v_{n}\right\|_{U} \rightarrow 0$. However, $v_{n}$ is contained in the closed unit ball of $V$ which is, by assumption precompact in $W$. Hence, there exists a subsequence, still denoted as $\left\{v_{n}\right\}$ such that $v_{n} \rightarrow v$ in $W$. But it was just determined that $v=0$ and so

$$
0 \geq \lim \sup _{n \rightarrow \infty}\left(\varepsilon+n\left\|v_{n}\right\|_{U}\right) \geq \varepsilon
$$

which is a contradiction.
Recall the following definition, this time for the space of continuous functions defined on a compact set with values in a Banach space.

Definition 21.10.2 Let $\mathscr{A} \subseteq C(K ; V)$ where the last symbol denotes the continuous functions defined on a compact set $K \subseteq X$ a metric space having values in $V$ a Banach space. Then $\mathscr{A}$ is equicontinuous if for every $\varepsilon>0$, there exists $\delta>0$ such that for every $f \in \mathscr{A}$, if $d(x, y)<\delta$, then

$$
\|f(x)-f(y)\|_{V}<\varepsilon
$$

Also $\mathscr{A} \subseteq C(K ; V)$ is uniformly bounded means

$$
\sup _{f \in \mathscr{A}}\|f\|_{\infty, V}<\infty \text { where }\|f\|_{\infty, V} \equiv \max _{x \in K}\|f(x)\|_{V}
$$

Here is a general version of the Ascoli Arzela theorem valid for Banach spaces.
Theorem 21.10.3 Let $V \subseteq W \subseteq U$ where the injection map of $V$ into $W$ is compact and $W$ embedds continuously into $U$, these being Banach spaces. Assume:

1. $\mathscr{A} \subseteq C(K ; U)$ where $K$ is compact and $\mathscr{A}$ is equicontinuous.
2. $\sup _{f \in \mathscr{A}}\|f\|_{\infty, V}<\infty$ where $\|f\|_{\infty, V} \equiv \max _{x \in K}\|f(x)\|_{V}$.

## Then

1. $\mathscr{A} \subseteq C(K ; W)$ and $\mathscr{A}$ is equicontinuous into $W$
2. $\mathscr{A}$ is pre-compact in $C(K ; W)$. Recall this means that $\overline{\mathscr{A}}$ is compact in $C(K ; W)$.

Proof: Let $C \equiv \sup _{f \in \mathscr{A}}\|f\|_{\infty, V}<\infty$. Let $\varepsilon>0$ be given. Then from Lemma 21.10.1,

$$
\|f(x)-f(y)\|_{W} \leq \frac{\varepsilon}{5 C}\|f(x)-f(y)\|_{V}+C_{\varepsilon}\|f(x)-f(y)\|_{U}
$$

$$
\leq \frac{2 \varepsilon}{5}+C_{\varepsilon}\|f(x)-f(y)\|_{U}
$$

By equicontinuity in $C(K, U)$, there exists a $\delta>0$ such that if $d(x, y)<\delta$, then for all $f \in \mathscr{A}$,

$$
C_{\varepsilon}\|f(x)-f(y)\|_{U}<\frac{2 \varepsilon}{5}
$$

Thus if $d(x, y)<\delta$, then $\|f(x)-f(y)\|_{W}<\varepsilon$ for all $f \in \mathscr{A}$.
It remains to verify that $\mathscr{A}$ is pre-compact in $C(K ; W)$. Since this space of continuous functions is complete, it suffices to verify that for all $\varepsilon>0, \mathscr{A}$ has an $\varepsilon$ net. Suppose then that for some $\varepsilon>0$ there is no $\varepsilon$ net. Thus there is an infinite sequence $\left\{f_{n}\right\}$ for which $\left\|f_{n}-f_{m}\right\|_{\infty, W} \geq \varepsilon$ whenever $m \neq n$. There exists $\delta>0$ such that if $d(x, y)<\delta$, then for all $f_{n}$,

$$
\left\|f_{n}(x)-f_{n}(y)\right\|_{W}<\frac{\varepsilon}{5}
$$

Let $\left\{x_{k}\right\}_{k=1}^{p}$ be a $\delta / 2$ net for $K$. This is where we use $K$ is compact. By compactness of the embedding of $V$ into $W$, there exists a further subsequence, still called $\left\{f_{n}\right\}$ such that each $\left\{f_{n}\left(x_{k}\right)\right\}_{n=1}^{\infty}$ converges, this for each $x_{k}$ in that $\delta / 2$ net. Thus there is a single $N$ such that if $n>N$, then for all $m, n>N$, and $k \leq p$,

$$
\left\|f_{n}\left(x_{k}\right)-f_{m}\left(x_{k}\right)\right\|_{W}<\frac{\varepsilon}{5}
$$

Now letting $x \in K$ be arbitrary, it is in $B\left(x_{k}, \delta / 2\right)$ for some $x_{k}$. Therefore, for $n, m$ larger than $N$,

$$
\begin{aligned}
\left\|f_{n}(x)-f_{m}(x)\right\|_{W} \leq & \left\|f_{n}(x)-f_{n}\left(x_{k}\right)\right\|_{W} \\
& +\left\|f_{n}\left(x_{k}\right)-f_{m}\left(x_{k}\right)\right\|_{W} \\
& +\left\|f_{m}\left(x_{k}\right)-f_{m}(x)\right\| \\
<\frac{\varepsilon}{5}+\frac{\varepsilon}{5}+ & \frac{\varepsilon}{5}=\frac{3 \varepsilon}{5}
\end{aligned}
$$

Taking the maximum for all $x$, for $m, n>N$,

$$
\left\|f_{n}-f_{m}\right\|_{W, \infty} \leq \frac{3 \varepsilon}{5}<\varepsilon
$$

contrary to the assumption that every pair is further apart than $\varepsilon$. Thus $\mathscr{A}$ is totally bounded so its closure would also be totally bounded and complete. In other words, $\mathscr{A}$ is precompact in $C(K ; W)$.

In the following theorem about compact subsets of an $L^{p}$ space, the measure will be Lebesgue measure. It depends on the above version of the Ascoli Arzela theorem. First note the following which I will use when convenient. For $a, b \geq 0$, and $p \geq 1$,

$$
(a+b)^{p} \leq 2^{p-1}\left(a^{p}+b^{p}\right)
$$

This follows from the convexity of $y=x^{p}$ for $x>0$. Also, for such $p$,

$$
(a+b)^{1 / p} \leq a^{1 / p}+b^{1 / p}
$$

Usually the thing of interest in this theorem is the case where $V=W=U=\mathbb{R}$. However, the more general version to be presented is interesting I think. Of course closed and bounded sets are compact in $\mathbb{R}$ so the usual case works as a special case of what is about to be presented.

Theorem 21.10.4 Let $V \subseteq W \subseteq U$ where these are Banach spaces such that the injection map of $V$ into $W$ is compact and the injection map of $W$ into $U$ is continuous. Let $\Omega$ be an open set in $\mathbb{R}^{m}$ and let $\mathscr{A}$ be a bounded subset of $L^{p}(\Omega ; V)$ and suppose that for all $\varepsilon>0$, there exist a $\delta>0$ such that if $|\mathbf{h}|<\delta$, then for $\tilde{u}$ denoting the zero extension of $u$ off $\Omega$,

$$
\begin{equation*}
\int_{\mathbb{R}^{m}}\|\widetilde{u}(\mathbf{x}+\mathbf{h})-\widetilde{u}(\mathbf{x})\|_{U}^{p} d x<\varepsilon^{p} \tag{21.10.42}
\end{equation*}
$$

Suppose also that for each $\varepsilon>0$ there exists an open set, $G_{\varepsilon} \subseteq \Omega$ such that $\overline{G_{\varepsilon}} \subseteq \Omega$ is compact and for all $u \in \mathscr{A}$,

$$
\begin{equation*}
\int_{\Omega \backslash \overline{G_{\varepsilon}}}\|u(\mathbf{x})\|_{W}^{p} d x<\varepsilon^{p} \tag{21.10.43}
\end{equation*}
$$

Then $\mathscr{A}$ is precompact in $L^{p}\left(\mathbb{R}^{n} ; W\right)$.
Proof: Let $\infty>M \geq \sup _{u \in L^{p}(\Omega ; V)}\|u\|_{L^{p}(\Omega ; V)}^{p}$. Let $\left\{\psi_{n}\right\}$ be a mollifier with support in $B(\mathbf{0}, 1 / n)$. I need to show that $\mathscr{A}$ has an $\eta$ net in $L^{p}(\Omega ; W)$ for every $\eta>0$. Suppose for some $\eta>0$ it fails to have an $\eta$ net. Without loss of generality, let $\eta<1$. Then by 21.10.43, it follows that for small enough $\varepsilon>0, \mathscr{A}_{\varepsilon} \equiv\left\{u \mathscr{X}_{\overline{G_{\varepsilon}}}: u \in \mathscr{A}\right\}$ fails to have an $\eta / 2$ net. Indeed, pick $\varepsilon$ small enough that for all $u \in \mathscr{A}$,

$$
\left\|u \mathscr{X}_{\overline{G_{\varepsilon}}}-u\right\|_{L^{p}(\Omega ; W)}<\frac{\eta}{5}
$$

Then if $\left\{u_{k} \mathscr{X}_{\overline{G_{\varepsilon}}}\right\}_{k=1}^{r}$ is an $\eta / 2$ net for $\mathscr{A}_{\varepsilon}$, so that $\cup_{k=1}^{r} B\left(u_{k} \mathscr{X}_{\overline{G_{\varepsilon}}}, \frac{\eta}{2}\right) \supseteq \mathscr{A}_{\varepsilon}$, then for $w \in \mathscr{A}, w \mathscr{X}_{\overline{G_{\varepsilon}}} \in B\left(u_{k} \mathscr{X}_{\overline{G_{\varepsilon}}}, \frac{\eta}{2}\right)$ for some $u_{k}$. Hence,

$$
\begin{gathered}
\left\|w-u_{k}\right\|_{L^{p}(\Omega ; W)} \leq\left\|w-w \mathscr{X}_{\overline{G_{\varepsilon}}}\right\|_{L^{p}(\Omega ; W)}+\left\|w \mathscr{X}_{\overline{G_{\varepsilon}}}-u_{k} \mathscr{X}_{\overline{G_{\varepsilon}}}\right\|_{L^{p}(\Omega ; W)} \\
+\left\|u_{k} \mathscr{X}_{\overline{G_{\varepsilon}}}-u_{k}\right\|_{L^{p}(\Omega ; W)} \\
\quad \leq \frac{\eta}{5}+\frac{\eta}{2}+\frac{\eta}{5}<\eta
\end{gathered}
$$

and so $\left\{u_{k}\right\}_{k=1}^{r}$ would be an $\eta$ net for $\mathscr{A}$ which is assumed to not exist.
Pick this $\varepsilon$ in all that follows. By compactness, Lemma 21.10.1, there exists $C_{\eta}$ such that for all $u \in V$,

$$
\begin{equation*}
\|u\|_{W}^{p} \leq \frac{\eta}{50\left(2^{p-1}\right) M}\|u\|_{V}^{p}+C_{\eta}\|u\|_{U}^{p} \tag{21.10.44}
\end{equation*}
$$

Let $\mathscr{A}_{\varepsilon n}$ consist of $\mathscr{A}_{\varepsilon n} \equiv\left\{u \mathscr{X}_{\overline{G_{\varepsilon}}} * \psi_{n}: u \in \mathscr{A}\right\}$. I want to show that $\mathscr{A}_{\varepsilon n}$ satisfies the conditions for Theorem 21.10.3.

Lemma 21.10.5 For each $n, \mathscr{A}_{\varepsilon n}$ satisfies the conditions of Theorem 21.10.3.
Proof: First consider the equicontinuity condition of that theorem. It suffices to show that if $\eta>0$ then there exists $\delta>0$ such that if $|\mathbf{h}|<\delta$, then for any $u \in \mathscr{A}$ and $\mathbf{x} \in \overline{G_{\mathcal{\varepsilon}}}$,

$$
\left\|u \mathscr{X}_{\overline{G_{\varepsilon}}} * \psi_{n}(\mathbf{x}+\mathbf{h})-u \mathscr{X}_{\overline{G_{\varepsilon}}} * \psi_{n}(\mathbf{x})\right\|_{U}<\eta
$$

Always assume $|\mathbf{h}|<\operatorname{dist}\left(\overline{G_{\varepsilon}}, \Omega^{C}\right)$, and $\mathbf{x} \in \overline{G_{\varepsilon}}$. Also assume that $|\mathbf{h}|$ is small enough that

$$
\begin{gather*}
\left(\int_{\mathbb{R}^{m}}\left|\left(\mathscr{X}_{\overline{G_{\varepsilon}}}(\mathbf{x}-\mathbf{y}+\mathbf{h})-\mathscr{X}_{\overline{G_{\varepsilon}}}(\mathbf{x}-\mathbf{y})\right) \psi_{n}(\mathbf{y})\right|^{p^{\prime}} d z\right)^{1 / p^{\prime}}= \\
\left(\int_{\mathbb{R}^{m}}\left|\left(\mathscr{X}_{\overline{G_{\varepsilon}}}(\mathbf{z}+\mathbf{h})-\mathscr{X}_{\overline{G_{\varepsilon}}}(\mathbf{z})\right) \psi_{n}(\mathbf{x}-\mathbf{z})\right|^{p^{\prime}} d z\right)^{1 / p^{\prime}}<\frac{\eta}{2 M} \tag{21.10.45}
\end{gather*}
$$

This can be obtained because by Holder's inequality,

$$
\begin{gathered}
\left(\int_{\mathbb{R}^{m}}\left|\left(\mathscr{X}_{\overline{G_{\varepsilon}}}(\mathbf{z}+\mathbf{h})-\mathscr{X}_{\overline{G_{\varepsilon}}}(\mathbf{z})\right) \psi_{n}(\mathbf{x}-\mathbf{z})\right|^{p^{\prime}} d z\right)^{1 / p^{\prime}} \\
\leq\left(\int_{\mathbb{R}^{m}}\left|\mathscr{X}_{\overline{G_{\varepsilon}}}(\mathbf{z}+\mathbf{h})-\mathscr{X}_{\overline{G_{\varepsilon}}}(\mathbf{z})\right|^{2 p^{\prime}} d z\right)^{\frac{1}{2 p^{\prime}}}\left(\int_{\mathbb{R}^{m}} \psi_{n}(\mathbf{x}-\mathbf{z})^{2 p^{\prime}} d z\right)^{\frac{1}{2 p^{\prime}}}
\end{gathered}
$$

which is small independent of $\mathbf{x}$ for $|\mathbf{h}|$ small enough, thanks to continuity of translation in $L^{2 p^{\prime}}\left(\mathbb{R}^{m}\right)$. Then

$$
\begin{aligned}
& \left\|u \mathscr{X}_{\overline{G_{\varepsilon}}} * \psi_{n}(\mathbf{x}+\mathbf{h})-u \mathscr{X}_{\overline{G_{\varepsilon}}} * \psi_{n}(\mathbf{x})\right\|_{U} \\
= & \left\|\int_{\mathbb{R}^{m}}\left(\tilde{u}(\mathbf{x}+\mathbf{h}-\mathbf{y}) \mathscr{X}_{\overline{G_{\varepsilon}}}(\mathbf{x}+\mathbf{h}-\mathbf{y})-\tilde{u}(\mathbf{x}-\mathbf{y}) \mathscr{X}_{\overline{G_{\varepsilon}}}(\mathbf{x}-\mathbf{y})\right) \psi_{n}(\mathbf{y}) d y\right\|_{U} \\
\leq & \int_{\mathbb{R}^{m}}\left\|\left(\tilde{u}(\mathbf{x}+\mathbf{h}-\mathbf{y}) \mathscr{X}_{\overline{G_{\varepsilon}}}(\mathbf{x}+\mathbf{h}-\mathbf{y})-\tilde{u}(\mathbf{x}-\mathbf{y}) \mathscr{X}_{\overline{G_{\varepsilon}}}(\mathbf{x}-\mathbf{y})\right)\right\|_{U} \psi_{n}(\mathbf{y}) d y
\end{aligned}
$$

Changing the variables,

$$
\begin{align*}
\leq & \int_{\mathbb{R}^{m}}\left\|\begin{array}{c}
(\tilde{u}(\mathbf{z}+\mathbf{h})-\tilde{u}(\mathbf{z})) \mathscr{X}_{\overline{G_{\varepsilon}}}(\mathbf{z}+\mathbf{h}) \\
+\tilde{u}(\mathbf{z})\left(\mathscr{X}_{\overline{G_{\varepsilon}}}(\mathbf{z}+\mathbf{h})-\mathscr{X}_{\overline{G_{\varepsilon}}}(\mathbf{z})\right)
\end{array}\right\|_{U} \psi_{n}(\mathbf{x}-\mathbf{z}) d z \\
\leq & \int_{\mathbb{R}^{m}}\left\|(\tilde{u}(\mathbf{z}+\mathbf{h})-\tilde{u}(\mathbf{z})) \mathscr{X}_{\overline{G_{\varepsilon}}}(\mathbf{z}+\mathbf{h})\right\|_{U} \psi_{n}(\mathbf{x}-\mathbf{z}) d z \\
& \quad+\int_{\mathbb{R}^{m}}\|\tilde{u}(\mathbf{z})\|_{U}\left|\mathscr{X}_{\overline{G_{\varepsilon}}}(\mathbf{z}+\mathbf{h})-\mathscr{X}_{\overline{G_{\varepsilon}}}(\mathbf{z})\right| \psi_{n}(\mathbf{x}-\mathbf{z}) d z \tag{21.10.46}
\end{align*}
$$

The first integral

$$
\leq\left(\int_{\mathbb{R}^{m}}\|\tilde{u}(\mathbf{z}+\mathbf{h})-\tilde{u}(\mathbf{z})\|_{U}^{p}\right)^{1 / p}\left(\int_{\mathbb{R}^{m}} \psi_{n}^{p^{\prime}}(\mathbf{x}-\mathbf{z}) d z\right)^{1 / p^{\prime}}
$$

You make the obvious change here in case $p=1$. Instead of the above, you would have

$$
\leq \int_{\mathbb{R}^{m}}\|\tilde{u}(\mathbf{z}+\mathbf{h})-\tilde{u}(\mathbf{z})\|_{U} d z 2\left\|\psi_{n}\right\|_{\infty}
$$

Since Lebesgue measure is translation independent, there is a constant $C_{n}$ such that the above is

$$
\leq C_{n}\left(\int_{\mathbb{R}^{m}}\|\tilde{u}(\mathbf{z}+\mathbf{h})-\tilde{u}(\mathbf{z})\|_{U}^{p}\right)^{1 / p}<\eta / 2
$$

and this holds for all $u \in \mathscr{A}$. As for the second integral in 21.10.46, from 21.10.45, it follows that this term is no larger than

$$
\leq\left(\int_{\mathbb{R}^{m}}\|\tilde{u}(\mathbf{z})\|_{U}^{p} d z\right)^{1 / p}\left(\int_{\mathbb{R}^{m}}\left(\left|\mathscr{X}_{\overline{G_{\varepsilon}}}(\mathbf{z}+\mathbf{h})-\mathscr{X}_{\overline{G_{\varepsilon}}}(\mathbf{z})\right| \psi_{n}(\mathbf{x}-\mathbf{z})\right)^{p^{\prime}} d z\right)^{1 / p^{\prime}}
$$

and by 21.10.45,

$$
<M \frac{\eta}{2 M}=\frac{\eta}{2}
$$

Thus, if $\delta<\operatorname{dist}\left(\overline{G_{\varepsilon}}, \Omega^{C}\right)$ and 21.10.45 holds, then for all $u \in \mathscr{A}$, when $|\mathbf{h}|<\delta$,

$$
\left\|u \mathscr{X}_{\overline{G_{\varepsilon}}} * \psi_{n}(\mathbf{x}+\mathbf{h})-u \mathscr{X}_{\overline{G_{\varepsilon}}} * \psi_{n}(\mathbf{x})\right\|_{U}<\eta
$$

and so the desired equicontinuity condition holds for $\mathscr{A}_{\varepsilon n}$. Note that $\delta$ does depend on $n$ but for each $n$, things work out well.

I also need to verify that the functions in $\mathscr{A}_{\varepsilon n}$ are uniformly bounded. For $\mathbf{x} \in \overline{G_{\varepsilon}}$ and $u \in \mathscr{A}$,

$$
\begin{gathered}
\left\|u \mathscr{X}_{\overline{G_{\varepsilon}}} * \psi_{n}(\mathbf{x})\right\|_{V} \leq \int_{\overline{G_{\varepsilon}}}\|u(\mathbf{z})\| \psi_{n}(\mathbf{x}-\mathbf{z}) d z \\
\leq\left(\int_{\Omega}\|u(\mathbf{z})\|^{p} d z\right)^{1 / p}\left(\int_{\Omega} \psi_{n}(\mathbf{x}-\mathbf{z})^{p^{\prime}}\right)^{1 / p^{\prime}} \leq M C_{n}
\end{gathered}
$$

Now is a general statement about norms, indicating that the $L^{p}$ norm is no more than a constant times the norm involving the maximum.

$$
\left(\int_{\overline{G_{\varepsilon}}}\|v(\mathbf{x})\|_{W}^{p} d x\right)^{1 / p} \leq \max _{\mathbf{x} \in \overline{G_{\varepsilon}}}\|v(\mathbf{x})\|_{W} m\left(\overline{G_{\varepsilon}}\right) \equiv m\left(\overline{G_{\varepsilon}}\right)\|v\|_{W, \infty}
$$

It follows from Theorem 21.10.3 that for every $\eta>0$, there exists a $\eta$ net in $C\left(\overline{G_{\varepsilon}} ; W\right)$ for $\mathscr{A}_{\varepsilon n}$, this for each $n$. Then from the above inequality, it follows that for each $\eta$, there exists an $\eta$ net in $L^{p}\left(\overline{G_{\varepsilon}} ; W\right)$ for $\mathscr{A}_{\varepsilon n}$.

Recall also, from the assumption that the theorem is not true, $\mathscr{A}_{\varepsilon} \equiv\left\{u \mathscr{X}_{\overline{G_{\varepsilon}}}: u \in \mathscr{A}\right\}$ has no $\eta / 2$ net in $L^{p}\left(\overline{G_{\varepsilon}} ; W\right)$. Next I estimate the distance in $L^{p}\left(\overline{G_{\varepsilon}} ; W\right)$ between $u \mathscr{X}_{\overline{G_{\varepsilon}}}$ for $u \in \mathscr{A}$ and $u \mathscr{X}_{\overline{G_{\varepsilon}}} * \psi_{n}$. The idea is that for each $n, \mathscr{A}_{\varepsilon n}$ has an $\eta / 8$ net and for $n$ large enough, $u \mathscr{X}_{\overline{G_{\varepsilon}}}$ is close to $u \mathscr{X}_{\overline{G_{\varepsilon}}} * \psi_{n}$ so a contradiction will result if the functions of the
second sort are totally bounded while those functions of the first sort don't. Assume always that $1 / n<\operatorname{dist}\left(\overline{G_{\varepsilon}}, \Omega^{C}\right)$. Using Minkowski's inequality,

$$
\begin{gathered}
\left\|u \mathscr{X}_{\overline{G_{\varepsilon}}}-u \mathscr{X}_{\overline{G_{\varepsilon}}} * \psi_{n}\right\|_{L^{p}\left(\overline{G_{\varepsilon}} ; W\right)}= \\
\left(\int_{\mathbb{R}^{m}}\left\|\int_{\mathbb{R}^{m}}\left(u \mathscr{X}_{\overline{G_{\varepsilon}}}(\mathbf{x})-u \mathscr{X}_{\overline{G_{\varepsilon}}}(\mathbf{x}-\mathbf{y})\right) \psi_{n}(\mathbf{y}) d y\right\|_{W}^{p} d x\right)^{1 / p} \\
\leq \int_{B(\mathbf{0}, 1 / n)} \psi_{n}(\mathbf{y})\left(\int_{\mathbb{R}^{m}}\left\|\left(u \mathscr{X}_{\overline{G_{\varepsilon}}}(\mathbf{x})-u \mathscr{X}_{\overline{G_{\varepsilon}}}(\mathbf{x}-\mathbf{y})\right)\right\|_{W}^{p} d x\right)^{1 / p} d y \\
\leq \int_{B(\mathbf{0}, 1 / n)} \psi_{n}(\mathbf{y})\left(\int_{\mathbb{R}^{m}}\|(\tilde{u}(\mathbf{x})-\tilde{u}(\mathbf{x}-\mathbf{y}))\|_{W}^{p} d x\right)^{1 / p} d y \\
\leq \int_{B(\mathbf{0}, 1 / n)} \psi_{n}(\mathbf{y})\binom{\int_{\mathbb{R}^{m}} \frac{\eta}{50\left(2^{p-1}\right) M}\|\tilde{u}(\mathbf{x})-\tilde{u}(\mathbf{x}-\mathbf{y})\|_{V}^{p}}{+C_{\eta}\|\tilde{u}(\mathbf{x})-\tilde{u}(\mathbf{x}-\mathbf{y})\|_{U}^{p} d x}^{1 / p} d y \\
\leq \int_{B\left(\mathbf{0}, \frac{1}{n}\right)} \psi_{n}(\mathbf{y})\left(\begin{array}{c}
\int_{\mathbb{R}^{m}} \frac{\eta}{5\left(C_{\eta} \int_{\mathbb{R}^{p}}^{p-1}\right) M} 2^{p-1} 2\left(\|\tilde{u}(\mathbf{x})-\tilde{u}(\mathbf{x})\|_{V}^{p}\right) d x \\
\leq
\end{array} \int_{B\left(\mathbf{x}, \frac{1}{n}\right)}^{1 / p} \psi_{n}(\mathbf{y})\left(\begin{array}{c}
\int_{U}^{p} d x
\end{array}\right)^{\int_{\mathbb{R}^{m}} \frac{\eta}{25 M}\left(\|\tilde{u}(\mathbf{x})\|_{V}^{p}\right) d x} \begin{array}{l}
\mathbb{R}^{m} C_{\eta}\|\tilde{u}(\mathbf{x})-\tilde{u}(\mathbf{x}-\mathbf{y})\|_{U}^{p} d x
\end{array}\right)^{1 / p} d y \\
\leq \int_{B\left(\mathbf{0}, \frac{1}{n}\right)} \psi_{n}(\mathbf{y})\left(\frac{\eta}{25}+\int_{\mathbb{R}^{m}} C_{\eta}\|\tilde{u}(\mathbf{x})-\tilde{u}(\mathbf{x}-\mathbf{y})\|_{U}^{p} d x\right)^{1 / p} d y
\end{gathered}
$$

By assumption 21.10.42, there exists $N$ such that if $n \geq N$, then $|\mathbf{y}|<\frac{1}{n}$ and for all $u \in \mathscr{A}$,

$$
\begin{aligned}
&\left\|u \mathscr{X}_{\overline{G_{\varepsilon}}}-u \mathscr{X}_{\overline{G_{\varepsilon}}} * \psi_{n}\right\|_{L^{p}\left(\overline{G_{\varepsilon}} ; W\right)} \leq \\
& \int_{B\left(\mathbf{0}, \frac{1}{n}\right)} \psi_{n}(\mathbf{y})\left(\frac{\eta}{25}+\frac{\eta^{p}}{8^{p}}\right)^{1 / p} d y \\
& \leq \int_{B\left(\mathbf{0}, \frac{1}{n}\right)} \psi_{n}(\mathbf{y})\left(\frac{\eta}{25}+\frac{\eta}{8}\right) d y \\
&= \frac{\eta}{25}+\frac{\eta}{8}
\end{aligned}
$$

Recall $\eta<1$.
Let $n$ be this large. Then let $\left\{u_{k} \mathscr{X}_{\overline{G_{\varepsilon}}} * \psi_{n}\right\}_{k=1}^{r}$ be a $\eta / 8$ net for $\mathscr{A}_{\varepsilon n}$ in $L^{p}\left(\overline{G_{\varepsilon}} ; W\right)$. Then consider the balls $B\left(u_{k} \mathscr{X}_{\overline{G_{\varepsilon}}}, \frac{\eta}{4}\right)$ in $L^{p}\left(\overline{G_{\varepsilon}} ; W\right)$. If $w \mathscr{X}_{\overline{G_{\varepsilon}}}$ is in $\mathscr{A}_{\varepsilon}$, is it in some $B\left(u_{k} \mathscr{X}_{\overline{G_{\varepsilon}}}, \frac{\eta}{2}\right)$ ? By what was just shown, there is $k$ such that

$$
\left\|w \mathscr{X}_{\overline{G_{\varepsilon}}} * \psi_{n}-u_{k} \mathscr{X}_{\overline{G_{\varepsilon}}} * \psi_{n}\right\|_{L^{p}\left(\overline{G_{\varepsilon}} ; W\right)}<\frac{\eta}{8}
$$

and also

$$
\begin{aligned}
\left\|w \mathscr{X}_{\overline{G_{\varepsilon}}}-w \mathscr{X}_{\overline{G_{\varepsilon}}} * \psi_{n}\right\|_{L^{p}\left(\overline{G_{\varepsilon}} ; W\right)}<\frac{\eta}{8}+\frac{\eta}{25} \\
\left\|u_{k} \mathscr{X}_{\overline{G_{\varepsilon}}}-u_{k} \mathscr{X}_{\overline{G_{\varepsilon}}} * \psi_{n}\right\|_{L^{p}\left(\overline{G_{\varepsilon}} ; W\right)}<\frac{\eta}{8}+\frac{\eta}{25}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\left\|w \mathscr{X}_{\overline{G_{\varepsilon}}}-u_{k} \mathscr{X}_{\overline{G_{\varepsilon}}}\right\|_{L^{p}\left(G_{\varepsilon} ; W\right)} \leq & \left\|w \mathscr{X}_{\overline{G_{\varepsilon}}}-w \mathscr{X}_{\overline{G_{\varepsilon}}} * \psi_{n}\right\|_{L^{p}\left(\overline{G_{\varepsilon}} ; W\right)} \\
& +\left\|w \mathscr{X}_{\overline{G_{\varepsilon}}} * \psi_{n}-u_{k} \mathscr{X}_{\overline{G_{\varepsilon}}} * \psi_{n}\right\|_{L^{p}\left(\overline{G_{\varepsilon}} ; W\right)} \\
& +\left\|u_{k} \mathscr{X}_{\overline{G_{\varepsilon}}} * \psi_{n}-u_{k} \mathscr{X}_{\overline{G_{\varepsilon}}}\right\|_{L^{p}\left(\overline{G_{\varepsilon}} ; W\right)} \\
< & \frac{3 \eta}{8}+\frac{2 \eta}{25}<\frac{\eta}{2}
\end{aligned}
$$

It follows that $\left\{u_{k} \mathscr{X}_{\overline{G_{\varepsilon}}}\right\}_{k=1}^{r}$ is a $\eta / 2$ net for $L^{p}\left(\overline{G_{\varepsilon}} ; W\right)$ contrary to the construction. Thus $\mathscr{A}$ has an $\eta$ net after all.

In case $\Omega$ is a closed interval, there are several versions of these sorts of embeddings which are enormously useful in the study of nonlinear evolution equations or inclusions.

The following theorem is an infinite dimensional version of the Ascoli Arzela theorem. It is like a well known result due to Simon [117]. It is an appropriate generalization when you do not necessarily have weak derivatives.
Theorem 21.10.6 Let $q>1$ and let $E \subseteq W \subseteq X$ where the injection map is continuous from $W$ to $X$ and compact from $E$ to $W$. Let $S$ be defined by

$$
\left\{u \text { such that }\|u(t)\|_{E} \leq R \text { for all } t \in[a, b], \text { and }\|u(s)-u(t)\|_{X} \leq R|t-s|^{1 / q}\right\}
$$

Thus $S$ is bounded in $L^{\infty}(a, b, E)$ and in addition, the functions are uniformly Holder continuous into $X$. Then $S \subseteq C([a, b] ; W)$ and if $\left\{u_{n}\right\} \subseteq S$, there exists a subsequence, $\left\{u_{n_{k}}\right\}$ which converges to a function $u \in C([a, b] ; W)$ in the following way.

$$
\lim _{k \rightarrow \infty}\left\|u_{n_{k}}-u\right\|_{\infty, W}=0
$$

Proof: First consider the issue of $S$ being a subset of $C([a, b] ; W)$. Let $\varepsilon>0$ be given. Then by Lemma 21.10.1, there exists a constant, $C_{\varepsilon}$ such that for all $u \in W$

$$
\|u\|_{W} \leq \frac{\varepsilon}{6 R}\|u\|_{E}+C_{\varepsilon}\|u\|_{X}
$$

Therefore, for all $u \in S$,

$$
\begin{align*}
\|u(t)-u(s)\|_{W} & \leq \frac{\varepsilon}{6 R}\|u(t)-u(s)\|_{E}+C_{\varepsilon}\|u(t)-u(s)\|_{X} \\
& \leq \frac{\varepsilon}{6 R}\left(\|u(t)\|_{E}+\|u(s)\|_{E}\right)+C_{\varepsilon}\|u(t)-u(s)\|_{X} \\
& \leq \frac{\varepsilon}{3}+C_{\varepsilon} R|t-s|^{1 / q} \tag{21.10.47}
\end{align*}
$$

Since $\varepsilon$ is arbitrary, it follows $u \in C([a, b] ; W)$.
Let $D=\mathbb{Q} \cap[a, b]$ so $D$ is a countable dense subset of $[a, b]$. Let $D=\left\{t_{n}\right\}_{n=1}^{\infty}$. By compactness of the embedding of $E$ into $W$, there exists a subsequence $u_{(n, 1)}$ such that as $n \rightarrow \infty, u_{(n, 1)}\left(t_{1}\right)$ converges to a point in $W$. Now take a subsequence of this, called $(n, 2)$ such that as $n \rightarrow \infty, u_{(n, 2)}\left(t_{2}\right)$ converges to a point in $W$. It follows that $u_{(n, 2)}\left(t_{1}\right)$ also converges to a point of $W$. Continue this way. Now consider the diagonal sequence, $u_{k} \equiv$ $u_{(k, k)}$ This sequence is a subsequence of $u_{(n, l)}$ whenever $k>l$. Therefore, $u_{k}\left(t_{j}\right)$ converges for all $t_{j} \in D$.

Claim: Let $\left\{u_{k}\right\}$ be as just defined, converging at every point of $D \equiv[a, b] \cap \mathbb{Q}$. Then $\left\{u_{k}\right\}$ converges at every point of $[a, b]$.

Proof of claim: Let $\varepsilon>0$ be given. Let $t \in[a, b]$. Pick $t_{m} \in D \cap[a, b]$ such that in 21.10.47 $C_{\varepsilon} R\left|t-t_{m}\right|<\varepsilon / 3$. Then there exists $N$ such that if $l, n>N$, then

$$
\left\|u_{l}\left(t_{m}\right)-u_{n}\left(t_{m}\right)\right\|_{X}<\varepsilon / 3 .
$$

It follows that for $l, n>N$,

$$
\begin{aligned}
\left\|u_{l}(t)-u_{n}(t)\right\|_{W} \leq & \left\|u_{l}(t)-u_{l}\left(t_{m}\right)\right\|_{W}+\left\|u_{l}\left(t_{m}\right)-u_{n}\left(t_{m}\right)\right\|_{W} \\
& +\left\|u_{n}\left(t_{m}\right)-u_{n}(t)\right\|_{W} \\
\leq & \frac{2 \varepsilon}{3}+\frac{\varepsilon}{3}+\frac{2 \varepsilon}{3}<2 \varepsilon
\end{aligned}
$$

Since $\varepsilon$ was arbitrary, this shows $\left\{u_{k}(t)\right\}_{k=1}^{\infty}$ is a Cauchy sequence. Since $W$ is complete, this shows this sequence converges.

Now for $t \in[a, b]$, it was just shown that if $\varepsilon>0$ there exists $N_{t}$ such that if $n, m>N_{t}$, then

$$
\left\|u_{n}(t)-u_{m}(t)\right\|_{W}<\frac{\varepsilon}{3}
$$

Now let $s \neq t$. Then

$$
\left\|u_{n}(s)-u_{m}(s)\right\|_{W} \leq\left\|u_{n}(s)-u_{n}(t)\right\|_{W}+\left\|u_{n}(t)-u_{m}(t)\right\|_{W}+\left\|u_{m}(t)-u_{m}(s)\right\|_{W}
$$

From 21.10.47

$$
\left\|u_{n}(s)-u_{m}(s)\right\|_{W} \leq 2\left(\frac{\varepsilon}{3}+C_{\varepsilon} R|t-s|^{1 / q}\right)+\left\|u_{n}(t)-u_{m}(t)\right\|_{W}
$$

and so it follows that if $\delta$ is sufficiently small and $s \in B(t, \delta)$, then when $n, m>N_{t}$

$$
\left\|u_{n}(s)-u_{m}(s)\right\|<\varepsilon
$$

Since $[a, b]$ is compact, there are finitely many of these balls, $\left\{B\left(t_{i}, \delta\right)\right\}_{i=1}^{p}$, such that for $s \in B\left(t_{i}, \boldsymbol{\delta}\right)$ and $n, m>N_{t_{i}}$, the above inequality holds. Let $N>\max \left\{N_{t_{1}}, \cdots, N_{t_{p}}\right\}$. Then if $m, n>N$ and $s \in[a, b]$ is arbitrary, it follows the above inequality must hold. Therefore, this has shown the following claim.

Claim: Let $\varepsilon>0$ be given. Then there exists $N$ such that if $m, n>N$, then

$$
\left\|u_{n}-u_{m}\right\|_{\infty, W}<\varepsilon
$$

Now let $u(t)=\lim _{k \rightarrow \infty} u_{k}(t)$.

$$
\begin{equation*}
\|u(t)-u(s)\|_{W} \leq\left\|u(t)-u_{n}(t)\right\|_{W}+\left\|u_{n}(t)-u_{n}(s)\right\|_{W}+\left\|u_{n}(s)-u(s)\right\|_{W} \tag{21.10.48}
\end{equation*}
$$

Let $N$ be in the above claim and fix $n>N$. Then

$$
\left\|u(t)-u_{n}(t)\right\|_{W}=\lim _{m \rightarrow \infty}\left\|u_{m}(t)-u_{n}(t)\right\|_{W} \leq \varepsilon
$$

and similarly, $\left\|u_{n}(s)-u(s)\right\|_{W} \leq \varepsilon$. Then if $|t-s|$ is small enough, 21.10 .47 shows the middle term in 21.10.48 is also smaller than $\varepsilon$. Therefore, if $|t-s|$ is small enough,

$$
\|u(t)-u(s)\|_{W}<3 \varepsilon
$$

Thus $u$ is continuous. Finally, let $N$ be as in the above claim. Then letting $m, n>N$, it follows that for all $t \in[a, b]$,

$$
\left\|u_{m}(t)-u_{n}(t)\right\|_{W}<\varepsilon
$$

Therefore, letting $m \rightarrow \infty$, it follows that for all $t \in[a, b]$,

$$
\left\|u(t)-u_{n}(t)\right\|_{W} \leq \varepsilon
$$

and so $\left\|u-u_{n}\right\|_{\infty, W} \leq \varepsilon$.
Here is an interesting corollary. Recall that for $E$ a Banach space $C^{0, \alpha}([0, T], E)$ is the space of continuous functions $u$ from $[0, T]$ to $E$ such that

$$
\|u\|_{\alpha, E} \equiv\|u\|_{\infty, E}+\rho_{\alpha, E}(u)<\infty
$$

where here

$$
\rho_{\alpha, E}(u) \equiv \sup _{t \neq s} \frac{\|u(t)-u(s)\|_{E}}{|t-s|^{\alpha}}
$$

Corollary 21.10.7 Let $E \subseteq W \subseteq X$ where the injection map is continuous from $W$ to $X$ and compact from $E$ to $W$. Then if $\gamma>\alpha$, the embedding of $C^{0, \gamma}([0, T], E)$ into $C^{0, \alpha}([0, T], X)$ is compact.

Proof: Let $\phi \in C^{0, \gamma}([0, T], E)$

$$
\begin{gathered}
\frac{\|\phi(t)-\phi(s)\|_{X}}{|t-s|^{\alpha}} \leq\left(\frac{\|\phi(t)-\phi(s)\|_{W}}{|t-s|^{\gamma}}\right)^{\alpha / \gamma}\|\phi(t)-\phi(s)\|_{W}^{1-(\alpha / \gamma)} \\
\leq\left(\frac{\|\phi(t)-\phi(s)\|_{E}}{|t-s|^{\gamma}}\right)^{\alpha / \gamma}\|\phi(t)-\phi(s)\|_{W}^{1-(\alpha / \gamma)} \leq \rho_{\gamma, E}(\phi)\|\phi(t)-\phi(s)\|_{W}^{1-(\alpha / \gamma)}
\end{gathered}
$$

Now suppose $\left\{u_{n}\right\}$ is a bounded sequence in $C^{0, \gamma}([0, T], E)$. By Theorem 21.10.6 above, there is a subsequence still called $\left\{u_{n}\right\}$ which converges in $C^{0}([0, T], W)$. Thus from the above inequality

$$
\begin{aligned}
& \frac{\left\|u_{n}(t)-u_{m}(t)-\left(u_{n}(s)-u_{m}(s)\right)\right\|_{X}}{|t-s|^{\alpha}} \\
\leq & \rho_{\gamma, E}\left(u_{n}-u_{m}\right)\left\|u_{n}(t)-u_{m}(t)-\left(u_{n}(s)-u_{m}(s)\right)\right\|_{W}^{1-(\alpha / \gamma)} \\
\leq & C\left(\left\{u_{n}\right\}\right)\left(2\left\|u_{n}-u_{m}\right\|_{\infty, W}\right)^{1-(\alpha / \gamma)}
\end{aligned}
$$

which converges to 0 as $n, m \rightarrow \infty$. Thus

$$
\rho_{\alpha, X}\left(u_{n}-u_{m}\right) \rightarrow 0 \text { as } n, m \rightarrow \infty
$$

Also $\left\|u_{n}-u_{m}\right\|_{\infty, X} \rightarrow 0$ as $n, m \rightarrow \infty$ so this is a Cauchy sequence in $C^{0, \alpha}([0, T], X)$.
The next theorem is a well known result probably due to Lions, Temam, or Aubin.
Theorem 21.10.8 Let $E \subseteq W \subseteq X$ where the injection map is continuous from $W$ to $X$ and compact from $E$ to $W$. Let $p \geq 1$, let $q>1$, and define

$$
\begin{gathered}
S \equiv\left\{u \in L^{p}([a, b] ; E): \text { for some } C,\|u(t)-u(s)\|_{X} \leq C|t-s|^{1 / q}\right. \\
\text { and } \left.\|u\|_{L^{p}([a, b] ; E)} \leq R\right\} .
\end{gathered}
$$

Thus $S$ is bounded in $L^{p}([a, b] ; E)$ and Holder continuous into $X$. Then $S$ is precompact in $L^{p}([a, b] ; W)$. This means that if $\left\{u_{n}\right\}_{n=1}^{\infty} \subseteq S$, it has a subsequence $\left\{u_{n_{k}}\right\}$ which converges in $L^{p}([a, b] ; W)$.

Proof: It suffices to show $S$ has an $\eta$ net in $L^{p}([a, b] ; W)$ for each $\eta>0$.
If not, there exists $\eta>0$ and a sequence $\left\{u_{n}\right\} \subseteq S$, such that

$$
\begin{equation*}
\left\|u_{n}-u_{m}\right\| \geq \eta \tag{21.10.49}
\end{equation*}
$$

for all $n \neq m$ and the norm refers to $L^{p}([a, b] ; W)$. Let

$$
a=t_{0}<t_{1}<\cdots<t_{k}=b, t_{i}-t_{i-1}=(b-a) / k
$$

Now define

$$
\bar{u}_{n}(t) \equiv \sum_{i=1}^{k} \bar{u}_{n_{i}} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t), \bar{u}_{n_{i}} \equiv \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} u_{n}(s) d s
$$

The idea is to show that $\bar{u}_{n}$ approximates $u_{n}$ well and then to argue that a subsequence of the $\left\{\bar{u}_{n}\right\}$ is a Cauchy sequence yielding a contradiction to 21.10.49.

Therefore,

$$
\begin{gathered}
u_{n}(t)-\bar{u}_{n}(t)=\sum_{i=1}^{k} u_{n}(t) \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)-\sum_{i=1}^{k} \bar{u}_{n_{i}} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t) \\
=\sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} u_{n}(t) d s \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)-\sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} u_{n}(s) d s \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t) \\
=\sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left(u_{n}(t)-u_{n}(s)\right) d s \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)
\end{gathered}
$$

It follows from Jensen's inequality that

$$
\begin{aligned}
& \left\|u_{n}(t)-\bar{u}_{n}(t)\right\|_{W}^{p} \\
= & \sum_{i=1}^{k}\left\|\frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left(u_{n}(t)-u_{n}(s)\right) d s\right\|_{W}^{p} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t) \\
\leq & \sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left\|u_{n}(t)-u_{n}(s)\right\|_{W}^{p} d s \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)
\end{aligned}
$$

and so

$$
\begin{align*}
& \int_{a}^{b}\left\|\left(u_{n}(t)-\bar{u}_{n}(s)\right)\right\|_{W}^{p} d s \\
\leq & \int_{a}^{b} \sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left\|u_{n}(t)-u_{n}(s)\right\|_{W}^{p} d s \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t) d t \\
= & \sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t_{i}}\left\|u_{n}(t)-u_{n}(s)\right\|_{W}^{p} d s d t . \tag{21.10.50}
\end{align*}
$$

From Lemma 21.10.1 if $\varepsilon>0$, there exists $C_{\varepsilon}$ such that

$$
\begin{gathered}
\left\|u_{n}(t)-u_{n}(s)\right\|_{W}^{p} \leq \varepsilon\left\|u_{n}(t)-u_{n}(s)\right\|_{E}^{p}+C_{\varepsilon}\left\|u_{n}(t)-u_{n}(s)\right\|_{X}^{p} \\
\leq 2^{p-1} \varepsilon\left(\left\|u_{n}(t)\right\|^{p}+\left\|u_{n}(s)\right\|^{p}\right)+C_{\varepsilon}|t-s|^{p / q}
\end{gathered}
$$

This is substituted in to 21.10 .50 to obtain

$$
\begin{gathered}
\int_{a}^{b}\left\|\left(u_{n}(t)-\bar{u}_{n}(s)\right)\right\|_{W}^{p} d s \leq \\
\\
\sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t_{i}}\left(2^{p-1} \varepsilon\left(\left\|u_{n}(t)\right\|^{p}+\left\|u_{n}(s)\right\|^{p}\right)+C_{\varepsilon}|t-s|^{p / q}\right) d s d t \\
=\sum_{i=1}^{k} 2^{p} \varepsilon \int_{t_{i-1}}^{t_{i}}\left\|u_{n}(t)\right\|_{W}^{p}+\frac{C_{\varepsilon}}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t_{i}}|t-s|^{p / q} d s d t \\
\leq \quad 2^{p} \varepsilon \int_{a}^{b}\left\|u_{n}(t)\right\|^{p} d t+C_{\varepsilon} \sum_{i=1}^{k} \frac{1}{\left(t_{i}-t_{i-1}\right)}\left(t_{i}-t_{i-1}\right)^{p / q} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t_{i}} d s d t \\
= \\
2^{p} \varepsilon \int_{a}^{b}\left\|u_{n}(t)\right\|^{p} d t+C_{\varepsilon} \sum_{i=1}^{k} \frac{1}{\left(t_{i}-t_{i-1}\right)}\left(t_{i}-t_{i-1}\right)^{p / q}\left(t_{i}-t_{i-1}\right)^{2} \\
\leq \\
\leq 2^{p} \varepsilon R^{p}+C_{\varepsilon} \sum_{i=1}^{k}\left(t_{i}-t_{i-1}\right)^{1+p / q}=2^{p} \varepsilon R^{p}+C_{\varepsilon} k\left(\frac{b-a}{k}\right)^{1+p / q} .
\end{gathered}
$$

Taking $\varepsilon$ so small that $2^{p} \varepsilon R^{p}<\eta^{p} / 8^{p}$ and then choosing $k$ sufficiently large, it follows

$$
\left\|u_{n}-\bar{u}_{n}\right\|_{L^{p}([a, b] ; W)}<\frac{\eta}{4} .
$$

Thus $k$ is fixed and $\bar{u}_{n}$ at a step function with $k$ steps having values in $E$. Now use compactness of the embedding of $E$ into $W$ to obtain a subsequence such that $\left\{\bar{u}_{n}\right\}$ is Cauchy in $L^{p}(a, b ; W)$ and use this to contradict 21.10.49. The details follow.

Suppose $\bar{u}_{n}(t)=\sum_{i=1}^{k} u_{i}^{n} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)$. Thus

$$
\left\|\bar{u}_{n}(t)\right\|_{E}=\sum_{i=1}^{k}\left\|u_{i}^{n}\right\|_{E} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)
$$

and so

$$
R \geq \int_{a}^{b}\left\|\bar{u}_{n}(t)\right\|_{E}^{p} d t=\frac{T}{k} \sum_{i=1}^{k}\left\|u_{i}^{n}\right\|_{E}^{p}
$$

Therefore, the $\left\{u_{i}^{n}\right\}$ are all bounded. It follows that after taking subsequences $k$ times there exists a subsequence $\left\{u_{n_{k}}\right\}$ such that $u_{n_{k}}$ is a Cauchy sequence in $L^{p}(a, b ; W)$. You simply get a subsequence such that $u_{i}^{n_{k}}$ is a Cauchy sequence in $W$ for each $i$. Then denoting this subsequence by $n$,

$$
\begin{aligned}
\left\|u_{n}-u_{m}\right\|_{L^{p}(a, b ; W)} \leq & \left\|u_{n}-\bar{u}_{n}\right\|_{L^{p}(a, b ; W)} \\
& +\left\|\bar{u}_{n}-\bar{u}_{m}\right\|_{L^{p}(a, b ; W)}+\left\|\bar{u}_{m}-u_{m}\right\|_{L^{p}(a, b ; W)} \\
\leq & \frac{\eta}{4}+\left\|\bar{u}_{n}-\bar{u}_{m}\right\|_{L^{p}(a, b ; W)}+\frac{\eta}{4}<\eta
\end{aligned}
$$

provided $m, n$ are large enough, contradicting 21.10.49.
You can give a different version of the above to include the case where there is, instead of a Holder condition, a bound on $u^{\prime}$ for $u \in S$. It is stated next. We are assuming a situation in which

$$
\int_{a}^{b} u^{\prime}(t) d t=u(b)-u(a)
$$

This happens, for example, if $u^{\prime}$ is the weak derivative. See [117].
Corollary 21.10.9 Let $E \subseteq W \subseteq X$ where the injection map is continuous from $W$ to $X$ and compact from $E$ to $W$. Let $p \geq 1$, let $q>1$, and define

$$
\begin{gathered}
S \equiv\left\{u \in L^{p}([a, b] ; E): \text { for some } C,\|u(t)-u(s)\|_{X} \leq C|t-s|^{1 / q}\right. \\
\text { and } \left.\|u\|_{L^{p}([a, b] ; E)} \leq R\right\}
\end{gathered}
$$

Thus $S$ is bounded in $L^{p}([a, b] ; E)$ and Holder continuous into $X$. Then $S$ is precompact in $L^{p}([a, b] ; W)$. This means that if $\left\{u_{n}\right\}_{n=1}^{\infty} \subseteq S$, it has a subsequence $\left\{u_{n_{k}}\right\}$ which converges in $L^{p}([a, b] ; W)$. The same conclusion can be drawn if it is known instead of the Holder condition that $\left\|u^{\prime}\right\|_{L^{1}([a, b] ; X)}$ is bounded.

Proof: The first part was done earlier. Therefore, we just prove the new stuff which involves a bound on the $L^{1}$ norm of the derivative. It suffices to show $S$ has an $\eta$ net in $L^{p}([a, b] ; W)$ for each $\eta>0$.

If not, there exists $\eta>0$ and a sequence $\left\{u_{n}\right\} \subseteq S$, such that

$$
\begin{equation*}
\left\|u_{n}-u_{m}\right\| \geq \eta \tag{21.10.51}
\end{equation*}
$$

for all $n \neq m$ and the norm refers to $L^{p}([a, b] ; W)$. Let

$$
a=t_{0}<t_{1}<\cdots<t_{k}=b, t_{i}-t_{i-1}=(b-a) / k
$$

Now define

$$
\bar{u}_{n}(t) \equiv \sum_{i=1}^{k} \bar{u}_{n_{i}} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t), \bar{u}_{n_{i}} \equiv \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} u_{n}(s) d s
$$

The idea is to show that $\bar{u}_{n}$ approximates $u_{n}$ well and then to argue that a subsequence of the $\left\{\bar{u}_{n}\right\}$ is a Cauchy sequence yielding a contradiction to 21.10.51.

Therefore,

$$
\begin{gathered}
u_{n}(t)-\bar{u}_{n}(t)=\sum_{i=1}^{k} u_{n}(t) \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)-\sum_{i=1}^{k} \bar{u}_{n_{i}} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t) \\
=\sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} u_{n}(t) d s \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)-\sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} u_{n}(s) d s \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t) \\
=\sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left(u_{n}(t)-u_{n}(s)\right) d s \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)
\end{gathered}
$$

It follows from Jensen's inequality that

$$
\begin{aligned}
& \left\|u_{n}(t)-\bar{u}_{n}(t)\right\|_{W}^{p} \\
= & \sum_{i=1}^{k}\left\|\frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left(u_{n}(t)-u_{n}(s)\right) d s\right\|_{W}^{p} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)
\end{aligned}
$$

And so

$$
\begin{align*}
& \int_{0}^{T}\left\|u_{n}(t)-\bar{u}_{n}(t)\right\|_{W}^{p} d t=\sum_{i=1}^{k} \int_{t_{i-1}}^{t_{i}}\left\|\frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left(u_{n}(t)-u_{n}(s)\right) d s\right\|_{W}^{p} d t \\
& \leq \sum_{i=1}^{k} \int_{t_{i-1}}^{t_{i}} \varepsilon\left\|\frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left(u_{n}(t)-u_{n}(s)\right) d s\right\|_{E}^{p} d t \\
&+C_{\varepsilon} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_{i}}\left\|\frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left(u_{n}(t)-u_{n}(s)\right) d s\right\|_{X}^{p} d t \tag{21.10.52}
\end{align*}
$$

Consider the second of these. It equals

$$
C_{\varepsilon} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_{i}}\left\|\frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} \int_{s}^{t} u_{n}^{\prime}(\tau) d \tau d s\right\|_{X}^{p} d t
$$

This is no larger than

$$
\begin{gathered}
\leq C_{\varepsilon} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_{i}}\left(\frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t_{i}}\left\|u_{n}^{\prime}(\tau)\right\|_{X} d \tau d s\right)^{p} d t \\
=C_{\varepsilon} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_{i}}\left(\int_{t_{i-1}}^{t_{i}}\left\|u_{n}^{\prime}(\tau)\right\|_{X} d \tau\right)^{p} d t \\
=C_{\varepsilon} \sum_{i=1}^{k}\left(\left(t_{i}-t_{i-1}\right)^{1 / p} \int_{t_{i-1}}^{t_{i}}\left\|u_{n}^{\prime}(\tau)\right\|_{X} d \tau\right)^{p}
\end{gathered}
$$

Since $\frac{b-a}{k}=t_{i}-t_{i-1}$,

$$
\begin{aligned}
& =C_{\varepsilon}\left(\sum_{i=1}^{k}\left(t_{i}-t_{i-1}\right)^{1 / p} \int_{t_{i-1}}^{t_{i}}\left\|u_{n}^{\prime}(\tau)\right\|_{X} d \tau\right)^{p} \\
& =C_{\varepsilon}\left(\sum_{i=1}^{k}\left(\frac{b-a}{k}\right)^{1 / p} \int_{t_{i-1}}^{t_{i}}\left\|u_{n}^{\prime}(\tau)\right\|_{X} d \tau\right)^{p} \\
& \leq \frac{C_{\varepsilon}(b-a)}{k}\left(\sum_{i=1}^{k} \int_{t_{i-1}}^{t_{i}}\left\|u_{n}^{\prime}(\tau)\right\|_{X} d \tau\right)^{p} \\
& =\frac{C_{\varepsilon}(b-a)}{k}\left(\left\|u_{n}^{\prime}\right\|_{L^{1}([a, b], X)}\right)^{p}<\frac{\eta^{p}}{10^{p}}
\end{aligned}
$$

if $k$ is chosen large enough. Now consider the first in 21.10.52. By Jensen's inequality

$$
\begin{gathered}
\sum_{i=1}^{k} \int_{t_{i-1}}^{t_{i}} \varepsilon\left\|\frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left(u_{n}(t)-u_{n}(s)\right) d s\right\|_{E}^{p} d t \leq \\
\sum_{i=1}^{k} \int_{t_{i-1}}^{t_{i}} \varepsilon \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left\|u_{n}(t)-u_{n}(s)\right\|_{E}^{p} d s d t \\
\leq \quad \varepsilon 2^{p-1} \sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t_{i}}\left(\left\|u_{n}(t)\right\|^{p}+\left\|u_{n}(s)\right\|^{p}\right) d s d t \\
=\quad 2 \varepsilon 2^{p-1} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_{i}}\left(\left\|u_{n}(t)\right\|^{p}\right) d t=\varepsilon(2)\left(2^{p-1}\right)\left\|u_{n}\right\|_{L^{p}([a, b], E)} \leq M \varepsilon
\end{gathered}
$$

Now pick $\varepsilon$ sufficiently small that $M \varepsilon<\frac{\eta^{p}}{10^{p}}$ and then $k$ large enough that the second term in 21.10 .52 is also less than $\eta^{p} / 10^{p}$. Then it will follow that

$$
\left\|\bar{u}_{n}-u_{n}\right\|_{L^{p}([a, b], W)}<\left(\frac{2 \eta^{p}}{10^{p}}\right)^{1 / p}=2^{1 / p} \frac{\eta}{10} \leq \frac{\eta}{5}
$$

Thus $k$ is fixed and $\bar{u}_{n}$ at a step function with $k$ steps having values in $E$. Now use compactness of the embedding of $E$ into $W$ to obtain a subsequence such that $\left\{\bar{u}_{n}\right\}$ is Cauchy in $L^{p}([a, b] ; W)$ and use this to contradict 21.10.51. The details follow.

Suppose $\bar{u}_{n}(t)=\sum_{i=1}^{k} u_{i}^{n} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)$. Thus

$$
\left\|\bar{u}_{n}(t)\right\|_{E}=\sum_{i=1}^{k}\left\|u_{i}^{n}\right\|_{E} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)
$$

and so

$$
R \geq \int_{a}^{b}\left\|\bar{u}_{n}(t)\right\|_{E}^{p} d t=\frac{T}{k} \sum_{i=1}^{k}\left\|u_{i}^{n}\right\|_{E}^{p}
$$

Therefore, the $\left\{u_{i}^{n}\right\}$ are all bounded. It follows that after taking subsequences $k$ times there exists a subsequence $\left\{u_{n_{k}}\right\}$ such that $u_{n_{k}}$ is a Cauchy sequence in $L^{p}([a, b] ; W)$. You simply get a subsequence such that $u_{i}^{n_{k}}$ is a Cauchy sequence in $W$ for each $i$. Then denoting this subsequence by $n$,

$$
\begin{aligned}
\left\|u_{n}-u_{m}\right\|_{L^{p}(a, b ; W)} \leq & \left\|u_{n}-\bar{u}_{n}\right\|_{L^{p}(a, b ; W)} \\
& +\left\|\bar{u}_{n}-\bar{u}_{m}\right\|_{L^{p}(a, b ; W)}+\left\|\bar{u}_{m}-u_{m}\right\|_{L^{p}(a, b ; W)} \\
\leq & \frac{\eta}{4}+\left\|\bar{u}_{n}-\bar{u}_{m}\right\|_{L^{p}(a, b ; W)}+\frac{\eta}{4}<\eta
\end{aligned}
$$

provided $m, n$ are large enough, contradicting 21.10.51.

### 21.11 Exercises

1. Show $L^{1}(\mathbb{R})$ is not reflexive. Hint: $L^{1}(\mathbb{R})$ is separable. What about $L^{\infty}(\mathbb{R})$ ?
2. If $f \in L^{1}\left(\mathbb{R}^{n} ; X\right)$ for $X$ a Banach space, does the usual fundamental theorem of calculus work? That is, can you say $\lim _{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} f(t) d m=f(x)$ a.e.?
3. Does the Vitali convergence theorem hold for Bochner integrable functions? If so, give a statement of the appropriate theorem and a proof.
4. Suppose $g \in L^{1}([a, b] ; X)$ where $X$ is a Banach space. Then if $\int_{a}^{b} g(t) \phi(t) d t=0$ for all $\phi \in C_{c}^{\infty}(a, b)$, then $g(t)=0$ a.e. Show that this is the case. Hint: It will likely depend on the regularity properties of Lebesgue measure.
5. Suppose $f \in L^{1}(a, b ; X)$ and for all $\phi \in C_{c}^{\infty}(a, b), \int_{a}^{b} f(t) \phi^{\prime}(t) d t=0$.Then there exists a constant, $a \in X$ such that $f(t)=a$ a.e. Hint: Let

$$
\psi_{\phi}(x) \equiv \int_{a}^{x}\left[\phi(t)-\left(\int_{a}^{b} \phi(y) d y\right) \phi_{0}(t)\right] d t, \phi_{0} \in C_{c}^{\infty}(a, b), \int_{a}^{b} \phi_{0}(x) d x=1
$$

Then explain why $\psi_{\phi} \in C_{c}^{\infty}(a, b), \psi_{\phi}^{\prime}=\phi-\left(\int_{a}^{b} \phi(y) d y\right) \phi_{0}$. Then use the assumption on $\psi_{\phi}$. Next use the above problem. Verify that

$$
f(y)=\int_{a}^{b} f(t) \phi_{0}(t) d t \text { a.e. } y
$$

6. Let $f \in L^{1}([a, b], X)$. Then we say that the weak derivative of $f$ is in $L^{1}([a, b], X)$ if there is a function denoted as $f^{\prime} \in L^{1}([a, b], X)$ such that for all $\phi \in C_{c}^{\infty}(a, b)$,

$$
-\int_{a}^{b} f(t) \phi^{\prime}(t) d t=\int_{a}^{b} f^{\prime}(t) \phi(t) d t
$$

Show that this definition is well defined. Next, using the above problems, show that if $f, f^{\prime} \in L^{1}([a, b], X)$, it follows that there is a continuous function, denoted by $t \rightarrow \hat{f}(t)$ such that $\hat{f}(t)=f(t)$ a.e. $t$ and

$$
\hat{f}(t)=\hat{f}(a)+\int_{0}^{t} f^{\prime}(s) d s
$$

Thus, unlike the classical definition of the derivative, when a function and its derivative are both in $L^{1}$, it has a representative $\hat{f}$ which equals the function a.e. such that $\hat{f}$ can be recovered from its derivative. Recall the well known example of this not working out which is based on the Cantor function which you should see in a real analysis course. This function had zero derivative a.e. and yet it climbed from 0 to 1 on the unit interval. Thus one could not recover it from integrating its classical derivative.

## Chapter 22

## The Derivative

### 22.1 Limits Of A Function

As in the case of scalar valued functions of one variable, a concept closely related to continuity is that of the limit of a function. The notion of limit of a function makes sense at points $\mathbf{x}$, which are limit points of $D(\mathbf{f})$ and this concept is defined next. In all that follows $(V,\|\cdot\|)$ and $(W,\|\cdot\|)$ are two normed linear spaces. Recall the definition of limit point first.

Definition 22.1.1 Let $A \subseteq W$ be a set. A point $\mathbf{x}$, is a limit point of $A$ if $B(\mathbf{x}, r)$ contains infinitely many points of A for every $r>0$.

Definition 22.1.2 Let $\mathbf{f}: D(\mathbf{f}) \subseteq V \rightarrow W$ be a function and let $\mathbf{x}$ be a limit point of $D(\mathbf{f})$. Then

$$
\lim _{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{f}(\mathbf{y})=\mathbf{L}
$$

if and only if the following condition holds. For all $\varepsilon>0$ there exists $\delta>0$ such that if

$$
0<\|\mathbf{y}-\mathbf{x}\|<\delta, \text { and } \mathbf{y} \in D(\mathbf{f})
$$

then,

$$
\|\mathbf{L}-\mathbf{f}(\mathbf{y})\|<\varepsilon
$$

Theorem 22.1.3 If $\lim _{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{f}(\mathbf{y})=\mathbf{L}$ and $\lim _{y \rightarrow x} \mathbf{f}(\mathbf{y})=\mathbf{L}_{1}$, then $\mathbf{L}=\mathbf{L}_{1}$.
Proof: Let $\varepsilon>0$ be given. There exists $\delta>0$ such that if $0<|\mathbf{y}-\mathbf{x}|<\delta$ and $\mathbf{y} \in D(\mathbf{f})$, then

$$
\|\mathbf{f}(\mathbf{y})-\mathbf{L}\|<\varepsilon,\left\|\mathbf{f}(\mathbf{y})-\mathbf{L}_{1}\right\|<\varepsilon .
$$

Pick such a $\mathbf{y}$. There exists one because $\mathbf{x}$ is a limit point of $D(\mathbf{f})$. Then

$$
\left\|\mathbf{L}-\mathbf{L}_{1}\right\| \leq\|\mathbf{L}-\mathbf{f}(\mathbf{y})\|+\left\|\mathbf{f}(\mathbf{y})-\mathbf{L}_{1}\right\|<\varepsilon+\varepsilon=2 \varepsilon
$$

Since $\varepsilon>0$ was arbitrary, this shows $\mathbf{L}=\mathbf{L}_{1}$.
As in the case of functions of one variable, one can define $\lim _{\mathbf{y} \rightarrow \mathbf{x}} f(\mathbf{x})= \pm \infty$.
Definition 22.1.4 If $f(\mathbf{x}) \in \mathbb{R}, \lim _{\mathbf{y} \rightarrow \mathbf{x}} f(\mathbf{x})=\infty$ if for every number $l$, there exists $\delta>0$ such that whenever $\|\mathbf{y}-\mathbf{x}\|<\delta$ and $\mathbf{y} \in D(\mathbf{f})$, then $f(\mathbf{x})>l . \lim _{\mathbf{y} \rightarrow \mathbf{x}} f(\mathbf{x})=-\infty$ if for every number $l$, there exists $\delta>0$ such that whenever $\|\mathbf{y}-\mathbf{x}\|<\delta$ and $\mathbf{y} \in D(\mathbf{f})$, then $f(\mathbf{x})<l$.

The following theorem is just like the one variable version of calculus.
Theorem 22.1.5 Suppose $\mathbf{f}: D(\mathbf{f}) \subseteq V \rightarrow \mathbb{F}^{m}$. Then for $\mathbf{x}$ a limit point of $D(\mathbf{f})$,

$$
\begin{equation*}
\lim _{y \rightarrow x} \mathbf{f}(\mathbf{y})=\mathbf{L} \tag{22.1.1}
\end{equation*}
$$

if and only if

$$
\begin{equation*}
\lim _{y \rightarrow x} f_{k}(\mathbf{y})=L_{k} \tag{22.1.2}
\end{equation*}
$$

where $\mathbf{f}(\mathbf{y}) \equiv\left(f_{1}(\mathbf{y}), \cdots, f_{p}(\mathbf{y})\right)$ and $\mathbf{L} \equiv\left(L_{1}, \cdots, L_{p}\right)$.
Suppose here that $f$ has values in $W$, a normed linear space and

$$
\lim _{y \rightarrow \mathbf{x}} f(y)=L, \lim _{y \rightarrow \mathbf{x}} g(y)=K
$$

where $K, L \in W$. Then if $a, b \in \mathbb{F}$,

$$
\begin{equation*}
\lim _{y \rightarrow x}(a f(y)+b g(y))=a L+b K \tag{22.1.3}
\end{equation*}
$$

If $W$ is an inner product space,

$$
\begin{equation*}
\lim _{y \rightarrow x}(f, g)(y)=(L, K) \tag{22.1.4}
\end{equation*}
$$

If $g$ is scalar valued with $\lim _{y \rightarrow x} g(y)=K$,

$$
\begin{equation*}
\lim _{y \rightarrow x} f(y) g(y)=L K \tag{22.1.5}
\end{equation*}
$$

Also, if $h$ is a continuous function defined near $L$, then

$$
\begin{equation*}
\lim _{y \rightarrow x} h \circ f(y)=h(L) . \tag{22.1.6}
\end{equation*}
$$

Suppose $\lim _{y \rightarrow \mathbf{x}} f(y)=L$. If $\|f(y)-b\| \leq r$ for all $y$ sufficiently close to $\mathbf{x}$, then $|L-b| \leq r$ also.

Proof: Suppose 22.1.1. Then letting $\varepsilon>0$ be given there exists $\delta>0$ such that if $0<\|y-x\|<\delta$, it follows

$$
\left|f_{k}(y)-L_{k}\right| \leq\|\mathbf{f}(y)-\mathbf{L}\|<\varepsilon
$$

which verifies 22.1.2.
Now suppose 22.1.2 holds. Then letting $\varepsilon>0$ be given, there exists $\delta_{k}$ such that if $0<\|y-x\|<\delta_{k}$, then

$$
\left|f_{k}(y)-L_{k}\right|<\varepsilon
$$

Let $0<\delta<\min \left(\delta_{1}, \cdots, \delta_{p}\right)$. Then if $0<\|y-x\|<\delta$, it follows

$$
\|\mathbf{f}(y)-\mathbf{L}\|_{\infty}<\varepsilon
$$

Any other norm on $\mathbb{F}^{m}$ would work out the same way because the norms are all equivalent.
Each of the remaining assertions follows immediately from the coordinate descriptions of the various expressions and the first part. However, I will give a different argument for these.

The proof of 22.1.3 is left for you. Now 22.1.4 is to be verified. Let $\varepsilon>0$ be given. Then by the triangle inequality,

$$
\begin{aligned}
|(f, g)(y)-(L, K)| & \leq|(f, g)(y)-(f(y), K)|+|(f(y), K)-(L, K)| \\
& \leq\|f(y)\|\|g(y)-K\|+\|K\|\|f(y)-L\|
\end{aligned}
$$

There exists $\delta_{1}$ such that if $0<\|y-x\|<\delta_{1}$ and $y \in D(f)$, then

$$
\|f(y)-L\|<1
$$

and so for such $y$, the triangle inequality implies, $\|f(y)\|<1+\|L\|$. Therefore, for $0<$ $\|y-x\|<\delta_{1}$,

$$
\begin{equation*}
|(f, g)(y)-(L, K)| \leq(1+\|K\|+\|L\|)[\|g(y)-K\|+\|f(y)-L\|] . \tag{22.1.7}
\end{equation*}
$$

Now let $0<\delta_{2}$ be such that if $y \in D(f)$ and $0<\|x-y\|<\delta_{2}$,

$$
\|f(y)-L\|<\frac{\varepsilon}{2(1+\|K\|+\|L\|)},\|g(y)-K\|<\frac{\varepsilon}{2(1+\|K\|+\|L\|)}
$$

Then letting $0<\boldsymbol{\delta} \leq \min \left(\boldsymbol{\delta}_{1}, \boldsymbol{\delta}_{2}\right)$, it follows from 22.1.7 that

$$
|(f, g)(y)-(L, K)|<\varepsilon
$$

and this proves 22.1.4.
The proof of 22.1.5 is left to you.
Consider 22.1.6. Since $h$ is continuous near $L$, it follows that for $\varepsilon>0$ given, there exists $\eta>0$ such that if $\|y-L\|<\eta$, then

$$
\|h(y)-h(L)\|<\varepsilon
$$

Now since $\lim _{y \rightarrow x} f(y)=L$, there exists $\delta>0$ such that if $0<\|y-x\|<\delta$, then

$$
\|f(y)-L\|<\eta
$$

Therefore, if $0<\|y-x\|<\delta$,

$$
\|h(f(y))-h(L)\|<\varepsilon
$$

It only remains to verify the last assertion. Assume $\|f(y)-b\| \leq r$. It is required to show that $\|L-b\| \leq r$. If this is not true, then $\|L-b\|>r$. Consider $B(L,\|L-b\|-r)$. Since $L$ is the limit of $f$, it follows $f(y) \in B(L,\|L-b\|-r)$ whenever $y \in D(f)$ is close enough to $x$. Thus, by the triangle inequality,

$$
\|f(y)-L\|<\|L-b\|-r
$$

and so

$$
\begin{aligned}
r & <\|L-b\|-\|f(y)-L\| \leq|\|b-L\|-\|f(y)-L\|| \\
& \leq\|b-f(y)\|
\end{aligned}
$$

a contradiction to the assumption that $\|b-f(y)\| \leq r$.
The relation between continuity and limits is as follows.

Theorem 22.1.6 For $f: D(f) \rightarrow W$ and $x \in D(f)$ a limit point of $D(f), f$ is continuous at $x$ if and only if

$$
\lim _{y \rightarrow x} f(y)=f(x)
$$

Proof: First suppose $f$ is continuous at $x$ a limit point of $D(f)$. Then for every $\varepsilon>0$ there exists $\delta>0$ such that if $\|x-y\|<\delta$ and $y \in D(f)$, then $|f(x)-f(y)|<\varepsilon$. In particular, this holds if $0<\|x-y\|<\delta$ and this is just the definition of the limit. Hence $f(x)=\lim _{y \rightarrow x} f(y)$.

Next suppose $x$ is a limit point of $D(f)$ and $\lim _{y \rightarrow x} f(y)=f(x)$. This means that if $\varepsilon>$ 0 there exists $\delta>0$ such that for $0<\|x-y\|<\delta$ and $y \in D(f)$, it follows $|f(y)-f(x)|<$ $\varepsilon$. However, if $y=x$, then $|f(y)-f(x)|=|f(x)-f(x)|=0$ and so whenever $y \in D(f)$ and $\|x-y\|<\delta$, it follows $|f(x)-f(y)|<\varepsilon$, showing $f$ is continuous at $x$.

Example 22.1.7 Find $\lim _{(x, y) \rightarrow(3,1)}\left(\frac{x^{2}-9}{x-3}, y\right)$.
It is clear that $\lim _{(x, y) \rightarrow(3,1)} \frac{x^{2}-9}{x-3}=6$ and $\lim _{(x, y) \rightarrow(3,1)} y=1$. Therefore, this limit equals $(6,1)$.

Example 22.1.8 Find $\lim _{(x, y) \rightarrow(0,0)} \frac{x y}{x^{2}+y^{2}}$.
First of all, observe the domain of the function is $\mathbb{R}^{2} \backslash\{(0,0)\}$, every point in $\mathbb{R}^{2}$ except the origin. Therefore, $(0,0)$ is a limit point of the domain of the function so it might make sense to take a limit. However, just as in the case of a function of one variable, the limit may not exist. In fact, this is the case here. To see this, take points on the line $y=0$. At these points, the value of the function equals 0 . Now consider points on the line $y=x$ where the value of the function equals $1 / 2$. Since, arbitrarily close to $(0,0)$, there are points where the function equals $1 / 2$ and points where the function has the value 0 , it follows there can be no limit. Just take $\varepsilon=1 / 10$ for example. You cannot be within $1 / 10$ of $1 / 2$ and also within $1 / 10$ of 0 at the same time.

Note it is necessary to rely on the definition of the limit much more than in the case of a function of one variable and there are no easy ways to do limit problems for functions of more than one variable. It is what it is and you will not deal with these concepts without suffering and anguish.

### 22.2 Basic Definitions

The concept of derivative generalizes right away to functions defined on a normed linear space. However, no attempt will be made to consider derivatives from one side or another. This is because there isn't a well defined side. However, it is certainly the case that there are more general notions which include such things. I will present a fairly general notion of the derivative of a function which is defined on a normed vector space which has values in a normed vector space.

In what follows, $X, Y$ will denote normed vector spaces. Recall that $\mathscr{L}(X, Y)$ will denote the bounded linear transformations from $X$ to $Y$.

Let $U$ be an open set in $X$, and let $\mathbf{f}: U \rightarrow Y$ be a function.

Definition 22.2.1 A function $\mathbf{g}$ is $\mathbf{O}(\mathbf{v})$ if

$$
\begin{equation*}
\lim _{\|\mathbf{v}\| \rightarrow 0} \frac{\mathbf{g}(\mathbf{v})}{\|\mathbf{v}\|}=\mathbf{0} \tag{22.2.8}
\end{equation*}
$$

A function $\mathbf{f}: U \rightarrow Y$ is differentiable at $\mathbf{x} \in U$ if there exists a linear transformation $L \in$ $\mathscr{L}(X, Y)$ such that

$$
\mathbf{f}(\mathbf{x}+\mathbf{v})=\mathbf{f}(\mathbf{x})+L \mathbf{v}+\mathbf{o}(\mathbf{v})
$$

This linear transformation $L$ is the definition of $D \mathbf{f}(\mathbf{x})$. This derivative is often called the Frechet derivative.

In finite dimensions, the question whether a given function is differentiable is independent of the norm used on the finite dimensional vector space. That is, a function is differentiable with one norm if and only if it is differentiable with another norm. This is because all norms are equivalent on a finite dimensional space.

The definition 22.2.8 means the error,

$$
\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})-L \mathbf{v}
$$

converges to $\mathbf{0}$ faster than $\|\mathbf{v}\|$. Thus the above definition is equivalent to saying

$$
\begin{equation*}
\lim _{\|\mathbf{v}\| \rightarrow 0} \frac{\|\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})-L \mathbf{v}\|}{\|\mathbf{v}\|}=0 \tag{22.2.9}
\end{equation*}
$$

or equivalently,

$$
\begin{equation*}
\lim _{\mathbf{y} \rightarrow \mathbf{x}} \frac{\|\mathbf{f}(\mathbf{y})-\mathbf{f}(\mathbf{x})-D \mathbf{f}(\mathbf{x})(\mathbf{y}-\mathbf{x})\|}{\|\mathbf{y}-\mathbf{x}\|}=0 \tag{22.2.10}
\end{equation*}
$$

The symbol $\mathbf{0}(\mathbf{v})$ should be thought of as an adjective. Thus, if $t$ and $k$ are constants,

$$
\mathbf{o}(\mathbf{v})=\mathbf{o}(\mathbf{v})+\mathbf{o}(\mathbf{v}), \mathbf{o}(t \mathbf{v})=\mathbf{o}(\mathbf{v}), k \mathbf{o}(\mathbf{v})=\mathbf{o}(\mathbf{v})
$$

and other similar observations hold.
Theorem 22.2.2 The derivative is well defined.
Proof: First note that for a fixed vector $\mathbf{v}, \mathbf{o}(t \mathbf{v})=\mathbf{o}(t)$. This is because

$$
\lim _{t \rightarrow 0} \frac{\mathbf{o}(t \mathbf{v})}{|t|}=\lim _{t \rightarrow 0}\|\mathbf{v}\| \frac{\mathbf{o}(t \mathbf{v})}{\|t \mathbf{v}\|}=\mathbf{0}
$$

Now suppose both $L_{1}$ and $L_{2}$ work in the above definition. Then let $\mathbf{v}$ be any vector and let $t$ be a real scalar which is chosen small enough that $t \mathbf{v}+\mathbf{x} \in U$. Then

$$
\mathbf{f}(\mathbf{x}+t \mathbf{v})=\mathbf{f}(\mathbf{x})+L_{1} t \mathbf{v}+\mathbf{o}(t \mathbf{v}), \mathbf{f}(\mathbf{x}+t \mathbf{v})=\mathbf{f}(\mathbf{x})+L_{2} t \mathbf{v}+\mathbf{o}(t \mathbf{v})
$$

Therefore, subtracting these two yields $\left(L_{2}-L_{1}\right)(t \mathbf{v})=\mathbf{o}(t \mathbf{v})=\mathbf{o}(t)$. Therefore, dividing by $t$ yields $\left(L_{2}-L_{1}\right)(\mathbf{v})=\frac{\mathbf{o}(t)}{t}$. Now let $t \rightarrow 0$ to conclude that $\left(L_{2}-L_{1}\right)(\mathbf{v})=0$. Since this is true for all $\mathbf{v}$, it follows $L_{2}=L_{1}$. This proves the theorem.

Lemma 22.2.3 Let $\mathbf{f}$ be differentiable at $\mathbf{x}$. Then $\mathbf{f}$ is continuous at $\mathbf{x}$ and in fact, there exists $K>0$ such that whenever $\|\mathbf{v}\|$ is small enough,

$$
\|\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})\| \leq K\|\mathbf{v}\|
$$

Also if $\mathbf{f}$ is differentiable at $\mathbf{x}$, then

$$
\mathbf{o}(\|\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})\|)=\mathbf{o}(\mathbf{v})
$$

Proof: From the definition of the derivative,

$$
\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})=D \mathbf{f}(\mathbf{x}) \mathbf{v}+\mathbf{o}(\mathbf{v}) .
$$

Let $\|\mathbf{v}\|$ be small enough that $\frac{\mathbf{o}(\|\mathbf{v}\|)}{\|\mathbf{v}\|}<1$ so that $\|\mathbf{o}(\mathbf{v})\| \leq\|\mathbf{v}\|$. Then for such $\mathbf{v}$,

$$
\begin{aligned}
\|\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})\| & \leq\|D \mathbf{f}(\mathbf{x}) \mathbf{v}\|+\|\mathbf{v}\| \\
& \leq(\|D \mathbf{f}(\mathbf{x})\|+1)\|\mathbf{v}\|
\end{aligned}
$$

This proves the lemma with $K=\|D \mathbf{f}(\mathbf{x})\|+1$. Recall the operator norm discussed in Definition 17.1.5.

The last assertion is implied by the first as follows. Define

$$
\mathbf{h}(\mathbf{v}) \equiv\left\{\begin{array}{l}
\frac{\mathbf{o}(\|\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})\|)}{\|\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})\|} \text { if }\|\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})\| \neq 0 \\
\mathbf{0} \text { if }\|\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})\|=0
\end{array}\right.
$$

Then $\lim _{\|\mathbf{v}\| \rightarrow 0} \mathbf{h}(\mathbf{v})=\mathbf{0}$ from continuity of $\mathbf{f}$ at $\mathbf{x}$ which is implied by the first part. Also from the above estimate,

$$
\left\|\frac{\mathbf{o}(\|\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})\|)}{\|\mathbf{v}\|}\right\|=\|\mathbf{h}(\mathbf{v})\| \frac{\|\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})\|}{\|\mathbf{v}\|} \leq\|\mathbf{h}(\mathbf{v})\|(\|D \mathbf{f}(\mathbf{x})\|+1)
$$

This establishes the second claim.
Here $\|D \mathbf{f}(\mathbf{x})\|$ is the operator norm of the linear transformation $D \mathbf{f}(\mathbf{x})$.

### 22.3 The Chain Rule

With the above lemma, it is easy to prove the chain rule.
Theorem 22.3.1 (The chain rule) Let $U$ and $V$ be open sets $U \subseteq X$ and $V \subseteq Y$. Suppose $\mathbf{f}: U \rightarrow V$ is differentiable at $\mathbf{x} \in U$ and suppose $\mathbf{g}: V \rightarrow \mathbb{F}^{q}$ is differentiable at $\mathbf{f}(\mathbf{x}) \in V$. Then $\mathbf{g} \circ \mathbf{f}$ is differentiable at $\mathbf{x}$ and

$$
D(\mathbf{g} \circ \mathbf{f})(\mathbf{x})=D \mathbf{g}(\mathbf{f}(\mathbf{x})) D \mathbf{f}(\mathbf{x})
$$

Proof: This follows from a computation. Let $B(\mathbf{x}, r) \subseteq U$ and let $r$ also be small enough that for $\|\mathbf{v}\| \leq r$, it follows that $\mathbf{f}(\mathbf{x}+\mathbf{v}) \in V$. Such an $r$ exists because $\mathbf{f}$ is continuous at $\mathbf{x}$. For $\|\mathbf{v}\|<r$, the definition of differentiability of $\mathbf{g}$ and $\mathbf{f}$ implies

$$
\mathbf{g}(\mathbf{f}(\mathbf{x}+\mathbf{v}))-\mathbf{g}(\mathbf{f}(\mathbf{x}))=
$$

$$
\begin{align*}
& D \mathbf{g}(\mathbf{f}(\mathbf{x}))(\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x}))+\mathbf{o}(\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})) \\
= & D \mathbf{g}(\mathbf{f}(\mathbf{x}))[D \mathbf{f}(\mathbf{x}) \mathbf{v}+\mathbf{o}(\mathbf{v})]+\mathbf{o}(\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})) \\
= & D \mathbf{g}(\mathbf{f}(\mathbf{x})) D \mathbf{f}(\mathbf{x}) \mathbf{v}+\mathbf{o}(\mathbf{v})+\mathbf{o}(\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x}))  \tag{22.3.11}\\
= & D \mathbf{g}(\mathbf{f}(\mathbf{x})) D \mathbf{f}(\mathbf{x}) \mathbf{v}+\mathbf{o}(\mathbf{v})
\end{align*}
$$

By Lemma 22.2.3. From the definition of the derivative $D(\mathbf{g} \circ \mathbf{f})(\mathbf{x})$ exists and equals $D \mathbf{g}(\mathbf{f}(\mathbf{x})) D \mathbf{f}(\mathbf{x})$.

### 22.4 The Derivative Of A Compact Mapping

Here is a little definition about compact mappings. It turns out that if you have a differentiable mapping which is also compact, then the derivative must also be compact.

Definition 22.4.1 Let $C \in \mathscr{L}(X, Y)$. It is said to be compact if it takes bounded sets to precompact sets. If $f$ is a function defined on an open subset $U$ of $X$, then $f$ is called compact if $f$ (bounded set $)=($ precompact $)$.

Theorem 22.4.2 Let $f: U \subseteq X \rightarrow Y$ where $f$ takes bounded sets to precompact sets. Then $D f(x)$ also takes bounded sets in $X$ to precompact sets in $Y$.

Proof: If this is not so, then there exists a bounded set $B$ in $X$ and for some $\varepsilon>0$ a sequence of points $D f(x) b_{n}$ such that all these points are further apart than $\varepsilon$. Without loss of generality, one can assume $B=B(0, r)$, a ball. In fact, one can assume that $r>0$ is as small as desired because if $D f(x) B(0, r)$ is precompact, then so is $D f(x) B(0, R), R>$ $r$. Just get an $\varepsilon \frac{r}{R}$ net $\left\{D f(x) x_{n}\right\}_{n=1}^{N}$ for $D f(x) B(0, r)$ and consider $\left\{\frac{R}{r} D f(x) x_{n}\right\}_{n=1}^{N}$. $\cup_{n} B\left(D f(x) x_{n}, \varepsilon \frac{r}{R}\right)$ covers $D f(x) B(0, R)$, so $\cup_{n} B\left(\frac{R}{r} D f(x) x_{n}, \varepsilon\right)$ covers $D f(x) B(0, R)$.

Choose $r$ very small so that $r<\varepsilon / 4$ and

$$
f\left(x+x_{n}\right)-f(x)=D f(x) x_{n}+o\left(x_{n}\right),\left\|o\left(x_{n}\right)\right\|<\left\|x_{n}\right\|
$$

and there are infinitely many $D f(x) x_{n}$ further apart than $\varepsilon, x_{n} \in B(0, r)$. Then consider $B(x, r)$ and $\left\{f\left(x+x_{n}\right)\right\}_{n=1}^{\infty}$.

$$
\begin{aligned}
\left\|f\left(x+x_{n}\right)-f\left(x+x_{m}\right)\right\| & \geq\left\|D f(x) x_{n}-D f(x) x_{m}\right\|-\left\|o\left(x_{n}\right)-o\left(x_{m}\right)\right\| \\
& \geq \varepsilon-2 \frac{\varepsilon}{4}=\frac{\varepsilon}{2}
\end{aligned}
$$

contradicting the assertion that $f$ takes bounded sets to precompact sets.

### 22.5 The Matrix Of The Derivative

The case of most interest here is the only one I will discuss. It is the case where $X=\mathbb{R}^{n}$ and $Y=\mathbb{R}^{m}$, the function being defined on an open subset of $\mathbb{R}^{n}$. Of course this all generalizes to arbitrary vector spaces and one considers the matrix taken with respect to various bases. As above, $\mathbf{f}$ will be defined and differentiable on an open set $U \subseteq \mathbb{R}^{n}$.

The matrix of $D \mathbf{f}(\mathbf{x})$ is the matrix having the $i^{t h}$ column equal to $D \mathbf{f}(\mathbf{x}) \mathbf{e}_{i}$ and so it is only necessary to compute this. Let $t$ be a small real number such that both

$$
\frac{\mathbf{f}\left(\mathbf{x}+t \mathbf{e}_{i}\right)-\mathbf{f}(\mathbf{x})-D \mathbf{f}(\mathbf{x})\left(t \mathbf{e}_{i}\right)}{t}=\frac{\mathbf{o}(t)}{t}
$$

Therefore,

$$
\frac{\mathbf{f}\left(\mathbf{x}+t \mathbf{e}_{i}\right)-\mathbf{f}(\mathbf{x})}{t}=D \mathbf{f}(\mathbf{x})\left(\mathbf{e}_{i}\right)+\frac{\mathbf{o}(t)}{t}
$$

The limit exists on the right and so it exists on the left also. Thus

$$
\frac{\partial \mathbf{f}(\mathbf{x})}{\partial x_{i}} \equiv \lim _{t \rightarrow 0} \frac{\mathbf{f}\left(\mathbf{x}+t \mathbf{e}_{i}\right)-\mathbf{f}(\mathbf{x})}{t}=D \mathbf{f}(\mathbf{x})\left(\mathbf{e}_{i}\right)
$$

and so the matrix of the derivative is just the matrix which has the $i^{t h}$ column equal to the $i^{\text {th }}$ partial derivative of $\mathbf{f}$. Note that this shows that whenever $\mathbf{f}$ is differentiable, it follows that the partial derivatives all exist. It does not go the other way however as discussed later.

Theorem 22.5.1 Let $\mathbf{f}: U \subseteq \mathbb{F}^{n} \rightarrow \mathbb{F}^{m}$ and suppose $\mathbf{f}$ is differentiable at $\mathbf{x}$. Then all the partial derivatives $\frac{\partial f_{i}(\mathbf{x})}{\partial x_{j}}$ exist and if $J \mathbf{f}(\mathbf{x})$ is the matrix of the linear transformation, $D \mathbf{f}(\mathbf{x})$ with respect to the standard basis vectors, then the $i j^{\text {th }}$ entry is given by $\frac{\partial f_{i}}{\partial x_{j}}(\mathbf{x})$ also denoted as $f_{i, j}$ or $f_{i, x_{j}}$. It is the matrix whose $i^{\text {th }}$ column is

$$
\frac{\partial \mathbf{f}(\mathbf{x})}{\partial x_{i}} \equiv \lim _{t \rightarrow 0} \frac{\mathbf{f}\left(\mathbf{x}+t \mathbf{e}_{i}\right)-\mathbf{f}(\mathbf{x})}{t} .
$$

Of course there is a generalization of this idea called the directional derivative.
Definition 22.5.2 In general, the symbol

$$
D_{\mathbf{v}} \mathbf{f}(\mathbf{x})
$$

is defined by

$$
\lim _{t \rightarrow 0} \frac{\mathbf{f}(\mathbf{x}+t \mathbf{v})-\mathbf{f}(\mathbf{x})}{t}
$$

where $t \in \mathbb{F}$. In case $|\mathbf{v}|=1$ and the norm is the standard Euclidean norm, this is called the directional derivative. More generally, with no restriction on the size of $\mathbf{v}$ and in any linear space, it is called the Gateaux derivative. $\mathbf{f}$ is said to be Gateaux differentiable at $\mathbf{x}$ if there exists $D_{\mathbf{v}} \mathbf{f}(\mathbf{x})$ such that

$$
\lim _{t \rightarrow 0} \frac{\mathbf{f}(\mathbf{x}+t \mathbf{v})-\mathbf{f}(\mathbf{x})}{t}=D_{\mathbf{v}} \mathbf{f}(\mathbf{x})
$$

where $\mathbf{v} \rightarrow D_{\mathbf{v}} \mathbf{f}(\mathbf{x})$ is linear. Thus we say it is Gateaux differentiable if the Gateaux derivative exists for each $\mathbf{v}$ and $\mathbf{v} \rightarrow D_{\mathbf{v}} \mathbf{f}(\mathbf{x})$ is linear. ${ }^{1}$

[^20]What if all the partial derivatives of $\mathbf{f}$ exist? Does it follow that $\mathbf{f}$ is differentiable? Consider the following function, $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$,

$$
f(x, y)=\left\{\begin{array}{l}
\frac{x y}{x^{2}+y^{2}} \text { if }(x, y) \neq(0,0) \\
0 \text { if }(x, y)=(0,0)
\end{array} .\right.
$$

Then from the definition of partial derivatives,

$$
\lim _{h \rightarrow 0} \frac{f(h, 0)-f(0,0)}{h}=\lim _{h \rightarrow 0} \frac{0-0}{h}=0
$$

and

$$
\lim _{h \rightarrow 0} \frac{f(0, h)-f(0,0)}{h}=\lim _{h \rightarrow 0} \frac{0-0}{h}=0
$$

However $f$ is not even continuous at $(0,0)$ which may be seen by considering the behavior of the function along the line $y=x$ and along the line $x=0$. By Lemma 22.2.3 this implies $f$ is not differentiable. Therefore, it is necessary to consider the correct definition of the derivative given above if you want to get a notion which generalizes the concept of the derivative of a function of one variable in such a way as to preserve continuity whenever the function is differentiable.

### 22.6 A Mean Value Inequality

The following theorem will be very useful in much of what follows. It is a version of the mean value theorem as is the next lemma.

Lemma 22.6.1 Let $Y$ be a normed vector space and suppose $\mathbf{h}:[0,1] \rightarrow Y$ is differentiable and satisfies

$$
\left\|\mathbf{h}^{\prime}(t)\right\| \leq M
$$

Then

$$
\|\mathbf{h}(1)-\mathbf{h}(0)\| \leq M
$$

Proof: Let $\varepsilon>0$ be given and let

$$
S \equiv\{t \in[0,1]: \text { for all } s \in[0, t],\|\mathbf{h}(s)-\mathbf{h}(0)\| \leq(M+\varepsilon) s\}
$$

Then $0 \in S$. Let $t=\sup S$. Then by continuity of $\mathbf{h}$ it follows

$$
\begin{equation*}
\|\mathbf{h}(t)-\mathbf{h}(0)\|=(M+\boldsymbol{\varepsilon}) t \tag{22.6.12}
\end{equation*}
$$

Suppose $t<1$. Then there exist positive numbers, $h_{k}$ decreasing to 0 such that

$$
\left\|\mathbf{h}\left(t+h_{k}\right)-\mathbf{h}(0)\right\|>(M+\varepsilon)\left(t+h_{k}\right)
$$

and now it follows from 22.6.12 and the triangle inequality that

$$
\begin{aligned}
& \left\|\mathbf{h}\left(t+h_{k}\right)-\mathbf{h}(t)\right\|+\|\mathbf{h}(t)-\mathbf{h}(0)\| \\
= & \left\|\mathbf{h}\left(t+h_{k}\right)-\mathbf{h}(t)\right\|+(M+\varepsilon) t>(M+\varepsilon)\left(t+h_{k}\right)
\end{aligned}
$$

and so

$$
\left\|\mathbf{h}\left(t+h_{k}\right)-\mathbf{h}(t)\right\|>(M+\varepsilon) h_{k}
$$

Now dividing by $h_{k}$ and letting $k \rightarrow \infty$

$$
\left\|\mathbf{h}^{\prime}(t)\right\| \geq M+\varepsilon
$$

a contradiction. Thus $t=1$.
Theorem 22.6.2 Suppose $U$ is an open subset of $X$ and $\mathbf{f}: U \rightarrow Y$ has the property that $D \mathbf{f}(\mathbf{x})$ exists for all $\mathbf{x}$ in $U$ and that, $\mathbf{x}+t(\mathbf{y}-\mathbf{x}) \in U$ for all $t \in[0,1]$. (The line segment joining the two points lies in $U$.) Suppose also that for all points on this line segment,

$$
\|D \mathbf{f}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))\| \leq M
$$

Then

$$
\|\mathbf{f}(\mathbf{y})-\mathbf{f}(\mathbf{x})\| \leq M\|\mathbf{y}-\mathbf{x}\|
$$

Proof: Let

$$
\mathbf{h}(t) \equiv \mathbf{f}(\mathbf{x}+t(\mathbf{y}-\mathbf{x})) .
$$

Then by the chain rule,

$$
\mathbf{h}^{\prime}(t)=D \mathbf{f}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))(\mathbf{y}-\mathbf{x})
$$

and so

$$
\begin{aligned}
\left\|\mathbf{h}^{\prime}(t)\right\| & =\|D \mathbf{f}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))(\mathbf{y}-\mathbf{x})\| \\
& \leq M\|\mathbf{y}-\mathbf{x}\|
\end{aligned}
$$

by Lemma 22.6.1

$$
\|\mathbf{h}(1)-\mathbf{h}(0)\|=\|\mathbf{f}(\mathbf{y})-\mathbf{f}(\mathbf{x})\| \leq M\|\mathbf{y}-\mathbf{x}\|
$$

Here is a little result which will help to tie the case of $\mathbb{R}^{n}$ in to the abstract theory presented for arbitrary spaces.

Theorem 22.6.3 Let $X$ be a normed vector space having basis $\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{n}\right\}$ and let $Y$ be another normed vector space having basis $\left\{\mathbf{w}_{1}, \cdots, \mathbf{w}_{m}\right\}$. Let $U$ be an open set in $X$ and let $\mathbf{f}: U \rightarrow Y$ have the property that the Gateaux derivatives,

$$
D_{\mathbf{v}_{k}} \mathbf{f}(\mathbf{x}) \equiv \lim _{t \rightarrow 0} \frac{\mathbf{f}\left(\mathbf{x}+t \mathbf{v}_{k}\right)-\mathbf{f}(\mathbf{x})}{t}
$$

exist and are continuous functions of $\mathbf{x}$. Then D $\mathbf{f}(\mathbf{x})$ exists and

$$
D \mathbf{f}(\mathbf{x}) \mathbf{v}=\sum_{k=1}^{n} D_{\mathbf{v}_{k}} \mathbf{f}(\mathbf{x}) a_{k}
$$

where

$$
\mathbf{v}=\sum_{k=1}^{n} a_{k} \mathbf{v}_{k}
$$

Furthermore, $\mathbf{x} \rightarrow$ Df( $\mathbf{x})$ is continuous; that is

$$
\lim _{\mathbf{y} \rightarrow \mathbf{x}}\|D \mathbf{f}(\mathbf{y})-D \mathbf{f}(\mathbf{x})\|=0
$$

Proof: Let $\mathbf{v}=\sum_{k=1}^{n} a_{k} \mathbf{v}_{k}$. Then

$$
\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})=\mathbf{f}\left(\mathbf{x}+\sum_{k=1}^{n} a_{k} \mathbf{v}_{k}\right)-\mathbf{f}(\mathbf{x})
$$

Then letting $\sum_{k=1}^{0} \equiv 0, \mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})$ is given by

$$
\begin{gather*}
\sum_{k=1}^{n}\left[\mathbf{f}\left(\mathbf{x}+\sum_{j=1}^{k} a_{j} \mathbf{v}_{j}\right)-\mathbf{f}\left(\mathbf{x}+\sum_{j=1}^{k-1} a_{j} \mathbf{v}_{j}\right)\right] \\
=\sum_{k=1}^{n}\left[\mathbf{f}\left(\mathbf{x}+a_{k} \mathbf{v}_{k}\right)-\mathbf{f}(\mathbf{x})\right]+ \\
\sum_{k=1}^{n}\left[\left(\mathbf{f}\left(\mathbf{x}+\sum_{j=1}^{k} a_{j} \mathbf{v}_{j}\right)-\mathbf{f}\left(\mathbf{x}+a_{k} \mathbf{v}_{k}\right)\right)-\left(\mathbf{f}\left(\mathbf{x}+\sum_{j=1}^{k-1} a_{j} \mathbf{v}_{j}\right)-\mathbf{f}(\mathbf{x})\right)\right] \tag{22.6.13}
\end{gather*}
$$

Consider the $k^{\text {th }}$ term in 22.6.13. Let

$$
\mathbf{h}(t) \equiv \mathbf{f}\left(\mathbf{x}+\sum_{j=1}^{k-1} a_{j} \mathbf{v}_{j}+t a_{k} \mathbf{v}_{k}\right)-\mathbf{f}\left(\mathbf{x}+t a_{k} \mathbf{v}_{k}\right)
$$

for $t \in[0,1]$. Then

$$
\begin{aligned}
\mathbf{h}^{\prime}(t)= & a_{k} \lim _{h \rightarrow 0} \frac{1}{a_{k} h}\left(\mathbf{f}\left(\mathbf{x}+\sum_{j=1}^{k-1} a_{j} \mathbf{v}_{j}+(t+h) a_{k} \mathbf{v}_{k}\right)-\mathbf{f}\left(\mathbf{x}+(t+h) a_{k} \mathbf{v}_{k}\right)\right. \\
& \left.-\left(\mathbf{f}\left(\mathbf{x}+\sum_{j=1}^{k-1} a_{j} \mathbf{v}_{j}+t a_{k} \mathbf{v}_{k}\right)-\mathbf{f}\left(\mathbf{x}+t a_{k} \mathbf{v}_{k}\right)\right)\right)
\end{aligned}
$$

and this equals

$$
\begin{equation*}
\left(D_{\mathbf{v}_{k}} \mathbf{f}\left(\mathbf{x}+\sum_{j=1}^{k-1} a_{j} \mathbf{v}_{j}+t a_{k} \mathbf{v}_{k}\right)-D_{\mathbf{v}_{k}} \mathbf{f}\left(\mathbf{x}+t a_{k} \mathbf{v}_{k}\right)\right) a_{k} \tag{22.6.14}
\end{equation*}
$$

Now without loss of generality, it can be assumed that the norm on $X$ is given by

$$
\|\mathbf{v}\| \equiv \max \left\{\left|a_{k}\right|: \mathbf{v}=\sum_{j=1}^{n} a_{k} \mathbf{v}_{k}\right\}
$$

because this is a finite dimensional space, all norms on $X$ are equivalent. Therefore, from 22.6.14 and the assumption that the Gateaux derivatives are continuous,

$$
\begin{aligned}
\left\|\mathbf{h}^{\prime}(t)\right\| & =\left\|\left(D_{\mathbf{v}_{k}} \mathbf{f}\left(\mathbf{x}+\sum_{j=1}^{k-1} a_{j} \mathbf{v}_{j}+t a_{k} \mathbf{v}_{k}\right)-D_{\mathbf{v}_{k}} \mathbf{f}\left(\mathbf{x}+t a_{k} \mathbf{v}_{k}\right)\right) a_{k}\right\| \\
& \leq \varepsilon\left|a_{k}\right| \leq \boldsymbol{\varepsilon}\|\mathbf{v}\|
\end{aligned}
$$

provided $\|\mathbf{v}\|$ is sufficiently small. Since $\varepsilon$ is arbitrary, it follows from Lemma 22.6.1 the expression in 22.6.13 is $\mathbf{0}(\mathbf{v})$ because this expression equals a finite sum of terms of the form $\mathbf{h}(1)-\mathbf{h}(0)$ where $\left\|\mathbf{h}^{\prime}(t)\right\| \leq \varepsilon\|\mathbf{v}\|$ whenever $\|\mathbf{v}\|$ is small enough. Thus

$$
\begin{gathered}
\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})=\sum_{k=1}^{n}\left[\mathbf{f}\left(\mathbf{x}+a_{k} \mathbf{v}_{k}\right)-\mathbf{f}(\mathbf{x})\right]+\mathbf{o}(\mathbf{v}) \\
=\sum_{k=1}^{n} D_{\mathbf{v}_{k}} \mathbf{f}(\mathbf{x}) a_{k}+\sum_{k=1}^{n}\left[\mathbf{f}\left(\mathbf{x}+a_{k} \mathbf{v}_{k}\right)-\mathbf{f}(\mathbf{x})-D_{\mathbf{v}_{k}} \mathbf{f}(\mathbf{x}) a_{k}\right]+\mathbf{o}(\mathbf{v})
\end{gathered}
$$

Consider the $k^{\text {th }}$ term in the second sum.

$$
\mathbf{f}\left(\mathbf{x}+a_{k} \mathbf{v}_{k}\right)-\mathbf{f}(\mathbf{x})-D_{\mathbf{v}_{k}} \mathbf{f}(\mathbf{x}) a_{k}=a_{k}\left(\frac{\mathbf{f}\left(\mathbf{x}+a_{k} \mathbf{v}_{k}\right)-\mathbf{f}(\mathbf{x})}{a_{k}}-D_{\mathbf{v}_{k}} \mathbf{f}(\mathbf{x})\right)
$$

where the expression in the parentheses converges to 0 as $a_{k} \rightarrow 0$. Thus whenever $\|\mathbf{v}\|$ is sufficiently small,

$$
\left\|\mathbf{f}\left(\mathbf{x}+a_{k} \mathbf{v}_{k}\right)-\mathbf{f}(\mathbf{x})-D_{\mathbf{v}_{k}} \mathbf{f}(\mathbf{x}) a_{k}\right\| \leq \boldsymbol{\varepsilon}\left|a_{k}\right| \leq \varepsilon\|\mathbf{v}\|
$$

which shows the second sum is also $\mathbf{0}(\mathbf{v})$. Therefore,

$$
\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})=\sum_{k=1}^{n} D_{\mathbf{v}_{k}} \mathbf{f}(\mathbf{x}) a_{k}+\mathbf{o}(\mathbf{v})
$$

Defining

$$
D \mathbf{f}(\mathbf{x}) \mathbf{v} \equiv \sum_{k=1}^{n} D_{\mathbf{v}_{k}} \mathbf{f}(\mathbf{x}) a_{k}
$$

where $\mathbf{v}=\sum_{k} a_{k} \mathbf{v}_{k}$, it follows $D \mathbf{f}(\mathbf{x}) \in \mathscr{L}(X, Y)$ and is given by the above formula.
It remains to verify $\mathbf{x} \rightarrow D \mathbf{f}(\mathbf{x})$ is continuous.

$$
\begin{aligned}
& \|(D \mathbf{f}(\mathbf{x})-D \mathbf{f}(\mathbf{y})) \mathbf{v}\| \\
\leq & \sum_{k=1}^{n}\left\|\left(D_{\mathbf{v}_{k}} \mathbf{f}(\mathbf{x})-D_{\mathbf{v}_{k}} \mathbf{f}(\mathbf{y})\right) a_{k}\right\| \\
\leq & \max \left\{\left|a_{k}\right|, k=1, \cdots, n\right\} \sum_{k=1}^{n}\left\|D_{\mathbf{v}_{k}} \mathbf{f}(\mathbf{x})-D_{\mathbf{v}_{k}} \mathbf{f}(\mathbf{y})\right\| \\
= & \|\mathbf{v}\| \sum_{k=1}^{n}\left\|D_{\mathbf{v}_{k}} \mathbf{f}(\mathbf{x})-D_{\mathbf{v}_{k}} \mathbf{f}(\mathbf{y})\right\|
\end{aligned}
$$

(Note that $\|\mathbf{v}\| \equiv \max \left\{\left|a_{k}\right|, k=1, \cdots, n\right\}$ where $\mathbf{v}=\sum_{k} a_{k} \mathbf{v}_{k}$ ) and so

$$
\|D \mathbf{f}(\mathbf{x})-D \mathbf{f}(\mathbf{y})\| \leq \sum_{k=1}^{n}\left\|D_{\mathbf{v}_{k}} \mathbf{f}(\mathbf{x})-D_{\mathbf{v}_{k}} \mathbf{f}(\mathbf{y})\right\|
$$

which proves the continuity of $D \mathbf{f}$ because of the assumption the Gateaux derivatives are continuous.

In particular, if $D_{\mathbf{v}_{k}} \mathbf{f}(\mathbf{x})$ exist and are continuous functions of $\mathbf{x}$, this shows that $\mathbf{f}$ is Gateaux differentiable and in fact the Gateaux derivatives are continuous. The following gives the corresponding result for functions defined on infinite dimensional spaces.

Theorem 22.6.4 Suppose $\mathbf{f}: U \rightarrow Y$ where $U$ is an open set in $X$, a normed linear space. Suppose that $\mathbf{f}$ is Gateaux differentiable on $U$ and that the Gateaux derivative is continuous on an open set containing $\mathbf{x}$. Then $\mathbf{f}$ is Frechet differentiable at $\mathbf{x}$.

Proof: Denote by $G(\mathbf{x}) \in \mathscr{L}(X, Y)$ the Gateaux derivative. Thus

$$
G(\mathbf{x}) \mathbf{v} \equiv \lim _{\lambda \rightarrow 0} \frac{\mathbf{f}(\mathbf{x}+\lambda \mathbf{v})-\mathbf{f}(\mathbf{x})}{\lambda}
$$

It is desired to show that $G(\mathbf{x})=D \mathbf{f}(\mathbf{x})$. Since $G$ is continuous, one can obtain

$$
\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})=\int_{0}^{1} G(\mathbf{x}+t \mathbf{v}) \mathbf{v} d t
$$

where this is the ordinary Riemann integral.

$$
\begin{aligned}
& \left\|\frac{\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})-G(\mathbf{x}) \mathbf{v}}{\|\mathbf{v}\|}\right\|=\left\|\frac{\int_{0}^{1} G(\mathbf{x}+t \mathbf{v}) \mathbf{v} d t-G(\mathbf{x}) \mathbf{v}}{\|\mathbf{v}\|}\right\| \\
= & \left\|\frac{\int_{0}^{1} G(\mathbf{x}+t \mathbf{v}) \mathbf{v}-G(\mathbf{x}) \mathbf{v} d t}{\|\mathbf{v}\|}\right\| \leq \frac{1}{\|\mathbf{v}\|} \int_{0}^{1}\|G(\mathbf{x}+t \mathbf{v})-G(\mathbf{x})\| d t\|\mathbf{v}\|
\end{aligned}
$$

which is small provided $\|\mathbf{v}\|$ is sufficiently small. Thus $G(\mathbf{x})=D \mathbf{f}(\mathbf{x})$ as hoped.
Recall the following.
Lemma 22.6.5 Let $\|x\|=\sup _{\left\|y^{*}\right\|_{X^{\prime}} \leq 1}\left|\left\langle y^{*}, x\right\rangle\right|$.
Proof: Let $f(k x)=k\|x\|$. Then

$$
\sup _{\|k x\| \leq 1}|\langle f, x\rangle|=\sup _{|k| \leq 1 /\|x\|}|k|\|x\|=1
$$

Then by Hahn Banach theorem, there is $y^{*} \in X^{\prime}$ which extends $f$ and $\left\|y^{*}\right\| \leq 1$. Then

$$
\|x\| \geq \sup _{\left\|z^{*}\right\|_{x^{\prime}} \leq 1}\left|\left\langle z^{*}, x\right\rangle\right| \geq\left|\left\langle y^{*}, x\right\rangle\right|=\|x\|
$$

One does not need continuity of $G$ near $\mathbf{x}$. It suffices to have continuity at $\mathbf{x}$. Let $\mathbf{y}^{*} \in Y^{\prime}$. Then by the mean value theorem,

$$
\left\langle\mathbf{y}^{*}, \mathbf{f}(\mathbf{x}+\mathbf{v})\right\rangle-\left\langle\mathbf{y}^{*}, \mathbf{f}(\mathbf{x})\right\rangle=\left\langle\mathbf{y}^{*}, G(\mathbf{x}+t \mathbf{v}) \mathbf{v}\right\rangle, t \in[0,1]
$$

Then

$$
\begin{aligned}
& \frac{1}{\|\mathbf{v}\|}\|\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})-G(\mathbf{x}) \mathbf{v}\|=\frac{1}{\|\mathbf{v}\|} \sup _{\left\|\mathbf{y}^{*}\right\| \leq 1}\left|\left\langle\mathbf{y}^{*}, \mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})-G(\mathbf{x}) \mathbf{v}\right\rangle\right| \\
& =\frac{1}{\|\mathbf{v}\|} \sup _{\left\|\mathbf{y}^{*}\right\| \leq 1}\left|\left\langle\mathbf{y}^{*}, G(\mathbf{x}+t \mathbf{v}) \mathbf{v}-G(\mathbf{x}) \mathbf{v}\right\rangle\right| \leq \sup _{|t| \leq 1}\|G(\mathbf{x}+t \mathbf{v})-G(\mathbf{x})\|_{\mathscr{L}(X, Y)}
\end{aligned}
$$

which converges to 0 as $\|\mathbf{v}\| \rightarrow 0$ thanks to continuity of $G$ at $\mathbf{x}$. This proves the following.

Theorem 22.6.6 Suppose $\mathbf{f}: U \rightarrow Y$ where $U$ is an open set in $X$, a normed linear space. Suppose that $\mathbf{f}$ is Gateaux differentiable on $U$ and that the Gateaux derivative is continuous at $\mathbf{x}$. Then $\mathbf{f}$ is Frechet differentiable at $\mathbf{x}$ and $D \mathbf{f}(\mathbf{x}) \mathbf{v}=D_{\mathbf{v}} \mathbf{f}(\mathbf{x})$.

Example 22.6.7 Let $X$ be $C_{0}^{2}(\bar{\Omega})$ where $\Omega$ is a bounded open set in $\mathbb{R}^{n}$ consisting of those functions which are twice continuously differentiable and vanish near $\partial \Omega$. The norm will be

$$
\|u\|_{X} \equiv\|u\|_{\infty}+\max \left\{\|u, i\|_{\infty}, i\right\}+\max \left\{\left\|u u_{, i j}\right\|_{\infty} i, j\right\}
$$

Then let $f: X \rightarrow \mathbb{R}$ be defined by

$$
f(u) \equiv \frac{1}{2} \int_{\Omega} \nabla u \cdot \nabla u d x
$$

Show $f$ is differentiable at $u \in X$.
Consider the Gateaux differentiability.

$$
\lim _{t \rightarrow 0} \frac{f(u+t v)-f(u)}{t}=\lim _{t \rightarrow 0} \frac{t \int_{\Omega} \nabla u \cdot \nabla v d x}{t}+t \frac{1}{2} \int_{\Omega} \nabla v \cdot \nabla v
$$

so it converges to

$$
\int_{\Omega} \nabla u \cdot \nabla v d x=-\int_{\Omega} \Delta u v d x
$$

the last step comes from the divergence theorem. Clearly $v \rightarrow-\int_{\Omega} \Delta u v d x$ is linear and $\mathbb{R}$ valued.

$$
\left|-\int_{\Omega} \Delta u v d x\right| \leq\|v\|_{X} \int_{\Omega}|\Delta u| d x \leq\|v\|_{X} m(\Omega)\|u\|_{X}
$$

Thus this appears to be in $\mathscr{L}(X, \mathbb{R})$. This also shows that,

$$
\sup _{\|v\| \leq 1}\left|D_{v} f(u)-D_{v} f(\hat{u})\right| \leq m(\Omega)\|u-\hat{u}\|_{X}
$$

and so $u \rightarrow D_{(\cdot)}(u)$ is continuous as a map from $X$ to $\mathscr{L}(X, \mathbb{R})$ so it seems that this is a differentiable function and

$$
D f(u)(v)=-\int_{\Omega} \Delta u v d x
$$

Definition 22.6.8 Let $\mathbf{f}: U \rightarrow Y$ where $U$ is an open set in $X$. Then $\mathbf{f}$ is called $C^{1}(U)$ if it Gateaux differentiable and the Gateaux derivative is continuous on $U$.

As shown, this implies $\mathbf{f}$ is differentiable and the Gateaux derivative is the Frechet derivative. It is good to keep in mind the following simple example or variations of it.

Example 22.6.9 Define

$$
f(x) \equiv\left\{\begin{array}{l}
x^{2} \sin \left(\frac{1}{x}\right) x \neq 0 \\
0 \text { if } x=0
\end{array}\right.
$$

This function has the property that it is differentiable everywhere but is not $C^{1}(\mathbb{R})$. In fact the derivative fails to be continuous at 0 .

### 22.7 Higher Order Derivatives

If $f: U \subseteq X \rightarrow Y$ for $U$ an open set, then

$$
\mathbf{x} \rightarrow D \mathbf{f}(\mathbf{x})
$$

is a mapping from $U$ to $\mathscr{L}(X, Y)$, a normed vector space. Therefore, it makes perfect sense to ask whether this function is also differentiable.

Definition 22.7.1 The following is the definition of the second derivative.

$$
D^{2} \mathbf{f}(\mathbf{x}) \equiv D(D \mathbf{f}(\mathbf{x}))
$$

Thus,

$$
D \mathbf{f}(\mathbf{x}+\mathbf{v})-D \mathbf{f}(\mathbf{x})=D^{2} \mathbf{f}(\mathbf{x}) \mathbf{v}+\mathbf{o}(\mathbf{v})
$$

This implies

$$
D^{2} \mathbf{f}(\mathbf{x}) \in \mathscr{L}(X, \mathscr{L}(X, Y)), D^{2} \mathbf{f}(\mathbf{x})(\mathbf{u})(\mathbf{v}) \in Y
$$

and the map

$$
(\mathbf{u}, \mathbf{v}) \rightarrow D^{2} \mathbf{f}(\mathbf{x})(\mathbf{u})(\mathbf{v})
$$

is a bilinear map having values in $Y$. In other words, the two functions,

$$
\mathbf{u} \rightarrow D^{2} \mathbf{f}(\mathbf{x})(\mathbf{u})(\mathbf{v}), \mathbf{v} \rightarrow D^{2} \mathbf{f}(\mathbf{x})(\mathbf{u})(\mathbf{v})
$$

are both linear.
The same pattern applies to taking higher order derivatives. Thus,

$$
D^{3} \mathbf{f}(\mathbf{x}) \equiv D\left(D^{2} \mathbf{f}(\mathbf{x})\right)
$$

and $D^{3} \mathbf{f}(\mathbf{x})$ may be considered as a trilinear map having values in $Y$. In general $D^{k} \mathbf{f}(\mathbf{x})$ may be considered a $k$ linear map. This means the function

$$
\left(\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}\right) \rightarrow D^{k} \mathbf{f}(\mathbf{x})\left(\mathbf{u}_{1}\right) \cdots\left(\mathbf{u}_{k}\right)
$$

has the property

$$
\mathbf{u}_{j} \rightarrow D^{k} \mathbf{f}(\mathbf{x})\left(\mathbf{u}_{1}\right) \cdots\left(\mathbf{u}_{j}\right) \cdots\left(\mathbf{u}_{k}\right)
$$

is linear.
Also, instead of writing

$$
D^{2} \mathbf{f}(\mathbf{x})(\mathbf{u})(\mathbf{v}), \text { or } D^{3} \mathbf{f}(\mathbf{x})(\mathbf{u})(\mathbf{v})(\mathbf{w})
$$

the following notation is often used.

$$
D^{2} \mathbf{f}(\mathbf{x})(\mathbf{u}, \mathbf{v}) \text { or } D^{3} \mathbf{f}(\mathbf{x})(\mathbf{u}, \mathbf{v}, \mathbf{w})
$$

with similar conventions for higher derivatives than 3. Another convention which is often used is the notation

$$
D^{k} \mathbf{f}(\mathbf{x}) \mathbf{v}^{k}
$$

instead of

$$
D^{k} \mathbf{f}(\mathbf{x})(\mathbf{v}, \cdots, \mathbf{v})
$$

Note that for every $k, D^{k} \mathbf{f}$ maps $U$ to a normed vector space. As mentioned above, $D \mathbf{f}(\mathbf{x})$ has values in $\mathscr{L}(X, Y), D^{2} \mathbf{f}(\mathbf{x})$ has values in $\mathscr{L}(X, \mathscr{L}(X, Y))$, etc. Thus it makes sense to consider whether $D^{k} \mathbf{f}$ is continuous. This is described in the following definition.

Definition 22.7.2 Let $U$ be an open subset of $X$, a normed vector space, and let $\mathbf{f}: U \rightarrow Y$. Then $\mathbf{f}$ is $C^{k}(U)$ if $\mathbf{f}$ and its first $k$ derivatives are all continuous. Also, $D^{k} \mathbf{f}(\mathbf{x})$ when it exists can be considered a $Y$ valued multi-linear function. Sometimes these are called tensors in case $\mathbf{f}$ has scalar values.

### 22.8 The Derivative And The Cartesian Product

There are theorems which can be used to get differentiability of a function based on existence and continuity of the partial derivatives. A generalization of this was given above. Here a function defined on a product space is considered. It is very much like what was presented above and could be obtained as a special case but to reinforce the ideas, I will do it from scratch because certain aspects of it are important in the statement of the implicit function theorem.

The following is an important abstract generalization of the concept of partial derivative presented above. Insead of taking the derivative with respect to one variable, it is taken with respect to several but not with respect to others. This vague notion is made precise in the following definition. First here is a lemma.

Lemma 22.8.1 Suppose $U$ is an open set in $X \times Y$. Then the set, $U_{\mathbf{y}}$ defined by

$$
U_{\mathbf{y}} \equiv\{\mathbf{x} \in X:(\mathbf{x}, \mathbf{y}) \in U\}
$$

is an open set in $X$. Here $X \times Y$ is a finite dimensional vector space in which the vector space operations are defined componentwise. Thus for $a, b \in \mathbb{F}$,

$$
a\left(\mathbf{x}_{1}, \mathbf{y}_{1}\right)+b\left(\mathbf{x}_{2}, \mathbf{y}_{2}\right)=\left(a \mathbf{x}_{1}+b \mathbf{x}_{2}, a \mathbf{y}_{1}+b \mathbf{y}_{2}\right)
$$

and the norm can be taken to be

$$
\|(\mathbf{x}, \mathbf{y})\| \equiv \max (\|\mathbf{x}\|,\|\mathbf{y}\|)
$$

Proof: In finite dimensions it doesn't matter how this norm is defined because all are equivalent. It obviously satisfies most axioms of a norm. The only one which is not obvious is the triangle inequality. I will show this now.

$$
\begin{aligned}
\left\|(\mathbf{x}, \mathbf{y})+\left(\mathbf{x}_{1}, \mathbf{y}_{1}\right)\right\| & \equiv\left\|\left(\mathbf{x}+\mathbf{x}_{1}, \mathbf{y}+\mathbf{y}_{1}\right)\right\| \equiv \max \left(\left\|\mathbf{x}+\mathbf{x}_{1}\right\|,\left\|\mathbf{y}+\mathbf{y}_{1}\right\|\right) \\
& \leq \max \left(\|\mathbf{x}\|+\left\|\mathbf{x}_{1}\right\|,\|\mathbf{y}\|+\left\|\mathbf{y}_{1}\right\|\right) \\
& \leq \max (\|\mathbf{x}\|,\|\mathbf{y}\|)+\max \left(\left\|\mathbf{x}_{1}\right\|,\left\|\mathbf{y}_{1}\right\|\right) \\
& \equiv\|(\mathbf{x}, \mathbf{y})\|+\left\|\left(\mathbf{x}_{1}, \mathbf{y}_{1}\right)\right\|
\end{aligned}
$$

Let $\mathbf{x} \in U_{\mathbf{y}}$. Then $(\mathbf{x}, \mathbf{y}) \in U$ and so there exists $r>0$ such that

$$
B((\mathbf{x}, \mathbf{y}), r) \in U
$$

This says that if $(\mathbf{u}, \mathbf{v}) \in X \times Y$ such that $\|(\mathbf{u}, \mathbf{v})-(\mathbf{x}, \mathbf{y})\|<r$, then $(\mathbf{u}, \mathbf{v}) \in U$. Thus if

$$
\|(\mathbf{u}, \mathbf{y})-(\mathbf{x}, \mathbf{y})\|=\|\mathbf{u}-\mathbf{x}\|<r
$$

then $(\mathbf{u}, \mathbf{y}) \in U$. This has just said that $B(\mathbf{x}, r)$, the ball taken in $X$ is contained in $U_{\mathbf{y}}$. This proves the lemma.

Or course one could also consider

$$
U_{\mathbf{x}} \equiv\{\mathbf{y}:(\mathbf{x}, \mathbf{y}) \in U\}
$$

in the same way and conclude this set is open in $Y$. Also, the generalization to many factors yields the same conclusion. In this case, for $\mathbf{x} \in \prod_{i=1}^{n} X_{i}$, let

$$
\|\mathbf{x}\| \equiv \max \left(\left\|\mathbf{x}_{i}\right\|_{X_{i}}: \mathbf{x}=\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{n}\right)\right)
$$

Then a similar argument to the above shows this is a norm on $\prod_{i=1}^{n} X_{i}$. Consider the triangle inequality.

$$
\begin{gathered}
\left\|\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{n}\right)+\left(\mathbf{y}_{1}, \cdots, \mathbf{y}_{n}\right)\right\|=\max _{i}\left(\left\|\mathbf{x}_{i}+\mathbf{y}_{i}\right\|_{X_{i}}\right) \leq \max _{i}\left(\left\|\mathbf{x}_{i}\right\|_{X_{i}}+\left\|\mathbf{y}_{i}\right\|_{X_{i}}\right) \\
\leq \max _{i}\left(\left\|\mathbf{x}_{i}\right\|_{X_{i}}\right)+\max _{i}\left(\left\|\mathbf{y}_{i}\right\|_{X_{i}}\right)
\end{gathered}
$$

Corollary 22.8.2 Let $U \subseteq \prod_{i=1}^{n} X_{i}$ be an open set and let

$$
U_{\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \cdots, \mathbf{x}_{n}\right)} \equiv\left\{\mathbf{x} \in \mathbb{F}^{r_{i}}:\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{i-1}, \mathbf{x}, \mathbf{x}_{i+1}, \cdots, \mathbf{x}_{n}\right) \in U\right\}
$$

Then $U_{\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \cdots, \mathbf{x}_{n}\right)}$ is an open set in $\mathbb{F}^{r_{i}}$.
Proof: Let $\mathbf{z} \in U_{\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \cdots, \mathbf{x}_{n}\right)}$. Then $\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{i-1}, \mathbf{z}, \mathbf{x}_{i+1}, \cdots, \mathbf{x}_{n}\right) \equiv \mathbf{x} \in U$ by definition. Therefore, since $U$ is open, there exists $r>0$ such that $B(\mathbf{x}, r) \subseteq U$. It follows that for $B(\mathbf{z}, r)_{X_{i}}$ denoting the ball in $X_{i}$, it follows that $B(\mathbf{z}, r)_{X_{i}} \subseteq U_{\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \cdots, \mathbf{x}_{n}\right)}$ because to say that $\|\mathbf{z}-\mathbf{w}\|_{X_{i}}<r$ is to say that

$$
\left\|\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{i-1}, \mathbf{z}, \mathbf{x}_{i+1}, \cdots, \mathbf{x}_{n}\right)-\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{i-1}, \mathbf{w}, \mathbf{x}_{i+1}, \cdots, \mathbf{x}_{n}\right)\right\|<r
$$

and so $\mathbf{w} \in U_{\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \cdots, \mathbf{x}_{n}\right)}$.
Next is a generalization of the partial derivative.
Definition 22.8.3 Let $\mathbf{g}: U \subseteq \prod_{i=1}^{n} X_{i} \rightarrow Y$, where $U$ is an open set. Then the map

$$
\mathbf{z} \rightarrow \mathbf{g}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{i-1}, \mathbf{z}, \mathbf{x}_{i+1}, \cdots, \mathbf{x}_{n}\right)
$$

is a function from the open set in $X_{i}$,

$$
\left\{\mathbf{z}: \mathbf{x}=\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{i-1}, \mathbf{z}, \mathbf{x}_{i+1}, \cdots, \mathbf{x}_{n}\right) \in U\right\}
$$

to $Y$. When this map is differentiable, its derivative is denoted by $D_{i} \mathbf{g}(\mathbf{x})$. To aid in the notation, for $\mathbf{v} \in X_{i}$, let $\theta_{i} \mathbf{v} \in \prod_{i=1}^{n} X_{i}$ be the vector $(\mathbf{0}, \cdots, \mathbf{v}, \cdots, \mathbf{0})$ where the $\mathbf{v}$ is in the $i^{\text {th }}$ slot and for $\mathbf{v} \in \prod_{i=1}^{n} X_{i}$, let $\mathbf{v}_{i}$ denote the entry in the $i^{\text {th }}$ slot of $\mathbf{v}$. Thus, by saying

$$
\mathbf{z} \rightarrow \mathbf{g}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{i-1}, \mathbf{z}, \mathbf{x}_{i+1}, \cdots, \mathbf{x}_{n}\right)
$$

is differentiable is meant that for $\mathbf{v} \in X_{i}$ sufficiently small,

$$
\mathbf{g}\left(\mathbf{x}+\theta_{i} \mathbf{v}\right)-\mathbf{g}(\mathbf{x})=D_{i} \mathbf{g}(\mathbf{x}) \mathbf{v}+\mathbf{o}(\mathbf{v}) .
$$

Note $D_{i} \mathbf{g}(\mathbf{x}) \in \mathscr{L}\left(X_{i}, Y\right)$.
Definition 22.8.4 Let $U \subseteq X$ be an open set. Then $\mathbf{f}: U \rightarrow Y$ is $C^{1}(U)$ if $\mathbf{f}$ is differentiable and the mapping

$$
\mathbf{x} \rightarrow D \mathbf{f}(\mathbf{x})
$$

is continuous as a function from $U$ to $\mathscr{L}(X, Y)$.
With this definition of partial derivatives, here is the major theorem. Note the resemblance with the matrix of the derivative of a function having values in $\mathbb{R}^{m}$ in terms of the partial derivatives.

Theorem 22.8.5 Let $\mathbf{g}, U, \prod_{i=1}^{n} X_{i}$, be given as in Definition 22.8.3. Then $\mathbf{g}$ is $C^{1}(U)$ if and only if $D_{i} \mathbf{g}$ exists and is continuous on $U$ for each $i$. In this case, $\mathbf{g}$ is differentiable and

$$
\begin{equation*}
D \mathbf{g}(\mathbf{x})(\mathbf{v})=\sum_{k} D_{k} \mathbf{g}(\mathbf{x}) \mathbf{v}_{k} \tag{22.8.15}
\end{equation*}
$$

where $\mathbf{v}=\left(\mathbf{v}_{1}, \cdots, \mathbf{v}_{n}\right)$.
Proof: Suppose then that $D_{i} \mathbf{g}$ exists and is continuous for each $i$. Note that

$$
\sum_{j=1}^{k} \theta_{j} \mathbf{v}_{j}=\left(\mathbf{v}_{1}, \cdots, \mathbf{v}_{k}, \mathbf{0}, \cdots, \mathbf{0}\right)
$$

Thus $\sum_{j=1}^{n} \theta_{j} \mathbf{v}_{j}=\mathbf{v}$ and define $\sum_{j=1}^{0} \theta_{j} \mathbf{v}_{j} \equiv \mathbf{0}$. Therefore,

$$
\begin{equation*}
\mathbf{g}(\mathbf{x}+\mathbf{v})-\mathbf{g}(\mathbf{x})=\sum_{k=1}^{n}\left[\mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k} \theta_{j} \mathbf{v}_{j}\right)-\mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k-1} \theta_{j} \mathbf{v}_{j}\right)\right] \tag{22.8.16}
\end{equation*}
$$

Consider the terms in this sum.

$$
\begin{gather*}
\mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k} \theta_{j} \mathbf{v}_{j}\right)-\mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k-1} \theta_{j} \mathbf{v}_{j}\right)=\mathbf{g}\left(\mathbf{x}+\theta_{k} \mathbf{v}_{k}\right)-\mathbf{g}(\mathbf{x})+  \tag{22.8.17}\\
\left(\mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k} \theta_{j} \mathbf{v}_{j}\right)-\mathbf{g}\left(\mathbf{x}+\theta_{k} \mathbf{v}_{k}\right)\right)-\left(\mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k-1} \theta_{j} \mathbf{v}_{j}\right)-\mathbf{g}(\mathbf{x})\right) \tag{22.8.18}
\end{gather*}
$$

and the expression in 22.8 .18 is of the form $\mathbf{h}\left(\mathbf{v}_{k}\right)-\mathbf{h}(\mathbf{0})$ where for small $\mathbf{w} \in X_{k}$,

$$
\mathbf{h}(\mathbf{w}) \equiv \mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k-1} \theta_{j} \mathbf{v}_{j}+\theta_{k} \mathbf{w}\right)-\mathbf{g}\left(\mathbf{x}+\theta_{k} \mathbf{w}\right)
$$

Therefore,

$$
D \mathbf{h}(\mathbf{w})=D_{k} \mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k-1} \theta_{j} \mathbf{v}_{j}+\theta_{k} \mathbf{w}\right)-D_{k} \mathbf{g}\left(\mathbf{x}+\theta_{k} \mathbf{w}\right)
$$

and by continuity, $\|D \mathbf{h}(\mathbf{w})\|<\varepsilon$ provided $\|\mathbf{v}\|$ is small enough. Therefore, by Theorem 22.6.2, the mean value inequality, whenever $\|\mathbf{v}\|$ is small enough,

$$
\left\|\mathbf{h}\left(\mathbf{v}_{k}\right)-\mathbf{h}(\mathbf{0})\right\| \leq \varepsilon\|\mathbf{v}\|
$$

which shows that since $\varepsilon$ is arbitrary, the expression in 6.13.24 is $\boldsymbol{o}(\mathbf{v})$. Now in 22.8.17

$$
\mathbf{g}\left(\mathbf{x}+\theta_{k} \mathbf{v}_{k}\right)-\mathbf{g}(\mathbf{x})=D_{k} \mathbf{g}(\mathbf{x}) \mathbf{v}_{k}+\mathbf{o}\left(\mathbf{v}_{k}\right)=D_{k} \mathbf{g}(\mathbf{x}) \mathbf{v}_{k}+\mathbf{o}(\mathbf{v}) .
$$

Therefore, referring to 22.8.16,

$$
\mathbf{g}(\mathbf{x}+\mathbf{v})-\mathbf{g}(\mathbf{x})=\sum_{k=1}^{n} D_{k} \mathbf{g}(\mathbf{x}) \mathbf{v}_{k}+\mathbf{o}(\mathbf{v})
$$

which shows $D \mathbf{g}(\mathbf{x})$ exists and equals the formula given in 22.8.15. Also $\mathbf{x} \rightarrow D \mathbf{g}(\mathbf{x})$ is continuous since each of the $D_{k} \mathbf{g}(\mathbf{x})$ are.

Next suppose $\mathbf{g}$ is $C^{1}$. I need to verify that $D_{k} \mathbf{g}(\mathbf{x})$ exists and is continuous. Let $\mathbf{v} \in X_{k}$ sufficiently small. Then

$$
\begin{aligned}
\mathbf{g}\left(\mathbf{x}+\theta_{k} \mathbf{v}\right)-\mathbf{g}(\mathbf{x}) & =D \mathbf{g}(\mathbf{x}) \theta_{k} \mathbf{v}+\mathbf{o}\left(\theta_{k} \mathbf{v}\right) \\
& =D \mathbf{g}(\mathbf{x}) \theta_{k} \mathbf{v}+\mathbf{o}(\mathbf{v})
\end{aligned}
$$

since $\left\|\theta_{k} \mathbf{v}\right\|=\|\mathbf{v}\|$. Then $D_{k} \mathbf{g}(\mathbf{x})$ exists and equals

$$
D \mathbf{g}(\mathbf{x}) \circ \theta_{k}
$$

Now $\mathbf{x} \rightarrow D \mathbf{g}(\mathbf{x})$ is continuous. It is clear that $\theta_{k}: X_{k} \rightarrow \prod_{i=1}^{n} X_{i}$ is also continuous because $\theta_{k} \mathbf{v}$ places $\mathbf{v}$ in the $k^{t h}$ position and $\mathbf{0}$ in every other position.

Note that the above argument also works at a single point $\mathbf{x}$. That is, continuity at $\mathbf{x}$ of the partials implies $D \mathbf{g}(\mathbf{x})$ exists and is continuous at $\mathbf{x}$.

### 22.9 Mixed Partial Derivatives

Let $U$ be an open set in $\prod_{i=1}^{n} X_{i}$ where the norm is the one described above and let $\mathbf{f}: U \rightarrow Y$ be a function for which the higher order partial derivatives of the sort described above exist. As in the case of functions defined on open sets of $\mathbb{R}^{n}$ one can ask whether the mixed partials are equal.

Results of this sort were known to Euler in around 1734. The theorem was proved by Clairaut some time later. It turns out that the mixed partial derivatives, if continuous will end up being equal. It will also work in the more general situation just described.

Theorem 22.9.1 Let $U$ be an open subset of $\prod_{i=1}^{n} X_{i}$ where each $X_{i}$ is a normed linear space and $\|\mathbf{x}\|=\max _{i}\left\|x_{i}\right\|_{i}$. Let $\mathbf{f}: U \rightarrow Y$ have mixed partial derivatives $D_{i} D_{j} \mathbf{f}$ and $D_{j} D_{i} \mathbf{f}$. Then if these are continuous at $\mathbf{x} \in U$, it follows they will be equal in the sense that $D_{j} D_{i} \mathbf{f}(\mathbf{x})(\mathbf{u}, \mathbf{v})=D_{i} D_{j} \mathbf{f}(\mathbf{x})(\mathbf{v}, \mathbf{u})$.

Proof: It suffices to assume that there are only two spaces and $U$ is an open subset of $X_{1} \times X_{2}$ because one simply specializes to two of the variables in the general case. We denote the variable for $X_{1}$ as $\mathbf{x}$ and the one from $X_{2}$ as $\mathbf{y}$. Also, to simplify this, first assume $\mathbf{f}$ has values in $\mathbb{R}$. Thus it will be denoted as $f$ rather than $\mathbf{f}$. Since $U$ is open, there exists $r>0$ such that $B((\mathbf{x}, \mathbf{y}), r) \subseteq U$. Now let $t, s$ be small real numbers and consider

$$
\Delta(s, t) \equiv \frac{1}{s t}\{\overbrace{f(\mathbf{x}+t \mathbf{u}, \mathbf{y}+s \mathbf{v})-f(\mathbf{x}+t \mathbf{u}, \mathbf{y})}^{h(t)}-\overbrace{(f(\mathbf{x}, \mathbf{y}+s \mathbf{v})-f(\mathbf{x}, \mathbf{y}))}^{h(0)}\}
$$

Then $h^{\prime}(t)=D_{1} f(\mathbf{x}+t \mathbf{u}, \mathbf{y}+s \mathbf{v})(\mathbf{u})-D_{1} f(\mathbf{x}+t \mathbf{u}, \mathbf{y})(\mathbf{u})$. By the mean value theorem,

$$
\Delta(s, t)=\frac{1}{s} h^{\prime}(\theta t)=\frac{1}{s}\left(D_{1} f(\mathbf{x}+\theta t \mathbf{u}, \mathbf{y}+s \mathbf{v})(\mathbf{u})-D_{1} f(\mathbf{x}+\theta t \mathbf{u}, \mathbf{y})(\mathbf{u})\right)
$$

where $\theta \in(0,1)$. Now use the mean value theorem again to obtain

$$
\Delta(s, t)=D_{2} D_{1} f(\mathbf{x}+\theta t \mathbf{u}, \mathbf{y}+\alpha s \mathbf{v})(\mathbf{u})(\mathbf{v}), \alpha \in(0,1)
$$

Similarly doing things in the other order writing

$$
\Delta(s, t)=\frac{1}{s t}\{(f(\mathbf{x}+t \mathbf{u}, \mathbf{y}+s \mathbf{v})-f(\mathbf{x}, \mathbf{y}+s \mathbf{v}))-(f(\mathbf{x}+t \mathbf{u}, \mathbf{y})-f(\mathbf{x}, \mathbf{y}))\}
$$

and taking the derivative first with respect to $s$ and next with respect to $t$, one can obtain

$$
\Delta(s, t)=D_{1} D_{2} f(\mathbf{x}+\hat{\theta} t \mathbf{u}, \mathbf{y}+\hat{\alpha} s \mathbf{v})(\mathbf{v})(\mathbf{u})
$$

where $\hat{\boldsymbol{\theta}}, \hat{\alpha}$ are also in $(0,1)$. Then letting $(s, t) \rightarrow(0,0)$ and using continuity of the mixed partial derivatives, one obtains that

$$
D_{2} D_{1} f(\mathbf{x}, \mathbf{y})(\mathbf{u})(\mathbf{v})=D_{1} D_{2} f(\mathbf{x}, \mathbf{y})(\mathbf{v})(\mathbf{u})
$$

Letting $\mathbf{v}=\mathbf{u}$ yields the desired result.
The general case follows right away by applying this result to $\left\langle\mathbf{y}^{*}, \mathbf{f}\right\rangle$. Thus one obtains

$$
\left\langle\mathbf{y}^{*}, D_{2} D_{1} f(\mathbf{x}, \mathbf{y})(\mathbf{u})(\mathbf{v})\right\rangle=\left\langle\mathbf{y}^{*}, D_{1} D_{2} f(\mathbf{x}, \mathbf{y})(\mathbf{v})(\mathbf{u})\right\rangle
$$

for every $\mathbf{y}^{*} \in Y^{\prime}$. Hence, since $Y^{\prime}$ separates the points, it follows that the mixed partials are equal.

It is necessary to assume the mixed partial derivatives are continuous in order to assert they are equal. The following is a well known example [6].

Example 22.9.2 Let

$$
f(x, y)=\left\{\begin{array}{l}
\frac{x y\left(x^{2}-y^{2}\right)}{x^{2}+y^{2}} \text { if }(x, y) \neq(0,0) \\
0 \text { if }(x, y)=(0,0)
\end{array}\right.
$$

From the definition of partial derivatives it follows immediately that

$$
f_{x}(0,0)=f_{y}(0,0)=0
$$

Using the standard rules of differentiation, for $(x, y) \neq(0,0)$,

$$
f_{x}=y \frac{x^{4}-y^{4}+4 x^{2} y^{2}}{\left(x^{2}+y^{2}\right)^{2}}, f_{y}=x \frac{x^{4}-y^{4}-4 x^{2} y^{2}}{\left(x^{2}+y^{2}\right)^{2}}
$$

Now

$$
f_{x y}(0,0) \equiv \lim _{y \rightarrow 0} \frac{f_{x}(0, y)-f_{x}(0,0)}{y}=\lim _{y \rightarrow 0} \frac{-y^{4}}{\left(y^{2}\right)^{2}}=-1
$$

while

$$
f_{y x}(0,0) \equiv \lim _{x \rightarrow 0} \frac{f_{y}(x, 0)-f_{y}(0,0)}{x}=\lim _{x \rightarrow 0} \frac{x^{4}}{\left(x^{2}\right)^{2}}=1
$$

showing that although the mixed partial derivatives do exist at $(0,0)$, they are not equal there.

Incidentally, the graph of this function appears very innocent. Its fundamental sickness is not apparent. It is like one of those whited sepulchers mentioned in the Bible.


### 22.10 Implicit Function Theorem

Recall the following notation. $\mathscr{L}(X, Y)$ is the space of bounded linear mappings from $X$ to $Y$ where here $\left(X,\|\cdot\|_{X}\right)$ and $\left(Y,\|\cdot\|_{Y}\right)$ are normed linear spaces. Recall that this means that for each $L \in \mathscr{L}(X, Y)$

$$
\|L\| \equiv \sup _{\|x\| \leq 1}\|L x\|<\infty
$$

As shown earlier, this makes $\mathscr{L}(X, Y)$ into a normed linear space. In case $X$ is finite dimensional, $\mathscr{L}(X, Y)$ is the same as the collection of linear maps from $X$ to $Y$. In what
follows $X, Y$ will be Banach spaces, complete normed linear spaces. Thus these are complete normed linear space and $\mathscr{L}(X, Y)$ is the space of bounded linear maps. I will also cease trying to write the vectors in bold face partly to emphasize that these are not in $\mathbb{R}^{n}$.

Definition 22.10.1 Let $\left(X,\|\cdot\|_{X}\right)$ and $\left(Y,\|\cdot\|_{Y}\right)$ be two normed linear spaces. Then $\mathscr{L}(X, Y)$ denotes the set of linear maps from $X$ to $Y$ which also satisfy the following condition. For $L \in \mathscr{L}(X, Y)$,

$$
\lim _{\|x\|_{X} \leq 1}\|L x\|_{Y} \equiv\|L\|<\infty
$$

Recall that this operator norm is less than infinity is always the case where $X$ is finite dimensional. However, if you wish to consider infinite dimensional situations, you assume the operator norm is finite as a qualification for being in $\mathscr{L}(X, Y)$. Then here is an important theorem.

Theorem 22.10.2 If $Y$ is a Banach space, then $\mathscr{L}(X, Y)$ is also a Banach space.
Proof: Let $\left\{L_{n}\right\}$ be a Cauchy sequence in $\mathscr{L}(X, Y)$ and let $x \in X$.

$$
\left\|L_{n} x-L_{m} x\right\| \leq\|x\|\left\|L_{n}-L_{m}\right\| .
$$

Thus $\left\{L_{n} x\right\}$ is a Cauchy sequence. Let

$$
L x=\lim _{n \rightarrow \infty} L_{n} x
$$

Then, clearly, $L$ is linear because if $x_{1}, x_{2}$ are in $X$, and $a, b$ are scalars, then

$$
\begin{aligned}
L\left(a x_{1}+b x_{2}\right) & =\lim _{n \rightarrow \infty} L_{n}\left(a x_{1}+b x_{2}\right) \\
& =\lim _{n \rightarrow \infty}\left(a L_{n} x_{1}+b L_{n} x_{2}\right) \\
& =a L x_{1}+b L x_{2}
\end{aligned}
$$

Also $L$ is bounded. To see this, note that $\left\{\left\|L_{n}\right\|\right\}$ is a Cauchy sequence of real numbers because $\left\|\mid L_{n}\right\|-\left\|L_{m}\right\|\|\leq\| L_{n}-L_{m} \|$. Hence there exists $K>\sup \left\{\left\|L_{n}\right\|: n \in \mathbb{N}\right\}$. Thus, if $x \in X$,

$$
\|L x\|=\lim _{n \rightarrow \infty}\left\|L_{n} x\right\| \leq K\|x\| .
$$

The following theorem is really nice. The series in this theorem is called the Neuman series.

Lemma 22.10.3 Let $(X,\|\cdot\|)$ is a Banach space, and if $A \in \mathscr{L}(X, X)$ and $\|A\|=r<1$, then

$$
(I-A)^{-1}=\sum_{k=0}^{\infty} A^{k} \in \mathscr{L}(X, X)
$$

where the series converges in the Banach space $\mathscr{L}(X, X)$. If $\mathscr{O}$ consists of the invertible maps in $\mathscr{L}(X, X)$, then $\mathscr{O}$ is open and if $\mathfrak{I}$ is the mapping which takes $A$ to $A^{-1}$, then $\mathfrak{I}$ is continuous.

Proof: First of all, why does the series make sense?

$$
\left\|\sum_{k=p}^{q} A^{k}\right\| \leq \sum_{k=p}^{q}\left\|A^{k}\right\| \leq \sum_{k=p}^{q}\|A\|^{k} \leq \sum_{k=p}^{\infty} r^{k} \leq \frac{r^{p}}{1-r}
$$

and so the partial sums are Cauchy in $\mathscr{L}(X, X)$. Therefore, the series converges to something in $\mathscr{L}(X, X)$ by completeness of this normed linear space. Now why is it the inverse?

$$
\sum_{k=0}^{\infty} A^{k}(I-A)=\lim _{n \rightarrow \infty} \sum_{k=0}^{n} A^{k}(I-A)=\lim _{n \rightarrow \infty}\left(\sum_{k=0}^{n} A^{k}-\sum_{k=1}^{n+1} A^{k}\right)=\lim _{n \rightarrow \infty}\left(I-A^{n+1}\right)=I
$$

because $\left\|A^{n+1}\right\| \leq\|A\|^{n+1} \leq r^{n+1}$. Similarly,

$$
(I-A) \sum_{k=0}^{\infty} A^{k}=\lim _{n \rightarrow \infty}\left(I-A^{n+1}\right)=I
$$

and so this shows that this series is indeed the desired inverse.
Next suppose $A \in \mathscr{O}$ so $A^{-1} \in \mathscr{L}(X, X)$. Then suppose $\|A-B\|<\frac{r}{1+\left\|A^{-1}\right\|}, r<1$. Does it follow that $B$ is also invertible?

$$
B=A-(A-B)=A\left[I-A^{-1}(A-B)\right]
$$

Then $\left\|A^{-1}(A-B)\right\| \leq\left\|A^{-1}\right\|\|A-B\|<r$ and so $\left[I-A^{-1}(A-B)\right]^{-1}$ exists. Hence

$$
B^{-1}=\left[I-A^{-1}(A-B)\right]^{-1} A^{-1}
$$

Thus $\mathscr{O}$ is open as claimed. As to continuity, let $A, B$ be as just described. Then using the Neuman series,

$$
\begin{aligned}
&\|\Im A-\Im B\|=\left\|A^{-1}-\left[I-A^{-1}(A-B)\right]^{-1} A^{-1}\right\| \\
&=\left\|A^{-1}-\sum_{k=0}^{\infty}\left(A^{-1}(A-B)\right)^{k} A^{-1}\right\|=\left\|\sum_{k=1}^{\infty}\left(A^{-1}(A-B)\right)^{k} A^{-1}\right\| \\
& \leq \sum_{k=1}^{\infty}\left\|A^{-1}\right\|^{k+1}\|A-B\|^{k}=\|A-B\|\left\|A^{-1}\right\|^{2} \sum_{k=0}^{\infty}\left\|A^{-1}\right\|^{k}\left(\frac{r}{1+\left\|A^{-1}\right\|}\right)^{k} \\
& \leq\|B-A\|\left\|A^{-1}\right\|^{2} \frac{1}{1-r} .
\end{aligned}
$$

Thus $\mathfrak{I}$ is continuous at $A \in \mathscr{O}$.
Lemma 22.10.4 Let

$$
\mathscr{O} \equiv\left\{A \in \mathscr{L}(X, Y): A^{-1} \in \mathscr{L}(Y, X)\right\}
$$

and let

$$
\mathfrak{I}: \mathscr{O} \rightarrow \mathscr{L}(Y, X), \Im A \equiv A^{-1}
$$

Then $\mathscr{O}$ is open and $\mathfrak{I}$ is in $C^{m}(\mathscr{O})$ for all $m=1,2, \cdots$. Also

$$
\begin{equation*}
D \mathfrak{I}(A)(B)=-\Im(A)(B) \Im(A) . \tag{22.10.19}
\end{equation*}
$$

In particular, $\mathfrak{I}$ is continuous.
Proof: Let $A \in \mathscr{O}$ and let $B \in \mathscr{L}(X, Y)$ with

$$
\|B\| \leq \frac{1}{2}\left\|A^{-1}\right\|^{-1}
$$

Then

$$
\left\|A^{-1} B\right\| \leq\left\|A^{-1}\right\|\|B\| \leq \frac{1}{2}
$$

and so by Lemma 22.10.3,

$$
\left(I+A^{-1} B\right)^{-1} \in \mathscr{L}(X, X)
$$

It follows that

$$
(A+B)^{-1}=\left(A\left(I+A^{-1} B\right)\right)^{-1}=\left(I+A^{-1} B\right)^{-1} A^{-1} \in \mathscr{L}(Y, X)
$$

Thus $\mathscr{O}$ is an open set.
Thus

$$
\begin{gathered}
(A+B)^{-1}=\left(I+A^{-1} B\right)^{-1} A^{-1}=\sum_{n=0}^{\infty}(-1)^{n}\left(A^{-1} B\right)^{n} A^{-1} \\
=\left[I-A^{-1} B+o(B)\right] A^{-1}
\end{gathered}
$$

which shows that $\mathscr{O}$ is open and, also,

$$
\begin{aligned}
\mathfrak{I}(A+B)-\mathfrak{I}(A) & =\sum_{n=0}^{\infty}(-1)^{n}\left(A^{-1} B\right)^{n} A^{-1}-A^{-1} \\
& =-A^{-1} B A^{-1}+o(B) \\
& =-\mathfrak{I}(A)(B) \mathfrak{I}(A)+o(B)
\end{aligned}
$$

which demonstrates 22.10.19. It follows from this that we can continue taking derivatives of $\mathfrak{I}$. For $\left\|B_{1}\right\|$ small,

$$
\begin{gathered}
-\left[D \mathfrak{I}\left(A+B_{1}\right)(B)-D \mathfrak{I}(A)(B)\right]= \\
\mathfrak{I}\left(A+B_{1}\right)(B) \mathfrak{I}\left(A+B_{1}\right)-\mathfrak{I}(A)(B) \mathfrak{I}(A) \\
= \\
\mathfrak{I}\left(A+B_{1}\right)(B) \mathfrak{I}\left(A+B_{1}\right)-\mathfrak{I}(A)(B) \mathfrak{I}\left(A+B_{1}\right)+ \\
\mathfrak{I}(A)(B) \mathfrak{I}\left(A+B_{1}\right)-\mathfrak{I}(A)(B) \mathfrak{I}(A) \\
\\
{\left[\mathfrak{I}(A)\left(B_{1}\right) \mathfrak{I}(A)+o\left(B_{1}\right)\right](B) \mathfrak{I}\left(A+B_{1}\right)+} \\
\mathfrak{I}(A)(B)\left[\mathfrak{I}(A)\left(B_{1}\right) \mathfrak{I}(A)+o\left(B_{1}\right)\right]
\end{gathered}
$$

$$
\begin{aligned}
= & {\left[\mathfrak{I}(A)\left(B_{1}\right) \mathfrak{I}(A)+o\left(B_{1}\right)\right](B)\left[A^{-1}-A^{-1} B_{1} A^{-1}+o\left(B_{1}\right)\right]+} \\
& \mathfrak{I}(A)(B)\left[\mathfrak{I}(A)\left(B_{1}\right) \mathfrak{I}(A)+o\left(B_{1}\right)\right] \\
= & \mathfrak{I}(A)\left(B_{1}\right) \mathfrak{I}(A)(B) \mathfrak{I}(A)+\mathfrak{I}(A)(B) \mathfrak{I}(A)\left(B_{1}\right) \mathfrak{I}(A)+o\left(B_{1}\right)
\end{aligned}
$$

and so

$$
D^{2} \mathfrak{I}(A)\left(B_{1}\right)(B)=\mathfrak{I}(A)\left(B_{1}\right) \mathfrak{I}(A)(B) \mathfrak{I}(A)+\mathfrak{I}(A)(B) \mathfrak{I}(A)\left(B_{1}\right) \mathfrak{I}(A)
$$

which shows $\mathfrak{I}$ is $C^{2}(\mathscr{O})$. Clearly we can continue in this way which shows $\mathfrak{I}$ is in $C^{m}(\mathscr{O})$ for all $m=1,2, \cdots$.

Here are the two fundamental results presented earlier which will make it easy to prove the implicit function theorem. First is the fundamental mean value inequality.

Theorem 22.10.5 Suppose $U$ is an open subset of $X$ and $f: U \rightarrow Y$ has the property that $D f(x)$ exists for all $x$ in $U$ and that, $x+t(y-x) \in U$ for all $t \in[0,1]$. (The line segment joining the two points lies in $U$.) Suppose also that for all points on this line segment,

$$
\|D f(x+t(y-x))\| \leq M .
$$

Then

$$
\|f(y)-f(x)\| \leq M|y-x|
$$

Next recall the following theorem about fixed points of a contraction map. It was Corollary 7.11.3.

Corollary 22.10.6 Let $B$ be a closed subset of the complete metric space $(X, d)$ and let $f: B \rightarrow X$ be a contraction map

$$
d(f(x), f(\hat{x})) \leq r d(x, \hat{x}), r<1
$$

Also suppose there exists $x_{0} \in B$ such that the sequence of iterates $\left\{f^{n}\left(x_{0}\right)\right\}_{n=1}^{\infty}$ remains in $B$. Then $f$ has a unique fixed point in $B$ which is the limit of the sequence of iterates. This is a point $x \in B$ such that $f(x)=x$. In the case that $B=\overline{B\left(x_{0}, \delta\right)}$, the sequence of iterates satisfies the inequality

$$
d\left(f^{n}\left(x_{0}\right), x_{0}\right) \leq \frac{d\left(x_{0}, f\left(x_{0}\right)\right)}{1-r}
$$

and so it will remain in $B$ if

$$
\frac{d\left(x_{0}, f\left(x_{0}\right)\right)}{1-r}<\delta
$$

The implicit function theorem deals with the question of solving, $f(\mathbf{x}, \mathbf{y})=0$ for $x$ in terms of $y$ and how smooth the solution is. It is one of the most important theorems in mathematics. The proof I will give holds with no change in the context of infinite dimensional complete normed vector spaces when suitable modifications are made on what is meant by $\mathscr{L}(X, Y)$. There are also even more general versions of this theorem than to normed vector spaces.

Recall that for $X, Y$ normed vector spaces, the norm on $X \times Y$ is of the form

$$
\|(x, y)\|=\max (\|x\|,\|y\|) .
$$

Theorem 22.10.7 (implicit function theorem) Let $X, Y, Z$ be Banach spaces and suppose $U$ is an open set in $X \times Y$. Let $f: U \rightarrow Z$ be in $C^{1}(U)$ and suppose

$$
\begin{equation*}
f\left(x_{0}, y_{0}\right)=0, D_{1} f\left(x_{0}, y_{0}\right)^{-1} \in \mathscr{L}(Z, X) \tag{22.10.20}
\end{equation*}
$$

Then there exist positive constants, $\delta, \eta$, such that for every $y \in B\left(y_{0}, \eta\right)$ there exists $a$ unique $x(y) \in B\left(x_{0}, \delta\right)$ such that

$$
\begin{equation*}
f(x(y), y)=0 . \tag{22.10.21}
\end{equation*}
$$

Furthermore, the mapping, $y \rightarrow x(y)$ is in $C^{1}\left(B\left(y_{0}, \eta\right)\right)$.
Proof: Let $T(x, y) \equiv x-D_{1} f\left(x_{0}, y_{0}\right)^{-1} f(x, y)$. Therefore,

$$
\begin{equation*}
D_{1} T(x, y)=I-D_{1} f\left(x_{0}, y_{0}\right)^{-1} D_{1} f(x, y) \tag{22.10.22}
\end{equation*}
$$

by continuity of the derivative which implies continuity of $D_{1} T$, it follows there exists $\delta>0$ such that if $\left\|x-x_{0}\right\|<\delta$ and $\left\|y-y_{0}\right\|<\delta$, then

$$
\begin{equation*}
\left\|D_{1} T(x, y)\right\|<\frac{1}{2}, D_{1} f(x, y)^{-1} \text { exists } \tag{22.10.23}
\end{equation*}
$$

The second claim follows from Lemma 22.10.4. By the mean value inequality, Theorem 22.10.5, whenever $x, x^{\prime} \in B\left(x_{0}, \boldsymbol{\delta}\right)$ and $y \in B\left(y_{0}, \boldsymbol{\delta}\right)$,

$$
\begin{equation*}
\left\|T(x, y)-T\left(x^{\prime}, y\right)\right\| \leq \frac{1}{2}\left\|x-x^{\prime}\right\| . \tag{22.10.24}
\end{equation*}
$$

Also, it can be assumed $\delta$ is small enough that for some $M$ and all such $(x, y)$,

$$
\begin{equation*}
\left\|D_{1} f\left(x_{0}, y_{0}\right)^{-1}\right\|\left\|D_{2} f(x, y)\right\|<M \tag{22.10.25}
\end{equation*}
$$

Next, consider only $y$ such that $\left\|y-y_{0}\right\|<\eta$ where $\eta$ is so small that

$$
\left\|T\left(x_{0}, y\right)-x_{0}\right\|<\frac{\delta}{3}
$$

Then for such $y$, consider the mapping $T_{y}(x)=T(x, y)$. Thus by Corollary 22.10.6, for each $n \in \mathbb{N}$,

$$
\delta>\frac{2}{3} \delta \geq \frac{\left\|T_{y}\left(x_{0}\right)-x_{0}\right\|}{1-(1 / 2)} \geq\left\|T_{y}^{n}\left(x_{0}\right)-x_{0}\right\|
$$

Then by 22.10 .24 , the sequence of iterations of this map $T_{y}$ converges to a unique fixed point $x(y)$ in the ball $B\left(x_{0}, \delta\right)$. Thus, from the definition of $T, f(x(y), y)=0$. This is the implicitly defined function.

Next we show that this function is Lipschitz continuous. For $y, \hat{y}$ in $B\left(y_{0}, \eta\right)$,

$$
\begin{gathered}
\|T(x, y)-T(x, \hat{y})\|= \\
\left\|D_{1} f\left(x_{0}, y_{0}\right)^{-1} f(x, y)-D_{1} f\left(x_{0}, y_{0}\right)^{-1} f(x, \hat{y})\right\| \leq M\|y-\hat{y}\|
\end{gathered}
$$

thanks to the above estimate 22.10 .25 and the mean value inequality, Theorem 22.10.5. Note how convexity of $B\left(y_{0}, \eta\right)$ which says that the line segment joining $y, \hat{y}$ is contained in $B\left(y_{0}, \eta\right)$ is important to use this theorem. Then from this,

$$
\begin{aligned}
\|x(y)-x(\hat{y})\|= & \|T(x(y), y)-T(x(\hat{y}), \hat{y})\| \leq\|T(x(y), y)-T(x(y), \hat{y})\| \\
+ & \|T(x(y), \hat{y})-T(x(\hat{y}), \hat{y})\| \\
& \leq M\|y-\hat{y}\|+\frac{1}{2}\|x(y)-x(\hat{y})\|
\end{aligned}
$$

Hence,

$$
\begin{equation*}
\|x(y)-x(\hat{y})\| \leq 2 M\|y-\hat{y}\| \tag{22.10.26}
\end{equation*}
$$

Finally consider the claim that this implicitly defined function is $C^{1}$.

$$
\begin{align*}
0= & f(x(y+u), y+u)-f(x(y), y) \\
= & D_{1} f(x(y), y)(x(y+u)-x(y))+D_{2} f(x(y), y) u \\
& +o(x(y+u)-x(y), u) \tag{22.10.27}
\end{align*}
$$

Consider the last term. $o(x(y+u)-x(y), u) /\|u\|$ equals

$$
\left\{\begin{array}{l}
\frac{o(x(y+u)-x(y), u)}{\|(x(y+u)-x(y), u)\|_{X \times Y}} \frac{\max (\|x(y+u)-x(y)\|,\|u\|)}{\|u\|} \text { if }\|(x(y+u)-x(y), u)\|_{X \times Y} \neq 0 \\
0 \text { if }\|(x(y+u)-x(y), u)\|_{X \times Y}=0
\end{array}\right.
$$

Now the Lipschitz condition just established shows that

$$
\frac{\max (\|x(y+u)-x(y)\|,\|u\|)}{\|u\|}
$$

is bounded for nonzero $u$ sufficiently small that $y, y+u \in B\left(y_{0}, \eta\right)$. Therefore,

$$
\lim _{u \rightarrow 0} \frac{o(x(y+u)-x(y), u)}{\|u\|}=0
$$

Then 22.10.27 shows that

$$
0=D_{1} f(x(y), y)(x(y+u)-x(y))+D_{2} f(x(y), y) u+o(u)
$$

Therefore, solving for $x(y+u)-x(y)$, it follows that

$$
\begin{aligned}
x(y+u)-x(y) & =-D_{1} f(x(y), y)^{-1} D_{2} f(x(y), y) u+D_{1} f(x(y), y)^{-1} o(u) \\
& =-D_{1} f(x(y), y)^{-1} D_{2} f(x(y), y) u+o(u)
\end{aligned}
$$

and now, the continuity of the partial derivatives $D_{1} f, D_{2} f$, continuity of the map $A \rightarrow A^{-1}$, along with the continuity of $y \rightarrow x(y)$ shows that $y \rightarrow x(y)$ is $C^{1}$ with derivative equal to $-D_{1} f(x(y), y)^{-1} D_{2} f(x(y), y)$.

It is easy to give a version of this theorem in which the function $f$ also depends on a parameter $\lambda \in \Lambda$, a metric space.

Corollary 22.10.8 Let $X, Y, Z$ be Banach spaces and suppose $U$ is an open set in $X \times Y$. Let $f: U \times \Lambda \rightarrow Z$ satisfy $f(\cdot, \cdot, \lambda)$ is in $C^{1}(U)$ and suppose for each $\lambda$,

$$
\begin{equation*}
f\left(x_{0}, y_{0}, \lambda\right)=0, D_{1} f\left(x_{0}, y_{0}, \lambda\right)^{-1} \in \mathscr{L}(Z, X) . \tag{22.10.28}
\end{equation*}
$$

Also suppose $(x, y) \rightarrow D_{1} f(x, y, \lambda)$ is continuous uniformly in $\lambda$ and $D_{2}(x, y, \lambda)$ is uniformly bounded in $\lambda$ for $(x, y)$ sufficiently close to $\left(x_{0}, y_{0}\right)$. Then there exist positive constants, $\delta, \eta$, such that for every $y \in B\left(y_{0}, \eta\right)$ there exists a unique $x(y, \lambda) \in B\left(x_{0}, \delta\right)$ such that

$$
\begin{equation*}
f(x(y, \lambda), y, \lambda)=0 \tag{22.10.29}
\end{equation*}
$$

Furthermore, the mapping, $y \rightarrow x(y, \lambda)$ is in $C^{1}\left(B\left(y_{0}, \eta\right)\right)$ and $\lambda \rightarrow x(y, \lambda)$ is continuous.
Proof: It is just a repeat of the above proof except you use the uniform contraction principle, Corollary 7.11.4 to get the fixed point.

The next theorem is a very important special case of the implicit function theorem known as the inverse function theorem. Actually one can also obtain the implicit function theorem from the inverse function theorem. It is done this way in [84], [96] and in [6].

Theorem 22.10.9 (inverse function theorem) Let $x_{0} \in U$, an open set in $X$, and let $f: U \rightarrow$ $Y$ where $X, Y$ are finite dimensional normed vector spaces. Suppose

$$
\begin{equation*}
f \text { is } C^{1}(U), \text { and } D f\left(x_{0}\right)^{-1} \in \mathscr{L}(Y, X) \tag{22.10.30}
\end{equation*}
$$

Then there exist open sets $W$, and $V$ such that

$$
\begin{gather*}
x_{0} \in W \subseteq U  \tag{22.10.31}\\
f: W \rightarrow V \text { is one to one and onto, }  \tag{22.10.32}\\
f^{-1} \text { is } C^{1} \tag{22.10.33}
\end{gather*}
$$

Proof: Apply the implicit function theorem to the function

$$
F(x, y) \equiv f(x)-y
$$

where $y_{0} \equiv f\left(x_{0}\right)$. Thus the function $y \rightarrow x(y)$ defined in that theorem is $f^{-1}$. Now let

$$
W \equiv B\left(x_{0}, \delta\right) \cap f^{-1}\left(B\left(y_{0}, \eta\right)\right)
$$

and

$$
V \equiv B\left(y_{0}, \eta\right)
$$

### 22.11 More Derivatives

When you consider a $C^{k}$ function $f$ defined on an open set $U$, you obtain the following

$$
D f(x) \in \mathscr{L}(X, Y), D^{2} f(x) \in \mathscr{L}(X, \mathscr{L}(X, Y)), D^{3} f(x) \in \mathscr{L}(X, \mathscr{L}(X, \mathscr{L}(X, Y)))
$$

and so forth. Thus they can each be considered as a linear transformation with values in some vector space. When you consider the vector spaces, you see that these can also be considered as multilinear functions on $X$ with values in $Y$. Now consider the product of two linear transformations $A(y) B(y) w$, where everything is given to make sense and here $w$ is an appropriate vector. Then if each of these linear transformations can be differentiated, you would do the following simple computation.

$$
\begin{gathered}
(A(y+u) B(y+u)-A(y) B(y))(w) \\
=(A(y+u) B(y+u)-A(y) B(y+u)+A(y) B(y+u)-A(y) B(y))(w) \\
=((D A(y) u+o(u)) B(y+u)+A(y)(D B(y) u+o(u)))(w) \\
=(D A(y)(u) B(y+u)+A(y) D B(y)(u)+o(u))(w) \\
=(D A(y)(u) B(y)+A(y) D B(y)(u)+o(u))(w)
\end{gathered}
$$

Then

$$
u \rightarrow(D A(y)(u) B(y)+A(y) D B(y)(u))(w)
$$

is clearly linear and

$$
(u, w) \rightarrow(D A(y)(u) B(y)+A(y) D B(y)(u))(w)
$$

is bilinear and continuous as a function of $y$. By this we mean that for a fixed choice of $(u, w)$ the resulting $Y$ valued function just described is continuous. Now if each of $A, B, D A, D B$ can be differentiated, you could replace $y$ with $y+\hat{u}$ and do a similar computation to obtain as many differentiations as desired, the $k^{t h}$ differentiation yielding a $k$ linear function. You can do this as long as $A$ and $B$ have derivatives. Now in the case of the implicit function theorem, you have

$$
\begin{equation*}
D x(y)=-D_{1} f(x(y), y)^{-1} D_{2} f(x(y), y) \tag{22.11.34}
\end{equation*}
$$

By Lemma 22.10.4 and the implicit function theorem and the chain rule, this is the situation just discussed. Thus $D^{2} x(y)$ can be obtained. Then the formula for it will only involve $D x$ which is known to be continuous. Thus one can continue in this way finding derivatives till $f$ fails to have them. The inverse map never creates difficulties because it is differentiable of order $m$ for any $m$ thanks to Lemma 22.10.4. Thus one can conclude the following corollary.

Corollary 22.11.1 In the implicit and inverse function theorems, you can replace $C^{1}$ with $C^{k}$ in the statements of the theorems for any $k \in \mathbb{N}$.

### 22.12 Lyapunov Schmidt Procedure

You have $f: X \times \Lambda \rightarrow Y$ where here $X, \Lambda$ are Banach spaces. Suppose $(0,0) \in X \times \Lambda$ and $f(0,0)=0$. Then if $D_{1} f(0,0)^{-1}$ is in $\mathscr{L}(Y, X)$, the implicit function theorem says that there exists $x(\lambda)$ a $C^{p}$ function such that locally $f(x(\lambda), \lambda)=0$. So what if $D_{1} f(0,0)$ fails to be one to one? Sometimes this case is also considered. It may be that $D_{1} f(0,0)$
is one to one on some subspace and other nice things happen. In particular, suppose the following.

Letting $X_{2} \equiv \operatorname{ker} D_{1} f(0,0)$ assume

$$
X=X_{1} \oplus X_{2}, \operatorname{dim}\left(X_{2}\right)<\infty
$$

where $X_{1}$ is a closed subspace. Thus $D_{1} f(0,0)$ is one to one on $X_{1}$. We let

$$
Y_{1}=D_{1} f(0,0)\left(X_{1}\right)
$$

and suppose that $Y=Y_{1} \oplus Y_{2}$ where $\operatorname{dim}\left(Y_{2}\right)<\infty, Y_{1}$ also a closed subspace.

$$
\begin{gathered}
X_{1} \xrightarrow{D_{1} f(0,0)} Y_{1}=D_{1} f(0,0)\left(X_{1}\right), Y_{1} \text { closed }<\infty \\
\quad Y=Y_{1} \oplus Y_{2}, \operatorname{dim}\left(Y_{2}\right)<\infty
\end{gathered}
$$

By the open mapping theorem, $D_{1} f(0,0)^{-1}$ is also continuous.
Let $Q$ be a continuous projection onto $Y_{1}$ which is assumed to exist ${ }^{2}$ so that $(I-Q)$ is a projection onto $Y_{2}$. Then the equation $f(x(\lambda), \lambda)=0$ can be written as the pair

$$
\begin{aligned}
Q f(x, \lambda) & =0 \\
(I-Q) f(x, \lambda) & =0
\end{aligned}
$$

Consider the top. For $x=x_{1}+x_{2}$ where $x_{i} \in X_{i}$, this is

$$
Q f\left(x_{1}+x_{2}, \lambda\right)=0
$$

Then if $g\left(x_{1}, x_{2}, \lambda\right)=Q f\left(x_{1}+x_{2}, \lambda\right)$, one has $g: X_{1} \times X_{2} \times \Lambda \rightarrow Y_{1}$

$$
D_{1} g\left(x_{1}, x_{2}, \lambda\right) h=D_{1} Q f\left(x_{1}+x_{2}, \lambda\right) h, h \in X_{1}
$$

Thus $D_{1} g(0,0,0)^{-1}$ is continuous by the open mapping theorem $\left(D_{1} f(0,0)\right.$ is one to one on $X_{1}$ ), and by the implicit function theorem, there is a solution to

$$
Q f\left(x_{1}+x_{2}, \lambda\right)=0
$$

for $x_{1}=x_{1}\left(x_{2}, \lambda\right)$. (Note how it is important that $X_{1}$ and $Y_{1}$ be Banach spaces.) Then the other equation yields

$$
(I-Q) f\left(x_{1}\left(x_{2}, \lambda\right)+x_{2}, \lambda\right)=0
$$

and so for fixed $\lambda$, this is a finite set of equations of a variable in a finite dimensional space.
This depends on being able to write $X=X_{1} \oplus X_{2}$ where $X_{1}$ is closed, $X_{2}=\operatorname{ker} D_{1} f(0,0)$, a similar situation for $Y=Y_{1} \oplus Y_{2}$. So when does this happen? Are there conditions on $D_{1} f(0,0)$ which will cause it to occur?

There are such conditions. For example, $D_{1} f(0,0)$ could be a Fredholm operator defined in Definition 17.6.7. The following are some easy examples in which all that nonsense about things being finite dimensional and part of a direct sum does not need to be considered.

[^21]Example 22.12.1 Say $X=\mathbb{R}^{2}$ and $\Lambda=\mathbb{R}$. Let $f(x, y, \lambda)=x+x y+y^{2}+\lambda$. Then

$$
D_{1} f(0,0,0)=(1,0)
$$

this $1 \times 2$ matrix mapping $\mathbb{R}^{2}$ to $\mathbb{R}$. Thus $X_{2}=(0, \alpha)^{T}: \alpha \in \mathbb{R}$ and $X_{1}=(\alpha, 0)^{T}: \alpha \in \mathbb{R}$. In this case, $Y_{1}=\mathbb{R}$ and so $Q=I$. Thus the above reduces to the single equation

$$
f((\alpha, 0)+(0, \beta), \lambda)=0
$$

and so since $D_{1} f(0,0,0)$ is one to one, $x_{1}=(\alpha, 0)=x_{1}((0, \beta), \lambda)$. Of course this is completely obvious because if you consider $f$ in the natural way as a function of three variables, then the implicit function theorem immediately gives $x=x(y, \lambda)$ which is essentially the same result. We just write $(\alpha, 0)$ in place of $\alpha$. The first independent variable is a function of the other two.

Example 22.12.2 Here is another easy example. $\mathbf{f}: \mathbb{R}^{2} \times \mathbb{R} \rightarrow \mathbb{R}^{2}$

$$
\mathbf{f}(x, y, \lambda)=\binom{x+x y+y^{2}+\sin (\lambda)}{x+y^{2}-x^{2}+\lambda}
$$

Then

$$
D_{1} \mathbf{f}(x, y, \lambda)=\left(\begin{array}{cc}
1+y & x+2 y \\
1-2 x & 2 y
\end{array}\right)
$$

So

$$
D_{1} \mathbf{f}((0,0), 0)=\left(\begin{array}{ll}
1 & 0 \\
1 & 0
\end{array}\right)
$$

Then

$$
X_{2}=\operatorname{ker} D_{1} \mathbf{f}((0,0), 0)=\left\{\binom{0}{\beta}: \beta \in \mathbb{R}\right\}
$$

and $X_{1}=\left\{\binom{\alpha}{0}: \alpha \in \mathbb{R}\right\}$ and clearly $D_{1} \mathbf{f}((0,0), 0)$ is indeed one to one on $X_{1}$.

$$
D_{1} \mathbf{f}(\mathbf{0}, 0)\left(X_{1}\right)=\left\{\binom{y}{y}: y \in \mathbb{R}\right\}=Y_{1}
$$

In this case, let

$$
Q\binom{\alpha}{\beta}=\binom{\frac{\alpha+\beta}{2}}{\frac{\alpha+\beta}{2}}=\left(\begin{array}{ll}
1 / 2 & 1 / 2 \\
1 / 2 & 1 / 2
\end{array}\right)\binom{\alpha}{\beta}
$$

so $(I-Q)=\left(\begin{array}{cc}1 / 2 & -1 / 2 \\ -1 / 2 & 1 / 2\end{array}\right)$. Thus the equations are

$$
\begin{aligned}
Q \mathbf{f}(\mathbf{x}, \lambda) & =0 \\
(I-Q) \mathbf{f}(\mathbf{x}, \lambda) & =0
\end{aligned}
$$

This reduces to

$$
\left.\left.\begin{array}{rl}
\left(-\frac{1}{2} x^{2}+\frac{1}{2} x y+x+y^{2}+\frac{1}{2} \lambda+\frac{1}{2} \sin \lambda\right. \\
-\frac{1}{2} x^{2}+\frac{1}{2} x y+x+y^{2}+\frac{1}{2} \lambda+\frac{1}{2} \sin \lambda
\end{array}\right)=\binom{0}{0} . \begin{array}{l}
\frac{1}{2} x^{2}+\frac{1}{2} y x-\frac{1}{2} \lambda+\frac{1}{2} \sin \lambda \\
-\frac{1}{2} x^{2}-\frac{1}{2} y x+\frac{1}{2} \lambda-\frac{1}{2} \sin \lambda
\end{array}\right)=\binom{0}{0} .
$$

Note how in both the top and the bottom, there is only one equation and one can solve for $x$ in terms of $y, \lambda$ near $(0,0,0)$ which is what the above general argument shows. Of course you can see this directly using the implicit function theorem. Then can you solve for $y=y(\lambda)$ ? This would involve trying to solve for $y$ as a function of $\lambda$ in the following where $x(y, \lambda)$ comes from the first equations.

$$
\frac{1}{2} x^{2}(y, \lambda)+\frac{1}{2} y x(y, \lambda)-\frac{1}{2} \lambda+\frac{1}{2} \sin \lambda=0
$$

If you can do this, then you would have found $(x, y)$ as a function of $\lambda$ for small $\lambda$.
In this example, in the top equation, at $(0,0,0), x_{y}=0$. Also $x_{\lambda}=-1$ so $x(y, \lambda) \approx-\lambda$ other than higher order terms for small $y, \lambda$. Then in the bottom equation, for all variables very small, you would have $\lambda^{2}+y(-\lambda)-\lambda+\sin (\lambda)=0, y(\lambda)=-1+\frac{\sin (\lambda)}{\lambda}+\lambda$ at least approximately. Thus it seems there is a nonzero solution to the equation $\mathbf{f}(x, y, \lambda)=0$ which is valid for small $\lambda, x, y$, this in addition to the zero solution. Note that for small nonzero $\lambda,-1+\frac{\sin (\lambda)}{\lambda}+\lambda \neq 0$. It equals approximately $\lambda-\frac{\lambda^{2}}{3!}$ for small $\lambda$ from the power series for $\sin$.

In the next example, the same procedure gives a solution to a problem $\mathbf{f}((x, y), \lambda)=\mathbf{0}$ such that for small $\boldsymbol{\lambda},(x, y)$ is a function of $\lambda$ which is nonzero and $\mathbf{f}((0,0), \lambda)=\mathbf{0}$. Thus for small $\lambda$, there are two solutions to the nonlinear system of equations.

Example 22.12.3 Let

$$
\mathbf{f}((x, y), \lambda)=\binom{x+x y+y^{2}+x \sin (\lambda)}{x+y^{2}-x^{2}+x \lambda}
$$

In this case $\mathbf{f}((0,0), \lambda)=\mathbf{0}$ even though $\lambda$ might not be 0 . The Lyapunov Schmidt procedure will be used to show that there are nonzero solutions $x(\boldsymbol{\lambda}), y(\lambda)$ such that

$$
f((x(\lambda), y(\lambda)), \lambda)=0
$$

At origin,

$$
D_{1} \mathbf{f}((0,0), 0)=\left(\begin{array}{ll}
1 & 0 \\
1 & 0
\end{array}\right)
$$

Thus $X_{1}=\operatorname{span}\left(e_{1}\right)$ and $X_{2}=\operatorname{span}\left(e_{2}\right)$. Then $Y_{1}=\operatorname{span}\left(e_{1}+e_{2}\right)$ and $Y_{2}=\operatorname{span}\left(e_{1}-e_{2}\right)$. Also $D_{1} \mathbf{f}((0,0), 0)$ is one to one on $X_{1}$ and its range is $Y_{1}$. Then let

$$
Q\binom{\alpha}{\beta}=\binom{\frac{\alpha+\beta}{2}}{\frac{\alpha+\beta}{2}}=\left(\begin{array}{ll}
1 / 2 & 1 / 2 \\
1 / 2 & 1 / 2
\end{array}\right)\binom{\alpha}{\beta}
$$

$$
(I-Q)=\left(\begin{array}{cc}
1 / 2 & -1 / 2 \\
-1 / 2 & 1 / 2
\end{array}\right)
$$

Then $Q \mathbf{f}=\mathbf{0}$ is yields the equation

$$
x+\frac{1}{2} x \lambda+\frac{1}{2} x \sin \lambda+\frac{1}{2} x y-\frac{1}{2} x^{2}+y^{2}=0
$$

Also $(I-Q) \mathbf{f}=\mathbf{0}$ yields the equation

$$
\frac{1}{2} x \sin \lambda-\frac{1}{2} x \lambda+\frac{1}{2} x y+\frac{1}{2} x^{2}=0
$$

Now consider $x_{y}$ and $x_{\lambda}$ at $(0,0)$ from the first equation. Both of these are easily seen to be 0 . Now consider $x_{y y}$. After some computations, this is seen to be $x_{y y}=-2$. Similarly, $x_{y \lambda}(0,0)=0, x_{\lambda \lambda}(0,0)=0$ also. Thus up to terms of degree 3,

$$
x(y, \lambda)=-y^{2}=\frac{1}{2}(-2) y^{2}
$$

Place this in the bottom equation.

$$
\frac{1}{2} y^{2} \lambda-\frac{1}{2} y^{2} \sin \lambda-\frac{1}{2} y^{3}+\frac{1}{2} y^{4}=0
$$

Now the idea is to find $y=y(\lambda)$, hopefully nonzero. Divide by $y^{2}$ and multiply by 2 .

$$
y^{2}-y+\lambda-\sin \lambda=0
$$

Then for small $\lambda$ this is approximately equal to

$$
y^{2}-y+\frac{\lambda^{3}}{6}=0
$$

Then a solution for $y$ for small $\lambda$ is

$$
y=\frac{1+\sqrt{1-\frac{2}{3} \lambda^{3}}}{2}
$$

Of course there is another solution as well, when you replace the + with a minus sign. This is the one we want because when $\lambda=0$ it reduces to $y=0$. This shows that there exist solutions to the equations $\mathbf{f}((x, y), \boldsymbol{\lambda})=\mathbf{0}$ which for small $\lambda$ are approximately

$$
(x(\lambda), y(\lambda))=\left(-y^{2}, \frac{1-\sqrt{1-\frac{2}{3} \lambda^{3}}}{2}\right)
$$

In terms of $\lambda$ very small,

$$
(x(\lambda), y(\lambda))=\left(\frac{1}{6} \lambda^{3}+\frac{1}{6} \sqrt{3} \sqrt{3-2 \lambda^{3}}-\frac{1}{2}, \frac{1-\sqrt{1-\frac{2}{3} \lambda^{3}}}{2}\right)
$$

Using a power series in $\lambda$ to approximate these functions, this reduces to

$$
(x(\lambda), y(\lambda))=\left(-\frac{1}{36} \lambda^{6}, \frac{1}{6} \lambda^{3}+\frac{1}{36} \lambda^{6}+\frac{1}{108} \lambda^{9}\right)
$$

where higher order terms are neglected. Thus there exist other solutions than the zero solution even though $\lambda$ may be nonzero. Note that in this example, $\mathbf{f}((0,0), \lambda)=0$.

### 22.13 Analytic Functions

In calculus, there was a difference between functions of a real variable and functions of a complex variable. In the latter case the existence of a single derivative implied the existence of all derivatives and in fact the Taylor series converged to the function. It is reasonable to ask if a similar phenomenon occurs in the case of complex Banach spaces versus real Banach spaces. This section presents a quick introduction to this topic based on the assumption that the reader has had some exposure to complex analysis. Some of the details involving questions of convergence and term by term differentiation are left to the reader. Also if $h$ maps an open subset of $\mathbb{C}$ to a complex Banach space $X$, and has a first derivative, then the usual Cauchy integral formula,

$$
h(z)=\frac{1}{2 \pi i} \int_{C} \frac{h(w)}{w-z} d w
$$

holds if $C$ is a circle contained, together with its interior, in the open set on which $h$ has a derivative. The integral can be defined as the ordinary Riemann integral using Riemann sums or it can be defined in terms of a Bochner integral. These details are routine and are left to the reader. There are several equivalent definitions of an analytic function defined on a complex Banach space. The following is the one we will use since it resembles the familiar definition encountered in undergraduate complex variable courses.

Definition 22.13.1 Let $X$ and $Y$ be complex Banach spaces and let $U \subseteq X$ be an open set. We say $f: U \rightarrow Y$ is analytic and bounded on $U$ if

$$
z \rightarrow f(x+z h) \text { is analytic for } x \in U, h \in X \text { and }|z| \text { small enough }
$$

exists for all $x \in U$ and also $\|f(x)\| \leq M<\infty$ for all $x \in U$. Here $z \in \mathbb{C}$ and $x, h \in X$.
Let $\mathbf{h} \in X^{l}$ and consider all $\mathbf{z} \in \mathbb{C}^{l}$ with $\|\mathbf{z}\|_{\mathbb{C}^{l}} \equiv \max \left(\left|z_{m}\right|, m=1, \cdots, l\right)$ sufficiently small. Let $C_{1}$ be a sufficiently small circle centered at 0 . Then consider

$$
z_{m} \rightarrow f\left(x+\sum_{m=1}^{l} z_{m} h_{m}\right)
$$

which is analytic on and inside $C_{1}$. Thus using the Cauchy integral formula,

$$
f\left(x+z_{1} h_{1}+\sum_{m=2}^{l} z_{m} h_{m}\right)
$$

$$
\begin{gathered}
=\frac{1}{2 \pi i} \int_{C_{1}} \frac{f\left(x+w_{1} h_{1}+\sum_{m=2}^{l} z_{m} h_{m}\right)}{\left(w_{1}-z_{1}\right)} d w_{1} \\
=\left(\frac{1}{2 \pi i}\right)^{2} \int_{C_{1}} \frac{1}{w_{1}-z_{1}} . \\
\int_{C_{1}} \frac{f\left(x+w_{1} h_{1}+w_{2} h_{2}+\sum_{m=3}^{l} z_{m} h_{m}\right)}{\left(w_{2}-z_{2}\right)} d w_{2} d w_{1}= \\
\left(\frac{1}{2 \pi i}\right)^{l} \int_{C_{1}} \cdots \int_{C_{1}} \frac{f\left(x+w_{1} h_{1}+w_{2} h_{2}+\cdots+w_{l} h_{l}\right)}{\prod_{m=1}^{l}\left(w_{m}-z_{m}\right)} d w_{l} \cdots d w_{1}
\end{gathered}
$$

Consider the case when $l=2$.

$$
\begin{gathered}
\left(\frac{1}{2 \pi i}\right)^{2} \int_{C_{1}} \int_{C_{1}} \frac{f\left(x+w_{1} h_{1}+w_{2} h_{2}\right)}{\left(w_{1}-z_{1}\right)\left(w_{2}-z_{2}\right)} d w_{2} d w_{1}= \\
\left(\frac{1}{2 \pi i}\right)^{2} \int_{C_{1}} \int_{C_{1}} f\left(x+w_{1} h_{1}+w_{2} h_{2}\right) \\
\sum_{k_{2}=0}^{\infty} \frac{z_{2}^{k_{2}}}{w_{2}^{k_{2}+1}} \sum_{k_{1}=0}^{\infty} \frac{z_{1}^{k_{1}}}{w_{1}^{k_{1}+1}} d w_{2} d w_{1}= \\
\left(\frac{1}{2 \pi i}\right)^{2} \sum_{k_{2}=0}^{\infty} \sum_{k_{1}=0}^{\infty}\left(\int_{C_{1}} \int_{C_{1}} \frac{f\left(x+w_{1} h_{1}+w_{2} h_{2}\right)}{w_{2}^{k_{2}+1} w_{1}^{k_{1}+1}} d w_{2} d w_{1}\right) z_{2}^{k_{2}} z_{1}^{k_{1}} .
\end{gathered}
$$

Similarly, for arbitrary $l$, and letting $C$ be any circle centered at 0 with radius smaller than $\frac{\delta}{l}$,

$$
\begin{equation*}
f\left(x+\sum_{m=1}^{l} z_{m} h_{m}\right)=\sum_{k_{l}=0}^{\infty} \cdots \sum_{k_{1}=0}^{\infty} a_{k_{1} \cdots k_{l}}\left(x, h_{l}, \cdots, h_{1}\right) z_{1}^{k_{1}} \cdots z_{l}^{k_{k}} \tag{22.13.35}
\end{equation*}
$$

where

$$
\begin{gather*}
a_{k_{1} \cdots k_{l}}\left(x, h_{l}, \cdots, h_{1}\right) \\
=\left(\frac{1}{2 \pi i}\right)^{l} \int_{C} \cdots \int_{C} \frac{f\left(x+\sum_{m=1}^{l} w_{m} h_{m}\right)}{\prod_{m=1}^{l} w_{m}^{k_{m}+1}} d w_{1} \cdots d w_{l} . \tag{22.13.36}
\end{gather*}
$$

Lemma 22.13.2 Let $l \geq 1$ and let $t_{m} \in \mathbb{C}$. Then if $\mathbf{h} \in X^{l}$, then whenever $|z|$ is small enough, 22.13.35 holds. Also the coefficients satisfy

$$
\begin{equation*}
a_{k_{1} \cdots k_{l}}\left(x, t_{l} h_{l}, \cdots, t_{1} h_{1}\right)=\left(\prod_{m=1}^{l} t_{m}^{k_{m}}\right) a_{k_{1} \cdots k_{l}}\left(x, h_{l}, \cdots, h_{1}\right) \tag{22.13.37}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|a_{k_{1} \cdots k_{l}}\left(x, h_{l}, \cdots, h_{1}\right)\right\| \leq C \prod_{m=1}^{l}\left\|h_{m}\right\| \tag{22.13.38}
\end{equation*}
$$

for some constant $C$.

Proof: Let $C$ be small enough that the circles $t_{m} C$ for all $m=1, \cdots, l$ and $C$ have radius less than $\frac{\delta}{l}$. First assume $t_{m} \neq 0$ for all $m$. Then

$$
\begin{gathered}
a_{k_{1} \cdots k_{l}}\left(x, t_{l} h_{l}, \cdots, t_{1} h_{1}\right) \\
=\left(\frac{1}{2 \pi i}\right)^{l} \int_{C} \cdots \int_{C} \frac{f\left(x+\sum_{m=1}^{l} w_{m} t_{m} h_{m}\right)}{\prod_{m=1}^{l}\left(w_{m} t_{m}\right)^{k_{m}+1}} . \\
\prod_{m=1}^{l} t_{m}^{k_{m}+1} d w_{1} \cdots d w_{l}
\end{gathered}
$$

Here we just multiplied and divided by $\prod_{m=1}^{l} t_{m}^{k_{m}+1}$.

$$
\begin{gathered}
=\left(\frac{1}{2 \pi i}\right)^{l} \int_{t_{l} C} \cdots \int_{t_{1} C} \frac{f\left(x+\sum_{m=1}^{l} u_{m} h_{m}\right)}{\prod_{m=1}^{l}\left(u_{m}\right)^{k_{m}+1}} d u_{1} \cdots d u_{l} \prod_{m=1}^{l} t_{m}^{k_{m}} \\
=a_{k_{1} \cdots k_{l}}\left(x, h_{l}, \cdots, h_{1}\right) \prod_{m=1}^{l} t_{m}^{k_{m}} .
\end{gathered}
$$

Formally, $w_{i} \in C$ and so $t_{i} w_{i} \equiv u_{i} \in t_{i} C$. Then $t_{i} d w_{i}=d u_{i}$ and so $d w_{i}=\left(1 / t_{i}\right) d u_{i}$. This is why $\prod_{m=1}^{l} t_{m}^{k_{m}+1}$ gets changed to $\prod_{m=1}^{l} t_{m}^{k_{m}}$.

If $t_{m}=0$ for any $m$, the result of both sides in the above equals zero due to the fact that

$$
\int_{C} \frac{1}{w_{m}^{k_{m}+1}} d w_{m}=0
$$

whenever $k_{m} \geq 1$.
To verify 22.13 .38 , use 22.13 .37 to conclude

$$
\begin{gathered}
\left\|a_{k_{1} \cdots k_{l}}\left(x, h_{l} \cdots h_{1}\right)\right\| \leq \\
\left\|a_{k_{1} \cdots k_{l}}\left(x, \frac{h_{l}}{\left\|h_{l}\right\|} \cdots \frac{h_{1}}{\left\|h_{1}\right\|}\right)\right\| \prod_{m=1}^{l}\left\|h_{m}\right\|^{k_{m}}
\end{gathered}
$$

and $\left\|a_{k_{1} \cdots k_{l}}\left(x, \frac{h_{l}}{\left\|h_{l}\right\|} \cdots \frac{h_{1}}{\left\|h_{1}\right\|}\right)\right\|$ is bounded by

$$
\frac{M}{(2 \pi)^{l}} \int_{C} \cdots \int_{C} \frac{1}{\prod_{m=1}^{l}\left|w_{m}\right|^{k_{m}+1}} d\left|w_{1}\right| \cdots d\left|w_{l}\right| \equiv C
$$

Lemma 22.13.3 Suppose

$$
g(x+z h)=g(x)+\sum_{m=1}^{\infty} b_{m}(x, h) z^{m}
$$

for all $z$ small enough. Then

$$
b_{1}\left(x, h_{1}+h_{2}\right)=b_{1}\left(x, h_{1}\right)+b_{1}\left(x, h_{2}\right) .
$$

Proof: Recall that

$$
f\left(x+\sum_{m=1}^{l} z_{m} h_{m}\right)=\sum_{k_{l}=0}^{\infty} \cdots \sum_{k_{1}=0}^{\infty} a_{k_{1} \cdots k_{l}}\left(x, h_{l}, \cdots, h_{1}\right) z_{1}^{k_{1}} \cdots z_{l}^{k_{k}}
$$

and so one can write the following where $g_{n m}$ is defined in the following expression.

$$
g\left(x+z_{1} h_{1}+z_{2} h_{2}\right)=\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} g_{m n}\left(x, h_{1}, h_{2}\right) z_{1}^{m} z_{2}^{n}
$$

Thus,

$$
\begin{gathered}
g\left(x+z_{1} h_{1}\right)=\sum_{m=0}^{\infty} g_{m 0}\left(x, h_{1}, h_{2}\right) z_{1}^{m}=g(x)+\sum_{m=1}^{\infty} b_{m}(x, h) z_{1}^{m}, \\
g\left(x+z_{2} h_{2}\right)=\sum_{n=0}^{\infty} g_{0 n}\left(x, h_{1}, h_{2}\right) z_{2}^{n}=g(x)+\sum_{n=1}^{\infty} b_{n}(x, h) z_{2}^{n}
\end{gathered}
$$

which implies

$$
g_{m 0}\left(x, h_{1}, h_{2}\right)=b_{m}\left(x, h_{1}\right), g_{0 n}\left(x, h_{1}, h_{2}\right)=b_{n}\left(x, h_{2}\right)
$$

Now let $z_{1}=z_{2}=z$. Then

$$
\begin{gathered}
g\left(x+z\left(h_{1}+h_{2}\right)\right)=g(x)+\sum_{n=0}^{\infty} b_{n}\left(x, h_{1}+h_{2}\right) z^{n} \\
=g(x)+z\left(g_{10}\left(x, h_{1}, h_{2}\right)+g_{01}\left(x, h_{1}, h_{2}\right)\right)+\text { higher order terms in } z .
\end{gathered}
$$

Therefore,

$$
\begin{aligned}
b_{1}\left(x, h_{1}+h_{2}\right) & =g_{10}\left(x, h_{1}, h_{2}\right)+g_{01}\left(x, h_{1} h_{2}\right) \\
& =b_{1}\left(x, h_{1}\right)+b_{1}\left(x, h_{2}\right)
\end{aligned}
$$

Lemma 22.13.4 Suppose $a\left(x, h_{l}, \cdots, h_{1}\right)$ is multilinear, $\left(h_{i} \rightarrow a\left(x, h_{l}, \cdots, h_{1}\right)\right.$ is linear $)$,

$$
\left\|a\left(x, h_{l}, \cdots, h_{1}\right)\right\| \leq C \prod_{m=1}^{l}\left\|h_{m}\right\|
$$

and

$$
\begin{gathered}
D^{l-1} f\left(x+h_{l}\right)\left(h_{l-1}\right) \cdots\left(h_{1}\right)-D^{l-1} f(x)\left(h_{l-1}\right) \cdots\left(h_{1}\right) \\
-a\left(x, h_{l}, \cdots, h_{1}\right)=o\left(\left\|h_{l}\right\|\right)
\end{gathered}
$$

Then $D^{l} f(x)$ exists and

$$
D^{l} f(x)\left(h_{l}\right)\left(h_{l-1}\right) \cdots\left(h_{1}\right)=a\left(x, h_{l}, \cdots, h_{1}\right) .
$$

Proof: If $l=1$, the conclusion is obvious and is nothing more than the definition of the derivative.

$$
f(x+h)-f(x)-a(x, h)=o(\|h\|)
$$

and so from the definition of the derivative, $a(x, h)=D f(x) h$.
Next let $n=2$. By assumption,

$$
D f(x+h)\left(h_{1}\right)-D f(x)\left(h_{1}\right)-a\left(x, h, h_{1}\right)=o(\|h\|) .
$$

Let $L(x)$ be defined by

$$
L(x)(h)\left(h_{1}\right) \equiv a\left(x, h, h_{1}\right)
$$

Then $L(x) \in \mathscr{L}(U, \mathscr{L}(X, Y))$ because

$$
\|L(x)\| \equiv \sup _{\|h\| \leq 1}\|L(x)(h)\| \equiv \sup _{\|h\| \leq 1\left\|h_{1}\right\| \leq 1} \sup _{\|}\left\|L(x)(h)\left(h_{1}\right)\right\| \leq C .
$$

Also

$$
\begin{gathered}
\|D f(x+h)-D f(x)-L(x) h\| \\
\equiv \sup _{\left\|h_{1}\right\| \leq 1}\left\|D f(x+h)\left(h_{1}\right)-D f(x)\left(h_{1}\right)-L(x)(h)\left(h_{1}\right)\right\| \\
=\sup _{\left\|h_{1}\right\| \leq 1}\left\|D f(x+h)\left(h_{1}\right)-D f(x)\left(h_{1}\right)-a\left(x, h, h_{1}\right)\right\|=o(\|h\|)
\end{gathered}
$$

and so $L(x)=D^{2} f(x)$. Continuing in this way, we verify the conclusion of the lemma.
Lemma 22.13.5 If $f$ is analytic on $U$, then $f \in C^{\infty}(U)$. Also
Proof: By Lemma 22.13.3 applied to $g=f$ and Lemma 22.13.2, $D f(x)$ exists and

$$
D f(x)(h)=a_{1}(x, h)
$$

These lemmas implied that $h \rightarrow a_{1}(x, h)$ was linear. Suppose $D^{l-1} f(x)$ exists for $l \geq 2$.

$$
f\left(x+\sum_{m=1}^{l} z_{m} h_{m}\right)=\sum_{n_{l}=0}^{\infty} \cdots \sum_{n_{1}=0}^{\infty} a_{n_{1} \cdots n_{l}}\left(x, h_{l}, \cdots, h_{1}\right) z_{1}^{n_{1}} \cdots z_{l}^{n_{l}}
$$

Differentiate with respect to $z_{1}, \cdots, z_{l-1}$ to obtain

$$
\begin{gathered}
D^{l-1} f\left(x+\sum_{m=1}^{l-1} z_{m} h_{m}+z_{l} h_{l}\right)\left(h_{l-1}\right) \cdots\left(h_{1}\right)= \\
\sum_{n_{l}=0}^{\infty} \sum_{n_{l-1}=1}^{\infty} \cdots \sum_{n_{1}=1}^{\infty} a_{n l n_{l-1} \cdots n_{1}}\left(x, h_{1} \cdots h_{l}\right)\left(\prod_{m=1}^{l-1} n_{m}\right) z_{1}^{n_{1}-1} \cdots z_{l-1}^{n_{l-1}-1} z_{l}^{n_{l}} .
\end{gathered}
$$

Take $z_{i}=0$ for $i=1, \cdots, l-1$. Then

$$
\begin{equation*}
D^{l-1} f\left(x+z_{l} h_{l}\right)\left(h_{l-1}\right) \cdots\left(h_{1}\right)=\sum_{n_{l}=0}^{\infty} a_{n_{l} 1 \cdots 1}\left(x, h_{l}, \cdots, h_{1}\right) z_{l}^{n_{l}} . \tag{22.13.39}
\end{equation*}
$$

Now we apply Lemma 22.13.3 to the function

$$
z_{l} \rightarrow D^{l-1} f\left(x+z_{l} h_{l}\right)\left(h_{l-1}\right) \cdots\left(h_{1}\right)
$$

and conclude

$$
h_{l} \rightarrow a_{1 \cdots 1}\left(x, h_{l}, \cdots, h_{1}\right)
$$

is linear. This involved taking $n_{l}=1$ to get $a_{1 \cdots 1}\left(x, h_{l}, \cdots, h_{1}\right)$. Thus from 22.13.39,

$$
\begin{gather*}
D^{l-1} f\left(x+z_{l} h_{l}\right)\left(h_{l-1}\right) \cdots\left(h_{1}\right)-D^{l-1} f(x)\left(h_{l-1}\right) \cdots\left(h_{1}\right) \\
=a_{1 \cdots 1}\left(x, h_{l}, \cdots, h_{1}\right) z_{l}+o\left(z_{l} h_{l}\right) . \tag{22.13.40}
\end{gather*}
$$

From this equation, it follows that

$$
\begin{gathered}
a_{1 \cdots 1}\left(x, h_{l} \cdots h_{i}+\widehat{h}_{i} \cdots h_{1}\right) z_{l}-a_{1 \cdots 1}\left(x, h_{l} \cdots h_{i} \cdots h_{1}\right) z_{l} \\
-a_{1 \cdots 1}\left(x, h_{l} \cdots \widehat{h}_{i} \cdots h_{1}\right) z_{l}=o\left(z_{l} h_{l}\right)
\end{gathered}
$$

because for each $z_{l}$, the left side of 22.13 .40 is linear in $h_{i}$ for each $i \leq l-1$. Dividing both sides of the above by $z_{l}$ and then letting $z_{l} \rightarrow 0$, we see that $a_{n_{l} 1 \cdots 1}$ is linear in each of the $h_{i}$. Denoting $z_{l} h_{l}$ by $h_{l}$,

$$
\begin{gathered}
D^{l-1} f\left(x+h_{l}\right)\left(h_{l-1}\right) \cdots\left(h_{1}\right)-D^{l-1} f(x)\left(h_{l-1}\right) \cdots\left(h_{1}\right) \\
=a_{1 \cdots 1}\left(x, h_{l}, \cdots, h_{1}\right)+o\left(\left\|h_{l}\right\|\right)
\end{gathered}
$$

and so by Lemma 22.13.4, $D^{l} f(x)$ exists and

$$
D^{l} f(x)\left(h_{l}\right) \cdots\left(h_{1}\right)=a_{1 \cdots 1}\left(x, h_{l}, \cdots, h_{1}\right)
$$

With these lemmas, the main result can be established. This is the generalization of the well known result for analytic functions.

Theorem 22.13.6 Let $X$ and $Y$ be two complex Banach spaces and let $U$ be an open set in $X$. Then $f: U \rightarrow Y$ is analytic on $U$ if and only if $D f(x)$ exists for each $x \in U$ and in this case, $f \in C^{\infty}(U)$, and if $h \in X$, then whenever $z$ is small enough,

$$
f(x+z h)=f(x)+\sum_{n=1}^{\infty} \frac{D^{n} f(x) h^{n} z^{n}}{n!} .
$$

Proof: We know

$$
f(x+z h)=f(x)+\sum_{n=1}^{\infty} a_{n}(x, h) z^{n} .
$$

Differentiating, we obtain

$$
D^{k} f(x+z h) h^{k}=k!a_{k}(x, h)+\sum_{n=k+1}^{\infty} n(n-1) \cdots(n-k+1) z^{n-k}
$$

Letting $z=0$ this shows

$$
D^{k} f(x) h^{k}=k!a_{k}(x, h)
$$

and this proves half the theorem.
Conversely, if $D f(x)$ exists on $U$, it is clear that $f$ is analytic on some ball, $B(x, r) \subseteq$ $U, z \rightarrow f(y+z h)$ is analytic for $y \in B(x, r)$ and small enough $z$. Therefore the formula involving the series follows.

### 22.14 Ordinary Differential Equations

In this section we give an application to ordinary differential equations. To begin with, here are two Banach spaces which will be of use. Let $Z$ be a complex Banach space and let $X$ be the space of functions mapping $\overline{B(0,1)} \equiv D_{1}$ to $Z$ such that the functions are continuous on $D_{1}$ and analytic on $B_{1} \equiv B(0,1)$, the derivative is the restriction to $B_{1}$ of a continuous function defined on $D_{1}$, and the function equals 0 at 0 .

$$
X \equiv\left\{\phi \in C\left(D_{1}, X\right): \phi(0)=0\right\}
$$

The norm on $X$ will be

$$
\|\phi\|_{X} \equiv\|\phi\|_{\infty}+\left\|\phi^{\prime}\right\|_{\infty}
$$

where

$$
\|\phi\|_{\infty} \equiv \sup \left\{\|\phi(t)\|_{Z}: t \in B_{1}\right\}
$$

(Note that for a function continuous on $D_{1}$ it does not matter in the above definition of $\|\cdot\|_{\infty}$ whether we use $B_{1}$ or $D_{1}$ in the definition.) We define $Y$ to be the space of continuous functions which are defined on $D_{1}$ having values in $Z$ which are also analytic on $B_{1}$. The norm on $Y$ is defined as

$$
\|\phi\|_{\infty} \equiv\|\phi\|_{Y} .
$$

Note that $B_{1}$ is in $\mathbb{C}$.
Lemma 22.14.1 The spaces $X$ and $Y$ with the given norms are Banach spaces and if $L$ : $X \rightarrow Y$ is defined as $L \phi(t)=\phi^{\prime}(t)$ for all $t \in B_{1}$, then $L$ is one to one, onto and continuous.

Proof: It is clear that $X$ and $Y$ are both normed linear spaces. It remains to show they are Banach spaces. Suppose $\left\{\phi_{n}\right\}$ is a Cauchy sequence in $X$. Then $\phi_{n} \rightarrow \phi$ uniformly and $\phi_{n}^{\prime} \rightarrow \psi$ uniformly where $\psi$ and $\phi$ are continuous on $D_{1}$. We need to verify that $\psi=\phi^{\prime}$ on $B_{1}$. Letting $C_{1}$ be the unit circle, the Cauchy integral formula implies for $t \in B_{1}$,

$$
\phi(t)=\lim _{n \rightarrow \infty} \phi_{n}(t)=\lim _{n \rightarrow \infty} \frac{1}{2 \pi i} \int_{C_{1}} \frac{\phi_{n}(w)}{w-t} d w=\frac{1}{2 \pi i} \int_{C_{1}} \frac{\phi(w)}{w-t} d w
$$

which shows $\phi^{\prime}(t)$ exists on $B_{1}$. Also for $t \in B_{1}$,

$$
\begin{aligned}
\psi(t)= & \lim _{n \rightarrow \infty} \phi_{n}^{\prime}(t)=\lim _{n \rightarrow \infty} \frac{1}{2 \pi i} \int_{C_{1}} \frac{\phi_{n}(w)}{(w-t)^{2}} d w \\
& =\frac{1}{2 \pi i} \int_{C_{1}} \frac{\phi(w)}{(w-t)^{2}} d w=\phi^{\prime}(t)
\end{aligned}
$$

This shows $X$ is a Banach space. A similar argument using the Cauchy integral theorem shows $Y$ is a Banach space also. It is obvious that $L$ is continuous. It remains to show $L$ is one to one and onto.

Let $\phi \in Y$. We need to show $\phi=L \psi$ for some $\psi \in X$. Let

$$
\psi(t) \equiv \int_{\Gamma} \phi(w) d w
$$

where $\Gamma$ is any piecewise smooth curve from 0 to $t$. By the Cauchy integral theorem, this definition is well defined and it is clear that $\psi(0)=0, \psi^{\prime}(t)=\phi(t)$, and $\psi$ is continuous on $D_{1}$. This shows $L$ is onto.

It only remains to show $L$ is one to one. Suppose $L \phi=0$. Since $\phi(0)=0$,

$$
\phi(t)=\int_{0}^{1} \phi^{\prime}(t s) t d s=0
$$

if $t \neq 0$. But $\phi(0)$ is given to equal zero. Thus $L$ is one to one as claimed.
Theorem 22.14.2 Let $\Lambda$ and $Z$ be complex Banach spaces and let $W$ be an open subset of $\mathbb{C} \times Z \times \Lambda$ containing $\left(0, y_{0}, \lambda\right)$. Also let $f: W \rightarrow Z$ be analytic. Then there exists a unique $y=y\left(y_{0}, \lambda\right)$ solving

$$
\begin{equation*}
y^{\prime}=f(t, y, \lambda), y(0)=y_{0} \tag{22.14.41}
\end{equation*}
$$

valid for $t \in D_{\alpha} \equiv \overline{B(0,|\alpha|)}$ where $\alpha=\alpha\left(y_{0}, \lambda\right)$. Furthermore, the map

$$
\left(t, y_{0}, \lambda\right) \rightarrow y\left(y_{0}, \lambda\right)(t)
$$

is analytic.
Proof: Let $\alpha s=t$ and define $\phi(s) \equiv y(t)-\underline{y_{0}}$. Then $y$ is a solution to 22.14.41 for $t \in D_{\alpha}$ if and only if $\phi$ is a solution for $s \in D_{1} \equiv \overline{B(0,1)}$ to the equations

$$
\phi^{\prime}(s)=\alpha f\left(\alpha s, \phi(s)+y_{0}, \lambda\right), \phi(0)=0 .
$$

Let $X, Y$, and $L$ be given above and define

$$
\begin{gathered}
\widetilde{W} \equiv\left\{\left(\alpha, \widehat{y_{0}}, \mu, \phi\right) \in \mathbb{C} \times Z \times \Lambda \times X:\right. \\
\text { for } \left.s \in D_{1},\left(s \alpha, \widehat{y_{0}}+\phi(s), \mu\right) \in W\right\} .
\end{gathered}
$$

For a given $\left(\alpha, \widehat{y_{0}}, \mu, \phi\right) \in \widetilde{W}$,

$$
\left\{\left(s \alpha, \widehat{y_{0}}+\phi(s), \mu\right): s \in D_{1}\right\}
$$

is a compact subset of $W$. This is because you have $s \rightarrow\left(\alpha, \widehat{y_{0}}+\phi(s), \mu\right)$ is the continuous image of a compact set which is assumed to be in $W$. Consequently, the distance from this set to $W^{C}$ is positive and so if $\left(\beta, y_{0}, \lambda, \psi\right)$ is sufficiently close to $\left(\alpha, \widehat{y_{0}}, \mu, \phi\right)$ in $\mathbb{C} \times Z \times$ $\Lambda \times X$ it follows $\left(\beta, y_{0}, \lambda, \psi\right)$ is also in $\widetilde{W}$. This shows $\widetilde{W}$ is an open subset of $\mathbb{C} \times Z \times \Lambda \times X$.

Now define $F: \widetilde{W} \rightarrow Y$ (Recall that $Y$ was a space of functions.) by

$$
F\left(\alpha, \widehat{y_{0}}, \mu, \phi\right)(s) \equiv L \phi(s)-\alpha f\left(\alpha s, \phi(s)+\widehat{y_{0}}, \mu\right)
$$

Then

$$
F\left(0, y_{0}, \lambda, 0\right)=L \phi=0
$$

and $F$ is analytic in $\widetilde{W}$. Also

$$
D_{4} F\left(0, y_{0}, \lambda, 0\right) \psi=L \psi=\psi^{\prime}
$$

and so $D_{4} F\left(0, y_{0}, \lambda, 0\right) \in \mathscr{L}(X, Y)$, is one to one, onto and continuous by Lemma 22.14.1.
By the open mapping theorem, its inverse is also continuous. Therefore, the conditions of the implicit function theorem are satisfied and so there exists $r>0$ such that if

$$
|\alpha|+\|\mu-\lambda\|+\left\|\widehat{y_{0}}-y_{0}\right\|<r
$$

then there exists a unique $\phi \in X$ such that

$$
F\left(\alpha, \widehat{y_{0}}, \mu, \phi\right)=0
$$

and $\phi$ is an analytic function of $\left(\alpha, \widehat{y_{0}}, \mu\right)$. Fixing $0<\alpha<r$, it follows

$$
\left(\widehat{y_{0}}, \mu\right) \rightarrow y\left(\widehat{y_{0}}, \mu\right)
$$

is analytic on an open subset of $Z \times \Lambda$. Also $t \rightarrow y\left(\widehat{y_{0}}, \mu\right)(t)$ is an analytic function because of the definition of $y$ in terms of $\phi, \phi(s) \equiv y(t)-y_{0}$. It follows that for $t \in B(0,|\alpha|)$,

$$
\left(\widehat{y_{0}}, \mu\right) \rightarrow y\left(\widehat{y_{0}}, \mu\right)(t) \text { and } t \rightarrow y\left(\widehat{y_{0}}, \mu\right)(t)
$$

are both analytic.

### 22.15 Exercises

1. Suppose $L \in \mathscr{L}(X, Y)$ where $X$ and $Y$ are two finite dimensional normed vector spaces and suppose $L$ is one to one. Show there exists $r>0$ such that for all $\mathbf{x} \in X$,

$$
\|L \mathbf{x}\| \geq r\|\mathbf{x}\|
$$

Hint: Show that $\|\mathbf{x}\| \equiv\|L \mathbf{x}\|$ is a norm. Now suppose $L \in \mathscr{L}(X, Y)$ is one to one and onto for $X, Y$ Banach spaces. Explain why the same result holds. Hint: Recall open mapping theorem.
2. Suppose $B$ is an open ball in $X$, a Banach space, and $f: B \rightarrow Y$ is differentiable. Suppose also there exists $L \in \mathscr{L}(X, Y)$ such that

$$
\|D f(x)-L\|<k
$$

for all $x \in B$. Show that if $x_{1}, x_{2} \in B$,

$$
\left\|f\left(x_{1}\right)-f\left(x_{2}\right)-L\left(x_{1}-x_{2}\right)\right\| \leq k\left\|x_{1}-x_{2}\right\|
$$

Hint: Consider $T x=f(x)-L x$ and argue $\|D T(x)\|<k$.
3. $\uparrow$ Let $U$ be an open subset of $X, f: U \rightarrow Y$ where $X, Y$ are finite dimensional normed linear spaces and suppose $f \in C^{1}(U)$ and $D f\left(x_{0}\right)$ is one to one. Then show $f$ is one to one near $x_{0}$. Hint: Show using the assumption that $f$ is $C^{1}$ that there exists $\delta>0$ such that if

$$
x_{1}, x_{2} \in B\left(x_{0}, \delta\right),
$$

then

$$
\begin{equation*}
\left|f\left(x_{1}\right)-f\left(x_{2}\right)-D f\left(x_{0}\right)\left(x_{1}-x_{2}\right)\right| \leq \frac{r}{2}\left|x_{1}-x_{2}\right| \tag{22.15.42}
\end{equation*}
$$

then use Problem 1. In case $X, Y$ are Banach spaces, assume $D f\left(x_{0}\right)$ is one to one and onto.
4. Suppose $U \subseteq X$ is an open subset of $X$ a Banach space and that $f: U \rightarrow Y$ is differentiable at $x_{0} \in U$ such that $D f\left(x_{0}\right)$ is one to one and onto from $X$ to $Y$. $\left(D f\left(x_{0}\right)^{-1} \in \mathscr{L}(Y, X)\right)$ Then show that $f(x) \neq f\left(x_{0}\right)$ for all $x$ sufficiently near but not equal to $x_{0}$. In this case, you only know the derivative exists at $x_{0}$.
5. Suppose $M \in \mathscr{L}(X, Y)$ where $X$ and $Y$ are finite dimensional linear spaces and suppose $M$ is onto. Show there exists $L \in \mathscr{L}(Y, X)$ such that

$$
L M \mathbf{x}=P \mathbf{x}
$$

where $P \in \mathscr{L}(X, X)$, and $P^{2}=P$. Also show $L$ is one to one and onto from $X_{1}$ to $Y$. Hint: Let $\left\{\mathbf{y}_{1} \cdots \mathbf{y}_{n}\right\}$ be a basis of $Y$ and let $M \mathbf{x}_{i}=\mathbf{y}_{i}$. Then define

$$
L \mathbf{y}=\sum_{i=1}^{n} \alpha_{i} \mathbf{x}_{i} \text { where } \mathbf{y}=\sum_{i=1}^{n} \alpha_{i} \mathbf{y}_{i}
$$

Show $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{n}\right\}$ is a linearly independent set and show you can obtain

$$
\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{n}, \cdots, \mathbf{x}_{m}\right\}
$$

a basis for $X$ in which $M \mathbf{x}_{j}=\mathbf{0}$ for $j>n$. Then let

$$
P \mathbf{x} \equiv \sum_{i=1}^{n} \alpha_{i} \mathbf{x}_{i}
$$

where

$$
\mathbf{x}=\sum_{i=1}^{m} \alpha_{i} \mathbf{x}_{i}
$$

6. $\uparrow$ Let $\mathbf{f}: U \subseteq X \rightarrow Y$, $\mathbf{f}$ is $C^{1}$, and $D \mathbf{f}(\mathbf{x})$ is onto for each $\mathbf{x} \in U$. Then show $\mathbf{f}$ maps open subsets of $U$ onto open sets in $Y$. Hint: Let $P=L D \mathbf{f}(\mathbf{x})$ as in Problem 5. Argue $L$ maps open sets from $Y$ to open sets of $X_{1} \equiv P X$ and $L^{-1}$ maps open sets from $X_{1}$ to open sets of $Y$. Then $L \mathbf{f}(\mathbf{x}+\mathbf{v})=L \mathbf{f}(\mathbf{x})+L D \mathbf{f}(\mathbf{x}) \mathbf{v}+\mathbf{o}(\mathbf{v})$. Now for $\mathbf{z} \in X_{1}$, let $\mathbf{h}(\mathbf{z})=L \mathbf{f}(\mathbf{x}+\mathbf{z})-L \mathbf{f}(\mathbf{x})$. Then $\mathbf{h}$ is $C^{1}$ on some small open subset of $X_{1}$ containing $\mathbf{0}$ and $D \mathbf{h}(\mathbf{0})=L D \mathbf{f}(\mathbf{x})$ which is seen to be one to one and onto and in $\mathscr{L}\left(X_{1}, X_{1}\right)$. Therefore, if $r$ is small enough, $\mathbf{h}(B(\mathbf{0}, r))$ equals an open set in $X_{1}, V$. This is by the inverse function theorem. Hence $L(\mathbf{f}(\mathbf{x}+B(\mathbf{0}, r))-\mathbf{f}(\mathbf{x}))=V$ and so $\mathbf{f}(\mathbf{x}+B(\mathbf{0}, r))-\mathbf{f}(\mathbf{x})=L^{-1}(V)$, an open set in $Y$.
7. Suppose $U \subseteq \mathbb{R}^{2}$ is an open set and $\mathbf{f}: U \rightarrow \mathbb{R}^{3}$ is $C^{1}$. Suppose $D \mathbf{f}\left(s_{0}, t_{0}\right)$ has rank two and

$$
\mathbf{f}\left(s_{0}, t_{0}\right)=\left(\begin{array}{c}
x_{0} \\
y_{0} \\
z_{0}
\end{array}\right)
$$

Show that for $(s, t)$ near $\left(s_{0}, t_{0}\right)$, the points $\mathbf{f}(s, t)$ may be realized in one of the following forms.

$$
\left\{(x, y, \phi(x, y)):(x, y) \text { near }\left(x_{0}, y_{0}\right)\right\}
$$

$$
\left\{(\phi(y, z), y, z):(y, z) \text { near }\left(y_{0}, z_{0}\right)\right\}
$$

or

$$
\left\{(x, \phi(x, z), z,):(x, z) \text { near }\left(x_{0}, z_{0}\right)\right\} .
$$

This shows that parametrically defined surfaces can be obtained locally in a particularly simple form.
8. Let $f: U \rightarrow Y, D f(x)$ exists for all $x \in U, B\left(x_{0}, \delta\right) \subseteq U$, and there exists $L \in$ $\mathscr{L}(X, Y)$, such that $L^{-1} \in \mathscr{L}(Y, X)$, and for all $x \in B\left(x_{0}, \delta\right)$

$$
\|D f(x)-L\|<\frac{r}{\left\|L^{-1}\right\|}, r<1 .
$$

Show that there exists $\varepsilon>0$ and an open subset of $B\left(x_{0}, \delta\right), V$, such that $f: V \rightarrow$ $B\left(f\left(x_{0}\right), \varepsilon\right)$ is one to one and onto. Also $D f^{-1}(y)$ exists for each $y \in B\left(f\left(x_{0}\right), \varepsilon\right)$ and is given by the formula

$$
D f^{-1}(y)=\left[D f\left(f^{-1}(y)\right)\right]^{-1}
$$

Hint: Let

$$
T_{y}(x) \equiv T(x, y) \equiv x-L^{-1}(f(x)-y)
$$

for $\left|y-f\left(x_{0}\right)\right|<\frac{(1-r) \delta}{2\left\|L^{-1}\right\|}$, consider $\left\{T_{y}^{n}\left(x_{0}\right)\right\}$. This is a version of the inverse function theorem for $f$ only differentiable, not $C^{1}$.
9. Denote by $C([0, T], X)$ the space of functions which are continuous having values in $X$ and define a norm on this linear space as follows.

$$
\|f\|_{\lambda} \equiv \max \left\{|f(t)| e^{\lambda t}: t \in[0, T]\right\}
$$

Show for each $\lambda \in \mathbb{R}$, this is a norm and that $C([0, T] ; X)$ is a complete normed linear space with this norm.
10. $\uparrow$ Let $f:[0, T] \times X \rightarrow X$ be continuous and suppose $f$ satisfies a Lipschitz condition,

$$
|f(t, x)-f(t, y)| \leq K|x-y|
$$

and let $x_{0} \in X$. Show there exists a unique solution to the Cauchy problem,

$$
x^{\prime}=f(t, x), x(0)=x_{0}
$$

for $t \in[0, T]$. Hint: Consider the map

$$
G: C([0, T] ; X) \rightarrow C([0, T] ; X)
$$

defined by

$$
G x(t) \equiv x_{0}+\int_{0}^{t} f(s, x(s)) d s
$$

where the integral is defined componentwise. Show $G$ is a contraction map for $\|\cdot\|_{\lambda}$ given in Problem 9 for a suitable choice of $\lambda$ and that therefore, it has a unique fixed point in $C([0, T] ; X)$. Next argue, using the fundamental theorem of calculus, that this fixed point is the unique solution to the Cauchy problem.
11. $\uparrow$ Use Theorem 7.11 .5 to give another proof of the above theorem. Hint: Use the same mapping and show that a large power is a contraction map.
12. Suppose you know that $u(t) \leq a+\int_{0}^{t} k(s) u(s) d s$ where $k(s) \geq 0$ and $k \in L^{1}([0, T])$. Show that then $u(t) \leq a \exp \left(\int_{0}^{t} k(s) d s\right)$. This is a version of Gronwall's inequality. Hint: Let $W(t)=\int_{0}^{t} k(s) u(s) d s$. Then explain why $W^{\prime}(t)-k(t) W(t) \leq a k(t)$. Now use the usual technique of an integrating factor you saw in beginning differential equations.
13. $\uparrow$ Use the above Gronwall's inequality to establish a result of continuous dependence on the initial condition and $f$ in the ordinary differential equation of Problem 10.
14. The existence of partial derivatives does not imply continuity as was shown in an example. However, much more can be said than this. Consider

$$
f(x, y)=\left\{\begin{array}{l}
\frac{\left(x^{2}-y^{4}\right)^{2}}{\left(x^{2}+y^{4}\right)^{2}} \text { if }(x, y) \neq(0,0), \\
1 \text { if }(x, y)=(0,0) .
\end{array}\right.
$$

Show the directional derivative of $f$ at $(0,0)$ exists and equals 0 for every direction. The directional derivative in the direction $\left(v_{1}, v_{2}\right)$ is defined as

$$
\lim _{t \rightarrow 0} \frac{f\left(x+t v_{1}, y+t v_{2}\right)-f(x, y)}{t}
$$

Now consider the curve $x^{2}=y^{4}$ and the curve $y=0$ to verify the function fails to be continuous at $(0,0)$.
15. Let

$$
f(x, y)=\left\{\begin{array}{l}
\frac{x^{2} y^{4}}{x^{2}+y^{8}} \text { if }(x, y) \neq(0,0) \\
0 \text { if }(x, y)=(0,0)
\end{array}\right.
$$

Show that this function is not continuous at $(0,0)$ but that it has all directional derivatives at $(0,0)$ and they all equal 0 .
16. Let $X_{i}$ be a normed linear space having norm $\|\cdot\|_{i}$. Then we can make $\prod_{i=1}^{n} X_{i}$ into a normed linear space by defining a norm on $\mathbf{x} \in \prod_{i=1}^{n} X_{i}$ by

$$
\|\mathbf{x}\| \equiv \max \left\{\left\|x_{i}\right\|_{i}: i=1, \cdots, n\right\}
$$

Show this is a norm on $\prod_{i=1}^{n} X_{i}$ as claimed.
17. Suppose $f: U \subseteq X \times Y \rightarrow Z$ and $D_{2} f\left(x_{0}, y_{0}\right)^{-1} \in \mathscr{L}(X, Y)$ exists and $f$ is $C^{1}$ so the conditions of the implicit function theorem are satisfied. Also suppose that all these are complex Banach spaces. Show that then the implicitly defined function $y=y(x)$ is analytic. Thus it has infinitely many derivatives and can be given as a power series as described above.

## Chapter 23

## Degree Theory

This chapter is on the Brouwer degree, a very useful concept with numerous and important applications. The degree can be used to prove some difficult theorems in topology such as the Brouwer fixed point theorem, the Jordan separation theorem, and the invariance of domain theorem. A couple of these big theorems have been presented earlier, but when you have degree theory, they get much easier. Degree theory is also used in bifurcation theory and many other areas in which it is an essential tool. The degree will be developed for $\mathbb{R}^{p}$ first. When this is understood, it is not too difficult to extend to versions of the degree which hold in Banach space. There is more on degree theory in the book by Deimling [38] and much of the presentation here follows this reference. Another more recent book which is really good is [43]. This is a whole book on degree theory.

The original reference for the approach given here, based on analysis, is [62] and dates from 1959. The degree was developed earlier by Brouwer and others using different methods.

To give you an idea what the degree is about, consider a real valued $C^{1}$ function defined on an interval $I$, and let $y \in f(I)$ be such that $f^{\prime}(x) \neq 0$ for all $x \in f^{-1}(y)$. In this case the degree is the sum of the signs of $f^{\prime}(x)$ for $x \in f^{-1}(y)$, written as $d(f, I, y)$.


In the above picture, $d(f, I, y)$ is 0 because there are two places where the sign is 1 and two where it is -1 .

The amazing thing about this is the number you obtain in this simple manner is a specialization of something which is defined for continuous functions and which has nothing to do with differentiability. An outline of the presentation is as follows. First define the degree for smooth functions at regular values and then extend to arbitrary values and finally to continuous functions. The reason this is possible is an integral expression for the degree which is insensitive to homotopy. It is very similar to the winding number of complex analysis. The difference between the two is that with the degree, the integral which ties it all together is taken over the open set while the winding number is taken over the boundary, although proofs of in the case of the winding number sometimes involve Green's theorem which involves an integral over the open set.

In this chapter $\Omega$ will refer to a bounded open set.
Definition 23.0.1 For $\Omega$ a bounded open set, denote by $C(\bar{\Omega})$ the set of functions which
are restrictions of functions in $C_{c}\left(\mathbb{R}^{p}\right)$, equivalently $C\left(\mathbb{R}^{p}\right)$ to $\bar{\Omega}$ and by $C^{m}(\bar{\Omega}), m \leq \infty$ the space of restrictions of functions in $C_{c}^{m}\left(\mathbb{R}^{p}\right)$, equivalently $C^{m}\left(\mathbb{R}^{p}\right)$ to $\bar{\Omega}$. If $f \in C(\bar{\Omega})$ the symbol $f$ will also be used to denote a function defined on $\mathbb{R}^{p}$ equalling $f$ on $\bar{\Omega}$ when convenient. The subscript c indicates that the functions have compact support. The norm in $C(\bar{\Omega})$ is defined as follows.

$$
\|f\|_{\infty, \bar{\Omega}}=\|f\|_{\infty} \equiv \sup \{|f(\mathbf{x})|: \mathbf{x} \in \bar{\Omega}\}
$$

If the functions take values in $\mathbb{R}^{p}$ write $C^{m}\left(\bar{\Omega} ; \mathbb{R}^{p}\right)$ or $C\left(\bar{\Omega} ; \mathbb{R}^{p}\right)$ for these functions if there is no differentiability assumed. The norm on $C\left(\bar{\Omega} ; \mathbb{R}^{p}\right)$ is defined in the same way as above,

$$
\|\mathbf{f}\|_{\infty, \bar{\Omega}}=\|\mathbf{f}\|_{\infty} \equiv \sup \{|\mathbf{f}(\mathbf{x})|: \mathbf{x} \in \bar{\Omega}\}
$$

Of course if $m=\infty$, the notation means that there are infinitely many derivatives. Also, $C\left(\Omega ; \mathbb{R}^{p}\right)$ consists of functions which are continuous on $\Omega$ that have values in $\mathbb{R}^{p}$ and $C^{m}\left(\Omega ; \mathbb{R}^{p}\right)$ denotes the functions which have $m$ continuous derivatives defined on $\Omega$. Also let $\mathscr{P}$ consist of functions $\mathbf{f}(\mathbf{x})$ such that $f_{k}(\mathbf{x})$ is a polynomial, meaning an element of the algebra of functions generated by $\left\{1, x_{1}, \cdots, x_{p}\right\}$. Thus a typical polynomial is of the form $\sum_{i_{1} \cdots i_{p}} a\left(i_{1} \cdots i_{p}\right) x^{i_{1}} \cdots x^{i_{p}}$ where the $i_{j}$ are nonnegative integers and $a\left(i_{1} \cdots i_{p}\right)$ is a real number.

Some of the theorems are simpler if you base them on the Weierstrass approximation theorem.

Note that, by applying the Tietze extension theorem to the components of the function, one can always extend a function continuous on $\bar{\Omega}$ to all of $\mathbb{R}^{p}$ so there is no loss of generality in simply regarding functions continuous on $\bar{\Omega}$ as restrictions of functions continuous on $\mathbb{R}^{p}$. Next is the idea of a regular value.
Definition 23.0.2 For $W$ an open set in $\mathbb{R}^{p}$ and $\mathbf{g} \in C^{1}\left(W ; \mathbb{R}^{p}\right) \mathbf{y}$ is called a regular value of $\mathbf{g}$ if whenever $\mathbf{x} \in \mathbf{g}^{-1}(\mathbf{y})$, $\operatorname{det}(D \mathbf{g}(\mathbf{x})) \neq 0$. Note that if $\mathbf{g}^{-1}(\mathbf{y})=\emptyset$, it follows that $\mathbf{y}$ is a regular value from this definition. That is, $\mathbf{y}$ is a regular value if and only if

$$
\mathbf{y} \notin \mathbf{g}(\{\mathbf{x} \in W: \operatorname{det} D \mathbf{g}(\mathbf{x})=0\})
$$

Denote by $S_{\mathbf{g}}$ the set of singular values of $\mathbf{g}$, those $\mathbf{y}$ such that $\operatorname{det}(\operatorname{dg}(\mathbf{x}))=0$ for some $\mathbf{x} \in \mathbf{g}^{-1}(\mathbf{y})$.

Also, $\partial \Omega$ will often be referred to. It is those points with the property that every open set (or open ball) containing the point contains points not in $\Omega$ and points in $\Omega$. Then the following simple lemma will be used frequently.

Lemma 23.0.3 Define $\partial U$ to be those points $\mathbf{x}$ with the property that for every $r>0$, $B(\mathbf{x}, r)$ contains points of $U$ and points of $U^{C}$. Then for $U$ an open set,

$$
\begin{equation*}
\partial U=\bar{U} \backslash U \tag{23.0.1}
\end{equation*}
$$

Let $C$ be a closed subset of $\mathbb{R}^{p}$ and let $\mathscr{K}$ denote the set of components of $\mathbb{R}^{p} \backslash C$. Then if $K$ is one of these components, it is open and

$$
\partial K \subseteq C
$$

Proof: First consider claim 23.0.1. Let $\mathbf{x} \in \bar{U} \backslash U$. If $B(\mathbf{x}, r)$ contains no points of $U$, then $\mathbf{x} \notin \bar{U}$. If $B(\mathbf{x}, r)$ contains no points of $U^{C}$, then $\mathbf{x} \in U$ and so $\mathbf{x} \notin \bar{U} \backslash U$. Therefore, $\bar{U} \backslash U \subseteq \partial U$. Now let $\mathbf{x} \in \partial U$. If $\mathbf{x} \in U$, then since $U$ is open there is a ball containing $\mathbf{x}$ which is contained in $U$ contrary to $\mathbf{x} \in \partial U$. Therefore, $\mathbf{x} \notin U$. If $\mathbf{x}$ is not a limit point of $U$, then some ball containing $\mathbf{x}$ contains no points of $U$ contrary to $\mathbf{x} \in \partial U$. Therefore, $\mathbf{x} \in \bar{U} \backslash U$ which shows the two sets are equal.

Why is $K$ open for $K$ a component of $\mathbb{R}^{p} \backslash C$ ? This follows from Theorem 7.13.10 and results from open balls being connected. Thus if $k \in K$, letting $B(k, r) \subseteq C^{C}$, it follows $K \cup B(k, r)$ is connected and contained in $C^{C}$ and therefore is contained in $K$ because $K$ is maximal with respect to being connected and contained in $C^{C}$.

Now for $K$ a component of $\mathbb{R}^{p} \backslash C$, why is $\partial K \subseteq C$ ? Let $\mathbf{x} \in \partial K$. If $\mathbf{x} \notin C$, then $\mathbf{x} \in K_{1}$, some component of $\mathbb{R}^{p} \backslash C$. If $K_{1} \neq K$ then $\mathbf{x}$ cannot be a limit point of $K$ and so it cannot be in $\partial K$. Therefore, $K=K_{1}$ but this also is a contradiction because if $\mathbf{x} \in \partial K$ then $\mathbf{x} \notin K$ thanks to 23.0.1.

Note that for an open set $U \subseteq \mathbb{R}^{p}$, and $\mathbf{h}: \bar{U} \rightarrow \mathbb{R}^{p}, \operatorname{dist}(\mathbf{h}(\partial U), \mathbf{y}) \geq \operatorname{dist}(\mathbf{h}(\bar{U}), \mathbf{y})$ because $\bar{U} \supseteq \partial U$.

The following lemma will be nice to keep in mind.
Lemma 23.0.4 $\mathbf{f} \in C\left(\bar{\Omega} \times[a, b] ; \mathbb{R}^{p}\right)$ if and only if $t \rightarrow \mathbf{f}(\cdot, t)$ is in $C\left([a, b] ; C\left(\bar{\Omega} ; \mathbb{R}^{p}\right)\right)$. Also

$$
\|\mathbf{f}\|_{\infty, \bar{\Omega} \times[a, b]}=\max _{t \in[a, b]}\left(\|\mathbf{f}(\cdot, t)\|_{\infty, \bar{\Omega}}\right)
$$

Proof: $\Rightarrow$ By uniform continuity, if $\varepsilon>0$ there is $\delta>0$ such that if $|t-s|<\delta$, then for all $\mathbf{x} \in \bar{\Omega},\|\mathbf{f}(\mathbf{x}, t)-\mathbf{f}(\mathbf{x}, s)\|<\frac{\varepsilon}{2}$. It follows that $\|\mathbf{f}(\cdot, t)-\mathbf{f}(\cdot, s)\|_{\infty} \leq \frac{\varepsilon}{2}<\varepsilon$.
$\Leftarrow$ Say $\left(\mathbf{x}_{n}, t_{n}\right) \rightarrow(\mathbf{x}, t)$. Does it follow that $\mathbf{f}\left(\mathbf{x}_{n}, t_{n}\right) \rightarrow \mathbf{f}(\mathbf{x}, t)$ ?

$$
\begin{aligned}
\left\|\mathbf{f}\left(\mathbf{x}_{n}, t_{n}\right)-\mathbf{f}(\mathbf{x}, t)\right\| & \leq\left\|\mathbf{f}\left(\mathbf{x}_{n}, t_{n}\right)-\mathbf{f}\left(\mathbf{x}_{n}, t\right)\right\|+\left\|\mathbf{f}\left(\mathbf{x}_{n}, t\right)-\mathbf{f}(\mathbf{x}, t)\right\| \\
& \leq\left\|\mathbf{f}\left(\cdot, t_{n}\right)-\mathbf{f}(\cdot, t)\right\|_{\infty}+\left\|\mathbf{f}\left(\mathbf{x}_{n}, t\right)-\mathbf{f}(\mathbf{x}, t)\right\|
\end{aligned}
$$

both terms converge to 0 , the first because $\mathbf{f}$ is continuous into $C\left(\bar{\Omega} ; \mathbb{R}^{p}\right)$ and the second because $\mathbf{x} \rightarrow \mathbf{f}(\mathbf{x}, t)$ is continuous.

The claim about the norms is next. Let $(\mathbf{x}, t)$ be such that $\|\mathbf{f}\|_{\infty, \bar{\Omega} \times[a, b]}<\|\mathbf{f}(\mathbf{x}, t)\|+\varepsilon$. Then

$$
\|\mathbf{f}\|_{\infty, \bar{\Omega} \times[a, b]}<\|\mathbf{f}(\mathbf{x}, t)\|+\varepsilon \leq \max _{t \in[a, b]}\left(\|\mathbf{f}(\cdot, t)\|_{\infty, \bar{\Omega}}\right)+\varepsilon
$$

and so $\|\mathbf{f}\|_{\infty, \bar{\Omega} \times[a, b]} \leq \max _{t \in[a, b]} \max \left(\|\mathbf{f}(\cdot, t)\|_{\infty, \bar{\Omega}}\right)$ because $\varepsilon$ is arbitrary. However, the same argument works in the other direction. There exists $t$ such that

$$
\|\mathbf{f}(\cdot, t)\|_{\infty, \bar{\Omega}}=\max _{t \in[a, b]}\left(\|\mathbf{f}(\cdot, t)\|_{\infty, \bar{\Omega}}\right)
$$

by compactness of the interval. Then by compactness of $\bar{\Omega}$, there is $\mathbf{x}$ such that $\|\mathbf{f}(\cdot, t)\|_{\infty, \bar{\Omega}}=$ $\|\mathbf{f}(\mathbf{x}, t)\| \leq\|\mathbf{f}\|_{\infty, \bar{\Omega} \times[a, b]}$ and so the two norms are the same.

### 23.1 Sard's Lemma and Approximation

First are easy assertions about approximation of continuous functions with smooth ones.
The following is the Weierstrass approximation theorem. It is Corollary 9.1.5 presented earlier.

Corollary 23.1.1 If $f \in C([a, b] ; X)$ where $X$ is a normed linear space, then there exists $a$ sequence of polynomials which converge uniformly to $f$ on $[a, b]$. The polynomials are of the form

$$
\begin{equation*}
\sum_{k=0}^{m}\binom{m}{k}\left(l^{-1}(t)\right)^{k}\left(1-l^{-1}(t)\right)^{m-k} f\left(l\left(\frac{k}{m}\right)\right) \tag{23.1.2}
\end{equation*}
$$

where $l$ is a linear one to one and onto map from $[0,1]$ to $[a, b]$.
Applying the Weierstrass approximation theorem, Theorem 9.2.9 or Theorem 9.2.5 to the components of a vector valued function yields the following corollary

Theorem 23.1.2 If $\mathbf{f} \in C\left(\bar{\Omega} ; \mathbb{R}^{p}\right)$ for $\Omega$ a bounded subset of $\mathbb{R}^{p}$, then for all $\varepsilon>0$, there exists $\mathbf{g} \in C^{\infty}\left(\bar{\Omega} ; \mathbb{R}^{p}\right)$ such that

$$
\|\mathbf{g}-\mathbf{f}\|_{\infty, \bar{\Omega}}<\varepsilon
$$

Recall Sard's lemma, shown earlier. It is Lemma 13.6.5. I am stating it here for convenience.

Lemma 23.1.3 (Sard) Let $U$ be an open set in $\mathbb{R}^{p}$ and let $\mathbf{h}: U \rightarrow \mathbb{R}^{p}$ be differentiable. Let

$$
S \equiv\{\mathbf{x} \in U: \operatorname{det} D \mathbf{h}(\mathbf{x})=0\}
$$

Then $m_{p}(\mathbf{h}(S))=0$.
First note that if $\mathbf{y} \notin \mathbf{g}(\Omega)$, then we are calling it a regular value because whenever $\mathbf{x} \in \mathbf{g}^{-1}(\mathbf{y})$ the desired conclusion follows vacuously. Thus, for $\mathbf{g} \in C^{\infty}\left(\bar{\Omega}, \mathbb{R}^{p}\right), \mathbf{y}$ is a regular value if and only if $\mathbf{y} \notin \mathbf{g}(\{\mathbf{x} \in \bar{\Omega}: \operatorname{det} D \mathbf{g}(\mathbf{x})=0\})$.

Observe that any uncountable set in $\mathbb{R}^{p}$ has a limit point. To see this, tile $\mathbb{R}^{p}$ with countably many congruent boxes. One of them has uncountably many points. Now subdivide this into $2^{p}$ congruent boxes. One has uncountably many points. Continue subdividing this way to obtain a limit point as the unique point in the intersection of a nested sequence of compact sets whose diameters converge to 0 .

Lemma 23.1.4 Let $\mathbf{g} \in C^{\infty}\left(\mathbb{R}^{p} ; \mathbb{R}^{p}\right)$ and let $\left\{\mathbf{y}_{i}\right\}_{i=1}^{\infty}$ be points of $\mathbb{R}^{p}$ and let $\eta>0$. Then there exists $\mathbf{e}$ with $\|\mathbf{e}\|<\eta$ and $\mathbf{y}_{i}+\mathbf{e}$ is a regular value for $\mathbf{g}$.

Proof: Let $S=\left\{\mathbf{x} \in \mathbb{R}^{p}: \operatorname{det} D \mathbf{g}(\mathbf{x})=0\right\}$. By Sard's lemma, $\mathbf{g}(S)$ has measure zero. Let $N \equiv \cup_{i=1}^{\infty}\left(\mathbf{g}(S)-\mathbf{y}_{i}\right)$. Thus $N$ has measure 0 . Pick $\mathbf{e} \in B(\mathbf{0}, \eta) \backslash N$. Then for each $i, \mathbf{y}_{i}+\mathbf{e} \notin \mathbf{g}(S)$.

Lemma 23.1.5 Let $\mathbf{f} \in C\left(\bar{\Omega} ; \mathbb{R}^{p}\right)$ and let $\left\{\mathbf{y}_{i}\right\}_{i=1}^{\infty}$ be points not in $\mathbf{f}(\partial \Omega)$ and let $\delta>0$. Then there exists $\mathbf{g} \in C^{\infty}\left(\bar{\Omega} ; \mathbb{R}^{p}\right)$ such that $\|\mathbf{g}-\mathbf{f}\|_{\infty, \bar{\Omega}}<\delta$ and $\mathbf{y}_{i}$ is a regular value for $\mathbf{g}$ for each i. That is, if $\mathbf{g}(\mathbf{x})=\mathbf{y}_{i}$, then $D \mathbf{g}(\mathbf{x})^{-1}$ exists. Also, if $\delta<\operatorname{dist}(\mathbf{f}(\partial \Omega), \mathbf{y})$ for some $\mathbf{y}$ a regular value of $\mathbf{g} \in C^{\infty}\left(\bar{\Omega} ; \mathbb{R}^{p}\right)$, then $\mathbf{g}^{-1}(\mathbf{y})$ is a finite set of points in $\Omega$. Also, if $\mathbf{y}$ is a regular value of $\mathbf{g} \in C^{\infty}\left(\mathbb{R}^{p}, \mathbb{R}^{p}\right)$, then $\mathbf{g}^{-1}(\mathbf{y})$ is countable.

Proof: Pick $\tilde{\mathbf{g}} \in C^{\infty}\left(\bar{\Omega} ; \mathbb{R}^{p}\right),\|\tilde{\mathbf{g}}-\mathbf{f}\|_{\infty, \bar{\Omega}}<\delta . \mathbf{g} \equiv \tilde{\mathbf{g}}-\mathbf{e}$ where $\mathbf{e}$ is from the above Lemma 23.1.4 and $\eta$ so small that $\|\mathbf{g}-\mathbf{f}\|_{\infty, \bar{\Omega}}<\delta$. Then if $\mathbf{g}(\mathbf{x})=\mathbf{y}_{i}$, you get $\tilde{\mathbf{g}}(\mathbf{x})=\mathbf{y}_{i}+\mathbf{e}$ a regular value of $\tilde{\mathbf{g}}$ and so $\operatorname{det}(D \mathbf{g}(\mathbf{x}))=\operatorname{det}(D \tilde{\mathbf{g}}(\mathbf{x})) \neq 0$ so this shows the first part.

It remains to verify the last claims. Since $\|\mathbf{g}-\mathbf{f}\|_{\bar{\Omega}, \infty}<\delta$, if $\mathbf{x} \in \partial \Omega$, then

$$
\|\mathbf{g}(\mathbf{x})-\mathbf{y}\| \geq\|\mathbf{f}(\mathbf{x})-\mathbf{y}\|-\|\mathbf{f}(\mathbf{x})-\mathbf{g}(\mathbf{x})\| \geq \operatorname{dist}(\mathbf{f}(\partial \Omega), \mathbf{y})-\delta>\delta-\delta=0
$$

and so $\mathbf{y} \notin \mathbf{g}(\partial \Omega)$, so if $\mathbf{g}(\mathbf{x})=\mathbf{y}$, then $\mathbf{x} \in \Omega$. If there are infinitely many points in $\mathbf{g}^{-1}(\mathbf{y})$, then there would be a subsequence converging to a point $\mathbf{x} \in \bar{\Omega}$. Thus $\mathbf{g}(\mathbf{x})=\mathbf{y}$ so $\mathbf{x} \notin \partial \Omega$. However, this would violate the inverse function theorem because $\mathbf{g}$ would fail to be one to one on an open ball containing $\mathbf{x}$ and contained in $\Omega$. Therefore, there are only finitely many points in $\mathbf{g}^{-1}(\mathbf{y})$ and at each point, the determinant of the derivative of $\mathbf{g}$ is nonzero. For $\mathbf{y}$ a regular value, $\mathbf{g}^{-1}(\mathbf{y})$ is countable since otherwise, there would be a limit point $\mathbf{x} \in \mathbf{g}^{-1}(\mathbf{y})$ and $\mathbf{g}$ would fail to be one to one near $\mathbf{x}$ contradicting the inverse function theorem.

Now with this, here is a definition of the degree.
Definition 23.1.6 Let $\Omega$ be a bounded open set in $\mathbb{R}^{p}$ and let $\mathbf{f}: \bar{\Omega} \rightarrow \mathbb{R}^{p}$ be continuous. Let $\mathbf{y} \notin \mathbf{f}(\partial \Omega)$. Then the degree is defined as follows: Let $\mathbf{g}$ be infinitely differentiable, $\|\mathbf{f}-\mathbf{g}\|_{\infty, \bar{\Omega}}<\operatorname{dist}(\mathbf{f}(\partial \Omega), \mathbf{y})$, and $\mathbf{y}$ is a regular value of $\mathbf{g}$.Then

$$
d(\mathbf{f}, \Omega, \mathbf{y}) \equiv \sum\left\{\operatorname{sgn}(\operatorname{det}(D \mathbf{g}(\mathbf{x}))): \mathbf{x} \in \mathbf{g}^{-1}(\mathbf{y})\right\}
$$

From Lemma 23.1.5 the definition at least makes sense because the sum is finite and such a $\mathbf{g}$ exists. The problem is whether the definition is well defined in the sense that we get the same answer if a different $\mathbf{g}$ is used. Suppose it is shown that the definition is well defined. If $\mathbf{y} \notin \mathbf{f}(\Omega)$, then you could pick $\mathbf{g}$ such that $\|\mathbf{g}-\mathbf{f}\|_{\bar{\Omega}}<\operatorname{dist}(\mathbf{y}, \mathbf{f}(\bar{\Omega}))$. However, this requires that $\mathbf{g}(\bar{\Omega})$ does not contain $\mathbf{y}$ because if $\mathbf{x} \in \bar{\Omega}$, then

$$
\begin{aligned}
\|\mathbf{g}(\mathbf{x})-\mathbf{y}\| & =\|(\mathbf{y}-\mathbf{f}(\mathbf{x}))-(\mathbf{g}(\mathbf{x})-\mathbf{f}(\mathbf{x}))\| \geq\|\mathbf{f}(\mathbf{x})-\mathbf{y}\|-\|\mathbf{g}(\mathbf{x})-\mathbf{f}(\mathbf{x})\| \\
& >\operatorname{dist}(\mathbf{y}, \mathbf{f}(\bar{\Omega}))-\operatorname{dist}(\mathbf{y}, \mathbf{f}(\bar{\Omega}))=0
\end{aligned}
$$

Therefore, $\mathbf{y}$ is a regular value of $\mathbf{g}$ because every point in $\mathbf{g}^{-1}(\mathbf{y})$ is such that the determinant of the derivative at this point is non zero since there are no such points. Thus if $\mathbf{f}^{-1}(\mathbf{y})=\emptyset, d(\mathbf{f}, \Omega, \mathbf{y})=0$.

If $\mathbf{f}(\mathbf{z})=\mathbf{y}$, then there is $\mathbf{g}$ having $\mathbf{y}$ a regular value and $\mathbf{g}(\mathbf{z})=\mathbf{y}$ by the above lemma.
Lemma 23.1.7 Suppose $\mathbf{g}, \hat{\mathbf{g}}$ both satisfy the above definition,

$$
\operatorname{dist}(\mathbf{f}(\partial \Omega), \mathbf{y})>\delta>\|\mathbf{f}-\mathbf{g}\|_{\infty, \bar{\Omega}}, \operatorname{dist}(\mathbf{f}(\partial \Omega), \mathbf{y})>\delta>\|\mathbf{f}-\hat{\mathbf{g}}\|_{\infty, \bar{\Omega}}
$$

Then for $t \in[0,1]$ so does $t \mathbf{g}+(1-t) \hat{\mathbf{g}}$. In particular, $\mathbf{y} \notin(t \mathbf{g}+(1-t) \hat{\mathbf{g}})(\partial \Omega)$. More generally, if

$$
\|\mathbf{h}-\mathbf{f}\|<\operatorname{dist}(\mathbf{f}(\partial \Omega), \mathbf{y})
$$

then $0<\operatorname{dist}(\mathbf{h}(\partial \Omega), \mathbf{y})$. Also $d(\mathbf{f}-\mathbf{y}, \Omega, \mathbf{0})=d(\mathbf{f}, \Omega, \mathbf{y})$.
Proof: From the triangle inequality, if $t \in[0,1]$,

$$
\|\mathbf{f}-(t \mathbf{g}+(1-t) \hat{\mathbf{g}})\|_{\infty} \leq t\|\mathbf{f}-\mathbf{g}\|_{\infty}+(1-t)\|\mathbf{f}-\hat{\mathbf{g}}\|_{\infty}<t \delta+(1-t) \delta=\delta .
$$

If $\|\mathbf{h}-\mathbf{f}\|_{\infty}<\delta<\operatorname{dist}(\mathbf{f}(\partial \Omega), \mathbf{y})$, as was just shown for $\mathbf{h} \equiv t \mathbf{g}+(1-t) \hat{\mathbf{g}}$, then if $\mathbf{x} \in \partial \Omega$,

$$
\|\mathbf{y}-\mathbf{h}(\mathbf{x})\| \geq\|\mathbf{y}-\mathbf{f}(\mathbf{x})\|-\|\mathbf{h}(\mathbf{x})-\mathbf{f}(\mathbf{x})\|>\operatorname{dist}(\mathbf{f}(\partial \Omega), \mathbf{y})-\delta \geq \delta-\delta=0
$$

Now consider the last claim. This follows because $\|\mathbf{g}-\mathbf{f}\|_{\infty}$ small is the same as $\|\mathbf{g}-\mathbf{y}-(\mathbf{f}-\mathbf{y})\|_{\infty}$ being small. They are the same. Also, $(\mathbf{g}-\mathbf{y})^{-1}(\mathbf{0})=\mathbf{g}^{-1}(\mathbf{y})$ and $D \mathbf{g}(\mathbf{x})=D(\mathbf{g}-\mathbf{y})(\mathbf{x})$.

First is an identity. It was Lemma 16.3.1 on Page 429.
Lemma 23.1.8 Let $\mathbf{g}: U \rightarrow \mathbb{R}^{p}$ be $C^{2}$ where $U$ is an open subset of $\mathbb{R}^{p}$. Then

$$
\sum_{j=1}^{p} \operatorname{cof}(D \mathbf{g})_{i j, j}=0
$$

where here $(D \mathbf{g})_{i j} \equiv g_{i, j} \equiv \frac{\partial g_{i}}{\partial x_{j}}$. Also, $\operatorname{cof}(D \mathbf{g})_{i j}=\frac{\partial \operatorname{det}(D \mathbf{g})}{\partial g_{i, j}}$.
Next is an integral representation of $\sum\left\{\operatorname{sgn}(\operatorname{det}(D \mathbf{g}(\mathbf{x}))): \mathbf{x} \in \mathbf{g}^{-1}(\mathbf{y})\right\}$ but first is a little lemma about disjoint sets.

Lemma 23.1.9 Let $K$ be a compact set and $C$ a closed set in $\mathbb{R}^{p}$ such that $K \cap C=\emptyset$. Then

$$
\operatorname{dist}(K, C) \equiv \inf \{\|\mathbf{k}-\mathbf{c}\|: \mathbf{k} \in K, \mathbf{c} \in C\}>0
$$

Proof: Let

$$
d \equiv \inf \{\|\mathbf{k}-\mathbf{c}\|: \mathbf{k} \in K, \mathbf{c} \in C\}
$$

Let $\left\{\mathbf{k}_{i}\right\},\left\{\mathbf{c}_{i}\right\}$ be such that

$$
d+\frac{1}{i}>\left\|\mathbf{k}_{i}-\mathbf{c}_{i}\right\|
$$

Since $K$ is compact, there is a subsequence still denoted by $\left\{\mathbf{k}_{i}\right\}$ such that $\mathbf{k}_{i} \rightarrow \mathbf{k} \in K$. Then also

$$
\left\|\mathbf{c}_{i}-\mathbf{c}_{m}\right\| \leq\left\|\mathbf{c}_{i}-\mathbf{k}_{i}\right\|+\left\|\mathbf{k}_{i}-\mathbf{k}_{m}\right\|+\left\|\mathbf{c}_{m}-\mathbf{k}_{m}\right\|
$$

If $d=0$, then as $m, i \rightarrow \infty$ it follows $\left\|\mathbf{c}_{i}-\mathbf{c}_{m}\right\| \rightarrow 0$ and so $\left\{\mathbf{c}_{i}\right\}$ is a Cauchy sequence which must converge to some $\mathbf{c} \in C$. But then $\|\mathbf{c}-\mathbf{k}\|=\lim _{i \rightarrow \infty}\left\|\mathbf{c}_{i}-\mathbf{k}_{i}\right\|=0$ and so $\mathbf{c}=\mathbf{k} \in C \cap K$, a contradiction to these sets being disjoint.

In particular the distance between a point and a closed set is always positive if the point is not in the closed set. Of course this is obvious even without the above lemma.

Definition 23.1.10 Let $\mathbf{g} \in C^{\infty}\left(\bar{\Omega} ; \mathbb{R}^{p}\right)$ where $\Omega$ is a bounded open set. Also let $\phi_{\varepsilon}$ be $a$ mollifier.

$$
\phi_{\varepsilon} \in C_{c}^{\infty}(B(\mathbf{0}, \varepsilon)), \phi_{\varepsilon} \geq 0, \int \phi_{\varepsilon} d x=1
$$

The idea is that $\varepsilon$ will converge to 0 to get suitable approximations.
First, here is a technical lemma which will be used to identify the degree with an integral.

Lemma 23.1.11 Let $\mathbf{y} \notin \mathbf{g}(\partial \Omega)$ for $\mathbf{g} \in C^{\infty}\left(\bar{\Omega} ; \mathbb{R}^{p}\right)$. Also suppose $\mathbf{y}$ is a regular value of g. Then for all positive $\varepsilon$ small enough,

$$
\int_{\Omega} \phi_{\varepsilon}(\mathbf{g}(\mathbf{x})-\mathbf{y}) \operatorname{det} D \mathbf{g}(\mathbf{x}) d x=\sum\left\{\operatorname{sgn}(\operatorname{det} D \mathbf{g}(\mathbf{x})): \mathbf{x} \in \mathbf{g}^{-1}(\mathbf{y})\right\}
$$

Proof: First note that the sum is finite from Lemma 23.1.5. It only remains to verify the equation.

I need to show the left side of this equation is constant for $\varepsilon$ small enough and equals the right side. By what was just shown, there are finitely many points, $\left\{\mathbf{x}_{i}\right\}_{i=1}^{m}=\mathbf{g}^{-1}(\mathbf{y})$. By the inverse function theorem, there exist disjoint open sets $U_{i}$ with $\mathbf{x}_{i} \in U_{i}$, such that $\mathbf{g}$ is one to one on $U_{i}$ with $\operatorname{det}(D \mathbf{g}(\mathbf{x}))$ having constant sign on $U_{i}$ and $\mathbf{g}\left(U_{i}\right)$ is an open set containing $\mathbf{y}$. Then let $\varepsilon$ be small enough that $B(\mathbf{y}, \boldsymbol{\varepsilon}) \subseteq \cap_{i=1}^{m} \mathbf{g}\left(U_{i}\right)$. Also, $\mathbf{y} \notin \mathbf{g}\left(\bar{\Omega} \backslash\left(\cup_{i=1}^{n} U_{i}\right)\right)$, a compact set. Let $\varepsilon$ be still smaller, if necessary, so that $B(\mathbf{y}, \boldsymbol{\varepsilon}) \cap \mathbf{g}\left(\bar{\Omega} \backslash\left(\cup_{i=1}^{n} U_{i}\right)\right)=\emptyset$ and let $V_{i} \equiv \mathbf{g}^{-1}(B(\mathbf{y}, \varepsilon)) \cap U_{i}$.


Therefore, for any $\varepsilon$ this small,

$$
\int_{\Omega} \phi_{\varepsilon}(\mathbf{g}(\mathbf{x})-\mathbf{y}) \operatorname{det} D \mathbf{g}(\mathbf{x}) d x=\sum_{i=1}^{m} \int_{V_{i}} \phi_{\varepsilon}(\mathbf{g}(\mathbf{x})-\mathbf{y}) \operatorname{det} D \mathbf{g}(\mathbf{x}) d x
$$

The reason for this is as follows. The integrand on the left is nonzero only if $\mathbf{g}(\mathbf{x})-$ $\mathbf{y} \in B(\mathbf{0}, \varepsilon)$ which occurs only if $\mathbf{g}(\mathbf{x}) \in B(\mathbf{y}, \varepsilon)$ which is the same as $\mathbf{x} \in \mathbf{g}^{-1}(B(\mathbf{y}, \varepsilon))$. Therefore, the integrand is nonzero only if $\mathbf{x}$ is contained in exactly one of the disjoint sets, $V_{i}$. Now using the change of variables theorem, $\left(\mathbf{z}=\mathbf{g}(\mathbf{x})-\mathbf{y}, \mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})=\mathbf{x}.\right)$

$$
=\sum_{i=1}^{m} \int_{\mathbf{g}\left(V_{i}\right)-\mathbf{y}} \phi_{\varepsilon}(\mathbf{z}) \operatorname{det} D \mathbf{g}\left(\mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})\right)\left|\operatorname{det} D \mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})\right| d z
$$

By the chain rule, $I=D \mathbf{g}\left(\mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})\right) D \mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})$ and so

$$
\begin{gathered}
\operatorname{det} D \mathbf{g}\left(\mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})\right)\left|\operatorname{det} D \mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})\right| \\
=\operatorname{sgn}\left(\operatorname{det} D \mathbf{g}\left(\mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})\right)\right)\left|\operatorname{det} D \mathbf{g}\left(\mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})\right)\right|\left|\operatorname{det} D \mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})\right| \\
=\operatorname{sgn}\left(\operatorname{det} D \mathbf{g}\left(\mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})\right)\right) \\
=\operatorname{sgn}(\operatorname{det} D \mathbf{g}(\mathbf{x}))=\operatorname{sgn}\left(\operatorname{det} D \mathbf{g}\left(\mathbf{x}_{i}\right)\right) .
\end{gathered}
$$

Therefore, this reduces to

$$
\begin{gathered}
\sum_{i=1}^{m} \operatorname{sgn}\left(\operatorname{det} D \mathbf{g}\left(\mathbf{x}_{i}\right)\right) \int_{\mathbf{g}\left(V_{i}\right)-\mathbf{y}} \phi_{\varepsilon}(\mathbf{z}) d z= \\
\sum_{i=1}^{m} \operatorname{sgn}\left(\operatorname{det} D \mathbf{g}\left(\mathbf{x}_{i}\right)\right) \int_{B(\mathbf{0}, \boldsymbol{\varepsilon})} \phi_{\varepsilon}(\mathbf{z}) d z=\sum_{i=1}^{m} \operatorname{sgn}\left(\operatorname{det} D \mathbf{g}\left(\mathbf{x}_{i}\right)\right)
\end{gathered}
$$

In case $\mathbf{g}^{-1}(\mathbf{y})=\emptyset$, there exists $\varepsilon>0$ such that $\mathbf{g}(\bar{\Omega}) \cap B(\mathbf{y}, \varepsilon)=\emptyset$ and so for $\varepsilon$ this small,

$$
\int_{\Omega} \phi_{\varepsilon}(\mathbf{g}(\mathbf{x})-\mathbf{y}) \operatorname{det} D \mathbf{g}(\mathbf{x}) d x=0
$$

As noted above, this will end up being $d(\mathbf{g}, \Omega, \mathbf{y})$ in this last case where $\mathbf{g}^{-1}(\mathbf{y})=\emptyset$.
Next is an important result on homotopy.
Lemma 23.1.12 If $\mathbf{h}$ is in $C^{\infty}\left(\bar{\Omega} \times[a, b], \mathbb{R}^{p}\right)$, and $\mathbf{0} \notin \mathbf{h}(\partial \Omega \times[a, b])$ then for $0<\varepsilon<$ $\operatorname{dist}(\mathbf{0}, \mathbf{h}(\partial \Omega \times[a, b]))$,

$$
t \rightarrow \int_{\Omega} \phi_{\varepsilon}(\mathbf{h}(\mathbf{x}, t)) \operatorname{det} D_{1} \mathbf{h}(\mathbf{x}, t) d x
$$

is constant for $t \in[a, b]$. As a special case, $d(\mathbf{f}, \Omega, \mathbf{y})$ is well defined. Also, if $\mathbf{y} \notin \mathbf{f}(\bar{\Omega})$, then $d(\mathbf{f}, \Omega, \mathbf{y})=0$.

Proof: By continuity of $\mathbf{h}, \mathbf{h}(\partial \Omega \times[a, b])$ is compact and so is at a positive distance from 0 . Let $\varepsilon>0$ be such that for all $t \in[a, b]$,

$$
\begin{equation*}
B(\mathbf{0}, \varepsilon) \cap \mathbf{h}(\partial \Omega \times[a, b])=\emptyset \tag{23.1.3}
\end{equation*}
$$

Define for $t \in(a, b)$,

$$
H(t) \equiv \int_{\Omega} \phi_{\varepsilon}(\mathbf{h}(\mathbf{x}, t)) \operatorname{det} D_{1} \mathbf{h}(\mathbf{x}, t) d x
$$

I will show that $H^{\prime}(t)=0$ on $(a, b)$. Then, since $H$ is continuous on $[a, b]$, it will follow from the mean value theorem that $H(t)$ is constant on $[a, b]$. If $t \in(a, b)$,

$$
H^{\prime}(t)=\int_{\Omega} \sum_{\alpha} \phi_{\varepsilon, \alpha}(\mathbf{h}(\mathbf{x}, t)) h_{\alpha, t}(\mathbf{x}, t) \operatorname{det} D_{1} \mathbf{h}(\mathbf{x}, t) d x
$$

$$
\begin{equation*}
+\int_{\Omega} \phi_{\varepsilon}(\mathbf{h}(\mathbf{x}, t)) \sum_{\alpha, j} \operatorname{det} D_{1}(\mathbf{h}(\mathbf{x}, t))_{, \alpha j} h_{\alpha, j t} d x \equiv \mathbf{A}+\mathbf{B} . \tag{23.1.4}
\end{equation*}
$$

In this formula, the function det is considered as a function of the $n^{2}$ entries in the $n \times n$ matrix and the, $\alpha j$ represents the derivative with respect to the $\alpha j^{\text {th }}$ entry $h_{\alpha, j}$. Now as in the proof of Lemma 16.3.1 on Page 429,

$$
\operatorname{det} D_{1}(\mathbf{h}(\mathbf{x}, t))_{, \alpha_{j}}=\left(\operatorname{cof} D_{1}(\mathbf{h}(\mathbf{x}, t))\right)_{\alpha_{j}}
$$

and so

$$
\mathbf{B}=\int_{\Omega} \sum_{\alpha} \sum_{j} \phi_{\varepsilon}(\mathbf{h}(\mathbf{x}, t))\left(\operatorname{cof} D_{1}(\mathbf{h}(\mathbf{x}, t))\right)_{\alpha j} h_{\alpha, j t} d x .
$$

By hypothesis

$$
\mathbf{x} \rightarrow \phi_{\varepsilon}(\mathbf{h}(\mathbf{x}, t))\left(\operatorname{cof} D_{1}(\mathbf{h}(\mathbf{x}, t))\right)_{\alpha_{j}} \text { for } \mathbf{x} \in \Omega
$$

is in $C_{c}^{\infty}(\Omega)$ because if $\mathbf{x} \in \partial \Omega$, it follows that for all $t \in[a, b], \mathbf{h}(\mathbf{x}, t) \notin B(\mathbf{0}, \varepsilon)$ and so $\phi_{\varepsilon}(\mathbf{h}(\mathbf{x}, t))=0$ off some compact set contained in $\Omega$. Therefore, integrate by parts and write

$$
\begin{aligned}
\mathbf{B}=- & \int_{\Omega} \sum_{\alpha} \sum_{j} \frac{\partial}{\partial x_{j}}\left(\phi_{\varepsilon}(\mathbf{h}(\mathbf{x}, t))\right)\left(\operatorname{cof} D_{1}(\mathbf{h}(\mathbf{x}, t))\right)_{\alpha j} h_{\alpha, t} d x+ \\
& -\int_{\Omega} \sum_{\alpha} \sum_{j} \phi_{\varepsilon}(\mathbf{h}(\mathbf{x}, t))(\operatorname{cof} D(\mathbf{h}(\mathbf{x}, t)))_{\alpha j, j} h_{\alpha, t} d x
\end{aligned}
$$

The second term equals zero by Lemma 23.1.8. Simplifying the first term yields

$$
\begin{aligned}
\mathbf{B} & =-\int_{\Omega} \sum_{\alpha} \sum_{j} \sum_{\beta} \phi_{\varepsilon, \beta}(\mathbf{h}(\mathbf{x}, t)) h_{\beta, j} h_{\alpha, t}\left(\operatorname{cof} D_{1}(\mathbf{h}(\mathbf{x}, t))\right)_{\alpha j} d x \\
& =-\int_{\Omega} \sum_{\alpha} \sum_{\beta} \phi_{\varepsilon, \beta}(\mathbf{h}(\mathbf{x}, t)) h_{\alpha, t} \sum_{j} h_{\beta, j}\left(\operatorname{cof} D_{1}(\mathbf{h}(\mathbf{x}, t))\right)_{\alpha_{j}} d x
\end{aligned}
$$

Now the sum on $j$ is the dot product of the $\beta^{\text {th }}$ row with the $\alpha^{t h}$ row of the cofactor matrix which equals zero unless $\beta=\alpha$ because it would be a cofactor expansion of a matrix with two equal rows. When $\beta=\alpha$, the sum on $j$ reduces to $\operatorname{det}\left(D_{1}(\mathbf{h}(\mathbf{x}, t))\right)$. Thus $\mathbf{B}$ reduces to

$$
=-\int_{\Omega} \sum_{\alpha} \phi_{\varepsilon, \alpha}(\mathbf{h}(\mathbf{x}, t)) h_{\alpha, t} \operatorname{det}\left(D_{1}(\mathbf{h}(\mathbf{x}, t))\right) d x
$$

Which is the same thing as $\mathbf{A}$, but with the opposite sign. Hence $\mathbf{A}+\mathbf{B}$ in 23.1.4 is 0 and $H^{\prime}(t)=0$ and so $H$ is a constant on $[a, b]$.

Finally consider the last claim. If $\mathbf{g}, \hat{\mathbf{g}}$ both work in the definition for the degree, then consider $\mathbf{h}(\mathbf{x}, t) \equiv \operatorname{tg}(\mathbf{x})+(1-t) \hat{\mathbf{g}}(\mathbf{x})-\mathbf{y}$ for $t \in[0,1]$. From Lemma 23.1.7, $\mathbf{h}$ satisfies what is needed for the first part of this lemma. Then from Lemma 23.1.11 and the first part of this lemma, if $0<\varepsilon<\operatorname{dist}(\mathbf{0}, \mathbf{h}(\partial \Omega \times[0,1]))$ is sufficiently small that the second and
last equations hold in what follows,

$$
\begin{aligned}
d(\mathbf{f}, \Omega, \mathbf{y}) & =\sum\left\{\operatorname{sgn}(\operatorname{det}(D \mathbf{g}(\mathbf{x}))): \mathbf{x} \in \mathbf{g}^{-1}(\mathbf{y})\right\} \\
& =\int_{\Omega} \phi_{\varepsilon}(\mathbf{h}(\mathbf{x}, 1)) \operatorname{det} D_{1} \mathbf{h}(\mathbf{x}, 1) d x \\
& =\int_{\Omega} \phi_{\varepsilon}(\mathbf{h}(\mathbf{x}, 0)) \operatorname{det} D_{1} \mathbf{h}(\mathbf{x}, 0) d x \\
& =\sum\left\{\operatorname{sgn}(\operatorname{det}(D \hat{\mathbf{g}}(\mathbf{x}))): \mathbf{x} \in \mathbf{g}^{-1}(\mathbf{y})\right\}
\end{aligned}
$$

The last claim was noted earlier. If $\mathbf{y} \notin \mathbf{f}(\bar{\Omega})$, then letting $\mathbf{g}$ be smooth with $\|\mathbf{f}-\mathbf{g}\|_{\infty, \bar{\Omega}}<$ $\delta<\operatorname{dist}(\mathbf{f}(\partial \Omega), \mathbf{y})$, it follows that if $\mathbf{x} \in \partial \Omega$,

$$
\|\mathbf{g}(\mathbf{x})-\mathbf{y}\| \geq\|\mathbf{y}-\mathbf{f}(\mathbf{x})\|-\|\mathbf{g}(\mathbf{x})-\mathbf{f}(\mathbf{x})\|>\operatorname{dist}(\mathbf{f}(\partial \Omega), \mathbf{y})-\delta \geq 0
$$

Thus, from the definition, $d(\mathbf{f}, \Omega, \mathbf{y})=0$.

### 23.2 Properties of the Degree

Now that the degree for a continuous function has been defined, it is time to consider properties of the degree. In particular, it is desirable to prove a theorem about homotopy invariance which depends only on continuity considerations.

Theorem 23.2.1 If $\mathbf{h}$ is in $C\left(\bar{\Omega} \times[a, b], \mathbb{R}^{p}\right)$, and $\mathbf{0} \notin \mathbf{h}(\partial \Omega \times[a, b])$ then

$$
t \rightarrow d(\mathbf{h}(\cdot, t), \Omega, \mathbf{0})
$$

is constant for $t \in[a, b]$.
Proof: Let $0<\delta<\min _{t \in[a, b]} \operatorname{dist}(\mathbf{h}(\partial \Omega \times[a, b]), \mathbf{0})$. By Corollary 23.1.1, there exists

$$
\mathbf{h}_{m}(\cdot, t)=\sum_{k=0}^{m} p_{k}(t) \mathbf{h}\left(\cdot, t_{k}\right)
$$

for $p_{k}(t)$ some polynomial in $t$ of degree $m$ such that

$$
\begin{equation*}
\max _{t \in[a, b]}\left\|\mathbf{h}_{m}(\cdot, t)-\mathbf{h}(\cdot, t)\right\|_{\infty, \bar{\Omega}}<\delta \tag{23.2.5}
\end{equation*}
$$

Letting $\psi_{n}$ be a mollifier,

$$
C_{c}^{\infty}\left(B\left(\mathbf{0}, \frac{1}{n}\right)\right), \int_{\mathbb{R}^{p}} \psi_{n}(\mathbf{u}) d u=1
$$

let

$$
\mathbf{g}_{n}(\cdot, t) \equiv \mathbf{h}_{m} * \psi_{n}(\cdot, t)
$$

Thus,

$$
\begin{aligned}
\mathbf{g}_{n}(\mathbf{x}, t) & \equiv \int_{\mathbb{R}^{p}} \mathbf{h}_{m}(\mathbf{x}-\mathbf{u}, t) \psi_{n}(\mathbf{u}) d u=\sum_{k=0}^{m} p_{k}(t) \int_{\mathbb{R}^{p}} \mathbf{h}\left(\mathbf{x}-\mathbf{u}, t_{k}\right) \psi_{n}(\mathbf{u}) d u \\
& =\sum_{k=0}^{m} p_{k}(t) \int_{\mathbb{R}^{p}} \mathbf{h}\left(\mathbf{u}, t_{k}\right) \psi_{n}(\mathbf{x}-\mathbf{u}) d u \equiv \sum_{k=0}^{m} p_{k}(t) \mathbf{h}\left(\cdot, t_{k}\right) * \psi_{n}(\mathbf{x})(23.2 .6)
\end{aligned}
$$

so $\mathbf{x} \rightarrow \mathbf{g}_{n}(\mathbf{x}, t)$ is in $C^{\infty}\left(\bar{\Omega} ; \mathbb{R}^{p}\right)$. Also,

$$
\left\|\mathbf{g}_{n}(\mathbf{x}, t)-\mathbf{h}_{m}(\mathbf{x}, t)\right\| \leq \int_{B\left(\mathbf{0}, \frac{1}{n}\right)}\|\mathbf{h}(\mathbf{x}-\mathbf{u}, t)-\mathbf{h}(\mathbf{z}, t)\| d u<\varepsilon
$$

provided $n$ is large enough. This follows from uniform continuity of $\mathbf{h}$ on the compact set $\bar{\Omega} \times[a, b]$. It follows that if $n$ is large enough, one can replace $\mathbf{h}_{m}$ in 23.2 .5 with $\mathbf{g}_{n}$ and obtain for large enough $n$

$$
\begin{equation*}
\max _{t \in[a, b]}\left\|\mathbf{g}_{n}(\cdot, t)-\mathbf{h}(\cdot, t)\right\|_{\infty, \bar{\Omega}}<\delta \tag{23.2.7}
\end{equation*}
$$

Now $\mathbf{g}_{n} \in C^{\infty}\left(\bar{\Omega} \times[a, b] ; \mathbb{R}^{p}\right)$ because all partial derivatives with respect to either $t$ or $\mathbf{x}$ are continuous. Here

$$
\mathbf{g}_{n}(\mathbf{x}, t)=\sum_{k=0}^{m} p_{k}(t) \mathbf{h}\left(\cdot, t_{k}\right) * \psi_{n}(\mathbf{x})
$$

Let $\tau \in(a, b]$. Let $\mathbf{g}_{a \tau}(\mathbf{x}, t) \equiv \mathbf{g}_{n}(\mathbf{x}, t)-\left(\frac{\tau-t}{\tau-a} \mathbf{y}_{a}+\mathbf{y}_{\tau} \frac{t-a}{\tau-a}\right)$ where $\mathbf{y}_{a}$ is a regular value of $\mathbf{g}_{n}(\cdot, a), \mathbf{y}_{\tau}$ is a regular value of $\mathbf{g}_{n}(\mathbf{x}, \tau)$ and both $\mathbf{y}_{a}, \mathbf{y}_{\tau}$ are so small that

$$
\begin{equation*}
\max _{t \in[a, b]}\left\|\mathbf{g}_{a \tau}(\cdot, t)-\mathbf{h}(\cdot, t)\right\|_{\infty, \bar{\Omega}}<\delta \tag{23.2.8}
\end{equation*}
$$

This uses Lemma 23.1.3. Thus if $\mathbf{g}_{a \tau}(\mathbf{x}, \tau)=\mathbf{0}$, then $\mathbf{g}_{n}(\mathbf{x}, \tau)-\mathbf{y}_{\tau}=\mathbf{0}$ so $\mathbf{0}$ is a regular value for $\mathbf{g}_{a \tau}(\cdot, \tau)$. If $\mathbf{g}_{a \tau}(\mathbf{x}, a)=\mathbf{0}$, then $\mathbf{g}_{n}(\mathbf{x}, t)=\mathbf{y}_{a}$ so $\mathbf{0}$ is also a regular value for $\mathbf{g}_{a \tau}(\cdot, a)$. From 23.2.8, $\operatorname{dist}\left(\mathbf{g}_{a \tau}(\partial \Omega \times[a, b]), \mathbf{0}\right)>0$ as in Lemma 23.1.5. Choosing $\varepsilon<$ $\operatorname{dist}\left(\mathbf{g}_{a \tau}(\partial \Omega \times[a, b]), \mathbf{0}\right)$, it follows from Lemma 23.1.12, the definition of the degree, and Lemma 23.1.11 in the first and last equations that for $\varepsilon$ small enough,

$$
\begin{aligned}
d(\mathbf{h}(\cdot, a), \Omega, \mathbf{0}) & =\int_{\Omega} \phi_{\varepsilon}\left(\mathbf{g}_{a \tau}(\mathbf{x}, a)\right) \operatorname{det} D_{1} \mathbf{g}_{a \tau}(\mathbf{x}, a) d x \\
& =\int_{\Omega} \phi_{\varepsilon}\left(\mathbf{g}_{a \tau}(\mathbf{x}, \tau)\right) \operatorname{det} D_{1} \mathbf{g}_{a \tau}(\mathbf{x}, \tau) d x=d(\mathbf{h}(\cdot, \tau), \Omega, \mathbf{0})
\end{aligned}
$$

Since $\tau$ is arbitrary, this proves the theorem.
Now the following theorem is a summary of the main result on properties of the degree.
Theorem 23.2.2 Definition 23.1.6 is well defined and the degree satisfies the following properties.

1. (homotopy invariance) If $\mathbf{h} \in C\left(\bar{\Omega} \times[0,1], \mathbb{R}^{p}\right)$ and $\mathbf{y}(t) \notin \mathbf{h}(\partial \Omega, t)$ for all $t \in[0,1]$ where $\mathbf{y}$ is continuous, then

$$
t \rightarrow d(\mathbf{h}(\cdot, t), \Omega, \mathbf{y}(t))
$$

is constant for $t \in[0,1]$.
2. If $\Omega \supseteq \Omega_{1} \cup \Omega_{2}$ where $\Omega_{1} \cap \Omega_{2}=\emptyset$, for $\Omega_{i}$ an open set, then if $\mathbf{y} \notin \mathbf{f}\left(\bar{\Omega} \backslash\left(\Omega_{1} \cup \Omega_{2}\right)\right)$, then

$$
d\left(\mathbf{f}, \Omega_{1}, \mathbf{y}\right)+d\left(\mathbf{f}, \Omega_{2}, \mathbf{y}\right)=d(\mathbf{f}, \Omega, \mathbf{y})
$$

3. $d(I, \Omega, \mathbf{y})=1$ if $\mathbf{y} \in \Omega$.
4. $d(\mathbf{f}, \Omega, \cdot)$ is continuous and constant on every connected component of $\mathbb{R}^{p} \backslash \mathbf{f}(\partial \Omega)$.
5. $d(\mathbf{g}, \Omega, \mathbf{y})=d(\mathbf{f}, \Omega, \mathbf{y})$ if $\left.\mathbf{g}\right|_{\partial \Omega}=\left.\mathbf{f}\right|_{\partial \Omega}$.
6. If $\mathbf{y} \notin \mathbf{f}(\partial \Omega)$, and if $(\mathbf{f}, \Omega, \mathbf{y}) \neq 0$, then there exists $\mathbf{x} \in \Omega$ such that $\mathbf{f}(\mathbf{x})=\mathbf{y}$.

Proof: That the degree is well defined follows from Lemma 23.1.12.
Consider 1., the first property about homotopy. This follows from Theorem 23.2.1 applied to $H(\mathbf{x}, t) \equiv \mathbf{h}(\mathbf{x}, t)-\mathbf{y}(t)$.

Consider 2. where $\mathbf{y} \notin \mathbf{f}\left(\bar{\Omega} \backslash\left(\Omega_{1} \cup \Omega_{2}\right)\right)$. Note that

$$
\operatorname{dist}\left(\mathbf{y}, \mathbf{f}\left(\bar{\Omega} \backslash\left(\Omega_{1} \cup \Omega_{2}\right)\right)\right) \leq \operatorname{dist}(\mathbf{y}, \mathbf{f}(\partial \Omega))
$$

Then let $\mathbf{g}$ be in $C\left(\bar{\Omega} ; \mathbb{R}^{p}\right)$ and

$$
\begin{aligned}
\|\mathbf{g}-\mathbf{f}\|_{\infty} & <\operatorname{dist}\left(\mathbf{y}, \mathbf{f}\left(\bar{\Omega} \backslash\left(\Omega_{1} \cup \Omega_{2}\right)\right)\right) \\
& \leq \min \left(\operatorname{dist}\left(\mathbf{y}, \mathbf{f}\left(\partial \Omega_{1}\right)\right), \operatorname{dist}\left(\mathbf{y}, \mathbf{f}\left(\partial \Omega_{2}\right)\right), \operatorname{dist}(\mathbf{y}, \mathbf{f}(\partial \Omega))\right)
\end{aligned}
$$

where $\mathbf{y}$ is a regular value of $\mathbf{g}$. Then by definition,

$$
\begin{aligned}
& d(\mathbf{f}, \Omega, \mathbf{y}) \equiv \sum\left\{\operatorname{det}(D \mathbf{g}(\mathbf{x})): \mathbf{x} \in \mathbf{g}^{-1}(\mathbf{y})\right\} \\
&= \sum\left\{\operatorname{det}(D \mathbf{g}(\mathbf{x})): \mathbf{x} \in \mathbf{g}^{-1}(\mathbf{y}), \mathbf{x} \in \Omega_{1}\right\} \\
&+\sum\left\{\operatorname{det}(D \mathbf{g}(\mathbf{x})): \mathbf{x} \in \mathbf{g}^{-1}(\mathbf{y}), \mathbf{x} \in \Omega_{2}\right\} \\
& \equiv d\left(\mathbf{f}, \Omega_{1}, \mathbf{y}\right)+d\left(\mathbf{f}, \Omega_{2}, \mathbf{y}\right)
\end{aligned}
$$

It is of course obvious that this can be extended by induction to any finite number of disjoint open sets $\Omega_{i}$.

Note that 3. is obvious because $I(\mathbf{x})=\mathbf{x}$ and so if $\mathbf{y} \in \Omega$, then $I^{-1}(\mathbf{y})=\mathbf{y}$ and $D I(\mathbf{x})=I$ for any $\mathbf{x}$ so the definition gives 3 .

Now consider 4. Let $U$ be a connected component of $\mathbb{R}^{p} \backslash \mathbf{f}(\partial \Omega)$. This is open as well as connected and arc wise connected by Theorem 7.13.10. Hence, if $\mathbf{u}, \mathbf{v} \in U$, there is a continuous function $\mathbf{y}(t)$ which is in $U$ such that $\mathbf{y}(0)=\mathbf{u}$ and $\mathbf{y}(1)=\mathbf{v}$. By homotopy invariance, it follows $d(\mathbf{f}, \Omega, \mathbf{y}(t))$ is constant. Thus $d(\mathbf{f}, \Omega, \mathbf{u})=d(\mathbf{f}, \Omega, \mathbf{v})$.

Next consider 5. When $\mathbf{f}=\mathbf{g}$ on $\partial \Omega$, it follows that if $\mathbf{y} \notin \mathbf{f}(\partial \Omega)$, then $\mathbf{y} \notin \mathbf{f}(\mathbf{x})+$ $t(\mathbf{g}(\mathbf{x})-\mathbf{f}(\mathbf{x}))$ for $t \in[0,1]$ and $\mathbf{x} \in \partial \Omega$ so

$$
d(\mathbf{f}+t(\mathbf{g}-\mathbf{f}), \Omega, \mathbf{y})
$$

is constant for $t \in[0,1]$ by homotopy invariance in part 1 . Therefore, let $t=0$ and then $t=1$ to obtain 5 .

Claim 6. follows from Lemma 23.1.12 which says that if $\mathbf{y} \notin \mathbf{f}(\bar{\Omega})$, then $d(\mathbf{f}, \Omega, \mathbf{y})=0$.
From the above, there is an easy corollary which gives related properties of the degree.

Corollary 23.2.3 The following additional properties of the degree are also valid.

1. If $\mathbf{y} \notin \mathbf{f}\left(\bar{\Omega} \backslash \Omega_{1}\right)$ and $\Omega_{1}$ is an open subset of $\Omega$, then $d(\mathbf{f}, \Omega, \mathbf{y})=d\left(\mathbf{f}, \Omega_{1}, \mathbf{y}\right)$.
2. $d(\cdot, \Omega, \mathbf{y})$ is defined and constant on

$$
\left\{\mathbf{g} \in C\left(\bar{\Omega} ; \mathbb{R}^{p}\right):\|\mathbf{g}-\mathbf{f}\|_{\infty}<r\right\}
$$

where $r=\operatorname{dist}(\mathbf{y}, \mathbf{f}(\partial \Omega))$.
3. If $\operatorname{dist}(\mathbf{y}, \mathbf{f}(\partial \Omega)) \geq \delta$ and $|\mathbf{z}-\mathbf{y}|<\delta$, then $d(\mathbf{f}, \Omega, \mathbf{y})=d(\mathbf{f}, \Omega, \mathbf{z})$.

Proof: Consider 1. You can take $\Omega_{2}=\emptyset$ in 2 of Theorem 23.2.2 or you can modify the proof of 2 slightly. Consider 2. To verify, let $\mathbf{h}(\mathbf{x}, t)=\mathbf{f}(\mathbf{x})+t(\mathbf{g}(\mathbf{x})-\mathbf{f}(\mathbf{x}))$. Then note that $\mathbf{y} \notin \mathbf{h}(\partial \Omega, t)$ and use Property 1 of Theorem 23.2.2. Finally, consider 3. Let $\mathbf{y}(t) \equiv(1-t) \mathbf{y}+t \mathbf{z}$. Then for $\mathbf{x} \in \partial \Omega$

$$
\begin{aligned}
|(1-t) \mathbf{y}+t \mathbf{z}-\mathbf{f}(\mathbf{x})| & =|\mathbf{y}-\mathbf{f}(\mathbf{x})+t(\mathbf{z}-\mathbf{y})| \\
& \geq \delta-t|\mathbf{z}-\mathbf{y}|>\delta-\delta=0
\end{aligned}
$$

Then by 1 of Theorem 23.2.2, $d(\mathbf{f}, \Omega,(1-t) \mathbf{y}+t \mathbf{z})$ is constant. When $t=0$ you get $d(\mathbf{f}, \Omega, \mathbf{y})$ and when $t=1$ you get $d(\mathbf{f}, \Omega, \mathbf{z})$.

Another simple observation is that if you have $\mathbf{y}_{1}, \cdots, \mathbf{y}_{r}$ in $\mathbb{R}^{p} \backslash \mathbf{f}(\partial \Omega)$, then if $\tilde{\mathbf{f}}$ has the property that $\|\tilde{\mathbf{f}}-\mathbf{f}\|_{\infty}<\min _{i \leq r} \operatorname{dist}\left(\mathbf{y}_{i}, \mathbf{f}(\partial \Omega)\right)$, then

$$
d\left(\mathbf{f}, \Omega, \mathbf{y}_{i}\right)=d\left(\tilde{\mathbf{f}}, \Omega, \mathbf{y}_{i}\right)
$$

for each $\mathbf{y}_{i}$. This follows right away from the above arguments and the homotopy invariance applied to each of the finitely many $\mathbf{y}_{i}$. Just consider $d\left(\mathbf{f}+t(\tilde{\mathbf{f}}-\mathbf{f}), \Omega, \mathbf{y}_{i}\right), t \in[0,1]$. If $\mathbf{x} \in \partial \Omega, \mathbf{f}+t(\tilde{\mathbf{f}}-\mathbf{f})(\mathbf{x}) \neq \mathbf{y}_{i}$ and so $d\left(\mathbf{f}+t(\tilde{\mathbf{f}}-\mathbf{f}), \Omega, \mathbf{y}_{i}\right)$ is constant on $[0,1]$, this for each $i$.

### 23.3 Borsuk's Theorem

In this section is an important theorem which can be used to verify that $d(\mathbf{f}, \Omega, \mathbf{y}) \neq 0$. This is significant because when this is known, it follows from Theorem 23.2.2 that $\mathbf{f}^{-1}(\mathbf{y}) \neq \emptyset$. In other words there exists $\mathbf{x} \in \Omega$ such that $\mathbf{f}(\mathbf{x})=\mathbf{y}$.

Definition 23.3.1 A bounded open set, $\Omega$ is symmetric if $-\Omega=\Omega$. A continuous function, $\mathbf{f}: \bar{\Omega} \rightarrow \mathbb{R}^{p}$ is odd if $\mathbf{f}(-\mathbf{x})=-\mathbf{f}(\mathbf{x})$.

Suppose $\Omega$ is symmetric and $\mathbf{g} \in C^{\infty}\left(\bar{\Omega} ; \mathbb{R}^{p}\right)$ is an odd map for which $\mathbf{0}$ is a regular value. Then the chain rule implies $D \mathbf{g}(-\mathbf{x})=D \mathbf{g}(\mathbf{x})$ and so $d(\mathbf{g}, \Omega, \mathbf{0})$ must equal an odd integer because if $\mathbf{x} \in \mathbf{g}^{-1}(\mathbf{0})$, it follows that $-\mathbf{x} \in \mathbf{g}^{-1}(\mathbf{0})$ also and since $D \mathbf{g}(-\mathbf{x})=D \mathbf{g}(\mathbf{x})$, it follows the overall contribution to the degree from $\mathbf{x}$ and $-\mathbf{x}$ must be an even integer. Also $\mathbf{0} \in \mathbf{g}^{-1}(\mathbf{0})$ and so the degree equals an even integer added to sgn $(\operatorname{det} D \mathbf{g}(\mathbf{0}))$, an odd integer, either -1 or 1 . It seems reasonable to expect that something like this would hold for an arbitrary continuous odd function defined on symmetric $\Omega$. In fact this is the
case and this is next. The following lemma is the key result used. This approach is due to Gromes [57]. See also Deimling [38] which is where I found this argument.

The idea is to start with a smooth odd map and approximate it with a smooth odd map which also has $\mathbf{0}$ a regular value. Note that $\mathbf{0}$ is achieved because $\mathbf{g}(\mathbf{0})=-\mathbf{g}(\mathbf{0})$.

Lemma 23.3.2 Let $\mathbf{g} \in C^{\infty}\left(\bar{\Omega} ; \mathbb{R}^{p}\right)$ be an odd map. Then for every $\varepsilon>0$, there exists $\mathbf{h}$ $\in C^{\infty}\left(\bar{\Omega} ; \mathbb{R}^{p}\right)$ such that $\mathbf{h}$ is also an odd map, $\|\mathbf{h}-\mathbf{g}\|_{\infty}<\varepsilon$, and $\mathbf{0}$ is a regular value of $\mathbf{h}, \mathbf{0} \notin \mathbf{g}(\partial \Omega)$. Here $\Omega$ is a symmetric bounded open set. In addition, $d(\mathbf{g}, \Omega, \mathbf{0})$ is an odd integer.

Proof: In this argument $\eta>0$ will be a small positive number. Let $\mathbf{h}_{0}(\mathbf{x})=\mathbf{g}(\mathbf{x})+\eta \mathbf{x}$ where $\eta$ is chosen such that $\operatorname{det} D \mathbf{h}_{0}(\mathbf{0}) \neq 0$. Just let $-\eta$ not be an eigenvalue of $D \mathbf{g}(\mathbf{0})$ also see Lemma 5.9.7. Note that $\mathbf{h}_{0}$ is odd and $\mathbf{0}$ is a value of $\mathbf{h}_{0}$ thanks to $\mathbf{h}_{0}(\mathbf{0})=\mathbf{0}$. This has taken care of $\mathbf{0}$. However, it is not known whether $\mathbf{0}$ is a regular value of $\mathbf{h}_{0}$ because there may be other $\mathbf{x}$ where $\mathbf{h}_{0}(\mathbf{x})=\mathbf{0}$. These other points must be accounted for. The eventual function will be of the form

$$
\mathbf{h}(\mathbf{x}) \equiv \mathbf{h}_{0}(\mathbf{x})-\sum_{j=1}^{p} \mathbf{y}^{j} x_{j}^{3}, \sum_{j=1}^{0} \mathbf{y}^{j} x_{j}^{3} \equiv 0
$$

Note that $\mathbf{h}(\mathbf{0})=\mathbf{0}$ and $\operatorname{det}(D \mathbf{h}(\mathbf{0}))=\operatorname{det}\left(D \mathbf{h}_{0}(\mathbf{0})\right) \neq 0$. This is because when you take the matrix of the derivative, those terms involving $x_{j}^{3}$ will vanish when $\mathbf{x}=\mathbf{0}$ because of the exponent 3 .

The idea is to choose small $\mathbf{y}^{j},\left\|\mathbf{y}^{j}\right\|<\eta$ in such a way that $\mathbf{0}$ is a regular value for $\mathbf{h}$ for each $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{x} \in \mathbf{h}^{-1}(\mathbf{0})$. As just noted, the case where $\mathbf{x}=\mathbf{0}$ creates no problems. Let

$$
\Omega_{i} \equiv\left\{\mathbf{x} \in \Omega: x_{i} \neq 0\right\}, \text { so } \cup_{j=1}^{p} \Omega_{j}=\left\{\mathbf{x} \in \mathbb{R}^{p}: \mathbf{x} \neq \mathbf{0}\right\}, \Omega_{0} \equiv\{\mathbf{0}\}
$$

Each $\Omega_{i}$ is a symmetric open set while $\Omega_{0}$ is the single point $\mathbf{0}$. Then let $\mathbf{h}_{1}(\mathbf{x}) \equiv \mathbf{h}_{0}(\mathbf{x})-$ $\mathbf{y}^{1} x_{1}^{3}$ on $\Omega_{1}$. If $\mathbf{h}_{1}(\mathbf{x})=\mathbf{0}$, then

$$
\begin{equation*}
\mathbf{h}_{0}(\mathbf{x})=\mathbf{y}^{1} x_{1}^{3} \text { so } \mathbf{y}^{1}=\frac{\mathbf{h}_{0}(\mathbf{x})}{x_{1}^{3}} \tag{23.3.9}
\end{equation*}
$$

The set of singular values of $\mathbf{x} \rightarrow \frac{\mathbf{h}_{0}(\mathbf{x})}{x_{1}^{3}}$ for $\mathbf{x} \in \Omega_{1}$ contains no open sets. Hence there exists a regular value $\mathbf{y}^{1}$ for $\mathbf{x} \rightarrow \frac{\mathbf{h}_{0}(\mathbf{x})}{x_{1}^{3}}$ such that $\left\|\mathbf{y}^{1}\right\|<\eta$. Then if $\mathbf{y}^{1}=\frac{\mathbf{h}_{0}(\mathbf{x})}{x_{1}^{3}}$, abusing notation a little, the matrix of the derivative at this $\mathbf{x}$ is

$$
\begin{aligned}
D \mathbf{h}_{1}(\mathbf{x})_{r s} & =\frac{h_{0 r, s}(\mathbf{x}) x_{1}^{3}-\left(x_{1}\right)_{, x_{s}}^{3} h_{0 r}(\mathbf{x})}{x_{1}^{6}}=\frac{h_{0 r, s}(\mathbf{x}) x_{1}^{3}-\left(x_{1}\right)_{, x_{s}}^{3} y_{r}^{1} x_{1}^{3}}{x_{1}^{6}} \\
& =\frac{h_{0 r, s}(\mathbf{x})-\left(x_{1}\right)_{, x_{s}}^{3} y_{r}^{1}}{x_{1}^{3}}
\end{aligned}
$$

and so, choosing $\mathbf{y}^{1}$ this way in 23.3.9, this derivative is non-singular if and only if

$$
\operatorname{det}\left(h_{0 r, s}(\mathbf{x})-\left(x_{1}\right)_{, x_{s}}^{3} y_{r}^{1}\right)=\operatorname{det} D\left(\mathbf{h}_{1}(\mathbf{x})\right) \neq 0
$$

which of course is exactly what is wanted. Thus there is a small $\mathbf{y}^{1},\left\|\mathbf{y}^{1}\right\|<\eta$ such that if $\mathbf{h}_{1}(\mathbf{x})=\mathbf{0}$, for $\mathbf{x} \in \Omega_{1}$, then $\operatorname{det} D\left(\mathbf{h}_{1}(\mathbf{x})\right) \neq 0$. That is, $\mathbf{0}$ is a regular value of $\mathbf{h}_{1}$. Thus, 23.3.9 is how to choose $\mathbf{y}^{1}$ for $\left\|\mathbf{y}^{1}\right\|<\eta$.

Then the theorem will be proved by doing the same process, going from $\Omega_{1}$ to $\Omega_{1} \cup \Omega_{2}$ and so forth. If for each $k \leq p$, there exist $\mathbf{y}^{i},\left\|\mathbf{y}^{i}\right\|<\eta, i \leq k$ and $\mathbf{0}$ is a regular value of

$$
\mathbf{h}_{k}(\mathbf{x}) \equiv \mathbf{h}_{0}(\mathbf{x})-\sum_{j=1}^{k} \mathbf{y}^{j} x_{j}^{3} \text { for } \mathbf{x} \in \cup_{i=0}^{k} \Omega_{i}
$$

Then the theorem will be proved by considering $\mathbf{h}_{p} \equiv \mathbf{h}$. For $k=0,1$ this is done. Suppose then that this holds for $k-1, k \geq 2$. Thus $\mathbf{0}$ is a regular value for

$$
\mathbf{h}_{k-1}(\mathbf{x}) \equiv \mathbf{h}_{0}(\mathbf{x})-\sum_{j=1}^{k-1} \mathbf{y}^{j} x_{j}^{3} \text { on } \cup_{i=0}^{k-1} \Omega_{i}
$$

What should $\mathbf{y}^{k}$ be? Keeping $\mathbf{y}^{1}, \cdots, \mathbf{y}^{k-1}$, it follows that if $x_{k}=0, \mathbf{h}_{k}(\mathbf{x})=\mathbf{h}_{k-1}(\mathbf{x})$. For $\mathbf{x} \in \Omega_{k}$ where $x_{k} \neq 0$,

$$
\mathbf{h}_{k}(\mathbf{x}) \equiv \mathbf{h}_{0}(\mathbf{x})-\sum_{j=1}^{k} \mathbf{y}^{j} x_{j}^{3}=\mathbf{0}
$$

if and only if

$$
\frac{\mathbf{h}_{0}(\mathbf{x})-\sum_{j=1}^{k-1} \mathbf{y}^{j} x_{j}^{3}}{x_{k}^{3}}=\mathbf{y}^{k}
$$

So let $\mathbf{y}^{k}$ be a regular value of $\frac{\mathbf{h}_{0}(\mathbf{x})-\sum_{j=1}^{k-1} \mathbf{y}^{j} x_{j}^{3}}{x_{k}^{3}}$ on $\Omega_{k},\left(x_{k} \neq 0\right)$ and also $\left\|\mathbf{y}^{k}\right\|<\eta$. This is the same reasoning as before, the set of singular values does not contain any open set. Then for such $\mathbf{x}$ satisfying

$$
\begin{equation*}
\frac{\mathbf{h}_{0}(\mathbf{x})-\sum_{j=1}^{k-1} \mathbf{y}^{j} x_{j}^{3}}{x_{k}^{3}}=\mathbf{y}^{k} \text { on } \Omega_{k}, \tag{23.3.10}
\end{equation*}
$$

and using the quotient rule as before,

$$
0 \neq \operatorname{det}\left(\frac{\left(h_{0 r, s}-\sum_{j=1}^{k-1} y_{r}^{j} \partial_{x_{s}}\left(x_{j}^{3}\right)\right) x_{k}^{3}-\left(h_{0 r}(\mathbf{x})-\sum_{j=1}^{k-1} y_{r}^{j} x_{j}^{3}\right) \partial_{x_{s}}\left(x_{k}^{3}\right)}{x_{k}^{6}}\right)
$$

and $h_{0 r}(\mathbf{x})-\sum_{j=1}^{k-1} y_{r}^{j} x_{j}^{3}=y_{r}^{k} x_{k}^{3}$ from 23.3.10 and so

$$
\begin{aligned}
& 0 \neq \operatorname{det}\left(\frac{\left(h_{0 r, s}-\sum_{j=1}^{k-1} y_{r}^{j} \partial_{x_{s}}\left(x_{j}^{3}\right)\right) x_{k}^{3}-y_{r}^{k} x_{k}^{3} \partial_{x_{s}}\left(x_{k}^{3}\right)}{x_{k}^{6}}\right) \\
& 0 \neq \operatorname{det}\left(\frac{\left(h_{0 r, s}-\sum_{j=1}^{k-1} y_{r}^{j} \partial_{x_{s}}\left(x_{j}^{3}\right)\right)-y_{r}^{k} \partial_{x_{s}}\left(x_{k}^{3}\right)}{x_{k}^{3}}\right)
\end{aligned}
$$

which implies

$$
0 \neq \operatorname{det}\left(\left(h_{0 r, s}-\sum_{j=1}^{k-1} y_{r}^{j} \partial_{x_{s}}\left(x_{j}^{3}\right)\right)-y_{r}^{k} \partial_{x_{s}}\left(x_{k}^{3}\right)\right)=\operatorname{det}\left(D\left(\mathbf{h}_{k}(\mathbf{x})\right)\right)
$$

If $\mathbf{x} \in \Omega_{k},\left(x_{k} \neq 0\right)$ this has shown that $\operatorname{det}\left(D \mathbf{h}_{k}(\mathbf{x})\right) \neq 0$ whenever $\mathbf{h}_{k}(\mathbf{x})=\mathbf{0}$ and $\mathbf{x}_{k} \in \Omega_{k}$. If $\mathbf{x} \notin \Omega_{k}$, then $x_{k}=0$ and so, since $\mathbf{h}_{k}(\mathbf{x})=\mathbf{h}_{0}(\mathbf{x})-\sum_{j=1}^{k} \mathbf{y}^{j} x_{j}^{3}$, because of the power of 3 in $x_{k}^{3}, D \mathbf{h}_{k}(\mathbf{x})=D \mathbf{h}_{k-1}(\mathbf{x})$ which has nonzero determinant by induction if $\mathbf{h}_{k-1}(\mathbf{x})=\mathbf{0}$. Thus for each $k \leq p$, there is a function $\mathbf{h}_{k}$ of the form described above, such that if $\mathbf{x} \in \cup_{j=1}^{k} \Omega_{k}$, with $\mathbf{h}_{k}(\mathbf{x})=\mathbf{0}$, it follows that $\operatorname{det}\left(D \mathbf{h}_{k}(\mathbf{x})\right) \neq 0$. Thus $\mathbf{0}$ is a regular value for $\mathbf{h}_{k}$ on $\cup_{j=1}^{k} \Omega_{j}$. Let $\mathbf{h} \equiv \mathbf{h}_{p}$. Then

$$
\begin{aligned}
\|\mathbf{h}-\mathbf{g}\|_{\infty, \bar{\Omega}} & \leq \max _{\mathbf{x} \in \Omega}\left\{\|\eta \mathbf{x}\|+\sum_{k=1}^{p}\left\|\mathbf{y}^{k}\right\|\|\mathbf{x}\|\right\} \\
& \leq \eta((p+1) \operatorname{diam}(\Omega))<\varepsilon<\operatorname{dist}(\mathbf{g}(\partial \Omega), \mathbf{0})
\end{aligned}
$$

provided $\eta$ was chosen sufficiently small to begin with.
So what is $d(\mathbf{h}, \Omega, \mathbf{0})$ ? Since $\mathbf{0}$ is a regular value and $\mathbf{h}$ is odd,

$$
\mathbf{h}^{-1}(\mathbf{0})=\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{r},-\mathbf{x}_{1}, \cdots,-\mathbf{x}_{r}, \mathbf{0}\right\} .
$$

So consider $D \mathbf{h}(\mathbf{x})$ and $D \mathbf{h}(-\mathbf{x})$.

$$
\begin{aligned}
D \mathbf{h}(-\mathbf{x}) \mathbf{u}+\mathbf{o}(\mathbf{u}) & =\mathbf{h}(-\mathbf{x}+\mathbf{u})-\mathbf{h}(-\mathbf{x}) \\
& =-\mathbf{h}(\mathbf{x}+(-\mathbf{u}))+\mathbf{h}(\mathbf{x}) \\
& =-(D \mathbf{h}(\mathbf{x})(-\mathbf{u}))+\mathbf{o}(-\mathbf{u}) \\
& =D \mathbf{h}(\mathbf{x})(\mathbf{u})+\mathbf{o}(\mathbf{u})
\end{aligned}
$$

Hence $D \mathbf{h}(\mathbf{x})=D \mathbf{h}(-\mathbf{x})$ and so the determinants of these two are the same. It follows from the definition that $d(\mathbf{g}, \Omega, \mathbf{0})=$

$$
\begin{aligned}
d(\mathbf{h}, \Omega, \mathbf{0})= & \sum_{i=1}^{r} \operatorname{sgn}\left(\operatorname{det}\left(D \mathbf{h}\left(\mathbf{x}_{i}\right)\right)\right)+\sum_{i=1}^{r} \operatorname{sgn}\left(\operatorname{det}\left(D \mathbf{h}\left(-\mathbf{x}_{i}\right)\right)\right) \\
& +\operatorname{sgn}(\operatorname{det}(D \mathbf{h}(\mathbf{0}))) \\
= & 2 m \pm 1 \text { some integer } m
\end{aligned}
$$

Theorem 23.3.3 (Borsuk) Let $\mathbf{f} \in C\left(\bar{\Omega} ; \mathbb{R}^{p}\right)$ be odd and let $\Omega$ be symmetric and $\mathbf{0} \notin \mathbf{f}(\partial \Omega)$. Then $d(\mathbf{f}, \Omega, \mathbf{0})$ equals an odd integer.

Proof: Let $\psi_{n}$ be a mollifier which is symmetric, $\psi(-\mathbf{x})=\psi(\mathbf{x})$. Also recall that $\mathbf{f}$ is the restriction to $\bar{\Omega}$ of a continuous function, still denoted as $\mathbf{f}$ which is defined on all of $\mathbb{R}^{p}$. Let $\mathbf{g}$ be the odd part of this function. That is,

$$
\mathbf{g}(\mathbf{x}) \equiv \frac{1}{2}(\mathbf{f}(\mathbf{x})-\mathbf{f}(-\mathbf{x}))
$$

Since $\mathbf{f}$ is odd, $\mathbf{g}=\mathbf{f}$ on $\bar{\Omega}$. Then

$$
\begin{gathered}
\mathbf{g}_{n}(-\mathbf{x}) \equiv \mathbf{g} * \psi_{n}(-\mathbf{x})=\int_{\Omega} \mathbf{g}(-\mathbf{x}-\mathbf{y}) \psi_{n}(\mathbf{y}) d y \\
=-\int_{\Omega} \mathbf{g}(\mathbf{x}+\mathbf{y}) \psi_{n}(\mathbf{y}) d y=-\int_{\Omega} \mathbf{g}(\mathbf{x}-(-\mathbf{y})) \psi_{n}(-\mathbf{y}) d y=-\mathbf{g}_{n}(\mathbf{x})
\end{gathered}
$$

Thus $\mathbf{g}_{n}$ is odd and is infinitely differentiable. Let $n$ be large enough that $\left\|\mathbf{g}_{n}-\mathbf{g}\right\|_{\infty, \bar{\Omega}}<$ $\delta<\operatorname{dist}(\mathbf{f}(\partial \Omega), \mathbf{0})$. Then by definition of the degree,

$$
d(\mathbf{f}, \Omega, \mathbf{0})=d(\mathbf{g}, \Omega, \mathbf{0})=d\left(\mathbf{g}_{n}, \Omega, \mathbf{0}\right)
$$

and by Lemma 23.3.2 this is an odd integer.

### 23.4 Applications

With these theorems it is possible to give easy proofs of some very important and difficult theorems.

Definition 23.4.1 If $\mathbf{f}: U \subseteq \mathbb{R}^{p} \rightarrow \mathbb{R}^{p}$ where $U$ is an open set. Then $\mathbf{f}$ is locally one to one if for every $\mathbf{x} \in U$, there exists $\delta>0$ such that $\mathbf{f}$ is one to one on $B(\mathbf{x}, \boldsymbol{\delta})$.

As a first application, consider the invariance of domain theorem. This result says that a one to one continuous map takes open sets to open sets. It is an amazing result which is essential to understand if you wish to study manifolds. In fact, the following theorem only requires $\mathbf{f}$ to be locally one to one. First here is a lemma which has the main idea.

Lemma 23.4.2 Let $\mathbf{g}: \overline{B(\mathbf{0}, r)} \rightarrow \mathbb{R}^{p}$ be one to one and continuous where here $B(\mathbf{0}, r)$ is the ball centered at $\mathbf{0}$ of radius $r$ in $\mathbb{R}^{p}$. Then there exists $\delta>0$ such that

$$
\mathbf{g}(\mathbf{0})+B(\mathbf{0}, \boldsymbol{\delta}) \subseteq \mathbf{g}(B(\mathbf{0}, r))
$$

The symbol on the left means: $\{\mathbf{g}(\mathbf{0})+\mathbf{x}: \mathbf{x} \in B(\mathbf{0}, \boldsymbol{\delta})\}$.
Proof: For $t \in[0,1]$, let

$$
\mathbf{h}(\mathbf{x}, t) \equiv \mathbf{g}\left(\frac{\mathbf{x}}{1+t}\right)-\mathbf{g}\left(\frac{-t \mathbf{x}}{1+t}\right)
$$

Then for $\mathbf{x} \in \partial B(\mathbf{0}, r), \mathbf{h}(\mathbf{x}, t) \neq \mathbf{0}$ because if this were so, the fact $\mathbf{g}$ is one to one implies

$$
\frac{\mathbf{x}}{1+t}=\frac{-t \mathbf{x}}{1+t}
$$

and this requires $\mathbf{x}=\mathbf{0}$ which is not the case since $\|\mathbf{x}\|=r$. Then $d(\mathbf{h}(\cdot, t), B(\mathbf{0}, r), \mathbf{0})$ is constant. Hence it is an odd integer for all $t$ thanks to Borsuk's theorem, because $\mathbf{h}(\cdot, 1)$ is odd. Now let $B(\mathbf{0}, \boldsymbol{\delta})$ be such that $B(\mathbf{0}, \boldsymbol{\delta}) \cap \mathbf{h}(\partial \Omega, 0)=\emptyset$. Then $d(\mathbf{h}(\cdot, 0), B(\mathbf{0}, r), \mathbf{0})=$
$d(\mathbf{h}(\cdot, 0), B(\mathbf{0}, r), \mathbf{z})$ for $\mathbf{z} \in B(\mathbf{0}, \boldsymbol{\delta})$ because the degree is constant on connected components of $\mathbb{R}^{p} \backslash \mathbf{h}(\partial \Omega, 0)$. Hence $\mathbf{z}=\mathbf{h}(\mathbf{x}, 0)=\mathbf{g}(\mathbf{x})-\mathbf{g}(\mathbf{0})$ for some $\mathbf{x} \in B(0, r)$. Thus

$$
\mathbf{g}(B(\mathbf{0}, r)) \supseteq \mathbf{g}(\mathbf{0})+B(\mathbf{0}, \boldsymbol{\delta})
$$

Now with this lemma, it is easy to prove the very important invariance of domain theorem.

A function $\mathbf{f}$ is locally one to one on an open set $\Omega$ if for every $\mathbf{x}_{0} \in \Omega$, there exists $B\left(\mathbf{x}_{0}, r\right) \subseteq \Omega$ such that $\mathbf{f}$ is one to one on $B\left(\mathbf{x}_{0}, r\right)$.

Theorem 23.4.3 (invariance of domain)Let $\Omega$ be any open subset of $\mathbb{R}^{p}$ and let $\mathbf{f}: \Omega \rightarrow \mathbb{R}^{p}$ be continuous and locally one to one. Then $\mathbf{f}$ maps open subsets of $\Omega$ to open sets in $\mathbb{R}^{p}$.

Proof: Let $\overline{B\left(\mathbf{x}_{0}, r\right)} \subseteq \Omega$ where $\mathbf{f}$ is one to one on $\overline{B\left(\mathbf{x}_{0}, r\right)}$. Let $\mathbf{g}$ be defined on $\overline{B(\mathbf{0}, r)}$ given by

$$
\mathbf{g}(\mathbf{x}) \equiv \mathbf{f}\left(\mathbf{x}+\mathbf{x}_{0}\right)
$$

Then $\mathbf{g}$ satisfies the conditions of Lemma 23.4.2, being one to one and continuous. It follows from that lemma there exists $\delta>0$ such that

$$
\begin{aligned}
\mathbf{f}(\Omega) & \supseteq \mathbf{f}\left(B\left(\mathbf{x}_{0}, r\right)\right)=\mathbf{f}\left(\mathbf{x}_{0}+B(\mathbf{0}, r)\right) \\
& =\mathbf{g}(B(\mathbf{0}, r)) \supseteq \mathbf{g}(\mathbf{0})+B(\mathbf{0}, \boldsymbol{\delta}) \\
& =\mathbf{f}\left(\mathbf{x}_{0}\right)+B(\mathbf{0}, \boldsymbol{\delta})=B\left(\mathbf{f}\left(\mathbf{x}_{0}\right), \delta\right)
\end{aligned}
$$

This shows that for any $\mathbf{x}_{0} \in \Omega, \mathbf{f}\left(\mathbf{x}_{0}\right)$ is an interior point of $\mathbf{f}(\Omega)$ which shows $\mathbf{f}(\Omega)$ is open.
With the above, one gets easily the following amazing result. It is something which is clear for linear maps but this is a statement about continuous maps.

Corollary 23.4.4 If $p>m$ there does not exist a continuous one to one map from $\mathbb{R}^{p}$ to $\mathbb{R}^{m}$.

Proof: Suppose not and let $\mathbf{f}$ be such a continuous map,

$$
\mathbf{f}(\mathbf{x}) \equiv\left(f_{1}(\mathbf{x}), \cdots, f_{m}(\mathbf{x})\right)^{T}
$$

Then let $\mathbf{g}(\mathbf{x}) \equiv\left(f_{1}(\mathbf{x}), \cdots, f_{m}(\mathbf{x}), 0, \cdots, 0\right)^{T}$ where there are $p-m$ zeros added in. Then $\mathbf{g}$ is a one to one continuous map from $\mathbb{R}^{p}$ to $\mathbb{R}^{p}$ and so $\mathbf{g}\left(\mathbb{R}^{p}\right)$ would have to be open from the invariance of domain theorem and this is not the case.

Corollary 23.4.5 If $\mathbf{f}$ is locally one to one and continuous, $\mathbf{f}: \mathbb{R}^{p} \rightarrow \mathbb{R}^{p}$, and

$$
\lim _{|\mathbf{x}| \rightarrow \infty}|\mathbf{f}(\mathbf{x})|=\infty
$$

then $\mathbf{f}$ maps $\mathbb{R}^{p}$ onto $\mathbb{R}^{p}$.

Proof: By the invariance of domain theorem, $\mathbf{f}\left(\mathbb{R}^{p}\right)$ is an open set. It is also true that $\mathbf{f}\left(\mathbb{R}^{p}\right)$ is a closed set. Here is why. If $\mathbf{f}\left(\mathbf{x}_{k}\right) \rightarrow \mathbf{y}$, the growth condition ensures that $\left\{\mathbf{x}_{k}\right\}$ is a bounded sequence. Taking a subsequence which converges to $\mathbf{x} \in \mathbb{R}^{p}$ and using the continuity of $\mathbf{f}$, it follows $\mathbf{f}(\mathbf{x})=\mathbf{y}$. Thus $\mathbf{f}\left(\mathbb{R}^{p}\right)$ is both open and closed which implies $\mathbf{f}$ must be an onto map since otherwise, $\mathbb{R}^{p}$ would not be connected.

The next theorem is the famous Brouwer fixed point theorem.
Theorem 23.4.6 (Brouwer fixed point) Let $B=\overline{B(\mathbf{0}, r)} \subseteq \mathbb{R}^{p}$ and let $\mathbf{f}: B \rightarrow B$ be continuous. Then there exists a point $\mathbf{x} \in B$, such that $\mathbf{f}(\mathbf{x})=\mathbf{x}$.

Proof: Assume there is no fixed point. Consider $\mathbf{h}(\mathbf{x}, t) \equiv t \mathbf{f}(\mathbf{x})-\mathbf{x}$ for $t \in[0,1]$. Then for $\|\mathbf{x}\|=r$,

$$
\mathbf{0} \notin t \mathbf{f}(\mathbf{x})-\mathbf{x}, t \in[0,1]
$$

By homotopy invariance,

$$
t \rightarrow d(t \mathbf{f}-I, B, \mathbf{0})
$$

is constant. But when $t=0$, this is $d(-I, B, \mathbf{0})=(-1)^{n} \neq 0$. Hence $d(\mathbf{f}-I, B, \mathbf{0}) \neq 0$ so there exists $\mathbf{x}$ such that $\mathbf{f}(\mathbf{x})-\mathbf{x}=\mathbf{0}$.

You can use standard stuff from Hilbert space to get this the fixed point theorem for a compact convex set. Let $K$ be a closed bounded convex set and let $\mathbf{f}: K \rightarrow K$ be continuous. Let $P$ be the projection map onto $K$ as in Problem 3 on Page 194. Then $P$ is continuous because $|P \mathbf{x}-P \mathbf{y}| \leq|\mathbf{x}-\mathbf{y}|$. Recall why this is. From the characterization of the projection $\operatorname{map} P,(\mathbf{x}-P \mathbf{x}, \mathbf{y}-P \mathbf{x}) \leq 0$ for all $\mathbf{y} \in K$. Therefore,

$$
(\mathbf{x}-P \mathbf{x}, P \mathbf{y}-P \mathbf{x}) \leq 0,(\mathbf{y}-P \mathbf{y}, P \mathbf{x}-P \mathbf{y}) \leq 0 \text { so }(\mathbf{y}-P \mathbf{y}, P \mathbf{y}-P \mathbf{x}) \geq 0
$$

Hence, subtracting the first from the last,

$$
(\mathbf{y}-P \mathbf{y}-(\mathbf{x}-P \mathbf{x}), P \mathbf{y}-P \mathbf{x}) \geq 0
$$

consequently,

$$
|\mathbf{x}-\mathbf{y}||P \mathbf{y}-P \mathbf{x}| \geq(\mathbf{y}-\mathbf{x}, P \mathbf{y}-P \mathbf{x}) \geq|P \mathbf{y}-P \mathbf{x}|^{2}
$$

and so $|P \mathbf{y}-P \mathbf{x}| \leq|\mathbf{y}-\mathbf{x}|$ as claimed.
Now let $r$ be so large that $K \subseteq B(\mathbf{0}, r)$. Then consider $\mathbf{f} \circ P$. This map takes $\overline{B(\mathbf{0}, r)} \rightarrow$ $\overline{B(\mathbf{0}, r)}$. In fact it maps $\overline{B(\mathbf{0}, r)}$ to $K$. Therefore, being the composition of continuous functions, it is continuous and so has a fixed point in $\overline{B(\mathbf{0}, r)}$ denoted as $\mathbf{x}$. Hence $\mathbf{f}(P(\mathbf{x}))=$ $\mathbf{x}$. Now, since $\mathbf{f}$ maps into $K$, it follows that $\mathbf{x} \in K$. Hence $P \mathbf{x}=\mathbf{x}$ and so $\mathbf{f}(\mathbf{x})=\mathbf{x}$. This has proved the following general Brouwer fixed point theorem.

Theorem 23.4.7 Let $\mathbf{f}: K \rightarrow K$ be continuous where $K$ is compact and convex and nonempty, $K \subseteq \mathbb{R}^{p}$. Then $\mathbf{f}$ has a fixed point.

Definition 23.4.8 $\mathbf{f}$ is a retract of $\overline{B(\mathbf{0}, r)}$ onto $\partial B(0, r)$ if $\mathbf{f}$ is continuous,

$$
\mathbf{f}(\overline{B(\mathbf{0}, r)}) \subseteq \partial B(\mathbf{0}, r)
$$

and $\mathbf{f}(\mathbf{x})=\mathbf{x}$ for all $\mathbf{x} \in \partial B(\mathbf{0}, r)$.

Theorem 23.4.9 There does not exist a retract of $\overline{B(\mathbf{0}, r)}$ onto its boundary, $\partial B(\mathbf{0}, r)$.
Proof: Suppose $\mathbf{f}$ were such a retract. Then for all $\mathbf{x} \in \partial B(\mathbf{0}, r), \mathbf{f}(\mathbf{x})=\mathbf{x}$ and so from the properties of the degree, the one which says if two functions agree on $\partial \Omega$, then they have the same degree,

$$
1=d(I, B(\mathbf{0}, r), \mathbf{0})=d(\mathbf{f}, B(\mathbf{0}, r), \mathbf{0})
$$

which is clearly impossible because $\mathbf{f}^{-1}(\mathbf{0})=\emptyset$ which implies $d(\mathbf{f}, B(\mathbf{0}, r), \mathbf{0})=0$.
You should now use this theorem to give another proof of the Brouwer fixed point theorem.

The proofs of the next two theorems make use of the Tietze extension theorem, Theorem 7.10.7.

Theorem 23.4.10 Let $\Omega$ be a symmetric open set in $\mathbb{R}^{p}$ such that $\mathbf{0} \in \Omega$ and let $\mathbf{f}: \partial \Omega \rightarrow V$ be continuous where $V$ is an $m$ dimensional subspace of $\mathbb{R}^{p}, m<p$. Then $\mathbf{f}(-\mathbf{x})=\mathbf{f}(\mathbf{x})$ for some $\mathbf{x} \in \partial \Omega$.

Proof: Suppose not. Using the Tietze extension theorem on components of the function, extend $\mathbf{f}$ to all of $\mathbb{R}^{p}, \mathbf{f}(\bar{\Omega}) \subseteq V$. (Here the extended function is also denoted by $\mathbf{f}$.) Let $\mathbf{g}(\mathbf{x})=\mathbf{f}(\mathbf{x})-\mathbf{f}(-\mathbf{x})$. Then $\mathbf{0} \notin \mathbf{g}(\partial \Omega)$ and so for some $r>0, B(\mathbf{0}, r) \subseteq \mathbb{R}^{p} \backslash \mathbf{g}(\partial \Omega)$. For $\mathbf{z} \in B(\mathbf{0}, r)$,

$$
d(\mathbf{g}, \Omega, \mathbf{z})=d(\mathbf{g}, \Omega, \mathbf{0}) \neq 0
$$

because $B(\mathbf{0}, r)$ is contained in a component of $\mathbb{R}^{p} \backslash \mathbf{g}(\partial \Omega)$ and Borsuk's theorem implies that $d(\mathbf{g}, \Omega, \mathbf{0}) \neq 0$ since $\mathbf{g}$ is odd. Hence

$$
V \supseteq \mathbf{g}(\Omega) \supseteq B(\mathbf{0}, r)
$$

and this is a contradiction because $V$ is $m$ dimensional.
This theorem is called the Borsuk Ulam theorem. Note that it implies there exist two points on opposite sides of the surface of the earth which have the same atmospheric pressure and temperature, assuming the earth is symmetric and that pressure and temperature are continuous functions. The next theorem is an amusing result which is like combing hair. It gives the existence of a "cowlick".
Theorem 23.4.11 Let $n$ be odd and let $\Omega$ be an open bounded set in $\mathbb{R}^{p}$ with $\mathbf{0} \in \Omega$. Suppose $\mathbf{f}: \partial \Omega \rightarrow \mathbb{R}^{p} \backslash\{\mathbf{0}\}$ is continuous. Then for some $\mathbf{x} \in \partial \Omega$ and $\lambda \neq 0, \mathbf{f}(\mathbf{x})=\lambda \mathbf{x}$.

Proof: Using the Tietze extension theorem, extend $\mathbf{f}$ to all of $\mathbb{R}^{p}$. Also denote the extended function by $\mathbf{f}$. Suppose for all $\mathbf{x} \in \partial \Omega, \mathbf{f}(\mathbf{x}) \neq \lambda \mathbf{x}$ for all $\lambda \in \mathbb{R}$. Then

$$
\begin{aligned}
\mathbf{0} \notin t \mathbf{f}(\mathbf{x})+(1-t) \mathbf{x}(\mathbf{x}, t) & \in \partial \Omega \times[0,1] \\
\mathbf{0} \notin t \mathbf{f}(\mathbf{x})-(1-t) \mathbf{x},(\mathbf{x}, t) & \in \partial \Omega \times[0,1]
\end{aligned}
$$

Thus there exists a homotopy of $\mathbf{f}$ and $I$ and a homotopy of $\mathbf{f}$ and $-I$. Then by the homotopy invariance of degree,

$$
d(\mathbf{f}, \Omega, \mathbf{0})=d(I, \Omega, \mathbf{0}), d(\mathbf{f}, \Omega, \mathbf{0})=d(-I, \Omega, \mathbf{0})
$$

But this is impossible because $d(I, \Omega, \mathbf{0})=1$ but $d(-I, \Omega, \mathbf{0})=(-1)^{n}=-1$.

### 23.5 Product Formula, Jordan Separation Theorem

This section is on the product formula for the degree which is used to prove the Jordan separation theorem. To begin with is a significant observation which is used without comment below. Recall that the connected components of an open set are open. The formula is all about the composition of continuous functions.

$$
\Omega \xrightarrow{\mathbf{f}} \mathbf{f}(\Omega) \subseteq \mathbb{R}^{p} \xrightarrow{\mathbf{g}} \mathbb{R}^{p}
$$

Lemma 23.5.1 Let $\left\{K_{i}\right\}_{i=1}^{\infty}$ be the connected components of $\mathbb{R}^{p} \backslash C$ where $C$ is a closed set. Then $\partial K_{i} \subseteq C$.

Proof: Since $K_{i}$ is a connected component of an open set, it is itself open. See Theorem 7.13.10. Thus $\partial K_{i}$ consists of all limit points of $K_{i}$ which are not in $K_{i}$. Let $\mathbf{p}$ be such a point. If it is not in $C$ then it must be in some other $K_{j}$ which is impossible because these are disjoint open sets. Thus if $\mathbf{x}$ is a point in $U$ it cannot be a limit point of $V$ for $V$ disjoint from $U$.

Definition 23.5.2 Let the connected components of $\mathbb{R}^{p} \backslash \mathbf{f}(\partial \Omega)$ be denoted by $K_{i}$. From the properties of the degree listed in Theorem 23.2.2, $d(\mathbf{f}, \Omega, \cdot)$ is constant on each of these components. Denote by $d\left(\mathbf{f}, \Omega, K_{i}\right)$ the constant value on the component $K_{i}$.

The following is the product formula. Note that if $K$ is an unbounded component of $\mathbf{f}(\partial \Omega)^{C}$, then $d(\mathbf{f}, \Omega, \mathbf{y})=0$ for all $\mathbf{y} \in K$ by homotopy invariance and the fact that for large enough $\|\mathbf{y}\|, \mathbf{f}^{-1}(\mathbf{y})=\emptyset$ since $\mathbf{f}(\bar{\Omega})$ is compact.

Theorem 23.5.3 (product formula)Let $\left\{K_{i}\right\}_{i=1}^{\infty}$ be the bounded components of $\mathbb{R}^{p} \backslash \mathbf{f}(\partial \Omega)$ for $\mathbf{f} \in C\left(\bar{\Omega} ; \mathbb{R}^{p}\right)$, let $\mathbf{g} \in C\left(\mathbb{R}^{p}, \mathbb{R}^{p}\right)$, and suppose that $\mathbf{y} \notin \mathbf{g}(\mathbf{f}(\partial \Omega))$ or in other words, $\mathbf{g}^{-1}(\mathbf{y}) \cap \mathbf{f}(\partial \Omega)=\emptyset$. Then

$$
\begin{equation*}
d(\mathbf{g} \circ \mathbf{f}, \Omega, \mathbf{y})=\sum_{i=1}^{\infty} d\left(\mathbf{f}, \Omega, K_{i}\right) d\left(\mathbf{g}, K_{i}, \mathbf{y}\right) \tag{23.5.11}
\end{equation*}
$$

All but finitely many terms in the sum are zero. If there are no bounded components of $\mathbf{f}(\partial \Omega)^{C}$, then $d(\mathbf{g} \circ \mathbf{f}, \Omega, \mathbf{y})=0$.

Proof: The compact set $\mathbf{f}(\bar{\Omega}) \cap \mathbf{g}^{-1}(\mathbf{y})$ is contained in $\mathbb{R}^{p} \backslash \mathbf{f}(\partial \Omega)$ and so, $\mathbf{f}(\bar{\Omega}) \cap$ $\mathbf{g}^{-1}(\mathbf{y})$ is covered by finitely many of the components $K_{j}$ one of which may be the unbounded component. Since these components are disjoint, the other components fail to intersect $\mathbf{f}(\bar{\Omega}) \cap \mathbf{g}^{-1}(\mathbf{y})$. Thus, if $K_{i}$ is one of these others, either it fails to intersect $\mathbf{g}^{-1}(\mathbf{y})$ or $K_{i}$ fails to intersect $\mathbf{f}(\bar{\Omega})$. Thus either $d\left(\mathbf{f}, \Omega, K_{i}\right)=0$ because $K_{i}$ fails to intersect $\mathbf{f}(\bar{\Omega})$ or $d\left(\mathbf{g}, K_{i}, \mathbf{y}\right)=0$ if $K_{i}$ fails to intersect $\mathbf{g}^{-1}(\mathbf{y})$. Thus the sum is always a finite sum. I am using Theorem 23.2.2, the part which says that if $\mathbf{y} \notin \mathbf{h}(\bar{\Omega})$, then $d(\mathbf{h}, \Omega, \mathbf{y})=0$. Note that $\partial K_{i} \subseteq \mathbf{f}(\partial \Omega)$ so $\mathbf{y} \notin \mathbf{g}\left(\partial K_{i}\right)$.

Let $\tilde{\mathbf{g}}$ be in $C^{\infty}\left(\mathbb{R}^{p}, \mathbb{R}^{p}\right)$ and let $\|\tilde{\mathbf{g}}-\mathbf{g}\|_{\infty}$ be so small that for each of the finitely many $K_{i}$ intersecting $\mathbf{f}(\bar{\Omega}) \cap \mathbf{g}^{-1}(\mathbf{y})$,

$$
\begin{align*}
d(\mathbf{g} \circ \mathbf{f}, \Omega, \mathbf{y}) & =d(\tilde{\mathbf{g}} \circ \mathbf{f}, \Omega, \mathbf{y}) \\
d\left(\mathbf{g}, K_{i}, \mathbf{y}\right) & =d\left(\tilde{\mathbf{g}}, K_{i}, \mathbf{y}\right) \tag{23.5.12}
\end{align*}
$$

Since $\partial K_{i} \subseteq \mathbf{f}(\partial \Omega)$, both conditions are obtained by letting

$$
\|\mathbf{g}-\tilde{\mathbf{g}}\|_{\infty, \mathbf{f}(\bar{\Omega})}<\operatorname{dist}(\mathbf{y}, \mathbf{g}(\mathbf{f}(\partial \Omega)))
$$

By Lemma 23.1.5, there exists $\tilde{\mathbf{g}}$ such that $\mathbf{y}$ is a regular value of $\tilde{\mathbf{g}}$ in addition to 23.5 .12 and $\tilde{\mathbf{g}}^{-1}(\mathbf{y}) \cap \mathbf{f}(\partial \Omega)=\emptyset$. Then $\tilde{\mathbf{g}}^{-1}(\mathbf{y})$ is contained in the union of the $K_{i}$ along with the unbounded component(s) and by Lemma 23.1.5 $\tilde{\mathbf{g}}^{-1}(\mathbf{y})$ is countable. As discussed there, $\tilde{\mathbf{g}}^{-1}(\mathbf{y}) \cap K_{i}$ is finite if $K_{i}$ is bounded. Let $\tilde{\mathbf{g}}^{-1}(\mathbf{y}) \cap K_{i}=\left\{\mathbf{x}_{j}^{i}\right\}_{j=1}^{m_{i}}, m_{i} \leq \infty . m_{i}$ could only be $\infty$ on the unbounded component.

Now use Lemma 23.1.5 again to get $\tilde{\mathbf{f}}$ in $C^{\infty}\left(\bar{\Omega} ; \mathbb{R}^{p}\right)$ such that each $\mathbf{x}_{j}^{i}$ is a regular value of $\tilde{\mathbf{f}}$ on $\Omega$ and also $\|\tilde{\mathbf{f}}-\mathbf{f}\|_{\infty}$ is very small, so small that

$$
d(\tilde{\mathbf{g}} \circ \tilde{\mathbf{f}}, \Omega, \mathbf{y})=d(\tilde{\mathbf{g}} \circ \mathbf{f}, \Omega, \mathbf{y})=d(\mathbf{g} \circ \mathbf{f}, \Omega, \mathbf{y})
$$

and

$$
d\left(\tilde{\mathbf{f}}, \Omega, \mathbf{x}_{j}^{i}\right)=d\left(\mathbf{f}, \Omega, \mathbf{x}_{j}^{i}\right)
$$

for each $i, j$.
Thus, from the above,

$$
\begin{aligned}
d(\mathbf{g} \circ \mathbf{f}, \Omega, \mathbf{y}) & =d(\tilde{\mathbf{g}} \circ \tilde{\mathbf{f}}, \Omega, \mathbf{y}) \\
d\left(\tilde{\mathbf{f}}, \Omega, \mathbf{x}_{j}^{i}\right) & =d\left(\mathbf{f}, \Omega, \mathbf{x}_{j}^{i}\right)=d\left(\mathbf{f}, \Omega, K_{i}\right) \\
d\left(\tilde{\mathbf{g}}, K_{i}, \mathbf{y}\right) & =d\left(\mathbf{g}, K_{i}, \mathbf{y}\right)
\end{aligned}
$$

Is $\mathbf{y}$ a regular value for $\tilde{\mathbf{g}} \circ \tilde{\mathbf{f}}$ on $\Omega$ ? Suppose $\mathbf{z} \in \Omega$ and $\mathbf{y}=\tilde{\mathbf{g}} \circ \tilde{\mathbf{f}}(\mathbf{z})$ so $\tilde{\mathbf{f}}(\mathbf{z}) \in \tilde{\mathbf{g}}^{-1}(\mathbf{y})$. Then $\tilde{\mathbf{f}}(\mathbf{z})=\mathbf{x}_{j}^{i}$ for some $i, j$ and $D \tilde{\mathbf{f}}(\mathbf{z})^{-1}$ exists. Hence $D(\tilde{\mathbf{g}} \circ \tilde{\mathbf{f}})(\mathbf{z})=D \tilde{\mathbf{g}}\left(\mathbf{x}_{j}^{i}\right) D \tilde{\mathbf{f}}(\mathbf{z})$, both linear transformations invertible. Thus $\mathbf{y}$ is a regular value of $\tilde{\mathbf{g}} \circ \tilde{\mathbf{f}}$ on $\Omega$.

What of $\mathbf{x}_{j}^{i}$ in $K_{i}$ where $K_{i}$ is unbounded? Then as observed above, the sum of the terms $\operatorname{sgn}(\operatorname{det} D \tilde{\mathbf{f}}(\mathbf{z}))$ for $\mathbf{z} \in \tilde{\mathbf{f}}^{-1}\left(\mathbf{x}_{j}^{i}\right)$ is $d\left(\tilde{\mathbf{f}}, \Omega, \mathbf{x}_{j}^{i}\right)$ and is 0 because the degree is constant on $K_{i}$ which is unbounded.

From the definition of the degree, the left side of $23.5 .11 d(\mathbf{g} \circ \mathbf{f}, \Omega, \mathbf{y})$ equals

$$
\sum\left\{\operatorname{sgn}(\operatorname{det} D \tilde{\mathbf{g}}(\tilde{\mathbf{f}}(\mathbf{z}))) \operatorname{sgn}(\operatorname{det} D \tilde{\mathbf{f}}(\mathbf{z})): \mathbf{z} \in \tilde{\mathbf{f}}^{-1}\left(\tilde{\mathbf{g}}^{-1}(\mathbf{y})\right)\right\}
$$

The $\tilde{\mathbf{g}}^{-1}(\mathbf{y})$ are the $\mathbf{x}_{j}^{i}$. Thus the above is of the form

$$
=\sum_{i} \sum_{j} \sum_{\mathbf{z} \in \tilde{\mathbf{f}}^{-1}\left(\mathbf{x}_{j}^{i}\right)} \operatorname{sgn}\left(\operatorname{det}\left(D \tilde{\mathbf{g}}\left(\mathbf{x}_{j}^{i}\right)\right)\right) \operatorname{sgn}(\operatorname{det}(D \tilde{\mathbf{f}}(\mathbf{z})))
$$

As mentioned, if $\mathbf{x}_{j}^{i} \in K_{i}$ an unbounded component, then

$$
\sum_{\mathbf{z} \in \tilde{\mathbf{f}}^{-1}\left(\mathbf{x}_{j}^{i}\right)} \operatorname{sgn}\left(\operatorname{det}\left(D \tilde{\mathbf{g}}\left(\mathbf{x}_{j}^{i}\right)\right)\right) \operatorname{sgn}(\operatorname{det}(D \tilde{\mathbf{f}}(\mathbf{z})))=0
$$

and so, it suffices to only consider bounded components in what follows and the sum makes sense because there are finitely many $\mathbf{x}_{j}^{i}$ in bounded $K_{i}$. This also shows that if there are no bounded components of $\mathbf{f}(\partial \Omega)^{C}$, then $d(\mathbf{g} \circ \mathbf{f}, \Omega, \mathbf{y})=0$. Thus $d(\mathbf{g} \circ \mathbf{f}, \Omega, \mathbf{y})$ equals

$$
\begin{aligned}
& =\sum_{i} \sum_{j} \operatorname{sgn}\left(\operatorname{det}\left(D \tilde{\mathbf{g}}\left(\mathbf{x}_{j}^{i}\right)\right)\right) \sum_{\mathbf{z} \in \tilde{\mathbf{f}}^{-1}\left(\mathbf{x}_{j}^{i}\right)} \operatorname{sgn}(\operatorname{det}(D \tilde{\mathbf{f}}(\mathbf{z}))) \\
& =\sum_{i} d\left(\tilde{\mathbf{g}}, K_{i}, \mathbf{y}\right) d\left(\tilde{\mathbf{f}}, \Omega, K_{i}\right)
\end{aligned}
$$

For the last step, $\sum_{\mathbf{z} \in \tilde{\mathbf{f}}^{-1}\left(\mathbf{x}_{j}^{i}\right)} \operatorname{sgn}(\operatorname{det}(D \tilde{\mathbf{f}}(\mathbf{z}))) \equiv d\left(\tilde{\mathbf{f}}, \Omega, \mathbf{x}_{j}^{i}\right)=d\left(\tilde{\mathbf{f}}, \Omega, K_{i}\right)$. This proves the product formula because $\tilde{\mathbf{g}}$ and $\tilde{\mathbf{f}}$ were chosen close enough to $\mathbf{f}, \mathbf{g}$ respectively that

$$
\sum_{i} d\left(\tilde{\mathbf{f}}, \Omega, K_{i}\right) d\left(\tilde{\mathbf{g}}, K_{i}, \mathbf{y}\right)=\sum_{i} d\left(\mathbf{f}, \Omega, K_{i}\right) d\left(\mathbf{g}, K_{i}, \mathbf{y}\right)
$$

Before the general Jordan separation theorem, I want to first consider the examples of most interest.

Recall that if a function $\mathbf{f}$ is continuous and one to one on a compact set $K$, then $\mathbf{f}$ is a homeomorphism of $K$ and $\mathbf{f}(K)$. Also recall that if $U$ is a nonempty open set, the boundary of $U$, denoted as $\partial U$ and meaning those points $\mathbf{x}$ with the property that for all $r>0 B(\mathbf{x}, r)$ intersects both $U$ and $U^{C}$, is $\bar{U} \backslash U$.

Proposition 23.5.4 Let $H$ be a compact set and let $\mathbf{f}: H \rightarrow \mathbb{R}^{p}, p \geq 2$ be one to one and continuous so that $H$ and $\mathbf{f}(H) \equiv C$ are homeomorphic. Suppose $H^{C}$ has only one connected component so $H^{C}$ is connected. Then $C^{C}$ also has only one component.

Proof: I want to show that $C^{C}$ has no bounded components so suppose it has a bounded component $K$. Extend $\mathbf{f}$ to all of $\mathbb{R}^{p}$ and let $\mathbf{g}$ be an extension of $\mathbf{f}^{-1}$ to all of $\mathbb{R}^{p}$. Then, by the above Lemma 23.5.1, $\partial K \subseteq C$. Since $\mathbf{f} \circ \mathbf{g}(\mathbf{x})=\mathbf{x}$ on $\partial K \subseteq C$, if $\mathbf{z} \in K$, it follows that $1=d(\mathbf{f} \circ \mathbf{g}, K, \mathbf{z})$. Now $\mathbf{g}(\partial K) \subseteq \mathbf{g}(C)=H$. Thus $\mathbf{g}(\partial K)^{C} \supseteq H^{C}$ and $H^{C}$ is unbounded and connected. If a component of $\mathbf{g}(\partial K)^{C}$ is bounded, then it cannot contain the unbounded $H^{C}$ which must be contained in the component of $\mathbf{g}(\partial K)^{C}$ which it intersects. Thus, the only bounded components of $\mathbf{g}(\partial K)^{C}$ must be contained in $H$. Let the set of such bounded components be denoted by $\mathscr{Q}$. By the product formula,

$$
d(\mathbf{f} \circ \mathbf{g}, K, \mathbf{z})=\sum_{Q \in \mathscr{Q}} d(\mathbf{g}, K, Q) d(\mathbf{f}, Q, \mathbf{z})
$$

However, $\mathbf{f}(\bar{Q}) \subseteq \mathbf{f}(H)=C$ and $\mathbf{z}$ is contained in a component of $C^{C}$ so for each $Q \in \mathscr{Q}$, $d(\mathbf{f}, Q, \mathbf{z})=0$. Hence, by the product formula, $d(\mathbf{f} \circ \mathbf{g}, K, \mathbf{z})=0$ which is a contradiction to $1=d(\mathbf{f} \circ \mathbf{g}, K, \mathbf{z})$. Thus there is no bounded component of $C^{C}$.

It is obvious that the unit sphere $S^{p-1}$ divides $\mathbb{R}^{p}$ into two disjoint open sets, the inside and the outside. The following shows that this also holds for any homeomorphic image of $S^{p-1}$.

Proposition 23.5.5 Let $B$ be the ball $B(\mathbf{0}, 1)$ with $S^{p-1}$ its boundary, $p \geq 2$. Suppose $\mathbf{f}: S^{p-1} \rightarrow C \equiv \mathbf{f}\left(S^{p-1}\right) \subseteq \mathbb{R}^{p}$ is a homeomorphism. Then $C^{C}$ also has exactly two components, a bounded and an unbounded.

Proof: I need to show there is only one bounded component of $C^{C}$. From Proposition 23.5.4, there is at least one. Otherwise, $S^{p-1}$ would have no bounded components which is obviously false.

Let $\mathbf{f}$ be a continuous extension of $\mathbf{f}$ off $S^{p-1}$ and let $\mathbf{g}$ be a continuous extension of $\mathbf{g}$ off $C$. Let $\left\{K_{i}\right\}$ be the bounded components of $C^{C}$. Now $\partial K_{i} \subseteq C$ and so $\mathbf{g}\left(\partial K_{i}\right) \subseteq \mathbf{g}(C)=$ $S^{p-1}$. Let $G$ be those points $\mathbf{x}$ with $|\mathbf{x}|>1$.

Let a bounded component of $\mathbf{g}\left(\partial K_{i}\right)^{C}$ be $H$. If $H$ intersects $B$ then $H$ must contain $B$ because $B$ is connected and contained in $\left(S^{p-1}\right)^{C} \subseteq \mathbf{g}\left(\partial K_{i}\right)^{C}$ which means that $B$ is contained in the union of components of $\mathbf{g}\left(\partial K_{i}\right)^{C}$. The same is true if $H$ intersects $G$. However, $H$ cannot contain the unbounded $G$ and so $H$ cannot intersect $G$. Since $H$ is open and cannot intersect $G$, it cannot intersect $S^{p-1}$ either. Thus $H=B$ and so there is only one bounded component of $\mathbf{g}\left(\partial K_{i}\right)^{C}$ and it is $B$. Now let $\mathbf{z} \in K_{i} . \mathbf{f} \circ \mathbf{g}=I$ on $\partial K_{i} \subseteq C$ and so if $\mathbf{z} \in K_{i}$, since the degree is determined by values on the boundary, the product formula implies

$$
1=d\left(\mathbf{f} \circ \mathbf{g}, K_{i}, \mathbf{z}\right)=d\left(\mathbf{g}, K_{i}, B\right) d(\mathbf{f}, B, \mathbf{z})=d\left(\mathbf{g}, K_{i}, B\right) d\left(\mathbf{f}, B, K_{i}\right)
$$

If there are $n$ of these $K_{i}$, it follows $\sum_{i} d\left(\mathbf{g}, K_{i}, B\right) d\left(\mathbf{f}, B, K_{i}\right)=n$. Now pick $\mathbf{y} \in B$. Then, since $\mathbf{g} \circ \mathbf{f}(\mathbf{x})=\mathbf{x}$ on $\partial B$, the product formula shows

$$
1=d(\mathbf{g} \circ \mathbf{f}, B, \mathbf{y})=\sum_{i} d\left(\mathbf{f}, B, K_{i}\right) d\left(\mathbf{g}, K_{i}, \mathbf{y}\right)=\sum_{i} d\left(\mathbf{f}, B, K_{i}\right) d\left(\mathbf{g}, K_{i}, B\right)=n
$$

Thus $n=1$ and there is only one bounded component of $C^{C}$.
It remains to show that, in the above $\mathbf{f}\left(S^{p-1}\right)$ is the boundary of both components, the bounded one and the unbounded one.

Theorem 23.5.6 Let $S^{p-1}$ be the unit sphere in $\mathbb{R}^{p}, p \geq 2$. Suppose $\gamma: S^{p-1} \rightarrow \Gamma \subseteq \mathbb{R}^{p}$ is one to one onto and continuous. Then $\mathbb{R}^{p} \backslash \Gamma$ consists of two components, a bounded component (called the inside) $U_{i}$ and an unbounded component (called the outside), $U_{o}$. Also the boundary of each of these two components of $\mathbb{R}^{p} \backslash \Gamma$ is $\Gamma$ and $\Gamma$ has empty interior.

Proof: $\gamma^{-1}$ is continuous since $S^{p-1}$ is compact and $\gamma$ is one to one. By the Jordan separation theorem, $\mathbb{R}^{p} \backslash \Gamma=U_{o} \cup U_{i} \quad$ where these on the right are the connected components of the set on the left, both open sets. Only one of them is bounded, $U_{i}$. Thus $\Gamma \cup U_{i} \cup U_{o}=\mathbb{R}^{p}$. Since both $U_{i}, U_{o}$ are open, $\partial U \equiv \bar{U} \backslash U$ for $U$ either $U_{o}$ or $U_{i}$. If $\mathbf{x} \in \Gamma$, and is not a limit point of $U_{i}$, then there is $B(\mathbf{x}, r)$ which contains no points of $U_{i}$. Let $S$ be those points $\mathbf{x}$ of $\Gamma$ for which $B(\mathbf{x}, r)$ contains no points of $U_{i}$ for some $r>0$. This $S$ is open in $\Gamma$. Let $\hat{\Gamma}$ be $\Gamma \backslash S$. Then if $\hat{C}=\gamma^{-1}(\hat{\Gamma})$, it follows that $\hat{C}$ is a closed set in $S^{p-1}$ and is a proper subset of $S^{p-1}$. It is obvious that taking a relatively open set from $S^{p-1}$ results in a compact set whose complement is an open connected set. By Proposition 23.5.4, $\mathbb{R}^{p} \backslash \hat{\Gamma}$ is also an open connected set. Start with $\mathbf{x} \in U_{i}$ and consider a continuous curve which goes from $\mathbf{x}$ to $\mathbf{y} \in U_{o}$ which is contained in $\mathbb{R}^{p} \backslash \hat{\Gamma}$. Thus the curve contains no points of $\hat{\Gamma}$.

However, it must contain points of $\Gamma$ which can only be in $S$. The first point of $\Gamma$ intersected by this curve is a point in $\overline{U_{i}}$ and so this point of intersection is not in $S$ after all because every ball containing it must contain points of $U_{i}$. Thus $S=\emptyset$ and every point of $\Gamma$ is in $\overline{U_{i}}$. Similarly, every point of $\Gamma$ is in $\overline{U_{o}}$. Thus $\Gamma \subseteq \overline{U_{i}} \backslash U_{i}$ and $\Gamma \subseteq \overline{U_{o}} \backslash U_{o}$. However, if $\mathbf{x} \in \overline{U_{i}} \backslash U_{i}$, then $\mathbf{x} \notin U_{o}$ because it is a limit point of $U_{i}$ and so $\mathbf{x} \in \Gamma$. It is similar with $U_{o}$. Thus $\Gamma=\overline{U_{i}} \backslash U_{i}$ and $\Gamma=\overline{U_{o}} \backslash U_{o}$. This could not happen if $\Gamma$ had an interior point. Such a point would be in $\Gamma$ but would fail to be in either $\partial U_{i}$ or $\partial U_{o}$.

When $p=2$, this theorem is called the Jordan curve theorem.
What if $\gamma$ maps $\bar{B}$ to $\mathbb{R}^{p}$ instead of just $S^{p-1}$ ? Obviously, one should be able to say a little more.

Corollary 23.5.7 Let $B$ be an open ball and let $\gamma: \bar{B} \rightarrow \mathbb{R}^{p}$ be one to one and continuous. Let $U_{i}, U_{o}$ be as in the above theorem, the bounded and unbounded components of $\gamma(\partial B)^{C}$. Then $U_{i}=\gamma(B)$.

Proof: By connectedness and the observation that $\gamma(B)$ contains no points of $C \equiv$ $\gamma(\partial B)$, it follows that $\gamma(B) \subseteq U_{i}$ or $U_{o}$. Suppose $\gamma(B) \subseteq U_{i}$. I want to show that $\gamma(B)$ is not a proper subset of $U_{i}$. If $\gamma(B)$ is a proper subset of $U_{i}$, then if $\mathbf{x} \in U_{i} \backslash \gamma(B)$, it follows also that $\mathbf{x} \in U_{i} \backslash \gamma(\bar{B})$ because $\mathbf{x} \notin C=\gamma(\partial B)$. Let $H$ be the component of $U_{i} \backslash \gamma(\bar{B})$ determined by $\mathbf{x}$. In particular $H$ and $U_{o}$ are separated since they are disjoint open sets. Now $\gamma(\bar{B})^{C}$ and $\bar{B}^{C}$ each have only one component. This is true for $\bar{B}^{C}$ and by the Jordan separation theorem, also true for $\gamma(\bar{B})^{C}$. However, $H$ is in $\gamma(\bar{B})^{C}$ as is $U_{o}$ and so $\gamma(\bar{B})^{C}$ has at least two components after all. If $\gamma(B) \subseteq U_{o}$ the argument works exactly the same. Thus either $\gamma(B)=U_{i}$ or $\gamma(B)=U_{o}$. The second alternative cannot take place because $\gamma(B)$ is bounded. Hence $\gamma(B)=U_{i}$.

Note that this essentially gives the invariance of domain theorem.

### 23.6 The Jordan Separation Theorem

What follows is the general Jordan separation theorem.
Lemma 23.6.1 Let $\Omega$ be a bounded open set in $\mathbb{R}^{p}, \mathbf{f} \in C\left(\bar{\Omega} ; \mathbb{R}^{p}\right)$, and suppose $\left\{\Omega_{i}\right\}_{i=1}^{\infty}$ are disjoint open sets contained in $\Omega$ such that

$$
\mathbf{y} \notin \mathbf{f}\left(\bar{\Omega} \backslash \cup_{j=1}^{\infty} \Omega_{j}\right)
$$

Then

$$
d(\mathbf{f}, \Omega, \mathbf{y})=\sum_{j=1}^{\infty} d\left(\mathbf{f}, \Omega_{j}, \mathbf{y}\right)
$$

where the sum has all but finitely many terms equal to 0 .
Proof: By assumption, the compact set $\mathbf{f}^{-1}(\mathbf{y}) \equiv\{\mathbf{x} \in \bar{\Omega}: \mathbf{f}(\mathbf{x})=\mathbf{y}\}$ has empty intersection with

$$
\bar{\Omega} \backslash \cup_{j=1}^{\infty} \Omega_{j}
$$

and so this compact set is covered by finitely many of the $\Omega_{j}$, say $\left\{\Omega_{1}, \cdots, \Omega_{n-1}\right\}$ and

$$
\mathbf{y} \notin \mathbf{f}\left(\cup_{j=n}^{\infty} \Omega_{j}\right) .
$$

By Theorem 23.2.2 and letting $O=\cup_{j=n}^{\infty} \Omega_{j}$,

$$
d(\mathbf{f}, \Omega, \mathbf{y})=\sum_{j=1}^{n-1} d\left(\mathbf{f}, \Omega_{j}, \mathbf{y}\right)+d(\mathbf{f}, O, \mathbf{y})=\sum_{j=1}^{\infty} d\left(\mathbf{f}, \Omega_{j}, \mathbf{y}\right)
$$

because $d(\mathbf{f}, O, \mathbf{y})=0$ as is $d\left(\mathbf{f}, \Omega_{j}, \mathbf{y}\right)$ for every $j \geq n$.
To help remember some symbols, here is a short diagram. I have a little trouble remembering what is what when I read the proof.

| bounded components $\mathbb{R}^{p} \backslash C$ | bounded components $\mathbb{R}^{p} \backslash \mathbf{f}(C)$ |
| :---: | :---: |
| $\mathscr{K}$ | $\mathscr{L}$ |

For $K \in \mathscr{K}, K$ a bounded component of $\mathbb{R}^{p} \backslash C$,

| bounded components of $\mathbb{R}^{p} \backslash \mathbf{f}(\partial K)$ |
| :---: |
| $\mathscr{H}$ |
| $\mathscr{L}_{H}$ sets of $\mathscr{L}$ contained in $H \in \mathscr{H}$ |
| $\mathscr{H}_{1}$ those sets of $\mathscr{H}$ which intersect a set of $\mathscr{L}$ |

Note that the sets of $\mathscr{H}$ will tend to be larger than the sets of $\mathscr{L}$. The following lemma relating to the above definitions is interesting.

Lemma 23.6.2 If a component $L$ of $\mathbb{R}^{p} \backslash \mathbf{f}(C)$ intersects a component $H$ of $\mathbb{R}^{p} \backslash \mathbf{f}(\partial K), K$ a component of $\mathbb{R}^{p} \backslash C$, then $L \subseteq H$.

Proof: First note that by Lemma 23.5.1 for $K \in \mathscr{K}, \partial K \subseteq C$.
Next suppose $L \in \mathscr{L}$ and $L \cap H \neq \emptyset$ where $\mathscr{H}$ is as described above. Suppose $\mathbf{y} \in L \backslash H$ so $L$ is not contained in $H$. Since $\mathbf{y} \in L$, it follows that $\mathbf{y} \notin \mathbf{f}(C)$ and so $\mathbf{y} \notin \mathbf{f}(\partial K)$ either, by Lemma 23.5.1. It follows that $\mathbf{y} \in \tilde{H}$ for some other $\tilde{H}$ a component of $\mathbb{R}^{p} \backslash \mathbf{f}(\partial K)$. Now $\mathbb{R}^{p} \backslash \mathbf{f}(\partial K) \supseteq \mathbb{R}^{p} \backslash \mathbf{f}(C)$ and so $\cup \mathscr{H} \supseteq L$ and so

$$
L=\cup\{L \cap H: H \in \mathscr{H}\}
$$

and these are disjoint open sets with two of them nonempty. Hence $L$ would not be connected which is a contradiction. Hence if $L \cap H \neq \emptyset$, then $L \subseteq H$.

The following is the Jordan separation theorem. It is based on a use of the product formula and splitting up the sets of $\mathscr{L}$ into $\mathscr{L}_{H}$ for various $H \in \mathscr{H}$, those in $\mathscr{L}_{H}$ being the ones which intersect $H$ thanks to the above lemma.

Theorem 23.6.3 (Jordan separation theorem) Let $\mathbf{f}$ be a homeomorphism of $C$ and $\mathbf{f}(C)$ where $C$ is a compact set in $\mathbb{R}^{p}$. Then $\mathbb{R}^{p} \backslash C$ and $\mathbb{R}^{p} \backslash \mathbf{f}(C)$ have the same number of connected components.

Proof: Denote by $\mathscr{K}$ the bounded components of $\mathbb{R}^{p} \backslash C$ and denote by $\mathscr{L}$, the bounded components of $\mathbb{R}^{p} \backslash \mathbf{f}(C)$. Also, using the Tietze extension theorem on components of a vector valued function, there exists $\overline{\mathbf{f}}$ an extension of $\mathbf{f}$ to all of $\mathbb{R}^{p}$ and let $\overline{\mathbf{f}^{-1}}$ be an extension of $\mathbf{f}^{-1}$ to all of $\mathbb{R}^{p}$. Pick $K \in \mathscr{K}$ and take $\mathbf{y} \in K$. Then $\partial K \subseteq C$ and so
$\mathbf{y} \notin \overline{\mathbf{f}^{-1}}(\overline{\mathbf{f}}(\partial K))$. Since $\overline{\mathbf{f}^{-1}} \circ \overline{\mathbf{f}}$ equals the identity $I$ on $\partial K$, it follows from the properties of the degree that

$$
1=d(I, K, \mathbf{y})=d\left(\overline{\mathbf{f}^{-1}} \circ \overline{\mathbf{f}}, K, \mathbf{y}\right)
$$

Recall that if two functions agree on the boundary, then they have the same degree. Let $\mathscr{H}$ denote the set of bounded components of $\mathbb{R}^{p} \backslash \mathbf{f}(\partial K)$. Then

$$
\cup \mathscr{H}=\mathbb{R}^{p} \backslash \mathbf{f}(\partial K) \supseteq \mathbb{R}^{p} \backslash \mathbf{f}(C)
$$

Thus if $L \in \mathscr{L}$, then $L \subseteq \cup \mathscr{H}$ and so it must intersect some set $H$ of $\mathscr{H}$. By the above Lemma 23.6.2, $L$ is contained in this set of $\mathscr{H}$ so it is in $\mathscr{L}_{H}$.

By the product formula,

$$
\begin{equation*}
1=d\left(\overline{\mathbf{f}^{-1}} \circ \overline{\mathbf{f}}, K, \mathbf{y}\right)=\sum_{H \in \mathscr{H}} d(\overline{\mathbf{f}}, K, H) d\left(\overline{\mathbf{f}^{-1}}, H, \mathbf{y}\right) \tag{23.6.13}
\end{equation*}
$$

the sum being a finite sum. That is, there are finitely many $H$ involved in the sum, the other terms being zero, this by the general result in the product formula.

What about those sets of $\mathscr{H}$ which contain no set of $\mathscr{L}$ ? These sets also have empty intersection with all sets of $\mathscr{L}$ and empty intersection with the unbounded component(s) of $\mathbb{R}^{p} \backslash \mathbf{f}(C)$ by Lemma 23.6.2. Therefore, for $H$ one of these, $H \subseteq \mathbf{f}(C)$ because $H$ has no points of $\mathbb{R}^{p} \backslash \mathbf{f}(C)$ which equals

$$
\cup \mathscr{L} \cup\left\{\text { unbounded component(s) of } \mathbb{R}^{p} \backslash \mathbf{f}(C)\right\} .
$$

Therefore,

$$
d\left(\overline{\mathbf{f}^{-1}}, H, \mathbf{y}\right)=d\left(\mathbf{f}^{-1}, H, \mathbf{y}\right)=0
$$

because $\mathbf{y} \in K$ a bounded component of $\mathbb{R}^{p} \backslash C$, but for $\mathbf{u} \in H \subseteq \mathbf{f}(C), \mathbf{f}^{-1}(\mathbf{u}) \in C$ so $\mathbf{f}^{-1}(\mathbf{u}) \neq \mathbf{y}$ implying that $d\left(\mathbf{f}^{-1}, H, \mathbf{y}\right)=0$. Thus in 23.6.13, all such terms are zero. Then letting $\mathscr{H}_{1}$ be those sets of $\mathscr{H}$ which contain (intersect) some sets of $\mathscr{L}$, the above sum reduces to

$$
1=\sum_{H \in \mathscr{H}_{1}} d(\overline{\mathbf{f}}, K, H) d\left(\overline{\mathbf{f}^{-1}}, H, \mathbf{y}\right)
$$

Note also that for $H \in \mathscr{H}_{1}, \bar{H} \backslash \cup \mathscr{L}_{H}=\bar{H} \backslash \cup \mathscr{L}$ and it has no points of $\mathbb{R}^{p} \backslash \mathbf{f}(C)=\cup \mathscr{L}$ so $\bar{H} \backslash \cup \mathscr{L}_{H}$ is contained in $\mathbf{f}(C)$. Thus $\mathbf{y} \notin \overline{\mathbf{f}^{-1}}\left(\bar{H} \backslash \cup \mathscr{L}_{H}\right) \subseteq C$ because $\mathbf{y} \notin C$.

It follows from Lemma 23.6.1 which comes from Theorem 23.2.2, the part about having disjoint open sets in $\Omega$ and getting a sum of degrees over these being equal to $d(\mathbf{f}, \Omega, \mathbf{y})$ that

$$
\begin{aligned}
\sum_{H \in \mathscr{H}_{1}} d(\overline{\mathbf{f}}, K, H) d\left(\overline{\mathbf{f}^{-1}}, H, \mathbf{y}\right) & =\sum_{H \in \mathscr{H}_{1}} d(\overline{\mathbf{f}}, K, H) \sum_{L \in \mathscr{L}_{H}} d\left(\overline{\mathbf{f}^{-1}}, L, \mathbf{y}\right) \\
& =\sum_{H \in \mathscr{H}_{1}} \sum_{L \in \mathscr{\mathscr { L }}_{H}} d(\overline{\mathbf{f}}, K, H) d\left(\overline{\mathbf{f}^{-1}}, L, \mathbf{y}\right)
\end{aligned}
$$

where $\mathscr{L}_{H}$ are those sets of $\mathscr{L}$ contained in $H$.

Now $d(\overline{\mathbf{f}}, K, H)=d(\overline{\mathbf{f}}, K, L)$ where $L \in \mathscr{L}_{H}$. This is because $L$ is an open connected subset of $H$ and $\mathbf{z} \rightarrow d(\overline{\mathbf{f}}, K, \mathbf{z})$ is constant on $H$. Therefore,

$$
\sum_{H \in \mathscr{H}_{1}} \sum_{L \in \mathscr{L}_{H}} d(\overline{\mathbf{f}}, K, H) d\left(\overline{\mathbf{f}^{-1}}, L, \mathbf{y}\right)=\sum_{H \in \mathscr{H}_{1}} \sum_{L \in \mathscr{L}_{H}} d(\overline{\mathbf{f}}, K, L) d\left(\overline{\mathbf{f}^{-1}}, L, \mathbf{y}\right)
$$

As noted above, there are finitely many $H \in \mathscr{H}$ which are involved.

$$
\mathbb{R}^{p} \backslash \mathbf{f}(C) \subseteq \mathbb{R}^{p} \backslash \mathbf{f}(\partial K)
$$

and so every $L$ must be contained in some $H \in \mathscr{H}_{1}$. It follows that the above reduces to

$$
\sum_{L \in \mathscr{L}} d(\overline{\mathbf{f}}, K, L) d\left(\overline{\mathbf{f}^{-1}}, L, \mathbf{y}\right)
$$

Where this is a finite sum because all but finitely many terms are 0 .
Thus from 23.6.13,

$$
\begin{equation*}
1=\sum_{L \in \mathscr{L}} d(\overline{\mathbf{f}}, K, L) d\left(\overline{\mathbf{f}^{-1}}, L, \mathbf{y}\right)=\sum_{L \in \mathscr{L}} d(\overline{\mathbf{f}}, K, L) d\left(\overline{\mathbf{f}^{-1}}, L, K\right) \tag{23.6.14}
\end{equation*}
$$

Let $|\mathscr{K}|$ denote the number of components in $\mathscr{K}$ and similarly, $|\mathscr{L}|$ denotes the number of components in $\mathscr{L}$. Thus

$$
|\mathscr{K}|=\sum_{K \in \mathscr{K}} 1=\sum_{K \in \mathscr{K}} \sum_{L \in \mathscr{L}} d(\overline{\mathbf{f}}, K, L) d\left(\overline{\mathbf{f}^{-1}}, L, K\right)
$$

Similarly, the argument taken another direction yields

$$
|\mathscr{L}|=\sum_{L \in \mathscr{L}} 1=\sum_{L \in \mathscr{L}} \sum_{K \in \mathscr{K}} d(\overline{\mathbf{f}}, K, L) d\left(\overline{\mathbf{f}^{-1}}, L, K\right)
$$

If $|\mathscr{K}|<\infty$, then $\sum_{K \in \mathscr{K}} \overbrace{\sum_{L \in \mathscr{L}} d(\overline{\mathbf{f}}, K, L) d\left(\overline{\mathbf{f}^{-1}}, L, K\right)}^{1}<\infty$. The summation which equals 1 is a finite sum and so is the outside sum. Hence we can switch the order of summation and get

$$
|\mathscr{K}|=\sum_{L \in \mathscr{L}} \sum_{K \in \mathscr{K}} d(\overline{\mathbf{f}}, K, L) d\left(\overline{\mathbf{f}^{-1}}, L, K\right)=|\mathscr{L}|
$$

A similar argument applies if $|\mathscr{L}|<\infty$. Thus if one of these numbers $|\mathscr{K}|,|\mathscr{L}|$ is finite, so is the other and they are equal. This proves the theorem because if $p>1$ there is exactly one unbounded component to both $\mathbb{R}^{p} \backslash C$ and $\mathbb{R}^{p} \backslash \mathbf{f}(C)$ and if $p=1$ there are exactly two unbounded components.

As an application, here is a very interesting little result. It has to do with $d(\mathbf{f}, \Omega, \mathbf{f}(\mathbf{x}))$ in the case where $\mathbf{f}$ is one to one and $\Omega$ is connected. You might imagine this should equal 1 or -1 based on one dimensional analogies. Recall a one to one map defined on an interval is either increasing or decreasing. It either preserves or reverses orientation. It is similar in $n$ dimensions and it is a nice application of the Jordan separation theorem and the product formula.

Proposition 23.6.4 Let $\Omega$ be an open connected bounded set in $\mathbb{R}^{p}, p \geq 1$ such that $\mathbb{R}^{p} \backslash$ $\partial \Omega$ consists of two, three if $n=1$, connected components. Let $\mathbf{f} \in C\left(\bar{\Omega} ; \mathbb{R}^{p}\right)$ be continuous and one to one. Then $\mathbf{f}(\Omega)$ is the bounded component of $\mathbb{R}^{p} \backslash \mathbf{f}(\partial \Omega)$ and for $\mathbf{y} \in \mathbf{f}(\Omega)$, $d(\mathbf{f}, \Omega, \mathbf{y})$ either equals 1 or -1 .

Proof: First suppose $n \geq 2$. By the Jordan separation theorem, $\mathbb{R}^{p} \backslash \mathbf{f}(\partial \Omega)$ consists of two components, a bounded component $B$ and an unbounded component $U$. Using the Tietze extention theorem, there exists $\mathbf{g}$ defined on $\mathbb{R}^{p}$ such that $\mathbf{g}=\mathbf{f}^{-1}$ on $\mathbf{f}(\bar{\Omega})$. Thus on $\partial \Omega, \mathbf{g} \circ \mathbf{f}=\mathrm{id}$. It follows from this and the product formula that

$$
\begin{aligned}
1 & =d(\mathrm{id}, \Omega, \mathbf{g}(\mathbf{y}))=d(\mathbf{g} \circ \mathbf{f}, \Omega, \mathbf{g}(\mathbf{y})) \\
& =d(\mathbf{g}, B, \mathbf{g}(\mathbf{y})) d(\mathbf{f}, \Omega, B)+d(\mathbf{f}, \Omega, U) d(\mathbf{g}, U, \mathbf{g}(\mathbf{y})) \\
& =d(\mathbf{g}, B, \mathbf{g}(\mathbf{y})) d(\mathbf{f}, \Omega, B)
\end{aligned}
$$

The reduction happens because $d(\mathbf{f}, \Omega, U)=0$ as explained above. Since $U$ is unbounded, there are points in $U$ which cannot be in the compact set $\mathbf{f}(\bar{\Omega})$. For such, the degree is 0 but the degree is constant on $U$, one of the components of $\mathbf{f}(\partial \Omega)$. Therefore, $d(\mathbf{f}, \Omega, B) \neq 0$ and so for every $\mathbf{z} \in B$, it follows $\mathbf{z} \in \mathbf{f}(\Omega)$. Thus $B \subseteq \mathbf{f}(\Omega)$. On the other hand, $\mathbf{f}(\Omega)$ cannot have points in both $U$ and $B$ because it is a connected set. Therefore $\mathbf{f}(\Omega) \subseteq B$ and this shows $B=\mathbf{f}(\Omega)$. Thus $d(\mathbf{f}, \Omega, B)=d(\mathbf{f}, \Omega, \mathbf{y})$ for each $\mathbf{y} \in B$ and the above formula shows this equals either 1 or -1 because the degree is an integer. In the case where $n=1$, the argument is similar but here you have 3 components in $\mathbb{R}^{1} \backslash \mathbf{f}(\partial \Omega)$ so there are more terms in the above sum although two of them give 0 .

Here is another nice application of the Jordan separation theorem to the Jordan curve theorem and generalizations to higher dimensions.

### 23.7 Uniqueness of the Degree

The degree exists and has the above properties which allow one to prove amazing theorems. This is plenty to justify it. However, one might wonder whether, if we have a degree function with these properties, it will be the same? The answer is yes. It is likely possible to give a shorter list of desired properties however. Nevertheless, we want all these properties. First are some simple applications of the product formula which is one of the items which it is assumed satisfied by the degree.
Lemma 23.7.1 Let $\mathbf{h}: \mathbb{R}^{p} \rightarrow \mathbb{R}^{p}$ be a homeomorphism. Then

$$
1=d(\mathbf{h}, \Omega, \mathbf{h}(\mathbf{y})) d\left(\mathbf{h}^{-1}, \mathbf{h}(\Omega), \mathbf{y}\right)
$$

whenever $\mathbf{y} \notin \partial \Omega$ for $\Omega$ a bounded open set. Similarly,

$$
1=d\left(\mathbf{h}^{-1}, \Omega, \mathbf{h}^{-1}(\mathbf{y})\right) d\left(\mathbf{h}, \mathbf{h}^{-1}(\Omega), \mathbf{y}\right)
$$

Thus both terms in the factor equal either 1 or -1 .
Proof: It is known that $\mathbf{y} \notin \partial \Omega$. Let $\mathscr{L}$ be the components of $\mathbb{R}^{p} \backslash \mathbf{h}(\partial \Omega)$. Thus $\mathbf{h}(\mathbf{y}) \notin$ $\mathbf{h}(\partial \Omega)$. The product formula gives

$$
1=d\left(\mathbf{h}^{-1} \circ \mathbf{h}, \Omega, \mathbf{y}\right)=\sum_{L \in \mathscr{L}} d(\mathbf{h}, \Omega, L) d\left(\mathbf{h}^{-1}, L, \mathbf{y}\right)
$$

$d\left(\mathbf{h}^{-1}, L, \mathbf{y}\right)$ might be nonzero if $\mathbf{y} \in \mathbf{h}^{-1}(L)$. But, since $\mathbf{h}^{-1}$ is one to one, if $\mathbf{y} \in \mathbf{h}^{-1}(L)$, then $\mathbf{y} \notin \mathbf{h}^{-1}(\tilde{L})$ for any other $\tilde{L} \in \mathscr{L}$. Thus there is only one term in the above sum.

$$
\begin{gathered}
1=d(\mathbf{h}, \Omega, L) d\left(\mathbf{h}^{-1}, L, \mathbf{y}\right), \mathbf{y} \in \mathbf{h}^{-1}(L) \\
1=d(\mathbf{h}, \Omega, \mathbf{h}(\mathbf{y})) d\left(\mathbf{h}^{-1}, \mathbf{h}(\Omega), \mathbf{y}\right)
\end{gathered}
$$

Since the degree is an integer, both factors are either 1 or -1 .
The following assumption is convenient and seems like something which should be true. It says just a little more than $d(\mathrm{id}, \Omega, \mathbf{y})=1$ for $\mathbf{y} \in \Omega$. Actually, you could prove it by adding in a succession of $\frac{\mathbf{z}}{n}$ to id for $n$ large till you get to $\mathbf{h}$ in the first argument and $\mathbf{h}_{\mathbf{z}}(\mathbf{y})$ in the last and use the property that the degree is continuous in the first and last places. However, this is such a reasonable thing to assume, that it seems like we might as well have assumed it instead of $d(\mathrm{id}, \Omega, \mathbf{y})=1$ for $\mathbf{y} \in \Omega$.

Assumption 23.7.2 If $\mathbf{h}_{\mathbf{z}}(\mathbf{x})=\mathbf{x}+\mathbf{z}$, then for $\mathbf{y} \notin \partial \Omega, \mathbf{y} \in \Omega$,

$$
d\left(\mathbf{h}_{\mathbf{z}}, \Omega, \mathbf{h}_{\mathbf{z}}(\mathbf{y})\right)=d(\mathrm{id}, \Omega, \mathbf{y})=1
$$

That is, $d(\cdot+\mathbf{z}, \Omega, \mathbf{z}+\mathbf{y})=1$. From the above lemma, $d\left(\mathbf{h}_{\mathbf{z}}, \mathbf{h}_{-\mathbf{z}}(\Omega), \mathbf{y}\right)=1$ also.
Now let $\mathbf{h}_{\mathbf{z}}(\mathbf{x}) \equiv \mathbf{x}+\mathbf{z}$. Let $\mathbf{y} \notin \mathbf{g}(\partial \Omega)$ and consider

$$
d\left(\mathbf{h}_{-\mathbf{y}} \circ \mathbf{g} \circ \mathbf{h}_{\mathbf{z}}, \mathbf{h}_{-\mathbf{z}}(\Omega), \mathbf{h}_{-\mathbf{y}}(\mathbf{y})\right)=d\left(\mathbf{h}_{-\mathbf{y}} \circ \mathbf{g} \circ \mathbf{h}_{\mathbf{z}}, \mathbf{h}_{-\mathbf{z}}(\Omega), \mathbf{0}\right)
$$

Let $\mathscr{L}$ be the components of $\mathbb{R}^{p} \backslash \mathbf{g} \circ \mathbf{h}_{\mathbf{z}}\left(\partial\left(\mathbf{h}_{-\mathbf{z}}(\Omega)\right)\right)=\mathbb{R}^{p} \backslash \mathbf{g}(\partial \Omega)$. Then the product rule gives $d\left(\mathbf{h}_{-\mathbf{y}} \circ \mathbf{g} \circ \mathbf{h}_{\mathbf{z}}, \mathbf{h}_{-\mathbf{z}}(\Omega), \mathbf{0}\right)=$

$$
\sum_{L \in \mathscr{L}} d\left(\mathbf{g} \circ \mathbf{h}_{\mathbf{z}}, \mathbf{h}_{-\mathbf{z}}(\Omega), L\right) d\left(\mathbf{h}_{-y}, L, \mathbf{0}\right)
$$

Now $\mathbf{h}_{-\mathbf{y}}$ is one to one and so if $\mathbf{0} \in \mathbf{h}_{-\mathbf{y}}(L)$, then this is true for only that single $L$ and so there is only one term in the sum. For a single $L$ where $\mathbf{0} \in \mathbf{h}_{-\mathbf{y}}(L)$ so $\mathbf{y} \in L$,

$$
d\left(\mathbf{h}_{-\mathbf{y}} \circ \mathbf{g} \circ \mathbf{h}_{\mathbf{z}}, \mathbf{h}_{-\mathbf{z}}(\Omega), \mathbf{h}_{-\mathbf{y}}(\mathbf{y})\right)=d\left(\mathbf{g} \circ \mathbf{h}_{\mathbf{z}}, \mathbf{h}_{-\mathbf{z}}(\Omega), L\right) d\left(\mathbf{h}_{-y}, L, \mathbf{h}_{-\mathbf{y}}(\mathbf{y})\right)
$$

Now from the assumption, this equals $d\left(\mathbf{g} \circ \mathbf{h}_{\mathbf{z}}, \mathbf{h}_{-\mathbf{z}}(\Omega), L\right)=d\left(\mathbf{g} \circ \mathbf{h}_{\mathbf{z}}, \mathbf{h}_{-\mathbf{z}}(\Omega), \mathbf{y}\right)$. Now we use the product rule on this. Letting $\mathscr{K}$ be the bounded components of

$$
\begin{gathered}
\mathbb{R}^{p} \backslash \mathbf{h}_{\mathbf{z}}\left(\partial \mathbf{h}_{-\mathbf{z}}(\Omega)\right)=\mathbb{R}^{p} \backslash \partial \Omega \\
d\left(\mathbf{g} \circ \mathbf{h}_{\mathbf{z}}, \mathbf{h}_{-\mathbf{z}}(\Omega), \mathbf{y}\right)=\sum_{K \in \mathscr{K}} d\left(\mathbf{h}_{\mathbf{z}}, \mathbf{h}_{-\mathbf{z}}(\Omega), K\right) d(\mathbf{g}, K, \mathbf{y})
\end{gathered}
$$

The points of $K$ are not in $\partial \Omega$ and so, from the assumption, $d\left(\mathbf{h}_{\mathbf{z}}, \mathbf{h}_{-\mathbf{z}}(\Omega), K\right)=1$. Thus this whole thing reduces to $\sum_{K \in \mathscr{K}} d(\mathbf{g}, K, \mathbf{y})=d(\mathbf{g}, \Omega, \mathbf{y})$. This shows the following lemma.

Lemma 23.7.3 The following formula holds

$$
d\left(\mathbf{h}_{-\mathbf{y}} \circ \mathbf{g} \circ \mathbf{h}_{\mathbf{z}}, \mathbf{h}_{-\mathbf{z}}(\Omega), \mathbf{h}_{-\mathbf{y}}(\mathbf{y})\right)=d\left(\mathbf{g} \circ \mathbf{h}_{\mathbf{z}}, \mathbf{h}_{-\mathbf{z}}(\Omega), L\right)=d(\mathbf{g}, \Omega, \mathbf{y})
$$

In other words, $d(\mathbf{g}, \Omega, \mathbf{y})=d(\mathbf{g}(\cdot+\mathbf{z}), \Omega-\mathbf{z}, \mathbf{y})=d(\mathbf{g}(\cdot+\mathbf{z})-\mathbf{y}, \Omega-\mathbf{z}, \mathbf{0})$.

Theorem 23.7.4 You have a function $d$ which has integer values $d(\mathbf{g}, \Omega, \mathbf{y}) \in \mathbb{Z}$ whenever $\mathbf{y} \notin \mathbf{g}(\partial \Omega)$ for $\mathbf{g} \in C\left(\bar{\Omega} ; \mathbb{R}^{p}\right)$. Assume it satisfies the following properties which the one above satisfies.

1. (homotopy invariance) If

$$
\mathbf{h} \in C\left(\bar{\Omega} \times[0,1], \mathbb{R}^{p}\right)
$$

and $\mathbf{y}(t) \notin \mathbf{h}(\partial \Omega, t)$ for all $t \in[0,1]$ where $\mathbf{y}$ is continuous, then

$$
t \rightarrow d(\mathbf{h}(\cdot, t), \Omega, \mathbf{y}(t))
$$

is constant for $t \in[0,1]$.
2. If $\Omega \supseteq \Omega_{1} \cup \Omega_{2}$ where $\Omega_{1} \cap \Omega_{2}=\emptyset$, for $\Omega_{i}$ an open set, then if

$$
\mathbf{y} \notin \mathbf{f}\left(\bar{\Omega} \backslash\left(\Omega_{1} \cup \Omega_{2}\right)\right)
$$

then

$$
d\left(\mathbf{f}, \Omega_{1}, \mathbf{y}\right)+d\left(\mathbf{f}, \Omega_{2}, \mathbf{y}\right)=d(\mathbf{f}, \Omega, \mathbf{y})
$$

3. $d(\mathrm{id}, \Omega, \mathbf{y})=1$ if $\mathbf{y} \in \Omega$.
4. $d(\mathbf{f}, \Omega, \cdot)$ is continuous and constant on every connected component of $\mathbb{R}^{p} \backslash \mathbf{f}(\partial \Omega)$.
5. If $\mathbf{y} \notin \mathbf{f}(\partial \Omega)$, and if $d(\mathbf{f}, \Omega, \mathbf{y}) \neq 0$, then there exists $\mathbf{x} \in \Omega$ such that $\mathbf{f}(\mathbf{x})=\mathbf{y}$.
6. Product formula, Assumption 23.7.2.

Then $d$ is the degree which was defined above. Thus, in a sense, the degree is unique if we want it to do these things.

Proof: First note that $\mathbf{h} \rightarrow d(\mathbf{h}, \Omega, \mathbf{y})$ is continuous on $C\left(\bar{\Omega}, \mathbb{R}^{p}\right)$. Say $\|\mathbf{g}-\mathbf{h}\|_{\infty}<$ $\operatorname{dist}(\mathbf{h}(\partial \Omega), \mathbf{y})$. Then if $\mathbf{x} \in \partial \Omega, t \in[0,1]$,

$$
\begin{aligned}
\|\mathbf{g}(\mathbf{x})+t(\mathbf{h}-\mathbf{g})(\mathbf{x})-\mathbf{y}\| & =\|\mathbf{g}(\mathbf{x})-\mathbf{h}(\mathbf{x})+t(\mathbf{h}-\mathbf{g})(\mathbf{x})+\mathbf{h}(\mathbf{x})-\mathbf{y}\| \\
& \geq \operatorname{dist}(\mathbf{h}(\partial \Omega), \mathbf{y})-(1-t)\|\mathbf{g}-\mathbf{h}\|_{\infty}>0
\end{aligned}
$$

By homotopy invariance, $d(\mathbf{g}, \Omega, \mathbf{y})=d(\mathbf{h}, \Omega, \mathbf{y})$. By the approximation lemma, if we can identify the degree for $\mathbf{g} \in C^{\infty}\left(\bar{\Omega}, \mathbb{R}^{p}\right)$ with $\mathbf{y}$ a regular value, $\mathbf{y} \notin \mathbf{g}(\partial \Omega)$ then we know what the degree is. Say $\mathbf{g}^{-1}(\mathbf{y})=\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{n}\right\}$. Using the inverse function theorem there are balls $B_{i r}$ containing $\mathbf{x}_{i}$ such that none of these balls of radius $r$ intersect and $\mathbf{g}$ is one to one on each. Then from the theorem on the fundamental properties assumed above,

$$
\begin{equation*}
d(\mathbf{g}, \Omega, \mathbf{y})=\sum_{i=1}^{n} d\left(\mathbf{g}, B_{i r}, \mathbf{y}\right) \tag{23.7.15}
\end{equation*}
$$

and by assumption, this is

$$
\sum_{i=1}^{n} d\left(\mathbf{g}\left(\cdot+\mathbf{x}_{i}\right), B(\mathbf{0}, r), \mathbf{y}\right)=\sum_{i=1}^{n} d\left(\mathbf{g}\left(\cdot+\mathbf{x}_{i}\right)-\mathbf{y}, B(\mathbf{0}, r), \mathbf{0}\right)=\sum_{i=1}^{n} d(\mathbf{h}, B(\mathbf{0}, r), \mathbf{0})
$$

where $\mathbf{h}(\mathbf{x}) \equiv \mathbf{g}\left(\mathbf{x}+\mathbf{x}_{i}\right)-\mathbf{y}$. There is no restriction on the size of $r$. Consider one of the terms in the sum. $d(\mathbf{h}, B(\mathbf{0}, r), \mathbf{y})$. Note $D \mathbf{h}(\mathbf{0})=D \mathbf{g}\left(\mathbf{x}_{i}\right)$.

$$
\begin{equation*}
\mathbf{h}(\mathbf{x})=D \mathbf{h}(\mathbf{0}) \mathbf{x}+\mathbf{o}(\mathbf{x}) \tag{23.7.16}
\end{equation*}
$$

Now let

$$
L\left(\mathbf{z}_{1}, \ldots, \mathbf{z}_{p}\right) \equiv\left(\begin{array}{c}
\nabla h_{1}\left(\mathbf{z}_{1}\right) \\
\vdots \\
\nabla h_{p}\left(\mathbf{z}_{p}\right)
\end{array}\right)
$$

By the mean value theorem applied to the components of $\mathbf{h}$, if $\mathbf{x} \neq \mathbf{0},\|\mathbf{x}\| \leq r$, there exist $\mathbf{z}_{1}, \ldots, \mathbf{z}_{p} \in \overline{B(\mathbf{0}, r)} \times \cdots \times \overline{B(\mathbf{0}, r)}$

$$
\begin{equation*}
\frac{\|\mathbf{h}(\mathbf{x})-\mathbf{h}(\mathbf{0})\|}{\|\mathbf{x}\|}=\frac{\|\mathbf{h}(\mathbf{x})-\mathbf{0}\|}{\|\mathbf{x}\|}=\frac{\left\|L\left(\mathbf{z}_{1}, \ldots, \mathbf{z}_{p}\right) \mathbf{x}\right\|}{\|\mathbf{x}\|}=\left\|L\left(\mathbf{z}_{1}, \ldots, \mathbf{z}_{n}\right)\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right)\right\| \tag{23.7.17}
\end{equation*}
$$

Since $D \mathbf{h}(\mathbf{0})^{-1}$ exists, we can choose $r$ small enough that for all $\left(\mathbf{z}_{1}, \ldots, \mathbf{z}_{p}\right) \in \overline{B(\mathbf{0}, r)} \times$ $\cdots \times \overline{B(\mathbf{0}, r)}$

$$
\operatorname{det}\left(L\left(\mathbf{z}_{1}, \ldots, \mathbf{z}_{p}\right)\right) \neq 0
$$

Hence, for such sufficiently small $r$, there is $\delta>0$ such that for $S$ the unit sphere,

$$
\begin{gathered}
\{\mathbf{x}:\|\mathbf{x}\|=1\} \\
\inf \left\{\left\|L\left(\mathbf{z}_{1}, \ldots, \mathbf{z}_{p}\right) \mathbf{x}\right\|:\left(\mathbf{z}_{1}, \ldots, \mathbf{z}_{p}, \mathbf{x}\right) \in \overline{B(\mathbf{0}, r)} \times \cdots \times \overline{B(\mathbf{0}, r)} \times S\right\}=\delta>0
\end{gathered}
$$

Then it follows from 23.7 .17 that if $\|\mathbf{x}\| \leq r, \frac{\|\mathbf{h}(\mathbf{x})\|}{\|\mathbf{x}\|} \geq \delta$ so $\|\mathbf{h}(\mathbf{x})-\mathbf{0}\| \geq \delta\|\mathbf{x}\|$. Letting $\|\mathbf{x}\|=r$ so $\mathbf{x} \in \mathbf{h}(\partial B(\mathbf{0}, r))$,

$$
\operatorname{dist}(\mathbf{h}(\partial B(\mathbf{0}, r)), \mathbf{0}) \geq \delta r
$$

Now pick $\varepsilon<\delta$. Making $r$ still smaller if necessary, it follows from 23.7.16 that if $\|\mathbf{x}\| \leq r$,

$$
\|\mathbf{h}(\mathbf{x})-D \mathbf{h}(\mathbf{0}) \mathbf{x}\| \leq \varepsilon\|\mathbf{x}\|<\delta r \leq \operatorname{dist}(\mathbf{h}(\partial B(\mathbf{0}, r)), \mathbf{0})
$$

Thus,

$$
\|\mathbf{h}-D \mathbf{h}(\mathbf{0})(\cdot)\|_{\infty, \overline{B(\mathbf{0}, r)}}<\operatorname{dist}(\mathbf{h}(\partial B(\mathbf{0}, r)), \mathbf{0})
$$

and so $d(\mathbf{h}, B(\mathbf{0}, r), \mathbf{0})=d(D \mathbf{h}(\mathbf{0})(\cdot), B(\mathbf{0}, r), \mathbf{0})$.
Say $A=D \mathbf{h}(\mathbf{0})$ an invertible matrix. Then if $U$ is any open set containing $\mathbf{0}$, it follows from the properties of the degree that $d(A, U, \mathbf{0})=d(A, B(\mathbf{0}, r), \mathbf{0})$ whenever $r$ is small enough. By the product formula and 3. above,

$$
\left.\begin{array}{rl} 
& 1=d(I, B(\mathbf{0}, r), \mathbf{0})=d\left(A^{-1} A, B(\mathbf{0}, r), \mathbf{0}\right) \\
= & d(A, B(\mathbf{0}, r), A B(\mathbf{0}, r)) d\left(A^{-1}, A B(\mathbf{0}, r), \mathbf{0}\right) \\
& +d(A, B(\mathbf{0}, r), \overline{A B(\mathbf{0}, r)} \\
C
\end{array}\right) d\left(A^{-1}, \overline{A B(\mathbf{0}, r)}^{c}, \mathbf{0}\right)
$$

From 5. above, this reduces to

$$
1=d(A, B(\mathbf{0}, r), A B(\mathbf{0}, r)) d\left(A^{-1}, A B(\mathbf{0}, r), \mathbf{0}\right)
$$

Thus, since the degree is an integer, either both $d(A, B(\mathbf{0}, r), \mathbf{0}), d\left(A^{-1}, A B(\mathbf{0}, r), \mathbf{0}\right)$ are 1 or they are both -1 . This would hold if when $A$ is invertible,

$$
d(A, B(\mathbf{0}, r), \mathbf{0})=\operatorname{sgn}(\operatorname{det}(A))=\operatorname{sgn}\left(\operatorname{det}\left(A^{-1}\right)\right)
$$

It would also hold if $d(A, B(\mathbf{0}, r), \mathbf{0})=-\operatorname{sgn}(\operatorname{det}(A))$. However, the latter of the two alternatives is not the one wanted because, doing something like the above,

$$
\begin{aligned}
d\left(A^{2}, B(\mathbf{0}, r), \mathbf{0}\right) & =d(A, B(\mathbf{0}, r), A B(\mathbf{0}, r)) d(A, A B(\mathbf{0}, r), \mathbf{0}) \\
& =d(A, B(\mathbf{0}, r), \mathbf{0}) d(A, A B(\mathbf{0}, r), \mathbf{0})
\end{aligned}
$$

If you had for a definition $d(A, B(\mathbf{0}, r), \mathbf{0})=-\operatorname{sgn} \operatorname{det}(A)$, then you would have

$$
-\operatorname{sgn} \operatorname{det}\left(A^{2}\right)=-1=(-\operatorname{sgn} \operatorname{det}(A))^{2}=1
$$

Hence the only reasonable definition is to let $d(A, B(\mathbf{0}, r), \mathbf{0})=\operatorname{sgn}(\operatorname{det}(A))$. It follows from 23.7.15 that $d(\mathbf{g}, \Omega, \mathbf{y})=\sum_{i=1}^{n} d\left(\mathbf{g}, B_{i r}, \mathbf{y}\right)=\sum_{i=1}^{n} \operatorname{sgn}\left(\operatorname{det}\left(D \mathbf{g}\left(\mathbf{x}_{i}\right)\right)\right)$. Thus, if the above conditions hold, then in whatever manner you construct the degree, it amounts to the definition given above in this chapter in the sense that for $\mathbf{y}$ a regular point of a smooth function, you get the definition of the chapter.

### 23.8 A Function With Values In Smaller Dimensions

Recall that we have the degree defined $d(f, \Omega, y)$ for continuous functions on $\bar{\Omega}$ and $y \notin$ $f(\partial \Omega)$. It had properties as follows.

1. $d(\mathrm{id}, \Omega, \mathbf{y})=1$ if $\mathbf{y} \in \Omega$.
2. If $\Omega_{i} \subseteq \Omega, \Omega_{i}$ open, and $\Omega_{1} \cap \Omega_{2}=\emptyset$ and if $\mathbf{y} \notin \mathbf{f}\left(\bar{\Omega} \backslash\left(\Omega_{1} \cup \Omega_{2}\right)\right)$, then $d\left(\mathbf{f}, \Omega_{1}, \mathbf{y}\right)+$ $d\left(\mathbf{f}, \Omega_{2}, \mathbf{y}\right)=d(\mathbf{f}, \Omega, \mathbf{y})$.
3. If $\mathbf{y} \notin \mathbf{f}\left(\bar{\Omega} \backslash \Omega_{1}\right)$ and $\Omega_{1}$ is an open subset of $\Omega$, then

$$
d(\mathbf{f}, \Omega, \mathbf{y})=d\left(\mathbf{f}, \Omega_{1}, \mathbf{y}\right)
$$

4. For $\mathbf{y} \in \mathbb{R}^{n} \backslash \mathbf{f}(\partial \Omega)$, if $d(\mathbf{f}, \Omega, \mathbf{y}) \neq 0$ then $\mathbf{f}^{-1}(\mathbf{y}) \cap \Omega \neq \emptyset$.
5. If $t \rightarrow \mathbf{y}(t)$ is continuous $\mathbf{h}: \bar{\Omega} \times[0,1] \rightarrow \mathbb{R}^{n}$ is continuous and if $\mathbf{y}(t) \notin \mathbf{h}(\partial \Omega, t)$ for all $t$, then $t \rightarrow d(\mathbf{h}(\cdot, t), \Omega, \mathbf{y}(t))$ is constant.
6. $d(\cdot, \Omega, \mathbf{y})$ is defined and constant on

$$
\left\{\mathbf{g} \in C\left(\bar{\Omega} ; \mathbb{R}^{n}\right):\|\mathbf{g}-\mathbf{f}\|_{\infty}<r\right\}
$$

where $r=\operatorname{dist}(\mathbf{y}, \mathbf{f}(\partial \Omega))$.
7. $d(\mathbf{f}, \Omega, \cdot)$ is constant on every connected component of $\mathbb{R}^{n} \backslash \mathbf{f}(\partial \Omega)$.
8. $d(\mathbf{g}, \Omega, \mathbf{y})=d(\mathbf{f}, \Omega, \mathbf{y})$ if $\left.\mathbf{g}\right|_{\partial \Omega}=\left.\mathbf{f}\right|_{\partial \Omega}$.

Theorem 23.8.1 Let $\Omega$ be a bounded open set in $\mathbb{R}^{n}$ and let $\mathbf{f} \in C\left(\bar{\Omega} ; \mathbb{R}_{m}^{n}\right)$ where $\mathbb{R}_{m}^{n}=$ $\left\{\mathbf{x} \in \mathbb{R}^{n}: x_{k}=0\right.$ for $\left.k>m\right\}$. Thus $\mathbf{x}$ concludes with a column of $n-m$ zeros. Let $\mathbf{y} \in \mathbb{R}_{m}^{n} \backslash$ $(\mathrm{id}-\mathbf{f})(\partial \Omega)$. Then $d(\mathrm{id}-\mathbf{f}, \Omega, \mathbf{y})=d\left(\left.(\mathrm{id}-\mathbf{f})\right|_{\Omega \cap \mathbb{R}_{m}^{n}}, \Omega \cap \mathbb{R}_{m}^{n}, \mathbf{y}\right)$.

Proof: To save space, let $\mathbf{g}=\mathrm{id}-\mathbf{f}$. Then there is no loss of generality in assuming at the outset that $\mathbf{y}$ is a regular value for $\mathbf{g}$. Indeed, everything above was reduced to this case. Then for $\mathbf{x} \in \mathbf{g}^{-1}(\mathbf{y})$ and letting $\mathbf{x}_{m}$ be the first $m$ variables for $\mathbf{x}$,

$$
D \mathbf{g}(\mathbf{x})=\left(\begin{array}{cc}
D_{\mathbf{x}_{m}} \mathbf{g}(\mathbf{x}) & * \\
0 & I_{n-m}
\end{array}\right)
$$

Then it follows that

$$
\begin{aligned}
0 & \neq \operatorname{det}(D \mathbf{g}(\mathbf{x}))=\operatorname{det}\left(\begin{array}{cc}
D_{\mathbf{x}_{m}} \mathbf{g}(\mathbf{x}) & * \\
0 & I_{n-m}
\end{array}\right) \\
& =\operatorname{det}\left(\begin{array}{cc}
D_{\mathbf{x}_{m}} \mathbf{g}(\mathbf{x}) & 0 \\
0 & I_{n-m}
\end{array}\right)=\operatorname{det}\left(D_{\mathbf{x}_{m}} \mathbf{g}(\mathbf{x})\right)
\end{aligned}
$$

This last is just the determinant of the derivative of the function which results from restricting $\mathbf{g}$ to the first $m$ variables. Now $\mathbf{y} \in \mathbb{R}_{m}^{n}$ and $\mathbf{f}$ also is given to have values in $\mathbb{R}_{m}^{n}$ so if $\mathbf{g}(\mathbf{x})=\mathbf{y}$, then you have $\mathbf{x}-\mathbf{f}(\mathbf{x})=\mathbf{y}$ which requires $\mathbf{x} \in \mathbb{R}_{m}^{n}$ also. Therefore, $\mathbf{g}^{-1}(\mathbf{y})$ consists of points in $\mathbb{R}_{m}^{n}$ only. Thus, $\mathbf{y}$ is also a regular value of the function which results from restricting $\mathbf{g}$ to $\overline{\mathbb{R}_{m}^{n} \cap \Omega}$.

$$
\begin{gathered}
d(\mathrm{id}-\mathbf{f}, \Omega, \mathbf{y})=d(\mathbf{g}, \Omega, \mathbf{y}) \\
=\sum_{\mathbf{x} \in \mathbf{g}^{-1}(\mathbf{y})} \operatorname{sign}(\operatorname{det}(D \mathbf{g}(\mathbf{x}))) \\
=\sum_{\mathbf{x} \in \mathbf{g}^{-1}(\mathbf{y})} \operatorname{sign}\left(\operatorname{det}\left(D_{\mathbf{x}_{m}} \mathbf{g}(\mathbf{x})\right)\right) \equiv d\left(\left.(\mathrm{id}-\mathbf{f})\right|_{\Omega \cap \mathbb{R}_{m}^{n}}, \Omega \cap \mathbb{R}_{m}^{n}, \mathbf{y}\right)
\end{gathered}
$$

Recall that for $\mathbf{g} \in C^{2}\left(\bar{\Omega} ; \mathbb{R}^{n}\right)$,

$$
d(\mathbf{g}, \Omega, \mathbf{y}) \equiv \lim _{\varepsilon \rightarrow 0} \int_{\Omega} \phi_{\varepsilon}(\mathbf{g}(\mathbf{x})-\mathbf{y}) \operatorname{det} D \mathbf{g}(\mathbf{x}) d x
$$

In fact, it can be shown that the degree is unique based on its Properties, $1,2,5$ above. It involves reducing to linear maps and then some complicated arguments involving linear algebra. It is done in [38]. Here we will be a little less ambitious. The following lemma will be useful when extending the degree to finite dimensional normed linear spaces and from there to Banach spaces. It is motivated by the following diagram.

$$
\begin{array}{lll}
\theta^{-1}(y) & \stackrel{\theta^{-1}}{\leftarrow} & y \\
\uparrow \theta^{-1} \circ g \circ \theta & & \uparrow g \\
\theta^{-1}(\Omega) & \xrightarrow{\theta} & \Omega
\end{array}
$$

Lemma 23.8.2 Let $\mathbf{y} \notin \mathbf{g}(\partial \Omega)$ and let $\theta$ be an isomorphism of $\mathbb{R}^{n}$. That is, $\theta$ is one to one onto and linear. Then

$$
d\left(\theta^{-1} \circ \mathbf{g} \circ \theta, \theta^{-1}(\Omega), \theta^{-1} \mathbf{y}\right)=d(\mathbf{g}, \Omega, \mathbf{y})
$$

Proof: It suffices to consider $\mathbf{g} \in C^{2}\left(\bar{\Omega} ; \mathbb{R}^{n}\right)$ for which $\mathbf{y}$ is a regular value because you can get such a $\hat{\mathbf{g}}$ with $\|\hat{\mathbf{g}}-\mathbf{g}\|_{\infty}<\delta$ where

$$
B(\mathbf{y}, 2 \delta) \cap \mathbf{g}(\partial \Omega)=\emptyset
$$

Thus $B(\mathbf{y}, \delta) \cap \hat{\mathbf{g}}(\partial \Omega)=\emptyset$ and so $d(\mathbf{g}, \Omega, \mathbf{y})=d(\hat{\mathbf{g}}, \Omega, \mathbf{y})$. One can assume similarly that $\|\hat{\mathbf{g}}-\mathbf{g}\|_{\infty}$ is sufficiently small that

$$
d\left(\theta^{-1} \mathbf{g} \circ \theta, \theta^{-1}(\Omega), \theta^{-1} \mathbf{y}\right)=d\left(\theta^{-1} \hat{\mathbf{g}} \circ \theta, \theta^{-1}(\Omega), \theta^{-1} \mathbf{y}\right)
$$

because both $\theta$ and $\theta^{-1}$ are continuous. Thus it suffices to consider at the outset $\mathbf{g} \in$ $C^{2}\left(\bar{\Omega} ; \mathbb{R}^{n}\right)$. Then from the definition of degree for $C^{2}$ maps,

$$
\begin{aligned}
& d\left(\theta^{-1} \circ \mathbf{g} \circ \theta, \theta^{-1}(\Omega), \theta^{-1} \mathbf{y}\right) \\
= & \lim _{\varepsilon \rightarrow 0} \int_{\theta^{-1} \Omega} \phi_{\varepsilon}\left(\left(\theta^{-1} \circ \mathbf{g} \circ \theta\right)(\mathbf{z})-\theta^{-1} \mathbf{y}\right) \operatorname{det} D\left(\theta^{-1} \circ \mathbf{g} \circ \theta\right)(\mathbf{z}) d z
\end{aligned}
$$

Now $D\left(\theta^{-1} \circ \mathbf{g} \circ \theta\right)(\mathbf{z})=\theta^{-1} D(\mathbf{g} \circ \theta)(\mathbf{z})=\theta^{-1} D \mathbf{g}(\theta(\mathbf{z})) \theta \mathbf{z}$. Changing the variables $\mathbf{x}=\theta \mathbf{z}, \mathbf{z}=\theta^{-1} \mathbf{x}$, this last integral equals

$$
\begin{aligned}
& \int_{\Omega} \phi_{\varepsilon}\left(\left(\theta^{-1} \mathbf{g} \circ \theta\right)\left(\theta^{-1}(\mathbf{x})\right)-\theta^{-1} \mathbf{y}\right) \operatorname{det} D \mathbf{g}(\mathbf{x})|\operatorname{det} \theta|\left|\operatorname{det} \theta^{-1}\right|^{2} d x \\
= & \int_{\Omega} \phi_{\varepsilon}\left(\theta^{-1} \mathbf{g}(\mathbf{x})-\theta^{-1} \mathbf{y}\right)\left|\operatorname{det} \theta^{-1}\right| \operatorname{det} D \mathbf{g}(\mathbf{x}) d x
\end{aligned}
$$

Recall that $\phi_{\varepsilon}$ is a mollifier which is nonzero only in $B(\mathbf{0}, \varepsilon)$. Now

$$
\mathbf{g}^{-1}(\mathbf{y})=\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{m}\right\}=\left(\theta^{-1} \mathbf{g}\right)^{-1}\left(\theta^{-1} \mathbf{y}\right)
$$

and so $\mathbf{g}\left(\mathbf{x}_{i}\right)=\mathbf{y}$ and $\theta^{-1} \mathbf{g}\left(\mathbf{x}_{i}\right)=\theta^{-1} \mathbf{y}$. By the inverse function theorem, there exist disjoint open sets $U_{i}$ with $\mathbf{x}_{i} \in U_{i}$, such that $\theta^{-1} \mathbf{g}$ is one to one on $U_{i}$ with $\operatorname{det}\left(D\left(\theta^{-1} \mathbf{g}\right)(\mathbf{x})\right)=$ $\operatorname{det}\left(\theta^{-1}\right) \operatorname{det} D \mathbf{g}(\mathbf{x})$ having constant sign on $U_{i}$ and $\theta^{-1} \mathbf{g}\left(U_{i}\right)$ is an open set containing $\theta^{-1} \mathbf{y}$. Then let $\varepsilon$ be small enough that $B\left(\theta^{-1} \mathbf{y}, \varepsilon\right) \subseteq \cap_{i=1}^{m} \theta^{-1} \mathbf{g}\left(U_{i}\right)$ and let

$$
V_{i} \equiv\left(\theta^{-1} \mathbf{g}\right)^{-1}\left(B\left(\theta^{-1} \mathbf{y}, \varepsilon\right)\right) \cap U_{i}
$$

Thus for small $\varepsilon$, the $V_{i}$ are disjoint open sets in $\Omega$ and

$$
\begin{aligned}
& \int_{\Omega} \phi_{\varepsilon}\left(\theta^{-1} \mathbf{g}(\mathbf{x})-\theta^{-1} \mathbf{y}\right) \operatorname{det} D \mathbf{g}(\mathbf{x})\left|\operatorname{det} \theta^{-1}\right| d x \\
= & \sum_{i=1}^{m} \int_{V_{i}} \phi_{\varepsilon}\left(\theta^{-1}(\mathbf{g}(\mathbf{x})-\mathbf{y})\right) \operatorname{det} D \mathbf{g}(\mathbf{x})\left|\operatorname{det} \theta^{-1}\right| d x
\end{aligned}
$$

Now just let $\mathbf{z}=\mathbf{g}(\mathbf{x})-\mathbf{y}$ and change the variables.

$$
=\sum_{i=1}^{m}\left|\operatorname{det} \theta^{-1}\right| \int_{\mathbf{g}\left(V_{i}\right)-\mathbf{y}} \phi_{\varepsilon}\left(\theta^{-1} \mathbf{z}\right) \operatorname{det} D \mathbf{g}\left(\mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})\right)\left|\operatorname{det} D \mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})\right| d z
$$

By the chain rule, $I=D \mathbf{g}\left(\mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})\right) D \mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})$ and so

$$
\begin{aligned}
& \operatorname{det} D \mathbf{g}\left(\mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})\right)\left|\operatorname{det} D \mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})\right| \\
= & \operatorname{sgn}\left(\operatorname{det} D \mathbf{g}\left(\mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})\right)\right) \\
& \left|\operatorname{det} D \mathbf{g}\left(\mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})\right)\right|\left|\operatorname{det} D \mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})\right| \\
= & \operatorname{sgn}\left(\operatorname{det} D \mathbf{g}\left(\mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})\right)\right) \\
= & \operatorname{sgn}(\operatorname{det} D \mathbf{g}(\mathbf{x}))=\operatorname{sgn}\left(\operatorname{det} D \mathbf{g}\left(\mathbf{x}_{i}\right)\right) .
\end{aligned}
$$

and so it all reduces to

$$
\begin{aligned}
& \sum_{i=1}^{m} \operatorname{sgn}\left(\operatorname{det} D \mathbf{g}\left(\mathbf{x}_{i}\right)\right) \int_{\mathbf{g}\left(V_{i}\right)-\mathbf{y}} \phi_{\varepsilon}\left(\theta^{-1} \mathbf{z}\right) d z \\
= & \sum_{i=1}^{m} \operatorname{sgn}\left(\operatorname{det} D \mathbf{g}\left(\mathbf{x}_{i}\right)\right) \int_{\theta B(\mathbf{0}, \varepsilon)}\left|\operatorname{det} \theta^{-1}\right| \phi_{\varepsilon}\left(\theta^{-1} \mathbf{z}\right) d z \\
= & \sum_{i=1}^{m} \operatorname{sgn}\left(\operatorname{det} D \mathbf{g}\left(\mathbf{x}_{i}\right)\right) \int_{B(\mathbf{0}, \boldsymbol{\varepsilon})} \phi_{\varepsilon}(\mathbf{w})\left|\operatorname{det} \theta^{-1}\right||\operatorname{det} \theta| d w \\
= & \sum_{i=1}^{m} \operatorname{sgn}\left(\operatorname{det} D \mathbf{g}\left(\mathbf{x}_{i}\right)\right)=d(\mathbf{g}, \Omega, \mathbf{y}) .
\end{aligned}
$$

What about functions which have values in finite dimensional vector spaces?
Theorem 23.8.3 Let $\Omega$ be an open bounded set in $V$ a real normed $n$ dimensional vector space. Then there exists a topological degree d $(f, \Omega, y)$ for $f \in C(\bar{\Omega}, V), y \notin f(\partial \Omega)$ which satisfies all the properties of the degree for functions having values in $\mathbb{R}^{n}$ described above,

1. $d(\mathrm{id}, \Omega, y)=1$ if $\mathbf{y} \in \Omega$.
2. If $\Omega_{i} \subseteq \Omega, \Omega_{i}$ open, and $\Omega_{1} \cap \Omega_{2}=\emptyset$ and if $y \notin f\left(\bar{\Omega} \backslash\left(\Omega_{1} \cup \Omega_{2}\right)\right)$, then $d\left(f, \Omega_{1}, y\right)+$ $d\left(f, \Omega_{2}, y\right)=d(f, \Omega, y)$.
3. If $y \notin f\left(\bar{\Omega} \backslash \Omega_{1}\right)$ and $\Omega_{1}$ is an open subset of $\Omega$, then

$$
d(f, \Omega, y)=d\left(f, \Omega_{1}, y\right)
$$

4. For $y \in \mathbb{R}^{n} \backslash f(\partial \Omega)$, if $d(f, \Omega, y) \neq 0$ then $f^{-1}(y) \cap \Omega \neq \emptyset$.
5. If $t \rightarrow \mathbf{y}(t)$ is continuous $\mathbf{h}: \bar{\Omega} \times[0,1] \rightarrow \mathbb{R}^{n}$ is continuous and if $\mathbf{y}(t) \notin \mathbf{h}(\partial \Omega, t)$ for all $t$, then $t \rightarrow d(\mathbf{h}(\cdot, t), \Omega, \mathbf{y}(t))$ is constant.
6. $d(\cdot, \Omega, y)$ is defined and constant on

$$
\left\{g \in C\left(\bar{\Omega} ; \mathbb{R}^{n}\right):\|g-f\|_{\infty}<r\right\}
$$

where $r=\operatorname{dist}(y, f(\partial \Omega))$.
7. $d(f, \Omega, \cdot)$ is constant on every connected component of $\mathbb{R}^{n} \backslash f(\partial \Omega)$.
8. $d(g, \Omega, y)=d(f, \Omega, y)$ if $\left.g\right|_{\partial \Omega}=\left.f\right|_{\partial \Omega}$.

Proof: There is an isomorphism $\theta: \mathbb{R}^{n} \rightarrow V$ which also preserves all topological properties. This follows from the properties of finite dimensional vector spaces. In fact, every algebraic isomorphism is automatically a homeomorphism preserving all topological properties. Then it is pretty easy to see what the degree should be.

$$
d(f, \Omega, y) \equiv d\left(\theta^{-1} \circ f \circ \theta, \theta^{-1}(\Omega), \theta^{-1} y\right)
$$

Then by standard material on finite dimensional vector spaces, the norm on $V$ is equivalent to the norm defined by $|v| \equiv\left|\theta^{-1} v\right|_{\mathbb{R}^{n}}$. Hence all of those properties hold. By Lemma 23.8.2 this definition does not depend on the particular isomorphism used. If $\hat{\theta}$ is another one, then one would need to verify that

$$
d\left(\theta^{-1} \circ f \circ \theta, \theta^{-1}(\Omega), \theta^{-1} y\right)=d\left(\hat{\theta}^{-1} \circ f \circ \hat{\theta}, \hat{\theta}^{-1}(\Omega), \hat{\theta}^{-1} y\right)
$$

However, you could use that lemma to conclude that

$$
\begin{aligned}
& d\left(\hat{\theta}^{-1} \circ f \circ \hat{\theta}, \hat{\theta}^{-1}(\Omega), \hat{\theta}^{-1} y\right) \\
= & d\left(\alpha^{-1} \circ \hat{\theta}^{-1} \circ f \circ \hat{\theta} \circ \alpha, \alpha^{-1} \hat{\theta}^{-1}(\Omega), \alpha^{-1} \hat{\theta}^{-1} y\right)
\end{aligned}
$$

where $\alpha$ is such that $\hat{\theta} \circ \alpha=\theta$. Then this verifies the appropriate equation.
Next one considers what happens when the function $I-f$ has values in a smaller dimensional subspace.

Theorem 23.8.4 Let $\Omega$ be a bounded open set in $V$ an $n$ dimensional normed linear space and let $f \in C\left(\bar{\Omega} ; V_{m}\right)$ where $V_{m}$ is an $m$ dimensional subspace. Let $y \in V_{m} \backslash(I-f)(\partial \Omega)$. Then $d(I-f, \Omega, y)=d\left(\left.(I-f)\right|_{\overline{\Omega \cap V_{m}}}, \Omega \cap V_{m}, y\right)$.

Proof: Letting $\left\{v_{1}, \cdots, v_{m}\right\}$ be a basis for $V_{m}$, let a basis for $V$ be

$$
\left\{v_{1}, \cdots, v_{m}, v_{m+1}, \cdots, v_{n}\right\}
$$

Let $\theta$ be the isomorphism which satisfies $\theta \mathbf{e}_{i}=v_{i}$ where the $\mathbf{e}_{i}$ denotes the standard basis vectors for $\mathbb{R}^{n}$. Then from the above,

$$
\begin{aligned}
d(I-f, \Omega, y) & \equiv d\left(\theta^{-1} \circ(I-f) \circ \theta, \theta^{-1} \Omega, \theta^{-1} y\right) \\
& =d\left(\left.\theta^{-1} \circ(I-f) \circ \theta\right|_{\overline{\theta^{-1}(\Omega) \cap \mathbb{R}_{m}^{n}}}, \theta^{-1}(\Omega) \cap \mathbb{R}_{m}^{n}, \theta^{-1} y\right) \\
& \equiv d\left(\left.(I-f)\right|_{\overline{\Omega \cap V_{m}}}, \Omega \cap V_{m}, y\right) .
\end{aligned}
$$

### 23.9 The Leray Schauder Degree

This is a very important generalization to Banach spaces. It turns out you can define the degree of $I-F$ where $F$ is a compact mapping. To recall what one of these is, here is the definition.

Definition 23.9.1 Let $\Omega$ be a bounded open set in $X$ a Banach space and let $F: \bar{\Omega} \rightarrow X$ be continuous. Then $F$ is called compact if $F(B)$ is precompact whenever $B$ is bounded. That is, if $\left\{x_{n}\right\}$ is a bounded sequence, then there is a subsequence $\left\{x_{n_{k}}\right\}$ such that $\left\{F\left(x_{n_{k}}\right)\right\}$ converges.

Theorem 23.9.2 Let $F: \bar{\Omega} \rightarrow X$ as above be compact. Then for each $\varepsilon>0$, there exists $F_{\varepsilon}: \bar{\Omega} \rightarrow X$ such that $F$ has values in a finite dimensional subspace of $X$ and

$$
\sup _{x \in \bar{\Omega}}\left\|F_{\varepsilon}(x)-F(x)\right\|<\varepsilon
$$

In addition to this, $(I-F)^{-1}$ (compact set) $=$ compact set. (This is called "proper".)
Proof: It is known that $\overline{F(\Omega)}$ is compact. Therefore, there is an $\varepsilon$ net for $F(\Omega)$, $\left\{F x_{k}\right\}_{k=1}^{n}$ satisfying

$$
\overline{F(\Omega)} \subseteq \cup_{k} B\left(F x_{k}, \varepsilon\right)
$$

Now let

$$
\phi_{k}(F x) \equiv\left(\varepsilon-\left\|F x-F x_{k}\right\|\right)^{+}
$$

Thus this is equal to 0 if $\left\|F x_{k}-F x\right\| \geq \varepsilon$ and is positive if $\left\|F x_{k}-F x\right\|<\varepsilon$. Then consider

$$
F_{\varepsilon}(x) \equiv \sum_{k=1}^{n} F\left(x_{k}\right) \frac{\phi_{k}(F x)}{\sum_{i} \phi_{i}(F x)}
$$

It clearly has values in $\operatorname{span}\left(\left\{F x_{k}\right\}_{k=1}^{n}\right)$. How close is it to $F(x)$ ? Say $F x \in B\left(F x_{k}, \boldsymbol{\varepsilon}\right)$. Then for such $x,\left\|F(x)-F\left(x_{k}\right)\right\|<\varepsilon$ by definition. Hence

$$
\begin{aligned}
\left\|F(x)-F_{\varepsilon}(x)\right\| & =\sum_{k:\left\|F(x)-F x_{k}\right\|<\varepsilon}\left\|F\left(x_{k}\right)-F(x)\right\| \frac{\phi_{k}(F x)}{\sum_{i} \phi_{i}(F x)} \\
& <\varepsilon \sum_{k} \frac{\phi_{k}(F x)}{\sum_{i} \phi_{i}(F x)}=\varepsilon
\end{aligned}
$$

Of course $x$ is arbitrary and so

$$
\sup _{x \in \bar{\Omega}}\left\|F_{\varepsilon}(x)-F(x)\right\|<\varepsilon
$$

Next consider the second claim. Let $K$ be compact. Consider

$$
\left\{x_{k}\right\} \subseteq(I-F)^{-1}(K)
$$

It is necessary to show that it has a convergent subsequence. Then $\left\{(I-F)\left(x_{k}\right)\right\}$ is a sequence in $K$ and so it has a convergent subsequence still denoted with subscript $k$ such that $(I-F)\left(x_{k}\right) \rightarrow y$. The $x_{k}$ are in a bounded set $\Omega$ and so, from compactness of $F$, there is a further subsequence, still denoted with subscript $k$ such that $F x_{k} \rightarrow z$. It follows that $x_{k} \rightarrow y-z$ and hence every sequence in $(I-F)^{-1}(K)$ has a convergent subsequence.

Corollary 23.9.3 Let $F: \bar{\Omega} \rightarrow X$ as above be compact. Then for each $\varepsilon>0$, there exists $F_{\varepsilon}: \bar{\Omega} \rightarrow X$ such that $F$ has values in a finite dimensional subspace of $X$ and

$$
\sup _{x \in \bar{\Omega}}\left\|F_{\varepsilon}(x)-F(x)\right\|<\varepsilon
$$

. . In addition to this, $(I-F)^{-1}($ compact set $)=$ compact set. (This is called "proper".) If $\Omega$ is symmetric and $F$ is odd $(F(-x)=-F(x))$ then one can also assume $F_{\varepsilon}$ is also odd.

Proof: Suppose $\Omega$ is symmetric in that $x \in \Omega$ iff $-x \in \Omega$. Suppose also that $F$ is odd. Thus $F(\Omega)$ is also symmetric. Thus $\overline{F(\Omega)}$ is compact and symmetric. If $y \in F(\Omega)$, then $y=F x$ and so $-y=-F(x)=F(-x) \in F(\Omega)$. Choose the $\varepsilon$ net to be symmetric. That is, you have $(F x)_{k}$ in the net if and only if $-(F x)_{k}$ is in the net. Just add them in if needed. Therefore, there is an $\varepsilon$ net for $F(\Omega),\left\{(F x)_{k}\right\}_{k=-m_{\varepsilon}}^{m_{\varepsilon}}$ satisfying

$$
\overline{F(\Omega)} \subseteq \cup_{k} B\left(F x_{k}, \varepsilon\right),\left\{F x_{k}\right\} \text { is symmetric. }
$$

Number these so that

$$
F x_{-k}=-F x_{k}=F\left(-x_{k}\right),|k| \leq m_{\varepsilon}
$$

Now let

$$
\begin{aligned}
\phi_{k}(F x) & \equiv\left(\varepsilon-\left\|F x-F x_{k}\right\|\right)^{+} \\
\phi_{-k}(F x) & \equiv\left(\varepsilon-\left\|F x-F x_{-k}\right\|\right)^{+} \\
& =\left(\varepsilon-\left\|F x+F x_{k}\right\|\right)^{+} \\
& =\left(\varepsilon-\left\|F x-\left(-F x_{k}\right)\right\|\right)^{+} \\
& =\phi_{k}(-F x)
\end{aligned}
$$

that is, $\phi_{-k}$ is centered at $-F x_{k}$ while $\phi_{k}$ is centered at $F x_{k}$, each function equal to 0 off $B\left(F x_{k}, \varepsilon\right)$ and is positive on $B\left(F x_{k}, \varepsilon\right)$. Then consider

$$
\begin{gathered}
F_{\varepsilon}(x) \equiv \sum_{k=-m_{\varepsilon}}^{m_{\varepsilon}} F\left(x_{k}\right) \frac{\phi_{k}(F x)}{\sum_{i} \phi_{i}(F x)} \\
F_{\varepsilon}(-x)=\sum_{k=-m_{\varepsilon}}^{m_{\varepsilon}} F\left(x_{k}\right) \frac{\phi_{k}(F(-x))}{\sum_{i} \phi_{i}(F(-x))}=\sum_{k=-m_{\varepsilon}}^{m_{\varepsilon}} F\left(x_{k}\right) \frac{\phi_{-k}(F(x))}{\sum_{i} \phi_{-i}(F(x))} \\
=-\sum_{k=-m_{\varepsilon}}^{m_{\varepsilon}} F\left(-x_{k}\right) \frac{\phi_{-k}(F(x))}{\sum_{i} \phi_{-i}(F(x))}=-\sum_{k=-m_{\varepsilon}}^{m_{\varepsilon}} F\left(x_{-k}\right) \frac{\phi_{-k}(F(x))}{\sum_{i} \phi_{-i}(F(x))} \\
=-\sum_{k=-m_{\varepsilon}}^{m_{\varepsilon}} F\left(x_{k}\right) \frac{\phi_{k}(F(x))}{\sum_{i} \phi_{i}(F(x))}=-F_{\varepsilon}(x)
\end{gathered}
$$

The rest of the argument is the same.
Now let $F: \bar{\Omega} \rightarrow X$ be compact and consider $I-F$. Is $(I-F)(\partial \Omega)$ closed? Suppose $(I-F) x_{k} \rightarrow y$. Then $K \equiv y \cup\left\{(I-F) x_{k}\right\}_{k=1}^{\infty}$ is a compact set because if you have any open
cover, one of the open sets contains $y$ and hence it contains all $(I-F) x_{k}$ except for finitely many which can then be covered by finitely many open sets in the open cover. Hence, since $(I-F)$ is proper, $(I-F)^{-1}(K)$ is compact. It follows that there is a subsequence, still called $x_{k}$ such that $x_{k} \rightarrow x \in(I-F)^{-1}(K)$. Then by continuity of $F$,

$$
\begin{aligned}
& (I-F)\left(x_{k}\right) \rightarrow(I-F)(x) \\
& (I-F)\left(x_{k}\right) \rightarrow y
\end{aligned}
$$

It follows $y=(I-F) x$ and so in fact $(I-F)(\partial \Omega)$ is closed.
Lemma 23.9.4 If $F: \bar{\Omega} \rightarrow X$ is compact and $\Omega$ is a bounded open set in $X$, then

$$
(I-F)(\partial \Omega)
$$

is closed.

## Justification for definition of Leray Schauder Degree

Now let $y \notin(I-F)(\partial \Omega)$, a closed set. Hence dist $(y,(I-F)(\partial \Omega))>4 \delta>0$. Now let $F_{k}$ be a sequence of approximations to $F$ which have values in an increasing sequence of finite dimensional subsets $V_{k}$ each of which contains $y$. Thus

$$
\lim _{k \rightarrow \infty}
$$

$\sup _{x \in \bar{\Omega}}\left\|F(x)-F_{k}(x)\right\|=0$. Consider

$$
d\left(I-\left.F_{k}\right|_{V_{k}}, \Omega \cap V_{k}, y\right)
$$

Each of these is a well defined integer according to Theorem 23.8.3. For all $k$ large enough,

$$
\sup _{x \in \bar{\Omega}}\left\|(I-F)(x)-\left(I-F_{k}\right)(x)\right\|<\delta
$$

Hence, for all such $k$,

$$
\begin{equation*}
B(y, 3 \delta) \cap\left(I-F_{k}\right)(\partial \Omega)=\emptyset, \text { that is } \operatorname{dist}\left(y,\left(I-F_{k}\right)(\partial \Omega)\right)>3 \delta \tag{23.9.18}
\end{equation*}
$$

Note that this implies

$$
\operatorname{dist}\left(y,\left(I-F_{k}\right)(\partial(\Omega \cap V))\right)>3 \delta
$$

for any subspace $V$. If $k<l$ are two such indices, then consider

$$
d\left(I-F_{k} \mid V_{k}, \Omega \cap V_{k}, y\right), d\left(I-\left.F_{l}\right|_{V_{l}}, \Omega \cap V_{l}, y\right)
$$

Are they equal? Let $V=V_{k}+V_{l}$. Then by Theorem 23.8.4,

$$
\begin{aligned}
& d\left(I-\left.F_{l}\right|_{V_{l}}, \Omega \cap V_{l}, y\right)=d\left(I-\left.F_{l}\right|_{V}, \Omega \cap V, y\right) \\
& d\left(I-\left.F_{k}\right|_{V_{k}}, \Omega \cap V_{k}, y\right)=d\left(I-\left.F_{k}\right|_{V}, \Omega \cap V, y\right)
\end{aligned}
$$

So what about $d\left(I-\left.F_{l}\right|_{V}, \Omega \cap V, y\right), d\left(I-\left.F_{k}\right|_{V}, \Omega \cap V, y\right)$ ? Are these equal?

$$
\sup _{x \in \overline{\Omega \cap V}}\left\|F_{l}(x)-F_{k}(x)\right\| \leq \sup _{x \in \overline{\Omega \cap V}}\left\|F_{l}(x)-F(x)\right\|+\sup _{x \in \overline{\Omega \cap V}}\left\|F(x)-F_{k}(x)\right\|<2 \delta
$$

This implies for

$$
h(x, t)=t\left(I-F_{l}\right)(x)+(1-t)\left(I-F_{k}\right)(x)
$$

and $x \in \overline{\Omega \cap V}, y \notin h(\partial(\Omega \cap V), t)$ for all $t \in[0,1]$. To see this, let $x \in \partial \Omega$

$$
\begin{aligned}
& \left\|t\left(I-F_{l}\right)(x)+(1-t)\left(I-F_{k}\right)(x)-y\right\| \\
= & \left\|t\left(I-F_{k}\right)(x)+t\left(F_{k} x-F_{l} x\right)+(1-t)\left(I-F_{k}\right)(x)-y\right\| \\
= & \left\|\left(I-F_{k}\right)(x)+t\left(F_{k} x-F_{l} x\right)\right\| \geq 3 \delta-t 2 \delta \geq \delta
\end{aligned}
$$

Hence

$$
d\left(I-\left.F_{l}\right|_{V}, \Omega \cap V, y\right)=d\left(I-\left.F_{k}\right|_{V}, \Omega \cap V, y\right)
$$

and so

$$
\lim _{k \rightarrow \infty} d\left(I-\left.F_{k}\right|_{V_{k}}, \Omega \cap V_{k}, y\right)
$$

exists. A similar argument shows that this limit is independent of the sequence $\left\{F_{k}\right\}$ of approximating functions having values in a finite dimensional space. Thus we have the following definition of the Leray Schauder degree.

Definition 23.9.5 Let $X$ be a Banach space and let $F: X \rightarrow X$ be compact. That is, $F(\Omega)$ is precompact whenever $\Omega$ is bounded. Let $\Omega$ be a bounded open set in $X$ and let $y \notin$ $(I-F)(\partial \Omega)$. Let $F_{k}$ be a sequence of operators which have values in finite dimensional spaces $V_{k}$ such that $V_{k} \subseteq V_{k+1} \cdots, y \in V_{k}$, and $\lim _{k \rightarrow \infty} \sup _{x \in \bar{\Omega}}\left\|F(x)-F_{k}(x)\right\|=0$. Then

$$
D(I-F, \Omega, y) \equiv \lim _{k \rightarrow \infty} d\left(I-\left.F_{k}\right|_{V_{k}}, \Omega \cap V_{k}, y\right)
$$

In fact, the sequence on the right is eventually constant. So

$$
D(I-F, \Omega, y) \equiv d\left(I-\left.F_{k}\right|_{V_{k}}, \Omega \cap V_{k}, y\right)
$$

for all $k$ sufficiently large.
The main properties of the Leray Schauder degree follow from the corresponding properties of Brouwer degree.

Theorem 23.9.6 Let $D$ be the Leray Schauder degree just defined and let $\Omega$ be a bounded open set $y \notin(I-F)(\partial \Omega)$ where $F$ is always a compact mapping. Then the following properties hold:

1. $D(I, \Omega, y)=1$
2. If $\Omega_{i} \subseteq \Omega$ where $\Omega_{i}$ is open, $\Omega_{1} \cap \Omega_{2}=\emptyset$, and $y \notin \bar{\Omega} \backslash\left(\Omega_{1} \cup \Omega_{2}\right)$ then

$$
D(I-F, \Omega, y)=D\left(I-F, \Omega_{1}, y\right)+D\left(I-F, \Omega_{2}, y\right)
$$

3. If $t \rightarrow y(t)$ is continuous $h: \bar{\Omega} \times[0,1] \rightarrow X$ is continuous, $(x, t) \rightarrow h(x, t)$ is compact, (It takes bounded subsets of $\bar{\Omega} \times[0,1]$ to precompact sets in $X$ ) and if $y(t) \notin$ $(I-h)(\partial \Omega, t)$ for all $t$, then $t \rightarrow D((I-h)(\cdot, t), \Omega, y(t))$ is constant.

Proof: The mapping $x \rightarrow 0$ is clearly compact. Then an approximating sequence is $F_{k}, F_{k} x=0$ for all $k$. Then

$$
D(I, \Omega, y)=\lim _{k \rightarrow \infty} d\left(\left.I\right|_{V_{k}}, \Omega \cap V_{k}, y\right)=1
$$

For the second part, let $k$ be large enough that for $U=\Omega, \Omega_{1}, \Omega_{2}$,

$$
D(I-F, U, y)=d\left(I-F_{k} \mid V_{k}, U \cap V_{k}, y\right)
$$

where $F_{k}$ is the sequence of approximating functions having finite dimensional range. Then the result follows from the Brouwer degree. In fact,

$$
\begin{aligned}
D(I-F, \Omega, y) & =d\left(I-F_{k} \mid V_{k}, \Omega \cap V_{k}, y\right) \\
& =d\left(I-F_{k} \mid V_{k}, \Omega_{1} \cap V_{k}, y\right)+d\left(I-F_{k} \mid V_{k}, \Omega_{2} \cap V_{k}, y\right) \\
& =D\left(I-F, \Omega_{1}, y\right)+D\left(I-F, \Omega_{2}, y\right)
\end{aligned}
$$

this does the second claim of the theorem. Now consider the third one about homotopy invariance.

Claim: If $\operatorname{dist}(y,(I-F) \partial \Omega) \geq 6 \delta$, and if $\|y-z\|<\delta$, then

$$
D(I-F, \Omega, y)=D(I-F, \Omega, z)
$$

Proof of claim: Let $F_{k}$ be the approximations and include both $y, z$ in all the finite dimensional subspaces $V_{k}$. Then for $k$ large enough, $\sup _{x \in \bar{\Omega}}\left\|F(x)-F_{k}(x)\right\|<\delta$ and also,

$$
\begin{aligned}
& D(I-F, \Omega, y)=d\left(\left.\left(I-F_{k}\right)\right|_{V_{k}}, \Omega \cap V_{k}, y\right) \\
& D(I-F, \Omega, z)=d\left(\left.\left(I-F_{k}\right)\right|_{V_{k}}, \Omega \cap V_{k}, z\right)
\end{aligned}
$$

Now for $x \in \partial\left(\Omega \cap V_{k}\right)$,

$$
\begin{aligned}
\left\|\left(I-F_{k}\right)(x)-y\right\| & \geq\left\|(I-F)(x)+\left(F(x)-F_{k}(x)\right)-y\right\| \\
& \geq\|(I-F)(x)-y\|-\left\|F(x)-F_{k}(x)\right\| \\
& >6 \delta-\delta=5 \delta
\end{aligned}
$$

Hence dist $\left(y,\left(I-F_{k}\right) \partial \Omega\right) \geq 5 \delta$ while $\|y-z\|<\delta$. Hence

$$
d\left(\left.\left(I-F_{k}\right)\right|_{V_{k}}, \Omega \cap V_{k}, y\right)=d\left(\left.\left(I-F_{k}\right)\right|_{V_{k}}, \Omega \cap V_{k}, z\right)
$$

by Theorem 23.2.2.
From compactness of $h$, there is an $\varepsilon$ net for $h(\bar{\Omega} \times[0,1]),\left\{h\left(x_{k}, t_{k}\right)\right\}$ such that

$$
h(\bar{\Omega} \times[0, T]) \subseteq \cup_{k=1}^{n} B\left(h\left(x_{k}, t_{k}\right), \varepsilon\right) .
$$

Say the $t_{k}$ are ordered. Then, as before,

$$
\begin{aligned}
& \phi_{k}(x) \equiv\left(\varepsilon-\left\|h(x, t)-h\left(x_{k}, t_{k}\right)\right\|\right)^{+} \\
& h_{\varepsilon}(x, t) \equiv \sum_{k=1}^{n} h\left(x_{k}, t_{k}\right) \frac{\phi_{k}(h(x, t))}{\sum_{i} \phi_{i}(h(x, t))}
\end{aligned}
$$

Then this is clearly continuous and has values in $\operatorname{span}\left(\left\{h\left(x_{k}, t_{k}\right)\right\}_{k=1}^{n}\right)$. How well does it approximate? Say $h(x, t) \in h(\bar{\Omega} \times[0, T])$. Then it is in some

$$
B\left(h\left(x_{k}, t_{k}\right), \varepsilon\right),
$$

maybe several. Thus letting $\mathscr{K}(x, t)$ be those indices $k$ such that

$$
\begin{aligned}
& h(x, t) \in B\left(h\left(x_{k}, t_{k}\right), \varepsilon\right) \\
\left\|h_{\varepsilon}(x, t)-h(x, t)\right\| & \leq \sum_{k \in \mathscr{K}(x, t)}\left\|h\left(x_{k}, t_{k}\right)-h(x, t)\right\| \frac{\phi_{k}(h(x, t))}{\sum_{i} \phi_{i}(h(x, t))} \\
& \leq \varepsilon \sum_{k=1}^{n} \frac{\phi_{k}(h(x, t))}{\sum_{i} \phi_{i}(h(x, t))}=\varepsilon
\end{aligned}
$$

Now here is a claim.
Claim: There exists $\delta>0$ such that for all $t \in[0,1]$,

$$
\operatorname{dist}(y(t),(I-h)(\partial \Omega, t))>6 \delta
$$

Proof of claim: If not, there is $\left(x_{n}, t_{n}\right) \in \partial \Omega \times[0,1]$ such that

$$
\left\|y\left(t_{n}\right)-(I-h)\left(x_{n}, t_{n}\right)\right\|<1 / n
$$

Then $h\left(x_{n}, t_{n}\right)$ is in a compact set because of compactness of $h$. Also, the $y\left(t_{n}\right)$ are in a compact set because $y$ is continuous and $y([0, T])$ must therefore be compact. It follows that $\left(x_{n}, t_{n}\right)$ must be in a compact subset of $\partial \Omega \times[0,1]$. It follows there is a subsequence, still denoted as $\left(x_{n}, t_{n}\right)$ which converges to $(x, t)$ in $\partial \Omega \times[0,1]$. then by continuity, $\|y(t)-(I-h)(x, t)\|=0$ contrary to assumption. This proves the claim.

As with $h$ there exists a sequence $\left\{y_{k}(t)\right\}$ such that $y_{k}(t) \rightarrow y(t)$ uniformly in $t \in[0,1]$ but $y_{k}$ has values in a finite dimensional subspace of $X, Y_{k}$. Choose $k_{0}$ large enough that for all $t \in[0,1],\left\|y(t)-y_{k_{0}}(t)\right\|<\delta$. Thus by the first claim,

$$
D(h(\cdot, t), \Omega, y(t))=D\left(h(\cdot, t), \Omega, y_{k_{0}}(t)\right)
$$

for all $t$. Also,

$$
\operatorname{dist}\left(y_{0}(t),(I-h)(\partial \Omega, t)\right)>5 \delta
$$

From the above, let $h_{k} \rightarrow h$ uniformly on $\bar{\Omega} \times[0,1]$ but $h_{k}$ has values in a finite dimensional subspace $V_{k}$. Let all the $V_{k}$ contain the values of $y_{k_{0}}$ and so, for all $k$ large enough,

$$
\sup _{\bar{\Omega} \times[0, T]}\left\|h(t, x)-h_{k}(t, x)\right\|<\delta
$$

so for such $k$,

$$
\begin{aligned}
& \operatorname{dist}\left(y_{0}(t),\left(I-h_{k}\right)\left(\partial\left(\Omega \cap V_{k}\right), t\right)\right) \\
\geq \quad & \operatorname{dist}\left(y_{0}(t),\left(I-h_{k}\right)(\partial \Omega, t)\right)>4 \delta
\end{aligned}
$$

Then

$$
\begin{aligned}
D(h(\cdot, t), \Omega, y(t)) & =D\left(h(\cdot, t), \Omega, y_{k_{0}}(t)\right) \\
& =\lim _{k \rightarrow \infty} d\left(\left.h_{k}(\cdot, t)\right|_{\overline{\Omega \cap V_{k}}}, \Omega \cap V_{k}, y_{k_{0}}(t)\right)
\end{aligned}
$$

and $d\left(\left.h_{k}(\cdot, t)\right|_{\overline{\Omega \cap V_{k}}}, \Omega \cap V_{k}, y_{k_{0}}(t)\right)$ is constant in $t$ for all large enough $k$. Thus

$$
D(h(\cdot, t), \Omega, y(t))=\lim _{k \rightarrow \infty} a_{k}, a_{k} \text { independent of } t .
$$

One of the nice results which follows right away from this is the Schauder fixed point theorem.

Theorem 23.9.7 Let $B=\overline{B(0, r)}$ and let $F: B \rightarrow B$ be compact. Then $F$ has a fixed point.
Proof: Suppose it does not. Then consider $D(I-t F, B(0, r), 0)$. If $t=1$, then $0 \notin$ $(I-t F)(\partial B)$ since otherwise, there would be a fixed point. If $t<1$ there is no point of $\partial B$ which $I-t F$ sends to 0 because if so,

$$
x-t F x=0,\|x\|=1,\|F x\| \leq 1
$$

Therefore, by homotopy invariance, $t \rightarrow D(I-t F, B(0, r), 0)$ is constant for $t \in[0,1]$. It must equal

$$
D(I-F, B(0, r), 0)=D(I, B(0, r), 0)=1 .
$$

Therefore, there exists $x \in B(0, r)$ such that $(I-F)(x)=0$ so $F$ which means $F$ has a fixed point after all.

One can get an improved version of this easily.
Theorem 23.9.8 Let $K$ be a closed bounded convex subset of a Banach space $X$ and suppose $F: K \rightarrow K$ is compact. Then $F$ has a fixed point.

Proof: By Theorem $16.2 .5, K$ is a retract. Thus there is a continuous function $R: X \rightarrow K$ which leaves points of $K$ unchanged. Then you consider $F \circ R$. It is still a compact mapping obviously. Let $B(0, r)$ be so large that it contains $K$. Then from the above theorem, it has a fixed point in $\overline{B(0, r)}$ denoted as $x$. Then $F(R(x))=x$. But $F(R(x)) \in K$ and so $x \in K$. Hence $R x=x$ and so $F x=x$.

There is an easy modification of the above which is often useful. If $F: X \rightarrow F(X)$ where $F(X)$ is bounded and in a compact set, and $F$ is a compact map, then you could consider $F$ : $\overline{\operatorname{conv}(F(X))} \rightarrow F(X) \subseteq \overline{\operatorname{conv}(F(X))}$ where here $\overline{\operatorname{conv}(F(X))}$ is a closed bounded convex subset of $X$. Then by the Schauder theorem, there is a fixed point for $F$.

Here is an easy application of this theorem to ordinary differential equations.

Theorem 23.9.9 Let $g:[0, T] \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be continuous. Let

$$
F: C\left([0, T] ; \mathbb{R}^{n}\right) \rightarrow C\left([0, T] ; \mathbb{R}^{n}\right)
$$

be given by

$$
F(y)(t)=y_{0}+\int_{0}^{t} g(s, y(s)) d s
$$

Suppose that whenever

$$
y(s)=F(y)(s), \text { for } s \leq t
$$

it follows that $\max _{s \in[0, t]}|y(s)|<M,\left|y_{0}\right|<M$. Then there exists a solution to the integral equation

$$
y(t)=y_{0}+\int_{0}^{t} g(s, y(s)) d s
$$

for $t \in[0, T]$.
Proof: Let $r_{M}$ be the radial projection in $\mathbb{R}^{n}$ onto $\overline{B(0, M)}$. Then $F \circ r_{M}$ is compact because $\left|g\left(s, r_{M} y\right)\right|$ is bounded. It also maps into a compact subset of $C\left([0, T] ; \mathbb{R}^{n}\right)$ thanks to the Arzela Ascoli theorem. Then by the Schauder fixed point theorem, there exists a solution $y=F \circ r_{M}$ to

$$
y(t)=y_{0}+\int_{0}^{t} g\left(s, r_{M} y(s)\right) d s
$$

Then for $s \in[0, \hat{T}]$ where $\hat{T}$ is the largest such that $\|y(s)\| \leq M$ for $s \in[0, \hat{T}]$. Thus on $[0, \hat{T}], r_{M}$ has no effect. If $\hat{T}<T$, then by the estimate, $|y(\hat{T})|<M$. Hence $\hat{T}$ is not really the last. Thus $\hat{T}=T$.

The Schauder alternative or Schaefer fixed point theorem is as follows [38].
Theorem 23.9.10 Let $f: X \rightarrow X$ be a compact map. Then either

1. There is a fixed point for $t f$ for all $t \in[0,1]$ or
2. For every $r>0$, there exists a solution to $x=t f(x)$ for $t \in(0,1)$ such that $\|x\|>r$.

Proof: Suppose there is $t_{0} \in[0,1]$ such that $t_{0} f$ has no fixed point. Then $t_{0} \neq 0 . t_{0} f$ obviously has a fixed point if $t_{0}=0$. Thus $t_{0} \in(0,1]$. Then let $r_{M}$ be the radial retraction onto $\overline{B(0, M)}$. By Schauder's theorem there exists $x \in \overline{B(0, M)}$ such that $t_{0} r_{M} f(x)=x$. Then if $\|f(x)\| \leq M, r_{M}$ has no effect and so $t_{0} f(x)=x$ which is assumed not to take place. Hence $\|f(x)\|>M$ and so $\left\|r_{M} f(x)\right\|=M$ so $\|x\|=t_{0} M$. Also $t_{0} r_{M} f(x)=t_{0} M \frac{f(x)}{\|f(x)\|}=$ $x$ and so $x=\hat{t} f(x), \hat{t}=t_{0} \frac{M}{\|f(x)\|}<1$. Since $M$ is arbitrary, it follows that the solutions to $x=t f(x)$ for $t \in(0,1)$ are unbounded. It was just shown that there is a solution to $x=\hat{t} f(x), \hat{t}<1$ such that $\|x\|=t_{0} M$ where $M$ is arbitrary. Thus the second of the two alternatives holds.

There is a lot more on degree theory in [43]. Here is a very interesting theorem from this reference which pertains specifically to infinite dimensional spaces.

Theorem 23.9.11 Let $X$ be an infinite dimensional Banach space and let $0 \notin \partial \Omega$ where $\Omega$ is an open bounded subset of $X$. Let $F: \bar{\Omega} \rightarrow X$ be compact. Suppose that $F x \neq \lambda x$ for all $x \in \partial \Omega$ and that $0 \notin \overline{F(\partial \Omega)}$. Then $D(I-F, \Omega, 0)=0$.

Proof: Recall that $D(I-F, \Omega, 0) \equiv \lim _{k \rightarrow \infty} d\left(I-F_{k}, \Omega \cap V_{k}, 0\right)$ where $F_{k}$ has values in a finite dimensional subspace $V_{k}$,

$$
\sup _{x \in \bar{\Omega}}\left\|F_{k}(x)-F(x)\right\|<1 / k
$$

Since the dimension of $X$ is infinite, it can always be assumed that $\operatorname{span}\left(-F_{k}(\bar{\Omega})\right)$ is a proper subspace of $V_{k}$ and this will be assumed. This is where it is significant that the dimension of $X$ is infinite. Also recall that in the limit, eventually $d\left(I-F_{k}, \Omega \cap V_{k}, 0\right)$ is a constant. Then the fact that $F x \neq \lambda x$ for all $x \in \partial \Omega$ will persist for $F_{k}$ for all $k$ large enough.

If not, then there exists $x_{k} \in \partial \Omega, F_{k} x_{k}=\lambda_{k} x_{k}$ for some $x_{k} \in \partial \Omega$ and $\lambda_{k} \in[0,1]$. Then there are subsequence $\lambda_{k} \rightarrow \lambda_{0} \in[0,1]$. Then

$$
F x_{k}-\lambda_{k} x_{k} \rightarrow 0
$$

because it is uniformly close to $F_{k} x_{k}-\lambda_{k} x_{k}$. Now by assumption $0 \notin \overline{F(\partial \Omega)}$. If $\lambda_{0}=0$, then you would have $F x_{k} \rightarrow 0$ which does not happen because 0 is at a positive distance from $\overline{F(\partial \Omega)}$. Hence for all $k$ large enough,

$$
F_{k} x \neq \lambda x
$$

for all $\lambda \in[0,1]$. Pick $k$ sufficiently large that in the limit for the Leray Schauder degree $d\left(I-F_{k}, \Omega \cap V_{k}, 0\right)$ remains constant. Then for $\lambda \in[0,1]$,

$$
d\left(\lambda I-F_{k}, \Omega \cap V_{k}, 0\right)=d\left(-F_{k}, \Omega \cap V_{k}, 0\right)=d\left(-F_{k}, \Omega \cap V_{k}, p\right)
$$

for all $p \notin \operatorname{span}\left(-F_{k}(\bar{\Omega})\right)$ which is also close enough to 0 . Hence, since the degree of this last equals 0 for such $p$, it follows that

$$
d\left(I-F_{k}, \Omega \cap V_{k}, 0\right)=0
$$

Hence $D(I-F, \Omega, 0)=0$ as claimed.
This theorem implies a very strange fixed point theorem. It is strange because it only applies to infinite dimensions.

Corollary 23.9.12 Let $X$ be an infinite dimensional Banach space. Let $0 \in \Omega_{0} \subseteq \Omega$ be two open sets. Let $F: \bar{\Omega} \rightarrow X$ be a compact mapping which satisfies

1. $\|F x\| \leq\|x\|$ for $x \in \partial \Omega_{0}$
2. $\|F x\| \geq\|x\|$ for $x \in \partial \Omega$

Then $F$ has a fixed point in $\overline{\Omega \backslash \Omega_{0}}$.

Proof: First note that $\overline{\Omega \backslash \Omega_{0}}$ is like an annulus with both edges included. Suppose $F$ does not have a fixed point in $\overline{\Omega \backslash \Omega_{0}}$. What if $t=1$ and $x \in \partial \Omega$ ? Could $0=(I-F)(x)$ ? If so, the fixed point is obtained so assume this is not so. Then for $t<1$ and $x \in \partial \Omega$, if you have $x=t^{-1} F x$, this would mean that $\|F x\|=t\|x\|$ and $\|F x\|<\|x\|$ which is assumed not to happen. Hence for $t \in(0,1]$ we can assume that $0 \notin\left(I-t^{-1} F\right)(\partial \Omega)$. If $(I-F)(x)=0$ for $x \notin \partial \Omega_{0}$ then the fixed point has been found. For $t \in[0,1)$, you can't have $(I-t F)(x)=$ 0 for $x \in \partial \Omega_{0}$ because then you would have $\|x\|=t\|F x\|$ and so $\|F x\|>\|x\|$ which is assumed not to happen. Therefore, we can assume that for $x \in \partial \Omega_{0},(I-t F)(x) \neq 0$. Therefore, $D\left(I-F, \Omega_{0}, 0\right)=D\left(I, \Omega_{0}, 0\right)=1$ by homotopy invariance. Also from properties of the degree,

$$
D\left(I-F, \Omega_{0}, 0\right)+D\left(I-F, \Omega \backslash \overline{\Omega_{0}}, 0\right)=D(I-F, \Omega, 0)
$$

Recall that this is true if $0 \notin(I-F)\left(\bar{\Omega}-\Omega_{0}\right)$ which is assumed to take place when we assume there is no fixed point. It is desired to use the above theorem so we need to consider $\overline{F(\partial \Omega)}$ and whether 0 is in this set. Condition 2 implies 0 is not in this set. Then Theorem 23.9.11 implies that $D(I-F, \Omega, 0)=0$ and so $D\left(I-F, \Omega \backslash \overline{\Omega_{0}}, 0\right)=-1$. Hence there is a fixed point in $\overline{\Omega \backslash \Omega_{0}}$ after all contrary to the assumption that there was no such thing.

This only works in infinite dimensions. Consider an annulus in $\mathbb{R}^{2}$ and let $F$ be a rotation through an angle of 30 degrees. It clearly has no fixed point but the above conditions are satisfied. This seems very interesting, something which happens in infinite dimensions but not in finite dimensions.

### 23.10 Exercises

1. Show the Brouwer fixed point theorem is equivalent to the nonexistence of a continuous retraction onto the boundary of $B(\mathbf{0}, r)$.
2. Using the Jordan separation theorem, prove the invariance of domain theorem $n \geq 2$. Thus an open ball goes to some open. Hint: You might consider $B(\mathbf{x}, r)$ and show $\mathbf{f}$ maps the inside to one of two components of $\mathbb{R}^{n} \backslash \mathbf{f}(\partial B(\mathbf{x}, r))$. etc.
3. Give a version of Proposition 23.6.4 which is valid for the case where $n=1$.
4. It was shown that if $\mathbf{f}$ is locally one to one and continuous, $\mathbf{f}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$, and

$$
\lim _{|\mathbf{x}| \rightarrow \infty}|\mathbf{f}(\mathbf{x})|=\infty
$$

then $\mathbf{f}$ maps $\mathbb{R}^{n}$ onto $\mathbb{R}^{n}$. Suppose you have $\mathbf{f}: \mathbb{R}^{m} \rightarrow \mathbb{R}^{n}$ where $\mathbf{f}$ is one to one and $\lim _{|\mathbf{x}| \rightarrow \infty}|\mathbf{f}(\mathbf{x})|=\infty$. Show that $\mathbf{f}$ cannot be onto.
5. Can there exists a one to one onto continuous map, $\mathbf{f}$ which takes the unit interval $[0,1]$ to the unit disk $\overline{B(\mathbf{0}, 1)}$ ? Hint: Think in terms of invariance of domain.
6. Let $m<n$ and let $B_{m}(\mathbf{0}, r)$ be the ball in $\mathbb{R}^{m}$ and $B_{n}(\mathbf{0}, r)$ be the ball in $\mathbb{R}^{n}$. Show that there is no one to one continuous map from $\overline{B_{m}(\mathbf{0}, r)}$ to $\overline{B_{n}(\mathbf{0}, r)}$. Hint: It is like the above problem.
7. Consider the unit disk,

$$
\left\{(x, y): x^{2}+y^{2} \leq 1\right\} \equiv D
$$

and the annulus

$$
\left\{(x, y): \frac{1}{2} \leq x^{2}+y^{2} \leq 1\right\} \equiv A
$$

Is it possible there exists a one to one onto continuous map $\mathbf{f}$ such that $\mathbf{f}(D)=A$ ? Thus $D$ has no holes and $A$ is really like $D$ but with one hole punched out. Can you generalize to different numbers of holes? Hint: Consider the invariance of domain theorem. The interior of $D$ would need to be mapped to the interior of $A$. Where do the points of the boundary of $A$ come from? Consider Theorem 7.13.3.
8. Suppose $C$ is a compact set in $\mathbb{R}^{n}$ which has empty interior and $\mathbf{f}: C \rightarrow \Gamma \subseteq \mathbb{R}^{n}$ is one to one onto and continuous with continuous inverse. Could $\Gamma$ have nonempty interior? Show also that if $\mathbf{f}$ is one to one and onto $\Gamma$ then if it is continuous, so is $\mathbf{f}^{-1}$.
9. Let $K$ be a nonempty closed and convex subset of $\mathbb{R}^{n}$. Recall $K$ is convex means that if $\mathbf{x}, \mathbf{y} \in K$, then for all $t \in[0,1], t \mathbf{x}+(1-t) \mathbf{y} \in K$. Show that if $\mathbf{x} \in \mathbb{R}^{n}$ there exists a unique $\mathbf{z} \in K$ such that

$$
|\mathbf{x}-\mathbf{z}|=\min \{|\mathbf{x}-\mathbf{y}|: \mathbf{y} \in K\} .
$$

This $\mathbf{z}$ will be denoted as $P \mathbf{x}$. Hint: First note you do not know $K$ is compact. Establish the parallelogram identity if you have not already done so,

$$
|\mathbf{u}-\mathbf{v}|^{2}+|\mathbf{u}+\mathbf{v}|^{2}=2|\mathbf{u}|^{2}+2|\mathbf{v}|^{2}
$$

Then let $\left\{\mathbf{z}_{k}\right\}$ be a minimizing sequence,

$$
\lim _{k \rightarrow \infty}\left|\mathbf{z}_{k}-\mathbf{x}\right|^{2}=\inf \{|\mathbf{x}-\mathbf{y}|: \mathbf{y} \in K\} \equiv \lambda
$$

Now using convexity, explain why

$$
\left|\frac{\mathbf{z}_{k}-\mathbf{z}_{m}}{2}\right|^{2}+\left|\mathbf{x}-\frac{\mathbf{z}_{k}+\mathbf{z}_{m}}{2}\right|^{2}=2\left|\frac{\mathbf{x}-\mathbf{z}_{k}}{2}\right|^{2}+2\left|\frac{\mathbf{x}-\mathbf{z}_{m}}{2}\right|^{2}
$$

and then use this to argue $\left\{\mathbf{z}_{k}\right\}$ is a Cauchy sequence. Then if $\mathbf{z}_{i}$ works for $i=1,2$, consider $\left(\mathbf{z}_{1}+\mathbf{z}_{2}\right) / 2$ to get a contradiction.
10. In Problem 9 show that $P \mathbf{x}$ satisfies the following variational inequality.

$$
(\mathbf{x}-P \mathbf{x}) \cdot(\mathbf{y}-P \mathbf{x}) \leq 0
$$

for all $\mathbf{y} \in K$. Then show that $\left|P \mathbf{x}_{1}-P \mathbf{x}_{2}\right| \leq\left|\mathbf{x}_{1}-\mathbf{x}_{2}\right|$. Hint: For the first part note that if $\mathbf{y} \in K$, the function $t \rightarrow|\mathbf{x}-(P \mathbf{x}+t(\mathbf{y}-P \mathbf{x}))|^{2}$ achieves its minimum on $[0,1]$ at $t=0$. For the second part,

$$
\left(\mathbf{x}_{1}-P \mathbf{x}_{1}\right) \cdot\left(P \mathbf{x}_{2}-P \mathbf{x}_{1}\right) \leq 0,\left(\mathbf{x}_{2}-P \mathbf{x}_{2}\right) \cdot\left(P \mathbf{x}_{1}-P \mathbf{x}_{2}\right) \leq 0
$$

Explain why

$$
\left(\mathbf{x}_{2}-P \mathbf{x}_{2}-\left(\mathbf{x}_{1}-P \mathbf{x}_{1}\right)\right) \cdot\left(P \mathbf{x}_{2}-P \mathbf{x}_{1}\right) \geq 0
$$

and then use a some manipulations and the Cauchy Schwarz inequality to get the desired inequality. Thus $P$ is called a retraction onto $K$.
11. Establish the Brouwer fixed point theorem for any convex compact set in $\mathbb{R}^{n}$. Hint: If $K$ is a compact and convex set, let $R$ be large enough that the closed ball, $D(\mathbf{0}, R) \supseteq K$. Let $P$ be the projection onto $K$ as in Problem 10 above. If $\mathbf{f}$ is a continuous map from $K$ to $K$, consider $\mathbf{f} \circ P$. You want to show $\mathbf{f}$ has a fixed point in $K$.
12. Suppose $D$ is a set which is homeomorphic to $\overline{B(\mathbf{0}, 1)}$. This means there exists a continuous one to one map, $\mathbf{h}$ such that $\mathbf{h}(\overline{B(\mathbf{0}, 1)})=D$ such that $\mathbf{h}^{-1}$ is also one to one. Show that if $\mathbf{f}$ is a continuous function which maps $D$ to $D$ then $\mathbf{f}$ has a fixed point. Now show that it suffices to say that $\mathbf{h}$ is one to one and continuous. In this case the continuity of $\mathbf{h}^{-1}$ is automatic. Sets which have the property that continuous functions taking the set to itself have at least one fixed point are said to have the fixed point property. Work Problem 7 using this notion of fixed point property. What about a solid ball and a donut? Could these be homeomorphic?
13. Suppose $\Omega$ is any open bounded subset of $\mathbb{R}^{n}$ which contains $\mathbf{0}$ and that $\mathbf{f}: \bar{\Omega} \rightarrow \mathbb{R}^{n}$ is continuous with the property that

$$
\mathbf{f}(\mathbf{x}) \cdot \mathbf{x} \geq 0
$$

for all $\mathbf{x} \in \partial \Omega$. Show that then there exists $\mathbf{x} \in \bar{\Omega}$ such that $\mathbf{f}(\mathbf{x})=\mathbf{0}$. Give a similar result in the case where the above inequality is replaced with $\leq$. Hint: You might consider the function

$$
\mathbf{h}(t, \mathbf{x}) \equiv t \mathbf{f}(\mathbf{x})+(1-t) \mathbf{x}
$$

14. Suppose $\Omega$ is an open set in $\mathbb{R}^{n}$ containing $\mathbf{0}$ and suppose that $\mathbf{f}: \bar{\Omega} \rightarrow \mathbb{R}^{n}$ is continuous and $|\mathbf{f}(\mathbf{x})| \leq|\mathbf{x}|$ for all $\mathbf{x} \in \partial \Omega$. Show $\mathbf{f}$ has a fixed point in $\bar{\Omega}$. Hint: Consider $\mathbf{h}(t, \mathbf{x}) \equiv t(\mathbf{x}-\mathbf{f}(\mathbf{x}))+(1-t) \mathbf{x}$ for $t \in[0,1]$. If $t=1$ and some $\mathbf{x} \in \partial \Omega$ is sent to $\mathbf{0}$, then you are done. Suppose therefore, that no fixed point exists on $\partial \Omega$. Consider $t<1$ and use the given inequality.
15. Let $\Omega$ be an open bounded subset of $\mathbb{R}^{n}$ and let $\mathbf{f}, \mathbf{g}: \bar{\Omega} \rightarrow \mathbb{R}^{n}$ both be continuous such that

$$
|\mathbf{f}(\mathbf{x})|-|\mathbf{g}(\mathbf{x})|>0
$$

for all $\mathbf{x} \in \partial \Omega$. Show that then

$$
d(\mathbf{f}-\mathbf{g}, \Omega, \mathbf{0})=d(\mathbf{f}, \Omega, \mathbf{0})
$$

Show that if there exists $\mathbf{x} \in \mathbf{f}^{-1}(\mathbf{0})$, then there exists $\mathbf{x} \in(\mathbf{f}-\mathbf{g})^{-1}(\mathbf{0})$. Hint: You might consider $\mathbf{h}(t, \mathbf{x}) \equiv(1-t) \mathbf{f}(\mathbf{x})+t(\mathbf{f}(\mathbf{x})-\mathbf{g}(\mathbf{x}))$ and argue $\mathbf{0} \notin \mathbf{h}(t, \partial \Omega)$ for $t \in[0,1]$.
16. Suppose $\mathbf{f}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ is continuous and satisfies

$$
|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})| \geq \alpha|\mathbf{x}-\mathbf{y}|, \alpha>0
$$

Show that $\mathbf{f}$ must map $\mathbb{R}^{n}$ onto $\mathbb{R}^{n}$. Hint: First show $\mathbf{f}$ is one to one. Then use invariance of domain. Next show, using the inequality, that the points not in $\mathbf{f}\left(\mathbb{R}^{n}\right)$ must form an open set because if $\mathbf{y}$ is such a point, then there can be no sequence $\left\{\mathbf{f}\left(\mathbf{x}_{n}\right)\right\}$ converging to it. Finally recall that $\mathbb{R}^{n}$ is connected.
It is obvious that $\mathbf{f}$ is one to one. This follows from the inequality. If $U$ are the points not in the image of $\mathbf{f}$, then $U$ must be open because if not, then for $\mathbf{y}$ one of these points, there would be a sequence $\mathbf{f}\left(\mathbf{x}_{n}\right) \rightarrow \mathbf{y}$. Then by the inequality, $\left\{\mathbf{x}_{n}\right\}$ is a Cauchy sequence and so it converges to $\mathbf{x}$. Thus $\mathbf{f}(\mathbf{x})=\lim _{n \rightarrow \infty} \mathbf{f}\left(\mathbf{x}_{n}\right)=\mathbf{y}$. Now by invariance of domain, $\mathbf{f}\left(\mathbb{R}^{n}\right)$ is open. However, $\mathbb{R}^{n}$ is connected and so in fact, $U$ is empty.
17. Let $f: \mathbb{C} \rightarrow \mathbb{C}$ where $\mathbb{C}$ is the field of complex numbers. Thus $f$ has a real and imaginary part. Letting $z=x+i y$,

$$
f(z)=u(x, y)+i v(x, y)
$$

Recall that the norm in $\mathbb{C}$ is given by $|x+i y|=\sqrt{x^{2}+y^{2}}$ and this is the usual norm in $\mathbb{R}^{2}$ for the ordered pair $(x, y)$. Thus complex valued functions defined on $\mathbb{C}$ can be considered as $\mathbb{R}^{2}$ valued functions defined on some subset of $\mathbb{R}^{2}$. Such a complex function is said to be analytic if the usual definition holds. That is

$$
f^{\prime}(z)=\lim _{h \rightarrow 0} \frac{f(z+h)-f(z)}{h}
$$

In other words,

$$
\begin{equation*}
f(z+h)=f(z)+f^{\prime}(z) h+o(h) \tag{23.10.19}
\end{equation*}
$$

at a point $z$ where the derivative exists. Let $f(z)=z^{n}$ where $n$ is a positive integer. Thus $z^{n}=p(x, y)+i q(x, y)$ for $p, q$ suitable polynomials in $x$ and $y$. Show this function is analytic. Next show that for an analytic function and $u$ and $v$ the real and imaginary parts, the Cauchy Riemann equations hold.

$$
u_{x}=v_{y}, u_{y}=-v_{x} .
$$

In terms of mappings show 23.10.19 has the form

$$
\begin{aligned}
& \binom{u\left(x+h_{1}, y+h_{2}\right)}{v\left(x+h_{1}, y+h_{2}\right)} \\
= & \binom{u(x, y)}{v(x, y)}+\left(\begin{array}{ll}
u_{x}(x, y) & u_{y}(x, y) \\
v_{x}(x, y) & v_{y}(x, y)
\end{array}\right)\binom{h_{1}}{h_{2}}+\mathbf{o}(\mathbf{h}) \\
= & \binom{u(x, y)}{v(x, y)}+\left(\begin{array}{cc}
u_{x}(x, y) & -v_{x}(x, y) \\
v_{x}(x, y) & u_{x}(x, y)
\end{array}\right)\binom{h_{1}}{h_{2}}+\mathbf{o}(\mathbf{h})
\end{aligned}
$$

where $\mathbf{h}=\left(h_{1}, h_{2}\right)^{T}$ and $h$ is given by $h_{1}+i h_{2}$. Thus the determinant of the above matrix is always nonnegative. Letting $B_{r}$ denote the ball $B(0, r)=B((0,0), r)$ show

$$
d\left(f, B_{r}, \mathbf{0}\right)=n .
$$

where $f(z)=z^{n}$. In terms of mappings on $\mathbb{R}^{2}$,

$$
\mathbf{f}(x, y)=\binom{u(x, y)}{v(x, y)}
$$

Thus show

$$
d\left(\mathbf{f}, B_{r}, \mathbf{0}\right)=n
$$

Hint: You might consider

$$
g(z) \equiv \prod_{j=1}^{n}\left(z-a_{j}\right)
$$

where the $a_{j}$ are small real distinct numbers and argue that both this function and $f$ are analytic but that $\mathbf{0}$ is a regular value for $\mathbf{g}$ although it is not so for $\mathbf{f}$. However, for each $a_{j}$ small but distinct $d\left(\mathbf{f}, B_{r}, \mathbf{0}\right)=d\left(\mathbf{g}, B_{r}, \mathbf{0}\right)$.
18. Using Problem 17, prove the fundamental theorem of algebra as follows. Let $p(z)$ be a nonconstant polynomial of degree $n$,

$$
p(z)=a_{n} z^{n}+a_{n-1} z^{n-1}+\cdots
$$

Show that for large enough $r,|p(z)|>\left|p(z)-a_{n} z^{n}\right|$ for all $z \in \partial B(0, r)$. Now from Problem 15 you can conclude $d\left(p, B_{r}, 0\right)=d\left(f, B_{r}, 0\right)=n$ where $f(z)=a_{n} z^{n}$.
19. The proof of Sard's lemma made use of the hard Vitali covering theorem. Here is another way to do something similar. Let $U$ be a bounded open set and let $\mathbf{f}: U \rightarrow \mathbb{R}^{n}$ be in $C^{1}(U)$. Let $S$ denote the set of $x \in U$ such that $D \mathbf{f}(x)$ has rank less than $n$. Thus it is a closed set. Let $U_{m}=\{\mathbf{x} \in U:\|D \mathbf{f}(\mathbf{x})\| \leq m\}$, a closed set. It suffices to show that for $S_{m} \equiv U_{m} \cap S, \mathbf{f}\left(S_{m}\right)$ has measure zero because $\mathbf{f}(S)=\cup_{m} \mathbf{f}\left(S_{m}\right)$ these sets increasing in $m$. By definition of differentiability,

$$
\lim _{k \rightarrow \infty} \sup _{\|\mathbf{v}\| \leq 1 / k} \frac{\|\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})-D \mathbf{f}(\mathbf{x}) \mathbf{v}\|}{\|\mathbf{v}\|}=0
$$

for each $\mathbf{x} \in U$. Explain why the above function of $\mathbf{x}$ is measurable. Now by Eggoroff's theorem, there is measurable set $A$ of measure less than $\frac{\varepsilon}{m^{n} 10^{n}}$ such that off $A$, the convergence is uniform. Let $\mathscr{C}_{k}$ be a countable union of non overlapping half open rectangles one of which is of the form $\prod_{i=1}^{n}\left(a_{i}, b_{i}\right]$ such that each has diameter less than $2^{-k}$. Consider the half open rectangles which have nonempty intersection with $S_{m} \backslash A, \mathscr{I}_{k}$. Then repeat the argument given in the first section of this chapter. Show that for $k$ large enough, the rank condition and uniform convergence above implies that $m_{n}\left(\cup\left\{\mathbf{f}(I): I \in \mathscr{I}_{k}\right\}\right)$ is less than $\varepsilon$. Now show that $\mathbf{f}(A)$ is contained in a set of measure no more than $m^{n} 10^{n} \frac{\varepsilon}{m^{n} 10^{n}}=2 \varepsilon$. Thus $\mathbf{f}\left(S_{m}\right)$ has measure no more than $3 \varepsilon$. Since $\varepsilon$ is arbitrary, this establishes the desired conclusion.
20. Let $X$ be a Banach space and let $\Omega$ be a symmetric and bounded open set. Let $F: \Omega \rightarrow X$ be odd and compact $0 \notin(I-F)(\partial \Omega)$. Show using Corollary 23.9.3 that $D(I-F, \Omega, 0)$ is an odd integer.
21. Let $F$ be compact. Suppose $I-F$ is one to one on $\overline{B(0, r)}$. Then using similar reasoning to the finite dimensional case, show that there is a $\delta>0$ such that

$$
(I-F)(0)+B(0, \delta) \subseteq(I-F)(B(0, r))
$$

22. Let $F$ be compact. Suppose $I-F$ is locally one to one on an open set $\Omega$. Show that $(I-F)$ maps open sets to open sets. This is a version of invariance of domain.
23. Suppose $(I-F)$ is locally one to one and $F$ is compact. Suppose also that

$$
\lim _{\|x\| \rightarrow \infty}\|(I-F) x\|=\infty
$$

Show that $(I-F)$ is onto.
24. As a variation of the above problem, suppose $F: X \rightarrow X$ is compact and

$$
\lim _{\|x\| \rightarrow \infty} \frac{\|F(x)\|}{\|x\|}=0
$$

Then $I-F$ is onto. Note that $I-F$ is not one to one.
25. Suppose $F$ is compact and $\|(I-F) x-(I-F) y\| \geq \alpha\|x-y\|$. Show that then $(I-F)$ is onto.
26. The Jordan curve theorem is: Let $C$ denote the unit circle,

$$
\left\{(x, y) \in \mathbb{R}^{2}: x^{2}+y^{2}=1\right\}
$$

Suppose $\gamma: C \rightarrow \Gamma \subseteq \mathbb{R}^{2}$ is one to one onto and continuous. Then $\mathbb{R}^{2} \backslash \Gamma$ consists of two components, a bounded component (called the inside) $U_{1}$ and an unbounded component (called the outside), $U_{2}$. Also the boundary of each of these two components of $\mathbb{R}^{2} \backslash \Gamma$ is $\Gamma$ and $\Gamma$ has empty interior. Using the Jordan separation theorem, prove this important result.
27. This problem is from [43] Recall Theorem 23.9.11. It allowed you to say that $D(I-F, \Omega, 0)=0$ provided $0 \notin \overline{F(\partial \Omega)}$ and $\lambda x \neq F x$ for all $x \in \partial \Omega, \lambda \in[0,1]$. This was for $F$ compact and defined on an infinite dimensional space $X$. Suppose now that $F$ is compact and $F: \bar{\Omega} \rightarrow X$ where $0 \in \Omega$ an open set in $X$. Suppose also that $F(0)=0$ and that

$$
\lim \inf _{x \rightarrow 0} \frac{\|F(x)\|}{\|x\|} \equiv \lim _{r \rightarrow 0+} \inf \left\{\frac{\|F(x)\|}{\|x\|}:\|x\| \leq r\right\}=\infty
$$

Show that there is a sequence $\alpha_{n} \rightarrow 0$ each $\alpha_{n} \neq 0$, and for some $x_{n} \neq 0, x_{n}-$ $\alpha_{n} F\left(x_{n}\right)=0$. Note that when $\alpha=0$, there is only one solution to $(I-\alpha F)(x)=0$,
but this says that there are many small $\alpha_{n} \neq 0$ for which there is a nonzero solution to $\left(I-\alpha_{n} F\right)(x)=0$. That is there exist arbitrarily small $\alpha_{n}$ such that $\left(I-\alpha_{n}\right) F\left(x_{n}\right)=$ 0 . This says that 0 is a bifurcation point for $I-\alpha F$. Hint: Let $\alpha_{n} \downarrow 0$ and pick $r_{n}$ such that for all $\|x\|=r_{n}$,

$$
\left\|\alpha_{n} F(x)\right\|>\|x\|
$$

Thus $0 \notin \overline{\alpha_{n} F\left(\partial B\left(0, r_{n}\right)\right)}$ and also $\alpha_{n} F(x) \neq \lambda x$ for all $x \in \partial B\left(0, r_{n}\right)$. Use the theorem to conclude that

$$
D\left(I-\alpha_{n} F, B\left(0, r_{n}\right), 0\right)=0
$$

and then consider the homotopy $I-\alpha_{n} t F$. If it sends no point of $\partial B\left(0, r_{n}\right)$ to 0 then you would have

$$
D\left(I-\alpha_{n} t F, B\left(0, r_{n}\right), 0\right)=D\left(I, B\left(0, r_{n}\right), 0\right)=1
$$

## Chapter 24

## Critical Points

### 24.1 Mountain Pass Theorem In Hilbert Space

This is from Evans [49]. It is an interesting theorem. See also [55] for more general versions. It has to do with differentiable functions defined on a Hilbert space $H$. Thus $I: H \rightarrow \mathbb{R}$ will be differentiable. Then the following is the Palais Smale condition.

Definition 24.1.1 A functional I satisfies the Palais Smale conditions means that if $\left\{I\left(u_{k}\right)\right\}$ is a bounded sequence and $I^{\prime}\left(u_{k}\right) \rightarrow 0$, then $\left\{u_{k}\right\}$ is precompact. That is, it has a subsequence which converges.

It will be assumed that $I$ is $C^{1}(H ; \mathbb{R})$ and also that $I^{\prime}$ is Lipschitz on bounded sets. By $I^{\prime}(u)$ is meant the element of $H$ such that

$$
I(u+v)=I(u)+\left(I^{\prime}(u), v\right)_{H}+o(v)
$$

Such exists because of the Riesz representation theorem. Note that, from the assumption that $I^{\prime}$ is Lipschitz continuous, it follows that $I^{\prime}$ is bounded on every bounded set.

First is a deformation theorem. The notation $[I(u) \in S]$ means $\{u: I(u) \in S\}$.
Theorem 24.1.2 Let I be $C^{1}$, I is non constant, satisfy the Palais Smale condition, and $I^{\prime}$ is Lipschitz continuous on bounded sets. Also suppose that $c \in \mathbb{R}$ is such that either $[I(u) \in[c-\delta, c+\delta]]=\emptyset$ for some $\delta>0$ or $[I(u) \in[c-\delta, c+\delta]] \neq \emptyset$ for all $\delta>0$ and IF $I(u)=c$, then $I^{\prime}(u) \neq 0$. Then for each sufficiently small $\varepsilon>0$, there is a constant $\delta \in(0, \varepsilon)$ and a function $\eta:[0,1] \times H \rightarrow H$ such that

1. $\eta(0, u)=u$
2. $\eta(1, u)=u$ on $[I(u) \notin(c-\varepsilon, c+\varepsilon)]$
3. $I(\eta(t, u)) \leq I(u)$
4. $\eta(1,[I(u) \leq c+\delta]) \subseteq[I(u) \leq c-\delta]$

The main part of this conclusion is the statement about $u \rightarrow \eta(1, u)$ contained in parts 2. and 4. The other two parts are there to facilitate these two although they are certainly interesting for their own sake.

Proof: Suppose $[I(u) \in[c-\delta, c+\delta]]=\emptyset$ for some $\delta>0$. Then $[I(u) \leq c+\delta / 2] \subseteq$ $[I(u) \leq c-\delta / 2]$ and you could take $\varepsilon=\delta$ and let $\eta(t, u)=u$. Therefore, assume

$$
[I(u) \in[c-\delta, c+\delta]] \neq \emptyset
$$

for all $\delta>0$. Since $I$ is nonconstant, $\varepsilon>0$ can be chosen small enough that

$$
[I(u) \notin(c-\varepsilon, c+\varepsilon)] \neq \emptyset
$$

Always let $\varepsilon$ be this small.

Claim 1: For all small enough $\varepsilon>0$, if $u \in[I(u) \in[c-\varepsilon, c+\varepsilon]], I^{\prime}(u) \neq 0$ and in fact, for such $\varepsilon$, there exists $\sigma(\varepsilon)>0$ such that $\sigma(\varepsilon)<\varepsilon,\left\|I^{\prime}(u)\right\|>\sigma(\varepsilon)$ for all $u \in$ $[I(u) \in[c-\varepsilon, c+\varepsilon]]$.

Proof of Claim 1: If claim is not so, then there is $\left\{u_{k}\right\}, \varepsilon_{k}, \sigma_{k} \rightarrow 0,\left\|I^{\prime}\left(u_{k}\right)\right\|<\sigma_{k}$, and $I\left(u_{k}\right) \in\left[c-\varepsilon_{k}, c+\varepsilon_{k}\right]$ but $\left\|I^{\prime}\left(u_{k}\right)\right\| \leq \sigma_{k}$. However, from the Palais Smale condition, there is a subsequence, still denoted as $u_{k}$ which converges to some $u$. Now $I\left(u_{k}\right) \in$ $\left[c-\varepsilon_{k}, c+\varepsilon_{k}\right]$ and so $I(u)=c$ while $I^{\prime}(u)=0$ contrary to the hypothesis. This proves Claim 1. From now on, $\varepsilon$ will be sufficiently small.

Now define for $\delta<\varepsilon$ (The description of small $\delta$ will be described later.)

$$
\begin{aligned}
A & \equiv[I(u) \notin(c-\varepsilon, c+\varepsilon)] \\
B & \equiv[I(u) \in[c-\delta, c+\delta]]
\end{aligned}
$$

Thus $A$ and $B$ are disjoint closed sets. Recall that it is assumed that $B \neq \emptyset$ since otherwise, there is nothing to prove. Also it is assumed throughout that $\varepsilon>0$ is such that $A \neq \emptyset$ thanks to $I$ not being constant. Thus these are nonempty sets and we do not have to fuss with worrying about meaning when one is empty.

Claim 2: For any $u, \operatorname{dist}(u, A)+\operatorname{dist}(u, B)>0$.
This is so because if not, then both would be zero and this requires that $u \in A \cap B$ since these sets are closed. But $A \cap B=\emptyset$.

Now define a function

$$
g(u) \equiv \frac{\operatorname{dist}(u, A)}{\operatorname{dist}(u, A)+\operatorname{dist}(u, B)}
$$

It is a continuous function of $u$ which has values in $[0,1]$. Consider the ordinary differential initial value problem

$$
\begin{gather*}
\eta^{\prime}(t, u)+g(u) h\left(\left\|I^{\prime}(\eta(t, u))\right\|\right) I^{\prime}(\eta(t, u))=0  \tag{24.1.1}\\
\eta(0, u)=u \tag{24.1.2}
\end{gather*}
$$

where $r \rightarrow h(r)$ is a decreasing function which has values in $(0,1]$ and equals 1 for $r \in[0,1]$ and equals $1 / r$ for $r>1$. Here $u$ is given and the $\eta^{\prime}$ is the time derivative is with respect to $t$. Thus, by assumption, the function

$$
\eta \rightarrow g(u) h\left(\left\|I^{\prime}(\eta)\right\|\right) I^{\prime}(\eta)
$$

is Lipschitz continuous on bounded sets and so there exists a solution to the above initial value problem valid for all $t \in[0,1]$. To see this, you can let $P$ be the projection map onto the closed ball of radius $M>\|u\|$ and the system

$$
\begin{gathered}
\eta^{\prime}(t, u)+g(u) h\left(\left\|I^{\prime}(P(\eta(t, u)))\right\|\right) I^{\prime}(P(\eta(t, u)))=0 \\
\eta(0, u)=u
\end{gathered}
$$

Then by Lipschitz continuity, there is a global solution for all $t \geq 0$. Hence there is a local solution to 24.1.1, 24.1.2. Note that

$$
\begin{aligned}
& \left\|g(u) h\left(\left\|I^{\prime}(P(\eta(t, u)))\right\|\right) I^{\prime}(P(\eta(t, u)))\right\| \\
= & g(u) h\left(\left\|I^{\prime}(P(\eta(t, u)))\right\|\right)\left\|I^{\prime}(P(\eta(t, u)))\right\| \leq 1
\end{aligned}
$$

Taking inner products with $\eta(t, u)$, and integrating $\int_{0}^{t}$ for this local solution,

$$
\begin{aligned}
& \frac{1}{2}\|\eta(t, u)\|^{2}-\frac{1}{2}\|u\|^{2}+\int_{0}^{t} g(u) h\left(\left\|I^{\prime}(P \eta(s, u))\right\|\right)\left(I^{\prime}(P \eta(s, u)), \eta(s, u)\right) d s=0 \\
& \frac{1}{2}\|\eta(t, u)\|^{2} \leq \frac{1}{2}\|u\|^{2}+\int_{0}^{t} g(u) h\left(\left\|I^{\prime}(P \eta(t, u))\right\|\right)\left\|I^{\prime}(P \eta(t, u))\right\|\|\eta(s, u)\| d s
\end{aligned}
$$

It follows that for $t \leq 1$,

$$
\begin{gathered}
\|\eta(t, u)\|^{2} \leq\|u\|^{2}+2 \int_{0}^{t}\|\eta(s, u)\| d s \\
\leq\|u\|^{2}+1+\int_{0}^{t}\|\eta(s, u)\|^{2} d s
\end{gathered}
$$

and so from Gronwall's inequality, for $t \leq 1$,

$$
\|\eta(t, u)\|^{2} \leq\left(\|u\|^{2}+1\right) e^{1}
$$

Thus we pick $M>e\left(\|u\|^{2}+1\right)$ and then we obtain that for $t \in[0,1]$, the projection map does not change anything. Hence there exists a solution to 24.1.1, 24.1.2 on $[0,1]$ as desired.

Then for this solution, $\eta(0, u)=u$ because of the above initial condition. If $u \in$ $[I(u) \notin[c-\varepsilon, c+\varepsilon]]$, then $u \in A$ and so $g(u)=0$ so $\eta(t, u)=u$ for all $t \in[0,1]$. This gives the first two conditions. Consider the third.

$$
\begin{aligned}
\frac{d}{d t}(I(\eta(t, u))) & =\left(I^{\prime}(\eta), \eta^{\prime}\right)=-\left(I^{\prime}(\eta), g(u) h\left(\left\|I^{\prime}(\eta)\right\|\right) I^{\prime}(\eta)\right) \\
& =-g(u) h\left(\left\|I^{\prime}(\eta)\right\|\right)\left\|I^{\prime}(\eta)\right\|^{2}
\end{aligned}
$$

and so this implies the third condition since it says that the function $t \rightarrow I(\eta(t, u))$ is decreasing.

It remains to consider the last condition. This involves choosing $\delta$ still smaller if necessary. It is desired to verify that

$$
\eta(1,[I(u) \leq c+\delta]) \subseteq[I(u) \leq c-\delta]
$$

Suppose it is not so. Then there exists $u \in[I(u) \leq c+\delta]$ but $I(\eta(1, u))>c-\delta$.

$$
\begin{aligned}
c-\delta & <I(\eta(1, u))=I(u)-\int_{0}^{1}\left(I^{\prime}(\eta), g(u) h\left(\left\|I^{\prime}(\eta)\right\|\right) I^{\prime}(\eta)\right) d t \\
& =I(u)-g(u) \int_{0}^{1} h\left(\left\|I^{\prime}(\eta)\right\|\right)\left\|I^{\prime}(\eta(t, u))\right\|^{2} d t \\
& <c+\delta-g(u) \int_{0}^{1} h\left(\left\|I^{\prime}(\eta)\right\|\right)\left\|I^{\prime}(\eta(t, u))\right\|^{2} d t
\end{aligned}
$$

Then

$$
c-2 \delta+g(u) \int_{0}^{1} h\left(\left\|I^{\prime}(\eta(t, u))\right\|\right)\left\|I^{\prime}(\eta(t, u))\right\|^{2} d t<c
$$

If $I(u) \leq c-\delta$, there is nothing to show because in this case $I(\eta(1, u)) \leq I(u) \leq c-\delta$. Hence we can assume that $I(u)>c-\delta$ and also that $I(u) \leq c+\delta$. Thus $u \in B$ and so $g(u)=1$. Thus

$$
c-2 \delta+\int_{0}^{1} h\left(\left\|I^{\prime}(\eta(t, u))\right\|\right)\left\|I^{\prime}(\eta(t, u))\right\|^{2} d t<c
$$

Also, it is being assumed that $I(\eta(1, u))>c-\delta$ and so by the third conclusion shown above, $\eta(t, u) \in B$ for $t \in[0,1]$. We also know that for such values of $\eta(t, u)$,

$$
\left\|I^{\prime}(\eta(t, u))\right\| \geq \sigma(\varepsilon)
$$

from Claim 1. If $\left\|I^{\prime}(\eta(t, u))\right\|>1$, the integrand equals

$$
\left\|I^{\prime}(\eta(t, u))\right\| \geq \sigma(\varepsilon)
$$

if $\left\|I^{\prime}(\eta(t, u))\right\| \leq 1$, the integrand is $\left\|I^{\prime}(\eta(t, u))\right\|^{2} \geq \sigma(\varepsilon)^{2}$. Thus

$$
c-2 \delta+\int_{0}^{1} \min \left(\sigma(\varepsilon), \sigma(\varepsilon)^{2}\right) d t<c
$$

and the only restriction on $\delta$ was that it should be smaller than $\varepsilon$. Although it was not mentioned above, $\delta$ was chosen so small that $-2 \delta+\min \left(\sigma(\varepsilon), \sigma(\varepsilon)^{2}\right)>0$. Hence this yields a contradiction. Thus the last conclusion is verified.

Imagine a valley surrounded by a ring of mountains. On the other side of this ring of moutains, there is another low place. Then there must be some path from the valley to the exterior low place which goes through a point where the gradient equals 0 , the gradient being the gradient of a function $f$ which gives the altitude of the land. This is the idea of the mountain pass theorem. The critical point where $\nabla f=0$ is the mountain pass.

Theorem 24.1.3 Let $H$ be a Hilbert space and let $I: H \rightarrow \mathbb{R}$ be a $C^{1}$ functional having $I^{\prime}$ Lipschitz continuous and such that I satisfies the Palais Smale condition. Suppose $I(0)=0$ and $I(u) \geq a>0$ for all $\|u\|=r$. Suppose also that there exists $v,\|v\|>r$ such that $I(v) \leq 0$. Then define

$$
\Gamma \equiv\{g \in C([0,1] ; H): g(0)=0, g(1)=v\}
$$

Let

$$
c \equiv \inf _{g \in \Gamma 0 \leq t \leq 1} \max _{0} I(g(t))
$$

Then $c$ is a critical value of I meaning that there exists $u$ such that $I(u)=c$ and $I^{\prime}(u)=0$. In particular, there is $u \neq 0$ such that $I^{\prime}(u)=0$.

Proof: First note that $c \geq a>0$. Suppose $c$ is not a critical value. Then by the deformation theorem, for $\varepsilon>0, \varepsilon$ sufficiently small, there is $\eta: H \rightarrow H$ and a $\delta<\varepsilon$ small enough that

$$
\eta([I(u) \leq c+\delta]) \subseteq[I(u) \leq c-\delta]
$$

and $\eta$ leaves unchanged $[I(u) \notin(c-\varepsilon, c+\varepsilon)]$. Then there is $g \in \Gamma$ such that

$$
\max _{t \in[0,1]} I(g(t))<c+\delta
$$

Then in particular, $I(g(t))<c+\delta$ for every $t$. Hence you look at $\eta \circ g$. We know that $g(0), g(1)$ are both in the set $[I(u) \notin(c-\varepsilon, c+\varepsilon)]$ because they are both 0 and so $\eta$ leaves these unchanged. Hence $\eta \circ g \in \Gamma$ and

$$
I(\eta \circ g(t)) \leq c-\delta
$$

for all $t \in[0,1]$. Thus
which is clearly a contradiction.
The Palais Smale conditions are pretty restrictive. For example, let $I(x)=\cos x$. Thus $I: \mathbb{R} \rightarrow \mathbb{R}$. Then let $u_{k}=k \pi$. Clearly $I\left(u_{k}\right)$ is bounded and $\lim _{k \rightarrow \infty} I\left(u_{k}\right)=0$ but $\left\{u_{k}\right\}$ is not precompact. However, here is a simple case which does satisfy the Palais Smale conditions.

Example 24.1.4 Let $I: \mathbb{R}^{d} \rightarrow \mathbb{R}$ satisfy $\lim _{|\mathbf{x}| \rightarrow \infty} I(\mathbf{x})=\infty$. Then I satisfies the Palais Smale conditions.

The growth condition implies that if $I\left(\mathbf{x}_{k}\right)$ is bounded, then so is $\left\{\mathbf{x}_{k}\right\}$ and so this sequence is precompact. Nothing needs to be said about $I^{\prime}\left(\mathbf{x}_{k}\right)$.

### 24.1.1 A Locally Lipschitz Selection, Pseudogradients

When you have a functional $\phi$ defined on a Banach space $X, \phi^{\prime}(u)$ is in $X^{\prime}$ and it isn't obvious how you can understand it in terms of an element in $X$ like what is done with Hilbert space using the Riesz representation theorem. However, there is something called a pseudogradient which is defined next.

Definition 24.1.5 Let $\phi: X \rightarrow \mathbb{R}$ be $C^{1}$. Then $v$ is a pseudogradient for $\phi$ at $x$ if the following hold.

1. $\|v\|_{X} \leq 2\left\|\phi^{\prime}(x)\right\|_{X^{\prime}}$
2. $\left\|\phi^{\prime}(x)\right\|_{X^{\prime}}^{2} \leq\left\langle\phi^{\prime}(x), v\right\rangle$

A pseudogradient field $V$ is a locally Lipschitz selection of $G(x)$ where $G(x)$ is defined to be the set of pseudogradients of $\phi$ at $x$. Thus $V(x) \in G(x)$ and $V(x)$ is a pseudogradient for $\phi$ at each $x$ a regular point of $\phi$.

Note how this generalizes the case of Hilbert space. In the Hilbert space case, you have $\phi^{\prime}(x)$ which technically is in $H^{\prime}$ and you have the gradient, written here as $\nabla \phi$ which is in $H$ such that

$$
(\nabla \phi(x), v)_{H} \equiv\left\langle\phi^{\prime}(x), v\right\rangle_{H^{\prime}, H}
$$

the existence of $\nabla \phi(x)$ coming from the Riesz representation theorem which also gives that $\nabla \phi(x)=R^{-1} \phi^{\prime}(x)$ and so $\|\nabla \phi(x)\|_{H}=\left\|\phi^{\prime}(x)\right\|_{H^{\prime}}$ so the above two conditions hold for the gradient field except for one thing. Why is $x \rightarrow \nabla \phi(x)$ locally Lipschitz. We don't know this, but with a pseudogradient field, we do. Also, the pseudogradient field is only required at regular points of $\phi$ where $\phi^{\prime}(x) \neq 0$. If you had strict inequalities holding in the above definition, then they would continue to hold for $\hat{x}$ near $x$. Thus if you had

$$
\|v\|_{X}<2\left\|\phi^{\prime}(x)\right\|_{X^{\prime}}, \quad\left\|\phi^{\prime}(x)\right\|_{X^{\prime}}^{2}<\left\langle\phi^{\prime}(x), v\right\rangle
$$

and $\Gamma(x)$ were the set of such $v$, then there would be an open set $U$ containing $x$ such that $\cap_{\hat{x} \in U} \Gamma(\hat{x}) \neq \emptyset$. In fact, the intersection would contain $v$.

This very nice lemma is from Gasinski L. and Papageorgiou N. [55]. It is a lovely application of Stone's theorem and partitions of unity for a metric space.

Lemma 24.1.6 Let $Y$ be a metric space and let $X$ be a normed linear space. (We will want to add in $X$.) Let $\Gamma: Y \rightarrow \mathscr{P}(X)$ such that $\Gamma(y)$ is a nonempty convex set. Suppose that for each $y \in Y$, there exists an open set $U$ containing $y$ such that

$$
\emptyset \neq \cap_{\hat{y} \in U} \Gamma(\hat{y})
$$

Then there exists a locally Lipschitz map $\gamma: Y \rightarrow X$ such that $\gamma(y) \in \Gamma(y)$ for all $y$.
Proof: Let $\mathscr{U}$ denote the collection of all open sets $U$ such that the nonempty intersection described above holds. Let $\mathscr{V}$ be a locally finite open refinement which also covers. Thus for any $V \in \mathscr{V}$

$$
\emptyset \neq \cap_{\hat{y} \in V} \Gamma(\hat{y})
$$

because it is a smaller intersection. Let $\left\{\phi_{V}\right\}_{V \in \mathscr{V}}$ be a partition of unity subordinate to the open covering $\mathscr{V}$. In fact, we can have $\phi_{V}$ locally Lipschitz. This follows from the above construction of the partition of unity in Theorem 16.1.1. Pick $x_{V} \in \cap_{\hat{y} \in V} \Gamma(\hat{y})$. Then consider

$$
\gamma(y)=\sum_{V \in \mathscr{V}} x_{V} \phi_{V}(y)
$$

It is clearly locally Lipschitz because near any point $y$, it is a finite sum of Lipschitz functions. Pick $y \in Y$. Then it is in some $V \in \mathscr{V}$. In fact, it is finitely many, $V_{1}, \cdots, V_{n}$ and for other $V \in \mathscr{V}, \phi_{V}(y)=0$. Therefore,

$$
\gamma(y)=\sum_{i=1}^{n} x_{V_{i}} \phi_{V_{i}}(y)
$$

which is a convex combination of the $x_{V_{i}}$. Now $x_{V_{i}} \in \cap_{\hat{y} \in V_{i}} \Gamma(\hat{y}) \subseteq \Gamma(y)$, this for each $i$. Hence this is a convex combination of points in a nonempty convex set $\Gamma(y)$. Thus $\gamma(y) \in$ $\Gamma(y)$.

The following lemma says that if $\phi$ is $C^{1}$ on $X$, then it has a pseudogradient field on $\left\{x: \phi^{\prime}(x) \neq 0\right\}$, the set of regular points.

Lemma 24.1.7 Let $\phi$ be a $C^{1}$ function defined on $X$ a Banach space. Then there exists a pseudogradient field for $\phi$ on the set of regular points. $(V(x) \in G(x)$ and $x \rightarrow V(x)$ is locally Lipschitz on the set of regular points.)

Proof: First consider whether $G(x)$, the set of pseudogradients of $\phi$ at $x$ is nonempty for $\phi^{\prime}(x) \neq 0$. From the definition of the operator norm, there exists $u$ such that $\|u\|_{X}=1$ and $\left\langle\phi^{\prime}(x), u\right\rangle \geq \delta\left\|\phi^{\prime}(x)\right\|_{X^{\prime}}$ where $\delta \in(0,1)$. Then let $v=r u\left\|\phi^{\prime}(x)\right\|_{X^{\prime}}$ where $r \in(1,2)$.

$$
\left\langle\phi^{\prime}(x), v\right\rangle=\left\langle\phi^{\prime}(x), r u\left\|\phi^{\prime}(x)\right\|\right\rangle=r\left\langle\phi^{\prime}(x), u\right\rangle\left\|\phi^{\prime}(x)\right\| \geq r \delta\left\|\phi^{\prime}(x)\right\|^{2}
$$

Then choose $r, \delta$ such that $r \delta>1$ and $r<2$. Then if these were chosen this way in the above reasoning, it follows that

$$
\|v\|<2\left\|\phi^{\prime}(x)\right\| \text { and }\left\langle\phi^{\prime}(x), v\right\rangle>\left\|\phi^{\prime}(x)\right\|^{2}
$$

That $\phi^{\prime}(x) \neq 0$ is needed to insure that the above strict inequalities hold.
Thus, letting $Y$ be the metric space consisting of the regular points of $\phi$, the continuity of $\phi^{\prime}$ implies that the above inequalities persist for all $y$ close enough to $x$. Thus there is an open set $U$ containing $x$ such that $v$ satisfies the above inequalities for $x$ replaced with arbitrary $y \in U$. Thus

$$
v \in \cap_{y \in U} G(y)
$$

Since it is clear that each $G(y)$ is convex, Lemma 24.1.6 implies the existence of a locally Lipschitz selection from $G$. That is $x \rightarrow V(x)$ is locally Lipschitz and $V(x) \in G(x)$ for all regular $x$.

It will be important to consider $y^{\prime}=f(y)$ where $f$ is locally Lipschitz and $y$ is just in a Banach space. This is more complicated than in Hilbert space because of the lack of a convenient projection map.

Theorem 24.1.8 Let $f: U \rightarrow X$ be locally Lipschitz, where $X$ is a Banach space and $U$ is an open set. Then there exists a unique local solution to the IVP

$$
y^{\prime}=f(y), \quad y(0)=y_{0} \in U
$$

Proof: Let $B$ be a closed ball of radius $R$ centered at $y_{0}$ such that $f$ has Lipschitz constant $K$ on $B$. Then

$$
y_{1}(t)=y_{0}+\int_{0}^{t} f\left(y_{0}\right) d s
$$

and if $y_{n}(t)$ has been obtained,

$$
\begin{equation*}
y_{n+1}(t)=y_{0}+\int_{0}^{t} f\left(y_{n}(s)\right) d s \tag{24.1.3}
\end{equation*}
$$

Now $t<T$ where $T$ is so small that $\left\|f\left(y_{0}\right)\right\| T e^{K T}<R$.
Claim: $\left\|y_{n}(t)-y_{n-1}(t)\right\| \leq\left\|f\left(y_{0}\right)\right\| t^{n} K^{n-1} \frac{1}{(n-1)!}$.
Proof of claim: First

$$
\left\|y_{1}(t)-y_{0}\right\| \leq \int_{0}^{t}\left\|f\left(y_{0}\right)\right\| d s \leq\left\|f\left(y_{0}\right)\right\| t
$$

Now suppose it is so for $n$. Then

$$
\left\|y_{n+1}(t)-y_{n}(t)\right\| \leq \int_{0}^{t}\left\|f\left(y_{n}(s)\right)-f\left(y_{n-1}(s)\right)\right\| d s
$$

By induction, $y_{n}(s), y_{n-1}(s)$ are still in $B$. This is because

$$
\begin{align*}
& \left\|y_{n}(t)-y_{0}\right\| \leq \sum_{k=1}^{n}\left\|y_{k}(t)-y_{k-1}(t)\right\| \\
& \quad \leq \sum_{k=1}^{n}\left\|f\left(y_{0}\right)\right\| \frac{1}{(k-1)!} t^{k} K^{k-1} \\
& \quad \leq\left\|f\left(y_{0}\right)\right\| t e^{K t}<R \tag{24.1.4}
\end{align*}
$$

showing that $y_{n}(t)$ stays in $B$. Then since all values of the iterates remain in $B$, induction gives

$$
\begin{aligned}
& \left\|y_{n+1}(t)-y_{n}(t)\right\| \leq \int_{0}^{t} K\left\|y_{n}(s)-y_{n-1}(s)\right\| d s \\
& \leq \quad K \int_{0}^{t}\left\|f\left(y_{0}\right)\right\| \frac{1}{(n-1)!} s^{n} K^{n-1} d s=K^{n} \frac{1}{(n-1)!}\left\|f\left(y_{0}\right)\right\| \int_{0}^{t} s^{n} d s \\
& =\quad K^{n} \frac{1}{n!}\left\|f\left(y_{0}\right)\right\| t^{n+1}
\end{aligned}
$$

which proves the claim. Since the inequality of the claim shows that $\left\|y_{n}-y_{n-1}\right\|$ is summable, it follows that $\left\{y_{n}\right\}$ is a Cauchy sequence in $C([0, T], X)$. It satsifies $\left\|y_{n}-y_{0}\right\|<R$ and so $y_{n}$ converges uniformly to some $y \in C([0, T], X)$. Hence one can pass to a limit in 24.1.3 and obtain

$$
y(t)=y_{0}+\int_{0}^{t} f(y(s)) d s
$$

for $t \in[0, T]$. Also $\left\|y(t)-y_{0}\right\| \leq R$ and on $B\left(y_{0}, R\right), f$ is Lipschitz continuous so Gronwall's inequality gives uniqueness of solutions which remain in $B$.

Here is an alternate proof which other than the ugly lemma, seems more elegant to me. However, it is a useful lemma.

## Lemma 24.1.9 Define

$$
\gamma(x) \equiv\left\{\begin{array}{c}
x \text { if }\left\|x-y_{0}\right\| \leq R \\
y_{0}+\frac{x-y_{0}}{\left\|x-y_{0}\right\|} R \text { if }\left\|x-y_{0}\right\|>R
\end{array}\right.
$$

Then $\|\gamma(x)-\gamma(y)\| \leq 3\|x-y\|$ for all $x, y \in X$. Thus

$$
\left\|\gamma(x)-y_{0}\right\| \leq R
$$

Proof: In case both of $x, y$ are in $B=\overline{B\left(y_{0}, R\right)}$, there is nothing to show. Suppose then that $\left\|y-y_{0}\right\| \leq R$ but $\left\|x-y_{0}\right\|>R$. Then, assuming $y-y_{0} \neq 0$,

$$
\|\gamma(x)-\gamma(y)\|=\left\|y_{0}+\frac{x-y_{0}}{\left\|x-y_{0}\right\|} R-y\right\|=\left\|\frac{x-y_{0}}{\left\|x-y_{0}\right\|} R-\left(y-y_{0}\right)\right\|
$$

$$
\begin{gathered}
=\left\|\frac{x-y_{0}}{\left\|x-y_{0}\right\|} R-\frac{\left\|y-y_{0}\right\|}{\left\|y-y_{0}\right\|}\left(y-y_{0}\right)\right\| \\
\leq\left\|\frac{x-y_{0}}{\left\|x-y_{0}\right\|} R-\frac{\left(y-y_{0}\right)}{\left\|y-y_{0}\right\|} R\right\|+ \\
+\left\|\frac{R}{\left\|y-y_{0}\right\|}\left(y-y_{0}\right)-\frac{\left\|y-y_{0}\right\|}{\left\|y-y_{0}\right\|}\left(y-y_{0}\right)\right\|=A+B
\end{gathered}
$$

Now

$$
\begin{gathered}
B=\left(R-\left\|y-y_{0}\right\|\right)<\left\|x-y_{0}\right\|-\left\|y-y_{0}\right\| \leq\|y-x\| \\
A \leq\left\|\frac{x-y_{0}}{\left\|x-y_{0}\right\|} R-\frac{\left(y-y_{0}\right)}{\left\|y-y_{0}\right\|} R\right\| \leq R \frac{\left\|\left(x-y_{0}\right)\right\| y-y_{0}\left\|-\left(y-y_{0}\right)\right\| x-y_{0}\| \|}{\left\|x-y_{0}\right\|\left\|y-y_{0}\right\|} \\
\leq \frac{R}{\left\|x-y_{0}\right\|\left\|y-y_{0}\right\|}\binom{\left\|\left(x-y_{0}\right)\right\| y-y_{0}\left\|-\left(y-y_{0}\right)\right\| y-y_{0}\| \|}{+\left\|\left(y-y_{0}\right)\right\| y-y_{0}\left\|-\left(y-y_{0}\right)\right\| x-y_{0}\| \|} \\
\leq \frac{R}{\left\|x-y_{0}\right\|\left\|y-y_{0}\right\|}\left(\left\|y-y_{0}\right\|\|x-y\|+\left\|y-y_{0}\right\|\|y-x\|\right) \\
\leq \frac{R}{\left\|x-y_{0}\right\|}(\|x-y\|+\|y-x\|)<2\|y-x\|
\end{gathered}
$$

In case $y=y_{0}$, you have

$$
\|\gamma(x)-\gamma(y)\|=\left\|\frac{x-y_{0}}{\left\|x-y_{0}\right\|} R\right\|=\left\|\frac{x-y}{\left\|x-y_{0}\right\|} R\right\|<\|x-y\|
$$

The only other case is where both $x, y$ are in $X \backslash B$. In this case, you get

$$
\begin{aligned}
\|\gamma(x)-\gamma(y)\| & =\left\|y_{0}+\frac{x-y_{0}}{\left\|x-y_{0}\right\|} R-\left(y_{0}+\frac{y-y_{0}}{\left\|y-y_{0}\right\|} R\right)\right\| \\
& =\left\|\frac{x-y_{0}}{\left\|x-y_{0}\right\|} R-\frac{y-y_{0}}{\left\|y-y_{0}\right\|} R\right\| \leq 2\|x-y\|
\end{aligned}
$$

by the same reasoning used above to estimate $A$.
Alternate Proof of Theorem 24.1.8: Let $B$ be a closed ball of radius $R$ centered at $y_{0}$ such that $f$ has Lipschitz constant $K$ on $B$. Let $\gamma$ be as in Lemma 24.1.9. Consider $g(x) \equiv f(\gamma(x))$. Then

$$
\|g(x)-g(y)\|=\|f(\gamma(x))-f(\gamma(y))\| \leq K\|\gamma(x)-\gamma(y)\| \leq 3 K\|x-y\|
$$

Now consider for $y \in C([0, T], X)$

$$
F y(t) \equiv y_{0}+\int_{0}^{t} g(y(s)) d s
$$

Then

$$
\|F y(t)-F z(t)\| \leq \int_{0}^{t} K\|y(s)-z(s)\| d s
$$

Thus, iterating this inequality, it follows that a large enough power of $F$ is a contraction map. Therefore, there is a unique fixed point. Now letting $y$ be this fixed point,

$$
\left\|y(t)-y_{0}\right\| \leq \int_{0}^{t} 3 K\left\|y(s)-y_{0}\right\| d s+\left\|f\left(y_{0}\right)\right\| T
$$

It follows that

$$
\left\|y(t)-y_{0}\right\| \leq\left\|f\left(y_{0}\right)\right\| T e^{3 K T}
$$

Choosing $T$ small enough, it follows that $\left\|y(t)-y_{0}\right\|<R$ on $[0, T]$ and so $\gamma$ has no effect. Thus this yields a local solution to the initial value problem.

In the case that $U=X$, the above argument shows that there exists a solution on some $[0, T)$ where $T$ is maximal.

$$
y(t)=y_{0}+\int_{0}^{t} f(y(s)) d s, t<T
$$

Suppose $T<\infty$. Suppose $\int_{0}^{T}\|f(y(s))\| d s<\infty$. Then you can consider $y_{0}+\int_{0}^{T} f(y(s)) d s$ as an initial condition for the equation and obtain a unique solution $z$ valid on $[T, T+\delta]$. Then one could consider $\hat{y}(t)=y(t)$ for $t<T$ and for $t \geq T, \hat{y}(t)=z(t)$. Then for $t \in$ $[T, T+\delta]$,

$$
\begin{aligned}
\hat{y}(t) & =z(t)=y_{0}+\int_{0}^{T} f(y(s)) d+\int_{T}^{t} f(\hat{y}(s)) d s \\
& =y_{0}+\int_{0}^{T} f(\hat{y}(s)) d+\int_{T}^{t} f(\hat{y}(s)) d s
\end{aligned}
$$

and so in fact, for all $t \in[0, T+\delta]$,

$$
\hat{y}(t)=y_{0}+\int_{0}^{t} f(\hat{y}(s)) d s
$$

contrary to the maximality of $T$. Hence it cannot be the case that $T<\infty$. Thus it must be the case that $\int_{0}^{T}\|f(y(s))\| d s=\infty$ if the solution is not global.

From the above observation, here is a corollary.
Corollary 24.1.10 Let $f: X \rightarrow X$ be locally Lipschitz where $X$ is a Banach space. Then there exists a unique local solution to the IVP

$$
y^{\prime}=f(y), \quad y(0)=y_{0}
$$

If $f$ is bounded, then in fact the solutions exists on $[0, T]$ for any $T>0$.
Proof: Say $\|f(x)\| \leq M$ for all $M$. Then letting $[0, \hat{T})$ be the maximal interval, it must be the case that $\int_{0}^{\hat{T}}\|f(y(t))\| d t=\infty$, but this does not happen if $f$ is bounded.

Note that this conclusion holds just as well if $f$ has linear growth, $\|f(u)\| \leq a+b\|u\|$ for $a, b \geq 0$. One just uses an application of Gronwall's inequality to verify a similar conclusion.

One can also give a simple modification of these theorems as follows.

Corollary 24.1.11 Suppose $f: X \rightarrow X$ is continuous and $f$ is locally Lipschitz on $U$, an open subset of $X$, a Banach space. Suppose also that $f(x)=0$ for all $x \notin U$ and that $\|f(x)\|<M$ for all $x \in X$. Then there exists a solution to the IVP

$$
y^{\prime}=f(y), \quad y(0)=y_{0}
$$

for $t \in[0, T]$ for any $T>0$.
Proof: Let $T$ be given. If $y_{0} \notin U$, there is nothing to show. The solution is $y(t) \equiv$ $y_{0}$. Suppose then that $y_{0} \in U$. Then by Theorem 24.1.8, there exists a unique solution to the initial value problem on an interval $[0, \hat{T})$ of maximal length. If $\hat{T}=T$, then as $t_{n} \rightarrow T,\left\{y\left(t_{n}\right)\right\}$ must converge. This is because for $t_{m}<t_{n}$,

$$
\left\|y\left(t_{n}\right)-y\left(t_{m}\right)\right\| \leq M\left|t_{n}-t_{m}\right|
$$

showing that this is a Cauchy sequence. Since all such sequences lead to a Cauchy sequence, it must be the case that $\lim _{t \rightarrow T} y(t)$ exists. Thus it equals

$$
y_{0}+\int_{0}^{T} f(y(t)) d t
$$

We let $y(T)$ equal the above and it follows from Gronwall's inequality that there is a unique solution to the IVP on $[0, T]$ so the claim is true in this case.

Otherwise, if $\hat{T}<T$, then one can define

$$
y(\hat{T}) \equiv y_{0}+\int_{0}^{\hat{T}} f(y(s)) d s
$$

If $y(\hat{T}) \in U$, then by the assumption that $f$ is bounded, one could consider a new initial condition and extend the solution further violating the maximality of the length of $[0, \hat{T})$. Therefore, it must be the case that $y(\hat{T}) \in U^{C}$. Then the solution is

$$
\hat{y}(t)=\left\{\begin{array}{c}
y(t), t<\hat{T} \\
y(\hat{T}), t>\hat{T}
\end{array}\right.
$$

because $f(y(\hat{T}))=0$ by assumption.
One could also change the above argument for Corollary 24.1.11 to include the case that $f$ has linear growth.

### 24.1.2 Mountain Pass Theorem In Banach Space

In this section, is a more general version of the mountain pass theorem. It is generalized in two ways. First, the space is not a Hilbert space and second, the derivative of the functional is not assumed to be Lipschitz. Instead of using $I^{\prime}$ one uses the pseudogradient in an appropriate differential equation. This is a significant generalization because there is no convenient projection map from $X^{\prime}$ to $X$ like there is in Hilbert space. This is why the use of the psedogradient is so interesting. For many more considerations of this sort of thing, see [55]. First is a deformation theorem. Here $I$ will be defined on a Banach space $X$ and $I^{\prime}(x) \in X^{\prime}$. First recall the Palais Smale conditions.

Definition 24.1.12 A functional I satisfies the Palais Smale conditions if $\left\{I\left(u_{k}\right)\right\}$ is a bounded sequence and $I^{\prime}\left(u_{k}\right) \rightarrow 0$, then $\left\{u_{k}\right\}$ is precompact. That is, it has a subsequence which converges.

Here is a picture which illustrates the main conclusion of the following theorem. The idea is that you modify the functional on some set making it smaller and leaving it unchanged off that set.


Theorem 24.1.13 Let I be $C^{1}$, I is non constant, satisfy the Palais Smale condition, and $I^{\prime}$ is bounded on bounded sets. Also suppose that $c \in \mathbb{R}$ is such that either $I^{-1}([c-\delta, c+\delta])=$ $\emptyset$ for some $\delta>0$ or $I^{-1}([c-\delta, c+\delta]) \neq \emptyset$ for all $\delta>0$ and $\boldsymbol{I F} I(u)=c$, then $I^{\prime}(u) \neq 0$. Then for each sufficiently small $\varepsilon>0$, there is a constant $\delta \in(0, \varepsilon)$ and a function $\eta$ : $[0,1] \times X \rightarrow X$ such that

1. $\eta(0, u)=u$
2. $\eta(1, u)=u$ on $I^{-1}(X \backslash(c-\varepsilon, c+\varepsilon))$
3. $I(\eta(t, u)) \leq I(u)$
4. $\eta\left(1, I^{-1}(-\infty, c+\delta]\right) \subseteq I^{-1}(-\infty, c-\delta]$, so $I(\eta(1, u)) \leq c-\delta$ if $I(u) \leq c+\delta$.

The main part of this conclusion is the statement about $u \rightarrow \eta(1, u)$ contained in parts 2. and 4. The other two parts are there to facilitate these two although they are certainly interesting for their own sake.

Proof: Suppose $I^{-1}([c-\hat{\delta}, c+\hat{\delta}])=\emptyset$ for some $\hat{\delta}>0$. Then

$$
I^{-1}\left(\left(-\infty, c+\frac{\hat{\delta}}{2}\right]\right) \subseteq I^{-1}\left(\left(-\infty, c-\frac{\hat{\delta}}{2}\right]\right)
$$

and you could take $\varepsilon=\delta$ and let $\eta(t, u)=u$. The conclusion holds with $\delta=\hat{\delta} / 2$.
Therefore, assume $I^{-1}([c-\delta, c+\delta]) \neq \emptyset$ for all $\delta>0$. Since $I$ is nonconstant, $\varepsilon>0$ can be chosen small enough that

$$
I^{-1}(X \backslash(c-\varepsilon, c+\varepsilon)) \neq \emptyset
$$

Always let $\varepsilon$ be this small. Note that $I$ nonconstant is part of the assumptions.
Claim 1: For all small enough $\varepsilon>0$, if $u \in I^{-1}([c-\varepsilon, c+\varepsilon])$, then $I^{\prime}(u) \neq 0$ and in fact, for such $\varepsilon$, there exists $\sigma(\varepsilon)>0$, such that $\sigma(\varepsilon)<\min (\varepsilon, 1),\left\|I^{\prime}(u)\right\|>\sigma(\varepsilon)$ for all $u \in I^{-1}([c-\varepsilon, c+\varepsilon])$.

Proof of Claim 1: If the claim is not so, then there is $\left\{u_{k}\right\}, \varepsilon_{k}, \sigma_{k} \rightarrow 0,\left\|I^{\prime}\left(u_{k}\right)\right\|_{X^{\prime}}<$ $\sigma_{k}$, and $I\left(u_{k}\right) \in\left[c-\varepsilon_{k}, c+\varepsilon_{k}\right]$ but $\left\|I^{\prime}\left(u_{k}\right)\right\|_{X^{\prime}} \leq \sigma_{k}$. However, from the Palais Smale condition, there is a subsequence, still denoted as $u_{k}$ which converges to some $u$. Now $I\left(u_{k}\right) \in\left[c-\varepsilon_{k}, c+\varepsilon_{k}\right]$ and so $I(u)=c$ while $I^{\prime}(u)=0$ contrary to the hypothesis. This proves Claim 1. From now on, $\varepsilon$ will be sufficiently small that this holds.

Now define for $\delta<\varepsilon$ (The precise description of small $\delta$ will be described later. However, it will be $\delta<\sigma(\varepsilon) / 2$, but this exact description is only used at the end.)

$$
\begin{aligned}
A & \equiv I^{-1}(X \backslash(c-\varepsilon, c+\varepsilon)) \\
B & \equiv I^{-1}([c-\delta, c+\delta])
\end{aligned}
$$

Thus $A$ and $B$ are disjoint closed sets. Recall that it is assumed that $B \neq \emptyset$ since otherwise, there is nothing to prove. Also it is assumed throughout that $\varepsilon>0$ is such that $A \neq \emptyset$ thanks to $I$ not being constant. Thus these are nonempty sets and we do not have to fuss with worrying about meaning when one is empty.

Claim 2: For any $u, \operatorname{dist}(u, A)+\operatorname{dist}(u, B)>0$.
This is so because if not, then both summands would be zero and this requires that $u \in A \cap B$ since these sets are closed. But $A \cap B=\emptyset$.

Now define a function

$$
g(u) \equiv \frac{\operatorname{dist}(u, A)}{\operatorname{dist}(u, A)+\operatorname{dist}(u, B)}
$$

It is a continuous function of $u$ which has values in $[0,1]$. It is 1 on $B$ and 0 on $A$. Also define $V(x)$ as a pseudogradient field for $I$ on the regular points of $I$. At points where $I^{\prime}(x)=0$, let $V(x)=0$. Recall what this means:

$$
\begin{equation*}
\left\|I^{\prime}(x)\right\|_{X^{\prime}}^{2} \leq\left\langle I^{\prime}(x), V(x)\right\rangle, \quad\|V(x)\|_{X} \leq 2\left\|I^{\prime}(x)\right\|_{X^{\prime}} \tag{24.1.5}
\end{equation*}
$$

and also $V$ is locally Lipschitz on the regular points of $I$. Thus $x \rightarrow V(x)$ is continuous on $X$, thanks to continuity of $I^{\prime}$, satisfies the above inequalities, and is locally Lipschitz on $U=\left\{x: I^{\prime}(x) \neq 0\right\}$. It exists because of Lemma 24.1.7. Consider the ordinary differential initial value problem

$$
\begin{gather*}
\eta^{\prime}(t, u)+g(u) h(\|V(\eta(t, u))\|) V(\eta(t, u))=0  \tag{24.1.6}\\
\eta(0, u)=u \tag{24.1.7}
\end{gather*}
$$

where $r \rightarrow h(r)$ is a decreasing function which has values in $(0,1]$ and equals 1 for $r \in[0,1]$ and equals $1 / r$ for $r>1$.


Here $u$ is given and the $\eta^{\prime}$ is the time derivative is with respect to $t$. By Corollary 24.1.11, there exists a solution to this for $t \in[0,1]$.

Then for this solution, $\eta(0, u)=u$ because of the above initial condition. If $u \in$ $I^{-1}(X \backslash[c-\varepsilon, c+\varepsilon])$, then $u \in A$ and so $g(u)=0$ so $\eta(t, u)=u$ for all $t \in[0,1]$. This gives the first two conditions. Consider the third.

$$
\begin{aligned}
\frac{d}{d t}(I(\eta(t, u))) & =\left\langle I^{\prime}(\eta), \eta^{\prime}\right\rangle=-\left\langle I^{\prime}(\eta), g(u) h(\|V(\eta(t, u))\|) V(\eta(t, u))\right\rangle \\
& =-g(u) h(\|V(\eta(t, u))\|)\left\langle I^{\prime}(\eta), V(\eta(t, u))\right\rangle \\
& \leq-g(u) h(\|V(\eta(t, u))\|)\left\|I^{\prime}(\eta)\right\|_{X^{\prime}}^{2} \leq 0
\end{aligned}
$$

this last inequality from the inequalities of 24.1 .5 , and so this implies the third condition since it says that the function $t \rightarrow I(\eta(t, u))$ is decreasing.

It remains to consider the last condition. This involves an appropriate choice of small $\delta$. It was chosen small and now it will be seen how small. It is desired to verify that

$$
\eta\left(1, I^{-1}((-\infty, c+\delta])\right) \subseteq I^{-1}((-\infty, c-\delta])
$$

Suppose it is not so. Then there exists $u$ such that $I(u) \in(c-\delta, c+\delta]$ but $I(\eta(1, u))>$ $c-\delta$. We can assume that $I(u) \in(c-\delta, c+\delta]$ because if $I(u) \leq c-\delta$, then so is $I(\eta(1, u))$ from what was just shown. Hence $g(u)=1$. Then using the fact that $g(u)=1$,

$$
\begin{aligned}
c & -\delta<I(\eta(1, u))=I(u)+\int_{0}^{1} \frac{d}{d t}(I(\eta)) d t \\
& =I(u)-\int_{0}^{1}\left\langle I^{\prime}(\eta), h(\|V(\eta)\|) V(\eta)\right\rangle d t \\
& =I(u)+\int_{0}^{1}-h(\|V(\eta)\|)\left\langle I^{\prime}(\eta), V(\eta)\right\rangle d t \\
& \leq c+\delta \\
& \leq(u)+\int_{0}^{1}\left(-h(\|V(\eta)\|)\left\|I^{\prime}(\eta)\right\|^{2}\right) d t
\end{aligned}
$$

Then

$$
c-\delta+\int_{0}^{1} h(\|V(\eta)\|)\left\|I^{\prime}(\eta)\right\|^{2} d t<I(u) \leq c+\delta
$$

Thus

$$
c-2 \delta+\int_{0}^{1} h(\|V(\eta)\|)\left\|I^{\prime}(\eta)\right\|^{2} d t<c
$$

Also, it is being assumed that $I(\eta(1, u))>c-\delta$ and so by the third conclusion shown above, $\eta(t, u) \in B$ for $t \in[0,1]$. We also know that for such values of $\eta(t, u)$,

$$
\left\|I^{\prime}(\eta(t, u))\right\| \geq \sigma(\varepsilon)
$$

from Claim 1. Now

$$
\left\|I^{\prime}(x)\right\|_{X^{\prime}}^{2} \leq\left\langle I^{\prime}(x), V(x)\right\rangle \leq\left\|I^{\prime}(x)\right\|\|V(x)\|
$$

and so

$$
\begin{equation*}
\|V(\eta(t, u))\|_{X} \geq\left\|I^{\prime}(\eta(t, u))\right\|_{X^{\prime}} \geq \sigma(\varepsilon) \tag{24.1.8}
\end{equation*}
$$

Thus the above inequality yields

$$
c-2 \delta+\int_{0}^{1} h(\|V(\eta)\|) \sigma(\varepsilon)^{2} d t<c
$$

Now what is the value of $h(\|V(\eta)\|)$ ? From 24.1.8

$$
h\left(\|V(\eta(t, u))\|_{X}\right) \leq h\left(\left\|I^{\prime}(\eta(t, u))\right\|_{X^{\prime}}\right) \leq h(\sigma(\varepsilon)) \leq \frac{1}{\sigma(\varepsilon)}
$$

In fact, $\sigma(\varepsilon)<1$ so $h(\sigma(\varepsilon))=1$ so the above estimate, while correct is sloppy. Hence

$$
c-2 \delta+\int_{0}^{1} \frac{1}{\sigma(\varepsilon)} \sigma(\varepsilon)^{2} d t<c
$$

So far it was only assumed $\delta<\varepsilon$. As indicated above, $\delta$ was chosen small enough that $-2 \delta+\sigma(\varepsilon)>0$. Hence this yields a contradiction. Thus the last conclusion is verified.

Imagine a valley surrounded by a ring of mountains. On the other side of this ring of moutains, there is another low place. Then there must be some path from the valley to the exterior low place which goes through a point where the gradient equals 0 , the gradient being the gradient of a function $f$ which gives the altitude of the land. This is the idea of the mountain pass theorem. The critical point where $\nabla f=0$ is the mountain pass.

Theorem 24.1.14 Let $X$ be a Banach space and let $I: X \rightarrow \mathbb{R}$ be a $C^{1}$ functional having $I^{\prime}$ bounded on bounded sets and such that I satisfies the Palais Smale condition. Suppose $I(0)=0$ and $I(u) \geq a>0$ for all $\|u\|=r$. Suppose also that there exists $v,\|v\|>r$ such that $I(v) \leq 0$. Then define

$$
\Gamma \equiv\{g \in C([0,1] ; X): g(0)=0, g(1)=v\}
$$

Let

$$
c \equiv \inf _{g \in \Gamma} \max _{0 \leq t \leq 1} I(g(t))
$$

Then $c$ is a critical value of I meaning that there exists $u$ such that $I(u)=c$ and $I^{\prime}(u)=0$. In particular, there is $u \neq 0$ such that $I^{\prime}(u)=0$.

Proof: First note that $c \geq a>0$. Suppose $c$ is not a critical value. Then either $I^{-1}((c-\delta, c+\delta))=\emptyset$ for some $\delta>0$ in which case the conclusion of the deformation theorem, (Theorem 24.1.13) holds, or for all $\delta>0, I^{-1}((c-\delta, c+\delta)) \neq \emptyset$ and if $I(u)=c$, then $I^{\prime}(u) \neq 0$ in which case the deformation theorem also holds. Then by this theorem, for $\varepsilon>0, \varepsilon$ sufficiently small, $\varepsilon<c$, there is $\eta: X \rightarrow X$ and a $\delta<\varepsilon$ small enough that

$$
\eta\left(I^{-1}((-\infty, c+\delta])\right) \subseteq I^{-1}((-\infty, c-\delta])
$$

and $\eta$ leaves unchanged $I^{-1}(X \backslash(c-\varepsilon, c+\varepsilon))$. Then there is $g \in \Gamma$ such that

$$
\max _{t \in[0,1]} I(g(t))<c+\delta
$$

Then in particular, $I(g(t))<c+\delta$ for every $t$. Hence you look at $\eta \circ g$. We know that $g(0), g(1)$ are both in the set $[I(u) \notin(c-\varepsilon, c+\varepsilon)]$ because they are both 0 or less than 0 and so $\eta$ leaves these unchanged. Hence $\eta \circ g \in \Gamma$ and

$$
I(\eta \circ g(t)) \leq c-\delta
$$

for all $t \in[0,1]$. Thus
which is clearly a contradiction.

## Chapter 25

## Nonlinear Operators

In this chapter, is a discussion of various kinds of nonlinear operators. Some standard references on these operators are [39], [40], [22], [24], [13], [91], [116], [25] and references listed there. The most important examples of these operators seem to be due to Brezis in the 1960's and these things have been generalized and used by many others since this time. I am following many of these, but the stuff about maximal monotone operators is mainly from Barbu [13]. I am trying to include all the necessary basic results such as fixed point theorems which are needed to prove the main theorems and also to re write in a manner understandable to me.

It seems like the main issue is the following. When does $\left\langle f_{n}, x_{n}\right\rangle$ converge to $\langle f, x\rangle$ given that $f_{n}$ and $x_{n}$ both converge weakly to $f$ and $x$ respectively? There is no problem in finite dimensions because in finite dimensions, there is only one meaning for convergence. However, in infinite dimensions, there certainly is a problem as can be instantly realized by consideration of the Riemann Lebesgue lemma, for example. You know that $\int_{-\pi}^{\pi} f(x) \sin (n x) d x \rightarrow 0$ so $\sin (n x)$ converges weakly to 0 but $\int_{-\pi}^{\pi} \sin ^{2}(n x) d x$ certainly does not converge to 0 .

The idea behind all of these considerations is that $f_{n}$ is to come from some nonlinear operator which has properties which will allow one to successfully pass to a limit. When the operator is linear, there usually is no problem because the graph is a subspace and so if it is closed, it will also be weakly closed. Thus, if $x_{n} \rightarrow x$ weakly and $L x_{n} \rightarrow f$ weakly, then $f=L x$. However, nothing like this happens with nonlinear operators. Consideration of when this happens is the purpose of this catalogue of nonlinear operators, and also to generalize to set valued operators. First is a section on single valued nonlinear operators and then the case of set valued nonlinear operators is discussed.

### 25.1 Some Nonlinear Single Valued Operators

Here is an assortment of nonlinear operators which are useful in applications to nonlinear partial differential equations. Generalizations of the notion of a pseudomonotone map will be presented later to include the case of set valued pseudomonotone maps. This is on the single valued version of some of these and these ideas originate with Brezis in the 1960's. A good description is given in Lions [91].

Definition 25.1.1 For $V$ a real Banach space, $A: V \rightarrow V^{\prime}$ is a pseudomonotone map if whenever

$$
\begin{equation*}
u_{n} \rightharpoonup u \tag{25.1.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\lim \sup _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-u\right\rangle \leq 0 \tag{25.1.2}
\end{equation*}
$$

it follows that for all $v \in V$,

$$
\begin{equation*}
\lim \inf _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-v\right\rangle \geq\langle A u, u-v\rangle \tag{25.1.3}
\end{equation*}
$$

The half arrows denote weak convergence.

If $V$ is finite dimensional, then pseudomonotone maps are continuous. Also the property of being pseudomonotone is preserved when restriction is made to finite dimensional spaces. The notation is explained in the following diagram.

$$
\begin{array}{ccc}
W^{\prime} & \stackrel{i^{*}}{\leftarrow} & V^{\prime} \\
W & \stackrel{\rightarrow}{\rightarrow} & V
\end{array}
$$

The map $i$ is just the inclusion map. $i w \equiv w$ and $i^{*}$ is the usual adjoint map. $\left\langle i^{*} f, w\right\rangle_{W^{\prime}, W} \equiv$ $\langle f, i w\rangle_{V^{\prime}, V}=\langle f, w\rangle_{V^{\prime}, V}$. Thus $i^{*} A i(w) \in W^{\prime}$ and it is defined by

$$
\left\langle i^{*} A i(w), z\right\rangle_{W^{\prime}, W} \equiv\langle A w, z\rangle_{V^{\prime}, V}
$$

in other words, you restrict $A$ to $W$ and only consider what the resulting functional does to things in $W$.

Proposition 25.1.2 Let $V$ be finite dimensional and let $A: V \rightarrow V^{\prime}$ be pseudomonotone and bounded (meaning A maps bounded sets to bounded sets). Then A is continuous. Also, if $A: V \rightarrow V^{\prime}$ is pseudomonotone and bounded, and if $W \subseteq V$ is a finite dimensional subspace, then $i^{*} A i$ is pseudomonotone as a map from $W$ to $W^{\prime}$.

Proof: Say $u_{n} \rightarrow u$. Does it follow that $A u_{n} \rightarrow A u$ ? If not, then there is a subsequence such that $A u_{n} \rightarrow \xi \neq A u$ thanks to $\left\{A u_{n}\right\}$ being bounded. Then the limsup condition holds obviously. In fact the limit of $\left\langle A u_{n}, u_{n}-u\right\rangle$ exists and equals 0 . Hence for all $v$,

$$
\lim _{n \rightarrow \infty} \inf _{n}\left\langle A u_{n}, u_{n}-v\right\rangle \geq\langle A u, u-v\rangle
$$

Therefore,

$$
\langle\xi, u-v\rangle \geq\langle A u, u-v\rangle
$$

for all $v$ and so in fact $\xi=A u$ after all. Thus $A$ must be continuous.
As to the second part of this proposition, if you have $w_{n} \rightharpoonup w$ in $W$, then in fact convergence takes place strongly because weak and strong convergence are the same in finite dimensions. Hence the same argument given above holds to show that $i^{*} A i$ is continuous.

Definition 25.1.3 $A: V \rightarrow V^{\prime}$ is monotone if for all $v, u \in V$,

$$
\langle A u-A v, u-v\rangle \geq 0
$$

and $A$ is Hemicontinuous if for all $v, u \in V$,

$$
\lim _{t \rightarrow 0+}\langle A(u+t(v-u)), u-v\rangle=\langle A u, u-v\rangle .
$$

Theorem 25.1.4 Let $V$ be a Banach space and let $A: V \rightarrow V^{\prime}$ be monotone and hemicontinuous. Then A is pseudomonotone.

Proof: Let $A$ be monotone and Hemicontinuous. First here is a claim.
Claim: If 25.1.1 and 25.1.2 hold, then $\lim _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-u\right\rangle=0$.

Proof of the claim: Since $A$ is monotone,

$$
\left\langle A u_{n}-A u, u_{n}-u\right\rangle \geq 0
$$

so

$$
\left\langle A u_{n}, u_{n}-u\right\rangle \geq\left\langle A u, u_{n}-u\right\rangle .
$$

Therefore,

$$
0=\lim \inf _{n \rightarrow \infty}\left\langle A u, u_{n}-u\right\rangle \leq \lim \inf _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-u\right\rangle \leq \lim \sup _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-u\right\rangle \leq 0
$$

Now using that $A$ is monotone again, then letting $t>0$,

$$
\left\langle A u_{n}-A(u+t(v-u)), u_{n}-u+t(u-v)\right\rangle \geq 0
$$

and so

$$
\left\langle A u_{n}, u_{n}-u+t(u-v)\right\rangle \geq\left\langle A(u+t(v-u)), u_{n}-u+t(u-v)\right\rangle .
$$

Taking the liminf on both sides and using the claim and $t>0$,

$$
t \lim _{n \rightarrow \infty} \inf _{n}\left\langle A u_{n}, u-v\right\rangle \geq t\langle A(u+t(v-u)),(u-v)\rangle
$$

Next divide by $t$ and use the Hemicontinuity of $A$ to conclude that

$$
\lim _{n \rightarrow \infty} \inf _{n \rightarrow}\left\langle A u_{n}, u-v\right\rangle \geq\langle A u, u-v\rangle
$$

From the claim,

$$
\begin{aligned}
\lim \inf _{n \rightarrow \infty}\left\langle A u_{n}, u-v\right\rangle & =\lim \inf _{n \rightarrow \infty}\left(\left\langle A u_{n}, u_{n}-v\right\rangle+\left\langle A u_{n}, u-u_{n}\right\rangle\right) \\
& =\lim \inf _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-v\right\rangle \geq\langle A u, u-v\rangle \text { ■ }
\end{aligned}
$$

Monotonicity is very important in the above proof. The next example shows that even if the operator is linear and bounded, it is not necessarily pseudomonotone.

Example 25.1.5 Let $H$ be any Hilbert space (complete inner product space, more on these later) and let $A: H \rightarrow H^{\prime}$ be given by

$$
\langle A x, y\rangle \equiv(-x, y)_{H}
$$

Then A fails to be pseudomonotone.
Proof: Let $\left\{x_{n}\right\}_{n=1}^{\infty}$ be an orthonormal set of vectors in $H$. Then Parsevall's inequality implies

$$
\|x\|^{2} \geq \sum_{n=1}^{\infty}\left|\left(x_{n}, x\right)\right|^{2}
$$

and so for any $x \in H, \lim _{n \rightarrow \infty}\left(x_{n}, x\right)=0$. Thus $x_{n} \rightharpoonup 0 \equiv x$. Also

$$
\lim \sup _{n \rightarrow \infty}\left\langle A x_{n}, x_{n}-x\right\rangle=
$$

$$
\lim \sup _{n \rightarrow \infty}\left\langle A x_{n}, x_{n}-0\right\rangle=\lim \sup _{n \rightarrow \infty}\left(-\left\|x_{n}\right\|^{2}\right)=-1 \leq 0
$$

If $A$ were pseudomonotone, we would need to be able to conclude that for all $y \in H$,

$$
\lim \inf _{n \rightarrow \infty}\left\langle A x_{n}, x_{n}-y\right\rangle \geq\langle A x, x-y\rangle=0
$$

However,

$$
\lim _{n \rightarrow \infty} \inf _{n \rightarrow 2}\left\langle A x_{n}, x_{n}-0\right\rangle=-1<0=\langle A 0,0-0\rangle
$$

The following proposition is useful.
Proposition 25.1.6 Suppose $A: V \rightarrow V^{\prime}$ is pseudomonotone and bounded where $V$ is separable. Then it must be demicontinuous. This means that if $u_{n} \rightarrow u$, then $A u_{n} \rightharpoonup A u$. In case that $V$ is reflexive, you don't need the assumption that $V$ is separable.

Proof: Since $u_{n} \rightarrow u$ is strong convergence and since $A u_{n}$ is bounded, it follows

$$
\lim \sup _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-u\right\rangle=\lim _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-u\right\rangle=0
$$

Suppose this is not so that $A u_{n}$ converges weakly to $A u$. Since $A$ is bounded, there exists a subsequence, still denoted by $n$ such that $A u_{n} \rightharpoonup \xi$ weak $*$. I need to verify $\xi=A u$. From the above, it follows that for all $v \in V$

$$
\begin{aligned}
\langle A u, u-v\rangle & \leq \lim \inf _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-v\right\rangle \\
& =\lim \inf _{n \rightarrow \infty}\left\langle A u_{n}, u-v\right\rangle=\langle\xi, u-v\rangle
\end{aligned}
$$

Hence $\xi=A u$.
There is another type of operator which is more general than pseudomonotone.
Definition 25.1.7 Let $A: V \rightarrow V^{\prime}$ be an operator. Then $A$ is called type $M$ if whenever $u_{n} \rightharpoonup u$ and $A u_{n} \rightharpoonup \xi$, and

$$
\lim _{\sup _{n \rightarrow \infty}}\left\langle A u_{n}, u_{n}\right\rangle \leq\langle\xi, u\rangle
$$

it follows that $A u=\xi$.
Proposition 25.1.8 If $A$ is pseudomonotone, then $A$ is type $M$.
Proof: Suppose $A$ is pseudomonotone and $u_{n} \rightharpoonup u$ and $A u_{n} \rightharpoonup \xi$, and

$$
\lim _{\sup _{n \rightarrow \infty}}\left\langle A u_{n}, u_{n}\right\rangle \leq\langle\xi, u\rangle
$$

Then

$$
\lim \sup _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-u\right\rangle=\lim \sup _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}\right\rangle-\langle\xi, u\rangle \leq 0
$$

Hence

$$
\lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-v\right\rangle \geq\langle A u, u-v\rangle
$$

for all $v \in V$. Consequently, for all $v \in V$,

$$
\begin{aligned}
\langle A u, u-v\rangle & \leq \lim _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-v\right\rangle \\
& =\lim _{n \rightarrow \infty}\left(\left\langle A u_{n}, u-v\right\rangle+\left\langle A u_{n}, u_{n}-u\right\rangle\right) \\
& =\langle\xi, u-v\rangle+\lim \inf _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-u\right\rangle \leq\langle\xi, u-v\rangle
\end{aligned}
$$

and so $A u=\xi$.
An interesting result is the following which states that a monotone linear function added to a type $M$ is also type $M$.

Proposition 25.1.9 Suppose $A: V \rightarrow V^{\prime}$ is type $M$ and suppose $L: V \rightarrow V^{\prime}$ is monotone, bounded and linear. Then $L+A$ is type $M$. Let $V$ be separable or reflexive so that the weak convergences in the following argument are valid.

Proof: Suppose $u_{n} \rightharpoonup u$ and $A u_{n}+L u_{n} \rightharpoonup \xi$ and also that

$$
\lim \sup _{n \rightarrow \infty}\left\langle A u_{n}+L u_{n}, u_{n}\right\rangle \leq\langle\xi, u\rangle
$$

Does it follow that $\xi=A u+L u$ ? Suppose not. There exists a further subsequence, still called $n$ such that $L u_{n} \rightharpoonup L u$. This follows because $L$ is linear and bounded. Then from monotonicity,

$$
\left\langle L u_{n}, u_{n}\right\rangle \geq\left\langle L u_{n}, u\right\rangle+\left\langle L(u), u_{n}-u\right\rangle
$$

Hence with this further subsequence, the lim sup is no larger and so

$$
\lim \sup _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}\right\rangle+\lim _{n \rightarrow \infty}\left(\left\langle L u_{n}, u\right\rangle+\left\langle L(u), u_{n}-u\right\rangle\right) \leq\langle\xi, u\rangle
$$

and so

$$
\lim _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}\right\rangle \leq\langle\xi-L u, u\rangle
$$

It follows since $A$ is type $M$ that $A u=\xi-L u$, which contradicts the assumption that $\xi \neq$ $A u+L u$.

There is also the following useful generalization of the above proposition.
Corollary 25.1.10 Suppose $A: V \rightarrow V^{\prime}$ is type $M$ and suppose $L: W \rightarrow W^{\prime}$ is monotone, bounded and linear where $V \subseteq W$ and $V$ is dense in $W$ so that $W^{\prime} \subseteq V^{\prime}$. Then for $u_{0} \in W$ define $M(u) \equiv L\left(u-u_{0}\right)$. Then $M+A$ is type $M$. Let $V$ be separable or reflexive so that the weak convergences in the following argument are valid.

Proof: Suppose $u_{n} \rightharpoonup u$ and $A u_{n}+M u_{n} \rightharpoonup \xi$ and also that

$$
\lim \sup _{n \rightarrow \infty}\left\langle A u_{n}+M u_{n}, u_{n}\right\rangle \leq\langle\xi, u\rangle
$$

Does it follow that $\xi=A u+M u$ ? Suppose not. By assumption, $u_{n} \rightharpoonup u$ and so,

$$
u_{n}-u_{0} \rightharpoonup u-u_{0} \text { weak convergence in } W
$$

since $L$ is bounded, there is a further subsequence, still called $n$ such that

$$
M u_{n}=L\left(u_{n}-u_{0}\right) \rightharpoonup L\left(u-u_{0}\right)=M u
$$

Since $M$ is monotone,

$$
\left\langle M u_{n}-M u, u_{n}-u\right\rangle \geq 0
$$

Thus

$$
\left\langle M u_{n}, u_{n}\right\rangle-\left\langle M u_{n}, u\right\rangle-\left\langle M u, u_{n}\right\rangle+\langle M u, u\rangle \geq 0
$$

and so

$$
\left\langle M u_{n}, u_{n}\right\rangle \geq\left\langle M u_{n}, u\right\rangle+\left\langle M u, u_{n}-u\right\rangle
$$

Hence with this further subsequence, the limsup is no larger and so

$$
\begin{gathered}
\langle\xi, u\rangle \geq \lim \sup _{n \rightarrow \infty}\left\langle A u_{n}+M u_{n}, u_{n}\right\rangle \\
\geq \lim \sup _{n \rightarrow \infty}\left(\left\langle A u_{n}, u_{n}\right\rangle+\left\langle M u_{n}, u\right\rangle+\left\langle M u, u_{n}-u\right\rangle\right) \\
=\lim \sup _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}\right\rangle+\lim _{n \rightarrow \infty}\left(\left\langle M u_{n}, u\right\rangle+\left\langle M(u), u_{n}-u\right\rangle\right) \leq\langle\xi, u\rangle
\end{gathered}
$$

and so

$$
\lim \sup _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}\right\rangle \leq\langle\xi-M u, u\rangle
$$

It follows since $A$ is type $M$ that $A u=\xi-M u$, which contradicts the assumption that $\xi \neq A u+M u$.

The following is Browder's lemma. It is a very interesting application of the Brouwer fixed point theorem.

Lemma 25.1.11 (Browder) Let $K$ be a convex closed and bounded set in $\mathbb{R}^{n}$ and let $A$ : $K \rightarrow \mathbb{R}^{n}$ be continuous and $\mathbf{f} \in \mathbb{R}^{n}$. Then there exists $\mathbf{x} \in K$ such that for all $\mathbf{y} \in K$,

$$
(\mathbf{f}-A \mathbf{x}, \mathbf{y}-\mathbf{x})_{\mathbb{R}^{n}} \leq 0
$$

If $K$ is convex, closed, bounded subset of $V$ a finite dimensional vector space, then the same conclusion holds. If $f \in V^{\prime}$, there exists $x \in K$ such that for all $y \in K$,

$$
\langle f-A x, y-x\rangle_{V^{\prime}, V} \leq 0
$$

Proof: Let $P_{K}$ denote the projection onto $K$. Thus $P_{K}$ is Lipschitz continuous.

$$
\mathbf{x} \rightarrow P_{K}(\mathbf{f}-A \mathbf{x}+\mathbf{x})
$$

is a continuous map from $K$ to $K$. By the Brouwer fixed point theorem, it has a fixed point $\mathbf{x} \in K$. Therefore, for all $\mathbf{y} \in K$,

$$
(\mathbf{f}-A \mathbf{x}+\mathbf{x}-\mathbf{x}, \mathbf{y}-\mathbf{x})=(\mathbf{f}-A \mathbf{x}, \mathbf{y}-\mathbf{x}) \leq 0
$$

As to the second claim. Consider the following diagram.

$$
\begin{array}{lll}
\mathbb{R}^{n} & \stackrel{\theta^{*}}{\leftarrow} & V^{\prime} \\
\mathbb{R}^{n} & \stackrel{\theta}{\rightarrow} & V
\end{array}
$$

where

$$
\theta(\mathbf{x})=\sum_{i=1}^{n} x_{i} v_{i}
$$

Thus $\theta$ and $\theta^{*}$ are both continuous linear and one to one and onto. Hence there is $\mathbf{x} \in \theta^{-1} K$ a closed convex and bounded subset of $\mathbb{R}^{n}$ such that $\mathbf{x}=\theta^{-1} u, u \in K$, and

$$
\left(\theta^{*} f-\theta^{*} A \theta\left(\theta^{-1} u\right), \theta^{-1} y-\theta^{-1} u\right)_{\mathbb{R}^{n}} \equiv\langle f-A u, y-u\rangle_{V^{\prime}, V} \leq 0
$$

for all $y \in K$.
From this lemma, there is an interesting theorem on surjectivity.
Proposition 25.1.12 Let $A: V \rightarrow V^{\prime}$ be continuous and coercive,

$$
\lim _{\|v\| \rightarrow \infty} \frac{\left\langle A\left(v+v_{0}\right), v\right\rangle}{\|v\|_{V}}=\infty
$$

for some $v_{0}$. Then for all $f \in V^{\prime}$, there exists $v \in V$ such that $A v=f$.
Proof: Define the closed convex sets $B_{n} \equiv \overline{B\left(v_{0}, n\right)}$. By Browder's lemma, there exists $\mathbf{x}_{n}$ such that

$$
\left(f-A v_{n}, y-v_{n}\right) \leq 0
$$

for all $y \in B_{n}$. Then taking $y=v_{0}$,

$$
\left\langle A v_{n}, v_{n}-v_{0}\right\rangle \leq\left\langle f, v_{n}-v_{0}\right\rangle
$$

letting $w_{n}=v_{n}-v_{0}$,

$$
\left\langle A\left(w_{n}+v_{0}\right), w_{n}\right\rangle \leq\left\langle f, w_{n}\right\rangle
$$

and so

$$
\frac{\left\langle A\left(w_{n}+v_{0}\right), w_{n}\right\rangle}{\left\|w_{n}\right\|} \leq\|f\|
$$

which implies that the $\left\|w_{n}\right\|$ and hence the $\left\|v_{n}\right\|$ are bounded. It follows that for large $n, v_{n}$ is an interior point of $B_{n}$. Therefore,

$$
\left\langle f-A v_{n}, z\right\rangle_{V^{\prime}, V} \leq 0
$$

for all $z$ in some open ball centered at $v_{0}$. Hence $f-A v_{n}=0$.
Lemma 25.1.13 Let $A: V \rightarrow V^{\prime}$ be type $M$ and bounded and suppose $V$ is reflexive or $V$ is separable. Then $A$ is demicontinuous.

Proof: Suppose $u_{n} \rightarrow u$ and $A u_{n}$ fails to converge weakly to $A u$. Then there is a further subsequence, still denoted as $u_{n}$ such that $A u_{n} \rightharpoonup \zeta \neq A u$. Then thanks to the strong convergence, you have

$$
\lim _{\sup _{n \rightarrow \infty}}\left\langle A u_{n}, u_{n}\right\rangle=\langle\zeta, u\rangle
$$

which implies $\zeta=A u$ after all.
With these lemmas and the above proposition, there is a very interesting surjectivity result.

Theorem 25.1.14 Let $A: V \rightarrow V^{\prime}$ be type $M$, bounded, and coercive

$$
\begin{equation*}
\lim _{\|u\| \rightarrow \infty} \frac{\left\langle A\left(u+u_{0}\right), u\right\rangle}{\|u\|}=\infty \tag{25.1.4}
\end{equation*}
$$

for some $u_{0}$, where $V$ is a separable reflexive Banach space. Then $A$ is surjective.
Proof: Since $V$ is separable, there exists an increasing sequence of finite dimensional subspaces $\left\{V_{n}\right\}$ such that $\overline{\cup_{n} V_{n}}=V$ and each $V_{n}$ contains $u_{0}$. Say $\operatorname{span}\left(v_{1}, \cdots, v_{n}\right)=V_{n}$. Then consider the following diagram.

$$
\begin{array}{lll}
V_{n}^{\prime} & \stackrel{i^{*}}{\leftarrow} & V^{\prime} \\
V_{n} & \xrightarrow{i} & V
\end{array}
$$

The map $i$ is the inclusion map. Consider the map $i^{*} A i$. By Lemma 25.1.13 this map is continuous.

$$
\frac{\left\langle i^{*} A i\left(v+u_{0}\right), v\right\rangle_{V_{n}^{\prime} V_{n}}}{\|v\|}=\frac{\left\langle A\left(v+u_{0}\right), v\right\rangle_{V^{\prime}, V}}{\|v\|}
$$

Hence $i^{*} A i$ is coercive. Let $f \in V^{\prime}$. Then from Proposition 25.1.12, there exists $\mathbf{x}_{n}$ such that

$$
i^{*} A i v_{n}=i^{*} f
$$

In other words,

$$
\begin{equation*}
\left\langle A v_{n}, y\right\rangle_{V^{\prime} V}=\langle f, y\rangle_{V^{\prime} V} \tag{25.1.5}
\end{equation*}
$$

for all $y \in V_{n}$. Letting $y \equiv v_{n}-u_{0} \equiv w_{n}$,

$$
\left\langle A\left(w_{n}+u_{0}\right), w_{n}\right\rangle=\left\langle f, w_{n}\right\rangle
$$

Then from the coercivity condition 25.1.4, the $w_{n}$ are bounded independent of $n$. Hence this is also true of the $v_{n}$. Since $V$ is reflexive, there is a subsequence, still called $\left\{v_{n}\right\}$ which converges weakly to $v \in V$. Since $A$ is bounded, it can also be assumed that $A v_{n} \rightharpoonup \zeta \in V^{\prime}$. Then

$$
\lim \sup _{n \rightarrow \infty}\left\langle A v_{n}, v_{n}\right\rangle=\lim \sup _{n \rightarrow \infty}\left\langle f, v_{n}\right\rangle=\langle f, v\rangle
$$

Also, passing to the limit in 25.1.5,

$$
\langle\zeta, y\rangle=\langle f, y\rangle
$$

for any $y \in V_{n}$, this for any $n$. Since the union of these $V_{n}$ is dense, it follows that the above equation holds for all $y \in V$. Therefore, $f=\zeta$ and so

$$
\lim \sup _{n \rightarrow \infty}\left\langle A v_{n}, v_{n}\right\rangle=\lim \sup _{n \rightarrow \infty}\left\langle f, v_{n}\right\rangle=\langle f, v\rangle=\langle\zeta, v\rangle
$$

Since $A$ is type $M, A v=\zeta=f$.
You can generalize pseudomonotone slightly without any trouble.
Definition 25.1.15 Let $V$ be a Banach space and let $K$ be a closed convex nonempty subset of $V$. Then $A: K \rightarrow V^{\prime}$ is pseudomonotone if similar conditions hold as above. That is, if

$$
\begin{equation*}
u_{n} \rightharpoonup u \tag{25.1.6}
\end{equation*}
$$

and

$$
\begin{equation*}
\lim \sup _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-u\right\rangle \leq 0 \tag{25.1.7}
\end{equation*}
$$

it follows that for all $v \in K$,

$$
\begin{equation*}
\lim \inf _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-v\right\rangle \geq\langle A u, u-v\rangle \tag{25.1.8}
\end{equation*}
$$

Then it is easy to give a nice result on variational inequalities.
Proposition 25.1.16 Let $K$ be a closed convex nonempty subset of $V$ a separable reflexive Banach space. Let $A: K \rightarrow V^{\prime}$ be pseudomonotone and bounded. Also assume that either $K$ is bounded or there is a coercivity condition

$$
\lim _{\|u\| \rightarrow \infty} \frac{\left\langle A u, u-u_{0}\right\rangle}{\|u\|}=\infty, u_{0} \in K
$$

then for $f \in V^{\prime}$, there exists $u \in K$ such that for all $v \in K$,

$$
\langle A u, u-v\rangle \leq\langle f, u-v\rangle
$$

Proof: Let $V_{n}$ be finite dimensional spaces whose union is dense in $V, \cdots V_{n} \subseteq V_{n+1} \cdots$, each containing $u_{0}, n>\left\|u_{0}\right\|$. By a repeat of the proof of Proposition 25.1.2, $i^{*} A i$ will be continuous on $K$. Therefore, by Browder's lemma, there exists $u_{n} \in K_{n} \equiv K \cap B(0, n) \cap V_{n}$ such that for all $v \in K_{n}$,

$$
\left\langle i^{*} f-i^{*} A i u_{n}, v-u_{n}\right\rangle_{V_{n}^{\prime}, V_{n}}=\left\langle f-A u_{n}, v-u_{n}\right\rangle_{V^{\prime}, V} \leq 0
$$

Now assume we don't know that $K$ is bounded. In case it is bounded, the argument simplifies. In the harder case, the coercivity condition implies that the $u_{n}$ are bounded in $V$. This follows from letting $v=u_{0}$ in the above inequality. Thus

$$
\left\langle f, u_{n}-u_{0}\right\rangle \geq\left\langle A u_{n}, u_{n}-u_{0}\right\rangle
$$

Hence

$$
\frac{\left\langle A u_{n}, u_{n}-u_{0}\right\rangle}{\left\|u_{n}\right\|} \leq \frac{\|f\|\left\|u_{n}-u_{0}\right\|}{\left\|u_{n}\right\|}
$$

The right side is bounded and so it follows that the left side is also bounded. Therefore, $\left\|u_{n}\right\|$ must be bounded. Taking a subsequence and using the assumption that $V$ is reflexive, we can obtain

$$
u_{n} \rightarrow u \text { weakly in } V
$$

By the fact that convex closed sets are weakly closed also, it follows that $u \in K$. Also, given $M$, eventually all $\left\|u_{n}\right\|$ and $\|u\|$ are less than $M$. Now from the inequality,

$$
\left\langle A u_{n}, u_{n}-v\right\rangle \leq\left\langle f, u_{n}-v\right\rangle
$$

Thus

$$
\left\langle A u_{n}, u_{n}-u\right\rangle+\left\langle A u_{n}, u-v\right\rangle \leq\left\langle f, u_{n}-u\right\rangle+\langle f, u-v\rangle
$$

Then taking limsup ${ }_{n \rightarrow \infty}$ one gets

$$
\lim \sup _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-u\right\rangle+\langle\xi, u-v\rangle \leq\langle f, u-v\rangle
$$

This holds for $v \in K_{m}$ where $m$ is arbitrary. Hence one could let $v_{m} \rightarrow u$. Thus eventually $\left\|v_{m}\right\|<M$ and so for large $m, v_{m} \in K_{m}$. Then it follows that

$$
\lim \sup _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-u\right\rangle \leq 0
$$

Consequently, by the assumption that $A$ is pseudomonotone on $K$, for every $v \in K$,

$$
\begin{equation*}
\langle A u, u-v\rangle \leq \lim \inf _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-v\right\rangle \tag{*}
\end{equation*}
$$

for all $v \in K$. Then from the inequality obtained from Browder's lemma,

$$
\left\langle A u_{n}, u_{n}-v\right\rangle_{V^{\prime}, V} \leq\left\langle f, u_{n}-v\right\rangle_{V^{\prime}, V}
$$

and so *implies on taking liminf that for all $v \in K$,

$$
\langle A u, u-v\rangle_{V^{\prime}, V} \leq\langle f, u-v\rangle_{V^{\prime}, V}
$$

### 25.2 Duality Maps

The duality map is an attempt to duplicate some of the features of the Riesz map in Hilbert space which is discussed in the chapter on Hilbert space.

Definition 25.2.1 A Banach space is said to be strictly convex if whenever $\|x\|=\|y\|$ and $x \neq y$, then

$$
\left\|\frac{x+y}{2}\right\|<\|x\|
$$

$F: X \rightarrow X^{\prime}$ is said to be a duality map if it satisfies the following: a.) $\|F(x)\|=\|x\|^{p-1}$. b.) $F(x)(x)=\|x\|^{p}$, where $p>1$.

Duality maps exist. Here is why. Let

$$
F(x) \equiv\left\{x^{*}:\left\|x^{*}\right\| \leq\|x\|^{p-1} \text { and } x^{*}(x)=\|x\|^{p}\right\}
$$

Then $F(x)$ is not empty because you can let $f(\alpha x)=\alpha\|x\|^{p}$. Then $f$ is linear and defined on a subspace of $X$. Also

$$
\sup _{\|\alpha x\| \leq 1}|f(\alpha x)|=\sup _{\|\alpha x\| \leq 1}|\alpha|\|x\|^{p} \leq\|x\|^{p-1}
$$

Also from the definition,

$$
f(x)=\|x\|^{p}
$$

and so, letting $x^{*}$ be a Hahn Banach extension, it follows $x^{*} \in F(x)$. Also, $F(x)$ is closed and convex. It is clearly closed because if $x_{n}^{*} \rightarrow x^{*}$, the condition on the norm clearly holds and also the other one does too. It is convex because

$$
\left\|x^{*} \lambda+(1-\lambda) y^{*}\right\| \leq \lambda\left\|x^{*}\right\|+(1-\lambda)\left\|y^{*}\right\| \leq \lambda\|x\|^{p-1}+(1-\lambda)\|x\|^{p-1}
$$

If the conditions hold for $x^{*}$, then we can show that in fact $\left\|x^{*}\right\|=\|x\|^{p-1}$. This is because

$$
\left\|x^{*}\right\| \geq\left|x^{*}\left(\frac{x}{\|x\|}\right)\right|=\frac{1}{\|x\|}\left|x^{*}(x)\right|=\|x\|^{p-1}
$$

Now how many things are in $F(x)$ assuming the norm on $X^{\prime}$ is strictly convex? Suppose $x_{1}^{*}$, and $x_{2}^{*}$ are two things in $F(x)$. Then by convexity, so is $\left(x_{1}^{*}+x_{2}^{*}\right) / 2$. Hence by strict convexity, if the two are different, then

$$
\left\|\frac{x_{1}^{*}+x_{2}^{*}}{2}\right\|=\|x\|^{p-1}<\frac{1}{2}\left\|x_{1}^{*}\right\|+\frac{1}{2}\left\|x_{2}^{*}\right\|=\|x\|^{p-1}
$$

which is a contradiction. Therefore, $F$ is an actual mapping.
What are some of its properties? First is one which is similar to the Cauchy Schwarz inequality. Since $p-1=p / p^{\prime}$,

$$
\sup _{\|y\| \leq 1}|\langle F x, y\rangle|=\|x\|^{p / p^{\prime}}
$$

and so for arbitrary $y \neq 0$,

$$
\begin{aligned}
|\langle F x, y\rangle| & =\|y\|\left|\left\langle F x, \frac{y}{\|y\|}\right\rangle\right| \leq\|y\|\|x\|^{p / p^{\prime}} \\
& =|\langle F y, y\rangle|^{1 / p}|\langle F x, x\rangle|^{1 / p^{\prime}}
\end{aligned}
$$

Next we can show that $F$ is monotone.

$$
\begin{aligned}
\langle F x-F y, x-y\rangle & =\langle F x, x\rangle-\langle F x, y\rangle-\langle F y, x\rangle+\langle F y, y\rangle \\
& \geq\|x\|^{p}+\|y\|^{p}-\|y\|\|x\|^{p / p^{\prime}}-\|y\|^{p / p^{\prime}}\|x\|
\end{aligned}
$$

$$
\geq\|x\|^{p}+\|y\|^{p}-\left(\frac{\|y\|^{p}}{p}+\frac{\|x\|^{p}}{p^{\prime}}\right)-\left(\frac{\|y\|^{p}}{p^{\prime}}+\frac{\|x\|^{p}}{p}\right)=0
$$

Next it can be shown that $F$ is hemicontinuous. By the construction, $F(x+t y)$ is bounded as $t \rightarrow 0$. Let $t \rightarrow 0$ be a subsequence such that

$$
F(x+t y) \rightarrow \xi \text { weak } *
$$

Then we ask: Does $\xi$ do what it needs to do in order to be $F(x)$ ? The answer is yes. First of all $\|F(x+t y)\|=\|x+t y\|^{p-1} \rightarrow\|x\|^{p-1}$. The set

$$
\left\{x^{*}:\left\|x^{*}\right\| \leq\|x\|^{p-1}+\varepsilon\right\}
$$

is closed and convex and so it is weak $*$ closed as well. For all small enough $t$, it follows $F(x+t y)$ is in this set. Therefore, the weak limit is also in this set and it follows $\|\xi\| \leq$ $\|x\|^{p-1}+\varepsilon$. Since $\varepsilon$ is arbitrary, it follows $\|\xi\| \leq\|x\|^{p-1}$. Is $\xi(x)=\|x\|^{p}$ ? We have

$$
\begin{aligned}
\|x\|^{p} & =\lim _{t \rightarrow 0}\|x+t y\|^{p}=\lim _{t \rightarrow 0}\langle F(x+t y), x+t y\rangle \\
& =\lim _{t \rightarrow 0}\langle F(x+t y), x\rangle=\langle\xi, x\rangle
\end{aligned}
$$

and so, $\xi$ does what it needs to do to be $F(x)$. This would be clear if $\|\xi\|=\|x\|^{p-1}$. However, $|\langle\xi, x\rangle|=\|x\|^{p}$ and so $\|\xi\| \geq\left|\left\langle\xi, \frac{x}{\|x\|}\right\rangle\right|=\|x\|^{p-1}$. Thus $\|\xi\|=\|x\|^{p-1}$ which shows $\xi$ does everyting it needs to do to equal $F(x)$ and so it is $F(x)$. Since this conclusion follows for any convergent sequence, it follows that $F(x+t y)$ converges to $F(x)$ weakly as $t \rightarrow 0$. This is what it means to be hemicontinuous. This proves the following theorem. One can show also that $F$ is demicontinuous which means strongly convergent sequences go to weakly convergent sequences. Here is a proof for the case where $p=2$. You can clearly do the same thing for arbitrary $p$.

Lemma 25.2.2 Let $F$ be a duality map for $p=2$ where $X, X^{\prime}$ are reflexive and have strictly convex norms. (If $X$ is reflexive, there is always an equivalent strictly convex norm [8].) Then $F$ is demicontinuous.

Proof: Say $x_{n} \rightarrow x$. Then does it follow that $F x_{n} \rightharpoonup F x$ ? Suppose not. Then there is a subsequence, still denoted as $x_{n}$ such that $x_{n} \rightarrow x$ but $F x_{n} \rightharpoonup y \neq F x$ where here $\rightharpoonup$ denotes weak convergence. This follows from the Eberlein Smulian theorem. Then

$$
\langle y, x\rangle=\lim _{n \rightarrow \infty}\left\langle F x_{n}, x_{n}\right\rangle=\lim _{n \rightarrow \infty}\left\|x_{n}\right\|^{2}=\|x\|^{2}
$$

Also, there exists $z,\|z\|=1$ and $\langle y, z\rangle \geq\|y\|-\varepsilon$. Then

$$
\|y\|-\varepsilon \leq\langle y, z\rangle=\lim _{n \rightarrow \infty}\left\langle F x_{n}, z\right\rangle \leq \lim \inf _{n \rightarrow \infty}\left\|F x_{n}\right\|=\lim \inf _{n \rightarrow \infty}\left\|x_{n}\right\|=\|x\|
$$

and since $\varepsilon$ is arbitrary, $\|y\| \leq\|x\|$. It follows from the above construction of $F x$, that $y=F x$ after all, a contradiction.

Theorem 25.2.3 Let $X$ be a reflexive Banach space with $X^{\prime}$ having strictly convex norm ${ }^{1}$. Then for $p>1$, there exists a mapping $F: X \rightarrow X^{\prime}$ which is bounded, monotone, hemicontinuous, coercive in the sense that $\lim _{|x| \rightarrow \infty}\langle F x, x\rangle /|x|=\infty$, which also satisfies the inequalities

$$
|\langle F x, y\rangle| \leq|\langle F x, x\rangle|^{1 / p^{\prime}}|\langle F y, y\rangle|^{1 / p}
$$

Note that these conclusions about duality maps show that they map onto the dual space.
The duality map was onto and it was monotone. This was shown above. Consider the form of a duality map for the $L^{p}$ spaces. Let $F: L^{p} \rightarrow\left(L^{p}\right)^{\prime}$ be the one which satisfies

$$
\|F f\|=\|f\|^{p-1},\langle F f, f\rangle=\|f\|^{p}
$$

Then in this case,

$$
F f=|f|^{p-2} \bar{f}
$$

This is because it does what it needs to do.

$$
\begin{aligned}
\|F f\|_{L^{p^{\prime}}} & =\left(\int_{\Omega}\left(|f|^{p-1}\right)^{p^{\prime}} d \mu\right)^{1 / p^{\prime}}=\left(\int_{\Omega}\left(|f|^{p / p^{\prime}}\right)^{p^{\prime}} d \mu\right)^{1 / p^{\prime}} \\
& =\left(\int_{\Omega}|f|^{p} d \mu\right)^{1-(1 / p)}=\left(\left(\int_{\Omega}|f|^{p} d \mu\right)^{1 / p}\right)^{p-1}=\|f\|_{L^{p}}^{p-1}
\end{aligned}
$$

while it is obvious that

$$
\langle F f, f\rangle=\int_{\Omega}|f|^{p} d \mu=\|f\|_{L^{p}(\Omega)}^{p}
$$

Now here is an interesting inequality which I will only consider in the case where the quantities are real valued.

Lemma 25.2.4 Let $p \geq 2$. Then for $a, b$ real numbers,

$$
\left(|a|^{p-2} a-|b|^{p-2} b\right)(a-b) \geq C|a-b|^{p}
$$

for some constant $C$ independent of $a, b$.
Proof: There is nothing to show if $a=b$. Without loss of generality, assume $a>b$. Also assume $p>2$. There is nothing to show if $p=2$. I want to show that there exists a constant $C$ such that for $a>b$,

$$
\begin{equation*}
\frac{|a|^{p-2} a-|b|^{p-2} b}{|a-b|^{p-1}} \geq C \tag{25.2.9}
\end{equation*}
$$

First assume also that $b \geq 0$. Now it is clear that as $a \rightarrow \infty$, the quotient above converges to 1. Take the derivative of this quotient. This yields

$$
(p-1)|a-b|^{p-2} \frac{|a|^{p-2}|a-b|-\left(|a|^{p-2} a-|b|^{p-2} b\right)}{|a-b|^{2 p-2}}
$$

[^22]Now remember $a>b$. Then the above reduces to

$$
(p-1)|a-b|^{p-2} b \frac{|b|^{p-2}-|a|^{p-2}}{|a-b|^{2 p-2}}
$$

Since $b \geq 0$, this is negative and so 1 would be a lower bound. Now suppose $b<0$. Then the above derivative is negative for $b<a \leq-b$ and then it is positive for $a>-b$. It equals 0 when $a=-b$. Therefore the quotient in 25.2.9 achieves its minimum value when $a=-b$. This value is

$$
\frac{|-b|^{p-2}(-b)-|b|^{p-2} b}{|-b-b|^{p-1}}=|b|^{p-2} \frac{-2 b}{|2 b|^{p-1}}=|b|^{p-2} \frac{1}{|2 b|^{p-2}}=\frac{1}{2^{p-2}} .
$$

Therefore, the conclusion holds whenever $p \geq 2$. That is

$$
\left(|a|^{p-2} a-|b|^{p-2} b\right)(a-b) \geq \frac{1}{2^{p-2}}|a-b|^{p}
$$

This proves the lemma.
However, in the context of strictly convex norms on the reflexive Banach space $X$, the following important result holds. I will give it first for the case where $p=2$ since this is the case of most interest.

Theorem 25.2.5 Let $X$ be a reflexive Banach space and $X, X^{\prime}$ have strictly convex norms as discussed above. Let $F$ be the duality map with $p=2$. Then $F$ is strictly monotone. This means

$$
\langle F u-F v, u-v\rangle \geq 0
$$

and it equals 0 if and only if $u-v$.
Proof: First why is it monotone? By definition of $F,\langle F(u), u\rangle=\|u\|^{2}$ and $\|F(u)\|=$ $\|u\|$. Then

$$
|\langle F u, v\rangle|=\left|\left\langle F u, \frac{v}{\|v\|}\right\rangle\right|\|v\| \leq\|F u\|\|v\|=\|u\|\|v\|
$$

Hence

$$
\begin{aligned}
\langle F u-F v, u-v\rangle & =\|u\|^{2}+\|v\|^{2}-\langle F u, v\rangle-\langle F v, u\rangle \\
& \geq\|u\|^{2}+\|v\|^{2}-2\|u\|\|v\| \geq 0
\end{aligned}
$$

Now suppose $\|x\|=\|y\|=1$ but $x \neq y$. Then

$$
\left\langle F x, \frac{x+y}{2}\right\rangle \leq\left\|\frac{x+y}{2}\right\|<\frac{\|x\|+\|y\|}{2}=1
$$

It follows that

$$
\frac{1}{2}\langle F x, x\rangle+\frac{1}{2}\langle F x, y\rangle=\frac{1}{2}+\frac{1}{2}\langle F x, y\rangle<1
$$

and so

$$
\langle F x, y\rangle<1
$$

For arbitrary $x, y, x /\|x\| \neq y /\|y\|$

$$
\langle F x, y\rangle=\|x\|\|y\|\left\langle F\left(\frac{x}{\|x\|}\right),\left(\frac{y}{\|y\|}\right)\right\rangle
$$

It is easy to check that $F(\alpha x)=\alpha F(x)$. Therefore,

$$
|\langle F x, y\rangle|=\|x\|\|y\|\left\langle F\left(\frac{x}{\|x\|}\right),\left(\frac{y}{\|y\|}\right)\right\rangle<\|x\|\|y\|
$$

Now say that $x \neq y$ and consider

$$
\langle F x-F y, x-y\rangle
$$

First suppose $x=\alpha y$. Then the above is

$$
\begin{aligned}
\langle F(\alpha y)-F y,(\alpha-1) y\rangle & =(\alpha-1)\left(\langle F(\alpha y), y\rangle-\|y\|^{2}\right) \\
& =(\alpha-1)\left(\langle\alpha F(y), y\rangle-\|y\|^{2}\right) \\
& =(\alpha-1)^{2}\|y\|^{2}>0
\end{aligned}
$$

The other case is that $x /\|x\| \neq y /\|y\|$ and in this case,

$$
\begin{gathered}
\langle F x-F y, x-y\rangle=\|x\|^{2}+\|y\|^{2}-\langle F x, y\rangle-\langle F y, x\rangle \\
>\|x\|^{2}+\|y\|^{2}-2\|x\|\|y\| \geq 0
\end{gathered}
$$

Thus $F$ is strictly monotone as claimed.
As mentioned, this will hold for any $p>1$. Here is a proof in the case that the Banach space is real which is the usual case of interest. First here is a simple observation.

Observation 25.2.6 Let $p>1$. Then $x \rightarrow|x|^{p-2} x$ is strictly monotone. Here $x \in \mathbb{R}$.
To verify this observation,

$$
\frac{d}{d x}\left(\left(x^{2}\right)^{\frac{p-2}{2}} x\right)=\frac{1}{x^{2}}(p-1)\left(x^{2}\right)^{\frac{1}{2} p}>0
$$

Theorem 25.2.7 Let $X$ be a real reflexive Banach space and $X, X^{\prime}$ have strictly convex norms as discussed above. Let $F$ be the duality map for $p>1$. Then $F$ is strictly monotone. This means

$$
\langle F u-F v, u-v\rangle \geq 0
$$

and it equals 0 if and only if $u-v$.
Proof: First why is it monotone? By definition of $F,\langle F(u), u\rangle=\|u\|^{p}$ and $\|F(u)\|=$ $\|u\|^{p-1}$. Then

$$
|\langle F u, v\rangle|=\left|\left\langle F u, \frac{v}{\|v\|}\right\rangle\right|\|v\| \leq\|F u\|\|v\|=\|u\|^{p-1}\|v\|
$$

Hence

$$
\begin{aligned}
\langle F u-F v, u-v\rangle & =\|u\|^{p}+\|v\|^{p}-\langle F u, v\rangle-\langle F v, u\rangle \\
& \geq\|u\|^{p}+\|v\|^{p}-\|u\|^{p-1}\|v\|-\|u\|\|v\|^{p-1} \\
\geq\|u\|^{p}+\|v\|^{p} & -\left(\frac{\|u\|^{p}}{p^{\prime}}+\frac{\|v\|^{p}}{p}\right)-\left(\frac{\|u\|^{p}}{p}+\frac{\|v\|^{p}}{p^{\prime}}\right)=0
\end{aligned}
$$

Now suppose $\|x\|=\|y\|=1$ but $x \neq y$. Then

$$
\left\langle F x, \frac{x+y}{2}\right\rangle \leq\|x\|^{p-1}\left\|\frac{x+y}{2}\right\|<\frac{\|x\|+\|y\|}{2}=1
$$

It follows that

$$
\frac{1}{2}\langle F x, x\rangle+\frac{1}{2}\langle F x, y\rangle=\frac{1}{2}+\frac{1}{2}\langle F x, y\rangle<1
$$

and so

$$
\langle F x, y\rangle<1
$$

It is easy to check that for nonzero $\alpha, F(\alpha x)=|\alpha|^{p-2} \alpha F(x)$. This is because

$$
\begin{gathered}
\left\||\alpha|^{p-2} \alpha F(x)\right\|=|\alpha|^{p-1}\|x\|^{p-1}=\|\alpha x\|^{p-1} \\
\left.\left.\langle | \alpha\right|^{p-2} \alpha F(x), \alpha x\right\rangle=|\alpha|^{p}\|x\|^{p}=\|\alpha x\|^{p}
\end{gathered}
$$

and so, since $|\alpha|^{p-2} \alpha F(x)$ acts like $F(\alpha x)$, it is $F(\alpha x)$. It follows that for arbitrary $x, y$, such that $x /\|x\| \neq y /\|y\|$

$$
\langle F x, y\rangle=\|x\|^{p-1}\|y\|\left\langle F\left(\frac{x}{\|x\|}\right),\left(\frac{y}{\|y\|}\right)\right\rangle
$$

Therefore,

$$
\begin{equation*}
\langle F x, y\rangle=\|x\|^{p-1}\|y\|\left\langle F\left(\frac{x}{\|x\|}\right),\left(\frac{y}{\|y\|}\right)\right\rangle<\|x\|^{p-1}\|y\| \tag{25.2.10}
\end{equation*}
$$

Now say that $x \neq y$ and consider

$$
\langle F x-F y, x-y\rangle
$$

First suppose $x=\alpha y$. This is the case where $x$ is a multiple of $y$. Then the above is

$$
\begin{gathered}
\langle F(\alpha y)-F y,(\alpha-1) y\rangle=(\alpha-1)\left(\langle F(\alpha y), y\rangle-\|y\|^{p}\right) \\
=(\alpha-1)\left(|\alpha|^{p-2} \alpha\|y\|^{p}-\|y\|^{p}\right)=(\alpha-1)\left(|\alpha|^{p-2} \alpha-1\right)\|y\|^{p}>0
\end{gathered}
$$

by the above observation that $x \rightarrow|x|^{p-2} x$ is strictly monotone. Similarly,

$$
\langle F x-F y, x-y\rangle>0
$$

if $y=\alpha x$ for $\alpha \neq 1$.
Thus the desired result holds in the case that one vector is a multiple of the other. The other case is that neither vector is a multiple of the other. Thus, in particular, $x /\|x\| \neq$ $y /\|y\|$, and in this case, it follows from 25.2.10

$$
\begin{gathered}
\langle F x-F y, x-y\rangle=\|x\|^{p}+\|y\|^{p}-\langle F x, y\rangle-\langle F y, x\rangle \\
>\|x\|^{p}+\|y\|^{p}-\|x\|^{p-1}\|y\|-\|y\|^{p-1}\|x\| \\
\geq\|x\|^{p}+\|y\|^{p}-\left(\frac{\|x\|^{p}}{p^{\prime}}+\frac{\|y\|^{p}}{p}\right)-\left(\frac{\|y\|^{p}}{p^{\prime}}+\frac{\|x\|^{p}}{p}\right)=0
\end{gathered}
$$

Thus $F$ is strictly monotone as claimed.
Another useful observation about duality maps for $p=2$ is that $\left\|F^{-1} y^{*}\right\|_{V}=\left\|y^{*}\right\|_{V^{\prime}}$. This is because

$$
\left\|y^{*}\right\|_{V^{\prime}}=\left\|F F^{-1} y^{*}\right\|_{V^{\prime}}=\left\|F^{-1} y^{*}\right\|_{V}
$$

also from similar reasoning,

$$
\left\langle y^{*}, F^{-1} y^{*}\right\rangle=\left\langle F F^{-1} y^{*}, F^{-1} y^{*}\right\rangle=\left\|F^{-1} y^{*}\right\|_{V}^{2}=\left\|y^{*}\right\|_{V^{\prime}}^{2}
$$

You can give specific inequalities in certain cases. Here is a nice little inequality which will allow this.

Theorem 25.2.8 Let $p \geq 2$ then for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n}$,

$$
\begin{equation*}
\left(|\mathbf{x}|^{p-2} \mathbf{x}-|\mathbf{y}|^{p-2} \mathbf{y}, \mathbf{x}-\mathbf{y}\right) \geq \frac{1}{2^{p-1}}|\mathbf{x}-\mathbf{y}|^{p} \tag{*}
\end{equation*}
$$

Proof: We have $(\mathbf{x}, \mathbf{y})=\frac{1}{2}\left(|\mathbf{x}|^{2}+|\mathbf{y}|^{2}-|\mathbf{x}-\mathbf{y}|^{2}\right)$. Consider the following.

$$
\frac{1}{2}\left(\frac{|\mathbf{x}|^{p-2}+|\mathbf{y}|^{p-2}}{|\mathbf{x}-\mathbf{y}|^{p-2}}\right)|\mathbf{x}-\mathbf{y}|^{p}+\frac{1}{2}\left(|\mathbf{x}|^{p-2}-|\mathbf{y}|^{p-2}\right)\left(|\mathbf{x}|^{2}-|\mathbf{y}|^{2}\right)
$$

multiplying this out gives

$$
\frac{1}{2}\left(|\mathbf{x}|^{p-2}+|\mathbf{y}|^{p-2}\right)\left(|\mathbf{x}|^{2}+|\mathbf{y}|^{2}-2(\mathbf{x}, \mathbf{y})\right)+\frac{1}{2}\left(|\mathbf{x}|^{p}-|\mathbf{x}|^{2}|\mathbf{y}|^{p-2}+|\mathbf{y}|^{p}-|\mathbf{x}|^{p-2}|\mathbf{y}|^{2}\right)
$$

thus this yields

$$
\begin{aligned}
& \frac{1}{2}\left[|\mathbf{x}|^{p}+|\mathbf{y}|^{p-2}|\mathbf{x}|^{2}+|\mathbf{x}|^{p-2}|\mathbf{y}|^{2}+|\mathbf{y}|^{p}-\left(2(\mathbf{x}, \mathbf{y})|\mathbf{x}|^{p-2}+2(\mathbf{x}, \mathbf{y})|\mathbf{y}|^{p-2}\right)\right] \\
& +\frac{1}{2}\left(|\mathbf{x}|^{p}+|\mathbf{y}|^{p}-\left(|\mathbf{x}|^{2}|\mathbf{y}|^{p-2}+|\mathbf{x}|^{p-2}|\mathbf{y}|^{2}\right)\right)
\end{aligned}
$$

It simplifies to

$$
|\mathbf{x}|^{p}+|\mathbf{y}|^{p}-2(\mathbf{x}, \mathbf{y})\left(|\mathbf{x}|^{p-2}+|\mathbf{y}|^{p-2}\right)
$$

On the left side of $*$, when you multiply it out, you get

$$
|\mathbf{x}|^{p}-|\mathbf{x}|^{p-2}(\mathbf{x}, \mathbf{y})-|\mathbf{y}|^{p-2}(\mathbf{x}, \mathbf{y})+|\mathbf{y}|^{p}
$$

which is exactly the same thing. Therefore,

$$
\begin{align*}
\left(|\mathbf{x}|^{p-2} \mathbf{x}-|\mathbf{y}|^{p-2} \mathbf{y}, \mathbf{x}-\mathbf{y}\right)= & \frac{1}{2}\left(\frac{|\mathbf{x}|^{p-2}+|\mathbf{y}|^{p-2}}{|\mathbf{x}-\mathbf{y}|^{p-2}}\right)|\mathbf{x}-\mathbf{y}|^{p}  \tag{**}\\
& +\frac{1}{2}\left(|\mathbf{x}|^{p-2}-|\mathbf{y}|^{p-2}\right)\left(|\mathbf{x}|^{2}-|\mathbf{y}|^{2}\right)
\end{align*}
$$

Suppose first that $p \geq 3$. Now $p \geq 3$ and so $|\mathbf{x}|^{p-2}$ is convex. Hence

$$
\left|\frac{\mathbf{x}+(-\mathbf{y})}{2}\right|^{p-2} \leq \frac{1}{2}\left(|\mathbf{x}|^{p-2}+|-\mathbf{y}|^{p-2}\right)
$$

and so

$$
\left(|\mathbf{x}|^{p-2} \mathbf{x}-|\mathbf{y}|^{p-2} \mathbf{y}, \mathbf{x}-\mathbf{y}\right) \geq\left|\frac{\mathbf{x}-\mathbf{y}}{2}\right|^{p-2} \frac{1}{|\mathbf{x}-\mathbf{y}|^{p-2}}|\mathbf{x}-\mathbf{y}|^{p}=\frac{1}{2^{p-2}}|\mathbf{x}-\mathbf{y}|^{p}
$$

Next suppose $p>2$. There is nothing to show if $p=2$. Then for a positive integer $m$, you can get $m(p-2)>1$. Then

$$
\left(|\mathbf{x}|^{p-2}+|\mathbf{y}|^{p-2}\right)^{m} \geq|\mathbf{x}|^{m(p-2)}+|\mathbf{y}|^{m(p-2)} \geq 2^{1-m(p-2)}|\mathbf{x}-\mathbf{y}|^{m(p-2)}
$$

Thus we can raise both sides of the above to $1 / \mathrm{m}$ and conclude

$$
|\mathbf{x}|^{p-2}+|\mathbf{y}|^{p-2} \geq 2^{1 / m-(p-2)}|\mathbf{x}-\mathbf{y}|^{p-2}
$$

Then we use this in $* *$ to obtain

$$
\begin{gathered}
\left(|\mathbf{x}|^{p-2} \mathbf{x}-|\mathbf{y}|^{p-2} \mathbf{y}, \mathbf{x}-\mathbf{y}\right) \geq \frac{1}{2}\left(\frac{|\mathbf{x}|^{p-2}+|\mathbf{y}|^{p-2}}{|\mathbf{x}-\mathbf{y}|^{p-2}}\right)|\mathbf{x}-\mathbf{y}|^{p} \\
\quad \geq \frac{1}{2} \frac{1}{2^{(p-2)-(1 / m)}}|\mathbf{x}-\mathbf{y}|^{p} \geq \frac{1}{2^{p-1}}|\mathbf{x}-\mathbf{y}|^{p}
\end{gathered}
$$

Thus, if you have the duality map $F$ for $p \geq 2$ for real valued $L^{p}(\Omega)$ to $L^{p^{\prime}}(\Omega)$, it is clear that $F f=|f|^{p-2} f$ and so

$$
\begin{aligned}
& \langle F f-F g, f-g\rangle=\int_{\Omega}\left(|f|^{p-2} f-|g|^{p-2} g\right)(f-g) d \mu \geq \frac{1}{2^{p-1}} \int_{\Omega}|f-g|^{p} d \mu \\
& \langle F f-F g, f-g\rangle \geq \frac{1}{2^{p-1}}\|f-g\|_{L^{p}(\Omega)}^{p}
\end{aligned}
$$

A similar result would hold for the duality map from $\left(L^{p}(\Omega)\right)^{n}$ to $\left(L^{p^{\prime}}(\Omega)\right)^{n}$.

### 25.3 Penalizaton And Projection Operators

In this section, $X$ will be a reflexive Banach space such that $X, X^{\prime}$ has a strictly convex norm. Let $K$ be a closed convex set in $X$. Then the following lemma is obtained.

Lemma 25.3.1 Let $K$ be closed and convex nonempty subset of $X$ a reflexive Banach space which has strictly convex norm. Then there exists a projection map $P$ such that $P x \in K$ and for all $y \in K$,

$$
\|y-x\| \geq\|x-P x\|
$$

Proof: Let $\left\{y_{n}\right\}$ be a minimizing sequence for $y \rightarrow\|y-x\|$ for $y \in K$. Thus

$$
d \equiv \inf \{\|y-x\|: y \in K\}=\lim _{n \rightarrow \infty}\left\|y_{n}-x\right\|
$$

Then obviously $\left\{y_{n}\right\}$ is bounded. Hence there is a subsequence, still denoted by $n$ such that $y_{n} \rightarrow w \in K$. Then

$$
\|w-x\| \leq \lim \inf _{n \rightarrow \infty}\left\|y_{n}-x\right\|=d
$$

How many closest points to $x$ are there? Suppose $w_{1}$ is another one. Then

$$
\left\|\frac{w_{1}+w}{2}-x\right\|=\left\|\frac{w_{1}-x+w-x}{2}\right\|<\left\|\frac{w_{1}-x}{2}\right\|+\left\|\frac{w-x}{2}\right\|=d
$$

contradicting the assumption that both $w, w_{1}$ are closest points to $x$. Therefore, $P x$ consists of a single point.

Denote by $F$ the duality map such that $\langle F x, x\rangle=\|x\|^{2}$. This is described earlier but there is also a very nice treatment which is somewhat different in [13]. Everything can be generalized and is in [91] but here I will only consider this case. First here is a useful result.

Proposition 25.3.2 Let $F$ be the duality map just described. Let $\phi(x) \equiv \frac{\|x\|^{2}}{2}$. Then $F(x)=$ $\partial \phi(x)$.

Proof: This follows from

$$
\begin{aligned}
\langle F x, y-x\rangle & \leq\langle F x, y\rangle-\langle F x, x\rangle \leq\langle F x, x\rangle^{1 / 2}\langle F y, y\rangle^{1 / 2}-\langle F x, x\rangle \\
& \leq \frac{\langle F y, y\rangle}{2}-\frac{\langle F x, x\rangle}{2}=\frac{\|y\|^{2}}{2}-\frac{\|x\|^{2}}{2} .
\end{aligned}
$$

Next is a really nice result about the characterization of $P x$ in terms of $F$.
Proposition 25.3.3 Let $K$ be a nonempty closed convex set in $X$ a reflexive Banach space in which both $X, X^{\prime}$ have strictly convex norms. Then $w \in K$ is equal to $P x$ if and only if

$$
\langle F(x-w), y-w\rangle \leq 0
$$

for every $y \in K$.

Proof: First suppose the condition. Then for $y \in K$, it follows from the above proposition about the subgradient,

$$
\frac{1}{2}\|x-y\|^{2}-\frac{1}{2}\|x-w\|^{2} \geq\langle F(x-w), w-y\rangle \geq 0
$$

and so since this holds for all $y$ it follows that

$$
\|x-y\| \geq\|x-w\|
$$

for all $y$ which says that $w=P x$.
Next, using the subgradient idea again, for $\theta \in[0,1]$, suppose $w=P x$ then for $y \in K$ arbitrary,

$$
0 \geq \frac{1}{2}\|x-w\|^{2}-\frac{1}{2}\|x-(w+\theta(y-w))\|^{2} \geq\langle F(x-(w+\theta(y-w))), \theta(y-x)\rangle
$$

Now divide by $\theta$ and let $\theta \downarrow 0$ and use the hemicontinuity of $F$ given above. Then

$$
0 \geq\langle F(x-w), y-x\rangle
$$

Definition 25.3.4 An operator of penalization is an operator $f: X \rightarrow X^{\prime}$ such that $f=0$ on $K, f$ is monotone and nonzero off $K$ as well as demicontinuous. (Strong convergence goes to weak convergence.) Actually, in applications, it is usually easy to give an ad hoc description of an appropriate penalization operator.

Proposition 25.3.5 Let $K$ be a closed convex nonempty subset of $X$ a reflexive Banach space such that $X, X^{\prime}$ have strictly convex norms. Then

$$
f(x) \equiv F(x-P x)
$$

is an operator of penalization. Here $P$ is the projection onto $K$. This operator of penalization is demicontinuous.

Proof: First, observe that $f(x)$ is 0 on $K$ and nonzero off $K$. Why is it monotone?

$$
\begin{aligned}
& \left\langle F(x-P x)-F\left(x_{1}-P x_{1}\right), x-x_{1}\right\rangle \\
& =\left\langle F(x-P x)-F\left(x_{1}-P x_{1}\right), x-P x-\left(x_{1}-P x_{1}\right)\right\rangle \\
& +\left\langle F(x-P x)-F\left(x_{1}-P x_{1}\right), P x-P x_{1}\right\rangle
\end{aligned}
$$

The first term is $\geq 0$ because $F$ is monotone. As to the second, it equals

$$
\left\langle F(x-P x), P x-P x_{1}\right\rangle+\left\langle F\left(x_{1}-P x_{1}\right), P x_{1}-P x\right\rangle
$$

and both of these are $\geq 0$ because of Proposition 25.3 .3 which characterizes the projection map.

Now why is this hemicontinuous? Let $x_{n} \rightarrow x$. Then $P x_{n}$ is clearly bounded. Taking a subsequence, it can be assumed that $P x_{n} \rightarrow \xi$ weakly. Is $\xi=P x$ ?

$$
\begin{aligned}
\|x-P x\| & \leq\left\|x-P x_{n}\right\| \leq\left\|x-x_{n}\right\|+\left\|x_{n}-P x_{n}\right\| \\
\left\|x_{n}-P x_{n}\right\| & \leq\left\|x_{n}-P x\right\| \leq\left\|x_{n}-x\right\|+\|x-P x\|
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \|x-P x\|-\left\|x_{n}-P x_{n}\right\| \leq\left\|x-x_{n}\right\| \\
& \left\|x_{n}-P x_{n}\right\|-\|x-P x\| \leq\left\|x-x_{n}\right\|
\end{aligned}
$$

Hence $\left\|x_{n}-P x_{n}\right\| \rightarrow\|x-P x\|$. However, from convexity and strong lower semicontinuity implying weak lower semicontinuity,

$$
\|x-\xi\| \leq \lim \inf _{n \rightarrow \infty}\left\|x_{n}-P x_{n}\right\|=\|x-P x\|
$$

and so $\xi=P x$ because there is only one value in $P x$. This has shown that, thanks to uniqueness of $P x, x_{n} \rightarrow x$ implies $P x_{n} \rightarrow P x$ weakly.

Next we show that $f$ is demicontinuous. Suppose $x_{n} \rightarrow x$. Then from what was just shown, $P x_{n} \rightarrow P x$ weakly. Thus $x_{n}-P x_{n} \rightarrow x-P x$ weakly. Then

$$
\begin{aligned}
& \lim \sup _{n \rightarrow \infty}\left\langle F\left(x_{n}-P x_{n}\right), x_{n}-P x_{n}-(x-P x)\right\rangle \\
= & \lim \sup _{n \rightarrow \infty}\left\langle F\left(x_{n}-P x_{n}\right), P x-P x_{n}\right\rangle \leq 0
\end{aligned}
$$

from Proposition 25.3 .3 which characterizes the projection map. It follows that, since $F$ is monotone hemicontinuous and bounded, it is also pseudomonotone and so for all $v$

$$
\begin{gathered}
\lim _{n \rightarrow \infty} \inf _{n}\left\langle F\left(x_{n}-P x_{n}\right),\left(x_{n}-P x_{n}\right)-v\right\rangle \\
\geq\langle F(x-P x),(x-P x)-v\rangle
\end{gathered}
$$

Now $F\left(x_{n}-P x_{n}\right)$ is bounded. If it converges to $\xi$, then

$$
\begin{gathered}
\lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty}\left\langle F\left(x_{n}-P x_{n}\right),\left(x_{n}-P x_{n}\right)-v\right\rangle \\
\leq \lim \sup _{n \rightarrow \infty}\left[\begin{array}{c}
\left\langle F\left(x_{n}-P x_{n}\right),\left(x_{n}-P x_{n}\right)-(x-P x)\right\rangle \\
+\left\langle F\left(x_{n}-P x_{n}\right),(x-P x)-v\right\rangle
\end{array}\right] \\
\leq\langle\xi,(x-P x)-v\rangle
\end{gathered}
$$

It follows that

$$
\begin{aligned}
\langle\xi,(x-P x)-v\rangle & \geq \lim _{n \rightarrow \infty}\left\langle F\left(x_{n}-P x_{n}\right),\left(x_{n}-P x_{n}\right)-v\right\rangle \\
& \geq\langle F(x-P x),(x-P x)-v\rangle
\end{aligned}
$$

Since $v$ is arbitrary, it follows that $\xi=F(x-P x)$. Hence $F\left(x_{n}-P x_{n}\right) \rightarrow F(x-P x)$ weakly. Thus this is demicontinuous.

### 25.4 Set-Valued Maps, Pseudomonotone Operators

In the abstract theory of partial differential equations and variational inequalities, it is important to consider set-valued maps from a Banach space to the power set of its dual. In this section we give an introduction to this theory by proving a general result on surjectivity for a class of such operators.

To begin with, if $A: X \rightarrow \mathscr{P}(Y)$ is a set-valued map, define the graph of $A$ by

$$
G(A) \equiv\{(x, y): y \in A x\} .
$$

First consider a map $A$ which maps $\mathbb{C}^{n}$ to $\mathscr{P}\left(\mathbb{C}^{n}\right)$ which satisfies
$A \mathbf{x}$ is compact and convex.
and also the condition that if $O$ is open and $O \supseteq A \mathbf{x}$, then there exists $\delta>0$ such that if

$$
\begin{equation*}
\mathbf{y} \in B(\mathbf{x}, \boldsymbol{\delta}), \text { then } A \mathbf{y} \subseteq O \tag{25.4.12}
\end{equation*}
$$

This last condition is sometimes referred to as upper semicontinuity. In words, $A$ is upper semicontinuous and has values which are compact and convex. As to the last condition of upper semi continuity, here is the formal definition.
Definition 25.4.1 Let $F: X \rightarrow \mathscr{P}(Y)$ be a set valued function. Then $F$ is upper semicontinuous at $x$ if for every open $V \supseteq F(x)$ there exists an open set $U$ containing $x$ such that whenever $\hat{x} \in U$, it follows that $F(\hat{x}) \subseteq V$.

Lemma 25.4.2 Let A satisfy 25.4.12. Then $A K$ is a subset of a compact set whenever $K$ is compact. Also the graph of $A$ is closed if $A \mathbf{x}$ is closed.

Proof: Let $\mathbf{x} \in K$. Then $A \mathbf{x}$ is compact and contained in some open set whose closure is compact, $U_{\mathbf{x}}$. By assumption 25.4.12 there exists an open set $V_{\mathbf{X}}$ containing $\mathbf{x}$ such that if $\mathbf{y} \in V_{\mathbf{x}}$, then $A \mathbf{y} \subseteq U_{\mathbf{x}}$. Let $V_{\mathbf{x}_{1}}, \cdots, V_{\mathbf{x}_{m}}$ cover $K$. Then $A K \subseteq \cup_{k=1}^{m} \bar{U}_{\mathbf{x}_{k}}$, a compact set.

To see the graph of $A$ is closed when $A \mathbf{x}$ is closed, let $\mathbf{x}_{k} \rightarrow \mathbf{x}, \mathbf{y}_{k} \rightarrow \mathbf{y}$ where $\mathbf{y}_{k} \in A \mathbf{x}_{k}$. Then letting $O=A \mathbf{x}+B(\mathbf{0}, r)$ it follows from 25.4.12 that $\mathbf{y}_{k} \in A \mathbf{x}_{k} \subseteq O$ for all $k$ large enough. Therefore, $\mathbf{y} \in A \mathbf{x}+B(\mathbf{0}, 2 r)$ and since $r>0$ is arbitrary and $A \mathbf{x}$ is closed it follows $\mathbf{y} \in A \mathbf{x}$.

Also, there is a general consideration relative to upper semicontinuous functions.
Lemma 25.4.3 If $\mathbf{f}$ is upper semicontinuous on some set $K$ and $\mathbf{g}$ is continuous and defined on $\mathbf{f}(K)$, then $\mathbf{g} \circ \mathbf{f}$ is also upper semicontinuous.

Proof: Let $\mathbf{x}_{n} \rightarrow \mathbf{x}$ in $K$. Let $U \supseteq \mathbf{g} \circ \mathbf{f}(\mathbf{x})$. Is $\mathbf{g} \circ \mathbf{f}\left(\mathbf{x}_{n}\right) \in U$ for all $n$ large enough? We have $\mathbf{f}(\mathbf{x}) \in \mathbf{g}^{-1}(U)$, an open set. Therefore, if $n$ is large enough, $\mathbf{f}\left(\mathbf{x}_{n}\right) \in \mathbf{g}^{-1}(U)$. It follows that for large enough $n, \mathbf{g} \circ \mathbf{f}\left(\mathbf{x}_{n}\right) \in U$ and so $\mathbf{g} \circ \mathbf{f}$ is upper semicontinuous on $K$.

The next theorem is an application of the Brouwer fixed point theorem. First define an $n$ simplex, denoted by $\left[\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right]$, to be the convex hull of the $n+1$ points, $\left\{\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right\}$ where $\left\{\mathbf{x}_{i}-\mathbf{x}_{0}\right\}_{i=1}^{n}$ are independent. Thus

$$
\left[\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right] \equiv\left\{\sum_{i=0}^{n} t_{i} \mathbf{x}_{i}: \sum_{i=0}^{n} t_{i}=1, t_{i} \geq 0\right\}
$$

Since $\left\{\mathbf{x}_{i}-\mathbf{x}_{0}\right\}_{i=1}^{n}$ is independent, the $t_{i}$ are uniquely determined. If two of them are

$$
\sum_{i=0}^{n} t_{i} \mathbf{x}_{i}=\sum_{i=0}^{n} s_{i} \mathbf{x}_{i}
$$

Then

$$
\sum_{i=0}^{n} t_{i}\left(\mathbf{x}_{i}-\mathbf{x}_{0}\right)=\sum_{i=0}^{n} s_{i}\left(\mathbf{x}_{i}-\mathbf{x}_{0}\right)
$$

so $t_{i}=s_{i}$ for $i \geq 1$. Since the $s_{i}$ and $t_{i}$ sum to 1 , it follows that also $s_{0}=t_{0}$. If $n \leq 2$, the simplex is a triangle, line segment, or point. If $n \leq 3$, it is a tetrahedron, triangle, line segment or point. To say that $\left\{\mathbf{x}_{i}-\mathbf{x}_{0}\right\}_{i=1}^{n}$ are independent is to say that $\left\{\mathbf{x}_{i}-\mathbf{x}_{r}\right\}_{i \neq r}$ are independent for each fixed $r$. Indeed, if $\mathbf{x}_{i}-\mathbf{x}_{r}=\sum_{j \neq i, r} c_{j}\left(\mathbf{x}_{j}-\mathbf{x}_{r}\right)$, then you would have

$$
\mathbf{x}_{i}-\mathbf{x}_{0}+\mathbf{x}_{0}-\mathbf{x}_{r}=\sum_{j \neq i, r} c_{j}\left(\mathbf{x}_{j}-\mathbf{x}_{0}\right)+\left(\sum_{j \neq i, r} c_{j}\right) \mathbf{x}_{0}
$$

and it follows that $\mathbf{x}_{i}-\mathbf{x}_{0}$ is a linear combination of the $\mathbf{x}_{j}-\mathbf{x}_{0}$ for $j \neq i$, contrary to assumption. A collection of simplices is a tiling of $\mathbb{R}^{n}$ if $\mathbb{R}^{n}$ is contained in their union and if $S_{1}, S_{2}$ are two simplices in the tiling, with

$$
S_{j}=\left[\mathbf{x}_{0}^{j}, \cdots, \mathbf{x}_{n}^{j}\right]
$$

then

$$
S_{1} \cap S_{2}=\left[\mathbf{x}_{k_{0}}, \cdots, \mathbf{x}_{k_{r}}\right]
$$

where

$$
\left\{\mathbf{x}_{k_{0}}, \cdots, \mathbf{x}_{k_{r}}\right\} \subseteq\left\{\mathbf{x}_{0}^{1}, \cdots, \mathbf{x}_{n}^{1}\right\} \cap\left\{\mathbf{x}_{0}^{2}, \cdots, \mathbf{x}_{n}^{2}\right\}
$$

or else the two simplices do not intersect. The collection of simplices is said to be locally finite if, for every point, there exists a ball containing that point which also intersects only finitely many of the simplices in the collection. It is left to the reader to verify that for each $\varepsilon>0$, there exists a locally finite tiling of $\mathbb{R}^{n}$ which is composed of simplices which have diameters less than $\varepsilon$. The local finiteness ensures that for each $\varepsilon$ the vertices have no limit point. To see how to do this, consider the case of $\mathbb{R}^{2}$. Tile the plane with identical small squares and then form the triangles indicated in the following picture. It is clear something similar can be done in any dimension. Making the squares identical ensures that the little triangles are locally finite.


In general, you could consider $[0,1]^{n}$. The point at the center is $(1 / 2, \cdots, 1 / 2)$. Then there are $2 n$ faces. Form the $2 n$ pyramids having this point along with the $2^{n-1}$ vertices of the face. Then use induction on each of these faces to form smaller dimensional simplices tiling that face. Corresponding to each of these $2 n$ pyramids, it is the union of the simplices whose vertices consist of the center point along with those of these new simplicies tiling the chosen face. In general, you can write any $n$ dimensional cube as the translate of a scaled $[0,1]^{n}$. Thus one can express each of identical cubes as a tiling of $m(n)$ simplices of the appropriate size and thereby obtain a tiling of $\mathbb{R}^{n}$ with simplices. A ball will intersect only finitely many of the cubes and hence finitely many of the simplices. To get their diameters small as desired, just use $[0, r]^{n}$ instead of $[0,1]^{n}$.

Thus one can give a function any value desired on these vertices and extend appropriately to the rest of the simplex and obtain a continuous function.

The Kakutani fixed point theorem is a generalization of the Brouwer fixed point theorem from continuous single valued maps to upper semicontinuous maps which have closed convex values.

Theorem 25.4.4 Let $K$ be a compact convex subset of $\mathbb{R}^{n}$ and let $A: K \rightarrow \mathscr{P}(K)$ such that $A \mathbf{x}$ is a closed convex subset of $K$ and $A$ is upper semicontinuous. Then there exists $\mathbf{x}$ such that $\mathbf{x} \in A \mathbf{x}$. This is the "fixed point".

Proof: Let there be a locally finite tiling of $\mathbb{R}^{n}$ consisting of simplices having diameter no more than $\varepsilon$. Let $P \mathbf{x}$ be the point in $K$ which is closest to $\mathbf{x}$. For each vertex $\mathbf{x}_{k}$, pick $A_{\varepsilon} \mathbf{x}_{k} \in A P \mathbf{x}_{k}$ and define $A_{\varepsilon}$ on all of $\mathbb{R}^{n}$ by the following rule. If

$$
\mathbf{x} \in\left[\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right]
$$

so $\mathbf{x}=\sum_{i=0}^{n} t_{i} \mathbf{x}_{i}, t_{i} \in[0,1], \sum_{i} t_{i}=1$, then

$$
A_{\mathcal{E}} \mathbf{x} \equiv \sum_{k=0}^{n} t_{k} A_{\mathcal{E}} \mathbf{x}_{k}
$$

Now by construction $A_{\varepsilon} \mathbf{x}_{k} \in A P \mathbf{x}_{k} \in K$ and so $A_{\varepsilon}$ is a continuous map defined on $\mathbb{R}^{n}$ with values in $K$ thanks to the local finiteness of the collection of simplices. By the Brouwer fixed point theorem $A_{\mathcal{\varepsilon}}$ has a fixed point $\mathbf{x}_{\varepsilon}$ in $K, A_{\varepsilon} \mathbf{x}_{\varepsilon}=\mathbf{x}_{\varepsilon}$.

$$
\mathbf{x}_{\varepsilon}=\sum_{k=0}^{n} t_{k}^{\varepsilon} A_{\varepsilon} \mathbf{x}_{k}^{\varepsilon}, A_{\varepsilon} \mathbf{x}_{k}^{\varepsilon} \in A P \mathbf{x}_{k}^{\varepsilon} \in K
$$

where a simplex containing $\mathbf{x}_{\varepsilon}$ is

$$
\left[\mathbf{x}_{0}^{\varepsilon}, \cdots, \mathbf{x}_{n}^{\varepsilon}\right], \mathbf{x}_{\varepsilon}=\sum_{k=0}^{n} t_{k}^{\varepsilon} \mathbf{x}_{k}^{\varepsilon}
$$

Also, $\mathbf{x}_{\varepsilon} \in K$ and is closer than $\varepsilon$ to each $\mathbf{x}_{k}^{\varepsilon}$ so each $\mathbf{x}_{k}^{\varepsilon}$ is within $\varepsilon$ of $K$. It follows that for each $k,\left|P \mathbf{x}_{k}^{\varepsilon}-\mathbf{x}_{k}^{\varepsilon}\right|<\varepsilon$ and so

$$
\lim _{\varepsilon \rightarrow 0}\left|P \mathbf{x}_{k}^{\varepsilon}-\mathbf{x}_{k}^{\varepsilon}\right|=0
$$

By compactness of $K$, there exists a subsequence, still denoted with the subscript of $\varepsilon$ such that for each $k$, the following convergences hold as $\varepsilon \rightarrow 0$

$$
t_{k}^{\varepsilon} \rightarrow t_{k}, A_{\varepsilon} \mathbf{x}_{k}^{\varepsilon} \rightarrow \mathbf{y}_{k}, P \mathbf{x}_{k}^{\varepsilon} \rightarrow \mathbf{z}_{k}, \mathbf{x}_{k}^{\varepsilon} \rightarrow \mathbf{z}_{k}
$$

Any pair of the $\mathbf{x}_{k}^{\varepsilon}$ are within $\varepsilon$ of each other. Hence, any pair of the $P \mathbf{x}_{k}^{\varepsilon}$ are within $\varepsilon$ of each other because $P$ reduces distances. Therefore, in fact, $\mathbf{z}_{k}$ does not depend on $k$.

$$
\lim _{\varepsilon \rightarrow 0} P \mathbf{x}_{k}^{\varepsilon}=\lim _{\varepsilon \rightarrow 0} \mathbf{x}_{k}^{\varepsilon}=\mathbf{z}, \quad \lim _{\varepsilon \rightarrow 0} \mathbf{x}_{\varepsilon}=\lim _{\varepsilon \rightarrow 0} \sum_{k=0}^{n} t_{k}^{\varepsilon} \mathbf{x}_{k}^{\varepsilon}=\sum_{k=0}^{n} t_{k} \mathbf{z}=\mathbf{z}
$$

By upper semicontinuity of $A$, for all $\varepsilon$ small enough,

$$
A P \mathbf{x}_{k}^{\varepsilon} \subseteq A \mathbf{z}+B(\mathbf{0}, r)
$$

In particular, since $A_{\varepsilon} \mathbf{x}_{k}^{\varepsilon} \in A P \mathbf{x}_{k}^{\varepsilon}$,

$$
A_{\varepsilon} \mathbf{x}_{k}^{\varepsilon} \in A \mathbf{z}+B(\mathbf{0}, r) \text { for } \varepsilon \text { small enough }
$$

Since $r$ is arbitrary and $A \mathbf{z}$ is closed, it follows

$$
\mathbf{y}_{k} \in A \mathbf{z}
$$

It follows that since $K$ is closed,

$$
\mathbf{x}_{\mathcal{E}} \rightarrow \mathbf{z}=\sum_{k=0}^{n} t_{k} \mathbf{y}_{k}, t_{k} \geq 0, \sum_{k=0}^{n} t_{k}=1
$$

Now by convexity of $A \mathbf{z}$ and the fact just shown that $\mathbf{y}_{k} \in A \mathbf{z}$,

$$
\mathbf{z}=\sum_{k=0}^{n} t_{k} \mathbf{y}_{k} \in A \mathbf{z}
$$

and so $\mathbf{z} \in A \mathbf{z}$. This is the fixed point.
One can replace $\mathbb{R}^{n}$ with $\mathbb{C}^{n}$ in the above theorem because it is essentially $\mathbb{R}^{2 n}$. Also the theorem holds with no change for any finite dimensional normed linear space since these are homeomorpic to $\mathbb{R}^{n}$ or $\mathbb{C}^{n}$.

Lemma 25.4.5 Suppose $A: \mathbb{C}^{n} \rightarrow \mathscr{P}\left(\mathbb{C}^{n}\right)$ satisfies $A \mathbf{x}$ is compact and convex, and $A$ is upper semicontinuous, 25.4.12 and $K$ is a nonempty compact convex set in $\mathbb{C}^{n}$. Then if $\mathbf{y} \in \mathbb{C}^{n}$ there exists $[\mathbf{x}, \mathbf{w}] \in G(A)$ such that $\mathbf{x} \in K$ and

$$
\operatorname{Re}(\mathbf{y}-\mathbf{w}, \mathbf{z}-\mathbf{x}) \leq 0
$$

for all $\mathbf{z} \in K$.
Proof: Tile $\mathbb{C}^{n}$ with $2 n$ simplices such that the collection is locally finite and each simplex has diameter less than $\varepsilon<1$. This collection of simplices is determined by a
countable collection of vertices. For each vertex $\mathbf{x}$, pick $A_{\mathcal{E}} \mathbf{x} \in A \mathbf{x}$ and define $A_{\mathcal{\varepsilon}}$ on all of $\mathbb{C}^{n}$ by the following rule. If

$$
\mathbf{x} \in\left[\mathbf{x}_{0}, \cdots, \mathbf{x}_{2 n}\right]
$$

so $\mathbf{x}=\sum_{i=0}^{2 n} t_{i} \mathbf{x}_{i}$, then

$$
A_{\mathcal{\varepsilon}} \mathbf{x} \equiv \sum_{k=0}^{2 n} t_{k} A_{\mathcal{\varepsilon}} \mathbf{x}_{k}
$$

Thus $A_{\varepsilon}$ is a continuous map defined on $\mathbb{C}^{n}$ thanks to the local finiteness of the collection of simplices. Let $P_{K}$ denote the projection on the convex set $K$. By the Brouwer fixed point theorem, there exists a fixed point, $\mathbf{x}_{\varepsilon} \in K$ such that

$$
P_{K}\left(\mathbf{y}-A_{\varepsilon} \mathbf{x}_{\varepsilon}+\mathbf{x}_{\varepsilon}\right)=\mathbf{x}_{\varepsilon}
$$

By Corollary 19.1.9 this requires

$$
\operatorname{Re}\left(\mathbf{y}-A_{\mathcal{E}} \mathbf{x}_{\mathcal{E}}, \mathbf{z}-\mathbf{x}_{\varepsilon}\right) \leq 0
$$

for all $\mathbf{z} \in K$.
Suppose $\mathbf{x}_{\varepsilon} \in\left[\mathbf{x}_{0}^{\varepsilon}, \cdots, \mathbf{x}_{2 n}^{\varepsilon}\right]$ so $\mathbf{x}_{\varepsilon}=\sum_{k=0}^{2 n} t_{k}^{\varepsilon} \mathbf{x}_{k}^{\varepsilon}$. Then since $\mathbf{x}_{\varepsilon}$ is contained in $K$, a compact set, and the diameter of each simplex is less than 1, it follows that $A_{\varepsilon} \mathbf{x}_{k}^{\varepsilon}$ is contained in $\overline{A(K+B(\mathbf{0}, 1)})$, which is contained in a compact set thanks to Lemma 25.4.2. The reason is that $A$ is assumed to take bounded sets to bounded sets and $K+B(0,1)$ is a bounded set.

From the Heine Borel theorem, there exists a sequence $\varepsilon \rightarrow 0$ such that

$$
t_{k}^{\varepsilon} \rightarrow t_{k}, \mathbf{x}_{\varepsilon} \rightarrow \mathbf{x} \in K, A_{\varepsilon} \mathbf{x}_{k}^{\varepsilon} \rightarrow \mathbf{y}_{k}
$$

for $k=0, \cdots, 2 n$. Since the diameter of the simplex containing $\mathbf{x}_{\varepsilon}$ converges to 0 , it follows

$$
\mathbf{x}_{k}^{\varepsilon} \rightarrow \mathbf{x}, A_{\varepsilon} \mathbf{x}_{k}^{\varepsilon} \rightarrow \mathbf{y}_{k}
$$

By upper semicontinuity, it follows that for all $r>0, A \mathbf{x}_{k}^{\varepsilon} \subseteq A \mathbf{x}+B(0, r)$ for all $\varepsilon$ small enough. Since $A_{\varepsilon} \mathbf{x}_{k}^{\varepsilon} \in A \mathbf{x}_{k}^{\varepsilon}$, and $A \mathbf{x}$ is closed, this implies $\mathbf{y}_{k} \in A \mathbf{x}$. Since $A \mathbf{x}$ is convex,

$$
\sum_{k=1}^{2 n} t_{k} \mathbf{y}_{k} \in A \mathbf{x}
$$

Hence for all $\mathbf{z} \in K$,

$$
\begin{gathered}
\operatorname{Re}\left(\mathbf{y}-\sum_{k=1}^{2 n} t_{k} \mathbf{y}_{k}, \mathbf{z}-\mathbf{x}\right)=\lim _{\varepsilon \rightarrow 0} \operatorname{Re}\left(\mathbf{y}-\sum_{k=1}^{2 n} t_{k}^{\varepsilon} A_{\varepsilon} \mathbf{x}_{k}^{\varepsilon}, \mathbf{z}-\mathbf{x}_{\varepsilon}\right) \\
=\lim _{\varepsilon \rightarrow 0} \operatorname{Re}\left(\mathbf{y}-A_{\varepsilon} \mathbf{x}_{\varepsilon}, \mathbf{z}-\mathbf{x}_{\varepsilon}\right) \leq 0
\end{gathered}
$$

Let $\mathbf{w}=\sum_{k=1}^{2 n} t_{k} \mathbf{y}_{k}$.
You could replace $A$ with $A \circ P_{K}$ in the above and assume only that $A$ is only defined on $K$. This is because $\mathbf{x} \in K$.

Lemma 25.4.6 Suppose in addition to 25.4.11 and 25.4.12, (compact convex valued and upper semicontinuous) $A$ is coercive,

$$
\lim _{|\mathbf{x}| \rightarrow \infty} \inf \left\{\frac{\operatorname{Re}(\mathbf{y}, \mathbf{x})}{|\mathbf{x}|}: \mathbf{y} \in A \mathbf{x}\right\}=\infty
$$

Then $A$ is onto.
Proof: Let $\mathbf{y} \in \mathbb{C}^{n}$ and let $K_{r} \equiv \overline{B(\mathbf{0}, r)}$. By Lemma 25.4.5 there exists $\mathbf{x}_{r} \in K_{r}$ and $\mathbf{w}_{r} \in A \mathbf{x}_{r}$ such that

$$
\begin{equation*}
\operatorname{Re}\left(\mathbf{y}-\mathbf{w}_{r}, \mathbf{z}-\mathbf{x}_{r}\right) \leq 0 \tag{25.4.13}
\end{equation*}
$$

for all $\mathbf{z} \in K_{r}$. Letting $\mathbf{z}=\mathbf{0}$,

$$
\operatorname{Re}\left(\mathbf{w}_{r}, \mathbf{x}_{r}\right) \leq \operatorname{Re}\left(\mathbf{y}, \mathbf{x}_{r}\right)
$$

Therefore,

$$
\inf \left\{\frac{\operatorname{Re}\left(\mathbf{w}, \mathbf{x}_{r}\right)}{\left|\mathbf{x}_{r}\right|}: \mathbf{w} \in A \mathbf{x}_{r}\right\} \leq|\mathbf{y}|
$$

It follows from the assumption of coercivity that $\left|\mathbf{x}_{r}\right|$ is bounded independent of $r$. Therefore, picking $r$ strictly larger than this bound, 25.4.13 implies

$$
\operatorname{Re}\left(\mathbf{y}-\mathbf{w}_{r}, \mathbf{v}\right) \leq 0
$$

for all $\mathbf{v}$ in some open ball containing $\mathbf{0}$. Therefore, for all $\mathbf{v}$ in this ball

$$
\operatorname{Re}\left(\mathbf{y}-\mathbf{w}_{r}, \mathbf{v}\right)=0
$$

and hence this holds for all $\mathbf{v} \in \mathbb{C}^{n}$ and so $\mathbf{y}=\mathbf{w}_{r} \in A \mathbf{x}_{r}$. This proves the lemma.
Lemma 25.4.7 Let $F$ be a finite dimensional Banach space of dimension $n$, and let $T$ be a mapping from $F$ to $\mathscr{P}\left(F^{\prime}\right)$ such that 25.4.11 and 25.4.12 both hold for $F^{\prime}$ in place of $\mathbb{C}^{n}$. Then if $T$ is also coercive,

$$
\begin{equation*}
\lim _{\|\mathbf{u}\| \rightarrow \infty} \inf \left\{\frac{\operatorname{Re} \mathbf{y}^{*}(\mathbf{u})}{\|\mathbf{u}\|}: \mathbf{y}^{*} \in T \mathbf{u}\right\}=\infty \tag{25.4.14}
\end{equation*}
$$

it follows $T$ is onto.
Proof: Let $|\cdot|$ be an equivalent norm for $F$ such that there is an isometry of $\mathbb{C}^{n}$ and $F, \theta$. Now define $A: \mathbb{C}^{n} \rightarrow \mathscr{P}\left(\mathbb{C}^{n}\right)$ by $A \mathbf{x} \equiv \theta^{*} T \theta \mathbf{x}$.

$$
\begin{array}{cll}
\mathscr{P}\left(F^{\prime}\right) & \stackrel{\theta^{*}}{\rightarrow} & \mathbb{C}^{n} \\
T \uparrow & \circ & \uparrow A \\
F & \stackrel{\theta}{\leftarrow} & \mathbb{C}^{n}
\end{array}
$$

Thus $\mathbf{y} \in A \mathbf{x}$ means that there exists $\mathbf{z}^{*} \in T \theta \mathbf{x}$ such that

$$
(\mathbf{w}, \mathbf{y})_{\mathbb{C}^{n}}=\mathbf{z}^{*}(\theta \mathbf{w})
$$

for all $\mathbf{w} \in \mathbb{C}^{n}$. Then $A$ satisfies the conditions of Lemma 25.4.6 and so $A$ is onto. Consequently $T$ is also onto.

With these lemmas, it is possible to prove a very useful result about a class of mappings which map a reflexive Banach space to the power set of its dual space. For more theorems about these mappings and their applications, see [99]. In the discussion below, we will use the symbol, $\rightharpoonup$, to denote weak convergence.

Definition 25.4.8 Let $V$ be a Reflexive Banach space. We say $T: V \rightarrow \mathscr{P}\left(V^{\prime}\right)$ is pseudomonotone if the following conditions hold.

> Tu is closed, nonempty, convex.

If $F$ is a finite dimensional subspace of $V$, then if $u \in F$ and $W \supseteq T u$ for $W$ a weakly open set in $V^{\prime}$, then there exists $\delta>0$ such that

$$
\begin{equation*}
v \in B(u, \delta) \cap F \text { implies } T v \subseteq W \tag{25.4.16}
\end{equation*}
$$

If $u_{k} \rightharpoonup u$ and if $u_{k}^{*} \in T u_{k}$ is such that

$$
\limsup _{k \rightarrow \infty} \operatorname{Re} u_{k}^{*}\left(u_{k}-u\right) \leq 0
$$

then for all $v \in V$, there exists $u^{*}(v) \in T u$ such that

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \operatorname{Re} u_{k}^{*}\left(u_{k}-v\right) \geq \operatorname{Re} u^{*}(v)(u-v) \tag{25.4.17}
\end{equation*}
$$

We say $T$ is coercive if

$$
\begin{equation*}
\lim _{\|v\| \rightarrow \infty} \inf \left\{\frac{\operatorname{Re} z^{*}(v)}{\|v\|}: z^{*} \in T v\right\}=\infty . \tag{25.4.18}
\end{equation*}
$$

In the case that $T$ takes bounded sets to bounded sets so it is a bounded set valued operator, it turns out you don't have to consider the second of the above conditions about the upper semicontinuity. It follows from the other conditions. It is convenient to use the notation

$$
\left\langle u^{*}, v\right\rangle \equiv u^{*}(v), u^{*} \in V^{\prime}, v \in V
$$

and this will be used interchangeably with the earlier notation from now on.
The next lemma has to do with upper semicontinuity being obtained from simpler conditions.

Lemma 25.4.9 Let $T: X \rightarrow \mathscr{P}\left(X^{\prime}\right)$ satisfy conditions 25.4.15 and 25.4.17 above and suppose $T$ is bounded (Tx for $x$ in a bounded set is bounded). Then if $x_{n} \rightarrow x$ in $X$, and if $U$ is a weakly open set containing $T x$, then $T x_{n} \subseteq U$ for all $n$ large enough. If fact the limit condition 25.4.17 can be weakened to the following more general condition: If $u_{k} \rightharpoonup u$, and

$$
\begin{equation*}
\limsup _{k \rightarrow \infty} \operatorname{Re} u_{k}^{*}\left(u_{k}-u\right) \leq 0 \tag{**}
\end{equation*}
$$

then there exists a subsequence still denoted as $\left\{u_{k}\right\}$, such that if $u_{k}^{*} \in T u_{k}$, then for all $v \in V$, there exists $u^{*}(v) \in T u$ such that

$$
\begin{equation*}
\underset{k \rightarrow \infty}{\lim } \inf _{k} \operatorname{Re} u_{k}^{*}\left(u_{k}-v\right) \geq \operatorname{Re} u^{*}(v)(u-v) \tag{25.4.19}
\end{equation*}
$$

(This weaker condition says that if the lim sup condition holds for the original sequence, then there is a subsequence such that the lim inf condition holds for all v. In particular, for this subsequence, the lim sup condition continues to hold.)

Proof: If this is not true, there exists $x_{n} \rightarrow x$, also a weakly open set $U$, containing $T x$ and $z_{n} \in T x_{n}$, but $z_{n} \notin U$. Then, taking a further subsequence, we can assume $z_{n} \rightarrow z$ weakly and $z \notin U$. Then the strong convergence implies

$$
\lim _{n \rightarrow \infty} \operatorname{sep}\left\langle z_{n}, x_{n}-x\right\rangle \leq 0
$$

By assumption, there is a subsequence still denoted with $n$ such that for any $y$,

$$
\lim _{k \rightarrow \infty} \inf \left\langle z_{n}, x_{n}-y\right\rangle \geq \operatorname{Re}\langle z(y), x-y\rangle, \text { some } z(y) \in T(x)
$$

Then in particular, for this subsequence,

$$
0 \geq \lim \sup _{n \rightarrow \infty} \operatorname{Re}\left\langle z_{n}, x_{n}-x\right\rangle \geq \lim _{n \rightarrow \infty} \operatorname{Re}\left\langle z_{n}, x_{n}-x\right\rangle \geq \operatorname{Re}\langle z(x), x-x\rangle=0
$$

so for this subsequence,

$$
\lim _{n \rightarrow \infty} \operatorname{Re}\left\langle z_{n}, x_{n}-x\right\rangle=0
$$

Therefore, if $y \in X$ there exists $z(y) \in T x$ such that

$$
\operatorname{Re}\langle z, x-y\rangle=\lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty} \operatorname{Re}\left\langle z_{n}, x_{n}-y\right\rangle \geq \operatorname{Re}\langle z(y), x-y\rangle
$$

Letting $w=x-y$, this shows, since $y \in X$ is arbitrary, that the following inequality holds for every $w \in X$. (If you have $w \in X$, then you just choose $y=x-w$.)

$$
\operatorname{Re}\langle z, w\rangle \geq \operatorname{Re}\langle z(x-w), w\rangle, z(x-w) \in T x
$$

In particular, we may replace $w$ with $-w$ and obtain

$$
\operatorname{Re}\langle z,-w\rangle \geq \operatorname{Re}\langle z(x+w),-w\rangle
$$

which implies

$$
\operatorname{Re}\langle z(x-w), w\rangle \leq \operatorname{Re}\langle z, w\rangle \leq \operatorname{Re}\langle z(x+w), w\rangle
$$

Therefore, there exists, $\lambda \in[0,1]$,

$$
z_{\lambda}(y) \equiv \lambda z(x-w)+(1-\lambda) z(x+w) \in A x
$$

such that

$$
\operatorname{Re}\langle z, w\rangle=\operatorname{Re}\left\langle z_{\lambda}(y), w\right\rangle
$$

But this is a contradiction to $z \notin T x$ because if $z \notin T x$, it follows from separation theorems there exists $w \in X$ such that for all $z_{1} \in T x$,

$$
\operatorname{Re}\langle z, w\rangle>\operatorname{Re}\left\langle z_{1}, w\right\rangle
$$

You pick that $w$ in the above. Therefore, $z \in T x$ which contradicts the assumption that $z_{n}$ and consequently $z$ are not contained in $U$.

What if $T: V \rightarrow \mathscr{P}\left(V^{\prime}\right)$ for $V$ a finite dimensional vector space such that $T$ is upper semicontinuous and bounded? If $u_{n} \rightarrow u$ weakly, then this also happens strongly because the weak convergence and strong convergence are the same in finite dimensions. Therefore, by upper semicontinuity, there is a subsequence still denoted with $n$ and $z_{n} \in T u_{n}$ such that $z_{n} \rightarrow z$ for some $z$. Then since $T u$ is closed, we must have $z \in T u$ thanks to upper semicontinuity. Then for this subsequence and arbitrary $v$,

$$
\lim \inf _{n \rightarrow \infty} \operatorname{Re} z_{n}\left(u_{n}-v\right)=\operatorname{Re} z(u-v)
$$

Also, this limit condition holds whenever $u_{n} \rightarrow u$ and $z_{n} \rightarrow z$ even without an assumption that $T$ is bounded. This mostly proves the following.

Proposition 25.4.10 Let $V$ be finite dimensional and let $T: V \rightarrow \mathscr{P}(V)$ be upper semicontinuous with closed values. Then if $u_{n} \rightarrow u$ and $z_{n} \in T u_{n}$ with $z_{n} \rightarrow z$, then

$$
\lim _{n \rightarrow \infty} \inf _{n} \operatorname{Re} z_{n}\left(u_{n}-v\right)=\operatorname{Re} z(u-v), z \in T u
$$

If $T$ is bounded, and $u_{n} \rightarrow u$, then if $v$ is given,

$$
\lim _{n \rightarrow \infty} \operatorname{Re} z_{n}\left(u_{n}-v\right)=\operatorname{Re} z(u-v), \text { some } z \in T u
$$

Proof: Consider the last claim and suppose the limit condition does not hold for some v. Then take a subsequence such that

$$
\lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty} \operatorname{Re} z_{n}\left(u_{n}-v\right)=\lim _{n \rightarrow \infty} \operatorname{Re} z_{n}\left(u_{n}-v\right)
$$

By boundedness, there is a further subsequence such that $z_{n} \rightarrow z$. Then from upper semicontinuity and $T u$ being closed, $z \in T u$ and so

$$
\lim _{n \rightarrow \infty} \operatorname{Re} z_{n}\left(u_{n}-v\right)=\lim \inf _{n \rightarrow \infty} \operatorname{Re} z_{n}\left(u_{n}-v\right)=\operatorname{Re} z(u-v)
$$

This more general limit condition is sometimes useful if not essential to use. The following is a definition of this more general condition used in the above lemma.

Definition 25.4.11 Say $T: V \rightarrow \mathscr{P}\left(V^{\prime}\right)$ is modified bounded pseudomonotone if the following conditions hold.

> Tu is closed, nonempty, convex.
$T$ is bounded meaning it takes bounded sets to bounded sets.
If $u_{k} \rightharpoonup u$ and if

$$
\limsup _{k \rightarrow \infty} \operatorname{Re} u_{k}^{*}\left(u_{k}-u\right) \leq 0
$$

then there exists a subsequence, still denoted as $\left\{u_{k}\right\}$ such that if $u_{k}^{*} \in T u_{k}$ then for all $v \in V$, there exists $u^{*}(v) \in T u$ such that

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \inf u_{k}^{*}\left(u_{k}-v\right) \geq \operatorname{Re} u^{*}(v)(u-v) \tag{25.4.21}
\end{equation*}
$$

In this limit condition, there is a subsequence which works for all $v$. However, the preservation of lower semicontinuity happens under even less.

Definition 25.4.12 Say $T: V \rightarrow \mathscr{P}\left(V^{\prime}\right)$ is generalized bounded pseudomonotone if the following conditions hold.
Tu is closed, nonempty, convex.
$T$ is bounded meaning it takes bounded sets to bounded sets.
If $u_{k} \rightharpoonup u$ and if

$$
\limsup _{k \rightarrow \infty} \operatorname{Re} u_{k}^{*}\left(u_{k}-u\right) \leq 0
$$

then if $v$ is given there exists a subsequence, still denoted as $\left\{u_{k}\right\}$ possibly depending on $v$ such that if $u_{k}^{*} \in T u_{k}$ then, there exists $u^{*}(v) \in T u$ such that

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \inf u_{k}^{*}\left(u_{k}-v\right) \geq \operatorname{Re} u^{*}(v)(u-v) \tag{25.4.23}
\end{equation*}
$$

This is more general because in this situation, the subsequence depends on the choice of $v$.
In case $T$ is single valued, this condition is equivalent to type $M$.
Proposition 25.4.13 A single valued bounded operator $T: V \rightarrow V^{\prime}, V$ reflexive is generalized bounded pseudomonotone then it is bounded and type $M$.

Proof: Suppose that $u_{n} \rightarrow u$ weakly and $T u_{n} \rightarrow \xi$ weakly and

$$
\lim \sup _{n \rightarrow \infty}\left\langle T u_{n}, u_{n}\right\rangle \leq\langle\xi, u\rangle
$$

Then

$$
\lim \sup _{n \rightarrow \infty}\left\langle T u_{n}, u_{n}-u\right\rangle \leq 0
$$

and so there is a subsequence depending on $v$ and a further one depending on $u$ such that

$$
\begin{aligned}
& \lim \inf _{n \rightarrow \infty}\left\langle T u_{n}, u_{n}-u\right\rangle \geq\langle T u, u-u\rangle=0 \\
& \lim \inf _{n \rightarrow \infty}\left\langle T u_{n}, u_{n}-v\right\rangle \geq\langle T u, u-v\rangle
\end{aligned}
$$

The first in the above shows with the limsup condition that $\lim _{n \rightarrow \infty}\left\langle T u_{n}, u_{n}-u\right\rangle=0$. Therefore, the second condition implies

$$
\langle\xi, u-v\rangle \geq\langle T u, u-v\rangle
$$

since $v$ is arbitrary, it follows that $\xi=T u$. Thus $T$ is type M .
If $T$ is generalized bounded pseudomonotone, then it is upper semicontinuous from the strong to the weak topology.

Lemma 25.4.14 Let $T: X \rightarrow \mathscr{P}\left(X^{\prime}\right)$ satisfy conditions 25.4 .15 and 25.4.17 above and suppose $T$ is bounded. Then if $x_{n} \rightarrow x$ in $X$, and if $U$ is a weakly open set containing $T x$, then $T x_{n} \subseteq U$ for all $n$ large enough. If fact the limit condition 25.4 .17 can be weakened to the following more general condition: If $u_{k} \rightharpoonup u$, and

$$
\begin{equation*}
\lim _{\sup _{k \rightarrow \infty}} \operatorname{Re} u_{k}^{*}\left(u_{k}-u\right) \leq 0 \tag{**}
\end{equation*}
$$

then for each $v$, there exists a subsequence still denoted as $\left\{u_{k}\right\}$, possibly depending on $v$ such that if $u_{k}^{*} \in T u_{k}$, then there exists $u^{*}(v) \in T u$ such that

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \operatorname{Re} u_{k}^{*}\left(u_{k}-v\right) \geq \operatorname{Re} u^{*}(v)(u-v) \tag{25.4.24}
\end{equation*}
$$

(This weaker condition says that if the lim sup condition holds for the original sequence, then for given $v$ there is a subsequence such that the lim inf condition holds for that $v$. In particular, for this subsequence, the lim sup condition continues to hold.)

Proof: If this is not true, there exists $x_{n} \rightarrow x$, and a weakly open set $U$, containing $T x$ and $z_{n} \in T x_{n}$, but $z_{n} \notin U$. Then, taking a further subsequence, we can assume $z_{n} \rightarrow z$ weakly and $z \notin U$. Then the strong convergence implies

$$
\lim _{n \rightarrow \infty} \sup \left\langle z_{n}, x_{n}-x\right\rangle \leq 0
$$

By separation theorems, there exists $w$ such that for all $\hat{w} \in T(x)$,

$$
\begin{equation*}
\operatorname{Re}\langle z, w\rangle<\operatorname{Re}\langle\hat{w}, w\rangle \tag{*}
\end{equation*}
$$

Thus, choose $y$ such that $w=x-y$. By assumption, there is a subsequence still denoted with $n$ such that

$$
\begin{gathered}
\lim _{k \rightarrow \infty} \inf \operatorname{Re}\left\langle z_{n}, x_{n}-y\right\rangle \geq \operatorname{Re}\langle z(y), x-y\rangle, \text { some } z(y) \in T(x) \\
\lim \inf _{k \rightarrow \infty} \operatorname{Re}\left\langle z_{n}, x_{n}-x\right\rangle \geq \operatorname{Re}\langle z(x), x-x\rangle=0, \text { some } z(x) \in T(x)
\end{gathered}
$$

To get this subsequence, get one which goes with $y$ and then note that the limsup only gets smaller when you go to a subsequence. Hence you can apply the condition to get a further subsequence which goes with $x$. By doing so, the liminf condition for $y$ is strengthened. Then in particular, for this subsequence,

$$
0 \geq \lim \sup _{n \rightarrow \infty} \operatorname{Re}\left\langle z_{n}, x_{n}-x\right\rangle \geq \lim _{n \rightarrow \infty} \operatorname{Re}\left\langle z_{n}, x_{n}-x\right\rangle \geq \operatorname{Re}\langle z(x), x-x\rangle=0
$$

so for this subsequence,

$$
\lim _{n \rightarrow \infty} \operatorname{Re}\left\langle z_{n}, x_{n}-x\right\rangle=0
$$

Therefore, from the assumed condition, there is a further subsequence such that

$$
\operatorname{Re}\langle z, x-y\rangle=\lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty} \operatorname{Re}\left\langle z_{n}, x_{n}-y\right\rangle \geq \operatorname{Re}\langle z(y), x-y\rangle, z(y) \in T(x)
$$

Since $w=x-y$,

$$
\operatorname{Re}\langle z, w\rangle \geq \operatorname{Re}\langle z(y), w\rangle
$$

where $z(y) \in T x$. which contradicts $*$. Thus $z \in U$ as claimed.

### 25.5 Sum Of Pseudomonotone Operators

One of the nice properties of pseudomonotone maps is that when you add two of them, you get another one. I will give a proof in the case that the two pseudomonotone maps are both bounded. It is probably true in general, but as just noted, it is less trouble to verify if you don't have to worry about as many conditions. I will also assume the spaces are all real so it will not be necessary to constantly write the real part. Actually, we do a slightly more general version which says that a bounded pseudomonotone added to a modified bounded pseudomonotone is a modified bounded pseudomonotone. First is the theorem about the sum of two bounded set valued pseudomonotone operators.

Theorem 25.5.1 Say $A, B$ are set valued bounded pseudomonotone operators. Then their sum is also a set valued bounded pseudomonotone operator. Also, if $u_{n} \rightarrow u$ weakly, $z_{n} \rightarrow z$ weakly, $z_{n} \in A\left(u_{n}\right)$, and $w_{n} \rightarrow w$ weakly with $w_{n} \in B\left(u_{n}\right)$, then if

$$
\limsup _{n \rightarrow \infty}\left\langle z_{n}+w_{n}, u_{n}-u\right\rangle \leq 0
$$

it follows that

$$
\liminf _{n \rightarrow \infty}\left\langle z_{n}+w_{n}, u_{n}-v\right\rangle \geq\langle z(v)+w(v), u-v\rangle, z(v) \in A(u), w(v) \in B(u)
$$

and $z \in A(u), w \in B(u)$.
Proof: Say $z_{n} \in A\left(u_{n}\right), w_{n} \in B\left(u_{n}\right), u_{n} \rightarrow u$ weakly and

$$
\lim \sup _{n \rightarrow \infty}\left\langle z_{n}+w_{n}, u_{n}-u\right\rangle \leq 0
$$

Claim: Both of $\lim \sup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle, \limsup _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle$ are no larger than 0 .
Proof of the claim: Suppose $\lim \sup _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle=\delta>0$. Then take a subsequence such that the limsup equals lim. The limsup only gets smaller when you go to a subsequence. Thus, continuing to denote the subsequence with $n$ we still have

$$
\limsup _{n \rightarrow \infty}\left\langle z_{n}+w_{n}, u_{n}-u\right\rangle \leq 0
$$

But from the fact that we just took a subsequence for which the lim sup $=\lim$,

$$
\begin{aligned}
\limsup _{n \rightarrow \infty}\left\langle z_{n}+w_{n}, u_{n}-u\right\rangle & =\lim \sup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle+\lim _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle \\
& =\lim \sup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle+\delta \leq 0
\end{aligned}
$$

and so $\lim \sup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle=-\delta<0$. Therefore by the limit condition,

$$
\lim _{n \rightarrow \infty} \inf _{n}\left\langle z_{n}, u_{n}-u\right\rangle \geq\langle z(u), u-u\rangle=0
$$

and so

$$
0>-\delta \geq \lim \sup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle \geq \lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle \geq 0
$$

a contradiction. Thus the claim is established. We have

$$
\lim \sup _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle \leq 0, \quad \lim \sup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle \leq 0
$$

Thus we can apply the limit condition to the two operators separately. This refers to the original sequence now. If $v$ is given, then

$$
\lim _{n \rightarrow \infty} \inf _{n}\left\langle w_{n}, u_{n}-v\right\rangle \geq\langle w(v), u-v\rangle, \lim \inf _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-v\right\rangle \geq\langle z(v), u-v\rangle
$$

where $w(v) \in B(u)$ and $z(v) \in A(u)$. Thus

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty}\left\langle z_{n}+w_{n}, u_{n}-v\right\rangle & \geq \lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-v\right\rangle+\lim \inf _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-v\right\rangle \\
& \geq\langle w(v)+z(v), u-v\rangle
\end{aligned}
$$

and $w(v)+z(v) \in A(u)+B(u)$ which shows that the sum is pseudomonotone.
In addition, from the claim, we know that

$$
\liminf \left\langle z_{n}, u_{n}-u\right\rangle \geq\langle z(u), u-u\rangle=0
$$

similar for $w_{n}$. Thus $\liminf \left\langle z_{n}, u_{n}-u\right\rangle \geq 0 \geq \limsup \left\langle z_{n}, u_{n}-u\right\rangle$ so $\lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle=0$, similar for $\left\langle w_{n}, u_{n}-u\right\rangle$. Therefore, if $z_{n} \rightarrow z$ weakly and $w_{n} \rightarrow w$ weakly,

$$
\begin{aligned}
\langle z, u-v\rangle & =\lim \left\langle z_{n}, u-v\right\rangle=\lim _{n \rightarrow \infty}\left[\left\langle z_{n}, u-u_{n}\right\rangle+\left\langle z_{n}, u_{n}-v\right\rangle\right] \\
& \geq \lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-v\right\rangle \geq\langle z(v), u-v\rangle, z(v) \in A(u)
\end{aligned}
$$

It follows that $\langle z, u-v\rangle \geq\langle z(v), u-v\rangle$ for all $v$ which could be violated using separation theorems if $z$ is not in $A(u)$. Thus $z \in A(u)$. Similarly $w \in B(u)$.

The above is the main result but we can attempt to see what happens if one of the operators is only modified pseudomonotone.

Note that if $B$ is bounded pseudomonotone, then it is certainly modified bounded pseudomonotone.

Theorem 25.5.2 Suppose $A, B: X \rightarrow \mathscr{P}\left(X^{\prime}\right)$ are both pseudomonotone and bounded. Then so is their sum. If $A$ is bounded pseudomonotone and $B$ is modified bounded pseudomonotone, then $A+B$ is modified bounded pseudomonotone.

Proof: It is clear that $A x+B x$ is closed and convex because this is true of both of the sets in the sum. It is also bounded because both terms in the sum are bounded. It only remains to verify the limit condition. Suppose then that

$$
u_{n} \rightarrow u \text { weakly }
$$

Will the limit condition hold for $A+B$ when applied to this further subsequence? Suppose $z_{n} \in A x_{n}, w_{n} \in B x_{n}$ and

$$
\begin{equation*}
\lim \sup _{n \rightarrow \infty}\left\langle z_{n}+w_{n}, u_{n}-u\right\rangle \leq 0 \tag{25.5.25}
\end{equation*}
$$

Is there a subsequence such that the liminf condition holds? From the above,

$$
\begin{equation*}
\lim \sup _{n \rightarrow \infty}\left\langle z_{n}+w_{n}, u_{n}-u\right\rangle \leq \lim \sup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle+\lim \sup _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle \tag{25.5.26}
\end{equation*}
$$

and so, if the second term $\leq 0$, since $B$ is modified bounded pseudomonotone, there is a subsequence, still denoted with $n$ for which

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-v\right\rangle \geq\langle w(v), u-v\rangle, w(v) \in B(u) \tag{*}
\end{equation*}
$$

for all $v$. In particular,

$$
\lim _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle \geq\langle w(u), u-u\rangle=0
$$

Hence you would have

$$
\lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle \geq 0 \geq \lim \sup _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle
$$

and so $\lim _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle=0$ for this subsequence still denoted with $n$. Hence for this subsequence,

$$
\lim \sup _{n \rightarrow \infty}\left\langle z_{n}+w_{n}, u_{n}-u\right\rangle=\lim \sup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle \leq 0
$$

Then using that $A$ is bounded pseudomonotone, $\lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle=0$ also. It follows for any $v$,

$$
\left.\lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty}, u_{n}-v\right\rangle \geq\langle z(v), u-v\rangle
$$

Then from this it is routine to establish the modified pseudomonotone limit condition for the sum $A+B$. For the subsequence just described, still denoted with $n$,

$$
\lim _{n \rightarrow \infty}\left\langle z_{n}+w_{n}, u_{n}-u\right\rangle \leq 0
$$

and $*$. In fact, you would have for any $v$,

$$
\begin{aligned}
\lim \inf _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-v\right\rangle & \geq\langle z(v), u-v\rangle, z(v) \in A(u) \\
\lim \inf _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-v\right\rangle & \geq\langle w(v), u-v\rangle, w(v) \in A(u)
\end{aligned}
$$

Then you would get

$$
\begin{aligned}
\lim _{n \rightarrow \infty}\left\langle z_{n}+w_{n}, u_{n}-v\right\rangle & =\lim _{n \rightarrow \infty}\left(\left\langle z_{n}, u_{n}-v\right\rangle+\left\langle w_{n}, u_{n}-v\right\rangle\right) \\
& \geq \lim _{n \rightarrow \infty}\left(\left\langle z_{n}, u_{n}-v\right\rangle\right)+\lim \inf _{n \rightarrow \infty}\left(\left\langle w_{n}, u_{n}-v\right\rangle\right) \\
& \geq\langle z(v), u-v\rangle+\langle w(v), u-v\rangle
\end{aligned}
$$

and $z(v)+w(v) \in(A+B)(u)$.
Returning to 25.5.26, the other case to consider is that

$$
\lim \sup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle \leq 0
$$

Then in this case, the assumption that $A$ is pseudomonotone implies that for any $v$,

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-v\right\rangle \geq\langle z(v), u-v\rangle, z(v) \in A(u) \tag{***}
\end{equation*}
$$

No subsequence here. However, if you use a subsequence, the inequality is only strengthened. In particular,

$$
0 \geq \lim \inf _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle=\langle z(u), u-u\rangle=0 \geq \lim \sup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle
$$

and so for the original sequence,

$$
\lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle=0
$$

Then back to 25.5.26,

$$
\lim \sup _{n \rightarrow \infty}\left\langle z_{n}+w_{n}, u_{n}-u\right\rangle=\lim \sup _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle \leq 0
$$

Now by assumption that $B$ is modified bounded pseudomonotone, there is a subsequence, still denoted with $n$ such that for any $v$

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-v\right\rangle \geq\langle w(v), u-v\rangle, w(v) \in B(u) \tag{****}
\end{equation*}
$$

In particular, for this subsequence,

$$
0 \geq \lim \sup _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle \geq \lim \inf _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle \geq\langle w(u), u-u\rangle=0
$$

and so for this subsequence, $\lim _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle=0$. Then for this subsequence, it follows from $* * *$, and $* * * *$,

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \inf _{n}\left\langle z_{n}+w_{n}, u_{n}-v\right\rangle & =\lim _{n \rightarrow \infty}\left(\left\langle z_{n}, u_{n}-v\right\rangle+\left\langle w_{n}, u_{n}-v\right\rangle\right) \\
& \geq \lim _{n \rightarrow \infty}\left(\left\langle z_{n}, u_{n}-v\right\rangle\right)+\lim \inf _{n \rightarrow \infty}\left(\left\langle w_{n}, u_{n}-v\right\rangle\right) \\
& \geq\langle z(v), u-v\rangle+\langle w(v), u-v\rangle
\end{aligned}
$$

We continue to be in the situation of 25.5 .25 and we are asking for a subsequence such that the liminf condition will hold for the subsequence. Suppose this liminf condition is not obtained for any subsequence. The desired liminf condition will hold for a subsequence if either $\limsup \sin _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle$ or $\lim \sup _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle$ is $\leq 0$. This was shown above. Therefore, if there is no subsequence yielding the liminf condition, you must have both of these strictly positive. Say $\delta>0$ is smaller than both. Let $n$ denote a subsequence such that

$$
\lim \sup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle=\lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle>\delta>0
$$

If, for this new subsequence, $\lim \sup _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle<0$, then, since the limsup gets smaller for a subsequence,

$$
\lim \sup _{n \rightarrow \infty}\left\langle z_{n}+w_{n}, u_{n}-u\right\rangle=\lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle+\lim \sup _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle \leq 0
$$

then you could apply the above argument and obtain a further subsequence for which the liminf condition would hold for the sum. Thus, we must have for this new subsequence,

$$
\lim \sup _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle \geq 0
$$

Then, using this subsequence,

$$
0 \geq \lim \sup _{n \rightarrow \infty}\left\langle z_{n}+w_{n}, u_{n}-u\right\rangle \geq \delta+\lim \sup _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle \geq \delta
$$

which is a contradiction. Thus the liminf condition must hold for some subsequence. In case both are bounded and pseudomonotone, things are easier. You don't have to take a subsequence.

It is not entirely clear whether the sum of modified bounded pseudomonotone operators is modified bounded pseudomonotone. This is because when you go to a subsequence, the limsup gets smaller and so it is not entirely clear whether the subsequence for $A$ will continue to yield the limit condition if a further subsequence is taken.

In fact, you can add a bounded pseudomonotone to a generalized bounded pseudomonotone and get a generalized bounded pseudomonotone. The proof is just like the above and is given next.

Theorem 25.5.3 Suppose $A, B: X \rightarrow \mathscr{P}\left(X^{\prime}\right)$. If $A$ is bounded pseudomonotone and $B$ is generalized bounded pseudomonotone, then $A+B$ is generalized bounded pseudomonotone.

Proof: It is clear that $A x+B x$ is closed and convex because this is true of both of the sets in the sum. It is also bounded because both terms in the sum are bounded. It only remains to verify the limit condition. Suppose then that

$$
u_{n} \rightarrow u \text { weakly }
$$

Will the limit condition hold for $A+B$ when applied to this further subsequence? Suppose $z_{n} \in A x_{n}, w_{n} \in B x_{n}$ and

$$
\begin{equation*}
\limsup _{n \rightarrow \infty}\left\langle z_{n}+w_{n}, u_{n}-u\right\rangle \leq 0 \tag{25.5.27}
\end{equation*}
$$

If $v$ is given, is there a subsequence such that the liminf condition holds? From the above,

$$
\begin{equation*}
\lim \sup _{n \rightarrow \infty}\left\langle z_{n}+w_{n}, u_{n}-u\right\rangle \leq \lim \sup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle+\lim \sup _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle \tag{25.5.28}
\end{equation*}
$$

and so, if the second term $\leq 0$, since $B$ is modified bounded pseudomonotone, there is a subsequence, still denoted with $n$ for which

$$
\begin{gather*}
\lim \inf _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-v\right\rangle \geq\langle w(v), u-v\rangle, w(v) \in B(u)  \tag{*}\\
\lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle \geq\langle w(u), u-u\rangle=0
\end{gather*}
$$

You just get a subsequence which works for $v$ and note that the limsup condition is only strengthened for the subsequence and then obtain a further subsequence which goes with
$u$ to get the second condition along with the first. Note that liminf gets bigger when you go to a subsequence so if $*$ holds for the first subsequence, then it holds even better for the second.

Hence you would have, for this subsequence depending on $v$

$$
\lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle \geq 0 \geq \lim \sup _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle
$$

and so $\lim _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle=0$ for this subsequence still denoted with $n$. Hence for this subsequence,

$$
\lim \sup _{n \rightarrow \infty}\left\langle z_{n}+w_{n}, u_{n}-u\right\rangle=\lim \sup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle \leq 0
$$

Then using that $A$ is bounded pseudomonotone, $\lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle=0$ also. It follows for any $v$,

$$
\lim _{n \rightarrow \infty} \inf _{n}\left\langle z_{n}, u_{n}-v\right\rangle \geq\langle z(v), u-v\rangle
$$

Then from this it is routine to establish the modified pseudomonotone limit condition for the sum $A+B$. For the subsequence just described, still denoted with $n$,

$$
\limsup _{n \rightarrow \infty}\left\langle z_{n}+w_{n}, u_{n}-u\right\rangle \leq 0
$$

and $*$. In fact, you would have

$$
\begin{aligned}
\lim \inf _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-v\right\rangle & \geq\langle z(v), u-v\rangle, z(v) \in A(u) \\
\lim \inf _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-v\right\rangle & \geq\langle w(v), u-v\rangle, w(v) \in A(u)
\end{aligned}
$$

Then you would get

$$
\begin{aligned}
\liminf _{n \rightarrow \infty}\left\langle z_{n}+w_{n}, u_{n}-v\right\rangle & =\lim _{n \rightarrow \infty}\left(\left\langle z_{n}, u_{n}-v\right\rangle+\left\langle w_{n}, u_{n}-v\right\rangle\right) \\
& \geq \lim _{n \rightarrow \infty}\left(\left\langle z_{n}, u_{n}-v\right\rangle\right)+\lim \inf _{n \rightarrow \infty}\left(\left\langle w_{n}, u_{n}-v\right\rangle\right) \\
& \geq\langle z(v), u-v\rangle+\langle w(v), u-v\rangle
\end{aligned}
$$

and $z(v)+w(v) \in(A+B)(u)$.
Returning to 25.5.28, the other case to consider is that

$$
\lim \sup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle \leq 0
$$

Then in this case, the assumption that $A$ is pseudomonotone implies that for any $v$,

$$
\begin{equation*}
\lim \inf _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-v\right\rangle \geq\langle z(v), u-v\rangle, z(v) \in A(u) \tag{***}
\end{equation*}
$$

No subsequence here. In particular,

$$
0 \geq \lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle=\langle z(u), u-u\rangle=0 \geq \lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle
$$

and so for the original sequence,

$$
\lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle=0
$$

Then back to 25.5.28,

$$
\lim \sup _{n \rightarrow \infty}\left\langle z_{n}+w_{n}, u_{n}-u\right\rangle=\lim \sup _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle \leq 0
$$

Now by assumption that $B$ is generalized bounded pseudomonotone, there is a subsequence, still denoted with $n$ such that for the given $v$,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \inf _{n}\left\langle w_{n}, u_{n}-v\right\rangle \geq\langle w(v), u-v\rangle, w(v) \in B(u) \tag{****}
\end{equation*}
$$

Then taking a further subsequence to go with $u$ in the third inequality below, the first inequality is preserved and

$$
0 \geq \lim \sup _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle \geq \lim _{n \rightarrow \infty} \inf _{n \rightarrow}\left\langle w_{n}, u_{n}-u\right\rangle \geq\langle w(u), u-u\rangle=0
$$

and so for this further subsequence, $\lim _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle=0$. Then for this subsequence, it follows from $* * *$, and $* * * *$,

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \inf _{n \rightarrow}\left\langle z_{n}+w_{n}, u_{n}-v\right\rangle & =\lim _{n \rightarrow \infty}\left(\left\langle z_{n}, u_{n}-v\right\rangle+\left\langle w_{n}, u_{n}-v\right\rangle\right) \\
& \geq \lim _{n \rightarrow \infty}\left(\left\langle z_{n}, u_{n}-v\right\rangle\right)+\lim \inf _{n \rightarrow \infty}\left(\left\langle w_{n}, u_{n}-v\right\rangle\right) \\
& \geq\langle z(v), u-v\rangle+\langle w(v), u-v\rangle
\end{aligned}
$$

We continue to be in the situation of 25.5 .27 and we are asking for a subsequence such that the liminf condition will hold for some subsequence depending on $v$. Suppose this liminf condition is not obtained for any subsequence. The desired liminf condition will hold for a subsequence if either $\limsup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle \leq$ or $\limsup _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle$ is $\leq 0$. This was shown above. Therefore, if there is no subsequence yielding the liminf condition, you must have both of these strictly positive. Say $\delta>0$ is smaller than both. Let $n$ denote a subsequence such that

$$
\lim \sup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle=\lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle>\delta>0
$$

If, for this new subsequence, $\lim \sup _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle<0$, then, since the limsup gets smaller for a subsequence,

$$
\lim _{n \rightarrow \infty}\left\langle z_{n}+w_{n}, u_{n}-u\right\rangle=\lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle+\lim _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle \leq 0
$$

then you could apply the above argument and obtain a further subsequence for which the liminf condition would hold for the sum. Thus, we must have for this new subsequence,

$$
\lim \sup _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle \geq 0
$$

Then, using this subsequence,

$$
0 \geq \lim \sup _{n \rightarrow \infty}\left\langle z_{n}+w_{n}, u_{n}-u\right\rangle \geq \delta+\lim \sup _{n \rightarrow \infty}\left\langle w_{n}, u_{n}-u\right\rangle \geq \delta
$$

which is a contradiction. Thus the liminf condition must hold for some subsequence.
The following is mostly in [99].
Theorem 25.5.4 Let $V$ be a reflexive Banach space and let $T: V \rightarrow \mathscr{P}\left(V^{\prime}\right)$ be pseudomonotone, bounded, and coercive. Then $T$ is onto. More generally, the same holds if $T$ is modified or generalized bounded pseudomonotone and coercive.

Proof: The proof is for modified bounded pseudomonotone since this is more general. Let $\mathscr{F}$ be the set of finite dimensional subspaces of $V$ and let $F \in \mathscr{F}$. Then define $T_{F}$ as

$$
T_{F} \equiv i_{F}^{*} T i_{F}
$$

where here $i_{F}$ is the identity map from $F$ to $V$. Then $T_{F}$ satisfies the conditions of Lemma 25.4.7 thanks to Lemma 25.4.9 or Lemma 25.4.14 and so $T_{F}$ is onto $\mathscr{P}\left(F^{\prime}\right)$. Let $w^{*} \in V^{\prime}$. Then since $T_{F}$ is onto, there exists $u_{F} \in F$ such that

$$
i_{F}^{*} w^{*} \in i_{F}^{*} T i_{F} u_{F}
$$

Thus for each finite dimensional subspace $F$, there exists $u_{F} \in F$ such that for all $v \in F$,

$$
\begin{equation*}
\left\langle w^{*}, v\right\rangle=\left\langle u_{F}^{*}, v\right\rangle, u_{F}^{*} \in T u_{F} . \tag{25.5.29}
\end{equation*}
$$

Replacing $v$ with $u_{F}$, in 25.5.29,

$$
\frac{\left\langle u_{F}^{*}, u_{F}\right\rangle}{\left\|u_{F}\right\|}=\frac{\left\langle w^{*}, u_{F}\right\rangle}{\left\|u_{F}\right\|} \leq\left\|w^{*}\right\| .
$$

Therefore, the assumption that $T$ is coercive implies $\left\{u_{F}: F \in \mathscr{F}\right\}$ is bounded in $V$. Now define

$$
W_{F} \equiv \cup\left\{u_{F^{\prime}}: F^{\prime} \supseteq F\right\}
$$

Then $W_{F}$ is bounded and if $\overline{W_{F}} \equiv$ weak closure of $W_{F}$, then

$$
\left\{\overline{W_{F}}: F \in \mathscr{F}\right\}
$$

is a collection of nonempty weakly compact (since $V$ is reflexive and the $u_{F}$ were just shown bounded) sets having the finite intersection property because $W_{F} \neq \emptyset$ for each $F$. (If $F_{i}, i=1, \cdots, n$ are finite dimensional subspaces, let $F$ be a finite dimensional subspace which contains all of these. Then $W_{F} \neq \emptyset$ and $W_{F} \subseteq \cap_{i=1}^{n} \overline{W_{F_{i}}}$.) Thus there exists

$$
u \in \cap\left\{\overline{W_{F}}: F \in \mathscr{F}\right\} .
$$

I will show $w^{*} \in T u$. If $w^{*} \notin T u$, a closed convex set, there exists $v \in V$ such that

$$
\begin{equation*}
\operatorname{Re}\left\langle w^{*}, u-v\right\rangle<\operatorname{Re}\left\langle u^{*}, u-v\right\rangle \tag{25.5.30}
\end{equation*}
$$

for all $u^{*} \in T u$. This follows from the separation theorems. (These theorems imply there exists $z \in V$ such that

$$
\operatorname{Re}\left\langle w^{*}, z\right\rangle<\operatorname{Re}\left\langle u^{*}, z\right\rangle
$$

for all $u^{*} \in T u$. Define $u-v \equiv z$.)
Now let $F \supseteq\{u, v\}$. Since $u \in \overline{W_{F}}$, a weakly sequentially compact set, there exists a sequence, $\left\{u_{k}\right\}$, such that

$$
u_{k} \rightharpoonup u, u_{k} \in W_{F}
$$

Then since $F \supseteq\{u, v\}$, there exists $u_{k}^{*} \in T u_{k}$ such that

$$
\left\langle u_{k}^{*}, u_{k}-u\right\rangle=\left\langle w^{*}, u_{k}-u\right\rangle
$$

Therefore,

It follows by the assumption that $T$ is modified bounded pseudomonotone or generalized bounded pseudomonotone and the pseudomonotone limit condition, a further subsequence corresponding to $v$ such that the following holds for the $v$ defined above in 25.5.30.

$$
\lim _{k \rightarrow \infty} \operatorname{Re}\left\langle u_{k}^{*}, u_{k}-v\right\rangle \geq \operatorname{Re}\left\langle u^{*}(v), u-v\right\rangle, u^{*}(v) \in T u
$$

But since $v \in F, \operatorname{Re}\left\langle u_{k}^{*}, u_{k}-v\right\rangle=\operatorname{Re}\left\langle w^{*}, u_{k}-v\right\rangle$ and so

$$
\lim \inf _{k \rightarrow \infty} \operatorname{Re}\left\langle u_{k}^{*}, u_{k}-v\right\rangle=\underset{k \rightarrow \infty}{\lim \inf _{k} \operatorname{Re}\left\langle w^{*}, u_{k}-v\right\rangle=\operatorname{Re}\left\langle w^{*}, u-v\right\rangle, ~ ;, ~}
$$

so from 25.5.30, $\operatorname{Re}\left\langle w^{*}, u-v\right\rangle<\operatorname{Re}\left\langle u^{*}, u-v\right\rangle$ for all $u^{*} \in T u$,

$$
\begin{aligned}
& \operatorname{Re}\left\langle w^{*}, u-v\right\rangle=\lim _{k \rightarrow \infty} \inf \left\langle u_{k}^{*}, u_{k}-v\right\rangle \\
& \geq \operatorname{Re}\left\langle u^{*}(v), u-v\right\rangle>\operatorname{Re}\left\langle w^{*}, u-v\right\rangle,
\end{aligned}
$$

a contradiction. Thus, $w^{*} \in T u$.
This is likely a good place to put an extremely interesting convergence theorem. It is a version of one in Aubin and Cellina [9]. It is a perfectly marvelous use of the fact that the weak and strong closures of a convex set are the same.

Proposition 25.5.5 Let $X, Y$ be Banach spaces, and let $F:(0, T) \times X \rightarrow \mathscr{P}(Y)$ be a multifunction such that

1. The values of $F$ are nonempty, closed and convex subsets of $Y$
2. For a.e. $t \in(0, T), F(t, \cdot)$ is upper semicontinuous from $X$ into $Y$ with the weak topology

Then let $x_{n}:(0, T) \rightarrow X, y_{n}:(0, T) \rightarrow Y$ be measurable functions such that the sequence $\left\{x_{n}\right\}$ converges a.e. on $(0, T)$ to a function $x:(0, T) \rightarrow X$ and $y_{n}$ converges weakly in $L^{1}(0, T ; Y)$ to $y \in L^{1}(0, T, Y)$. If $y_{n}(t) \in F\left(t, x_{n}(t)\right)$ for all $n \in \mathbb{N}$ and a.e.t, then $y(t) \in$ $F(t, x(t))$ for a.e. $t \in(0, T)$.

Proof: It is given that $y_{n} \rightarrow y$ weakly in $L^{1}(0, T ; Y)$. It is too bad that this does not confer pointwise convergence of some subsequence. However, what can be said is this:

$$
\text { weak closure of } \operatorname{co}\left(\cup_{k=n}^{\infty} y_{k}\right)=\text { strong closure of } \operatorname{co}\left(\cup_{k=n}^{\infty} y_{k}\right)
$$

Here co signifies the convex hull. Thus something is in co $\left(\cup_{k=n}^{\infty} y_{k}\right)$ means it is of the form

$$
\begin{equation*}
v_{n}=\sum_{k=n}^{\infty} c_{k}^{n} y_{k} \tag{*}
\end{equation*}
$$

where all but finitely many of the $c_{k}^{n}$ are zero and they sum to 1 , each being a number in $[0,1]$. Now it is given that $y$ is in the weak closure of $\operatorname{co}\left(\cup_{k=n}^{\infty} y_{k}\right)$. In fact $y_{n}$ converges weakly to $y$. Therefore, from the above observation, $y$ is in the strong closure of co $\left(\cup_{k=n}^{\infty} y_{k}\right)$. Let $v_{n}$ be of the form in $*$ and let it converge in $L^{1}(0, T ; Y)$ to $y$. Then there is a subsequence, still denoted as $v_{n}$ such that for a.e. $t$,

$$
v_{n}(t) \rightarrow y(t) \text { in } Y
$$

Pick such a $t$.
If $y(t) \notin F(t, x(t))$, there exist numbers $k>l$ and $y^{*} \in Y^{\prime}$ such that

$$
\left\langle y^{*}, y(t)\right\rangle>k>l>\left\langle y^{*}, z\right\rangle \text { for all } z \in F(t, x(t))
$$

This follows from separation theorems due to the assumption that $F(t, x(t))$ is a closed convex set. Thus for all $n$ large enough,

$$
\begin{equation*}
\left\langle y^{*}, v_{n}(t)\right\rangle>k>l>\left\langle y^{*}, z\right\rangle \text { for all } z \in F(t, x(t)) \tag{**}
\end{equation*}
$$

Let $k-l>2 \varepsilon>0$. Consider

$$
F(t, x(t))+B_{y_{*}}(0, \varepsilon)
$$

where the ball signifies all $z \in Y$ such that

$$
\left|\left\langle y^{*}, z\right\rangle\right|<\varepsilon
$$

By the weak upper semicontinuity assumption of $F(t, \cdot)$ and $x_{n}(t) \rightarrow x(t)$, it follows that for $k$ large enough,

$$
y_{k}(t) \in F\left(t, x_{k}(t)\right) \subseteq F(t, x(t))+B_{y_{*}}(0, \varepsilon)
$$

Now $v_{n}$ is a convex combination of $y_{k}$ for $k \geq n$ and so it follows that for $n$ large enough,

$$
v_{n}(t) \in F(t, x(t))+B_{y_{*}}(0, \varepsilon)
$$

which says that there exists $z_{n} \in F(t, x(t))$ such that

$$
\left|\left\langle y^{*}, v_{n}(t)\right\rangle-\left\langle y^{*}, z_{n}\right\rangle\right|<\varepsilon
$$

However, this is a contradiction to $* *$ because it says two things are closer than $\varepsilon$ and also farther than $k-l>2 \varepsilon$. Thus $y(t) \in F(t, x(t))$.

It does not use that the measure space is Lebesgue measure on $[0, T]$ that I can see. I think it appears to work for $[0, T]$ replaced with $\Omega$ and $t$ replaced with $\omega \in \Omega$ where $(\Omega, \mathscr{F}, \mu)$ is just some measure space.

### 25.6 Generalized Gradients

This is an interesting theorem, but one might wonder if there are easy to verify examples of such possibly set valued mappings. In what follows consider only real spaces because the essential ideas are included in this case which is also the case of most use in applications. Of course, you might with some justification, make the claim that the following is not really very easy to verify any more than the original definition.

Definition 25.6.1 Let $V$ be a real reflexive Banach space and let $f: V \rightarrow \mathbb{R}$ be a locally Lipschitz function, meaning that $f$ is Lipschitz near every point of $V$ although $f$ need not be Lipschitz on all of $V$. Under these conditions,

$$
\begin{equation*}
f^{0}(x, y) \equiv \lim \sup _{\mu \rightarrow 0+h \rightarrow 0} \frac{f(x+h+\mu y)-f(x+h)}{\mu} \tag{25.6.31}
\end{equation*}
$$

and $\partial f(x) \subseteq X^{\prime}$ is defined by

$$
\begin{equation*}
\partial f(x) \equiv\left\{x^{*} \in X^{\prime}: x^{*}(y) \leq f^{0}(x, y) \text { for all } y \in X\right\} \tag{25.6.32}
\end{equation*}
$$

The set just described is called the generalized gradient. In 25.6.31 we mean the following by the right hand side.

$$
\lim _{(r, \delta) \rightarrow(0,0)} \sup \left\{\frac{f(x+h+\mu y)-f(x+h)}{\mu}: \mu \in(0, r), h \in B(0, \delta)\right\}
$$

I will show, following [99], that these generalized gradients of locally Lipschitz functions are sometimes pseudomonotone. First here is a lemma.

Lemma 25.6.2 Let $f$ be as described in the above definition. Then $\partial f(x)$ is a closed, bounded, convex, and non empty subset of $V^{\prime}$. Furthermore, for $x^{*} \in \partial f(x)$,

$$
\begin{equation*}
\left\|x^{*}\right\| \leq \operatorname{Lip}_{x}(f) \tag{25.6.33}
\end{equation*}
$$

Proof: It is left as an exercise to verify the assertions that $\partial f(x)$ is closed, and convex. It follows directly from the definition. To verify this set is bounded, let $\operatorname{Lip}_{x}(f)$ denote a Lipschitz constant valid near $x \in V$ and let $x^{*} \in \partial f(x)$. Then choosing $y$ with $\|y\|=1$ and $x^{*}(y) \geq \frac{1}{2}\left\|x^{*}\right\|$,

$$
\begin{equation*}
\frac{1}{2}\left\|x^{*}\right\|=x^{*}(y) \leq f^{0}(x, y) \tag{25.6.34}
\end{equation*}
$$

Also, for small $\mu$ and $h$,

$$
\left|\frac{f(x+h+\mu y)-f(x+h)}{\mu}\right| \leq \operatorname{Lip}_{x}(f)\|y\|=\operatorname{Lip}_{x}(f)
$$

Therefore, $f^{0}(x, y) \leq \operatorname{Lip}_{x}(f)$ and so 25.6 .34 shows $\left\|x^{*}\right\| \leq 2 L i p_{x}(f)$.
The interesting part of this Lemma is that $\partial f(x) \neq \emptyset$. To verify this first note that the definition of $f^{0}$ implies that $y \rightarrow f^{0}(x, y)$ is a gauge function. Now fix $y \in V$ and define on $\mathbb{R} y$ a linear map $x_{0}^{*}$ by

$$
x_{0}^{*}(\alpha y) \equiv \alpha f^{0}(x, y)
$$

Then if $\alpha \geq 0$,

$$
x_{0}^{*}(\alpha y)=\alpha f^{0}(x, y)=f^{0}(x, \alpha y)
$$

If $\alpha<0$,

$$
\begin{gathered}
x_{0}^{*}(\alpha y) \equiv \alpha f^{0}(x, y)= \\
\lim _{\mu \rightarrow 0+h \rightarrow 0} \inf \frac{(-\alpha) f(x+h)-(-\alpha) f(x+h+\mu y)}{\mu}= \\
(-\alpha) \lim _{\mu \rightarrow 0+h \rightarrow 0} \inf ^{\mu} \frac{f(x+h-\mu y)-f(x+h)}{\mu} \leq \\
(-\alpha) f^{0}(x,-y)=f^{0}(x, \alpha y)
\end{gathered}
$$

Therefore, $x_{0}^{*}(\alpha y) \leq f^{0}(x, \alpha y)$ for all $\alpha$. By the Hahn Banach theorem there is an extension of $x_{0}^{*}$ to all of $V, x^{*}$ which satisfies,

$$
x^{*}(y) \leq f^{0}(x, y)
$$

for all $y$. It remains to verify $x^{*}$ is continuous. This follows easily from

$$
\begin{gathered}
\left|x^{*}(y)\right|=\max \left(x^{*}(-y), x^{*}(y)\right) \leq \\
\max \left(f^{0}(x, y), f^{0}(x,-y)\right) \leq \operatorname{Lip}_{x}(f)\|y\|,
\end{gathered}
$$

which verifies 25.6 .33 and proves the lemma.
This lemma has verified the first condition needed in the definition of pseudomonotone. The next lemma verifies that these generalized subgradients satisfy the second of the conditions needed in the definition. In fact somewhat more than is needed in the definition is shown.

Lemma 25.6.3 Let $U$ be weakly open in $V^{\prime}$ and suppose $\partial f(x) \subseteq U$. Then $\partial f(z) \subseteq U$ whenever $z$ is close enough to $x$.

Proof: Suppose to the contrary there exists $z_{n} \rightarrow x$ but $z_{n}^{*} \in \partial f\left(z_{n}\right) \backslash U$. From the first lemma, we may assert that $\left\|z_{n}^{*}\right\| \leq 2 \operatorname{Lip}(f)$ for all $n$ large enough. Therefore, there is a subsequence, still denoted by $n$ such that $z_{n}^{*}$ converges weakly to $z^{*} \notin U$.

Claim: $f^{0}(x, y) \geq \limsup _{n \rightarrow \infty} f^{0}\left(x_{n}, y\right)$.
Proof of the claim: There exists $\delta>0$ such that if $\mu,\|h\|<\delta$, then

$$
\varepsilon+f^{0}(x, y) \geq \frac{f(x+h+\mu y)-f(x+h)}{\mu}
$$

Thus, for $\|h\|<\delta$,

$$
\varepsilon+f^{0}(x, y) \geq \frac{f\left(x_{n}+\left(x-x_{n}\right)+h+\mu y\right)-f\left(x_{n}+\left(x-x_{n}\right)+h\right)}{\mu}
$$

Now let $\left\|h^{\prime}\right\|<\frac{\delta}{2}$ and let $n$ be so large that $\left\|x-x_{n}\right\|<\frac{\delta}{2}$. Suppose $\left\|h^{\prime}\right\|<\frac{\delta}{2}$. Then choosing $h \equiv h^{\prime}-\left(x-x_{n}\right)$, it follows the above inequality holds because $\|h\|<\delta$. Therefore, if $\left\|h^{\prime}\right\|<\frac{\delta}{2}$, and $n$ is sufficiently large,

$$
\varepsilon+f^{0}(x, y) \geq \frac{f\left(x_{n}+h^{\prime}+\mu y\right)-f\left(x_{n}+h^{\prime}\right)}{\mu}
$$

Consequently, for all $n$ large enough,

$$
\varepsilon+f^{0}(x, y) \geq f^{0}\left(x_{n}, y\right)
$$

which proves the claim.
Now with the claim,

$$
z^{*}(y)=\lim \sup _{n \rightarrow \infty} z_{n}^{*}(y) \leq \lim \sup _{n \rightarrow \infty} f^{0}\left(x_{n}, y\right) \leq f^{0}(x, y)
$$

so $z^{*} \in \partial f(x)$ contradicting the assumption that $z^{*} \notin U$. This proves the lemma.
It is necessary to assume more on $f^{0}$ in order to obtain the third axiom defining pseudomonotone. The following theorem describes the situation.

Theorem 25.6.4 Let $f: V \rightarrow V^{\prime}$ be locally Lipschitz and suppose it satisfies the condition that whenever

$$
x_{n} \text { converges weakly to } x
$$

and

$$
\lim _{n \rightarrow \infty} \sup _{n \rightarrow \infty} f^{0}\left(x_{n}, x-x_{n}\right) \geq 0
$$

it follows that

$$
\lim \sup _{n \rightarrow \infty} f^{0}\left(x_{n}, z-x_{n}\right) \leq f^{0}(x, z-x)
$$

for all $z \in V$. Then $\partial f$ is pseudomonotone.
Proof: 25.4.15 and 25.4.16 both are satisfied thanks to Lemmas 25.6.1 and 25.6.2. It remains to verify 25.4.17. To do so, I will adopt the convention that $x^{*} \in \partial f(x)$. Suppose

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} x_{n}^{*}\left(x_{n}-x\right) \leq 0 \tag{25.6.35}
\end{equation*}
$$

This implies $\liminf _{n \rightarrow \infty} x_{n}^{*}\left(x-x_{n}\right) \geq 0$. Thus,

$$
\begin{gathered}
0 \leq \lim \inf _{n \rightarrow \infty} x_{n}^{*}\left(x-x_{n}\right) \leq \liminf f^{0}\left(x_{n}, x-x_{n}\right) \\
\leq \lim \sup _{n \rightarrow \infty} f^{0}\left(x_{n}, x-x_{n}\right)
\end{gathered}
$$

which implies, by the above assumption that for all $z$,

$$
\begin{equation*}
\limsup x_{n}^{*}\left(z-x_{n}\right) \leq \limsup f^{0}\left(x_{n}, z-x_{n}\right) \leq f^{0}(x, z-x) \tag{25.6.36}
\end{equation*}
$$

In particular, this holds for $z=x$ and this implies $\limsup x_{n}^{*}\left(x-x_{n}\right) \leq 0$ which along with 25.6.35 yields

$$
\begin{equation*}
\lim _{n \rightarrow \infty} x_{n}^{*}\left(x_{n}-x\right)=0 \tag{25.6.37}
\end{equation*}
$$

Now let $z$ be arbitrary. There exists a subsequence, $n_{k}$, depending on $z$ such that

$$
\lim _{k \rightarrow \infty} x_{n_{k}}^{*}\left(x_{n_{k}}-z\right)=\liminf x_{n_{k}}^{*}\left(x_{n_{k}}-z\right)
$$

Now from Lemma 25.6.2 and its proof, the $\left\|x_{n}^{*}\right\|$ are all bounded by $\operatorname{Lip}_{x}(f)$ whenever $n$ is large enough. Therefore, there is a further subsequence, still denoted by $n_{k}$ such that

$$
x_{n_{k}}^{*} \text { converges weakly to } x^{*}(z) .
$$

We need to verify that $x^{*}(z) \in \partial f(x)$. To do so, let $y$ be arbitrary. Then from the definition,

$$
\begin{equation*}
x_{n}^{*}\left(y-x_{n}\right) \leq f^{0}\left(x_{n}, y-x_{n}\right) . \tag{25.6.38}
\end{equation*}
$$

From 25.6.37, we can take the limsup of both sides and obtain, using 25.6.36

$$
x^{*}(z)(y-x) \leq \lim \sup f^{0}\left(x_{n}, y-x_{n}\right) \leq f^{0}(x, y-x) .
$$

Since $y$ is arbitrary, this shows $x^{*}(z) \in \partial f(x)$ and proves the theorem.

### 25.7 Maximal Monotone Operators

Here it is assumed that the spaces are all real spaces to simplify the presentation.
Definition 25.7.1 Let $A: D(A) \subseteq X \rightarrow \mathscr{P}(X)$ be a set valued map. It is said to be monotone if whenever $y_{i} \in A x_{i}$,

$$
\left\langle y_{1}-y_{2}, x_{1}-x_{2}\right\rangle \geq 0
$$

Denote by $\mathscr{G}(A)$ the graph of $A$ consisting of all pairs $(x, y)$ where $y \in A x$. Such a monotone operator is said to be maximal monotone if

$$
F+A
$$

is onto where $F$ is the duality map with $p=2$.
Actually, it is more usual to say that the graph is maximal monotone if the graph is monotone and there is no monotone graph which properly contains the given graph. However, the two conditions are equivalent and I am more used to using the version in the above definition.

There is a fundamental result about these which is given next.
Theorem 25.7.2 Let $X, X^{\prime}$ be reflexive and have strictly convex norms. Let A be a monotone set valued map as just described. Then if $\lambda F+A$ is onto for some $\lambda>0$, then whenever

$$
\langle y-z, x-u\rangle \geq 0 \text { for all }[x, y] \in \mathscr{G}(A)
$$

it follows that $z \in A u$ and $u \in D(A)$. That is, the graph is maximal.

Proof: Suppose that for all $[x, y] \in \mathscr{G}(A)$,

$$
\langle y-z, x-t\rangle \geq 0
$$

Does it follow that $z \in A t$ ? By assumption, $z+\lambda F(t)=\lambda F \hat{x}+\hat{\xi}, \hat{\xi} \in A \hat{x}$. Then replacing $y$ with $\hat{\xi}$ and $x$ with $\hat{x}$,

$$
\langle\hat{\xi}-(\lambda F \hat{x}+\hat{\xi}-\lambda F t), \hat{x}-t\rangle \geq 0
$$

and so

$$
\lambda\langle F t-F \hat{x}, t-\hat{x}\rangle \leq 0
$$

which implies from Theorem 25.2.5 that $t=\hat{x}$ and so the graph of $A$ is indeed maximal monotone.

$$
z+\lambda F(t)=\lambda F \hat{x}+\hat{\xi} \Rightarrow z=\hat{\xi} \in A \hat{x}=A t
$$

Note that this would have worked with no change if the duality map had been for arbitrary $p>1$.

### 25.7.1 The min max Theorem

In fact, these two conditions are equivalent. This is shown in [13]. We give a proof of this here. First it is necessary to prove a min max theorem. The proof given follows Brezis [24] which is where I found $i$. Here is the min max theorem. A function $f$ is convex if

$$
f(\lambda x+(1-\lambda) y) \leq \lambda f(x)+(1-\lambda) f(y)
$$

It is concave if the inequality is turned around. It can be shown that in finite dimensions, convex functions are automatically continuous, similar for concave functions. Recall the following definition of upper and lower semicontinuous functions defined on a metric space and having values in $[-\infty, \infty]$.

Definition 25.7.3 A function is upper semicontinuous if whenever $x_{n} \rightarrow x$, it follows that $f(x) \geq \limsup _{n \rightarrow \infty} f\left(x_{n}\right)$ and it is lower semicontinuous if $f(x) \leq \liminf _{n \rightarrow \infty} f\left(x_{n}\right)$.

Lemma 25.7.4 If $\mathscr{F}$ is a set of functions which are upper semicontinuous, then $g(x) \equiv$ $\inf \{f(x): f \in \mathscr{F}\}$ is also upper semicontinuous. Similarly, if $\mathscr{F}$ is a set offunctions which are lower semicontinuous, then if $g(x) \equiv \sup \{f(x): f \in \mathscr{F}\}$ it follows that $g$ is lower semicontinuous.

Proof: Let $f \in \mathscr{F}$ where these functions are upper semicontinuous. Then if $x_{n} \rightarrow x$, and $g(x) \equiv \inf \{f(x): f \in \mathscr{F}\}$,

$$
f(x) \geq \lim \sup _{n \rightarrow \infty} f\left(x_{n}\right) \geq \lim \sup _{n \rightarrow \infty} g\left(x_{n}\right)
$$

Since this is true for each $f \in \mathscr{F}$, then it follows that you can take the infimum and obtain $g(x) \geq \lim \sup _{n \rightarrow \infty} g\left(x_{n}\right)$. Similarly, lower semicontinuity is preserved on taking sup.

Note that in a metric space, the above definitions up upper and lower semicontinuity in terms of sequences are equivalent to the definitions that

$$
\begin{aligned}
& f(x) \geq \lim _{r \rightarrow 0} \sup \{f(y): y \in B(x, r)\} \\
& f(x) \leq \lim _{r \rightarrow 0} \inf \{f(y): y \in B(x, r)\}
\end{aligned}
$$

respectively.
Here is a technical lemma which will make the proof shorter. It seems fairly interesting also.

Lemma 25.7.5 Suppose $H: A \times B \rightarrow \mathbb{R}$ is strictly convex in the first argument and concave in the second argument where $A, B$ are compact convex nonempty subsets of Banach spaces $E, F$ respectively and $x \rightarrow H(x, y)$ is lower semicontinuous while $y \rightarrow H(x, y)$ is upper semicontinuous. Let

$$
H(g(y), y) \equiv \min _{x \in A} H(x, y)
$$

Then $g(y)$ is uniquely defined and also for $t \in[0,1]$,

$$
\lim _{t \rightarrow 0} g(y+t(z-y))=g(y)
$$

Proof: First suppose both $z, w$ yield the definition of $g(y)$. Then

$$
H\left(\frac{z+w}{2}, y\right)<\frac{1}{2} H(z, y)+\frac{1}{2} H(w, y)
$$

which contradicts the definition of $g(y)$. As to the existence of $g(y)$ this is nothing more than the theorem that a lower semicontinuous function defined on a compact set achieves its minimum.

Now consider the last claim about "hemicontinuity". For all $x \in A$, it follows from the definition of $g$ that

$$
H(g(y+t(z-y)), y+t(z-y)) \leq H(x, y+t(z-y))
$$

By concavity of $H$ in the second argument,

$$
\begin{align*}
& (1-t) H(g(y+t(z-y)), y)+t H(g(y+t(z-y)), z)  \tag{25.7.39}\\
\leq & H(x, y+t(z-y)) \tag{25.7.40}
\end{align*}
$$

Now let $t_{n} \rightarrow 0$. Does $g\left(y+t_{n}(z-y)\right) \rightarrow g(y)$ ? Suppose not. By compactness, the expression $g\left(y+t_{n}(z-y)\right)$ is in a compact set and so there is a further subsequence, still denoted by $t_{n}$ such that

$$
g\left(y+t_{n}(z-y)\right) \rightarrow \hat{x} \in A
$$

Then passing to a limit in 25.7.40, one obtains, using the upper semicontinuity in one and lower semicontinuity in the other the following inequality.

$$
H(\hat{x}, y) \leq \lim _{n \rightarrow \infty} \inf _{n}\left(1-t_{n}\right) H\left(g\left(y+t_{n}(z-y)\right), y\right)+
$$

$$
\begin{gathered}
\lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty} H\left(g\left(y+t_{n}(z-y)\right), z\right) \\
\leq \lim \inf _{n \rightarrow \infty}\binom{\left(1-t_{n}\right) H\left(g\left(y+t_{n}(z-y)\right), y\right)}{+t_{n} H\left(g\left(y+t_{n}(z-y)\right), z\right)} \\
\leq \lim _{n \rightarrow \infty} H\left(x, y+t_{n}(z-y)\right) \leq H(x, y)
\end{gathered}
$$

This shows that $\hat{x}=g(y)$ because this holds for every $x$. Since $t_{n} \rightarrow 0$ was arbitrary, this shows that in fact

$$
\lim _{t \rightarrow 0+} g(y+t(z-y))=g(y)
$$

Now with this preparation, here is the min-max theorem. A norm is called strictly convex if whenever $x \neq y,\left\|\frac{x+y}{2}\right\|<\frac{\|x\|}{2}+\frac{\|y\|}{2}$.

Theorem 25.7.6 Let $E, F$ be Banach spaces with $E$ having a strictly convex norm. Also suppose that $A \subseteq E, B \subseteq F$ are compact and convex sets and that $H: A \times B \rightarrow \mathbb{R}$ is such that

$$
\begin{gathered}
x \rightarrow H(x, y) \text { is convex } \\
y \rightarrow H(x, y) \text { is concave }
\end{gathered}
$$

Thus $H$ is continuous in each variable in the case of finite dimensional spaces. Here assume that $x \rightarrow H(x, y)$ is lower semicontinuous and $y \rightarrow H(x, y)$ is upper semicontinuous. Then

$$
\min _{x \in A} \max _{y \in B} H(x, y)=\max _{y \in B} \min _{x \in A} H(x, y)
$$

This condition is equivalent to the existence of $\left(x_{0}, y_{0}\right) \in A \times B$ such that

$$
\begin{equation*}
H\left(x_{0}, y\right) \leq H\left(x_{0}, y_{0}\right) \leq H\left(x, y_{0}\right) \text { for all } x, y \tag{25.7.41}
\end{equation*}
$$

Proof: One part of the main equality is obvious.

$$
\max _{y \in B} H(x, y) \geq H(x, y) \geq \min _{x \in A} H(x, y)
$$

and so for each $x$,

$$
\max _{y \in B} H(x, y) \geq \max _{y \in B} \min _{x \in A} H(x, y)
$$

and so

$$
\begin{equation*}
\min _{x \in A} \max _{y \in B} H(x, y) \geq \max _{y \in B} \min _{x \in A} H(x, y) \tag{25.7.42}
\end{equation*}
$$

Next consider the other direction.
Define $H_{\mathcal{\varepsilon}}(x, y) \equiv H(x, y)+\varepsilon\|x\|^{2}$ where $\varepsilon>0$. Then $H_{\varepsilon}$ is strictly convex in the first variable. This results from the observation that

$$
\left\|\frac{x+y}{2}\right\|^{2}<\left(\frac{\|x\|+\|y\|}{2}\right)^{2} \leq \frac{1}{2}\left(\|x\|^{2}+\|y\|^{2}\right)
$$

Then by Lemma 25.7.5 there exists a unique $x \equiv g(y)$ such that

$$
H_{\varepsilon}(g(y), y) \equiv \min _{x \in A} H_{\varepsilon}(x, y)
$$

and also, whenever $y, z \in A$,

$$
\lim _{t \rightarrow 0+} g(y+t(z-y))=g(y) .
$$

Thus $H_{\mathcal{E}}(g(y), y)=\min _{x \in A} H_{\mathcal{E}}(x, y)$. But also this shows that $y \rightarrow H_{\mathcal{E}}(g(y), y)$ is the minimum of functions which are upper semicontinuous and so this function is also upper semicontinuous. Hence there exists $y^{*}$ such that

$$
\begin{equation*}
\max _{y \in B} H_{\varepsilon}(g(y), y)=H_{\varepsilon}\left(g\left(y^{*}\right), y^{*}\right)=\max _{y \in B} \min _{x \in A} H_{\varepsilon}(x, y) \tag{25.7.43}
\end{equation*}
$$

Thus from concavity in the second argument and what was just defined, for $t \in(0,1)$,

$$
\begin{gather*}
H_{\varepsilon}\left(g\left(y^{*}\right), y^{*}\right) \geq H_{\varepsilon}\left(g\left((1-t) y^{*}+t y\right),(1-t) y^{*}+t y\right) \\
\geq(1-t) H_{\varepsilon}\left(g\left((1-t) y^{*}+t y\right), y^{*}\right)+t H_{\varepsilon}\left(g\left((1-t) y^{*}+t y\right), y\right) \\
\geq(1-t) H_{\varepsilon}\left(g\left(y^{*}\right), y^{*}\right)+t H_{\varepsilon}\left(g\left((1-t) y^{*}+t y\right), y\right) \tag{25.7.44}
\end{gather*}
$$

This is because $\min _{x} H_{\varepsilon}\left(x, y^{*}\right) \equiv H_{\varepsilon}\left(g\left(y^{*}\right), y^{*}\right)$ so

$$
H_{\varepsilon}\left(g\left((1-t) y^{*}+t y\right), y^{*}\right) \geq H_{\varepsilon}\left(g\left(y^{*}\right), y^{*}\right)
$$

Then subtracting the first term on the right, one gets

$$
t H_{\mathcal{E}}\left(g\left(y^{*}\right), y^{*}\right) \geq t H_{\mathcal{E}}\left(g\left((1-t) y^{*}+t y\right), y\right)
$$

and cancelling the $t$,

$$
H_{\mathcal{E}}\left(g\left(y^{*}\right), y^{*}\right) \geq H_{\mathcal{E}}\left(g\left((1-t) y^{*}+t y\right), y\right)
$$

Now apply Lemma 25.7.5 and let $t \rightarrow 0+$. This along with lower semicontinuity yields

$$
\begin{equation*}
H_{\varepsilon}\left(g\left(y^{*}\right), y^{*}\right) \geq \lim \inf _{t \rightarrow 0+} H_{\varepsilon}\left(g\left((1-t) y^{*}+t y\right), y\right)=H_{\varepsilon}\left(g\left(y^{*}\right), y\right) \tag{25.7.45}
\end{equation*}
$$

Hence for every $x, y$

$$
H_{\mathcal{E}}\left(x, y^{*}\right) \geq H_{\mathcal{E}}\left(g\left(y^{*}\right), y^{*}\right) \geq H_{\mathcal{E}}\left(g\left(y^{*}\right), y\right)
$$

Thus

$$
\min _{x} H_{\mathcal{E}}\left(x, y^{*}\right) \geq H_{\mathcal{E}}\left(g\left(y^{*}\right), y^{*}\right) \geq \max _{y} H_{\mathcal{E}}\left(g\left(y^{*}\right), y\right)
$$

and so

$$
\begin{aligned}
\max _{y \in B} \min _{x \in A} H_{\varepsilon}(x, y) & \geq \min _{x} H_{\varepsilon}\left(x, y^{*}\right) \geq H_{\mathcal{\varepsilon}}\left(g\left(y^{*}\right), y^{*}\right) \\
& \geq \max _{y} H_{\varepsilon}\left(g\left(y^{*}\right), y\right) \geq \min _{x \in A} \max _{y \in B} H_{\varepsilon}(x, y)
\end{aligned}
$$

Thus, letting $C \equiv \max \{\|x\|: x \in A\}$

$$
\varepsilon C^{2}+\max _{y \in B} \min _{x \in A} H(x, y) \geq \min _{x \in A} \max _{y \in B} H(x, y)
$$

Since $\varepsilon$ is arbitrary, it follows that

$$
\max _{y \in B} \min _{x \in A} H(x, y) \geq \min _{x \in A} \max _{y \in B} H(x, y)
$$

This proves the first part because it was shown above in 25.7.42 that

$$
\min _{x \in A} \max _{y \in B} H(x, y) \geq \max _{y \in B} \min _{x \in A} H(x, y)
$$

Now consider 25.7.41 about the existence of a "saddle point" given the equality of min max and max min. Let

$$
\alpha=\max _{y \in B} \min _{x \in A} H(x, y)=\min _{x \in A} \max _{y \in B} H(x, y)
$$

Then from

$$
y \rightarrow \min _{x \in A} H(x, y) \text { and } x \rightarrow \max _{y \in B} H(x, y)
$$

being upper semicontinuous and lower semicontinuous respectively, there exist $y_{0}$ and $x_{0}$ such that

$$
\alpha=\min _{x \in A} H\left(x, y_{0}\right)=\max _{y \in B} \min _{x \in A}^{\operatorname{minimum} \text { of u.s.c }} H(x, y)=\min _{x \in A} \operatorname{maximum~of~l.s.c.~}_{\max _{y \in B} H(x, y)}=\max _{y \in B} H\left(x_{0}, y\right)
$$

Then

$$
\begin{aligned}
\alpha & =\max _{y \in B} H\left(x_{0}, y\right) \geq H\left(x_{0}, y_{0}\right) \\
\alpha & =\min _{x \in A} H\left(x, y_{0}\right) \leq H\left(x_{0}, y_{0}\right)
\end{aligned}
$$

so in fact $\alpha=H\left(x_{0}, y_{0}\right)$ and from the above equalities,

$$
\begin{aligned}
& H\left(x_{0}, y_{0}\right)=\alpha=\min _{x \in A} H\left(x, y_{0}\right) \leq H\left(x, y_{0}\right) \\
& H\left(x_{0}, y_{0}\right)=\alpha=\max _{y \in B} H\left(x_{0}, y\right) \geq H\left(x_{0}, y\right)
\end{aligned}
$$

and so

$$
H\left(x_{0}, y\right) \leq H\left(x_{0}, y_{0}\right) \leq H\left(x, y_{0}\right)
$$

Thus if the min max condition holds, then there exists a saddle point, namely $\left(x_{0}, y_{0}\right)$.
Finally suppose there is a saddle point $\left(x_{0}, y_{0}\right)$ where

$$
H\left(x_{0}, y\right) \leq H\left(x_{0}, y_{0}\right) \leq H\left(x, y_{0}\right)
$$

Then

$$
\min _{x \in A} \max _{y \in B} H(x, y) \leq \max _{y \in B} H\left(x_{0}, y\right) \leq H\left(x_{0}, y_{0}\right) \leq \min _{x \in A} H\left(x, y_{0}\right) \leq \max _{y \in B} \min _{x \in A} H(x, y)
$$

However, as noted above, it is always the case that

$$
\max _{y \in B} \min _{x \in A} H(x, y) \leq \min _{x \in A} \max _{y \in B} H(x, y)
$$

Of course all of this works with no change if you have $E, F$ reflexive Banach spaces and the sets $A, B$ are just closed and bounded and convex. Then you just use the fact that the functional is weakly lower semicontinuous in the first variable and weakly upper semicontinuous in the second. Recall that lower semicontinuous and convex implies weakly lower semicontinuity. Then just use weak convergence instead of strong convergence in the above argument. Recall that closed bounded and convex sets with the weak topology can be considered metric spaces. I think the above is most interesting in finite dimensions. Of course in this case, you can simply assume the norm is the standard Euclidean norm and there is then no need to assume one of the norms is strictly convex. It comes automatically. Just use an equivalent norm which is strictly convex.

### 25.7.2 Equivalent Conditions For Maximal Monotone

Next is the theorem about the graph being maximal being equivalent to the operator being maximal monotone. It is a very convenient result to have. The proof is a modified version of one in Barbu [13]. It is based on the following lemma also in Barbu. This is a little like the Browder lemma but is based on the min max theorem above. It is also a very interesting argument.

Lemma 25.7.7 Let $E$ be a finite dimensional Banach space and let $K$ be a convex and compact subset of $E . \operatorname{Let} \mathscr{G}(A)$ be a monotone subset of $E \times E^{\prime}$ such that $D(A) \subseteq K$ and $B$ is a single valued monotone and continuous operator from $E$ to $E^{\prime}$. Then there exists $x \in K$ such that

$$
\langle B x+v, u-x\rangle_{E^{\prime}, E} \geq 0 \text { for all }[u, v] \in \mathscr{G}(A) .
$$

If $B$ is coercive

$$
\lim _{\|x\| \rightarrow \infty} \frac{\langle B x, x\rangle}{\|x\|}=\infty
$$

and $0 \in D(A)$, then one can assume only that $K$ is convex and closed.
Proof: Let $T: E \rightarrow K$ be the multivalued operator defined by

$$
T y \equiv\left\{x \in K:\langle B y+v, u-x\rangle_{E^{\prime}, E} \geq 0 \text { for all }[u, v] \in \mathscr{G}(A)\right\}
$$

Here $y \in E$ and it is desired to show that $T y \neq \emptyset$ for all $y \in K$. For $[u, v] \in \mathscr{G}(A)$, let

$$
K_{u, v}=\left\{x \in K:\langle B y+v, u-x\rangle_{E^{\prime}, E} \geq 0\right\}
$$

Then $K_{u, v}$ is a closed, hence compact subset of $K$. The thing to do is to show that

$$
\cap_{[u, v] \in \mathscr{G}(A)} K_{u, v} \equiv T y \neq \emptyset
$$

whenever $y \in K$. Then one argues that $T$ is set valued, has convex compact values and is upper semicontinuous. Then one applies the Kakutani fixed point theorem to get $x \in T x$.

Since these sets $K_{u, v}$ are compact, it suffices to show that they satisfy the finite intersection property. Thus for $\left\{\left[u_{i}, v_{i}\right]\right\}_{i=1}^{n}$ a finite set of elements of $\mathscr{G}(A)$, it is necessary to show that there exists a solution $x$ to the inequalities

$$
\left\langle u_{i}-x, B y+v_{i}\right\rangle \geq 0, i=1,2, \cdots, n
$$

and then it follows from finite intersection property that there exists

$$
x \in \cap_{[u, v] \in \mathscr{G}(A)} K_{u, v}
$$

which is what was desired. Let $P_{n}$ be all $\vec{\lambda}=\left(\lambda_{1}, \cdots, \lambda_{n}\right)$ such that each $\lambda_{k} \geq 0$ and $\sum_{k=1}^{n} \lambda_{k}=1$. Let $H: P_{n} \times P_{n} \rightarrow \mathbb{R}$ be given by

$$
\begin{equation*}
H(\vec{\mu}, \vec{\lambda}) \equiv \sum_{i=1}^{n} \mu_{i}\left\langle B y+v_{i}, \sum_{j=1}^{n} \lambda_{j} u_{j}-u_{i}\right\rangle \tag{25.7.46}
\end{equation*}
$$

Then this is both convex and concave in both $\vec{\lambda}, \vec{\mu}$ and so by Theorem 25.7.6, there exists $\vec{\mu}_{0}, \vec{\lambda}_{0}$ both in $P_{n}$ such that for all $\vec{\mu}, \vec{\lambda}$,

$$
\begin{equation*}
H\left(\vec{\mu}, \vec{\lambda}_{0}\right) \leq H\left(\vec{\mu}_{0}, \vec{\lambda}_{0}\right) \leq H\left(\vec{\mu}_{0}, \vec{\lambda}\right) \tag{25.7.47}
\end{equation*}
$$

However, plugging in $\vec{\mu}=\vec{\lambda}$ in 25.7.46,

$$
\begin{gathered}
H(\vec{\lambda}, \vec{\lambda})=\sum_{i=1}^{n} \lambda_{i}\left\langle B y+v_{i}, \sum_{j=1}^{n} \lambda_{j} u_{j}-u_{i}\right\rangle \\
=\sum_{i=1}^{n}\left\langle B y+v_{i}, \sum_{j=1}^{n} \lambda_{i} \lambda_{j} u_{j}-\lambda_{i} u_{i}\right\rangle \\
=\sum_{i=1}^{n}\left\langle B y+v_{i}, \sum_{j=1}^{n}\left(\lambda_{i} \lambda_{j} u_{j}-\lambda_{i} \lambda_{j} u_{i}\right)\right\rangle \\
=\overbrace{\left\langle B y, \sum_{i=1}^{n} \sum_{j=1}^{n}\left(\lambda_{i} \lambda_{j} u_{j}-\lambda_{i} \lambda_{j} u_{i}\right)\right\rangle}^{=0}+\sum_{i=1}^{n}\left\langle v_{i}, \sum_{j=1}^{n}\left(\lambda_{i} \lambda_{j} u_{j}-\lambda_{i} \lambda_{j} u_{i}\right)\right\rangle
\end{gathered}
$$

The first term obviously equals 0 . Consider the second. This term equals

$$
\sum_{i} \sum_{j} \lambda_{i} \lambda_{j}\left\langle v_{i},\left(u_{j}-u_{i}\right)\right\rangle
$$

The terms equal 0 when $j=i$ or they come in pairs

$$
\begin{aligned}
& \lambda_{i} \lambda_{j}\left\langle v_{i},\left(u_{j}-u_{i}\right)\right\rangle+\lambda_{i} \lambda_{j}\left\langle v_{j},\left(u_{i}-u_{j}\right)\right\rangle \\
= & \lambda_{i} \lambda_{j}\left(\left\langle v_{i},\left(u_{j}-u_{i}\right)\right\rangle-\left\langle v_{j},\left(u_{j}-u_{i}\right)\right\rangle\right) \\
= & \lambda_{i} \lambda_{j}\left(\left\langle v_{i},\left(u_{j}-u_{i}\right)\right\rangle-\left\langle v_{j},\left(u_{j}-u_{i}\right)\right\rangle\right) \leq 0
\end{aligned}
$$

by monotonicity of $A$. Hence $H(\vec{\lambda}, \vec{\lambda}) \leq 0$. Then from 25.7.47, for all $\vec{\mu}$

$$
H\left(\vec{\mu}, \vec{\lambda}_{0}\right) \leq H\left(\vec{\mu}_{0}, \vec{\lambda}_{0}\right) \leq H\left(\vec{\mu}_{0}, \vec{\mu}_{0}\right) \leq 0
$$

It follows that

$$
\begin{aligned}
& \sum_{i=1}^{m} \mu_{i}\left\langle B y+v_{i}, \sum_{j=1}^{n} \lambda_{j}^{0} u_{j}-u_{i}\right\rangle \leq 0 \\
& \sum_{i=1}^{m} \mu_{i}\left\langle B y+v_{i}, u_{i}-\sum_{j=1}^{n} \lambda_{j}^{0} u_{j}\right\rangle \geq 0
\end{aligned}
$$

where $\vec{\lambda}_{0} \equiv\left(\lambda_{1}^{0}, \cdots, \lambda_{n}^{0}\right)$. This is true for any choice of $\vec{\mu}$. In particular, you could let $\vec{\mu}$ equal 1 in the $i^{t h}$ position and 0 elsewhere and conclude that for all $i=1, \cdots, n$,

$$
\left\langle B y+v_{i}, u_{i}-\sum_{j=1}^{n} \lambda_{j}^{0} u_{j}\right\rangle \geq 0
$$

so you let $x=\sum_{j=1}^{n} \lambda_{j}^{0} u_{j}$ and this shows that $T y \neq \emptyset$ because the sets $K_{u, v}$ have the finite intersection property.

Thus $T: K \rightarrow \mathscr{P}(K)$ and for each $y \in K, T y \neq \emptyset$. In fact this is true for any $y$ but we are only considering $y \in K$. Now $T y$ is clearly a closed subset of $K$. It is also clearly convex. Is it upper semicontinuous? Let $y_{k} \rightarrow y$ and consider $T y+B(0, r)$. Is $T y_{k} \in T y+B(0, r)$ for all $k$ large enough? If not, then there is a subsequence, denoted as $z_{k} \in T y_{k}$ which is outside this open set $T y+B(0, r)$. Then taking a further subsequence, still denoted as $z_{k}$, it follows that $z_{k} \rightarrow z \notin T y+B(0, r)$. Now

$$
\left\langle B y_{k}+v, u-z_{k}\right\rangle \geq 0 \text { all }[u, v] \in \mathscr{G}(A)
$$

Therefore, from continuity of $B$,

$$
\langle B y+v, u-z\rangle \geq 0 \text { all }[u, v] \in \mathscr{G}(A)
$$

which means $z \in T y$ contrary to the assumption that $T$ is not upper semicontinuous. Since $T$ is upper semicontinuous and maps to compact convex sets, it follows from Theorem 25.4.4 that $T$ has a fixed point $x \in T x$. Hence there exists a solution $x$ to

$$
\langle B x+v, u-x\rangle \geq 0 \text { all }[u, v] \in \mathscr{G}(A)
$$

Next suppose that $K$ is only closed and convex but $B$ is coercive and $0 \in D(A)$. Then let $K_{n} \equiv \overline{B(0, n)} \cap K$ and let $A_{n}$ be the restriction of $A$ to $\overline{B(0, n)}$. It follows that there exists $x_{n} \in K_{n}$ such that for all $[u, v] \in \mathscr{G}\left(A_{n}\right)$,

$$
\left\langle B x_{n}+v, u-x_{n}\right\rangle \geq 0
$$

Then since $0 \in D(A)$, one can pick $v_{0} \in A 0$ and obtain

$$
\left\langle B x_{n}+v_{0},-x_{n}\right\rangle \geq 0,\left\langle v_{0},-x_{n}\right\rangle \geq\left\langle B x_{n}, x_{n}\right\rangle
$$

from which it follows from coercivity of $B$ that the $x_{n}$ are bounded independent of $n$. Say $\left\|x_{n}\right\|<C$. Then there is a subsequence still denoted as $x_{n}$ such that $x_{n} \rightarrow x \in K$, thanks to the assumption that $K$ is closed and convex. Let $[u, v] \in \mathscr{G}(A)$. Then for all $n$ large enough $\|u\|<n$ and so

$$
\left\langle B x_{n}+v, u-x_{n}\right\rangle \geq 0
$$

Then letting $n \rightarrow \infty$ and using the continuity of $B$,

$$
\langle B x+v, u-x\rangle \geq 0
$$

Since $[u, v]$ was arbitrary, this proves the lemma.
Observation 25.7.8 If you have a monotone set valued function, then its graph can always be considered a subset of the graph of a maximal monotone graph. If $A$ is monotone, then let $\mathscr{F}$ be $\mathscr{G}(B)$ such that $\mathscr{G}(B) \supseteq \mathscr{G}(A)$ and $B$ is monotone. Partially order by set inclusion. Then let $\mathscr{C}$ be a maximal chain. Let $\mathscr{G}(\hat{A})=\cup \mathscr{C}$. If $\left[x_{i}, y_{i}\right] \in \mathscr{G}(\hat{A})$, then both are in some $B \in \mathscr{C}$. Hence $\left(y_{1}-y_{2}, x_{1}-x_{2}\right) \geq 0$ so monotone and must be maximal monotone because if $\langle z-v, x-u\rangle \geq 0$ for all $[u, v] \in \mathscr{G}(\hat{A})$ and $[x, z] \notin \hat{A}$, then you could include this ordered pair and contradict maximality of the chain $\mathscr{C}$.

Next is an interesting theorem which comes from this lemma. It is an infinite dimensional version of the above lemma.

Theorem 25.7.9 Let $X$ be a reflexive Banach space and let $K$ be a closed convex subset of $X$. Let $A, B$ be monotone such that

1. $D(A) \subseteq K, 0 \in D(A)$.
2. $B$ is single valued, hemicontinuous, bounded and coercive mapping $X$ to $X^{\prime}$.

Then there exists $x \in K$ such that

$$
\langle B x+v, u-x\rangle_{X^{\prime}, X} \geq 0 \text { for all }[u, v] \in \mathscr{G}(A)
$$

Before giving the proof, here is an easy lemma.
Lemma 25.7.10 Let $E$ be finite dimensional and let $B: E \rightarrow E^{\prime}$ be monotone and hemicontinuous. Then B is continuous.

Proof: The space can be considered a finite dimensional Hilbert space $\left(\mathbb{R}^{n}\right)$ and so weak and strong convergence are exactly the same. First it is desired to show that $B$ is bounded. Suppose it is not. Then there exists $\left\|x_{k}\right\|_{E}=1$ but $\left\|B x_{n}\right\|_{E^{\prime}} \rightarrow \infty$. Since finite dimensional, there is a subsequence still denoted as $x_{k}$ such that $x_{k} \rightarrow x,\|x\|_{E}=1$.

$$
\left\langle B x_{k}-B x, x_{k}-x\right\rangle \geq 0
$$

Hence

$$
\left\langle\frac{B x_{k}-B x}{\left\|B x_{k}\right\|_{E^{\prime}}}, x_{k}-x\right\rangle \geq 0
$$

Then taking another subsequence, written with index $k$, it can be assumed that

$$
B x_{k} /\left\|B x_{k}\right\| \rightarrow y^{*} \in E^{\prime},\left\|y^{*}\right\|_{E^{\prime}}=1
$$

Hence,

$$
\left\langle y^{*}, x_{k}-x\right\rangle \geq 0
$$

for all $x \in E$, but this requires that $y^{*}=0$, a contradiction. Thus $B$ is monotone, hemicontinuous, and bounded. It follows from Theorem 25.1 .4 which says that monotone and hemicontinuous operators are pseudomonotone and Proposition 25.1 .6 which says that bounded pseudomonotone operators are demicontinuous that $B$ is demicontinuous, hence continuous because, as just noted above, weak and strong convergence are the same for finite dimensional spaces. In case $B$ is bounded, then this follows from Proposition 25.1.6 above. It is pseudomonotone and bounded hence demicontinuous and weak and strong convergence is the same in finite dimensions.

Proof of Theorem 25.7.9: Let $\left\{X_{n}\right\}$ be an increasing sequence of finite dimensional subspaces. Let $\hat{A}$ be maximal monotone on $\cup_{n} X_{n}$ and extending $A$. By this is meant that the graph of $\hat{A}$ contains the graph of $A$ restricted to $\cup_{n} X_{n}, \hat{A}$ is monotone and there is no other larger graph with these properties. See the above observation. Let $j_{n}: X_{n} \rightarrow X$ be the inclusion map and $j_{n}^{*}: X^{\prime} \rightarrow X_{n}^{\prime}$ be the dual map. Then $j_{n}^{*} \hat{A} j_{n} \equiv A_{n}$ and $j_{n}^{*} B j_{n} \equiv B_{n}$ have monotone graphs from $X_{n}$ to $\mathscr{P}\left(X_{n}^{\prime}\right)$ with $B_{n}$ being continuous and single valued. This follows from the hemicontinuity and the above lemma which states that on finite dimensional spaces, hemicontinuity and monotonicity imply continuity. Then

$$
[u, v] \in \mathscr{G}\left(A_{n}\right)
$$

means

$$
u \in D(A) \cap X_{n} \text { and } v \in j_{n}^{*} \hat{A} j_{n}(u)=j_{n}^{*} \hat{A}(u) \text { since } u \in X_{n}
$$

Then from Lemma 25.7.7, there exists $x_{n} \in X_{n}$ such that

$$
\left\langle B_{n} x_{n}+v_{n}, u_{n}-x_{n}\right\rangle_{X^{\prime}, X} \geq 0 \text { all }\left[u_{n}, v_{n}\right] \in \mathscr{G}\left(A_{n}\right)
$$

That is, there exists $x_{n} \in K \cap X_{n}$ such that for all $u \in D(\hat{A}) \cap X_{n},[u, v] \in \mathscr{G}(\hat{A})$

$$
\begin{equation*}
\left\langle B x_{n}+v, u-x_{n}\right\rangle_{X^{\prime}, X} \geq 0 \tag{25.7.48}
\end{equation*}
$$

Then

$$
\begin{equation*}
\left\langle v, u-x_{n}\right\rangle \geq\left\langle B x_{n}, x_{n}-u\right\rangle \tag{25.7.49}
\end{equation*}
$$

From the assumption that $0 \in D(\hat{A})$, one can let $u=0$ and then pick $v_{0} \in \hat{A} 0$. Then the above reduces to

$$
\left\langle v_{0},-x_{n}\right\rangle \geq\left\langle B x_{n}, x_{n}\right\rangle
$$

By coercivity of $B$, these $x_{n}$ are all bounded and so by the Eberlien Smulian theorem, there is a subsequence $\left\{x_{n}\right\}$ which satisfies

$$
\begin{aligned}
x_{n} & \rightarrow x \text { weakly in } X \\
B x_{n} & \rightarrow y \text { weakly in } X^{\prime}
\end{aligned}
$$

Then from 25.7.48

$$
\left\langle v, u-x_{n}\right\rangle+\left\langle B x_{n}, u\right\rangle \geq\left\langle B x_{n}, x_{n}\right\rangle
$$

Then it follows that

$$
\left\langle v, u-x_{n}\right\rangle+\left\langle B x_{n}, u\right\rangle-\left\langle B x_{n}, x\right\rangle \geq\left\langle B x_{n}, x_{n}-x\right\rangle
$$

It follows that

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \sup _{n}\left\langle B x_{n}, x_{n}-x\right\rangle & \leq\langle v, u-x\rangle+\langle y, u\rangle-\langle y, x\rangle \\
& =\langle v+y, u-x\rangle
\end{aligned}
$$

Claim: limsup $\operatorname{sim}_{n \rightarrow \infty}\left\langle B x_{n}, x_{n}-x\right\rangle \leq 0$.
Proof of claim: This is so if $\langle v+y, u-x\rangle \leq 0$ for some $[u, v] \in \mathscr{G}(\hat{A})$.If $\langle v+y, u-x\rangle$ is greater than 0 for all $[u, v]$, then since $\hat{A}$ is maximal, it would follow that $-y \in \hat{A} x$. Now consider 25.7.49.

$$
\langle v, u-x\rangle \geq \lim \sup _{n \rightarrow \infty}\left\langle B x_{n}, x_{n}\right\rangle-\langle y, u\rangle
$$

Since $x \in D(\hat{A})$, you could put in $u=x$ in the above and obtain

$$
0 \geq \lim \sup _{n \rightarrow \infty}\left\langle B x_{n}, x_{n}\right\rangle-\langle y, x\rangle=\lim \sup _{n \rightarrow \infty}\left\langle B x_{n}, x_{n}-x\right\rangle
$$

which shows the claim is true.
Since $B$ is monotone and hemicontinuous, it satisfies the pseudomonotone condition, Theorem 25.1.4. Hence for any $z$,

$$
\begin{aligned}
\langle y, x-z\rangle & \geq \lim \sup _{n \rightarrow \infty}\left\langle B x_{n}, x_{n}-x\right\rangle+\lim \sup _{n \rightarrow \infty}\left\langle B x_{n}, x-z\right\rangle \\
& \geq \lim \sup _{n \rightarrow \infty}\left(\left\langle B x_{n}, x_{n}-x\right\rangle+\left\langle B x_{n}, x-z\right\rangle\right) \\
& \geq \lim _{n \rightarrow \infty}\left(\left\langle B x_{n}, x_{n}-z\right\rangle\right) \geq\langle B x, x-z\rangle
\end{aligned}
$$

Since $z$ is arbitrary, this shows that $y=B x$. It follows from 25.7.48 that for any $[u, v] \in$ $\mathscr{G}(\hat{A})$,

$$
\begin{gathered}
\left\langle B x_{n}+v, u-x_{n}\right\rangle=\left\langle B x_{n}+v, u-x\right\rangle+\left\langle B x_{n}+v, x-x_{n}\right\rangle \geq 0 \\
\left\langle B x_{n}+v, u-x\right\rangle \geq\left\langle B x_{n}, x_{n}-x\right\rangle \geq\left\langle B x, x_{n}-x\right\rangle
\end{gathered}
$$

Now take a limit of both sides and use the fact that $y=B x$ to obtain

$$
\langle B x+v, u-x\rangle \geq 0
$$

for all $[u, v] \in \mathscr{G}(\hat{A})$. Here $\hat{A}$ extends $A$ on $\cup_{n} X_{n}$. Why does it follow from this that there exists an $x$ such that the inequality holds for all $[u, v] \in \mathscr{G}(A)$ ?

Let $V$ be a finite dimensional subspace.

$$
K_{V} \equiv\left\{x \in K:\langle B x+v, u-x\rangle_{X^{\prime}, X} \geq 0 \text { for all }[u, v] \in \mathscr{G}(A), u \in V\right\}
$$

Then from the above argument, $K_{V} \neq \emptyset$. You just choose your subspaces $X_{n}$ to all include $V$. Also, from coercivity of $B$ and the above argument, these $K_{V}$ are all bounded and weakly closed. Hence they are weakly compact. Then if you have finitely many of them, you can let your subspaces include each $V$ and conclude that these $K_{V}$ have finite intersection property and so there exists $x \in \cap_{V} K_{V}$ which gives the desired $x$.

Note that there is only one place where $0 \in D(A)$ was used and it was to get the estimate. In the argument,

$$
\left\langle v, u-x_{n}\right\rangle \geq\left\langle B x_{n}, x_{n}-u\right\rangle
$$

and it was convenient to be able to take $u=0$. However, you could also assume other things on $B$ such as that it satisfies an estimate of the form

$$
\|B x\| \leq C\|x\|+C
$$

and if you did this, you could also obtain the necessary estimate as follows.

$$
\begin{aligned}
\left\langle v, u-x_{n}\right\rangle & \geq\left\langle B x_{n}, x_{n}-u\right\rangle \\
\left\langle v, u-x_{n}\right\rangle+\left\langle B x_{n}, u\right\rangle & \geq\left\langle B x_{n}, x_{n}\right\rangle \\
\|v\|\left(\|u\|+\left\|x_{n}\right\|\right)+\left(\left[C\left\|x_{n}\right\|+C\right]\|u\|\right) & \geq\left\langle B x_{n}, x_{n}\right\rangle
\end{aligned}
$$

and then pick some $[u, v]$. Thus the following corollary comes right away. This would have worked just as well if you had an estimate of the form

$$
\|B x\| \leq C\|x\|^{p-1}+C, p>1
$$

Corollary 25.7.11 Let $X$ be a reflexive Banach space and let $K$ be a closed convex subset of $X$. Let $A, B$ be monotone such that

1. $D(A) \subseteq K$
2. $B$ is single valued, hemicontinuous, bounded and coercive mapping $X$ to $X^{\prime}$ which satisfies the estimate

$$
\|B x\| \leq C\|x\|+C \text { or more generally }\|B x\| \leq C\|x\|^{p-1}+C, p>1
$$

Then there exists $x \in K$ such that

$$
\langle B x+v, u-x\rangle_{X^{\prime}, X} \geq 0 \text { for all }[u, v] \in \mathscr{G}(A)
$$

Now here is the equivalence between maximal monotone graph and having $F+A$ be onto. It was already shown that if $\lambda F+A$ is onto, then the graph of $A$ is maximal monotone in the sense that there is no monotone operator whose graph properly contains the graph of A. This was Theorem 25.7.2 above which is stated here as a reminder of what it said.

Theorem 25.7.12 Let $X, X^{\prime}$ be reflexive and have strictly convex norms. Let A be a monotone set valued map as just described. Then if

$$
\lambda F+A \text { onto }
$$

for some $\lambda>0$, then whenever

$$
\langle y-z, x-u\rangle \geq 0 \text { for all }[x, y] \in \mathscr{G}(A)
$$

it follows that $z \in A u$ and $u \in D(A)$. That is, the graph is maximal.
Theorem 25.7.13 Let $X$ be a strictly convex reflexive Banach space. Suppose the graph of $A: X \rightarrow \mathscr{P}(X)$ is maximal monotone in the sense that it is monotone and no monotone graph can properly contain the graph of $A$. Then for all $\lambda>0, \lambda F+A$ is onto. Conversely, if for some $\lambda>0, \lambda F+A$ is onto, then the graph of $A$ is maximal with respect to being monotone.

Proof: In Theorem 25.7.9, let $B x \equiv \lambda F(x)-y_{0}$. Then from the properties of the duality map, Theorem 25.2.3 above, it follows that $B$ satisfies the necessary conditions to use the result of Corollary 25.7.11 with $K=X$. This $B$ is monotone hemicontinuous, and coercive. Thus there exists $x$ such that for all $[u, v] \in \mathscr{G}(A)$,

$$
\begin{aligned}
\left\langle\lambda F(x)-y_{0}+v, u-x\right\rangle_{X^{\prime}, X} & \geq 0 \\
\left\langle v-\left(y_{0}-\lambda F(x)\right), u-x\right\rangle_{X^{\prime}, X} & \geq 0
\end{aligned}
$$

By maximality of the graph, it follows that $x \in D(A)$ and

$$
y_{0}-\lambda F(x) \in A(x), \quad y_{0}=\lambda F(x)+A(x)
$$

so $\lambda F+A$ is onto as claimed. The converse was proved in Theorem 25.7.2.
Note that this theorem holds if $F$ is a duality map for $p>1$. That is, $\langle F x, x\rangle=$ $\|x\|^{p},\|F x\|=\|x\|^{p-1}$.

Suppose $A: X \rightarrow \mathscr{P}(X)$ is maximal monotone. Then let $z \in X$ and define a new mapping $\hat{A}$ as follows.

$$
D(\hat{A}) \equiv\left\{x: x-x_{0} \in D(A)\right\}, \hat{A}(x) \equiv A\left(x-x_{0}\right)
$$

Proposition 25.7.14 Let $A, \hat{A}$ be as just defined. Then $\hat{A}$ is also maximal monotone.
Proof: From Theorem 25.7.13 it suffices to show that graph of $\hat{A}$ is monotone and is maximal. Suppose then that $x_{i}^{*} \in \hat{A} x_{i}$. Then

$$
\left\langle x_{1}^{*}-x_{2}^{*}, x_{1}-x_{2}\right\rangle=\left\langle x_{1}^{*}-x_{2}^{*}, x_{1}-x_{0}-\left(x_{2}-x_{0}\right)\right\rangle
$$

by definition, $x_{i}^{*} \in A\left(x_{i}-x_{0}\right)$ and so the above is $\geq 0$. Next suppose for all $\left[x, x^{*}\right] \in \mathscr{G}(\hat{A})$,

$$
\left\langle x^{*}-z^{*}, x-z\right\rangle \geq 0
$$

Does it follow that $\left[z, z^{*}\right] \in \mathscr{G}(\hat{A})$ ? The above says that

$$
\left\langle x^{*}-z^{*}, x-x_{0}-\left(z-x_{0}\right)\right\rangle \geq 0
$$

whenever $x-x_{0} \in D(A)$ and $x^{*} \in A\left(x-x_{0}\right)$. Hence, since $A$ is given to be maximal monotone, $z-x_{0} \in D(A)$ and $z^{*} \in A\left(z-x_{0}\right)$ which says that $z^{*} \in \hat{A}(z)$. Thus $\hat{A}$ is maximal monotone by the Theorem 25.7.13.

### 25.7.3 Surjectivity Theorems

As an interesting example of this theorem, here is another result in Barbu [13]. It is interesting because it is not assumed $B$ is bounded.

Theorem 25.7.15 Let $B: X \rightarrow X^{\prime}$ be monotone hemicontinuous. Then $B$ is maximal monotone. If $B$ is coercive, then $B$ is also onto. Here $X$ is a strictly convex reflexive Banach space.

Proof: Suppose $B$ is not maximal monotone. Then there exists $\left(x_{0}, x_{0}^{*}\right) \in X \times X^{\prime}$ such that for all $x$,

$$
\left\langle B x-x_{0}^{*}, x-x_{0}\right\rangle \geq 0
$$

and yet $x_{0}^{*} \neq B x_{0}$. This is going to be a contradiction. Let $u \in X$ and consider $x_{t} \equiv t x_{0}+$ $(1-t) u, t \in(0,1)$. Then consider

$$
\left\langle B x_{t}-x_{0}^{*}, x_{t}-x_{0}\right\rangle
$$

However, $x_{t}-x_{0}=t x_{0}+(1-t) u-x_{0}=(1-t)\left(u-x_{0}\right)$ and so, for each $t \in(0,1)$,

$$
0 \leq\left\langle B x_{t}-x_{0}^{*}, x_{t}-x_{0}\right\rangle=(1-t)\left\langle B x_{t}-x_{0}^{*}, u-x_{0}\right\rangle
$$

Divide by $(1-t)$ and then let $t \uparrow 1$. This yields the following by hemicontinuity.

$$
\left\langle B x_{0}-x_{0}^{*}, u-x_{0}\right\rangle \geq 0
$$

which holds for all $u$. Hence $B x_{0}=x_{0}^{*}$ after all. Thus $B$ is indeed maximal monotone.
Next suppose $B$ is coercive. Let $F$ be the duality map (or the duality map for arbitrary $p>1$ ). Then from Theorem 25.7.13 there exists a solution $x_{\lambda}$ to

$$
\begin{equation*}
\lambda F x_{\lambda}+B x_{\lambda}=x_{0}^{*} \in X^{\prime} \tag{25.7.50}
\end{equation*}
$$

Then the $x_{\lambda}$ are bounded because, doing both sides to $x_{\lambda}$,

$$
\lambda\left\|x_{\lambda}\right\|^{2}+\left\langle B x_{\lambda}, x_{\lambda}\right\rangle=\left\langle x_{0}^{*}, x_{\lambda}\right\rangle
$$

and so

$$
\frac{\left\langle B x_{\lambda}, x_{\lambda}\right\rangle}{\left\|x_{\lambda}\right\|} \leq\left\|x_{0}^{*}\right\|
$$

Thus the coercivity of $B$ implies that the $x_{\lambda}$ are bounded. There exists a subsequence such that

$$
x_{\lambda} \rightarrow x \text { weakly. }
$$

Then from the equation 25.7.50 $\left\|\lambda F x_{\lambda}\right\|=\lambda\left\|x_{\lambda}\right\|$ and so,

$$
B x_{\lambda} \rightarrow x_{0}^{*} \text { strongly. }
$$

Since $B$ is monotone and hemicontinuous, it satisfies the pseudomonotone condition, Theorem 25.1.4. The above strong convergence implies

$$
\lim _{\lambda \rightarrow 0}\left\langle B x_{\lambda}, x_{\lambda}-x\right\rangle=0
$$

Hence for all $y$,

$$
\lim \inf _{\lambda \rightarrow 0}\left\langle B x_{\lambda}, x_{\lambda}-y\right\rangle=\lim \inf _{\lambda \rightarrow 0}\left\langle B x_{\lambda}, x-y\right\rangle=\left\langle x_{0}^{*}, x-y\right\rangle \geq\langle B x, x-y\rangle
$$

Since $y$ is arbitrary, this shows that $x_{0}^{*}=B x$ and so $B$ is onto as claimed.
Again, note that it really didn't matter about the particular duality map used, although the usual one was featured in the argument.

There are some more things which can be said about maximal monotone operators. To include some of these, here is a very interesting lemma found in [13].

Lemma 25.7.16 Let $X$ be a Banach space and suppose that

$$
x_{n} \rightarrow 0, \quad\left\|x_{n}^{*}\right\| \rightarrow \infty
$$

Then denoting by $D_{r}$ the closed disk centered at 0 with radius $r$. It follows that for every $D_{r}$, there exists $y_{0} \in D_{r}$ and a subsequence with index $n_{k}$ such that

$$
\left\langle x_{n_{k}}^{*}, x_{n_{k}}-y_{0}\right\rangle \rightarrow-\infty
$$

Proof: Suppose this is not true. Then there exists $D_{r}$ which has the property that for all $u \in D_{r}$,

$$
\left\langle x_{n}^{*}, x_{n}-u\right\rangle \geq C_{u}
$$

for all $n$. Now let

$$
E_{k} \equiv\left\{y \in D_{r}:\left\langle x_{n}^{*}, x_{n}-y\right\rangle \geq-k \text { for all } n\right\}
$$

Then this is a closed set, being the intersection of closed sets. Also, by assumption, the union of these $E_{k}$ equals $D_{r}$ which is a complete metric space. Hence one of these $E_{k}$ must have nonempty interior by the Bair category theorem, say for $k_{0}$. Say $B(y, \varepsilon) \subseteq D_{r}$. Then for all $\|u-y\|<\varepsilon$,

$$
\left\langle x_{n}^{*}, x_{n}-u\right\rangle \geq-k_{0} \text { for all } n
$$

Of course $-y \in D_{r}$ also, and so there is $C$ such that

$$
\left\langle x_{n}^{*}, x_{n}+y\right\rangle \geq C \text { for all } n
$$

Then

$$
\left\langle x_{n}^{*}, 2 x_{n}+y-u\right\rangle \geq C-k_{0} \text { for all } n
$$

whenever $\|y-u\|<\varepsilon$. Now recall that $x_{n} \rightarrow 0$. Consider only $u$ such that $\|y-u\|<\varepsilon / 2$. Therefore, for all $n$ large enough, the expression $2 x_{n}+y-u$ for such $u$ contains a small ball centered at the origin, say $D_{\delta}$. (The set of all $y-u$ for $u$ closer to $y$ than $\varepsilon / 2$ is the ball $B(0, \varepsilon / 2)$ and then the $2 x_{n}$ does not move it by much provided $n$ is large enough.) Therefore,

$$
\left\langle x_{n}^{*}, v\right\rangle \geq C-k_{0}
$$

for all $\|v\| \leq \delta$. This contradicts the assumption that $\left\|x_{n}^{*}\right\| \rightarrow \infty$.

Corollary 25.7.17 Let $X$ be a Banach space and suppose that

$$
x_{n} \rightarrow x, \quad\left\|x_{n}^{*}\right\| \rightarrow \infty
$$

Then denoting by $D_{r}$ the closed disk centered at $x$ with radius $r$. It follows that for every $D_{r}$, there exists $y_{0} \in D_{r}$ and a subsequence with index $n_{k}$ such that

$$
\left\langle x_{n_{k}}^{*}, x_{n_{k}}-y_{0}\right\rangle \rightarrow-\infty
$$

Proof: It follows that $x_{n}-x \rightarrow 0$. Therefore, from Lemma 25.7.16, for every $r>0$, there exists $\hat{y}_{0} \in \overline{B(0, r)}$ and a subsequence $x_{n_{k}}$ such that

$$
\left\langle x_{n_{k}}^{*},\left(x_{n_{k}}-x\right)-\hat{y}_{0}\right\rangle \rightarrow-\infty
$$

Thus

$$
\left\langle x_{n_{k}}^{*}, x_{n_{k}}-\left(x+\hat{y}_{0}\right)\right\rangle \rightarrow-\infty
$$

Just let $y_{0}=x+\hat{y}_{0}$. Then $y_{0} \in D_{r}$ and satisfies the desired conditions.
Definition 25.7.18 $A$ set valued mapping $A: D(A) \rightarrow \mathscr{P}(X)$ is locally bounded at $x \in$ $\overline{D(A)}$ if whenever $x_{n} \rightarrow x, x_{n} \in D(A)$ it follows that

$$
\limsup _{n \rightarrow \infty}\left\{\left\|x_{n}^{*}\right\|: x_{n}^{*} \in A x_{n}\right\}<\infty
$$

Lemma 25.7.19 $A$ set valued operator $A$ is locally bounded at $x \in \overline{D(A)}$ if and only if there exists $r>0$ such that $A$ is bounded on $\overline{B(x, r)} \cap D(A)$.

Proof: Say the limit condition holds. Then if no such $r$ exists, it follows that $A$ is unbounded on every $B(x, r) \cap D(A)$. Hence, you can let $r_{n} \rightarrow 0$ and pick $x_{n} \in B\left(x, r_{n}\right) \cap$ $D(A)$ with $x_{n}^{*} \in A x_{n}$ such that $\left\|x_{n}^{*}\right\|>n$, violating the limit condition. Hence some $r$ exists such that $A$ is bounded on $\overline{B(x, r)} \cap D(A)$. Conversely, suppose $A$ is bounded on $\overline{B(x, r)} \cap$ $D(A)$ by $M$. Then if $x_{n} \rightarrow x$, it follows that for all $n$ large enough, $x_{n} \in B(x, r)$ and so if $x_{n}^{*} \in A x_{n},\left\|x_{n}^{*}\right\| \leq M$. Hence $\lim \sup _{n \rightarrow \infty}\left\{\left\|x_{n}^{*}\right\|: x_{n}^{*} \in A x_{n}\right\} \leq M<\infty$ which verifies the limit condition.

With this definition, here is a very interesting result.
Theorem 25.7.20 Let $A: D(A) \rightarrow X^{\prime}$ be monotone. Then if $x$ is an interior point of $D(A)$, it follows that $A$ is locally bounded at $x$.

Proof: You could use Corollary 25.7.17. If $x$ is an interior point of $D(A)$, and $A$ is not locally bounded, then there exists $x_{n} \rightarrow x$ and $x_{n}^{*} \in A x_{n}$ such that $\left\|x_{n}^{*}\right\| \rightarrow \infty$. Then by Corollary 25.7.17, there exists $y_{0}$ close to $x$, in $D(A)$ and a subsequence $x_{n_{k}}$ such that

$$
\left\langle x_{n_{k}}^{*}, x_{n_{k}}-y_{0}\right\rangle \rightarrow-\infty
$$

Letting $y_{0}^{*} \in A y_{0}$,

$$
\left\langle x_{n_{k}}^{*}-y_{0}^{*}, x_{n_{k}}-y_{0}\right\rangle \geq 0
$$

and so

$$
\left\langle x_{n_{k}}^{*}, x_{n_{k}}-y_{0}\right\rangle \geq\left\langle y_{0}^{*}, x_{n_{k}}-y_{0}\right\rangle
$$

and the right side is bounded below because it converges to $\left\langle y_{0}^{*}, x-y_{0}\right\rangle$ and this is a contradiction.

Does the same proof work if $x$ is a limit point of $D(A)$ ? No. Suppose $x$ is a limit point of $D(A)$. If $A$ is not locally bounded, then there exists $x_{n} \rightarrow x, x_{n} \in D(A)$ and $x_{n}^{*} \in A x_{n}$ and $\left\|x_{n}^{*}\right\| \rightarrow \infty$. Then there is $y_{0}$ close to $x$ such that $\left\langle x_{n_{k}}^{*}, x_{n_{k}}-y_{0}\right\rangle \rightarrow-\infty$ but now everything crashes in flames because it is not known that $y_{0} \in D(A)$.

It follows from the above theorem that if $A$ is defined on all of $X$ and is maximal monotone, then it is locally bounded everywhere. Now here is a very interesting result which is like the one which involves monotone and hemicontinuous conditions. It is in [55].

Theorem 25.7.21 Let $A: X \rightarrow \mathscr{P}\left(X^{\prime}\right)$ be monotone and satisfies the following conditions:

1. If $\lambda_{n} \rightarrow \lambda, \lambda_{n} \in[0,1]$ and $z_{n} \in A\left(u+\lambda_{n}(v-u)\right)$, then if $B$ is any weakly open set containing $0, z_{n} \in A(u)+B$ for all n large enough. (Upper semicontinuous into weak topology along a line segment)
2. $A(x)$ is closed and convex.

Then one can conclude that A is maximal monotone.
Proof: Let $\hat{A}$ be a monotone extension of $A$. Let $[\hat{u}, \hat{w}]$ be such that $\hat{w} \in \hat{A}(\hat{u})$. Now also by assumption, $A(x)$ is not just convex but also closed.

If $[\hat{u}, \hat{w}]$ is not in the graph of $A$, then by separation theorems, there is $u$ such that

$$
\left\langle x^{*}, u\right\rangle<\langle\hat{w}, u\rangle \text { for all } x^{*} \in A(\hat{u})
$$

Then for $\lambda>0$, let $x_{\lambda} \equiv \hat{u}+\lambda u, x_{\lambda}^{*} \in A\left(x_{\lambda}\right)$. Then from monotonicity of $\hat{A}$,

$$
0 \leq\left\langle x_{\lambda}^{*}-\hat{w}, x_{\lambda}-\hat{u}\right\rangle=\lambda\left\langle x_{\lambda}^{*}-\hat{w}, u\right\rangle
$$

Thus

$$
\left\langle x_{\lambda}^{*}-\hat{w}, u\right\rangle \geq 0
$$

By Theorem 25.7.20, the monotonicity of $A$ on $X$ implies $A$ is locally bounded also. Thus in particular, $A x_{\lambda}$ for small $\lambda$ is contained in a bounded set. Now by that hemicontinuity assumption, you can get a subsequence $\lambda_{n} \rightarrow 0$ for which $x_{\lambda_{n}}^{*}$ converges weakly to $x^{*} \in A \hat{u}$. Therefore, passing to the limit in the above, we get

$$
\begin{gathered}
\left\langle x^{*}-\hat{w}, u\right\rangle \geq 0 \\
\left\langle x^{*}, u\right\rangle \geq\langle\hat{w}, u\rangle>\left\langle x^{*}, u\right\rangle
\end{gathered}
$$

a contradiction. Thus there is no proper extension and this shows that $A$ is maximal monotone.

Recall the definition of a pseudomonotone operator.

Definition 25.7.22 $A$ set valued operator $B$ is quasi-bounded if whenever $x \in D(B)$ and $x^{*} \in B x$ are such that

$$
\left|\left\langle x^{*}, x\right\rangle\right|,\|x\| \leq M
$$

it follows that $\left\|x^{*}\right\| \leq K_{M}$. Bounded would mean that if $\|x\| \leq M$, then $\left\|x^{*}\right\| \leq K_{M}$. Here you only know this if there is another condition.

By Proposition 25.7.23 an example of a quasi-bounded operator is a maximal monotone operator $G$ for which $0 \in \operatorname{int}(D(G))$.

Then there is a useful result which gives examples of quasi-bounded operators [25].
Proposition 25.7.23 Let $A: D(A) \subseteq X \rightarrow \mathscr{P}\left(X^{\prime}\right)$ be maximal monotone and suppose $0 \in$ $\operatorname{int}(D(A))$. Then $A$ is quasi-bounded.

Proof: From local boundedness, Theorem 25.7.20, there exists $\delta, C>0$ such that

$$
\sup \left\{\left\|x^{*}\right\|: x^{*} \in A(x) \text { for }\|x\| \leq \delta\right\}<C
$$

Now suppose that $\|x\|,\left|\left\langle x^{*}, x\right\rangle\right| \leq M$. Then letting $\|y\| \leq \delta, y^{*} \in A y$,

$$
0 \leq\left\langle x^{*}-y^{*}, x-y\right\rangle=\left\langle x^{*}, x\right\rangle-\left\langle x^{*}, y\right\rangle-\left\langle y^{*}, x\right\rangle+\left\langle y^{*}, y\right\rangle
$$

and so for $\|y\| \leq \delta$,

$$
\left\langle x^{*}, y\right\rangle \leq\left\langle x^{*}, x\right\rangle-\left\langle y^{*}, x\right\rangle+\left\langle y^{*}, y\right\rangle \leq M+M C+C \delta
$$

Hence, $\left\|x^{*}\right\| \leq M+M C+C \delta \equiv K_{M}$.
This is actually quite a restrictive requirement and leaves out a lot which would be interesting.

Definition 25.7.24 Let $V$ be a Reflexive Banach space. We say $T: V \rightarrow \mathscr{P}\left(V^{\prime}\right)$ is pseudomonotone if the following conditions hold.

> Tu is closed, nonempty, convex.

If $F$ is a finite dimensional subspace of $V$, then if $u \in F$ and $W \supseteq T u$ for $W$ a weakly open set in $V^{\prime}$, then there exists $\delta>0$ such that

$$
\begin{equation*}
v \in B(u, \delta) \cap F \text { implies } T v \subseteq W \tag{25.7.52}
\end{equation*}
$$

If $u_{k} \rightharpoonup u$ and if $u_{k}^{*} \in T u_{k}$ is such that

$$
\limsup _{k \rightarrow \infty} u_{k}^{*}\left(u_{k}-u\right) \leq 0
$$

then for all $v \in V$, there exists $u^{*}(v) \in T u$ such that

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \inf _{k} u_{k}^{*}\left(u_{k}-v\right) \geq u^{*}(v)(u-v) \tag{25.7.53}
\end{equation*}
$$

Then here is an interesting result [39].

Theorem 25.7.25 Suppose $A: X \rightarrow \mathscr{P}\left(X^{\prime}\right)$ is maximal monotone. That is, $D(A)=X$. Then A is pseudomonotone.

Proof: Consider the first condition. Say $x_{i}^{*} \in A x$. Let $u^{*} \in A u$. For $\lambda \in[0,1]$,

$$
\begin{aligned}
& \left\langle\lambda x_{1}^{*}+(1-\lambda) x_{2}^{*}-u^{*}, x-u\right\rangle \\
= & \lambda\left\langle x_{1}^{*}-u^{*}, x-u\right\rangle+(1-\lambda)\left\langle x_{2}^{*}-u^{*}, x-u\right\rangle \geq 0
\end{aligned}
$$

and so, since $\left[u, u^{*}\right]$ is arbitrary, it follows that $\lambda x_{1}^{*}+(1-\lambda) x_{2}^{*} \in A x$. Thus $A x$ is convex. Is it closed? Say $x_{n}^{*} \in A x$ and $x_{n}^{*} \rightarrow x^{*}$. Is it the case that $x^{*} \in D(A)$ ? Let $\left[u, u^{*}\right] \in \mathscr{G}(A)$ be arbitrary. Then

$$
\left\langle x^{*}-u^{*}, x-u\right\rangle=\lim _{n \rightarrow \infty}\left\langle x_{n}^{*}-u^{*}, x_{n}-u\right\rangle \geq 0
$$

and so $A x$ is also closed.
Consider the second condition. It is to show that if $x_{n} \rightarrow x$ in $V$ a finite dimensional subspace and if $U$ is a weakly open set containing 0 , then eventually $A x_{n} \subseteq A x+U$. Suppose then that this is not the case. Then there exists $x_{n}^{*}$ outside of $A x+U$ but in $A x_{n}$. Since $A$ is locally bounded at $x$, it follows that the $\left\|x_{n}^{*}\right\|$ are bounded. Thus there is a subsequence, still denoted as $x_{n}$ and $x_{n}^{*}$ such that $x_{n}^{*} \rightarrow x^{*}$ weakly and $x^{*} \notin A x+U$. Now let $\left[u, u^{*}\right] \in \mathscr{G}(A)$.

$$
\left\langle x^{*}-u^{*}, x-u\right\rangle=\lim _{n \rightarrow \infty}\left\langle x_{n}^{*}-u^{*}, x_{n}-u\right\rangle \geq 0
$$

and since $\left[u, u^{*}\right]$ is arbitrary, it follows that $x^{*} \in A x$ and so is inside $A x+U$. Thus the second condition holds also.

Consider the third. Say $x_{k} \rightarrow x$ weakly and letting $x_{k}^{*} \in A x_{k}$,suppose

$$
\limsup _{k \rightarrow \infty}\left\langle x_{k}^{*}, x_{k}-x\right\rangle \leq 0
$$

Is it the case that there exists $x^{*}(y) \in A x$ such that

$$
\lim _{k \rightarrow \infty}\left\langle x_{k}^{*}, x_{k}-y\right\rangle \geq\left\langle x^{*}(y), x-y\right\rangle ?
$$

The proof goes just like it did earlier in the case of single valued pseudomonotone operators. It is just a little more complicated. First, let $x^{*} \in A x$.

$$
\left\langle x_{k}^{*}-x^{*}, x_{k}-x\right\rangle \geq 0
$$

and so

$$
\lim _{k \rightarrow \infty}\left\langle x_{k}^{*}, x_{k}-x\right\rangle \geq \lim \inf _{k \rightarrow \infty}\left\langle x^{*}, x_{k}-x\right\rangle=0 \geq \limsup \left\langle x_{k}^{*}, x_{k}-x\right\rangle
$$

Thus

$$
\lim _{k \rightarrow \infty}\left\langle x_{k}^{*}, x_{k}-x\right\rangle=0
$$

Now let $x_{t}^{*} \in A(x+t(y-x)), t \in(0,1)$, where here $y$ is arbitrary. Then

$$
\left\langle x_{n}^{*}-x_{t}^{*}, x_{n}-x+t(x-y)\right\rangle \geq 0
$$

Hence

$$
\lim \inf _{n \rightarrow \infty}\left\langle x_{n}^{*}, x_{n}-x+t(x-y)\right\rangle \geq \lim \inf _{n \rightarrow \infty}\left\langle x_{t}^{*}, x_{n}-x+t(x-y)\right\rangle
$$

and so from the above limit,

$$
t \lim _{n \rightarrow \infty}\left\langle x_{n}^{*}, x-y\right\rangle \geq t\left\langle x_{t}^{*}, x-y\right\rangle
$$

Cancel the $t$.

$$
\lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty}\left\langle x_{n}^{*}, x-y\right\rangle=\lim \inf _{n \rightarrow \infty}\left\langle x_{n}^{*}, x_{n}-y\right\rangle \geq\left\langle x_{t}^{*}, x-y\right\rangle
$$

Now you have a fixed $y$ and $x_{t}^{*} \in A(x+t(y-x))$. The subspace determined by $x, y$ is finite dimensional. Also it was shown above that $A$ is locally bounded at $x$ and so there is a subsequence, still denoted as $x_{t}^{*}$ such that $x_{t}^{*} \rightarrow x^{*}(y)$ weakly. Now from the upper semicontinuity on finite dimensional spaces shown above, for every $S$ a finite subset of $X$ and $\varepsilon>0$, it follows that for all $t$ small enough,

$$
x_{t}^{*} \in A x+B_{S}(0, \varepsilon)
$$

Thus $x^{*}(y) \in A x$. Hence, there exists $x^{*}(y) \in A x$ such that

$$
\lim _{n \rightarrow \infty}\left\langle x_{n}^{*}, x_{n}-y\right\rangle \geq\left\langle x^{*}(y), x-y\right\rangle
$$

I found this in a paper by Peng. It is a very nice result.
Proposition 25.7.26 Let $X$ and $Y$ be reflexive Banach spaces with $Y \subseteq X^{\prime}$. Let $1<p<\infty$ and let $q=\frac{p}{p-1}=p^{\prime}$ so $\frac{1}{p}+\frac{1}{q}=1$. Let $F:[0, T] \times X \rightarrow \mathscr{P}(Y)$ be multivalued and satisfies.

1. $F(\cdot, x)$ has a measurable selection for each $x \in X$
2. $F(t, \cdot)$ is maximal monotone for a.e. $t \in[0, T]$
3. $\|y\|_{Y} \leq \rho_{1}(t)+\rho_{2}\|x\|_{X}^{p-1}$ where $y \in F(t, x)$ for a.e. $t$, and where $\rho_{1} \in L^{q}(0, T)$ and $\rho_{2}>0$

Let $0 \leq a<b \leq T$ with $b-a=\tau$. Define

$$
F_{\tau} x \equiv\left\{\begin{array}{c}
\frac{1}{\tau} \int_{a}^{b} y(t) d t: t \rightarrow y(t) \text { is measurable } \\
\text { and } y(t) \in F(t, x) \text { a.e. } t \in(a, b)
\end{array}\right\}
$$

Then $F_{\tau}: X \rightarrow \mathscr{P}(Y)$ is maximal monotone.
Proof: First note that $F_{\tau} x$ is convex and nonempty. To see this, say $z, \hat{z}$ are in $F_{\tau} x$. and let $y, \hat{y}$ be the corresponding functions. Then for $\lambda \in[0,1]$

$$
\lambda z+(1-\lambda) \hat{z}=\frac{1}{\tau} \int_{a}^{b}(\lambda y(t)+(1-\lambda) \hat{y}(t)) d t
$$

and $\lambda y(t)+(1-\lambda) \hat{y}(t) \in F(t, x)$ because $F(t, \cdot)$ is maximal monotone which implies that the set values are convex.

Next is a claim that $F_{\tau} x$ is closed and also has the property that if $z_{n} \in F_{\tau} x_{n}$ and if $x_{n} \rightarrow x$ strongly in $X$ and $z_{n} \rightarrow z$ weakly in $Y$, then $z \in F_{\tau} x$. Let $y_{n}(t) \in F\left(t, x_{n}\right)$ a.e. such that $z_{n}=\frac{1}{\tau} \int_{a}^{b} y_{n}(t) d t$. These $x_{n}$ are bounded and so by the assumed estimate, it follows that the $y_{n}$ are bounded in $L^{q}(0, T ; Y)$. Therefore, there is a subsequence, still denoted with $n$ such that $y_{n} \rightarrow \hat{y}$ weakly in $L^{q}(0, T ; Y)$. Now this means that $\hat{y}$ is in the weak closure of the convex hull of $\left\{y_{k}: k \geq n\right\}$. However, this is the same as the strong closure because convex and closed is the same as convex and weakly closed. Therefore, there are functions

$$
\lim _{n \rightarrow \infty} \sum_{k=n}^{\infty} c_{k}^{n} y_{k}=\hat{y} \text { strongly in } L^{q}(0, T ; Y) \text { where } \sum_{k=n}^{\infty} c_{k}^{n}=1, c_{k}^{n} \geq 0
$$

and only finitely many are nonzero. Thus a subsequence still denoted with subscript $n$ also converges to $\hat{y}(t)$ for each $t$ off a set of measure zero. The function $F(t, \cdot)$ is maximal monotone and defined on $X$ and so it is pseudomonotone by Theorem 25.7.25. The estimate also shows that it is bounded. Therefore, as shown in the section on set valued pseudomonotone operators, $x \rightarrow F(t, x)$ is upper semicontinuous from strong to weak topology. Thus, for large $n$ depending on $t$, all of the $F\left(t, x_{n}\right)$ are contained in $F(t, x)+B_{S}(0, r / 2)$ where $S$ is a finite subset of points of $X$.

$$
B \equiv B_{S}(0, r / 2) \equiv\left\{w^{*}:\left|w^{*}(x)\right|<\frac{r}{2} \text { for all } x \in S\right\}
$$

Thus, for a fixed $t$ not in the exceptional set, off which the above pointwise convergence takes place, $\hat{y}(t) \in F(t, x)+D$ where

$$
D \equiv\left\{w^{*}:\left|w^{*}(x)\right| \leq r \text { for all } x \in S\right\}
$$

Since $S, r$ are arbitrary, separation theorems imply that $\hat{y}(t) \in F(t, x)$ for $t$ off a set of measure zero: If not, there would exist $u \in X$ such that $\hat{y}(t)(u)>l>l-\delta>p(u)$ for all $p \in F(t, x)$. But then you could take $r=\delta / 2$ and $B_{u}(0, \delta / 2)$ and find that $\hat{y}(t)=p+w^{*}$ where $p \in F(t, x)$ and $\left|w^{*}(u)\right| \leq \delta$. Hence $p(u)=\hat{y}(t)(u)-w^{*}(u)>l-\delta>p(u)$ an obvious contradiction. Is $z \in F_{\tau} x$ ? Certainly so if $z=\frac{1}{\tau} \int_{a}^{b} \hat{y}(t) d t$. Letting $\phi \in X$,

$$
\begin{aligned}
\langle z, \phi\rangle & =\lim _{n \rightarrow \infty}\left\langle z_{n}, \phi\right\rangle=\lim _{n \rightarrow \infty}\left\langle\frac{1}{\tau} \int_{a}^{b} y_{n}(t) d t, \phi\right\rangle=\lim _{n \rightarrow \infty}\left\langle\frac{1}{\tau} \int_{a}^{b} \sum_{k=n}^{\infty} c_{k}^{n} y_{k}(t) d t, \phi\right\rangle \\
& =\left\langle\frac{1}{\tau} \int_{a}^{b} \hat{y}(t) d t, \phi\right\rangle
\end{aligned}
$$

Since $\phi$ is arbitrary, it follows that $z=\frac{1}{\tau} \int_{a}^{b} \hat{y}(t) d t$ and so $z \in F_{\tau} x$.
Is $F_{\tau}$ monotone? Say $z, \hat{z}$ are in $F_{\tau}(x), F_{\tau}(\hat{x})$ respectively. Consider

$$
\langle z-\hat{z}, x-\hat{x}\rangle=\left\langle\frac{1}{\tau} \int_{a}^{b} y(t)-\hat{y}(t) d t, x-\hat{x}\right\rangle=\frac{1}{\tau} \int_{a}^{b}\langle y(t)-\hat{y}(t), x-\hat{x}\rangle d t
$$

but $y(t) \in F(t, x)$ similar for $\hat{y}$ and so the above is $\geq 0$. Thus $F_{\tau}$ is indeed monotone. This has also shown that $F_{\tau}$ satisfies the necessary modified hemicontinuity condition of

Theorem 25.7.21 to conclude that $F_{\tau}$ is indeed maximal monotone because it has convex closed values, the hemicontinuity condition, and is monotone.

Suppose $T$ is a bounded pseudomonotone operator and $S$ is a maximal monotone operator, both defined on a strictly convex reflexive Banach space. What of their sum? Is $(T+S)(x)$ convex and closed? Say $t_{i} \in T x$ and $s_{i} \in S x$ is it the case that $\theta\left(s_{1}+t_{1}\right)+$ $(1-\theta)\left(s_{2}+t_{2}\right) \in(T+S)(x)$ whenever $\theta \in[0,1]$ ? Of course this is so. Thus $T+S$ has convex values. Does it have closed values? Suppose $\left\{s_{n}+t_{n}\right\}$ converges to $z \in X^{\prime}, s_{n} \in$ $S x, t_{n} \in T x$. Is $z \in(T+S)(x)$ ? Taking a subsequence, and using the assumption that $T$ is bounded, it can be assumed that $t_{n} \rightarrow t \in T x$ weakly. Therefore, $s_{n}$ must also converge weakly and so it converges to some $s=z-t \in S x$. Convex and closed implies weakly closed. Thus $T+S$ has closed convex values. Is it upper semicontinuous on finite dimensional subspaces? Suppose $x_{n} \rightarrow x$ in a finite dimensional subspace $F$. Does it follow that

$$
(S+T) x_{n} \subseteq(S+T) x+B(0, r)
$$

for all $n$ sufficiently large? It is known that $S x_{n} \subseteq S x+B(0, r / 2)$ and $T x_{n} \subseteq T x+B(0, r / 2)$ whenever $n$ is sufficiently large and so it follows that

$$
(S+T) x_{n} \subseteq(S+T) x+B(0, r / 2)+B(0, r / 2) \subseteq(S+T) x+B(0, r)
$$

whenever $n$ is large enough.
What of the pseudomonotone condition? Suppose

$$
\limsup _{n \rightarrow \infty}\left\langle u_{n}^{*}+v_{n}^{*}, x_{n}-x\right\rangle \leq 0
$$

where $u_{n}^{*} \in S x_{n}$ and $v_{n}^{*} \in T x_{n}$ where $x_{n} \rightarrow x$ weakly. Is it the case that for every $y$, there exists $u^{*} \in S x$ and $v^{*} \in T x$ such that

$$
\lim _{n \rightarrow \infty}\left\langle u_{n}^{*}+v_{n}^{*}, x_{n}-y\right\rangle \geq\left\langle u^{*}+v^{*}, x-y\right\rangle ?
$$

By monotonicity,

$$
\begin{aligned}
0 & \geq \lim \sup _{n \rightarrow \infty}\left\langle u_{n}^{*}+v_{n}^{*}, x_{n}-x\right\rangle \geq \lim \sup _{n \rightarrow \infty}\left\langle u^{*}+v_{n}^{*}, x_{n}-x\right\rangle \\
& =\lim \sup _{n \rightarrow \infty}\left\langle v_{n}^{*}, x_{n}-x\right\rangle
\end{aligned}
$$

Hence

$$
\lim \sup _{n \rightarrow \infty}\left\langle v_{n}^{*}, x_{n}-x\right\rangle \leq 0
$$

which implies

$$
\lim _{n \rightarrow \infty} \inf _{n \rightarrow}\left\langle v_{n}^{*}, x_{n}-x\right\rangle \geq\left\langle\hat{v}^{*}, x-x\right\rangle=0 \geq \lim _{n \rightarrow \infty}\left\langle v_{n}^{*}, x_{n}-x\right\rangle
$$

showing that

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\langle v_{n}^{*}, x_{n}-x\right\rangle=0 \tag{25.7.54}
\end{equation*}
$$

It follows that if $y$ is given, there exists $v^{*} \in T(x)$ such that

$$
\lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty}\left\langle v_{n}^{*}, x_{n}-y\right\rangle \geq\left\langle v^{*}, x-y\right\rangle
$$

Now let $u_{t}^{*} \in S(x+t(y-x))$ for $t>0$. Thus

$$
\begin{gathered}
\left\langle u_{n}^{*}-u_{t}^{*}, x_{n}-x+t(x-y)\right\rangle \geq 0 \\
\left\langle u_{n}^{*}, x_{n}-x+t(x-y)\right\rangle \geq\left\langle u_{t}^{*}, x_{n}-x+t(x-y)\right\rangle
\end{gathered}
$$

Then using the above and the convergence in 25.7.54,

$$
\begin{gathered}
\lim _{n \rightarrow \infty}\left\langle u_{n}^{*}+v_{n}^{*}, x_{n}-y\right\rangle \geq \lim _{n \rightarrow \infty}\left\langle u_{t}^{*}+v_{n}^{*}, x_{n}-y\right\rangle \\
=\left\langle u_{t}^{*}, x-y\right\rangle+\left\langle v^{*}, x-y\right\rangle
\end{gathered}
$$

Now as before where it was shown that maximal monotone and defined on $X$ implied pseudomonotone, and the theorem which says that maximal monotone operators are locally bounded on the interior of their domains, it follows that there exists a sequence, still denoted as $u_{t}^{*}$ which converges to something called $u^{*}$. Then as before, the subspace spanned by $x, y$ is finite dimensional and so from upper semicontinuity, for all $t$ small enough,

$$
u_{t}^{*} \in S(x)+B(0, r)
$$

Note that weak convergence is the same as strong on finite dimensional spaces. Since this is true for all $r$ and $S(x)$ is closed, it follows that $u^{*} \in S(x)$. Thus, passing to a limit as $t \rightarrow 0$ one gets $u^{*} \in S(x), v^{*} \in T(x)$, and

$$
\lim _{n \rightarrow \infty}\left\langle u_{n}^{*}+v_{n}^{*}, x_{n}-y\right\rangle \geq\left\langle u^{*}+v^{*}, x-y\right\rangle
$$

This proves the following generalization of Theorem 25.7.25.
Theorem 25.7.27 Let $T, S: X \rightarrow \mathscr{P}\left(X^{\prime}\right)$ where $X$ is a strictly convex reflexive Banach space and suppose $T$ is bounded and pseudomonotone while $S$ is maximal monotone. Then $T+S$ is pseudomonotone.

Also, there is an interesting result which is based on the obvious observation that if $A$ is maximal monotone, then so is $\hat{A}(x) \equiv A\left(x_{0}+x\right)$.

Lemma 25.7.28 Let A be maximal monotone. Then for each $\lambda>0$,

$$
x \rightarrow \lambda F\left(x-x_{0}\right)+A x
$$

is onto.
Proof: Let $\hat{A}(x) \equiv A\left(x_{0}+x\right)$ so as earlier, $\hat{A}$ is maximal monotone. Then let $y^{*} \in X^{\prime}$. Then there exists $y$ such that $\hat{A}(y)+\lambda F(y) \ni y^{*}$. Now define $x \equiv y+x_{0}$. Then

$$
\hat{A}(y)+\lambda F(y) \ni y^{*}, \hat{A}\left(x-x_{0}\right)+\lambda F\left(x-x_{0}\right) \ni y^{*}, A(x)+\lambda F\left(x-x_{0}\right) \ni y^{*}
$$

Definition 25.7.29 Let $A: D(A) \rightarrow \mathscr{P}\left(X^{\prime}\right)$ be maximal monotone. Let $A^{-1}: A(D(A)) \rightarrow$ $\mathscr{P}\left(X^{\prime}\right)$ be defined as follows.

$$
x \in A^{-1} x^{*} \text { if and only if } x^{*} \in A x
$$

Observation 25.7.30 $A^{-1}$ is also maximal monotone. This is easily seen as follows. $[x, y] \in$ $\mathscr{G}(A)$ if and only if $[y, x] \in \mathscr{G}\left(A^{-1}\right)$.

Earlier, it was shown that if $B$ is monotone and hemicontinuous and coercive, then it was onto. It was not necessary to assume that $B$ is bounded. The same thing holds for $A$ maximal monotone. This will follow from the next result. Recall that a maximal monotone operator is locally bounded at every interior point of its domain which was shown above. Also it appears to not be possible to show that a maximal monotone operator is locally bounded at a limit point of $D(A)$. The following result is in [13] although he claims a better result than what I am proving here in which it is only necessary to verify $A^{-1}$ is locally bounded at every point of $A(D(A))$. However, I was unable to follow the argument and so I am proving another theorem with the same argument he uses. It looks like a typo to me but I often have trouble following hard theorems so I am not sure. Anyway, the following is the best I can do. I think it is still a very interesting result.

Theorem 25.7.31 Suppose $A^{-1}$ is locally bounded at every point of $\overline{A(D(A))}$. Then in fact $A(D(A))=X^{\prime}$ and in fact $\overline{A(D(A))}=A(D(A))$.

Proof: This is done by showing that $A(D(A))$ is both open and closed. Since it is nonempty, it must be all of $X^{\prime}$ because $X^{\prime}$ is connected. First it is shown that $A(D(A))$ is closed. Suppose $y_{n} \in A x_{n}$ and $y_{n} \rightarrow y$. Does it follow that $y \in A(D(A))$ ? Since $y$ is a limit point of $A(D(A))$, it follows that $A^{-1}$ is locally bounded at $y$. Thus there is a subsequence still denoted by $y_{n}$ such that $y_{n} \rightarrow y$ and for $x_{n} \in A^{-1} y_{n}$ or in other words, $y_{n} \in A x_{n}$, it follows that $x_{n}$ is bounded. Hence there exists a subsequence, still denoted with the subscript $n$ such that $x_{n} \rightarrow x$ weakly and $y_{n} \rightarrow y$ strongly. Hence if $[u, v] \in \mathscr{G}(A)$,

$$
\langle y-v, x-u\rangle=\lim _{n \rightarrow \infty}\left\langle y_{n}-v, x_{n}-x\right\rangle \geq 0
$$

Since $[u, v]$ is arbitrary and $A$ is maximal monotone, it follows that $y \in A x$ or in other words, $x \in A^{-1} y$ and $y \in A(D(A))$. Thus $A(D(A))$ is closed.

Next consider why $A(D(A))$ is open. Let $y_{0} \in A(D(A))$. Then there exists $D_{r} \equiv$ $\overline{B\left(y_{0}, r\right)}$ centered at $y_{0}$ such that $A^{-1}$ is bounded on $D_{r}$. Since $A$ is maximal montone, for each $y \in X^{\prime}$ there is a solution $x_{\varepsilon}$ to the inclusion

$$
y \in \varepsilon F\left(x_{\varepsilon}-x_{0}\right)+A x_{\varepsilon}, y_{\varepsilon} \equiv y-\varepsilon F\left(x_{\varepsilon}-x_{0}\right) \in A x_{\varepsilon}
$$

Consider only $y \in B\left(y_{0}, \frac{r}{2}\right)$.

$$
\left\langle\left(y-\varepsilon F\left(x_{\varepsilon}-x_{0}\right)\right)-y_{0}, x_{\varepsilon}-x_{0}\right\rangle \geq 0
$$

Then using $\langle F z, z\rangle=\|z\|^{2}$,

$$
\left\|y-y_{0}\right\|\left\|x_{\varepsilon}-x_{0}\right\| \geq\left\langle y-y_{0}, x_{\varepsilon}-x_{0}\right\rangle \geq \varepsilon\left\|x_{\varepsilon}-x_{0}\right\|^{2}
$$

and so $\varepsilon\left\|x_{\varepsilon}-x_{0}\right\|=\varepsilon\left\|F\left(x_{\varepsilon}-x_{0}\right)\right\| \leq\left\|y-y_{0}\right\|<r / 2$. Thus $y_{\varepsilon}$ stays in $B\left(y_{0}, r\right)$. This is because $y$ is closer to $y_{0}$ than $r / 2$ while $y_{\varepsilon}$ is within $r / 2$ of $y$. It follows that the $x_{\varepsilon}$ are bounded and so $x_{\varepsilon}-x_{0}$ is bounded and so $\varepsilon F\left(x_{\varepsilon}-x_{0}\right) \rightarrow 0$. Thus $y_{\varepsilon} \rightarrow y$ strongly. Since the $x_{\varepsilon}$ are bounded, there exists a further subsequence, still denoted as $x_{\varepsilon}$ such that $x_{\varepsilon} \rightarrow x$, some point of $X$. Then if $[u, v] \in \mathscr{G}(A)$,

$$
\left\langle y_{\varepsilon}-v, x_{\varepsilon}-u\right\rangle \geq 0
$$

and letting $\varepsilon \rightarrow 0$ using the strong convergence of $y_{\varepsilon}$ one obtains

$$
\langle y-v, x-u\rangle \geq 0
$$

which shows that $y \in A x$. Thus $B\left(y_{0}, \frac{r}{2}\right) \subseteq A(D(A)) \equiv D\left(A^{-1}\right)$ and so $A(D(A))$ is open.
The proof featured the usual duality map.
Note that as part of the proof $A(D(A))$ was shown to be closed so although it was assumed at the outset that $A^{-1}$ was locally bounded on $\overline{A(D(A))}$, this is the same as saying that $A^{-1}$ is locally bounded on $A(D(A))$.

Corollary 25.7.32 Suppose $A: D(A) \rightarrow \mathscr{P}\left(X^{\prime}\right)$ is maximal monotone and coercive. Then $A$ is onto.

Proof: From Theorem 25.7 .31 it suffics to show that $A^{-1}$ is locally bounded at $y^{*} \in$ $\overline{A(D(A))}$. The case of an interior point follows from Theorem 25.7.20. Assume then that $y^{*}$ is a limit point of $A(D(A))$. Of course this includes the case of interior points. Then there exists $y_{n}^{*} \rightarrow y^{*}$ where $y_{n}^{*} \in A x_{n}$. Then

$$
\frac{\left\langle y_{n}^{*}, x_{n}\right\rangle}{\left\|x_{n}\right\|} \leq\left\|y_{n}^{*}\right\|
$$

and the right side is bounded. Hence by coercivity, so is $\left\|x_{n}\right\|$. Therefore, there is a further subsequence, still denoted as $x_{n}$ such that $x_{n} \rightarrow x$ weakly while $y_{n}^{*} \rightarrow y^{*}$ strongly. Then letting $\left[u, v^{*}\right] \in \mathscr{G}(A)$,

$$
\left\langle y^{*}-v^{*}, x-u\right\rangle=\lim _{n \rightarrow \infty}\left\langle y_{n}^{*}-v^{*}, x_{n}-u\right\rangle \geq 0
$$

Hence $y^{*} \in A x$ and $y^{*} \in A(D(A))$. Thus $A^{-1}$ is locally bounded on $\overline{A(D(A))}$ and so $A$ is onto from the above theorem.

### 25.7.4 Approximation Theorems

This section continues following Barbu [13]. Always it is assumed that the situation is of a real reflexive Banach space $X$ having strictly convex norm and its dual $X^{\prime}$. As observed earlier, there exists a solution $x_{\lambda}$ to the inclusion

$$
0 \in F\left(x_{\lambda}-x\right)+\lambda A\left(x_{\lambda}\right)
$$

To see this, you consider $\hat{A}(y) \equiv A(x+y)$. Then $\hat{A}$ is also maximal monotone and so there exists a solution to

$$
0 \in F(\hat{x})+\lambda \hat{A}(\hat{x})=F(\hat{x})+\lambda A(x+\hat{x})
$$

Now let $x_{\lambda}=x+\hat{x}$ so $\hat{x}=x_{\lambda}-x$. Hence

$$
0 \in F\left(x_{\lambda}-x\right)+\lambda A x_{\lambda}
$$

Here you could have $F$ the duality map for any given $p>1$.
The symbol $\lim \sup _{n, n \rightarrow \infty} a_{m n}$ means $\lim _{N \rightarrow \infty}\left(\sup _{m \geq N, n \geq N} a_{m n}\right)$. Then here is a simple observation.

Lemma 25.7.33 Suppose $\limsup _{n, n \rightarrow \infty} a_{m n} \leq 0$. Then

$$
\lim \sup _{m \rightarrow \infty}\left(\lim \sup _{n \rightarrow \infty} a_{m n}\right) \leq 0
$$

Proof: There exists $N$ such that if both $m, n \geq N, a_{m n} \leq \varepsilon$. Then

$$
\limsup _{n \rightarrow \infty} a_{m n}=\lim \sup _{n \rightarrow \infty, n>N} a_{m n} \leq \varepsilon
$$

Thus also

$$
\lim \sup _{m \rightarrow \infty}\left(\lim \sup _{n \rightarrow \infty} a_{m n}\right)=\lim \sup _{m \rightarrow \infty, m \geq N}\left(\lim \sup _{n \rightarrow \infty} a_{m n}\right) \leq \varepsilon
$$

The argument will be based on the following lemma.
Lemma 25.7.34 Let $A: D(A) \rightarrow \mathscr{P}\left(X^{\prime}\right)$ be maximal monotone and let $v_{n} \in A u_{n}$ and

$$
u_{n} \rightarrow u, v_{n} \rightarrow v \text { weakly. }
$$

Also suppose that

$$
\lim \sup _{m, n \rightarrow \infty}\left\langle v_{n}-v_{m}, u_{n}-u_{m}\right\rangle \leq 0
$$

or

$$
\lim \sup _{n \rightarrow \infty}\left\langle v_{n}-v, u_{n}-u\right\rangle \leq 0
$$

Then $[u, v] \in \mathscr{G}(A)$ and $\left\langle v_{n}, u_{n}\right\rangle \rightarrow\langle v, u\rangle$.
Proof: By monotonicity,

$$
\lim _{m, n \rightarrow \infty}\left\langle v_{n}-v_{m}, u_{n}-u_{m}\right\rangle=0
$$

Suppose then that $\left\langle v_{n}, u_{n}\right\rangle$ fails to converge to $\langle v, u\rangle$. Then there is a subsequence, still denoted with subscript $n$ such that $\left\langle v_{n}, u_{n}\right\rangle \rightarrow \mu \neq\langle v, u\rangle$. Let $\varepsilon>0$. Then there exists $M$ such that if $n, m>M$, then

$$
\left|\left\langle v_{n}, u_{n}\right\rangle-\mu\right|<\varepsilon,\left|\left\langle v_{n}-v_{m}, u_{n}-u_{m}\right\rangle\right|<\varepsilon
$$

Then if $m, n>M$,

$$
\left|\left\langle v_{n}-v_{m}, u_{n}-u_{m}\right\rangle\right|=\left|\left\langle v_{n}, u_{n}\right\rangle+\left\langle v_{m}, u_{m}\right\rangle-\left\langle v_{n}, u_{m}\right\rangle-\left\langle v_{m}, u_{n}\right\rangle\right|<\varepsilon
$$

Hence it is also true that

$$
\left|\left\langle v_{n}, u_{n}\right\rangle+\left\langle v_{m}, u_{m}\right\rangle-\left\langle v_{n}, u_{m}\right\rangle-\left\langle v_{m}, u_{n}\right\rangle\right| \leq\left|2 \mu-\left(\left\langle v_{n}, u_{m}\right\rangle+\left\langle v_{m}, u_{n}\right\rangle\right)\right|<3 \varepsilon
$$

Now take a limit first with respect to $n$ and then with respect to $m$ to obtain

$$
|2 \mu-(\langle v, u\rangle+\langle v, u\rangle)|<3 \varepsilon
$$

Since $\boldsymbol{\varepsilon}$ is arbitrary, $\mu=\langle v, u\rangle$ after all. Hence the claim that $\left\langle v_{n}, u_{m}\right\rangle \rightarrow\langle v, u\rangle$ is verified. Next suppose $[x, y] \in \mathscr{G}(A)$ and consider

$$
\begin{aligned}
& \langle v-y, u-x\rangle=\langle v, u\rangle-\langle v, x\rangle-\langle y, u\rangle+\langle y, x\rangle \\
& =\lim _{n \rightarrow \infty}\left(\left\langle v_{n}, u_{n}\right\rangle-\left\langle v_{n}, x\right\rangle-\left\langle y, u_{n}\right\rangle+\langle y, x\rangle\right) \\
& =\lim _{n \rightarrow \infty}\left\langle v_{n}-y, u_{n}-x\right\rangle \geq 0
\end{aligned}
$$

and since $[x, y]$ is arbitrary, it follows that $v \in A u$.
Next suppose $\lim \sup _{n \rightarrow \infty}\left\langle v_{n}-v, u_{n}-u\right\rangle \leq 0$. It is not known that $[u, v] \in \mathscr{G}(A)$.

$$
\begin{array}{r}
\limsup _{n \rightarrow \infty}\left[\left\langle v_{n}, u_{n}\right\rangle-\left\langle v, u_{n}\right\rangle-\left\langle v_{n}, u\right\rangle+\langle v, u\rangle\right] \leq 0 \\
\lim \sup _{n \rightarrow \infty}\left\langle v_{n}, u_{n}\right\rangle-\langle v, u\rangle \leq 0
\end{array}
$$

Thus $\lim \sup _{n \rightarrow \infty}\left\langle v_{n}, u_{n}\right\rangle \leq\langle v, u\rangle$. Now let $[x, y] \in \mathscr{G}(A)$

$$
\begin{aligned}
& \langle v-y, u-x\rangle=\langle v, u\rangle-\langle v, x\rangle-\langle y, u\rangle+\langle y, x\rangle \\
& \geq \lim \sup _{n \rightarrow \infty}\left[\left\langle v_{n}, u_{n}\right\rangle-\left\langle v_{n}, x\right\rangle-\left\langle y, u_{n}\right\rangle+\langle y, x\rangle\right] \\
& \geq \lim _{n \rightarrow \infty}\left[\left\langle v_{n}-y, u_{n}-x\right\rangle\right] \geq 0
\end{aligned}
$$

Hence $[u, v] \in \mathscr{G}(A)$. Now

$$
\lim \sup _{n \rightarrow \infty}\left\langle v_{n}-v, u_{n}-u\right\rangle \leq 0 \leq \lim _{n \rightarrow \infty}\left\langle v_{n}-v, u_{n}-u\right\rangle
$$

the second coming from monotonicity and the fact that $v \in A u$. Therefore,

$$
\lim _{n \rightarrow \infty}\left\langle v_{n}-v, u_{n}-u\right\rangle=0
$$

which shows that $\lim _{n \rightarrow \infty}\left\langle v_{n}, u_{n}\right\rangle=\langle v, u\rangle$.

Definition 25.7.35 Let $x_{\lambda}$ just defined

$$
0 \in F\left(x_{\lambda}-x\right)+\lambda A x_{\lambda}
$$

be denoted by $J_{\lambda} x$ and define also

$$
A_{\lambda}(x)=-\lambda^{-(p-1)} F\left(x_{\lambda}-x\right)=-\lambda^{-(p-1)} F\left(J_{\lambda} x-x\right)
$$

This is for $F$ a duality map with $p>1$. Thus for the usual duality map, you would have

$$
A_{\lambda}(x)=-\lambda^{-1} F\left(J_{\lambda} x-x\right)
$$

Recall how this $x_{\lambda}$ is defined. In general,

$$
0 \in F\left(J_{\lambda} x-x\right)+\lambda^{p-1} A x_{\lambda}
$$

Thus, from the definition,

$$
A_{\lambda}(x) \in A\left(J_{\lambda} x\right)
$$

Formally, and to help remember what is going on, you are looking at a generalization of

$$
A_{\lambda} x=\frac{A}{1+\lambda A} x=\frac{1}{\lambda}\left(x-(I+\lambda A)^{-1} x\right)
$$

This is in the case where $F=I$ to keep things simpler. You have $0=x_{\lambda}-x+\lambda A x_{\lambda}$ and so formally $x_{\lambda}=(I+\lambda A)^{-1} x$. Thus you are looking at $\frac{1}{\lambda}\left(x-x_{\lambda}\right)=\frac{1}{\lambda}\left(x-(I+\lambda A)^{-1} x\right)=$ $A_{\lambda} x$. In fact, this is exactly what you do when you are in a single Hilbert space. This is just a generalization to mappings between Banach spaces and their duals.

Then there are some things which can be said about these operators. It is presented for the general duality map for $p>1$.

Theorem 25.7.36 The following hold. Here $X$ is a reflexive Banach space with strictly convex norm. $A: D(A) \rightarrow \mathscr{P}\left(X^{\prime}\right)$ is maximal monotone. Then

1. $J_{\lambda}$ and $A_{\lambda}$ are bounded single valued operators defined on $X$. Bounded means they take bounded sets to bounded sets. Also $A_{\lambda}$ is a monotone operator.
2. $A_{\lambda}, J_{\lambda}$ are demicontinuous. That is, strongly convergent sequences are mapped to weakly convergent sequences.
3. For every $x \in D(A)$,

$$
\left\|A_{\lambda}(x)\right\| \leq|A x| \equiv \inf \left\{\left\|y^{*}\right\|: y^{*} \in A x\right\}
$$

For every $x \in \overline{\operatorname{conv}(D(A))}$, it follows that $\lim _{\lambda \rightarrow 0} J_{\lambda}(x)=x$. The new symbol means the closure of the convex hull. It is the closure of the set of all convex combinations of points of $D(A)$.

Proof: 1.) It is clear that these are single valued operators. What about the assertion that they are bounded? Let $y^{*} \in A x_{\lambda}$ such that the inclusion defining $x_{\lambda}$ becomes an equality. Thus

$$
F\left(x_{\lambda}-x\right)+\lambda^{p-1} y^{*}=0
$$

Then let $x_{0} \in D(A)$ be given.

$$
\left\langle F\left(x_{\lambda}-x\right), x_{\lambda}-x\right\rangle+\lambda^{p-1}\left\langle y^{*}, x_{\lambda}-x_{0}\right\rangle+\lambda^{p-1}\left\langle y^{*}, x_{0}-x\right\rangle=0
$$

Then by monotonicity of $A$,

$$
\left\|x_{\lambda}-x\right\|^{p}+\lambda^{p-1}\left\langle y_{0}^{*}, x_{\lambda}-x_{0}\right\rangle+\lambda^{p-1}\left\langle y^{*}, x_{0}-x\right\rangle \leq 0
$$

It follows that

$$
\left\|x_{\lambda}-x\right\|^{p} \leq \lambda^{p-1}\left\|y_{0}^{*}\right\|\left\|x_{\lambda}-x_{0}\right\|+\lambda^{p-1}\left\|y^{*}\right\|\left\|x_{0}-x\right\|
$$

Hence if $x$ is in a bounded set, it follows the resulting $x_{\lambda}=J_{\lambda} x$ remain in a bounded set. Now from the definition of $A_{\lambda}$, it follows that this is also a bounded operator.

Why is $A_{\lambda}$ monotone?

$$
\begin{aligned}
0= & \left\langle A_{\lambda} x-A_{\lambda} y, x-y\right\rangle=\left\langle A_{\lambda} x-A_{\lambda} y, x-J_{\lambda} x-\left(y-J_{\lambda} y\right)\right\rangle \\
& +\left\langle A_{\lambda} x-A_{\lambda} y, J_{\lambda} x-J_{\lambda} y\right\rangle \\
= & \left\langle\lambda^{-(p-1)} F\left(J_{\lambda} x-x\right)-\lambda^{-(p-1)} F\left(J_{\lambda} y-y\right), J_{\lambda} x-x-\left(J_{\lambda} y-y\right)\right\rangle \\
& +\left\langle A_{\lambda} x-A_{\lambda} y, J_{\lambda} x-J_{\lambda} y\right\rangle
\end{aligned}
$$

and both terms are nonnegative, the first because $F$ is monotone so indeed $A_{\lambda}$ is monotone.
2.) What of the demicontinuity of $A_{\lambda}$ ? This one is really tricky. Suppose $x_{n} \rightarrow x$. Does it follow that $A_{\lambda} x_{n} \rightarrow A_{\lambda} x$ weakly? The proof will be based on a pair of equations. These are

$$
\lim _{m, n \rightarrow \infty}\left\langle F\left(J_{\lambda} x_{n}-x_{n}\right)-F\left(J_{\lambda} x_{m}-x_{m}\right), J_{\lambda} x_{n}-x_{n}-\left(J_{\lambda} x_{m}-x_{m}\right)\right\rangle=0
$$

and

$$
\lim _{m, n \rightarrow \infty}\left\langle A_{\lambda}\left(x_{n}\right)-A_{\lambda}\left(x_{m}\right), J_{\lambda} x_{n}-J_{\lambda} x_{m}\right\rangle=0
$$

When these have been established, Lemma 25.7 .34 is used to get the desired result for a subsequence. It will be shown that every sequence has a subsequence which gives the right sort of weak convergence and from this the desired weak convergence of $A_{\lambda} x_{n}$ to $A_{\lambda} x$ follows.

$$
\begin{gathered}
0 \in F\left(J_{\lambda} x_{n}-x_{n}\right)+\lambda^{p-1} A\left(J_{\lambda} x_{n}\right) \\
0 \in F\left(J_{\lambda} x-x\right)+\lambda^{p-1} A\left(J_{\lambda} x\right) \\
-\lambda^{-(p-1)} F\left(J_{\lambda} x-x\right) \equiv A_{\lambda}(x) \in A\left(J_{\lambda} x\right) \\
-\lambda^{-(p-1)} F\left(J_{\lambda} x_{n}-x_{n}\right) \equiv A_{\lambda}\left(x_{n}\right) \in A\left(J_{\lambda} x_{n}\right)
\end{gathered}
$$

Note also that for a given $x$ there is only one solution $J_{\lambda} x$ to $0 \in F\left(J_{\lambda} x-x\right)+\lambda^{p-1} A\left(J_{\lambda} x\right)$. By monotonicity of $F$,

$$
0 \leq\left\langle F\left(J_{\lambda} x_{n}-x_{n}\right)-F\left(J_{\lambda} x_{m}-x_{m}\right), x_{m}-x_{n}+J_{\lambda} x_{n}-J_{\lambda} x_{m}\right\rangle
$$

Then from the above,

$$
\begin{aligned}
& \left\langle F\left(J_{\lambda} x_{n}-x_{n}\right)-F\left(J_{\lambda} x_{m}-x_{m}\right), x_{n}-x_{m}\right\rangle \\
\leq & \left\langle F\left(J_{\lambda} x_{n}-x_{n}\right)-F\left(J_{\lambda} x_{m}-x_{m}\right), J_{\lambda} x_{n}-J_{\lambda} x_{m}\right\rangle
\end{aligned}
$$

Now from the boundedness of these operators, the left side of the above inequality converges to 0 as $n, m \rightarrow \infty$. Thus

$$
\begin{equation*}
\lim _{m, n \rightarrow \infty} \inf \left\langle F\left(J_{\lambda} x_{n}-x_{n}\right)-F\left(J_{\lambda} x_{m}-x_{m}\right), J_{\lambda} x_{n}-J_{\lambda} x_{m}\right\rangle \geq 0 \tag{25.7.55}
\end{equation*}
$$

$$
\begin{aligned}
& \lim \inf _{m, n \rightarrow \infty}\left\langle-\lambda^{p-1} A_{\lambda}\left(x_{n}\right)-\left(-\lambda^{p-1} A_{\lambda}\left(x_{m}\right)\right), J_{\lambda} x_{n}-J_{\lambda} x_{m}\right\rangle \geq 0 \\
& \lim _{\inf _{m, n}\langle }\langle\lambda^{p-1} \overbrace{A_{\lambda}\left(x_{m}\right)}^{\in A\left(J_{\lambda} x_{m}\right)}-\lambda^{p-1} \overbrace{A_{\lambda}\left(x_{n}\right)}^{A\left(J_{\lambda} x_{n}\right)}, J_{\lambda} x_{n}-J_{\lambda} x_{m}\rangle \geq 0
\end{aligned}
$$

The expression on the left in the above is non positive. Multiplying by -1 ,

$$
\begin{align*}
0 & \geq \lim \sup _{m, n \rightarrow \infty}\left\langle A_{\lambda}\left(x_{n}\right)-A_{\lambda}\left(x_{m}\right), J_{\lambda} x_{n}-J_{\lambda} x_{m}\right\rangle \\
& \geq \lim _{m, n \rightarrow \infty}\left\langle A_{\lambda}\left(x_{n}\right)-A_{\lambda}\left(x_{m}\right), J_{\lambda} x_{n}-J_{\lambda} x_{m}\right\rangle \geq 0 \tag{25.7.56}
\end{align*}
$$

Thus, in fact, the expression in 25.7 .55 converges to 0 . By boundedness considerations and the strong convergence given,

$$
\begin{equation*}
\lim _{m, n \rightarrow \infty}\left\langle F\left(J_{\lambda} x_{n}-x_{n}\right)-F\left(J_{\lambda} x_{m}-x_{m}\right), J_{\lambda} x_{n}-x_{n}-\left(J_{\lambda} x_{m}-x_{m}\right)\right\rangle=0 \tag{25.7.57}
\end{equation*}
$$

From boundedness again, there is a subsequence still denoted with the subscript $n$ such that

$$
J_{\lambda} x_{n}-x_{n} \rightarrow a-x, F\left(J_{\lambda} x_{n}-x_{n}\right) \rightarrow b \text { both weakly. }
$$

Since $F$ is maximal monotone, (Theorem 25.7.9) it follows from Lemma 25.7.34 that $[a-x, b] \in \mathscr{G}(F)$ and so in fact $F(a-x)=b$. Thus this has just shown that $F\left(J_{\lambda} x_{n}-x_{n}\right) \rightarrow$ $F(a-x)$. Next consider 25.7.56. We have $J_{\lambda} x_{n} \rightarrow a$ weakly and

$$
A_{\lambda}\left(x_{n}\right)=-\lambda^{-(p-1)} F\left(J_{\lambda} x_{n}-x_{n}\right) \rightarrow-\lambda^{-(p-1)} b
$$

weakly. Then from Lemma 25.7 .34 again, $\left[a,-\lambda^{-(p-1)} b\right] \in \mathscr{G}(A)$ so $-\lambda^{-(p-1)} b \in A(a)$ so $b \in-\lambda^{p-1} A(a)$. But it was just shown that $b=F(a-x)$ and so

$$
F(a-x) \in-\lambda^{p-1} A(a) \text { so } 0 \in F(a-x)+\lambda^{p-1} A(a), \text { so } a=J_{\lambda} x
$$

As noted at the beginning, there is only one solution to this inclusion for a given $x$ and it is $a=J_{\lambda} x$. This has shown that in terms of weak convergence,

$$
A_{\lambda}\left(x_{n}\right) \rightarrow-\lambda^{-(p-1)} b=-\lambda^{-(p-1)} F(a-x)=-\lambda^{-(p-1)} F\left(J_{\lambda} x-x\right) \equiv A_{\lambda}(x)
$$

This has shown that $A_{\lambda}$ is demicontinuous. Also it has shown that $J_{\lambda}$ is also demicontinuous. (This result is a lot nicer in Hilbert space. )
3.) Why is $\left\|A_{\lambda}(x)\right\| \leq|A x|$ whenever $x \in D(A)$ ?

$$
A_{\lambda}(x)=-\lambda^{-(p-1)} F\left(J_{\lambda} x-x\right)
$$

where $0 \in F\left(J_{\lambda} x-x\right)+\lambda^{p-1} A\left(J_{\lambda} x\right)$. Therefore, $A_{\lambda}(x) \in A\left(J_{\lambda} x\right)$. Then letting $[u, v] \in$ $\mathscr{G}(A)$,

$$
0 \leq\left\langle v-A_{\lambda}(x), u-J_{\lambda} x\right\rangle
$$

In particular, if $y \in A x$

$$
0 \leq\left\langle y-A_{\lambda}(x), x-J_{\lambda} x\right\rangle=\left\langle y+\lambda^{-(p-1)} F\left(J_{\lambda} x-x\right), x-J_{\lambda} x\right\rangle
$$

Hence

$$
\lambda^{-(p-1)}\left\|J_{\lambda} x-x\right\|^{p} \leq\|y\|\left\|J_{\lambda} x-x\right\|
$$

and so

$$
\lambda^{-(p-1)}\left\|J_{\lambda} x-x\right\|^{p-1}=\lambda^{-(p-1)}\left\|F\left(J_{\lambda} x-x\right)\right\|=\left\|A_{\lambda}(x)\right\| \leq\|y\|
$$

and since $y \in A x$ is arbitrary, $\left\|A_{\lambda}(x)\right\| \leq|A x| \equiv \inf \{\|y\|: y \in A x\}$.
Next consider the claim that for all $x \in \overline{\operatorname{conv}(D(A))}$, it follows that

$$
\lim _{\lambda \rightarrow 0} J_{\lambda}(x)=x
$$

Let $[u, v] \in \mathscr{G}(A)$ and $x$ is arbitrary.

$$
\begin{gathered}
0 \leq\left\langle v-A_{\lambda}(x), u-J_{\lambda} x\right\rangle=\left\langle v+\lambda^{-(p-1)} F\left(J_{\lambda} x-x\right), u-J_{\lambda} x\right\rangle \\
=\left\langle v+\lambda^{-(p-1)} F\left(J_{\lambda} x-x\right), u-x\right\rangle+\left\langle v+\lambda^{-(p-1)} F\left(J_{\lambda} x-x\right), x-J_{\lambda} x\right\rangle
\end{gathered}
$$

Thus

$$
\begin{equation*}
\left\|J_{\lambda} x-x\right\|^{p} \leq \lambda^{p-1}\langle v, u-x\rangle+\left\langle F\left(J_{\lambda} x-x\right), u-x\right\rangle+\lambda^{p-1}\left\langle v, x-J_{\lambda} x\right\rangle \tag{25.7.58}
\end{equation*}
$$

for $x$ arbitrary and $u$ anything in $D(A)$. It follows that 25.7 .58 holds for any $u \in \operatorname{conv}(D(A))$. Say $u=x_{n} \in \operatorname{conv}(D(A))$ where $x_{n} \rightarrow x$. Then

$$
\begin{gathered}
\left\|J_{\lambda} x-x\right\|^{p} \leq \lambda^{p-1}\left\langle v, x_{n}-x\right\rangle+\left\langle F\left(J_{\lambda} x-x\right), x_{n}-x\right\rangle+\lambda^{p-1}\left\langle v, x-J_{\lambda} x\right\rangle \\
\leq \lambda^{p-1}\|v\|\left\|x_{n}-x\right\|+\left\|J_{\lambda} x-x\right\|^{p-1}\left\|x_{n}-x\right\|+\lambda^{p-1}\|v\|\left\|J_{\lambda} x-x\right\|
\end{gathered}
$$

You have something like this: $y_{\lambda}=\left\|J_{\lambda} x-x\right\|, a_{n}=\left\|x_{n}-x\right\|$,

$$
y_{\lambda}^{p} \leq \lambda^{p-1}\|v\| a_{n}+y_{\lambda}^{p-1} a_{n}+\lambda^{p-1}\|v\| y_{\lambda}, \quad y_{\lambda} \geq 0
$$

where $p>1$ and $a_{n} \rightarrow 0$. Then

$$
\lim \sup _{\lambda \rightarrow 0} y_{\lambda}^{p} \leq \lim \sup _{\lambda \rightarrow 0} y_{\lambda}^{p-1} a_{n}
$$

and so,

$$
\limsup _{\lambda \rightarrow 0} y_{\lambda} \leq a_{n}
$$

Hence

$$
\lim \sup _{\lambda \rightarrow 0}\left\|J_{\lambda} x-x\right\| \leq\left\|x_{n}-x\right\|
$$

Since $x_{n}$ is arbitrary, it follows that for every $\varepsilon>0$,

$$
\lim \sup _{\lambda \rightarrow 0}\left\|J_{\lambda} x-x\right\| \leq \varepsilon
$$

and so in fact, $\limsup _{\lambda \rightarrow \infty}\left\|J_{\lambda} x-x\right\|=0$.
Now here is an interesting corollary.

Corollary 25.7.37 Let A be maximal monotone. $A: X \rightarrow X^{\prime}$ where $X$ is a strictly convex reflexive Banach space. Then $\overline{D(A)}$ is convex.

Proof: It is known that $J_{\lambda}: X \rightarrow D(A)$ for any $\lambda$. Also, if $x \in \overline{\operatorname{conv}(D(A))}$, then it was shown that $J_{\lambda} x \rightarrow x$. Clearly

$$
\overline{\operatorname{conv}(D(A))} \supseteq \overline{D(A)}
$$

Now if $x$ is in the set on the left, $J_{\lambda} x \rightarrow x$ and so in fact, since $J_{\lambda} x \in D(A)$, it must be the case that $x \in \overline{D(A)}$. Thus the two sets are the same and so in fact, $\overline{D(A)}$ is closed and convex.

Note that this implies that $\overline{A(D(A))}$ is also convex. This is because $A^{-1}$ described above, is maximal monotone with domain $A(D(A))$.

Next is a useful generalization of some of the earlier material used to establish the above results on approximation. It will include the general case of $F$ a duality map for $p>1$.

Proposition 25.7.38 Suppose $A: X \rightarrow \mathscr{P}\left(X^{\prime}\right)$ where $X$ is a reflexive Banach space with strictly convex norm. Suppose also that $A$ is maximal monotone. Then if $\lambda_{n} \rightarrow 0$ and if $x_{n} \rightarrow x$ weakly, $A_{\lambda_{n}} x_{n} \rightarrow x^{*}$ weakly, and

$$
\lim \sup _{n, m \rightarrow \infty}\left\langle A_{\lambda_{n}} x_{n}-A_{\lambda_{m}} x_{m}, x_{n}-x_{m}\right\rangle \leq 0
$$

Then

$$
\lim _{n, m \rightarrow \infty}\left\langle A_{\lambda_{n}} x_{n}-A_{\lambda_{m}} x_{m}, x_{n}-x_{m}\right\rangle=0
$$

$\left[x, x^{*}\right] \in \mathscr{G}(A)$, and $\left\langle A_{\lambda_{n}} x_{n}, x_{n}\right\rangle \rightarrow\left\langle x^{*}, x\right\rangle$.

Proof: Let $\alpha=\limsup \sin _{n \rightarrow \infty}\left\langle A_{\lambda_{n}} x_{n}, x_{n}\right\rangle$. It is finite because the expression is bounded independent of $n$. Then

$$
\lim \sup _{m \rightarrow \infty}\left({\lim \sup _{n \rightarrow \infty}}\binom{\left\langle A_{\lambda_{n}} x_{n}, x_{n}\right\rangle+\left\langle A_{\lambda_{m}} x_{m}, x_{m}\right\rangle}{-\left[\left\langle A_{\lambda_{n}} x_{n}, x_{m}\right\rangle+\left\langle A_{\lambda_{m}} x_{m}, x_{n}\right\rangle\right]}\right) \leq 0
$$

Thus

$$
\lim _{m \rightarrow \infty} \sup _{m}\left(\alpha+\left\langle A_{\lambda_{m}} x_{m}, x_{m}\right\rangle-\left[\left\langle x^{*}, x_{m}\right\rangle+\left\langle A_{\lambda_{m}} x_{m}, x\right\rangle\right]\right) \leq 0
$$

and so

$$
2 \alpha-2\left\langle x^{*}, x\right\rangle \leq 0
$$

The next simple observation is that

$$
\left\|A_{\lambda_{n}} x_{n}\right\|=\left\|\lambda_{n}^{-(p-1)} F\left(J_{\lambda_{n}} x_{n}-x_{n}\right)\right\| \leq C
$$

due to the weak convergence. Hence $\lambda_{n}^{-(p-1)}\left\|J_{\lambda_{n}} x_{n}-x_{n}\right\|^{p-1} \leq C$ and so

$$
\begin{equation*}
\left\|J_{\lambda_{n}} x_{n}-x_{n}\right\| \leq \lambda_{n} C^{1 /(p-1)} \tag{25.7.59}
\end{equation*}
$$

Thus if $\left[u, u^{*}\right] \in \mathscr{G}(A)$,

$$
\lim \inf _{n \rightarrow \infty}\left\langle A_{\lambda_{n}} x_{n}-u^{*}, x_{n}-u\right\rangle=\lim \inf _{n \rightarrow \infty}\left\langle A_{\lambda_{n}} x_{n}-u^{*}, J_{\lambda_{n}} x_{n}-u\right\rangle \geq 0
$$

because $A_{\lambda} x \in A J_{\lambda} x$. However, the left side satisfies

$$
\begin{aligned}
0 & \leq \lim \inf _{n \rightarrow \infty}\left\langle A_{\lambda_{n}} x_{n}-u^{*}, x_{n}-u\right\rangle \leq \lim \sup _{n \rightarrow \infty}\left\langle A_{\lambda_{n}} x_{n}-u^{*}, x_{n}-u\right\rangle \\
& =\lim \sup _{n \rightarrow \infty}\left[\left\langle A_{\lambda_{n}} x_{n}, x_{n}\right\rangle-\left\langle A_{\lambda_{n}} x_{n}, u\right\rangle-\left\langle u^{*}, x_{n}\right\rangle+\left\langle u^{*}, u\right\rangle\right] \\
& =\alpha-\left\langle x^{*}, u\right\rangle-\left\langle u^{*}, x\right\rangle+\left\langle u^{*}, u\right\rangle \leq\left\langle x^{*}, x\right\rangle-\left\langle x^{*}, u\right\rangle-\left\langle u^{*}, x\right\rangle+\left\langle u^{*}, u\right\rangle \\
& =\left\langle x^{*}-u^{*}, x-u\right\rangle
\end{aligned}
$$

and this shows that $\left[x, x^{*}\right] \in \mathscr{G}(A)$ since $\left[u, u^{*}\right]$ was arbitrary.
Next let $\left[u, u^{*}\right] \in \mathscr{G}(A)$. Then thanks to 25.7.59,

$$
\begin{aligned}
0 & \leq \lim \inf _{n \rightarrow \infty}\left\langle A_{\lambda_{n}} x_{n}-u^{*}, J_{\lambda_{n}} x_{n}-u\right\rangle=\lim \inf _{n \rightarrow \infty}\left\langle A_{\lambda_{n}} x_{n}-u^{*}, x_{n}-u\right\rangle \\
& \leq \lim \sup _{n \rightarrow \infty}\left\langle A_{\lambda_{n}} x_{n}-u^{*}, x_{n}-u\right\rangle \\
& =\lim \sup _{n \rightarrow \infty}\left(\left\langle A_{\lambda_{n}} x_{n}, x_{n}\right\rangle-\left\langle A_{\lambda_{n}} x_{n}, u\right\rangle-\left\langle u^{*}, x_{n}\right\rangle+\left\langle u^{*}, u\right\rangle\right) \\
& =\lim \sup _{n \rightarrow \infty}\left\langle A_{\lambda_{n}} x_{n}, x_{n}\right\rangle-\left\langle x^{*}, u\right\rangle-\left\langle u^{*}, x\right\rangle+\left\langle u^{*}, u\right\rangle \\
& \leq\left\langle x^{*}, x\right\rangle-\left\langle x^{*}, u\right\rangle-\left\langle u^{*}, x\right\rangle+\left\langle u^{*}, u\right\rangle=\left\langle x^{*}-u^{*}, x-u\right\rangle
\end{aligned}
$$

In particular, you could let $\left[u, u^{*}\right]=\left[x, x^{*}\right]$ and conclude that

$$
\begin{aligned}
\lim _{n \rightarrow \infty}\left\langle A_{\lambda_{n}} x_{n}-x^{*}, x_{n}-x\right\rangle & =\lim _{n \rightarrow \infty}\left(\left\langle A_{\lambda_{n}} x_{n}, x_{n}\right\rangle-\left\langle A_{\lambda_{n}} x_{n}, x\right\rangle+\left\langle x^{*}, x\right\rangle-\left\langle x^{*}, x_{n}\right\rangle\right) \\
& =\lim _{n \rightarrow \infty}\left\langle A_{\lambda_{n}} x_{n}, x_{n}\right\rangle-\left\langle x^{*}, x\right\rangle+\left\langle x^{*}, x\right\rangle-\left\langle x^{*}, x\right\rangle=0
\end{aligned}
$$

which shows that $\lim _{n \rightarrow \infty}\left\langle A_{\lambda_{n}} x_{n}, x_{n}\right\rangle=\left\langle x^{*}, x\right\rangle$. Then it follows from this that

$$
\lim _{n, m \rightarrow \infty}\left\langle A_{\lambda_{n}} x_{n}-A_{\lambda_{m}} x_{m}, x_{n}-x_{m}\right\rangle=0
$$

For the rest of this, the usual duality map for $p=2$ will be used. It may be that one could change this, but I don't have a need to do it right now so from now on, $F$ will be the usual thing.

### 25.7.5 Sum Of Maximal Monotone Operators

To begin with, here is a nice lemma.
Lemma 25.7.39 Let $0 \in D(A)$ and let $A$ be maximal monotone and let $B: X \rightarrow X^{\prime}$ be monotone hemicontinuous, bounded, and coercive. Then $B+A$ is also maximal monotone. Also $B+A$ is onto.

Proof: By Theorem 25.7.9, there exists $x \in \overline{D(A)}$ such that for all $\left[u, u^{*}\right] \in \mathscr{G}(A)$,

$$
\left\langle B x+F x-y^{*}+u^{*}, u-x\right\rangle \geq 0
$$

Hence for all $\left[u, u^{*}\right]$,

$$
\left\langle u^{*}-\left(y^{*}-(B x+F x)\right), u-x\right\rangle \geq 0
$$

It follows that

$$
y^{*}-(B x+F x) \in A x
$$

and so $y^{*} \in B x+A x+F x$ showing that $B+A$ is maximal monotone because it added to $F$ is onto. As to the last claim, just don't add in $F$ in the argument. Thus for all $\left[u, u^{*}\right]$,

$$
\left\langle B x-y^{*}+u^{*}, u-x\right\rangle \geq 0
$$

Then the rest is as before. You find that $y^{*}-B x \in A x$.
Corollary 25.7.40 Suppose instead of $0 \in D(A)$, it is known that $x_{0} \in D(A)$ and

$$
\lim _{\|x\| \rightarrow \infty} \frac{\left\langle B\left(x_{0}+x\right), x\right\rangle}{\|x\|}=\infty
$$

Then if $B$ is monotone and hemicontinuous and $A$ is maximal monotone, then $B+A$ is onto.
Proof: Let $\hat{A}(x) \equiv A\left(x_{0}+x\right)$ so in fact $0 \in D(\hat{A})$. Then letting $\hat{B}$ be defined similarly, it follows from the above lemma that if $y^{*} \in X^{\prime}$, there exists $x$ such that

$$
y^{*} \in \hat{A} x+\hat{B} x \equiv A\left(x_{0}+x\right)+B\left(x_{0}+x\right)
$$

Lemma 25.7.41 Let 0 be on the interior of $D(A)$ and also in $D(B)$. Also let $0 \in B(0)$ and $0 \in A(0)$. Then if $A, B$ are maximal monotone, so is $A+B$.

Proof: Note that, since $0 \in A(0)$, if $x^{*} \in A x$, then $\left\langle x^{*}, x\right\rangle \geq 0$. Also note that $\left\|B_{\lambda}(0)\right\| \leq$ $|B(0)|=0$ and so also $\left\langle B_{\lambda} x, x\right\rangle \geq 0$. It is necessary to show that $F+A+B$ is onto. However, $B_{\lambda}$ is monotone hemicontinuous, bounded and coercive. Hence, by Lemma 25.7.39, $B_{\lambda}+A$ is maximal monotone. If $x^{*} \in X^{\prime}$ is given, there exists a solution to

$$
x^{*} \in F x_{\lambda}+B_{\lambda} x_{\lambda}+A x_{\lambda}
$$

Do both sides to $x_{\lambda}$ and let $x_{\lambda}^{*} \in A x_{\lambda}$ be such that equality holds in the above.

$$
\begin{equation*}
x^{*}=F x_{\lambda}+B_{\lambda} x_{\lambda}+x_{\lambda}^{*} \tag{25.7.60}
\end{equation*}
$$

Then

$$
\left\langle x^{*}, x_{\lambda}\right\rangle=\left\|x_{\lambda}\right\|^{2}+\left\langle x_{\lambda}^{*}, x_{\lambda}\right\rangle
$$

It follows that

$$
\begin{equation*}
\left\|x_{\lambda}\right\| \leq\left\|x^{*}\right\|, \quad\left\langle x_{\lambda}^{*}, x_{\lambda}\right\rangle \leq\left\langle x^{*}, x_{\lambda}\right\rangle \leq\left\|x^{*}\right\|\left\|x_{\lambda}\right\| \leq\left\|x^{*}\right\|^{2} \tag{25.7.61}
\end{equation*}
$$

Next, 0 is on the interior of $D(A)$ and so from Theorem 25.7.20, there exists $\rho>0$ such that if $y^{*} \in A x$ for $\|x\| \leq \rho$, then $\left\|y^{*}\right\|<M$ and in fact, all such $x$ are in $D(A)$. Now let

$$
y_{\lambda}=\frac{1}{2\left\|x_{\lambda}^{*}\right\|} F^{-1}\left(x_{\lambda}^{*}\right) \text { so }\left\|y_{\lambda}\right\|<\rho
$$

Thus $y_{\lambda} \in D(A)$ and if $y_{\lambda}^{*} \in A y_{\lambda}$, then $\left\|y_{\lambda}^{*}\right\|<M$. Then for such bounded $y_{\lambda}^{*}$,

$$
0 \leq\left\langle y_{\lambda}^{*}-x_{\lambda}^{*}, y_{\lambda}-x_{\lambda}\right\rangle=\left\langle y_{\lambda}^{*}, y_{\lambda}\right\rangle-\left\langle x_{\lambda}^{*}, y_{\lambda}\right\rangle-\left\langle y_{\lambda}^{*}, x_{\lambda}\right\rangle+\left\langle x_{\lambda}^{*}, x_{\lambda}\right\rangle
$$

Then

$$
\begin{gathered}
\frac{1}{2}\left\|x_{\lambda}^{*}\right\|=\left\langle x_{\lambda}^{*}, \frac{1}{2\left\|x_{\lambda}^{*}\right\|} F^{-1}\left(x_{\lambda}^{*}\right)\right\rangle=\left\langle x_{\lambda}^{*}, y_{\lambda}\right\rangle \leq\left\langle y_{\lambda}^{*}, y_{\lambda}\right\rangle-\left\langle y_{\lambda}^{*}, x_{\lambda}\right\rangle+\left\langle x_{\lambda}^{*}, x_{\lambda}\right\rangle \\
\leq M \rho+M\left\|x_{\lambda}\right\|+\left\langle x_{\lambda}^{*}, x_{\lambda}\right\rangle
\end{gathered}
$$

From 25.7.61,

$$
\left\|x_{\lambda}^{*}\right\| \leq 2\left(M \rho+M\left\|x^{*}\right\|+\left\|x^{*}\right\|^{2}\right)
$$

Thus from 25.7.61, $x_{\lambda}, x_{\lambda}^{*}, F x_{\lambda}$ are all bounded. Hence it follows from 25.7.60 that $B_{\lambda} x_{\lambda}$ is also bounded. Therefore, there is a sequence, $\lambda_{n} \rightarrow 0$ such that

$$
\begin{gathered}
x_{\lambda_{n}} \rightarrow z \text { weakly } \\
x_{\lambda}^{*} \rightarrow w^{*} \text { weakly } \\
F x_{\lambda} \rightarrow u^{*} \text { weakly } \\
B_{\lambda_{n}} x_{\lambda_{n}} \rightarrow b^{*} \text { weakly }
\end{gathered}
$$

Using 25.7.60, it follows that

$$
\left\langle F x_{\lambda_{n}}+x_{\lambda_{n}}^{*}+B_{\lambda_{n}} x_{\lambda_{n}}-\left(F x_{\lambda_{m}}+x_{\lambda_{m}}^{*}+B_{\lambda_{m}} x_{\lambda_{m}}\right), x_{\lambda_{n}}-x_{\lambda_{m}}\right\rangle=0
$$

Thus

$$
\begin{equation*}
\left\langle F x_{\lambda_{n}}+x_{\lambda_{n}}^{*}-\left(F x_{\lambda_{m}}+x_{\lambda_{m}}^{*}\right), x_{\lambda_{n}}-x_{\lambda_{m}}\right\rangle+\left\langle B_{\lambda_{n}} x_{\lambda_{n}}-B_{\lambda_{m}} x_{\lambda_{m}}, x_{\lambda_{n}}-x_{\lambda_{m}}\right\rangle=0 \tag{25.7.62}
\end{equation*}
$$

Now $F+A$ is surely monotone and so

$$
\lim \sup _{m, n \rightarrow \infty}\left\langle B_{\lambda_{n}} x_{\lambda_{n}}-B_{\lambda_{m}} x_{\lambda_{m}}, x_{\lambda_{n}}-x_{\lambda_{m}}\right\rangle \leq 0
$$

By Proposition 25.7.38, $b^{*} \in B z$ and

$$
\lim _{m, n \rightarrow \infty}\left\langle B_{\lambda_{n}} x_{\lambda_{n}}-B_{\lambda_{m}} x_{\lambda_{m}}, x_{\lambda_{n}}-x_{\lambda_{m}}\right\rangle=0
$$

Then returning to 25.7.62,

$$
\lim \sup _{m, n \rightarrow \infty}\left\langle F x_{\lambda_{n}}+x_{\lambda_{n}}^{*}-\left(F x_{\lambda_{m}}+x_{\lambda_{m}}^{*}\right), x_{\lambda_{n}}-x_{\lambda_{m}}\right\rangle \leq 0
$$

Now from Lemma 25.7.39, $F+A$ is maximal monotone. Hence Proposition 25.7.38 applies again and it follows that $u^{*}+w^{*} \in F z+A z$. Then passing to the limit as $n \rightarrow \infty$ in

$$
x^{*}=F x_{\lambda_{n}}+B_{\lambda_{n}} x_{\lambda_{n}}+x_{\lambda_{n}}^{*}
$$

it follows that

$$
x^{*}=u^{*}+b^{*}+w^{*}=F z+A z+B z
$$

and this shows that $A+B$ is maximal monotone because $x^{*}$ was arbitrary.
You don't need to assume all that stuff about $0 \in A(0), 0 \in B(0), 0$ on interior of $D(A)$ and so forth.

Theorem 25.7.42 Suppose $A, B$ are maximal monotone and the interior of $D(A)$ has nonempty intersection with $D(B)$. Then $A+B$ is maximal monotone.

Proof: Let $x_{0}$ be on the interior of $D(A)$ and also in $D(B)$. Let $\hat{A}(x)=A\left(x_{0}+x\right)-x_{0}^{*}$ where $x_{0}^{*} \in A\left(x_{0}\right)$. Thus $0 \in D(\hat{A})$ and $0 \in \hat{A}(0)$. Do the same thing for $B$ to get $\hat{B}$ defined similarly. Are these still maximal monotone? Suppose for all $\left[u, u^{*}\right] \in \mathscr{G}(\hat{A})$

$$
\left\langle y^{*}-u^{*}, y-u\right\rangle \geq 0
$$

Does it follow that $y^{*} \in \hat{A} y$ ? It is given that $u^{*} \in A\left(x_{0}+u\right)$. The above implies for all $\left[u, u^{*}\right] \in \mathscr{G}(\hat{A})$

$$
\left\langle y^{*}+x_{0}^{*}-\left(u^{*}+x_{0}^{*}\right),\left(y+x_{0}\right)-\left(u+x_{0}\right)\right\rangle \geq 0
$$

and since $u+x_{0}$ is a generic element of $D(A)$ for $u \in D(\hat{A})$, the above implies $y^{*}+x_{0}^{*} \in$ $A\left(y+x_{0}\right)$ and so $y \in A\left(y+x_{0}\right)-x_{0}^{*} \equiv \hat{A}(y)$. Hence the graph is maximal. Similar for $\hat{B}$. Thus the lemma can be applied to $\hat{A}, \hat{B}$ to conclude that the sum of these is maximal
monotone. Now a repeat of the above reasoning which shows that $\hat{A}$ is maximal monotone shows that the fact that $\hat{A}+\hat{B}$ is maximal monotone implies that $A+B$ is also. You just shift with $-x_{0}$ instead of $x_{0}$. It amounts to nothing more than the observation that maximal graphs don't lose their maximality by shifting their ranges and domains.

Suppose $B, A$ are maximal monotone. Does there always exist a solution $x$ to

$$
\begin{equation*}
x^{*} \in F x+B_{\lambda} x+A x ? \tag{25.7.63}
\end{equation*}
$$

Consider the monotone hemicontinuous and bounded operator $F+B_{\lambda}$.Is $\hat{F}+\hat{B}_{\lambda}$ defined by

$$
\left(\hat{F}+\hat{B}_{\lambda}\right)(x) \equiv\left(\hat{F}+\hat{B}_{\lambda}\right)\left(x+x_{0}\right)
$$

also coercive for some $x_{0} \in D(A)$ ? If so, the existence of the desired solution to the above inclusion follows from Corollary 25.7.40. Then for all $\|x\|$ large enough that $\left\|x+x_{0}\right\|>$ $\left\|x_{0}\right\|$,

$$
\begin{gathered}
\frac{\left\langle F\left(x+x_{0}\right)+B_{\lambda}\left(x+x_{0}\right), x\right\rangle}{\|x\|} \\
=\frac{\left\langle F\left(x+x_{0}\right), x\right\rangle}{\|x\|}+\frac{\left\langle B_{\lambda}\left(x+x_{0}\right)-B_{\lambda}\left(x_{0}\right), x\right\rangle}{\|x\|}+\frac{\left\langle B_{\lambda}\left(x_{0}\right), x\right\rangle}{\|x\|} \\
\geq \frac{1}{2} \frac{\left\langle F\left(x+x_{0}\right), x\right\rangle}{\left\|x+x_{0}\right\|}-\left\|B_{\lambda}\left(x_{0}\right)\right\| \\
\geq \frac{1}{2} \frac{\left\langle F\left(x+x_{0}\right), x+x_{0}\right\rangle}{\left\|x+x_{0}\right\|}-\frac{1}{2} \frac{\left\langle F\left(x+x_{0}\right), x_{0}\right\rangle}{\left\|x+x_{0}\right\|}-\left\|B_{\lambda}\left(x_{0}\right)\right\| \\
\geq \frac{1}{2} \frac{\left\langle F\left(x+x_{0}\right), x+x_{0}\right\rangle}{\left\|x+x_{0}\right\|}-\frac{1}{2} \frac{\left\langle F\left(x+x_{0}\right), x_{0}\right\rangle}{\left\|x_{0}\right\|}-\left\|B_{\lambda}\left(x_{0}\right)\right\| \\
\geq \frac{1}{2} \frac{\left\langle F\left(x+x_{0}\right), x+x_{0}\right\rangle}{\left\|x+x_{0}\right\|}-\frac{1}{2}\left\|x+x_{0}\right\|-\left\|B_{\lambda}\left(x_{0}\right)\right\| \\
=\frac{1}{2}\left\|x+x_{0}\right\|^{2}-\frac{1}{2}\left\|x+x_{0}\right\|-\left\|B_{\lambda}\left(x_{0}\right)\right\|
\end{gathered}
$$

which shows that

$$
\lim _{\|x\| \rightarrow \infty} \frac{\left\langle F\left(x+x_{0}\right)+B_{\lambda}\left(x+x_{0}\right), x\right\rangle}{\|x\|}=\infty
$$

and so by Corollary 25.7.40, there exists a solution to 25.7.63. This shows half of the following interesting theorem which is another version of the above major result.

Theorem 25.7.43 Suppose $A, B$ are maximal monotone operators. Then for each $x^{*} \in X^{\prime}$, there exists a solution $x_{\lambda}$ to

$$
\begin{equation*}
x^{*} \in F x_{\lambda}+B x_{\lambda} x_{\lambda}+A x_{\lambda}, \lambda>0 \tag{25.7.64}
\end{equation*}
$$

If for $\lambda \in(0, \delta),\left\{B_{\lambda} x_{\lambda}\right\}$ is bounded, then there exists a solution $x$ to

$$
x^{*} \in F x+B x+A x
$$

Proof: The existence of a solution to the inclusion 25.7 .64 comes from the above discussion. The last claim follows from almost a repeat of the last part of the proof of the above theorem. Since $\left\{B_{\lambda} x_{\lambda}\right\}$ is given to be bounded for $\lambda \in(0, \delta)$, there is a sequence, $\lambda_{n} \rightarrow 0$ such that

$$
\begin{gathered}
x_{\lambda_{n}} \rightarrow z \text { weakly } \\
x_{\lambda}^{*} \rightarrow w^{*} \text { weakly } \\
F x_{\lambda} \rightarrow u^{*} \text { weakly } \\
B_{\lambda_{n}} x_{\lambda_{n}} \rightarrow b^{*} \text { weakly }
\end{gathered}
$$

Using 25.7.64, it follows that

$$
\left\langle F x_{\lambda_{n}}+x_{\lambda_{n}}^{*}+B_{\lambda_{n}} x_{\lambda_{n}}-\left(F x_{\lambda_{m}}+x_{\lambda_{m}}^{*}+B_{\lambda_{m}} x_{\lambda_{m}}\right), x_{\lambda_{n}}-x_{\lambda_{m}}\right\rangle=0
$$

Thus

$$
\begin{gather*}
\left\langle F x_{\lambda_{n}}+x_{\lambda_{n}}^{*}-\left(F x_{\lambda_{m}}+x_{\lambda_{m}}^{*}\right), x_{\lambda_{n}}-x_{\lambda_{m}}\right\rangle \\
\quad+\left\langle B_{\lambda_{n}} x_{\lambda_{n}}-B_{\lambda_{m}} x_{\lambda_{m}}, x_{\lambda_{n}}-x_{\lambda_{m}}\right\rangle=0 \tag{25.7.65}
\end{gather*}
$$

Now $F+A$ is surely monotone and so

$$
\lim \sup _{m, n \rightarrow \infty}\left\langle B_{\lambda_{n}} x_{\lambda_{n}}-B_{\lambda_{m}} x_{\lambda_{m}}, x_{\lambda_{n}}-x_{\lambda_{m}}\right\rangle \leq 0
$$

By Proposition 25.7.38, $b^{*} \in B z$ and

$$
\lim _{m, n \rightarrow \infty}\left\langle B_{\lambda_{n}} x_{\lambda_{n}}-B_{\lambda_{m}} x_{\lambda_{m}}, x_{\lambda_{n}}-x_{\lambda_{m}}\right\rangle=0
$$

Then returning to 25.7 .65 ,

$$
\lim \sup _{m, n \rightarrow \infty}\left\langle F x_{\lambda_{n}}+x_{\lambda_{n}}^{*}-\left(F x_{\lambda_{m}}+x_{\lambda_{m}}^{*}\right), x_{\lambda_{n}}-x_{\lambda_{m}}\right\rangle \leq 0
$$

Now from Corollary 25.7.40, $F+A$ is maximal monotone (In fact, $F+A$ is onto). Hence Proposition 25.7.38 applies again and it follows that $u^{*}+w^{*} \in F z+A z$. Then passing to the limit as $n \rightarrow \infty$ in

$$
x^{*}=F x_{\lambda_{n}}+B_{\lambda_{n}} x_{\lambda_{n}}+x_{\lambda_{n}}^{*}
$$

it follows that

$$
x^{*}=u^{*}+b^{*}+w^{*}=F z+A z+B z
$$

### 25.7.6 Convex Functions, An Example

As before, $X$ will be a Banach space in what follows. Sometimes it will be a reflexive Banach space and in this case, it will be assumed that the norm is strictly convex.

Definition 25.7.44 Let $\phi: X \rightarrow(-\infty, \infty]$. Then $\phi$ is convex if whenever $t \in[0,1], x, y \in X$,

$$
\phi(t x+(1-t) y) \leq t \phi(x)+(1-t) \phi(y)
$$

The epigraph of $\phi$ is defined by

$$
\operatorname{epi}(\phi) \equiv\{(x, y): y \geq \phi(x)\}
$$

When epi $(\phi)$ is closed in $X \times(-\infty, \infty]$, we say that $\phi$ is lower semicontinuous, l.s.c. The function is called proper if $\phi(x)<\infty$ for some $x$. The collection of all such $x$ is called $D(\phi)$, the domain of $\phi$.

This definition of lower semicontinuity is equivalent to the usual definition.
Lemma 25.7.45 The above definition of lower semicontinuity is equivalent to the assertion that whenever $x_{n} \rightarrow x$, it follows that $\phi(x) \leq \liminf _{n \rightarrow \infty} \phi\left(x_{n}\right)$. In case that $\phi$ is convex, lower semicontinuity is equivalent to weak lower semicontinuity. That is epi $(\phi)$ is closed if and only if epi $(\phi)$ is weakly closed. In this case, the limit condition: If $x_{x} \rightarrow x$ weakly, then $\phi(x) \leq \liminf _{n \rightarrow \infty} \phi\left(x_{n}\right)$ is valid.

Proof: Suppose the limit condition holds. Why is epi $(\phi)$ closed? Why is $X \times(-\infty, \infty] \backslash$ epi $(\phi) \equiv$ epi $(\phi)^{C}$ open? Let $(x, \alpha) \in \operatorname{epi}(\phi)^{C}$. Then $\alpha<\phi(x), \alpha+\delta<\phi(x)$. Consider $B(x, r) \times\left(\alpha-\frac{\delta}{2}, \alpha+\frac{\delta}{2}\right)$. If every such open set contains a point of epi $(\phi)$, then there exists $x_{n} \rightarrow x, y_{n}<\alpha+\frac{\delta}{2}, y_{n} \geq \phi\left(x_{n}\right)$. Hence, from the limit condition,

$$
\phi(x) \leq \lim \inf _{n \rightarrow \infty} \phi\left(x_{n}\right) \leq \lim \inf _{n \rightarrow \infty} y_{n} \leq \alpha+\frac{\delta}{2}<\alpha+\delta<\phi(x)
$$

a contradiction. It follows that there exists $r>0$ such that $B(x, r) \times\left(\alpha-\frac{\delta}{2}, \alpha+\frac{\delta}{2}\right) \cap$ epi $(\phi)=\emptyset$. Since epi $(\phi)^{C}$ is open, it follows that epi $(\phi)$ is closed.

Next suppose epi $(\phi)$ is closed. Why does the limit condition hold? Suppose $x_{n} \rightarrow x$. Then $\left(x_{n}, \phi\left(x_{n}\right)\right) \in \operatorname{epi}(\phi)$. There is a subsequence such that

$$
\alpha \equiv \lim \inf _{n \rightarrow \infty} \phi\left(x_{n}\right)=\lim _{k \rightarrow \infty} \phi\left(x_{n_{k}}\right)
$$

and so $\left(x_{n_{k}}, \phi\left(x_{n_{k}}\right)\right) \rightarrow(x, \alpha)$. Since epi $(\phi)$ is closed, this means $(x, \alpha) \in \operatorname{epi}(\phi)$. Hence

$$
\alpha \equiv \lim \inf _{n \rightarrow \infty} \phi\left(x_{n}\right) \geq \phi(x)
$$

Consider the last claim. In this case, epi $(\phi)$ is convex. If it is closed, then it is weakly closed thanks to separation theorems: If $(x, \alpha) \in \operatorname{epi}(\phi)^{C}$, then $\alpha<\infty$ and so there exists $\left(x^{*}, \beta\right) \in(X \times \mathbb{R})^{\prime}$ and $l$ such that for all $(t, \gamma) \in \operatorname{epi}(\phi)$,

$$
x^{*}(t)+\beta \gamma>l>x^{*}(x)+\alpha \beta
$$

Then $B_{\left(x^{*}, \beta\right)}((x, \alpha), \delta)$ is a weakly open set containing $(x, \alpha)$. For $\delta$ small enough, it does not intersect epi $(\phi)$ since if not so, there would exist $\left(t_{n}, \gamma_{n}\right) \in \operatorname{epi}(\phi) \cap B_{\left(x^{*}, \beta\right)}\left((x, \alpha), \frac{1}{n}\right)$ and so

$$
x^{*}\left(t_{n}\right)+\beta \gamma_{n} \rightarrow x^{*}(x)+\alpha \beta
$$

contrary to the above inequality. Thus epi $(\phi)$ is weakly closed. Also, if epi $(\phi)$ is weakly closed, then it is obviously strongly closed.

What of the limit condition using weak convergence instead of strong convergence? Say $x_{n} \rightarrow x$ weakly. Does it follow that if epi $(\phi)$ is weakly closed that $\phi(x) \leq \liminf _{n \rightarrow \infty} \phi\left(x_{n}\right)$ ? It is just as above. There is a subsequence such that

$$
\alpha \equiv \lim \inf _{n \rightarrow \infty} \phi\left(x_{n}\right)=\lim _{k \rightarrow \infty} \phi\left(x_{n_{k}}\right)
$$

and so $\left(x_{n_{k}}, \phi\left(x_{n_{k}}\right)\right) \rightarrow(x, \alpha)$ weakly. Since epi $(\phi)$ is weakly closed, this means $(x, \alpha) \in$ epi $(\phi)$. Hence

$$
\alpha \equiv \lim \inf _{n \rightarrow \infty} \phi\left(x_{n}\right) \geq \phi(x)
$$

There is also another convenient characterization of what it means for a function to be lower semicontinuous.

Lemma 25.7.46 Let $\phi: X \rightarrow(-\infty, \infty]$. Then $\phi$ is lower semicontinuous if and only if $\phi^{-1}((a, \infty])$ is open for any $a \in \mathbb{R}$.

Proof: Suppose first that epi $(\phi)$ is closed. Consider $x \in \phi^{-1}((a, \infty])$. Thus $\phi(x)>a$. Thus $(x, a) \in \operatorname{epi}(\phi)^{C}$ because $a<\phi(x)$. Since epi $(\phi)$ is closed, there exists $r, \varepsilon>0$ such that

$$
B(x, r) \times(a-\varepsilon, a+\varepsilon) \subseteq \operatorname{epi}(\phi)^{C}
$$

Hence if $y \in B(x, r)$, it follows that $\phi(y) \geq a+\varepsilon$ since otherwise there would be a point of epi $(\phi)^{C}$ in this open set $B(x, r) \times(a-\varepsilon, a+\varepsilon)$. Hence $B(x, r) \subseteq \phi^{-1}((a, \infty])$.

Conversely, suppose $\phi^{-1}((a, \infty])$ is open for any $a$ and let $(x, b) \in \operatorname{epi}(\phi)^{C}$. Then $\phi(x)>b$. Thus there exists $B(x, r)$ such that for $y \in B(x, r)$, it follows that $\phi(y)>b$. That is, $y \in \phi^{-1}((b, \infty])$. So consider $B(x, r) \times(-\infty, b)$. If $(y, \alpha) \in B(x, r) \times(-\infty, b)$, then since $\phi(y)>b, \alpha<\phi(y)$ and so there is no point of intersection between epi $(\phi)$ and this open set $B(x, r) \times(-\infty, b)$.

Of course one can define upper semicontinuous the same way that $\phi^{-1}(-\infty, a)$ is open. Thus a function is continuous if and only if it is both upper and lower semicontinuous.

In case $X$ is reflexive, the limit condition implies that epi $(\phi)$ is weakly closed. Suppose $(x, \alpha)$ is a weak limit point of epi $(\phi)$. Then by the Eberlein Smulian theorem, there is a subsequence of points of $X,\left(x_{n}, \alpha_{n}\right)$ which converges weakly to $(x, \alpha)$. Thus if the limit condition holds,

$$
\phi(x) \leq \lim \inf _{n \rightarrow \infty} \phi\left(x_{n}\right) \leq \lim \inf _{n \rightarrow \infty} \alpha_{n}=\alpha
$$

and so $(x, \alpha) \in \mathrm{epi}(\phi)$. If $X$ is not reflexive, this isn't all that clear because it is not clear that a limit point is the limit of a sequence. However, one could consider a limit condition involving nets and get a similar result.

Definition 25.7.47 Let $\phi: X \rightarrow(-\infty, \infty]$ be convex lower semicontinuous, and proper. Then

$$
\partial \phi(x) \equiv\left\{x^{*}: \phi(y)-\phi(x) \geq\left\langle x^{*}, y-x\right\rangle \text { for all } y\right\}
$$

The domain of $\partial \phi$, denoted as $D(\partial \phi)$ is just the set of all $x$ for which $\partial \phi(x) \neq \emptyset$. Note that $D(\partial \phi) \subseteq D(\phi)$ since if $x \notin D(\phi)$, the defining inequality could not hold for all y because the left side would be $-\infty$ for some $y$.

Theorem 25.7.48 For $X$ a real Banach space, let $\phi(x) \equiv \frac{1}{2}\|x\|^{2}$. Then $F(x)=\partial \phi(x)$. Here $F$ was the set valued map satisfying $x^{*} \in F x$ means

$$
\left\|x^{*}\right\|=\|F x\|,\langle F x, x\rangle=\|x\|^{2}
$$

Proof: Let $x^{*} \in F(x)$. Then

$$
\begin{aligned}
\left\langle x^{*}, y-x\right\rangle & =\left\langle x^{*}, y\right\rangle-\left\langle x^{*}, x\right\rangle \\
& \leq\|x\|\|y\|-\|x\|^{2} \leq \frac{1}{2}\|y\|^{2}-\frac{1}{2}\|x\|^{2} .
\end{aligned}
$$

This shows $F(x) \subseteq \partial \phi(x)$.
Now let $x^{*} \in \partial \phi(x)$. Then for all $t \in \mathbb{R}$,

$$
\begin{equation*}
\left\langle x^{*}, t y\right\rangle=\left\langle x^{*},(t y+x)-x\right\rangle \leq \frac{1}{2}\left(\|x+t y\|^{2}-\|x\|^{2}\right) \tag{25.7.66}
\end{equation*}
$$

Now if $t>0$, divide both sides by $t$. This yields

$$
\begin{aligned}
\left\langle x^{*}, y\right\rangle & \leq \frac{1}{2 t}\left((\|x\|+t\|y\|)^{2}-\|x\|^{2}\right) \\
& =\frac{1}{2 t}\left(2 t\|x\|\|y\|+t^{2}\|y\|^{2}\right)
\end{aligned}
$$

Letting $t \rightarrow 0$,

$$
\begin{equation*}
\left\langle x^{*}, y\right\rangle \leq\|x\|\|y\| . \tag{25.7.67}
\end{equation*}
$$

Next suppose $t=-s$, where $s>0$ in 25.7.66. Then, since when you divide by a negative, you reverse the inequality, for $s>0$

$$
\begin{gather*}
\left\langle x^{*}, y\right\rangle \geq \frac{1}{2 s}\left[\|x\|^{2}-\|x-s y\|^{2}\right] \geq \\
\frac{1}{2 s}\left[\|x-s y\|^{2}-2\|x-s y\|\|s y\|+\|s y\|^{2}-\|x-s y\|\right]^{2}  \tag{25.7.68}\\
=\frac{1}{2 s}\left[-2\|x-s y\|\|s y\|+\|s y\|^{2}\right] \tag{25.7.69}
\end{gather*}
$$

Taking a limit as $s \rightarrow 0$ yields

$$
\begin{equation*}
\left\langle x^{*}, y\right\rangle \geq-\|x\|\|y\| . \tag{25.7.70}
\end{equation*}
$$

It follows from 25.7.70 and 25.7.67 that

$$
\left|\left\langle x^{*}, y\right\rangle\right| \leq\|x\|\|y\|
$$

and that, therefore, $\left\|x^{*}\right\| \leq\|x\|$ and $\left|\left\langle x^{*}, x\right\rangle\right| \leq\|x\|^{2}$. Now return to 25.7.69 and let $y=x$. Then

$$
\begin{aligned}
\left\langle x^{*}, x\right\rangle & \geq \frac{1}{2 s}\left[-2\|x-s x\|\|s x\|+\|s x\|^{2}\right] \\
& =-\|x\|^{2}(1-s)+s\|x\|^{2}
\end{aligned}
$$

Letting $s \rightarrow 1$,

$$
\left\langle x^{*}, x\right\rangle \geq\|x\|^{2}
$$

Since it was already shown that $\left|\left\langle x^{*}, x\right\rangle\right| \leq\|x\|^{2}$, this shows $\left\langle x^{*}, x\right\rangle=\|x\|^{2}$ and also $\left\|x^{*}\right\| \leq$ $\|x\|$. Thus

$$
\left\|x^{*}\right\| \geq\left\langle x^{*} \frac{x}{\|x\|}\right\rangle=\|x\|
$$

so in fact $x^{*} \in F(x)$.
The next result gives conditions under which the subgradient is onto. This means that if $y^{*} \in X^{\prime}$, then there exists $x \in X$ such that $y^{*} \in \partial \phi(x)$.

Theorem 25.7.49 Suppose $X$ is a reflexive Banach space and suppose $\phi: X \rightarrow(-\infty, \infty]$ is convex, proper, l.s.c., and for all $y^{*} \in X^{\prime}, x \rightarrow \phi(x)-\left\langle y^{*}, x\right\rangle$ is coercive,

$$
\lim _{\|x\| \rightarrow \infty} \phi(x)-\left\langle y^{*}, x\right\rangle=\infty
$$

Then $\partial \phi$ is onto.
Proof: The function $x \rightarrow \phi(x)-y^{*}(x) \equiv \psi(x)$ is convex, proper, l.s.c., and coercive. Let

$$
\lambda \equiv \inf \left\{\phi(x)-\left\langle y^{*}, x\right\rangle: x \in X\right\}
$$

and let $\left\{x_{n}\right\}$ be a minimizing sequence satisfying

$$
\lambda=\lim _{n \rightarrow \infty} \phi\left(x_{n}\right)-\left\langle y^{*}, x_{n}\right\rangle
$$

By coercivity,

$$
\lim _{\|x\| \rightarrow \infty} \phi(x)-\left\langle y^{*}, x\right\rangle=\infty
$$

and so this minimizing sequence is bounded. By the Eberlein Smulian theorem, Theorem 17.5.12, there is a weakly convergent subsequence $x_{n_{k}} \rightarrow x$. By Lemma 25.7.45,

$$
\lambda=\phi(x)-\left\langle y^{*}, x\right\rangle \leq \lim \inf _{k \rightarrow \infty} \phi\left(x_{n_{k}}\right)-\left\langle y^{*}, x_{n_{k}}\right\rangle=\lambda
$$

so there exists $x$ which minimizes $x \rightarrow \phi(x)-\left\langle y^{*}, x\right\rangle \equiv \psi(x)$. Therefore, $0 \in \partial \psi(x)$ because

$$
\psi(y)-\psi(x) \geq 0=\langle 0, y-x\rangle
$$

Thus, $0 \in \partial \psi(x)=\partial \phi(x)-y^{*}$.
Now let $\phi$ be a convex proper lower semicontinuous function defined on $X$ where $X$ is a reflexive Banach space with strictly convex norm. Consider $\partial \phi$. Is it maximal monotone?

Is it the case that $F+\partial \phi$ is onto? First of all, is $\partial \phi$ monotone? Let $x^{*} \in \partial \phi(x), y^{*} \in \partial \phi(y)$. Then

$$
\begin{aligned}
\phi(y)-\phi(x) & \geq\left\langle x^{*}, y-x\right\rangle \\
\phi(x)-\phi(y) & \geq\left\langle y^{*}, x-y\right\rangle
\end{aligned}
$$

Hence adding these yields

$$
\left\langle y^{*}-x^{*}, x-y\right\rangle \leq 0,\left\langle y^{*}-x^{*}, y-x\right\rangle \geq 0 .
$$

Yes, $\partial \phi$ is certainly monotone. Is it maximal monotone?
Theorem 25.7.50 Let $\phi$ be convex, proper, and lower semicontinuous on $X$ where $X$ is a reflexive Banach space having strictly convex norm. Then $\partial \phi$ is maximal monotone.

Proof: It is necessary to show that $F+\partial \phi$ is onto. To do this, let

$$
\psi(x) \equiv \frac{1}{2}\|x\|^{2}+\phi(x)-\left\langle y^{*}, x\right\rangle
$$

where $y^{*}$ is a given element of $X^{\prime}$ and the idea is to show that $y^{*} \in F(x)+\partial \phi(x)$ for some $x$. Then by separation theorems, $\phi(x) \geq b+\left\langle z^{*}, x\right\rangle$ for some $b, z^{*}$. Hence it is clear that $\psi$ is convex, lower semicontinuous and coercive in the sense that

$$
\lim _{\|x\| \rightarrow \infty} \psi(x)=\infty
$$

It follows that any minimizing sequence for $\psi$ is bounded. Hence by the weak lower semicontinuity, this function has a minimum at $x_{0}$ say. Thus

$$
\frac{1}{2}\left\|x_{0}\right\|^{2}+\phi\left(x_{0}\right)-\left\langle y^{*}, x_{0}\right\rangle \leq \frac{1}{2}\|x\|^{2}+\phi(x)-\left\langle y^{*}, x\right\rangle
$$

for all $x$. Then

$$
\frac{1}{2}\left\|x_{0}\right\|^{2}-\frac{1}{2}\|x\|^{2}+\left\langle y^{*}, x-x_{0}\right\rangle \leq \phi(x)-\phi\left(x_{0}\right)
$$

Now from Theorem 25.7.48,

$$
\left\langle F(x), x_{0}-x\right\rangle \leq \frac{1}{2}\left\|x_{0}\right\|^{2}-\frac{1}{2}\|x\|^{2}
$$

and so, the above reduces to

$$
\left\langle F(x), x_{0}-x\right\rangle+\left\langle y^{*}, x-x_{0}\right\rangle \leq \phi(x)-\phi\left(x_{0}\right)
$$

Next let $x=x_{0}+t\left(z-x_{0}\right), t \in(0,1)$, where $z$ is arbitary. Then

$$
-t\left\langle F\left(x_{0}+t\left(z-x_{0}\right)\right), z-x_{0}\right\rangle+t\left\langle y^{*}, z-x_{0}\right\rangle \leq \phi\left(x_{0}+t\left(z-x_{0}\right)\right)-\phi\left(x_{0}\right)
$$

and so, by convexity,

$$
-t\left\langle F\left(x_{0}+t\left(z-x_{0}\right)\right), z-x_{0}\right\rangle+t\left\langle y^{*}, z-x_{0}\right\rangle \leq(1-t) \phi\left(x_{0}\right)+t \phi(z)-\phi\left(x_{0}\right)
$$

$$
t\left\langle y^{*}, z-x_{0}\right\rangle \leq t\left(\phi(z)-\phi\left(x_{0}\right)\right)+t\left\langle F\left(x_{0}+t\left(z-x_{0}\right)\right), z-x_{0}\right\rangle
$$

Now cancel the $t$ on both sides to obtain

$$
\left\langle y^{*}, z-x_{0}\right\rangle \leq\left(\phi(z)-\phi\left(x_{0}\right)\right)+\left\langle F\left(x_{0}+t\left(z-x_{0}\right)\right), z-x_{0}\right\rangle
$$

By the fact that $F$ is hemicontinuous, actually demicontinuous, one can let $t \downarrow 0$ and obtain

$$
\left\langle y^{*}, z-x_{0}\right\rangle \leq\left(\phi(z)-\phi\left(x_{0}\right)\right)+\left\langle F\left(x_{0}\right), z-x_{0}\right\rangle
$$

This says that $y^{*}-F\left(x_{0}\right) \in \partial \phi\left(x_{0}\right)$ from the definition of what $\partial \phi\left(x_{0}\right)$ means.
There is a much harder approach to this theorem which is based on a theorem about when the subgradient of a sum equals the sum of the subgradients. This major theorem is given next. Much of the above is in [13] but I don't remember where I found the following proof.

Theorem 25.7.51 Let $\phi_{1}$ and $\phi_{2}$ be convex, l.s.c. and proper having values in $(-\infty, \infty]$. Then

$$
\begin{equation*}
\partial\left(\lambda \phi_{i}\right)(x)=\lambda \partial \phi_{i}(x), \partial\left(\phi_{1}+\phi_{2}\right)(x) \supseteq \partial \phi_{1}(x)+\partial \phi_{2}(x) \tag{25.7.71}
\end{equation*}
$$

if $\lambda>0$. If there exists $\bar{x} \in \operatorname{dom}\left(\phi_{1}\right) \cap \operatorname{dom}\left(\phi_{2}\right)$ and $\phi_{1}$ is continuous at $\bar{x}$ then for all $x \in X$,

$$
\begin{equation*}
\partial\left(\phi_{1}+\phi_{2}\right)(x)=\partial \phi_{1}(x)+\partial \phi_{2}(x) . \tag{25.7.72}
\end{equation*}
$$

Proof: 25.7 .71 is obvious so we only need to show 25.7.72. Suppose $\bar{x}$ is as described. It is clear 25.7.72 holds whenever $x \notin \operatorname{dom}\left(\phi_{1}\right) \cap \operatorname{dom}\left(\phi_{2}\right)$ since then $\partial\left(\phi_{1}+\phi_{2}\right)=\emptyset$. Therefore, assume

$$
x \in \operatorname{dom}\left(\phi_{1}\right) \cap \operatorname{dom}\left(\phi_{2}\right)
$$

in what follows. Let $x^{*} \in \partial\left(\phi_{1}+\phi_{2}\right)(x)$. Is $x^{*}$ is the sum of an element of $\partial \phi_{1}(x)$ and $\partial \phi_{2}(x)$ ? Does there exist $x_{1}^{*}$ and $x_{2}^{*}$ such that for every $y$,

$$
\begin{aligned}
x^{*}(y-x) & =x_{1}^{*}(y-x)+x_{2}^{*}(y-x) \\
& \leq \phi_{1}(y)-\phi_{1}(x)+\phi_{2}(y)-\phi_{2}(x) ?
\end{aligned}
$$

If so, then

$$
\phi_{1}(y)-\phi_{1}(x)-x^{*}(y-x) \geq \phi_{2}(x)-\phi_{2}(y)
$$

Define

$$
\begin{aligned}
C_{1} \equiv & \left\{(y, a) \in X \times \mathbb{R}: \phi_{1}(y)-\phi_{1}(x)-x^{*}(y-x) \leq a\right\}, \\
& C_{2} \equiv\left\{(y, a) \in X \times \mathbb{R}: a \leq \phi_{2}(x)-\phi_{2}(y)\right\} .
\end{aligned}
$$

I will show $\operatorname{int}\left(C_{1}\right) \cap C_{2}=\emptyset$ and then by Theorem 18.2.14 there exists an element of $X^{\prime}$ which does something interesting.

Both $C_{1}$ and $C_{2}$ are convex and nonempty. Say $y_{1}, y_{2} \in C_{1}$ and $t \in[0,1]$. Then

$$
\phi_{1}\left(\left(t y_{1}\right)+(1-t) y_{2}\right)-\phi_{1}(x)-x^{*}\left(\left(\left(t y_{1}\right)+(1-t) y_{2}\right)-x\right)
$$

$$
\begin{gathered}
\leq t \phi\left(y_{1}\right)+(1-t) \phi\left(y_{2}\right)-\left(t \phi_{1}(x)+(1-t) \phi(x)\right) \\
-\left(t x^{*}\left(y_{1}-x\right)+(1-t) x^{*}\left(y_{2}-x\right)\right) \\
\leq t a+(1-t) a=a
\end{gathered}
$$

so $C_{1}$ is indeed convex. The case of $C_{2}$ is similar.
$C_{1}$ is nonempty because it contains $\left(\bar{x}, \phi_{1}(\bar{x})-\phi_{1}(x)-x^{*}(\bar{x}-x)\right)$ since

$$
\phi_{1}(\bar{x})-\phi_{1}(x)-x^{*}(\bar{x}-x) \leq \phi_{1}(\bar{x})-\phi_{1}(x)-x^{*}(\bar{x}-x)
$$

$C_{2}$ is also nonempty because it contains $\left(\bar{x}, \phi_{2}(x)-\phi_{2}(\bar{x})\right)$ since

$$
\phi_{2}(x)-\phi_{2}(\bar{x}) \leq \phi_{2}(x)-\phi_{2}(\bar{x})
$$

In addition to this,

$$
\left(\bar{x}, \phi_{1}(\bar{x})-x^{*}(\bar{x}-x)-\phi_{1}(x)+1\right) \in \operatorname{int}\left(C_{1}\right)
$$

due to the assumed continuity of $\phi_{1}$ at $\bar{x}$ and so $\operatorname{int}\left(C_{1}\right) \neq \emptyset$. If $(y, a) \in \operatorname{int}\left(C_{1}\right)$ then

$$
\phi_{1}(y)-x^{*}(y-x)-\phi_{1}(x) \leq a-\varepsilon
$$

whenever $\varepsilon$ is small enough. Therefore, if $(y, a)$ is also in $C_{2}$, the assumption that $x^{*} \in$ $\partial\left(\phi_{1}+\phi_{2}\right)(x)$ implies

$$
a-\varepsilon \geq \phi_{1}(y)-x^{*}(y-x)-\phi_{1}(x) \geq \phi_{2}(x)-\phi_{2}(y) \geq a
$$

a contradiction. Therefore $\operatorname{int}\left(C_{1}\right) \cap C_{2}=\emptyset$ and so by Theorem 18.2.14, there exists $\left(w^{*}, \boldsymbol{\beta}\right) \in X^{\prime} \times \mathbb{R}$ with

$$
\begin{equation*}
\left(w^{*}, \beta\right) \neq(0,0) \tag{25.7.73}
\end{equation*}
$$

and

$$
\begin{equation*}
w^{*}(y)+\beta a \geq w^{*}\left(y_{1}\right)+\beta a_{1}, \tag{25.7.74}
\end{equation*}
$$

whenever $(y, a) \in C_{1}$ and $\left(y_{1}, a_{1}\right) \in C_{2}$.
Claim: $\beta>0$.
Proof of claim: If $\beta<0$ let

$$
\begin{gathered}
a=\phi_{1}(\bar{x})-x^{*}(\bar{x}-x)-\phi_{1}(x)+1, \\
a_{1}=\phi_{2}(x)-\phi_{2}(\bar{x}), \text { and } y=y_{1}=\bar{x} .
\end{gathered}
$$

Then from 25.7.74

$$
\beta\left(\phi_{1}(\bar{x})-x^{*}(\bar{x}-x)-\phi_{1}(x)+1\right) \geq \beta\left(\phi_{2}(x)-\phi_{2}(\bar{x})\right) .
$$

Dividing by $\beta$ yields

$$
\phi_{1}(\bar{x})-x^{*}(\bar{x}-x)-\phi_{1}(x)+1 \leq \phi_{2}(x)-\phi_{2}(\bar{x})
$$

and so

$$
\begin{aligned}
\phi_{1}(\bar{x}) & +\phi_{2}(\bar{x})-\left(\phi_{1}(x)+\phi_{2}(x)\right)+1 \leq x^{*}(\bar{x}-x) \\
& \leq \phi_{1}(\bar{x})+\phi_{2}(\bar{x})-\left(\phi_{1}(x)+\phi_{2}(x)\right),
\end{aligned}
$$

a contradiction. Therefore, $\beta \geq 0$.
Now suppose $\beta=0$. Letting

$$
\begin{gathered}
a=\phi_{1}(\bar{x})-x^{*}(\bar{x}-x)-\phi_{1}(x)+1, \\
(\bar{x}, a) \in \operatorname{int}\left(C_{1}\right),
\end{gathered}
$$

and so there exists an open set $U$ containing 0 and $\eta>0$ such that

$$
\bar{x}+U \times(a-\eta, a+\eta) \subseteq C_{1} .
$$

Therefore, 25.7.74 applied to $(\bar{x}+z, a) \in C_{1}$ and $\left(\bar{x}, \phi_{2}(x)-\phi_{2}(\bar{x})\right) \in C_{2}$ for $z \in U$ yields

$$
w^{*}(\bar{x}+z) \geq w^{*}(\bar{x})
$$

for all $z \in U$. Hence $w^{*}(z)=0$ on $U$ which implies $w^{*}=0$, contradicting 25.7.73. This proves the claim.

Now with the claim, it follows $\beta>0$ and so, letting $z^{*}=w^{*} / \beta, 25.7 .74$ and Lemma 18.2.15 implies

$$
\begin{equation*}
z^{*}(y)+a \geq z^{*}\left(y_{1}\right)+a_{1} \tag{25.7.75}
\end{equation*}
$$

whenever $(y, a) \in C_{1}$ and $\left(y_{1}, a_{1}\right) \in C_{2}$. In particular,

$$
\begin{equation*}
\left(y, \phi_{1}(y)-\phi_{1}(x)-x^{*}(y-x)\right) \in C_{1} \tag{25.7.76}
\end{equation*}
$$

because

$$
\phi_{1}(y)-\phi_{1}(x)-x^{*}(y-x) \leq \phi_{1}(y)-x^{*}(y-x)-\phi_{1}(x)
$$

and

$$
\begin{equation*}
\left(y_{1}, \phi_{2}(x)-\phi_{2}\left(y_{1}\right)\right) \in C_{2} . \tag{25.7.77}
\end{equation*}
$$

by similar reasoning so letting $y=x$,

$$
z^{*}(x)+(\overbrace{\phi_{1}(x)-x^{*}(x-x)-\phi_{1}(x)}^{=0}) \geq z^{*}\left(y_{1}\right)+\phi_{2}(x)-\phi_{2}\left(y_{1}\right) .
$$

Therefore,

$$
z^{*}\left(y_{1}-x\right) \leq \phi_{2}\left(y_{1}\right)-\phi_{2}(x)
$$

for all $y_{1}$ and so $z^{*} \in \partial \phi_{2}(x)$. Now let $y_{1}=x$ in 25.7.77 and using 25.7.75 and 25.7.76, it follows

$$
\begin{gathered}
z^{*}(y)+\phi_{1}(y)-x^{*}(y-x)-\phi_{1}(x) \geq z^{*}(x) \\
\phi_{1}(y)-\phi_{1}(x) \geq x^{*}(y-x)-z^{*}(y-x)
\end{gathered}
$$

and so $x^{*}-z^{*} \in \partial \phi_{1}(x)$ so $x^{*}=z^{*}+\left(x^{*}-z^{*}\right) \in \partial \phi_{2}(x)+\partial \phi_{1}(x)$.

Corollary 25.7.52 Let $\phi: X \rightarrow(-\infty, \infty]$ be convex, proper, and lower semicontinuous. Here $X$ is a Banach space. Then $\partial \phi$ is maximal monotone.

Proof: Let $\psi(x)=\frac{1}{2}\|x\|^{2}$. There exists $x^{*}$ and some number $b$ such that $\phi(x) \geq$ $b+\left\langle x^{*}, x\right\rangle$. Therefore, $\psi+\phi$ is convex, lower semicontinuous, and bounded. It follows $\partial(\psi+\phi)$ is onto by Theorem 25.7.49. However, $\psi$ is continuous everywhere, in particular at every point of the domain of $\phi$. Therefore, $\partial \psi+\partial \phi=\partial(\phi+\psi)$ and by Theorem 25.7.48, this shows that $F+\partial \phi$ is onto.

It seems to me that the above are the most important results about convex proper lower semicontinuous functions. However, there are many other very interesting properties known.

Proposition 25.7.53 Let $\phi: X \rightarrow(-\infty, \infty]$ be convex proper and lower semicontinuous. Then $D(\partial \phi)$ is dense in $D(\phi)$ and so $\overline{D(\partial \phi)}=\overline{D(\phi)}$.

Proof: Let $x_{\lambda}$ be the solution to $0 \in F\left(x_{\lambda}-x\right)+\lambda \partial \phi\left(x_{\lambda}\right)$. Here $x \in D(\phi)$. Say $u_{\lambda}^{*} \in$ $\partial \phi\left(x_{\lambda}\right)$ such that the inclusion becomes an equality. Then

$$
\begin{aligned}
0 & =\left\langle F\left(x_{\lambda}-x\right)+\lambda u_{\lambda}^{*}, x_{\lambda}-x\right\rangle=\left\|x_{\lambda}-x\right\|^{2}-\lambda\left\langle u_{\lambda}^{*}, x-x_{\lambda}\right\rangle \\
& \geq\left\|x_{\lambda}-x\right\|^{2}-\lambda\left(\phi(x)-\phi\left(x_{\lambda}\right)\right)
\end{aligned}
$$

Hence, letting $z^{*}, b$ be such that $\phi(y) \geq b+\left\langle z^{*}, y-x\right\rangle$,

$$
\begin{gathered}
\lambda\left(\phi(x)-\left[b+\left\langle z^{*}, x_{\lambda}-x\right\rangle\right]\right) \geq \lambda\left(\phi(x)-\phi\left(x_{\lambda}\right)\right) \geq\left\|x_{\lambda}-x\right\|^{2} \\
\lambda \phi(x)-\lambda b \geq\left\|x_{\lambda}-x\right\|^{2}-\lambda\left\|z^{*}\right\|\left\|x_{\lambda}-x\right\| \\
\geq\left\|x_{\lambda}-x\right\|^{2}-\lambda\left(\frac{\left\|z^{*}\right\|^{2}}{2}+\frac{\left\|x_{\lambda}-x\right\|^{2}}{2}\right)
\end{gathered}
$$

Thus

$$
\lambda \phi(x)-\lambda b+\lambda \frac{\left\|z^{*}\right\|^{2}}{2} \geq\left(1-\frac{\lambda}{2}\right)\left\|x_{\lambda}-x\right\|^{2}
$$

It follows that $x_{\lambda} \rightarrow x$. This shows that $D(\phi) \subseteq \overline{D(\partial \phi)}$ and so $\overline{D(\phi)} \subseteq \overline{D(\partial \phi)} \subseteq \overline{D(\phi)}$.
There is a really amazing theorem, Moreau's theorem. It is in [24], [13] and [116]. It involves approximating a convex function with one which is differentiable, at least in the case where you have a Hilbert space. In the general case considered in this chapter, the function is continuous.

Theorem 25.7.54 Let $\phi$ be a convex lower semicontinuous proper function defined on $X$. Define $A \equiv \partial \phi, A_{\lambda}=(\partial \phi)_{\lambda}$

$$
\phi_{\lambda}(x) \equiv \min _{y \in X}\left(\frac{1}{2 \lambda}\|x-y\|^{2}+\phi(y)\right)
$$

Then the function is well defined, convex, Gateaux differentiable,

$$
D_{z} \phi_{\lambda}(x) \equiv \lim _{t \downarrow 0} \frac{\phi_{\lambda}(x+t z)-\phi_{\lambda}(x)}{t}=\left\langle A_{\lambda} x, z\right\rangle
$$

so the Gateaux derivative is just $A_{\lambda} x$ and for all $x \in X$,

$$
\lim _{\lambda \rightarrow 0} \phi_{\lambda}(x)=\phi(x)
$$

In addition,

$$
\begin{equation*}
\phi_{\lambda}(x)=\frac{1}{2 \lambda}\left\|x-J_{\lambda} x\right\|^{2}+\phi\left(J_{\lambda}(x)\right) \tag{25.7.78}
\end{equation*}
$$

where $J_{\lambda} x$ is as before, the solution to

$$
0 \in F\left(J_{\lambda} x-x\right)+\lambda \partial \phi\left(J_{\lambda} x\right)
$$

Proof: First of all, why does the minimum take place? By the convexity, closed epigraph, and assumption that $\phi$ is proper, separation theorems apply and one can say that there exists $z^{*}$ such that for all $y \in H$,

$$
\begin{equation*}
\frac{1}{2 \lambda}\|x-y\|^{2}+\phi(y) \geq \frac{1}{2 \lambda}\|x-y\|^{2}+\left(z^{*}, y\right)+c \tag{25.7.79}
\end{equation*}
$$

It follows easily that a minimizing sequence is bounded and so from lower semicontinuity which implies weak lower semicontinuity due to convexity, there exists $y_{x}$ such that

$$
\min _{y \in H}\left(\frac{1}{2 \lambda}\|x-y\|^{2}+\phi(y)\right)=\left(\frac{1}{2 \lambda}\left\|x-y_{x}\right\|^{2}+\phi\left(y_{x}\right)\right)
$$

Why is $\phi_{\lambda}$ convex? For $\theta \in[0,1]$,

$$
\begin{gathered}
\phi_{\lambda}(\theta x+(1-\theta) z) \equiv \frac{1}{2 \lambda}\left\|\theta x+(1-\theta) z-y_{(\theta x+(1-\theta) z)}\right\|^{2}+\phi\left(y_{\theta x+(1-\theta) z}\right) \\
\leq \frac{1}{2 \lambda}\left|\theta x+(1-\theta) z-\left(\theta y_{x}+(1-\theta) y_{z}\right)\right|^{2}+\phi\left(\theta y_{x}+(1-\theta) y_{z}\right) \\
\leq \frac{\theta}{2 \lambda}\left|x-y_{x}\right|^{2}+\frac{1-\theta}{2 \lambda}\left|z-y_{z}\right|^{2}+\theta \phi\left(y_{x}\right)+(1-\theta) \phi\left(y_{z}\right) \\
=\theta \phi_{\lambda}(x)+(1-\theta) \phi_{\lambda}(z)
\end{gathered}
$$

So is there a formula for $y_{x}$ ? Since it involves minimization of the functional, it follows that

$$
0 \in-\frac{1}{\lambda} F\left(x-y_{x}\right)+\partial \phi\left(y_{x}\right)=\frac{1}{\lambda} F\left(y_{x}-x\right)+\partial \phi\left(y_{x}\right)
$$

Recall that if $\psi(x)=\frac{1}{2}\|x\|^{2}$, then $\partial \psi(x)=F(x)$. Thus

$$
y_{x}=J_{\lambda} x
$$

because this was how $J_{\lambda} x$ was defined. Therefore,

$$
\phi_{\lambda}(x)=\frac{1}{2 \lambda}\left\|x-J_{\lambda} x\right\|^{2}+\phi\left(J_{\lambda}(x)\right)=\frac{\lambda}{2}\left\|A_{\lambda} x\right\|^{2}+\phi\left(J_{\lambda} x\right), A=\partial \phi
$$

It follows from this equation that

$$
\begin{equation*}
\phi\left(J_{\lambda} x\right) \leq \phi_{\lambda}(x) \leq \phi(x) \tag{25.7.80}
\end{equation*}
$$

the second inequality following from taking $y=x$ in the definition of $\phi_{\lambda}$.
Next consider the claim about $\phi_{\lambda}(x) \uparrow \phi(x)$. First suppose that $x \in D(\phi)$. Then from Proposition 25.7.53, $x \in \overline{D(\partial \phi)}$ and so from the material on approximations, Theorem 25.7.36, it follows that $J_{\lambda} x \rightarrow x$. Hence from 25.7.80 and lower semicontinuity of $\phi$,

$$
\phi(x) \leq \lim \inf _{\lambda \rightarrow 0} \phi\left(J_{\lambda} x\right) \leq \lim \inf _{\lambda \rightarrow 0} \phi_{\lambda}(x) \leq \lim \sup _{\lambda \rightarrow 0} \phi_{\lambda}(x) \leq \phi(x)
$$

showing that in this case, $\lim _{\lambda \rightarrow 0} \phi_{\lambda}(x)=\phi(x)$. Next suppose $x \notin D(\phi)$ so that $\phi(x)=\infty$. Why does $\phi_{\lambda}(x) \rightarrow \infty$ ? Suppose not. Then from the description of $\phi_{\lambda}$ given above and using the fact that the epigraph is closed and convex, there would exist a subsequence, still denoted as $\lambda$ such that

$$
C \geq \phi_{\lambda}(x)=\frac{1}{2 \lambda}\left\|x-J_{\lambda} x\right\|^{2}+\phi\left(J_{\lambda}(x)\right) \geq \frac{1}{2 \lambda}\left\|x-J_{\lambda} x\right\|^{2}+\left\langle z^{*}, x-J_{\lambda} x\right\rangle+b
$$

Then multiplying by $\lambda$, it follows that for a suitable constant $M$,

$$
\left\|x-J_{\lambda} x\right\|^{2} \leq M \lambda+\lambda M\left\|x-J_{\lambda} x\right\|
$$

and so a use of the quadratic formula implies

$$
\left\|x-J_{\lambda} x\right\| \leq \frac{M}{2}(1+\sqrt{5}) \lambda
$$

Hence $J_{\lambda} x \rightarrow x$ and so in 25.7.80 it follows from lower semicontinuity again that

$$
\infty=\phi(x) \leq \lim \inf _{\lambda \rightarrow 0} \phi\left(J_{\lambda} x\right) \leq \lim \inf _{\lambda \rightarrow 0} \phi_{\lambda}(x) \leq \lim \sup _{\lambda \rightarrow 0} \phi_{\lambda}(x) \leq \phi(x)
$$

and so again, $\lim _{\lambda \rightarrow 0} \phi_{\lambda}(x)=\infty$. Also note that if $\lambda>\mu$, then

$$
\min _{y \in X}\left(\frac{1}{2 \lambda}\|x-y\|^{2}+\phi(y)\right) \leq \min _{y \in X}\left(\frac{1}{2 \mu}\|x-y\|^{2}+\phi(y)\right)
$$

because for a given $y, \frac{1}{2 \lambda}\|x-y\|^{2}+\phi(y) \leq \frac{1}{2 \mu}\|x-y\|^{2}+\phi(y)$. Thus $\phi_{\lambda}(x) \uparrow \phi(x)$.
Next consider the claim about the Gateaux differentiability. Using the description 25.7.78

$$
\begin{gather*}
\phi_{\lambda}(y)-\phi_{\lambda}(x)= \\
\frac{1}{2 \lambda}\left\|y-J_{\lambda} y\right\|^{2}+\phi\left(J_{\lambda}(y)\right)-\left(\frac{1}{2 \lambda}\left\|x-J_{\lambda} x\right\|^{2}+\phi\left(J_{\lambda}(x)\right)\right) \tag{25.7.81}
\end{gather*}
$$

Using the fact that if $\psi(x)=\|x\|^{2}$, then $\partial \psi(x)=F x$, and that $A_{\lambda} x \in \partial \phi\left(J_{\lambda} x\right)$,

$$
\begin{aligned}
& \geq \lambda^{-1}\left\langle F\left(x-J_{\lambda}(x)\right),\left(y-J_{\lambda} y\right)-\left(x-J_{\lambda} x\right)\right\rangle+\left\langle A_{\lambda} x, J_{\lambda}(y)-J_{\lambda}(x)\right\rangle \\
& =\left\langle A_{\lambda}(x),\left(y-J_{\lambda} y\right)-\left(x-J_{\lambda} x\right)\right\rangle+\left\langle A_{\lambda} x, J_{\lambda}(y)-J_{\lambda}(x)\right\rangle=\left\langle A_{\lambda} x, y-x\right\rangle
\end{aligned}
$$

Hence

$$
\left(\phi_{\lambda}(y)-\phi_{\lambda}(x)\right)-\left\langle A_{\lambda} x, y-x\right\rangle \geq 0
$$

Also from 25.7.81

$$
\begin{aligned}
& \frac{1}{2 \lambda}\left\|y-J_{\lambda} y\right\|^{2}-\frac{1}{2 \lambda}\left\|x-J_{\lambda} x\right\|^{2}=-\left(\frac{1}{2 \lambda}\left\|x-J_{\lambda} x\right\|^{2}-\frac{1}{2 \lambda}\left\|y-J_{\lambda} y\right\|^{2}\right) \\
& \leq-\frac{1}{\lambda}\left\langle F\left(y-J_{\lambda} y\right),\left(x-J_{\lambda} x\right)-\left(y-J_{\lambda} y\right)\right\rangle=\left\langle A_{\lambda} y,\left(y-J_{\lambda} y\right)-\left(x-J_{\lambda} x\right)\right\rangle
\end{aligned}
$$

Similarly, from 25.7.81,

$$
\begin{gathered}
\phi\left(J_{\lambda}(y)\right)-\phi\left(J_{\lambda}(x)\right)=-\left(\phi\left(J_{\lambda}(x)\right)-\phi\left(J_{\lambda}(y)\right)\right) \\
\leq-\left\langle A_{\lambda}(y), J_{\lambda}(x)-J_{\lambda}(y)\right\rangle=\left\langle A_{\lambda}(y), J_{\lambda}(y)-J_{\lambda}(x)\right\rangle
\end{gathered}
$$

It follows that

$$
\begin{aligned}
& \left\langle A_{\lambda}(y), J_{\lambda}(y)-J_{\lambda}(x)\right\rangle+\left\langle A_{\lambda} y,\left(y-J_{\lambda} y\right)-\left(x-J_{\lambda} x\right)\right\rangle \\
\geq & \left(\phi_{\lambda}(y)-\phi_{\lambda}(x)\right) \geq\left\langle A_{\lambda} x, y-x\right\rangle
\end{aligned}
$$

and so

$$
\left\langle A_{\lambda}(y), y-x\right\rangle \geq\left(\phi_{\lambda}(y)-\phi_{\lambda}(x)\right) \geq\left\langle A_{\lambda} x, y-x\right\rangle
$$

Therefore,

$$
\left\langle A_{\lambda}(y)-A_{\lambda}(x), y-x\right\rangle \geq\left(\phi_{\lambda}(y)-\phi_{\lambda}(x)\right)-\left\langle A_{\lambda} x, y-x\right\rangle \geq 0
$$

Next let $y=x+t z$ for $t>0$. Then

$$
t\left\langle A_{\lambda}(x+t z)-A_{\lambda}(x), z\right\rangle \geq\left(\phi_{\lambda}(x+t z)-\phi_{\lambda}(x)\right)-t\left\langle A_{\lambda} x, z\right\rangle \geq 0
$$

Using the demicontinuity of $A_{\lambda}$, you can divide by $t$ and pass to a limit to obtain

$$
\lim _{t \downarrow 0} \frac{\phi_{\lambda}(x+t z)-\phi_{\lambda}(x)}{t}=\left\langle A_{\lambda} x, z\right\rangle
$$

A much better theorem is available in case $X=X^{\prime}=H$ a Hilbert space. In this case $\phi_{\lambda}$ is also Frechet differentiable. See Theorem 35.3 .24 which is presented later. Everything is much nicer in the Hilbert space setting because $F$ is just replaced with the identity and the approximations are defined more easily.

$$
\begin{aligned}
0 & \in J_{\lambda} x-x+\lambda A J_{\lambda} x, \\
x & \in J_{\lambda} x+\lambda A J_{\lambda} x=(I+\lambda A) J_{\lambda} x \\
J_{\lambda} x & =(I+\lambda A)^{-1} x
\end{aligned}
$$

Then one can show that $J_{\lambda}$ is Lipschitz continuous and many other nice things happen.
Next is an interesting result about when the sum of a maximal monotone operator and a subgradient is also maximal monotone. A version of this is well known in the case of a
single Hilbert space. In the case of a single Hilbert space, this result can be used to produce very regular solutions to evolution equations for functions which have values in the Hilbert space. You would get this by letting $X=X^{\prime}$ equal to a Hilbert space and your maximal monotone operator $A$ would be defined on $L^{2}(0, T ; H)=X$ a space of Hilbert space valued functions which are square integrable. Then you could take $L u=u^{\prime}$ with domain equal to those functions in $X$ which are equal to 0 at the left end of the interval for example. This is done more generally later. In this case the duality map is just the identity. The next theorem includes the case of two different spaces. I am not sure whether this is a useful result at this time, in terms of evolution equations. However, it is good to have conditions which show that the sum of two maximal monotone operators is maximal monotone.

Theorem 25.7.55 Let $X$ be a reflexive Banach space with strictly convex norm and let $\Phi$ be non negative, convex, proper, and lower semicontinuous. Suppose also that $A: D(A) \rightarrow$ $\mathscr{P}\left(X^{\prime}\right)$ is a maximal monotone operator and there exists

$$
\begin{equation*}
\xi \in D(A) \cap D(\Phi) . \tag{25.7.82}
\end{equation*}
$$

Suppose also that

$$
\begin{equation*}
\Phi\left(J_{\lambda} x\right) \leq \Phi(x)+C \lambda \tag{25.7.83}
\end{equation*}
$$

Then $A+\partial \Phi$ is maximal monotone.
Proof: Recall that

$$
A_{\lambda} x=-\lambda^{-1} F\left(J_{\lambda} x-x\right), \text { where } 0 \in F\left(J_{\lambda} x-x\right)+\lambda \partial A\left(J_{\lambda} x\right)
$$

Let $y^{*} \in X^{\prime}$. From Theorem 25.7.43 there exists $x_{\lambda} \in H$ such that

$$
y^{*} \in F x_{\lambda}+A_{\lambda} x_{\lambda}+\partial \Phi\left(x_{\lambda}\right)
$$

It is desired to show that $A_{\lambda} x_{\lambda}$ is bounded. From the above,

$$
\begin{equation*}
y^{*}-F x_{\lambda}-A_{\lambda} x_{\lambda} \in \partial \Phi\left(x_{\lambda}\right) \tag{25.7.84}
\end{equation*}
$$

and so

$$
\begin{equation*}
\left\langle y^{*}-F x_{\lambda}-A_{\lambda} x_{\lambda}, J_{\lambda} x_{\lambda}-x_{\lambda}\right\rangle \leq \Phi\left(J_{\lambda} x_{\lambda}\right)-\Phi\left(x_{\lambda}\right) \leq C \lambda \tag{25.7.85}
\end{equation*}
$$

which implies

$$
\left\langle y^{*}-F x_{\lambda}-A_{\lambda} x_{\lambda},(-\lambda) F^{-1}\left(A_{\lambda} x\right)\right\rangle \leq \Phi\left(J_{\lambda} x_{\lambda}\right)-\Phi\left(x_{\lambda}\right) \leq C \lambda
$$

and so

$$
\left\langle y^{*}-F x_{\lambda}-A_{\lambda} x_{\lambda},-F^{-1}\left(A_{\lambda} x\right)\right\rangle \leq C
$$

Hence

$$
\begin{equation*}
\left\langle y^{*}-F x_{\lambda},-F^{-1}\left(A_{\lambda} x_{\lambda}\right)\right\rangle+\left\|A_{\lambda} x_{\lambda}\right\|^{2} \leq C \tag{25.7.86}
\end{equation*}
$$

I claim $\left\{\left\|x_{\lambda}\right\|\right\}$ are bounded independent of $\lambda$.
By 25.7.84 and monotonicity of $A_{\lambda}$,

$$
\Phi(\xi)-\Phi\left(x_{\lambda}\right) \geq\left\langle y^{*}-F x_{\lambda}-A_{\lambda} x_{\lambda}, \xi-x_{\lambda}\right\rangle
$$

$$
\begin{gathered}
\geq\left\langle y^{*}-F x_{\lambda}, \xi-x_{\lambda}\right\rangle-\left\langle A_{\lambda} x_{\lambda}, \xi-x_{\lambda}\right\rangle \\
\geq\left\langle y^{*}-F x_{\lambda}, \xi-x_{\lambda}\right\rangle-\left\langle A_{\lambda} \xi, \xi-x_{\lambda}\right\rangle \\
=\left\langle y^{*}, \xi\right\rangle-\left\langle y^{*}, x_{\lambda}\right\rangle-\left\langle F x_{\lambda}, \xi\right\rangle+\left\|x_{\lambda}\right\|^{2}-\left\|\xi-x_{\lambda}\right\|\left\|A_{\lambda} \xi\right\| \\
\geq-\left\|y^{*}\right\|\|\xi\|-\left\|y^{*}\right\|\left\|x_{\lambda}\right\|-\left\|x_{\lambda}\right\|\|\xi\|-\|\xi\||A \xi|-\left\|x_{\lambda}\right\||A \xi|+\left\|x_{\lambda}\right\|^{2}
\end{gathered}
$$

Therefore, there exist constants, $C_{1}$ and $C_{2}$, depending on $\xi$ and $y^{*}$ but not on $\lambda$ such that

$$
\Phi(\xi) \geq \Phi\left(x_{\lambda}\right)+\left\|x_{\lambda}\right\|^{2}-C_{1}\left\|x_{\lambda}\right\|-C_{2} .
$$

Since $\Phi \geq 0$, the above shows that $\left\|x_{\lambda}\right\|$ is indeed bounded. Now from 25.7.86 it follows that $\left\{A_{\lambda} x_{\lambda}\right\}$ is bounded for small positive $\lambda$. By Theorem 25.7.43, there exists a solution $x$ to

$$
y^{*} \in F x+A x+\partial \Phi(x)
$$

and since $y^{*}$ is arbitrary, this shows that $A+\partial \Phi$ is maximal monotone.

### 25.8 Perturbation Theorems

In this section gives surjectivity of the sum of a pseudomonotone set valued map with a linear maximal monotone map and also with another maximal monotone operator added in. It generalizes the surjectivity results given earlier because one could have 0 for the maximal monotone linear operator. The theorems developed here lead to nice results on evolution equations because the linear maximal monotone operator can be something like a time derivative and $X$ can be some sort of an $L^{p}$ space for functions having values in a suitable Banach space. This is presented later in the material on Bochner integrals.

The notation $\left\langle z^{*}, u\right\rangle_{V^{\prime}, V}$ will mean $z^{*}(u)$ in this section. We will not worry about the order either. Thus

$$
\left\langle u, z^{*}\right\rangle \equiv z^{*}(u) \equiv\left\langle z^{*}, u\right\rangle
$$

This is just convenient in writing things down. Also, it is assumed that all Banach spaces are real to simplify the presentation. It is also usually assumed that the Banach spaces are reflexive. Thus we can regard

$$
\left(V \times V^{\prime}\right)^{\prime}=V^{\prime} \times V
$$

and $\left\langle\left(y^{*}, x\right),\left(u, v^{*}\right)\right\rangle \equiv\left\langle y^{*}, u\right\rangle+\left\langle x, v^{*}\right\rangle$. It is known [8] that for a reflexive Banach space, there is always an equivalent strictly convex norm. It is therefore, assumed that the norm for the reflexive Banach space is strictly convex.

Definition 25.8.1 Let $L: D(L) \subseteq V \rightarrow V^{\prime}$ be a linear map where we always assume $D(L)$ is dense in $V$. Then

$$
D\left(L^{*}\right) \equiv\{u \in V:|\langle L v, u\rangle| \leq C\|v\| \text { for all } v \in D(L)\}
$$

For such $u$, it follows that on a dense subset of $V$, namely $D(L), v \rightarrow\langle L z, u\rangle$ is a continuous linear map. Hence there exists a unique element of $V^{\prime}$, denoted as $L^{*} u$ such that for all $v \in D(L)$,

$$
\langle L v, u\rangle_{V^{\prime}, V}=\left\langle L^{*} u, v\right\rangle_{V^{\prime}, V}
$$

Thus

$$
\begin{aligned}
& L: D(L) \subseteq V \rightarrow V^{\prime} \\
& L^{*}: D\left(L^{*}\right) \subseteq V \rightarrow V^{\prime}
\end{aligned}
$$

There is an interesting description of $L^{*}$ in terms of $L$ which will be quite useful.
Proposition 25.8.2 Let $\tau: V \times V^{\prime} \rightarrow V^{\prime} \times V$ be given by $\tau(a, b) \equiv(-b, a)$. Also for $S \subseteq X$ a reflexive Banach space,

$$
S^{\perp} \equiv\left\{z^{*} \in X^{\prime}:\left\langle z^{*}, s\right\rangle=0 \text { for all } s \in S\right\}
$$

Also denote by $\mathscr{G}(L) \equiv\{(x, L x): x \in D(L)\}$. Then

$$
\mathscr{G}\left(L^{*}\right)=(\tau \mathscr{G}(L))^{\perp}
$$

Proof: Let $\left(x, L^{*} x\right) \in \mathscr{G}\left(L^{*}\right)$. This means that

$$
|\langle L y, x\rangle| \leq C\|y\| \text { for all } y \in D(L)
$$

and $\langle L y, x\rangle=\left\langle L^{*} x, y\right\rangle$ for all $y \in D(L)$. Let $(y, L y) \in \mathscr{G}(L)$. Then $\tau(y, L y)=(-L y, y)$. Then

$$
\left\langle\left(x, L^{*} x\right),(-L y, y)\right\rangle=\langle x,-L y\rangle+\left\langle L^{*} x, y\right\rangle=-\langle x, L y\rangle+\left\langle x, L^{*} y\right\rangle=0
$$

Thus $\mathscr{G}\left(L^{*}\right) \subseteq(\tau \mathscr{G}(L))^{\perp}$. Next suppose $\left(x, y^{*}\right) \in(\tau \mathscr{G}(L))^{\perp}$. This means that if $(u, L u) \in$ $\mathscr{G}(L)$, then

$$
\left\langle\left(x, y^{*}\right),(-L u, u)\right\rangle \equiv\langle x,-L u\rangle+\left\langle y^{*}, u\right\rangle=0
$$

and so for all $u \in D(L)$,

$$
\left\langle y^{*}, u\right\rangle=\langle x, L u\rangle
$$

and so $x \in D\left(L^{*}\right)$. Hence for all $u \in D(L)$,

$$
\left\langle y^{*}, u\right\rangle=\langle x, L u\rangle=\left\langle L^{*} x, u\right\rangle
$$

Then, since $D(L)$ is dense, it follows that $y^{*}=L^{*} x$ and so $(x, y) \in \mathscr{G}\left(L^{*}\right)$. Thus these are the same.

Theorem 25.5 .4 is a very nice surjectivity result for set valued pseudomonotone operators. We recall what it said here. Recall the meaning of coercive.

$$
\lim _{\|v\| \rightarrow \infty} \inf \left\{\frac{\left\langle z^{*}, v\right\rangle}{\|v\|}: z^{*} \in T v\right\}=\infty
$$

In this section, we use the convenient notation $\left\langle z^{*}, x\right\rangle_{V^{\prime}, V} \equiv z^{*}(x)$.
Theorem 25.8.3 Let $V$ be a reflexive Banach space and let $T: V \rightarrow \mathscr{P}\left(V^{\prime}\right)$ be pseudomonotone, bounded and coercive. Then $T$ is onto. More generally, this continues to hold if $T$ is modified bounded pseudomonotone.

Recall the definition of pseudomonotone.

Definition 25.8.4 For $X$ a reflexive Banach space, we say $A: X \rightarrow \mathscr{P}\left(X^{\prime}\right)$ is pseudomonotone if the following hold.

1. The set $A u$ is nonempty, closed and convex for all $u \in X$.
2. If $F$ is a finite dimensional subspace of $X, u \in F$, and if $U$ is a weakly open set in $V^{\prime}$ such that $A u \subseteq U$, then there exists a $\delta>0$ such that if $v \in B_{\delta}(u) \cap F$ then $A v \subseteq U$. (Weakly upper semicontinuous on finite dimensional subspaces.)
3. If $u_{i} \rightarrow u$ weakly in $X$ and $u_{i}^{*} \in A u_{i}$ is such that

$$
\begin{equation*}
\limsup _{i \rightarrow \infty}\left\langle u_{i}^{*}, u_{i}-u\right\rangle \leq 0 \tag{25.8.87}
\end{equation*}
$$

then, for each $v \in X$, there exists $u^{*}(v) \in A u$ such that

$$
\begin{equation*}
\liminf _{i \rightarrow \infty}\left\langle u_{i}^{*}, u_{i}-v\right\rangle \geq\left\langle u^{*}(v), u-v\right\rangle \tag{25.8.88}
\end{equation*}
$$

Also recall the definition of modified bounded pseudomonotone. It is just the above except that the limit condition is replaced with the following condition: If $u_{i} \rightarrow u$ weakly in $X$ and

$$
\begin{equation*}
\limsup _{i \rightarrow \infty}\left\langle u_{i}^{*}, u_{i}-u\right\rangle \leq 0 \tag{25.8.89}
\end{equation*}
$$

then there exists a subsequence, still denoted as $\left\{u_{i}\right\}$ such that for each $v \in X$, there exists $u^{*}(v) \in A u$ such that

$$
\begin{equation*}
\lim \inf _{i \rightarrow \infty}\left\langle u_{i}^{*}, u_{i}-v\right\rangle \geq\left\langle u^{*}(v), u-v\right\rangle \tag{25.8.90}
\end{equation*}
$$

Also recall that this more general limit condition along with the assumption 1 and the assumption that $A$ is bounded is sufficient to obtain condition 2. This was Lemma 25.4.9 proved earlier and stated here for convenience.

Lemma 25.8.5 Let $A: X \rightarrow \mathscr{P}\left(X^{\prime}\right)$ satisfy conditions 1 and 3 above and suppose $A$ is bounded. Also suppose the condition that if $x_{n} \rightarrow x$ weakly and

$$
\lim _{n \rightarrow \infty}\left\langle z_{n}, x_{n}-x\right\rangle \leq 0
$$

implies there exists a subsequence $\left\{x_{n_{k}}\right\}$ such that for any $y$,

$$
\lim \inf _{n \rightarrow \infty}\left\langle z_{n_{k}}, x_{n_{k}}-y\right\rangle \geq\langle z(y), x-y\rangle
$$

for $z(y)$ some element of $A x$. Then if this weaker condition holds, you have that if $U$ is a weakly open set containing $A x$, then $A x_{n} \subseteq U$ for all $n$ large enough.

Definition 25.8.6 Now let $L: D(L) \subseteq V \rightarrow V^{\prime}$ such that $L$ is linear, monotone, $D(L)$ is dense in $V, L$ is closed, and $L^{*}$ is monotone. Let $A: V \rightarrow \mathscr{P}\left(V^{\prime}\right)$ be a bounded operator. Then $A$ is called $L$ pseudomonotone if $A v$ is closed and convex in $V^{\prime}$ and for any sequence $\left\{u_{n}\right\} \subseteq D(L)$ such that $u_{n} \rightarrow u$ weakly in $V$ and $L u_{n} \rightarrow L u$ weakly in $V^{\prime}$, and for $z_{n}^{*} \in A u_{n}$,

$$
\lim _{n \rightarrow \infty}\left\langle z_{n}^{*}, u_{n}-u\right\rangle \leq 0
$$

then for every $v \in V$, there exists $z^{*}(v) \in A u$ such that

$$
\lim \inf _{n \rightarrow \infty}\left\langle z_{n}^{*}, u_{n}-v\right\rangle \geq\left\langle z^{*}(v), u-v\right\rangle
$$

It is called L modified bounded pseudomonotone if the above liminf condition holds for some subsequence whenever $u_{n} \rightarrow u$ weakly and $L u_{n} \rightarrow L u$ weakly and

$$
\limsup _{n \rightarrow \infty}\left\langle z_{n}^{*}, u_{n}-u\right\rangle \leq 0
$$

Lemma 25.8.7 Suppose $X$ is the Banach space

$$
X=D(L),\|u\|_{X} \equiv\|u\|_{V}+\|L u\|_{V^{\prime}}
$$

where $L$ is as described in the above definition. Also assume that $A$ is bounded. Then if $A$ is $L$ pseudomonotone, it follows that $A$ is pseudomonotone as a map from $X$ to $\mathscr{P}\left(X^{\prime}\right)$. If A is L modified bounded pseudomonotone, then $A$ is modified bounded pseudomonotone as a map from $X$ to $\mathscr{P}\left(X^{\prime}\right)$.

Proof: Is $A$ bounded? Of course, because the norm of $X$ is stronger than the norm on $V$. Is $A u$ convex and closed? This also follows because $X \subseteq V$. It is clear that $A u$ is convex. If $\left\{z_{n}\right\} \subseteq A u$ and $z_{n} \rightarrow z$ in $X^{\prime}$, then does it follow that $z \in A u$ ? Since $A$ is bounded, there is a further subsequence which converges weakly to $w$ in $V^{\prime}$. However, $A u$ is convex and closed so it is weakly closed. Hence $w \in A u$ and also $w=z$. It only remains to verify the pseudomonotone limit condition. Suppose then that $u_{n} \rightarrow u$ weakly in $X$ and for $z_{n}^{*} \in A u_{n}$,

$$
\lim \sup _{n \rightarrow \infty}\left\langle z_{n}^{*}, u_{n}-u\right\rangle \leq 0
$$

Then it follows that $L u_{n} \rightarrow L u$ weakly in $V^{\prime}$ and $u_{n} \rightarrow u$ weakly in $V$ so $u \in X$. Hence the assumption that $A$ is $L$ pseudomonotone implies that for every $v \in V$, and for every $v \in X$, there exists $z^{*}(v) \in A u \subseteq V^{\prime} \subseteq X^{\prime}$ such that

$$
\lim \inf _{n \rightarrow \infty}\left\langle z_{n}^{*}, u_{n}-v\right\rangle \geq\left\langle z^{*}(v), u-v\right\rangle
$$

The last claim goes the same way. You just have to take a subsequence.
Then we have the following major surjectivity result. In this theorem, we will assume for simplicity that all spaces are real spaces. Versions of this appear to be due to Brezis [23] and Lions [91]. Of course the theorem holds for complex spaces as well. You just need to use $\operatorname{Re}\rangle$ instead of $\rangle$.

Theorem 25.8.8 Let $L: D(L) \subseteq V \rightarrow V^{\prime}$ where $D(L)$ is dense, $L$ is monotone, $L$ is closed, and $L^{*}$ is monotone, L a linear map. Let $A: V \rightarrow \mathscr{P}\left(V^{\prime}\right)$ be L pseudomonotone, bounded, coercive. Then $L+A$ is onto. Here $V$ is a reflexive Banach space such that the norms for $V$ and $V^{\prime}$ are strictly convex. In case that $A$ is strictly monotone $(\langle A u-A v, u-v\rangle>0$ implies $u \neq v$ ) the solution $u$ to $f \in L u+A u$ is unique. If, in addition to this, $\langle A u-A v, u-v\rangle \geq$ $r\left(\|u-v\|_{U}\right)$ where $U$ is some Banach space containing $V$, and $r$ is a positive strictly increasing function for which $\lim _{t \rightarrow 0+} r(t)=0$, then the map $f \rightarrow u$ where $f \in L u+A u$ is continuous as a map from $V^{\prime}$ to $U$. The conclusion holds if $A$ is only $L$ modified bounded pseudomonotone.

Proof: Let $F$ be the duality map for $p=2$. Consider the Banach space $X$ given by

$$
X=D(L),\|u\|_{X} \equiv\|u\|_{V}+\|L u\|_{V^{\prime}}
$$

This is isometric with the graph of $L$ with the graph norm and so $X$ is reflexive. Now define a set valued map $G_{\varepsilon}$ on $X$ as follows. $z^{*} \in G_{\varepsilon}(u)$ means there exists $w^{*} \in A u$ such that.

$$
\left\langle z^{*}, v\right\rangle_{X^{\prime}, X}=\varepsilon\left\langle L v, F^{-1}(L u)\right\rangle_{V^{\prime}, V}+\langle L u, v\rangle_{V^{\prime}, V}+\left\langle w^{*}, v\right\rangle_{V^{\prime}, V}
$$

It follows from Lemma 25.8.7 that $G_{\varepsilon}$ is the sum of a set valued $L$ modified bounded pseudomonotone operator with an operator which is demicontinuous, bounded, and monotone, hence pseudomonotone. Thus by Lemma 25.5.2 it is $L$ modified bounded pseudomonotone. Is it coercive?

$$
\lim _{\|u\|_{X} \rightarrow \infty} \inf \left\{\frac{\left\langle z^{*}, u\right\rangle+\varepsilon\left\langle L u, F^{-1}(L u)\right\rangle_{V^{\prime}, V}+\langle L u, u\rangle_{V^{\prime}, V}}{\|u\|_{X}}: z^{*} \in A u\right\}=\infty ?
$$

It equals

$$
\lim _{\|u\|_{X} \rightarrow \infty} \inf \left\{\frac{\left\langle z^{*}, u\right\rangle+\varepsilon\left\langle F F^{-1}(L u), F^{-1}(L u)\right\rangle_{V^{\prime}, V}+\langle L u, u\rangle_{V^{\prime}, V}}{\|u\|_{X}}: z^{*} \in A u\right\}
$$

and this is

$$
\begin{aligned}
& \geq \lim _{\|u\|_{X} \rightarrow \infty} \inf \left\{\frac{\left\langle z^{*}, u\right\rangle+\varepsilon\left\|F^{-1}(L u)\right\|_{V}^{2}}{\|u\|_{X}}: z^{*} \in A u\right\} \\
& =\lim _{\|u\|_{X} \rightarrow \infty} \inf \left\{\frac{\left\langle z^{*}, u\right\rangle+\varepsilon\|L u\|_{V^{\prime}}^{2}}{\|u\|_{V}+\|L u\|_{V^{\prime}}}: z^{*} \in A u\right\}
\end{aligned}
$$

because $L$ is monotone. Now let $M$ be an arbitrary positive number. By assumption, there exists $R$ such that if $\|u\|_{V}>R$, then

$$
\inf \left\{\frac{\left\langle z^{*}, u\right\rangle}{\|u\|_{V}}: z^{*} \in A u\right\}>M
$$

and so for every $z^{*} \in A u$,

$$
\frac{\left\langle z^{*}, u\right\rangle}{\|u\|_{V}}>M,\left\langle z^{*}, u\right\rangle>M\|u\|_{V}
$$

Thus if $\|u\|_{V}>R$,

$$
\inf \left\{\frac{\left\langle z^{*}, u\right\rangle+\varepsilon\|L u\|_{V^{\prime}}^{2}}{\|u\|_{V}+\|L u\|_{V^{\prime}}}: z^{*} \in A u\right\} \geq \frac{M\|u\|_{V}+\varepsilon\|L u\|_{V^{\prime}}^{2}}{\|u\|_{V}+\|L u\|_{V^{\prime}}}
$$

I claim that if $\|u\|_{X}$ is large enough, the above is larger than $M / 2$. If not, then there exists $\left\{u_{n}\right\}$ such that $\left\|u_{n}\right\|_{X} \rightarrow \infty$ but the right side is less than $M / 2$. First say $\left\|L u_{n}\right\|$ is bounded.
then there is an obvious contradiction since the right hand side then converges to $M$. Thus it can be assumed that $\left\|L u_{n}\right\|_{V^{\prime}} \rightarrow \infty$. Hence, for all $n$ large enough, $\varepsilon\|L u\|_{V^{\prime}}^{2}>M\left\|L u_{n}\right\|_{V^{\prime}}$. However, this implies the right side is larger than

$$
\frac{M\left\|u_{n}\right\|_{V}+M\left\|L u_{n}\right\|_{V^{\prime}}}{\left\|u_{n}\right\|_{V}+\left\|L u_{n}\right\|_{V^{\prime}}}=M>M / 2
$$

This is a contradiction. Hence the right side is larger than $M / 2$ for all $n$ large enough. It follows since $M$ is arbitrary, that

$$
\lim _{\|u\|_{X} \rightarrow \infty} \inf \left\{\frac{\left\langle z^{*}, u\right\rangle+\varepsilon\|L u\|_{V^{\prime}}^{2}}{\|u\|_{V}+\|L u\|_{V^{\prime}}}: z^{*} \in A u\right\}=\infty
$$

It follows from Theorem 25.5.4 that if $f \in V^{\prime}$, there exists $u_{\varepsilon}$ such that for all $v \in$ $D(L)=X$,

$$
\begin{equation*}
\varepsilon\left\langle L v, F^{-1}\left(L u_{\varepsilon}\right)\right\rangle_{V^{\prime}, V}+\left\langle L u_{\varepsilon}, v\right\rangle_{V^{\prime}, V}+\left\langle w_{\varepsilon}^{*}, v\right\rangle_{V^{\prime}, V}=\langle f, v\rangle, w_{\varepsilon}^{*} \in A u_{\varepsilon} \tag{25.8.91}
\end{equation*}
$$

First we get an estimate.

$$
\begin{gathered}
\varepsilon\left\langle L u_{\varepsilon}, F^{-1}\left(L u_{\varepsilon}\right)\right\rangle_{V^{\prime}, V}+\left\langle L u_{\varepsilon}, u_{\varepsilon}\right\rangle_{V^{\prime}, V}+\left\langle w_{\varepsilon}^{*}, u_{\varepsilon}\right\rangle_{V^{\prime}, V}=\left\langle f, u_{\varepsilon}\right\rangle \\
\varepsilon\left\|L u_{\varepsilon}\right\|_{V^{\prime}}^{2}+\left\langle L u_{\varepsilon}, u_{\varepsilon}\right\rangle_{V^{\prime}, V}+\left\langle w_{\varepsilon}^{*}, u_{\varepsilon}\right\rangle_{V^{\prime}, V}=\left\langle f, u_{\varepsilon}\right\rangle
\end{gathered}
$$

Hence it follows from the coercivity of $A$ that $\left\|u_{\varepsilon}\right\|_{V}$ is bounded independent of $\varepsilon$. Thus the $w_{\varepsilon}^{*}$ are also bounded in $V^{\prime}$ because it is assumed that $A$ is bounded. Now from the equation solved 25.8.91, it follows that $F^{-1}\left(L u_{\varepsilon}\right) \in D\left(L^{*}\right)$. Thus the first term is just $\varepsilon\left\langle L^{*}\left(F^{-1}\left(L u_{\varepsilon}\right)\right), v\right\rangle_{V^{\prime}, V}$. It follows, since $D(L)=X$ is dense in $V$ that

$$
\begin{equation*}
\varepsilon L^{*}\left(F^{-1}\left(L u_{\varepsilon}\right)\right)+L u_{\varepsilon}+w_{\varepsilon}^{*}=f \tag{25.8.92}
\end{equation*}
$$

Then act on $F^{-1}\left(L u_{\varepsilon}\right)$ on both sides. From monotonicity of $L^{*}$, this yields $\left\|L u_{\varepsilon}\right\|_{V^{\prime}}$ is bounded independent of $\varepsilon>0$. Thus there is a subsequence still denoted with a subscript of $\varepsilon$ such that

$$
\begin{gathered}
u_{\varepsilon} \rightharpoonup u \text { in } V \\
L u_{\varepsilon} \rightharpoonup L u \text { in } V^{\prime}
\end{gathered}
$$

This because of the fact that the graph of $L$ is closed, hence weakly closed. Thus $u \in X$. Also

$$
w_{\varepsilon}^{*} \rightharpoonup w^{*} \text { in } V^{\prime} .
$$

It follows that we can pass to a limit in 25.8.92 and obtain

$$
\begin{equation*}
L u+w^{*}=f \tag{25.8.93}
\end{equation*}
$$

Now by assumption on $A$, it is $L$ modified bounded pseudomonotone and so there is a subsequence, still denoted as $u_{\mathcal{\varepsilon}}$ such that the liminf pseudomonotone limit condition holds. This will be what is referred to in what follows. Then

$$
\left\langle\varepsilon L^{*}\left(F^{-1}\left(L u_{\varepsilon}\right)\right), u_{\varepsilon}-u\right\rangle+\left\langle L u_{\varepsilon}, u_{\varepsilon}-u\right\rangle+\left\langle w_{\varepsilon}^{*}, u_{\varepsilon}-u\right\rangle=\left\langle f, u_{\varepsilon}-u\right\rangle
$$

and so,

$$
\varepsilon\left\langle F^{-1}\left(L u_{\varepsilon}\right), L u_{\varepsilon}-L u\right\rangle+\left\langle L u_{\varepsilon}, u_{\varepsilon}-u\right\rangle+\left\langle w_{\varepsilon}^{*}, u_{\varepsilon}-u\right\rangle=\left\langle f, u_{\varepsilon}-u\right\rangle
$$

using the monotonicity of $L$,

$$
\begin{gathered}
\varepsilon\left\langle L u_{\varepsilon}-L u, F^{-1}\left(L u_{\varepsilon}\right)-F^{-1}(L u)\right\rangle+\varepsilon\left\langle L u_{\varepsilon}-L u, F^{-1}(L u)\right\rangle \\
+\left\langle L u, u_{\varepsilon}-u\right\rangle+\left\langle w_{\varepsilon}^{*}, u_{\varepsilon}-u\right\rangle \leq\left\langle f, u_{\varepsilon}-u\right\rangle
\end{gathered}
$$

Now using monotonicity of $F^{-1}$,

$$
\varepsilon\left\langle L u_{\varepsilon}-L u, F^{-1}(L u)\right\rangle+\left\langle L u, u_{\varepsilon}-u\right\rangle+\left\langle w_{\varepsilon}^{*}, u_{\varepsilon}-u\right\rangle \leq\left\langle f, u_{\varepsilon}-u\right\rangle
$$

and so, passing to a limit as $\varepsilon \rightarrow 0$,

$$
\limsup _{\varepsilon \rightarrow 0}\left\langle w_{\varepsilon}^{*}, u_{\varepsilon}-u\right\rangle \leq 0
$$

It follows that for all $v \in X=D(L)$ there exists $w^{*}(v) \in A u$

$$
\lim \inf _{\varepsilon \rightarrow 0}\left\langle w_{\varepsilon}^{*}, u_{\varepsilon}-v\right\rangle \geq\left\langle w^{*}(v), u-v\right\rangle
$$

But the left side equals

$$
\begin{aligned}
& \lim \inf _{\varepsilon \rightarrow 0}\left[\left\langle w_{\varepsilon}^{*}, u_{\varepsilon}-u\right\rangle+\left\langle w_{\varepsilon}^{*}, u-v\right\rangle\right] \\
\leq \quad & \lim \sup _{\varepsilon \rightarrow 0}\left\langle w_{\varepsilon}^{*}, u_{\varepsilon}-u\right\rangle+\left\langle w^{*}, u-v\right\rangle \leq\left\langle w^{*}, u-v\right\rangle
\end{aligned}
$$

and so

$$
\left\langle w^{*}, u-v\right\rangle \geq\left\langle w^{*}(v), u-v\right\rangle
$$

for all $v$.
Is $w^{*} \in A u$ ? Suppose not. Then $A u$ is a closed convex set and $w^{*}$ is not in it. Hence, since $V$ is reflexive, there exists $z \in V$ such that whenever $y^{*} \in A u,\left\langle w^{*}, z\right\rangle<\left\langle y^{*}, z\right\rangle$. Now simply choose $v$ such that $u-v=z$ and it follows that

$$
\left\langle w^{*}(v), u-v\right\rangle>\left\langle w^{*}, u-v\right\rangle \geq\left\langle w^{*}(v), u-v\right\rangle
$$

which is clearly a contradiction. Hence $w^{*} \in A u$. Thus from 25.8.93, this has shown that $L+A$ is onto.

Consider the claim about uniqueness and continuous dependence. Say you have $f_{i} \in$ $L u_{i}+A u_{i}, i=1,2$. Let $z_{i}^{*} \in A u_{i}$ be such that equality holds in the two inclusions. Then

$$
f_{1}-f_{2}=z_{1}^{*}-z_{2}^{*}+L u_{1}-L u_{2}
$$

It follows that

$$
\left\langle f_{1}-f_{2}, u_{1}-u_{2}\right\rangle=\left\langle z_{1}^{*}-z_{2}^{*}+L u_{1}-L u_{2}, u_{1}-u_{2}\right\rangle \geq r\left(\left\|u_{1}-u_{2}\right\|\right)
$$

Thus if $f_{1}=f_{2}$, then $u_{1}=u_{2}$. If $f_{n} \rightarrow f$ in $V^{\prime}$, then $r\left(\left\|u-u_{n}\right\|\right) \rightarrow 0$ where $u_{n}$ goes with $f_{n}$ and $u$ with $f$ as just described, and so $u_{n} \rightarrow u$ because the coercivity estimate given above shows that the $u_{n}$ and $u$ are all bounded. Thus the map just described is continuous.

The following lemma is interesting in terms of the hypotheses of the above theorem. [23]

Lemma 25.8.9 Let $L: D(L) \rightarrow X^{\prime}$ where $D(L)$ is dense and $L$ is a closed operator. Then $L$ is maximal monotone if and only if both $L, L^{*}$ are monotone.

Proof: Suppose both $L, L^{*}$ are monotone. One must show that $\lambda F+L$ is onto. However, $F$ is monotone and hemicontinuous (actually demicontinuous) and coercive. Hence the fact that $\lambda F+L$ is onto follows from Theorem 25.8.8. Next suppose $L$ is maximal monotone. If $L$ is maximal monotone, then for every $\varepsilon>0$ there exists a solution $u_{\varepsilon}$ such that $\varepsilon L u_{\varepsilon}+$ $F\left(u_{\varepsilon}-u\right)=0$. Here $u \in D\left(L^{*}\right)$. This is from Lemma 25.7.28. It is originally due to Browder [26]. Then

$$
\varepsilon\left\langle L u_{\varepsilon}, u_{\varepsilon}\right\rangle+\left\langle F\left(u_{\varepsilon}-u\right), u_{\varepsilon}\right\rangle=0
$$

and so $\left\langle F\left(u_{\varepsilon}-u\right), u_{\varepsilon}\right\rangle \leq 0$. Then

$$
\left\langle F\left(u_{\varepsilon}-u\right), u_{\varepsilon}-u\right\rangle \leq\left\langle F\left(u_{\varepsilon}-u\right), u\right\rangle
$$

so $\left\|u_{\varepsilon}-u\right\|^{2} \leq\left\|u_{\varepsilon}-u\right\|\|u\|$ and so

$$
\left\|u_{\varepsilon}-u\right\| \leq\|u\|
$$

Thus the $u_{\varepsilon}$ are bounded.
Next let $v \in D(L)$.

$$
\begin{gathered}
\left\|u_{\varepsilon}-u\right\|^{2}=\left\langle F\left(u_{\varepsilon}-u\right), u_{\varepsilon}-u\right\rangle=\left\langle F\left(u_{\varepsilon}-u\right), u_{\varepsilon}-v\right\rangle+\left\langle F\left(u_{\varepsilon}-u\right), v-u\right\rangle \\
\leq \varepsilon\left\langle L u_{\varepsilon}, v-u_{\varepsilon}\right\rangle+\left\langle F\left(u_{\varepsilon}-u\right), v-u\right\rangle \leq \varepsilon\left\langle L v, v-u_{\varepsilon}\right\rangle+\left\langle F\left(u_{\varepsilon}-u\right), v-u\right\rangle
\end{gathered}
$$

Hence

$$
\begin{aligned}
\lim \sup _{\varepsilon \rightarrow 0}\left\|u_{\varepsilon}-u\right\|^{2} & \leq \lim \sup _{\varepsilon \rightarrow 0}\left(\varepsilon\left\langle L v, v-u_{\varepsilon}\right\rangle+\left\langle F\left(u_{\varepsilon}-u\right), v-u\right\rangle\right) \\
& \leq \lim \sup _{\varepsilon \rightarrow 0}\left\langle F\left(u_{\varepsilon}-u\right), v-u\right\rangle \leq \lim \sup _{\varepsilon \rightarrow 0}\left\|u_{\varepsilon}-u\right\|\|v-u\|
\end{aligned}
$$

and so $u_{\varepsilon} \rightarrow u$ strongly. Also

$$
\left\langle F\left(u_{\varepsilon}-u\right), u_{\varepsilon}\right\rangle=-\varepsilon\left\langle L u_{\varepsilon}, u_{\varepsilon}\right\rangle \leq 0
$$

Then

$$
\begin{gathered}
\left\langle L^{*} u, u\right\rangle=\lim _{\varepsilon \rightarrow 0}\left\langle L^{*} u, u_{\varepsilon}\right\rangle=\lim _{\varepsilon \rightarrow 0}\left\langle L u_{\varepsilon}, u\right\rangle=\lim _{\varepsilon \rightarrow 0} \frac{1}{\varepsilon}\left\langle-F\left(u_{\varepsilon}-u\right), u\right\rangle \\
=\lim _{\varepsilon \rightarrow 0} \frac{1}{\varepsilon}\left\langle F\left(u_{\varepsilon}-u\right), u_{\varepsilon}-u\right\rangle-\lim _{\varepsilon \rightarrow 0} \frac{1}{\varepsilon}\left\langle F\left(u_{\varepsilon}-u\right), u_{\varepsilon}\right\rangle
\end{gathered}
$$

Both of these last terms are nonnegative, the first obviously and the second from the above where it was shown that $\left\langle F\left(u_{\varepsilon}-u\right), u_{\varepsilon}\right\rangle \leq 0$.

In the hypotheses of Theorem 25.8 .8 , one could have simply said that $L$ is closed, linear, densely defined and maximal monotone. One can also show that if $L$ is maximal monotone, then it must be densely defined. This is done in [23].

One can go further in obtaining a perturbation theorem like the above. Let linear $L$ be densely defined with $L$ closed and $L, L^{*}$ monotone. In short, $L$ is densely defined and
maximal monotone, $L: X \rightarrow X^{\prime}$. Let $A$ be a set valued $L$ pseudomonotone operator which is coercive and bounded. Also let $B: D(B) \rightarrow \mathscr{P}(X)$ be maximal monotone. It is of interest to consider whether $L+A+B$ is onto $X^{\prime}$. In considering this, I will add further assumptions as needed. First note that $\langle L x, x\rangle=\langle L x-L 0, x-0\rangle \geq 0$.

Definition 25.8.10 Define $\limsup _{m, n \rightarrow \infty} a_{m, n} \equiv \lim _{k \rightarrow \infty} \sup \left\{a_{m, n}: \min (m, n) \geq k\right\}$
Then

$$
\lim \sup _{m, n \rightarrow \infty} a_{m, n} \geq \lim \sup _{m \rightarrow \infty}\left(\lim \sup _{n \rightarrow \infty} a_{m, n}\right)
$$

To see this, suppose $a>\limsup _{m, n \rightarrow \infty} a_{m, n}$. Then there exist $k$ such that whenever $m, n>k$,

$$
a_{m, n}<a
$$

It follows that for $m \geq k$,

$$
\lim \sup _{n \rightarrow \infty} a_{m, n} \leq a
$$

Hence

$$
\lim \sup _{m \rightarrow \infty}\left(\lim \sup _{n \rightarrow \infty} a_{m, n}\right) \leq a
$$

Since $a>\limsup _{m, n \rightarrow \infty} a_{m, n}$ is arbitrary, it follows that

$$
\lim \sup _{m \rightarrow \infty}\left(\lim \sup _{n \rightarrow \infty} a_{m, n}\right) \leq \lim \sup _{m, n \rightarrow \infty} a_{m, n}
$$

Then the following lemma is useful. I found this result in a paper by Gasinski, Migorski and Ochal [54]. They begin with the following interesting lemma or something like it which is similar to some of the ideas used in the section on approximation of maximal monotone operators.

Lemma 25.8.11 Suppose $A$ is a set valued operator, $A: X \rightarrow \mathscr{P}(X)$ and $u_{n}^{*} \in A u_{n}$. Suppose also that $u_{n} \rightarrow u$ weakly and $u_{n}^{*} \rightarrow u^{*}$ weakly. Suppose also that

$$
\lim \sup _{m, n \rightarrow \infty}\left\langle u_{n}^{*}-u_{m}^{*}, u_{n}-u_{m}\right\rangle \leq 0
$$

Then one can conclude that

$$
\lim \sup _{n \rightarrow \infty}\left\langle u_{n}^{*}, u_{n}-u\right\rangle \leq 0
$$

Proof: Let $\alpha \equiv \lim \sup _{n \rightarrow \infty}\left\langle u_{n}^{*}, u_{n}\right\rangle$. It is a finite number because these sequences are bounded. Then using the weak convergence,

$$
\begin{aligned}
0 & \geq \lim \sup _{m \rightarrow \infty}\left(\lim \sup _{n \rightarrow \infty}\left\langle u_{n}^{*}-u_{m}^{*}, u_{n}-u_{m}\right\rangle\right) \\
& =\lim \sup _{m \rightarrow \infty}\left(\lim \sup _{n \rightarrow \infty}\left(\left\langle u_{n}^{*}, u_{n}\right\rangle+\left\langle u_{m}^{*}, u_{m}\right\rangle-\left\langle u_{n}^{*}, u_{m}\right\rangle-\left\langle u_{m}^{*}, u_{n}\right\rangle\right)\right) \\
& =\lim \sup _{m \rightarrow \infty}\left(\alpha+\left\langle u_{m}^{*}, u_{m}\right\rangle-\left\langle u^{*}, u_{m}\right\rangle-\left\langle u_{m}^{*}, u\right\rangle\right) \\
& =\left(\alpha+\alpha-\left\langle u^{*}, u\right\rangle-\left\langle u^{*}, u\right\rangle\right)=2 \alpha-2\left\langle u^{*}, u\right\rangle
\end{aligned}
$$

Now

$$
\lim \sup _{n \rightarrow \infty}\left\langle u_{n}^{*}, u_{n}-u\right\rangle=\alpha-\left\langle u^{*}, u\right\rangle \leq 0
$$

To begin with, consider the approximate problem which is to determine whether $L+$ $A+B_{\lambda}$ is onto. Here $B_{\lambda} x=-\lambda^{-1} F\left(x_{\lambda}-x\right)$ where $0 \in F\left(x_{\lambda}-x\right)+\lambda B x$. In the notation given above, $B_{\lambda} x=-\lambda^{-1} F\left(J_{\lambda} x-x\right)$. Then by Theorem 25.7.36, $B_{\lambda}$ is monotone, demicontinuous, and bounded. In addition, we assume $0 \in D(B)$. Then

$$
\begin{equation*}
\left\langle B_{\lambda} x, x\right\rangle \geq\left\langle B_{\lambda} 0, x\right\rangle \geq-|B(0)|\|x\| \tag{25.8.94}
\end{equation*}
$$

Lemma 25.8.12 Let A be pseudomonotone, bounded and coercive and let $0 \in D(B)$. Then if $y^{*} \in X^{\prime}$, there exists a solution $x_{\lambda}$ to

$$
y^{*} \in L x_{\lambda}+A x_{\lambda}+B_{\lambda} x_{\lambda}
$$

Proof: From the inequality $25.8 .94, A+B_{\lambda}$ is coercive. It is also bounded and pseudomonotone. It is pseudomonotone from Theorem 25.7.27. Therefore, there exists a solution $x_{\lambda}$ by Theorem 25.8.8.

Acting on $x_{\lambda}$ and using the inequality 25.8 .94 , it follows that these solutions $x_{\lambda}$ lie in a bounded set. The details follow. Letting $z_{\lambda}^{*} \in A x_{\lambda}$ be such that equality holds in the above inclusion,

$$
\begin{align*}
& y^{*}=L x_{\lambda}+z_{\lambda}^{*}+B_{\lambda} x_{\lambda}  \tag{25.8.95}\\
&\left\|y^{*}\right\| \geq \frac{\left\langle y^{*}, x_{\lambda}\right\rangle}{\left\|x_{\lambda}\right\|}=\frac{\left\langle L x_{\lambda}, x_{\lambda}\right\rangle+\left\langle z_{\lambda}^{*}, x_{\lambda}\right\rangle+\left\langle B_{\lambda} x_{\lambda}, x_{\lambda}\right\rangle}{\left\|x_{\lambda}\right\|} \\
& \geq \frac{\left\langle L x_{\lambda}, x_{\lambda}\right\rangle+\left\langle z_{\lambda}^{*}, x_{\lambda}\right\rangle-|B(0)|\left\|x_{\lambda}\right\|}{\left\|x_{\lambda}\right\|} \\
& \geq \frac{\left\langle z_{\lambda}^{*}, x_{\lambda}\right\rangle}{\left\|x_{\lambda}\right\|}-|B(0)|
\end{align*}
$$

Thus, from coercivity, $\left\|x_{\lambda}\right\|$ are bounded. Then since $A$ is bounded, the $z_{\lambda}^{*}$ are all bounded also independent of $\lambda$. The top line shows also that

$$
\begin{align*}
\left\langle y^{*}, x_{\lambda}\right\rangle & =\left\langle L x_{\lambda}, x_{\lambda}\right\rangle+\left\langle z_{\lambda}^{*}, x_{\lambda}\right\rangle+\left\langle B_{\lambda} x_{\lambda}, x_{\lambda}\right\rangle \geq\left\langle z_{\lambda}^{*}, x_{\lambda}\right\rangle+\left\langle B_{\lambda} x_{\lambda}, x_{\lambda}\right\rangle \\
& \geq\left\langle B_{\lambda} x_{\lambda}, x_{\lambda}\right\rangle-\hat{M} \geq-|B(0)|\left\|x_{\lambda}\right\|-\hat{M} \tag{25.8.96}
\end{align*}
$$

where $\left|\left\langle z_{\lambda}^{*}, x_{\lambda}\right\rangle\right| \leq \hat{M}$ for all $\lambda$. Hence there is a constant $M$ such that

$$
\left|\left\langle B_{\lambda} x_{\lambda}, x_{\lambda}\right\rangle\right| \leq M
$$

Definition 25.8.13 A set valued operator $B$ is quasi-bounded if whenever $x \in D(B)$ and $x^{*} \in B x$ are such that

$$
\left|\left\langle x^{*}, x\right\rangle\right|,\|x\| \leq M
$$

it follows that $\left\|x^{*}\right\| \leq K_{M}$. Bounded would mean that if $\|x\| \leq M$, then $\left\|x^{*}\right\| \leq K_{M}$. Here you only know this if there is another condition.

Lemma 25.8.14 In the above situation, suppose the maximal monotone operator $B$ is quasi-bounded and $\left|\left\langle B_{\lambda} x_{\lambda}, x_{\lambda}\right\rangle\right| \leq M$. Then the $B_{\lambda} x_{\lambda}$ are bounded. Also

$$
\left\|J_{\lambda} x_{\lambda}-x_{\lambda}\right\|^{2} \leq M \lambda
$$

Proof: Now $B_{\lambda} x_{\lambda} \in B J_{\lambda} x_{\lambda}$

$$
\begin{gathered}
-|B(0)|\left\|x_{\lambda}\right\| \leq\left\langle B_{\lambda} x_{\lambda}, x_{\lambda}\right\rangle=\left\langle B_{\lambda} x_{\lambda}, J_{\lambda} x_{\lambda}\right\rangle+\left\langle B_{\lambda} x_{\lambda}, x_{\lambda}-J_{\lambda} x_{\lambda}\right\rangle \\
=\left\langle B_{\lambda} x_{\lambda}, J_{\lambda} x_{\lambda}\right\rangle+\left\langle\lambda^{-1} F\left(J_{\lambda} x_{\lambda}-x_{\lambda}\right), J_{\lambda} x_{\lambda}-x_{\lambda}\right\rangle \\
=\left\langle B_{\lambda} x_{\lambda}, J_{\lambda} x_{\lambda}\right\rangle+\lambda^{-1}\left\|J_{\lambda} x_{\lambda}-x_{\lambda}\right\|^{2} \leq M
\end{gathered}
$$

This inequality shows that $J_{\lambda} x_{\lambda}-x_{\lambda} \rightarrow 0$ and so $J_{\lambda} x_{\lambda}$ is bounded as is $x_{\lambda}$ which was shown above. Also $B_{\lambda} x_{\lambda} \in B J_{\lambda} x_{\lambda}$ and since $B$ is quasi-bounded, it follows that $B_{\lambda} x_{\lambda}$ is bounded.

Assume from now on that $B$ is quasi-bounded. Then the estimate 25.8.96 and this lemma shows that $B_{\lambda} x_{\lambda}$ is also bounded independent of $\lambda$. Thus, adjusting the constants, there exists an estimate of the form

$$
\begin{equation*}
\left\|x_{\lambda}\right\|+\left\|J_{\lambda} x_{\lambda}\right\|+\left\|B_{\lambda} x_{\lambda}\right\|+\left\|z_{\lambda}^{*}\right\|+\left\|L x_{\lambda}\right\| \leq C, \quad\left\|x_{\lambda}-J_{\lambda} x_{\lambda}\right\| \leq \sqrt{\lambda} M \tag{25.8.97}
\end{equation*}
$$

Let $\lambda=1 / n$. Also denote by $J_{n}$ the the operator $J_{1 / n}$ to save notation. There exists a subsequence

$$
\begin{aligned}
& x_{n} \rightarrow x \text { weakly } \\
& J_{n} x_{n} \rightarrow x \text { weakly } \\
& B_{n} x_{n} \rightarrow g^{*} \text { weakly } \\
& z_{n}^{*} \rightarrow z^{*} \text { weakly } \\
& L x_{n} \rightarrow L x \text { weakly }
\end{aligned}
$$

Now from the inclusion satisfied,

$$
\begin{equation*}
0=\left\langle z_{n}^{*}-z_{m}^{*}, x_{n}-x_{m}\right\rangle+\left\langle B_{n} x_{n}-B_{m} x_{m}, x_{n}-x_{m}\right\rangle \tag{25.8.98}
\end{equation*}
$$

Consider that last term. $B_{n} x_{n} \in B J_{n} x_{n}$ similar for $B_{m} x_{m}$. Hence this term is of the form

$$
\begin{aligned}
& \left\langle B_{n} x_{n}-B_{m} x_{m}, x_{n}-x_{m}\right\rangle=\overbrace{\left\langle B_{n} x_{n}-B_{m} x_{m}, J_{n} x_{n}-J_{m} x_{m}\right\rangle}^{\geq 0} \\
& \quad+\left\langle B_{n} x_{n}-B_{m} x_{m},\left(x_{n}-J_{n} x_{n}\right)-\left(x_{m}-J_{m} x_{m}\right)\right\rangle
\end{aligned}
$$

From the estimate 25.8.97,

$$
\left\langle B_{n} x_{n}-B_{m} x_{m}, x_{n}-x_{m}\right\rangle \geq\left\langle B_{n} x_{n}-B_{m} x_{m},\left(x_{n}-J_{n} x_{n}\right)-\left(x_{m}-J_{m} x_{m}\right)\right\rangle
$$

and

$$
\left|\left\langle B_{n} x_{n}-B_{m} x_{m},\left(x_{n}-J_{n} x_{n}\right)-\left(x_{m}-J_{m} x_{m}\right)\right\rangle\right| \leq 2 C\left(\sqrt{\frac{1}{n}}+\sqrt{\frac{1}{m}}\right)
$$

Then from 25.8.98,

$$
0 \geq\left\langle z_{n}^{*}-z_{m}^{*}, x_{n}-x_{m}\right\rangle+e_{n, m}
$$

where $e_{n, m} \rightarrow 0$ as $n, m \rightarrow \infty$. Hence

$$
\lim \sup _{m, n \rightarrow \infty}\left\langle z_{n}^{*}-z_{m}^{*}, x_{n}-x_{m}\right\rangle \leq 0
$$

From Lemma 25.8.11,

$$
\limsup _{n \rightarrow \infty}\left\langle z_{n}^{*}, x_{n}-x\right\rangle \leq 0
$$

Hence, since $A$ is pseudomonotone, for every $y$, there exists $z^{*}(y) \in A x$ such that

$$
\lim \inf _{n \rightarrow \infty}\left\langle z_{n}^{*}, x_{n}-y\right\rangle \geq\left\langle z^{*}(y), x-y\right\rangle
$$

In particular, if $x=y$, this shows that

$$
\lim \inf _{n \rightarrow \infty}\left\langle z_{n}^{*}, x_{n}-x\right\rangle \geq 0 \geq \lim _{n \rightarrow \infty}\left\langle z_{n}^{*}, x_{n}-x\right\rangle
$$

showing that

$$
\lim _{n \rightarrow \infty}\left\langle z_{n}^{*}, x_{n}\right\rangle=\left\langle z^{*}, x\right\rangle
$$

Next, returning to the inclusion solved,

$$
0=L x_{n}+z_{n}^{*}+B_{n} x_{n}
$$

Act on $\left(x_{n}-x\right)$. Then from monotonicity of $L$,

$$
0 \geq\left\langle L x, x_{n}-x\right\rangle+\left\langle z_{n}^{*}, x_{n}-x\right\rangle+\left\langle B_{n} x_{n}, x_{n}-x\right\rangle
$$

Thus, taking limsup of both sides,

$$
\lim \sup _{n \rightarrow \infty}\left\langle B_{n} x_{n}, x_{n}-x\right\rangle=\lim \sup _{n \rightarrow \infty}\left\langle B_{n} x_{n}, J_{n} x_{n}-x\right\rangle \leq 0
$$

Hence

$$
\lim \sup _{n \rightarrow \infty}\left\langle B_{n} x_{n}, J_{n} x_{n}\right\rangle \leq\left\langle g^{*}, x\right\rangle
$$

Letting $\left[a, b^{*}\right] \in \mathscr{G}(B)$,

$$
\left\langle B_{n} x_{n}-b^{*}, J_{n} x_{n}-a\right\rangle=\left\langle B_{n} x_{n}, J_{n} x_{n}\right\rangle-\left\langle B_{n} x_{n}, a\right\rangle-\left\langle b^{*}, J_{n} x_{n}\right\rangle+\left\langle b^{*}, a\right\rangle
$$

Then taking limsup,

$$
\begin{aligned}
0 & \leq \lim \sup _{n \rightarrow \infty}\left\langle B_{n} x_{n}-b^{*}, J_{n} x_{n}-a\right\rangle \\
& \leq\left\langle g^{*}, x\right\rangle-\left\langle g^{*}, a\right\rangle-\left\langle b^{*}, x\right\rangle+\left\langle b^{*}, a\right\rangle=\left\langle g^{*}-b^{*}, x-a\right\rangle
\end{aligned}
$$

It follows that $g^{*} \in B(x)$ and $x \in D(B)$.
Thus, passing to the limit in the equation 25.8 .95 where, as explained $\lambda=1 / n$, one obtains

$$
y^{*}=L u+z^{*}+g^{*}
$$

where $z^{*} \in A x$ and $g^{*} \in B x$. This proves the following nice generalization of the above perturbation theorem.

Theorem 25.8.15 Let $B$ be maximal monotone from $X$ to $\mathscr{P}\left(X^{\prime}\right), 0 \in D(B)$, and $B$ is quasi-bounded as explained above. Let $A: X \rightarrow \mathscr{P}\left(X^{\prime}\right)$ be pseudomonotone, bounded, and coercive. Also let $L$ be a densely defined linear operator such that both $L$ and $L^{*}$ are monotone. (That is, $L$ is linear and maximal monotone.) Then $L+A+B$ is onto $X^{\prime}$.

## Chapter 26

## Integrals And Derivatives

### 26.1 The Fundamental Theorem Of Calculus

The version of the fundamental theorem of calculus found in Calculus has already been referred to frequently. It says that if $f$ is a Riemann integrable function, the function

$$
x \rightarrow \int_{a}^{x} f(t) d t
$$

has a derivative at every point where $f$ is continuous. It is natural to ask what occurs for $f$ in $L^{1}$. It is an amazing fact that the same result is obtained aside from a set of measure zero even though $f$, being only in $L^{1}$ may fail to be continuous anywhere. Proofs of this result are based on some form of the Vitali covering theorem presented above. In what follows, the measure space is $\left(\mathbb{R}^{n}, \mathscr{S}, m\right)$ where $m$ is $n$-dimensional Lebesgue measure although the same theorems can be proved for arbitrary Radon measures [84]. To save notation, $m$ is written in place of $m_{n}$.

By Lemma 12.1.9 on Page 278 and the completeness of $m$, the Lebesgue measurable sets are exactly those measurable in the sense of Caratheodory. Also, to save on notation $m$ is also the name of the outer measure defined on all of $\mathscr{P}\left(\mathbb{R}^{n}\right)$ which is determined by $m_{n}$. Recall

$$
\begin{equation*}
B(\mathbf{p}, r)=\{\mathbf{x}:|\mathbf{x}-\mathbf{p}|<r\} . \tag{26.1.1}
\end{equation*}
$$

Also define the following.

$$
\begin{equation*}
\text { If } B=B(\mathbf{p}, r), \text { then } \widehat{B}=B(\mathbf{p}, 5 r) \tag{26.1.2}
\end{equation*}
$$

The first version of the Vitali covering theorem presented above will now be used to establish the fundamental theorem of calculus. The space of locally integrable functions is the most general one for which the maximal function defined below makes sense.

Definition 26.1.1 $f \in L_{l o c}^{1}\left(\mathbb{R}^{n}\right)$ means $f \mathscr{X}_{B(0, R)} \in L^{1}\left(\mathbb{R}^{n}\right)$ for all $R>0$. For $f \in L_{\text {loc }}^{1}\left(\mathbb{R}^{n}\right)$, the Hardy Littlewood Maximal Function, $M f$, is defined by

$$
M f(\mathbf{x}) \equiv \sup _{r>0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})| d y .
$$

Theorem 26.1.2 If $f \in L^{1}\left(\mathbb{R}^{n}\right)$, then for $\alpha>0$,

$$
\bar{m}([M f>\alpha]) \leq \frac{5^{n}}{\alpha}\|f\|_{1}
$$

(Here and elsewhere, $[M f>\alpha] \equiv\left\{\mathbf{x} \in \mathbb{R}^{n}: M f(\mathbf{x})>\alpha\right\}$ with other occurrences of [] being defined similarly.)

Proof: Let $S \equiv[M f>\alpha]$. For $\mathbf{x} \in S$, choose $r_{\mathbf{x}}>0$ with

$$
\frac{1}{m\left(B\left(\mathbf{x}, r_{\mathbf{x}}\right)\right)} \int_{B\left(\mathbf{x}, r_{\mathbf{x}}\right)}|f| d m>\alpha
$$

The $r_{\mathbf{x}}$ are all bounded because

$$
m\left(B\left(\mathbf{x}, r_{\mathbf{x}}\right)\right)<\frac{1}{\alpha} \int_{B\left(\mathbf{x}, r_{\mathbf{x}}\right)}|f| d m<\frac{1}{\alpha}\|f\|_{1}
$$

By the Vitali covering theorem, there are disjoint balls $B\left(\mathbf{x}_{i}, r_{i}\right)$ such that

$$
S \subseteq \cup_{i=1}^{\infty} B\left(\mathbf{x}_{i}, 5 r_{i}\right)
$$

and

$$
\frac{1}{m\left(B\left(\mathbf{x}_{i}, r_{i}\right)\right)} \int_{B\left(\mathbf{x}_{i}, r_{i}\right)}|f| d m>\alpha
$$

Therefore

$$
\begin{aligned}
\bar{m}(S) & \leq \sum_{i=1}^{\infty} m\left(B\left(\mathbf{x}_{i}, 5 r_{i}\right)\right)=5^{n} \sum_{i=1}^{\infty} m\left(B\left(\mathbf{x}_{i}, r_{i}\right)\right) \\
& \leq \frac{5^{n}}{\alpha} \sum_{i=1}^{\infty} \int_{B\left(\mathbf{x}_{i}, r_{i}\right)}|f| d m \\
& \leq \frac{5^{n}}{\alpha} \int_{\mathbb{R}^{n}}|f| d m
\end{aligned}
$$

the last inequality being valid because the balls $B\left(\mathbf{x}_{i}, r_{i}\right)$ are disjoint. This proves the theorem.

Note that at this point it is unknown whether $S$ is measurable. This is why $\bar{m}(S)$ and not $m(S)$ is written.

The following is the fundamental theorem of calculus from elementary calculus.
Lemma 26.1.3 Suppose $g$ is a continuous function. Then for all $\mathbf{x}$,

$$
\lim _{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} g(\mathbf{y}) d y=g(\mathbf{x})
$$

Proof: Note that

$$
g(\mathbf{x})=\frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} g(\mathbf{x}) d y
$$

and so

$$
\begin{aligned}
& \left|g(\mathbf{x})-\frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} g(\mathbf{y}) d y\right| \\
= & \left|\frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}(g(\mathbf{y})-g(\mathbf{x})) d y\right| \\
\leq & \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|g(\mathbf{y})-g(\mathbf{x})| d y .
\end{aligned}
$$

Now by continuity of $g$ at $\mathbf{x}$, there exists $r>0$ such that if $|\mathbf{x}-\mathbf{y}|<r,|g(\mathbf{y})-g(\mathbf{x})|<\varepsilon$. For such $r$, the last expression is less than

$$
\frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} \varepsilon d y<\varepsilon
$$

This proves the lemma.

Definition 26.1.4 Let $f \in L^{1}\left(\mathbb{R}^{k}, m\right)$. A point, $\mathbf{x} \in \mathbb{R}^{k}$ is said to be a Lebesgue point if

$$
\limsup \sup _{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d m=0
$$

Note that if $\mathbf{x}$ is a Lebesgue point, then

$$
\lim _{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} f(\mathbf{y}) d m=f(\mathbf{x})
$$

and so the symmetric derivative exists at all Lebesgue points.
Theorem 26.1.5 (Fundamental Theorem of Calculus) Let $f \in L^{1}\left(\mathbb{R}^{k}\right)$. Then there exists a set of measure $0, N$, such that if $\mathbf{x} \notin N$, then

$$
\lim _{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d y=0
$$

Proof: Let $\lambda>0$ and let $\varepsilon>0$. By density of $C_{c}\left(\mathbb{R}^{k}\right)$ in $L^{1}\left(\mathbb{R}^{k}, m\right)$ there exists $g \in$ $C_{c}\left(\mathbb{R}^{k}\right)$ such that $\|g-f\|_{L^{1}\left(\mathbb{R}^{k}\right)}<\varepsilon$. Now since $g$ is continuous,

$$
\begin{aligned}
& \lim \sup _{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d m \\
= & \lim \sup _{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d m \\
& -\lim _{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|g(\mathbf{y})-g(\mathbf{x})| d m
\end{aligned}
$$

$$
=\lim \sup _{r \rightarrow 0}\left(\frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})|-|g(\mathbf{y})-g(\mathbf{x})| d m\right)
$$

$$
\leq \quad \lim \sup _{r \rightarrow 0}\left(\frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}\|f(\mathbf{y})-f(\mathbf{x})|-| g(\mathbf{y})-g(\mathbf{x})\| d m\right)
$$

$$
\leq \quad \lim \sup _{r \rightarrow 0}\left(\frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-g(\mathbf{y})-(f(\mathbf{x})-g(\mathbf{x}))| d m\right)
$$

$$
\leq \quad \lim \sup _{r \rightarrow 0}\left(\frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-g(\mathbf{y})| d m\right)+|f(\mathbf{x})-g(\mathbf{x})|
$$

$$
\leq M([f-g])(\mathbf{x})+|f(\mathbf{x})-g(\mathbf{x})|
$$

Therefore,

$$
\begin{aligned}
& {\left[\mathbf{x}: \lim _{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d m>\lambda\right] } \\
\subseteq & {\left[M([f-g])>\frac{\lambda}{2}\right] \cup\left[|f-g|>\frac{\lambda}{2}\right] }
\end{aligned}
$$

Now

$$
\begin{aligned}
\varepsilon & >\int|f-g| d m \geq \int_{\left[|f-g|>\frac{\lambda}{2}\right]}|f-g| d m \\
& \geq \frac{\lambda}{2} m\left(\left[|f-g|>\frac{\lambda}{2}\right]\right)
\end{aligned}
$$

This along with the weak estimate of Theorem 26.1.2 implies

$$
\begin{aligned}
& m\left(\left[\mathbf{x}: \lim _{r \rightarrow 0} \sup _{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d m>\lambda\right]\right) \\
< & \left(\frac{2}{\lambda} 5^{k}+\frac{2}{\lambda}\right)\|f-g\|_{L^{1}\left(\mathbb{R}^{k}\right)} \\
< & \left(\frac{2}{\lambda} 5^{k}+\frac{2}{\lambda}\right) \varepsilon .
\end{aligned}
$$

Since $\varepsilon>0$ is arbitrary, it follows

$$
m_{n}\left(\left[\mathbf{x}: \lim _{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d m>\lambda\right]\right)=0
$$

Now let

$$
N=\left[\mathbf{x}: \limsup _{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d m>0\right]
$$

and

$$
N_{n}=\left[\mathbf{x}: \limsup _{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d m>\frac{1}{n}\right]
$$

It was just shown that $m\left(N_{n}\right)=0$. Also, $N=\cup_{n=1}^{\infty} N_{n}$. Therefore, $m(N)=0$ also. It follows that for $\mathbf{x} \notin N$,

$$
\limsup \sup _{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d m=0
$$

and this proves a.e. point is a Lebesgue point.
Of course it is sufficient to assume $f$ is only in $L_{l o c}^{1}\left(\mathbb{R}^{k}\right)$.
Corollary 26.1.6 (Fundamental Theorem of Calculus) Let $f \in L_{l o c}^{1}\left(\mathbb{R}^{k}\right)$. Then there exists a set of measure $0, N$, such that if $\mathbf{x} \notin N$, then

$$
\lim _{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d y=0
$$

Proof: Consider $B(\mathbf{0}, n)$ where $n$ is a positive integer. Then $f_{n} \equiv f \mathscr{X}_{B(\mathbf{0}, n)} \in L^{1}\left(\mathbb{R}^{k}\right)$ and so there exists a set of measure $0, N_{n}$ such that if $\mathbf{x} \in B(\mathbf{0}, n) \backslash N_{n}$, then

$$
\begin{aligned}
& \lim _{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}\left|f_{n}(\mathbf{y})-f_{n}(\mathbf{x})\right| d y \\
= & \lim _{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d y=0 .
\end{aligned}
$$

Let $N=\cup_{n=1}^{\infty} N_{n}$. Then if $\mathbf{x} \notin N$, the above equation holds.

Corollary 26.1.7 If $f \in L_{l o c}^{1}\left(\mathbb{R}^{n}\right)$, then

$$
\begin{equation*}
\lim _{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} f(\mathbf{y}) d y=f(\mathbf{x}) \text { a.e. } \mathbf{x} \text {. } \tag{26.1.3}
\end{equation*}
$$

Proof:

$$
\begin{aligned}
& \left|\frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} f(\mathbf{y}) d y-f(\mathbf{x})\right| \\
\leq & \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d y
\end{aligned}
$$

and the last integral converges to 0 a.e. $\mathbf{x}$.
Definition 26.1.8 For $N$ the set of Theorem 26.1.5 or Corollary 26.1.6, $N^{C}$ is called the Lebesgue set or the set of Lebesgue points.

The next corollary is a one dimensional version of what was just presented.
Corollary 26.1.9 Let $f \in L^{1}(\mathbb{R})$ and let

$$
F(x)=\int_{-\infty}^{x} f(t) d t
$$

Then for a.e. $x, F^{\prime}(x)=f(x)$.
Proof: For $h>0$

$$
\frac{1}{h} \int_{x}^{x+h}|f(y)-f(x)| d y \leq 2\left(\frac{1}{2 h}\right) \int_{x-h}^{x+h}|f(y)-f(x)| d y
$$

By Theorem 26.1.5, this converges to 0 a.e. Similarly

$$
\frac{1}{h} \int_{x-h}^{x}|f(y)-f(x)| d y
$$

converges to 0 a.e. $x$.

$$
\begin{equation*}
\left|\frac{F(x+h)-F(x)}{h}-f(x)\right| \leq \frac{1}{h} \int_{x}^{x+h}|f(y)-f(x)| d y \tag{26.1.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|\frac{F(x)-F(x-h)}{h}-f(x)\right| \leq \frac{1}{h} \int_{x-h}^{x}|f(y)-f(x)| d y . \tag{26.1.5}
\end{equation*}
$$

Now the expression on the right in 26.1.4 and 26.1.5 converges to zero for a.e. $x$. Therefore, by 26.1.4, for a.e. $x$ the derivative from the right exists and equals $f(x)$ while from 26.1.5 the derivative from the left exists and equals $f(x)$ a.e. It follows

$$
\lim _{h \rightarrow 0} \frac{F(x+h)-F(x)}{h}=f(x) \text { a.e. } x
$$

This proves the corollary.

### 26.2 Absolutely Continuous Functions

Definition 26.2.1 Let $[a, b]$ be a closed and bounded interval and let $F:[a, b] \rightarrow \mathbb{R}$. Then $F$ is said to be absolutely continuous if for every $\varepsilon>0$ there exists $\delta>0$ such that if $\sum_{i=1}^{m}\left|y_{i}-x_{i}\right|<\delta$ where the intervals $\left(x_{i}, y_{i}\right)$ are non-overlapping, then

$$
\sum_{i=1}^{m}\left|F\left(y_{i}\right)-F\left(x_{i}\right)\right|<\varepsilon
$$

Definition 26.2.2 A finite subset, $P$ of $[a, b]$ is called a partition of $[x, y] \subseteq[a, b]$ if $P=$ $\left\{x_{0}, x_{1}, \cdots, x_{n}\right\}$ where

$$
x=x_{0}<x_{1}<\cdots,<x_{n}=y .
$$

For $f:[a, b] \rightarrow \mathbb{R}$ and $P=\left\{x_{0}, x_{1}, \cdots, x_{n}\right\}$ define

$$
V_{P}[x, y] \equiv \sum_{i=1}^{n}\left|f\left(x_{i}\right)-f\left(x_{i-1}\right)\right| .
$$

Denoting by $\mathscr{P}[x, y]$ the set of all partitions of $[x, y]$ define

$$
V[x, y] \equiv \sup _{P \in \mathscr{P}[x, y]} V_{P}[x, y] .
$$

For simplicity, $V[a, x]$ will be denoted by $V(x)$. It is called the total variation of the function, $f$.

There are some simple facts about the total variation of an absolutely continuous function, $f$ which are contained in the next lemma.

Lemma 26.2.3 Let $f$ be an absolutely continuous function defined on $[a, b]$ and let $V$ be its total variation function as described above. Then $V$ is an increasing bounded function. Also if $P$ and $Q$ are two partitions of $[x, y]$ with $P \subseteq Q$, then $V_{P}[x, y] \leq V_{Q}[x, y]$ and if $[x, y] \subseteq[z, w]$,

$$
\begin{equation*}
V[x, y] \leq V[z, w] \tag{26.2.6}
\end{equation*}
$$

If $P=\left\{x_{0}, x_{1}, \cdots, x_{n}\right\}$ is a partition of $[x, y]$, then

$$
\begin{equation*}
V[x, y]=\sum_{i=1}^{n} V\left[x_{i}, x_{i-1}\right] \tag{26.2.7}
\end{equation*}
$$

Also if $y>x$,

$$
\begin{equation*}
V(y)-V(x) \geq|f(y)-f(x)| \tag{26.2.8}
\end{equation*}
$$

and the function, $x \rightarrow V(x)-f(x)$ is increasing. The total variation function, $V$ is absolutely continuous.

Proof: The claim that $V$ is increasing is obvious as is the next claim about $P \subseteq Q$ leading to $V_{P}[x, y] \leq V_{Q}[x, y]$. To verify this, simply add in one point at a time and verify that from the triangle inequality, the sum involved gets no smaller. The claim that $V$ is
increasing consistent with set inclusion of intervals is also clearly true and follows directly from the definition.

Now let $t<V[x, y]$ where $P_{0}=\left\{x_{0}, x_{1}, \cdots, x_{n}\right\}$ is a partition of $[x, y]$. There exists a partition, $P$ of $[x, y]$ such that $t<V_{P}[x, y]$. Without loss of generality it can be assumed that $\left\{x_{0}, x_{1}, \cdots, x_{n}\right\} \subseteq P$ since if not, you can simply add in the points of $P_{0}$ and the resulting sum for the total variation will get no smaller. Let $P_{i}$ be those points of $P$ which are contained in $\left[x_{i-1}, x_{i}\right]$. Then

$$
t<V_{p}[x, y]=\sum_{i=1}^{n} V_{P_{i}}\left[x_{i-1}, x_{i}\right] \leq \sum_{i=1}^{n} V\left[x_{i-1}, x_{i}\right] .
$$

Since $t<V[x, y]$ is arbitrary,

$$
\begin{equation*}
V[x, y] \leq \sum_{i=1}^{n} V\left[x_{i}, x_{i-1}\right] \tag{26.2.9}
\end{equation*}
$$

Note that 26.2.9 does not depend on $f$ being absolutely continuous. Suppose now that $f$ is absolutely continuous. Let $\delta$ correspond to $\varepsilon=1$. Then if $[x, y]$ is an interval of length no larger than $\delta$, the definition of absolute continuity implies

$$
V[x, y]<1
$$

Then from 26.2.9

$$
V[a, n \delta] \leq \sum_{i=1}^{n} V[a+(i-1) \delta, a+i \delta]<\sum_{i=1}^{n} 1=n
$$

Thus $V$ is bounded on $[a, b]$. Now let $P_{i}$ be a partition of $\left[x_{i-1}, x_{i}\right]$ such that

$$
V_{P_{i}}\left[x_{i-1}, x_{i}\right]>V\left[x_{i-1}, x_{i}\right]-\frac{\varepsilon}{n}
$$

Then letting $P=\cup P_{i}$,

$$
-\varepsilon+\sum_{i=1}^{n} V\left[x_{i-1}, x_{i}\right]<\sum_{i=1}^{n} V_{P_{i}}\left[x_{i-1}, x_{i}\right]=V_{P}[x, y] \leq V[x, y] .
$$

Since $\varepsilon$ is arbitrary, 26.2.7 follows from this and 26.2.9.
Now let $x<y$

$$
\begin{aligned}
V(y)-f(y)-(V(x)-f(x)) & =V(y)-V(x)-(f(y)-f(x)) \\
& \geq V(y)-V(x)-|f(y)-f(x)| \geq 0
\end{aligned}
$$

It only remains to verify that $V$ is absolutely continuous.
Let $\varepsilon>0$ be given and let $\delta$ correspond to $\varepsilon / 2$ in the definition of absolute continuity applied to $f$. Suppose $\sum_{i=1}^{n}\left|y_{i}-x_{i}\right|<\delta$ and consider $\sum_{i=1}^{n}\left|V\left(y_{i}\right)-V\left(x_{i}\right)\right|$. By 26.2.9 this last is no larger than $\sum_{i=1}^{n} V\left[x_{i}, y_{i}\right]$. Now let $P_{i}$ be a partition of $\left[x_{i}, y_{i}\right]$ such that $V_{P_{i}}\left[x_{i}, y_{i}\right]+$ $\frac{\varepsilon}{2 n}>V\left[x_{i}, y_{i}\right]$. Then by the definition of absolute continuity,

$$
\begin{aligned}
\sum_{i=1}^{n}\left|V\left(y_{i}\right)-V\left(x_{i}\right)\right| & =\sum_{i=1}^{n} V\left[x_{i}, y_{i}\right] \\
& \leq \sum_{i=1}^{n} V_{P_{i}}\left[x_{i}, y_{i}\right]+\eta<\varepsilon / 2+\varepsilon / 2=\varepsilon
\end{aligned}
$$

and shows $V$ is absolutely continuous as claimed.
Lemma 26.2.4 Suppose $f:[a, b] \rightarrow \mathbb{R}$ is absolutely continuous and increasing. Then $f^{\prime}$ exists a.e., is in $L^{1}([a, b])$, and

$$
f(x)=f(a)+\int_{a}^{x} f^{\prime}(t) d t
$$

Proof: Define $L$, a positive linear functional on $C([a, b])$ by

$$
L g \equiv \int_{a}^{b} g d f
$$

where this integral is the Riemann Stieltjes integral with respect to the integrating function, $f$. By the Riesz representation theorem for positive linear functionals, there exists a unique Radon measure, $\mu$ such that $L g=\int g d \mu$. Now consider the following picture for $g_{n} \in$ $C([a, b])$ in which $g_{n}$ equals 1 for $x$ between $x+1 / n$ and $y$.


Then $g_{n}(t) \rightarrow \mathscr{X}_{(x, y]}(t)$ pointwise. Therefore, by the dominated convergence theorem,

$$
\mu((x, y])=\lim _{n \rightarrow \infty} \int g_{n} d \mu
$$

However,

$$
\begin{aligned}
& \left(f(y)-f\left(x+\frac{1}{n}\right)\right) \\
\leq & \int g_{n} d \mu=\int_{a}^{b} g_{n} d f \leq\left(f\left(y+\frac{1}{n}\right)-f(y)\right) \\
& +\left(f(y)-f\left(x+\frac{1}{n}\right)\right)+\left(f\left(x+\frac{1}{n}\right)-f(x)\right)
\end{aligned}
$$

and so as $n \rightarrow \infty$ the continuity of $f$ implies

$$
\mu((x, y])=f(y)-f(x) .
$$

Similarly, $\mu(x, y)=f(y)-f(y)$ and $\mu([x, y])=f(y)-f(x)$, the argument used to establish this being very similar to the above. It follows in particular that

$$
f(x)-f(a)=\int_{[a, x]} d \mu
$$

Note that up till now, no referrence has been made to the absolute continuity of $f$. Any increasing continuous function would be fine.

Now if $E$ is a Borel set such that $m(E)=0$, Then the outer regularity of $m$ implies there exists an open set, $V$ containing $E$ such that $m(V)<\delta$ where $\delta$ corresponds to $\varepsilon$ in the definition of absolute continuity of $f$. Then letting $\left\{I_{k}\right\}$ be the connected components of $V$ it follows $E \subseteq \cup_{k=1}^{\infty} I_{k}$ with $\sum_{k} m\left(I_{k}\right)=m(V)<\delta$. Therefore, from absolute continuity of $f$, it follows that for $I_{k}=\left(a_{k}, b_{k}\right)$ and each $n$

$$
\mu\left(\cup_{k=1}^{n} I_{k}\right)=\sum_{k=1}^{n} \mu\left(I_{k}\right)=\sum_{k=1}^{n}\left|f\left(b_{k}\right)-f\left(a_{k}\right)\right|<\varepsilon
$$

and so letting $n \rightarrow \infty$,

$$
\mu(E) \leq \mu(V)=\sum_{k=1}^{\infty}\left|f\left(b_{k}\right)-f\left(a_{k}\right)\right| \leq \varepsilon
$$

Since $\varepsilon$ is arbitrary, it follows $\mu(E)=0$. Therefore, $\mu \ll m$ and so by the Radon Nikodym theorem there exists a unique $h \in L^{1}([a, b])$ such that

$$
\mu(E)=\int_{E} h d m
$$

In particular,

$$
\mu([a, x])=f(x)-f(a)=\int_{[a, x]} h d m
$$

From the fundamental theorem of calculus $f^{\prime}(x)=h(x)$ at every Lebesgue point of $h$. Therefore, writing in usual notation,

$$
f(x)=f(a)+\int_{a}^{x} f^{\prime}(t) d t
$$

as claimed. This proves the lemma.
With the above lemmas, the following is the main theorem about absolutely continuous functions.

Theorem 26.2.5 Let $f:[a, b] \rightarrow \mathbb{R}$ be absolutely continuous if and only if $f^{\prime}(x)$ exists a.e., $f^{\prime} \in L^{1}([a, b])$ and

$$
f(x)=f(a)+\int_{a}^{x} f^{\prime}(t) d t
$$

Proof: Suppose first that $f$ is absolutely continuous. By Lemma 26.2.3 the total variation function, $V$ is absolutely continuous and $f(x)=V(x)-(V(x)-f(x))$ where both $V$ and $V-f$ are increasing and absolutely continuous. By Lemma 26.2.4

$$
\begin{aligned}
f(x)-f(a) & =V(x)-V(a)-[(V(x)-f(x))-(V(a)-f(a))] \\
& =\int_{a}^{x} V^{\prime}(t) d t-\int_{a}^{x}(V-f)^{\prime}(t) d t
\end{aligned}
$$

Now $f^{\prime}$ exists and is in $L^{1}$ becasue $f=V-(V-f)$ and $V$ and $V-f$ have derivatives in $L^{1}$. Therefore, $(V-f)^{\prime}=V^{\prime}-f^{\prime}$ and so the above reduces to

$$
f(x)-f(a)=\int_{a}^{x} f^{\prime}(t) d t
$$

This proves one half of the theorem.
Now suppose $f^{\prime} \in L^{1}$ and $f(x)=f(a)+\int_{a}^{x} f^{\prime}(t) d t$. It is necessary to verify that $f$ is absolutely continuous. But this follows easily from Lemma 11.5.2 on Page 256 which implies that a single function, $f^{\prime}$ is uniformly integrable. This lemma implies that if $\sum_{i}\left|y_{i}-x_{i}\right|$ is sufficiently small then

$$
\sum_{i}\left|\int_{x_{i}}^{y_{i}} f^{\prime}(t) d t\right|=\sum_{i}\left|f\left(y_{i}\right)-f\left(x_{i}\right)\right|<\varepsilon .
$$

The following simple corollary is a case of Rademacher's theorem.
Corollary 26.2.6 Suppose $f:[a, b] \rightarrow \mathbb{R}$ is Lipschitz continuous,

$$
|f(x)-f(y)| \leq K|x-y|
$$

Then $f^{\prime}(x)$ exists a.e. and

$$
f(x)=f(a)+\int_{a}^{x} f^{\prime}(t) d t
$$

Proof: It is easy to see that $f$ is absolutely continuous. Therefore, Theorem 26.2.5 applies.

### 26.3 Weak Derivatives

A related concept is that of weak derivatives. Let $\Omega \subseteq \mathbb{R}^{n}$. A distribution on $\Omega$ is defined to be a linear functional on $C_{c}^{\infty}(\Omega)$, called the space of test functions. The space of all such linear functionals will be denoted by $\mathscr{D}^{*}(\Omega)$. Actually, more is sometimes done here. One imposes a topology on $C_{c}^{\infty}(\Omega)$ making it into a topological vector space, and when this has been done, $\mathscr{D}^{\prime}(\Omega)$ is defined as the dual space of this topological vector space. To see this, consult the book by Yosida [127] or the book by Rudin [114].

Example: The space $L_{l o c}^{1}(\Omega)$ may be considered as a subset of $\mathscr{D}^{*}(\Omega)$ as follows.

$$
f(\phi) \equiv \int_{\Omega} f(\mathbf{x}) \phi(\mathbf{x}) d x
$$

for all $\phi \in C_{c}^{\infty}(\Omega)$. Recall that $f \in L_{l o c}^{1}(\Omega)$ if $f \mathscr{X}_{K} \in L^{1}(\Omega)$ whenever $K$ is compact.
The following lemma is the main result which makes this identification possible.
Lemma 26.3.1 Suppose $f \in L_{l o c}^{1}\left(\mathbb{R}^{n}\right)$ and suppose

$$
\int f \phi d x=0
$$

for all $\phi \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$. Then $f(\mathbf{x})=0$ a.e. $\mathbf{x}$.

Proof: Without loss of generality $f$ is real-valued. Let

$$
E \equiv\{\mathbf{x}: f(\mathbf{x})>\varepsilon\}
$$

and let

$$
E_{m} \equiv E \cap B(0, m)
$$

We show that $m\left(E_{m}\right)=0$. If not, there exists an open set, $V$, and a compact set $K$ satisfying

$$
\begin{gathered}
K \subseteq E_{m} \subseteq V \subseteq B(0, m), m(V \backslash K)<4^{-1} m\left(E_{m}\right), \\
\int_{V \backslash K}|f| d x<\varepsilon 4^{-1} m\left(E_{m}\right) .
\end{gathered}
$$

Let $H$ and $W$ be open sets satisfying

$$
K \subseteq H \subseteq \bar{H} \subseteq W \subseteq \bar{W} \subseteq V
$$

and let

$$
\bar{H} \prec g \prec W
$$

where the symbol, $\prec$, has the same meaning as it does in Chapter 12. Then let $\phi_{\delta}$ be a mollifier and let $h \equiv g * \phi_{\delta}$ for $\delta$ small enough that

$$
K \prec h \prec V .
$$

Thus

$$
\begin{aligned}
0 & =\int f h d x=\int_{K} f d x+\int_{V \backslash K} f h d x \\
& \geq \varepsilon m(K)-\varepsilon 4^{-1} m\left(E_{m}\right) \\
& \geq \varepsilon\left(m\left(E_{m}\right)-4^{-1} m\left(E_{m}\right)\right)-\varepsilon 4^{-1} m\left(E_{m}\right) \\
& \geq 2^{-1} \varepsilon m\left(E_{m}\right)
\end{aligned}
$$

Therefore, $m\left(E_{m}\right)=0$, a contradiction. Thus

$$
m(E) \leq \sum_{m=1}^{\infty} m\left(E_{m}\right)=0
$$

and so, since $\varepsilon>0$ is arbitrary,

$$
m(\{\mathbf{x}: f(\mathbf{x})>0\})=0 .
$$

Similarly $m(\{\mathbf{x}: f(\mathbf{x})<0\})=0$. This proves the lemma.
Example: $\delta_{x} \in \mathscr{D}^{*}(\Omega)$ where $\delta_{\mathbf{x}}(\phi) \equiv \phi(\mathbf{x})$.
It will be observed from the above two examples and a little thought that $\mathscr{D}^{*}(\Omega)$ is truly enormous. We shall define the derivative of a distribution in such a way that it agrees with the usual notion of a derivative on those distributions which are also continuously differentiable functions. With this in mind, let $f$ be the restriction to $\Omega$ of a smooth function defined on $\mathbb{R}^{n}$. Then $D_{x_{i}} f$ makes sense and for $\phi \in C_{c}^{\infty}(\Omega)$

$$
D_{x_{i}} f(\phi) \equiv \int_{\Omega} D_{x_{i}} f(\mathbf{x}) \phi(\mathbf{x}) d x=-\int_{\Omega} f D_{x_{i}} \phi d x=-f\left(D_{x_{i}} \phi\right)
$$

Motivated by this, here is the definition of a weak derivative.

Definition 26.3.2 For $T \in \mathscr{D}^{*}(\Omega)$

$$
D_{x_{i}} T(\phi) \equiv-T\left(D_{x_{i}} \phi\right)
$$

Of course one can continue taking derivatives indefinitely. Thus,

$$
D_{x_{i} x_{j}} T \equiv D_{x_{i}}\left(D_{x_{j}} T\right)
$$

and it is clear that all mixed partial derivatives are equal because this holds for the functions in $C_{c}^{\infty}(\Omega)$. Thus one can differentiate virtually anything, even functions that may be discontinuous everywhere. However the notion of "derivative" is very weak, hence the name, "weak derivatives".

Example: Let $\Omega=\mathbb{R}$ and let

$$
H(x) \equiv\left\{\begin{array}{l}
1 \text { if } x \geq 0 \\
0 \text { if } x<0
\end{array}\right.
$$

Then

$$
D H(\phi)=-\int H(x) \phi^{\prime}(x) d x=\phi(0)=\delta_{0}(\phi)
$$

Note that in this example, $D H$ is not a function.
What happens when $D f$ is a function?
Theorem 26.3.3 Let $\Omega=(a, b)$ and suppose that $f$ and $D f$ are both in $L^{1}(a, b)$. Then $f$ is equal to a continuous function a.e., still denoted by $f$ and

$$
f(x)=f(a)+\int_{a}^{x} D f(t) d t
$$

The proof of Theorem 26.3.3 depends on the following lemma.
Lemma 26.3.4 Let $T \in \mathscr{D}^{*}(a, b)$ and suppose $D T=0$. Then there exists a constant $C$ such that

$$
T(\phi)=\int_{a}^{b} C \phi d x
$$

Proof: $T(D \phi)=0$ for all $\phi \in C_{c}^{\infty}(a, b)$ from the definition of $D T=0$. Let

$$
\phi_{0} \in C_{c}^{\infty}(a, b), \int_{a}^{b} \phi_{0}(x) d x=1
$$

and let

$$
\psi_{\phi}(x)=\int_{a}^{x}\left[\phi(t)-\left(\int_{a}^{b} \phi(y) d y\right) \phi_{0}(t)\right] d t
$$

for $\phi \in C_{c}^{\infty}(a, b)$. Thus $\psi_{\phi} \in C_{c}^{\infty}(a, b)$ and

$$
D \psi_{\phi}=\phi-\left(\int_{a}^{b} \phi(y) d y\right) \phi_{0}
$$

Therefore,

$$
\phi=D \psi_{\phi}+\left(\int_{a}^{b} \phi(y) d y\right) \phi_{0}
$$

and so

$$
T(\phi)=T\left(D \psi_{\phi}\right)+\left(\int_{a}^{b} \phi(y) d y\right) T\left(\phi_{0}\right)=\int_{a}^{b} T\left(\phi_{0}\right) \phi(y) d y
$$

Let $C=T \phi_{0}$. This proves the lemma.
Proof of Theorem 36.2.2 Since $f$ and $D f$ are both in $L^{1}(a, b)$,

$$
D f(\phi)-\int_{a}^{b} D f(x) \phi(x) d x=0
$$

Consider

$$
f(\cdot)-\int_{a}^{(\cdot)} D f(t) d t
$$

and let $\phi \in C_{c}^{\infty}(a, b)$.

$$
\begin{gathered}
D\left(f(\cdot)-\int_{a}^{(\cdot)} D f(t) d t\right)(\phi) \\
\equiv-\int_{a}^{b} f(x) \phi^{\prime}(x) d x+\int_{a}^{b}\left(\int_{a}^{x} D f(t) d t\right) \phi^{\prime}(x) d x \\
=D f(\phi)+\int_{a}^{b} \int_{t}^{b} D f(t) \phi^{\prime}(x) d x d t \\
=D f(\phi)-\int_{a}^{b} D f(t) \phi(t) d t=0
\end{gathered}
$$

By Lemma 36.2.3, there exists a constant, $C$, such that

$$
\left(f(\cdot)-\int_{a}^{(\cdot)} D f(t) d t\right)(\phi)=\int_{a}^{b} C \phi(x) d x
$$

for all $\phi \in C_{c}^{\infty}(a, b)$. Thus

$$
\int_{a}^{b}\left\{\left(f(x)-\int_{a}^{x} D f(t) d t\right)-C\right\} \phi(x) d x=0
$$

for all $\phi \in C_{c}^{\infty}(a, b)$. It follows from Lemma 26.3.1 in the next section that

$$
f(x)-\int_{a}^{x} D f(t) d t-C=0 \text { a.e. } x
$$

Thus we let $f(a)=C$ and write

$$
f(x)=f(a)+\int_{a}^{x} D f(t) d t
$$

This proves Theorem 36.2.2.

Theorem 36.2.2 says that

$$
f(x)=f(a)+\int_{a}^{x} D f(t) d t
$$

whenever it makes sense to write $\int_{a}^{x} D f(t) d t$, if $D f$ is interpreted as a weak derivative. Somehow, this is the way it ought to be. It follows from the fundamental theorem of calculus that $f^{\prime}(x)$ exists for a.e. $x$ in the classical sense where the derivative is taken in the sense of a limit of difference quotients and $f^{\prime}(x)=D f(x)$. This raises an interesting question. Suppose $f$ is continuous on $[a, b]$ and $f^{\prime}(x)$ exists in the classical sense for a.e. $x$. Does it follow that

$$
f(x)=f(a)+\int_{a}^{x} f^{\prime}(t) d t ?
$$

The answer is no. You can build such an example from the Cantor function which is increasing and has a derivative a.e. which equals 0 a.e. and yet climbs from 0 to 1 . Thus this function is not recovered from integrating its classical derivative. Thus, in a sense weak derivatives are more agreeable than the classical ones.

### 26.4 Lipschitz Functions

Definition 26.4.1 A function $f:[a, b] \rightarrow \mathbb{R}$ is Lipschitz if there is a constant $K$ such that for all $x, y$,

$$
|f(x)-f(y)| \leq K|x-y|
$$

More generally, $f$ is Lipschitz on a subset of $\mathbb{R}^{n}$ iffor all $\mathbf{x}, \mathbf{y}$ in this set,

$$
|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})| \leq K|\mathbf{x}-\mathbf{y}|
$$

Lemma 26.4.2 Suppose $f:[a, b] \rightarrow \mathbb{R}$ is Lipschitz continuous and increasing. Then $f^{\prime}$ exists a.e., is in $L^{1}([a, b])$, and

$$
f(x)=f(a)+\int_{a}^{x} f^{\prime}(t) d t
$$

If $f: \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz, then it is in $L_{\text {loc }}^{1}(\mathbb{R})$.
Proof: The Dini derivates are defined as follows.

$$
\begin{aligned}
D^{+} f(x) & \equiv \lim \sup _{h \rightarrow 0+} \frac{f(x+h)-f(x)}{h}, D_{+} f(x) \equiv \lim \inf _{h \rightarrow 0+} \frac{f(x+h)-f(x)}{h} \\
D^{-} f(x) & \equiv \lim \sup _{h \rightarrow 0+} \frac{f(x)-f(x-h)}{h}, D_{-} f(x) \equiv \lim \inf _{h \rightarrow 0+} \frac{f(x)-f(x-h)}{h}
\end{aligned}
$$

For convenience, just let $f$ equal $f(a)$ for $x<a$ and equal $f(b)$ for $x>b$. Let $(a, b)$ be an open interval and let

$$
N_{a b} \equiv\left\{x \in(a, b): D^{+} f(x)>q>p>D_{+} f(x)\right\}
$$

Let $V \subseteq(a, b)$ be an open set containing $N_{p q}$ such that $m(V)<m\left(N_{p q}\right)+\varepsilon$. By assumption, if $x \in N_{p q}$, there exist arbitrarily small $h$ such that

$$
\frac{f(x+h)-f(x)}{h}<p
$$

These intervals $[x, x+h]$ are then a Vitali covering of $N_{p q}$. It follows from Corollary 13.4.6 that there is a disjoint union of countably many, $\left\{\left[x_{i}, x_{i}+h_{i}\right]\right\}_{i=1}^{\infty}$ which cover all of $N_{p q}$ except for a set of measure zero. Thus also the open intervals $\left\{\left(x_{i}, x_{i}+h_{i}\right)\right\}_{i=1}^{\infty}$ also cover all of $N_{p q}$ except for a set of measure zero. Now for points $x^{\prime}$ of $N_{p q}$ so covered, there are arbitrarily small $h$ such that

$$
\frac{f\left(x^{\prime}+h^{\prime}\right)-f\left(x^{\prime}\right)}{h^{\prime}}>q
$$

and $\left[x^{\prime}, x^{\prime}+h^{\prime}\right]$ is contained in one of these original open intervals $\left(x_{i}, x_{i}+h_{i}\right)$. By the Vitali covering theorem again, Corollary 13.4.6, it follows that there exists a countable disjoint sequence $\left\{\left[x_{j}^{\prime}, x_{j}^{\prime}+h_{j}^{\prime}\right]\right\}_{j=1}^{\infty}$ which covers all of $N_{p q}$ except for a set of measure zero, each of these $\left[x_{j}^{\prime}, x_{j}^{\prime}+h_{j}^{\prime}\right]$ being contained in some $\left(x_{i}, x_{i}+h_{i}\right)$. Then it follows that

$$
\begin{aligned}
q m\left(N_{p q}\right) & \leq q \sum_{j} h_{j}^{\prime} \leq \sum_{j} f\left(x_{j}^{\prime}+h_{j}^{\prime}\right)-f\left(x_{j}^{\prime}\right) \leq \sum_{i} f\left(x_{i}+h_{i}\right)-f\left(x_{i}\right) \\
& \leq p \sum_{i} h_{i} \leq p m(V) \leq p\left(m\left(N_{p q}\right)+\varepsilon\right)
\end{aligned}
$$

Since $\varepsilon>0$ is arbitrary, this shows that $q m\left(N_{p q}\right) \leq p m\left(N_{p q}\right)$ and so $m\left(N_{p q}\right)=0$. Now taking the union of all $N_{p q}$ for $p, q \in \mathbb{Q}$, it follows that for a.e. $x, D^{+} f(x)=D_{+} f(x)$ and so the derivative from the right exists. Similar reasoning shows that off a set of measure zero the derivative from the left also exists. You just do the same argument using $D^{-} f(x)$ and $D_{-} f(x)$ to obtain the existence of a derivative from the left. Next you can use the same argument to verify that $D^{-} f(x)=D_{+} f(x)$ off a set of measure zero. This is outlined next. Define a new $N_{p q}$,

$$
N_{p q} \equiv\left\{x \in(a, b): D_{+} f(x)>q>p>D^{-} f(x)\right\}
$$

Let $V$ be an open set containing $N_{p q}$ such that $m(V)<m\left(N_{p q}\right)+\varepsilon$. For each $x \in N_{p q}$ there are arbitrarily small $h$ such that

$$
\frac{f(x)-f(x-h)}{h}<p
$$

Then as before, there is a countable disjoint sequence of closed intervals contained in $V,\left\{\left[x_{i}-h_{i}, x_{i}\right]\right\}_{i=1}^{\infty}$ such that their union includes all of $N_{p q}$ except a set of measure zero. Thus this is also true of the open intervals $\left\{\left(x_{i}-h_{i}, x_{i}\right)\right\}_{i=1}^{\infty}$. Then for the points of $N_{p q}$ covered by these open intervals $x^{\prime}$, there are arbitrarily small $h^{\prime}$ such that

$$
\frac{f\left(x^{\prime}+h^{\prime}\right)-f\left(x^{\prime}\right)}{h^{\prime}}>q
$$

and each $\left[x^{\prime}, x^{\prime}+h^{\prime}\right]$ is contained in an interval $\left(x_{i}-h_{i}, x_{i}\right)$. Then by the Vitali covering theorem again, Corollary 13.4.6 there are countably many disjoint closed intervals $\left\{\left[x_{j}^{\prime}, x_{j}^{\prime}+h_{j}^{\prime}\right]\right\}_{j=1}^{\infty}$ whose union includes all of $N_{p q}$ except for a set of measure zero such that each of these is contained in some $\left(x_{i}-h_{i}, x_{i}\right)$ described earlier. Then as before,

$$
\begin{aligned}
q m\left(N_{p q}\right) & \leq q \sum_{j} h_{j}^{\prime} \leq \sum_{j} f\left(x_{j}^{\prime}+h_{j}^{\prime}\right)-f\left(x_{j}^{\prime}\right) \leq \sum_{i} f\left(x_{i}\right)-f\left(x_{i}-h_{i}\right) \\
& \leq p \sum_{i} h_{i} \leq p m(V) \leq p\left(m\left(N_{p q}\right)+\varepsilon\right)
\end{aligned}
$$

Then as before, this shows that $q m\left(N_{p q}\right) \leq p m\left(N_{p q}\right)$ and so $m\left(N_{p q}\right)=0$. Then taking the union of all such for $p, q \in \mathbb{Q}$ yields $D_{+} f(x)=D^{-} f(x)$ for a.e. $x$. Taking the union of all these sets of measure zero and considering points not in this union, it follows that $f^{\prime}(x)$ exists for a.e. $x$. Thus $f^{\prime}(t) \geq 0$ and is a limit of measurable even continuous functions for a.e. $x$ so $f^{\prime}$ is clearly measurable. The issue is whether $f(y)-f(x)=\int \mathscr{X}_{[x, y]}(t) f^{\prime}(t) d m$. Up to now, the only thing used has been that $f$ is increasing.

Let $h>0$.

$$
\begin{aligned}
\int_{a}^{x} \frac{f(t)-f(t-h)}{h} d t & =\frac{1}{h} \int_{a}^{x} f(t) d t-\frac{1}{h} \int_{a}^{x} f(t-h) d t \\
& =\frac{1}{h} \int_{a}^{x} f(t) d t-\frac{1}{h} \int_{a-h}^{x-h} f(t) d t \\
& =\frac{1}{h} \int_{x-h}^{x} f(t) d t-\frac{1}{h} \int_{a-h}^{a} f(t) d t \\
& =\frac{1}{h} \int_{x-h}^{x} f(t) d t-f(a)
\end{aligned}
$$

Therefore, by continuity of $f$ it follows from Fatou's lemma that

$$
\int_{a}^{x} D_{-} f(t) d t=\int_{a}^{x} f^{\prime}(t) d t \leq \lim \inf _{h \rightarrow 0+} \int_{a}^{x} \frac{f(t)-f(t-h)}{h} d t=f(x)-f(a)
$$

and this shows that $f^{\prime}$ is in $L^{1}$. This part only used the fact that $f$ is increasing and continuous. That $f$ is Lipschitz has not been used.

If it were known that there is a dominating function for $t \rightarrow \frac{f(t)-f(t-h)}{h}$, then you could simply apply the dominated convergence theorem in the above inequality instead of Fatou's lemma and get the desired result. But from Lipschitz continuity, you have

$$
\left|\frac{f(t)-f(t-h)}{h}\right| \leq K
$$

and so one can indeed apply the dominated convergence theorem and conclude that

$$
\int_{a}^{x} f^{\prime}(t) d t=f(x)-f(a)
$$

The last claim follows right away from consideration of intervals since the restriction of a Lipschitz function is Lipschitz.

With the above lemmas, the following is the main theorem about absolutely continuous functions.

The following simple corollary is a case of Rademacher's theorem.
Corollary 26.4.3 Suppose $f:[a, b] \rightarrow \mathbb{R}$ is Lipschitz continuous,

$$
|f(x)-f(y)| \leq K|x-y|
$$

Then $f^{\prime}(x)$ exists a.e. and

$$
f(x)=f(a)+\int_{a}^{x} f^{\prime}(t) d t
$$

Proof: If $f$ were increasing, this would follow from the above lemma. Let $g(x)=$ $2 K x-f(x)$. Then $g$ is Lipschitz with a different Lipschitz constant and also if $x<y$,

$$
\begin{aligned}
& g(y)-g(x)=2 K y-f(y)-(2 K x-f(x)) \\
& \quad \geq 2 K(y-x)-K|y-x|=k|y-x| \geq 0
\end{aligned}
$$

and so Lemma 26.4.2 applies to $g$ and this shows that $f^{\prime}(t)$ exists for a.e. $t$ and $g^{\prime}(x)=$ $2 K-f^{\prime}(x)$. Also

$$
\begin{aligned}
& 2 K(x-a)-(f(x)-f(a)) \\
= & g(x)-g(a)=2 K x-f(x)-(2 K a-f(a))=\int_{a}^{x}\left(2 K-f^{\prime}(t)\right) \\
= & 2 K(x-a)-\int_{a}^{x} f^{\prime}(t) d t
\end{aligned}
$$

showing that $f(x)-f(a)=\int_{a}^{x} f^{\prime}(t) d t$.

### 26.5 Rademacher's Theorem

To begin with is a useful proposition which says the the set where a sequence converges is a measurable set.

Proposition 26.5.1 Let $\left\{f_{n}\right\}$ be measurable with values in a complete normed vector space. Let $A \equiv\left\{\omega:\left\{f_{n}(\omega)\right\}\right.$ converges $\}$. Then $A$ is measurable.

Proof: The set $A$ is the same as the set on which $\left\{f_{n}(\omega)\right\}$ is a Cauchy sequence. This set is

$$
\cap_{n=1}^{\infty} \cup_{m=1}^{\infty} \cap_{p, q>m}\left[\left\|f_{p}(\omega)-f_{q}(\omega)\right\|<\frac{1}{n}\right]
$$

which is a measurable set thanks to the measurability of each $f_{n}$.
It turns out that Lipschitz functions on $\mathbb{R}^{p}$ can be differentiated a.e. This is called Rademacher's theorem. It also can be shown to follow from the Lebesgue theory of differentiation. We denote $D_{\mathbf{v}} f(\mathbf{x})$ the directional derivative of $f$ in the direction $\mathbf{v}$. Here $\mathbf{v}$ is a unit vector. In the following lemma, notation is abused slightly. The symbol $f(\mathbf{x}+t \mathbf{v})$ will mean $t \rightarrow f(\mathbf{x}+t \mathbf{v})$ and $\frac{d}{d t} f(\mathbf{x}+t \mathbf{v})$ will refer to the derivative of this function of $t$.

Lemma 26.5.2 Let $u: \mathbb{R}^{p} \rightarrow \mathbb{R}$ be Lipschitz with Lipschitz constant $K$. Let $u_{n} \equiv u * \phi_{n}$ where $\left\{\phi_{n}\right\}$ is a mollifier,

$$
\phi_{n}(\mathbf{y}) \equiv n^{p} \phi(n \mathbf{y}), \int \phi(\mathbf{y}) d m_{p}(\mathbf{y})=1, \phi(\mathbf{y}) \geq 0, \phi \in C_{c}^{\infty}(B(\mathbf{0}, 1))
$$

Then

$$
\begin{equation*}
\nabla u_{n}(\mathbf{x})=\nabla u * \phi_{n}(\mathbf{x}) \tag{26.5.10}
\end{equation*}
$$

where $\nabla u$ is defined almost everywhere according to Corollary 26.2.6. In fact,

$$
\begin{equation*}
\int_{a}^{b} \frac{\partial u}{\partial x_{i}}\left(\mathbf{x}+t \mathbf{e}_{i}\right) d t=u\left(\mathbf{x}+b \mathbf{e}_{i}\right)-u\left(\mathbf{x}+a \mathbf{e}_{i}\right) \tag{26.5.11}
\end{equation*}
$$

and $\left|\frac{\partial u}{\partial x_{i}}\right| \leq K$. Also, $u_{n}(\mathbf{x}) \rightarrow u(\mathbf{x})$ uniformly on $\mathbb{R}^{p}$ and for a suitable subsequence, still denoted with $n, \nabla u_{n}(\mathbf{x}) \rightarrow \nabla u(\mathbf{x})$ for a.e. $\mathbf{x}$.

Proof: To get the existence of the gradient satisfying the condition given in 26.5.11, apply the corollary to each variable. Now

$$
\begin{aligned}
\frac{u_{n}\left(\mathbf{x}+h \mathbf{e}_{i}\right)-u_{n}(\mathbf{x})}{h} & =\int_{\mathbb{R}^{p}}\left(\frac{u\left(\mathbf{x}+h \mathbf{e}_{i}-\mathbf{y}\right)-u(\mathbf{x}-\mathbf{y})}{h}\right) \phi_{n}(\mathbf{y}) d m_{p}(\mathbf{y}) \\
& =\int_{B\left(\mathbf{0}, \frac{1}{n}\right)}\left(\frac{u\left(\mathbf{x}+h \mathbf{e}_{i}-\mathbf{y}\right)-u(\mathbf{x}-\mathbf{y})}{h}\right) \phi_{n}(\mathbf{y}) d m_{p}(\mathbf{y})
\end{aligned}
$$

Now that difference quotient converges to $\frac{\partial u}{\partial x_{i}}(\mathbf{x}-\mathbf{y})$ for $y_{i}$ off a set of measure zero $N(\hat{\mathbf{y}})$ where $m_{1}(N)=0$ and $\hat{\mathbf{y}} \in \mathbb{R}^{p-1}$. You just use Corollary 26.2 .6 on the $i^{t h}$ variable. Also, the difference quotients are bounded thanks to the Lipshitz condition. Therefore, you can apply the dominated convergence theorem to get

$$
\begin{aligned}
& \lim _{h \rightarrow 0} \int_{\mathbb{R}^{p-1}} \int_{\mathbb{R}}\left(\frac{u\left(\mathbf{x}+h \mathbf{e}_{i}-\mathbf{y}\right)-u(\mathbf{x}-\mathbf{y})}{h}\right) \phi_{n}(\mathbf{y}) d m_{1}\left(y_{i}\right) d m_{p-1}(\hat{\mathbf{y}}) \\
= & \int_{\mathbb{R}^{p-1}} \int_{\mathbb{R}} \frac{\partial u(\mathbf{x}-\mathbf{y})}{\partial x_{i}} \phi_{n}(\mathbf{y}) d m_{1}\left(y_{i}\right) d m_{p-1}(\hat{\mathbf{y}})=\frac{\partial u}{\partial x_{i}} * \phi_{n}(\mathbf{x})
\end{aligned}
$$

The set of $\mathbf{y}$ in $\mathbb{R}^{p}$ where $\frac{u\left(\mathbf{x}+h \mathbf{e}_{i}-\mathbf{y}\right)-u(\mathbf{x}-\mathbf{y})}{h}$ converges as $h \rightarrow 0$ through a sequence of values is a measurable set thanks to Proposition 26.5.1 and for each $\hat{\mathbf{y}}$, the convergence takes place off a set of $m_{1}$ measure zero. Thus there are no measurability issues here and off a set of measure zero, the difference quotient converges to the partial derivative. This proves 26.5.10.

$$
\left\|u_{n}(\mathbf{x})-u(\mathbf{x})\right\| \leq \int_{\mathbb{R}^{p}}\|u(\mathbf{x}-\mathbf{y})-u(\mathbf{x})\| \phi_{n}(\mathbf{y}) d m_{p}(\mathbf{y})
$$

by uniform continuity of $u$ coming from the Lipschitz condition, when $n$ is large enough, this is no larger than

$$
\int_{\mathbb{R}^{p}} \varepsilon \psi_{n}(\mathbf{y}) d m_{p}(\mathbf{y})=\varepsilon
$$

and so uniform convergence holds.
Now consider the last claim. From the first part,

$$
\begin{aligned}
&\left\|u_{n x_{i}}(\mathbf{x})-u_{x_{i}}(\mathbf{x})\right\|=\left\|\int_{B\left(\mathbf{0}, \frac{1}{n}\right)} u_{x_{i}}(\mathbf{x}-\mathbf{y}) \phi_{n}(\mathbf{y}) d m_{p}(\mathbf{y})-u_{x_{i}}(\mathbf{x})\right\| \\
&=\left\|\int_{B\left(\mathbf{x}, \frac{1}{n}\right)} u_{x_{i}}(\mathbf{z}) \phi_{n}(\mathbf{x}-\mathbf{z}) d m_{p}(\mathbf{z})-u_{x_{i}}(\mathbf{x})\right\| \\
&\left\|u_{n x_{i}}(\mathbf{x})-u_{x_{i}}(\mathbf{x})\right\| \leq \leq \int_{\mathbb{R}^{p}}\left\|u_{x_{i}}(\mathbf{x}-\mathbf{y})-u_{x_{i}}(\mathbf{x})\right\| \phi_{n}(\mathbf{y}) d m_{p}(\mathbf{y}) \\
&=\int_{B\left(\mathbf{0}, \frac{1}{n}\right)}\left\|u_{x_{i}}(\mathbf{x}-\mathbf{y})-u_{x_{i}}(\mathbf{x})\right\| \phi_{n}(\mathbf{y}) d m_{p}(\mathbf{y})
\end{aligned}
$$

Now $\phi_{n}(\mathbf{y})=n^{p} \phi(n \mathbf{y})=\frac{m_{p}(B(\mathbf{0}, 1))}{m_{p}\left(B\left(\mathbf{0}, \frac{1}{n}\right)\right)} \phi(n \mathbf{y})$. Therefore, the above equals

$$
\begin{aligned}
& =\frac{m_{p}(B(\mathbf{0}, 1))}{m_{p}\left(B\left(\mathbf{0}, \frac{1}{n}\right)\right)} \int_{B\left(\mathbf{0}, \frac{1}{n}\right)}\left\|u_{x_{i}}(\mathbf{x}-\mathbf{y})-u_{x_{i}}(\mathbf{x})\right\| \phi(n \mathbf{y}) d m_{p}(\mathbf{y}) \\
& =\frac{m_{p}(B(\mathbf{0}, 1))}{m_{p}\left(B\left(\mathbf{0}, \frac{1}{n}\right)\right)} \int_{B\left(\mathbf{x}, \frac{1}{n}\right)}\left\|u_{x_{i}}(\mathbf{z})-u_{x_{i}}(\mathbf{x})\right\| \phi(n(\mathbf{x}-\mathbf{z})) d m_{p}(\mathbf{z}) \\
& \leq C \frac{m_{p}(B(\mathbf{0}, 1))}{m_{p}\left(B\left(\mathbf{x}, \frac{1}{n}\right)\right)} \int_{B\left(\mathbf{x}, \frac{1}{n}\right)}\left\|u_{x_{i}}(\mathbf{z})-u_{x_{i}}(\mathbf{x})\right\| d m_{p}(\mathbf{z})
\end{aligned}
$$

which converges to 0 for a.e. $\mathbf{x}$, in fact at any Lebesgue point. This is because $u_{x_{i}}$ is bounded by $K$ and so is in $L_{l o c}^{1}$.

The following lemma gives an interesting inequality due to Morrey. To simplify notation $d z$ will mean $d m_{p}(\mathbf{z})$.

Lemma 26.5.3 Let $u$ be a $C^{1}$ function on $\mathbb{R}^{p}$. Then there exists a constant $C$, depending only on $p$ such that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{p}$,

$$
\begin{gather*}
|u(\mathbf{x})-u(\mathbf{y})| \\
\leq C\left(\int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)}|\nabla u(\mathbf{z})|^{q} d z\right)^{1 / q}\left(|\mathbf{x}-\mathbf{y}|^{(1-p / q)}\right) \tag{26.5.12}
\end{gather*}
$$

Here $q>p$.
Proof: In the argument $C$ will be a generic constant which depends on $p$. Consider the following picture.


This is a picture of two balls of radius $r$ in $\mathbb{R}^{p}, U$ and $V$ having centers at $\mathbf{x}$ and $\mathbf{y}$ respectively, which intersect in the set $W$. The center of $U$ is on the boundary of $V$ and the center of $V$ is on the boundary of $U$ as shown in the picture. There exists a constant, $C$, independent of $r$ depending only on $p$ such that

$$
\frac{m(W)}{m(U)}=\frac{m(W)}{m(V)}=\frac{1}{C}
$$

You could compute this constant if you desired but it is not important here.
Then

$$
\begin{aligned}
|u(\mathbf{x})-u(\mathbf{y})| & =\frac{1}{m(W)} \int_{W}|u(\mathbf{x})-u(\mathbf{y})| d z \\
& \leq \frac{1}{m(W)} \int_{W}|u(\mathbf{x})-u(\mathbf{z})| d z+\frac{1}{m(W)} \int_{W}|u(\mathbf{z})-u(\mathbf{y})| d z \\
& =\frac{C}{m(U)}\left[\int_{W}|u(\mathbf{x})-u(\mathbf{z})| d z+\int_{W}|u(\mathbf{z})-u(\mathbf{y})| d z\right] \\
& \leq \frac{C}{m(U)}\left[\int_{U}|u(\mathbf{x})-u(\mathbf{z})| d z+\int_{V}|u(\mathbf{y})-u(\mathbf{z})| d z\right]
\end{aligned}
$$

Now consider these two terms. Let $q>p$
Using spherical coordinates and letting $U_{0}$ denote the ball of the same radius as $U$ but with center at $\mathbf{0}$,

$$
\begin{aligned}
& \frac{1}{m(U)} \int_{U}|u(\mathbf{x})-u(\mathbf{z})| d z \\
= & \frac{1}{m\left(U_{0}\right)} \int_{U_{0}}|u(\mathbf{x})-u(\mathbf{z}+\mathbf{x})| d z
\end{aligned}
$$

Now using spherical coordinates, Section 13.9, and letting $C$ denote a generic constant which depends on $p$,

$$
\begin{aligned}
&=\frac{1}{m\left(U_{0}\right)} \int_{0}^{r} \rho^{p-1} \int_{S^{p-1}}|u(\mathbf{x})-u(\rho \mathbf{w}+\mathbf{x})| d \sigma(w) d \rho \\
& \leq \frac{1}{m\left(U_{0}\right)} \int_{0}^{r} \rho^{p-1} \int_{S^{p-1}} \int_{0}^{\rho}\left|D_{\mathbf{w}} u(\mathbf{x}+t \mathbf{w})\right| d t d \sigma(w) d \rho \\
&=\frac{1}{m\left(U_{0}\right)} \int_{0}^{r} \rho^{p-1} \int_{S^{p-1}} \int_{0}^{\rho}|\nabla u(\mathbf{x}+t \mathbf{w}) \cdot \mathbf{w}| d t d \sigma(w) d \rho \\
& \leq \frac{1}{m\left(U_{0}\right)} \int_{0}^{r} \rho^{p-1} \int_{S^{p-1}} \int_{0}^{r}|\nabla u(\mathbf{x}+t \mathbf{w}) \cdot \mathbf{w}| d t d \sigma(w) d \rho \\
&=\frac{1}{m\left(U_{0}\right)} \int_{S^{p-1}} \int_{0}^{r}|\nabla u(\mathbf{x}+t \mathbf{w}) \cdot \mathbf{w}| \int_{0}^{r} \rho^{p-1} d \rho d t d \sigma(w) \\
&=C \int_{0}^{r} \int_{S^{p-1}}|\nabla u(\mathbf{x}+t \mathbf{w})| d \sigma(w) d t=C \int_{0}^{r} \int_{S^{p-1}} \frac{|\nabla u(\mathbf{x}+t \mathbf{w})|}{t^{p-1}} t^{p-1} d \sigma(w) d t
\end{aligned}
$$

But this is just the polar coordinates description of what follows.

$$
\begin{gathered}
=C \int_{U_{0}} \frac{|\nabla u(\mathbf{x}+\mathbf{z})|}{|\mathbf{z}|^{p-1}} d z \\
\leq C\left(\int_{U_{0}}|\nabla u(\mathbf{x}+\mathbf{z})|^{q} d z\right)^{1 / q}\left(\int_{U_{0}}|\mathbf{z}|^{q^{\prime}-p q^{\prime}}\right)^{1 / q^{\prime}} \\
=C\left(\int_{U}|\nabla u(\mathbf{z})|^{q} d z\right)^{1 / q}\left(\int_{S^{p-1}} \int_{0}^{r} \rho^{q^{\prime}-p q^{\prime}} \rho^{p-1} d \rho d \sigma\right)^{(q-1) / q} \\
=C\left(\int_{U}|\nabla u(\mathbf{z})|^{q} d z\right)^{1 / q}\left(\int_{S^{p-1}} \int_{0}^{r} \frac{1}{\left.\rho^{\frac{p-1}{q-1}} d \rho d \sigma\right)^{(q-1) / q}}\right. \\
=C\left(\frac{q-1}{q-p}\right)^{(q-1) / q}\left(\int_{U}|\nabla u(\mathbf{z})|^{q} d z\right)^{1 / q} r^{1-\frac{p}{q}} \\
=C\left(\frac{q-1}{q-p}\right)^{(q-1) / q}\left(\int_{U}|\nabla u(\mathbf{z})|^{q} d z\right)^{1 / q}|\mathbf{x}-\mathbf{y}|^{1-\frac{p}{q}}
\end{gathered}
$$

Similarly,

$$
\frac{1}{m(V)} \int_{U}|u(\mathbf{y})-u(\mathbf{z})| d z \leq C\left(\frac{q-1}{q-p}\right)^{(q-1) / q}\left(\int_{V}|\nabla u(\mathbf{z})|^{q} d z\right)^{1 / q}|\mathbf{x}-\mathbf{y}|^{1-\frac{p}{q}}
$$

Therefore,

$$
|u(\mathbf{x})-u(\mathbf{y})| \leq C\left(\frac{q-1}{q-p}\right)^{(q-1) / q}\left(\int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)}|\nabla u(\mathbf{z})|^{q} d z\right)^{1 / q}|\mathbf{x}-\mathbf{y}|^{1-\frac{p}{q}}
$$

because $B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|) \supseteq V \cup U$.
Corollary 26.5.4 Let $u$ be Lipschitz on $\mathbb{R}^{p}$ with constant $K$. Then there is a constant $C$ depending only on $p$ such that

$$
\begin{equation*}
|u(\mathbf{x})-u(\mathbf{y})| \leq C\left(\int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)}|\nabla u(\mathbf{z})|^{q} d z\right)^{1 / q}\left(|\mathbf{x}-\mathbf{y}|^{(1-p / q)}\right) \tag{26.5.13}
\end{equation*}
$$

Here $q>p$.
Proof: Let $u_{n}=u * \phi_{n}$ where $\left\{\phi_{n}\right\}$ is a mollifier as in Lemma 26.5.2. Then from Lemma 26.5.3, there is a constant depending only on $p$ such that

$$
\left|u_{n}(\mathbf{x})-u_{n}(\mathbf{y})\right| \leq C\left(\int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)}\left|\nabla u_{n}(\mathbf{z})\right|^{q} d z\right)^{1 / q}\left(|\mathbf{x}-\mathbf{y}|^{(1-p / q)}\right)
$$

Now $\left|\nabla u_{n}\right|=\left|\nabla u * \phi_{n}\right|$ by Lemma 26.5.2 and this last is bounded. Also, by this lemma, $\nabla u_{n}(\mathbf{z}) \rightarrow \nabla u(\mathbf{z})$ a.e. and $u_{n}(\mathbf{x}) \rightarrow u(\mathbf{x})$ for all $\mathbf{x}$. Therefore, we can pass to a limit in the above and obtain 26.5.13.

Note you can write 26.5.13 in the form

$$
\begin{aligned}
|u(\mathbf{x})-u(\mathbf{y})| & \leq C\left(\frac{1}{|\mathbf{x}-\mathbf{y}|^{p}} \int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)}|\nabla u(\mathbf{z})|^{q} d z\right)^{1 / q}|\mathbf{x}-\mathbf{y}| \\
& =\hat{C}\left(\frac{1}{m_{p}(B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|))} \int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)}|\nabla u(\mathbf{z})|^{q} d z\right)^{1 / q}|\mathbf{x}-\mathbf{y}|
\end{aligned}
$$

Before leaving this remarkable formula, note that if you are in any situation where the above formula holds and $\nabla u$ exists in some sense and is in $L^{q}, q>p$, then $u$ would need to be continuous. This is the basis for the Sobolev embedding theorem.

Here is Rademacher's theorem.
Theorem 26.5.5 Suppose $u$ is Lipschitz with constant $K$ then if $\mathbf{x}$ is a point where $\nabla u(\mathbf{x})$ exists,

$$
|u(\mathbf{y})-u(\mathbf{x})-\nabla u(\mathbf{x}) \cdot(\mathbf{y}-\mathbf{x})|
$$

$$
\begin{equation*}
\leq C\left(\frac{1}{m(B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|))} \int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)}|\nabla u(\mathbf{z})-\nabla u(\mathbf{x})|^{q} d z\right)^{1 / q}|\mathbf{x}-\mathbf{y}| \tag{26.5.14}
\end{equation*}
$$

Also $u$ is differentiable at a.e. $\mathbf{x}$ and also

$$
\begin{equation*}
u(\mathbf{x}+t \mathbf{v})-u(\mathbf{x})=\int_{0}^{t} D_{\mathbf{v}} u(\mathbf{x}+s \mathbf{v}) d s \tag{26.5.15}
\end{equation*}
$$

Proof: This follows easily from letting $g(\mathbf{y}) \equiv u(\mathbf{y})-u(\mathbf{x})-\nabla u(\mathbf{x}) \cdot(\mathbf{y}-\mathbf{x})$. As explained above, $|\nabla u(\mathbf{x})| \leq \sqrt{p} K$ at every point where $\nabla u$ exists, the exceptional points being in a set of measure zero. Then $g(\mathbf{x})=0$, and $\nabla g(\mathbf{y})=\nabla u(\mathbf{y})-\nabla u(\mathbf{x})$ at the points $\mathbf{y}$ where the gradient of $g$ exists. From Corollary 26.5.4,

$$
\begin{aligned}
& |u(\mathbf{y})-u(\mathbf{x})-\nabla u(\mathbf{x}) \cdot(\mathbf{y}-\mathbf{x})| \\
= & |g(\mathbf{y})|=|g(\mathbf{y})-g(\mathbf{x})| \\
\leq & C\left(\int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)}|\nabla u(\mathbf{z})-\nabla u(\mathbf{x})|^{q} d z\right)^{1 / q}|\mathbf{x}-\mathbf{y}|^{1-\frac{p}{q}} \\
= & C\left(\int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)}|\nabla u(\mathbf{z})-\nabla u(\mathbf{x})|^{q} d z\right)^{1 / q} \frac{1}{|\mathbf{x}-\mathbf{y}|^{p}} \\
& \frac{1}{q} \\
= & C\left(\frac{1}{m(B(\mathbf{x}, 2|\mathbf{y}|}\right. \\
& \left.|\nabla u(\mathbf{y})-\nabla u(\mathbf{x})|^{q} d z\right)^{1 / q}|\mathbf{x}-\mathbf{y}| .
\end{aligned}
$$

Now this is no larger than

$$
\leq C\left(\frac{1}{m(B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|))} \int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)}|\nabla u(\mathbf{z})-\nabla u(\mathbf{x})|(2 \sqrt{p} K)^{q-1} d z\right)^{1 / q}|\mathbf{x}-\mathbf{y}|
$$

It follows that at Lebesgue points of $\nabla u$, the above expression is $o(|\mathbf{x}-\mathbf{y}|)$ and so at all such points $u$ is differentiable. As to 26.5.15, this follows from an application of Corollary 26.2.6 to $f(t)=u(\mathbf{x}+t \mathbf{v})$.

Note that for a.e. $\mathbf{x}, D_{\mathbf{v}} u(\mathbf{x})=\nabla u(\mathbf{x}) \cdot \mathbf{v}$. If you have a line with direction vector $\mathbf{v}$, does it follow that $D u(\mathbf{x}+t \mathbf{v})$ exists for a.e. $t$ ? We know the directional derivative exists a.e. $t$ but it might not be clear that it is $\nabla u(\mathbf{x}) \cdot \mathbf{v}$.

For $|\mathbf{w}|=1$, denote the measure of Section 13.9 defined on the unit sphere $S^{p-1}$ as $\sigma$. Let $N_{\mathbf{w}}$ be defined as those $t \in[0, \infty)$ for which $D_{\mathbf{w}} u(\mathbf{x}+t \mathbf{w}) \neq \nabla u(\mathbf{x}+t \mathbf{w}) \cdot \mathbf{w}$.

$$
B \equiv\left\{\mathbf{w} \in S^{p-1}: N_{\mathbf{w}} \text { has positive measure }\right\}
$$

This is contained in the set of points of $\mathbb{R}^{p}$ where the derivative of $v(\cdot) \equiv u(\mathbf{x}+\cdot)$ fails to exist.Thus from Section 13.9 the measure of this set is

$$
\int_{B} \int_{N_{\mathrm{w}}} \rho^{n-1} d \rho d \sigma(w)
$$

This must equal zero from what was just shown about the derivative of the Lipschitz function $v$ existing a.e. and so $\sigma(B)=0$. The claimed formula follows from this. Thus we obtain the following corollary.

Corollary 26.5.6 Let $u$ be Lipschitz. Then for any $\mathbf{x}$ and $\mathbf{v} \in S^{p-1} \backslash B_{\mathbf{x}}$ where $\sigma\left(B_{\mathbf{x}}\right)=0$, it follows that for all $t$,

$$
u(\mathbf{x}+t \mathbf{v})-u(\mathbf{x})=\int_{0}^{t} D_{\mathbf{v}} u(\mathbf{x}+s \mathbf{v}) d s=\int_{0}^{t} \nabla u(\mathbf{x}+s \mathbf{v}) \cdot \mathbf{v} d s
$$

In the all of the above, the function $u$ is defined on all of $\mathbb{R}^{p}$. However, it is always the case that Lipschitz functions can be extended off a given set. Thus if a Lipschitz function is defined on some set $\Omega$, then it can always be considered the restriction to $\Omega$ of a Lipschitz map defined on all of $\mathbb{R}^{p}$.

Theorem 26.5.7 If $\mathbf{h}: \Omega \rightarrow \mathbb{R}^{m}$ is Lipschitz, then there exists $\overline{\mathbf{h}}: \mathbb{R}^{p} \rightarrow \mathbb{R}^{m}$ which extends h and is also Lipschitz.

Proof: It suffices to assume $m=1$ because if this is shown, it may be applied to the components of $\mathbf{h}$ to get the desired result. Suppose

$$
\begin{equation*}
|h(\mathbf{x})-h(\mathbf{y})| \leq K|\mathbf{x}-\mathbf{y}| . \tag{26.5.16}
\end{equation*}
$$

Define

$$
\begin{equation*}
\bar{h}(\mathbf{x}) \equiv \inf \{h(\mathbf{w})+K|\mathbf{x}-\mathbf{w}|: \mathbf{w} \in \Omega\} . \tag{26.5.17}
\end{equation*}
$$

If $\mathbf{x} \in \Omega$, then for all $\mathbf{w} \in \Omega$,

$$
h(\mathbf{w})+K|\mathbf{x}-\mathbf{w}| \geq h(\mathbf{x})
$$

by 26.5.16. This shows $h(\mathbf{x}) \leq \bar{h}(\mathbf{x})$. But also you could take $\mathbf{w}=\mathbf{x}$ in 26.5.17 which yields $\bar{h}(\mathbf{x}) \leq h(\mathbf{x})$. Therefore $\bar{h}(\mathbf{x})=h(\mathbf{x})$ if $\mathbf{x} \in \Omega$.

Now suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{p}$ and consider $|\bar{h}(\mathbf{x})-\bar{h}(\mathbf{y})|$. Without loss of generality assume $\bar{h}(\mathbf{x}) \geq \bar{h}(\mathbf{y})$. (If not, repeat the following argument with $\mathbf{x}$ and $\mathbf{y}$ interchanged.) Pick $\mathbf{w} \in \Omega$ such that

$$
h(\mathbf{w})+K|\mathbf{y}-\mathbf{w}|-\varepsilon<\bar{h}(\mathbf{y})
$$

Then

$$
\begin{aligned}
|\bar{h}(\mathbf{x})-\bar{h}(\mathbf{y})|=\bar{h}(\mathbf{x})-\bar{h}(\mathbf{y}) & \leq h(\mathbf{w})+K|\mathbf{x}-\mathbf{w}|- \\
{[h(\mathbf{w})+K|\mathbf{y}-\mathbf{w}|-\boldsymbol{\varepsilon}] } & \leq K|\mathbf{x}-\mathbf{y}|+\boldsymbol{\varepsilon}
\end{aligned}
$$

Since $\varepsilon$ is arbitrary,

$$
|\bar{h}(\mathbf{x})-\bar{h}(\mathbf{y})| \leq K|\mathbf{x}-\mathbf{y}|
$$

### 26.6 Rademacher's Theorem

It turns out that Lipschitz functions on $\mathbb{R}^{n}$ can be differentiated a.e. This is called Rademacher's theorem. It also can be shown to follow from the Lebesgue theory of differentiation.

### 26.6.1 Morrey's Inequality

The following inequality will be called Morrey's inequality. It relates an expression which is given pointwise to an integral of the $p^{t h}$ power of the derivative.

Lemma 26.6.1 Let $u \in C^{1}\left(\mathbb{R}^{n}\right)$ and $p>n$. Then there exists a constant, $C$, depending only on $n$ such that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n}$,

$$
\begin{gather*}
|u(\mathbf{x})-u(\mathbf{y})| \\
\leq C\left(\int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)}|\nabla u(\mathbf{z})|^{p} d z\right)^{1 / p}\left(|\mathbf{x}-\mathbf{y}|^{(1-n / p)}\right) \tag{26.6.18}
\end{gather*}
$$

Proof: In the argument $C$ will be a generic constant which depends on $n$. Consider the following picture.


This is a picture of two balls of radius $r$ in $\mathbb{R}^{n}, U$ and $V$ having centers at $\mathbf{x}$ and $\mathbf{y}$ respectively, which intersect in the set, $W$. The center of $U$ is on the boundary of $V$ and the center of $V$ is on the boundary of $U$ as shown in the picture. There exists a constant, $C$, independent of $r$ depending only on $n$ such that

$$
\frac{m(W)}{m(U)}=\frac{m(W)}{m(V)}=C
$$

You could compute this constant if you desired but it is not important here.

Define the average of a function over a set, $E \subseteq \mathbb{R}^{n}$ as follows.

$$
f_{E} f d x \equiv \frac{1}{m(E)} \int_{E} f d x
$$

Then

$$
\begin{aligned}
|u(\mathbf{x})-u(\mathbf{y})| & =f_{W}|u(\mathbf{x})-u(\mathbf{y})| d z \\
& \leq f_{W}|u(\mathbf{x})-u(\mathbf{z})| d z+f_{W}|u(\mathbf{z})-u(\mathbf{y})| d z \\
& =\frac{C}{m(U)}\left[\int_{W}|u(\mathbf{x})-u(\mathbf{z})| d z+\int_{W}|u(\mathbf{z})-u(\mathbf{y})| d z\right] \\
& \leq C\left[f_{U}|u(\mathbf{x})-u(\mathbf{z})| d z+f_{V}|u(\mathbf{y})-u(\mathbf{z})| d z\right]
\end{aligned}
$$

Now consider these two terms. Using spherical coordinates and letting $U_{0}$ denote the ball of the same radius as $U$ but with center at $\mathbf{0}$,

$$
\begin{aligned}
& f_{U}|u(\mathbf{x})-u(\mathbf{z})| d z \\
= & \frac{1}{m\left(U_{0}\right)} \int_{U_{0}}|u(\mathbf{x})-u(\mathbf{z}+\mathbf{x})| d z \\
= & \frac{1}{m\left(U_{0}\right)} \int_{0}^{r} \rho^{n-1} \int_{S^{n-1}}|u(\mathbf{x})-u(\rho \mathbf{w}+\mathbf{x})| d \sigma(w) d \rho \\
\leq & \frac{1}{m\left(U_{0}\right)} \int_{0}^{r} \rho^{n-1} \int_{S^{n-1}} \int_{0}^{\rho}|\nabla u(\mathbf{x}+t \mathbf{w}) \cdot \mathbf{w}| d t d \sigma d \rho \\
\leq & \frac{1}{m\left(U_{0}\right)} \int_{0}^{r} \rho^{n-1} \int_{S^{n-1}} \int_{0}^{\rho}|\nabla u(\mathbf{x}+t \mathbf{w})| d t d \sigma d \rho \\
\leq & C \frac{1}{r} \int_{0}^{r} \int_{S^{n-1}} \int_{0}^{r}|\nabla u(\mathbf{x}+t \mathbf{w})| d t d \sigma d \rho \\
= & C \frac{1}{r} \int_{0}^{r} \int_{S^{n-1}} \int_{0}^{r} \frac{|\nabla u(\mathbf{x}+t \mathbf{w})|}{t^{n-1}} t^{n-1} d t d \sigma d \rho \\
= & C \int_{S^{n-1}} \int_{0}^{r} \frac{|\nabla u(\mathbf{x}+t \mathbf{w})|}{t^{n-1}} t^{n-1} d t d \sigma \\
= & C \int_{U_{0}} \frac{|\nabla u(\mathbf{x}+\mathbf{z})|}{|\mathbf{z}|^{n-1}} d z \\
\leq & C\left(\int_{U_{0}}|\nabla u(\mathbf{x}+\mathbf{z})|^{p} d z\right)^{1 / p}\left(\int_{U}|\mathbf{z}|^{p^{\prime}-n p^{\prime}}\right){ }^{1 / p^{\prime}}
\end{aligned}
$$

$$
\begin{gathered}
=C\left(\int_{U}|\nabla u(\mathbf{z})|^{p} d z\right)^{1 / p}\left(\int_{S^{n-1}} \int_{0}^{r} \rho^{p^{\prime}-n p^{\prime}} \rho^{n-1} d \rho d \sigma\right)^{(p-1) / p} \\
=C\left(\int_{U}|\nabla u(\mathbf{z})|^{p} d z\right)^{1 / p}\left(\int_{S^{n-1}} \int_{0}^{r} \frac{1}{\rho^{\frac{n-1}{p-1}}} d \rho d \sigma\right)^{(p-1) / p} \\
=C\left(\frac{p-1}{p-n}\right)^{(p-1) / p}\left(\int_{U}|\nabla u(\mathbf{z})|^{p} d z\right)^{1 / p} r^{1-\frac{n}{p}} \\
=C\left(\frac{p-1}{p-n}\right)^{(p-1) / p}\left(\int_{U}|\nabla u(\mathbf{z})|^{p} d z\right)^{1 / p}|\mathbf{x}-\mathbf{y}|^{1-\frac{n}{p}}
\end{gathered}
$$

Similarly,

$$
f_{V}|u(\mathbf{y})-u(\mathbf{z})| d z \leq C\left(\frac{p-1}{p-n}\right)^{(p-1) / p}\left(\int_{V}|\nabla u(\mathbf{z})|^{p} d z\right)^{1 / p}|\mathbf{x}-\mathbf{y}|^{1-\frac{n}{p}}
$$

Therefore,

$$
|u(\mathbf{x})-u(\mathbf{y})| \leq C\left(\frac{p-1}{p-n}\right)^{(p-1) / p}\left(\int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)}|\nabla u(\mathbf{z})|^{p} d z\right)^{1 / p}|\mathbf{x}-\mathbf{y}|^{1-\frac{n}{p}}
$$

because $B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|) \supseteq V \cup U$. This proves the lemma.
The following corollary is also interesting
Corollary 26.6.2 Suppose $u \in C^{1}\left(\mathbb{R}^{n}\right)$. Then

$$
\begin{gather*}
|u(\mathbf{y})-u(\mathbf{x})-\nabla u(\mathbf{x}) \cdot(\mathbf{y}-\mathbf{x})| \\
\leq C\left(\frac{1}{m(B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|))} \int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)}|\nabla u(\mathbf{z})-\nabla u(\mathbf{x})|^{p} d z\right)^{1 / p}|\mathbf{x}-\mathbf{y}| \tag{26.6.19}
\end{gather*}
$$

Proof: This follows easily from letting $g(\mathbf{y}) \equiv u(\mathbf{y})-u(\mathbf{x})-\nabla u(\mathbf{x}) \cdot(\mathbf{y}-\mathbf{x})$. Then $g \in C^{1}\left(\mathbb{R}^{n}\right), g(\mathbf{x})=0$, and $\nabla g(\mathbf{z})=\nabla u(\mathbf{z})-\nabla u(\mathbf{x})$. From Lemma 26.6.1,

$$
\begin{aligned}
& |u(\mathbf{y})-u(\mathbf{x})-\nabla u(\mathbf{x}) \cdot(\mathbf{y}-\mathbf{x})| \\
= & |g(\mathbf{y})|=|g(\mathbf{y})-g(\mathbf{x})| \\
\leq & C\left(\int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)}|\nabla u(\mathbf{x})-\nabla u(\mathbf{x})|^{p} d z\right)^{1 / p}|\mathbf{x}-\mathbf{y}|^{1-\frac{n}{p}} \\
= & C\left(\frac{1}{m(B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|))} \int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)}|\nabla u(\mathbf{z})-\nabla u(\mathbf{x})|^{p} d z\right)^{1 / p}|\mathbf{x}-\mathbf{y}| .
\end{aligned}
$$

This proves the corollary.
It may be interesting at this point to recall the definition of differentiability on Page 117. If you knew the above inequality held for $\nabla u$ having components in $L_{l o c}^{1}\left(\mathbb{R}^{n}\right)$, then at Lebesgue points of $\nabla u$, the above would imply $D u(\mathbf{x})$ exists.

### 26.6.2 Rademacher's Theorem

Lemma 26.6.3 Let u be a Lipschitz continuous function which vanishes outside some compact set. Then there exists a unique $u_{, i} \in L^{\infty}\left(\mathbb{R}^{n}\right)$ such that

$$
\lim _{h \rightarrow 0} \frac{u(\cdot+h)-u(\cdot)}{h}=u_{, i} \text { weak } * \operatorname{in} L^{\infty}\left(\mathbb{R}^{n}\right)
$$

Proof: By the Lipschitz condition, the above difference quotient is bounded in $L^{\infty}$ by $K$ the Lipschitz constant of $u$. It follows from the Banach Aloglu theorem and Corollary 17.5.6 on Page 463 that there exists a subsequence $h_{k} \rightarrow 0$ and $g \in L^{\infty}\left(\mathbb{R}^{n}\right)$ such that

$$
\frac{u\left(\cdot+h_{k}\right)-u(\cdot)}{h_{k}} \rightarrow g \text { weak } * \text { in } L^{\infty}\left(\mathbb{R}^{n}\right)
$$

Letting $\phi \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$, it follows

$$
\int g \phi d x=\lim _{k \rightarrow \infty} \int \frac{u\left(\cdot+h_{k}\right)-u(\cdot)}{h_{k}} \phi d x=-\int u \phi_{, i} d x
$$

This also shows that $g$ must vanish outside some compact set because the integral on the right shows that if $\operatorname{spt} \phi$ does not intersect $\operatorname{spt} u$, then $\int g \phi d x=0$. Thus $g \in L^{2}\left(\mathbb{R}^{n}\right)$. If $g_{1}$ is a weak $*$ limit of another subsequence $h_{j} \rightarrow 0$, the same result follows. Thus for any $\phi \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$

$$
\int\left(g-g_{1}\right) \phi d x=0
$$

and since $C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$ is dense in $L^{2}\left(\mathbb{R}^{n}\right)$, this requires $g=g_{1}$ in $L^{2}$ and so they are equal a.e. Since every sequence of $h \rightarrow 0$ has a subsequence which when applied to the difference quotient, always converges to the same thing, it follows the claimed limit exists. This is called $u_{, i}$. This proves the lemma.

Lemma 26.6.4 Let u be a Lipschitz continuous function which vanishes outside a compact set and let $u_{, i}$ be described above. For $\phi_{\varepsilon}$ a mollifier and $u_{\varepsilon} \equiv u * \phi_{\varepsilon}$,

$$
u_{\varepsilon, i}=u_{, i} * \phi_{\varepsilon}
$$

where the symbol $u_{\varepsilon, i}$ means the usual partial derivative with respect to the $i^{\text {th }}$ variable. Also for any $p>n$,

$$
u_{\varepsilon, i} \rightarrow u_{, i} \text { in } L^{p}\left(\mathbb{R}^{n}\right)
$$

Proof: This follows from a computation and Lemma 26.6.3.

$$
\begin{aligned}
u_{\varepsilon, i}(\mathbf{x}) & \equiv \lim _{h \rightarrow 0} \int \frac{u\left(\mathbf{x}-\mathbf{y}+h \mathbf{e}_{i}\right)-u(\mathbf{x}-\mathbf{y})}{h} \phi_{\varepsilon}(\mathbf{y}) d y \\
& =\lim _{h \rightarrow 0} \int \frac{u\left(\mathbf{z}+h \mathbf{e}_{i}\right)-u(\mathbf{z})}{h} \phi_{\varepsilon}(\mathbf{x}-\mathbf{z}) d z \\
& =\int u_{i i}(\mathbf{z}) \phi_{\varepsilon}(\mathbf{x}-\mathbf{z}) d z=u_{i,} * \phi_{\varepsilon}(\mathbf{x})
\end{aligned}
$$

It remains to verify the last assertion. Note that $u_{, i} \in L^{p}\left(\mathbb{R}^{n}\right)$ for any $p>1$ because it is bounded and vanishes outside some compact set. By the first part,

$$
\left(\int\left|u_{\varepsilon, i}-u_{, i}\right|^{p} d x\right)^{1 / p}=\left(\int\left|\int\left(u_{, i}(\mathbf{x}-\mathbf{y})-u_{, i}(\mathbf{x})\right) \phi_{\varepsilon}(\mathbf{y}) d y\right|^{p} d x\right)^{1 / p}
$$

and by Minkowski's inequality,

$$
\begin{aligned}
& \leq \int \phi_{\varepsilon}(\mathbf{y})\left(\int\left|\left(u_{, i}(\mathbf{x}-\mathbf{y})-u_{, i}(\mathbf{x})\right)\right|^{p} d x\right)^{1 / p} d y \\
& =\int_{B(\mathbf{0}, \boldsymbol{\varepsilon})} \phi_{\varepsilon}(\mathbf{y})\left\|\left(u_{, i}\right)_{\mathbf{y}}-u_{i, i}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)} d y
\end{aligned}
$$

which converges to 0 from continuity of translation. This proves the lemma.
Now from Corollary 26.6.2 applied to $u_{\varepsilon}$ just described and letting $\mathbf{y}-\mathbf{x}=\mathbf{v}$

$$
\begin{gathered}
\left|u_{\varepsilon}(\mathbf{x}+\mathbf{v})-u_{\varepsilon}(\mathbf{x})-\nabla u_{\varepsilon}(\mathbf{x}) \cdot \mathbf{v}\right| \\
\leq C\left(\frac{1}{m(B(\mathbf{x}, 2|\mathbf{v}|))} \int_{B(\mathbf{x}, 2|\mathbf{v}|)}\left|\nabla u_{\mathcal{\varepsilon}}(\mathbf{z})-\nabla u_{\varepsilon}(\mathbf{x})\right|^{p} d z\right)^{1 / p}|\mathbf{v}| .
\end{gathered}
$$

From Lemma 26.6.4, there is a subsequence, still denoted as $\varepsilon$ such that for each $i, u_{\varepsilon, i} \rightarrow u_{, i}$ pointwise a.e. and in $L^{p}\left(\mathbb{R}^{n}\right)$ where $p>n$ is given. $\nabla u$ is the vector $\left(u_{, 1}, u_{, 2}, \cdots, u_{, n}\right)^{T}$. Then passing to the limit as $\varepsilon \rightarrow 0$, for a.e. $\mathbf{x}$,

$$
\begin{gathered}
|u(\mathbf{x}+\mathbf{v})-u(\mathbf{x})-\nabla u(\mathbf{x}) \cdot \mathbf{v}| \\
\leq C\left(\frac{1}{m(B(\mathbf{x}, 2|\mathbf{v}|))} \int_{B(\mathbf{x}, 2|\mathbf{v}|)}|\nabla u(\mathbf{z})-\nabla u(\mathbf{x})|^{p} d z\right)^{1 / p}|\mathbf{v}| .
\end{gathered}
$$

At every Lebesgue point $\mathbf{x}$ of $\nabla u$, the above shows $u(\mathbf{x}+\mathbf{v})-u(\mathbf{x})-\nabla u(\mathbf{x}) \cdot \mathbf{v}=\mathbf{o}(\mathbf{v})$. Thus this has proved the following.

Lemma 26.6.5 Let u be Lipschitz continuous and vanish outside some bounded set. Then $D u(\mathbf{x})$ exists for a.e. $\mathbf{x}$.

This is a good result but it is easy to give an even easier to use result. First here is a theorem which says you can extend a Lipschitz map.

Theorem 26.6.6 If $\mathbf{h}: \Omega \rightarrow \mathbb{R}^{m}$ is Lipschitz, then there exists $\overline{\mathbf{h}}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ which extends h and is also Lipschitz.

Proof: It suffices to assume $m=1$ because if this is shown, it may be applied to the components of $\mathbf{h}$ to get the desired result. Suppose

$$
\begin{equation*}
|h(\mathbf{x})-h(\mathbf{y})| \leq K|\mathbf{x}-\mathbf{y}| . \tag{26.6.20}
\end{equation*}
$$

Define

$$
\begin{equation*}
\bar{h}(\mathbf{x}) \equiv \inf \{h(\mathbf{w})+K|\mathbf{x}-\mathbf{w}|: \mathbf{w} \in \Omega\} \tag{26.6.21}
\end{equation*}
$$

If $\mathbf{x} \in \Omega$, then for all $\mathbf{w} \in \Omega$,

$$
h(\mathbf{w})+K|\mathbf{x}-\mathbf{w}| \geq h(\mathbf{x})
$$

by 26.6.20. This shows $h(\mathbf{x}) \leq \bar{h}(\mathbf{x})$. But also you could take $\mathbf{w}=\mathbf{x}$ in 26.6.21 which yields $\bar{h}(\mathbf{x}) \leq h(\mathbf{x})$. Therefore $\bar{h}(\mathbf{x})=h(\mathbf{x})$ if $\mathbf{x} \in \Omega$.

Now suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n}$ and consider $|\bar{h}(\mathbf{x})-\bar{h}(\mathbf{y})|$. Without loss of generality assume $\bar{h}(\mathbf{x}) \geq \bar{h}(\mathbf{y})$. (If not, repeat the following argument with $\mathbf{x}$ and $\mathbf{y}$ interchanged.) Pick $\mathbf{w} \in \Omega$ such that

$$
h(\mathbf{w})+K|\mathbf{y}-\mathbf{w}|-\varepsilon<\bar{h}(\mathbf{y})
$$

Then

$$
\begin{aligned}
|\bar{h}(\mathbf{x})-\bar{h}(\mathbf{y})|=\bar{h}(\mathbf{x})-\bar{h}(\mathbf{y}) & \leq h(\mathbf{w})+K|\mathbf{x}-\mathbf{w}|- \\
{[h(\mathbf{w})+K|\mathbf{y}-\mathbf{w}|-\varepsilon] } & \leq K|\mathbf{x}-\mathbf{y}|+\varepsilon .
\end{aligned}
$$

Since $\varepsilon$ is arbitrary,

$$
|\bar{h}(\mathbf{x})-\bar{h}(\mathbf{y})| \leq K|\mathbf{x}-\mathbf{y}|
$$

and this proves the theorem.
With this theorem, here is the main result called Rademacher's theorem.
Theorem 26.6.7 Let $\mathbf{h}: \Omega \rightarrow \mathbb{R}^{m}$ be Lipschitz on $\Omega$ where $\Omega$ is some nonempty measurable set in $\mathbb{R}^{n}$. Then $D \mathbf{h}(\mathbf{x})$ exists for a.e. $\mathbf{x} \in \Omega$. If $\Omega=\mathbb{R}^{n}$, then for each $\mathbf{e}_{i}$,

$$
\lim _{h \rightarrow 0} \frac{\mathbf{h}\left(\cdot+h \mathbf{e}_{i}\right)-\mathbf{h}(\cdot)}{h}=\mathbf{h}_{, i} \text { weak } * \operatorname{in} L^{\infty}\left(\mathbb{R}^{n}\right)
$$

and whenever $\phi_{\varepsilon}$ is a mollifier,

$$
\left(\mathbf{h} * \phi_{\varepsilon}\right)_{, i} \rightarrow \mathbf{h}_{, i} \text { in } L^{p}\left(\mathbb{R}^{n} ; \mathbb{R}^{m}\right)
$$

Proof: The last two claims follow from the above argument applied to the components of $\mathbf{h}$. By Theorem 26.6.6 the function can be extended to a Lipschitz function defined on all of $\mathbb{R}^{n}$, still denoted as $\mathbf{h}$. Let $\Omega_{r} \equiv \Omega \cap B(\mathbf{0}, r)$. Now let $\psi \in C_{c}^{\infty}(B(\mathbf{0}, 2 r))$ such that $\psi=1$ on $B\left(\mathbf{0}, \frac{3}{2} r\right)$. Then $\psi \mathbf{h}$ is Lipschitz on $\mathbb{R}^{n}$ and vanishes off a bounded set. It follows from Lemma 26.6.5 applied to the components of $\mathbf{h}$ that this function has a derivative off a set of measure zero $N_{r}$. If $\mathbf{x} \in \Omega_{r} \backslash N_{r}$ it follows since $\psi=1$ near $\mathbf{x}$ that $D \mathbf{h}(\mathbf{x})$ exists. Letting $N=\cup_{r=1}^{\infty} N_{r}$, it follows that if $\mathbf{x} \in \Omega \backslash N$, then $D \mathbf{h}(\mathbf{x})$ exists. This proves the theorem.

For $u$ Lipschitz as described above, the limit of the difference quotient $u_{, i}$ is called the weak partial derivative of $u$. For $p>n$ and an assertion that the difference quotients are bounded in $L^{p}$ everything done above would work out the same way and one can therefore generalize parts of the above theorem. The extension is problematic but one can give the following results with essentially the same proof as the above.
Lemma 26.6.8 Let $u \in L^{p}\left(\mathbb{R}^{n}\right)$. There exists $u_{, i} \in L^{p}\left(\mathbb{R}^{n}\right)$ such that

$$
\lim _{h \rightarrow 0} \frac{u\left(\cdot+h \mathbf{e}_{i}\right)-u(\cdot)}{h}=u_{, i} \text { weakly in } L^{p}\left(\mathbb{R}^{n}\right)
$$

if and only if the difference quotients $\frac{u\left(\cdot+h \mathbf{e}_{i}\right)-u(\cdot)}{h}$ are bounded in $L^{p}\left(\mathbb{R}^{n}\right)$ for all nonzero $h$.

Proof: If the weak limit exists, then the difference quotients must be bounded. This follows from the uniform boundedness theorem, Theorem 17.1.8. Here is why. Denote the difference quotient by $D_{h}$ to save space. Weak convergence requires $\int D_{h} f \rightarrow \int u_{i} f$ for all $f \in L^{p^{\prime}}$. Could there exist $h_{k}$ such that $\left\|D_{h_{k}}\right\|_{L^{p}} \rightarrow \infty$ ? Not unless a subsequence satisfies $h_{k} \rightarrow 0$ because if this sequence is bounded away from 0 , the formula for $D_{h}$ will yield the difference quotients are bounded. However, if $h_{k} \rightarrow 0$, then for each $f \in L^{p^{\prime}}$,

$$
\sup _{k} \int D_{h_{k}} f<\infty
$$

because in fact, $\lim _{k \rightarrow \infty} \int D_{h_{k}} f$ exists so it must be bounded. Now $D_{h_{k}}$ can be considered in $\left(L^{p^{\prime}}\right)^{\prime}$ and this shows it is pointwise bounded on $L^{p^{\prime}}$. Therefore, $D_{h_{k}}$ is bounded in $\left(L^{p^{\prime}}\right)^{\prime}$ but the norm on this is the same as the norm in $L^{p}$. Thus $D_{h_{k}}$ is bounded after all.

Conversely, if the difference quotients are bounded, the same argument used earlier, involving convergence of a subsequence, this time coming from the Eberlein Smulian theorem, Theorem 17.5.12 and showing that every subsequence converges to the same thing, shows the difference quotients converge weakly in $L^{p}\left(\mathbb{R}^{n}\right)$ to something we can call $u_{, i}$. This proves the lemma.

Definition 26.6.9 A function $f \in L^{p}\left(\mathbb{R}^{n}\right)$ is said to have weak partial derivatives in $L^{p}\left(\mathbb{R}^{n}\right)$ if the difference quotients $\frac{u\left(\cdot+h \mathbf{e}_{i}\right)-u(\cdot)}{h}$ for each $i=1,2, \cdots, n$ are bounded for $h \neq 0$. If $\mathbf{f} \in L^{p}\left(\mathbb{R}^{n} ; \mathbb{R}^{m}\right)$, it has weak partial derivatives in $L^{p}\left(\mathbb{R}^{n} ; \mathbb{R}^{m}\right)$ if each component function has weak partial derivatives in $L^{p}\left(\mathbb{R}^{n}\right)$.

This following theorem may also be referred to as Rademacher's theorem.
Theorem 26.6.10 Let $\mathbf{h}$ be in $L^{p}\left(\mathbb{R}^{n} ; \mathbb{R}^{m}\right), p>n$, and suppose it has weak derivatives $\mathbf{h}_{, i} \in L^{p}\left(\mathbb{R}^{n} ; \mathbb{R}^{m}\right)$ for $i=1, \cdots, n$. Then $D \mathbf{h}(\mathbf{x})$ exists a.e. and $\mathbf{h}$ is almost everywhere equal to a continuous function. Also if $\phi_{\varepsilon}$ is a mollifier,

$$
\left(\mathbf{h} * \phi_{\varepsilon}\right)_{, i}=\mathbf{h}_{, i} * \phi_{\varepsilon},\left(\mathbf{h} * \phi_{\varepsilon}\right)_{, i} \rightarrow \mathbf{h}_{, i}
$$

in $L^{p}\left(\mathbb{R}^{n} ; \mathbb{R}^{m}\right)$.
Proof: As before,

$$
\begin{aligned}
& \left(\mathbf{h} * \phi_{\varepsilon}\right)_{, i}(\mathbf{x}) \equiv \lim _{h \rightarrow 0} \int \frac{\mathbf{h}\left(\mathbf{x}+h \mathbf{e}_{i}-\mathbf{y}\right)-\mathbf{h}(\mathbf{x}-\mathbf{y})}{h} \phi_{\varepsilon}(\mathbf{y}) d y \\
= & \lim _{h \rightarrow 0} \int \frac{\mathbf{h}\left(\mathbf{z}+h \mathbf{e}_{i}\right)-\mathbf{h}(\mathbf{z})}{h} \phi_{\varepsilon}(\mathbf{x}-\mathbf{z}) d y \equiv \int \mathbf{h}_{, i}(\mathbf{z}) \phi_{\varepsilon}(\mathbf{x}-\mathbf{y}) d y \\
= & \mathbf{h}_{, i} * \phi_{\varepsilon}(\mathbf{x})
\end{aligned}
$$

and now $\left(\mathbf{h} * \phi_{\varepsilon}\right)_{, i} \rightarrow \mathbf{h}_{, i}$ follows as before from a use of Minkowski's inequality. Letting $u$ be one of the component functions of $\mathbf{h}$, Morrey's inequality holds for $u_{\varepsilon} \equiv u * \phi_{\varepsilon}$. Thus

$$
\left|u_{\varepsilon}(\mathbf{x})-u_{\varepsilon}(\mathbf{y})\right| \leq C\left(\int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)}\left|\nabla u_{\varepsilon}(\mathbf{z})\right|^{p} d z\right)^{1 / p}\left(|\mathbf{x}-\mathbf{y}|^{1-n / p}\right)
$$

Now there exists a subsequence such that $u_{\varepsilon} \rightarrow u$ pointwise a.e. and also each $u_{\varepsilon, i} \rightarrow u_{, i}$ pointwise a.e. as well as in $L^{p}$. Therefore, for $\mathbf{x}, \mathbf{y}$ not in a set of measure zero,

$$
|u(\mathbf{x})-u(\mathbf{y})| \leq C\left(\int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)}|\nabla u(\mathbf{z})|^{p} d z\right)^{1 / p}\left(|\mathbf{x}-\mathbf{y}|^{1-n / p}\right)
$$

which shows the claim about $u$ being equal to a continuous function off a set of measure zero. Thus $\mathbf{h}$ is also continuous off a set of measure zero.

As before, letting $g(\mathbf{y}) \equiv u_{\varepsilon}(\mathbf{y})-u_{\varepsilon}(\mathbf{x})-\nabla u_{\varepsilon}(\mathbf{x}) \cdot(\mathbf{y}-\mathbf{x})$ and writing Morrey's inequality,

$$
\begin{aligned}
& \left|u_{\varepsilon}(\mathbf{y})-u_{\varepsilon}(\mathbf{x})-\nabla u_{\varepsilon}(\mathbf{x}) \cdot(\mathbf{y}-\mathbf{x})\right| \\
\leq & C\left(\int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)}\left|\nabla u_{\varepsilon}(\mathbf{z})-\nabla u_{\varepsilon}(\mathbf{x})\right|^{p} d z\right)^{1 / p}\left(|\mathbf{x}-\mathbf{y}|^{1-n / p}\right)
\end{aligned}
$$

Then taking a suitable subsequence and passing to the limit while also letting $\mathbf{v}=\mathbf{y}-\mathbf{x}$, it follows

$$
\begin{aligned}
& |u(\mathbf{x}+\mathbf{v})-u(\mathbf{x})-\nabla u(\mathbf{x}) \cdot \mathbf{v}| \\
\leq & C\left(\int_{B(\mathbf{x}, 2|\mathbf{v}|)}|\nabla u(\mathbf{z})-\nabla u(\mathbf{x})|^{p} d z\right)^{1 / p}\left(|\mathbf{v}|^{1-n / p}\right) \\
= & C\left(\frac{1}{|\mathbf{v}|^{n}} \int_{B(\mathbf{x}, 2|\mathbf{v}|)}|\nabla u(\mathbf{z})-\nabla u(\mathbf{x})|^{p} d z\right)^{1 / p}|\mathbf{v}| \\
= & C^{\prime}\left(\frac{1}{B(\mathbf{x}, 2|\mathbf{v}|)} \int_{B(\mathbf{x}, 2|\mathbf{v}|)}|\nabla u(\mathbf{z})-\nabla u(\mathbf{x})|^{p} d z\right)^{1 / p}|\mathbf{v}|
\end{aligned}
$$

for all $\mathbf{x}, \mathbf{x}+\mathbf{v} \notin N$, a set of measure zero. Defining $u, \nabla u$ at the points of $N$ so that the inequality continues to hold, $D u(\mathbf{x})$ exists at every Lebesgue point of $\nabla u$. Also $D \mathbf{h}$ exists a.e. because this is true of the component functions. This proves the theorem.

### 26.7 Differentiation Of Measures

Recall the Vitali covering theorem in Corollary 13.4.5 on Page 350.
Corollary 26.7.1 Let $E \subseteq \mathbb{R}^{n}$ and let $\mathscr{F}$, be a collection of open balls of bounded radii such that $\mathscr{F}$ covers $E$ in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathscr{F},\left\{B_{j}\right\}_{j=1}^{\infty}$, such that $\bar{m}\left(E \backslash \cup_{j=1}^{\infty} B_{j}\right)=0$.

Definition 26.7.2 Let $\mu$ be a Radon measure defined on $\mathbb{R}^{n}$. Then

$$
\frac{d \mu}{d m}(\mathbf{x}) \equiv \lim _{r \rightarrow 0} \frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))}
$$

whenever this limit exists.
It turns out this limit exists for $m$ a.e. $\mathbf{x}$. To verify this here is another definition.

Definition 26.7.3 Let $f(r)$ be a function having values in $[-\infty, \infty]$. Then

$$
\begin{aligned}
\lim _{\sup _{r \rightarrow 0+}} f(r) & \equiv \lim _{r \rightarrow 0}(\sup \{f(t): t \in[0, r]\}) \\
\lim _{r \rightarrow 0+} f(r) & \equiv \lim _{r \rightarrow 0}(\inf \{f(t): t \in[0, r]\})
\end{aligned}
$$

This is well defined because the function $r \rightarrow \inf \{f(t): t \in[0, r]\}$ is increasing and $r \rightarrow$ $\sup \{f(t): t \in[0, r]\}$ is decreasing. Also note that $\lim _{r \rightarrow 0+} f(r)$ exists if and only if

$$
\lim \sup _{r \rightarrow 0+} f(r)=\lim \inf _{r \rightarrow 0+} f(r)
$$

and if this happens

$$
\lim _{r \rightarrow 0+} f(r)=\lim \inf _{r \rightarrow 0+} f(r)=\lim \sup _{r \rightarrow 0+} f(r) .
$$

The claims made in the above definition follow immediately from the definition of what is meant by a limit in $[-\infty, \infty]$ and are left for the reader.

Theorem 26.7.4 Let $\mu$ be a Borel measure on $\mathbb{R}^{n}$ then $\frac{d \mu}{d m}(\mathbf{x})$ exists in $[-\infty, \infty] m$ a.e.
Proof:Let $p<q$ and let $p, q$ be rational numbers. Define

$$
\begin{aligned}
N_{p q}(M) \equiv & \left\{\mathbf{x} \in \mathbb{R}^{n} \text { such that } \lim _{r \rightarrow 0+} \sup _{r \rightarrow 0} \frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))}>q\right. \\
& \left.>p>\lim _{r \rightarrow 0+} \frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))}\right\} \cap B(\mathbf{0}, M) \\
N_{p q} \equiv & \left\{\mathbf{x} \in \mathbb{R}^{n} \text { such that } \lim _{r \rightarrow 0+} \sup \frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))}>q\right. \\
& \left.>p>\lim _{r \rightarrow 0+} \frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))}\right\} \\
N \equiv & \left\{\mathbf{x} \in \mathbb{R}^{n} \text { such that } \lim _{r \rightarrow 0+} \frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))}>\right. \\
& \left.\lim \inf _{r \rightarrow 0+} \frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))}\right\}
\end{aligned}
$$

I will show $\bar{m}\left(N_{p q}(M)\right)=0$. Use outer regularity to obtain an open set, $V$ containing $N_{p q}(M)$ such that

$$
\bar{m}\left(N_{p q}(M)\right)+\varepsilon>m(V) .
$$

From the definition of $N_{p q}(M)$, it follows that for each $\mathbf{x} \in N_{p q}(M)$ there exist arbitrarily small $r>0$ such that

$$
\frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))}<p
$$

Only consider those $r$ which are small enough to be contained in $B(\mathbf{0}, M)$ so that the collection of such balls has bounded radii. This is a Vitali cover of $N_{p q}(M)$ and so by Corollary 26.7.1 there exists a sequence of disjoint balls of this sort, $\left\{B_{i}\right\}_{i=1}^{\infty}$ such that

$$
\begin{equation*}
\mu\left(B_{i}\right)<p m\left(B_{i}\right), \bar{m}\left(N_{p q}(M) \backslash \cup_{i=1}^{\infty} B_{i}\right)=0 \tag{26.7.22}
\end{equation*}
$$

Now for $\mathbf{x} \in N_{p q}(M) \cap\left(\cup_{i=1}^{\infty} B_{i}\right)$ (most of $N_{p q}(M)$ ), there exist arbitrarily small balls, $B(\mathbf{x}, r)$, such that $B(\mathbf{x}, r)$ is contained in some set of $\left\{B_{i}\right\}_{i=1}^{\infty}$ and

$$
\frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))}>q
$$

This is a Vitali cover of $N_{p q}(M) \cap\left(\cup_{i=1}^{\infty} B_{i}\right)$ and so there exists a sequence of disjoint balls of this sort, $\left\{B_{j}^{\prime}\right\}_{j=1}^{\infty}$ such that

$$
\begin{equation*}
\bar{m}\left(\left(N_{p q}(M) \cap\left(\cup_{i=1}^{\infty} B_{i}\right)\right) \backslash \cup_{j=1}^{\infty} B_{j}^{\prime}\right)=0, \mu\left(B_{j}^{\prime}\right)>q m\left(B_{j}^{\prime}\right) \tag{26.7.23}
\end{equation*}
$$

It follows from 26.7.22 and 26.7.23 that

$$
\begin{equation*}
\bar{m}\left(N_{p q}(M)\right) \leq \bar{m}\left(\left(N_{p q}(M) \cap\left(\cup_{i=1}^{\infty} B_{i}\right)\right)\right) \leq m\left(\cup_{j=1}^{\infty} B_{j}^{\prime}\right) \tag{26.7.24}
\end{equation*}
$$

Therefore,

$$
\begin{aligned}
\sum_{j} \mu\left(B_{j}^{\prime}\right) & >q \sum_{j} m\left(B_{j}^{\prime}\right) \geq q \bar{m}\left(N_{p q}(M) \cap\left(\cup_{i} B_{i}\right)\right)=q \bar{m}\left(N_{p q}(M)\right) \\
& \geq p \bar{m}\left(N_{p q}(M)\right) \geq p(m(V)-\varepsilon) \geq p \sum_{i} m\left(B_{i}\right)-p \varepsilon \\
& \geq \sum_{i} \mu\left(B_{i}\right)-p \varepsilon \geq \sum_{j} \mu\left(B_{j}^{\prime}\right)-p \varepsilon
\end{aligned}
$$

It follows

$$
p \varepsilon \geq(q-p) \bar{m}\left(N_{p q}(M)\right)
$$

Since $\varepsilon$ is arbitrary, $m\left(N_{p q}(M)\right)=0$. Now $N_{p q} \subseteq \cup_{M=1}^{\infty} N_{p q}(M)$ and so $m\left(N_{p q}\right)=0$. Now

$$
N=\cup_{p . q \in \mathbb{Q}} N_{p q}
$$

and since this is a countable union of sets of measure zero, $m(N)=0$ also. This proves the theorem.

From Theorem 20.2.5 on Page 605 it follows that if $\mu$ is a complex measure then $|\mu|$ is a finite measure. This makes possible the following definition.

Definition 26.7.5 Let $\mu$ be a real measure. Define the following measures. For $E$ a measurable set,

$$
\begin{aligned}
\mu^{+}(E) & \equiv \frac{1}{2}(|\mu|+\mu)(E) \\
\mu^{-}(E) & \equiv \frac{1}{2}(|\mu|-\mu)(E)
\end{aligned}
$$

These are measures thanks to Theorem 20.2.3 on Page 603 and $\mu^{+}-\mu^{-}=\mu$. These measures have values in $[0, \infty)$. They are called the positive and negative parts of $\mu$ respectively. For $\mu$ a complex measure, define $\operatorname{Re} \mu$ and $\operatorname{Im} \mu$ by

$$
\begin{aligned}
\operatorname{Re} \mu(E) & \equiv \frac{1}{2}(\mu(E)+\overline{\mu(E)}) \\
\operatorname{Im} \mu(E) & \equiv \frac{1}{2 i}(\mu(E)-\overline{\mu(E)})
\end{aligned}
$$

Then $\operatorname{Re} \mu$ and $\operatorname{Im} \mu$ are both real measures. Thus for $\mu$ a complex measure,

$$
\begin{aligned}
\mu & =\operatorname{Re} \mu^{+}-\operatorname{Re} \mu^{-}+i\left(\operatorname{Im} \mu^{+}-\operatorname{Im} \mu^{-}\right) \\
& =v_{1}-v_{1}+i\left(v_{3}-v_{4}\right)
\end{aligned}
$$

where each $v_{i}$ is a real measure having values in $[0, \infty)$.
Then there is an obvious corollary to Theorem 26.7.4.
Corollary 26.7.6 Let $\mu$ be a complex Borel measure on $\mathbb{R}^{n}$. Then $\frac{d \mu}{d m}(\mathbf{x})$ exists a.e.
Proof: Letting $v_{i}$ be defined in Definition 26.7.5. By Theorem 26.7.4, for $m$ a.e. $\mathbf{x}$, $\frac{d v_{i}}{d m}(\mathbf{x})$ exists. This proves the corollary because $\mu$ is just a finite sum of these $v_{i}$.

Theorem 20.1.2 on Page 597, the Radon Nikodym theorem, implies that if you have two finite measures, $\mu$ and $\lambda$, you can write $\lambda$ as the sum of a measure absolutely continuous with respect to $\mu$ and one which is singular to $\mu$ in a unique way. The next topic is related to this. It has to do with the differentiation of a measure which is singular with respect to Lebesgue measure.

Theorem 26.7.7 Let $\mu$ be a Radon measure on $\mathbb{R}^{n}$ and suppose there exists a $\mu$ measurable set, $N$ such that for all Borel sets, $E, \mu(E)=\mu(E \cap N)$ where $\bar{m}(N)=0$. Then

$$
\frac{d \mu}{d m}(\mathbf{x})=0 m \text { a.e. }
$$

Proof: For $k \in \mathbb{N}$, let

$$
\begin{aligned}
B_{k}(M) & \equiv\left\{\mathbf{x} \in N^{C}: \lim \sup _{r \rightarrow 0+} \frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))}>\frac{1}{k}\right\} \cap B(\mathbf{0}, M) \\
B_{k} & \equiv\left\{\mathbf{x} \in N^{C}: \lim \sup _{r \rightarrow 0+} \frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))}>\frac{1}{k}\right\} \\
B & \equiv\left\{\mathbf{x} \in N^{C}: \lim _{\sup _{r \rightarrow 0+}} \frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))}>0\right\}
\end{aligned}
$$

Let $\varepsilon>0$. Since $\mu$ is regular, there exists $H$, a compact set such that $H \subseteq N \cap B(\mathbf{0}, M)$ and

$$
\mu(N \cap B(\mathbf{0}, M) \backslash H)<\varepsilon
$$



For each $\mathbf{x} \in B_{k}(M)$, there exist arbitrarily small $r>0$ such that $B(\mathbf{x}, r) \subseteq B(\mathbf{0}, M) \backslash H$ and

$$
\begin{equation*}
\frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))}>\frac{1}{k} \tag{26.7.25}
\end{equation*}
$$

Two such balls are illustrated in the above picture. This is a Vitali cover of $B_{k}(M)$ and so there exists a sequence of disjoint balls of this sort, $\left\{B_{i}\right\}_{i=1}^{\infty}$ such that $\bar{m}\left(B_{k}(M) \backslash \cup_{i} B_{i}\right)=0$. Therefore,

$$
\begin{aligned}
\bar{m}\left(B_{k}(M)\right) & \leq \bar{m}\left(B_{k}(M) \cap\left(\cup_{i} B_{i}\right)\right) \leq \sum_{i} \bar{m}\left(B_{i}\right) \leq k \sum_{i} \mu\left(B_{i}\right) \\
& =k \sum_{i} \mu\left(B_{i} \cap N\right)=k \sum_{i} \mu\left(B_{i} \cap N \cap B(\mathbf{0}, M)\right) \\
& \leq k \mu(N \cap B(\mathbf{0}, M) \backslash H)<\varepsilon k
\end{aligned}
$$

Since $\varepsilon$ was arbitrary, this shows $\bar{m}\left(B_{k}(M)\right)=0$.
Therefore,

$$
\bar{m}\left(B_{k}\right) \leq \sum_{M=1}^{\infty} \bar{m}\left(B_{k}(M)\right)=0
$$

and $\bar{m}(B) \leq \sum_{k} \bar{m}\left(B_{k}\right)=0$. Since $\bar{m}(N)=0$, this proves the theorem.
It is easy to obtain a different version of the above theorem. This is done with the aid of the following lemma.

Lemma 26.7.8 Suppose $\mu$ is a Borel measure on $\mathbb{R}^{n}$ having values in $[0, \infty)$. Then there exists a Radon measure, $\mu_{1}$ such that $\mu_{1}=\mu$ on all Borel sets.

Proof: By assumption, $\mu\left(\mathbb{R}^{n}\right)<\infty$ and so it is possible to define a positive linear functional, $L$ on $C_{c}\left(\mathbb{R}^{n}\right)$ by

$$
L f \equiv \int f d \mu
$$

By the Riesz representation theorem for positive linear functionals of this sort, there exists a unique Radon measure, $\mu_{1}$ such that for all $f \in C_{c}\left(\mathbb{R}^{n}\right)$,

$$
\int f d \mu_{1}=L f=\int f d \mu
$$

Now let $V$ be an open set and let $K_{k} \equiv\left\{\mathbf{x} \in V\right.$ : $\left.\operatorname{dist}\left(\mathbf{x}, V^{C}\right) \leq 1 / k\right\} \cap \overline{B(\mathbf{0}, k)}$. Then $\left\{K_{k}\right\}$ is an incresing sequence of compact sets whose union is $V$. Let $K_{k} \prec f_{k} \prec V$. Then $f_{k}(\mathbf{x}) \rightarrow$ $\mathscr{X}_{V}(\mathbf{x})$ for every $\mathbf{x}$. Therefore,

$$
\mu_{1}(V)=\lim _{k \rightarrow \infty} \int f_{k} d \mu_{1}=\lim _{k \rightarrow \infty} \int f_{k} d \mu=\mu(V)
$$

and so $\mu=\mu_{1}$ on open sets. Now if $K$ is a compact set, let

$$
V_{k} \equiv\left\{\mathbf{x} \in \mathbb{R}^{n}: \operatorname{dist}(\mathbf{x}, K)<1 / k\right\}
$$

Then $V_{k}$ is an open set and $\cap_{k} V_{k}=K$. Letting $K \prec f_{k} \prec V_{k}$, it follows that $f_{k}(\mathbf{x}) \rightarrow \mathscr{X}_{K}(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^{n}$. Therefore, by the dominated convergence theorem with a dominating function, $\mathscr{X}_{\mathbb{R}^{n}}$

$$
\mu_{1}(K)=\lim _{k \rightarrow \infty} \int f_{k} d \mu_{1}=\lim _{k \rightarrow \infty} \int f_{k} d \mu=\mu(K)
$$

and so $\mu$ and $\mu_{1}$ are equal on all compact sets. It follows $\mu=\mu_{1}$ on all countable unions of compact sets and countable intersections of open sets.

Now let $E$ be a Borel set. By regularity of $\mu_{1}$, there exist sets, $H$ and $G$ such that $H$ is the countable union of an increasing sequence of compact sets, $G$ is the countable intersection of a decreasing sequence of open sets, $H \subseteq E \subseteq G$, and $\mu_{1}(H)=\mu_{1}(G)=$ $\mu_{1}(E)$. Therefore,

$$
\mu_{1}(H)=\mu(H) \leq \mu(E) \leq \mu(G)=\mu_{1}(G)=\mu_{1}(E)=\mu_{1}(H)
$$

therefore, $\mu(E)=\mu_{1}(E)$ and this proves the lemma.
Corollary 26.7.9 Suppose $\mu$ is a complex Borel measure defined on $\mathbb{R}^{n}$ for which there exists a $\mu$ measurable set, $N$ such that for all Borel sets, $E, \mu(E)=\mu(E \cap N)$ where $\bar{m}(N)=0$. Then

$$
\frac{d \mu}{d m}(\mathbf{x})=0 m \text { a.e. }
$$

Proof: Each of $\operatorname{Re} \mu^{+}, \operatorname{Re} \mu^{-}, \operatorname{Im} \mu^{+}$, and $\operatorname{Im} \mu^{-}$are real measures having values in $[0, \infty)$ and so by Lemma 26.7 .8 each is a Radon measure having the same property that $\mu$ has in terms of being supported on a set of $m$ measure zero. Therefore, for $v$ equal to any of these, $\frac{d v}{d m}(\mathbf{x})=0 m$ a.e. This proves the corollary.

### 26.8 Exercises

1. Suppose $A$ and $B$ are sets of positive Lebesgue measure in $\mathbb{R}^{n}$. Show that $A-B$ must contain $B(\mathbf{c}, \varepsilon)$ for some $\mathbf{c} \in \mathbb{R}^{n}$ and $\varepsilon>0$.

$$
A-B \equiv\{\mathbf{a}-\mathbf{b}: \mathbf{a} \in A \text { and } \mathbf{b} \in B\}
$$

Hint: First assume both sets are bounded. This creates no loss of generality. Next there exist $\mathbf{a}_{0} \in A, \mathbf{b}_{0} \in B$ and $\delta>0$ such that

$$
\int_{B\left(a_{0}, \delta\right)} \mathscr{X}_{A}(t) d t>\frac{3}{4} m\left(B\left(\mathbf{a}_{0}, \boldsymbol{\delta}\right)\right), \int_{B\left(b_{0}, \delta\right)} \mathscr{X}_{B}(t) d t>\frac{3}{4} m\left(B\left(\mathbf{b}_{0}, \boldsymbol{\delta}\right)\right) .
$$

Now explain why this implies

$$
m\left(A-\mathbf{a}_{0} \cap B(\mathbf{0}, \boldsymbol{\delta})\right)>\frac{3}{4} m(B(\mathbf{0}, \boldsymbol{\delta}))
$$

and

$$
m\left(B-\mathbf{b}_{0} \cap B(\mathbf{0}, \boldsymbol{\delta})\right)>\frac{3}{4} m(B(\mathbf{0}, \boldsymbol{\delta}))
$$

Explain why

$$
m\left(\left(A-\mathbf{a}_{0}\right) \cap\left(B-\mathbf{b}_{0}\right)\right)>\frac{1}{2} m(B(\mathbf{0}, \delta))>0
$$

Let

$$
f(\mathbf{x}) \equiv \int \mathscr{X}_{A-\mathbf{a}_{0}}(\mathbf{x}+\mathbf{t}) \mathscr{X}_{B-\mathbf{b}_{0}}(\mathbf{t}) d t
$$

Explain why $f(\mathbf{0})>0$. Next explain why $f$ is continuous and why $f(\mathbf{x})>0$ for all $\mathbf{x} \in B(\mathbf{0}, \varepsilon)$ for some $\varepsilon>0$. Thus if $|\mathbf{x}|<\varepsilon$, there exists $\mathbf{t}$ such that $\mathbf{x}+\mathbf{t} \in A-\mathbf{a}_{0}$ and $\mathbf{t} \in B-\mathbf{b}_{0}$. Subtract these.
2. Show $M f$ is Borel measurable by verifying that $[M f>\lambda] \equiv E_{\lambda}$ is actually an open set. Hint: If $\mathbf{x} \in E_{\lambda}$ then for some $r, \int_{B(\mathbf{x}, r)}|f| d m>\lambda m(B(\mathbf{x}, r))$. Then for $\delta$ a small enough positive number, $\int_{B(\mathbf{x}, r)}|f| d m>\lambda m(B(\mathbf{x}, r+2 \delta))$. Now pick $\mathbf{y} \in B(\mathbf{x}, \delta)$ and argue that $B(\mathbf{y}, \delta+r) \supseteq B(\mathbf{x}, r)$. Therefore show that,

$$
\int_{B(\mathbf{y}, \delta+r)}|f| d m>\int_{B(\mathbf{x}, \mathbf{r})}|f| d m>\lambda B(\mathbf{x}, r+2 \delta) \geq \lambda m(B(\mathbf{y}, r+\delta))
$$

Thus $B(\mathbf{x}, \delta) \subseteq E_{\lambda}$.
3. Consider the following nested sequence of compact sets, $\left\{P_{n}\right\}$. Let $P_{1}=[0,1], P_{2}=$ $\left[0, \frac{1}{3}\right] \cup\left[\frac{2}{3}, 1\right]$, etc. To go from $P_{n}$ to $P_{n+1}$, delete the open interval which is the middle third of each closed interval in $P_{n}$. Let $P=\cap_{n=1}^{\infty} P_{n}$. By the finite intersection property of compact sets, $P \neq \emptyset$. Show $m(P)=0$. If you feel ambitious also show there is a one to one onto mapping of $[0,1]$ to $P$. The set $P$ is called the Cantor set. Thus, although $P$ has measure zero, it has the same number of points in it as $[0,1]$ in the sense that there is a one to one and onto mapping from one to the other. Hint: There are various ways of doing this last part but the most enlightenment is obtained by exploiting the topological properties of the Cantor set rather than some silly representation in terms of sums of powers of two and three. All you need to do is use the Schroder Bernstein theorem and show there is an onto map from the Cantor set to $[0,1]$. If you do this right and remember the theorems about characterizations of compact metric spaces, Proposition 7.6 .5 on Page 144, you may get a pretty good idea why every compact metric space is the continuous image of the Cantor set.
4. Consider the sequence of functions defined in the following way. Let $f_{1}(x)=x$ on $[0,1]$. To get from $f_{n}$ to $f_{n+1}$, let $f_{n+1}=f_{n}$ on all intervals where $f_{n}$ is constant. If $f_{n}$ is nonconstant on $[a, b]$, let $f_{n+1}(a)=f_{n}(a), f_{n+1}(b)=f_{n}(b), f_{n+1}$ is piecewise linear and equal to $\frac{1}{2}\left(f_{n}(a)+f_{n}(b)\right)$ on the middle third of $[a, b]$. Sketch a few of these and you will see the pattern. The process of modifying a nonconstant section of the graph of this function is illustrated in the following picture.


Show $\left\{f_{n}\right\}$ converges uniformly on $[0,1]$. If $f(x)=\lim _{n \rightarrow \infty} f_{n}(x)$, show that $f(0)=$ $0, f(1)=1, f$ is continuous, and $f^{\prime}(x)=0$ for all $x \notin P$ where $P$ is the Cantor set of Problem 3. This function is called the Cantor function.It is a very important example to remember. Note it has derivative equal to zero a.e. and yet it succeeds in climbing from 0 to 1 . Explain why this interesting function is not absolutely continuous although it is continuous. Hint: This isn't too hard if you focus on getting a careful estimate on the difference between two successive functions in the list considering only a typical small interval in which the change takes place. The above picture should be helpful.
5. A function, $f:[a, b] \rightarrow \mathbb{R}$ is Lipschitz if $|f(x)-f(y)| \leq K|x-y|$. Show that every Lipschitz function is absolutely continuous. Thus every Lipschitz function is differentiable a.e., $f^{\prime} \in L^{1}$, and $f(y)-f(x)=\int_{x}^{y} f^{\prime}(t) d t$.
6. Suppose $f, g$ are both absolutely continuous on $[a, b]$. Show the product of these functions is also absolutely continuous. Explain why $(f g)^{\prime}=f^{\prime} g+g^{\prime} f$ and show the usual integration by parts formula

$$
f(b) g(b)-f(a) g(a)-\int_{a}^{b} f g^{\prime} d t=\int_{a}^{b} f^{\prime} g d t
$$

7. In Problem $4 f^{\prime}$ failed to give the expected result for $\int_{a}^{b} f^{\prime} d x^{1}$ but at least $f^{\prime} \in L^{1}$. Suppose $f^{\prime}$ exists for $f$ a continuous function defined on $[a, b]$. Does it follow that $f^{\prime}$ is measurable? Can you conclude $f^{\prime} \in L^{1}([a, b])$ ?
8. A sequence of sets, $\left\{E_{i}\right\}$ containing the point $\mathbf{x}$ is said to shrink to $\mathbf{x}$ nicely if there exists a sequence of positive numbers, $\left\{r_{i}\right\}$ and a positive constant, $\alpha$ such that $r_{i} \rightarrow 0$ and

$$
m\left(E_{i}\right) \geq \alpha m\left(B\left(\mathbf{x}, r_{i}\right)\right), E_{i} \subseteq B\left(\mathbf{x}, r_{i}\right)
$$

Show the above theorems about differentiation of measures with respect to Lebesgue measure all have a version valid for $E_{i}$ replacing $B(\mathbf{x}, r)$.
9. Suppose $F(x)=\int_{a}^{x} f(t) d t$. Using the concept of nicely shrinking sets in Problem 8 show $F^{\prime}(x)=f(x)$ a.e.

[^23]10. A random variable, $X$ is a measurable real valued function defined on a measure space, $(\Omega, \mathscr{S}, P)$ where $P$ is just a measure with $P(\Omega)=1$ called a probability measure. The distribution function for $X$ is the function, $F(x) \equiv P([X \leq x])$ in words, $F(x)$ is the probability that $X$ has values no larger than $x$. Show that $F$ is a right continuous increasing function with the property that $\lim _{x \rightarrow-\infty} F(x)=0$ and $\lim _{x \rightarrow \infty} F(x)=1$.
11. Suppose $F$ is an increasing right continuous function.
(a) Show that $L f \equiv \int_{a}^{b} f d F$ is a well defined positive linear functional on $C_{c}(\mathbb{R})$ where here $[a, b]$ is a closed interval containing the support of $f \in C_{c}(\mathbb{R})$.
(b) Using the Riesz representation theorem for positive linear functionals on $C_{c}(\mathbb{R})$, let $\mu$ denote the Radon measure determined by $L$. Show that $\mu((a, b])=F(b)-$ $F(a)$ and $\mu(\{b\})=F(b)-F(b-)$ where $F(b-) \equiv \lim _{x \rightarrow b-} F(x)$.
(c) Review Corollary 20.1.4 on Page 601 at this point. Show that the conditions of this corollary hold for $\mu$ and $m$. Consider $\mu_{\perp}+\mu_{\|}$, the Lebesgue decomposition of $\mu$ where $\mu_{\|} \ll m$ and there exists a set of $m$ measure zero, $N$ such that $\mu_{\perp}(E)=\mu_{\perp}(E \cap N)$. Show $\mu((0, x])=\mu_{\perp}((0, x])+\int_{0}^{x} h(t) d t$ for some $h \in L^{\perp}(m)$. Using Theorem 26.7.7 show $h(x)=F^{\prime}(x) m$ a.e. Explain why $F(x)=F(0)+S(x)+\int_{0}^{x} F^{\prime}(t) d t$ for some function, $S(x)$ which is increasing but has $S^{\prime}(x)=0$ a.e. Note this shows in particular that a right continuous increasing function has a derivative a.e.
12. Suppose now that $G$ is just an increasing function defined on $\mathbb{R}$. Show that $G^{\prime}(x)$ exists a.e. Hint: You can mimic the proof of Theorem 26.7.4. The Dini derivates are defined as
\[

$$
\begin{aligned}
D_{+} G(x) & \equiv \lim _{\inf _{h \rightarrow 0+}} \frac{G(x+h)-G(x)}{h} \\
D^{+} G(x) & \equiv \lim \sup _{h \rightarrow 0+} \frac{G(x+h)-G(x)}{h} \\
D_{-} G(x) & \equiv \lim \inf _{h \rightarrow 0+} \frac{G(x)-G(x-h)}{h} \\
D^{-} G(x) & \equiv \lim \sup _{h \rightarrow 0+} \frac{G(x)-G(x-h)}{h}
\end{aligned}
$$
\]

When $D_{+} G(x)=D^{+} G(x)$ the derivative from the right exists and when $D^{-} G(x)=$ $D_{-} G(x)$, then the derivative from the left exists. Let $(a, b)$ be an open interval and let

$$
N_{p q} \equiv\left\{x \in(a, b): D^{+} G(x)>q>p>D_{+} G(x)\right\} .
$$

Let $V \subseteq(a, b)$ be an open set containing $N_{p q}$ such that $m(V)<m\left(N_{p q}\right)+\varepsilon$. Show using a Vitali covering theorem there is a disjoint sequence of intervals contained in
$V,\left\{\left(x_{i}, x_{i}+h_{i}\right)\right\}_{i=1}^{\infty}$ such that

$$
\frac{G\left(x_{i}+h_{i}\right)-G\left(x_{i}\right)}{h_{i}}<p
$$

Next show there is a disjoint sequence of intervals $\left\{\left(x_{i}^{\prime}, x_{j}^{\prime}+h_{j}^{\prime}\right)\right\}_{j=1}^{\infty}$ such that each of these is contained in one of the former intervals and

$$
\frac{G\left(x_{j}^{\prime}+h_{j}^{\prime}\right)-G\left(x_{j}^{\prime}\right)}{h_{j}^{\prime}}>q, \sum_{j} h_{j}^{\prime} \geq m\left(N_{p q}\right) .
$$

Then

$$
\begin{aligned}
q m\left(N_{p q}\right) & \leq q \sum_{j} h_{j}^{\prime} \leq \sum_{j} G\left(x_{j}^{\prime}+h_{j}^{\prime}\right)-G\left(x_{j}^{\prime}\right) \leq \sum_{i} G\left(x_{i}+h_{i}\right)-G\left(x_{i}\right) \\
& \leq p \sum_{i} h_{i} \leq p m(V) \leq p\left(m\left(N_{p q}\right)+\varepsilon\right)
\end{aligned}
$$

Since $\varepsilon$ was arbitrary, this shows $m\left(N_{p q}\right)=0$. Taking a union of all $N_{p q}$ for $p, q$ rational, shows the derivative from the right exists a.e. Do a similar argument to show the derivative from the left exists a.e. and then show the derivative from the left equals the derivative from the right a.e. using a simlar argument. Thus $G^{\prime}(x)$ exists on $(a, b)$ a.e. and so it exists a.e. on $\mathbb{R}$ because $(a, b)$ was arbitrary.

## Chapter 27

## Orlitz Spaces

### 27.1 Basic Theory

All the theorems about the $L^{p}$ spaces have generalizations to something called an Orlitz space. [1], [94] Instead of the convex function, $A(t)=t^{p} / p$, one considers a more general convex increasing function called an $N$ function.

Definition 27.1.1 $A:[0, \infty) \rightarrow[0, \infty)$ is an $N$ function if the following two conditions hold.

$$
\begin{align*}
& A \text { is convex and strictly increasing }  \tag{27.1.1}\\
& \lim _{t \rightarrow 0+} \frac{A(t)}{t}=0, \lim _{t \rightarrow \infty} \frac{A(t)}{t}=\infty . \tag{27.1.2}
\end{align*}
$$

For $A$ an $N$ function,

$$
\begin{equation*}
\widetilde{A}(s) \equiv \max \{s t-A(t): t \geq 0\} \tag{27.1.3}
\end{equation*}
$$

As an example see the following picture of a typical $N$ function.


Note that from the assumption, 27.1.2 the maximum in the definition of $\widetilde{A}$ must exist. This is because for $t \neq 0$

$$
(s-A(t) / t) t
$$

is negative for all $t$ large enough. On the other hand, it equals 0 when $t=0$ and so it suffices to consider only $t$ in a compact set.

Lemma 27.1.2 Let $\phi: \mathbb{R} \rightarrow \mathbb{R}$ be a convex function. Then $\phi$ is Lipschitz continuous on $[a, b]$.

Proof: Since it is convex, the difference quotients,

$$
\frac{\phi(t)-\phi(a)}{t-a}
$$

are increasing because by convexity, if $a<t<x$

$$
\frac{t-a}{x-a} \phi(x)+\left(1-\frac{t-a}{x-a}\right) \phi(a) \geq \phi(t)
$$

and this reduces to

$$
\frac{\phi(t)-\phi(a)}{t-a} \leq \frac{\phi(x)-\phi(a)}{x-a}
$$

Also these difference quotients are bounded below by

$$
\frac{\phi(a)-\phi(a-1)}{1}=\phi(a)-\phi(a-1) .
$$

Let

$$
A \equiv \inf \left\{\frac{\phi(t)-\phi(a)}{t-a}: t \in(a, b)\right\}
$$

Then $A$ is some finite real number. Similarly there exists a real number $B$ such that for all $t \in(a, b)$,

$$
B \geq \frac{\phi(b)-\phi(t)}{b-t}
$$

Now let $a \leq s<t \leq b$. Then

$$
\frac{\phi(t)-\phi(s)}{t-s} \geq \frac{\phi(t)-\theta \phi(a)-(1-\theta) \phi(t)}{t-s}
$$

where $\theta$ is such that $\theta a+(1-\theta) t=s$. Thus

$$
\theta=\frac{t-s}{t-t_{1}}
$$

and so the above implies

$$
\frac{\phi(t)-\phi(s)}{t-s} \geq \frac{t-s}{t-t_{1}} \frac{\phi(t)-\phi(a)}{t-s}=\frac{\phi(t)-\phi(a)}{t-t_{1}} \geq A .
$$

Similarly,

$$
\begin{aligned}
\frac{\phi(t)-\phi(s)}{t-s} & \leq \frac{\theta \phi(b)+(1-\theta) \phi(s)-\phi(s)}{t-s} \\
& =\frac{t-s}{b-s} \frac{\phi(b)-\phi(s)}{t-s} \leq B
\end{aligned}
$$

It follows

$$
|\phi(t)-\phi(s)| \leq(|A|+|B|)|t-s|
$$

and this proves the lemma.
The following is like the inequality, $s t \leq t^{p} / p+s^{q} / q$, important in the study of $L^{p}$ spaces.

Proposition 27.1.3 If $A$ is an $N$ function, then so is $\widetilde{A}$ and

$$
\begin{equation*}
A(t)=\max \{t s-\widetilde{A}(s): s \geq 0\} \tag{27.1.4}
\end{equation*}
$$

so $\widetilde{\widetilde{A}}=A$. Also

$$
\begin{equation*}
s t \leq A(t)+\widetilde{A}(s) \text { for all } s, t \geq 0 \tag{27.1.5}
\end{equation*}
$$

and for all $s>0$,

$$
\begin{equation*}
A\left(\frac{\widetilde{A}(s)}{s}\right) \leq \widetilde{A}(s) \tag{27.1.6}
\end{equation*}
$$

Proof: First consider the claim $\widetilde{A}$ is convex. Let $\lambda \in[0,1]$.

$$
\begin{aligned}
& \widetilde{A}\left(\lambda s_{1}+(1-\lambda) s_{2}\right) \equiv \max \left\{\left[s_{1} \lambda+(1-\lambda) s_{2}\right] t-A(t): t \geq 0\right\} \\
& \leq \lambda \max \left\{s_{1} t-A(t): t \geq 0\right\}+(1-\lambda) \max \left\{s_{2} t-A(t): t \geq 0\right\} \\
& =\lambda \widetilde{A}\left(s_{1}\right)+(1-\lambda) \widetilde{A}\left(s_{2}\right)
\end{aligned}
$$

It is obvious $\widetilde{A}$ is stictly increasing because $s t$ is strictly increasing in $s$. Next consider 27.1.2.

For $s>0$ let $t_{s}$ denote the number where the maximum is achieved. That is,

$$
\widetilde{A}(s) \equiv s t_{s}-A\left(t_{s}\right)
$$

Thus

$$
\begin{equation*}
\frac{\widetilde{A}(s)}{s}=t_{s}-\frac{A\left(t_{s}\right)}{s} \geq 0 \tag{27.1.7}
\end{equation*}
$$

It follows from this that

$$
\lim _{s \rightarrow 0+} t_{s}=0
$$

since otherwise, a contradiction results to 27.1.7, the expression becoming negative for small enough $s$. Thus

$$
t_{s} \geq \frac{\widetilde{A}(s)}{s} \geq 0
$$

and this shows

$$
\lim _{s \rightarrow 0+} \frac{\widetilde{A}(s)}{s}=0
$$

which shows 27.1.2.
To verify the second part of 27.1.2, let $t_{s}$ be as just described. Then for any $t>0$

$$
\frac{\widetilde{A}(s)}{s}=t_{s}-\frac{A\left(t_{s}\right)}{s} \geq t-\frac{A(t)}{s}
$$

It follows

$$
\lim \inf _{s \rightarrow \infty} \frac{\widetilde{A}(s)}{s} \geq t
$$

Since $t$ is arbitrary, this proves the second part of 27.1.2.
The inequality 27.1.5 follows from the definition of $\widetilde{A}(s)$.

Next consider 27.1.4. It must be shown that

$$
A\left(t_{0}\right)=\max \left\{t_{0} s-\widetilde{A}(s): s \geq 0\right\}
$$

To do so, first note

$$
\widetilde{A}(s)=\max \{s t-A(t): t \geq 0\} \geq s t_{0}-A\left(t_{0}\right)
$$

Hence

$$
\max \left\{t_{0} s-\widetilde{A}(s): s \geq 0\right\} \leq \max \left\{t_{0} s-\left[s t_{0}-A\left(t_{0}\right)\right]\right\}=A\left(t_{0}\right)
$$

Now let

$$
s_{0} \equiv \inf \left\{\frac{A(t)-A\left(t_{0}\right)}{t-t_{0}}: t>t_{0}\right\}
$$

By convexity, the above difference quotients are nondecreasing in $t$ and so

$$
s_{0}\left(t-t_{0}\right) \leq A(t)-A\left(t_{0}\right)
$$

for all $t \neq t_{0}$. Hence for all $t$,

$$
s_{0} t-A(t) \leq s_{0} t_{0}-A\left(t_{0}\right)
$$

and so

$$
\widetilde{A}\left(s_{0}\right)=s_{0} t_{0}-A\left(t_{0}\right)
$$

implying

$$
A\left(t_{0}\right)=s_{0} t_{0}-\widetilde{A}\left(s_{0}\right) \leq \max \left\{s t_{0}-\widetilde{A}(s): s \geq 0\right\} \leq A\left(t_{0}\right)
$$

Therefore, 27.1.4 holds.
Consider 27.1.6 next. To do so, let $a=A^{\prime}$ so that

$$
A(t)=\int_{0}^{t} a(r) d r, a \text { increasing. }
$$

This is possible by Rademacher's theorem, Corollary 26.4.3 and the fact that since $A$ is convex, it is locally Lipshitz found in Lemma 27.1.2 above. That $a$ is increasing follows from convexity of $A$. Here is why. For a.e. $s, t \geq 0$, and letting $\lambda \in[0,1]$,

$$
\begin{aligned}
\frac{A(s+\lambda(t-s))-A(s)}{\lambda} & \leq \frac{(1-\lambda) A(s)+\lambda A(t)-A(s)}{\lambda} \\
& =A(t)-A(s)
\end{aligned}
$$

Then passing to a limit as $\lambda \rightarrow 0+$,

$$
a(s)(t-s) \leq A(t)-A(s)
$$

Similarly

$$
a(t)(s-t) \leq A(s)-A(t)
$$

and so

$$
(a(t)-a(s))(t-s) \geq 0
$$

(If you like, you can simply assume from the beginning that $A(t)$ is given this way as an integral of a positive increasing function, $a$, and verify directly that such an $A$ is convex and satisfies the properties of an $N$ function. There is no loss of generality in doing so.) Thus geometrically, $A(t)$ equals the area under the curve defined by $a$ and above the $x$ axis from $x=0$ to $x=t$. In the definition of $\widetilde{A}(s)$ let $t_{s}$ be the point where the maximum is achieved. Then

$$
\widetilde{A}(s)=s t_{s}-A\left(t_{s}\right)
$$

and so at this point, $\widetilde{A}(s)+A\left(t_{s}\right)=s t_{s}$. This means that $\tilde{A}(s)$ is the area to the left of the graph of $a$ which is to the right of the $y$ axis for $y$ between 0 and $a\left(t_{s}\right)$ and that in fact $a\left(t_{s}\right)=s$. The following picture illustrates the reasoning which follows.


Therefore,

$$
\begin{aligned}
\frac{\tilde{A}(s)}{s} & =t_{s}-\frac{A\left(t_{s}\right)}{s}=t_{s}-\frac{1}{s} \int_{0}^{t_{s}} a(r) d r \\
& =t_{s}-\frac{1}{a\left(t_{s}\right)} \int_{0}^{t_{s}} a(r) d r=\frac{1}{a\left(t_{s}\right)}\left(t_{s} s-\int_{0}^{t_{s}} a(r) d r\right)
\end{aligned}
$$

and so

$$
\begin{aligned}
A\left(\frac{\widetilde{A}(s)}{s}\right) & =\int_{0}^{\widetilde{A}(s) / s} a(r) d r=\int_{0}^{\frac{1}{a\left(t_{s}\right)} \int_{0}^{t_{s}}(s-a(r)) d r} a(\tau) d \tau \\
& \leq \int_{0}^{t_{s}} s-a(r) d r=s t_{s}-A\left(t_{s}\right)=\widetilde{A}(s)
\end{aligned}
$$

The inequality results from replacing $a(\tau)$ with $a\left(t_{s}\right)$ in the last integral on the top line.
An example of an $N$ function is $A(t)=\frac{t^{p}}{p}$ for $t \geq 0$ and $p>1$. For this example, $\widetilde{A}(s)=\frac{s^{p^{\prime}}}{p^{\prime}}$ where $\frac{1}{p}+\frac{1}{p^{\prime}}=1$.

Definition 27.1.4 Let $A$ be an $N$ function and let $(\Omega, \mathscr{S}, \mu)$ be a measure space. Define

$$
\begin{equation*}
K_{A}(\Omega) \equiv\left\{\text { u measurable such that } \int_{\Omega} A(|u|) d \mu<\infty\right\} \tag{27.1.8}
\end{equation*}
$$

This is called the Orlitz class. Also define

$$
\begin{equation*}
L_{A}(\Omega) \equiv\left\{\lambda u: u \in K_{A}(\Omega) \text { and } \lambda \in \mathbb{F}\right\} \tag{27.1.9}
\end{equation*}
$$

where $\mathbb{F}$ is the field of scalars, assumed to be either $\mathbb{R}$ or $\mathbb{C}$.
The pair $(A, \Omega)$ is called $\Delta$ regular if either of the following conditions hold.

$$
\begin{equation*}
A(r x) \leq K_{r} A(x) \text { for all } x \in[0, \infty) \tag{27.1.10}
\end{equation*}
$$

or $\mu(\Omega)<\infty$ and for all $r>0$, there exists $M_{r}$ and $K_{r}>0$ such that

$$
\begin{equation*}
A(r x) \leq K_{r} A(x) \text { for all } x \geq M_{r} \tag{27.1.11}
\end{equation*}
$$

Note there are $N$ functions which are not $\Delta$ regular. For example, consider

$$
A(x) \equiv e^{x^{2}}-1
$$

It can't be $\Delta$ regular because

$$
\lim _{r \rightarrow \infty} \frac{e^{r^{2} x^{2}}-1}{e^{x^{2}}-1}=\infty
$$

However, functions like $x^{p} / p$ for $p>1$ are $\Delta$ regular.
Then the following proposition is important.
Proposition 27.1.5 If $(A, \Omega)$ is $\Delta$ regular, then $K_{A}(\Omega)=L_{A}(\Omega)$. In any case, $L_{A}(\Omega)$ is a vector space and $K_{A}(\Omega) \subseteq L_{A}(\Omega)$.

Proof: Suppose $(A, \Omega)$ is $\Delta$ regular. Then I claim $K_{A}(\Omega)$ is a vector space. This will verify $K_{A}(\Omega)=L_{A}(\Omega)$. Let $f, g \in K_{A}(\Omega)$ and suppose 27.1.10. Then

$$
A(|f+g|)=A\left(2\left(\frac{|f+g|}{2}\right)\right) \leq K_{2} A\left(\frac{|f+g|}{2}\right) \leq K_{2} \frac{1}{2}[A(|f|)+A(|g|)]
$$

so $f+g \in K_{A}(\Omega)$ in this case. Now suppose 27.1.11

$$
\begin{gathered}
\int_{\Omega} A(|f+g|) d \mu=\int_{\left[|f+g| \leq M_{2}\right]} A(|f+g|) d \mu+\int_{\left[|f+g|>M_{2}\right]} A(|f+g|) d \mu \\
\leq A\left(M_{2}\right) \mu(\Omega)+\int_{\Omega} \frac{K_{2}}{2}(A(|f|)+A(|g|)) d \mu<\infty
\end{gathered}
$$

Thus $f+g \in K_{A}(\Omega)$ in this case also.
Next consider scalar multiplication. First consider the case of 27.1.10. If $f \in K_{A}(\Omega)$ and $\alpha \in \mathbb{F}$,

$$
\int_{\Omega} A(|\alpha||f|) d \mu \leq K_{|\alpha|} \int_{\Omega} A(f) d \mu
$$

so in the case of 27.1.10 $\alpha f \in K_{A}(\Omega)$ whenever $f \in K_{A}(\Omega)$. In the case of 27.1.11,

$$
\begin{aligned}
\int_{\Omega} A(|\alpha||f|) d \mu & =\int_{\left[|\alpha||f| \leq M_{|\alpha|}\right]} A(|\alpha||f|) d \mu+\int_{\left[|\alpha||f|>M_{|\alpha|}\right]} A(|\alpha||f|) d \mu \\
& \leq A\left(M_{|\alpha|}\right) \mu(\Omega)+\int_{\Omega} K_{|\alpha|} A(|f|) d \mu<\infty
\end{aligned}
$$

This establishes the first part of the proposition.
Next consider the claim that $L_{A}(\Omega)$ is always a vector space. First note $K_{A}(\Omega)$ is always convex due to convexity of $A$. Let $\lambda u, \alpha v \in L_{A}(\Omega)$ where $u, v \in K_{A}(\Omega)$ and let $a, b$ be scalars in $\mathbb{F}$. Then

$$
a \lambda u+b \alpha v=|a \lambda| \omega u+|b \alpha| \theta v
$$

where $|\omega|=|\theta|=1$. Then

$$
=(|a \lambda|+|b \alpha|)\left(\frac{|a \lambda| \omega u+|b \alpha| \theta v}{|a \lambda|+|b \alpha|}\right)
$$

which exhibits $a \lambda u+b \alpha v$ as a multiple of a convex combination of two elements of $K_{A}(\Omega), \omega u$ and $\theta v$. Thus $L_{A}(\Omega)$ is closed with respect to linear combinations. This shows it is a vector space. This proves the proposition.

The following norm for $L_{A}(\Omega)$ is due to Luxemburg [94]. You might compare this to the definition of a Minkowski functional. The definition of $L_{A}(\Omega)$ above was cooked up so that the following norm does make sense.

## Definition 27.1.6 Define

$$
\|u\|_{A}=\|u\|_{A, \Omega} \equiv \inf \left\{t>0: \int_{\Omega} A\left(\frac{|u(x)|}{t}\right) d \mu \leq 1\right\}
$$

If two functions of $L_{A}(\Omega)$ are equal a.e. they are considered to be the same in the usual way.

Proposition 27.1.7 The number defined in Definition 27.1.6 is a norm on $L_{A}(\Omega)$. Also, if $\Omega_{1} \subseteq \Omega$, then

$$
\|u\|_{A, \Omega_{1}} \leq\|u\|_{A, \Omega}
$$

Proof: Clearly $\|u\|_{A} \geq 0$. Is $\|u\|_{A}$ finite for $u \in L_{A}(\Omega)$ ? Let $u \in L_{A}(\Omega)$ so $u=\lambda v$ where $v \in K_{A}(\Omega)$. Then for $s>0$

$$
\int_{\Omega} A\left(\frac{|u|}{s|\lambda|}\right) d \mu=\int_{\Omega} A\left(\frac{|v|}{s}\right) d \mu<\infty
$$

whenever $s>1$. Therefore, from the dominated convergence theorem, if $s$ is large enough,

$$
\int_{\Omega} A\left(\frac{|u|}{s|\lambda|}\right) d \mu \leq 1
$$

and this shows there are values of $t>0$ such that

$$
\int_{\Omega} A\left(\frac{|u(x)|}{t}\right) d \mu \leq 1
$$

Thus $\|u\|_{A}$ is finite as hoped.

Now suppose $\|u\|_{A}=0$ and let

$$
E_{n} \equiv\left\{x:|u(x)| \geq \frac{1}{n}\right\}
$$

Then for arbitrarily small values of $t$,

$$
\int_{E_{n}} A\left(\frac{(1 / n)}{t}\right) d \mu \leq \int_{\Omega} A\left(\frac{|u(x)|}{t}\right) d \mu \leq 1
$$

and so for arbitrarily small values of $t$,

$$
A\left(\frac{(1 / n)}{t}\right) \mu\left(E_{n}\right) \leq 1
$$

Letting $t \rightarrow 0+$ yields a contradiction unless $\mu\left(E_{n}\right)=0$. Now

$$
\mu([|u(x)|>0]) \leq \sum_{n=1}^{\infty} \mu\left(E_{n}\right)=0
$$

Thus $u=0$ as claimed.
Consider the other axioms of a norm. Let $u, v \in L_{A}(\Omega)$ and let $\alpha, \beta$ be scalars. Then

$$
\|\alpha u+\beta v\|_{A} \equiv \inf \left\{t>0: \int_{\Omega} A\left(\frac{|u(x)+v(x)|}{t}\right) d \mu \leq 1\right\}
$$

Without loss of generality $\|u\|_{A},\|v\|_{A}<\infty$ since otherwise there is nothing to prove.

$$
\begin{aligned}
& \|u+v\|_{A} \equiv \inf \left\{t>0: \int_{\Omega} A\left(\frac{|\alpha u(x)+\beta v(x)|}{t}\right) d \mu \leq 1\right\} \\
& \leq \inf \left\{t>0: \int_{\Omega} A\left(\frac{|\alpha||u|+|\beta||v|}{t}\right) d \mu \leq 1\right\} \\
& =\inf \left\{t>0: \int_{\Omega} A\left(\frac{|\alpha| \frac{(|\alpha|+|\beta|)|u|}{t}+|\beta| \frac{(|\alpha|+|\beta|)|v|}{t}}{(|\alpha|+|\beta|)}\right) d \mu \leq 1\right\} \\
& \leq \quad \inf \left\{t>0: \frac{|\alpha|}{(|\alpha|+|\beta|)} \int_{\Omega} A\left(\frac{|u|}{t /(|\alpha|+|\beta|)}\right) d \mu \leq 1\right\} \\
& \quad+\inf \left\{t>0: \frac{|\beta|}{(|\alpha|+|\beta|)} \int_{\Omega} A\left(\frac{|v|}{t /(|\alpha|+|\beta|)}\right) d \mu \leq 1\right\} \\
& =\quad|\alpha| \inf \left\{t /(|\alpha|+|\beta|)>0: \int_{\Omega} A\left(\frac{|u|}{t /(|\alpha|+|\beta|)}\right) d \mu \leq 1\right\} \\
& \quad+|\beta| \inf \left\{t /(|\alpha|+|\beta|)>0: \int_{\Omega} A\left(\frac{|v|}{t /(|\alpha|+|\beta|)}\right) d \mu \leq 1\right\}
\end{aligned}
$$

$$
=|\alpha|\|u\|_{A}+|\beta|\|v\|_{A}
$$

Now let $\Omega_{1} \subseteq \Omega$.

$$
\begin{aligned}
\|u\|_{A, \Omega_{1}} & \equiv \inf \left\{t>0: \int_{\Omega_{1}} A\left(\frac{|u(x)|}{t}\right) d \mu \leq 1\right\} \\
& \leq \inf \left\{t>0: \int_{\Omega} A\left(\frac{|u(x)|}{t}\right) d \mu \leq 1\right\} \equiv\|u\|_{A, \Omega}
\end{aligned}
$$

This occurs because if $t$ is in the second set, then it is in the first so the infimum of the second is no smaller than that of the first. This proves the proposition.

Next it is shown that $L_{A}(\Omega)$ is a Banach space.
Theorem 27.1.8 $L_{A}(\Omega)$ is a Banach space and every Cauchy sequence has a subsequence which also converges pointwise a.e.

Proof: Let $\left\{f_{n}\right\}$ be a Cauchy sequence in $L_{A}(\Omega)$ and select a subsequence $\left\{f_{n_{k}}\right\}$ such that

$$
\left\|f_{n_{k+1}}-f_{n_{k}}\right\|_{A} \leq 2^{-k}
$$

Thus

$$
f_{n_{m}}(x)=f_{n_{1}}(x)+\sum_{k=1}^{m-1} f_{n_{k+1}}(x)-f_{n_{k}}(x) .
$$

Let

$$
g_{m}(x) \equiv\left|f_{n_{1}}(x)\right|+\sum_{k=1}^{m-1}\left|f_{n_{k+1}}(x)-f_{n_{k}}(x)\right|
$$

Then

$$
\left\|g_{m}\right\|_{A} \leq\left\|f_{n_{1}}\right\|_{A}+\sum_{k=1}^{\infty} 2^{-k} \equiv K<\infty
$$

Let

$$
g(x) \equiv \lim _{m \rightarrow \infty} g_{m}(x) \equiv\left|f_{n_{1}}(x)\right|+\sum_{k=1}^{\infty}\left|f_{n_{k+1}}(x)-f_{n_{k}}(x)\right|
$$

Now $K>\left\|g_{m}\right\|_{A}$ so

$$
1 \geq \int_{\Omega} A\left(\frac{\left|g_{m}(x)\right|}{K}\right) d \mu
$$

By the monotone convergence theorem,

$$
1 \geq \int_{\Omega} A\left(\frac{|g(x)|}{K}\right) d \mu
$$

showing $g(x)<\infty$ a.e., say for all $x \notin E$ where $E$ is a measurable set having measure zero. Let

$$
\begin{aligned}
f(x) & \equiv \mathscr{X}_{E^{C}}(x)\left(f_{n_{1}}(x)+\sum_{k=1}^{\infty}\left(f_{n_{k+1}}(x)-f_{n_{k}}(x)\right)\right) \\
& =\lim _{m \rightarrow \infty} \mathscr{X}_{E^{C}}(x) f_{n_{m}}(x)
\end{aligned}
$$

Thus $f$ is measurable and $f_{n_{m}}(x) \rightarrow f(x)$ a.e. as $m \rightarrow \infty$.
For $l>k$,

$$
\left\|f_{n_{k}}-f_{n_{l}}\right\|_{A}<\frac{1}{2^{k-2}}
$$

and so

$$
1 \geq \int_{\Omega} A\left(\frac{\left|f_{n_{l}}(x)-f_{n_{k}}(x)\right|}{\left(\frac{1}{2^{k-2}}\right)}\right) d \mu
$$

By Fatou's lemma, let $l \rightarrow \infty$ and obtain

$$
1 \geq \int_{\Omega} A\left(\frac{\left|f(x)-f_{n_{k}}(x)\right|}{\left(\frac{1}{2^{k-2}}\right)}\right) d \mu
$$

and so $\left(f-f_{n_{k}}\right) 2^{k-2} \in K_{A}(\Omega)$ and so $f-f_{n_{k}} \in L_{A}(\Omega), f_{n_{k}} \in L_{A}(\Omega)$. Since $L_{A}(\Omega)$ is a vector space, this shows $f \in L_{A}(\Omega)$. Also

$$
\left\|f-f_{n_{k}}\right\|_{A} \leq \frac{1}{2^{k-2}}
$$

showing that $f_{n_{k}} \rightarrow f$ in $L_{A}(\Omega)$. Since a subsequence converges in $L_{A}(\Omega)$, it follows the original Cauchy sequence also converges to $f$ in $L_{A}(\Omega)$. This proves the theorem.

Next consider the space, $E_{A}(\Omega)$ which will be a subspace of the Orlitz class, $K_{A}(\Omega)$ just as $L_{A}(\Omega)$ is a vector space containing the Orlitz class.

Definition 27.1.9 Let $\mathbb{S}$ denote the set of simple functions, $s$, such that

$$
\mu(\{x: s(x) \neq 0\})<\infty .
$$

Then define

$$
E_{A}(\Omega) \equiv \text { the closure in } L_{A}(\Omega) \text { of } \mathbb{S} .
$$

Proposition 27.1.10 $E_{A}(\Omega) \subseteq K_{A}(\Omega) \subseteq L_{A}(\Omega)$ and they are all equal if $(A, \Omega)$ is $\Delta$ regular.

Proof: First note that $\mathbb{S} \subseteq K_{A}(\Omega) \cap E_{A}(\Omega)$. Let $f \in E_{A}(\Omega)$. Then by the definition of $E_{A}(\Omega)$, there exists $s_{n} \in \mathbb{S}$ such that

$$
\left\|s_{n}-f\right\|_{A} \rightarrow 0
$$

Therefore, for $n$ large enough,

$$
\left\|s_{n}-f\right\|_{A}<\frac{1}{2}
$$

and so

$$
\int_{\Omega} A\left(\frac{\left|f-s_{n}\right|}{\left(\frac{1}{2}\right)}\right) d \mu=\int_{\Omega} A\left(\left|2 f-2 s_{n}\right|\right) d \mu<\infty
$$

Since $\mathbb{S} \subseteq K_{A}(\Omega)$,

$$
\int_{\Omega} A\left(2\left|s_{n}\right|\right) d \mu<\infty
$$

Therefore, $2 f-2 s_{n} \in K_{A}(\Omega)$ and $2 s_{n} \in K_{A}(\Omega)$ and so, since $K_{A}(\Omega)$ is convex,

$$
\frac{2 f-2 s_{n}}{2}+\frac{2 s_{n}}{2}=f \in K_{A}(\Omega)
$$

This shows $E_{A}(\Omega) \subseteq K_{A}(\Omega)$.
Next consider the claim these spaces are all equal in the case that $(A, \Omega)$ is $\Delta$ regular. It was already shown in Proposition 27.1.5 that in this case,

$$
K_{A}(\Omega)=L_{A}(\Omega)
$$

so it remains to show $E_{A}(\Omega)=K_{A}(\Omega)$. Is every $f \in K_{A}(\Omega)$ the limit in $L_{A}(\Omega)$ of functions from $\mathbb{S}$ ? First suppose $\mu(\Omega)=\infty$. Then $A(r|f|) \leq K_{r} A(|f|)$ and so $A(r|f|) \in L^{1}(\Omega)$ for any $r$. Let $\varepsilon>0$ be given and let

$$
\Omega_{\delta} \equiv\{x:|f(x)| \geq \delta\}
$$

Then by the dominated convergence theorem,

$$
\lim _{\delta \rightarrow 0+} \int_{\Omega \backslash \Omega_{\delta}} A\left(\frac{|f|}{\varepsilon}\right) d \mu=0
$$

Choose $\delta$ such that

$$
\int_{\Omega \backslash \Omega_{\delta}} A\left(\frac{|f|}{\varepsilon}\right) d \mu<\frac{1}{2}
$$

and let $s_{n} \rightarrow f \mathscr{X}_{\Omega_{\delta}}$ pointwise with $\left|s_{n}\right| \leq\left|f \mathscr{X}_{\Omega_{\delta}}\right|$. Then $s_{n}=0$ on $\Omega \backslash \Omega_{\delta}$ and so

$$
\begin{aligned}
& \int_{\Omega} A\left(\frac{\left|f-s_{n}\right|}{\varepsilon}\right) d \mu=\int_{\Omega_{\delta}} A\left(\frac{\left|f-s_{n}\right|}{\varepsilon}\right) d \mu \\
+ & \int_{\Omega \backslash \Omega_{\delta}} A\left(\frac{|f|}{\varepsilon}\right) d \mu \leq \int_{\Omega_{\delta}} A\left(\frac{\left|f-s_{n}\right|}{\varepsilon}\right) d \mu+\frac{1}{2} .
\end{aligned}
$$

By the dominated convergence theorem,

$$
\int_{\Omega_{\delta}} A\left(\frac{\left|f-s_{n}\right|}{\varepsilon}\right) d \mu<\frac{1}{2}
$$

for all $n$ large enough. Therefore, for such $n$,

$$
\int_{\Omega} A\left(\frac{\left|f-s_{n}\right|}{\varepsilon}\right) d \mu<1
$$

and so $\left\|f-s_{n}\right\|_{A} \leq \varepsilon$ showing that in this case $E_{A}(\Omega) \supseteq K_{A}(\Omega)$ since $\varepsilon>0$ is arbitrary.

Now suppose $\mu(\Omega)<\infty$. In this case, only assume $A(r t) \leq K_{r} A(t)$ for $t$ large enough, say for $t \geq M_{r}$. However, this is enough to conclude $A(r|f|) \in L^{1}(\Omega)$ for any $r>0$ because $\mu(\Omega)<\infty$ and $f \in K_{A}(\Omega)$. Let $s_{n} \rightarrow f$ pointwise with $\left|s_{n}\right| \leq|f|$, and $s$ simple. Then

$$
A\left(\frac{\left|f-s_{n}\right|}{\varepsilon}\right) \leq A\left(\frac{2}{\varepsilon}|f|\right) \in L^{1}(\Omega)
$$

and so the dominated convergence theorem implies

$$
\lim _{n \rightarrow \infty} \int_{\Omega} A\left(\frac{\left|f-s_{n}\right|}{\varepsilon}\right) d \mu=0
$$

Hence

$$
\int_{\Omega} A\left(\frac{\left|f-s_{n}\right|}{\varepsilon}\right) d \mu<1
$$

for all $n$ large enough and so for such $n$,

$$
\left\|f-s_{n}\right\|_{A} \leq \varepsilon
$$

which proves the proposition.
It turns out $E_{A}(\Omega)$ is the largest linear subspace of $K_{A}(\Omega)$.
Proposition 27.1.11 $E_{A}(\Omega)$ is the maximal linear subspace of $K_{A}(\Omega)$.
Proof: Let $M$ be a subspace of $K_{A}(\Omega)$. Is $M \subseteq E_{A}(\Omega)$ ? For $f \in M, f / \varepsilon \in K_{A}(\Omega)$ for all $\varepsilon>0$ because of the fact that $M$ is a subspace and $f \in M$. Thus $A(|f| / \varepsilon)$ is in $L^{1}(\Omega)$. Let $\varepsilon>0$ be given, choose $\delta>0$ and let

$$
F_{\delta} \equiv\{x:|f(x)| \leq \delta\}
$$

By the dominated convergence theorem there exists $\delta$ small enough that

$$
\int_{F_{\delta}} A\left(\frac{2|f|}{\varepsilon}\right) d \mu<\frac{1}{2}
$$

Let $\left|s_{n}\right| \leq|f| \mathscr{X}_{F_{\delta}^{C}}$ and $s_{n} \rightarrow f \mathscr{X}_{F_{\delta}^{C}}$ pointwise for $s_{n}$ a simple function. Thus $s_{n}=0$ on $F_{\delta}$ and so $s_{n} \in \mathbb{S}$ because $\mu\left(F_{\delta}^{C}\right)<\infty$. Now

$$
\begin{aligned}
\int_{\Omega} A\left(\frac{\left|f-s_{n}\right|}{\varepsilon}\right) d \mu & =\int_{F_{\delta}} A\left(\frac{|f|}{\varepsilon}\right) d \mu+\int_{F_{\delta}^{C}} A\left(\frac{\left|f-s_{n}\right|}{\varepsilon}\right) d \mu \\
& <\frac{1}{2}+\int_{F_{\delta}^{C}} A\left(\frac{\left|f-s_{n}\right|}{\varepsilon}\right) d \mu
\end{aligned}
$$

The integrand in the last integral is no larger than $\frac{2|f|}{\varepsilon}$ and so by the dominated convergence theorem, this integral converges to 0 as $n \rightarrow \infty$. In particular, it is eventually less than $\frac{1}{2}$. Therefore, for such $n$,

$$
\left\|f-s_{n}\right\|_{A} \leq r
$$

Since $r$ is arbitrary, this shows that $f \in E_{A}(\Omega)$ which proves the proposition.
Next is a comparison of these function spaces for different choices of the $N$ function. The notation $X \hookrightarrow Y$ for two normed linear spaces means $X$ is a subset of $Y$ and the identity map is continuous.

Proposition 27.1.12 $L_{B}(\Omega) \hookrightarrow L_{A}(\Omega)$ if either

$$
\begin{equation*}
B(t) \geq A(t) \text { for all } t \geq 0 \tag{27.1.12}
\end{equation*}
$$

or if

$$
\begin{equation*}
B(t) \geq A(t) \text { for all } t>M \tag{27.1.13}
\end{equation*}
$$

and $\mu(\Omega)<\infty$.
Proof: Let $f \in L_{B}(\Omega)$ and let

$$
\int_{\Omega} B\left(\frac{|f|}{t}\right) d \mu \leq 1
$$

Then if 27.1.12 holds, it follows

$$
\int_{\Omega} A\left(\frac{|f|}{t}\right) d \mu \leq 1
$$

Thus if $t \geq\|f\|_{B}$ then $t \geq\|f\|_{A}$ which implies $\|f\|_{B} \geq\|f\|_{A}$.
Now suppose 27.1.13 holds and $\mu(\Omega)<\infty$. Then $\max (A, B)$ is an $N$ function dominating both $A$ and $B$ for all $t$. By what was just shown $L_{\max (A, B)}(\Omega) \hookrightarrow L_{B}(\Omega)$. Then let $f \in L_{B}(\Omega)$ and let

$$
\int_{\Omega} B\left(\frac{|f|}{t}\right) d \mu<1
$$

Then

$$
\begin{gathered}
\int_{\Omega} \max (A, B)\left(\frac{|f|}{t}\right) d \mu=\int_{\left[\frac{|f|}{t}>M\right]} B\left(\frac{|f|}{t}\right) d \mu \\
\quad+\int_{\left[\frac{|f|}{t} \leq M\right]} \max (A, B)\left(\frac{|f|}{t}\right) d \mu \\
\leq \int_{\Omega} B\left(\frac{|f|}{t}\right) d \mu+\mu(\Omega) \max (A, B)(M)<\infty .
\end{gathered}
$$

It follows $\frac{|f|}{t} \in K_{\max (A, B)}(\Omega)$ and so $f \in L_{\max (A, B)}(\Omega)$. Hence $L_{B}(\Omega)=L_{\max (A, B)}(\Omega)$ and the identity map from $L_{\max (A, B)}(\Omega)$ to $L_{B}(\Omega)$ is continuous. Therefore, by the open mapping theorem, the norms $\left\|\|_{B}\right.$ and $\| \|_{\max (A, B)}$ are equivalent. Hence for $f \in L_{B}(\Omega)$,

$$
\|f\|_{A} \leq\|f\|_{\max (A, B)} \leq C\|f\|_{B} .
$$

This proves the proposition.

Corollary 27.1.13 Suppose there exists $C>0$, a constant such that either

$$
C B(t) \geq A(t)
$$

for all $t \geq 0$ or

$$
C B(t) \geq A(t)
$$

for all $t>M$ and $\mu(\Omega)<\infty$. Then

$$
L_{B}(\Omega) \hookrightarrow L_{A}(\Omega)
$$

Proof: If $f \in L_{B}(\Omega)$ then $f=\lambda u$ where $u \in K_{B}(\Omega)=K_{C B}(\Omega)$. Hence $L_{C B}(\Omega)=$ $L_{B}(\Omega)$ and the two norms on $L_{B}(\Omega)$,

$$
\left\|\|_{C B}, \text { and }\right\| \|_{B}
$$

are equivalent norms by the open mapping theorem. Hence by the Proposition 27.1.12, if $f \in L_{B}(\Omega)$,

$$
\|f\|_{A} \leq C_{1}\|f\|_{C B} \leq C_{2}\|f\|_{B}
$$

which proves the corollary.
Definition 27.1.14 A increases essentially more slowly than $B$ if for all $a>0$,

$$
\lim _{t \rightarrow \infty} \frac{A(a t)}{B(t)}=0
$$

The next theorem gives added information on how these spaces are related in case that one $N$ function increases essentially more slowly than the other.

Theorem 27.1.15 Suppose $\mu(\Omega)<\infty$ and $A$ increases essentially more slowly than $B$. Then

$$
L_{B}(\Omega) \hookrightarrow E_{A}(\Omega)
$$

Proof: Let $f \in L_{B}(\Omega)$. Then there exists $\lambda>0$ such that

$$
\int_{\Omega} B\left(\frac{|f|}{\lambda}\right) d \mu \leq 1
$$

Let $r$ be such that for $t \geq r$,

$$
A(|\lambda| t) \leq B(t)
$$

Then

$$
\begin{aligned}
\int_{\Omega} A(|f|) d \mu & =\int_{[|f| \geq r]} A(|f|) d \mu+\int_{[|f|<r]} A(|f|) d \mu \\
& \leq \int_{\Omega} B\left(\frac{|f|}{|\lambda|}\right) d \mu+A(r) \mu(\Omega) \\
& <1+A(r) \mu(\Omega)
\end{aligned}
$$

Therefore, $L_{B}(\Omega)$ is a linear space contained in $K_{A}(\Omega)$. It follows from Proposition 27.1.11 that $L_{B}(\Omega) \subseteq E_{A}(\Omega)$. This proves the theorem.

The norm of $E_{A}(\Omega)$ is the same as the norm on $L_{A}(\Omega)$ so this shows $L_{B}(\Omega) \hookrightarrow L_{A}(\Omega)$.
Note that for $1<p<q$ and $A(t)=t^{p} / p, B(t)=t^{q} / q$,

$$
A(r t)=r^{p} t^{p}
$$

and

$$
\lim _{t \rightarrow \infty} \frac{A(r t)}{B(t)}=0
$$

showing this case is covered by the above theorem.
If $A$ is increasing essentially more slowly than $B$ and $\mu(\Omega)<\infty$, this has shown the following inclusions

$$
E_{B}(\Omega) \subseteq K_{B}(\Omega) \subseteq L_{B}(\Omega) \hookrightarrow E_{A}(\Omega) \subseteq K_{A}(\Omega) \subseteq L_{A}(\Omega)
$$

In the case of $A(t)=t^{p} / p, B(t)=t^{q} / q$ both $(A, \Omega)$ and $(B, \Omega)$ are $\Delta$ regular and so in this case or any other case where the $N$ functions are $\Delta$ regular, the above sequence of inclusions reduces to

$$
E_{B}(\Omega)=K_{B}(\Omega)=L_{B}(\Omega) \hookrightarrow E_{A}(\Omega)=K_{A}(\Omega)=L_{A}(\Omega)
$$

### 27.2 Dual Spaces In Orlitz Space

Recall that for $s, t \geq 0$,

$$
s t \leq A(t)+\widetilde{A}(s)
$$

Let $v \in L_{\widetilde{A}}(\Omega)$ and $u \in L_{A}(\Omega)$. Then there is a version of Holder's inequality as follows. For $\varepsilon>0$,

$$
\frac{|v|}{\|v\|_{\widetilde{A}}+\varepsilon} \in K_{\widetilde{A}}(\Omega), \frac{|u|}{\|u\|_{A}+\varepsilon} \in K_{A}(\Omega)
$$

Therefore,

$$
\begin{aligned}
\int_{\Omega}\left(\frac{|u|}{\|u\|_{A}+\varepsilon}\right. & )\left(\frac{|v|}{\|v\|_{\widetilde{A}}+\varepsilon}\right) d \mu \leq \int_{\Omega} A\left(\frac{|u|}{\|u\|_{A}+\varepsilon}\right) d \mu \\
& +\int_{\Omega} \widetilde{A}\left(\frac{|v|}{\|v\|_{\tilde{A}}+\varepsilon}\right) d \mu \leq 2
\end{aligned}
$$

and so $u v \in L^{1}(\Omega)$ and

$$
\left|\int_{\Omega} u v d \mu\right| \leq \int_{\Omega}|u||v| d \mu \leq 2\left(\|v\|_{\tilde{A}}+\varepsilon\right)\left(\|u\|_{A}+\varepsilon\right)
$$

Since $\varepsilon$ is arbitrary this shows

$$
\begin{equation*}
\left|\int_{\Omega} u v d \mu\right| \leq \int_{\Omega}|u||v| d \mu \leq 2\|v\|_{\tilde{A}}\|u\|_{A} \tag{27.2.14}
\end{equation*}
$$

Defining $L_{v}$ for $v \in \widetilde{A}$ by

$$
L_{v}(u) \equiv \int_{\Omega} u v d \mu
$$

it follows $L_{v} \in L_{A}(\Omega)^{\prime}$. From now on assume the measure space is $\sigma$ finite. That is, there exist measurable sets, $\Omega_{k}$ satisfying the following:

$$
\Omega=\cup_{k=1}^{\infty} \Omega_{k}, \mu\left(\Omega_{k}\right)<\infty, \Omega_{k} \subseteq \Omega_{k+1}
$$

Then
Proposition 27.2.1 For $v \in L_{\widetilde{A}}(\Omega)$, the following inequality holds.

$$
\|v\|_{\tilde{A}} \leq\left\|L_{v}\right\| \leq 2\|v\|_{\widetilde{A}}
$$

Here $L_{v}$ is considered as either an element of $E_{A}(\Omega)^{\prime}$ or $L_{A}(\Omega)^{\prime}$ and $\left\|L_{v}\right\|$ refers to the operator norm in either dual space.

Proof: The inequality 27.2 .14 implies $\left\|L_{v}\right\| \leq 2\|v\|_{\tilde{A}}$. It remains to show the other half of the inequality. If $L_{v}=0$ there is nothing to show because this would imply that $v=0$ so assume $\left\|L_{v}\right\|>0$. Define a measurable function, $u$, as follows. Letting $r \in(0,1)$,

$$
u(x) \equiv\left\{\begin{array}{l}
\widetilde{A}\left(\frac{r|v(x)|}{\left\|L_{v}\right\|}\right) / \frac{v(x)}{\left\|L_{v}\right\|} \text { if } v(x) \neq 0  \tag{27.2.15}\\
0 \text { if } v(x)=0 .
\end{array}\right.
$$

Now let

$$
\begin{equation*}
F_{n} \equiv\{x:|u(x)| \leq n\} \cap \Omega_{n} \cap\{x: v(x) \neq 0\} \tag{27.2.16}
\end{equation*}
$$

and define

$$
\begin{equation*}
u_{n}(x) \equiv u(x) \mathscr{X}_{F_{n}}(x) . \tag{27.2.17}
\end{equation*}
$$

Thus $u_{n}$ is bounded and equals zero off a set which has finite measure. It follows that

$$
A\left(\frac{\left|u_{n}\right|}{\alpha}\right) \in L^{1}(\Omega)
$$

for all $\alpha>0$. I claim that $\left\|u_{n}\right\|_{A} \leq 1$. If not, there exists $\varepsilon>0$ such that $\left\|u_{n}\right\|_{A}-\varepsilon>1$. Then since $A$ is convex,

$$
1<\int_{\Omega} A\left(\frac{\left|u_{n}\right|}{\left\|u_{n}\right\|_{A}-\varepsilon}\right) d \mu \leq \frac{1}{\left\|u_{n}\right\|_{A}-\varepsilon} \int_{\Omega} A\left(\left|u_{n}\right|\right) d \mu
$$

Taking $\varepsilon \rightarrow 0+$, using 27.1.6, and convexity of $A$ along with 27.2.15 and 27.2.17,

$$
\begin{aligned}
\left\|u_{n}\right\|_{A} & \leq \int_{\Omega} A\left(\left|u_{n}\right|\right) d \mu=\int_{F_{n}} A\left(r \widetilde{A}\left(\frac{r|v(x)|}{\left\|L_{v}\right\|}\right) / \frac{r v(x)}{\left\|L_{v}\right\|}\right) d \mu \\
& \leq r \int_{F_{n}} A\left(\widetilde{A}\left(\frac{r|v(x)|}{\left\|L_{v}\right\|}\right) / \frac{r v(x)}{\left\|L_{v}\right\|}\right) d \mu \leq r \int_{F_{n}} \widetilde{A}\left(\frac{r|v(x)|}{\left\|L_{v}\right\|}\right) d \mu \\
& =r \frac{1}{\left\|L_{v}\right\|} \int_{\Omega} u_{n}(x) v(x) d \mu \leq r \frac{1}{\left\|L_{v}\right\|}\left\|u_{n}\right\|_{A}\left\|L_{v}\right\|=r\left\|u_{n}\right\|_{A}
\end{aligned}
$$

a contradiction since $r<1$. Therefore, from 27.2.15,

$$
\left\|L_{v}\right\| \geq\left|L_{v}\left(u_{n}\right)\right| \equiv \int_{F_{n}} v(x) u(x) d \mu=\left\|L_{v}\right\| \int_{F_{n}} \widetilde{A}\left(\frac{r|v(x)|}{\left\|L_{v}\right\|}\right) d \mu
$$

and so

$$
1 \geq \int_{F_{n}} \tilde{A}\left(\frac{r|v(x)|}{\left\|L_{v}\right\|}\right) d \mu
$$

By the monotone convergence theorem, letting $n \rightarrow \infty$,

$$
1 \geq \int_{\Omega} \widetilde{A}\left(\frac{r|v(x)|}{\left\|L_{v}\right\|}\right) d \mu
$$

showing that

$$
\|v\|_{\tilde{A}} \leq \frac{\left\|L_{v}\right\|}{r}
$$

Since this holds for all $r \in(0,1)$, it follows $\left\|L_{v}\right\| \geq\|v\|_{\widetilde{A}}$ as claimed. This proves the proposition.

Now what follows is the Riesz representation theorem for the dual space of $E_{A}(\Omega)$.
Theorem 27.2.2 Suppose $\mu(\Omega)<\infty$ and suppose $L \in E_{A}(\Omega)^{\prime}$. Then the map $v \rightarrow L_{v}$ from $L_{\widetilde{A}}(\Omega)$ to $E_{A}(\Omega)^{\prime}$ is one to one continuous, linear, and onto. If $(\Omega, A)$ is $\Delta$ regular then $v \rightarrow L_{v}$ is one to one, linear, onto and continuous as a map from $L_{\widetilde{A}}(\Omega)$ to $L_{A}(\Omega)^{\prime}$.

Proof: It is obvious this map is linear. From Proposition 27.2.1 it is continuous and one to one. It remains only to verify that it is onto. Let $L \in E_{A}(\Omega)^{\prime}$ and define a complex valued function, $\lambda$, mapping the measurable sets to $\mathbb{C}$ as follows.

$$
\lambda(F) \equiv L\left(\mathscr{X}_{F}\right)
$$

In case $\mu(F) \neq 0$,

$$
\begin{aligned}
\int_{\Omega} A\left(\mathscr{X}_{F}(x) A^{-1}\left(\frac{1}{\mu(F)}\right)\right) d \mu & =\int_{F} A\left(A^{-1}\left(\frac{1}{\mu(F)}\right)\right) d \mu \\
& =\int_{F} \frac{1}{\mu(F)} d \mu=1
\end{aligned}
$$

and so

$$
\begin{equation*}
\left\|\mathscr{X}_{F}\right\|_{A} \leq \frac{1}{A^{-1}\left(\frac{1}{\mu(F)}\right)} \tag{27.2.18}
\end{equation*}
$$

In fact, $\lambda$ is actually a complex measure. To see this, suppose $F_{i} \uparrow F$. Then from the formula just derived,

$$
\left\|\mathscr{X}_{F_{i}}-\mathscr{X}_{F}\right\|_{A}=\left\|\mathscr{X}_{F \backslash F_{i}}\right\|_{A} \leq \frac{1}{A^{-1}\left(\frac{1}{\mu\left(F \backslash F_{i}\right)}\right)}
$$

which converges to zero as $i \rightarrow \infty$. Therefore, if the $F_{i}$ are disjoint and $F=\cup_{i=1}^{\infty} F_{i}$, let $S_{m} \equiv \cup_{i=1}^{m} F_{i}$ so that $S_{m} \uparrow F$. Then since $\mathscr{X}_{S_{m}} \rightarrow \mathscr{X}_{F}$ in $E_{A}(\Omega)$ and $L$ is continuous,

$$
\begin{aligned}
\lambda(F) & \equiv L\left(\mathscr{X}_{F}\right)=\lim _{m \rightarrow \infty} L\left(\mathscr{X}_{S_{m}}\right) \\
& =\lim _{m \rightarrow \infty} \sum_{i=1}^{m} L\left(\mathscr{X}_{F_{i}}\right)=\sum_{i=1}^{\infty} \lambda\left(F_{i}\right) .
\end{aligned}
$$

Next observe that $\lambda$ is absolutely continuous with respect to $\mu$. To see this, suppose $\mu(F)=0$. Then if $t>0$,

$$
\int_{\Omega} A\left(\frac{\mathscr{X}_{F}(x)}{t}\right) d \mu=0<1
$$

for all $t>0$ and so $\left\|\mathscr{X}_{F}\right\|_{A}=0$. Therefore, $\lambda(F) \equiv L\left(\mathscr{X}_{F}\right)=0$.
It follows by the Radon Nikodym theorem there exists $v \in L^{1}(\Omega)$ such that

$$
L\left(\mathscr{X}_{F}\right)=\lambda(F)=\int_{F} v d \mu
$$

Therefore, for all $s \in \mathbb{S}$,

$$
\begin{equation*}
L(s)=\int_{F} s v d \mu \tag{27.2.19}
\end{equation*}
$$

I need to show that $v$ is actually in $L_{\widetilde{A}}(\Omega)$. If $v=0$ a.e., there is nothing to prove so assume this is not so. Let $u$ be defined by.

$$
u(x) \equiv\left\{\begin{array}{l}
\widetilde{A}\left(\frac{r \mid v(x) \|}{\|L\|}\right) / \frac{v(x)}{\|L\|} \text { if } v(x) \neq 0  \tag{27.2.20}\\
0 \text { if } v(x)=0
\end{array}\right.
$$

for $r \in(0,1)$. Now let

$$
\begin{equation*}
F_{n} \equiv\{x:|u(x)| \leq n\} \cap\{x: v(x) \neq 0\} \tag{27.2.21}
\end{equation*}
$$

and define

$$
\begin{equation*}
u_{n}(x) \equiv u(x) \mathscr{X}_{F_{n}}(x) \tag{27.2.22}
\end{equation*}
$$

I claim $\left\|u_{n}\right\|_{A} \leq 1$. It is clear that since $\mu(\Omega)<\infty, u_{n} \in E_{A}(\Omega)$. If $\left\|u_{n}\right\|_{A}>1$, Then for $\varepsilon$ small enough,

$$
\left\|u_{n}\right\|_{A}-\varepsilon>1
$$

and so, by convexity of $A$ and the fact that $A(0)=0$,

$$
1<\int_{\Omega} A\left(\frac{\left|u_{n}(x)\right|}{\left\|u_{n}\right\|_{A}-\varepsilon}\right) d \mu \leq \frac{1}{\left\|u_{n}\right\|_{A}-\varepsilon} \int_{\Omega} A\left(\left|u_{n}(x)\right|\right) d \mu
$$

and so, letting $\varepsilon \rightarrow 0+$ and using 27.1.6 and convexity of $A$ as in the proof of the preceeding proposition,

$$
\begin{align*}
\left\|u_{n}\right\|_{A} & \leq \int_{\Omega} A\left(\left|u_{n}(x)\right|\right) d \mu \leq \int_{F_{n}} A\left(r \widetilde{A}\left(\frac{r|v(x)|}{\|L\|}\right) / \frac{r|v(x)|}{\|L\|}\right) d \mu \\
& \leq r \int_{F_{n}} A\left(\widetilde{A}\left(\frac{r|v(x)|}{\|L\|}\right) / \frac{r|v(x)|}{\|L\|}\right) d \mu \leq r \int_{F_{n}} \widetilde{A}\left(\frac{r|v(x)|}{\|L\|}\right) d \mu \\
& \leq r \frac{1}{\|L\|} \int_{F_{n}} u(x) v(x) d \mu=\frac{r}{\|L\|} \int_{\Omega} u_{n} v d \mu \tag{27.2.23}
\end{align*}
$$

Now by Theorem 11.3.9 applied to the positive and negative parts of real and imaginary parts, there exists a uniformly bounded sequence of simple functions, $\left\{s_{k}\right\}$ converging uniformly to $u_{n}$, implying convergence in $E_{A}(\Omega)$, and so

$$
\begin{equation*}
L u_{n}=\lim _{k \rightarrow \infty} L s_{k}=\lim _{k \rightarrow \infty} \int_{\Omega} s_{k} v d \mu=\int_{\Omega} u_{n} v d \mu \tag{27.2.24}
\end{equation*}
$$

therefore, from 27.2.23,

$$
\left\|u_{n}\right\|_{A} \leq \frac{r}{\|L\|} \int_{\Omega} u_{n} v d \mu=\frac{r}{\|L\|} L\left(u_{n}\right) \leq \frac{r}{\|L\|}\|L\|\left\|u_{n}\right\|_{A}
$$

which is a contradiction since $r<1$. Therefore, $\left\|u_{n}\right\|_{A} \leq 1$ and from 27.2.23,

$$
\begin{aligned}
\|L\| & \geq\|L\|\left\|u_{n}\right\|_{A} \geq\left|L u_{n}\right|=\int_{\Omega} u_{n} v d \mu \\
& \geq\|L\| \int_{F_{n}} \widetilde{A}\left(\frac{r|v(x)|}{\|L\|}\right) d \mu
\end{aligned}
$$

Letting $n \rightarrow \infty$ the monotone convergence theorem and the above imply

$$
\int_{\Omega} \widetilde{A}\left(\frac{r|v(x)|}{\|L\|}\right) d \mu \leq 1
$$

which shows that $v \in L_{\widetilde{A}}(\Omega)$ and $\|v\|_{\widetilde{A}} \leq \frac{\|L\|}{r}$ for all $r \in(0,1)$. Therefore, $\|v\|_{\widetilde{A}} \leq\|L\|$.
Since $v \in L_{\widetilde{A}}(\Omega)$ it follows $L_{v}=L$ on $\mathbb{S}$ and so $L_{v}=L$ because $\mathbb{S}$ is dense in the set $E_{A}(\Omega)$. The last assertion follows from Proposition 27.1.10. This completes the proof.

## Chapter 28

## Hausdorff Measure

### 28.1 The Definition

This chapter is on Hausdorff measures. First I will discuss some outer measures. In all that is done here, $\alpha(n)$ will be the volume of the ball in $\mathbb{R}^{n}$ which has radius 1 .

Definition 28.1.1 For a set $E$, denote by $r(E)$ the number which is half the diameter of $E$. Thus

$$
r(E) \equiv \frac{1}{2} \sup \{|\mathbf{x}-\mathbf{y}|: \mathbf{x}, \mathbf{y} \in E\} \equiv \frac{1}{2} \operatorname{diam}(E)
$$

Let $E \subseteq \mathbb{R}^{n}$.

$$
\begin{gathered}
\mathscr{H}_{\delta}^{s}(E) \equiv \inf \left\{\sum_{j=1}^{\infty} \beta(s)\left(r\left(C_{j}\right)\right)^{s}: E \subseteq \cup_{j=1}^{\infty} C_{j}, \operatorname{diam}\left(C_{j}\right) \leq \delta\right\} \\
\mathscr{H}^{s}(E) \equiv \lim _{\delta \rightarrow 0+} \mathscr{H}_{\delta}^{s}(E)
\end{gathered}
$$

Note that $\mathscr{H}_{\delta}^{s}(E)$ is increasing as $\delta \rightarrow 0+$ so the limit clearly exists.
In the above definition, $\beta(s)$ is an appropriate positive constant depending on $s$. It will turn out that for $n$ an integer, $\beta(n)=\alpha(n)$ where $\alpha(n)$ is the Lebesgue measure of the unit ball, $B(\mathbf{0}, 1)$ where the usual norm is used to determine this ball.

Lemma 28.1.2 $\mathscr{H}^{s}$ and $\mathscr{H}_{\delta}^{s}$ are outer measures.
Proof: It is clear that $\mathscr{H}^{s}(\emptyset)=0$ and if $A \subseteq B$, then $\mathscr{H}^{s}(A) \leq \mathscr{H}^{s}(B)$ with similar assertions valid for $\mathscr{H}_{\delta}^{s}$. Suppose $E=\cup_{i=1}^{\infty} E_{i}$ and $\mathscr{H}_{\delta}^{s}\left(E_{i}\right)<\infty$ for each $i$. Let $\left\{C_{j}^{i}\right\}_{j=1}^{\infty}$ be a covering of $E_{i}$ with

$$
\sum_{j=1}^{\infty} \beta(s)\left(r\left(C_{j}^{i}\right)\right)^{s}-\varepsilon / 2^{i}<\mathscr{H}_{\delta}^{s}\left(E_{i}\right)
$$

and $\operatorname{diam}\left(C_{j}^{i}\right) \leq \delta$. Then

$$
\begin{aligned}
\mathscr{H}_{\delta}^{s}(E) & \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \beta(s)\left(r\left(C_{j}^{i}\right)\right)^{s} \\
& \leq \sum_{i=1}^{\infty} \mathscr{H}_{\delta}^{s}\left(E_{i}\right)+\varepsilon / 2^{i} \\
& \leq \varepsilon+\sum_{i=1}^{\infty} \mathscr{H}_{\delta}^{s}\left(E_{i}\right) .
\end{aligned}
$$

It follows that since $\varepsilon>0$ is arbitrary,

$$
\mathscr{H}_{\delta}^{s}(E) \leq \sum_{i=1}^{\infty} \mathscr{H}_{\delta}^{s}\left(E_{i}\right)
$$

which shows $\mathscr{H}_{\delta}^{s}$ is an outer measure. Now notice that $\mathscr{H}_{\delta}^{s}(E)$ is increasing as $\delta \rightarrow 0$. Picking a sequence $\delta_{k}$ decreasing to 0 , the monotone convergence theorem implies

$$
\mathscr{H}^{s}(E) \leq \sum_{i=1}^{\infty} \mathscr{H}^{s}\left(E_{i}\right)
$$

The outer measure $\mathscr{H}^{s}$ is called $s$ dimensional Hausdorff measure when restricted to the $\sigma$ algebra of $\mathscr{H}^{s}$ measurable sets.

Next I will show the $\sigma$ algebra of $\mathscr{H}^{s}$ measurable sets includes the Borel sets. This is done by the following very interesting condition known as Caratheodory's criterion.

### 28.1.1 Properties

Definition 28.1.3 For two sets, $A, B$ in a metric space, we define

$$
\operatorname{dist}(A, B) \equiv \inf \{d(x, y): x \in A, y \in B\}
$$

Theorem 28.1.4 Let $\mu$ be an outer measure on the subsets of $(X, d)$, a metric space. If

$$
\mu(A \cup B)=\mu(A)+\mu(B)
$$

whenever $\operatorname{dist}(A, B)>0$, then the $\sigma$ algebra of measurable sets contains the Borel sets.
Proof: It suffices to show that closed sets are in $\mathscr{S}$, the $\sigma$-algebra of measurable sets, because then the open sets are also in $\mathscr{S}$ and consequently $\mathscr{S}$ contains the Borel sets. Let $K$ be closed and let $S$ be a subset of $\Omega$. Is $\mu(S) \geq \mu(S \cap K)+\mu(S \backslash K)$ ? It suffices to assume $\mu(S)<\infty$. Let

$$
K_{n} \equiv\left\{x: \operatorname{dist}(x, K) \leq \frac{1}{n}\right\}
$$

By Lemma 7.1 .7 on Page $136, x \rightarrow \operatorname{dist}(x, K)$ is continuous and so $K_{n}$ is closed. By the assumption of the theorem,

$$
\begin{equation*}
\mu(S) \geq \mu\left((S \cap K) \cup\left(S \backslash K_{n}\right)\right)=\mu(S \cap K)+\mu\left(S \backslash K_{n}\right) \tag{28.1.1}
\end{equation*}
$$

since $S \cap K$ and $S \backslash K_{n}$ are a positive distance apart. Now

$$
\begin{equation*}
\mu\left(S \backslash K_{n}\right) \leq \mu(S \backslash K) \leq \mu\left(S \backslash K_{n}\right)+\mu\left(\left(K_{n} \backslash K\right) \cap S\right) \tag{28.1.2}
\end{equation*}
$$

If $\lim _{n \rightarrow \infty} \mu\left(\left(K_{n} \backslash K\right) \cap S\right)=0$ then the theorem will be proved because this limit along with 28.1.2 implies $\lim _{n \rightarrow \infty} \mu\left(S \backslash K_{n}\right)=\mu(S \backslash K)$ and then taking a limit in 28.1.1, $\mu(S) \geq$ $\mu(S \cap K)+\mu(S \backslash K)$ as desired. Therefore, it suffices to establish this limit.

Since $K$ is closed, a point, $x \notin K$ must be at a positive distance from $K$ and so

$$
K_{n} \backslash K=\cup_{k=n}^{\infty} K_{k} \backslash K_{k+1}
$$

Therefore

$$
\begin{equation*}
\mu\left(S \cap\left(K_{n} \backslash K\right)\right) \leq \sum_{k=n}^{\infty} \mu\left(S \cap\left(K_{k} \backslash K_{k+1}\right)\right) . \tag{28.1.3}
\end{equation*}
$$

If

$$
\begin{equation*}
\sum_{k=1}^{\infty} \mu\left(S \cap\left(K_{k} \backslash K_{k+1}\right)\right)<\infty \tag{28.1.4}
\end{equation*}
$$

then $\mu\left(S \cap\left(K_{n} \backslash K\right)\right) \rightarrow 0$ because it is dominated by the tail of a convergent series so it suffices to show 28.1.4.

$$
\begin{gather*}
\sum_{k=1}^{M} \mu\left(S \cap\left(K_{k} \backslash K_{k+1}\right)\right)= \\
\sum_{k \text { even }, k \leq M} \mu\left(S \cap\left(K_{k} \backslash K_{k+1}\right)\right)+\sum_{k o d d, k \leq M} \mu\left(S \cap\left(K_{k} \backslash K_{k+1}\right)\right) . \tag{28.1.5}
\end{gather*}
$$

By the construction, the distance between any pair of sets, $S \cap\left(K_{k} \backslash K_{k+1}\right)$ for different even values of $k$ is positive and the distance between any pair of sets, $S \cap\left(K_{k} \backslash K_{k+1}\right)$ for different odd values of $k$ is positive. Therefore,

$$
\begin{aligned}
& \sum_{k \text { even, } k \leq M} \mu\left(S \cap\left(K_{k} \backslash K_{k+1}\right)\right)+\sum_{\text {kodd, } k \leq M} \mu\left(S \cap\left(K_{k} \backslash K_{k+1}\right)\right) \leq \\
& \mu\left(\bigcup_{\text {keven }} S \cap\left(K_{k} \backslash K_{k+1}\right)\right)+\mu\left(\bigcup_{\text {kodd }} S \cap\left(K_{k} \backslash K_{k+1}\right)\right) \leq 2 \mu(S)<\infty
\end{aligned}
$$

and so for all $M, \sum_{k=1}^{M} \mu\left(S \cap\left(K_{k} \backslash K_{k+1}\right)\right) \leq 2 \mu(S)$ showing 28.1.4
With the above theorem, the following theorem is easy to obtain.
Theorem 28.1.5 The $\sigma$ algebra of $\mathscr{H}^{s}$ measurable sets contains the Borel sets and $\mathscr{H}^{s}$ has the property that for all $E \subseteq \mathbb{R}^{n}$, there exists a Borel set $F \supseteq E$ such that $\mathscr{H}^{s}(F)=$ $\mathscr{H}^{s}(E)$.

Proof: Let $\operatorname{dist}(A, B)=2 \delta_{0}>0$. Is it the case that

$$
\mathscr{H}^{s}(A)+\mathscr{H}^{s}(B)=\mathscr{H}^{s}(A \cup B) ?
$$

This is what is needed to use Caratheodory's criterion.
Let $\left\{C_{j}\right\}_{j=1}^{\infty}$ be a covering of $A \cup B$ such that $\operatorname{diam}\left(C_{j}\right) \leq \delta<\delta_{0}$ for each $j$ and

$$
\mathscr{H}_{\delta}^{s}(A \cup B)+\varepsilon>\sum_{j=1}^{\infty} \beta(s)\left(r\left(C_{j}\right)\right)^{s}
$$

Thus

$$
\mathscr{H}_{\delta}^{s}(A \cup B)+\varepsilon>\sum_{j \in J_{1}} \beta(s)\left(r\left(C_{j}\right)\right)^{s}+\sum_{j \in J_{2}} \beta(s)\left(r\left(C_{j}\right)\right)^{s}
$$

where

$$
J_{1}=\left\{j: C_{j} \cap A \neq \emptyset\right\}, J_{2}=\left\{j: C_{j} \cap B \neq \emptyset\right\} .
$$

Recall $\operatorname{dist}(A, B)=2 \delta_{0}, J_{1} \cap J_{2}=\emptyset$. It follows

$$
\mathscr{H}_{\delta}^{s}(A \cup B)+\varepsilon>\mathscr{H}_{\delta}^{s}(A)+\mathscr{H}_{\delta}^{s}(B)
$$

Letting $\delta \rightarrow 0$, and noting $\varepsilon>0$ was arbitrary, yields

$$
\mathscr{H}^{s}(A \cup B) \geq \mathscr{H}^{s}(A)+\mathscr{H}^{s}(B)
$$

Equality holds because $\mathscr{H}^{s}$ is an outer measure. By Caratheodory's criterion, $\mathscr{H}^{s}$ is a Borel measure.

To verify the second assertion, note first there is no loss of generality in letting $\mathscr{H}^{s}(E)<$ $\infty$. Let

$$
E \subseteq \cup_{j=1}^{\infty} C_{j}, r\left(C_{j}\right)<\delta
$$

and

$$
\mathscr{H}_{\delta}^{s}(E)+\delta>\sum_{j=1}^{\infty} \beta(s)\left(r\left(C_{j}\right)\right)^{s}
$$

Let

$$
F_{\delta}=\cup_{j=1}^{\infty} \overline{C_{j}}
$$

Thus $F_{\delta} \supseteq E$ and

$$
\begin{aligned}
\mathscr{H}_{\delta}^{s}(E) & \leq \mathscr{H}_{\delta}^{s}\left(F_{\delta}\right) \leq \sum_{j=1}^{\infty} \beta(s)\left(r\left(\overline{C_{j}}\right)\right)^{s} \\
& =\sum_{j=1}^{\infty} \beta(s)\left(r\left(C_{j}\right)\right)^{s}<\delta+\mathscr{H}_{\delta}^{s}(E) .
\end{aligned}
$$

Let $\delta_{k} \rightarrow 0$ and let $F=\cap_{k=1}^{\infty} F_{\delta_{k}}$. Then $F \supseteq E$ and

$$
\mathscr{H}_{\delta_{k}}^{s}(E) \leq \mathscr{H}_{\delta_{k}}^{s}(F) \leq \mathscr{H}_{\delta_{k}}^{s}\left(F_{\delta}\right) \leq \boldsymbol{\delta}_{k}+\mathscr{H}_{\delta_{k}}^{s}(E)
$$

Letting $k \rightarrow \infty$,

$$
\mathscr{H}^{s}(E) \leq \mathscr{H}^{s}(F) \leq \mathscr{H}^{s}(E)
$$

A measure satisfying the conclusion of Theorem 28.1.5 is called a Borel regular measure.

## $28.2 \quad \mathscr{H}^{p}$ and $m_{p}$

Next I will compare $\mathscr{H}^{p}$ and $m_{p}$. To do this, recall the following covering theorem which is a summary of Corollary 13.4.5 found on Page 350.

Theorem 28.2.1 Let $E \subseteq \mathbb{R}^{p}$ and let $\mathscr{F}$ be a collection of balls of bounded radii such that $\mathscr{F}$ covers $E$ in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathscr{F},\left\{B_{j}\right\}_{j=1}^{\infty}$, such that $\overline{m_{p}}\left(E \backslash \cup_{j=1}^{\infty} B_{j}\right)=0$.

In the next lemma, the balls are the usual balls taken with respect to the usual distance in $\mathbb{R}^{p}$ coming from the Euclidean norm.

Lemma 28.2.2 If $S \subseteq \mathbb{R}^{p}$ and $m_{p}(S)=0$, then $\mathscr{H}^{p}(S)=\mathscr{H}_{\delta}^{p}(S)=0$. Also, there exists a constant $k$ such that $\mathscr{H}^{p}(E) \leq k m_{p}(E)$ for all $E$ Borel $k \equiv \frac{\beta(p)}{\alpha(p)}$. Also, if $Q_{0} \equiv[0,1)^{p}$, the unit cube, then $\mathscr{H}^{p}\left([0,1)^{p}\right)>0$.

Proof: Suppose first $m_{p}(S)=0$. Without loss of generality, $S$ is bounded. Then by outer regularity, there exists a bounded open $V$ containing $S$ and $m_{p}(V)<\varepsilon$. For each $\mathbf{x} \in S$, there exists a ball $B_{\mathbf{x}}$ such that $\widehat{B_{\mathbf{x}}} \subseteq V$ and $\delta>r\left(\widehat{B_{\mathbf{x}}}\right)$. By the Vitali covering theorem there is a sequence of disjoint balls $\left\{B_{k}\right\}$ such that $\left\{\widehat{B_{k}}\right\}$ covers $S$. Here $\widehat{B_{k}}$ has the same center as $B_{k}$ but 5 times the radius. Then letting $\alpha(p)$ be the Lebesgue measure of the unit ball in $\mathbb{R}^{p}$

$$
\begin{aligned}
\mathscr{H}_{\delta}^{p}(S) & \leq \sum_{k} \beta(p) r\left(\widehat{B_{k}}\right)^{p}=\frac{\beta(p)}{\alpha(p)} 5^{p} \sum_{k} \alpha(p) r\left(B_{k}\right)^{p} \\
& \leq \frac{\beta(p)}{\alpha(p)} 5^{p} m_{p}(V)<\frac{\beta(p)}{\alpha(p)} 5^{p} \varepsilon
\end{aligned}
$$

Since $\varepsilon$ is arbitrary, this shows $\mathscr{H}_{\delta}^{p}(S)=0$ and now it follows that

$$
\mathscr{H}^{p}(S) \equiv \lim _{\delta \rightarrow 0} \mathscr{H}_{\delta}^{p}(S)=0
$$

Letting $U$ be an open set and $\delta>0$, consider all balls $B$ contained in $U$ which have diameters less than $\delta$. This is a Vitali covering of $U$ and therefore by Theorem 28.2.1, there exists $\left\{B_{i}\right\}$, a sequence of disjoint balls of radii less than $\delta$ contained in $U$ such that $\cup_{i=1}^{\infty} B_{i}$ differs from $U$ by a set of Lebesgue measure zero. Let $\alpha(p)$ be the Lebesgue measure of the unit ball in $\mathbb{R}^{p}$. Then from what was just shown,

$$
\begin{aligned}
\mathscr{H}_{\delta}^{p}(U) & =\mathscr{H}_{\delta}^{p}\left(\cup_{i} B_{i}\right) \leq \sum_{i=1}^{\infty} \beta(p) r\left(B_{i}\right)^{p}=\frac{\beta(p)}{\alpha(p)} \sum_{i=1}^{\infty} \alpha(p) r\left(B_{i}\right)^{p} \\
& =\frac{\beta(p)}{\alpha(p)} \sum_{i=1}^{\infty} m_{p}\left(B_{i}\right)=\frac{\beta(p)}{\alpha(p)} m_{p}(U) \equiv k m_{p}(U), k \equiv \frac{\beta(p)}{\alpha(p)}
\end{aligned}
$$

Now letting $E$ be Lebesgue measurable, it follows from the outer regularity of $m_{p}$ there exists a decreasing sequence of open sets, $\left\{V_{i}\right\}$ containing $E$ such such that $m_{p}\left(V_{i}\right) \rightarrow$ $m_{p}(E)$. Then from the above,

$$
\mathscr{H}_{\delta}^{p}(E) \leq \lim _{i \rightarrow \infty} \mathscr{H}_{\delta}^{p}\left(V_{i}\right) \leq \lim _{i \rightarrow \infty} k m_{p}\left(V_{i}\right)=k m_{p}(E)
$$

Since $\delta>0$ is arbitrary, it follows that also $\mathscr{H}^{p}(E) \leq k m_{p}(E)$. This proves the first part of the lemma.

To verify the second part, note that it is obvious $\mathscr{H}_{\delta}^{p}$ and $\mathscr{H}^{p}$ are translation invariant because diameters of sets do not change when translated. Therefore, if $\mathscr{H}^{p}\left([0,1)^{p}\right)=0$, it follows $\mathscr{H}^{p}\left(\mathbb{R}^{p}\right)=0$ because $\mathbb{R}^{p}$ is the countable union of translates of $Q_{0} \equiv[0,1)^{p}$. Since each $\mathscr{H}_{\delta}^{p}$ is no larger than $\mathscr{H}^{p}, \mathscr{H}_{\delta}^{p}\left([0,1)^{p}\right)=0$. Therefore, there exists a sequence of sets, $\left\{C_{i}\right\}$ each having diameter less than $\delta$ such that the union of these sets equals $\mathbb{R}^{p}$ but $1>\sum_{i=1}^{\infty} \beta(p) r\left(C_{i}\right)^{p}$. Now let $B_{i}$ be a ball having radius $r_{i}$ equal to diam $\left(C_{i}\right)=2 r\left(C_{i}\right)$ which contains $C_{i}$. These $B_{i}$ cover $\mathbb{R}^{p}, \frac{1}{2} r_{i}=r\left(C_{i}\right)$. It follows that

$$
1>\sum_{i=1}^{\infty} \beta(p) r\left(C_{i}\right)^{p}=\sum_{i=1}^{\infty} \frac{\beta(p)}{\alpha(p) 2^{p}} m_{p}\left(B_{i}\right)=\infty
$$

a contradiction.

Lemma 28.2.3 Every open set $U$ in $\mathbb{R}^{p}$ is a countable disjoint union of half open boxes of the form $Q \equiv \prod_{i=1}^{p}\left[a_{i}, a_{i}+2^{-k}\right)$ where $a_{i}=l 2^{-k}$ for $l$ some integer.

Proof: It is clear that there exists $\mathscr{Q}_{k}$ a countable disjoint collection of these half open boxes each of sides of length $2^{-k}$ whose union is all of $\mathbb{R}^{p}$. Let $\mathscr{B}_{1}$ be those sets of $\mathscr{Q}_{1}$ which are contained in $U$, if any. Having chosen $\mathscr{B}_{k-1}$, let $\mathscr{B}_{k}$ consist of those sets of $\mathscr{Q}_{k}$ which are contained in $U$ such that none of these are contained in $\mathscr{B}_{k-1}$. Then $\cup_{k=1}^{\infty} \mathscr{B}_{k}$ is a countable collection of disjoint boxes of the right sort whose union is $U$. This is because if $R$ is a box of $\mathscr{Q}_{k}$ and $\hat{R}$ is a box of $\mathscr{Q}_{k-1}$, then either $R \subseteq \hat{R}$ or $R \cap \hat{R}=\emptyset$.

Theorem 28.2.4 By choosing $\beta(p)$ properly, one can obtain $\mathscr{H}^{p}=m_{p}$ on all Lebesgue measurable sets.

Proof: I will show $\mathscr{H}^{p}$ is a positive multiple of $m_{p}$ for any choice of $\beta(p)$. Define $k=$ $\frac{m_{p}\left(Q_{0}\right)}{\mathscr{H}^{p}\left(Q_{0}\right)}$ where $Q_{0}=[0,1)^{p}$ is the half open unit cube in $\mathbb{R}^{p}$. I will show $k \mathscr{H}^{p}(E)=m_{p}(E)$ for any Lebesgue measurable set. When this is done, it will follow that by adjusting $\beta(p)$ the multiple can be taken to be 1 .

Let $Q=\prod_{i=1}^{p}\left[a_{i}, a_{i}+2^{-k}\right)$ be a half open box where $a_{i}=l 2^{-k}$. Thus $Q_{0}$ is the union of $\left(2^{k}\right)^{p}$ of these identical half open boxes. By translation invariance, of $\mathscr{H}^{p}$ and $m_{p}$,

$$
\left(2^{k}\right)^{p} \mathscr{H}^{p}(Q)=\mathscr{H}^{p}\left(Q_{0}\right)=\frac{1}{k} m_{p}\left(Q_{0}\right)=\frac{1}{k}\left(2^{k}\right)^{p} m_{p}(Q) .
$$

Therefore, $k \mathscr{H}^{p}(Q)=m_{p}(Q)$ for any such half open box and by translation invariance, for the translation of any such half open box. It follows that $k \mathscr{H}^{p}(U)=m_{p}(U)$ for all open sets because each open set is a countable disjoint union of such half open boxes. It follows immediately, since every compact set is the countable intersection of open sets that $k \mathscr{H}^{p}=m_{p}$ on compact sets. Therefore, $k \mathscr{H}^{p}=m_{p}$ on all closed sets because every closed set is the countable union of compact sets. Now let $F$ be an arbitrary Lebesgue measurable set. I will show that $F$ is $\mathscr{H}^{p}$ measurable and that $k \mathscr{H}^{p}(F)=m_{p}(F)$. Let $F_{l}=B(\mathbf{0}, l) \cap F$. By Proposition 11.7.3, there exists $H$ a countable union of compact sets and $G$ a countable intersection of open sets such that $H \subseteq F_{l} \subseteq G$ and $m_{p}(G \backslash H)=0$ which implies by Lemma 28.2.2 that $m_{p}(G \backslash H)=k \mathscr{H}^{p}(G \backslash H)=0$. Then by completeness of $\mathscr{H}^{p}$ it follows $F_{l}$ is $\mathscr{H}^{p}$ measurable and $k \mathscr{H}^{p}\left(F_{l}\right)=k \mathscr{H}^{p}(H)=m_{p}(H)=m_{p}\left(F_{l}\right)$. Now taking $l \rightarrow \infty$, it follows $F$ is $\mathscr{H}^{p}$ measurable and $k \mathscr{H}^{p}(F)=m_{p}(F)$. Therefore, adjusting $\beta(p)$ it can be assumed the constant $k$ is 1 .

The exact determination of $\beta(p)$ is more technical.

### 28.3 Technical Considerations

Let $\alpha(n)$ be the volume of the unit ball in $\mathbb{R}^{n}$. Thus the volume of $B(\mathbf{0}, r)$ in $\mathbb{R}^{n}$ is $\alpha(n) r^{n}$ from the change of variables formula. There is a very important and interesting inequality known as the isodiametric inequality which says that if $A$ is any set in $\mathbb{R}^{n}$, then

$$
\bar{m}(A) \leq \alpha(n)\left(2^{-1} \operatorname{diam}(A)\right)^{n}=\alpha(n) r(A)^{n}
$$

This inequality may seem obvious at first but it is not really. The reason it is not is that there are sets which are not subsets of any sphere having the same diameter as the set. For example, consider an equilateral triangle.

Lemma 28.3.1 Let $f: \mathbb{R}^{n-1} \rightarrow[0, \infty)$ be Borel measurable and let

$$
S=\{(\mathbf{x}, y):|y|<f(\mathbf{x})\} .
$$

Then $S$ is a Borel set in $\mathbb{R}^{n}$.
Proof: Set $s_{k}$ be an increasing sequence of Borel measurable functions converging pointwise to $f$.

$$
s_{k}(\mathbf{x})=\sum_{m=1}^{N_{k}} c_{m}^{k} \mathscr{X}_{E_{m}^{k}}(\mathbf{x})
$$

Let

$$
S_{k}=\cup_{m=1}^{N_{k}} E_{m}^{k} \times\left(-c_{m}^{k}, c_{m}^{k}\right)
$$

Then $(\mathbf{x}, y) \in S_{k}$ if and only if $f(\mathbf{x})>0$ and $|y|<s_{k}(\mathbf{x}) \leq f(\mathbf{x})$. It follows that $S_{k} \subseteq S_{k+1}$ and

$$
S=\cup_{k=1}^{\infty} S_{k} .
$$

But each $S_{k}$ is a Borel set and so $S$ is also a Borel set. This proves the lemma.
Let $P_{i}$ be the projection onto

$$
\operatorname{span}\left(\mathbf{e}_{1}, \cdots, \mathbf{e}_{i-1}, \mathbf{e}_{i+1}, \cdots, \mathbf{e}_{n}\right)
$$

where the $\mathbf{e}_{k}$ are the standard basis vectors in $\mathbb{R}^{n}, \mathbf{e}_{k}$ being the vector having a 1 in the $k^{t h}$ slot and a 0 elsewhere. Thus $P_{i} \mathbf{x} \equiv \sum_{j \neq i} x_{j} \mathbf{e}_{j}$. Also let

 function defined on $P_{i}\left(\mathbb{R}^{n}\right)$.

Proof: Let $\mathscr{K}$ be the $\pi$ system consisting of sets of the form $\prod_{j=1}^{n} A_{j}$ where $A_{i}$ is Borel. Also let $\mathscr{G}$ denote those Borel sets of $\mathbb{R}^{n}$ such that if $A \in \mathscr{G}$ then

$$
P_{i} \mathbf{x} \rightarrow m\left(\left(A \cap R_{k}\right)_{P_{i} \mathbf{x}}\right) \text { is Borel measurable. }
$$

where $R_{k}=(-k, k)^{n}$. Thus $\mathscr{K} \subseteq \mathscr{G}$. If $A \in \mathscr{G}$

$$
P_{i} \mathbf{x} \rightarrow m\left(\left(A^{C} \cap R_{k}\right)_{P_{i} \mathbf{x}}\right)
$$

is Borel measurable because it is of the form

$$
m\left(\left(R_{k}\right)_{P_{i} \mathbf{x}}\right)-m\left(\left(A \cap R_{k}\right)_{P_{i} \mathbf{x}}\right)
$$

and these are Borel measurable functions of $P_{i} \mathbf{x}$. Also, if $\left\{A_{i}\right\}$ is a disjoint sequence of sets in $\mathscr{G}$ then

$$
m\left(\left(\cup_{i} A_{i} \cap R_{k}\right)_{P_{i} \mathbf{x}}\right)=\sum_{i} m\left(\left(A_{i} \cap R_{k}\right)_{P_{i} \mathbf{x}}\right)
$$

and each function of $P_{i} \mathbf{x}$ is Borel measurable. Thus by the lemma on $\pi$ systems, Lemma 12.12.3, $\mathscr{G}=\mathscr{B}\left(\mathbb{R}^{n}\right)$ and this proves the lemma.

Now let $A \subseteq \mathbb{R}^{n}$ be Borel. Let $P_{i}$ be the projection onto

$$
\operatorname{span}\left(\mathbf{e}_{1}, \cdots, \mathbf{e}_{i-1}, \mathbf{e}_{i+1}, \cdots, \mathbf{e}_{n}\right)
$$

and as just described,

$$
A_{P_{i} \mathbf{x}}=\left\{y \in \mathbb{R}: P_{i} \mathbf{x}+y \mathbf{e}_{i} \in A\right\}
$$

Thus for $\mathbf{x}=\left(x_{1}, \cdots, x_{n}\right)$,

$$
A_{P_{i} \mathbf{x}}=\left\{y \in \mathbb{R}:\left(x_{1}, \cdots, x_{i-1}, y, x_{i+1}, \cdots, x_{n}\right) \in A\right\} .
$$

Since $A$ is Borel, it follows from Lemma 28.3.1 that

$$
P_{i} \mathbf{x} \rightarrow m\left(A_{P_{i} \mathbf{x}}\right)
$$

is a Borel measurable function on $P_{i} \mathbb{R}^{n}=\mathbb{R}^{n-1}$.

### 28.3.1 Steiner Symmetrization

Define

$$
S\left(A, \mathbf{e}_{i}\right) \equiv\left\{\mathbf{x}=P_{i} \mathbf{x}+y \mathbf{e}_{i}:|y|<2^{-1} m\left(A_{P_{i} \mathbf{x}}\right)\right\}
$$

Lemma 28.3.3 Let $A$ be a Borel subset of $\mathbb{R}^{n}$. Then $S\left(A, \mathbf{e}_{i}\right)$ satisfies

$$
\begin{gather*}
P_{i} \mathbf{x}+y \mathbf{e}_{i} \in S\left(A, \mathbf{e}_{i}\right) \text { if and only if } P_{i} \mathbf{x}-y \mathbf{e}_{i} \in S\left(A, \mathbf{e}_{i}\right), \\
S\left(A, \mathbf{e}_{i}\right) \text { is a Borel set in } \mathbb{R}^{n}, \\
m_{n}\left(S\left(A, \mathbf{e}_{i}\right)\right)=m_{n}(A)  \tag{28.3.6}\\
\operatorname{diam}\left(S\left(A, \mathbf{e}_{i}\right)\right) \leq \operatorname{diam}(A) \tag{28.3.7}
\end{gather*}
$$

Proof: The first assertion is obvious from the definition. The Borel measurability of $S\left(A, \mathbf{e}_{i}\right)$ follows from the definition and Lemmas 28.3.2 and 28.3.1. To show Formula 28.3.6,

$$
\begin{aligned}
m_{n}\left(S\left(A, \mathbf{e}_{i}\right)\right) & =\int_{P_{i} \mathbb{R}^{n}} \int_{-2^{-1} m\left(A A_{P_{i} x}\right)}^{2^{-1} m\left(A P_{i}\right)} d x_{i} d x_{1} \cdots d x_{i-1} d x_{i+1} \cdots d x_{n} \\
& =\int_{P_{i} \mathbb{R}^{n}} m\left(A_{P_{i} \mathbf{x}}\right) d x_{1} \cdots d x_{i-1} d x_{i+1} \cdots d x_{n} \\
& =m(A) .
\end{aligned}
$$

Now suppose $\mathbf{x}_{1}$ and $\mathbf{x}_{2} \in S\left(A, \mathbf{e}_{i}\right)$

$$
\mathbf{x}_{1}=P_{i} \mathbf{x}_{1}+y_{1} \mathbf{e}_{i}, \mathbf{x}_{2}=P_{i} \mathbf{x}_{2}+y_{2} \mathbf{e}_{i} .
$$

For $\mathbf{x} \in A$ define

$$
\begin{aligned}
& l(\mathbf{x})=\sup \left\{y: P_{i} \mathbf{x}+y \mathbf{e}_{i} \in A\right\} . \\
& g(\mathbf{x})=\inf \left\{y: P_{i} \mathbf{x}+y \mathbf{e}_{i} \in A\right\} .
\end{aligned}
$$

Then it is clear that

$$
\begin{align*}
& l\left(\mathbf{x}_{1}\right)-g\left(\mathbf{x}_{1}\right) \geq m\left(A_{P_{i} \mathbf{x}_{1}}\right) \geq 2\left|y_{1}\right|  \tag{28.3.8}\\
& l\left(\mathbf{x}_{2}\right)-g\left(\mathbf{x}_{2}\right) \geq m\left(A_{P_{i} \mathbf{x}_{2}}\right) \geq 2\left|y_{2}\right| \tag{28.3.9}
\end{align*}
$$

Claim: $\left|y_{1}-y_{2}\right| \leq\left|l\left(\mathbf{x}_{1}\right)-g\left(\mathbf{x}_{2}\right)\right|$ or $\left|y_{1}-y_{2}\right| \leq\left|l\left(\mathbf{x}_{2}\right)-g\left(\mathbf{x}_{1}\right)\right|$.
Proof of Claim: If not,

$$
\begin{gathered}
2\left|y_{1}-y_{2}\right|>\left|l\left(\mathbf{x}_{1}\right)-g\left(\mathbf{x}_{2}\right)\right|+\left|l\left(\mathbf{x}_{2}\right)-g\left(\mathbf{x}_{1}\right)\right| \\
\geq\left|l\left(\mathbf{x}_{1}\right)-g\left(\mathbf{x}_{1}\right)+l\left(\mathbf{x}_{2}\right)-g\left(\mathbf{x}_{2}\right)\right| \\
=l\left(\mathbf{x}_{1}\right)-g\left(\mathbf{x}_{1}\right)+l\left(\mathbf{x}_{2}\right)-g\left(\mathbf{x}_{2}\right) . \\
\geq 2\left|y_{1}\right|+2\left|y_{2}\right|
\end{gathered}
$$

by 28.3.8 and 28.3.9 contradicting the triangle inequality.
Now suppose $\left|y_{1}-y_{2}\right| \leq\left|l\left(\mathbf{x}_{1}\right)-g\left(\mathbf{x}_{2}\right)\right|$. From the claim,

$$
\begin{aligned}
\left|\mathbf{x}_{1}-\mathbf{x}_{2}\right| & =\left(\left|P_{i} \mathbf{x}_{1}-P_{i} \mathbf{x}_{2}\right|^{2}+\left|y_{1}-y_{2}\right|^{2}\right)^{1 / 2} \\
& \leq\left(\left|P_{i} \mathbf{x}_{1}-P_{i} \mathbf{x}_{2}\right|^{2}+\left|l\left(\mathbf{x}_{1}\right)-g\left(\mathbf{x}_{2}\right)\right|^{2}\right)^{1 / 2} \\
& \leq\left(\left|P_{i} \mathbf{x}_{1}-P_{i} \mathbf{x}_{2}\right|^{2}+\left(\left|z_{1}-z_{2}\right|+2 \varepsilon\right)^{2}\right)^{1 / 2} \\
& \leq \operatorname{diam}(A)+O(\sqrt{\varepsilon})
\end{aligned}
$$

where $z_{1}$ and $z_{2}$ are such that $P_{i} \mathbf{x}_{1}+z_{1} \mathbf{e}_{i} \in A, P_{i} \mathbf{x}_{2}+z_{2} \mathbf{e}_{i} \in A$, and

$$
\left|z_{1}-l\left(\mathbf{x}_{1}\right)\right|<\varepsilon \text { and }\left|z_{2}-g\left(\mathbf{x}_{2}\right)\right|<\varepsilon .
$$

If $\left|y_{1}-y_{2}\right| \leq\left|l\left(\mathbf{x}_{2}\right)-g\left(\mathbf{x}_{1}\right)\right|$, then we use the same argument but let

$$
\left|z_{1}-g\left(\mathbf{x}_{1}\right)\right|<\varepsilon \text { and }\left|z_{2}-l\left(\mathbf{x}_{2}\right)\right|<\varepsilon
$$

Since $\mathbf{x}_{1}, \mathbf{x}_{2}$ are arbitrary elements of $S\left(A, \mathbf{e}_{i}\right)$ and $\varepsilon$ is arbitrary, this proves 28.3.7.
The next lemma says that if $A$ is already symmetric with respect to the $j^{\text {th }}$ direction, then this symmetry is not destroyed by taking $S\left(A, \mathbf{e}_{i}\right)$.

Lemma 28.3.4 Suppose $A$ is a Borel set in $\mathbb{R}^{n}$ such that $P_{j} \mathbf{x}+\mathbf{e}_{j} x_{j} \in A$ if and only if $P_{j} \mathbf{x}+\left(-x_{j}\right) \mathbf{e}_{j} \in A$. Then if $i \neq j, P_{j} \mathbf{x}+\mathbf{e}_{j} x_{j} \in S\left(A, \mathbf{e}_{i}\right)$ if and only if $P_{j} \mathbf{x}+\left(-x_{j}\right) \mathbf{e}_{j} \in S\left(A, \mathbf{e}_{i}\right)$.

Proof: By definition,

$$
P_{j} \mathbf{x}+\mathbf{e}_{j} x_{j} \in S\left(A, \mathbf{e}_{i}\right)
$$

if and only if

$$
\left|x_{i}\right|<2^{-1} m\left(A_{P_{i}\left(P_{j} \mathbf{x}+\mathbf{e}_{j} x_{j}\right)}\right) .
$$

Now

$$
x_{i} \in A_{P_{i}\left(P_{j} \mathbf{x}+\mathbf{e}_{j} x_{j}\right)}
$$

if and only if

$$
x_{i} \in A_{P_{i}\left(P_{j} \mathbf{x}+\left(-x_{j}\right) \mathbf{e}_{j}\right)}
$$

by the assumption on $A$ which says that $A$ is symmetric in the $\mathbf{e}_{j}$ direction. Hence

$$
P_{j} \mathbf{x}+\mathbf{e}_{j} x_{j} \in S\left(A, \mathbf{e}_{i}\right)
$$

if and only if

$$
\left|x_{i}\right|<2^{-1} m\left(A_{P_{i}\left(P_{j} \mathbf{x}+\left(-x_{j}\right) \mathbf{e}_{j}\right)}\right)
$$

if and only if

$$
P_{j} \mathbf{x}+\left(-x_{j}\right) \mathbf{e}_{j} \in S\left(A, \mathbf{e}_{i}\right)
$$

This proves the lemma.

### 28.3.2 The Isodiametric Inequality

The next theorem is called the isodiametric inequality. It is the key result used to compare Lebesgue and Hausdorff measures.
Theorem 28.3.5 Let A be any Lebesgue measurable set in $\mathbb{R}^{n}$. Then

$$
m_{n}(A) \leq \alpha(n)(r(A))^{n}
$$

Proof: Suppose first that $A$ is Borel. Let $A_{1}=S\left(A, \mathbf{e}_{1}\right)$ and let $A_{k}=S\left(A_{k-1}, \mathbf{e}_{k}\right)$. Then by the preceding lemmas, $A_{n}$ is a Borel set, $\operatorname{diam}\left(A_{n}\right) \leq \operatorname{diam}(A), m_{n}\left(A_{n}\right)=m_{n}(A)$, and $A_{n}$ is symmetric. Thus $\mathbf{x} \in A_{n}$ if and only if $-\mathbf{x} \in A_{n}$. It follows that

$$
A_{n} \subseteq \overline{B\left(\mathbf{0}, r\left(A_{n}\right)\right)}
$$

(If $\mathbf{x} \in A_{n} \backslash \overline{B\left(\mathbf{0}, r\left(A_{n}\right)\right)}$, then $-\mathbf{x} \in A_{n} \backslash \overline{B\left(\mathbf{0}, r\left(A_{n}\right)\right)}$ and so $\operatorname{diam}\left(A_{n}\right) \geq 2|\mathbf{x}|>\operatorname{diam}\left(A_{n}\right)$.) Therefore,

$$
m_{n}\left(A_{n}\right) \leq \alpha(n)\left(r\left(A_{n}\right)\right)^{n} \leq \alpha(n)(r(A))^{n}
$$

It remains to establish this inequality for arbitrary measurable sets. Letting $A$ be such a set, let $\left\{K_{n}\right\}$ be an increasing sequence of compact subsets of $A$ such that

$$
m(A)=\lim _{k \rightarrow \infty} m\left(K_{k}\right)
$$

Then

$$
\begin{aligned}
m(A) & =\lim _{k \rightarrow \infty} m\left(K_{k}\right) \leq \lim \sup _{k \rightarrow \infty} \alpha(n)\left(r\left(K_{k}\right)\right)^{n} \\
& \leq \alpha(n)(r(A))^{n}
\end{aligned}
$$

This proves the theorem.

### 28.4 The Proper Value Of $\beta(n)$

I will show that the proper determination of $\beta(n)$ is $\alpha(n)$, the volume of the unit ball. Since $\beta(n)$ has been adjusted such that $k=1, m_{n}(B(\mathbf{0}, 1))=\mathscr{H}^{n}(B(\mathbf{0}, 1))$. There exists a covering of $B(\mathbf{0}, 1)$ of sets of radii less than $\delta,\left\{C_{i}\right\}_{i=1}^{\infty}$ such that

$$
\mathscr{H}_{\delta}^{n}(B(\mathbf{0}, 1))+\varepsilon>\sum_{i} \beta(n) r\left(C_{i}\right)^{n}
$$

Then by Theorem 28.3.5, the isodiametric inequality,

$$
\begin{aligned}
\mathscr{H}_{\delta}^{n}(B(\mathbf{0}, 1))+\varepsilon & >\sum_{i} \beta(n) r\left(C_{i}\right)^{n}=\frac{\beta(n)}{\alpha(n)} \sum_{i} \alpha(n) r\left(\bar{C}_{i}\right)^{n} \\
& \geq \frac{\beta(n)}{\alpha(n)} \sum_{i} m_{n}\left(\bar{C}_{i}\right) \geq \frac{\beta(n)}{\alpha(n)} m_{n}(B(\mathbf{0}, 1))=\frac{\beta(n)}{\alpha(n)} \mathscr{H}^{n}(B(\mathbf{0}, 1))
\end{aligned}
$$

Now taking the limit as $\delta \rightarrow 0$,

$$
\mathscr{H}^{n}(B(\mathbf{0}, 1))+\varepsilon \geq \frac{\beta(n)}{\alpha(n)} \mathscr{H}^{n}(B(\mathbf{0}, 1))
$$

and since $\varepsilon>0$ is arbitrary, this shows $\alpha(n) \geq \beta(n)$.
By the Vitali covering theorem, there exists a sequence of disjoint balls, $\left\{B_{i}\right\}$ such that

$$
B(\mathbf{0}, 1)=\left(\cup_{i=1}^{\infty} B_{i}\right) \cup N
$$

where $m_{n}(N)=0$. Then $\mathscr{H}_{\delta}^{n}(N)=0$ can be concluded because $\mathscr{H}_{\delta}^{n} \leq \mathscr{H}^{n}$ and Lemma 28.2.2. Using $m_{n}(B(\mathbf{0}, 1))=\mathscr{H}^{n}(B(\mathbf{0}, 1))$ again,

$$
\begin{aligned}
\mathscr{H}_{\delta}^{n}(B(\mathbf{0}, 1)) & =\mathscr{H}_{\delta}^{n}\left(\cup_{i} B_{i}\right) \leq \sum_{i=1}^{\infty} \beta(n) r\left(B_{i}\right)^{n} \\
& =\frac{\beta(n)}{\alpha(n)} \sum_{i=1}^{\infty} \alpha(n) r\left(B_{i}\right)^{n}=\frac{\beta(n)}{\alpha(n)} \sum_{i=1}^{\infty} m_{n}\left(B_{i}\right) \\
& =\frac{\beta(n)}{\alpha(n)} m_{n}\left(\cup_{i} B_{i}\right)=\frac{\beta(n)}{\alpha(n)} m_{n}(B(\mathbf{0}, 1))=\frac{\beta(n)}{\alpha(n)} \mathscr{H}^{n}(B(\mathbf{0}, 1))
\end{aligned}
$$

which implies $\alpha(n) \leq \beta(n)$ and so the two are equal. This proves that if $\alpha(n)=\beta(n)$, then the $\mathscr{H}^{n}=m_{n}$ on the measurable sets of $\mathbb{R}^{n}$.

This gives another way to think of Lebesgue measure which is a particularly nice way because it is coordinate free, depending only on the notion of distance.

For $s<n$, note that $\mathscr{H}^{s}$ is not a Radon measure because it will not generally be finite on compact sets. For example, let $n=2$ and consider $\mathscr{H}^{1}(L)$ where $L$ is a line segment joining $(0,0)$ to $(1,0)$. Then $\mathscr{H}^{1}(L)$ is no smaller than $\mathscr{H}^{1}(L)$ when $L$ is considered a subset of $\mathbb{R}^{1}, n=1$. Thus by what was just shown, $\mathscr{H}^{1}(L) \geq 1$. Hence $\mathscr{H}^{1}([0,1] \times[0,1])=\infty$. The situation is this: $L$ is a one-dimensional object inside $\mathbb{R}^{2}$ and $\mathscr{H}^{1}$ is giving a onedimensional measure of this object. In fact, Hausdorff measures can make such heuristic remarks as these precise. Define the Hausdorff dimension of a set, $A$, as

$$
\operatorname{dim}(A)=\inf \left\{s: \mathscr{H}^{s}(A)=0\right\}
$$

### 28.4.1 A Formula For $\alpha(n)$

What is $\alpha(n)$ ? Recall the gamma function which makes sense for all $p>0$.

$$
\Gamma(p) \equiv \int_{0}^{\infty} e^{-t} t^{p-1} d t
$$

Lemma 28.4.1 The following identities hold.

$$
\begin{gathered}
p \Gamma(p)=\Gamma(p+1) \\
\Gamma(p) \Gamma(q)=\left(\int_{0}^{1} x^{p-1}(1-x)^{q-1} d x\right) \Gamma(p+q) \\
\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}
\end{gathered}
$$

Proof: Using integration by parts,

$$
\begin{aligned}
\Gamma(p+1) & =\int_{0}^{\infty} e^{-t} t^{p} d t=-\left.e^{-t} t^{p}\right|_{0} ^{\infty}+p \int_{0}^{\infty} e^{-t} t^{p-1} d t \\
& =p \Gamma(p)
\end{aligned}
$$

Next

$$
\begin{aligned}
\Gamma(p) \Gamma(q) & =\int_{0}^{\infty} e^{-t} t^{p-1} d t \int_{0}^{\infty} e^{-s} s^{q-1} d s \\
& =\int_{0}^{\infty} \int_{0}^{\infty} e^{-(t+s)} t^{p-1} s^{q-1} d t d s \\
& =\int_{0}^{\infty} \int_{s}^{\infty} e^{-u}(u-s)^{p-1} s^{q-1} d u d s \\
& =\int_{0}^{\infty} \int_{0}^{u} e^{-u}(u-s)^{p-1} s^{q-1} d s d u \\
& =\int_{0}^{\infty} \int_{0}^{1} e^{-u}(u-u x)^{p-1}(u x)^{q-1} u d x d u \\
& =\int_{0}^{\infty} \int_{0}^{1} e^{-u} u^{p+q-1}(1-x)^{p-1} x^{q-1} d x d u \\
& =\Gamma(p+q)\left(\int_{0}^{1} x^{p-1}(1-x)^{q-1} d x\right)
\end{aligned}
$$

It remains to find $\Gamma\left(\frac{1}{2}\right)$.

$$
\Gamma\left(\frac{1}{2}\right)=\int_{0}^{\infty} e^{-t} t^{-1 / 2} d t=\int_{0}^{\infty} e^{-u^{2}} \frac{1}{u} 2 u d u=2 \int_{0}^{\infty} e^{-u^{2}} d u
$$

Now

$$
\begin{aligned}
\left(\int_{0}^{\infty} e^{-x^{2}} d x\right)^{2} & =\int_{0}^{\infty} e^{-x^{2}} d x \int_{0}^{\infty} e^{-y^{2}} d y=\int_{0}^{\infty} \int_{0}^{\infty} e^{-\left(x^{2}+y^{2}\right)} d x d y \\
& =\int_{0}^{\infty} \int_{0}^{\pi / 2} e^{-r^{2}} r d \theta d r=\frac{1}{4} \pi
\end{aligned}
$$

and so

$$
\Gamma\left(\frac{1}{2}\right)=2 \int_{0}^{\infty} e^{-u^{2}} d u=\sqrt{\pi}
$$

This proves the lemma.
Next let $n$ be a positive integer.
Theorem 28.4.2 $\alpha(n)=\pi^{n / 2}(\Gamma(n / 2+1))^{-1}$ where $\Gamma(s)$ is the gamma function

$$
\Gamma(s)=\int_{0}^{\infty} e^{-t} t^{s-1} d t
$$

Proof: First let $n=1$.

$$
\Gamma\left(\frac{3}{2}\right)=\frac{1}{2} \Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2}
$$

Thus

$$
\pi^{1 / 2}(\Gamma(1 / 2+1))^{-1}=\frac{2}{\sqrt{\pi}} \sqrt{\pi}=2=\alpha(1)
$$

and this shows the theorem is true if $n=1$.
Assume the theorem is true for $n$ and let $B_{n+1}$ be the unit ball in $\mathbb{R}^{n+1}$. Then by the result in $\mathbb{R}^{n}$,

$$
\begin{gathered}
m_{n+1}\left(B_{n+1}\right)=\int_{-1}^{1} \alpha(n)\left(1-x_{n+1}^{2}\right)^{n / 2} d x_{n+1} \\
=2 \alpha(n) \int_{0}^{1}\left(1-t^{2}\right)^{n / 2} d t
\end{gathered}
$$

Doing an integration by parts and using Lemma 28.4.1

$$
\begin{aligned}
& =2 \alpha(n) n \int_{0}^{1} t^{2}\left(1-t^{2}\right)^{(n-2) / 2} d t \\
& =2 \alpha(n) n \frac{1}{2} \int_{0}^{1} u^{1 / 2}(1-u)^{n / 2-1} d u \\
& =n \alpha(n) \int_{0}^{1} u^{3 / 2-1}(1-u)^{n / 2-1} d u \\
& =n \alpha(n) \Gamma(3 / 2) \Gamma(n / 2)(\Gamma((n+3) / 2))^{-1} \\
& =n \pi^{n / 2}(\Gamma(n / 2+1))^{-1}(\Gamma((n+3) / 2))^{-1} \Gamma(3 / 2) \Gamma(n / 2) \\
& =n \pi^{n / 2}(\Gamma(n / 2)(n / 2))^{-1}(\Gamma((n+1) / 2+1))^{-1} \Gamma(3 / 2) \Gamma(n / 2) \\
& =2 \pi^{n / 2} \Gamma(3 / 2)(\Gamma((n+1) / 2+1))^{-1} \\
& =\pi^{(n+1) / 2}(\Gamma((n+1) / 2+1))^{-1}
\end{aligned}
$$

This proves the theorem.
From now on, in the definition of Hausdorff measure, it will always be the case that $\beta(s)=\alpha(s)$. As shown above, this is the right thing to have $\beta(s)$ to equal if $s$ is a positive integer because this yields the important result that Hausdorff measure is the same as Lebesgue measure. Note the formula, $\pi^{s / 2}(\Gamma(s / 2+1))^{-1}$ makes sense for any $s \geq 0$.

### 28.4.2 Hausdorff Measure And Linear Transformations

Hausdorff measure makes possible a unified development of $n$ dimensional area. As in the case of Lebesgue measure, the first step in this is to understand basic considerations related to linear transformations. Recall that for $L \in \mathscr{L}\left(\mathbb{R}^{k}, \mathbb{R}^{l}\right), L^{*}$ is defined by

$$
(L \mathbf{u}, \mathbf{v})=\left(\mathbf{u}, L^{*} \mathbf{v}\right)
$$

Also recall Theorem 5.9.6 on Page 94 which is stated here for convenience. This theorem says you can write a linear transformation as the composition of two linear transformations, one which preserves length and the other which distorts, the right polar decomposition. The one which distorts is the one which will have a nontrivial interaction with Hausdorff measure while the one which preserves lengths does not change Hausdorff measure. These ideas are behind the following theorems and lemmas.

Theorem 28.4.3 Let $F$ be an $n \times m$ matrix where $m \geq n$. Then there exists an $m \times n$ matrix $R$ and $a n \times n$ matrix $U$ such that

$$
F=R U, U=U^{*}
$$

all eigenvalues of $U$ are non negative,

$$
U^{2}=F^{*} F, R^{*} R=I
$$

and $|R \mathbf{x}|=|\mathbf{x}|$.
Lemma 28.4.4 Let $R \in \mathscr{L}\left(\mathbb{R}^{n}, \mathbb{R}^{m}\right), n \leq m$, and $R^{*} R=I$. Then if $A \subseteq \mathbb{R}^{n}$,

$$
\mathscr{H}^{n}(R A)=\mathscr{H}^{n}(A)
$$

In fact, if $P: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ satisfies $|P \mathbf{x}-P \mathbf{y}|=|\mathbf{x}-\mathbf{y}|$, then

$$
\mathscr{H}^{n}(P A)=\mathscr{H}^{n}(A)
$$

Proof: Note that

$$
|R(\mathbf{x}-\mathbf{y})|^{2}=(R(\mathbf{x}-\mathbf{y}), R(\mathbf{x}-\mathbf{y}))=\left(R^{*} R(\mathbf{x}-\mathbf{y}), \mathbf{x}-\mathbf{y}\right)=|\mathbf{x}-\mathbf{y}|^{2}
$$

Thus $R$ preserves lengths.
Now let $P$ be an arbitrary mapping which preserves lengths and let $A$ be bounded, $P(A) \subseteq \cup_{j=1}^{\infty} C_{j}, r\left(C_{j}\right)<\delta$, and

$$
\mathscr{H}_{\delta}^{n}(P A)+\varepsilon>\sum_{j=1}^{\infty} \alpha(n)\left(r\left(C_{j}\right)\right)^{n}
$$

Since $P$ preserves lengths, it follows $P$ is one to one on $P\left(\mathbb{R}^{n}\right)$ and $P^{-1}$ also preserves lengths on $P\left(\mathbb{R}^{n}\right)$. Replacing each $C_{j}$ with $C_{j} \cap(P A)$,

$$
\begin{aligned}
\mathscr{H}_{\delta}^{n}(P A)+\varepsilon & >\sum_{j=1}^{\infty} \alpha(n) r\left(C_{j} \cap(P A)\right)^{n} \\
& =\sum_{j=1}^{\infty} \alpha(n) r\left(P^{-1}\left(C_{j} \cap(P A)\right)\right)^{n} \\
& \geq \mathscr{H}_{\delta}^{n}(A) .
\end{aligned}
$$

Thus $\mathscr{H}_{\delta}^{n}(P A) \geq \mathscr{H}_{\delta}^{n}(A)$.
Now let $A \subseteq \cup_{j=1}^{\infty} C_{j}, \operatorname{diam}\left(C_{j}\right) \leq \delta$, and

$$
\mathscr{H}_{\delta}^{n}(A)+\varepsilon \geq \sum_{j=1}^{\infty} \alpha(n)\left(r\left(C_{j}\right)\right)^{n}
$$

Then

$$
\begin{aligned}
\mathscr{H}_{\delta}^{n}(A)+\varepsilon & \geq \sum_{j=1}^{\infty} \alpha(n)\left(r\left(C_{j}\right)\right)^{n} \\
& =\sum_{j=1}^{\infty} \alpha(n)\left(r\left(P C_{j}\right)\right)^{n} \geq \mathscr{H}_{\delta}^{n}(P A)
\end{aligned}
$$

Hence $\mathscr{H}_{\delta}^{n}(P A)=\mathscr{H}_{\delta}^{n}(A)$. Letting $\delta \rightarrow 0$ yields the desired conclusion in the case where $A$ is bounded. For the general case, let $A_{r}=A \cap B(0, r)$. Then $\mathscr{H}^{n}\left(P A_{r}\right)=\mathscr{H}^{n}\left(A_{r}\right)$. Now let $r \rightarrow \infty$.

Lemma 28.4.5 Let $F \in \mathscr{L}\left(\mathbb{R}^{n}, \mathbb{R}^{m}\right), n \leq m$, and let $F=R U$ where $R$ and $U$ are described in Theorem 5.9.6 on Page 94. Then if $A \subseteq \mathbb{R}^{n}$ is Lebesgue measurable,

$$
\mathscr{H}^{n}(F A)=\operatorname{det}(U) m_{n}(A)
$$

Proof: Using Theorem 13.5.7 on Page 354 and Theorem 28.2.4,

$$
\begin{gathered}
\mathscr{H}^{n}(F A)=\mathscr{H}^{n}(R U A) \\
=\mathscr{H}^{n}(U A)=m_{n}(U A)=\operatorname{det}(U) m_{n}(A) .
\end{gathered}
$$

Definition 28.4.6 Define $J$ to equal $\operatorname{det}(U)$. Thus

$$
J=\operatorname{det}\left(\left(F^{*} F\right)^{1 / 2}\right)=\left(\operatorname{det}\left(F^{*} F\right)\right)^{1 / 2}
$$

## Chapter 29

## The Area Formula

I am grateful to those who have found errors in this material, some of which were egregious. I would not have found these mistakes because I never teach this material and I don't use it in my research. I do think it is wonderful mathematics however.

To begin with is a simple theorem about extending Lipschitz functions.
Theorem 29.0.1 If $\mathbf{h}: \Omega \rightarrow \mathbb{R}^{m}$ is Lipschitz, then there exists $\overline{\mathbf{h}}: \mathbb{R}^{p} \rightarrow \mathbb{R}^{m}$ which extends h and is also Lipschitz.

Proof: It suffices to assume $m=1$ because if this is shown, it may be applied to the components of $\mathbf{h}$ to get the desired result. Suppose

$$
\begin{equation*}
|h(\mathbf{x})-h(\mathbf{y})| \leq K|\mathbf{x}-\mathbf{y}| \tag{29.0.1}
\end{equation*}
$$

Define

$$
\begin{equation*}
\bar{h}(\mathbf{x}) \equiv \inf \{h(\mathbf{w})+K|\mathbf{x}-\mathbf{w}|: \mathbf{w} \in \Omega\} \tag{29.0.2}
\end{equation*}
$$

If $\mathbf{x} \in \Omega$, then for all $\mathbf{w} \in \Omega$,

$$
h(\mathbf{w})+K|\mathbf{x}-\mathbf{w}| \geq h(\mathbf{x})
$$

by 29.0.1. This shows $h(\mathbf{x}) \leq \bar{h}(\mathbf{x})$. But also you could take $\mathbf{w}=\mathbf{x}$ in 29.0.2 which yields $\bar{h}(\mathbf{x}) \leq h(\mathbf{x})$. Therefore $\bar{h}(\mathbf{x})=h(\mathbf{x})$ if $\mathbf{x} \in \Omega$.

Now suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{p}$ and consider $|\bar{h}(\mathbf{x})-\bar{h}(\mathbf{y})|$. Without loss of generality assume $\bar{h}(\mathbf{x}) \geq \bar{h}(\mathbf{y})$. (If not, repeat the following argument with $\mathbf{x}$ and $\mathbf{y}$ interchanged.) Pick $\mathbf{w} \in \Omega$ such that

$$
h(\mathbf{w})+K|\mathbf{y}-\mathbf{w}|-\varepsilon<\bar{h}(\mathbf{y})
$$

Then

$$
\begin{aligned}
|\bar{h}(\mathbf{x})-\bar{h}(\mathbf{y})|=\bar{h}(\mathbf{x})-\bar{h}(\mathbf{y}) & \leq h(\mathbf{w})+K|\mathbf{x}-\mathbf{w}|- \\
{[h(\mathbf{w})+K|\mathbf{y}-\mathbf{w}|-\varepsilon] } & \leq K|\mathbf{x}-\mathbf{y}|+\varepsilon .
\end{aligned}
$$

Since $\varepsilon$ is arbitrary,

$$
|\bar{h}(\mathbf{x})-\bar{h}(\mathbf{y})| \leq K|\mathbf{x}-\mathbf{y}|
$$

### 29.1 Estimates for Hausdorff Measure

It was shown in Lemma 28.4.5 that

$$
\mathscr{H}^{n}(F A)=\operatorname{det}(U) m_{n}(A)
$$

where $F=R U$ with $R$ preserving distances and $U$ a symmetric matrix having all positive eigenvalues. The area formula gives a generalization of this simple relationship to the case where $F$ is replaced by a nonlinear mapping $\mathbf{h}$. It contains as a special case the earlier change of variables formula. There are two parts to this development. The first part is to generalize Lemma 28.4.5 to the case of nonlinear maps. When this is done, the area formula can be presented.

In the first version of the area formula $\mathbf{h}$ will be a Lipschitz function,

$$
|\mathbf{h}(\mathbf{x})-\mathbf{h}(\mathbf{y})| \leq K|\mathbf{x}-\mathbf{y}|
$$

defined on $\mathbb{R}^{n}$. This is no loss of generality because of Theorem 29.0.1.
The following lemma states that Lipschitz maps take sets of measure zero to sets of measure zero. It also gives a convenient estimate. It involves the consideration of $\mathscr{H}^{n}$ as an outer measure. Thus it is not necessary to know the set $B$ is measurable.

Lemma 29.1.1 If $\mathbf{h}$ is Lipschitz with Lipschitz constant $K$ then

$$
\mathscr{H}^{n}(\mathbf{h}(B)) \leq K^{n} \mathscr{H}^{n}(B)
$$

Also, if $T$ is a set in $\mathbb{R}^{n}, m_{n}(T)=0$, then $\mathscr{H}^{n}(\mathbf{h}(T))=0$. It is not necessary that $\mathbf{h}$ be one to one.

Proof: Let $\left\{C_{i}\right\}_{i=1}^{\infty}$ cover $B$ with each having diameter less than $\delta$ and let this cover be such that

$$
\sum_{i} \beta(n) \frac{1}{2} \operatorname{diam}\left(C_{i}\right)^{n}<\mathscr{H}_{\delta}^{n}(B)+\varepsilon
$$

Then $\left\{\mathbf{h}\left(C_{i}\right)\right\}$ covers $\mathbf{h}(B)$ and each set has diameter no more than $K \boldsymbol{\delta}$. Then

$$
\begin{aligned}
\mathscr{H}_{K \delta}^{n}(\mathbf{h}(B)) & \leq \sum_{i} \beta(n)\left(\frac{1}{2} \operatorname{diam}\left(\mathbf{h}\left(C_{i}\right)\right)\right)^{n} \\
& \leq K^{n} \sum_{i} \beta(n)\left(\frac{1}{2} \operatorname{diam}\left(C_{i}\right)\right)^{n} \leq K^{n}\left(\mathscr{H}_{\delta}^{n}(B)+\varepsilon\right)
\end{aligned}
$$

Since $\varepsilon$ is arbitrary, this shows that

$$
\mathscr{H}_{K \delta}^{n}(\mathbf{h}(B)) \leq K^{n} \mathscr{H}_{\delta}^{n}(B)
$$

Now take a limit as $\delta \rightarrow 0$. The second claim follows from $m_{n}=\mathscr{H}^{n}$ on Lebesgue measurable sets of $\mathbb{R}^{n}$.

Lemma 29.1.2 If $S$ is a Lebesgue measurable set and $\mathbf{h}$ is Lipschitz then $\mathbf{h}(S)$ is $\mathscr{H}^{n}$ measurable. Also, if $\mathbf{h}$ is Lipschitz with constant $K$,

$$
\mathscr{H}^{n}(\mathbf{h}(S)) \leq K^{n} m_{n}(S)
$$

It is not necessary that $\mathbf{h}$ be one to one.
Proof: The estimate follows from Lemma 29.1.1 and the observation that, as shown before, Theorem 28.2.4, if $S$ is Lebesgue measurable in $\mathbb{R}^{n}$, then $\mathscr{H}^{n}(S)=m_{n}(S)$. The estimate also shows that $\mathbf{h}$ maps sets of Lebesgue measure zero to sets of $\mathscr{H}^{n}$ measure zero. Why is $\mathbf{h}(S) \mathscr{H}^{n}$ measurable if $S$ is Lebesgue measurable? This follows from completeness of $\mathscr{H}^{n}$. Indeed, let $F$ be $F_{\sigma}$ and contained in $S$ with $m_{n}(S \backslash F)=0$. Then

$$
\mathbf{h}(S)=\mathbf{h}(S \backslash F) \cup \mathbf{h}(F)
$$

The second set is Borel and the first has $\mathscr{H}^{n}$ measure zero. By completeness of $\mathscr{H}^{n}, \mathbf{h}(S)$ is $\mathscr{H}^{n}$ measurable.

By Theorem 5.9.6 on Page 94 , when $D \mathbf{h}(\mathbf{x})$ exists,

$$
D \mathbf{h}(\mathbf{x})=R(\mathbf{x}) U(\mathbf{x})
$$

where $(U(\mathbf{x}) \mathbf{u}, \mathbf{v})=(U(\mathbf{x}) \mathbf{v}, \mathbf{u}),(U(\mathbf{x}) \mathbf{u}, \mathbf{u}) \geq 0$ and $R^{*} R=I$ so $R$ preserves lengths. This convention will be used in what follows.

Lemma 29.1.3 In this situation where $R^{*} R=I,\left|R^{*} \mathbf{u}\right| \leq|\mathbf{u}|$.
Proof: First note that

$$
\begin{aligned}
\left(\mathbf{u}-R R^{*} \mathbf{u}, R R^{*} \mathbf{u}\right) & =\left(\mathbf{u}, R R^{*} \mathbf{u}\right)-\left|R R^{*} \mathbf{u}\right|^{2} \\
& =\left|R^{*} \mathbf{u}\right|^{2}-\left|R^{*} \mathbf{u}\right|^{2}=0
\end{aligned}
$$

and so

$$
\begin{aligned}
|\mathbf{u}|^{2} & =\left|\mathbf{u}-R R^{*} \mathbf{u}+R R^{*} \mathbf{u}\right|^{2} \\
& =\left|\mathbf{u}-R R^{*} \mathbf{u}\right|^{2}+\left|R R^{*} \mathbf{u}\right|^{2} \\
& =\left|\mathbf{u}-R R^{*} \mathbf{u}\right|^{2}+\left|R^{*} \mathbf{u}\right|^{2} .
\end{aligned}
$$

Then the following corollary follows from Lemma 29.1.3.
Corollary 29.1.4 Let $T \subseteq \mathbb{R}^{m}$. Then

$$
\mathscr{H}^{n}(T) \geq \mathscr{H}^{n}\left(R^{*} T\right)
$$

### 29.2 Comparison Theorems

First is a simple lemma which is fairly interesting which involves comparison of two linear transformations.

Lemma 29.2.1 Suppose $S, T$ are linear defined on a finite dimensional normed linear space, $S^{-1}$ exists and let $\delta \in(0,1)$. Then whenever $\|S-T\|$ is small enough, it follows that

$$
\begin{equation*}
\frac{|T \mathbf{v}|}{|S \mathbf{v}|} \in(1-\delta, 1+\delta) \tag{29.2.3}
\end{equation*}
$$

for all $\mathbf{v} \neq \mathbf{0}$. Similarly if $T^{-1}$ exists and $\|S-T\|$ is small enough,

$$
\frac{|T \mathbf{v}|}{|S \mathbf{v}|} \in(1-\delta, 1+\delta)
$$

Proof: Say $S^{-1}$ exists. Then $\mathbf{v} \rightarrow|S \mathbf{v}|$ is a norm. Then by equivalence of norms, Theorem 8.4.9, there exists $\eta>0$ such that for all $\mathbf{v},|S \mathbf{v}| \geq \eta|\mathbf{v}|$. Say $\|T-S\|<r<\delta \eta$

$$
\frac{|S \mathbf{v}|-\|T-S\||\mathbf{v}|}{|S \mathbf{v}|} \leq \frac{|T \mathbf{v}|}{|S \mathbf{v}|}=\frac{|[S+(T-S)] \mathbf{v}|}{|S \mathbf{v}|} \leq \frac{|S \mathbf{v}|+\|T-S\||\mathbf{v}|}{|S \mathbf{v}|}
$$

and so

$$
\begin{aligned}
1-\delta & \leq 1-\frac{r|\mathbf{v}|}{\eta|\mathbf{v}|} \leq \frac{|S \mathbf{v}|-\|T-S\||\mathbf{v}|}{|S \mathbf{v}|} \leq \frac{|T \mathbf{v}|}{|S \mathbf{v}|} \\
& \leq \frac{|S \mathbf{v}|+\|T-S\||\mathbf{v}|}{|S \mathbf{v}|} \leq 1+\frac{r|\mathbf{v}|}{\eta|\mathbf{v}|}=1+\delta
\end{aligned}
$$

The last assertion follows by noting that if $T^{-1}$ is given to exist and $S$ is close to $T$ then

$$
\frac{|S \mathbf{v}|}{|T \mathbf{v}|} \in(1-\delta, 1+\delta) \text { so } \frac{|T \mathbf{v}|}{|S \mathbf{v}|} \in\left(\frac{1}{1+\delta}, \frac{1}{1-\delta}\right) \subseteq(1-\hat{\delta}, 1+\hat{\delta})
$$

By choosing $\delta$ appropriately, one can achieve the last inclusion for given $\hat{\delta}$.
In short, the above lemma says that if one of $S, T$ is invertible and the other is close to it, then it is also invertible and the quotient of $|S \mathbf{v}|$ and $|T \mathbf{v}|$ is close to 1 . Then the following lemma is fairly obvious.

Lemma 29.2.2 Let $S, T$ be $n \times n$ matrices which are invertible. Then

$$
\mathbf{o}(T \mathbf{v})=\mathbf{o}(S \mathbf{v})=\mathbf{o}(\mathbf{v})
$$

and if $L$ is a continuous linear transformation such that for $a<b$,

$$
\sup _{\mathbf{v} \neq \mathbf{0}} \frac{|L \mathbf{v}|}{|S \mathbf{v}|}<b, \inf _{\mathbf{v} \neq \mathbf{0}} \frac{|L \mathbf{v}|}{|S \mathbf{v}|}>a
$$

If $\|S-T\|$ is small enough, it follows that the same inequalities hold with $S$ replaced with $T$. Here $\|\cdot\|$ denotes the operator norm.

Proof: Consider the first claim. For

$$
\frac{|o(T \mathbf{v})|}{|\mathbf{v}|}=\frac{|o(T \mathbf{v})|}{|T \mathbf{v}|} \frac{|T \mathbf{v}|}{|\mathbf{v}|} \leq \frac{|o(T \mathbf{v})|}{|T \mathbf{v}|}\|T\|
$$

Thus $o(T \mathbf{v})=o(\mathbf{v})$. It is similar for $T$ replaced with $S$.
Consider the second claim. Pick $\delta$ sufficiently small. Then by Lemma 29.2.1

$$
\sup _{\mathbf{v} \neq \mathbf{0}} \frac{|L \mathbf{v}|}{|T \mathbf{v}|}=\sup _{\mathbf{v} \neq \mathbf{0}} \frac{|L \mathbf{v}|}{|S \mathbf{v}|} \frac{|S \mathbf{v}|}{|T \mathbf{v}|} \leq(1+\delta) \sup _{\mathbf{v} \neq \mathbf{0}} \frac{|L \mathbf{v}|}{|S \mathbf{v}|}<b
$$

if $\delta$ is small enough. The other inequality is shown exactly similar.

### 29.3 A Decomposition

This follows [47] which is where I encountered this material. Assume the following:

$$
\begin{equation*}
D \mathbf{h}(\mathbf{x}) \text { exists at } a . e . \mathbf{x} \in G \text { say at all } \mathbf{x} \in A \subseteq G \tag{29.3.4}
\end{equation*}
$$

By regularity, we can and will assume $A$ is a Borel set. Of course this is automatic if $\mathbf{h}$ is Lipschitz. I have in mind the assumption that $\mathbf{h}$ is Lipschitz. This makes things very convenient because then $\mathbf{h}(E)$ is $\mathscr{H}^{n}$ Hausdorff measurable whenever $E$ is $n$ dimensional Lebesgue measurable. However, there are interesting things which don't depend on Lipschitz continuity. Initially, I will only assume that $\mathbf{h}$ is continuous on $G$ and differentiable on $A$.

For $\mathbf{x} \in A$, let $D \mathbf{h}(\mathbf{x}) \equiv R(\mathbf{x}) U(\mathbf{x})$ where $R(\mathbf{x})$ preserves lengths and

$$
U(\mathbf{x}) \equiv\left(D \mathbf{h}(\mathbf{x})^{*} D \mathbf{h}(\mathbf{x})\right)^{1 / 2}
$$

Let $A^{+}$denote those points of $A$ for which $U(\mathbf{x})^{-1}$ exists. Thus this is a measurable subset of $A$.

Let $B$ be a Borel measurable subset of $A^{+}$and let $\mathbf{b} \in B$. Let $\mathscr{S}$ be a countable dense subset of the space of symmetric invertible matrices and let $\mathscr{C}$ be a countable dense subset of $B$. The idea is to decompose $B$ into countably many Borel sets $E$ on which $\mathbf{h}$ is one to one and Lipschitz with $\mathbf{h}^{-1}$ Lipschitz on $\mathbf{h}(E)$. This will be done by establishing 29.3.10 given below where $T$ is an invertible symmetric transformation.

Let $\varepsilon$ be a small number. Since $U(\mathbf{b})$ is invertible, Lemma 29.2.2 implies $\mathbf{o}(\mathbf{a}-\mathbf{b})=$ $\mathbf{o}(U(\mathbf{b})(\mathbf{a}-\mathbf{b}))$ and so

$$
\begin{equation*}
|\mathbf{h}(\mathbf{a})-\mathbf{h}(\mathbf{b})-D \mathbf{h}(\mathbf{b})(\mathbf{a}-\mathbf{b})|<\varepsilon|U(\mathbf{b})(\mathbf{a}-\mathbf{b})| \tag{29.3.5}
\end{equation*}
$$

provided that $\mathbf{a} \in B\left(\mathbf{b}, \frac{2}{i}\right)$ for $i$ sufficiently large. By Lemma 29.2.1,

$$
\begin{equation*}
|\mathbf{h}(\mathbf{a})-\mathbf{h}(\mathbf{b})-D \mathbf{h}(\mathbf{b})(\mathbf{a}-\mathbf{b})|<\varepsilon|T(\mathbf{a}-\mathbf{b})| \tag{29.3.6}
\end{equation*}
$$

where $U(\mathbf{b})$ is replaced by another linear one to one and onto symmetric mapping $T$ provided $T$ is sufficiently close to $U(\mathbf{b})$.

Now let $\mathbf{c} \in \mathscr{C}$ be close enough to $\mathbf{b}$ that $\mathbf{b} \in B\left(\mathbf{c}, \frac{1}{i}\right)$. Thus $\mathbf{b} \in E(T, \mathbf{c}, i)$ where for $i \in \mathbb{N}, \mathbf{c} \in \mathscr{C}, T \in \mathscr{S}, E(T, \mathbf{c}, i)$ consists of those $\mathbf{b} \in B\left(\mathbf{c}, \frac{1}{i}\right)$ such that for all $\mathbf{a} \in B\left(\mathbf{b}, \frac{2}{i}\right)$, 29.3.6 holds and also

$$
\begin{align*}
& \inf _{\mathbf{v} \neq \mathbf{0}} \frac{|D \mathbf{h}(\mathbf{b}) \mathbf{v}|}{|T \mathbf{v}|}=\inf _{\mathbf{v} \neq \mathbf{0}} \frac{|U(\mathbf{b}) \mathbf{v}|}{|T \mathbf{v}|}>1-\varepsilon,  \tag{29.3.7}\\
& \sup _{\mathbf{v} \neq \mathbf{0}} \frac{|D \mathbf{h}(\mathbf{b}) \mathbf{v}|}{|T \mathbf{v}|}=\sup _{\mathbf{v} \neq \mathbf{0}} \frac{|U(\mathbf{b}) \mathbf{v}|}{|T \mathbf{v}|}<1+\varepsilon \tag{29.3.8}
\end{align*}
$$

It follows then from the above inequalities and 29.3.6 that for all $\mathbf{a} \in B\left(\mathbf{b}, \frac{2}{i}\right)$,

$$
\begin{align*}
|\mathbf{h}(\mathbf{a})-\mathbf{h}(\mathbf{b})| & \leq(1+2 \varepsilon)|T(\mathbf{a}-\mathbf{b})| \\
|\mathbf{h}(\mathbf{a})-\mathbf{h}(\mathbf{b})| & \geq(1-2 \varepsilon)|T(\mathbf{a}-\mathbf{b})| \tag{29.3.9}
\end{align*}
$$

and so

$$
\begin{equation*}
(1-2 \varepsilon)|T(\mathbf{a}-\mathbf{b})| \leq|\mathbf{h}(\mathbf{a})-\mathbf{h}(\mathbf{b})| \leq(1+2 \varepsilon)|T(\mathbf{a}-\mathbf{b})| \tag{29.3.10}
\end{equation*}
$$

Then if $\mathbf{a}, \mathbf{b} \in E(T, \mathbf{c}, i), 29.3 .10$ holds for these two $\mathbf{a}, \mathbf{b}$ because $|\mathbf{a}-\mathbf{b}|<2 / i$.


Note that this proves that on $E(T, \mathbf{c}, i)$ the function $\mathbf{h}$ is one to one and $T$ is a close approximation to $U(\mathbf{b})$ for each $\mathbf{b} \in E(T, \mathbf{c}, i)$. It also shows the Lipschitz continuity of $\mathbf{h}$ and $\mathbf{h}^{-1}$ on $E$ by comparison with $T$. What has just been shown is a very interesting result for its own sake. It is summarized in the following lemma.

### 29.4 Estimates and a Limit

Lemma 29.4.1 Let $\mathbf{h}$ be differentiable on $A \subseteq G$ and let $A^{+}$consist of those points $\mathbf{x}$ where $\operatorname{det}\left(D \mathbf{h}(\mathbf{x})^{*} D \mathbf{h}(\mathbf{x})\right)>0$. Then if B is any Borel subset of $A^{+}$, there is a disjoint sequence of Borel sets and invertible symmetric transformations $T_{k},\left\{\left(E_{k}, T_{k}\right)\right\}, \cup_{k} E_{k}=B$ such that $\mathbf{h}$ is Lipschitz on $E_{k}$ and $\mathbf{h}^{-1}$ is Lipschitz on $\mathbf{h}(E(T, \mathbf{c}, i))$. Also for any $\mathbf{b} \in E_{k}, 29.3 .7$ and 29.3.8 both hold. Also, for $\mathbf{b} \in E_{k}$

$$
\begin{equation*}
(1-\boldsymbol{\varepsilon})\left|T_{k} \mathbf{v}\right|<|D \mathbf{h}(\mathbf{b}) \mathbf{v}|=|U(\mathbf{b}) \mathbf{v}|<(1+\boldsymbol{\varepsilon})\left|T_{k} \mathbf{v}\right| \tag{29.4.11}
\end{equation*}
$$

One can also conclude that for $\mathbf{b} \in E_{k}$,

$$
\begin{equation*}
(1-\varepsilon)^{-n}\left|\operatorname{det}\left(T_{k}\right)\right| \leq \operatorname{det}(U(\mathbf{b})) \leq(1+\varepsilon)^{n}\left|\operatorname{det}\left(T_{k}\right)\right| \tag{29.4.12}
\end{equation*}
$$

Proof: It follows from 29.3.10 that for $\mathbf{x}, \mathbf{y} \in T(E(T, \mathbf{c}, i))$

$$
\begin{equation*}
\left|\mathbf{h}\left(T^{-1}(\mathbf{x})\right)-\mathbf{h}\left(T^{-1}(\mathbf{y})\right)\right| \leq(1+2 \varepsilon)|\mathbf{x}-\mathbf{y}| \tag{29.4.13}
\end{equation*}
$$

and for $\mathbf{x}, \mathbf{y}$ in $\mathbf{h}(E(T, \mathbf{c}, i))$,

$$
\begin{equation*}
\left|T\left(\mathbf{h}^{-1}(\mathbf{x})\right)-T\left(\mathbf{h}^{-1}(\mathbf{y})\right)\right| \leq \frac{1}{(1-2 \varepsilon)}|\mathbf{x}-\mathbf{y}| \tag{29.4.14}
\end{equation*}
$$

The symbol $\mathbf{h}^{-1}$ refers to the restriction to $\mathbf{h}(E(T, \mathbf{c}, i))$ of the inverse image of $\mathbf{h}$. Thus, on this set, $\mathbf{h}^{-1}$ is actually a function even though $\mathbf{h}$ might not be one to one. This also shows that $\mathbf{h}^{-1}$ is Lipschitz on $\mathbf{h}(E(T, \mathbf{c}, i))$ and $\mathbf{h}$ is Lipschitz on $E(T, \mathbf{c}, i)$. Indeed, from 29.4.13, letting $T^{-1}(\mathbf{x})=\mathbf{a}$ and $T^{-1}(\mathbf{y})=\mathbf{b}$,

$$
\begin{equation*}
|\mathbf{h}(\mathbf{a})-\mathbf{h}(\mathbf{b})| \leq(1+2 \varepsilon)|T(\mathbf{a})-T(\mathbf{b})| \leq(1+2 \varepsilon)\|T\||\mathbf{a}-\mathbf{b}| \tag{29.4.15}
\end{equation*}
$$

and using the fact that $T$ is one to one, there is $\delta>0$ such that $|T \mathbf{z}| \geq \delta|\mathbf{z}|$ so 29.4.14 implies that

$$
\begin{equation*}
\left|\mathbf{h}^{-1}(\mathbf{x})-\mathbf{h}^{-1}(\mathbf{y})\right| \leq \frac{1}{\delta(1-2 \varepsilon)}|\mathbf{x}-\mathbf{y}| \tag{29.4.16}
\end{equation*}
$$

Now let $\left(E_{k}, T_{k}\right)$ result from a disjoint union of measurable subsets of the countably many $E(T, \mathbf{c}, i)$ such that $B=\cup_{k} E_{k}$. Thus the above Lipschitz conditions 29.4.13 and
29.4.14 hold for $T_{k}$ in place of $T$. It is not necessary to assume $\mathbf{h}$ is one to one in this lemma. $\mathbf{h}^{-1}$ refers to the inverse image of $\mathbf{h}$ restricted to $\mathbf{h}\left(E_{k}\right)$ as discussed above.

Finally, consider 29.4.12. 29.4.11 implies that

$$
(1-\varepsilon)|\mathbf{v}|<\left|U(\mathbf{b}) T_{k}^{-1} \mathbf{v}\right|<(1+\boldsymbol{\varepsilon})|\mathbf{v}|
$$

A generic vector in $B(\mathbf{0}, 1-\varepsilon)$ is $(1-\varepsilon) \mathbf{v}$ where $|\mathbf{v}|<1$. Thus, the above inequality implies

$$
B(\mathbf{0}, 1-\varepsilon) \subseteq U(\mathbf{b}) T_{k}^{-1} B(\mathbf{0}, 1) \subseteq B(\mathbf{0}, 1+\varepsilon)
$$

This implies

$$
\alpha(n)(1-\varepsilon)^{n} \leq \operatorname{det}\left(U(\mathbf{b}) T_{k}^{-1}\right) \alpha(n) \leq \alpha(n)(1+\varepsilon)^{n}
$$

and so $(1-\varepsilon)^{n} \leq \operatorname{det}(U(\mathbf{b})) \operatorname{det}\left(T_{k}^{-1}\right) \leq(1+\varepsilon)^{n}$ and so for $\mathbf{b} \in E_{k}$,

$$
(1-\varepsilon)^{n}\left|\operatorname{det}\left(T_{k}\right)\right| \leq \operatorname{det}(U(\mathbf{b})) \leq(1+\varepsilon)^{n}\left|\operatorname{det}\left(T_{k}\right)\right|
$$

Recall that $B$ was a Borel measurable subset of $A^{+}$the set where $U(\mathbf{x})^{-1}$ exists. Now the above estimates can be used to estimate $\mathscr{H}^{n}\left(\mathbf{h}\left(E_{k}\right)\right)$. There is no problem about measurability of $\mathbf{h}\left(E_{k}\right)$ due to Lipschitz continuity of $\mathbf{h}$ on $E_{k}$. From Lemma 29.1.1 about the relationship between Hausdorff measure and Lipschitz mappings, it follows from 29.4.13 and 29.4.12,

$$
\begin{aligned}
\mathscr{H}^{n}\left(\mathbf{h}\left(E_{k}\right)\right) & =\mathscr{H}^{n}\left(\mathbf{h} \circ T_{k}^{-1}\left(T_{k}\left(E_{k}\right)\right)\right) \leq(1+2 \varepsilon)^{n} \mathscr{H}^{n}\left(T_{k}\left(E_{k}\right)\right) \\
& =(1+2 \varepsilon)^{n} m_{n}\left(T_{k}\left(E_{k}\right)\right) \leq(1+2 \varepsilon)^{n}\left|\operatorname{det}\left(T_{k}\right)\right| m_{n}\left(E_{k}\right)
\end{aligned}
$$

also,

$$
\begin{equation*}
m_{n}\left(T_{k}\left(E_{k}\right)\right)=\mathscr{H}^{n}\left(\left(T_{k} \circ \mathbf{h}^{-1}\left(\mathbf{h}\left(E_{k}\right)\right)\right)\right) \leq\left(\frac{1}{1-2 \varepsilon}\right)^{n} \mathscr{H}^{n}\left(\mathbf{h}\left(E_{k}\right)\right) \tag{29.4.17}
\end{equation*}
$$

Summarizing,

$$
\left(\frac{1}{1-2 \varepsilon}\right)^{n} \mathscr{H}^{n}\left(\mathbf{h}\left(E_{k}\right)\right) \geq m_{n}\left(T_{k}\left(E_{k}\right)\right) \geq \frac{1}{(1+2 \varepsilon)^{n}} \mathscr{H}^{n}\left(\mathbf{h}\left(E_{k}\right)\right)
$$

Then the above inequality and 29.4.12, 29.4.17 imply the following.

$$
\begin{gather*}
\frac{1}{(1+2 \varepsilon)^{n}} \mathscr{H}^{n}\left(\mathbf{h}\left(E_{k}\right)\right) \leq m_{n}\left(T_{k}\left(E_{k}\right)\right) \leq\left(\frac{1}{1-2 \varepsilon}\right)^{n}\left|\operatorname{det}\left(T_{k}\right)\right| m_{n}\left(E_{k}\right) \\
\leq\left(\frac{1}{1-2 \varepsilon}\right)^{n}(1-\varepsilon)^{n} \int_{E_{k}} \operatorname{det}(U(\mathbf{x})) d m_{n} \leq\left(\frac{1}{1-2 \varepsilon}\right)^{n}(1+\varepsilon)^{n}\left|\operatorname{det}\left(T_{k}\right)\right| m_{n}\left(E_{k}\right) \\
\leq \frac{(1+2 \varepsilon)^{n}}{(1-2 \varepsilon)^{n}} m_{n}\left(T_{k} E_{k}\right) \leq \frac{(1+2 \varepsilon)^{n}}{(1-2 \varepsilon)^{n}}\left(\frac{1}{1-2 \varepsilon}\right)^{n} \mathscr{H}^{n}\left(\mathbf{h}\left(E_{k}\right)\right) \tag{29.4.18}
\end{gather*}
$$

Assume now that $\mathbf{h}$ is one to one on $B$. Summing over all $E_{k}$ yields the following thanks to the assumption that $\mathbf{h}$ is one to one.

$$
\frac{1}{(1+2 \varepsilon)^{n}} \mathscr{H}^{n}(\mathbf{h}(B)) \leq(1-2 \varepsilon)^{-n}(1-\varepsilon)^{n} \int_{B} \operatorname{det}(U(\mathbf{x})) d x
$$

$$
\leq \frac{(1+2 \varepsilon)^{n}}{(1-2 \varepsilon)^{n}}\left(\frac{1}{1-2 \varepsilon}\right)^{n} \mathscr{H}^{n}(\mathbf{h}(B))
$$

$\varepsilon$ was arbitrary and so when $\mathbf{h}$ is one to one on $B$,

$$
\mathscr{H}^{n}(\mathbf{h}(B)) \leq \int_{B} \operatorname{det}(U(\mathbf{x})) d x \leq \mathscr{H}^{n}(\mathbf{h}(B))
$$

Now $B$ was completely arbitrary. Let it equal $B(\mathbf{x}, r) \cap A^{+}$where $\mathbf{x} \in A^{+}$. Then for $\mathbf{x} \in A^{+}$,

$$
\mathscr{H}^{n}\left(\mathbf{h}\left(B(\mathbf{x}, r) \cap A^{+}\right)\right)=\int_{B(\mathbf{x}, r)} \mathscr{X}_{A^{+}}(\mathbf{y}) \operatorname{det}(U(\mathbf{y})) d y
$$

Divide by $m_{n}(B(\mathbf{x}, r))$ and use the fundamental theorem of calculus. This yields that for $\mathbf{x}$ off a set of $m_{n}$ measure zero,

$$
\begin{equation*}
\lim _{r \rightarrow 0} \frac{\mathscr{H}^{n}\left(\mathbf{h}\left(B(\mathbf{x}, r) \cap A^{+}\right)\right)}{m_{n}(B(\mathbf{x}, r))}=\mathscr{X}_{A^{+}}(\mathbf{x}) \operatorname{det}(U(\mathbf{x})) \tag{29.4.19}
\end{equation*}
$$

This has proved the following lemma.
Lemma 29.4.2 Let $\mathbf{h}$ be continuous on $G$ and differentiable on $A \subseteq G$ and one to one on $A^{+}$which is as defined above. There is a set of measure zero $N$ such that for $\mathbf{x} \in A^{+} \backslash N$,

$$
\lim _{r \rightarrow 0+} \frac{\mathscr{H}^{n}\left(\mathbf{h}\left(B(\mathbf{x}, r) \cap A^{+}\right)\right)}{m_{n}(B(\mathbf{x}, r))}=\operatorname{det}(U(\mathbf{x}))
$$

The next theorem removes the assumption that $U(\mathbf{x})^{-1}$ exists and replaces $A^{+}$with A. From now on $J_{*}(\mathbf{x}) \equiv \operatorname{det}(U(\mathbf{x}))$. Also note that if $F$ is measurable and a subset of $A^{+}, \mathbf{h}\left(E_{k} \cap F\right)$ is Hausdorff measurable because of the Lipschitz continuity of $\mathbf{h}$ on $E_{k}$.

Theorem 29.4.3 Let $\mathbf{h}: G \subseteq \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ for $n \leq m, G$ an open set in $\mathbb{R}^{n}$, and suppose $\mathbf{h}$ is continuous on $G$ differentiable and one to one on $A$. Then for a.e. $\mathbf{x} \in A$, the set in $G$ where Dh(x) exists,

$$
\begin{equation*}
J_{*}(\mathbf{x})=\lim _{r \rightarrow 0} \frac{\mathscr{H}^{n}(\mathbf{h}(B(\mathbf{x}, r) \cap A))}{m_{n}(B(\mathbf{x}, r))} \tag{29.4.20}
\end{equation*}
$$

where $J_{*}(\mathbf{x}) \equiv \operatorname{det}(U(\mathbf{x}))=\operatorname{det}\left(D \mathbf{h}(\mathbf{x})^{*} D \mathbf{h}(\mathbf{x})\right)^{1 / 2}$.
Proof: The above argument shows that the conclusion of the theorem holds when $J_{*}(\mathbf{x}) \neq 0$ at least with $A$ replaced with $A^{+}$. I will apply this to a modified function in which the corresponding $U(\mathbf{x})$ always has an inverse. Let $\mathbf{k}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m} \times \mathbb{R}^{n}$ be defined as

$$
\mathbf{k}(\mathbf{x}) \equiv\binom{\mathbf{h}(\mathbf{x})}{\varepsilon \mathbf{x}}
$$

in which dependence of $\mathbf{k}$ on $\varepsilon$ is suppressed. Then $D \mathbf{k}(\mathbf{x})^{*} D \mathbf{k}(\mathbf{x})=D \mathbf{h}(\mathbf{x})^{*} D \mathbf{h}(\mathbf{x})+\varepsilon^{2} I_{n}$ and so

$$
J_{*} \mathbf{k}(\mathbf{x})^{2} \equiv \operatorname{det}\left(D \mathbf{h}(\mathbf{x})^{*} D \mathbf{h}(\mathbf{x})+\varepsilon^{2} I_{n}\right)=\operatorname{det}\left(Q^{*} D Q+\varepsilon^{2} I_{n}\right)>0
$$

where $D$ is a diagonal matrix having the nonnegative eigenvalues of $D \mathbf{h}(\mathbf{x})^{*} D \mathbf{h}(\mathbf{x})$ down the main diagonal, $Q$ an orthogonal matrix. $A$ is where $\mathbf{h}$ is differentiable. However, it is $A^{+}$when referring to $\mathbf{k}$. Then Lemma 29.4.2 implies

$$
\lim _{r \rightarrow 0} \frac{\mathscr{H}^{n}(\mathbf{k}(B(\mathbf{x}, r) \cap A))}{m_{n}(B(\mathbf{x}, r))}=J_{*} \mathbf{k}(\mathbf{x})
$$

This is true for each choice of $\varepsilon>0$. Pick such an $\varepsilon$ small enough that $J_{*} \mathbf{k}(\mathbf{x})<J_{*} \mathbf{h}(\mathbf{x})+\delta$.
Let

$$
\begin{gathered}
T \equiv\left\{(\mathbf{h}(\mathbf{w}), \mathbf{0})^{T}: \mathbf{w} \in B(\mathbf{x}, r) \cap A\right\} \\
T_{\varepsilon} \equiv\left\{(\mathbf{h}(\mathbf{w}), \varepsilon \mathbf{w})^{T}: \mathbf{w} \in B(\mathbf{x}, r) \cap A\right\} \equiv \mathbf{k}(B(\mathbf{x}, r) \cap A)
\end{gathered}
$$

then $T=\left(\begin{array}{ll}P T_{\varepsilon} & \mathbf{0}\end{array}\right)^{T}$ where $P$ is the projection map defined by $P\binom{\mathbf{x}}{\mathbf{y}} \equiv \mathbf{x}$. Since $P$ decreases distances, it follows from Lemma 29.1.1

$$
\mathscr{H}^{n}(\mathbf{h}(B(\mathbf{x}, r) \cap A))=\mathscr{H}^{n}\left(P T_{\varepsilon}\right) \leq \mathscr{H}^{n}\left(T_{\varepsilon}\right)=\mathscr{H}^{n}(\mathbf{k}(B(\mathbf{x}, r)) \cap A)
$$

Thus for a.e. $\mathbf{x} \in A^{+}$,

$$
\begin{align*}
& J_{*} \mathbf{h}(\mathbf{x})+\delta \geq J_{*} \mathbf{k}(\mathbf{x})=\lim _{r \rightarrow 0} \frac{\mathscr{H}^{n}(\mathbf{k}(B(\mathbf{x}, r) \cap A))}{m_{n}(B(\mathbf{x}, r))} \geq \lim _{r \rightarrow 0} \frac{\mathscr{H}^{n}(\mathbf{h}(B(\mathbf{x}, r) \cap A))}{m_{n}(B(\mathbf{x}, r))} \\
& \geq \lim \inf _{r \rightarrow 0} \frac{\mathscr{H}^{n}(\mathbf{h}(B(\mathbf{x}, r) \cap A))}{m_{n}(B(\mathbf{x}, r))} \geq \lim _{r \rightarrow 0} \frac{\mathscr{H}^{n}\left(\mathbf{h}\left(B(\mathbf{x}, r) \cap A^{+}\right)\right)}{m_{n}(B(\mathbf{x}, r))}=J_{*} \mathbf{h}(\mathbf{x}) \tag{29.4.21}
\end{align*}
$$

Thus, since $\delta$ is arbitrary, $\lim _{r \rightarrow 0} \frac{\mathscr{H}^{n}(\mathbf{h}(B(\mathbf{x}, r) \cap A))}{m_{n}(B(\mathbf{x}, r))}=\operatorname{det}\left(D \mathbf{h}(\mathbf{x})^{*} D \mathbf{h}(\mathbf{x})\right)^{1 / 2}$ when $\mathbf{x} \in A^{+}$. If $\mathbf{x} \notin A^{+}$, the above 29.4.21 shows that

$$
J_{*} \mathbf{h}(\mathbf{x})=0 \geq \lim \sup _{r \rightarrow 0} \frac{\mathscr{H}^{n}(\mathbf{h}(B(\mathbf{x}, r) \cap A))}{m_{n}(B(\mathbf{x}, r))} \geq 0
$$

and so this has shown that for a.e. $\mathbf{x} \in A$,

$$
\lim _{r \rightarrow 0} \frac{\mathscr{H}^{n}(\mathbf{h}(B(\mathbf{x}, r) \cap A))}{m_{n}(B(\mathbf{x}, r))}=J_{*}(\mathbf{x})
$$

Another good idea is in the following lemma.
Lemma 29.4.4 Let $\mathbf{k}$ be as defined above. Let A be the set of points where Dhexists so $A=A^{+}$relative to $\mathbf{k}$. Then if $F$ is Lebesgue measurable, $\mathbf{h}(F \cap A)$ is $\mathscr{H}^{n}$ measurable. Also $\mathscr{H}^{n}(\mathbf{h}(N \cap A))=0$ if $m_{n}(N)=0$.

Proof: By Lemma 29.4.1, there are disjoint Borel sets $E_{k}$ such that $\mathbf{k}$ is Lipschitz on each $E_{k}$ and $\cup_{k} E_{k}=A=A^{+}$where $A^{+}$refers to $\mathbf{k}$. Thus

$$
P \mathbf{k}\left(E_{k} \cap F \cap A\right)=\mathbf{h}\left(E_{k} \cap F \cap A\right)
$$

is $H^{n}$ measurable by Lemma 29.1.2. Hence $\mathbf{h}(F \cap A)=\cup_{k} \mathbf{h}\left(E_{k} \cap F \cap A\right)$ is $\mathscr{H}^{n}$ measurable. The last claim follows from Lemma 29.1.2.

$$
\mathscr{H}^{n}\left(\mathbf{h}\left(E_{k} \cap N \cap A\right)\right) \leq \mathscr{H}^{n}\left(\mathbf{k}\left(E_{k} \cap N \cap A\right)\right)=0
$$

and so the result follows.

### 29.5 The Area Formula

At this point, I will begin assuming $\mathbf{h}$ is Lipschitz continuous to avoid the fuss with whether sets are appropriately measurable and to ensure that the measure considered below is absolutely continuous without any heroics. Let $\mathbf{h}$ be Lipschitz continuous on $G$, an open set containing $A$, the set where $\mathbf{h}$ is differentiable. Suppose also that $\mathbf{h}$ is one to one on $A$. Then $\mathscr{H}^{n}(\mathbf{h}(G \backslash A))=0$ because $m_{n}(G \backslash A)=0$ by Rademacher's theorem. Hence Lemma 29.1.2 applies.

Lemma 29.5.1 Let $\mathbf{h}$ be Lipschitz. Let $\mathbf{h}$ be one to one and differentiable on $A$ with $m_{n}(G \backslash A)=0$. If $N \subseteq G$ has measure zero, then $\mathbf{h}(N)$ has $\mathscr{H}^{n}$ measure zero and if $E$ is Lebesgue measurable subset of $G$, then $\mathbf{h}(E)$ is $\mathscr{H}^{n}$ measurable subset of $\mathbb{R}^{m}$. If $v(E) \equiv \mathscr{H}^{n}(\mathbf{h}(E))$, then $v \ll m_{n}$.

Proof: Lemma 29.1.2 implies $\mathbf{h}(N)=0$ if $m_{n}(N)=0$.Also from this lemma, $\mathbf{h}(E)$ is $\mathscr{H}^{n}$ measurable if $E$ is. Is $v$ a measure? Suppose $\left\{E_{i}\right\}$ are disjoint Lebesgue measurable subsets of $G$. Then for $A$ the set where $D \mathbf{h}$ exists as above,

$$
\begin{aligned}
v\left(\cup_{i} E_{i}\right) & \equiv \mathscr{H}^{n}\left(\mathbf{h}\left(\cup_{i} E_{i}\right)\right) \leq \mathscr{H}^{n}\left(\mathbf{h}\left(\cup_{i} E_{i} \cap A\right) \cup \mathbf{h}(G \backslash A)\right) \\
& \leq \sum_{i} \mathscr{H}^{n}\left(\mathbf{h}\left(E_{i} \cap A\right)\right)+\mathscr{H}^{n}(\mathbf{h}(G \backslash A))=\sum_{i} \mathscr{H}^{n}\left(\mathbf{h}\left(E_{i} \cap A\right)\right)=\sum_{i} v\left(E_{i}\right)
\end{aligned}
$$

Thus $v \ll m_{n}$.
It follows from the Radon Nikodym theorem for Radon measures, that

$$
v(E) \equiv \mathscr{H}^{n}(\mathbf{h}(E))=\int_{E} D_{m_{n}} v d m_{n}
$$

but $D_{m_{n}} v \equiv \lim _{r \rightarrow 0} \frac{\mathscr{H}^{n}(\mathbf{h}(B(\mathbf{x}, r)))}{m_{n}(\mathbf{x}, r)}=\lim _{r \rightarrow 0} \frac{\mathscr{H}^{n}(\mathbf{h}(B(\mathbf{x}, r)) \cap A)}{m_{n}(\mathbf{x}, r)}=J_{*}(\mathbf{x})$ for a.e. $\mathbf{x}$ from Theorem 29.4.3. Also, $v$ is finite on closed balls so it is regular thanks to Corollary 11.6.8. This shows from the Radon Nikodym theorem, Theorem 31.3.5 that

$$
\int \mathscr{X}_{\mathbf{h}(E)} d \mathscr{H}^{n}=\int_{E} D_{m_{n}} v d m_{n}=\int \mathscr{X}_{E}(\mathbf{x}) J_{*} d m_{n}
$$

Note also that, since $A^{C}$ has measure zero,

$$
\int_{\mathbf{h}(A)} \mathscr{X}_{\mathbf{h}(E)} d \mathscr{H}^{n}=\int_{A} \mathscr{X}_{E}(\mathbf{x}) J_{*} d m_{n}
$$

Now let $F$ be a Borel set in $\mathbb{R}^{m}$. Recall this implies $F$ is $\mathscr{H}^{n}$ measurable. Then

$$
\begin{align*}
\int_{\mathbf{h}(A)} \mathscr{X}_{F}(\mathbf{y}) d \mathscr{H}^{n} & =\int \mathscr{X}_{F \cap \mathbf{h}(A)}(\mathbf{y}) d \mathscr{H}^{n}=\mathscr{H}^{n}\left(\mathbf{h}\left(\mathbf{h}^{-1}(F) \cap A\right)\right) \\
& =v\left(\mathbf{h}^{-1}(F)\right)=\int \mathscr{X}_{A \cap \mathbf{h}^{-1}(F)}(\mathbf{x}) J_{*}(\mathbf{x}) d m_{n} \\
& =\int_{A} \mathscr{X}_{F}(\mathbf{h}(\mathbf{x})) J_{*}(\mathbf{x}) d m_{n} \tag{29.5.22}
\end{align*}
$$

Note there are no measurability questions in the above formula because $\mathbf{h}^{-1}(F)$ is a Borel set due to the continuity of $\mathbf{h}$. The Borel measurability of $J_{*}(\mathbf{x})$ also follows from the observation that $\mathbf{h}$ is continuous and therefore, the partial derivatives are Borel measurable, being the limit of continuous functions. Then $J_{*}(\mathbf{x})$ is just a continuous function of these partial derivatives. However, things are not so clear if $F$ is only assumed $\mathscr{H}^{n}$ measurable. Is there a similar formula for $F$ only $\mathscr{H}^{n}$ measurable?

Let $\lambda(E) \equiv \mathscr{H}^{n}(E \cap \mathbf{h}(A))$ for $E$ an arbitrary bounded $\mathscr{H}^{n}$ measurable set. This measure is finite on finite balls from what was shown above. Therefore, from Proposition 11.7.3, there exists an $F_{\sigma}$ set $F$ and a $G_{\delta}$ set $H$ such that $F \subseteq E \subseteq H$ and $\lambda(H \backslash F)=0$. Thus

$$
\mathscr{X}_{F}(\mathbf{h}(\mathbf{x})) J_{*}(\mathbf{x}) \leq \mathscr{X}_{E}(\mathbf{h}(\mathbf{x})) J_{*}(\mathbf{x}) \leq \mathscr{X}_{H}(\mathbf{h}(\mathbf{x})) J_{*}(\mathbf{x})
$$

where the functions on the ends are measurable. Then

$$
\begin{aligned}
& \int_{A}\left(\mathscr{X}_{H}(\mathbf{h}(\mathbf{x}))-\mathscr{X}_{F}(\mathbf{h}(\mathbf{x})) J_{*}(\mathbf{x})\right) J_{*}(\mathbf{x}) d m_{n} \\
= & \lambda(H)-\lambda(F)=0
\end{aligned}
$$

and so $\mathscr{X}_{E}(\mathbf{h}(\mathbf{x})) J_{*}(\mathbf{x})=\mathscr{X}_{F}(\mathbf{h}(\mathbf{x})) J_{*}(\mathbf{x})=\mathscr{X}_{F}(\mathbf{h}(\mathbf{x})) J_{*}(\mathbf{x})$ off a set of Lebesgue measure zero showing by completeness of Lebesgue measure that $\mathbf{x} \rightarrow \mathscr{X}_{E}(\mathbf{h}(\mathbf{x})) J_{*}(\mathbf{x})$ is Lebesgue measurable. Then

$$
\begin{aligned}
\int_{\mathbf{h}(A)} \mathscr{X}_{F}(\mathbf{y}) d \mathscr{H}^{n} & =\int_{A} \mathscr{X}_{F}(\mathbf{h}(\mathbf{x})) J_{*}(\mathbf{x}) d m_{n}=\int_{A} \mathscr{X}_{E}(\mathbf{h}(\mathbf{x})) J_{*}(\mathbf{x}) d m_{n} \\
& =\int_{A} \mathscr{X}_{H}(\mathbf{h}(\mathbf{x})) J_{*}(\mathbf{x}) d m_{n}=\int_{\mathbf{h}(A)} \mathscr{X}_{H}(\mathbf{y}) d \mathscr{H}^{n} \\
& =\int_{\mathbf{h}(A)} \mathscr{X}_{E}(\mathbf{y}) d \mathscr{H}^{n}=\int_{\mathbf{h}(A)} \mathscr{X}_{F}(\mathbf{y}) d \mathscr{H}^{n}
\end{aligned}
$$

If $E$ is not bounded, then replace with $E_{r} \equiv E \cap B(\mathbf{0}, r)$ and pass to a limit using the monotone convergence theorem. This proves the following lemma.

Lemma 29.5.2 Whenever $E$ is Lebesgue measurable,

$$
\begin{equation*}
\int_{\mathbf{h}(A)} \mathscr{X}_{E}(\mathbf{y}) d \mathscr{H}^{n}=\int_{A} \mathscr{X}_{E}(\mathbf{h}(\mathbf{x})) J_{*}(\mathbf{x}) d m_{n} \tag{29.5.23}
\end{equation*}
$$

From this, it follows that if $s$ is a nonnegative, $\mathscr{H}^{n}$ measurable simple function, 29.5.23 continues to be valid with $s$ in place of $\mathscr{X}_{E}$. Then approximating an arbitrary nonnegative $\mathscr{H}^{n}$ measurable function $g$ by an increasing sequence of simple functions, it follows that 29.5.23 holds with $g$ in place of $\mathscr{X}_{E}$ and there are no measurability problems because $\mathbf{x} \rightarrow g(\mathbf{h}(\mathbf{x})) J_{*}(\mathbf{x})$ is Lebesgue measurable. This proves the following theorem which is the area formula.

Theorem 29.5.3 Let $\mathbf{h}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ be Lipschitz continuous for $m \geq n$. Let $A \subseteq G$ for $G$ an open set be the set of $\mathbf{x} \in G$ on which $D \mathbf{h}(\mathbf{x})$ exists, and let $g: \mathbf{h}(A) \rightarrow[0, \infty]$ be $\mathscr{H}^{n}$ measurable. Then

$$
\mathbf{x} \rightarrow(g \circ \mathbf{h})(\mathbf{x}) J_{*}(\mathbf{x})
$$

is Lebesgue measurable and

$$
\int_{\mathbf{h}(A)} g(\mathbf{y}) d \mathscr{H}^{n}=\int_{A} g(\mathbf{h}(\mathbf{x})) J_{*}(\mathbf{x}) d m_{n}
$$

where $J_{*}(\mathbf{x})=\operatorname{det}(U(\mathbf{x}))=\operatorname{det}\left(D \mathbf{h}(\mathbf{x})^{*} D \mathbf{h}(\mathbf{x})\right)^{1 / 2}$.
Since $\mathscr{H}^{n}=m_{n}$ on $\mathbb{R}^{n}$, this is just a generalization of the usual change of variables formula. This is much better because it is not limited to $\mathbf{h}$ having values in $\mathbb{R}^{n}$. Also note that you could replace $A$ with $G$ since they differ by a set of measure zero thanks to Rademacher's theorem. Note that if you assume that $\mathbf{h}$ is Lipschitz on $G$ then it has a Lipschitz extension to $\mathbb{R}^{n}$. The conclusion has to do with integrals over $G$. It is not really necessary to have $\mathbf{h}$ be Lipschitz continuous on $\mathbb{R}^{n}$, but you might as well assume this because of the existence of the Lipschitz extension. Here is another interesting change of variables theorem.

Theorem 29.5.4 Let $\mathbf{h}: G \subseteq \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ be continuous where $G$ is an open set and let $A \subseteq G$ where $A$ is the Borel measurable set consisting of $\mathbf{x}$ where Dh( $\mathbf{x})$ exists. Suppose $\mathbf{h}$ is differentiable and one to one on $A$. Also let $g: \mathbf{h}(G) \rightarrow[0, \infty]$ be $\mathscr{H}^{n}$ measurable. Then

$$
\mathbf{x} \rightarrow(g \circ \mathbf{h})(\mathbf{x}) J_{*}(\mathbf{x})
$$

is Lebesgue measurable and

$$
\begin{equation*}
\int_{\mathbf{h}(A)} g(\mathbf{y}) d \mathscr{H}^{n}=\int_{A} g(\mathbf{h}(\mathbf{x})) J_{*}(\mathbf{x}) d m_{n} \tag{29.5.24}
\end{equation*}
$$

where $J_{*}(\mathbf{x})=\operatorname{det}(U(\mathbf{x}))=\operatorname{det}\left(D \mathbf{h}(\mathbf{x})^{*} D \mathbf{h}(\mathbf{x})\right)^{1 / 2}$.
Proof: By Lemma 29.4.4, $v(E) \equiv \mathscr{H}^{n}(\mathbf{h}(E \cap A))$ is a measure defined on the Lebesgue measurable sets contained in $G$ and $v \ll m_{n}$. The reason it is a measure is

$$
\begin{aligned}
v\left(\cup_{i} E_{i}\right) & \equiv \mathscr{H}^{n}\left(\mathbf{h}\left(\cup_{i} E_{i} \cap A\right)\right)=\mathscr{H}^{n}\left(\mathbf{h}\left(\cup_{i} E_{i} \cap A\right)\right) \\
& =\mathscr{H}^{n}\left(\cup_{i} \mathbf{h}\left(E_{i} \cap A\right)\right)=\sum_{i} \mathscr{H}^{n}\left(E_{i} \cap A\right)=\sum_{i} v\left(E_{i}\right)
\end{aligned}
$$

This measure is finite on compact sets. Therefore, by Corollary 11.6.8, it is a regular measure. By the Radon Nikodym theorem for Radon measures, Theorem 31.3.5,

$$
v(E)=\int_{E} D_{m_{n}} v d m_{n}
$$

where $D_{m_{n}} v$ is the symmetric derivative given by

$$
\lim _{r \rightarrow 0} \frac{\mathscr{H}^{n}(B(\mathbf{x}, r) \cap A)}{m_{n}(B(\mathbf{x}, r))}
$$

However, from Theorem 29.4.3 this limit equals $J_{*}(\mathbf{x})$ described above as

$$
\operatorname{det}\left(D \mathbf{h}(\mathbf{x})^{*} D \mathbf{h}(\mathbf{x})\right)^{1 / 2}
$$

Now the rest of the argument is identical to that presented above leading to Theorem 29.5.3.

Note that from 29.5.24, $\mathscr{H}^{n}\left(\mathbf{h}\left(A \backslash A^{+}\right)\right)=0$ so this also gives a generalization of Sard's theorem used earlier in the case that $\mathbf{h}$ is one to one.

### 29.6 Mappings that are not One to One

Let $\mathbf{h}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ be Lipschitz. We drop the requirement that $\mathbf{h}$ be one to one. Again, let $A$ be the set on which $D \mathbf{h}(\mathbf{x})$ exists. Let $\mathbf{k}$ be as used earlier in Theorem 29.4.3. Thus $J \mathbf{k}(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in A$ the set where $D \mathbf{h}(\mathbf{x})$ exists. Thus there is a sequence of disjoint Borel sets $\left\{E_{k}\right\}$ whose union is $A$ such that $\mathbf{k}$ is Lipschitz on $E_{k}$. Let $S$ be given by

$$
S \equiv\left\{\mathbf{x} \in A, \text { such that } U(\mathbf{x})^{-1} \text { does not exist }\right\}
$$

Then $S$ is a Borel set and so letting $S_{k j} \equiv S \cap E_{k} \cap B(\mathbf{0}, j)$, the change of variables formula above implies

$$
\mathscr{H}^{n}\left(\mathbf{h}\left(S_{k j}\right)\right) \leq \mathscr{H}^{n}\left(\mathbf{k}\left(S_{k j}\right)\right)=\int_{\mathbf{k}\left(S_{k j}\right)} d \mathscr{H}^{n}=\int_{A} \mathscr{X}_{S_{k j}}(\mathbf{x}) J_{*} \mathbf{k}(\mathbf{x}) d m_{n} \leq \delta m_{n}\left(S_{k j}\right)
$$

where $\mathbf{k}$ is chosen with $\varepsilon$ small enough that $J_{*} \mathbf{k}(\mathbf{x})<\delta . \delta$ is arbitrary, so $\mathscr{H}^{n}\left(\mathbf{h}\left(S_{k j}\right)\right)=0$ and so $\mathscr{H}^{n}\left(\mathbf{h}\left(S \cap E_{k}\right)\right)=0$. Consequently $\mathscr{H}^{n}(\mathbf{h}(S))=0$. This is stated as the following lemma. Note how this includes the earlier Sard's theorem.

Lemma 29.6.1 For $S$ defined above, $\mathscr{H}^{n}(\mathbf{h}(S))=0$.
Thus $m_{n}(N)=0$ where $N$ is the set where $D \mathbf{h}(\mathbf{x})$ does not exist. Then by Lemma 29.1.2

$$
\begin{equation*}
\mathscr{H}^{n}(\mathbf{h}(S \cup N)) \leq \mathscr{H}^{n}(\mathbf{h}(S))+\mathscr{H}^{n}(\mathbf{h}(N))=0 \tag{29.6.25}
\end{equation*}
$$

Let $B \equiv \mathbb{R}^{n} \backslash(S \cup N)$.
Recall Lemma 29.4.1 above which said that for each $\mathbf{x} \in A^{+}$the set where $U(\mathbf{x})$ is invertible there is a Borel set $F$ containing $\mathbf{x}$ on which $\mathbf{h}$ is one to one. In fact it was one of countably many sets of the form $E(T, \mathbf{c}, i)$. By enumerating these sets as done earlier, referring to them as $E_{k}$, one can let $F_{1} \equiv E_{1}$, and if $F_{1}, \cdots, F_{n}$ have been chosen, $F_{n+1} \equiv$ $E_{n+1} \backslash \cup_{i=1}^{n} F_{i}$ to obtain the result of the following lemma.

Lemma 29.6.2 There exists a sequence of disjoint measurable sets, $\left\{F_{i}\right\}$, such that

$$
\cup_{i=1}^{\infty} F_{i}=B \subseteq A^{+}
$$

and $\mathbf{h}$ is one to one on $F_{i}$.
The following corollary will not be needed right away but it is of interest. Recall that $A$ is the set where $\mathbf{h}$ is differentiable and $A^{+}$is the set where $\operatorname{det}\left(D \mathbf{h}(\mathbf{x})^{*} D \mathbf{h}(\mathbf{x})\right)>0$. Part of Lemma 29.4.1 is reviewed in the following corollary.

Corollary 29.6.3 For each $F_{i}$ in Lemma 29.6.2, $\mathbf{h}^{-1}$ is Lipschitz on $\mathbf{h}\left(F_{i}\right)$.
Now let $g: \mathbf{h}\left(\mathbb{R}^{n}\right) \rightarrow[0, \infty]$ be $\mathscr{H}^{n}$ measurable. By Theorem 29.5.3,

$$
\begin{equation*}
\int_{\mathbf{h}(A)} \mathscr{X}_{\mathbf{h}\left(F_{i}\right)}(\mathbf{y}) g(\mathbf{y}) d \mathscr{H}^{n}=\int_{F_{i}} g(\mathbf{h}(\mathbf{x})) J_{*}(\mathbf{x}) d m \tag{29.6.26}
\end{equation*}
$$

Now define

$$
\mathfrak{n}(\mathbf{y})=\sum_{i=1}^{\infty} \mathscr{X}_{\mathbf{h}\left(F_{i}\right)}(\mathbf{y})
$$

By Lemma 29.1.2, $\mathbf{h}\left(F_{i}\right)$ is $\mathscr{H}^{n}$ measurable and so $\mathfrak{n}$ is a $\mathscr{H}^{n}$ measurable function. For each $\mathbf{y} \in B, \mathfrak{n}(\mathbf{y})$ gives the number of elements in $\mathbf{h}^{-1}(\mathbf{y}) \cap B$. From 29.6.26,

$$
\begin{equation*}
\int_{\mathbf{h}\left(\mathbb{R}^{n}\right)} \mathfrak{n}(\mathbf{y}) g(\mathbf{y}) d \mathscr{H}^{n}=\int_{B} g(\mathbf{h}(\mathbf{x})) J_{*}(\mathbf{x}) d m \tag{29.6.27}
\end{equation*}
$$

Now define

$$
\#(\mathbf{y}) \equiv \text { number of elements in } \mathbf{h}^{-1}(\mathbf{y})
$$

Theorem 29.6.4 Let $\mathbf{h}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ be Lipschitz. Then the function $\mathbf{y} \rightarrow \#(\mathbf{y})$ is $\mathscr{H}^{n}$ measurable and if

$$
g: \mathbf{h}\left(\mathbb{R}^{n}\right) \rightarrow[0, \infty]
$$

is $\mathscr{H}^{n}$ measurable, then

$$
\int_{\mathbf{h}\left(\mathbb{R}^{n}\right)} g(\mathbf{y}) \#(\mathbf{y}) d \mathscr{H}^{n}=\int_{\mathbb{R}^{n}} g(\mathbf{h}(\mathbf{x})) J_{*}(\mathbf{x}) d m
$$

Proof: If $\mathbf{y} \notin \mathbf{h}(S \cup N)$, then $\mathfrak{n}(\mathbf{y})=\#(\mathbf{y})$. By 29.6.25

$$
\mathscr{H}^{n}(\mathbf{h}(S \cup N))=0
$$

and so $\mathfrak{n}(\mathbf{y})=\#(\mathbf{y})$ a.e. Since $\mathscr{H}^{n}$ is a complete measure, \#(.) is $\mathscr{H}^{n}$ measurable. Letting

$$
G \equiv \mathbf{h}\left(\mathbb{R}^{n}\right) \backslash \mathbf{h}(S \cup N)
$$

29.6.27 implies

$$
\begin{aligned}
\int_{\mathbf{h}\left(\mathbb{R}^{n}\right)} g(\mathbf{y}) \#(\mathbf{y}) d \mathscr{H}^{n} & =\int_{G} g(\mathbf{y}) \mathfrak{n}(\mathbf{y}) d \mathscr{H}^{n}=\int_{B} g(\mathbf{h}(\mathbf{x})) J_{*}(\mathbf{x}) d m \\
& =\int_{\mathbb{R}^{n}} g(\mathbf{h}(\mathbf{x})) J_{*}(\mathbf{x}) d m_{n} .
\end{aligned}
$$

Note that the same argument would hold if $\mathbf{h}: G \rightarrow \mathbb{R}^{m}$ is continuous and if $A$ is the set where $\mathbf{h}$ is differentiable and $\mathscr{H}^{n}(\mathbf{h}(G \backslash A))=0$, then for $g$ as above,

$$
\int_{\mathbf{h}(A)} g(\mathbf{y}) \#(\mathbf{y}) d \mathscr{H}^{n}=\int_{A} g(\mathbf{h}(\mathbf{x})) J_{*}(\mathbf{x}) d m_{n}
$$

The details are left to the reader.

### 29.7 The Divergence Theorem

As an important application of the area formula I will give a general version of the divergence theorem for sets in $\mathbb{R}^{p}$. It will always be assumed $p \geq 2$. Actually it is not necessary to make this assumption but what results in the case where $p=1$ is nothing more than the fundamental theorem of calculus and the considerations necessary to draw this conclusion seem unneccessarily tedious. You have to consider $\mathscr{H}^{0}$, zero dimensional Hausdorff measure. It is left as an exercise but I will not present it.

It will be convenient to have some lemmas and theorems in hand before beginning the proof. First recall the Tietze extension theorem on Page 158. It is stated next for convenience.

Theorem 29.7.1 Let $M$ be a closed nonempty subset of a metric space $(X, d)$ and let $f$ : $M \rightarrow[a, b]$ be continuous at every point of $M$. Then there exists a function, $g$ continuous on all of $X$ which coincides with $f$ on $M$ such that $g(X) \subseteq[a, b]$.

The next topic needed is the concept of an infinitely differentiable partition of unity. This was discussed earlier in Lemma 37.1.6.

Definition 29.7.2 Let $\mathfrak{C}$ be a set whose elements are subsets of $\mathbb{R}^{p} .{ }^{1}$ Then $\mathfrak{C}$ is said to be locally finite if for every $\mathbf{x} \in \mathbb{R}^{p}$, there exists an open set, $U_{\mathbf{x}}$ containing $\mathbf{x}$ such that $U_{\mathbf{x}}$ has nonempty intersection with only finitely many sets of $\mathfrak{C}$.

The following was proved mostly in Theorem 7.5.5.
Lemma 29.7.3 Let $\mathfrak{C}$ be a set whose elements are open subsets of $\mathbb{R}^{p}$ and suppose $\cup \mathfrak{C} \supseteq H$, a closed set. Then there exists a countable list of open sets, $\left\{U_{i}\right\}_{i=1}^{\infty}$ such that each $U_{i}$ is bounded, each $U_{i}$ is a subset of some set of $\mathfrak{C}$, and $\cup_{i=1}^{\infty} U_{i} \supseteq H$. One can also assume that $\left\{U_{i}\right\}_{i=1}^{\infty}$ is locally finite.

Proof: The first part was proved earlier. Since $\mathbb{R}^{p}$ is separable, it is completely separable with a countable basis of balls called $\mathscr{B}$. For each $\mathbf{x} \in H$, let $U$ be a ball from $\mathscr{B}$ having diameter no more than 1 which is contained in some set of $\mathfrak{C}$. This collection of balls is countable because $\mathscr{B}$ is. Let $H_{m} \equiv \overline{B(\mathbf{0}, m) \cap H} \backslash(B(\mathbf{0}, m-1) \cap H)$ where $H_{0} \equiv \emptyset$. Thus each $H_{m}$ is compact closed and bounded. Let $\left\{U_{i}\right\}_{i=1}^{k_{m}} \equiv \mathscr{U}_{m}$ be a finite subset of $\left\{U_{i}\right\}_{i=1}^{\infty} \equiv \mathscr{U}$ which have nonempty intersection with $H_{m}$ and whose union includes $H_{m}$. Thus $\cup_{k=1}^{\infty} \mathscr{U}_{k}$ is a locally finite cover of $H$. To see this, if $\mathbf{x}$ is any point, consider $B\left(\mathbf{x}, \frac{1}{4}\right)$. Can it intersect a set of $\mathscr{U}_{m}$ for arbitrarily large $m$ ? If so, $\mathbf{x}$ would need to be within 2 of $H_{m}$ for arbitrarily large $m$. However, this is not possible because it would require that $\|\mathbf{x}\| \geq m-3$ for infinitely many $m$. Thus this ball can intersect only finitely many sets of $\cup_{k=1}^{\infty} \mathscr{U}_{k}$.

Recall Corollary 11.6.8 and Proposition 11.7.3. What is needed is listed here for convenience.

Lemma 29.7.4 Let $\Omega$ be a complete separable metric space and suppose $\mu$ is a complete measure defined on a $\sigma$ algebra which contains the Borel sets of $\Omega$ which is finite on balls, the closures of these balls being compact. Then $\mu$ must be both inner and outer regular.

[^24]One more lemma will be useful. It involves approximating a continuous function uniformly with one which is infinitely differentiable.

Lemma 29.7.5 Let $V$ be a bounded open set and let $X$ be the closed subspace of $C(\bar{V})$, the space of continuous functions defined on $\bar{V}$, which is given by the following.

$$
X=\{u \in C(\bar{V}): u(\mathbf{x})=0 \text { on } \partial V\} .
$$

Then $C_{c}^{\infty}(V)$ is dense in $X$ with respect to the norm given by

$$
\|u\|=\max \{|u(x)|: x \in \bar{V}\}
$$

Proof: Let $O \subseteq \bar{O} \subseteq W \subseteq \bar{W} \subseteq V$ be such that $\operatorname{dist}\left(\bar{O}, V^{C}\right)<\eta$ and let $\psi_{\delta}(\cdot)$ be a mollifier. Let $u \in X$ and consider $\mathscr{X}_{W} u * \psi_{\delta}$. Let $\varepsilon>0$ be given and let $\eta$ be small enough that $|u(\mathbf{x})|<\varepsilon / 2$ whenever $\mathbf{x} \in V \backslash \bar{O}$. Then if $\delta$ is small enough $\left|\mathscr{X}_{W} u * \psi_{\delta}(\mathbf{x})-u(\mathbf{x})\right|<\varepsilon$ for all $\mathbf{x} \in \bar{O}$ and $\mathscr{X}_{W} u * \psi_{\delta}$ is in $C_{c}^{\infty}(V)$. For $\mathbf{x} \in V \backslash \bar{O},\left|\mathscr{X}_{W} u * \psi_{\delta}(\mathbf{x})\right| \leq \varepsilon / 2$ and so for such $\mathbf{x}$,

$$
\left|\mathscr{X}_{W} u * \psi_{\delta}(\mathbf{x})-u(\mathbf{x})\right| \leq \varepsilon .
$$

This proves the lemma since $\varepsilon$ was arbitrary.
Lemma 29.7.6 Let $\alpha_{1}, \cdots, \alpha_{p}$ be real numbers and let $A\left(\alpha_{1}, \cdots, \alpha_{p}\right)$ be the matrix which has $1+\alpha_{i}^{2}$ in the $i i^{\text {th }}$ slot and $\alpha_{i} \alpha_{j}$ in the $i j^{\text {th }}$ slot when $i \neq j$. Then

$$
\operatorname{det} A=1+\sum_{i=1}^{p} \alpha_{i}^{2}
$$

Proof of the claim: The matrix, $A\left(\alpha_{1}, \cdots, \alpha_{p}\right)$ is of the form

$$
A\left(\alpha_{1}, \cdots, \alpha_{p}\right)=\left(\begin{array}{cccc}
1+\alpha_{1}^{2} & \alpha_{1} \alpha_{2} & \cdots & \alpha_{1} \alpha_{p} \\
\alpha_{1} \alpha_{2} & 1+\alpha_{2}^{2} & & \alpha_{2} \alpha_{p} \\
\vdots & & \ddots & \vdots \\
\alpha_{1} \alpha_{p} & \alpha_{2} \alpha_{p} & \cdots & 1+\alpha_{p}^{2}
\end{array}\right)
$$

Now consider the product of a matrix and its transpose, $B^{T} B$ below.

$$
\left(\begin{array}{ccccc}
1 & 0 & \cdots & 0 & \alpha_{1}  \tag{29.7.28}\\
0 & 1 & & 0 & \alpha_{2} \\
\vdots & & \ddots & & \vdots \\
0 & & & 1 & \alpha_{p} \\
-\alpha_{1} & -\alpha_{2} & \cdots & -\alpha_{p} & 1
\end{array}\right)\left(\begin{array}{ccccc}
1 & 0 & \cdots & 0 & -\alpha_{1} \\
0 & 1 & & 0 & -\alpha_{2} \\
\vdots & & \ddots & & \vdots \\
0 & & & 1 & -\alpha_{p} \\
\alpha_{1} & \alpha_{2} & \cdots & \alpha_{p} & 1
\end{array}\right)
$$

This product equals a matrix of the form

$$
\left(\begin{array}{cc}
A\left(\alpha_{1}, \cdots, \alpha_{p}\right) & \mathbf{0} \\
\mathbf{0} & 1+\sum_{i=1}^{p} \alpha_{i}^{2}
\end{array}\right)
$$

Therefore, $\left(1+\sum_{i=1}^{p} \alpha_{i}^{2}\right) \operatorname{det}\left(A\left(\alpha_{1}, \cdots, \alpha_{p}\right)\right)=\operatorname{det}(B)^{2}=\operatorname{det}\left(B^{T}\right)^{2}$. However, using row operations,

$$
\operatorname{det} B^{T}=\operatorname{det}\left(\begin{array}{ccccc}
1 & 0 & \cdots & 0 & \alpha_{1} \\
0 & 1 & & 0 & \alpha_{2} \\
\vdots & & \ddots & & \vdots \\
0 & & & 1 & \alpha_{p} \\
0 & 0 & \cdots & 0 & 1+\sum_{i=1}^{p} \alpha_{i}^{2}
\end{array}\right)=1+\sum_{i=1}^{p} \alpha_{i}^{2}
$$

and therefore,

$$
\left(1+\sum_{i=1}^{p} \alpha_{i}^{2}\right) \operatorname{det}\left(A\left(\alpha_{1}, \cdots, \alpha_{p}\right)\right)=\left(1+\sum_{i=1}^{p} \alpha_{i}^{2}\right)^{2}
$$

which shows $\operatorname{det}\left(A\left(\alpha_{1}, \cdots, \alpha_{p}\right)\right)=\left(1+\sum_{i=1}^{p} \alpha_{i}^{2}\right)$.
Definition 29.7.7 A bounded open set, $U \subseteq \mathbb{R}^{p}$ is said to have a Lipschitz boundary and to lie on one side of its boundary if the following conditions hold. There exist open boxes, $Q_{1}, \cdots, Q_{N}$,

$$
Q_{i}=\prod_{j=1}^{p}\left(a_{j}^{i}, b_{j}^{i}\right)
$$

such that $\partial U \equiv \bar{U} \backslash U$ is contained in their union. Also, for each $Q_{i}$, there exists $k$ and $a$ Lipschitz function, $g_{i}$ such that $U \cap Q_{i}$ is of the form

$$
\left\{\begin{array}{c}
\mathbf{x}:\left(x_{1}, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{p}\right) \in \prod_{j=1}^{k-1}\left(a_{j}^{i}, b_{j}^{i}\right) \times  \tag{29.7.29}\\
\prod_{j=k+1}^{p}\left(a_{j}^{i}, b_{j}^{i}\right) \text { and } a_{k}^{i}<x_{k}<g_{i}\left(x_{1}, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{p}\right)
\end{array}\right\}
$$

or else of the form

$$
\left\{\begin{array}{c}
\mathbf{x}:\left(x_{1}, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{p}\right) \in \prod_{j=1}^{k-1}\left(a_{j}^{i}, b_{j}^{i}\right) \times  \tag{29.7.30}\\
\prod_{j=k+1}^{p}\left(a_{j}^{i}, b_{j}^{i}\right) \text { and } g_{i}\left(x_{1}, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{p}\right)<x_{k}<b_{j}^{i}
\end{array}\right\}
$$

The function, $g_{i}$ has a derivative on $A_{i} \subseteq \prod_{j=1}^{k-1}\left(a_{j}^{i}, b_{j}^{i}\right) \times \prod_{j=k+1}^{p}\left(a_{j}^{i}, b_{j}^{i}\right)$ where

$$
m_{p-1}\left(\prod_{j=1}^{k-1}\left(a_{j}^{i}, b_{j}^{i}\right) \times \prod_{j=k+1}^{p}\left(a_{j}^{i}, b_{j}^{i}\right) \backslash A_{i}\right)=0
$$

Also, there exists an open set, $Q_{0}$ such that $Q_{0} \subseteq \overline{Q_{0}} \subseteq U$ and $\bar{U} \subseteq Q_{0} \cup Q_{1} \cup \cdots \cup Q_{N}$.
Note that since there are only finitely many $Q_{i}$ and each $g_{i}$ is Lipschitz, it follows from an application of Lemma 29.1.1 that $\mathscr{H}^{p-1}(\partial U)<\infty$. Also from Lemma 29.7.4 $\mathscr{H}^{p-1}$ is inner and outer regular on $\partial U$. In the following, $d x$ will be used in place of $d m_{p}$ to conform with more standard notation from calculus.

Lemma 29.7.8 Suppose $U$ is a bounded open set as described above. Then there exists a unique function in $L^{\infty}\left(\partial U, \mathscr{H}^{p-1}\right)^{p}, \mathbf{n}(\mathbf{y})$ for $\mathbf{y} \in \partial U$ such that $|\mathbf{n}(\mathbf{y})|=1, \mathbf{n}$ is $\mathscr{H}^{p-1}$ measurable, (meaning each component of $\mathbf{n}$ is $\mathscr{H}^{p-1}$ measurable) and for every $\mathbf{w} \in \mathbb{R}^{p}$ satisfying $|\mathbf{w}|=1$, and for every $f \in C_{c}^{1}\left(\mathbb{R}^{p}\right)$,

$$
\lim _{t \rightarrow 0} \int_{U} \frac{f(\mathbf{x}+t \mathbf{w})-f(\mathbf{x})}{t} d x=\int_{\partial U} f(\mathbf{n} \cdot \mathbf{w}) d \mathscr{H}^{p-1}
$$

Proof: Let $\bar{U} \subseteq V \subseteq \bar{V} \subseteq \cup_{i=0}^{N} Q_{i}$ and let $\left\{\psi_{i}\right\}_{i=0}^{N}$ be a $C^{\infty}$ partition of unity on $\bar{V}$ such that $\operatorname{spt}\left(\psi_{i}\right) \subseteq Q_{i}$. Then for all $t$ small enough and $\mathbf{x} \in U$,

$$
\frac{f(\mathbf{x}+t \mathbf{w})-f(\mathbf{x})}{t}=\frac{1}{t} \sum_{i=0}^{N} \psi_{i} f(\mathbf{x}+t \mathbf{w})-\psi_{i} f(\mathbf{x})
$$

Thus using the dominated convergence theorem and Rademacher's theorem,

$$
\begin{gather*}
\lim _{t \rightarrow 0} \int_{U} \frac{f(\mathbf{x}+t \mathbf{w})-f(\mathbf{x})}{t} d x \\
=\lim _{t \rightarrow 0} \int_{U}\left(\frac{1}{t} \sum_{i=0}^{N} \psi_{i} f(\mathbf{x}+t \mathbf{w})-\psi_{i} f(\mathbf{x})\right) d x \\
=\int_{U} \sum_{i=0}^{N} \sum_{j=1}^{p} D_{j}\left(\psi_{i} f\right)(\mathbf{x}) w_{j} d x \\
=\int_{U} \sum_{j=1}^{p} D_{j}\left(\psi_{0} f\right)(\mathbf{x}) w_{j} d x+\sum_{i=1}^{N} \int_{U} \sum_{j=1}^{p} D_{j}\left(\psi_{i} f\right)(\mathbf{x}) w_{j} d x \tag{29.7.31}
\end{gather*}
$$

Since $\operatorname{spt}\left(\psi_{0}\right) \subseteq Q_{0}$, it follows the first term in the above equals zero. In the second term, fix $i$. Without loss of generality, suppose the $k$ in the above definition equals $p$ and 29.7.29 holds. This just makes things a little easier to write. Thus $g_{i}$ is a function of

$$
\left(x_{1}, \cdots, x_{p-1}\right) \in \prod_{j=1}^{p-1}\left(a_{j}^{i}, b_{j}^{i}\right) \equiv B_{i}
$$

Then

$$
\begin{aligned}
& \int_{U} \sum_{j=1}^{p} D_{j}\left(\psi_{i} f\right)(\mathbf{x}) w_{j} d x \\
= & \int_{B_{i}} \int_{a_{p}^{i}}^{g_{i}\left(x_{1}, \cdots, x_{p-1}\right)} \sum_{j=1}^{p} D_{j}\left(\psi_{i} f\right)(\mathbf{x}) w_{j} d x_{p} d x_{1} \cdots d x_{p-1} \\
= & \int_{B_{i}} \int_{-\infty}^{g_{i}\left(x_{1}, \cdots, x_{p-1}\right)} \sum_{j=1}^{p} D_{j}\left(\psi_{i} f\right)(\mathbf{x}) w_{j} d x_{p} d x_{1} \cdots d x_{p-1}
\end{aligned}
$$

Letting $x_{p}=y+g_{i}\left(x_{1}, \cdots, x_{p-1}\right)$ and changing the variable, this equals

$$
\begin{aligned}
= & \int_{B_{i}} \int_{-\infty}^{0} \sum_{j=1}^{p} D_{j}\left(\psi_{i} f\right)\left(x_{1}, \cdots, x_{p-1}, y+g_{i}\left(x_{1}, \cdots, x_{p-1}\right)\right) . \\
& w_{j} d y d x_{1} \cdots d x_{p-1} \\
= & \int_{A_{i}} \int_{-\infty}^{0} \sum_{j=1}^{p} D_{j}\left(\psi_{i} f\right)\left(x_{1}, \cdots, x_{p-1}, y+g_{i}\left(x_{1}, \cdots, x_{p-1}\right)\right) . \\
& w_{j} d y d x_{1} \cdots d x_{p-1}
\end{aligned}
$$

Recall $A_{i}$ is all of $B_{i}$ except for the set of measure zero where the derivative does not exist. Also $D_{j}$ refers to the partial derivative taken with respect to the entry in the $j^{t h}$ slot. In the $p^{t h}$ slot is found not just $x_{p}$ but $y+g_{i}\left(x_{1}, \cdots, x_{p-1}\right)$ so a differentiation with respect to $x_{j}$ will not be the same as $D_{j}$. In fact, it will introduce another term involving $g_{i, j}$. Thus from the chain rule,

$$
\begin{gather*}
=\int_{A_{i}} \int_{-\infty}^{0} \sum_{j=1}^{p-1} \frac{\partial}{\partial x_{j}}\left(\psi_{i} f\left(x_{1}, \cdots, x_{p-1}, y+g_{i}\left(x_{1}, \cdots, x_{p-1}\right)\right)\right) w_{j}- \\
D_{p}\left(\psi_{i} f\right)\left(x_{1}, \cdots, x_{p-1}, y+g_{i}\left(x_{1}, \cdots, x_{p-1}\right)\right) \cdot g_{i, j}\left(x_{1}, \cdots, x_{p-1}\right) w_{j} d y d x_{1} \cdots d x_{p-1} \\
+\int_{A_{i}} \int_{-\infty}^{0} D_{p}\left(\psi_{i} f\right)\left(x_{1}, \cdots, x_{p-1}, y+g_{i}\left(x_{1}, \cdots, x_{p-1}\right)\right) w_{p} d y d x_{1} \cdots d x_{p-1} \tag{29.7.32}
\end{gather*}
$$

Consider the term

$$
\int_{A_{i}} \int_{-\infty}^{0} \sum_{j=1}^{p-1} \frac{\partial}{\partial x_{j}}\left(\psi_{i} f\left(x_{1}, \cdots, x_{p-1}, y+g_{i}\left(x_{1}, \cdots, x_{p-1}\right)\right)\right) w_{j} d y d x_{1} \cdots d x_{p-1}
$$

This equals

$$
\int_{B_{i}} \int_{-\infty}^{0} \sum_{j=1}^{p-1} \frac{\partial}{\partial x_{j}}\left(\psi_{i} f\left(x_{1}, \cdots, x_{p-1}, y+g_{i}\left(x_{1}, \cdots, x_{p-1}\right)\right)\right) w_{j} d y d x_{1} \cdots d x_{p-1}
$$

and now interchanging the order of integration and using the fact that $\operatorname{spt}\left(\psi_{i}\right) \subseteq Q_{i}$, it follows this term equals zero. The reason this is valid is that

$$
x_{j} \rightarrow \psi_{i} f\left(x_{1}, \cdots, x_{p-1}, y+g_{i}\left(x_{1}, \cdots, x_{p-1}\right)\right)
$$

is the composition of Lipschitz functions and is therefore Lipschitz. Therefore, this function can be recovered by integrating its derivative, Lemma 26.2.6.

Then, changing the variable back to $x_{p}$ it follows 29.7.32 reduces to

$$
-\int_{A_{i}} \int_{-\infty}^{g_{i}\left(x_{1}, \cdots, x_{p-1}\right)}\binom{\sum_{j=1}^{p-1} D_{p}\left(\psi_{i} f\right)\left(x_{1}, \cdots, x_{p-1}, x_{p}\right)}{\cdot g_{i, j}\left(x_{1}, \cdots, x_{p-1}\right) w_{j}} d x_{p} d x_{1} \cdots d x_{p-1}
$$

$$
+\int_{A_{i}} \int_{-\infty}^{g_{i}\left(x_{1}, \cdots, x_{p-1}\right)} D_{p}\left(\psi_{i} f\left(x_{1}, \cdots, x_{p-1}, x_{p}\right)\right) w_{p} d x_{p} d x_{1} \cdots d x_{p-1}
$$

Doing the integrals using the observation that $g_{i, j}\left(x_{1}, \cdots, x_{p-1}\right)$ does not depend on $x_{p}$, this reduces further to

$$
\begin{equation*}
\int_{A_{i}}\left(\psi_{i} f\right)\left(x_{1}, \cdots, x_{p-1}, x_{p}\right) \mathbf{N}_{i}\left(x_{1}, \cdots, x_{p-1}, g_{i}\left(x_{1}, \cdots, x_{p-1}\right)\right) \cdot \mathbf{w} d m_{p-1} \tag{29.7.33}
\end{equation*}
$$

where $\mathbf{N}_{i}\left(x_{1}, \cdots, x_{p-1}, g_{i}\left(x_{1}, \cdots, x_{p-1}\right)\right)$ is given by

$$
\begin{equation*}
\left(-g_{i, 1}\left(x_{1}, \cdots, x_{p-1}\right),-g_{i, 2}\left(x_{1}, \cdots, x_{p-1}\right), \cdots,-g_{i, p-1}\left(x_{1}, \cdots, x_{p-1}\right), 1\right) \tag{29.7.34}
\end{equation*}
$$

At this point I need a technical lemma which will allow the use of the area formula. The part of the boundary of $U$ which is contained in $Q_{i}$ is the image of the map, $\mathbf{h}_{i}\left(x_{1}, \cdots, x_{p-1}\right)$ given by $\left(x_{1}, \cdots, x_{p-1}, g_{i}\left(x_{1}, \cdots, x_{p-1}\right)\right)$ for $\left(x_{1}, \cdots, x_{p-1}\right) \in A_{i}$. I need a formula for

$$
\operatorname{det}\left(D \mathbf{h}_{i}\left(x_{1}, \cdots, x_{p-1}\right)^{*} D \mathbf{h}_{i}\left(x_{1}, \cdots, x_{p-1}\right)\right)^{1 / 2}
$$

To avoid interupting the argument, I will state the lemma here and prove it later.

## Lemma 29.7.9

$$
\begin{gathered}
\operatorname{det}\left(D \mathbf{h}_{i}\left(x_{1}, \cdots, x_{p-1}\right)^{*} D \mathbf{h}_{i}\left(x_{1}, \cdots, x_{p-1}\right)\right)^{1 / 2} \\
=\sqrt{1+\sum_{j-1}^{p-1} g_{i, j}\left(x_{1}, \cdots, x_{p-1}\right)^{2}} \equiv J_{* i}\left(x_{1}, \cdots, x_{p-1}\right)
\end{gathered}
$$

For

$$
\mathbf{y}=\left(x_{1}, \cdots, x_{p-1}, g_{i}\left(x_{1}, \cdots, x_{p-1}\right)\right) \in \partial U \cap Q_{i}
$$

and $\mathbf{n}$ defined by

$$
\mathbf{n}_{i}(\mathbf{y})=\frac{1}{J_{* i}\left(x_{1}, \cdots, x_{p-1}\right)} \mathbf{N}_{i}(\mathbf{y})
$$

it follows from the description of $J_{* i}\left(x_{1}, \cdots, x_{p-1}\right)$ given in the above lemma, that $\mathbf{n}_{i}$ is a unit vector. All components of $\mathbf{n}_{i}$ are continuous functions of limits of continuous functions. Therefore, $\mathbf{n}_{i}$ is Borel measurable and so it is $\mathscr{H}^{p-1}$ measurable. Now 29.7.33 reduces to

$$
\begin{aligned}
& \int_{A_{i}}\left(\psi_{i} f\right)\left(x_{1}, \cdots, x_{p-1}, g_{i}\left(x_{1}, \cdots, x_{p-1}\right)\right) \times \\
& \mathbf{n}_{i}\left(x_{1}, \cdots, x_{p-1}, g_{i}\left(x_{1}, \cdots, x_{p-1}\right)\right) \cdot \mathbf{w} J_{* i}\left(x_{1}, \cdots, x_{p-1}\right) d m_{p-1}
\end{aligned}
$$

By the area formula this equals

$$
\int_{\mathbf{h}\left(A_{i}\right)} \psi_{i} f(\mathbf{y}) \mathbf{n}_{i}(\mathbf{y}) \cdot \mathbf{w} d \mathscr{H}^{p-1}
$$

Now by Lemma 29.1.1 and the equality of $m_{p-1}$ and $\mathscr{H}^{p-1}$ on $\mathbb{R}^{p-1}$, the above integral equals

$$
\int_{\partial U \cap Q_{i}} \psi_{i} f(\mathbf{y}) \mathbf{n}_{i}(\mathbf{y}) \cdot \mathbf{w} d \mathscr{H}^{p-1}=\int_{\partial U} \psi_{i} f(\mathbf{y}) \mathbf{n}_{i}(\mathbf{y}) \cdot \mathbf{w} d \mathscr{H}^{p-1}
$$

Similar arguments apply to the other terms and therefore,

$$
\begin{aligned}
& \lim _{t \rightarrow 0} \int_{U} \frac{f(\mathbf{x}+t \mathbf{w})-f(\mathbf{x})}{t} d m_{p}=\sum_{i=1}^{N} \int_{\partial U} \psi_{i} f(\mathbf{y}) \mathbf{n}_{i}(\mathbf{y}) \cdot \mathbf{w} d \mathscr{H}^{p-1} \\
= & \int_{\partial U} f(\mathbf{y}) \sum_{i=1}^{N} \psi_{i}(\mathbf{y}) \mathbf{n}_{i}(\mathbf{y}) \cdot \mathbf{w} d \mathscr{H}^{p-1}=\int_{\partial U} f(\mathbf{y}) \mathbf{n}(\mathbf{y}) \cdot \mathbf{w} d \mathscr{H}^{p-1}(29.7 .35)
\end{aligned}
$$

Then let $\mathbf{n}(\mathbf{y}) \equiv \sum_{i=1}^{N} \psi_{i}(\mathbf{y}) \mathbf{n}_{i}(\mathbf{y})$.
I need to show first there is no other $\mathbf{n}$ which satisfies 29.7 .35 and then I need to show that $|\mathbf{n}(\mathbf{y})|=1$. Note that it is clear $|\mathbf{n}(\mathbf{y})| \leq 1$ because each $\mathbf{n}_{i}$ is a unit vector and this is just a convex combination of these. Suppose then that $\mathbf{n}_{1} \in L^{\infty}\left(\partial U, \mathscr{H}^{p-1}\right)$ also works in 29.7.35. Then for all $f \in C_{c}^{1}\left(\mathbb{R}^{p}\right)$,

$$
\int_{\partial U} f(\mathbf{y}) \mathbf{n}(\mathbf{y}) \cdot \mathbf{w} d \mathscr{H}^{p-1}=\int_{\partial U} f(\mathbf{y}) \mathbf{n}_{1}(\mathbf{y}) \cdot \mathbf{w} d \mathscr{H}^{p-1}
$$

Suppose $h \in C(\partial U)$. Then by the Tietze extension theorem, there exists $f \in C_{c}\left(\mathbb{R}^{p}\right)$ such that the restriction of $f$ to $\partial U$ equals $h$. Now by Lemma 29.7.5 applied to a bounded open set containing the support of $f$, there exists a sequence $\left\{f_{m}\right\}$ of functions in $C_{c}^{1}\left(\mathbb{R}^{p}\right)$ converging uniformly to $f$. Therefore,

$$
\begin{aligned}
& \int_{\partial U} h(\mathbf{y}) \mathbf{n}(\mathbf{y}) \cdot \mathbf{w} d \mathscr{H}^{p-1}=\lim _{m \rightarrow \infty} \int_{\partial U} f_{m}(\mathbf{y}) \mathbf{n}(\mathbf{y}) \cdot \mathbf{w} d \mathscr{H}^{p-1} \\
= & \lim _{m \rightarrow \infty} \int_{\partial U} f_{m}(\mathbf{y}) \mathbf{n}_{1}(\mathbf{y}) \cdot \mathbf{w} d \mathscr{H}^{p-1}=\int_{\partial U} h(\mathbf{y}) \mathbf{n}_{1}(\mathbf{y}) \cdot \mathbf{w} d \mathscr{H}^{p-1} .
\end{aligned}
$$

Now $\mathscr{H}^{p-1}$ is a Radon measure on $\partial U$ and so the continuous functions on $\partial U$ are dense in $L^{1}\left(\partial U, \mathscr{H}^{p-1}\right)$. It follows $\mathbf{n} \cdot \mathbf{w}=\mathbf{n}_{1} \cdot \mathbf{w}$ a.e. Now let $\left\{\mathbf{w}_{m}\right\}_{m=1}^{\infty}$ be a countable dense subset of the unit sphere. From what was just shown, $\mathbf{n} \cdot \mathbf{w}_{m}=\mathbf{n}_{1} \cdot \mathbf{w}_{m}$ except for a set of measure zero, $N_{m}$. Letting $N=\cup_{m} N_{m}$, it follows that for $\mathbf{y} \notin N, \mathbf{n}(\mathbf{y}) \cdot \mathbf{w}_{m}=\mathbf{n}_{1}(\mathbf{y}) \cdot \mathbf{w}_{m}$ for all $m$. Since the set is dense, it follows $\mathbf{n}(\mathbf{y}) \cdot \mathbf{w}=\mathbf{n}_{1}(\mathbf{y}) \cdot \mathbf{w}$ for all $\mathbf{y} \notin N$ and for all $\mathbf{w}$ a unit vector. Therefore, $\mathbf{n}(\mathbf{y})=\mathbf{n}_{1}(\mathbf{y})$ for all $\mathbf{y} \notin N$ and this shows $\mathbf{n}$ is unique. In particular, although it appears to depend on the partition of unity $\left\{\psi_{i}\right\}$ from its definition, this is not the case.

It only remains to verify $|\mathbf{n}(\mathbf{y})|=1$ a.e. I will do this by showing how to compute $\mathbf{n}$. In particular, I will show that $\mathbf{n}=\mathbf{n}_{i}$ a.e. on $\partial U \cap Q_{i}$. Let $W \subseteq \bar{W} \subseteq Q_{i} \cap \partial U$ where $W$ is open in $\partial U$. Let $O$ be an open set such that $O \cap \partial U=W$ and $\overline{\bar{O}} \subseteq Q_{i}$. Using Corollary 16.1.2 there exists a $C^{\infty}$ partition of unity $\left\{\psi_{m}\right\}$ such that $\psi_{i}=1$ on $\bar{O}$. Therefore, if $m \neq i, \psi_{m}=0$ on $\bar{O}$. Then if $f \in C_{c}^{1}(O)$,

$$
\begin{aligned}
& \int_{W} f \mathbf{w} \cdot \mathbf{n} d \mathscr{H}^{p-1}=\int_{\partial U} f \mathbf{w} \cdot \mathbf{n} d \mathscr{H}^{p-1} \\
= & \int_{U} \nabla f \cdot \mathbf{w} d m_{p}=\int_{U} \nabla\left(\psi_{i} f\right) \cdot \mathbf{w} d m_{p}
\end{aligned}
$$

which by the first part of the argument given above equals

$$
\int_{W} \psi_{i} f \mathbf{n}_{i} \cdot \mathbf{w} d \mathscr{H}^{p-1}=\int_{W} f \mathbf{w} \cdot \mathbf{n}_{i} d \mathscr{H}^{p-1}
$$

Thus for all $f \in C_{c}^{1}(O)$,

$$
\begin{equation*}
\int_{W} f \mathbf{w} \cdot \mathbf{n} d \mathscr{H}^{p-1}=\int_{W} f \mathbf{w} \cdot \mathbf{n}_{i} d \mathscr{H}^{p-1} \tag{29.7.36}
\end{equation*}
$$

Since $C_{c}^{1}(O)$ is dense in $C_{c}(O)$, the above equation is also true for all $f \in C_{c}(O)$. Now letting $h \in C_{c}(W)$, the Tietze extension theorem implies there exists $f_{1} \in C(\bar{O})$ whose restriction to $\bar{W}$ equals $h$. Let $f$ be defined by

$$
f_{1}(\mathbf{x}) \frac{\operatorname{dist}\left(\mathbf{x}, O^{C}\right)}{\operatorname{dist}(\mathbf{x}, \operatorname{spt}(h))+\operatorname{dist}\left(\mathbf{x}, O^{C}\right)}=f(\mathbf{x}) .
$$

Then $f=h$ on $W$ and so this has shown that for all $h \in C_{c}(W), 29.7 .36$ holds for $h$ in place of $f$. But as observed earlier, $\mathscr{H}^{p-1}$ is outer and inner regular on $\partial U$ and so $C_{c}(W)$ is dense in $L^{1}\left(W, \mathscr{H}^{p-1}\right)$ which implies $\mathbf{w} \cdot \mathbf{n}(\mathbf{y})=\mathbf{w} \cdot \mathbf{n}_{i}(\mathbf{y})$ for a.e. $\mathbf{y}$. Considering a countable dense subset of the unit sphere as above, this implies $\mathbf{n}(\mathbf{y})=\mathbf{n}_{i}(\mathbf{y})$ a.e. $\mathbf{y}$. This proves $|\mathbf{n}(\mathbf{y})|=1$ a.e. and in fact $\mathbf{n}(\mathbf{y})$ can be computed by using the formula for $\mathbf{n}_{i}(\mathbf{y})$.

It remains to prove Lemma 29.7.9.
Proof of Lemma 29.7.9: Let $\mathbf{h}(\mathbf{x})=\left(x_{1}, \cdots, x_{p-1}, g\left(x_{1}, \cdots, x_{p-1}\right)\right)^{T}$

$$
D \mathbf{h}(\mathbf{x})=\left(\begin{array}{lll}
1 & & 0 \\
\vdots & \ddots & \vdots \\
0 & & 1 \\
g_{, x_{1}} & \cdots & g_{, x_{p-1}}
\end{array}\right)
$$

Then,

$$
J_{*}(\mathbf{x})=\left(\operatorname{det}\left(D \mathbf{h}(\mathbf{x})^{*} D \mathbf{h}(\mathbf{x})\right)\right)^{1 / 2}
$$

Therefore, $J_{*}(\mathbf{x})$ is the square root of the determinant of the following $(p-1) \times(p-1)$ matrix.

$$
\left(\begin{array}{llll}
1+\left(g_{, x_{1}}\right)^{2} & g_{, x_{1}} g_{, x_{2}} & \cdots & g_{, x_{1}} g_{, x_{p-1}}  \tag{29.7.37}\\
g_{, x_{2}} g_{, x_{1}} & 1+\left(g_{, x_{2}}\right)^{2} & \cdots & g_{, x_{2}} g_{, x_{p-1}} \\
\vdots & & \ddots & \vdots \\
g_{, x_{p-1}} g_{, x_{1}} & g_{, x_{p-1}} g_{, x_{2}} & \cdots & 1+\left(g_{, x_{p-1}}\right)^{2}
\end{array}\right)
$$

By Lemma 29.7.6, this determinant is $1+\sum_{i=1}^{p-1}\left(g_{, x_{i}}(\mathbf{x})\right)^{2}$
Now Lemma 29.7.8 implies the divergence theorem.
Theorem 29.7.10 Let $U$ be a bounded open set with a Lipschitz boundary which lies on one side of its boundary. Then if $f \in C_{c}^{1}\left(\mathbb{R}^{p}\right)$,

$$
\begin{equation*}
\int_{U} f_{, k}(\mathbf{x}) d m_{p}=\int_{\partial U} f n_{k} d \mathscr{H}^{p-1} \tag{29.7.38}
\end{equation*}
$$

where $\mathbf{n}=\left(n_{1}, \cdots, n_{n}\right)$ is the $\mathscr{H}^{p-1}$ measurable unit vector of Lemma 29.7.8. Also, if $\mathbf{F}$ is a vector field such that each component is in $C_{c}^{1}\left(\mathbb{R}^{p}\right)$, then

$$
\begin{equation*}
\int_{U} \nabla \cdot \mathbf{F}(\mathbf{x}) d m_{p}=\int_{\partial U} \mathbf{F} \cdot \mathbf{n} d \mathscr{H}^{p-1} \tag{29.7.39}
\end{equation*}
$$

Proof: To obtain 29.7.38 apply Lemma 29.7.8 to $\mathbf{w}=\mathbf{e}_{k}$. Then to obtain 29.7.39 from this,

$$
\begin{aligned}
& \int_{U} \nabla \cdot \mathbf{F}(\mathbf{x}) d m_{p} \\
= & \sum_{j=1}^{p} \int_{U} F_{j, j} d m_{p}=\sum_{j=1}^{p} \int_{\partial U} F_{j} n_{j} d \mathscr{H}^{p-1} \\
= & \int_{\partial U} \sum_{j=1}^{p} F_{j} n_{j} d \mathscr{H}^{p-1}=\int_{\partial U} \mathbf{F} \cdot \mathbf{n} d \mathscr{H}^{p-1} .
\end{aligned}
$$

What is the geometric significance of the vector, $\mathbf{n}$ ? Recall that in the part of the boundary contained in $Q_{i}$, this vector points in the same direction as the vector

$$
\mathbf{N}_{i}\left(x_{1}, \cdots, x_{p-1}, g_{i}\left(x_{1}, \cdots, x_{p-1}\right)\right)
$$

given by

$$
\begin{equation*}
\left(-g_{i, 1}\left(x_{1}, \cdots, x_{p-1}\right),-g_{i, 2}\left(x_{1}, \cdots, x_{p-1}\right), \cdots,-g_{i, p-1}\left(x_{1}, \cdots, x_{p-1}\right), 1\right) \tag{29.7.40}
\end{equation*}
$$

in the case where $k=p$. This vector is the gradient of the function,

$$
x_{p}-g_{i}\left(x_{1}, \cdots, x_{p-1}\right)
$$

and so is perpendicular to the level surface given by

$$
x_{p}-g_{i}\left(x_{1}, \cdots, x_{p-1}\right)=0
$$

in the case where $g_{i}$ is $C^{1}$. It also points away from $U$ so the vector $\mathbf{n}$ is the unit outer normal. The other cases work similarly.

### 29.8 The Reynolds Transport Formula

Next is an interesting version of the chain rule for Lipschitz maps. The proof of this theorem is based on the following lemma.

Lemma 29.8.1 If $\mathbf{h}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ is Lipschitz, then if $\mathbf{h}(\mathbf{x})=\mathbf{0}$ for all $\mathbf{x} \in A$, then

$$
\operatorname{det}(D \mathbf{h}(\mathbf{x}))=0 \text { a.e } . \mathbf{x} \in A
$$

Proof: By the area formula, $0=\int_{\{0\}} \#(\mathbf{y}) d y=\int_{A}|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d x$, and so it follows that $\operatorname{det}(D \mathbf{h}(\mathbf{x}))=0$ a.e.

Theorem 29.8.2 Let $\mathbf{f}, \mathbf{g}$ be Lipschitz mappings from $\mathbb{R}^{n}$ to $\mathbb{R}^{n}$ with $\mathbf{g}(\mathbf{f}(\mathbf{x}))=\mathbf{x}$ on $A$, a measurable set. Then for a.e. $\mathbf{x} \in A, D \mathbf{g}(\mathbf{f}(\mathbf{x})), D \mathbf{f}(\mathbf{x})$, and $D(\mathbf{g} \circ \mathbf{f})(\mathbf{x})$ all exist and $I=D(\mathbf{g} \circ \mathbf{f})(\mathbf{x})=D \mathbf{g}(\mathbf{f}(\mathbf{x})) D \mathbf{f}(\mathbf{x})$.

Proof: By Lemma 29.8.1 there is a set of measure zero $N_{1}$ off which $\operatorname{det}(D(\mathbf{g} \circ \mathbf{f})(\mathbf{x})-I)=$ 0 and in particular $D(\mathbf{g} \circ \mathbf{f})(\mathbf{x})$ exists. Let $N_{2}$ be the set of measure zero off which $\mathbf{f}$ is differentiable. Let $M$ be the set of points in $\mathbf{f}\left(\mathbb{R}^{n} \backslash N_{2}\right)$ where, $\mathbf{g}$ fails to be differentiable. What about $\mathbf{f}^{-1}(M)$ ? If $\mathbf{x} \in \mathbf{f}^{-1}(M)$ then $D \mathbf{g}(\mathbf{f}(\mathbf{x}))$ fails to exist and so $\mathbf{x}$ is in the first exceptional set $N_{1}$ or else in $N_{2}$ because $D(\mathbf{g} \circ \mathbf{f})(\mathbf{x})$ will fail to exist. Thus $\mathbf{f}^{-1}(M)$ is a set of measure zero. So let $\mathbf{x} \notin N_{1} \cup N_{2}$. Then for such $\mathbf{x}, D(\mathbf{g} \circ \mathbf{f})(\mathbf{x}), D \mathbf{g}(\mathbf{f}(\mathbf{x})), D \mathbf{f}(\mathbf{x})$ all exist and $I=D \mathbf{g}(\mathbf{f}(\mathbf{x})) D \mathbf{f}(\mathbf{x})$.

You could give a generalization to the above by essentially repeating the argument.
Corollary 29.8.3 Suppose $\mathbf{h}$ is differentiable on A, a measurable set and that $\mathbf{f}, \mathbf{g}$ are Lipschitz with $\mathbf{g}(\mathbf{f}(\mathbf{x}))=\mathbf{h}(\mathbf{x})$ for $\mathbf{x} \in A$. Then for a.e. $\mathbf{x} \in A$,

$$
D \mathbf{h}(\mathbf{x})=D \mathbf{g}(\mathbf{f}(\mathbf{x})) D \mathbf{f}(\mathbf{x})
$$

In other words, the chain rule holds off a set of measure zero.
The Reynolds transport formula is an interesting application of the divergence theorem which is a generalization of the formula for taking the derivative under an integral.

$$
\frac{d}{d t} \int_{a(t)}^{b(t)} f(x, t) d x=\int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(x, t) d x+f(b(t), t) b^{\prime}(t)-f(a(t), t) a^{\prime}(t)
$$

First is an interesting lemma about the determinant. A $p \times p$ matrix can be thought of as a vector in $\mathbb{C}^{p^{2}}$. Just imagine stringing it out into one long list of numbers. In fact, a way to give the norm of a matrix is just $\sum_{i} \sum_{j}\left|A_{i j}\right|^{2} \equiv\|A\|^{2}$. This is called the Frobenius norm for a matrix. It makes no difference since all norms are equivalent, but this one is convenient in what follows. Also recall that det maps $p \times p$ matrices to $\mathbb{C}$. It makes sense to ask for the derivative of det on the set of invertible matrices, an open subset of $\mathbb{C}^{p^{2}}$ with the norm measured as just described because $A \rightarrow \operatorname{det}(A)$ is continuous, so the set where $\operatorname{det}(A) \neq 0$ would be an open set. Recall from linear algebra that the sum of the entries on the main diagonal satisfies trace $(A B)=$ trace $(B A)$ whenever both products make sense. Indeed, $\operatorname{trace}(A B) \equiv \sum_{i} \sum_{j} A_{i j} B_{j i}=\operatorname{trace}(B A)$

This next lemma is a very interesting observation about the determinant of a matrix added to the identity.

Lemma 29.8.4 $\operatorname{det}(I+U)=1+\operatorname{trace}(U)+o(U)$ where $o(U)$ is defined in terms of the Frobenius norm for $p \times p$ matrices.

Proof: This is obvious if $p=1$ or 2 . Assume true for $n-1$. Then for $U$ an $n \times n$, expand the matrix along the last column and use induction on the cofactor of $1+U_{n n}$.

With this lemma, it is easy to find $D \operatorname{det}(F)$ whenever $F$ is invertible.

$$
\begin{aligned}
\operatorname{det}(F+U) & =\operatorname{det}\left(F\left(I+F^{-1} U\right)\right)=\operatorname{det}(F) \operatorname{det}\left(I+F^{-1} U\right) \\
& =\operatorname{det}(F)\left(1+\operatorname{trace}\left(F^{-1} U\right)+o(U)\right) \\
& =\operatorname{det}(F)+\operatorname{det}(F) \operatorname{trace}\left(F^{-1} U\right)+o(U)
\end{aligned}
$$

Therefore,

$$
\operatorname{det}(F+U)-\operatorname{det}(F)=\operatorname{det}(F) \operatorname{trace}\left(F^{-1} U\right)+o(U)
$$

This proves the following.
Proposition 29.8.5 Let $F^{-1}$ exist. Then $D \operatorname{det}(F)(U)=\operatorname{det}(F) \operatorname{trace}\left(F^{-1} U\right)$.
From this, suppose $F(t)$ is a $p \times p$ matrix and all entries are differentiable. Then the following describes $\frac{d}{d t} \operatorname{det}(F)(t)$.

Proposition 29.8.6 Let $F(t)$ be a $p \times p$ matrix and all entries are differentiable. Then for a.e. $t$

$$
\begin{align*}
\frac{d}{d t} \operatorname{det}(F)(t) & =\operatorname{det}(F(t)) \operatorname{trace}\left(F^{-1}(t) F^{\prime}(t)\right) \\
& =\operatorname{det}(F(t)) \operatorname{trace}\left(F^{\prime}(t) F^{-1}(t)\right) \tag{29.8.41}
\end{align*}
$$

Let $\mathbf{y}=\mathbf{h}(t, \mathbf{x})$ with $F=F(t, \mathbf{x})=D_{2} \mathbf{h}(t, \mathbf{x})$. I will write $\nabla_{\mathbf{y}}$ to indicate the gradient with respect to the $\mathbf{y}$ variables and $F^{\prime}$ to indicate $\frac{\partial}{\partial t} F(t, \mathbf{x})$. Note that $\mathbf{h}(t, \mathbf{x})=\mathbf{y}$ and so by the inverse function theorem, this defines $\mathbf{x}$ as a function of $\mathbf{y}$, also as smooth as $\mathbf{h}$ because it is always assumed $\operatorname{det} F>0$.

Now let $V_{t}$ be $\mathbf{h}\left(t, V_{0}\right)$ where $V_{0}$ is an open bounded set. Let $V_{0}$ have a Lipschitz boundary so one can use the divergence theorem on $V_{0}$. Let $(t, \mathbf{y}) \rightarrow \mathbf{f}(t, \mathbf{y})$ be Lipschitz. The idea is to simplify $\frac{d}{d t} \int_{V_{t}} \mathbf{f}(t, \mathbf{y}) d m_{p}(y)$. This will involve the change of variables in which the Jacobian will be $\operatorname{det}(F)$ which is assumed positive. In applications of this theory, $\operatorname{det}(F) \leq 0$ is not physically possible. Since $\mathbf{h}(t, \cdot)$ is Lipschitz and the boundary of $V_{0}$ is Lipschitz, $V_{t}$ will be such that one can use the divergence theorem because the composition of Lipschitz functions is Lipschitz. Then, using the dominated convergence theorem as needed along with the area formula,

$$
\begin{align*}
& \frac{d}{d t} \int_{V_{t}} \mathbf{f}(t, \mathbf{y}) d m_{p}(y)=\frac{d}{d t} \int_{V_{0}} \mathbf{f}(t, \mathbf{h}(t, \mathbf{x})) \operatorname{det}(F) d m_{p}(x)  \tag{29.8.42}\\
&=\int_{V_{0}} \frac{\partial}{\partial t} \mathbf{f}(\cdot, \mathbf{h}(\cdot, \mathbf{x})) \operatorname{det}(F) d m_{p}(x)+\int_{V_{0}} \mathbf{f}(t, \mathbf{h}(t, \mathbf{x})) \frac{\partial}{\partial t}(\operatorname{det}(F)) d m_{p}(x) \\
&= \int_{V_{0}} \frac{\partial}{\partial t}(\mathbf{f}(t, \mathbf{h}(t, \mathbf{x}))) \operatorname{det}(F) d m_{p}(x) \\
&+\int_{V_{0}} \mathbf{f}(t, \mathbf{h}(t, \mathbf{x})) \operatorname{trace}\left(F^{\prime} F^{-1}\right) \operatorname{det}(F) d m_{p}(x) \\
&= \int_{V_{0}}\left(\frac{\partial}{\partial t} \mathbf{f}(t, \mathbf{h}(t, \mathbf{x}))+\sum_{i} \frac{\partial \mathbf{f}}{\partial y_{i}} \frac{\partial y_{i}}{\partial t}\right) \operatorname{det}(F) d m_{p}(x) \\
&+\int_{V_{0}} \mathbf{f}(t, \mathbf{h}(t, \mathbf{x})) \operatorname{trace}\left(F^{\prime} F^{-1}\right) \operatorname{det}(F) d m_{p}(x)
\end{align*}
$$

$$
=\int_{V_{t}} \frac{\partial}{\partial t} \mathbf{f}(t, \mathbf{y}) d m_{p}(y)+\int_{V_{t}} \sum_{i} \frac{\partial \mathbf{f}}{\partial y_{i}} \frac{\partial y_{i}}{\partial t}+\mathbf{f}(t, \mathbf{y}) \operatorname{trace}\left(F^{\prime} F^{-1}\right) d m_{p}(y)
$$

Now $\mathbf{v} \equiv \frac{\partial}{\partial t} \mathbf{h}(t, \mathbf{x})$ and also, as noted above, $\mathbf{y} \equiv \mathbf{h}(t, \mathbf{x})$ defines $\mathbf{y}$ as a function of $\mathbf{x}$ and so $\operatorname{trace}\left(F^{\prime} F^{-1}\right)=\sum_{\alpha} \frac{\partial v_{i}}{\partial x_{\alpha}} \frac{\partial x_{\alpha}}{\partial y_{i}}$. Hence the double sum $\sum_{\alpha, i} \frac{\partial v_{i}}{\partial x_{\alpha}} \frac{\partial x_{\alpha}}{\partial y_{i}}$ is $\frac{\partial v_{i}}{\partial y_{i}}=\nabla_{\mathbf{y}} \cdot \mathbf{v}$. The above then gives

$$
\begin{align*}
& \int_{V_{t}} \frac{\partial}{\partial t} \mathbf{f}(t, \mathbf{y}) d m_{p}(y)+\int_{V_{t}}\left(\sum_{i} \frac{\partial \mathbf{f}}{\partial y_{i}} \frac{\partial y_{i}}{\partial t}+\mathbf{f}(t, \mathbf{y}) \nabla_{\mathbf{y}} \cdot \mathbf{v}\right) d m_{p}(y) \\
= & \int_{V_{t}} \frac{\partial}{\partial t} \mathbf{f}(t, \mathbf{y}) d m_{p}(y)+\int_{V_{t}}\left(D_{2} \mathbf{f}(t, \mathbf{y}) \mathbf{v}+\mathbf{f}(t, \mathbf{y}) \nabla_{\mathbf{y}} \cdot \mathbf{v}\right) d m_{p}(y) \tag{29.8.43}
\end{align*}
$$

Now consider the $i^{\text {th }}$ component of the second integral in the above. It is

$$
\begin{aligned}
& \int_{V_{t}} \nabla_{\mathbf{y}} f_{i}(t, \mathbf{y}) \cdot \mathbf{v}+f_{i}(t, \mathbf{y}) \nabla_{\mathbf{y}} \cdot \mathbf{v} d m_{p}(y) \\
= & \int_{V_{t}} \nabla_{\mathbf{y}} \cdot\left(f_{i}(t, \mathbf{y}) \mathbf{v}\right) d m_{p}(y)
\end{aligned}
$$

At this point, use the divergence theorem to get this equals $=\int_{\partial V_{t}} f_{i}(t, \mathbf{y}) \mathbf{v} \cdot \mathbf{n} d \mathscr{H}^{p-1}$. Therefore, from 29.8.43 and 29.8.42,

$$
\begin{equation*}
\frac{d}{d t} \int_{V_{t}} \mathbf{f}(t, \mathbf{y}) d m_{p}(y)=\int_{V_{t}} \frac{\partial}{\partial t} \mathbf{f}(t, \mathbf{y}) d m_{p}(y)+\int_{\partial V_{t}} \mathbf{f}(t, \mathbf{y}) \mathbf{v} \cdot \mathbf{n} d A \tag{29.8.44}
\end{equation*}
$$

this is the Reynolds transport formula.
Proposition 29.8.7 Let $\mathbf{y}=\mathbf{h}(t, \mathbf{x})$ where $\mathbf{h}$ is Lipschitz continuous and let $\mathbf{f}$ also be Lipschitz continuous and let $V_{t} \equiv \mathbf{h}\left(t, V_{0}\right)$ where $V_{0}$ is a bounded open set which is on one side of a Lipschitz boundary so that the divergence theorem holds for $V_{0}$. Then 29.8.44 is obtained.

### 29.9 The Coarea Formula

The area formula was discussed above. This formula implies that for $E$ a measurable set

$$
\mathscr{H}^{n}(\mathbf{f}(E))=\int \mathscr{X}_{E}(\mathbf{x}) J_{*}(\mathbf{x}) d m
$$

where $\mathbf{f}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ for $\mathbf{f}$ a Lipschitz mapping and $m \geq n$. It is a version of the change of variables formula for multiple integrals. The coarea formula is a statement about the Hausdorff measure of a set which involves the inverse image of $\mathbf{f}$. It is somewhat reminiscent of Fubini's theorem. Recall that if $n>m$ and $\mathbb{R}^{n}=\mathbb{R}^{m} \times \mathbb{R}^{n-m}$, we may take a product measurable set, $E \subseteq \mathbb{R}^{n}$, and obtain its Lebesgue measure by the formula

$$
\begin{aligned}
m_{n}(E) & =\int_{\mathbb{R}^{m}} \int_{\mathbb{R}^{n-m}} \mathscr{X}_{E}(\mathbf{y}, \mathbf{x}) d m_{n-m} d m_{m} \\
& =\int_{\mathbb{R}^{m}} m_{n-m}\left(E^{\mathbf{y}}\right) d m_{m}=\int_{\mathbb{R}^{m}} \mathscr{H}^{n-m}\left(E^{\mathbf{y}}\right) d m_{m}
\end{aligned}
$$

Let $\pi_{1}$ and $\pi_{2}$ be defined by $\pi_{2}(\mathbf{y}, \mathbf{x})=\mathbf{x}, \pi_{1}(\mathbf{y}, \mathbf{x})=\mathbf{y}$. Then $E^{\mathbf{y}}=\pi_{2}\left(\pi_{1}^{-1}(\mathbf{y}) \cap E\right)$ and so

$$
\begin{align*}
m_{n}(E) & =\int_{\mathbb{R}^{m}} \mathscr{H}^{n-m}\left(\pi_{2}\left(\pi_{1}^{-1}(\mathbf{y}) \cap E\right)\right) d m_{m} \\
& =\int_{\mathbb{R}^{m}} \mathscr{H}^{n-m}\left(\pi_{1}^{-1}(\mathbf{y}) \cap E\right) d m_{m} \tag{29.9.45}
\end{align*}
$$

Thus, the notion of product measure yields a formula for the measure of a set in terms of the inverse image of one of the projection maps onto a smaller dimensional subspace. The coarea formula gives a generalization of 29.9.45 in the case where $\pi_{1}$ is replaced by an arbitrary Lipschitz function mapping $\mathbb{R}^{n}$ to $\mathbb{R}^{m}$. In general, we will take $m<n$ in this presentation. Whereas in the area formula the Lipschitz function has $m \geq n$.

It is possible to obtain the coarea formula as a computation involving the area formula and some simple linear algebra and this is the approach taken here. I found this formula in [47]. This is a good place to obtain a slightly different proof. This argument follows [84] which came from [47]. I find this material very hard, so I hope what follows doesn't have grievous errors. I have never had occasion to use this coarea formula, but I think it is obviously of enormous significance and gives a very interesting geometric assertion. I will use the form of the chain rule in Theorem 29.8.2 as needed.

To begin with we give the linear algebra identity which will be used. Recall that for a real matrix $A^{*}$ is just the transpose of $A$. Thus $A A^{*}$ and $A^{*} A$ are symmetric.
Theorem 29.9.1 Let $A$ be an $m \times n$ matrix and let $B$ be an $n \times m$ matrix for $m \leq n$. Then for I an appropriate size identity matrix,

$$
\operatorname{det}(I+A B)=\operatorname{det}(I+B A)
$$

Proof: Use block multiplication to write

$$
\begin{aligned}
& \left(\begin{array}{cc}
I+A B & 0 \\
B & I
\end{array}\right)\left(\begin{array}{cc}
I & A \\
0 & I
\end{array}\right)=\left(\begin{array}{cc}
I+A B & A+A B A \\
B & B A+I
\end{array}\right) \\
& \left(\begin{array}{cc}
I & A \\
0 & I
\end{array}\right)\left(\begin{array}{cc}
I & 0 \\
B & I+B A
\end{array}\right)=\left(\begin{array}{cc}
I+A B & A+A B A \\
B & I+B A
\end{array}\right)
\end{aligned}
$$

Hence

$$
\left(\begin{array}{cc}
I+A B & 0 \\
B & I
\end{array}\right)\left(\begin{array}{cc}
I & A \\
0 & I
\end{array}\right)=\left(\begin{array}{cc}
I & A \\
0 & I
\end{array}\right)\left(\begin{array}{cc}
I & 0 \\
B & I+B A
\end{array}\right)
$$

so

$$
\left(\begin{array}{cc}
I & A \\
0 & I
\end{array}\right)^{-1}\left(\begin{array}{cc}
I+A B & 0 \\
B & I
\end{array}\right)\left(\begin{array}{cc}
I & A \\
0 & I
\end{array}\right)=\left(\begin{array}{cc}
I & 0 \\
B & I+B A
\end{array}\right)
$$

which shows that the two matrices

$$
\left(\begin{array}{cc}
I+A B & 0 \\
B & I
\end{array}\right),\left(\begin{array}{cc}
I & 0 \\
B & I+B A
\end{array}\right)
$$

are similar and so they have the same determinant. Thus

$$
\operatorname{det}(I+A B)=\operatorname{det}(I+B A)
$$

Note that the two matrices are different sizes.

Corollary 29.9.2 Let $A$ be an $m \times n$ real matrix. Then

$$
\operatorname{det}\left(I+A A^{*}\right)=\operatorname{det}\left(I+A^{*} A\right)
$$

It is convenient to define the following [47] for a measure space $(\Omega, \mathscr{S}, \mu)$ and $f: \Omega \rightarrow$ $[0, \infty]$, an arbitrary function, maybe not measurable.

$$
\int^{*} f d \mu \equiv \int_{\Omega}^{*} f d \mu \equiv \inf \left\{\int_{\Omega} g d \mu: g \geq f, \text { and } g \text { measurable }\right\}
$$

This is just like an outer measure. It resembles an old idea found in Hobson [66] called a generalized Stieltjes integral.

Lemma 29.9.3 Suppose $f_{n} \geq 0$ and $\limsup _{n \rightarrow \infty} \int^{*} f_{n} d \mu=0$. Then there is a subsequence $f_{n_{k}}$ such that $f_{n_{k}}(\omega) \rightarrow 0$ a.e. $\omega$.

Proof: For $n$ large enough, $\int^{*} f_{n} d \mu<\infty$. Let $n$ be this large and pick $g_{n} \geq f_{n}, g_{n}$ measurable, such that

$$
\int^{*} f_{n} d \mu+n^{-1}>\int g_{n} d \mu
$$

Thus

$$
\limsup \int g_{n} d \mu=\liminf \int g_{n} d \mu=\lim \int g_{n} d \mu=0
$$

If $n=1,2, \cdots$, let $k_{n}>\max \left(k_{n-1}, n\right)$ be such that $\int g_{k_{n}} d \mu<2^{-n}$. Thus

$$
\mu\left(\left[g_{k_{n}} \geq n^{-1}\right]\right) \leq 2^{-n} n \text { and } \sum_{n=1}^{\infty} \mu\left(\left[g_{k_{n}} \geq n^{-1}\right]\right)<\infty
$$

so for all $N$,

$$
\mu\left(\cap_{n=1}^{\infty} \cup_{m \geq n}\left[g_{k_{m}} \geq m^{-1}\right]\right) \leq \sum_{n=N}^{\infty} \mu\left(\left[g_{k_{m}} \geq m^{-1}\right]\right) \leq \sum_{n=N}^{\infty} n 2^{-n}
$$

Thus $\mu\left(\cap_{n=1}^{\infty} \cup_{m \geq n}\left[g_{k_{m}} \geq m^{-1}\right]\right)=0$. Therefore, for $\omega \notin \cap_{n=1}^{\infty} \cup_{m \geq n}\left[g_{k_{m}} \geq m^{-1}\right]$, a set of measure zero, for all $m$ large enough, $\left[g_{k_{m}}<m^{-1}\right]$ and so $g_{k_{m}}(\omega) \rightarrow 0$ a.e. $\omega$. Since $f_{k_{m}}(\omega) \leq g_{k_{m}}(\omega)$, this proves the lemma.

It might help a little before proceeding further to recall the concept of a level surface of a function of $n$ variables. If $f: U \subseteq \mathbb{R}^{n} \rightarrow \mathbb{R}$, such a level surface is of the form $f^{-1}(y)$ and we would expect it to be an $n-1$ dimensional thing in some sense. In the next lemma, consider a more general construction in which the function has values in $\mathbb{R}^{m}, m \leq n$. In this more general case, one would expect $\mathbf{f}^{-1}(\mathbf{y})$ to be something which is in some sense $n-m$ dimensional. As earlier, sets will not be assumed measurable and $\mathscr{H}^{k}$ will refer to an outer measure.

Lemma 29.9.4 Let $A \subseteq \mathbb{R}^{p}$ and let $\mathbf{f}: \mathbb{R}^{p} \rightarrow \mathbb{R}^{m}$ be Lipschitz. Then

$$
\int_{\mathbb{R}^{m}}^{*} \mathscr{H}^{s}\left(A \cap \mathbf{f}^{-1}(\mathbf{y})\right) d \mathscr{H}^{m} \leq \frac{\beta(s) \beta(m)}{\beta(s+m)}(\operatorname{Lip}(\mathbf{f}))^{m} \mathscr{H}^{s+m}(A) .
$$

Proof: The formula is obvious if $\mathscr{H}^{s+m}(A)=\infty$ so assume $\mathscr{H}^{s+m}(A)<\infty$. The diameter of the closure of a set is the same as the diameter of the set and so one can assume

$$
A \subseteq \cup_{i=1}^{\infty} B_{i}^{j}, r\left(B_{i}^{j}\right) \leq j^{-1}, B_{i}^{j} \text { is closed }
$$

and

$$
\begin{equation*}
\mathscr{H}_{j^{-1}}^{s+m}(A)+j^{-1} \geq \sum_{i=1}^{\infty} \beta(s+m)\left(r\left(B_{i}^{j}\right)\right)^{s+m} \tag{29.9.46}
\end{equation*}
$$

Now define $g_{i}^{j}(\mathbf{y}) \equiv \beta(s)\left(r\left(B_{i}^{j}\right)\right)^{s} \mathscr{X}_{\mathbf{f}\left(B_{i}^{j}\right)}(\mathbf{y})$. If $\mathbf{f}^{-1}(\mathbf{y}) \notin B_{i}^{j}$, this indicator function $\mathscr{X}_{\mathbf{f}\left(B_{i}^{j}\right)}$ just gives 0 . If $\mathbf{f}^{-1}(\mathbf{y}) \in B_{i}^{j}$ then $\mathbf{y} \in B_{i}^{j}$. Thus

$$
\mathscr{H}_{j^{-1}}^{s}\left(A \cap \mathbf{f}^{-1}(\mathbf{y})\right) \leq \sum_{i=1}^{\infty} \beta(s)\left(r\left(B_{i}^{j}\right)\right)^{s} \mathscr{X}_{\mathbf{f}\left(B_{i}^{j}\right)}(\mathbf{y})=\sum_{i=1}^{\infty} g_{i}^{j}(\mathbf{y}),
$$

a Borel measurable function. It follows,

$$
\begin{aligned}
\int_{\mathbb{R}^{m}}^{*} \mathscr{H}^{s}\left(A \cap \mathbf{f}^{-1}(\mathbf{y})\right) d \mathscr{H}^{m} & =\int_{\mathbb{R}^{m}}^{*} \lim _{j \rightarrow \infty} \mathscr{H}_{j^{-1}}^{s}\left(A \cap \mathbf{f}^{-1}(\mathbf{y})\right) d \mathscr{H}^{m} \\
& \leq \int_{\mathbb{R}^{m}}^{*} \lim _{j \rightarrow \infty} \inf _{j=1}^{\infty} g_{i}^{j}(\mathbf{y}) d \mathscr{H}^{m}
\end{aligned}
$$

By Borel measurability of the integrand, the last term is no more than

$$
\int_{\mathbb{R}^{m}} \lim \inf _{j \rightarrow \infty} \sum_{i=1}^{\infty} g_{i}^{j}(\mathbf{y}) d \mathscr{H}^{m}
$$

By Fatou's lemma,

$$
\begin{gathered}
\leq \lim \inf _{j \rightarrow \infty} \int_{\mathbb{R}^{m}} \sum_{i=1}^{\infty} g_{i}^{j}(\mathbf{y}) d \mathscr{H}^{m}=\lim \inf _{j \rightarrow \infty} \sum_{i=1}^{\infty} \beta(s)\left(r\left(B_{i}^{j}\right)\right)^{s} \int_{\mathbb{R}^{m}} \mathscr{X}_{\mathbf{f}\left(B_{i}^{j}\right)}(\mathbf{y}) d \mathscr{H}^{m} \\
=\lim \inf _{j \rightarrow \infty} \sum_{i=1}^{\infty} \beta(s)\left(r\left(B_{i}^{j}\right)\right)^{s} \mathscr{H}^{m}\left(\mathbf{f}\left(B_{i}^{j}\right)\right)
\end{gathered}
$$

Recall the equality of $\mathscr{H}^{m}$ and Lebesgue measure on $\mathbb{R}^{m}$, (Recall this was how $\beta(m)$ was chosen. Theorem 28.2.4) Then the above is

$$
\begin{gathered}
\leq \lim \inf _{j \rightarrow \infty} \sum_{i=1}^{\infty} \beta(s)\left(r\left(B_{i}^{j}\right)\right)^{s} m_{m}\left(\mathbf{f}\left(B_{i}^{j}\right)\right) \\
\leq \lim \inf _{j \rightarrow \infty} \sum_{i=1}^{\infty} \beta(s) \alpha(m) \operatorname{Lip}(\mathbf{f})^{m} r\left(B_{i}^{j}\right)^{m}\left(r\left(B_{i}^{j}\right)\right)^{s} \\
=\operatorname{Lip}(\mathbf{f})^{m} \beta(s) \alpha(m) \lim \inf _{j \rightarrow \infty} \sum_{i=1}^{\infty} r\left(B_{i}^{j}\right)^{m}\left(r\left(B_{i}^{j}\right)\right)^{s}
\end{gathered}
$$

$$
\begin{gathered}
=\operatorname{Lip}(\mathbf{f})^{m} \frac{\beta(s) \alpha(m)}{\beta(m+s)} \lim \inf _{j \rightarrow \infty} \sum_{i=1}^{\infty} \beta(m+s) r\left(B_{i}^{j}\right)^{m+s} \\
\leq \operatorname{Lip}(\mathbf{f})^{m} \frac{\beta(s) \alpha(m)}{\beta(m+s)} \mathscr{H}^{s+m}(A)
\end{gathered}
$$

from 29.9.46. However, it was shown earlier that $\alpha(m)=\beta(m)$.
This last identification that $\alpha(m)=\beta(m)$ depended on the technical material involving isodiametric inequality. It isn't all that important to know the exact value of this constant $\frac{\beta(s) \alpha(m)}{\beta(m+s)}$ so one could simply write $C(s, m)$ in its place and suffer no lack of utility.

Can one change $\int^{*}$ to $\int$ ? The next lemma will enable the change in notation.
Lemma 29.9.5 Let $A \subseteq \mathbb{R}^{n}$ be Lebesgue measurable and

$$
\mathbf{f}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}
$$

be Lipschitz, $m<n$. Then

$$
\mathbf{y} \rightarrow \mathscr{H}^{n-m}\left(A \cap \mathbf{f}^{-1}(\mathbf{y})\right)
$$

is Lebesgue measurable. If A is compact, this function is Borel measurable.
Proof: Suppose first that $A$ is compact. Then $A \cap \mathbf{f}^{-1}(\mathbf{y})$ is also and so it is $\mathscr{H}^{n-m}$ measurable. Suppose $\mathscr{H}^{n-m}\left(A \cap \mathbf{f}^{-1}(\mathbf{y})\right)<t$. Then for all $\delta>0$,

$$
\mathscr{H}_{\delta}^{n-m}\left(A \cap \mathbf{f}^{-1}(\mathbf{y})\right)<t
$$

and so there exist sets $S_{i}$, satisfying

$$
r\left(S_{i}\right)<\delta, A \cap \mathbf{f}^{-1}(\mathbf{y}) \subseteq \cup_{i=1}^{\infty} S_{i}, \sum_{i=1}^{\infty} \beta(n-m)\left(r\left(S_{i}\right)\right)^{n-m}<t
$$

Replacing $S_{i}$ with the open set $\hat{S}_{i} \equiv S_{i}+B\left(\mathbf{0}, \eta_{i}\right)$ where the $\hat{S}_{i}$ satisfy the above inequality, it can be assumed each $S_{i}$ is open.

Claim: If $\mathbf{z}$ is close enough to $\mathbf{y}$, then $A \cap \mathbf{f}^{-1}(\mathbf{z}) \subseteq \cup_{i=1}^{\infty} S_{i}$.
Proof: If not, then there exists a sequence $\left\{\mathbf{z}_{k}\right\}$ such that $\mathbf{z}_{k} \rightarrow \mathbf{y}$, and $\mathbf{x}_{k} \in(A \cap$ $\left.\mathbf{f}^{-1}\left(\mathbf{z}_{k}\right)\right) \backslash \cup_{i=1}^{\infty} S_{i}$. Thus $\mathbf{f}\left(\mathbf{x}_{k}\right)=\mathbf{z}_{k}$. Taking a subsequence still denoted by $k$ we can have

$$
\mathbf{z}_{k} \rightarrow \mathbf{y}, \mathbf{x}_{k} \rightarrow \mathbf{x} \in A \backslash \cup_{i=1}^{\infty} S_{i}
$$

Hence $\mathbf{f}(\mathbf{x})=\lim _{k \rightarrow \infty} \mathbf{f}\left(\mathbf{x}_{k}\right)=\lim _{k \rightarrow \infty} \mathbf{z}_{k}=\mathbf{y}$, so $\mathbf{x} \in \mathbf{f}^{-1}(\mathbf{y}) \cap A \backslash \cup_{i=1}^{\infty} S_{i}$ contrary to the assumption that $A \cap \mathbf{f}^{-1}(\mathbf{y}) \subseteq \cup_{i=1}^{\infty} S_{i}$.

It follows from this claim that whenever $\mathbf{z}$ is close enough to $\mathbf{y}$,

$$
\mathscr{H}_{\delta}^{n-m}\left(A \cap \mathbf{f}^{-1}(\mathbf{z})\right)<t .
$$

Thus if

$$
U_{\delta} \equiv\left\{\mathbf{z}: \mathscr{H}_{\delta}^{n-m}\left(A \cap \mathbf{f}^{-1}(\mathbf{z})\right)<t+\delta\right\}
$$

then $U_{\delta}$ is open. Hence, letting $\delta_{i} \rightarrow 0+$,

$$
\left\{\mathbf{z}: \mathscr{H}^{n-m}\left(A \cap \mathbf{f}^{-1}(\mathbf{z})\right) \leq t\right\}=\cap_{i=1}^{\infty} U_{\delta_{i}}=\text { Borel set. }
$$

Thus, if $A$ is compact, then for each $\mathbf{y} \in \mathbb{R}^{m}, A \cap \mathbf{f}^{-1}(\mathbf{y})$ is $\mathscr{H}^{n-m}$ measurable and also the function

$$
\mathbf{y} \rightarrow \mathscr{H}^{n-m}\left(A \cap \mathbf{f}^{-1}(\mathbf{y})\right)
$$

is a Borel measurable function, hence Lebesgue measurable.
Let $A$ be Lebesgue measurable not just compact. Then by regularity, there exists $F \subseteq A$ where $F$ is the countable union of compact sets and $m_{m}(A \backslash F)=0$. Say $F=\cup_{k} F_{k}, F_{k+1} \supseteq$ $F_{k}$ and each $F_{k}$ is compact. Then $\mathscr{H}^{n-m}\left(F \cap \mathbf{f}^{-1}(\mathbf{y})\right)=\lim _{n \rightarrow \infty} \mathscr{H}^{n-m}\left(F_{n} \cap \mathbf{f}^{-1}(\mathbf{y})\right)$ so $\mathbf{y} \rightarrow \mathscr{H}^{n-m}\left(F \cap \mathbf{f}^{-1}(\mathbf{y})\right)$ is Lebesgue measurable.

$$
\int_{\mathbb{R}^{m}}^{*} \mathscr{H}^{n-m}\left((A \backslash F) \cap \mathbf{f}^{-1}(\mathbf{y})\right) d \mathscr{H}^{n-m} \leq C_{m, n} \mathscr{H}^{n}(A \backslash F)=C_{m, n} m_{n}(A \backslash F)=0
$$

From Lemma 29.9.3 $\mathscr{H}^{n-m}\left((A \backslash F) \cap \mathbf{f}^{-1}(\mathbf{y})\right)=0$ for $\mathscr{H}^{n-m}$ a.e. y. Hence, regarding $\mathscr{H}^{n-m}$ as an outer measure,

$$
\begin{aligned}
\mathscr{H}^{n-m}\left(F \cap \mathbf{f}^{-1}(\mathbf{y})\right) & \leq \mathscr{H}^{n-m}\left(A \cap \mathbf{f}^{-1}(\mathbf{y})\right) \\
& \leq \mathscr{H}^{n-m}\left((A \backslash F) \cap \mathbf{f}^{-1}(\mathbf{y})\right)+\mathscr{H}^{n-m}\left(F \cap \mathbf{f}^{-1}(\mathbf{y})\right) \\
& =\mathscr{H}^{n-m}\left(F \cap \mathbf{f}^{-1}(\mathbf{y})\right)
\end{aligned}
$$

and so $\mathbf{y} \rightarrow \mathscr{H}^{n-m}\left(A \cap \mathbf{f}^{-1}(\mathbf{y})\right)=\mathscr{H}^{n-m}\left(F \cap \mathbf{f}^{-1}(\mathbf{y})\right)$ is Lebesgue measurable.
With this lemma proved, it is possible to obtain the following useful inequality which will be used repeatedly.

Lemma 29.9.6 If $A \subseteq \mathbb{R}^{n}$ is Lebesgue measurable, then

$$
\begin{gathered}
\int_{\mathbb{R}^{m}} \mathscr{H}^{n-m}\left(A \cap \mathbf{f}^{-1}(\mathbf{y})\right) d y \\
\leq C(n, m)(\operatorname{Lip}(\mathbf{f}))^{m} m_{n}(A), C(n, m)=\frac{\beta(n-m) \beta(m)}{\beta(n)}
\end{gathered}
$$

Proof: This follows from Lemma 29.9.4 and Lemma 29.9.5. Since

$$
\mathbf{y} \rightarrow \mathscr{H}^{n-m}\left(A \cap \mathbf{f}^{-1}(\mathbf{y})\right)
$$

is measurable,

$$
\int_{\mathbb{R}^{m}} \mathscr{H}^{n-m}\left(A \cap \mathbf{f}^{-1}(\mathbf{y})\right) d y=\int_{\mathbb{R}^{m}}^{*} \mathscr{H}^{n-m}\left(A \cap \mathbf{f}^{-1}(\mathbf{y})\right) d y
$$

Now let $p=n$, and $s=n-m$ in Lemma 29.9.4.
With these lemmas it is now possible to establish the coarea formula. First we define $\Lambda(n, m)$ as all possible ordered lists of $m$ numbers taken from $\{1,2, \ldots, n\}$. Recall $\mathbf{x} \in$
$\mathbb{R}^{n}$ and $\mathbf{f}(\mathbf{x}) \in \mathbb{R}^{m}$ where $m \leq n$. Recall that this was part of the Binet Cauchy theorem, Theorem 30.2.1,

$$
\operatorname{det}\left(D \mathbf{f}(\mathbf{x}) D \mathbf{f}(\mathbf{x})^{*}\right)=\sum_{\mathbf{i} \in \Lambda(n, m)}\left(\operatorname{det} D_{\mathbf{x}_{\mathbf{i}}} \mathbf{f}(\mathbf{x})\right)^{2}
$$

Now let $\mathbf{i}_{c} \in \Lambda(n, n-m)$ consist of the remaining indices taken in order where $\mathbf{i} \in \Lambda(n, m)$. For $\mathbf{i}=\left(i_{1}, \cdots, i_{m}\right)$, define $\mathbf{x}_{\mathbf{i}} \equiv\left(x_{i_{1}}, \ldots, x_{i_{m}}\right)$ and $\mathbf{x}_{\mathbf{i}_{c}}$ to be the other components of $\mathbf{x}$ taken in order. Then let

$$
\mathbf{f}^{\mathbf{i}}(\mathbf{x}) \equiv\binom{\mathbf{f}(\mathbf{x})}{\mathbf{x}_{\mathbf{i}_{c}}}
$$

Thus there are $C(n, n-m)=D(n, m)$ different $\mathbf{f}^{\mathbf{i}}$.
Example 29.9.7 Say $\mathbf{f}: \mathbb{R}^{4} \rightarrow \mathbb{R}^{2}$. Here are some examples for $\mathbf{f}^{\mathbf{i}}$ :

$$
\left(\begin{array}{c}
f_{1}\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \\
f_{2}\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \\
x_{2} \\
x_{4}
\end{array}\right),\left(\begin{array}{c}
f_{1}\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \\
f_{2}\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \\
x_{1} \\
x_{2}
\end{array}\right),\left(\begin{array}{c}
f_{1}\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \\
f_{2}\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \\
x_{3} \\
x_{4}
\end{array}\right)
$$

Thus $\mathbf{f}^{\mathbf{i}}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$. For example, if $\mathbf{i}$ consists of the first $m$ of these indices, you have

$$
D \mathbf{f}^{\mathbf{i}}(\mathbf{x})=\left(\begin{array}{cc}
D_{\mathbf{x}_{\mathbf{i}}} \mathbf{f}(\mathbf{x}) & * \\
0 & I
\end{array}\right)
$$

and so

$$
\begin{equation*}
\operatorname{det} D \mathbf{f}^{\mathbf{i}}(\mathbf{x})=\operatorname{det} D_{\mathbf{x}_{\mathbf{i}}} \mathbf{f}(\mathbf{x}) \tag{29.9.47}
\end{equation*}
$$

It is the same with other $\mathbf{i} \in \Lambda(n, m)$, except you may have a minus sign. This will not matter here.

Earlier with the area formula, we integrated $J_{*}(\mathbf{x}) \equiv \operatorname{det}\left(D \mathbf{f}(\mathbf{x})^{*} D \mathbf{f}(\mathbf{x})\right)^{1 / 2}$. With the coarea formula, we integrate $J^{*}(\mathbf{x}) \equiv \operatorname{det}\left(D \mathbf{f}(\mathbf{x}) D \mathbf{f}(\mathbf{x})^{*}\right)^{1 / 2}$. This proof involves doing this integration and seeing what happens.

Theorem 29.9.8 Let A be a measurable set in $\mathbb{R}^{n}$ and let $\mathbf{f}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ be a Lipschitz map. Then the following formula holds along with all measurability assertions needed for it to make sense.

$$
\begin{equation*}
\int_{\mathbb{R}^{m}} \mathscr{H}^{n-m}\left(A \cap \mathbf{f}^{-1}(\mathbf{y})\right) d y=\int_{A} J^{*}(\mathbf{x}) d x \tag{29.9.48}
\end{equation*}
$$

where

$$
J^{*}(\mathbf{x}) \equiv \operatorname{det}\left(D \mathbf{f}(\mathbf{x}) D \mathbf{f}(\mathbf{x})^{*}\right)^{1 / 2}
$$

Proof: First note that $\operatorname{det}\left(D \mathbf{f}(\mathbf{x}) D \mathbf{f}(\mathbf{x})^{*}\right)=\sum_{\mathbf{i} \in \Lambda(n, m)} \operatorname{det}\left(D \mathbf{f}^{\mathbf{i}}(\mathbf{x})\right)^{2}$ by the Binet Cauchy theorem. Let $S \equiv\left\{\mathbf{x}: J^{*}(\mathbf{x})=0\right\}$. For each $\mathbf{i}, \mathbf{f}^{\mathbf{i}}\left(\left\{\mathbf{x}: \operatorname{det}\left(D \mathbf{f}^{\mathbf{i}}(\mathbf{x})\right)=0\right\}\right)$ has measure zero due to Sard's theorem and so it will follow from the argument presented below that $S^{\mathbf{i}} \equiv \mathbf{f}^{\mathbf{i}}\left(\left\{\mathbf{x}: \operatorname{det}\left(D \mathbf{f}^{\mathbf{i}}(\mathbf{x})\right)=0\right\}\right)$ has measure zero. Thus $S^{\mathbf{i}} \backslash S$ can be neglected. For $N \equiv$ $\{\mathbf{x}: D \mathbf{f}(\mathbf{x})$ does not exist $\}, m_{n}(N)=0$ by Rademacher's theorem. Thus in what follows, we can always assume that either $D \mathbf{f}^{\mathbf{i}}(\mathbf{x})$ does not exist or $\operatorname{det}\left(D \mathbf{f}^{\mathbf{i}}(\mathbf{x})\right)$ exists and is not 0 . This
will be clear from the argument. Let $A$ be a closed subset of $\mathbb{R}^{n} \backslash\{S \cup N\}$. By Lemma 29.4.1, there exist disjoint Borel measurable sets $\left\{F_{j}^{\mathbf{i}}\right\}_{j=1}^{\infty}$ such that $\mathbf{f}^{\mathbf{i}}$ is one to one on $F_{j}^{\mathbf{i}}$, $\left(\mathbf{f}^{\mathbf{i}}\right)^{-1}$ is Lipschitz on $\mathbf{f}^{\mathbf{i}}\left(F_{j}^{\mathbf{i}}\right)$, and

$$
\cup_{j=1}^{\infty} F_{j}^{\mathbf{i}}=\left\{\mathbf{x}: D \mathbf{f}^{\mathbf{i}}(\mathbf{x}) \text { exists and } \operatorname{det} D \mathbf{f}^{\mathbf{i}}(\mathbf{x}) \neq 0\right\} .
$$

If $\mathbf{x} \in \mathbb{R}^{n} \backslash\{S \cup N\}$, it follows $\mathbf{x} \in F_{j}^{\mathbf{i}}$ for some $\mathbf{i}$ and $j$. Hence $\cup_{\mathbf{i}, j} F_{j}^{\mathbf{i}} \supseteq A$.
Now let $\left\{E_{j}^{\mathbf{i}}\right\}$ be measurable sets such that $E_{j}^{\mathbf{i}} \subseteq F_{k}^{\mathbf{i}}$ for some $k$, the sets $E_{j}^{\mathbf{i}}$ are disjoint, and their union coincides with $\cup_{\mathbf{i}, j} F_{j}^{\mathbf{i}}$. Let $\mathbf{g}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be a Lipschitz function which equals $\left(\mathbf{f}^{\mathbf{i}}\right)^{-1}$ on $\mathbf{f}^{\mathbf{i}}\left(E_{j}^{\mathbf{i}}\right)$. I am supressing the dependence on $\mathbf{i}$. Then for any $\mathbf{x} \in E_{j}^{\mathbf{i}}, \mathbf{g}\left(\mathbf{f}^{\mathbf{i}}(\mathbf{x})\right)=\mathbf{x}$. In particular, $\mathbf{g}_{\mathbf{i}_{c}}\left(\mathbf{f}^{\mathbf{1}}(\mathbf{x})\right)=\mathbf{x}_{\mathbf{i}_{c}}$ where

$$
\mathbf{g}_{\mathbf{i}}(\mathbf{y}) \equiv\left(\begin{array}{lll}
g_{i_{1}}(\mathbf{y}) & \cdots & g_{i_{m}}(\mathbf{y})
\end{array}\right)^{T}
$$

for $\mathbf{i} \equiv\left(i_{1}, \cdots, i_{m}\right)$ with $\mathbf{g}_{\mathbf{i}_{c}}(\mathbf{y})$ defined similarly and $\mathbf{x} \in E_{j}^{\mathbf{i}}$, with

$$
\begin{gather*}
\mathbf{y} \equiv\binom{\mathbf{y}_{1}}{\mathbf{y}_{2}} \equiv\binom{\mathbf{f}(\mathbf{x})}{\mathbf{x}_{\mathbf{i}_{c}}} \equiv \mathbf{f}^{\mathbf{i}}(\mathbf{x}) \in \mathbf{f}^{\mathbf{i}}\left(E_{j}^{\mathbf{i}}\right), \\
\mathbf{x}_{\mathbf{i}}=\mathbf{g}_{\mathbf{i}}\left(\mathbf{f}^{\mathbf{i}}(\mathbf{x})\right), \mathbf{y}_{2} \equiv \mathbf{x}_{\mathbf{i}_{c}}=\mathbf{g}_{\mathbf{i}_{c}}\left(\mathbf{f}^{\mathbf{i}}(\mathbf{x})\right) \tag{29.9.49}
\end{gather*}
$$

Then, by definition,

$$
\begin{equation*}
\int_{A} J^{*}(\mathbf{x}) d x \equiv \int_{A} \operatorname{det}\left(D \mathbf{f}(\mathbf{x}) D \mathbf{f}(\mathbf{x})^{*}\right)^{1 / 2} d x \tag{29.9.50}
\end{equation*}
$$

First, using Theorem 29.8.2, and the fact that Lipschitz mappings take sets of measure zero to sets of measure zero, replace $E_{j}^{\mathbf{i}}$ with $\widetilde{E}_{j}^{\mathbf{i}} \subseteq E_{j}^{\mathbf{i}}$ such that $E_{j}^{\mathbf{i}} \backslash \widetilde{E}_{j}^{\mathbf{i}}$ has measure zero and

$$
\begin{equation*}
D \mathbf{f}^{\mathbf{i}}(\mathbf{g}(\mathbf{y})) D \mathbf{g}(\mathbf{y})=I,|\operatorname{det}(D \mathbf{g}(\mathbf{y}))|=\left|\operatorname{det} D \mathbf{f}^{\mathbf{i}}(\mathbf{g}(\mathbf{y}))\right|^{-1} \tag{29.9.51}
\end{equation*}
$$

on $\mathbf{f}^{\mathbf{i}}\left(\widetilde{E}_{j}^{\mathbf{i}}\right)$. Changing the variables using the area formula and 29.9.51, the expression in 29.9.50 equals

$$
\begin{align*}
& \int_{A} J^{*}(\mathbf{x}) d x=\sum_{j=1}^{\infty} \sum_{\mathbf{i} \in \Lambda(n, m)} \int_{\widetilde{E}_{j}^{\mathbf{i}} \cap A}\left(\operatorname{det}\left(D \mathbf{f}(\mathbf{x}) D \mathbf{f}(\mathbf{x})^{*}\right)\right)^{1 / 2} d x \\
&=\sum_{j=1}^{\infty} \sum_{\mathbf{i} \in \Lambda(n, m)} \int_{E_{j}^{\mathbf{i}} \cap A}\left(\operatorname{det}\left(D \mathbf{f}(\mathbf{x}) D \mathbf{f}(\mathbf{x})^{*}\right)\right)^{1 / 2} d x \\
&=\sum_{j=1}^{\infty} \sum_{\mathbf{i} \in \Lambda(n, m)} \int_{\mathbf{f}^{\mathbf{i}}\left(E_{j}^{\mathbf{i}} \cap A\right)}\left(\operatorname{det}\left(D \mathbf{f}(\mathbf{g}(\mathbf{y})) D \mathbf{f}(\mathbf{g}(\mathbf{y}))^{*}\right)\right)^{1 / 2}\left|\operatorname{det} D \mathbf{f}^{\mathbf{i}}(\mathbf{g}(\mathbf{y}))\right|^{-1} d y \tag{29.9.52}
\end{align*}
$$

Recall everything is Borel measurable. I will consider one of the integrals in the sum. For convenience, replace $E_{j}^{\mathbf{i}}$ with a compact set, $K_{j}^{\mathbf{i}}$ contained in it to obtain Borel measurability in what follows.

$$
\begin{gather*}
\int_{K_{j}^{\mathrm{i}} \cap A} \operatorname{det}\left(D \mathbf{f}(\mathbf{x}) D \mathbf{f}(\mathbf{x})^{*}\right)^{1 / 2} d x  \tag{29.9.53}\\
=\int_{\mathbf{f}^{\mathrm{i}}\left(K_{j}^{\mathbf{i}} \cap A\right)} \operatorname{det}\left(D \mathbf{f}(\mathbf{g}(\mathbf{y})) D \mathbf{f}(\mathbf{g}(\mathbf{y}))^{*}\right)^{1 / 2}|D \mathbf{g}(\mathbf{y})| d y \\
=\int_{\mathbf{f}^{\mathbf{i}}\left(K_{j}^{\mathrm{i}} \cap A\right)} \operatorname{det}\left(D \mathbf{f}(\mathbf{g}(\mathbf{y})) D \mathbf{f}(\mathbf{g}(\mathbf{y}))^{*}\right)^{1 / 2}\left|\operatorname{det} D \mathbf{f}^{\mathbf{i}}(\mathbf{g}(\mathbf{y}))\right|^{-1} d y
\end{gather*}
$$

Now $\binom{\mathbf{y}_{1}}{\mathbf{y}_{2}}=\mathbf{f}^{\mathbf{i}}(\mathbf{x})=\binom{\mathbf{f}(\mathbf{x})}{\mathbf{x}_{\mathbf{i}_{c}}}$ for $\mathbf{x} \in K_{j}^{\mathbf{i}} \cap A$ if and only if $\mathbf{x}$ is also in $\mathbf{f}^{-1}\left(\mathbf{y}_{1}\right)$ which recall is a vector in $\mathbb{R}^{n}$. Therefore, by 29.9.47, the above equals the following iterated integral.

$$
\begin{equation*}
=\int_{\mathbb{R}^{m}} \int_{\mathbf{f}^{-1}\left(\mathbf{y}_{1}\right) \cap K_{j}^{\mathbf{i}} \cap A} \operatorname{det}\left(D \mathbf{f}(\mathbf{g}(\mathbf{y})) D \mathbf{f}(\mathbf{g}(\mathbf{y}))^{*}\right)^{1 / 2}\left|\operatorname{det} D \mathbf{f}_{\mathbf{x}_{\mathbf{i}}}(\mathbf{g}(\mathbf{y}))\right|^{-1} d y_{2} d y_{1} \tag{29.9.54}
\end{equation*}
$$

where $\mathbf{y}_{1}=\mathbf{f}(\mathbf{x})$ and $\mathbf{y}_{2}=\mathbf{x}_{\mathbf{i}_{c}}$. Since $\mathbf{y}_{1}$ is fixed in the inner integral of 29.9.54, and $\mathbf{y}_{1}=$ $\mathbf{f}(\mathbf{g}(\mathbf{y}))$, and by definition $\mathbf{g}_{\mathbf{i}_{c}}\left(\mathbf{f}^{\mathbf{i}}(\mathbf{x})\right)=\mathbf{y}_{2}$, one can take the partial derivative of $\mathbf{y}_{1}=$ $\mathbf{f}(\mathbf{g}(\mathbf{y}))$ with respect to $\mathbf{y}_{2}$ to obtain

$$
\begin{align*}
\mathbf{0} & =D_{\mathbf{x}_{\mathbf{i}}} \mathbf{f}(\mathbf{g}(\mathbf{y})) D_{\mathbf{y}_{2}} \mathbf{g}_{\mathbf{i}}(\mathbf{y})+D_{\mathbf{x}_{\mathbf{i}_{c}}} \mathbf{f}(\mathbf{g}(\mathbf{y})) D_{\mathbf{y}_{2}} \mathbf{g}_{\mathbf{i}_{c}}(\mathbf{y}) \\
& =D_{\mathbf{x}_{\mathbf{i}}} \mathbf{f}(\mathbf{g}(\mathbf{y})) D_{\mathbf{y}_{2}} \mathbf{g}_{\mathbf{i}}(\mathbf{y})+D_{\mathbf{x}_{\mathbf{i}_{c}}} \mathbf{f}(\mathbf{g}(\mathbf{y})) \tag{29.9.55}
\end{align*}
$$

Now consider the inner integral in 29.9.54 in which $\mathbf{y}_{1}$ is fixed. The integrand equals

$$
\begin{equation*}
\operatorname{det}\left[\left(D_{\mathbf{x}_{\mathbf{i}}} \mathbf{f}(\mathbf{g}(\mathbf{y})) D_{\mathbf{x}_{\mathbf{i}_{c}}} \mathbf{f}(\mathbf{g}(\mathbf{y}))\right)\binom{D_{\mathbf{x}_{\mathbf{i}}} \mathbf{f}(\mathbf{g}(\mathbf{y}))^{*}}{D_{\mathbf{x}_{\mathbf{i}_{c}}} \mathbf{f}(\mathbf{g}(\mathbf{y}))^{*}}\right]^{1 / 2}\left|\operatorname{det} D \mathbf{f}_{\mathbf{x}_{\mathbf{i}}}(\mathbf{g}(\mathbf{y}))\right|^{-1} . \tag{29.9.56}
\end{equation*}
$$

Let $A \equiv D_{\mathbf{x}_{\mathbf{i}}} \mathbf{f}(\mathbf{g}(\mathbf{y}))$ so $A$ is $m \times m$ and $B \equiv D_{\mathbf{y}_{2}} \mathbf{g}_{\mathbf{i}}(\mathbf{y})$ an $m \times(n-m)$, and using 29.9.55, 29.9.56 is of the form

$$
\begin{aligned}
& \operatorname{det}\left[\left(\begin{array}{ll}
A & -A B
\end{array}\right)\binom{A^{*}}{-B^{*} A^{*}}\right]^{1 / 2}|\operatorname{det} A|^{-1} \\
= & \operatorname{det}\left[A A^{*}+A B B^{*} A^{*}\right]^{1 / 2}|\operatorname{det} A|^{-1} \\
= & \operatorname{det}\left[A\left(I+B B^{*}\right) A^{*}\right]^{1 / 2}|\operatorname{det} A|^{-1}=\operatorname{det}\left(I+B B^{*}\right)^{1 / 2}
\end{aligned}
$$

which, by Corollary 29.9.2, equals $\operatorname{det}\left(I+B^{*} B\right)^{1 / 2}$. (Note the size of the identity changes in these two expressions.) Since $B=D_{\mathbf{y}_{2}} \mathbf{g}_{\mathbf{i}}(\mathbf{y})$ and $D_{\mathbf{y}_{2}} \mathbf{g}_{\mathbf{g}_{c}}(\mathbf{y})=I$, the above reduces to

$$
\begin{gathered}
\operatorname{det}\left(I+B^{*} B\right)^{1 / 2}=\operatorname{det}\left[\left(\begin{array}{cc}
B^{*} & I
\end{array}\right)\binom{B}{I}\right]^{1 / 2}= \\
\operatorname{det}\left[\left(\begin{array}{ll}
D_{\mathbf{y}_{2}} \mathbf{g}_{\mathbf{i}}(\mathbf{y})^{*} & D_{\mathbf{y}_{2}} \mathbf{g}_{\mathbf{i}_{c}}(\mathbf{y})^{*}
\end{array}\right)\binom{D_{\mathbf{y}_{2}} \mathbf{g}_{\mathbf{i}}(\mathbf{y})}{D_{\mathbf{y}_{2}} \mathbf{g}_{c}(\mathbf{y})}\right]^{1 / 2}=\operatorname{det}\left(D_{\mathbf{y}_{2}} \mathbf{g}(\mathbf{y})^{*} D_{\mathbf{y}_{2}} \mathbf{g}(\mathbf{y})\right)^{1 / 2}
\end{gathered}
$$

Therefore, 29.9.53 reduces to $\int_{K_{j}^{\mathbf{i}} \cap A} \operatorname{det}\left(D \mathbf{f}(\mathbf{x}) D \mathbf{f}(\mathbf{x})^{*}\right)^{1 / 2} d x=$

$$
\begin{equation*}
\int_{\mathbb{R}^{m}} \int_{\mathbf{f}^{-1}\left(\mathbf{y}_{1}\right) \cap K_{j}^{\mathbf{i}} \cap A} \operatorname{det}\left(D_{\mathbf{y}_{2}} \mathbf{g}(\mathbf{y})^{*} D_{\mathbf{y}_{2}} \mathbf{g}(\mathbf{y})\right)^{1 / 2} d y_{2} d y_{1} \tag{29.9.57}
\end{equation*}
$$

Then $\mathbf{z} \in \mathbf{g}\left(\mathbf{y}_{1}, \mathbf{f}^{-1}\left(\mathbf{y}_{1}\right) \cap K_{j}^{\mathbf{i}} \cap A\right)$ if and only if

$$
\mathbf{f}^{\mathbf{i}}(\mathbf{z})=\binom{\mathbf{f}(\mathbf{z})}{\mathbf{z}_{\mathbf{i}_{c}}} \in\binom{\mathbf{y}_{1}}{\mathbf{f}^{-1}\left(\mathbf{y}_{1}\right) \cap K_{j}^{\mathbf{i}} \cap A}
$$

if and only if $\mathbf{z} \in \mathbf{f}^{-1}\left(\mathbf{y}_{1}\right)$ and $\mathbf{z}_{\mathbf{i}_{c}} \in \mathbf{f}^{-1}\left(\mathbf{y}_{1}\right) \cap K_{j}^{\mathbf{i}} \cap A$. Letting $\hat{\mathbf{g}}$ be the function $\mathbf{y}_{2} \rightarrow$ $\mathbf{g}\left(\mathbf{y}_{1}, \mathbf{y}_{2}\right)$, this shows that $\mathbf{z} \in \hat{\mathbf{g}}\left(\mathbf{f}^{-1}\left(\mathbf{y}_{1}\right) \cap K_{j}^{\mathbf{i}} \cap A\right)$ if and only if $\mathbf{y}_{2}=\mathbf{z}_{\mathbf{i}_{c}} \in \mathbf{f}^{-1}\left(\mathbf{y}_{1}\right) \cap K_{j}^{\mathbf{i}} \cap A$ and so $\hat{\mathbf{g}}\left(\mathbf{f}^{-1}\left(\mathbf{y}_{1}\right) \cap K_{j}^{\mathbf{i}} \cap A\right)=\mathbf{f}^{-1}\left(\mathbf{y}_{1}\right) \cap K_{j}^{\mathbf{i}} \cap A$. Of course $\hat{\mathbf{g}}$ actually depends on $\mathbf{y}_{1}$ but this is suppressed here. Therefore,

$$
\mathbf{g}\left(\mathbf{y}_{1}, \mathbf{f}^{-1}\left(\mathbf{y}_{1}\right) \cap K_{j}^{\mathbf{i}} \cap A\right)=\mathbf{f}^{-1}\left(\mathbf{y}_{1}\right) \cap K_{j}^{\mathbf{i}} \cap A
$$

By this observation and the area formula, the equations 29.9.53, 29.9.57 imply

$$
\int_{K_{j}^{\mathrm{i}} \cap A} \operatorname{det}\left(D \mathbf{f}(\mathbf{x}) D \mathbf{f}(\mathbf{x})^{*}\right)^{1 / 2} d x=\int_{\mathbb{R}^{m}} \mathscr{H}^{n-m}\left(\mathbf{f}^{-1}\left(\mathbf{y}_{1}\right) \cap K_{j}^{\mathbf{i}} \cap A\right) d y_{1} .
$$

Using Lemmas 29.9 .6 and 29.9.5, along with the inner regularity of Lebesgue measure, $K_{j}^{\mathbf{i}}$ can be replaced with $E_{j}^{\mathbf{i}}$. Therefore, summing the terms over all $\mathbf{i}$ and $j$,

$$
\int_{A} \operatorname{det}\left(D \mathbf{f}(\mathbf{x}) D \mathbf{f}(\mathbf{x})^{*}\right)^{1 / 2} d x=\int_{\mathbb{R}^{m}} \mathscr{H}^{n-m}\left(\mathbf{f}^{-1}(\mathbf{y}) \cap A\right) d y
$$

which verifies the coarea formula whenever $A$ is a closed subset of $\mathbb{R}^{n} \backslash\{S \cup N\}$.
By Lemma 29.9.6 again, this formula is true for all $A$ a closed subset of $\mathbb{R}^{n} \backslash S$. Using the same two lemmas again, we see this coarea formula holds for all $A$ a measurable subset of $\mathbb{R}^{n} \backslash S$.

It remains to verify the formula for all measurable sets $A$, regardless of whether they intersect $S$. Recall

$$
S \equiv\left\{\mathbf{x}: \sum_{\mathbf{i}} \operatorname{det}\left(D \mathbf{f}^{\mathbf{i}}(\mathbf{x})\right)^{2}=0\right\}=\left\{\mathbf{x}: \operatorname{det} U(\mathbf{x}) \equiv J^{*}(\mathbf{x})=0\right\}
$$

Consider the case where $A \subseteq S$. Let $A$ be compact so that by Lemma 29.9.5, $\mathbf{y} \rightarrow$ $\mathscr{H}^{n-m}\left(A \cap \mathbf{f}^{-1}(\mathbf{y})\right)$ is Borel measurable. For $\varepsilon>0$, define $\mathbf{k}_{\varepsilon}, \mathbf{p}: \mathbb{R}^{n} \times \mathbb{R}^{m} \rightarrow \mathbb{R}^{m}$ by

$$
\mathbf{k}_{\varepsilon}(\mathbf{x}, \mathbf{z}) \equiv \mathbf{f}(\mathbf{x})+\varepsilon \mathbf{z}, \mathbf{p}(\mathbf{x}, \mathbf{z}) \equiv \mathbf{z}
$$

Then $D \mathbf{k}_{\varepsilon}(\mathbf{x}, \mathbf{z})=\left(\begin{array}{ll}D \mathbf{f}(\mathbf{x}) & \varepsilon I\end{array}\right)=\left(\begin{array}{ll}U R & \varepsilon I\end{array}\right)$ where the dependence of $U$ and $R$ on $\mathbf{x}$ has been suppressed. Here $R R^{*}=I$ and $U$ is a non-negative symmetric transformation. Thus

$$
\left(\begin{array}{ll}
\left.J^{*} \mathbf{k}_{\varepsilon}\right)^{2}=\operatorname{det}\left(\begin{array}{ll}
U R & \varepsilon I
\end{array}\right)\binom{R^{*} U}{\varepsilon I}=\operatorname{det}\left(U^{2}+\varepsilon^{2} I\right), ~\left(\begin{array}{ll}
\end{array}\right) .
\end{array}\right.
$$

$$
\begin{gather*}
=\operatorname{det}\left(Q^{*} D Q Q^{*} D Q+\varepsilon^{2} I\right)=\operatorname{det}\left(D^{2}+\varepsilon^{2} I\right) \\
=\prod_{i=1}^{m}\left(\lambda_{i}^{2}+\varepsilon^{2}\right) \in\left[\varepsilon^{2 m}, C^{2} \varepsilon^{2}\right] \tag{29.9.58}
\end{gather*}
$$

since one of the $\lambda_{i}$ equals 0 due to $\operatorname{det}(U)=0$. All the eigenvalues of $U$ must be bounded independent of $\mathbf{x}$, since $\|D \mathbf{f}(\mathbf{x})\|$ is bounded independent of $\mathbf{x}$ due to the assumption that $\mathbf{f}$ is Lipschitz. Since the corresponding $S=\emptyset$, the first part of the argument implies

$$
\begin{gather*}
\varepsilon C m_{n+m}(A \times \overline{B(\mathbf{0}, 1)}) \geq \int_{A \times B \overline{(\mathbf{0}, 1)}}\left|J^{*} \mathbf{k}_{\varepsilon}\right| d m_{n+m} \\
=\int_{\mathbb{R}^{m}} \mathscr{H}^{n}\left(\mathbf{k}_{\varepsilon}^{-1}(\mathbf{y}) \cap A \times \overline{B(\mathbf{0}, 1)}\right) d y \tag{29.9.59}
\end{gather*}
$$

Now it is clear that $\mathbf{k}_{\varepsilon}^{-1}(\underline{y}) \supseteq \mathbf{f}^{-1}(\mathbf{y})$. Indeed, if $\mathbf{f}(\mathbf{x})=\mathbf{y}$, then $\mathbf{f}(\mathbf{x})+\varepsilon \mathbf{0}=\mathbf{y}$.
Note that $A \subseteq \mathbb{R}^{n}$ and $\overline{B(\mathbf{0}, 1)} \in \mathbb{R}^{m}$. By Lemma 29.9.4, and what was just noted,

$$
\begin{gathered}
\mathscr{H}^{n}\left(\mathbf{k}_{\varepsilon}^{-1}(\mathbf{y}) \cap A \times \overline{B(\mathbf{0}, 1)}\right) \geq \\
C_{n m} \frac{1}{(\operatorname{Lip}(\mathbf{p}))^{m}} \int_{\mathbb{R}^{m}} \mathscr{H}^{n-m}\left(\mathbf{k}_{\varepsilon}^{-1}(\mathbf{y}) \cap \mathbf{p}^{-1}(\mathbf{w}) \cap A \times \overline{B(\mathbf{0}, 1)}\right) d w
\end{gathered}
$$

Therefore, from 29.9.59,

$$
\begin{gather*}
\varepsilon C m_{n+m}(A \times \overline{B(\mathbf{0}, 1)}) \geq \\
C_{n m} \int_{\mathbb{R}^{m}} \int_{\mathbb{R}^{m}} \mathscr{H}^{n-m}\left(\mathbf{k}_{\varepsilon}^{-1}(\mathbf{y}) \cap \mathbf{p}^{-1}(\mathbf{w}) \cap A \times \overline{B(\mathbf{0}, 1)}\right) d w d y  \tag{29.9.60}\\
\geq C_{n m} \int_{\mathbb{R}^{m}} \int_{\mathbb{R}^{m}} \mathscr{H}^{n-m}\left(\mathbf{f}^{-1}(\mathbf{y}) \cap \mathbf{p}^{-1}(\mathbf{w}) \cap A \times \overline{B(\mathbf{0}, 1)}\right) d w d y
\end{gather*}
$$

The inside set is $\mathbf{f}^{-1}(\mathbf{y}) \cap \mathbf{p}^{-1}(\mathbf{w}) \cap A \times \overline{B(\mathbf{0}, 1)}$. That is,

$$
\{(\mathbf{x}, \mathbf{w}) \in A \times \overline{B(\mathbf{0}, 1)}: \mathbf{f}(\mathbf{x})=\mathbf{y}\}
$$

thus the set inside $\mathscr{H}^{n-m}$ is $\left(\mathbf{f}^{-1}(\mathbf{y}) \cap A\right) \times \overline{B(\mathbf{0}, 1)}$, then continuing the chain of inequalities,

$$
\begin{aligned}
& \geq C_{n m} \int_{\mathbb{R}^{m}} \int_{\mathbb{R}^{m}} \mathscr{H}^{n-m}\left(\left(\mathbf{f}^{-1}(\mathbf{y}) \cap A\right) \times \overline{B(\mathbf{0}, 1)}\right) d w d y \\
& \geq C_{n m} \int_{\mathbb{R}^{m}} \int_{\overline{B(\mathbf{0}, 1)}} \mathscr{H}^{n-m}\left(\mathbf{f}^{-1}(\mathbf{y}) \cap A\right) d w d y \\
& \quad=\quad C_{n m} \int_{\overline{B(\mathbf{0}, 1)}} \int_{\mathbb{R}^{m}} \mathscr{H}^{n-m}\left(\mathbf{f}^{-1}(\mathbf{y}) \cap A\right) d y d w \\
& \quad=C_{n m} \alpha_{n} \int_{\mathbb{R}^{m}} \mathscr{H}^{n-m}\left(\mathbf{f}^{-1}(\mathbf{y}) \cap A\right) d y
\end{aligned}
$$

Since $\varepsilon$ is arbitrary in 29.9.60, this shows that $\int_{\mathbb{R}^{m}} \mathscr{H}^{n-m}\left(\mathbf{f}^{-1}(\mathbf{y}) \cap A\right) d y=0=\int_{A} J^{*}(\mathbf{x}) d x$. Since this holds for arbitrary compact sets in $S$, it follows from Lemma 29.9.6 and inner regularity of Lebesgue measure that the equation holds for all measurable subsets of $S$. This completes the proof of the coarea formula.

There is a simple corollary to this theorem in the case of locally Lipschitz maps.
Corollary 29.9.9 Let $\mathbf{f}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ where $m \leq n$ and $\mathbf{f}$ is locally Lipschitz. This means that for each $r>0, \mathbf{f}$ is Lipschitz on $B(\mathbf{0}, r)$. Then the coarea formula, 29.9.48, holds for $\mathbf{f}$.

Proof: Let $A \subseteq B(\mathbf{0}, r)$ and let $\mathbf{f}_{r}$ be Lipschitz with $\mathbf{f}(\mathbf{x})=\mathbf{f}_{r}(\mathbf{x})$ for $\mathbf{x} \in B(\mathbf{0}, r+1)$. Then

$$
\begin{gathered}
\int_{A} J^{*} \mathbf{f}(\mathbf{x}) d x=\int_{A} J\left(D \mathbf{f}_{r}(\mathbf{x})\right) d x=\int_{\mathbb{R}^{m}} \mathscr{H}^{n-m}\left(A \cap \mathbf{f}_{r}^{-1}(\mathbf{y})\right) d y \\
=\int_{\mathbf{f}_{r}(A)} \mathscr{H}^{n-m}\left(A \cap \mathbf{f}_{r}^{-1}(\mathbf{y})\right) d y=\int_{\mathbf{f}(A)} \mathscr{H}^{n-m}\left(A \cap \mathbf{f}^{-1}(\mathbf{y})\right) d y \\
=\int_{\mathbb{R}^{m}} \mathscr{H}^{n-m}\left(A \cap \mathbf{f}^{-1}(\mathbf{y})\right) d y
\end{gathered}
$$

Now for arbitrary measurable $A$ the above shows for $k=1,2, \cdots$

$$
\int_{A \cap B(\mathbf{0}, k)} J^{*} \mathbf{f}(\mathbf{x}) d x=\int_{\mathbb{R}^{m}} \mathscr{H}^{n-m}\left(A \cap B(\mathbf{0}, k) \cap \mathbf{f}^{-1}(\mathbf{y})\right) d y
$$

Use the monotone convergence theorem to obtain 29.9.48.
From the definition of Hausdorff measure, it is easy to verify that $\mathscr{H}^{0}(E)$ equals the number of elements in $E$. Thus, if $n=m$, the Coarea formula implies

$$
\int_{A} J^{*} \mathbf{f}(\mathbf{x}) d x=\int_{\mathbf{f}(A)} \mathscr{H}^{0}\left(A \cap \mathbf{f}^{-1}(\mathbf{y})\right) d y=\int_{\mathbf{f}(A)} \#(y) d y
$$

Note also that this gives a version of Sard's theorem by letting $S=A$.

### 29.10 Change of Variables

We say that the coarea formula holds for $\mathbf{f}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}, n \geq m$ if whenever $A$ is a Lebesgue measurable subset of $\mathbb{R}^{n}, 29.9 .48$ holds. Note this is the same as

$$
\int_{A} J^{*}(\mathbf{x}) d x=\int_{\mathbf{f}(A)} \mathscr{H}^{n-m}\left(A \cap \mathbf{f}^{-1}(\mathbf{y})\right) d y, J^{*}(\mathbf{x}) \equiv \operatorname{det}\left(D \mathbf{f}(\mathbf{x}) D \mathbf{f}(\mathbf{x})^{*}\right)^{1 / 2}
$$

Now let $s(\mathbf{x})=\sum_{i=1}^{p} c_{i} \mathscr{X}_{E_{i}}(\mathbf{x})$ where $E_{i}$ is measurable and $c_{i} \geq 0$. Then

$$
\begin{gather*}
\int_{\mathbb{R}^{n}} s(\mathbf{x}) J^{*} \mathbf{f}(\mathbf{x}) d x=\sum_{i=1}^{p} c_{i} \int_{E_{i}} J^{*} \mathbf{f}(\mathbf{x}) d x=\sum_{i=1}^{p} c_{i} \int_{\mathbf{f}\left(E_{i}\right)} \mathscr{H}^{n-m}\left(E_{i} \cap \mathbf{f}^{-1}(\mathbf{y})\right) d y \\
=\int_{\mathbf{f}\left(\mathbb{R}^{n}\right)} \sum_{i=1}^{p} c_{i} \mathscr{H}^{n-m}\left(E_{i} \cap \mathbf{f}^{-1}(\mathbf{y})\right) d y=\int_{\mathbf{f}\left(\mathbb{R}^{n}\right)}\left[\int_{\mathbf{f}^{-1}(\mathbf{y})} s d \mathscr{H}^{n-m}\right] d y \\
=\int_{\mathbf{f}\left(\mathbb{R}^{n}\right)}\left[\int_{\mathbf{f}^{-1}(\mathbf{y})} s d \mathscr{H}^{n-m}\right] d y \tag{29.10.61}
\end{gather*}
$$

Theorem 29.10.1 Let $g \geq 0$ be Lebesgue measurable and let

$$
\mathbf{f}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}, n \geq m
$$

satisfy the Coarea formula. Then

$$
\int_{\mathbb{R}^{n}} g(\mathbf{x}) J^{*} \mathbf{f}(\mathbf{x}) d x=\int_{\mathbf{f}\left(\mathbb{R}^{n}\right)}\left[\int_{\mathbf{f}^{-1}(\mathbf{y})} g d \mathscr{H}^{n-m}\right] d y .
$$

Proof: Let $s_{i} \uparrow g$ where $s_{i}$ is a simple function satisfying 29.10.61. Then let $i \rightarrow \infty$ and use the monotone convergence theorem to replace $s_{i}$ with $g$. This proves the change of variables formula.

Note that this formula is a nonlinear version of Fubini's theorem. The " $n-m$ dimensional surface", $\mathbf{f}^{-1}(\mathbf{y})$, plays the role of $\mathbb{R}^{n-m}$ and $\mathscr{H}^{n-m}$ is like $n-m$ dimensional Lebesgue measure. The term, $J^{*} \mathbf{f}(\mathbf{x})$, corrects for the error occurring because of the lack of flatness of $\mathbf{f}^{-1}(\mathbf{y})$.

The following is an easy example of the use of the coarea formula to give a familiar relation.

Example 29.10.2 Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be given by $f(\mathbf{x}) \equiv|\mathbf{x}|$. Then $J^{*}(\mathbf{x})$ ends up being 1 . Then by the coarea formula,

$$
\int_{B(\mathbf{0}, r)} d m_{n}=\int_{0}^{r} \mathscr{H}^{n-1}\left(B(\mathbf{0}, r) \cap f^{-1}(y)\right) d y=\int_{0}^{r} \mathscr{H}^{n-1}(\partial B(\mathbf{0}, y)) d y
$$

Then $m_{n}(B(\mathbf{0}, r)) \equiv \alpha_{n} r^{n}=\int_{0}^{r} \mathscr{H}^{n-1}(\partial B(\mathbf{0}, y)) d y$. Then differentiate both sides to obtain $n \alpha_{n} r^{n-1}=\mathscr{H}^{n-1}(\partial B(\mathbf{0}, r))$. In particular $\mathscr{H}^{2}(\partial B(\mathbf{0}, r))=3 \frac{4}{3} \pi r^{2}=4 \pi r^{2}$. Of course $\alpha_{n}$ was computed earlier. Recall from Theorem 28.4.2 on Page 1005

$$
\alpha_{n}=\pi^{n / 2}(\Gamma(n / 2+1))^{-1}
$$

Therefore, the $n-1$ dimensional Hausdorf measure of the boundary of the ball of radius $r$ in $\mathbb{R}^{n}$ is $n \pi^{p / 2}(\Gamma(n / 2+1))^{-1} r^{n-1}$.

I think it is clear that you could generalize this to other more complicated situations. The above is nice because $J^{*}(\mathbf{x})=1$. This won't be so in general when considering other level surfaces.

### 29.11 Integration and the Degree

There is a very interesting application of the degree to integration [52]. Recall Lemma 23.1.11. I want to generalize this to the case where $\mathbf{h}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ is only Lipschitz continuous, vanishing outside a bounded set. In the following proposition, let $\phi_{\varepsilon}$ be a symmetric nonnegative mollifier,

$$
\phi_{\varepsilon}(\mathbf{x}) \equiv \frac{1}{\varepsilon^{n}} \phi\left(\frac{\mathbf{x}}{\varepsilon}\right), \operatorname{spt} \phi \subseteq B(\mathbf{0}, 1) .
$$

$\Omega$ will be a bounded open set. By Theorem 26.6.7, h satisfies

$$
\begin{equation*}
D \mathbf{h}(\mathbf{x}) \text { exists a.e., } \tag{29.11.62}
\end{equation*}
$$

For any $p>n$,

$$
\begin{equation*}
\lim _{m \rightarrow \infty} D\left(\mathbf{h} * \psi_{m}\right)=D \mathbf{h} \text { in } L^{p}\left(\mathbb{R}^{n} ; \mathbb{R}^{n \times n}\right) \tag{29.11.63}
\end{equation*}
$$

where $\psi_{m}$ is a mollifier.
Proposition 29.11.1 Let $S \subseteq \mathbf{h}(\partial \Omega)^{C}$ such that

$$
\operatorname{dist}(S, \mathbf{h}(\partial \Omega))>0
$$

where $\Omega$ is a bounded open set and also let $\mathbf{h}$ be Lipschitz continuous, vanishing outside some bounded set. Then whenever $\varepsilon>0$ is small enough,

$$
d(\mathbf{h}, \Omega, \mathbf{y})=\int_{\Omega} \phi_{\varepsilon}(\mathbf{h}(\mathbf{x})-\mathbf{y}) \operatorname{det} D \mathbf{h}(\mathbf{x}) d x
$$

for all $\mathbf{y} \in S$.
Proof: Let $\varepsilon_{0}>0$ be small enough that for all $\mathbf{y} \in S$,

$$
B\left(\mathbf{y}, 5 \varepsilon_{0}\right) \cap \mathbf{h}(\partial \Omega)=\emptyset
$$

Now let $\psi_{m}$ be a mollifier as $m \rightarrow \infty$ with support in $B\left(\mathbf{0}, m^{-1}\right)$ and let

$$
\mathbf{h}_{m} \equiv \mathbf{h} * \psi_{m}
$$

Thus $\mathbf{h}_{m} \in C^{\infty}\left(\bar{\Omega} ; \mathbb{R}^{n}\right)$ and for any $p>n$,

$$
\begin{equation*}
\left\|\mathbf{h}_{m}-\mathbf{h}\right\|_{L^{\infty}(\Omega)},\left\|D \mathbf{h}_{m}-D \mathbf{h}\right\|_{L^{p}(\Omega)} \rightarrow 0 \tag{29.11.64}
\end{equation*}
$$

as $m \rightarrow \infty$. The first claim above is obvious and the second follows by 29.11.63. Choose $M$ such that for $m \geq M$,

$$
\begin{equation*}
\left\|\mathbf{h}_{m}-\mathbf{h}\right\|_{\infty}<\varepsilon_{0} \tag{29.11.65}
\end{equation*}
$$

Thus $\mathbf{h}_{m} \in \mathscr{U}_{\mathbf{y}} \cap C^{2}\left(\bar{\Omega} ; \mathbb{R}^{n}\right)$ for all $\mathbf{y} \in S$.
For $\mathbf{y} \in S$, let $\mathbf{z} \in B(\mathbf{y}, \varepsilon)$ where $\varepsilon<\varepsilon_{0}$ and suppose $\mathbf{x} \in \partial \Omega$, and $k, m \geq M$. Then for $t \in[0,1]$,

$$
\begin{aligned}
\left|(1-t) \mathbf{h}_{m}(\mathbf{x})+\mathbf{h}_{k}(\mathbf{x}) t-\mathbf{z}\right| & \geq\left|\mathbf{h}_{m}(\mathbf{x})-\mathbf{z}\right|-t\left|\mathbf{h}_{k}(\mathbf{x})-\mathbf{h}_{m}(\mathbf{x})\right| \\
& >2 \varepsilon_{0}-t 2 \varepsilon_{0} \geq 0
\end{aligned}
$$

showing that for each $\mathbf{y} \in S, B(\mathbf{y}, \varepsilon) \cap\left((1-t) \mathbf{h}_{m}+t \mathbf{h}_{k}\right)(\partial \Omega)=\emptyset$. By Lemma 23.1.11, for all $\mathbf{y} \in S$,

$$
\begin{gather*}
\int_{\Omega} \phi_{\varepsilon}\left(\mathbf{h}_{m}(\mathbf{x})-\mathbf{y}\right) \operatorname{det}\left(D \mathbf{h}_{m}(\mathbf{x})\right) d x= \\
\int_{\Omega} \phi_{\varepsilon}\left(\mathbf{h}_{k}(\mathbf{x})-\mathbf{y}\right) \operatorname{det}\left(D \mathbf{h}_{k}(\mathbf{x})\right) d x \tag{29.11.66}
\end{gather*}
$$

for all $k, m \geq M$. By this lemma again, which says that for small enough $\varepsilon$ the integral is constant and the definition of the degree in Definition 23.1.10,

$$
\begin{equation*}
d\left(\mathbf{y}, \Omega, \mathbf{h}_{m}\right)=\int_{\Omega} \phi_{\varepsilon}\left(\mathbf{h}_{m}(\mathbf{x})-\mathbf{y}\right) \operatorname{det}\left(D \mathbf{h}_{m}(\mathbf{x})\right) d x \tag{29.11.67}
\end{equation*}
$$

for all $\varepsilon$ small enough. For $\mathbf{x} \in \partial \Omega, \mathbf{y} \in S$, and $t \in[0,1]$,

$$
\begin{aligned}
\left|(1-t) \mathbf{h}(\mathbf{x})+\mathbf{h}_{m}(\mathbf{x}) t-\mathbf{y}\right| & \geq|\mathbf{h}(\mathbf{x})-\mathbf{y}|-t\left|\mathbf{h}(\mathbf{x})-\mathbf{h}_{m}(\mathbf{x})\right| \\
& >3 \varepsilon_{0}-t 2 \varepsilon_{0}>0
\end{aligned}
$$

and so by Theorem 23.2.2, the part about homotopy, for each $\mathbf{y} \in S$,

$$
\begin{gathered}
d(\mathbf{y}, \Omega, \mathbf{h})=d\left(\mathbf{y}, \Omega, \mathbf{h}_{m}\right)= \\
\int_{\Omega} \phi_{\varepsilon}\left(\mathbf{h}_{m}(\mathbf{x})-\mathbf{y}\right) \operatorname{det}\left(D \mathbf{h}_{m}(\mathbf{x})\right) d x
\end{gathered}
$$

whenever $\varepsilon$ is small enough. Fix such an $\varepsilon<\varepsilon_{0}$ and use 29.11.66 to conclude the right side of the above equation is independent of $m>M$.

By 29.11.64, there exists a subsequence still denoted by $m$ such that $D \mathbf{h}_{m}(\mathbf{x}) \rightarrow D \mathbf{h}(\mathbf{x})$ a.e. Since $p>n$, $\operatorname{det}\left(D \mathbf{h}_{m}\right)$ is bounded in $L^{r}(\Omega)$ for some $r>1$ and so the integrands in the following are uniformly integrable. By the Vitali convergence theorem, one can pass to the limit as follows.

$$
\begin{aligned}
d(\mathbf{y}, \Omega, \mathbf{h}) & =\lim _{m \rightarrow \infty} \int_{\Omega} \phi_{\varepsilon}\left(\mathbf{h}_{m}(\mathbf{x})-\mathbf{y}\right) \operatorname{det}\left(D \mathbf{h}_{m}(\mathbf{x})\right) d x \\
& =\int_{\Omega} \phi_{\varepsilon}(\mathbf{h}(\mathbf{x})-\mathbf{y}) \operatorname{det}(D \mathbf{h}(\mathbf{x})) d x
\end{aligned}
$$

This proves the proposition.
Next is an interesting change of variables theorem. Let $\Omega$ be a bounded open set with the property that $\partial \Omega$ has measure zero and let $\mathbf{h}$ be Lipschitz continuous on $\mathbb{R}^{n}$. Then from Lemma 29.1.1, $\mathbf{h}(\partial \Omega)$ also has measure zero.

Now suppose $f \in C_{c}\left(\mathbf{h}(\partial \Omega)^{C}\right)$. There are finitely many components of $\mathbf{h}(\partial \Omega)^{C}$ which have nonempty intersection with $\operatorname{spt}(f)$. From the Proposition above,

$$
\int f(\mathbf{y}) d(\mathbf{y}, \Omega, \mathbf{h}) d y=\int f(\mathbf{y}) \lim _{\varepsilon \rightarrow 0} \int_{\Omega} \phi_{\varepsilon}(\mathbf{h}(\mathbf{x})-\mathbf{y}) \operatorname{det} D \mathbf{h}(\mathbf{x}) d x d y
$$

Actually, there exists an $\varepsilon$ small enough that for all $\mathbf{y} \in \operatorname{spt}(f)$,

$$
\begin{aligned}
\lim _{\varepsilon \rightarrow 0} \int_{\Omega} \phi_{\varepsilon}(\mathbf{h}(\mathbf{x})-\mathbf{y}) \operatorname{det} D \mathbf{h}(\mathbf{x}) d x & =\int_{\Omega} \phi_{\varepsilon}(\mathbf{h}(\mathbf{x})-\mathbf{y}) \operatorname{det} D \mathbf{h}(\mathbf{x}) d x \\
& =d(\mathbf{y}, \Omega, \mathbf{h})
\end{aligned}
$$

This is because $\operatorname{spt}(f)$ is at a positive distance from the compact set $\mathbf{h}(\partial \Omega)^{C}$. Therefore, for all $\varepsilon$ small enough,

$$
\begin{aligned}
\int f(\mathbf{y}) d(\mathbf{y}, \Omega, \mathbf{h}) d y & =\iint_{\Omega} f(\mathbf{y}) \phi_{\varepsilon}(\mathbf{h}(\mathbf{x})-\mathbf{y}) \operatorname{det} D \mathbf{h}(\mathbf{x}) d x d y \\
& =\int_{\Omega} \operatorname{det} D \mathbf{h}(\mathbf{x}) \int f(\mathbf{y}) \phi_{\varepsilon}(\mathbf{h}(\mathbf{x})-\mathbf{y}) d y d x
\end{aligned}
$$

Using the uniform continuity of $f$, you can now pass to a limit and obtain using the fact that $\operatorname{det} D \mathbf{h}(\mathbf{x})$ is in $L^{r}\left(\mathbb{R}^{n}\right)$ for some $r>1$,

$$
\int f(\mathbf{y}) d(\mathbf{y}, \Omega, \mathbf{h}) d y=\int_{\Omega} f(\mathbf{h}(\mathbf{x})) \operatorname{det} D \mathbf{h}(\mathbf{x}) d x
$$

This has proved the following interesting lemma.
Lemma 29.11.2 Let $f \in C_{c}\left(\mathbf{h}(\partial \Omega)^{C}\right)$ for $\Omega$ a bounded open set and let $\mathbf{h}$ be Lipschitz on $\mathbb{R}^{n}$. Say $\partial \Omega$ has measure zero so that $\mathbf{h}(\partial \Omega)$ has measure zero. Then everything is measurable which needs to be and

$$
\int f(\mathbf{y}) d(\mathbf{y}, \Omega, \mathbf{h}) d y=\int_{\Omega} \operatorname{det}(D \mathbf{h}(\mathbf{x})) f(\mathbf{h}(\mathbf{x})) d x
$$

Note that $\mathbf{h}$ is not necessarily one to one. Next is a simple corollary which replaces $C_{c}\left(\mathbb{R}^{n}\right)$ with $L_{l o c}^{1}\left(\mathbb{R}^{n}\right)$ in the case that $\mathbf{h}$ is one to one. Also another assumption is made on there being finitely many components.

Corollary 29.11.3 Let $f \in L_{l o c}^{1}\left(\mathbb{R}^{n}\right)$ and let $\mathbf{h}$ be one to one and satisfy 29.11.62-29.11.63, $\partial \Omega$ has measure zero for $\Omega$ a bounded open set and $\mathbf{h}(\partial \Omega)^{C}$ has finitely many components. Then everything is measurable which needs to be and

$$
\int f(\mathbf{y}) d(\mathbf{y}, \Omega, \mathbf{h}) d y=\int_{\Omega} \operatorname{det} D \mathbf{h}(\mathbf{x}) f(\mathbf{h}(\mathbf{x})) d x
$$

Proof: Since $d(\mathbf{y}, \Omega, \mathbf{h})=0$ for all $|\mathbf{y}|$ large enough due to $\mathbf{y} \notin \mathbf{h}(\Omega)$ for large $\mathbf{y}$, there is no loss of generality in assuming $f$ is in $L^{1}\left(\mathbb{R}^{n}\right)$. For all $\mathbf{y} \notin \mathbf{h}(\partial \Omega)$, a set of measure zero, $d(\mathbf{y}, \Omega, \mathbf{h})$ is bounded by some constant, depending on the maximum of the degree on the various components of $\mathbf{h}(\partial \Omega)^{C}$. Then from Proposition 29.11.1

$$
\begin{equation*}
\int f(\mathbf{y}) d(\mathbf{y}, \Omega, \mathbf{h}) d y=\int f(\mathbf{y}) \lim _{\varepsilon \rightarrow 0} \int_{\Omega} \phi_{\varepsilon}(\mathbf{h}(\mathbf{x})-\mathbf{y}) \operatorname{det} D \mathbf{h}(\mathbf{x}) d x d y \tag{29.11.68}
\end{equation*}
$$

This time, use the area formula to write

$$
\begin{gathered}
\left|\int_{\Omega} \phi_{\varepsilon}(\mathbf{h}(\mathbf{x})-\mathbf{y}) \operatorname{det} D \mathbf{h}(\mathbf{x}) d x\right| \leq \int_{\mathbb{R}^{n}} \phi_{\varepsilon}(\mathbf{h}(\mathbf{x})-\mathbf{y})|\operatorname{det} D \mathbf{h}(\mathbf{x})| d x \\
\leq K \int_{\mathbb{R}^{n}} \phi_{\varepsilon}(\mathbf{z}-\mathbf{y}) d z<\infty
\end{gathered}
$$

and so using the dominated convergence theorem in 29.11.68, it equals

$$
\begin{aligned}
& \lim _{\varepsilon \rightarrow 0} \int_{\Omega} \operatorname{det} D \mathbf{h}(\mathbf{x}) \int f(\mathbf{y}) \phi_{\varepsilon}(\mathbf{h}(\mathbf{x})-\mathbf{y}) d y d x \\
= & \lim _{\varepsilon \rightarrow 0} \int_{\Omega} \operatorname{det} D \mathbf{h}(\mathbf{x}) \int f(\mathbf{h}(\mathbf{x})-\mathbf{y}) \phi_{\varepsilon}(\mathbf{y}) d y d x \\
= & \lim _{\varepsilon \rightarrow 0} \int_{\Omega} \operatorname{det} D \mathbf{h}(\mathbf{x}) \int_{B(0,1)} f(\mathbf{h}(\mathbf{x})-\varepsilon \mathbf{u}) \phi(\mathbf{u}) d u d x
\end{aligned}
$$

Now

$$
\begin{gathered}
\mid \int_{\Omega} \operatorname{det} D \mathbf{h}(\mathbf{x}) \int_{B(\mathbf{0}, 1)} f(\mathbf{h}(\mathbf{x})-\varepsilon \mathbf{u}) \phi(\mathbf{u}) d u d x \\
-\int_{\Omega} \operatorname{det} D \mathbf{h}(\mathbf{x}) f(\mathbf{h}(\mathbf{x})) d x \mid \leq \\
\left|\int_{B(\mathbf{0}, 1)} \int_{\Omega}\right| \operatorname{det} D \mathbf{h}(\mathbf{x})||f(\mathbf{h}(\mathbf{x})-\varepsilon \mathbf{u})-f(\mathbf{h}(\mathbf{x}))| d x \phi(\mathbf{u}) d u|
\end{gathered}
$$

which needs to converge to 0 as $\varepsilon \rightarrow 0$. However, from the area formula, Theorem 29.5.3 applied to the inside integral, the above equals

$$
\int_{B(\mathbf{0}, 1)} \int_{\mathbf{h}(\Omega)}|f(\mathbf{y}-\varepsilon \mathbf{u})-f(\mathbf{y})| d y \phi(\mathbf{u}) d u \leq \int_{B(\mathbf{0}, 1)}\left\|f_{\varepsilon \mathbf{u}}-f\right\|_{L^{1}\left(\mathbb{R}^{n}\right)} \phi(\mathbf{u}) d u
$$

which converges to 0 by continuity of translation in $L^{1}\left(\mathbb{R}^{n}\right)$. Thus as in the lemma,

$$
\begin{aligned}
\int f(\mathbf{y}) d(\mathbf{y}, \Omega, \mathbf{h}) d y & =\lim _{\varepsilon \rightarrow 0} \int f(\mathbf{y}) \int_{\Omega} \phi_{\varepsilon}(\mathbf{h}(\mathbf{x})-\mathbf{y}) \operatorname{det} D \mathbf{h}(\mathbf{x}) d x d y \\
& =\lim _{\varepsilon \rightarrow 0} \int_{\Omega} \operatorname{det} D \mathbf{h}(\mathbf{x}) \int_{B(\mathbf{0}, 1)} f(\mathbf{h}(\mathbf{x})-\varepsilon \mathbf{u}) \phi(\mathbf{u}) d u d x \\
& =\int_{\Omega} \operatorname{det} D \mathbf{h}(\mathbf{x}) f(\mathbf{h}(\mathbf{x})) d x
\end{aligned}
$$

and this proves the corollary.
Note that in this corollary $\mathbf{h}$ is one to one.

### 29.12 The Case Of $W^{1, p}$

There is a very interesting application of the degree to integration [52]. Recall Lemma 23.1.11. I want to generalize this to the case where $\mathbf{h}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ has the property that its weak partial derivatives and $\mathbf{h}$ are in $L^{p}\left(\mathbb{R}^{n} ; \mathbb{R}^{n}\right), p>n$. This is denoted by saying

$$
\mathbf{h} \in W^{1, p}\left(\mathbb{R}^{n} ; \mathbb{R}^{n}\right)
$$

In the following proposition, let $\phi_{\varepsilon}$ be a symmetric nonnegative mollifier,

$$
\phi_{\varepsilon}(\mathbf{x}) \equiv \frac{1}{\varepsilon^{n}} \phi\left(\frac{\mathbf{x}}{\varepsilon}\right), \operatorname{spt} \phi \subseteq B(\mathbf{0}, 1)
$$

$\Omega$ will be a bounded open set. By Theorem 26.6.10, h may be considered continuous and it satisfies

$$
\begin{equation*}
D \mathbf{h}(\mathbf{x}) \text { exists a.e., } \tag{29.12.69}
\end{equation*}
$$

For any $p>n$,

$$
\begin{equation*}
\lim _{m \rightarrow \infty} D\left(\mathbf{h} * \psi_{m}\right)=D \mathbf{h} \text { in } L^{p}\left(\mathbb{R}^{n} ; \mathbb{R}^{n \times n}\right) \tag{29.12.70}
\end{equation*}
$$

where $\psi_{m}$ is a mollifier. Here $\mathbb{R}^{n \times n}$ denotes the $n \times n$ matrices with any norm you like.

Proposition 29.12.1 Let $S \subseteq \mathbf{h}(\partial \Omega)^{C}$ such that

$$
\operatorname{dist}(S, \mathbf{h}(\partial \Omega))>0
$$

where $\Omega$ is a bounded open set and also let $\mathbf{h}$ be in $W^{1, p}\left(\mathbb{R}^{n} ; \mathbb{R}^{n}\right)$. Then whenever $\varepsilon>0$ is small enough,

$$
d(\mathbf{h}, \Omega, \mathbf{y})=\int_{\Omega} \phi_{\varepsilon}(\mathbf{h}(\mathbf{x})-\mathbf{y}) \operatorname{det} D \mathbf{h}(\mathbf{x}) d x
$$

for all $\mathbf{y} \in S$.
Proof: Let $\varepsilon_{0}>0$ be small enough that for all $\mathbf{y} \in S$,

$$
B\left(\mathbf{y}, 3 \varepsilon_{0}\right) \cap \mathbf{h}(\partial \Omega)=\emptyset
$$

Now let $\psi_{m}$ be a mollifier as $m \rightarrow \infty$ with support in $B\left(\mathbf{0}, m^{-1}\right)$ and let

$$
\mathbf{h}_{m} \equiv \mathbf{h} * \psi_{m}
$$

Thus $\mathbf{h}_{m} \in C^{\infty}\left(\bar{\Omega} ; \mathbb{R}^{n}\right)$ and,

$$
\begin{equation*}
\left\|\mathbf{h}_{m}-\mathbf{h}\right\|_{L^{\infty}(\Omega)},\left\|D \mathbf{h}_{m}-D \mathbf{h}\right\|_{L^{p}(\Omega)} \rightarrow 0 \tag{29.12.71}
\end{equation*}
$$

as $m \rightarrow \infty$. The first claim above follows from the definition of convolution and the uniform continuity of $\mathbf{h}$ on the compact set $\bar{\Omega}$ and the second follows by 29.12.70. Choose $M$ such that for $m \geq M$,

$$
\begin{equation*}
\left\|\mathbf{h}_{m}-\mathbf{h}\right\|_{L^{\infty}(\Omega)}<\varepsilon_{0} \tag{29.12.72}
\end{equation*}
$$

Thus $\mathbf{h}_{m} \in \mathscr{U}_{\mathbf{y}} \cap C^{2}\left(\bar{\Omega} ; \mathbb{R}^{n}\right)$ for all $\mathbf{y} \in S$.
For $\mathbf{y} \in S$, let $\mathbf{z} \in B(\mathbf{y}, \boldsymbol{\varepsilon})$ where $\varepsilon<\varepsilon_{0}$ and suppose $\mathbf{x} \in \partial \Omega$, and $k, m \geq M$. Then for $t \in[0,1]$,

$$
\begin{aligned}
\left|(1-t) \mathbf{h}_{m}(\mathbf{x})+\mathbf{h}_{k}(\mathbf{x}) t-\mathbf{z}\right| & \geq\left|\mathbf{h}_{m}(\mathbf{x})-\mathbf{z}\right|-t\left|\mathbf{h}_{k}(\mathbf{x})-\mathbf{h}_{m}(\mathbf{x})\right| \\
& >2 \varepsilon_{0}-t 2 \varepsilon_{0} \geq 0
\end{aligned}
$$

showing that for each $\mathbf{y} \in S, B(\mathbf{y}, \varepsilon) \cap\left((1-t) \mathbf{h}_{m}+t \mathbf{h}_{k}\right)(\partial \Omega)=\emptyset$. By Lemma 23.1.11, for all $\mathbf{y} \in S$,

$$
\begin{gather*}
\int_{\Omega} \phi_{\varepsilon}\left(\mathbf{h}_{m}(\mathbf{x})-\mathbf{y}\right) \operatorname{det}\left(D \mathbf{h}_{m}(\mathbf{x})\right) d x= \\
\int_{\Omega} \phi_{\varepsilon}\left(\mathbf{h}_{k}(\mathbf{x})-\mathbf{y}\right) \operatorname{det}\left(D \mathbf{h}_{k}(\mathbf{x})\right) d x \tag{29.12.73}
\end{gather*}
$$

for all $k, m \geq M$. By this lemma again, which says that for small enough $\varepsilon$ the integral is constant and the definition of the degree in Definition 23.1.10,

$$
\begin{equation*}
d\left(\mathbf{y}, \Omega, \mathbf{h}_{m}\right)=\int_{\Omega} \phi_{\varepsilon}\left(\mathbf{h}_{m}(\mathbf{x})-\mathbf{y}\right) \operatorname{det}\left(D \mathbf{h}_{m}(\mathbf{x})\right) d x \tag{29.12.74}
\end{equation*}
$$

for all $\varepsilon$ small enough. For $\mathbf{x} \in \partial \Omega, \mathbf{y} \in S$, and $t \in[0,1]$,

$$
\begin{aligned}
\left|(1-t) \mathbf{h}(\mathbf{x})+\mathbf{h}_{m}(\mathbf{x}) t-\mathbf{y}\right| & \geq|\mathbf{h}(\mathbf{x})-\mathbf{y}|-t\left|\mathbf{h}(\mathbf{x})-\mathbf{h}_{m}(\mathbf{x})\right| \\
& >3 \varepsilon_{0}-t 2 \varepsilon_{0}>0
\end{aligned}
$$

and so by Theorem 23.2.2, the part about homotopy, for each $\mathbf{y} \in S$,

$$
\begin{gathered}
d(\mathbf{y}, \Omega, \mathbf{h})=d\left(\mathbf{y}, \Omega, \mathbf{h}_{m}\right)= \\
\int_{\Omega} \phi_{\varepsilon}\left(\mathbf{h}_{m}(\mathbf{x})-\mathbf{y}\right) \operatorname{det}\left(D \mathbf{h}_{m}(\mathbf{x})\right) d x
\end{gathered}
$$

whenever $\varepsilon$ is small enough. Fix such an $\varepsilon<\varepsilon_{0}$ and use 29.12.73 to conclude the right side of the above equation is independent of $m>M$.

By 29.12.71, there exists a subsequence still denoted by $m$ such that $D \mathbf{h}_{m}(\mathbf{x}) \rightarrow D \mathbf{h}(\mathbf{x})$ a.e. Since $p>n, \operatorname{det}\left(D \mathbf{h}_{m}\right)$ is bounded in $L^{r}(\Omega)$ for some $r>1$ and so the integrands in the following are uniformly integrable. By the Vitali convergence theorem, one can pass to the limit as follows.

$$
\begin{aligned}
d(\mathbf{y}, \Omega, \mathbf{h}) & =\lim _{m \rightarrow \infty} \int_{\Omega} \phi_{\varepsilon}\left(\mathbf{h}_{m}(\mathbf{x})-\mathbf{y}\right) \operatorname{det}\left(D \mathbf{h}_{m}(\mathbf{x})\right) d x \\
& =\int_{\Omega} \phi_{\varepsilon}(\mathbf{h}(\mathbf{x})-\mathbf{y}) \operatorname{det}(D \mathbf{h}(\mathbf{x})) d x
\end{aligned}
$$

This proves the proposition.
Next is an interesting change of variables theorem. Let $\Omega$ be a bounded open set and let $\mathbf{h} \in W^{1, p}\left(\mathbb{R}^{n}\right)$. Also assume

$$
m(\mathbf{h}(\partial \Omega))=0
$$

From Proposition 29.12.1, for $\mathbf{y} \notin \mathbf{h}(\partial \Omega)$,

$$
d(\mathbf{y}, \Omega, \mathbf{h})=\lim _{\varepsilon \rightarrow 0} \int_{\Omega} \phi_{\varepsilon}(\mathbf{h}(\mathbf{x})-\mathbf{y}) \operatorname{det} D \mathbf{h}(\mathbf{x}) d x
$$

showing that $\mathbf{y} \rightarrow d(\mathbf{y}, \Omega, \mathbf{h})$ is a measurable function since it is the limit of continuous functions off the set of measure zero $\mathbf{h}(\partial \Omega)$.

Now suppose $f \in C_{c}\left(\mathbf{h}(\partial \Omega)^{C}\right)$. There are finitely many components of $\mathbf{h}(\partial \Omega)^{C}$ which have nonempty intersection with $\operatorname{spt}(f)$. From the Proposition above,

$$
\int f(\mathbf{y}) d(\mathbf{y}, \Omega, \mathbf{h}) d y=\int f(\mathbf{y}) \lim _{\varepsilon \rightarrow 0} \int_{\Omega} \phi_{\varepsilon}(\mathbf{h}(\mathbf{x})-\mathbf{y}) \operatorname{det} D \mathbf{h}(\mathbf{x}) d x d y
$$

Actually, from Proposition 29.12.1 there exists an $\varepsilon$ small enough that for all $\mathbf{y} \in \operatorname{spt}(f)$,

$$
\begin{aligned}
\lim _{\varepsilon \rightarrow 0} \int_{\Omega} \phi_{\varepsilon}(\mathbf{h}(\mathbf{x})-\mathbf{y}) \operatorname{det} D \mathbf{h}(\mathbf{x}) d x & =\int_{\Omega} \phi_{\varepsilon}(\mathbf{h}(\mathbf{x})-\mathbf{y}) \operatorname{det} D \mathbf{h}(\mathbf{x}) d x \\
& =d(\mathbf{y}, \Omega, \mathbf{h})
\end{aligned}
$$

This is because $\operatorname{spt}(f)$ is at a positive distance from $\mathbf{h}(\partial \Omega)^{C}$. Therefore, for all $\varepsilon$ small enough,

$$
\begin{aligned}
\int f(\mathbf{y}) d(\mathbf{y}, \Omega, \mathbf{h}) d y & =\iint_{\Omega} f(\mathbf{y}) \phi_{\varepsilon}(\mathbf{h}(\mathbf{x})-\mathbf{y}) \operatorname{det} D \mathbf{h}(\mathbf{x}) d x d y \\
& =\int_{\Omega} \operatorname{det} D \mathbf{h}(\mathbf{x}) \int f(\mathbf{y}) \phi_{\varepsilon}(\mathbf{h}(\mathbf{x})-\mathbf{y}) d y d x \\
& =\int_{\Omega} \operatorname{det} D \mathbf{h}(\mathbf{x}) \int f(\mathbf{h}(\mathbf{x})-\varepsilon \mathbf{u}) \phi(\mathbf{u}) d u d x
\end{aligned}
$$

Using the uniform continuity of $f$, you can now pass to a limit as $\varepsilon \rightarrow 0$ and obtain, using the fact that $\operatorname{det} D \mathbf{h}(\mathbf{x})$ is in $L^{r}\left(\mathbb{R}^{n}\right)$ for some $r>1$,

$$
\int f(\mathbf{y}) d(\mathbf{y}, \Omega, \mathbf{h}) d y=\int_{\Omega} f(\mathbf{h}(\mathbf{x})) \operatorname{det} D \mathbf{h}(\mathbf{x}) d x
$$

This has proved the following interesting lemma.
Lemma 29.12.2 Let $f \in C_{c}\left(\mathbf{h}(\partial \Omega)^{C}\right)$ and let $\mathbf{h} \in W^{1, p}\left(\mathbb{R}^{n} ; \mathbb{R}^{n}\right), p>n, \mathbf{h}(\partial \Omega)$ has measure zero for $\Omega$ a bounded open set. Then everything is measurable which needs to be and

$$
\int f(\mathbf{y}) d(\mathbf{y}, \Omega, \mathbf{h}) d y=\int_{\Omega} \operatorname{det}(D \mathbf{h}(\mathbf{x})) f(\mathbf{h}(\mathbf{x})) d x
$$

Note that $\mathbf{h}$ is not necessarily one to one. The difficult issue is handling $d(\mathbf{y}, \Omega, \mathbf{h})$ which has integer values constant on each component of $\mathbf{h}(\partial \Omega)^{C}$ and the difficulty arrises in not knowing how many components there are. What if there are infinitely many, for example, and what if the degree changes sign. If this happens, it is hard to exploit convergence theorems to get generalizations of $f \in C_{c}\left(\mathbf{h}(\partial \Omega)^{C}\right)$. One way around this is to insist $\mathbf{h}$ be one to one and that $\Omega$ be connected having a boundary which separates $\mathbb{R}^{n}$ into two components, three if $n=1$. That way, you can use the Jordan separation theorem and assert $\mathbf{h}(\partial \Omega)$ also separates $\mathbb{R}^{n}$ into the same number of components with $\mathbf{h}(\Omega)$ being the only one on which the degree is nonzero.

First recall the following proposition.
Proposition 29.12.3 Let $\Omega$ be an open connected bounded set in $\mathbb{R}^{n}, n \geq 1$ such that $\mathbb{R}^{n} \backslash$ $\partial \Omega$ consists of two, three if $n=1$, connected components. Let $\mathbf{f} \in C\left(\bar{\Omega} ; \mathbb{R}^{n}\right)$ be continuous and one to one. Then $\mathbf{f}(\Omega)$ is the bounded component of $\mathbb{R}^{n} \backslash \mathbf{f}(\partial \Omega)$ and for $\mathbf{y} \in \mathbf{f}(\Omega)$, $d(\mathbf{f}, \Omega, \mathbf{y})$ either equals 1 or -1 .

Proof: First suppose $n \geq 2$. By the Jordan separation theorem, $\mathbb{R}^{n} \backslash \mathbf{f}(\partial \Omega)$ consists of two components, a bounded component $B$ and an unbounded component $U$. Using the Tietze extention theorem, there exists $\mathbf{g}$ defined on $\mathbb{R}^{n}$ such that $\mathbf{g}=\mathbf{f}^{-1}$ on $\mathbf{f}(\bar{\Omega})$. Thus on $\partial \Omega, \mathbf{g} \circ \mathbf{f}=\mathrm{id}$. It follows from this and the product formula that

$$
\begin{aligned}
1 & =d(\mathrm{id}, \Omega, \mathbf{g}(\mathbf{y}))=d(\mathbf{g} \circ \mathbf{f}, \Omega, \mathbf{g}(\mathbf{y})) \\
& =d(\mathbf{g}, B, \mathbf{g}(\mathbf{y})) d(\mathbf{f}, \Omega, B)+d(\mathbf{f}, \Omega, U) d(\mathbf{g}, U, \mathbf{g}(\mathbf{y})) \\
& =d(\mathbf{g}, B, \mathbf{g}(\mathbf{y})) d(\mathbf{f}, \Omega, B)
\end{aligned}
$$

Therefore, $d(\mathbf{f}, \Omega, B) \neq 0$ and so for every $\mathbf{z} \in B$, it follows $\mathbf{z} \in \mathbf{f}(\Omega)$. Thus $B \subseteq \mathbf{f}(\Omega)$. On the other hand, $\mathbf{f}(\Omega)$ cannot have points in both $U$ and $B$ because it is a connected set. Therefore $\mathbf{f}(\Omega) \subseteq B$ and this shows $B=\mathbf{f}(\Omega)$. Thus $d(\mathbf{f}, \Omega, B)=d(\mathbf{f}, \Omega, \mathbf{y})$ for each $\mathbf{y} \in B$ and the above formula shows this equals either 1 or -1 because the degree is an integer. In the case where $n=1$, the argument is similar but here you have 3 components in $\mathbb{R}^{1} \backslash \mathbf{f}(\partial \Omega)$ so there are more terms in the above sum although two of them give 0 . This proves the proposition.

The following is a version of the area formula.

Lemma 29.12.4 Let $\mathbf{h} \in W^{1, p}\left(\mathbb{R}^{n} ; \mathbb{R}^{n}\right), p>n$ where $\mathbf{h}$ is one to one, $\mathbf{h}(\partial \Omega), \partial \Omega$ have measure zero for $\Omega$ a bounded open connected set in $\mathbb{R}^{n}$. Then $\mathbf{h}(\partial \Omega)^{C}$ has two components, three if $n=1$, and for $\mathbf{y} \in \mathbf{h}(\Omega)$, and $f \in C_{c}\left(\mathbb{R}^{n}\right)$.

$$
\int_{\mathbf{h}(\Omega)} f(\mathbf{y}) d y=\int_{\Omega}|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| f(\mathbf{h}(\mathbf{x})) d x
$$

If $O$ is an open set, it is also true that

$$
\int_{\mathbf{h}(\Omega)} \mathscr{X}_{O}(\mathbf{y}) d y=\int_{\Omega}|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| \mathscr{X}_{O}(\mathbf{h}(\mathbf{x})) d x
$$

Also if $f$ is any nonnegative Borel measurable function

$$
\int_{\mathbf{h}(\Omega)} f(\mathbf{y}) d y=\int_{\Omega}|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| f(\mathbf{h}(\mathbf{x})) d x
$$

Proof: Consider the first claim. Let $\delta$ be such that $\overline{B\left(\mathbf{x}_{1}, \delta\right)} \subseteq \Omega$ and let $\left\{f_{j}(\mathbf{y})\right\}_{j=1}^{\infty}$ be nonnegative, increasing in $j$ and converging pointwise to $\mathscr{X}_{\mathbf{h}\left(B\left(\mathbf{x}_{1}, \delta\right)\right)}(\mathbf{y})$. This can be done because $\mathbf{h}\left(B\left(\mathbf{x}_{1}, \boldsymbol{\delta}\right)\right)$ is an open bounded set thanks to invariance of domain, Theorem 23.4.3. By Proposition 29.12.3, $d(\mathbf{y}, \Omega, \mathbf{h})$ either equals 1 or -1 . Suppose it equals -1 . Then from Lemma 29.11.2

$$
\int_{\mathbf{h}(\Omega)} f_{j}(\mathbf{y}) d y=-\int_{\Omega} \operatorname{det}(D \mathbf{h}(\mathbf{x})) f_{j}(\mathbf{h}(\mathbf{x})) d x
$$

The integrand on the right is uniformly integrable thanks to the fact the $f_{j}$ are bounded and $\operatorname{det}(D \mathbf{h}(\mathbf{x}))$ is in $L^{r}(\Omega)$ for some $r>1$. Therefore, by the Vitali convergence theorem and the monotone convergence theorem,

$$
\int_{\mathbf{h}(\Omega)} \mathscr{X}_{\mathbf{h}\left(B\left(\mathbf{x}_{1}, \delta\right)\right)}(\mathbf{y}) d y=-\int_{\Omega} \operatorname{det}(D \mathbf{h}(\mathbf{x})) \mathscr{X}_{B\left(\mathbf{x}_{1}, \delta\right)}(\mathbf{x}) d x
$$

so

$$
m\left(\mathbf{h}\left(B\left(\mathbf{x}_{1}, \boldsymbol{\delta}\right)\right)\right) \frac{1}{m\left(B\left(\mathbf{x}_{1}, \boldsymbol{\delta}\right)\right)}=-\frac{1}{m\left(B\left(\mathbf{x}_{1}, \boldsymbol{\delta}\right)\right)} \int_{B\left(\mathbf{x}_{1}, \boldsymbol{\delta}\right)} \operatorname{det}(D \mathbf{h}(\mathbf{x})) d x
$$

If $\mathbf{x}_{1}$ is a Lebesgue point of $\operatorname{det}(D \mathbf{h}(\mathbf{x}))$, then you can pass to the limit as $\delta \rightarrow 0$ and conclude

$$
-\operatorname{det}\left(D \mathbf{h}\left(\mathbf{x}_{1}\right)\right) \geq 0
$$

Since a.e. point is a Lebesgue point, it follows that in the case where $d(\mathbf{y}, \Omega, \mathbf{h})=-1$,

$$
-\operatorname{det}(D \mathbf{h}(\mathbf{x}))=|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| \text { a.e. } \mathbf{x} \in \Omega
$$

The case where the degree equals 1 is similar. Thus $\operatorname{det}(D \mathbf{h}(\mathbf{x}))$ has the same sign on $\mathbf{h}(\Omega)$.

Now let $O$ be an open set. Then by invariance of domain, $\mathbf{h}(O)$ is also an open set. Let $V_{k}$ denote a decreasing sequence of open sets, $V_{k} \supseteq \overline{V_{k+1}}$ whose intersection is the compact
set $\mathbf{h}(\partial \Omega)$ such that $m\left(\overline{V_{k}}\right)<1 / k$. Then if $f \prec \mathbf{h}(O) \backslash \overline{V_{k}}$, it follows since $\mathbf{h}(O) \backslash \overline{V_{k}}$ is an open set which is at a positive distance from $\mathbf{h}(\partial \Omega)$, Lemma 29.11.2 implies

$$
\int_{\mathbf{h}(\Omega)} f(\mathbf{y}) d y=\int_{\Omega}|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| f(\mathbf{h}(\mathbf{x})) d x
$$

Taking a sequence of such $f$ increasing to $\mathscr{X}_{\mathbf{h}(O) \backslash \overline{V_{k}}}$, it follows from monotone convergence theorem in the above that

$$
\begin{aligned}
\int_{\mathbf{h}(\Omega)} \mathscr{X}_{\mathbf{h}(O) \backslash \overline{V_{k}}}(\mathbf{y}) d y & =\int_{\Omega}|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| \mathscr{X}_{\mathbf{h}(O) \backslash \overline{V_{k}}}(\mathbf{h}(\mathbf{x})) d x \\
& =\int_{\Omega}|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| \mathscr{X}_{O \backslash \mathbf{h}^{-1}\left(\overline{V_{k}}\right)}(\mathbf{x}) d x
\end{aligned}
$$

Now letting $k \rightarrow \infty$, it follows from the monotone convergence theorem that

$$
\int_{\mathbf{h}(\Omega)} \mathscr{X}_{\mathbf{h}(O) \backslash \mathbf{h}(\partial \Omega)}(\mathbf{y}) d y=\int_{\Omega}|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| \mathscr{X}_{O \backslash \partial \Omega}(\mathbf{x}) d x
$$

Since both $\partial \Omega$ and $\mathbf{h}(\partial \Omega)$ have measure zero, this implies

$$
\int_{\mathbf{h}(\Omega)} \mathscr{X}_{\mathbf{h}(O)}(\mathbf{y}) d y=\int_{\Omega}|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| \mathscr{X}_{O}(\mathbf{x}) d x
$$

Now let $\mathscr{G}$ denote the Borel sets $E$ with the property that

$$
\int_{\mathbf{h}(\Omega)} \mathscr{X}_{\mathbf{h}(E)}(\mathbf{y}) d y=\int_{\Omega}|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| \mathscr{X}_{E}(\mathbf{x}) d x
$$

It follows easily that if $E \in \mathscr{G}$ then so does $E^{C}$. This is because $\mathbf{h}(\Omega)$ has finite measure and $|\operatorname{det}(D \mathbf{h}(\mathbf{x}))|$ is in $L^{1}(\Omega)$. If $E_{i}$ is a sequence of disjoint sets of $\mathscr{G}$ then the monotone convergence theorem implies $\cup E_{i}$ is also in $\mathscr{G}$. It was shown above that the $\pi$ system of open sets is contained in $\mathscr{G}$. Therefore, it follows from the lemma on $\pi$ systems, Lemma 12.12 .3 on Page $329, \mathscr{G}$ equals the Borel sets. Now the desired result follows from approximating $f \geq 0$ and Borel measurable with a sequence of Borel simple functions which converge pointwise to $f$. This proves the theorem.

The following corollary follows right away by splitting $f$ into positive and negative parts of real and imaginary parts.

Corollary 29.12.5 Let $\mathbf{h}$ be one to one and in $W^{1, p}\left(\mathbb{R}^{n} ; \mathbb{R}^{n}\right), p>n$. Let $\Omega$ be a bounded, open, connected set in $\mathbb{R}^{n}$ and suppose $\partial \Omega, \mathbf{h}(\partial \Omega)$ have measure zero. Let $f \in L^{1}(\mathbf{h}(\Omega))$ where $f$ is also Borel measurable. Then

$$
\int_{\mathbf{h}(\Omega)} f(\mathbf{y}) d y=\int_{\Omega}|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| f(\mathbf{h}(\mathbf{x})) d x
$$

It can also be written in the form

$$
\int f(\mathbf{y}) d(\mathbf{y}, \Omega, \mathbf{h}) d y=\int_{\Omega} \operatorname{det}(D \mathbf{h}(\mathbf{x})) f(\mathbf{h}(\mathbf{x})) d x
$$

Note this is a general area formula under somewhat more restrictive hypotheses than the usual area formula because it involves an assumption that $\Omega$ is connected and a troublesome condition on the measure of $\mathbf{h}(\partial \Omega), \partial \Omega$ being zero, but it does not require $\mathbf{h}$ to be Lipschitz. It looks like a strange result because $|\operatorname{det} D \mathbf{h}(\mathbf{x})|$ is not in $L^{\infty}$ and so it is not clear why the integral on the right should even be finite just because $f$ is in $L^{1}$. If the result is correct, it is surprising.

The condition on the measure of $\partial \Omega$ and $\mathbf{h}(\partial \Omega)$ is not necessary. Neither is it necessary to assume $\Omega$ is connected. This is shown next.

Theorem 29.12.6 Let $\mathbf{h}$ be one to one on $\bar{\Omega}$ and in $W^{1, p}\left(\mathbb{R}^{n} ; \mathbb{R}^{n}\right), p>n$. Let $\Omega$ be a bounded, open set in $\mathbb{R}^{n}$. Let $f \in L^{1}(\mathbf{h}(\Omega))$ where $f$ is also Borel measurable. Then

$$
\int_{\mathbf{h}(\Omega)} f(\mathbf{y}) d y=\int_{\Omega}|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| f(\mathbf{h}(\mathbf{x})) d x
$$

It can also be written in the form

$$
\int f(\mathbf{y}) d(\mathbf{y}, \Omega, \mathbf{h}) d y=\int_{\Omega} \operatorname{det}(D \mathbf{h}(\mathbf{x})) f(\mathbf{h}(\mathbf{x})) d x
$$

Proof: Let $\bar{\Omega} \subseteq[-R, R]^{n} \equiv Q$. Let $p_{i}(\mathbf{x}) \equiv x_{i}$ where $\mathbf{x}=\left(x_{1}, \cdots, x_{n}\right)^{T}$. Then here is a claim.

Claim: Let $b$ be given. There exists $a,|a-b| \leq 4^{-k}$ such that

$$
m\left(\mathbf{h}\left(\left[p_{i} \mathbf{x}=a\right] \cap Q\right)\right)=0
$$

Here $\left[p_{i} \mathbf{x}=a\right]$ is short for $\left\{\mathbf{x}: p_{i} \mathbf{x}=a\right\}$.
Proof of claim: If this is not so, then for every $a$ in an interval centered at $b$,

$$
m\left(\mathbf{h}\left(\left[p_{i} \mathbf{x}=a\right] \cap Q\right)\right)>0
$$

However,

$$
m\left(\cup_{a \in\left[b-4^{-k}, b+4^{-k}\right]} \mathbf{h}\left(\left[p_{i} \mathbf{x}=a\right] \cap Q\right)\right)=m\left(\mathbf{h}\left(\prod_{j} A_{j}\right)\right)
$$

where $A_{j}=[-R, R]$ if $j \neq i$ and $A_{i}=\left[b-4^{-k}, b+4^{-k}\right]$. This is finite because $\mathbf{h}\left(\Pi_{j} A_{j}\right)$ is a compact set, being the continuous image of such a set. Since $\mathbf{h}$ is one to one, this compact set would then be the union of uncountably many disjoint sets, each having positive measure. Thus for some $1 / l>0$ there must be infinitely many of these disjoint sets having measure larger than $1 / l, l \in \mathbb{N}$ a contradiction to the set having finite measure. This proves the claim.

Now from the claim, consider $b_{k}, k \in \mathbb{Z}$ given by $b_{k}=k 2^{-(m+1)}$ and for each $i=1, \cdots, n$, let $a_{i k}$ denote a value of the claim, $\left|a_{i k}-b_{k}\right| \leq 4^{-m}$. Thus

$$
m\left(\mathbf{h}\left(\left[p_{i} \mathbf{x}=a_{i k}\right] \cap Q\right)\right)=0, i=1, \cdots, n
$$

Thus also

$$
\left|a_{i k}-a_{i(k+1)}\right| \leq\left|a_{i k}-b_{k}\right|+\left|b_{k+1}-a_{i(k+1)}\right|+\left|b_{k+1}-b_{k}\right|
$$

$$
\leq 4^{-m}+4^{-m}+2^{-(m+1)}<2^{-m}
$$

Consider boxes of the form $\prod_{i=1}^{n}\left[a_{i k}, a_{i(k+1)}\right]$. Denote these boxes as $\mathscr{B}_{m}$. Thus they are non overlapping boxes the sides of which are of length less than $2^{-m}$. Let $\Omega_{1}$ denote the union of the finitely many boxes of $\mathscr{B}_{1}$ which are contained in $\Omega$. Next let $\Omega_{2}$ denote the union of the boxes of $\mathscr{B}_{2} \cup \mathscr{B}_{1}$ which are contained in $\Omega$ and so forth. Then $\cup_{k=1}^{\infty} \Omega_{k} \subseteq \Omega$. Suppose now that $\mathbf{p} \in \Omega$. Then it is at positive distance from $\partial \Omega$. Let $k$ be the first such that $\mathbf{p}$ is contained in a box of $\mathscr{B}_{k}$ which is contained in $\Omega$. Then $\mathbf{p} \in \Omega_{k}$. Therefore, this has shown that $\Omega$ is a countable union of non overlapping closed boxes $B$ which have the property that $\partial B, \mathbf{h}(\partial B)$ have measure zero. Denote these boxes as $\left\{B_{k}\right\}$.

First assume $f$ is nonnegative and Borel measurable. Then from Corollary 29.12.5,

$$
\int_{\mathbf{h}\left(B_{k}\right)} f(\mathbf{y}) d y=\int_{B_{k}}|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| f(\mathbf{h}(\mathbf{x})) d x
$$

Since $\mathbf{h}\left(\partial B_{k}\right)$ has measure zero,

$$
\begin{aligned}
\int_{\mathbf{h}\left(\cup_{k=1}^{m} B_{k}\right)} f(\mathbf{y}) d y & =\sum_{k=1}^{m} \int_{\mathbf{h}\left(B_{k}\right)} f(\mathbf{y}) d y \\
& =\sum_{k=1}^{m} \int_{B_{k}}|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| f(\mathbf{h}(\mathbf{x})) d x \\
& =\int_{\cup_{k=1}^{m} B_{k}}|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| f(\mathbf{h}(\mathbf{x})) d x
\end{aligned}
$$

and now letting $m \rightarrow \infty$ and using the monotone convergence theorem,

$$
\begin{equation*}
\int_{\mathbf{h}(\Omega)} f(\mathbf{y}) d y=\int_{\Omega}|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| f(\mathbf{h}(\mathbf{x})) d x \tag{29.12.75}
\end{equation*}
$$

Next assume in addition that $f$ is also in $L^{1}(\mathbf{h}(\Omega))$. Recall that from properties of the degree, $d(\mathbf{y}, U, \mathbf{h})$ is constant on $\mathbf{h}(U)$ for $U$ a component of $\Omega$. Since $\mathbf{h}$ is one to one, Proposition 23.6.4 implies this constant is either -1 or 1 . Let the components of $\Omega$ be $\left\{U_{i}\right\}_{i=1}^{\infty}$. Also from Theorem 23.2.2 and the assumption that $\mathbf{h}$ is one to one, if $\mathbf{y} \in \mathbf{h}\left(U_{i}\right)$, then $\mathbf{y} \notin \mathbf{h}\left(\cup_{j \neq i} U_{j}\right)$ and

$$
d\left(\mathbf{y}, U_{i}, \mathbf{h}\right)+d\left(\mathbf{y}, \cup_{j \neq i} U_{j}, \mathbf{h}\right)=d(\mathbf{y}, \Omega, \mathbf{h})
$$

Since $\mathbf{y} \notin \mathbf{h}\left(\cup_{j \neq i} U_{j}\right)$ the second term on the left is 0 and so $d\left(\mathbf{y}, U_{i}, \mathbf{h}\right)=d(\mathbf{y}, \Omega, \mathbf{h})$. Therefore, by Corollary 29.12.5,

$$
\begin{gather*}
\int_{\mathbf{h}(\Omega)} f(\mathbf{y}) d(\mathbf{y}, \Omega, \mathbf{h}) d y=\sum_{i=1}^{\infty} \int_{\mathbf{h}\left(U_{i}\right)} f(\mathbf{y}) d(\mathbf{y}, \Omega, \mathbf{h}) d y \\
=\sum_{i=1}^{\infty} \int_{\mathbf{h}\left(U_{i}\right)} f(\mathbf{y}) d\left(\mathbf{y}, U_{i}, \mathbf{h}\right) d y=\sum_{i=1}^{\infty} \int \mathscr{X}_{U_{i}}(\mathbf{x}) f(\mathbf{h}(\mathbf{x})) \operatorname{det}(D \mathbf{h}(\mathbf{x})) d x \tag{29.12.76}
\end{gather*}
$$

From 29.12.75

$$
\begin{aligned}
& \sum_{i=1}^{\infty} \int \mathscr{X}_{U_{i}}(\mathbf{x}) f(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d x \\
= & \int \sum_{i=1}^{\infty} \mathscr{X}_{U_{i}}(\mathbf{x}) f(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d x \\
= & \int \mathscr{X}_{\Omega} f(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d x<\infty
\end{aligned}
$$

and so by Fubini's theorem, the sum and the integral may be interchanged in 29.12.76 to obtain from the dominated convergence theorem,

$$
\begin{aligned}
& \int \sum_{i=1}^{\infty} \mathscr{X}_{U_{i}}(\mathbf{x}) f(\mathbf{h}(\mathbf{x})) \operatorname{det}(D \mathbf{h}(\mathbf{x})) d x \\
= & \int_{\Omega} f(\mathbf{h}(\mathbf{x})) \operatorname{det}(D \mathbf{h}(\mathbf{x})) d x
\end{aligned}
$$

which shows

$$
\begin{equation*}
\int_{\mathbf{h}(\Omega)} f(\mathbf{y}) d(\mathbf{y}, \Omega, \mathbf{h}) d y=\int_{\Omega} f(\mathbf{h}(\mathbf{x})) \operatorname{det}(D \mathbf{h}(\mathbf{x})) d x \tag{29.12.77}
\end{equation*}
$$

Now if $f$ is Borel measurable and in $L^{1}(\Omega)$, the above may be applied to the positive parts of the real and imaginary parts of $f$ to obtain 29.12.77 for such $f$. This proves the theorem.

Not surprisingly, it is not necessary to assume $f$ is Borel measurable.
Corollary 29.12.7 Let $\mathbf{h}$ be one to one on $\bar{\Omega}$ and in $W^{1, p}\left(\mathbb{R}^{n} ; \mathbb{R}^{n}\right), p>n$. Let $\Omega$ be a bounded, open set in $\mathbb{R}^{n}$. Let $f \in L^{1}(\mathbf{h}(\Omega))$ where $f$ is Lebesgue measurable. Then $\mathbf{x} \rightarrow|\operatorname{det}(\operatorname{Dh}(\mathbf{x}))| f(\mathbf{h}(\mathbf{x}))$ is Lebesgue measurable and

$$
\begin{equation*}
\int_{\mathbf{h}(\Omega)} f(\mathbf{y}) d y=\int_{\Omega}|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| f(\mathbf{h}(\mathbf{x})) d x \tag{29.12.78}
\end{equation*}
$$

It can also be written in the form

$$
\begin{equation*}
\int f(\mathbf{y}) d(\mathbf{y}, \Omega, \mathbf{h}) d y=\int_{\Omega} \operatorname{det}(D \mathbf{h}(\mathbf{x})) f(\mathbf{h}(\mathbf{x})) d x \tag{29.12.79}
\end{equation*}
$$

Proof: Let $E$ be a Lebesgue measurable subset of $\mathbf{h}(\Omega)$. By regularity of the measure, there exist Borel sets $F \subseteq E \subseteq G$ such that $F$ and $G$ are both Borel measurable sets contained in $\mathbf{h}(\Omega)$ with $m(G \backslash F)=0$. Then by Theorem 29.12.6

$$
\begin{align*}
\int_{\Omega}|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| \mathscr{X}_{F}(\mathbf{h}(\mathbf{x})) d x & =\int_{\mathbf{h}(\Omega)} \mathscr{X}_{F}(\mathbf{y}) d y \\
& =\int_{\mathbf{h}(\Omega)} \mathscr{X}_{E}(\mathbf{y}) d y=\int_{\mathbf{h}(\Omega)} \mathscr{X}_{G}(\mathbf{y}) d y \\
& =\int_{\Omega}|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| \mathscr{X}_{G}(\mathbf{h}(\mathbf{x})) d x \tag{29.12.80}
\end{align*}
$$

which shows that

$$
\begin{aligned}
|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| \mathscr{X}_{F}(\mathbf{h}(\mathbf{x})) & =|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| \mathscr{X}_{G}(\mathbf{h}(\mathbf{x})) \\
& =|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| \mathscr{X}_{E}(\mathbf{h}(\mathbf{x}))
\end{aligned}
$$

a.e. and so, by completeness, it follows $\mathbf{x} \rightarrow|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| \mathscr{X}_{E}(\mathbf{h}(\mathbf{x}))$ must be Lebesgue measurable. This is because the function $\mathbf{x} \rightarrow|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| \mathscr{X}_{G}(\mathbf{h}(\mathbf{x}))$ is Borel measurable due to the continuity of $\mathbf{h}$ which forces $\mathbf{x} \rightarrow \operatorname{det}(D \mathbf{h}(\mathbf{x}))$ to be Borel measurable, and the other function in the product is of the form $\mathscr{X}_{\mathbf{h}^{-1}(G)}(\mathbf{x})$ and since $G$ is Borel, so is $\mathbf{h}^{-1}(G)$. Now the desired result follows because

$$
\int_{\Omega}|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| \mathscr{X}_{E}(\mathbf{h}(\mathbf{x})) d x
$$

is between the ends of 29.12 .80 . The rest of the argument involves the usual technique of approximating a nonnegative function with an increasing sequence of simple functions followed by consideration of the positive and negative parts of the real and imaginary parts of an arbitrary function in $L^{1}(\Omega)$. The other version of the formula follows as in the proof of Theorem 29.12.6. This proves the corollary.

## Chapter 30

## Integration Of Differential Forms

### 30.1 Manifolds

Manifolds are sets which resemble $\mathbb{R}^{n}$ locally. To make the concept of a manifold more precise, here is a definition.

Definition 30.1.1 Let $\Omega \subseteq \mathbb{R}^{m}$. A set, $U$, is open in $\Omega$ if it is the intersection of an open set from $\mathbb{R}^{m}$ with $\Omega$. Equivalently, a set, $U$ is open in $\Omega$ if for every point, $\mathbf{x} \in U$, there exists $\delta>0$ such that if $|\mathbf{x}-\mathbf{y}|<\delta$ and $\mathbf{y} \in \Omega$, then $\mathbf{y} \in U$. A set, $H$, is closed in $\Omega$ if it is the intersection of a closed set from $\mathbb{R}^{m}$ with $\Omega$. Equivalently, a set, $H$, is closed in $\Omega$ if whenever, $\mathbf{y}$ is a limit point of $H$ and $\mathbf{y} \in \Omega$, it follows $\mathbf{y} \in H$.

Recall the following definition.
Definition 30.1.2 Let $V \subseteq \mathbb{R}^{n} . C^{k}\left(\bar{V} ; \mathbb{R}^{m}\right)$ is the set of functions which are restrictions to $V$ of some function defined on $\mathbb{R}^{n}$ which has $k$ continuous derivatives and compact support. When $k=0$, it means the restriction to $V$ of continuous functions with compact support.

Definition 30.1.3 A closed and bounded subset of $\mathbb{R}^{m}, \Omega$, will be called an $n$ dimensional manifold with boundary, $n \geq 1$, if there are finitely many sets, $U_{i}$, open in $\Omega$ and continuous one to one functions, $\mathbf{R}_{i} \in C^{0}\left(\overline{U_{i}}, \mathbb{R}^{n}\right)$ such that $\mathbf{R}_{i} U_{i}$ is relatively open in $\mathbb{R}_{\leq}^{n} \equiv\left\{\mathbf{u} \in \mathbb{R}^{n}: u_{1} \leq 0\right\}, \mathbf{R}_{i}^{-1}$ is continuous. These mappings, $\mathbf{R}_{i}$, together with their domains, $U_{i}$, are called charts and the totality of all the charts, $\left(U_{i}, \mathbf{R}_{i}\right)$ just described is called an atlas for the manifold. Define

$$
\operatorname{int}(\Omega) \equiv\left\{\mathbf{x} \in \Omega: \text { for some } i, \mathbf{R}_{i} \mathbf{x} \in \mathbb{R}_{<}^{n}\right\}
$$

where $\mathbb{R}_{<}^{n} \equiv\left\{\mathbf{u} \in \mathbb{R}^{n}: u_{1}<0\right\}$. Also define

$$
\partial \Omega \equiv\left\{\mathbf{x} \in \Omega: \text { for some } i, \mathbf{R}_{i} \mathbf{x} \in \mathbb{R}_{0}^{n}\right\}
$$

where

$$
\mathbb{R}_{0}^{n} \equiv\left\{\mathbf{u} \in \mathbb{R}^{n}: u_{1}=0\right\}
$$

and $\partial \Omega$ is called the boundary of $\Omega$. Note that if $n=1, \mathbb{R}_{0}^{n}$ is just the single point 0 . By convention, we will consider the boundary of such a 0 dimensional manifold to be empty.

This definition is a little too restrictive. In general the collection of sets, $U_{i}$ is not finite. However, in the case where $\Omega$ is closed and bounded, compactness of $\Omega$ can be used to get a finite covering and since this is the case of most interest here, the assumption that the collection of sets, $U_{i}$, is finite is made. However, most of what is presented here can be generalized to the case of a locally finite atlas.

Theorem 30.1.4 Let $\partial \Omega$ and $\operatorname{int}(\Omega)$ be as defined above. Then $\operatorname{int}(\Omega)$ is open in $\Omega$ and $\partial \Omega$ is closed in $\Omega$. Furthermore, $\partial \Omega \cap \operatorname{int}(\Omega)=\emptyset, \Omega=\partial \Omega \cup \operatorname{int}(\Omega)$, and for $n \geq 2, \partial \Omega$ is an $n-1$ dimensional manifold for which $\partial(\partial \Omega)=\emptyset$. The property of being in int $(\Omega)$ or $\partial \Omega$ does not depend on the choice of atlas.

Proof: It is clear that $\Omega=\partial \Omega \cup \operatorname{int}(\Omega)$. First consider the claim that $\partial \Omega \cap \operatorname{int}(\Omega)=\emptyset$. Suppose this does not happen. Then there would exist $\mathbf{x} \in \partial \Omega \cap \operatorname{int}(\Omega)$. Therefore, there would exist two mappings $\mathbf{R}_{i}$ and $\mathbf{R}_{j}$ such that $\mathbf{R}_{j} \mathbf{x} \in \mathbb{R}_{0}^{n}$ and $\mathbf{R}_{i} \mathbf{x} \in \mathbb{R}_{<}^{n}$ with $\mathbf{x} \in U_{i} \cap U_{j}$. Now consider the map, $\mathbf{R}_{j} \circ \mathbf{R}_{i}^{-1}$, a continuous one to one map from $\mathbb{R}_{\leq}^{n}$ to $\mathbb{R}_{\leq}^{n}$ having a continuous inverse. By continuity, there exists $r>0$ small enough that,

$$
\mathbf{R}_{i}^{-1} B\left(\mathbf{R}_{i} \mathbf{x}, r\right) \subseteq U_{i} \cap U_{j}
$$

Therefore, $\mathbf{R}_{j} \circ \mathbf{R}_{i}^{-1}\left(B\left(\mathbf{R}_{i} \mathbf{x}, r\right)\right) \subseteq \mathbb{R}_{\leq}^{n}$ and contains a point on $\mathbb{R}_{0}^{n}, \mathbf{R}_{j} \mathbf{x}$. However, this cannot occur because it contradicts the theorem on invariance of domain, Theorem 23.4.3, which requires that $\mathbf{R}_{j} \circ \mathbf{R}_{i}^{-1}\left(B\left(\mathbf{R}_{i} \mathbf{x}, r\right)\right)$ must be an open subset of $\mathbb{R}^{n}$ and this one isn't because of the point on $\mathbb{R}_{0}^{n}$. Therefore, $\partial \Omega \cap \operatorname{int}(\Omega)=\emptyset$ as claimed. This same argument shows that the property of being in $\operatorname{int}(\Omega)$ or $\partial \Omega$ does not depend on the choice of the atlas.

To verify that $\partial(\partial \Omega)=\emptyset$, let $\mathbf{S}_{i}$ be the restriction of $\mathbf{R}_{i}$ to $\partial \Omega \cap U_{i}$. Thus

$$
\mathbf{S}_{i}(\mathbf{x})=\left(0,\left(\mathbf{R}_{i} \mathbf{x}\right)_{2}, \cdots,\left(\mathbf{R}_{i} \mathbf{x}\right)_{n}\right)
$$

and the collection of such points for $\mathbf{x} \in \partial \Omega \cap U_{i}$ is an open bounded subset of

$$
\left\{\mathbf{u} \in \mathbb{R}^{n}: u_{1}=0\right\}
$$

identified with $\mathbb{R}^{n-1}$. $\mathbf{S}_{i}\left(\partial \Omega \cap U_{i}\right)$ is bounded because $\mathbf{S}_{i}$ is the restriction of a continuous function defined on $\mathbb{R}^{m}$ and $\partial \Omega \cap U_{i} \equiv V_{i}$ is contained in the compact set $\Omega$. Thus if $\mathbf{S}_{i}$ is modified slightly, to be of the form

$$
\mathbf{S}_{i}^{\prime}(\mathbf{x})=\left(\left(\mathbf{R}_{i} \mathbf{x}\right)_{2}-k_{i}, \cdots,\left(\mathbf{R}_{i} \mathbf{x}\right)_{n}\right)
$$

where $k_{i}$ is chosen sufficiently large enough that $\left(\mathbf{R}_{i}\left(V_{i}\right)\right)_{2}-k_{i}<0$, it follows that $\left\{\left(V_{i}, \mathbf{S}_{i}^{\prime}\right)\right\}$ is an atlas for $\partial \Omega$ as an $n-1$ dimensional manifold such that every point of $\partial \Omega$ is sent to to $\mathbb{R}_{<}^{n-1}$ and none gets sent to $\mathbb{R}_{0}^{n-1}$. It follows $\partial \Omega$ is an $n-1$ dimensional manifold with empty boundary. In case $n=1$, the result follows by definition of the boundary of a 0 dimensional manifold.

Next consider the claim that $\operatorname{int}(\Omega)$ is open in $\Omega$. If $\mathbf{x} \in \operatorname{int}(\Omega)$, are all points of $\Omega$ which are sufficiently close to $\mathbf{x}$ also in $\operatorname{int}(\Omega)$ ? If this were not true, there would exist $\left\{\mathbf{x}_{n}\right\}$ such that $\mathbf{x}_{n} \in \partial \Omega$ and $\mathbf{x}_{n} \rightarrow \mathbf{x}$. Since there are only finitely many charts of interest, this would imply the existence of a subsequence, still denoted by $\mathbf{x}_{n}$ and a single map, $\mathbf{R}_{i}$ such that $\mathbf{R}_{i}\left(\mathbf{x}_{n}\right) \in \mathbb{R}_{0}^{n}$. But then $\mathbf{R}_{i}\left(\mathbf{x}_{n}\right) \rightarrow \mathbf{R}_{i}(\mathbf{x})$ and so $\mathbf{R}_{i}(\mathbf{x}) \in \mathbb{R}_{0}^{n}$ showing $\mathbf{x} \in \partial \Omega$, a contradiction to $\operatorname{int}(\Omega) \cap \partial \Omega=\emptyset$. Now it follows that $\partial \Omega$ is closed in $\Omega$ because $\partial \Omega=\Omega \backslash \operatorname{int}(\Omega)$. This proves the Theorem.

Definition 30.1.5 An $n$ dimensional manifold with boundary, $\Omega$ is a $C^{k}$ manifold with boundary for some $k \geq 1$ if

$$
\mathbf{R}_{j} \circ \mathbf{R}_{i}^{-1} \in C^{k}\left(\overline{\mathbf{R}_{i}\left(U_{i} \cap U_{j}\right)} ; \mathbb{R}^{n}\right)
$$

and $\mathbf{R}_{i}^{-1} \in C^{k}\left(\overline{\mathbf{R}_{i} U_{i}} ; \mathbb{R}^{m}\right)$. It is called a continuous or Lipschitz manifold with boundary if the mappings, $\mathbf{R}_{j} \circ \mathbf{R}_{i}^{-1}, \mathbf{R}_{i}^{-1}, \mathbf{R}_{i}$ are respectively continuous or Lipschitz continuous. In
the case where $\Omega$ is a $C^{k}, k \geq 1$ manifold, it is called orientable if in addition to this there exists an atlas, $\left(U_{r}, \mathbf{R}_{r}\right)$, such that whenever $U_{i} \cap U_{j} \neq \emptyset$,

$$
\begin{equation*}
\operatorname{det}\left(D\left(\mathbf{R}_{j} \circ \mathbf{R}_{i}^{-1}\right)\right)(\mathbf{u})>0 \text { for all } \mathbf{u} \in \mathbf{R}_{i}\left(U_{i} \cap U_{j}\right) \tag{30.1.1}
\end{equation*}
$$

The mappings, $\mathbf{R}_{i} \circ \mathbf{R}_{j}^{-1}$ are called the overlap maps. In the case where $k=0$, the $\mathbf{R}_{i}$ are only assumed continuous so there is no differentiability available and in this case, the manifold is oriented if whenever $A$ is an open connected subset of $\operatorname{int}\left(\mathbf{R}_{i}\left(U_{i} \cap U_{j}\right)\right)$ whose boundary has measure zero and separates $\mathbb{R}^{n}$ into two components,

$$
\begin{equation*}
d\left(\mathbf{y}, A, \mathbf{R}_{j} \circ \mathbf{R}_{i}^{-1}\right) \in\{1,0\} \tag{30.1.2}
\end{equation*}
$$

depending on whether $\mathbf{y} \in \mathbf{R}_{j} \circ \mathbf{R}_{i}^{-1}(A)$. An atlas satisfying 30.1.1 or more generally 30.1.2 is called an oriented atlas.

It follows from Proposition 23.6.4 the degree in 30.1.2 is either undefined if $\mathbf{y} \in \mathbf{R}_{j} \circ$ $\mathbf{R}_{i}^{-1} \partial A$ or it is $1,-1$, or 0 .

The study of manifolds is really a generalization of something with which everyone who has taken a normal calculus course is familiar. We think of a point in three dimensional space in two ways. There is a geometric point and there are coordinates associated with this point. There are many different coordinate systems which describe a point. There are spherical coordinates, cylindrical coordinates and rectangular coordinates to name the three most popular coordinate systems. These coordinates are like the vector $\mathbf{u}$. The point, $\mathbf{x}$ is like the geometric point although it is always assumed $\mathbf{x}$ has rectangular coordinates in $\mathbb{R}^{m}$ for some $m$. Under fairly general conditions, it can be shown there is no loss of generality in making such an assumption. Next is some algebra.

### 30.2 The Binet Cauchy Formula

The Binet Cauchy formula is a generalization of the theorem which says the determinant of a product is the product of the determinants. The situation is illustrated in the following picture.


Theorem 30.2.1 Let $A$ be an $n \times m$ matrix with $n \geq m$ and let $B$ be a $m \times n$ matrix. Also let $A_{i}$

$$
i=1, \cdots, C(n, m)
$$

be the $m \times m$ submatrices of $A$ which are obtained by deleting $n-m$ rows and let $B_{i}$ be the $m \times m$ submatrices of $B$ which are obtained by deleting corresponding $n-m$ columns. Then

$$
\operatorname{det}(B A)=\sum_{k=1}^{C(n, m)} \operatorname{det}\left(B_{k}\right) \operatorname{det}\left(A_{k}\right)
$$

Proof: This follows from a computation. By Corollary 5.4.5 on Page 71, $\operatorname{det}(B A)=$

$$
\begin{gathered}
\frac{1}{m!} \sum_{\left(i_{1} \cdots i_{m}\right)} \sum_{\left(j_{1} \cdots j_{m}\right)} \operatorname{sgn}\left(i_{1} \cdots i_{m}\right) \operatorname{sgn}\left(j_{1} \cdots j_{m}\right)(B A)_{i_{1} j_{1}}(B A)_{i_{2} j_{2}} \cdots(B A)_{i_{m} j_{m}} \\
\frac{1}{m!} \sum_{\left(i_{1} \cdots i_{m}\right)} \sum_{\left(j_{1} \cdots j_{m}\right)} \operatorname{sgn}\left(i_{1} \cdots i_{m}\right) \operatorname{sgn}\left(j_{1} \cdots j_{m}\right) \\
\sum_{r_{1}=1}^{n} B_{i_{1} r_{1}} A_{r_{1} j_{1}} \sum_{r_{2}=1}^{n} B_{i_{2} r_{2}} A_{r_{2} j_{2}} \cdots \sum_{r_{m}=1}^{n} B_{i_{m} r_{m}} A_{r_{m} j_{m}}
\end{gathered}
$$

Now denote by $I_{k}$ one subsets of $\{1, \cdots, n\}$ having $m$ elements. Thus there are $C(n, m)$ of these. Then the above equals

$$
\begin{aligned}
= & \sum_{k=1}^{C(n, m)} \sum_{\left\{r_{1}, \cdots, r_{m}\right\}=I_{k}} \frac{1}{m!} \sum_{\left(i_{1} \cdots i_{m}\right)} \sum_{\left(j_{1} \cdots j_{m}\right)} \operatorname{sgn}\left(i_{1} \cdots i_{m}\right) \operatorname{sgn}\left(j_{1} \cdots j_{m}\right) . \\
& B_{i_{1} r_{1}} A_{r_{1} j_{1} B_{1}} B_{i_{2} r_{2}} A_{r_{2} j_{2}} \cdots B_{i_{m} r_{m}} A_{r_{m} j_{m}} \\
& \sum_{k=1}^{C(n, m)} \sum_{\left\{r_{1}, \cdots, r_{m}\right\}=I_{k}} \frac{1}{m!} \sum_{\left(i_{1} \cdots i_{m}\right)} \operatorname{sgn}\left(i_{1} \cdots i_{m}\right) B_{i_{1} r_{1}} B_{i_{2} r_{2}} \cdots B_{i_{m} r_{m}} \\
& \sum_{\left(j_{1} \cdots j_{m}\right)} \operatorname{sgn}\left(j_{1} \cdots j_{m}\right) A_{r_{1} j_{1}} A_{r_{2} j_{2}} \cdots A_{r_{m} j_{m}} \\
= & \sum_{k=1}^{C(n, m)} \sum_{\left\{r_{1}, \cdots, r_{m}\right\}=I_{k}} \frac{1}{m!} \operatorname{sgn}\left(r_{1} \cdots r_{m}\right)^{2} \operatorname{det}\left(B_{k}\right) \operatorname{det}\left(A_{k}\right) \\
= & \sum_{k=1}^{C(n, m)} \operatorname{det}\left(B_{k}\right) \operatorname{det}\left(A_{k}\right)
\end{aligned}
$$

since there are $m$ ! ways of arranging the indices $\left\{r_{1}, \cdots, r_{m}\right\}$.

### 30.3 Integration Of Differential Forms On Manifolds

This section presents the integration of differential forms on manifolds. This topic is a higher dimensional version of what is done in calculus in finding the work done by a force field on an object which moves over some path. There you evaluated line integrals. Differential forms are just a higher dimensional version of this idea and it turns out they are what it makes sense to integrate on manifolds. The following lemma, on Page 429 used in establishing the definition of the degree and in giving a proof of the Brouwer fixed point theorem is also a fundamental result in discussing the integration of differential forms.

Lemma 30.3.1 Let $\mathbf{g}: U \rightarrow V$ be $C^{2}$ where $U$ and $V$ are open subsets of $\mathbb{R}^{n}$. Then

$$
\sum_{j=1}^{n}(\operatorname{cof}(D \mathbf{g}))_{i j, j}=0
$$

where here $(D \mathbf{g})_{i j} \equiv g_{i, j} \equiv \frac{\partial g_{i}}{\partial x_{j}}$.
Also recall the interesting relation of the degree to integration in Corollary 29.11.3
Corollary 30.3.2 Let $f \in L_{\text {loc }}^{p}\left(\mathbb{R}^{n}\right)$ for $p \geq 1$ and let $\mathbf{h}$ be Lipschitz where $\partial U$ has measure zero for $U$ a bounded open set and $\mathbf{h}(\partial U)^{C}$ has finitely many components. Then everything is measurable which needs to be and

$$
\int f(\mathbf{y}) d(\mathbf{y}, U, \mathbf{h}) d y=\int_{U} \operatorname{det} D \mathbf{h}(\mathbf{x}) f(\mathbf{h}(\mathbf{x})) d x
$$

(Recall that if $\mathbf{y} \notin \mathbf{h}(U)$, then $d(\mathbf{y}, U, \mathbf{h})=0$.)
Recall Proposition 23.6.4.
Proposition 30.3.3 Let $\Omega$ be an open connected bounded set in $\mathbb{R}^{n}$ such that $\mathbb{R}^{n} \backslash \partial \Omega$ consists of two, three if $n=1$, connected components. Let $\mathbf{f} \in C\left(\bar{\Omega} ; \mathbb{R}^{n}\right)$ be continuous and one to one. Then $\mathbf{f}(\Omega)$ is the bounded component of $\mathbb{R}^{n} \backslash \mathbf{f}(\partial \Omega)$ and for $\mathbf{y} \in \mathbf{f}(\Omega)$, $d(\mathbf{f}, \Omega, \mathbf{y})$ either equals 1 or -1 .

Also recall the following fundamental lemma on partitions of unity in Corollary 15.5.9.
Lemma 30.3.4 Let $K$ be a compact set in $\mathbb{R}^{n}$ and let $\left\{U_{i}\right\}_{i=1}^{\infty}$ be an open cover of $K$. Then there exist functions, $\psi_{k} \in C_{c}^{\infty}\left(U_{i}\right)$ such that $\psi_{i} \prec U_{i}$ and for all $\mathbf{x} \in K$,

$$
\sum_{i=1}^{\infty} \psi_{i}(\mathbf{x})=1
$$

If $K$ is a compact subset of $U_{1}\left(U_{i}\right)$ there exist such functions such that also $\psi_{1}(\mathbf{x})=1$ ( $\psi_{i}(\mathbf{x})=1$ ) for all $\mathbf{x} \in K$.

With the above, what follows is the definition of what a differential form is and how to integrate one.

Definition 30.3.5 Let I denote an ordered list of $n$ indices taken from the set, $\{1, \cdots, m\}$. Thus $I=\left(i_{1}, \cdots, i_{n}\right)$. It is an ordered list because the order matters. A differential form of order $n$ in $\mathbb{R}^{m}$ is a formal expression,

$$
\omega=\sum_{I} a_{I}(\mathbf{x}) d \mathbf{x}^{I}
$$

where $a_{I}$ is at least Borel measurable $d \mathbf{x}^{I}$ is short for the expression

$$
d x^{i_{1}} \wedge \cdots \wedge d x^{i_{n}}
$$

and the sum is taken over all ordered lists of indices taken from the set, $\{1, \cdots, m\}$. For $\Omega$ an orientable $n$ dimensional manifold with boundary, define

$$
\begin{equation*}
\int_{\Omega} \omega \tag{30.3.3}
\end{equation*}
$$

according to the following procedure in which it is assumed the integrals which occur make sense. Let $\left\{\left(U_{i}, \mathbf{R}_{i}\right)\right\}$ be an oriented atlas for $\Omega$. Each $U_{i}$ is the intersection of an open set in $\mathbb{R}^{m}, O_{i}$, with $\Omega$ and so there exists a $C^{\infty}$ partition of unity subordinate to the open cover, $\left\{O_{i}\right\}$ which sums to 1 on $\Omega$. Thus $\psi_{i} \in C_{c}^{\infty}\left(O_{i}\right)$, has values in $[0,1]$ and satisfies $\sum_{i} \psi_{i}(\mathbf{x})=1$ for all $\mathbf{x} \in \Omega$. Then define 30.3.3 by

$$
\begin{equation*}
\int_{\Omega} \omega \equiv \sum_{i=1}^{p} \sum_{I} \int_{\mathbf{R}_{i} U_{i}} \psi_{i}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) a_{I}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) \frac{\partial\left(x_{i_{1}} \cdots x_{i_{n}}\right)}{\partial\left(u_{1} \cdots u_{n}\right)} d u \tag{30.3.4}
\end{equation*}
$$

where that symbol at the end denotes

$$
\operatorname{det}\left(\begin{array}{cccc}
x_{i_{1}, u_{1}} & x_{i_{1}, u_{2}} & \cdots & x_{i_{1}, u_{n}} \\
x_{i_{2}, u_{1}} & x_{i_{2}, u_{2}} & \cdots & x_{i_{2}, u_{2}} \\
\vdots & \vdots & \ddots & \vdots \\
x_{i_{n}, u_{1}} & x_{i_{n}, u_{2}} & \cdots & x_{i_{n}, u_{n}}
\end{array}\right)(\mathbf{u})
$$

for $\left(x_{1}, x_{2}, \cdots, x_{n}\right)=\mathbf{R}_{i}^{-1}(\mathbf{u})$.
Of course there are all sorts of questions related to whether this definition is well defined. The formula 30.3.3 makes no mention of partitions of unity or a particular atlas. What if you had a different atlas and a different partition of unity? Would $\int_{\Omega} \omega$ change? In general, the answer is yes. However, there is a sense in which 30.3 .3 is well defined. This involves the concept of orientation. This looks a lot like the concept of an oriented manifold.

Definition 30.3.6 Suppose $\Omega$ is an $n$ dimensional orientable manifold with boundary and let $\left(U_{i}, \mathbf{R}_{i}\right)$ and $\left(V_{i}, \mathbf{S}_{i}\right)$ be two oriented atlass of $\Omega$. They have the same orientation if for all open connected sets $A \subseteq \mathbf{S}_{j}\left(V_{j} \cap U_{i}\right)$ with $\partial A$ having measure zero and separating $\mathbb{R}^{n}$ into two components,

$$
d\left(\mathbf{u}, \mathbf{R}_{i} \circ \mathbf{S}_{j}^{-1}, A\right) \in\{0,1\}
$$

depending on whether $\mathbf{u} \in \mathbf{R}_{i} \circ \mathbf{S}_{j}^{-1}(A)$.
The above definition of $\int_{\Omega} \omega$ is well defined in the sense that any two atlass which have the same orientation deliver the same value for this symbol.

Theorem 30.3.7 Suppose $\Omega$ is an $n$ dimensional Lipschitz orientable manifold with boundary and let $\left(U_{i}, \mathbf{R}_{i}\right)$ and $\left(V_{i}, \mathbf{S}_{i}\right)$ be two oriented atlass of $\Omega$. Suppose the two atlass have the same orientation. Then if $\int_{\Omega} \omega$ is computed with respect to the two atlass the same number is obtained.

Proof: Let $\left\{\psi_{i}\right\}$ be a partition of unity as described in Lemma 30.3.4 which is associated with the atlas $\left(U_{i}, \mathbf{R}_{i}\right)$ and let $\left\{\eta_{i}\right\}$ be a partition of unity associated in the same manner with the atlas $\left(V_{i}, \mathbf{S}_{i}\right)$. First note the following.

$$
\begin{gather*}
\sum_{I} \int_{\mathbf{R}_{i} U_{i}} \psi_{i}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) a_{I}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) \frac{\partial\left(x_{i_{1}} \cdots x_{i_{n}}\right)}{\partial\left(u_{1} \cdots u_{n}\right)} d u  \tag{30.3.5}\\
=\sum_{j=1}^{q} \sum_{I} \int_{\mathbf{R}_{i}\left(U_{i} \cap V_{j}\right)} \eta_{j}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) \psi_{i}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) a_{I}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) \frac{\partial\left(x_{i_{1}} \cdots x_{i_{n}}\right)}{\partial\left(u_{1} \cdots u_{n}\right)} d u \\
=\sum_{j=1}^{q} \sum_{I} \int_{\operatorname{int} \mathbf{R}_{i}\left(U_{i} \cap V_{j}\right)} \eta_{j}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) \psi_{i}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) a_{I}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) \frac{\partial\left(x_{i_{1}} \cdots x_{i_{n}}\right)}{\partial\left(u_{1} \cdots u_{n}\right)} d u
\end{gather*}
$$

The reason this can be done is that points not on the interior of $\mathbf{R}_{i}\left(U_{i} \cap V_{j}\right)$ are on the plane $u_{1}=0$ which is a set of measure zero.

Now let $A$ be an open connected set contained in $\mathbf{S}_{j}\left(U_{i} \cap V_{j}\right)$ whose boundary $\partial A$ separates $\mathbb{R}^{n}$ into two components. Then by assumption,

$$
\begin{align*}
& \int_{\mathbf{R}_{i} \mathbf{S}_{j}^{-1}(A)} \eta_{j}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) \psi_{i}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) a_{I}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) \frac{\partial\left(x_{i_{1}} \cdots x_{i_{n}}\right)}{\partial\left(u_{1} \cdots u_{n}\right)} d u  \tag{30.3.6}\\
&= \int_{\mathbf{R}_{i} \mathbf{S}_{j}^{-1}(A)} \eta_{j}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) \psi_{i}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) a_{I}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) \\
& \cdot \frac{\partial\left(x_{\left.i_{1} \cdots x_{i_{n}}\right)}^{\partial\left(u_{1} \cdots u_{n}\right)} d\left(\mathbf{u}, A, \mathbf{R}_{i} \circ \mathbf{S}_{j}^{-1}\right) d u\right.}{}
\end{align*}
$$

because that degree is given to be 1 . (Unless $\mathbf{u} \in \mathbf{R}_{i} \circ \mathbf{S}_{j}^{-1}(A)$, the above degree equals 0 .) By Corollary 30.3.2, this equals

$$
\begin{aligned}
& \int_{A} \eta_{j}\left(\mathbf{S}_{j}^{-1}(\mathbf{v})\right) \psi_{i}\left(\mathbf{S}_{j}^{-1}(\mathbf{v})\right) a_{I}\left(\mathbf{S}_{j}^{-1}(\mathbf{v})\right) \\
& \frac{\partial\left(x_{i_{1}} \cdots x_{i_{n}}\right)}{\partial\left(u_{1} \cdots u_{n}\right)}\left(\mathbf{R}_{i} \circ \mathbf{S}_{j}^{-1}(\mathbf{v})\right) \operatorname{det}\left(D\left(\mathbf{R}_{i} \circ \mathbf{S}_{j}^{-1}\right)(\mathbf{v})\right) d v
\end{aligned}
$$

and by the chain rule and Rademacher's theorem, Theorem 26.6.7, this equals

$$
\begin{equation*}
\int_{A} \eta_{j}\left(\mathbf{S}_{j}^{-1}(\mathbf{v})\right) \psi_{i}\left(\mathbf{S}_{j}^{-1}(\mathbf{v})\right) a_{I}\left(\mathbf{S}_{j}^{-1}(\mathbf{v})\right) \frac{\partial\left(x_{i_{1}} \cdots x_{i_{n}}\right)}{\partial\left(v_{1} \cdots v_{n}\right)} d v \tag{30.3.7}
\end{equation*}
$$

Thus for every open $A$ of the sort described $30.3 .7=30.3 .6$. By the Vitali covering theorem, there exists a sequence of disjoint open balls $\left\{B_{k}\right\}$ whose union fills up int $\left(\mathbf{S}_{j}\left(U_{i} \cap V_{j}\right)\right)$ except for a set of measure zero $N$. Since $\mathbf{R}_{i} \circ \mathbf{S}_{j}^{-1}$ is Lipschitz, it follows $\mathbf{R}_{i} \circ \mathbf{S}_{j}^{-1}(N)$ also has measure zero. Therefore,

$$
\begin{equation*}
\int_{\mathbf{R}_{i}\left(U_{i} \cap V_{j}\right)} \eta_{j}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) \psi_{i}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) a_{I}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) \frac{\partial\left(x_{i_{1}} \cdots x_{i_{n}}\right)}{\partial\left(u_{1} \cdots u_{n}\right)} d u \tag{30.3.8}
\end{equation*}
$$

$$
\begin{align*}
& =\int_{\operatorname{int}^{2}\left(\mathbf{R}_{i} \cap V_{j}\right)} \eta_{j}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) \psi_{i}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) a_{I}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) \frac{\partial\left(x_{i_{1}} \cdots x_{i_{n}}\right)}{\partial\left(u_{1} \cdots u_{n}\right)} d u \\
& =\sum_{k=1}^{\infty} \int_{\mathbf{R}_{i} \circ \mathbf{S}_{j}^{-1}\left(B_{k}\right)} \eta_{j}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) \psi_{i}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) a_{I}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) \frac{\partial\left(x_{i_{1}} \cdots x_{i_{n}}\right)}{\partial\left(u_{1} \cdots u_{n}\right)} d u \\
& =\sum_{k=1}^{\infty} \int_{B_{k}} \eta_{j}\left(\mathbf{S}_{j}^{-1}(\mathbf{v})\right) \psi_{i}\left(\mathbf{S}_{j}^{-1}(\mathbf{v})\right) a_{I}\left(\mathbf{S}_{j}^{-1}(\mathbf{v})\right) \frac{\partial\left(x_{i_{1}} \cdots x_{i_{n}}\right)}{\partial\left(v_{1} \cdots v_{n}\right)} d v \\
& =\int_{\mathrm{int}^{\mathbf{S}}\left(U_{i} \cap V_{j}\right)} \eta_{j}\left(\mathbf{S}_{j}^{-1}(\mathbf{v})\right) \psi_{i}\left(\mathbf{S}_{j}^{-1}(\mathbf{v})\right) a_{I}\left(\mathbf{S}_{j}^{-1}(\mathbf{v})\right) \frac{\partial\left(x_{i_{1}} \cdots x_{i_{n}}\right)}{\partial\left(v_{1} \cdots v_{n}\right)} d v \\
& =\int_{\mathbf{S}_{j}\left(U_{i} \cap V_{j}\right)} \eta_{j}\left(\mathbf{S}_{j}^{-1}(\mathbf{v})\right) \psi_{i}\left(\mathbf{S}_{j}^{-1}(\mathbf{v})\right) a_{I}\left(\mathbf{S}_{j}^{-1}(\mathbf{v})\right) \frac{\partial\left(x_{i_{1}} \cdots x_{i_{n}}\right)}{\partial\left(v_{1} \cdots v_{n}\right)} d v \tag{30.3.9}
\end{align*}
$$

The equality of 30.3 .8 and 30.3.9 was the goal. With this, the definition of $\int \omega$ using the atlas $\left(U_{i}, \mathbf{R}_{i}\right)$ and partition of unity $\left\{\psi_{i}\right\}_{i=1}^{p}$ given in 30.3 .5 is

$$
\begin{gathered}
\sum_{i=1}^{p} \sum_{I} \int_{\mathbf{R}_{i} U_{i}} \psi_{i}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) a_{I}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) \frac{\partial\left(x_{i_{1} \cdots x_{i_{n}}}\right)}{\partial\left(u_{1} \cdots u_{n}\right)} d u \\
=\sum_{j=1}^{q} \sum_{i=1}^{p} \sum_{I} \int_{\mathbf{R}_{i}\left(U_{i} \cap V_{j}\right)} \eta_{j}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) \psi_{i}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) a_{I}\left(\mathbf{R}_{i}^{-1}(\mathbf{u})\right) \frac{\partial\left(x_{i_{1}} \cdots x_{i_{n}}\right)}{\partial\left(u_{1} \cdots u_{n}\right)} d u
\end{gathered}
$$

and from 30.3.8-30.3.9, this equals

$$
\begin{gathered}
=\sum_{j=1}^{q} \sum_{i=1}^{p} \sum_{I} \int_{\mathbf{S}_{j}\left(U_{i} \cap V_{j}\right)} \eta_{j}\left(\mathbf{S}_{j}^{-1}(\mathbf{v})\right) \psi_{i}\left(\mathbf{S}_{j}^{-1}(\mathbf{v})\right) a_{I}\left(\mathbf{S}_{j}^{-1}(\mathbf{v})\right) \frac{\partial\left(x_{i_{1}} \cdots x_{i_{n}}\right)}{\partial\left(v_{1} \cdots v_{n}\right)} d v \\
=\sum_{j=1}^{q} \sum_{I} \int_{\mathbf{S}_{j}\left(V_{j}\right)} \eta_{j}\left(\mathbf{S}_{j}^{-1}(\mathbf{v})\right) a_{I}\left(\mathbf{S}_{j}^{-1}(\mathbf{v})\right) \frac{\partial\left(x_{i_{1}} \cdots x_{i_{n}}\right)}{\partial\left(v_{1} \cdots v_{n}\right)} d v
\end{gathered}
$$

which is the definition of $\int \omega$ using the other atlas and partition of unity. This proves the theorem.

### 30.3.1 The Derivative Of A Differential Form

The derivative of a differential form is defined next.
Definition 30.3.8 Let $\omega=\sum_{I} a_{I}(\mathbf{x}) d x_{i_{1}} \wedge \cdots \wedge d x_{i_{n-1}}$ be a differential form of order $n-1$ where $a_{I}$ has weak partial derivatives. Then define $d \omega$, a differential form of order $n$ by replacing $a_{I}(\mathbf{x})$ with

$$
\begin{equation*}
d a_{I}(\mathbf{x}) \equiv \sum_{k=1}^{m} \frac{\partial a_{I}(\mathbf{x})}{\partial x_{k}} d x_{k} \tag{30.3.10}
\end{equation*}
$$

and putting $a$ wedge after the $d x_{k}$. Therefore,

$$
\begin{equation*}
d \omega \equiv \sum_{I} \sum_{k=1}^{m} \frac{\partial a_{I}(\mathbf{x})}{\partial x_{k}} d x_{k} \wedge d x_{i_{1}} \wedge \cdots \wedge d x_{i_{n-1}} \tag{30.3.11}
\end{equation*}
$$

### 30.4 Stoke's Theorem And The Orientation Of $\partial \Omega$

Here $\Omega$ will be an $n$ dimensional orientable Lipschitz manifold with boundary in $\mathbb{R}^{m}$. Let an oriented manifold for it be $\left\{U_{i}, \mathbf{R}_{i}\right\}_{i=1}^{p}$ and let a $C^{\infty}$ partition of unity be $\left\{\psi_{i}\right\}_{i=1}^{p}$. Also let

$$
\omega=\sum_{I} a_{I}(\mathbf{x}) d x_{i_{1}} \wedge \cdots \wedge d x_{i_{n-1}}
$$

be a differential form such that $a_{I}$ is $C^{1}(\bar{\Omega})$. Since $\sum \psi_{i}(\mathbf{x})=1$ on $\Omega$,

$$
d \omega=\sum_{I} \sum_{k=1}^{m} \sum_{j=1}^{p} \frac{\partial\left(\psi_{j} a_{I}\right)}{\partial x_{k}}(\mathbf{x}) d x_{k} \wedge d x_{i_{1}} \wedge \cdots \wedge d x_{i_{n-1}}
$$

It follows

$$
\begin{gather*}
\int d \omega=\sum_{I} \sum_{k=1}^{m} \sum_{j=1}^{p} \int_{\mathbf{R}_{j}\left(U_{j}\right)} \frac{\partial\left(\psi_{j} a_{I}\right)}{\partial x_{k}}\left(\mathbf{R}_{j}^{-1}(\mathbf{u})\right) \frac{\partial\left(x_{k}, x_{i_{1}} \cdots x_{i_{n-1}}\right)}{\partial\left(u_{1}, \cdots, u_{n}\right)} d \mathbf{u} \\
=\sum_{I} \sum_{k=1}^{m} \sum_{j=1}^{p} \int_{\mathbf{R}_{j}\left(U_{j}\right)} \frac{\partial\left(\psi_{j} a_{I}\right)}{\partial x_{k}}\left(\mathbf{R}_{j \varepsilon}^{-1}(\mathbf{u})\right) \frac{\partial\left(x_{k \varepsilon}, x_{i_{1} \varepsilon} \cdots x_{i_{n-1} \varepsilon}\right)}{\partial\left(u_{1}, \cdots, u_{n}\right)} d \mathbf{u}+ \\
\quad \sum_{I} \sum_{k=1}^{m} \sum_{j=1}^{p} \int_{\mathbf{R}_{j}\left(U_{j}\right)} \frac{\partial\left(\psi_{j} a_{I}\right)}{\partial x_{k}}\left(\mathbf{R}_{j}^{-1}(\mathbf{u})\right) \frac{\partial\left(x_{k}, x_{i_{1}} \cdots x_{i_{n-1}}\right)}{\partial\left(u_{1}, \cdots, u_{n}\right)} d \mathbf{u} \\
-\sum_{I} \sum_{k=1}^{m} \sum_{j=1}^{p} \int_{\mathbf{R}_{j}\left(U_{j}\right)} \frac{\partial\left(\psi_{j} a_{I}\right)}{\partial x_{k}}\left(\mathbf{R}_{j \varepsilon}^{-1}(\mathbf{u})\right) \frac{\partial\left(x_{k \varepsilon}, x_{i_{1} \varepsilon} \cdots x_{i_{n-1} \varepsilon}\right)}{\partial\left(u_{1}, \cdots, u_{n}\right)} d \mathbf{u} \tag{30.4.12}
\end{gather*}
$$

where those last two expressions sum to $e(\varepsilon)$ which converges to 0 as $\varepsilon \rightarrow 0$ for a suitable subsequence. Here is why.

$$
\frac{\partial\left(\psi_{j} a_{I}\right)}{\partial x_{k}}\left(\mathbf{R}_{j \varepsilon}^{-1}(\mathbf{u})\right) \rightarrow \frac{\partial\left(\psi_{j} a_{I}\right)}{\partial x_{k}}\left(\mathbf{R}_{j}^{-1}(\mathbf{u})\right)
$$

because of the uniform convergence of $\mathbf{R}_{j \varepsilon}^{-1}$ to $\mathbf{R}_{j}^{-1}$. In addition to this,

$$
\frac{\partial\left(x_{k \varepsilon}, x_{i_{1} \varepsilon} \varepsilon x_{i_{n-1}} \varepsilon\right)}{\partial\left(u_{1}, \cdots, u_{n}\right)} \rightarrow \frac{\partial\left(x_{k}, x_{i_{1}} \cdots x_{i_{n-1}}\right)}{\partial\left(u_{1}, \cdots, u_{n}\right)}
$$

in $L^{r}\left(\mathbf{R}_{j}\left(U_{j}\right)\right)$ for any $r>0$ and so a suitable subsequence converges pointwise. The integrands are also uniformly integrable. Thus the Vitali convergence theorem can be applied to each of the integrals in the above sum and obtain that for a suitable subsequence, $e(\varepsilon) \rightarrow 0$.

Then 30.4.12 equals

$$
=\sum_{I} \sum_{k=1}^{m} \sum_{j=1}^{p} \int_{\mathbf{R}_{j}\left(U_{j}\right)} \frac{\partial\left(\psi_{j} a_{I}\right)}{\partial x_{k}}\left(\mathbf{R}_{j \varepsilon}^{-1}(\mathbf{u})\right) \sum_{l=1}^{m} \frac{\partial x_{k \varepsilon}}{\partial u_{l}} A_{1 l} d \mathbf{u}+e(\varepsilon)
$$

where $A_{1 l}$ is the $1 l^{\text {th }}$ cofactor for the determinant

$$
\frac{\partial\left(x_{k \varepsilon}, x_{i_{1} \varepsilon} \cdots x_{i_{n-1}} \varepsilon\right)}{\partial\left(u_{1}, \cdots, u_{n}\right)}
$$

which is determined by a particular $I$. I am suppressing the $\varepsilon$ for the sake of notation. Then the above reduces to

$$
\begin{align*}
& =\sum_{I} \sum_{j=1}^{p} \int_{\mathbf{R}_{j}\left(U_{j}\right)} \sum_{l=1}^{n} A_{1 l} \sum_{k=1}^{m} \frac{\partial\left(\psi_{j} a_{I}\right)}{\partial x_{k}}\left(\mathbf{R}_{j \varepsilon}^{-1}(\mathbf{u})\right) \frac{\partial x_{k \varepsilon}}{\partial u_{l}} d \mathbf{u}+e(\varepsilon) \\
& =\sum_{I} \sum_{j=1}^{p} \sum_{l=1}^{n} \int_{\mathbf{R}_{j}\left(U_{j}\right)} A_{1 l} \frac{\partial}{\partial u_{l}}\left(\psi_{j} a_{I} \circ \mathbf{R}_{j \varepsilon}^{-1}\right)(\mathbf{u}) d \mathbf{u}+e(\varepsilon) \tag{30.4.13}
\end{align*}
$$

(Note $l$ goes up to $n$ not $m$.) Recall $\mathbf{R}_{j}\left(U_{j}\right)$ is relatively open in $\mathbb{R}_{\leq}^{n}$. Consider the integral where $l>1$. Integrate first with respect to $u_{l}$. In this case the boundary term vanishes because of $\psi_{j}$ and you get

$$
\begin{equation*}
-\int_{\mathbf{R}_{j}\left(U_{j}\right)} A_{1 l, l}\left(\psi_{j} a_{I} \circ \mathbf{R}_{j \varepsilon}^{-1}\right)(\mathbf{u}) d \mathbf{u} \tag{30.4.14}
\end{equation*}
$$

Next consider the case where $l=1$. Integrating first with respect to $u_{1}$, the term reduces to

$$
\begin{equation*}
\int_{\mathbf{R}_{j} V_{j}} \psi_{j} a_{I} \circ \mathbf{R}_{j \varepsilon}^{-1}\left(0, u_{2}, \cdots, u_{n}\right) A_{11} d \mathbf{u}_{1}-\int_{\mathbf{R}_{j}\left(U_{j}\right)} A_{11,1}\left(\psi_{j} a_{I} \circ \mathbf{R}_{j \varepsilon}^{-1}\right)(\mathbf{u}) d \mathbf{u} \tag{30.4.15}
\end{equation*}
$$

where $\mathbf{R}_{j} V_{j}$ is an open set in $\mathbb{R}^{n-1}$ consisting of

$$
\left\{\left(u_{2}, \cdots, u_{n}\right) \in \mathbb{R}^{n-1}:\left(0, u_{2}, \cdots, u_{n}\right) \in \mathbf{R}_{j}\left(U_{j}\right)\right\}
$$

and $d \mathbf{u}_{1}$ represents $d u_{2} d u_{3} \cdots d u_{n}$ on $\mathbf{R}_{j} V_{j}$ for short. Thus $V_{j}$ is just the part of $\partial \Omega$ which is in $U_{j}$ and the mappings $\mathbf{S}_{j}^{-1}$ given on $\mathbf{R}_{j} V_{j}=\mathbf{R}_{j}\left(U_{j} \cap \partial \Omega\right)$ by

$$
\mathbf{S}_{j}^{-1}\left(u_{2}, \cdots, u_{n}\right) \equiv \mathbf{R}_{j}^{-1}\left(0, u_{2}, \cdots, u_{n}\right)
$$

are such that $\left\{\left(\mathbf{S}_{j}, V_{j}\right)\right\}$ is an atlas for $\partial \Omega$. Then if 30.4.14 and 30.4.15 are placed in 30.4.13, then it follows from Lemma 30.3.1 that this reduces to

$$
\sum_{I} \sum_{j=1}^{p} \int_{\mathbf{R}_{j} V_{j}} \psi_{j} a_{I} \circ \mathbf{R}_{j \varepsilon}^{-1}\left(0, u_{2}, \cdots, u_{n}\right) A_{11} d \mathbf{u}_{1}+e(\varepsilon)
$$

Now as before, there exists a subsequence, still denoted as $\varepsilon$ such that each $\partial x_{s \varepsilon} / \partial u_{r}$ converges pointwise to $\partial x_{s} / \partial u_{r}$ and then using that these are bounded in every $L^{p}$, one can use the Vitali convergence theorem to pass to a limit obtaining finally

$$
\begin{aligned}
& \sum_{I} \sum_{j=1}^{p} \int_{\mathbf{R}_{j} V_{j}} \psi_{j} a_{I} \circ \mathbf{R}_{j}^{-1}\left(0, u_{2}, \cdots, u_{n}\right) A_{11} d \mathbf{u}_{1} \\
= & \sum_{I} \sum_{j=1}^{p} \int_{\mathbf{S}_{j} V_{j}} \psi_{j} a_{I} \circ \mathbf{S}_{j}^{-1}\left(u_{2}, \cdots, u_{n}\right) A_{11} d \mathbf{u}_{1}
\end{aligned}
$$

$$
\begin{equation*}
=\sum_{I} \sum_{j=1}^{p} \int_{\mathbf{S}_{j} V_{j}} \psi_{j} a_{I} \circ \mathbf{S}_{j}^{-1}\left(u_{2}, \cdots, u_{n}\right) \frac{\partial\left(x_{i_{1}} \cdots x_{i_{n-1}}\right)}{\partial\left(u_{2}, \cdots, u_{n}\right)}\left(0, u_{2}, \cdots, u_{n}\right) d \mathbf{u}_{1} \tag{30.4.16}
\end{equation*}
$$

This of course is the definition of $\int_{\partial \Omega} \omega$ provided $\partial \Omega$ is orientable. This is shown next.
What if spt $a_{I} \subseteq K \subseteq U_{i} \cap U_{j}$ for each $I$ ? Then using Lemma 30.3.4 it can be shown that $\int d \omega=$

$$
\sum_{I} \int_{\mathbf{S}_{j}\left(V_{j} \cap V_{j}\right)} a_{I} \circ \mathbf{S}_{j}^{-1}\left(u_{2}, \cdots, u_{n}\right) \frac{\partial\left(x_{i_{1}} \cdots x_{i_{n-1}}\right)}{\partial\left(u_{2}, \cdots, u_{n}\right)}\left(0, u_{2}, \cdots, u_{n}\right) d \mathbf{u}_{1}
$$

This is done by using a partition of unity which has the property that $\psi_{j}$ equals 1 on $K$ which forces all the other $\psi_{k}$ to equal zero there. Using the same trick involving a judicious choice of the partition of unity, $\int d \omega$ is also equal to

$$
\sum_{I} \int_{\mathbf{S}_{i}\left(V_{j} \cap V_{j}\right)} a_{I} \circ \mathbf{S}_{i}^{-1}\left(v_{2}, \cdots, v_{n}\right) \frac{\partial\left(x_{i_{1}} \cdots x_{i_{n-1}}\right)}{\partial\left(v_{2}, \cdots, v_{n}\right)}\left(0, v_{2}, \cdots, v_{n}\right) d \mathbf{v}_{1}
$$

Similarly if $A$ is an open connected subset of $\mathbf{S}_{i}\left(V_{j} \cap V_{j}\right)$ whose measure zero boundary separates $\mathbb{R}^{n}$ into two components, and $K$ is a compact subset of $\mathbf{S}_{i}^{-1}(A)$, containing spt $a_{I}$ for all $I, \int d \omega$ equals each of 30.4.18 and 30.4.17 below.

$$
\begin{array}{r}
\sum_{I} \int_{A} a_{I} \circ \mathbf{S}_{i}^{-1}\left(v_{2}, \cdots, v_{n}\right) \frac{\partial\left(x_{i_{1}} \cdots x_{i_{n-1}}\right)}{\partial\left(v_{2}, \cdots, v_{n}\right)}\left(0, v_{2}, \cdots, v_{n}\right) d \mathbf{v}_{1} \\
\sum_{I} \int_{\mathbf{S}_{j} \circ \mathbf{S}_{i}^{-1}(A)} a_{I} \circ \mathbf{S}_{j}^{-1}\left(u_{2}, \cdots, u_{n}\right) \frac{\partial\left(x_{i_{1}} \cdots x_{i_{n-1}}\right)}{\partial\left(u_{2}, \cdots, u_{n}\right)} d \mathbf{u}_{1} \tag{30.4.18}
\end{array}
$$

By Corollary 30.3.2 applied to $\mathbf{S}_{j} \circ \mathbf{S}_{i}^{-1}\left(\mathbf{v}_{1}\right)=\mathbf{u}_{1}$, the expression in 30.4.17 equals

$$
\sum_{I} \int_{\mathbf{S}_{j} \circ \mathbf{S}_{i}^{-1}(A)} a_{I} \circ \mathbf{S}_{j}^{-1}\left(u_{2}, \cdots, u_{n}\right) \frac{\partial\left(x_{i_{1}} \cdots x_{i_{n-1}}\right)}{\partial\left(u_{2}, \cdots, u_{n}\right)} d\left(\mathbf{u}_{1}, A, \mathbf{S}_{j} \circ \mathbf{S}_{i}^{-1}\right) d \mathbf{u}_{1}
$$

and so, subtracting 30.4.18 and the above,

$$
\begin{gathered}
\sum_{I} \int_{\mathbf{S}_{j} \circ \mathbf{S}_{i}^{-1}(A)} a_{I} \circ \mathbf{S}_{j}^{-1}\left(u_{2}, \cdots, u_{n}\right) \frac{\partial\left(x_{i_{1}} \cdots x_{i_{n-1}}\right)}{\partial\left(u_{2}, \cdots, u_{n}\right)} \\
\left(1-d\left(\mathbf{u}_{1}, A, \mathbf{S}_{j} \circ \mathbf{S}_{i}^{-1}\right)\right) d \mathbf{u}_{1}=0
\end{gathered}
$$

Now by invariance of domain, it follows $\mathbf{S}_{j} \circ \mathbf{S}_{i}^{-1}(A)$ is an open connected set contained in a single component of $\left(\mathbf{S}_{j} \circ \mathbf{S}_{i}^{-1}(\partial A)\right)^{C}$ and so the above degree is constant on $\mathbf{S}_{j} \circ \mathbf{S}_{i}^{-1}(A)$. If this degree is not 1 then it follows that for any choice of the $a_{I}$ having compact support in $\mathbf{S}_{i}^{-1}(A)$,

$$
\begin{equation*}
\sum_{I} \int_{\mathbf{S}_{j} \circ \mathbf{S}_{i}^{-1}(A)} a_{I} \circ \mathbf{S}_{j}^{-1}\left(u_{2}, \cdots, u_{n}\right) \frac{\partial\left(x_{i_{1}} \cdots x_{i_{n-1}}\right)}{\partial\left(u_{2}, \cdots, u_{n}\right)} d \mathbf{u}_{1}=0 \tag{30.4.19}
\end{equation*}
$$

Next let $I$ always denote an increasing list of indices. Note that $\mathbf{S}_{j} \circ \mathbf{S}_{i}^{-1}$ maps the open set $A$ to an open set which therefore has positive Lebesgue measure. It follows from the area formula that

$$
\begin{equation*}
\operatorname{det}\left(D\left(\mathbf{S}_{j} \circ \mathbf{S}_{i}^{-1}\right)\right)=\operatorname{det}(\overbrace{D\left(\mathbf{S}_{j}\left(\mathbf{S}_{i}^{-1}(\mathbf{u})\right)\right) D \mathbf{S}_{i}^{-1}(\mathbf{u})}^{n \times m}) \tag{30.4.20}
\end{equation*}
$$

must be nonzero on a set of positive measure. It follows that at least some

$$
\frac{\partial\left(x_{i_{1}} \cdots x_{i_{n-1}}\right)}{\partial\left(u_{2}, \cdots, u_{n}\right)}
$$

must be nonzero since by the Binet Cauchy theorem, the above determinant in 30.4.20 is the sum of products of these multiplied by other determinants which come from deleting corresponding columns in the matrix for $D\left(\mathbf{S}_{j}\left(\mathbf{S}_{i}^{-1}(\mathbf{u})\right)\right)$. It follows that

$$
\sum_{I}\left(\frac{\partial\left(x_{i_{1}} \cdots x_{i_{n-1}}\right)}{\partial\left(u_{2}, \cdots, u_{n}\right)}\right)^{2}
$$

is positive on a set of positive measure. Let

$$
\lim _{p \rightarrow \infty} a_{I p} \circ \mathbf{S}_{j}^{-1}=\frac{\partial\left(x_{i_{1}} \cdots x_{i_{n-1}}\right)}{\partial\left(u_{2}, \cdots, u_{n}\right)}
$$

in $L^{2}\left(\mathbf{S}_{j} \circ \mathbf{S}_{i}^{-1}(A)\right)$ for each $I=\left(i_{1}, \cdots, i_{n-1}\right)$. Replacing $a_{I} \circ \mathbf{S}_{j}^{-1}$ with $a_{I p} \circ \mathbf{S}_{j}^{-1}$ in 30.4.19 and passing to the limit, it follows

$$
\begin{aligned}
0 & =\lim _{p \rightarrow \infty} \int_{\mathbf{S}_{j} \mathbf{S}_{i}^{-1}(A)} \sum_{I} a_{I p} \circ \mathbf{S}_{j}^{-1}\left(\mathbf{u}_{1}\right) \frac{\partial\left(x_{i_{1}} \cdots x_{i_{n-1}}\right)}{\partial\left(u_{2}, \cdots, u_{n}\right)} d \mathbf{u}_{1} \\
& =\int_{\mathbf{S}_{j} \mathbf{S}_{i}^{-1}(A)} \sum_{I}\left(\frac{\partial\left(x_{i_{1}} \cdots x_{i_{n-1}}\right)}{\partial\left(u_{2}, \cdots, u_{n}\right)}\right)^{2} d \mathbf{u}_{1}>0
\end{aligned}
$$

a contradiction. Therefore, $d\left(\mathbf{u}_{1}, A, \mathbf{S}_{j} \circ \mathbf{S}_{i}^{-1}\right)=1$ and this shows the atlas is an oriented atlas for $\partial \Omega$. This has proved a general Stokes theorem.

Theorem 30.4.1 Let $\Omega$ be an oriented Lipschitz manifold and let

$$
\omega=\sum_{I} a_{I}(\mathbf{x}) d x_{i_{1}} \wedge \cdots \wedge d x_{i_{n-1}}
$$

where each $a_{I}$ is $C^{1}(\bar{\Omega})$. For $\left\{U_{j}, \mathbf{R}_{j}\right\}_{j=0}^{p}$ an oriented atlas for $\Omega$ where $\mathbf{R}_{j}\left(U_{j}\right)$ is a relatively open set in

$$
\left\{\mathbf{u} \in \mathbb{R}^{n}: u_{1} \leq 0\right\}
$$

define an atlas for $\partial \Omega,\left\{V_{j}, \mathbf{S}_{j}\right\}$ where $V_{j} \equiv \partial \Omega \cap U_{j}$ and $\mathbf{S}_{j}$ is just the restriction of $\mathbf{R}_{j}$ to $V_{j}$. Then this is an oriented atlas for $\partial \Omega$ and

$$
\int_{\partial \Omega} \omega=\int_{\Omega} d \omega
$$

where the two integrals are taken with respect to the given oriented atlass.
What if $a_{I}$ is only the restriction to $\Omega$ of a function in $W^{1, p}\left(\mathbb{R}^{m}\right), p>1$ ? Would the same formula still hold? Let $\phi_{\varepsilon}$ be a mollifier and let $a_{I \varepsilon} \equiv a_{I} * \phi_{\varepsilon}$. Then Stoke's theorem applies to the mollified situation and it follows

$$
\begin{aligned}
& \int_{\Omega} d \omega_{\varepsilon} \\
= & \sum_{I} \sum_{k=1}^{m} \sum_{j=1}^{p} \int_{\mathbf{R}_{j}\left(U_{j}\right)} \frac{\partial\left(\psi_{j} a_{I \varepsilon}\right)}{\partial x_{k}}\left(\mathbf{R}_{j}^{-1}(\mathbf{u})\right) \frac{\partial\left(x_{k}, x_{i_{1}} \cdots x_{i_{n-1}}\right)}{\partial\left(u_{1}, \cdots, u_{n}\right)} d \mathbf{u} \\
= & \sum_{I} \sum_{j=1}^{p} \int_{\mathbf{s}_{j} V_{j}} \psi_{j} a_{I \varepsilon} \circ \mathbf{S}_{j}^{-1}\left(u_{2}, \cdots, u_{n}\right) \frac{\partial\left(x_{i_{1}} \cdots x_{i_{n-1}}\right)}{\partial\left(u_{2}, \cdots, u_{n}\right)}\left(0, u_{2}, \cdots, u_{n}\right) d \mathbf{u}_{1} \\
\equiv & \int_{\partial \Omega} \omega_{\varepsilon}
\end{aligned}
$$

Now if you let $\varepsilon \rightarrow 0$, it follows from the definition of convolution that

$$
\frac{\partial\left(\psi_{j} a_{I \varepsilon}\right)}{\partial x_{k}} \rightarrow \frac{\partial\left(\psi_{j} a_{I}\right)}{\partial x_{k}} \text { in } L^{p}\left(\mathbb{R}^{m}\right)
$$

and so there is a subsequence such that for each $k$,

$$
\frac{\partial\left(\psi_{j} a_{I \varepsilon}\right)}{\partial x_{k}}(\mathbf{x}) \rightarrow \frac{\partial\left(\psi_{j} a_{I}\right)}{\partial x_{k}}(\mathbf{x})
$$

pointwise a.e. Since $\mathbf{R}_{j}^{-1}, \mathbf{R}_{j}$ are Lipschitz, they take sets of measure zero to sets of measure zero. Hence

$$
\frac{\partial\left(\psi_{j} a_{I \varepsilon}\right)}{\partial x_{k}} \circ \mathbf{R}_{j}^{-1} \rightarrow \frac{\partial\left(\psi_{j} a_{I}\right)}{\partial x_{k}} \circ \mathbf{R}_{j}^{-1}
$$

pointwise a.e. on $\mathbf{R}_{j}\left(U_{j}\right)$. Similar considerations apply to $a_{I \varepsilon}$. Using the Vitali convergence theorem in $\int_{\Omega} d \omega_{\varepsilon}, \int_{\Omega} \omega_{\varepsilon}$, it is possible to pass to the limit. This is because the integrands are bounded in $L^{p}$ and so they are uniformly integrable. This proves the following corollary.

Corollary 30.4.2 Let $\Omega$ be an oriented Lipschitz manifold and let

$$
\omega=\sum_{I} a_{I}(\mathbf{x}) d x_{i_{1}} \wedge \cdots \wedge d x_{i_{n-1}}
$$

where each $a_{I}$ is in $W^{1, p}\left(\mathbb{R}^{m}\right)$ where $p>1$. For $\left\{U_{j}, \mathbf{R}_{j}\right\}_{j=0}^{p}$ an oriented atlas for $\Omega$ where $\mathbf{R}_{j}\left(U_{j}\right)$ is a relatively open set in

$$
\left\{\mathbf{u} \in \mathbb{R}^{n}: u_{1} \leq 0\right\}
$$

define an atlas for $\partial \Omega,\left\{V_{j}, \mathbf{S}_{j}\right\}$ where $V_{j} \equiv \partial \Omega \cap U_{j}$ and $\mathbf{S}_{j}$ is just the restriction of $\mathbf{R}_{j}$ to $V_{j}$. Then this is an oriented atlas for $\partial \Omega$ and

$$
\int_{\partial \Omega} \omega=\int_{\Omega} d \omega
$$

where the two integrals are taken with respect to the given oriented atlass.

### 30.5 Green's Theorem

Green's theorem is a well known result in calculus and it pertains to a region in the plane. I am going to generalize to an open set in $\mathbb{R}^{n}$ with sufficiently smooth boundary using the methods of differential forms described above.

### 30.5.1 An Oriented Manifold

A bounded open subset, $\Omega$, of $\mathbb{R}^{n}, n \geq 2$ has Lipschitz boundary and lies locally on one side of its boundary if it satisfies the following conditions.

For each $p \in \partial \Omega \equiv \bar{\Omega} \backslash \Omega$, there exists an open set, $Q$, containing $p$, an open interval $(a, b)$, a bounded open set $B \subseteq \mathbb{R}^{n-1}$, and an orthogonal transformation $R$ such that $\operatorname{det} R=$ 1,

$$
B \times(a, b)=R Q
$$

and letting $W=Q \cap \Omega$,

$$
R W=\left\{\mathbf{u} \in \mathbb{R}^{n}: a<u_{1}<g\left(u_{2}, \cdots, u_{n}\right),\left(u_{2}, \cdots, u_{n}\right) \in B\right\}
$$

where $g$ is Lipschitz continuous on $\mathbb{R}^{n-1}, g\left(u_{2}, \cdots, u_{n}\right)<b$ for $\left(u_{2}, \cdots, u_{n}\right) \in B$, and $g$ vanishing outside some compact set in $\mathbb{R}^{n-1}$.

$$
R(\partial \Omega \cap Q)=\left\{\mathbf{u} \in \mathbb{R}^{n}: u_{1}=g\left(u_{2}, \cdots, u_{n}\right),\left(u_{2}, \cdots, u_{n}\right) \in B\right\}
$$

Note that finitely many of these sets $Q$ cover $\partial \Omega$ because $\partial \Omega$ is compact. The following picture describes the situation.


Define $\mathbf{P}_{1}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n-1}$ by

$$
\mathbf{P}_{1} \mathbf{u} \equiv\left(u_{2}, \cdots, u_{n}\right)
$$

and $\Sigma: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ given by

$$
\begin{gathered}
\Sigma \mathbf{u} \equiv \mathbf{u}-g\left(\mathbf{P}_{1} \mathbf{u}\right) \mathbf{e}_{1} \\
\equiv \mathbf{u}-g\left(u_{2}, \cdots, u_{n}\right) \mathbf{e}_{1} \\
\equiv\left(u_{1}-g\left(u_{2}, \cdots, u_{n}\right), u_{2}, \cdots, u_{n}\right)
\end{gathered}
$$

Thus $\Sigma$ is invertible and

$$
\begin{gathered}
\Sigma^{-1} \mathbf{u}=\mathbf{u}+g\left(\mathbf{P}_{1} \mathbf{u}\right) \mathbf{e}_{1} \\
\equiv\left(u_{1}+g\left(u_{2}, \cdots, u_{n}\right), u_{2}, \cdots, u_{n}\right)
\end{gathered}
$$

For $\mathbf{x} \in \partial \Omega \cap Q$, it follows the first component of $R \mathbf{x}$ is $g\left(\mathbf{P}_{1}(R \mathbf{x})\right)$. Now define $\mathbf{R}: W \rightarrow \mathbb{R}_{\leq}^{n}$ as

$$
\mathbf{u} \equiv \mathbf{R} \mathbf{x} \equiv R \mathbf{x}-g\left(\mathbf{P}_{1}(R \mathbf{x})\right) \mathbf{e}_{1} \equiv \Sigma R \mathbf{x}
$$

and so it follows

$$
\mathbf{R}^{-1}=R^{*} \Sigma^{-1}
$$

These mappings $\mathbf{R}$ involve first a rotation followed by a variable sheer in the direction of the $u_{1}$ axis.

Since $\partial \Omega$ is compact, there are finitely many of these open sets, $Q_{1}, \cdots, Q_{p}$ which cover $\partial \Omega$. Let the orthogonal transformations and other quantities described above also be indexed by $k$ for $k=1, \cdots, p$. Also let $Q_{0}$ be an open set with $\overline{Q_{0}} \subseteq \Omega$ and $\bar{\Omega}$ is covered by $Q_{0}, Q_{1}, \cdots, Q_{p}$. Let $\mathbf{u} \equiv \mathbf{R}_{0} \mathbf{x} \equiv \mathbf{x}-k \mathbf{e}_{1}$ where $k$ is large enough that $\mathbf{R}_{0} Q_{0} \subseteq \mathbb{R}_{<}^{n}$. Thus in this case, the orthogonal transformation $R_{0}$ equals $I$ and $\Sigma_{0} \mathbf{x} \equiv \mathbf{x}-k \mathbf{e}_{1}$. I claim $\Omega$ is an oriented manifold with boundary and the charts are $\left(W_{i}, \mathbf{R}_{i}\right)$.

Letting $A$ be an open set contained in $\mathbf{R}_{i}\left(W_{i} \cap W_{j}\right)$ such that $\partial A$ has measure 0 and $\partial A$ separates $\mathbb{R}^{n}$ into two components, consider

$$
d\left(\mathbf{u}, A, \mathbf{R}_{j} \circ \mathbf{R}_{i}^{-1}\right), \mathbf{u} \notin \mathbf{R}_{j} \circ \mathbf{R}_{i}^{-1}(\partial A)
$$

By convolving $g$ with a mollifier, there exists a sequence of infinitely differentiable functions $g_{\varepsilon}$ which converge uniformly to $g$ on all of $\mathbb{R}^{n-1}$ as $\varepsilon \rightarrow 0$. Therefore, letting $\Sigma_{\varepsilon}$ be the corresponding functions defined above with $g$ replaced with $g_{\varepsilon}$, it follows the $\Sigma_{\varepsilon}$ will converge uniformly to $\Sigma$ and $\Sigma_{\varepsilon}^{-1}$ will converge uniformly to $\Sigma^{-1}$. Thus from the above descriptions of $\mathbf{R}_{j}^{-1}$, it follows $\mathbf{R}_{j \varepsilon}^{-1}$ converges uniformly to $\mathbf{R}_{j}^{-1}$ for each $j$. Therefore, if $\varepsilon$ is small enough, $\mathbf{u} \notin\left(t \mathbf{R}_{j \varepsilon} \circ \mathbf{R}_{i \varepsilon}^{-1}+(1-t) \mathbf{R}_{j} \circ \mathbf{R}_{i}^{-1}\right)(\partial A)$ and so from properties of the degree, the mappings $\mathbf{R}_{j}$ and $\mathbf{R}_{i}^{-1}$ can be replaced with smooth ones in computing the degree. To save on notation, I will drop the $\varepsilon$. The mapping involved is

$$
\Sigma_{j} R_{j} R_{i}^{*} \Sigma_{i}^{-1}
$$

and it is a one to one mapping. What is the determinant of its derivative? By the chain rule,

$$
D\left(\Sigma_{j} R_{j} R_{i}^{*} \Sigma_{i}^{-1}\right)=D \Sigma_{j}\left(R_{j} R_{i}^{*} \Sigma_{i}^{-1}\right) D R_{j}\left(R_{i}^{*} \Sigma_{i}^{-1}\right) D R_{i}^{*}\left(\Sigma_{i}^{-1}\right) D \Sigma_{i}^{-1}
$$

However,

$$
\operatorname{det}\left(D \Sigma_{j}\right)=1=\operatorname{det}\left(D \Sigma_{j}^{-1}\right)
$$

and $\operatorname{det}\left(R_{i}\right)=\operatorname{det}\left(R_{i}^{*}\right)=1$ by assumption. Therefore, if $\mathbf{u} \in\left(\mathbf{R}_{j} \circ \mathbf{R}_{i}^{-1}\right)(A)$, the above degree is 1 and if $\mathbf{u}$ is not in this set, the above degree is 0 or undefined if $\mathbf{u}$ is on $\left(\mathbf{R}_{j} \circ \mathbf{R}_{i}^{-1}\right)(\partial A)$. By Definition 30.1.5 $\Omega$ is indeed an oriented manifold.

### 30.5.2 Green's Theorem

The general Green's theorem is the following. It follows from Corollary 30.4.2.
Theorem 30.5.1 Let $\Omega$ be a bounded open set having Lipschitz boundary as described above. Also let

$$
\omega=\sum_{I} a_{I}(\mathbf{x}) d x_{i_{1}} \wedge \cdots \wedge d x_{i_{n-1}}
$$

be a differential form where $a_{I}$ is assumed to be the restriction to $\Omega$ of a function in $W^{1, p}\left(\mathbb{R}^{n}\right), p>1$. Then

$$
\int_{\partial \Omega} \omega=\int_{\Omega} d \omega
$$

It can be shown that, since the boundary is Lipschitz, it would have sufficed to assume $u \in W^{1, p}(\Omega)$ and then it is automatically the restriction of one in $W^{1, p}\left(\mathbb{R}^{n}\right)$. However, these terms have not all been defined and the necessary results are not proved till the topic of Sobolev spaces is discussed.

Another thing to notice is that, while the above result is pretty general, including the usual calculus result in the plane as a special case, it does not have the generality of the best results in the plane which involve only a rectifiable simple closed curve. The issue whether $\partial \Omega$ is an oriented manifold was dealt with in the general Stokes theorem described above.

Next is a general version of the divergence theorem which comes from choosing the differential form in an auspicious manner.

### 30.6 The Divergence Theorem

From Green's theorem, one can quickly obtain a general Divergence theorem for $\Omega$ as described above in Section 30.5.1. First note that from the above description of the $\mathbf{R}_{j}$,

$$
\frac{\partial\left(x_{k}, x_{i_{1}}, \cdots x_{i_{n-1}}\right)}{\partial\left(u_{1}, \cdots, u_{n}\right)}=\operatorname{sgn}\left(k, i_{1} \cdots, i_{n-1}\right)
$$

So let $\mathbf{F}(\mathbf{x})$ be a Lipschitz vector field. Say $\mathbf{F}=\left(F_{1}, \cdots, F_{n}\right)$. Consider the differential form

$$
\omega(\mathbf{x}) \equiv \sum_{k=1}^{n} F_{k}(\mathbf{x})(-1)^{k-1} d x_{1} \wedge \cdots \wedge \widehat{d x_{k}} \wedge \cdots \wedge d x_{n}
$$

where the hat means $d x_{k}$ is being left out. Here it is assumed $F_{k}$ is the restriction to $\Omega$ of a function in $W^{1, p}\left(\mathbb{R}^{n}\right)$ where $p>1$. Then

$$
\begin{aligned}
d \omega(\mathbf{x}) & =\sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\partial F_{k}}{\partial x_{j}}(-1)^{k-1} d x_{j} \wedge d x_{1} \wedge \cdots \wedge \widehat{d x_{k}} \wedge \cdots \wedge d x_{n} \\
& =\sum_{k=1}^{n} \frac{\partial F_{k}}{\partial x_{k}} d x_{1} \wedge \cdots \wedge d x_{k} \wedge \cdots \wedge d x_{n} \\
& \equiv \operatorname{div}(\mathbf{F}) d x_{1} \wedge \cdots \wedge d x_{k} \wedge \cdots \wedge d x_{n}
\end{aligned}
$$

The assertion between the first and second lines follows right away from properties of determinants and the definition of the integral of the above wedge products in terms of determinants. From Green's theorem and the change of variables formula applied to the individual terms in the description of $\int_{\Omega} d \omega$

$$
\begin{gathered}
\int_{\Omega} \operatorname{div}(\mathbf{F}) d x= \\
\sum_{j=1}^{p} \int_{B_{j}} \sum_{k=1}^{n}(-1)^{k-1} \frac{\partial\left(x_{1}, \cdots \widehat{x_{k}} \cdots, x_{n}\right)}{\partial\left(u_{2}, \cdots, u_{n}\right)}\left(\psi_{j} F_{k}\right) \circ \mathbf{R}_{j}^{-1}\left(0, u_{2}, \cdots, u_{n}\right) d \mathbf{u}_{1}
\end{gathered}
$$

$d \mathbf{u}_{1}$ short for $d u_{2} d u_{3} \cdots d u_{n}$. Also, this shows the result on the right of the equal sign does not depend on the choice of partition of unity or on the atlas.

I want to write this in a more attractive manner which will give more insight in terms of the Hausdorff measure on $\partial \Omega$. The above involves a particular partition of unity, the functions being the $\psi_{i}$. Replace $\mathbf{F}$ in the above with $\psi_{s} \mathbf{F}$. Next let $\left\{\eta_{j}\right\}$ be a partition of unity $\eta_{j} \prec O_{j}$ such that $\eta_{s}=1$ on spt $\psi_{s}$. This partition of unity exists by Lemma 30.3.4. Then

$$
\begin{gather*}
\int_{\Omega} \operatorname{div}\left(\psi_{s} \mathbf{F}\right) d x= \\
\sum_{j=1}^{p} \int_{B_{j}} \sum_{k=1}^{n}(-1)^{k-1} \frac{\partial\left(x_{1}, \cdots \widehat{x_{k}} \cdots, x_{n}\right)}{\partial\left(u_{2}, \cdots, u_{n}\right)}\left(\eta_{j} \psi_{s} F_{k}\right) \circ \mathbf{R}_{j}^{-1}\left(0, u_{2}, \cdots, u_{n}\right) d \mathbf{u}_{1} \\
=\int_{B_{s}} \sum_{k=1}^{n}(-1)^{k-1} \frac{\partial\left(x_{1}, \cdots \widehat{x}_{k} \cdots, x_{n}\right)}{\partial\left(u_{2}, \cdots, u_{n}\right)}\left(\psi_{s} F_{k}\right) \circ \mathbf{R}_{s}^{-1}\left(0, u_{2}, \cdots, u_{n}\right) d \mathbf{u}_{1} \tag{30.6.21}
\end{gather*}
$$

because since $\eta_{s}=1$ on $\operatorname{spt} \psi_{s}$, it follows all the other $\eta_{j}$ equal zero there. Consider the vector defined for $\mathbf{u}_{1} \in \mathbf{R}_{s}\left(W_{s}\right) \cap \mathbb{R}_{0}^{n}$ whose $k^{t h}$ component is

$$
\begin{equation*}
(-1)^{k-1} \frac{\partial\left(x_{1}, \cdots \widehat{x_{k}} \cdots, x_{n}\right)}{\partial\left(u_{2}, \cdots, u_{n}\right)}=(-1)^{k+1} \frac{\partial\left(x_{1}, \cdots \widehat{x_{k}} \cdots, x_{n}\right)}{\partial\left(u_{2}, \cdots, u_{n}\right)} \tag{30.6.22}
\end{equation*}
$$

Suppose you dot this vector with a "tangent" vector $\partial \mathbf{R}_{s}^{-1} / \partial u_{i}$. For each $j$ this yields

$$
\sum_{j}(-1)^{k+1} \frac{\partial\left(x_{1}, \cdots \widehat{x_{k}} \cdots, x_{n}\right)}{\partial\left(u_{2}, \cdots, u_{n}\right)} \frac{\partial x_{k}}{\partial u_{i}}=0
$$

because it is the expansion of

$$
\left|\begin{array}{cccc}
x_{1, i} & x_{1,2} & \cdots & x_{1, n} \\
x_{2, i} & x_{2,2} & \cdots & x_{2, n} \\
\vdots & \vdots & \ddots & \vdots \\
x_{n, i} & x_{n, 2} & \cdots & x_{n, n}
\end{array}\right|
$$

a determinant with two equal columns. Thus this vector is at least in some sense normal to $\Omega$. Since it works in the divergence theorem, it is called the exterior normal.

One could normalize the vector of 30.6 .22 by dividing by its magnitude. Then it would be the unit exterior normal $\mathbf{n}$. Letting $J\left(\mathbf{u}_{1}\right)$ be its usual Euclidean norm, this equals

$$
J\left(\mathbf{u}_{1}\right)^{2}=\sum_{k=1}^{n}\left(\frac{\partial\left(x_{1}, \cdots \widehat{x_{k}} \cdots, x_{n}\right)}{\partial\left(u_{2}, \cdots, u_{n}\right)}\right)^{2}
$$

and by the Binet Cauchy theorem this equals

$$
\operatorname{det}\left(D \mathbf{R}_{s}^{-1}(\mathbf{u})^{*} D \mathbf{R}_{s}^{-1}(\mathbf{u})\right)^{1 / 2}
$$

Thus the expression in 30.6.21 reduces to

$$
\int_{B_{s}}\left(\psi_{s} \mathbf{F} \circ \mathbf{R}_{s}^{-1}\left(\mathbf{u}_{1}\right)\right) \cdot \mathbf{n}\left(\mathbf{R}_{s}^{-1}\left(\mathbf{u}_{1}\right)\right) J\left(\mathbf{u}_{1}\right) d \mathbf{u}_{1} .
$$

By the area formula, Theorem 29.5.3, this reduces to

$$
\int_{\partial \Omega \cap W_{s}} \psi_{s} \mathbf{F} \cdot \mathbf{n} d \mathscr{H}^{n-1}=\int_{\partial \Omega} \psi_{s} \mathbf{F} \cdot \mathbf{n} d \mathscr{H}^{n-1}
$$

It follows upon summing over $s$ and using that the $\psi_{s}$ add to 1 ,

$$
\begin{aligned}
& \int_{\partial \Omega} \mathbf{F} \cdot \mathbf{n} d \mathscr{H}^{n-1}=\int_{\Omega} \sum_{s=1}^{p} \operatorname{div}\left(\psi_{s} \mathbf{F}\right) d x \\
& =\int_{\Omega} \sum_{s=1}^{p} \psi_{s, k} F_{k}+\psi_{s} \operatorname{div}(\mathbf{F}) d x=\int_{\Omega} F_{k}\left(\sum_{s=1}^{p} \psi_{s}\right)_{, k}+\psi_{s} \operatorname{div}(\mathbf{F}) d x \\
& =\int_{\Omega} \operatorname{div}(\mathbf{F}) d x
\end{aligned}
$$

This proves the following general divergence theorem.
Theorem 30.6.1 Let $\Omega$ be a bounded open set having Lipschitz boundary as described above. Also let $\mathbf{F}$ be a vector field with the property that for each component function of $\mathbf{F}, F_{k}$ is the restriction to $\Omega$ of a function in $W^{1, p}\left(\mathbb{R}^{n}\right), p>1$. Then there exists a normal vector $\mathbf{n}$ which is defined a.e. on $\partial \Omega$ such that

$$
\int_{\partial \Omega} \mathbf{F} \cdot \mathbf{n} d \mathscr{H}^{n-1}=\int_{\Omega} \operatorname{div}(\mathbf{F}) d x
$$

It is clear $\mathbf{n}$ is unique $\mathscr{H}^{n-1}$ a.e. since if there were two, then a simple manipulation shows for all such $\mathbf{F}$,

$$
\int_{\partial \Omega} \mathbf{F} \cdot\left(\mathbf{n}-\mathbf{n}_{1}\right) d \mathscr{H}^{n-1}=0
$$

Thus $\mathbf{n}-\mathbf{n}_{1}=\mathbf{0}$ a.e.

## Chapter 31

## Differentiation, Radon Measures

This is a brief chapter on certain important topics on the differentiation theory for general Radon measures. For different proofs and some results which are not discussed here, a good source is [47] which is where I first read some of these things.

### 31.1 Fundamental Theorem Of Calculus

In this section the Besicovitch covering theorem will be used to give a generalization of the Lebesgue differentiation theorem to general Radon measures. In what follows, $\mu$ will be a Radon measure,

$$
Z \equiv\left\{\mathbf{x} \in \mathbb{R}^{n}: \mu(B(\mathbf{x}, r))=0 \text { for some } r>0\right\}
$$

Lemma 31.1.1 $Z$ is measurable and $\mu(Z)=0$.
Proof: For each $\mathbf{x} \in Z$, there exists a ball $B(\mathbf{x}, r)$ with $\mu(B(\mathbf{x}, r))=0$. Let $\mathscr{C}$ be the collection of these balls. Since $\mathbb{R}^{n}$ has a countable basis, a countable subset, $\widetilde{\mathscr{C}}$, of $\mathscr{C}$ also covers $Z$. Let

$$
\widetilde{\mathscr{C}}=\left\{B_{i}\right\}_{i=1}^{\infty}
$$

Then letting $\bar{\mu}$ denote the outer measure determined by $\mu$,

$$
\bar{\mu}(Z) \leq \sum_{i=1}^{\infty} \bar{\mu}\left(B_{i}\right)=\sum_{i=1}^{\infty} \mu\left(B_{i}\right)=0
$$

Therefore, $Z$ is measurable and has measure zero as claimed.
Let $M f: \mathbb{R}^{n} \rightarrow[0, \infty]$ by

$$
M f(\mathbf{x}) \equiv\left\{\begin{array}{l}
\sup _{r \leq 1} \frac{1}{\mu(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f| d \mu \text { if } \mathbf{x} \notin Z \\
0 \text { if } \mathbf{x} \in Z
\end{array}\right.
$$

Theorem 31.1.2 Let $\mu$ be a Radon measure and let $f \in L^{1}\left(\mathbb{R}^{n}, \mu\right)$. Then for a.e. $\mathbf{x}$,

$$
\lim _{r \rightarrow 0} \frac{1}{\mu(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d \mu(y)=0
$$

Proof: First consider the following claim which is a weak type estimate of the same sort used when differentiating with respect to Lebesgue measure.

Claim 1: The following inequality holds for $N_{n}$ the constant of the Besicovitch covering theorem.

$$
\bar{\mu}([M f>\varepsilon]) \leq N_{n} \varepsilon^{-1}\|f\|_{1}
$$

Proof: First note $[M f>\varepsilon] \cap Z=\emptyset$ and without loss of generality, you can assume $\bar{\mu}([M f>\varepsilon])>0$. Next, for each $\mathbf{x} \in[M f>\varepsilon]$ there exists a ball $B_{\mathbf{x}}=B\left(\mathbf{x}, r_{\mathbf{x}}\right)$ with $r_{\mathbf{x}} \leq 1$ and

$$
\mu\left(B_{\mathbf{x}}\right)^{-1} \int_{B\left(\mathbf{x}, r_{\mathbf{x}}\right)}|f| d \mu>\varepsilon
$$

Let $\mathscr{F}$ be this collection of balls so that $[M f>\varepsilon]$ is the set of centers of balls of $\mathscr{F}$. By the Besicovitch covering theorem,

$$
[M f>\varepsilon] \subseteq \cup_{i=1}^{N_{n}}\left\{B: B \in \mathscr{G}_{i}\right\}
$$

where $\mathscr{G}_{i}$ is a collection of disjoint balls of $\mathscr{F}$. Now for some $i$,

$$
\bar{\mu}([M f>\varepsilon]) / N_{n} \leq \mu\left(\cup\left\{B: B \in \mathscr{G}_{i}\right\}\right)
$$

because if this is not so, then

$$
\begin{aligned}
\bar{\mu}([M f>\varepsilon]) & \leq \sum_{i=1}^{N_{n}} \mu\left(\cup\left\{B: B \in \mathscr{G}_{i}\right\}\right) \\
& <\sum_{i=1}^{N_{n}} \frac{\bar{\mu}([M f>\varepsilon])}{N_{n}}=\bar{\mu}([M f>\varepsilon])
\end{aligned}
$$

a contradiction. Therefore for this $i$,

$$
\begin{aligned}
\frac{\bar{\mu}([M f>\varepsilon])}{N_{n}} & \leq \mu\left(\cup\left\{B: B \in \mathscr{G}_{i}\right\}\right)=\sum_{B \in \mathscr{G}_{i}} \mu(B) \leq \sum_{B \in \mathscr{G}_{i}} \varepsilon^{-1} \int_{B}|f| d \mu \\
& \leq \varepsilon^{-1} \int_{\mathbb{R}^{n}}|f| d \mu=\varepsilon^{-1}| | f \|_{1}
\end{aligned}
$$

This shows Claim 1.
Claim 2: If $g$ is any continuous function defined on $\mathbb{R}^{n}$, then for $\mathbf{x} \notin Z$,

$$
\lim _{r \rightarrow 0} \frac{1}{\mu(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|g(\mathbf{y})-g(\mathbf{x})| d \mu(y)=0
$$

and

$$
\begin{equation*}
\lim _{r \rightarrow 0} \frac{1}{\mu(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} g(\mathbf{y}) d \mu(y)=g(\mathbf{x}) \tag{31.1.1}
\end{equation*}
$$

Proof: Since $g$ is continuous at $\mathbf{x}$, whenever $r$ is small enough,

$$
\frac{1}{\mu(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|g(\mathbf{y})-g(\mathbf{x})| d \mu(y) \leq \frac{1}{\mu(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} \varepsilon d \mu(y)=\varepsilon .
$$

31.1.1 follows from the above and the triangle inequality. This proves the claim.

Now let $g \in C_{c}\left(\mathbb{R}^{n}\right)$ and $\mathbf{x} \notin Z$. Then from the above observations about continuous functions,

$$
\begin{align*}
& \bar{\mu}\left(\left[\mathbf{x} \notin Z: \lim _{r \rightarrow 0} \sup _{r \rightarrow 0} \frac{1}{\mu(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d \mu(y)>\varepsilon\right]\right)  \tag{31.1.2}\\
& \quad \leq \\
& \quad \bar{\mu}\left(\left[\mathbf{x} \notin Z: \lim \sup _{r \rightarrow 0} \frac{1}{\mu(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-g(\mathbf{y})| d \mu(y)>\frac{\varepsilon}{2}\right]\right) \\
& \quad+\bar{\mu}\left(\left[\mathbf{x} \notin Z:|g(\mathbf{x})-f(\mathbf{x})|>\frac{\varepsilon}{2}\right]\right) .
\end{align*}
$$

$$
\begin{equation*}
\leq \bar{\mu}\left(\left[M(f-g)>\frac{\varepsilon}{2}\right]\right)+\bar{\mu}\left(\left[|f-g|>\frac{\varepsilon}{2}\right]\right) \tag{31.1.3}
\end{equation*}
$$

Now

$$
\int_{\left[|f-g|>\frac{\varepsilon}{2}\right]}|f-g| d \mu \geq \frac{\varepsilon}{2} \bar{\mu}\left(\left[|f-g|>\frac{\varepsilon}{2}\right]\right)
$$

and so from Claim 1 31.1.3 and hence 31.1.2 is dominated by

$$
\left(\frac{2}{\varepsilon}+\frac{N_{n}}{\varepsilon}\right)\|f-g\|_{L^{1}\left(\mathbb{R}^{n}, \mu\right)}
$$

But by regularity of Radon measures, $C_{c}\left(\mathbb{R}^{n}\right)$ is dense in $L^{1}\left(\mathbb{R}^{n}, \mu\right)$, and so since $g$ in the above is arbitrary, this shows 31.1.2 equals 0 . Now

$$
\begin{aligned}
& \bar{\mu}\left(\left[\mathbf{x} \notin Z:{\lim \sup _{r \rightarrow 0}}^{\mu(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d \mu(y)>0\right]\right) \\
\leq & \sum_{k=1}^{\infty} \bar{\mu}\left(\left[\mathbf{x} \notin Z: \lim _{r \rightarrow 0} \frac{1}{\mu(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d \mu(y)>\frac{1}{k}\right]\right)=0
\end{aligned}
$$

By completeness of $\mu$ this implies

$$
\left[\mathbf{x} \notin Z: \lim \sup _{r \rightarrow 0} \frac{1}{\mu(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d \mu(y)>0\right]
$$

is a set of $\mu$ measure zero
The following corollary is the main result referred to as the Lebesgue Besicovitch Differentiation theorem.

Corollary 31.1.3 If $f \in L_{\text {loc }}^{1}\left(\mathbb{R}^{n}, \mu\right)$, then for a.e. $\mathbf{x} \notin Z$,

$$
\begin{equation*}
\lim _{r \rightarrow 0} \frac{1}{\mu(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d \mu(y)=0 \tag{31.1.4}
\end{equation*}
$$

Proof: If $f$ is replaced by $f \mathscr{X}_{B(\mathbf{0}, k)}$ then the conclusion 31.1.4 holds for all $\mathbf{x} \notin F_{k}$ where $F_{k}$ is a set of $\mu$ measure 0 . Letting $k=1,2, \cdots$, and $F \equiv \cup_{k=1}^{\infty} F_{k}$, it follows that $F$ is a set of measure zero and for any $\mathbf{x} \notin F$, and $k \in\{1,2, \cdots\}, 31.1 .4$ holds if $f$ is replaced by $f \mathscr{X}_{B(\mathbf{0}, k)}$. Picking any such $\mathbf{x}$, and letting $k>|\mathbf{x}|+1$, this shows

$$
\begin{gathered}
\lim _{r \rightarrow 0} \frac{1}{\mu(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d \mu(y) \\
=\lim _{r \rightarrow 0} \frac{1}{\mu(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}\left|f \mathscr{X}_{B(\mathbf{0}, k)}(\mathbf{y})-f \mathscr{X}_{B(\mathbf{0}, k)}(\mathbf{x})\right| d \mu(y)=0 .
\end{gathered}
$$

### 31.2 Slicing Measures

Let $\mu$ be a finite Radon measure. I will show here that a formula of the following form holds.

$$
\mu(F)=\int_{F} d \mu=\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{m}} \mathscr{X}_{F}(\mathbf{x}, \mathbf{y}) d v_{\mathbf{x}}(y) d \alpha(x)
$$

where $\alpha(E)=\mu\left(E \times \mathbb{R}^{m}\right)$. When this is done, the measures, $v_{\mathbf{x}}$, are called slicing measures and this shows that an integral with respect to $\mu$ can be written as an iterated integral in terms of the measure $\alpha$ and the slicing measures, $v_{\mathbf{x}}$. This is like going backwards in the construction of product measure. One starts with a measure $\mu$, defined on the Cartesian product and produces $\alpha$ and an infinite family of slicing measures from it whereas in the construction of product measure, one starts with two measures and obtains a new measure on a $\sigma$ algebra of subsets of the Cartesian product of two spaces. These slicing measures are dependent on $\mathbf{x}$. Later, this will be tied to the concept of independence or not of random variables. First here are two technical lemmas.

Lemma 31.2.1 The space $C_{c}\left(\mathbb{R}^{m}\right)$ with the norm

$$
\|f\| \equiv \sup \left\{|f(\mathbf{y})|: \mathbf{y} \in \mathbb{R}^{m}\right\}
$$

is separable.
Proof: Let $\mathscr{D}_{l}$ consist of all functions which are of the form

$$
\sum_{|\alpha| \leq N} a_{\alpha} \mathbf{y}^{\alpha}\left(\operatorname{dist}\left(\mathbf{y}, B(\mathbf{0}, l+1)^{C}\right)\right)^{n_{\alpha}}
$$

where $a_{\alpha} \in \mathbb{Q}, \alpha$ is a multi-index, and $n_{\alpha}$ is a positive integer. Consider $\mathscr{D} \equiv \cup_{l} \mathscr{D}_{l}$. Then $\mathscr{D}$ is countable. If $f \in C_{c}\left(\mathbb{R}^{n}\right)$, then choose $l$ large enough that $\operatorname{spt}(f) \subseteq B(\mathbf{0}, l+1)$, a locally compact space, $f \in C_{0}(B(\mathbf{0}, l+1))$. Then since $\mathscr{D}_{l}$ separates the points of $B(\mathbf{0}, l+1)$ is closed with respect to conjugates, and annihilates no point, it is dense in $C_{0}(B(\mathbf{0}, l+1))$ by the Stone Weierstrass theorem. Alternatively, $\mathscr{D}$ is dense in $C_{0}\left(\mathbb{R}^{n}\right)$ by Stone Weierstrass and $C_{c}\left(\mathbb{R}^{n}\right)$ is a subspace so it is also separable. So is $C_{c}\left(\mathbb{R}^{n}\right)^{+}$, the nonnegative functions in $C_{c}\left(\mathbb{R}^{n}\right)$.

From the regularity of Radon measures, the following lemma follows.
Lemma 31.2.2 If $\mu$ and $v$ are two Radon measures defined on $\sigma$ algebras, $\mathscr{S}_{\mu}$ and $\mathscr{S}_{v}$, of subsets of $\mathbb{R}^{n}$ and if $\mu(V)=v(V)$ for all $V$ open, then $\mu=v$ and $\mathscr{S}_{\mu}=\mathscr{S}_{v}$.

Proof: Every compact set is a countable intersection of open sets so the two measures agree on every compact set. Hence it is routine that the two measures agree on every $G_{\delta}$ and $F_{\sigma}$ set. (Recall $G_{\delta}$ sets are countable intersections of open sets and $F_{\sigma}$ sets are countable unions of closed sets.) Now suppose $E \in \mathscr{S}_{v}$ is a bounded set. Then by regularity of $v$ there exists $G$ a $G_{\delta}$ set and $F$, an $F_{\sigma}$ set such that $F \subseteq E \subseteq G$ and $v(G \backslash F)=0$. Then it is also true that $\mu(G \backslash F)=0$. Hence $E=F \cup(E \backslash F)$ and $E \backslash F$ is a subset of $G \backslash F$, a set of $\mu$ measure zero. By completeness of $\mu$, it follows $E \in \mathscr{S}_{\mu}$ and

$$
\mu(E)=\mu(F)=v(F)=v(E)
$$

If $E \in \mathscr{S}_{v}$ not necessarily bounded, let $E_{m}=E \cap B(0, m)$ and then $E_{m} \in \mathscr{S}_{\mu}$ and $\mu\left(E_{m}\right)=$ $v\left(E_{m}\right)$. Letting $m \rightarrow \infty, E \in \mathscr{S}_{\mu}$ and $\mu(E)=v(E)$. Similarly, $\mathscr{S}_{\mu} \subseteq \mathscr{S}_{v}$ and the two measures are equal on $\mathscr{S}_{\mu}$.

The main result in the section is the following theorem.
Theorem 31.2.3 Let $\mu$ be a finite Radon measure on $\mathbb{R}^{n+m}$ defined on a $\sigma$ algebra, $\mathscr{F}$. Then there exists a unique finite Radon measure $\alpha$, defined on a $\sigma$ algebra $\mathscr{S}$, of sets of $\mathbb{R}^{n}$ which satisfies

$$
\begin{equation*}
\alpha(E)=\mu\left(E \times \mathbb{R}^{m}\right) \tag{31.2.5}
\end{equation*}
$$

for all E Borel. There also exists a Borel set of $\alpha$ measure zero $N$, such that for each $\mathbf{x} \notin N$, there exists a Radon probability measure $v_{\mathbf{x}}$ such that if $f$ is a nonnegative $\mu$ measurable function or a $\mu$ measurable function in $L^{1}(\mu)$,

$$
\begin{gather*}
\mathbf{y} \rightarrow f(\mathbf{x}, \mathbf{y}) \text { is } v_{\mathbf{x}} \text { measurable } \alpha \text { a.e. } \\
\mathbf{x} \rightarrow \int_{\mathbb{R}^{m}} f(\mathbf{x}, \mathbf{y}) d v_{\mathbf{x}}(y) \text { is } \alpha \text { measurable } \tag{31.2.6}
\end{gather*}
$$

and

$$
\begin{equation*}
\int_{\mathbb{R}^{n+m}} f(\mathbf{x}, \mathbf{y}) d \mu=\int_{\mathbb{R}^{n}}\left(\int_{\mathbb{R}^{m}} f(\mathbf{x}, \mathbf{y}) d v_{\mathbf{x}}(y)\right) d \alpha(x) \tag{31.2.7}
\end{equation*}
$$

If $\widehat{v}_{\mathbf{x}}$ is any other collection of Radon measures satisfying 31.2.6 and 31.2.7, then $\widehat{v}_{\mathbf{x}}=v_{\mathbf{x}}$ for $\alpha$ a.e. $\mathbf{x}$.

## Proof:

## Existence and uniqueness of $\alpha$

First consider the uniqueness of $\alpha$. Suppose $\alpha_{1}$ is another Radon measure satisfying 31.2.5. Then in particular, $\alpha_{1}$ and $\alpha$ agree on open sets and so the two measures are the same by Lemma 31.2.2.

To establish the existence of $\alpha$, define $\alpha_{0}$ on Borel sets by

$$
\alpha_{0}(E)=\mu\left(E \times \mathbb{R}^{m}\right)
$$

Thus $\alpha_{0}$ is a finite Borel measure and so it is finite on compact sets. Lemma 14.2.3 on Page 388 implies the existence of the Radon measure $\alpha$ extending $\alpha_{0}$.

## Uniqueness of $v_{\mathbf{x}}$

Next consider the uniqueness of $v_{\mathbf{x}}$. Suppose $v_{\mathbf{x}}$ and $\widehat{v}_{\mathbf{x}}$ satisfy all conclusions of the theorem with exceptional sets denoted by $N$ and $\widehat{N}$ respectively. Then, enlarging $N$ and $\widehat{N}$, one may also assume, using Lemma 31.1.1, that for $\mathbf{x} \notin N \cup \widehat{N}, \alpha(B(\mathbf{x}, r))>0$ whenever $r>0$. Now let

$$
A=\prod_{i=1}^{m}\left(a_{i}, b_{i}\right]
$$

where $a_{i}$ and $b_{i}$ are rational. Thus there are countably many such sets. Then from the conclusion of the theorem, if $\mathbf{x}_{0} \notin N \cup \widehat{N}$,

$$
\begin{aligned}
& \frac{1}{\alpha\left(B\left(\mathbf{x}_{0}, r\right)\right)} \int_{B\left(\mathbf{x}_{0}, r\right)} \int_{\mathbb{R}^{m}} \mathscr{X}_{A}(\mathbf{y}) d v_{\mathbf{x}}(y) d \alpha \\
= & \frac{1}{\alpha\left(B\left(\mathbf{x}_{0}, r\right)\right)} \int_{B\left(\mathbf{x}_{0}, r\right)} \int_{\mathbb{R}^{m}} \mathscr{X}_{A}(\mathbf{y}) d \widehat{v}_{\mathbf{x}}(y) d \alpha
\end{aligned}
$$

and by the Lebesgue Besicovitch Differentiation theorem, there exists a set of $\alpha$ measure zero, $E_{A}$, such that if $\mathbf{x}_{0} \notin E_{A} \cup N \cup \widehat{N}$, then the limit in the above exists as $r \rightarrow 0$ and yields

$$
v_{\mathbf{x}_{0}}(A)=\widehat{v}_{\mathbf{x}_{0}}(A)
$$

Letting $E$ denote the union of all the sets $E_{A}$ for $A$ as described above, it follows that $E$ is a set of measure zero and if $\mathbf{x}_{0} \notin E \cup N \cup \widehat{N}$ then $V_{\mathbf{x}_{0}}(A)=\widehat{v}_{\mathbf{x}_{0}}(A)$ for all such sets $A$. But every open set can be written as a disjoint union of sets of this form and so for all such $\mathbf{x}_{0}, v_{\mathbf{x}_{0}}(V)=\widehat{v}_{\mathbf{x}_{0}}(V)$ for all $V$ open. By Lemma 31.2.2 this shows the two measures are equal and proves the uniqueness assertion for $v_{\mathbf{x}}$. It remains to show the existence of the measures $v_{\mathbf{x}}$.

## Existence of $v_{\mathbf{x}}$

For $f \geq 0, f, g \in C_{c}\left(\mathbb{R}^{m}\right)$ and $C_{c}\left(\mathbb{R}^{n}\right)$ respectively, define

$$
g \rightarrow \int_{\mathbb{R}^{n+m}} g(\mathbf{x}) f(\mathbf{y}) d \mu
$$

Since $f \geq 0$, this is a positive linear functional on $C_{c}\left(\mathbb{R}^{n}\right)$. Therefore, there exists a unique Radon measure $v_{f}$ such that for all $g \in C_{c}\left(\mathbb{R}^{n}\right)$,

$$
\int_{\mathbb{R}^{n+m}} g(\mathbf{x}) f(\mathbf{y}) d \mu=\int_{\mathbb{R}^{n}} g(\mathbf{x}) d v_{f}
$$

I claim that $v_{f} \ll \alpha$, the two being considered as measures on $\mathscr{B}\left(\mathbb{R}^{n}\right)$. Suppose then that $K$ is a compact set and $\alpha(K)=0$. Then let $K \prec g \prec V$ where $V$ is open.

$$
\begin{gathered}
v_{f}(K)=\int_{\mathbb{R}^{n}} \mathscr{X}_{K}(\mathbf{x}) d v_{f}(\mathbf{x}) \leq \int_{\mathbb{R}^{n}} g(\mathbf{x}) d v_{f}(\mathbf{x})=\int_{\mathbb{R}^{n+m}} g(\mathbf{x}) f(\mathbf{y}) d \mu \\
\quad \leq \int_{\mathbb{R}^{m+m}} \mathscr{X}_{V \times \mathbb{R}^{m}}(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d \mu \leq\|f\|_{\infty} \mu\left(V \times \mathbb{R}^{m}\right)=\|f\|_{\infty} \alpha(V)
\end{gathered}
$$

Then for any $\varepsilon>0$, one can choose $V$ such that the right side is less than $\varepsilon$. Therefore, $v_{f}(K)=0$ also. By regularity considerations, $v_{f} \ll \alpha$ as claimed.

It follows from the Radon Nikodym theorem the existence of a function $h_{f} \in L^{1}(\alpha)$ such that for all $g \in C_{c}\left(\mathbb{R}^{n}\right)$,

$$
\begin{equation*}
\int_{\mathbb{R}^{n+m}} g(\mathbf{x}) f(\mathbf{y}) d \mu=\int_{\mathbb{R}^{n}} g(\mathbf{x}) d v_{f}=\int_{\mathbb{R}^{n}} g(\mathbf{x}) h_{f}(\mathbf{x}) d \alpha \tag{31.2.8}
\end{equation*}
$$

It is obvious from the formula that the map from $f \in C_{c}\left(\mathbb{R}^{m}\right)$ to $L^{1}(\alpha)$ given by $f \rightarrow h_{f}$ is linear. However, this is not sufficiently specific because functions in $L^{1}(\alpha)$ are only determined a.e. However, for $h_{f} \in L^{1}(\alpha)$, you can specify a particular representative $\alpha$ a.e. By the fundamental theorem of calculus,

$$
\begin{equation*}
\widehat{h_{f}}(\mathbf{x}) \equiv \lim _{r \rightarrow 0} \frac{1}{\alpha(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} h_{f}(\mathbf{z}) d \alpha(z) \tag{31.2.9}
\end{equation*}
$$

exists off some set of measure zero $Z_{f}$. Note that since this involves the integral over a ball, it does not matter which representative of $h_{f}$ is placed in the formula. Therefore, $\widehat{h_{f}}(\mathbf{x})$ is well defined pointwise for all $\mathbf{x}$ not in some set of measure zero $Z_{f}$. Since $\widehat{h_{f}}=h_{f}$ a.e. it follows that $\widehat{h_{f}}$ is well defined and will work in the formula 31.2.8. Let

$$
Z=\cup\left\{Z_{f}: f \in \mathscr{D}\right\}
$$

where $\mathscr{D}$ is a countable dense subset of $C_{c}\left(\mathbb{R}^{m}\right)^{+}$. Of course it is desired to have the limit 31.2.9 hold for all $f$, not just $f \in \mathscr{D}$. We will show that this limit holds for all $\mathbf{x} \notin Z$. Thus, we will have $\mathbf{x} \rightarrow \widehat{h_{f}}(\mathbf{x})$ defined by the above limit off $Z$ and so, since $\widehat{h_{f}}(\mathbf{x})=h_{f}(\mathbf{x})$ a.e., it follows that

$$
\int_{\mathbb{R}^{n+m}} g(\mathbf{x}) f(\mathbf{y}) d \mu=\int_{\mathbb{R}^{n}} g(\mathbf{x}) d v_{f}=\int_{\mathbb{R}^{n}} g(\mathbf{x}) \widehat{h_{f}}(\mathbf{x}) d \alpha
$$

One could then take $\widehat{h_{f}}(\mathbf{x})$ to be defined as 0 for $\mathbf{x} \notin Z$.
For $f$ an arbitrary function in $C_{c}\left(\mathbb{R}^{m}\right)^{+}$and $f^{\prime} \in \mathscr{D}$, a dense countable subset of $C_{c}\left(\mathbb{R}^{n}\right)^{+}$, it follows from 31.2.8,

$$
\left|\int_{\mathbb{R}^{n}} g(\mathbf{x})\left(h_{f}(\mathbf{x})-h_{f^{\prime}}(\mathbf{x})\right) d \alpha\right| \leq\left|\left|f-f^{\prime} \|_{\infty} \int_{\mathbb{R}^{n+m}}\right| g(\mathbf{x})\right| d \mu
$$

Let $g_{k}(\mathbf{x}) \uparrow \mathscr{X}_{B(\mathbf{z}, r)}(\mathbf{x})$ where $\mathbf{z} \notin Z$. Then by the dominated convergence theorem, the above implies

$$
\left|\int_{B(\mathbf{z}, r)}\left(h_{f}(\mathbf{x})-h_{f^{\prime}}(\mathbf{x})\right) d \alpha\right| \leq\left\|f-f^{\prime}\right\|_{\infty} \int_{B(\mathbf{z}, r) \times \mathbb{R}^{m}} d \mu=\left\|f-f^{\prime}\right\|_{\infty} \alpha(B(\mathbf{z}, r)) .
$$

Dividing by $\alpha(B(\mathbf{z}, r))$, it follows that if $\alpha(B(\mathbf{z}, r))>0$ for all $r>0$, then for all $r>0$,

$$
\left|\frac{1}{\alpha(B(\mathbf{z}, r))} \int_{B(\mathbf{z}, r)}\left(h_{f}(\mathbf{x})-h_{f^{\prime}}(\mathbf{x})\right) d \alpha\right| \leq\left\|f-f^{\prime}\right\|_{\infty}
$$

It follows that for $f \in C_{c}\left(\mathbb{R}^{m}\right)^{+}$arbitrary and $\mathbf{z} \notin Z$,

$$
\begin{aligned}
& \lim \sup _{r \rightarrow 0} \frac{1}{\alpha(B(\mathbf{z}, r))} \int_{B(\mathbf{z}, r)} h_{f}(\mathbf{x}) d \alpha-\lim _{r \rightarrow 0} \inf _{r(B(\mathbf{z}, r))} \frac{1}{B(\mathbf{z}, r)} h_{f}(\mathbf{x}) d \alpha \\
= & \lim \sup _{r \rightarrow 0} \frac{1}{\alpha(B(\mathbf{z}, r))} \int_{B(\mathbf{z}, r)}\left(h_{f}(\mathbf{x})-h_{f^{\prime}}(\mathbf{x})\right) d \alpha(x) \\
& -\liminf _{r \rightarrow 0} \frac{1}{\alpha(B(\mathbf{z}, r))} \int_{B(\mathbf{z}, r)}\left(h_{f}(\mathbf{x})-h_{f^{\prime}}(\mathbf{x})\right) d \alpha(x) \\
\leq & \left|\limsup _{r \rightarrow 0} \frac{1}{\alpha(B(\mathbf{z}, r))} \int_{B(\mathbf{z}, r)}\left(h_{f}(\mathbf{x})-h_{f^{\prime}}(\mathbf{x})\right) d \alpha(x)\right| \\
& +\left|\lim _{r \rightarrow 0} \frac{1}{\alpha(B(\mathbf{z}, r))} \int_{B(\mathbf{z}, r)}\left(h_{f}(\mathbf{x})-h_{f^{\prime}}(\mathbf{x})\right) d \alpha(x)\right| \\
\leq & 2\left|\left|f-f^{\prime}\right|\right|_{\infty}
\end{aligned}
$$

and since $f^{\prime}$ is arbitrary, it follows that the limit of 31.2 .9 holds for all $f \in C_{c}\left(\mathbb{R}^{m}\right)^{+}$whenever $\mathbf{z} \notin Z$, the above set of measure zero.

Now for $f$ an arbitrary real valued function of $C_{c}\left(\mathbb{R}^{n}\right)$, simply apply the above result to positive and negative parts to obtain $h_{f} \equiv h_{f^{+}}-h_{f^{-}}$and $\widehat{h_{f}} \equiv \widehat{h_{f^{+}}}-\widehat{h_{f^{-}}}$. Then it follows that for all $f \in C_{c}\left(\mathbb{R}^{m}\right)$ and $g \in C_{c}\left(\mathbb{R}^{m}\right)$

$$
\int_{\mathbb{R}^{n+m}} g(\mathbf{x}) f(\mathbf{y}) d \mu=\int_{\mathbb{R}^{n}} g(\mathbf{x}) \widehat{h_{f}}(\mathbf{x}) d \alpha
$$

It is obvious from the description given above that for each $\mathbf{x} \notin Z$, the set of measure zero given above, that $f \rightarrow \widehat{h_{f}}(\mathbf{x})$ is a positive linear functional. It is clear that it acts like a linear map for nonnegative $f$ and so the usual trick just described above is well defined and delivers a positive linear functional. Hence by the Riesz representation theorem, there exists a unique $v_{\mathbf{x}}$ such that for all $\mathbf{x}$

$$
\widehat{h_{f}}(\mathbf{x})=\int_{\mathbb{R}^{m}} f(\mathbf{y}) d v_{\mathbf{x}}(y)
$$

It follows that

$$
\begin{equation*}
\int_{\mathbb{R}^{n+m}} g(\mathbf{x}) f(\mathbf{y}) d \mu=\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{m}} g(\mathbf{x}) f(\mathbf{y}) d v_{\mathbf{x}}(y) d \alpha(x) \tag{31.2.10}
\end{equation*}
$$

and $\mathbf{x} \rightarrow \int_{\mathbb{R}^{m}} f(\mathbf{y}) d v_{\mathbf{x}}$ is $\alpha$ measurable and $v_{\mathbf{x}}$ is a Radon measure.
Now let $f_{k} \uparrow \mathscr{X}_{\mathbb{R}^{m}}$ and $g \geq 0$. Then by monotone convergence theorem,

$$
\int_{\mathbb{R}^{n+m}} g(\mathbf{x}) d \mu=\int_{\mathbb{R}^{n}} g(\mathbf{x}) \int_{\mathbb{R}^{m}} d v_{\mathbf{x}} d \alpha
$$

If $g_{k} \uparrow \mathscr{X}_{\mathbb{R}^{n}}$, the monotone convergence theorem shows that $\mathbf{x} \rightarrow \int_{\mathbb{R}^{m}} d v_{\mathbf{x}}$ is $L^{1}(\alpha)$.
Next let $g_{k} \uparrow \mathscr{X}_{B(\mathbf{x}, r)}$ and use monotone convergence theorem to write

$$
\alpha(B(\mathbf{x}, r)) \equiv \int_{B(\mathbf{x}, r) \times \mathbb{R}^{m}} d \mu=\int_{B(\mathbf{x}, r)} \int_{\mathbb{R}^{m}} d v_{\mathbf{x}} d \alpha
$$

Then dividing by $\alpha(B(\mathbf{x}, r))$ and taking a limit as $r \rightarrow 0$, it follows that for $\alpha$ a.e. $\mathbf{x}$, $1=v_{\mathbf{x}}\left(\mathbb{R}^{m}\right)$, so these $v_{\mathbf{x}}$ are probability measures off a set of $\alpha$ measure zero. Letting $g_{k}(\mathbf{x}) \uparrow \mathscr{X}_{A}(\mathbf{x}), f_{k}(\mathbf{y}) \uparrow \mathscr{X}_{B}(\mathbf{y})$ for $A, B$ open, it follows that 31.2.10 is valid for $g(\mathbf{x})$ replaced with $\mathscr{X}_{A}(\mathbf{x})$ and $f(\mathbf{y})$ replaced with $\mathscr{X}_{B}(\mathbf{y})$.

Now let $\mathscr{G}$ denote the Borel sets $F$ of $\mathbb{R}^{n+m}$ such that

$$
\int_{\mathbb{R}^{n+m}} \mathscr{X}_{F}(\mathbf{x}, \mathbf{y}) d \mu(x, y)=\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{m}} \mathscr{X}_{F}(\mathbf{x}, \mathbf{y}) d v_{\mathbf{x}}(y) d \alpha(x)
$$

and that all the integrals make sense. As just explained, this includes all Borel sets of the form $F=A \times B$ where $A, B$ are open. It is clear that $\mathscr{G}$ is closed with respect to countable disjoint unions and complements, while sets of the form $A \times B$ for $A, B$ open form a $\pi$ system. Therefore, by Lemma $12.12 .3, \mathscr{G}$ contains the Borel sets which is the smallest $\sigma$ algebra which contains such products of open sets. It follows from the usual approximation with simple functions that if $f \geq 0$ and is Borel measurable, then

$$
\int_{\mathbb{R}^{n+m}} f(\mathbf{x}, \mathbf{y}) d \mu(x, y)=\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{m}} f(\mathbf{x}, \mathbf{y}) d v_{\mathbf{x}}(y) d \alpha(x)
$$

with all the integrals making sense.
This proves the theorem in the case where $f$ is Borel measurable and nonnegative. It just remains to extend this to the case where $f$ is only $\mu$ measurable. However, from regularity of $\mu$ there exist Borel measurable functions $g, h, g \leq f \leq h$ such that

$$
\begin{aligned}
\int_{\mathbb{R}^{n+m}} f(\mathbf{x}, \mathbf{y}) d \mu(x, y) & =\int_{\mathbb{R}^{n+m}} g(\mathbf{x}, \mathbf{y}) d \mu(x, y) \\
& =\int_{\mathbb{R}^{n+m}} h(\mathbf{x}, \mathbf{y}) d \mu(x, y)
\end{aligned}
$$

It follows

$$
\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{m}} g(\mathbf{x}, \mathbf{y}) d v_{\mathbf{x}}(y) d \alpha(x)=\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{m}} h(\mathbf{x}, \mathbf{y}) d v_{\mathbf{x}}(y) d \alpha(x)
$$

and so, since for $\alpha$ a.e. $\mathbf{x}, \mathbf{y} \rightarrow \mathbf{g}(\mathbf{x}, \mathbf{y})$ and $\mathbf{y} \rightarrow \mathbf{h}(\mathbf{x}, \mathbf{y})$ are $v_{\mathbf{x}}$ measurable with

$$
0=\int_{\mathbb{R}^{m}}(h(\mathbf{x}, \mathbf{y})-g(\mathbf{x}, \mathbf{y})) d v_{\mathbf{x}}(y)
$$

and $v_{\mathbf{x}}$ is a Radon measure, hence complete, it follows for $\alpha$ a.e. $\mathbf{x}, \mathbf{y} \rightarrow f(\mathbf{x}, \mathbf{y})$ must be $v_{\mathbf{x}}$ measurable because it is equal to $\mathbf{y} \rightarrow g(\mathbf{x}, \mathbf{y}), v_{\mathbf{x}}$ a.e. Therefore, for $\alpha$ a.e. $\mathbf{x}$, it makes sense to write

$$
\int_{\mathbb{R}^{m}} f(\mathbf{x}, \mathbf{y}) d v_{\mathbf{x}}(y)
$$

Similar reasoning applies to the above function of $\mathbf{x}$ being $\alpha$ measurable due to $\alpha$ being complete. It follows

$$
\begin{aligned}
\int_{\mathbb{R}^{n+m}} f(\mathbf{x}, \mathbf{y}) d \mu(x, y) & =\int_{\mathbb{R}^{n+m}} g(\mathbf{x}, \mathbf{y}) d \mu(x, y) \\
& =\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{m}} g(\mathbf{x}, \mathbf{y}) d v_{\mathbf{x}}(y) d \alpha(x) \\
& =\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{m}} f(\mathbf{x}, \mathbf{y}) d v_{\mathbf{x}}(y) d \alpha(x)
\end{aligned}
$$

with everything making sense.

### 31.3 Differentiation of Radon Measures

This section is a generalization of earlier ideas in which differentiation was with respect to Lebesgue measure. Here an arbitrary Radon measure, not necessarily an integral with respect to Lebesuge measure, will be differentiated with respect to another arbitrary Radon measure. This requires a more sophisticated covering theorem. In this section, $B(\mathbf{x}, r)$ will denote a closed ball with center $\mathbf{x}$ and radius $r$. Also, let $\lambda$ and $\mu$ be Radon measures and as above, $Z$ will denote a $\mu$ measure zero set off of which $\mu(B(\mathbf{x}, r))>0$ for all $r>0$.

Definition 31.3.1 For $\mathbf{x} \notin Z$, define the upper and lower symmetric derivatives as

$$
\bar{D}_{\mu} \lambda(\mathbf{x}) \equiv \lim \sup _{r \rightarrow 0} \frac{\lambda(B(\mathbf{x}, r))}{\mu(B(\mathbf{x}, r))}, \underline{D}_{\mu} \lambda(\mathbf{x}) \equiv \lim _{r \rightarrow 0} \inf \frac{\lambda(B(\mathbf{x}, r))}{\mu(B(\mathbf{x}, r))}
$$

respectively. Also define

$$
D_{\mu} \lambda(\mathbf{x}) \equiv \bar{D}_{\mu} \lambda(\mathbf{x})=\underline{D}_{\mu} \lambda(\mathbf{x})
$$

in the case when both the upper and lower derivatives are equal.
Lemma 31.3.2 Let $\lambda$ and $\mu$ be Radon measures. If $A$ is a bounded subset of

$$
\left\{\mathbf{x} \notin Z: \bar{D}_{\mu} \lambda(\mathbf{x}) \geq a\right\}
$$

then

$$
\bar{\lambda}(A) \geq a \bar{\mu}(A)
$$

and if $A$ is a bounded subset of $\left\{\mathbf{x} \notin Z: \underline{D}_{\mu} \lambda(\mathbf{x}) \leq a\right\}$, then

$$
\bar{\lambda}(A) \leq a \bar{\mu}(A)
$$

The same conclusion holds even if A is not necessarily bounded.
Proof: Suppose first that $A$ is a bounded subset of $\left\{\mathbf{x} \notin Z: \bar{D}_{\mu} \lambda(\mathbf{x}) \geq a\right\}$, let $\varepsilon>0$, and let $V$ be a bounded open set with $V \supseteq A$ and $\lambda(V)-\varepsilon<\bar{\lambda}(A), \mu(V)-\varepsilon<\bar{\mu}(A)$. Then if $\mathbf{x} \in A$,

$$
\frac{\lambda(B(\mathbf{x}, r))}{\mu(B(\mathbf{x}, r))}>a-\varepsilon, B(\mathbf{x}, r) \subseteq V
$$

for infinitely many values of $r$ which are arbitrarily small. Thus the collection of such balls constitutes a Vitali cover for $A$. By Corollary 13.14.3 there is a disjoint sequence of these closed balls $\left\{B_{i}\right\}$ such that

$$
\begin{equation*}
\bar{\mu}\left(A \backslash \cup_{i=1}^{\infty} B_{i}\right)=0 \tag{31.3.11}
\end{equation*}
$$

Therefore,

$$
(a-\varepsilon) \sum_{i=1}^{\infty} \mu\left(B_{i}\right)<\sum_{i=1}^{\infty} \lambda\left(B_{i}\right) \leq \lambda(V)<\varepsilon+\bar{\lambda}(A)
$$

and so

$$
\begin{align*}
a \sum_{i=1}^{\infty} \mu\left(B_{i}\right) & \leq \varepsilon+\varepsilon \mu(V)+\bar{\lambda}(A) \\
& \leq \varepsilon+\varepsilon(\bar{\mu}(A)+\varepsilon)+\bar{\lambda}(A) \tag{31.3.12}
\end{align*}
$$

Now

$$
\bar{\mu}\left(A \backslash \cup_{i=1}^{\infty} B_{i}\right)+\bar{\mu}\left(\cup_{i=1}^{\infty} B_{i}\right) \geq \bar{\mu}(A)
$$

and so by 31.3.11 and the fact the $B_{i}$ are disjoint, it follows from 31.3.12,

$$
\begin{align*}
a \bar{\mu}(A) & \leq a \bar{\mu}\left(\cup_{i=1}^{\infty} B_{i}\right)=a \sum_{i=1}^{\infty} \mu\left(B_{i}\right) \\
& \leq \varepsilon+\varepsilon(\bar{\mu}(A)+\varepsilon)+\bar{\lambda}(A) \tag{31.3.13}
\end{align*}
$$

Hence $a \bar{\mu}(A) \leq \bar{\lambda}(A)$ since $\varepsilon>0$ was arbitrary.
Now suppose $A$ is a bounded subset of $\left\{\mathbf{x} \notin Z: \underline{D}_{\mu} \lambda(\mathbf{x}) \leq a\right\}$ and let $V$ be a bounded open set containing $A$ with $\mu(V)-\varepsilon<\bar{\mu}(A)$. Then if $\mathbf{x} \in A$,

$$
\frac{\lambda(B(\mathbf{x}, r))}{\mu(B(\mathbf{x}, r))}<a+\varepsilon, B(\mathbf{x}, r) \subseteq V
$$

for values of $r$ which are arbitrarily small. Therefore, by Corollary 13.14.3 again, there exists a disjoint sequence of these balls, $\left\{B_{i}\right\}$ satisfying this time,

$$
\bar{\lambda}\left(A \backslash \cup_{i=1}^{\infty} B_{i}\right)=0
$$

Then by arguments similar to the above,

$$
\bar{\lambda}(A) \leq \sum_{i=1}^{\infty} \lambda\left(B_{i}\right)<(a+\varepsilon) \mu(V)<(a+\varepsilon)(\bar{\mu}(A)+\varepsilon) .
$$

Since $\varepsilon$ was arbitrary, this proves the lemma in case $A$ is bounded. In general, for

$$
A \in\left\{\mathbf{x} \notin Z: \bar{D}_{\mu} \lambda(\mathbf{x}) \geq a\right\}
$$

One obtains from the first part

$$
\lambda(A \cap B(\mathbf{0}, n)) \geq a \mu(A \cap B(\mathbf{0}, n))
$$

Then, passing to a limit as $n \rightarrow \infty$ gives the desired result. The case where

$$
A \subseteq\left\{\mathbf{x} \notin Z: \underline{D}_{\mu} \lambda(\mathbf{x}) \leq a\right\}
$$

is similar.
Theorem 31.3.3 There exists a set of measure zero $N$ containing $Z$ such that for $\mathbf{x} \notin$ $N, D_{\mu} \lambda(\mathbf{x})$ exists and also $\mathscr{X}_{N^{C}}(\cdot) D_{\mu} \lambda(\cdot)$ is a $\mu$ measurable function. Furthermore, $D_{\mu} \lambda(\mathbf{x})<\infty \mu$ a.e.

Proof: First I show $D_{\mu} \lambda(\mathbf{x})$ exists a.e. Let $0 \leq a<b<\infty$ and let $A$ be any bounded subset of

$$
N(a, b) \equiv\left\{\mathbf{x} \notin Z: \bar{D}_{\mu} \lambda(\mathbf{x})>b>a>\underline{D}_{\mu} \lambda(\mathbf{x})\right\} .
$$

By Lemma 31.3.2,

$$
a \bar{\mu}(A) \geq \bar{\lambda}(A) \geq b \bar{\mu}(A)
$$

and so $\mu(A)=0$ and $A$ is $\mu$ measurable. It follows $\mu(N(a, b))=0$ because

$$
\mu(N(a, b)) \leq \sum_{m=1}^{\infty} \mu(N(a, b) \cap B(\mathbf{0}, m))=0 .
$$

Define

$$
N_{0} \equiv\left\{\mathbf{x} \notin Z: \bar{D}_{\mu} \lambda(\mathbf{x})>\underline{D}_{\mu} \lambda(\mathbf{x})\right\}
$$

Thus $\mu\left(N_{0}\right)=0$ because

$$
N_{0} \subseteq \cup\{N(a, b): 0 \leq a<b, \text { and } a, b \in \mathbb{Q}\}
$$

Therefore, $N_{0}$ is also $\mu$ measurable and has $\mu$ measure zero. Letting $N \equiv N_{0} \cup Z$, it follows $D_{\mu} \lambda(\mathbf{x})$ exists on $N^{C}$. We can assume also that $N$ is a $G_{\delta}$ set. It remains to verify $\mathscr{X}_{N^{C}}(\cdot) D_{\mu} \lambda(\cdot)$ is finite a.e. and is $\mu$ measurable.

Let

$$
I=\left\{\mathbf{x}: D_{\mu} \lambda(\mathbf{x})=\infty\right\}
$$

Then by Lemma 31.3.2

$$
\bar{\lambda}(I \cap B(\mathbf{0}, m)) \geq a \bar{\mu}(I \cap B(\mathbf{0}, m))
$$

for all $a$ and since $\lambda$ is finite on bounded sets, the above implies $\bar{\mu}(I \cap B(\mathbf{0}, m))=0$ for each $m$ which implies that $I$ is $\mu$ measurable and has $\mu$ measure zero since

$$
I=\cup_{m=1}^{\infty} I \cap B(\mathbf{0}, m)
$$

Now the issue is measurability. Let $\lambda$ be an arbitrary Radon measure. I need show that $\mathbf{x} \rightarrow \lambda(B(\mathbf{x}, r))$ is measurable. Here is where it is convenient to have the balls be closed balls. If $V$ is an open set containing $B(\mathbf{x}, r)$, then for $\mathbf{y}$ close enough to $\mathbf{x}, B(\mathbf{y}, r) \subseteq V$ also and so,

$$
\limsup _{\mathbf{y} \rightarrow \mathbf{x}} \lambda(B(\mathbf{y}, r)) \leq \lambda(V)
$$

However, since $V$ is arbitrary and $\lambda$ is outer regular or observing that $B(\mathbf{x}, r)$ the closed ball is the intersection of nested open sets, it follows that

$$
\lim \sup _{\mathbf{y} \rightarrow \mathbf{x}} \lambda(B(\mathbf{y}, r)) \leq \lambda(B(\mathbf{x}, r))
$$

Thus $\mathbf{x} \rightarrow \lambda(B(\mathbf{x}, r))$ is upper semicontinuous and so,

$$
\mathbf{x} \rightarrow \frac{\lambda(B(\mathbf{x}, r))}{\mu(B(\mathbf{x}, r))}
$$

is measurable. Hence $\mathscr{X}_{N^{C}}(\mathbf{x}) D_{\mu}(\lambda)(\mathbf{x})=\lim _{r_{i} \rightarrow 0} \mathscr{X}_{N^{C}}(\mathbf{x}) \frac{\lambda(B(\mathbf{x}, r))}{\mu(B(\mathbf{x}, r))}$ is also measurable.
Typically I will write $D_{\mu} \lambda(\mathbf{x})$ rather than the more precise $\mathscr{X}_{N^{C}}(\mathbf{x}) D_{\mu} \lambda(\mathbf{x})$ since the values on the set of measure zero $N$ are not important due to the completeness of the measure $\mu$.

### 31.3.1 Radon Nikodym Theorem for Radon Measures

The Radon Nikodym theorem is an abstract result but this will be a special version for Radon measures which is based on these covering theorems and related theory.

Definition 31.3.4 Let $\lambda, \mu$ be two Radon measures defined on $\mathscr{F}$, a $\sigma$ algebra of subsets of an open set $U$. Then $\lambda \ll \mu$ means that whenever $\mu(E)=0$, it follows that $\lambda(E)=0$.

Next is a representation theorem for $\lambda$ in terms of an integral involving $D_{\mu} \lambda$.
Theorem 31.3.5 Let $\lambda$ and $\mu$ be Radon measures defined on $\mathscr{F}$ a $\sigma$ algebra of the open set $U$ then there exists a set of $\mu$ measure zero $N$ such that $D_{\mu} \lambda(\mathbf{x})$ exists off $N$ and if $E \subseteq N^{C}, E \in \mathscr{F}$, then

$$
\lambda(E)=\int_{U}\left(D_{\mu} \lambda\right) \mathscr{X}_{E} d \mu
$$

If $\lambda \ll \mu, \lambda(E)=\int_{E} D_{\mu} \lambda d \mu$. In any case, $\lambda(E) \geq \int_{E} D_{\mu} \lambda d \mu$.
Proof: The proof is based on Lemma 31.3.2. Let $E \subseteq N^{C}$ where $N$ has measure 0 and includes the set $Z$ along with the set where the symmetric derivative does not exist. It can be assumed that $N$ is a $G_{\delta}$ set. Assume $E$ is bounded to begin with. Then $E \cap$ $\left\{\mathbf{x} \in N^{C}: D_{\mu} \lambda(\mathbf{x})=0\right\}$ has measure zero. This is because by Lemma 31.3.2,

$$
\begin{aligned}
\lambda\left(E \cap\left\{\mathbf{x} \in N^{C}: D_{\mu} \lambda(\mathbf{x})=0\right\}\right) & \leq a \mu\left(E \cap\left\{\mathbf{x} \in N^{C}: D_{\mu} \lambda(\mathbf{x})=0\right\}\right) \\
& \leq a \mu(E), \mu(E)<\infty
\end{aligned}
$$

for all positive $a$ and so

$$
\lambda\left(E \cap\left\{\mathbf{x} \in N^{C}: D_{\mu} \lambda(\mathbf{x})=0\right\}\right)=0
$$

Thus, the set where $D_{\mu} \lambda(\mathbf{x})=0$ can be ignored.
Let $\left\{a_{k}^{n}\right\}_{k=1}^{\infty}$ be positive numbers such that $\left|a_{k}^{n}-a_{k+1}^{n}\right|=2^{-n}$. Specifically, let $\left\{a_{k}^{n}\right\}_{k=0}^{\infty}$ be given by

$$
0,2^{-n}, 2\left(2^{-n}\right), 3\left(2^{-n}\right), 4\left(2^{-n}\right), \ldots
$$

Define disjoint half open intervals whose union is all of $(0, \infty), I_{k}^{n}$ having end points $a_{k-1}^{n}$ and $a_{k}^{n}$. Say $I_{k}^{n}=\left(a_{k-1}^{n}, a_{k}^{n}\right]$.

$$
E_{k}^{n} \equiv E \cap\left\{\mathbf{x} \in \mathbb{R}^{p}: D_{\mu} \lambda(\mathbf{x}) \in I_{k}^{n}\right\} \equiv E \cap\left(D_{\mu} \lambda\right)^{-1}\left(I_{k}^{n}\right)
$$

Since the intervals are Borel sets, $\left(D_{\mu} \lambda\right)^{-1}\left(I_{k}^{n}\right)$ is measurable. Thus $\cup_{k=1}^{\infty} E_{k}^{n}=E$ and the $k \rightarrow E_{k}^{n}$ are disjoint measurable sets. From Lemma 31.3.2,

$$
\mu\left(E_{k}^{n}\right) a_{k}^{n} \geq \lambda\left(E_{k}^{n}\right) \geq a_{k-1}^{n} \mu\left(E_{k}^{n}\right)
$$

Then

$$
\sum_{k=1}^{\infty} a_{k}^{n} \mu\left(E_{k}^{n}\right) \geq \lambda(E)=\sum_{k=1}^{\infty} \lambda\left(E_{k}^{n}\right) \geq \sum_{k=1}^{\infty} a_{k-1}^{n} \mu\left(E_{k}^{n}\right)
$$

Let $l_{n}(\mathbf{x}) \equiv \sum_{k=1}^{\infty} a_{k-1}^{n} \mathscr{X}_{\left(D_{\mu} \lambda\right)^{-1}\left(l_{k}^{n}\right)}(\mathbf{x})$ and $u_{n}(\mathbf{x}) \equiv \sum_{k=1}^{\infty} a_{k}^{n} \mathscr{X}_{\left(D_{\mu} \lambda\right)^{-1}\left(l_{k}^{n}\right)}(\mathbf{x})$. Then the above implies

$$
\lambda(E) \in\left[\int_{E} l_{n} d \mu, \int_{E} u_{n} d \mu\right]
$$

Now both $l_{n}$ and $u_{n}$ converge to $D_{\mu} \lambda(\mathbf{x})$ which is nonnegative and measurable as shown earlier. The construction shows that $l_{n}$ increases to $D_{\mu} \lambda(\mathbf{x})$. Also, $u_{n}(\mathbf{x})-l_{n}(\mathbf{x})=2^{-n}$. Thus

$$
\lambda(E) \in\left[\int_{E} l_{n} d \mu, \int_{E} l_{n} d \mu+2^{-n} \mu(E)\right]
$$

By the monotone convergence theorem, this shows $\lambda(E)=\int_{E} D_{\mu} \lambda d \mu$.
Now if $E$ is an arbitrary set in $N^{C}$, maybe not bounded, the above shows

$$
\lambda(E \cap B(\mathbf{0}, n))=\int_{E \cap B(\mathbf{0}, n)} D_{\mu} \lambda d \mu
$$

Let $n \rightarrow \infty$ and use the monotone convergence theorem. Thus for all $E \subseteq N^{C}, \lambda(E)=$ $\int_{E} D_{\mu} \lambda d \mu$. For the last claim, $\int_{E} D_{\mu} \lambda d \mu=\int_{E \cap N^{C}} D_{\mu} \lambda d \mu=\lambda\left(E \cap N^{C}\right) \leq \lambda(E)$.

In case, $\lambda \ll \mu$, it does not matter that $E \subseteq N^{C}$ because, since $\mu(N)=0$, so is $\lambda(N)$ and so

$$
\lambda(E)=\lambda\left(E \cap N^{C}\right)=\int_{E \cap N^{C}} D_{\mu} \lambda d \mu=\int_{E} D_{\mu} \lambda d \mu
$$

for any $E \in \mathscr{F}$.
What if $\lambda$ and $\mu$ are just two arbitrary Radon measures defined on $\mathscr{F}$ ? What then? It was shown above that $D_{\mu} \lambda(\mathbf{x})$ exists for $\mu$ a.e. $\mathbf{x}$, off a $G_{\delta}$ set $N$ of $\mu$ measure 0 which includes $Z$, the set of $\mathbf{x}$ where $\mu(B(\mathbf{x}, r))=0$ for some $r>0$. Also, it was shown above that if $E \subseteq N^{C}$, then $\lambda(E)=\int_{E} D_{\mu} \lambda(\mathbf{x}) d \mu$. Define for arbitrary $E \in \mathscr{F}$,

$$
\lambda_{\mu}(E) \equiv \lambda\left(E \cap N^{C}\right), \lambda_{\perp}(E) \equiv \lambda(E \cap N)
$$

Then

$$
\begin{aligned}
\lambda(E) & =\lambda(E \cap N)+\lambda\left(E \cap N^{C}\right)=\lambda_{\perp}(E)+\lambda_{\mu}(E) \\
& =\lambda(E \cap N)+\int_{E \cap N^{C}} D_{\mu} \lambda(\mathbf{x}) d \mu \\
& =\lambda(E \cap N)+\int_{E} D_{\mu} \lambda(\mathbf{x}) d \mu \equiv \lambda(E \cap N)+\lambda_{\mu}(E) \\
& \equiv \lambda_{\perp}(E)+\lambda_{\mu}(E)
\end{aligned}
$$

This shows most of the following corollary.
Corollary 31.3.6 Let $\mu, \lambda$ be two Radon measures. Then there exist two measures, $\lambda_{\mu}, \lambda_{\perp}$ such that

$$
\lambda_{\mu} \ll \mu, \lambda=\lambda_{\mu}+\lambda_{\perp}
$$

and a set of $\mu$ measure zero $N$ such that

$$
\lambda_{\perp}(E)=\lambda(E \cap N)
$$

Also $\lambda_{\mu}$ is given by the formula

$$
\lambda_{\mu}(E) \equiv \int_{E} D_{\mu} \lambda(\mathbf{x}) d \mu
$$

Proof: If $\mathbf{x} \in N$, this could happen two ways, either $\mathbf{x} \in Z$ or $D_{\mu} \lambda(\mathbf{x})$ fails to exist. It only remains to verify that $\lambda_{\mu}$ given above satisfies $\lambda_{\mu} \ll \mu$. However, this is obvious because if $\mu(E)=0$, then clearly $\int_{E} D_{\mu} \lambda(\mathbf{x}) d \mu=0$.

Since $D_{\mu} \lambda(\mathbf{x})=D_{\mu} \lambda(\mathbf{x}) \mathscr{X}_{N^{C}}(\mathbf{x})$, it doesn't matter which we use but maybe $D_{\mu} \lambda(\mathbf{x})$ doesn't exist at some points of $N$.

This is sometimes called the Lebesgue decomposition.

## Chapter 32

## Fourier Transforms

### 32.1 An Algebra Of Special Functions

First recall the following definition of a polynomial.
Definition 32.1.1 $\alpha=\left(\alpha_{1}, \cdots, \alpha_{n}\right)$ for $\alpha_{1} \cdots \alpha_{n}$ positive integers is called a multi-index. For $\alpha$ a multi-index, $|\alpha| \equiv \alpha_{1}+\cdots+\alpha_{n}$ and if $\mathbf{x} \in \mathbb{R}^{n}$,

$$
\mathbf{x}=\left(x_{1}, \cdots, x_{n}\right)
$$

and $f$ a function, define

$$
\mathbf{x}^{\alpha} \equiv x_{1}^{\alpha_{1}} x_{2}^{\alpha_{2}} \cdots x_{n}^{\alpha_{n}}
$$

A polynomial in $n$ variables of degree $m$ is a function of the form

$$
p(\mathbf{x})=\sum_{|\alpha| \leq m} a_{\alpha} \mathbf{x}^{\alpha}
$$

Here $\alpha$ is a multi-index as just described and $a_{\alpha} \in \mathbb{C}$. Also define for $\alpha=\left(\alpha_{1}, \cdots, \alpha_{n}\right)$ a multi-index

$$
D^{\alpha} f(\mathbf{x}) \equiv \frac{\partial^{|\alpha|} f}{\partial x_{1}^{\alpha_{1}} \partial x_{2}^{\alpha_{2}} \cdots \partial x_{n}^{\alpha_{n}}}
$$

Definition 32.1.2 Define $\mathscr{G}_{1}$ to be the functions of the form $p(\mathbf{x}) e^{-a|\mathbf{x}|^{2}}$ where $a>0$ and $p(\mathbf{x})$ is a polynomial. Let $\mathscr{G}$ be all finite sums of functions in $\mathscr{G}_{1}$. Thus $\mathscr{G}$ is an algebra of functions which has the property that if $f \in \mathscr{G}$ then $\bar{f} \in \mathscr{G}$.

It is always assumed, unless stated otherwise that the measure will be Lebesgue measure.

Lemma 32.1.3 $\mathscr{G}$ is dense in $C_{0}\left(\mathbb{R}^{n}\right)$ with respect to the norm,

$$
\|f\|_{\infty} \equiv \sup \left\{|f(\mathbf{x})|: \mathbf{x} \in \mathbb{R}^{n}\right\}
$$

Proof: By the Weierstrass approximation theorem, it suffices to show $\mathscr{G}$ separates the points and annihilates no point. It was already observed in the above definition that $\bar{f} \in \mathscr{G}$ whenever $f \in \mathscr{G}$. If $\mathbf{y}_{1} \neq \mathbf{y}_{2}$ suppose first that $\left|\mathbf{y}_{1}\right| \neq\left|\mathbf{y}_{2}\right|$. Then in this case, you can let $f(\mathbf{x}) \equiv e^{-|\mathbf{x}|^{2}}$ and $f \in \mathscr{G}$ and $f\left(\mathbf{y}_{1}\right) \neq f\left(\mathbf{y}_{2}\right)$. If $\left|\mathbf{y}_{1}\right|=\left|\mathbf{y}_{2}\right|$, then suppose $y_{1 k} \neq y_{2 k}$. This must happen for some $k$ because $\mathbf{y}_{1} \neq \mathbf{y}_{2}$. Then let $f(\mathbf{x}) \equiv x_{k} e^{-|\mathbf{x}|^{2}}$. Thus $\mathscr{G}$ separates points. Now $e^{-|\mathbf{x}|^{2}}$ is never equal to zero and so $\mathscr{G}$ annihilates no point of $\mathbb{R}^{n}$. This proves the lemma.

These functions are clearly quite specialized. Therefore, the following theorem is somewhat surprising.

Theorem 32.1.4 For each $p \geq 1, p<\infty, \mathscr{G}$ is dense in $L^{p}\left(\mathbb{R}^{n}\right)$.

Proof: Let $f \in L^{p}\left(\mathbb{R}^{n}\right)$. Then there exists $g \in C_{c}\left(\mathbb{R}^{n}\right)$ such that $\|f-g\|_{p}<\varepsilon$. Now let $b>0$ be large enough that

$$
\int_{\mathbb{R}^{n}}\left(e^{-b|\mathbf{x}|^{2}}\right)^{p} d x<\varepsilon^{p}
$$

Then $\mathbf{x} \rightarrow g(\mathbf{x}) e^{b|\mathbf{x}|^{2}}$ is in $C_{c}\left(\mathbb{R}^{n}\right) \subseteq C_{0}\left(\mathbb{R}^{n}\right)$. Therefore, from Lemma 32.1.3 there exists $\psi \in \mathscr{G}$ such that

$$
\left\|g e^{b|\cdot|^{2}}-\psi\right\|_{\infty}<1
$$

Therefore, letting $\phi(\mathbf{x}) \equiv e^{-b|\mathbf{x}|^{2}} \psi(\mathbf{x})$ it follows that $\phi \in \mathscr{G}$ and for all $\mathbf{x} \in \mathbb{R}^{n}$,

$$
|g(\mathbf{x})-\phi(\mathbf{x})|<e^{-b|\mathbf{x}|^{2}}
$$

Therefore,

$$
\left(\int_{\mathbb{R}^{n}}|g(\mathbf{x})-\phi(\mathbf{x})|^{p} d x\right)^{1 / p} \leq\left(\int_{\mathbb{R}^{n}}\left(e^{-b|\mathbf{x}|^{2}}\right)^{p} d x\right)^{1 / p}<\varepsilon
$$

It follows

$$
\|f-\phi\|_{p} \leq\|f-g\|_{p}+\|g-\phi\|_{p}<2 \varepsilon
$$

Since $\varepsilon>0$ is arbitrary, this proves the theorem.
The following lemma is also interesting even if it is obvious.
Lemma 32.1.5 For $\psi \in \mathscr{G}$, p a polynomial, and $\alpha, \beta$ multiindices, $D^{\alpha} \psi \in \mathscr{G}$ and $p \psi \in \mathscr{G}$. Also

$$
\sup \left\{\left|\mathbf{x}^{\beta} D^{\alpha} \psi(\mathbf{x})\right|: \mathbf{x} \in \mathbb{R}^{n}\right\}<\infty
$$

### 32.2 Fourier Transforms Of Functions In $\mathscr{G}$

Definition 32.2.1 For $\psi \in \mathscr{G}$ Define the Fourier transform, $F$ and the inverse Fourier transform, $F^{-1}$ by

$$
\begin{aligned}
& F \psi(\mathbf{t}) \equiv(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \cdot \mathbf{x}} \psi(\mathbf{x}) d x \\
& F^{-1} \psi(\mathbf{t}) \equiv(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{i \mathbf{t} \cdot \mathbf{x}} \psi(\mathbf{x}) d x
\end{aligned}
$$

where $\mathbf{t} \cdot \mathbf{x} \equiv \sum_{i=1}^{n} t_{i} x_{i}$.Note there is no problem with this definition because $\psi$ is in $L^{1}\left(\mathbb{R}^{n}\right)$ and therefore,

$$
\left|e^{i \mathbf{t} \cdot \mathbf{x}} \psi(\mathbf{x})\right| \leq|\psi(\mathbf{x})|,
$$

an integrable function.
One reason for using the functions, $\mathscr{G}$ is that it is very easy to compute the Fourier transform of these functions. The first thing to do is to verify $F$ and $F^{-1} \operatorname{map} \mathscr{G}$ to $\mathscr{G}$ and that $F^{-1} \circ F(\psi)=\psi$.

Lemma 32.2.2 The following formulas are true. $(c>0)$

$$
\begin{gather*}
\int_{\mathbb{R}} e^{-c t^{2}} e^{-i s t} d t=\int_{\mathbb{R}} e^{-c t^{2}} e^{i s t} d t=e^{-\frac{s^{2}}{4 c}} \frac{\sqrt{\pi}}{\sqrt{c}}  \tag{32.2.1}\\
\int_{\mathbb{R}^{n}} e^{-c|t|^{2}} e^{-i s \cdot \mathbf{t}} d t=\int_{\mathbb{R}^{n}} e^{-c|\mathbf{t}|^{2}} e^{i s \cdot \mathbf{t}} d t=e^{-\frac{|s|^{2}}{4 c}}\left(\frac{\sqrt{\pi}}{\sqrt{c}}\right)^{n} . \tag{32.2.2}
\end{gather*}
$$

Proof: Consider the first one. Let $h(s)$ be given by the left side. Then

$$
H(s) \equiv \int_{\mathbb{R}} e^{-c t^{2}} e^{-i s t} d t=\int_{\mathbb{R}} e^{-c t^{2}} \cos (s t) d t
$$

Then using the dominated convergence theorem to differentiate,

$$
H^{\prime}(s)=\int_{\mathbb{R}}-e^{-c t^{2}} t \sin (s t) d t=\left.\frac{e^{-c t^{2}}}{2 c} \sin (s t)\right|_{-\infty} ^{\infty}-\frac{s}{2 c} \int_{\mathbb{R}} e^{-c t^{2}} \cos (s t) d t=-\frac{s}{2 c} H(s)
$$

Also $H(0)=\int_{\mathbb{R}} e^{-c t^{2}} d t$. Thus $H(0)=\int_{\mathbb{R}} e^{-c x^{2}} d x \equiv I$ and so

$$
I^{2}=\int_{\mathbb{R}^{2}} e^{-c\left(x^{2}+y^{2}\right)} d x d y=\int_{0}^{\infty} \int_{0}^{2 \pi} e^{-c r^{2}} r d \theta d r=\frac{\pi}{c}
$$

Hence

$$
H^{\prime}(s)+\frac{s}{2 c} H(s)=0, H(0)=\sqrt{\frac{\pi}{c}}
$$

It follows that $H(s)=e^{-\frac{s^{2}}{4 c}} \frac{\sqrt{\pi}}{\sqrt{c}}$. The second formula follows right away from Fubini's theorem.

With these formulas, it is easy to verify $F, F^{-1} \operatorname{map} \mathscr{G}$ to $\mathscr{G}$ and $F \circ F^{-1}=F^{-1} \circ F=i d$.
Theorem 32.2.3 Each of $F$ and $F^{-1} \operatorname{map} \mathscr{G}$ to $\mathscr{G}$. Also $F^{-1} \circ F(\psi)=\psi$ and $F \circ F^{-1}(\psi)=$ $\psi$.

Proof: The first claim will be shown if it is shown that $F \psi \in \mathscr{G}$ for $\psi(\mathbf{x})=\mathbf{x}^{\alpha} e^{-b|\mathbf{x}|^{2}}$ because an arbitrary function of $\mathscr{G}$ is a finite sum of scalar multiples of functions such as $\psi$. Using Lemma 32.2.2,

$$
\begin{aligned}
F \psi(\mathbf{t}) & \equiv\left(\frac{1}{2 \pi}\right)^{n / 2} \int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \cdot \mathbf{x}} \mathbf{x}^{\alpha} e^{-b|\mathbf{x}|^{2}} d x \\
& =\left(\frac{1}{2 \pi}\right)^{n / 2}(i)^{-|\alpha|} D_{t}^{\alpha}\left(\int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \cdot \mathbf{x}} e^{-b|\mathbf{x}|^{2}} d x\right) \\
& =\left(\frac{1}{2 \pi}\right)^{n / 2}(i)^{-|\alpha|} D_{t}^{\alpha}\left(e^{-\frac{|\mathbf{t}|^{2}}{4 b}}\left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^{n}\right)
\end{aligned}
$$

and this is clearly in $\mathscr{G}$ because it equals a polynomial times $e^{-\frac{|t|^{2}}{4 b}}$.

It remains to verify the other assertion. As in the first case, it suffices to consider $\psi(\mathbf{x})=\mathbf{x}^{\alpha} e^{-b|\mathbf{x}|^{2}}$.

$$
\begin{aligned}
& F^{-1} \circ F(\psi)(\mathbf{s}) \equiv\left(\frac{1}{2 \pi}\right)^{n / 2} \int_{\mathbb{R}^{n}} e^{i s \cdot \mathbf{t}} F(\psi)(\mathbf{t}) d t \\
= & \left(\frac{1}{2 \pi}\right)^{n / 2} \int_{\mathbb{R}^{n}} e^{i \mathbf{s} \cdot \mathbf{t}}\left(\frac{1}{2 \pi}\right)^{n / 2}(i)^{-|\alpha|} D_{t}^{\alpha}\left(e^{-\frac{|t|^{2}}{4 b}}\left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^{n}\right) d t \\
= & \left(\frac{1}{2 \pi}\right)^{n}(i)^{-|\alpha|} \int_{\mathbb{R}^{n}} e^{i \mathbf{s} \cdot \mathbf{t}} D_{t}^{\alpha}\left(e^{-\frac{|\mathbf{t}|^{2}}{4 b}}\left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^{n}\right) d t \\
= & \left(\frac{1}{2 \pi}\right)^{n}(i)^{-|\alpha|} \int_{\mathbb{R}^{n}}(i)^{|\alpha|} \mathbf{s}^{\alpha} e^{i \mathbf{s} \cdot \mathbf{t}}\left(e^{-\frac{|\mathbf{t}|^{2}}{4 b}}\left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^{n}\right) d t
\end{aligned}
$$

and by Lemma 32.2.2,

$$
\begin{aligned}
& =\left(\frac{1}{2 \pi}\right)^{n}\left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^{n} \int_{\mathbb{R}^{n}} \mathbf{s}^{\alpha} e^{i \mathbf{s} \cdot \mathbf{t}}\left(e^{-\frac{|\mathrm{t}|^{2}}{4 b}}\right) d t \\
& =\overbrace{\left(\frac{1}{2 \pi}\right)^{n}\left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^{n}\left(\frac{\sqrt{\pi}}{\sqrt{1 / 4 b}}\right)^{n}}^{=1} \mathbf{s}^{\alpha} e^{-b|\mathbf{s}|^{2}}=\psi(\mathbf{s}) .
\end{aligned}
$$

### 32.3 Fourier Transforms Of Just About Anything

### 32.3.1 Fourier Transforms Of $\mathscr{G}^{*}$

Definition 32.3.1 Let $\mathscr{G}^{*}$ denote the vector space of linear functions defined on $\mathscr{G}$ which have values in $\mathbb{C}$. Thus $T \in \mathscr{G}^{*}$ means $T: \mathscr{G} \rightarrow \mathbb{C}$ and $T$ is linear,

$$
T(a \psi+b \phi)=a T(\psi)+b T(\phi) \text { for all } a, b \in \mathbb{C}, \quad \psi, \phi \in \mathscr{G}
$$

Let $\psi \in \mathscr{G}$. Then we can regard $\psi$ as an element of $\mathscr{G}^{*}$ by defining

$$
\psi(\phi) \equiv \int_{\mathbb{R}^{n}} \psi(\mathbf{x}) \phi(\mathbf{x}) d x
$$

Then we have the following important lemma.

Lemma 32.3.2 The following is obtained for all $\phi, \psi \in \mathscr{G}$.

$$
F \psi(\phi)=\psi(F \phi), F^{-1} \psi(\phi)=\psi\left(F^{-1} \phi\right)
$$

Also if $\psi \in \mathscr{G}$ and $\psi=0$ in $\mathscr{G}^{*}$ so that $\psi(\phi)=0$ for all $\phi \in \mathscr{G}$, then $\psi=0$ as a function.

Proof:

$$
\begin{aligned}
F \psi(\phi) & \equiv \int_{\mathbb{R}^{n}} F \psi(\mathbf{t}) \phi(\mathbf{t}) d t \\
& =\int_{\mathbb{R}^{n}}\left(\frac{1}{2 \pi}\right)^{n / 2} \int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \cdot \mathbf{x}} \psi(\mathbf{x}) d x \phi(\mathbf{t}) d t \\
& =\int_{\mathbb{R}^{n}} \psi(\mathbf{x})\left(\frac{1}{2 \pi}\right)^{n / 2} \int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \cdot \mathbf{x}} \phi(\mathbf{t}) d t d x \\
& =\int_{\mathbb{R}^{n}} \psi(\mathbf{x}) F \phi(\mathbf{x}) d x \equiv \psi(F \phi)
\end{aligned}
$$

The other claim is similar.
Suppose now $\psi(\phi)=0$ for all $\phi \in \mathscr{G}$. Then

$$
\int_{\mathbb{R}^{n}} \psi \phi d x=0
$$

for all $\phi \in \mathscr{G}$. Therefore, this is true for $\phi=\psi$ and so $\psi=0$.
This lemma suggests a way to define the Fourier transform of something in $\mathscr{G}^{*}$.
Definition 32.3.3 For $T \in \mathscr{G}^{*}$, define $F T, F^{-1} T \in \mathscr{G}^{*}$ by

$$
F T(\phi) \equiv T(F \phi), F^{-1} T(\phi) \equiv T\left(F^{-1} \phi\right)
$$

Lemma 32.3.4 $F$ and $F^{-1}$ are both one to one, onto, and are inverses of each other.
Proof: First note $F$ and $F^{-1}$ are both linear. This follows directly from the definition. Suppose now $F T=0$. Then $F T(\phi)=T(F \phi)=0$ for all $\phi \in \mathscr{G}$. But $F$ and $F^{-1}$ map $\mathscr{G}$ onto $\mathscr{G}$ because if $\psi \in \mathscr{G}$, then as shown above, $\psi=F\left(F^{-1}(\psi)\right)$. Therefore, $T=0$ and so $F$ is one to one. Similarly $F^{-1}$ is one to one. Now

$$
F^{-1}(F T)(\phi) \equiv(F T)\left(F^{-1} \phi\right) \equiv T\left(F\left(F^{-1}(\phi)\right)\right)=T \phi
$$

Therefore, $F^{-1} \circ F(T)=T$. Similarly, $F \circ F^{-1}(T)=T$. Thus both $F$ and $F^{-1}$ are one to one and onto and are inverses of each other as suggested by the notation.

Probably the most interesting things in $\mathscr{G}^{*}$ are functions of various kinds. The following lemma will be useful in considering this situation.

Lemma 32.3.5 If $f \in L_{\text {loc }}^{1}\left(\mathbb{R}^{n}\right)$ and $\int_{\mathbb{R}^{n}} f \phi d x=0$ for all $\phi \in C_{c}\left(\mathbb{R}^{n}\right)$, then $f=0$ a.e.
Proof: For $r>0$, let

$$
E \equiv\{\mathbf{x}: f(\mathbf{x}) \geq r\}, E_{R} \equiv E \cap B(\mathbf{0}, R)
$$

Let $K_{m}$ be an increasing sequence of compact sets, and let $V_{m}$ be a decreasing sequence of open sets satisfying

$$
K_{m} \subseteq E_{R} \subseteq V_{m}, m_{n}\left(V_{m}\right) \leq m_{n}\left(K_{m}\right)+2^{-m}, V_{1} \subseteq B(\mathbf{0}, R)
$$

Therefore,

$$
m_{n}\left(V_{m} \backslash K_{m}\right) \leq 2^{-m}
$$

Let

$$
\phi_{m} \in C_{c}\left(V_{m}\right), K_{m} \prec \phi_{m} \prec V_{m} .
$$

The statement $K_{m} \prec \phi_{m} \prec V_{m}$ means that $\phi_{m}$ equals 1 on $K_{m}$, has compact support in $V_{m}$, maps into $[0,1]$, and is continuous. Then $\phi_{m}(\mathbf{x}) \rightarrow \mathscr{X}_{E_{R}}(\mathbf{x})$ a.e. because the set where $\phi_{m}(\mathbf{x})$ fails to converge to this set is contained in the set of all $\mathbf{x}$ which are in infinitely many of the sets $V_{m} \backslash K_{m}$. This set has measure zero because

$$
\sum_{m=1}^{\infty} m_{n}\left(V_{m} \backslash K_{m}\right)<\infty
$$

Thus $\phi_{m}$ converges pointwise $a . e$ to $\mathscr{X}_{E_{R}}$ and so, by the dominated convergence theorem,

$$
0=\lim _{m \rightarrow \infty} \int_{\mathbb{R}^{n}} f \phi_{m} d x=\lim _{m \rightarrow \infty} \int_{V_{1}} f \phi_{m} d x=\int_{E_{R}} f d x \geq r m\left(E_{R}\right)
$$

Thus, $m_{n}\left(E_{R}\right)=0$ and therefore $m_{n}(E)=\lim _{R \rightarrow \infty} m_{n}\left(E_{R}\right)=0$. Since $r>0$ is arbitrary, it follows

$$
\begin{aligned}
m_{n}([f>0]) & =\cup_{k=1}^{\infty} m_{n}\left(\left[f>k^{-1}\right]\right) \\
& =\cup_{k=1}^{\infty} m_{n}\left(\left[f^{+}>k^{-1}\right]\right)=m_{n}\left(\left[f^{+}>0\right]\right)=0
\end{aligned}
$$

Hence $f^{+}=0$ a.e. It follows that $\int f^{-} \phi d x=0$ for all $\phi \in C_{c}\left(\mathbb{R}^{n}\right)$ because

$$
\int f^{-} \phi d x=\int f^{+} \phi-\int f \phi=0
$$

Thus from what was just shown, with $f^{-}$taking the place of $f$, it follows $\frac{\left|f^{-}\right|+f^{-}}{2}=0$ and so $f^{-}=0$ a.e. also.

Corollary 32.3.6 Let $f \in L^{1}\left(\mathbb{R}^{n}\right)$ and suppose

$$
\int_{\mathbb{R}^{n}} f(\mathbf{x}) \phi(\mathbf{x}) d x=0
$$

for all $\phi \in \mathscr{G}$. Then $f=0$ a.e.
Proof: Let $\psi \in C_{c}\left(\mathbb{R}^{n}\right)$. Then by the Stone Weierstrass approximation theorem, there exists a sequence of functions, $\left\{\phi_{k}\right\} \subseteq \mathscr{G}$ such that $\phi_{k} \rightarrow \psi$ uniformly. Then by the dominated convergence theorem,

$$
\int f \psi d x=\lim _{k \rightarrow \infty} \int f \phi_{k} d x=0
$$

By Lemma 32.3.5 $f=0$.
The next theorem is the main result of this sort.

Theorem 32.3.7 Let $f \in L^{p}\left(\mathbb{R}^{n}\right), p \geq 1$, or suppose $f$ is measurable and has polynomial growth,

$$
|f(\mathbf{x})| \leq K\left(1+|\mathbf{x}|^{2}\right)^{m}
$$

for some $m \in \mathbb{N}$. Then if

$$
\int f \psi d x=0
$$

for all $\psi \in \mathscr{G}$, then it follows $f=0$.
Proof: First note that if $f \in L^{p}\left(\mathbb{R}^{n}\right)$ or has polynomial growth, then it makes sense to write the integral $\int f \psi d x$ described above. This is obvious in the case of polynomial growth. In the case where $f \in L^{p}\left(\mathbb{R}^{n}\right)$ it also makes sense because

$$
\int|f||\psi| d x \leq\left(\int|f|^{p} d x\right)^{1 / p}\left(\int|\psi|^{p^{\prime}} d x\right)^{1 / p^{\prime}}<\infty
$$

due to the fact mentioned above that all these functions in $\mathscr{G}$ are in $L^{p}\left(\mathbb{R}^{n}\right)$ for every $p \geq 1$. Suppose now that $f \in L^{p}, p \geq 1$. The case where $f \in L^{1}\left(\mathbb{R}^{n}\right)$ was dealt with in Corollary 32.3.6. Suppose $f \in L^{p}\left(\mathbb{R}^{n}\right)$ for $p>1$. Then

$$
|f|^{p-2} \bar{f} \in L^{p^{\prime}}\left(\mathbb{R}^{n}\right),\left(p^{\prime}=q, \frac{1}{p}+\frac{1}{q}=1\right)
$$

and by density of $\mathscr{G}$ in $L^{p^{\prime}}\left(\mathbb{R}^{n}\right)$ (Theorem 32.1.4), there exists a sequence $\left\{g_{k}\right\} \subseteq \mathscr{G}$ such that

$$
\left\|g_{k}-|f|^{p-2} \bar{f}\right\|_{p^{\prime}} \rightarrow 0
$$

Then

$$
\begin{aligned}
\int_{\mathbb{R}^{n}}|f|^{p} d x & =\int_{\mathbb{R}^{n}} f\left(|f|^{p-2} \bar{f}-g_{k}\right) d x+\int_{\mathbb{R}^{n}} f g_{k} d x \\
& =\int_{\mathbb{R}^{n}} f\left(|f|^{p-2} \bar{f}-g_{k}\right) d x \\
& \leq\|f\|_{L^{p}}\left|\left\|g_{k}-|f|^{p-2} \bar{f}\right\|_{p^{\prime}}\right.
\end{aligned}
$$

which converges to 0 . Hence $f=0$.
It remains to consider the case where $f$ has polynomial growth. Thus $\mathbf{x} \rightarrow f(\mathbf{x}) e^{-|\mathbf{x}|^{2}} \in$ $L^{1}\left(\mathbb{R}^{n}\right)$. Therefore, for all $\psi \in \mathscr{G}$,

$$
0=\int f(\mathbf{x}) e^{-|\mathbf{x}|^{2}} \psi(\mathbf{x}) d x
$$

because $e^{-|\mathbf{x}|^{2}} \psi(\mathbf{x}) \in \mathscr{G}$. Therefore, by the first part, $f(\mathbf{x}) e^{-|\mathbf{x}|^{2}}=0$ a.e.
The following theorem shows that you can consider most functions you are likely to encounter as elements of $\mathscr{G}^{*}$.

Theorem 32.3.8 Let $f$ be a measurable function with polynomial growth,

$$
|f(\mathbf{x})| \leq C\left(1+|\mathbf{x}|^{2}\right)^{N} \quad \text { for some } N
$$

or let $f \in L^{p}\left(\mathbb{R}^{n}\right)$ for some $p \in[1, \infty]$. Then $f \in \mathscr{G}^{*}$ if

$$
f(\phi) \equiv \int f \phi d x
$$

Proof: Let $f$ have polynomial growth first. Then the above integral is clearly well defined and so in this case, $f \in \mathscr{G}^{*}$.

Next suppose $f \in L^{p}\left(\mathbb{R}^{n}\right)$ with $\infty>p \geq 1$. Then it is clear again that the above integral is well defined because of the fact that $\phi$ is a sum of polynomials times exponentials of the form $e^{-c|\mathbf{x}|^{2}}$ and these are in $L^{p^{\prime}}\left(\mathbb{R}^{n}\right)$. Also $\phi \rightarrow f(\phi)$ is clearly linear in both cases.

This has shown that for nearly any reasonable function, you can define its Fourier transform as described above. You could also define the Fourier transform of a finite Borel measure $\mu$ because for such a measure

$$
\psi \rightarrow \int_{\mathbb{R}^{n}} \psi d \mu
$$

is a linear functional on $\mathscr{G}$. This includes the very important case of probability distribution measures. The theoretical basis for this assertion will be given a little later.

### 32.3.2 Fourier Transforms Of Functions In $L^{1}\left(\mathbb{R}^{n}\right)$

First suppose $f \in L^{1}\left(\mathbb{R}^{n}\right)$.
Theorem 32.3.9 Let $f \in L^{1}\left(\mathbb{R}^{n}\right)$. Then $F f(\phi)=\int_{\mathbb{R}^{n}} g \phi d t$ where

$$
g(\mathbf{t})=\left(\frac{1}{2 \pi}\right)^{n / 2} \int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \cdot \mathbf{x}} f(\mathbf{x}) d x
$$

and $F^{-1} f(\phi)=\int_{\mathbb{R}^{n}} g \phi d t$ where $g(\mathbf{t})=\left(\frac{1}{2 \pi}\right)^{n / 2} \int_{\mathbb{R}^{n}} e^{i \cdot \mathbf{x}} f(\mathbf{x}) d x$. In short,

$$
\begin{aligned}
& F f(\mathbf{t}) \equiv(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \cdot \mathbf{x}} f(\mathbf{x}) d x \\
& F^{-1} f(\mathbf{t}) \equiv(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{i \mathbf{t} \cdot \mathbf{x}} f(\mathbf{x}) d x
\end{aligned}
$$

Proof: From the definition and Fubini's theorem,

$$
\begin{aligned}
F f(\phi) & \equiv \int_{\mathbb{R}^{n}} f(\mathbf{t}) F \phi(\mathbf{t}) d t=\int_{\mathbb{R}^{n}} f(\mathbf{t})\left(\frac{1}{2 \pi}\right)^{n / 2} \int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \cdot \mathbf{x}} \phi(\mathbf{x}) d x d t \\
& =\int_{\mathbb{R}^{n}}\left(\left(\frac{1}{2 \pi}\right)^{n / 2} \int_{\mathbb{R}^{n}} f(\mathbf{t}) e^{-i \mathbf{t} \cdot \mathbf{x}} d t\right) \phi(\mathbf{x}) d x
\end{aligned}
$$

Since $\phi \in \mathscr{G}$ is arbitrary, it follows from Theorem 32.3.7 that $F f(\mathbf{x})$ is given by the claimed formula. The case of $F^{-1}$ is identical.

Here are interesting properties of these Fourier transforms of functions in $L^{1}$.

Theorem 32.3.10 If $f \in L^{1}\left(\mathbb{R}^{n}\right)$ and $\left\|f_{k}-f\right\|_{1} \rightarrow 0$, then $F f_{k}$ and $F^{-1} f_{k}$ converge uniformly to $F f$ and $F^{-1} f$ respectively. If $f \in L^{1}\left(\mathbb{R}^{n}\right)$, then $F^{-1} f$ and $F f$ are both continuous and bounded. Also,

$$
\begin{equation*}
\lim _{|\mathbf{x}| \rightarrow \infty} F^{-1} f(\mathbf{x})=\lim _{|\mathbf{x}| \rightarrow \infty} F f(\mathbf{x})=0 \tag{32.3.3}
\end{equation*}
$$

Furthermore, for $f \in L^{1}\left(\mathbb{R}^{n}\right)$ both $F f$ and $F^{-1} f$ are uniformly continuous.
Proof: The first claim follows from the following inequality.

$$
\begin{aligned}
\left|F f_{k}(\mathbf{t})-F f(\mathbf{t})\right| & \leq(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}}\left|e^{-i \boldsymbol{t} \cdot \mathbf{x}} f_{k}(\mathbf{x})-e^{-i \mathbf{t} \cdot \mathbf{x}} f(\mathbf{x})\right| d x \\
& =(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}}\left|f_{k}(\mathbf{x})-f(\mathbf{x})\right| d x \\
& =(2 \pi)^{-n / 2}\left\|f-f_{k}\right\|_{1}
\end{aligned}
$$

which a similar argument holding for $F^{-1}$.
Now consider the second claim of the theorem.

$$
\left|F f(\mathbf{t})-F f\left(\mathbf{t}^{\prime}\right)\right| \leq(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}}\left|e^{-i \mathbf{t} \cdot \mathbf{x}}-e^{-i \mathbf{t}^{\prime} \cdot \mathbf{x}}\right||f(\mathbf{x})| d x
$$

The integrand is bounded by $2|f(\mathbf{x})|$, a function in $L^{1}\left(\mathbb{R}^{n}\right)$ and converges to 0 as $\mathbf{t}^{\prime} \rightarrow \mathbf{t}$ and so the dominated convergence theorem implies $F f$ is continuous. To see $F f(\mathbf{t})$ is uniformly bounded,

$$
|F f(\mathbf{t})| \leq(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}}|f(\mathbf{x})| d x<\infty
$$

A similar argument gives the same conclusions for $F^{-1}$.
It remains to verify 32.3 .3 and the claim that $F f$ and $F^{-1} f$ are uniformly continuous.

$$
|F f(\mathbf{t})| \leq\left|(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \cdot \mathbf{x}} f(\mathbf{x}) d x\right|
$$

Now let $\varepsilon>0$ be given and let $g \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$ such that $(2 \pi)^{-n / 2}\|g-f\|_{1}<\varepsilon / 2$. Then

$$
\begin{aligned}
|F f(\mathbf{t})| \leq & (2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}}|f(\mathbf{x})-g(\mathbf{x})| d x \\
& +\left|(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \cdot \mathbf{x}} g(\mathbf{x}) d x\right| \\
\leq & \varepsilon / 2+\left|(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \cdot \mathbf{x}} g(\mathbf{x}) d x\right|
\end{aligned}
$$

Now integrating by parts, it follows that for $\|\mathbf{t}\|_{\infty} \equiv \max \left\{\left|t_{j}\right|: j=1, \cdots, n\right\}>0$

$$
\begin{equation*}
|F f(\mathbf{t})| \leq \varepsilon / 2+(2 \pi)^{-n / 2}\left|\frac{1}{\|\mathbf{t}\|_{\infty}} \int_{\mathbb{R}^{n}} \sum_{j=1}^{n}\right| \frac{\partial g(\mathbf{x})}{\partial x_{j}}|d x| \tag{32.3.4}
\end{equation*}
$$

and this last expression converges to zero as $\|\mathbf{t}\|_{\infty} \rightarrow \infty$. The reason for this is that if $t_{j} \neq 0$, integration by parts with respect to $x_{j}$ gives

$$
(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \cdot \mathbf{x}} g(\mathbf{x}) d x=(2 \pi)^{-n / 2} \frac{1}{-i t_{j}} \int_{\mathbb{R}^{n}} e^{-i \boldsymbol{t} \cdot \mathbf{x}} \frac{\partial g(\mathbf{x})}{\partial x_{j}} d x
$$

Therefore, choose the $j$ for which $\|\mathbf{t}\|_{\infty}=\left|t_{j}\right|$ and the result of 32.3.4 holds. Therefore, from 32.3.4, if $\|\mathbf{t}\|_{\infty}$ is large enough, $|F f(\mathbf{t})|<\varepsilon$. Similarly, $\lim _{\|\mathbf{t}\| \rightarrow \infty} F^{-1}(\mathbf{t})=0$. Consider the claim about uniform continuity. Let $\varepsilon>0$ be given. Then there exists $R$ such that if $\|\mathbf{t}\|_{\infty}>R$, then $|F f(\mathbf{t})|<\frac{\varepsilon}{2}$. Since $F f$ is continuous, it is uniformly continuous on the compact set $[-R-1, R+1]^{n}$. Therefore, there exists $\delta_{1}$ such that if $\left\|\mathbf{t}-\mathbf{t}^{\prime}\right\|_{\infty}<\delta_{1}$ for $\mathbf{t}^{\prime}, \mathbf{t} \in[-R-1, R+1]^{n}$, then

$$
\begin{equation*}
\left|F f(\mathbf{t})-F f\left(\mathbf{t}^{\prime}\right)\right|<\varepsilon / 2 . \tag{32.3.5}
\end{equation*}
$$

Now let $0<\delta<\min \left(\delta_{1}, 1\right)$ and suppose $\left\|\mathbf{t}-\mathbf{t}^{\prime}\right\|_{\infty}<\delta$. If both $\mathbf{t}, \mathbf{t}^{\prime}$ are contained in $[-R, R]^{n}$, then 32.3.5 holds. If $\mathbf{t} \in[-R, R]^{n}$ and $\mathbf{t}^{\prime} \notin[-R, R]^{n}$, then both are contained in $[-R-1, R+1]^{n}$ and so this verifies 32.3.5 in this case. The other case is that neither point is in $[-R, R]^{n}$ and in this case,

$$
\begin{aligned}
\left|F f(\mathbf{t})-F f\left(\mathbf{t}^{\prime}\right)\right| & \leq|F f(\mathbf{t})|+\left|F f\left(\mathbf{t}^{\prime}\right)\right| \\
& <\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon .
\end{aligned}
$$

There is a very interesting relation between the Fourier transform and convolutions.
Theorem 32.3.11 Let $f, g \in L^{1}\left(\mathbb{R}^{n}\right)$. Then $f * g \in L^{1}$ and $F(f * g)=(2 \pi)^{n / 2} F f F g$.
Proof: Consider

$$
\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}}|f(\mathbf{x}-\mathbf{y}) g(\mathbf{y})| d y d x
$$

The function, $(\mathbf{x}, \mathbf{y}) \rightarrow|f(\mathbf{x}-\mathbf{y}) g(\mathbf{y})|$ is Lebesgue measurable and so by Fubini's theorem,

$$
\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}}|f(\mathbf{x}-\mathbf{y}) g(\mathbf{y})| d y d x=\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}}|f(\mathbf{x}-\mathbf{y}) g(\mathbf{y})| d x d y=\|f\|_{1}\|g\|_{1}<\infty .
$$

It follows that for a.e. $\mathbf{x}, \int_{\mathbb{R}^{n}}|f(\mathbf{x}-\mathbf{y}) g(\mathbf{y})| d y<\infty$ and for each of these values of $\mathbf{x}$, it follows that $\int_{\mathbb{R}^{n}} f(\mathbf{x}-\mathbf{y}) g(\mathbf{y}) d y$ exists and equals a function of $\mathbf{x}$ which is in $L^{1}\left(\mathbb{R}^{n}\right), f *$ $g(\mathbf{x})$. Now

$$
\begin{aligned}
& F(f * g)(\mathbf{t}) \equiv(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \cdot \mathbf{x}} f * g(\mathbf{x}) d x \\
= & (2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{-i \boldsymbol{t} \cdot \mathbf{x}} \int_{\mathbb{R}^{n}} f(\mathbf{x}-\mathbf{y}) g(\mathbf{y}) d y d x \\
= & (2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \cdot \mathbf{y}} g(\mathbf{y}) \int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \cdot(\mathbf{x}-\mathbf{y})} f(\mathbf{x}-\mathbf{y}) d x d y \\
= & (2 \pi)^{n / 2} F f(\mathbf{t}) F g(\mathbf{t}) .
\end{aligned}
$$

There are other considerations involving Fourier transforms of functions in $L^{1}\left(\mathbb{R}^{n}\right)$.

### 32.3.3 Fourier Transforms Of Functions In $L^{2}\left(\mathbb{R}^{n}\right)$

Consider $F f$ and $F^{-1} f$ for $f \in L^{2}\left(\mathbb{R}^{n}\right)$. First note that the formula given for $F f$ and $F^{-1} f$ when $f \in L^{1}\left(\mathbb{R}^{n}\right)$ will not work for $f \in L^{2}\left(\mathbb{R}^{n}\right)$ unless $f$ is also in $L^{1}\left(\mathbb{R}^{n}\right)$. Recall that $\overline{a+i b}=a-i b$.

Theorem 32.3.12 For $\phi \in \mathscr{G},\|F \phi\|_{2}=\left\|F^{-1} \phi\right\|_{2}=\|\phi\|_{2}$.
Proof: First note that for $\psi \in \mathscr{G}$,

$$
\begin{equation*}
F(\bar{\psi})=\overline{F^{-1}(\psi)}, F^{-1}(\bar{\psi})=\overline{F(\psi)} \tag{32.3.6}
\end{equation*}
$$

This follows from the definition. For example,

$$
\begin{aligned}
F \bar{\psi}(\mathbf{t}) & =\frac{(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \cdot \mathbf{x}} \bar{\psi}(\mathbf{x}) d x}{} \\
= & (2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{i \mathbf{t} \cdot \mathbf{x}} \psi(\mathbf{x}) d x
\end{aligned}
$$

Let $\phi, \psi \in \mathscr{G}$. It was shown above that

$$
\int_{\mathbb{R}^{n}}(F \phi) \psi(\mathbf{t}) d t=\int_{\mathbb{R}^{n}} \phi(F \psi) d x
$$

Similarly,

$$
\begin{equation*}
\int_{\mathbb{R}^{n}} \phi\left(F^{-1} \psi\right) d x=\int_{\mathbb{R}^{n}}\left(F^{-1} \phi\right) \psi d t \tag{32.3.7}
\end{equation*}
$$

Now, 32.3.6-32.3.7 imply

$$
\begin{aligned}
\int_{\mathbb{R}^{n}}|\phi|^{2} d x & =\int_{\mathbb{R}^{n}} \phi \overline{F^{-1}(F \phi)} d x=\int_{\mathbb{R}^{n}} \phi F(\overline{F \phi}) d x \\
& =\int_{\mathbb{R}^{n}} F \phi(\overline{F \phi}) d x=\int_{\mathbb{R}^{n}}|F \phi|^{2} d x
\end{aligned}
$$

Similarly

$$
\|\phi\|_{2}=\left\|F^{-1} \phi\right\|_{2} .
$$

Lemma 32.3.13 Let $f \in L^{2}\left(\mathbb{R}^{n}\right)$ and let $\phi_{k} \rightarrow f$ in $L^{2}\left(\mathbb{R}^{n}\right)$ where $\phi_{k} \in \mathscr{G}$. (Such a sequence exists because of density of $\mathscr{G}$ in $L^{2}\left(\mathbb{R}^{n}\right)$.) Then $F f$ and $F^{-1} f$ are both in $L^{2}\left(\mathbb{R}^{n}\right)$ and the following limits take place in $L^{2}$.

$$
\lim _{k \rightarrow \infty} F\left(\phi_{k}\right)=F(f), \lim _{k \rightarrow \infty} F^{-1}\left(\phi_{k}\right)=F^{-1}(f)
$$

Proof: Let $\psi \in \mathscr{G}$ be given. Then

$$
\begin{aligned}
F f(\psi) & \equiv f(F \psi) \equiv \int_{\mathbb{R}^{n}} f(\mathbf{x}) F \psi(\mathbf{x}) d x \\
& =\lim _{k \rightarrow \infty} \int_{\mathbb{R}^{n}} \phi_{k}(\mathbf{x}) F \psi(\mathbf{x}) d x=\lim _{k \rightarrow \infty} \int_{\mathbb{R}^{n}} F \phi_{k}(\mathbf{x}) \psi(\mathbf{x}) d x
\end{aligned}
$$

Also by Theorem 32.3.12 $\left\{F \phi_{k}\right\}_{k=1}^{\infty}$ is Cauchy in $L^{2}\left(\mathbb{R}^{n}\right)$ and so it converges to some $h \in L^{2}\left(\mathbb{R}^{n}\right)$. Therefore, from the above,

$$
F f(\psi)=\int_{\mathbb{R}^{n}} h(\mathbf{x}) \psi(\mathbf{x})
$$

which shows that $F(f) \in L^{2}\left(\mathbb{R}^{n}\right)$ and $h=F(f)$. The case of $F^{-1}$ is entirely similar.
Since $F f$ and $F^{-1} f$ are in $L^{2}\left(\mathbb{R}^{n}\right)$, this also proves the following theorem.
Theorem 32.3.14 If $f \in L^{2}\left(\mathbb{R}^{n}\right), F f$ and $F^{-1} f$ are the unique elements of $L^{2}\left(\mathbb{R}^{n}\right)$ such that for all $\phi \in \mathscr{G}$,

$$
\begin{align*}
\int_{\mathbb{R}^{n}} F f(\mathbf{x}) \phi(\mathbf{x}) d x & =\int_{\mathbb{R}^{n}} f(\mathbf{x}) F \phi(\mathbf{x}) d x  \tag{32.3.8}\\
\int_{\mathbb{R}^{n}} F^{-1} f(\mathbf{x}) \phi(\mathbf{x}) d x & =\int_{\mathbb{R}^{n}} f(\mathbf{x}) F^{-1} \phi(\mathbf{x}) d x \tag{32.3.9}
\end{align*}
$$

Theorem 32.3.15 (Plancherel)

$$
\begin{equation*}
\|f\|_{2}=\|F f\|_{2}=\left\|F^{-1} f\right\|_{2} \tag{32.3.10}
\end{equation*}
$$

Proof: Use the density of $\mathscr{G}$ in $L^{2}\left(\mathbb{R}^{n}\right)$ to obtain a sequence, $\left\{\phi_{k}\right\}$ converging to $f$ in $L^{2}\left(\mathbb{R}^{n}\right)$. Then by Lemma 32.3.13

$$
\|F f\|_{2}=\lim _{k \rightarrow \infty}\left\|F \phi_{k}\right\|_{2}=\lim _{k \rightarrow \infty}\left\|\phi_{k}\right\|_{2}=\|f\|_{2}
$$

Similarly,

$$
\|f\|_{2}=\left\|F^{-1} f\right\|_{2} .
$$

The following corollary is a simple generalization of this. To prove this corollary, use the following simple lemma which comes as a consequence of the Cauchy Schwarz inequality.

Lemma 32.3.16 Suppose $f_{k} \rightarrow f$ in $L^{2}\left(\mathbb{R}^{n}\right)$ and $g_{k} \rightarrow g$ in $L^{2}\left(\mathbb{R}^{n}\right)$. Then

$$
\lim _{k \rightarrow \infty} \int_{\mathbb{R}^{n}} f_{k} g_{k} d x=\int_{\mathbb{R}^{n}} f g d x
$$

Proof:

$$
\begin{gathered}
\left|\int_{\mathbb{R}^{n}} f_{k} g_{k} d x-\int_{\mathbb{R}^{n}} f g d x\right| \leq\left|\int_{\mathbb{R}^{n}} f_{k} g_{k} d x-\int_{\mathbb{R}^{n}} f_{k} g d x\right|+ \\
\left|\int_{\mathbb{R}^{n}} f_{k} g d x-\int_{\mathbb{R}^{n}} f g d x\right| \\
\leq\left\|f_{k}\right\|_{2}| | g-g_{k}\left\|_{2}+\right\| g\left\|_{2}\right\| f_{k}-f \|_{2} .
\end{gathered}
$$

Now $\left\|f_{k}\right\|_{2}$ is a Cauchy sequence and so it is bounded independent of $k$. Therefore, the above expression is smaller than $\varepsilon$ whenever $k$ is large enough.

Corollary 32.3.17 For $f, g \in L^{2}\left(\mathbb{R}^{n}\right)$,

$$
\int_{\mathbb{R}^{n}} f \bar{g} d x=\int_{\mathbb{R}^{n}} F f \overline{F g} d x=\int_{\mathbb{R}^{n}} F^{-1} f \overline{F^{-1} g} d x
$$

Proof: First note the above formula is obvious if $f, g \in \mathscr{G}$. To see this, note

$$
\begin{aligned}
& \int_{\mathbb{R}^{n}} F f \overline{F g} d x=\int_{\mathbb{R}^{n}} F f(\mathbf{x}) \overline{\frac{1}{(2 \pi)^{n / 2}} \int_{\mathbb{R}^{n}} e^{-i \mathbf{x} \cdot \mathbf{t}} g(\mathbf{t}) d t d x} \\
= & \int_{\mathbb{R}^{n}} \frac{1}{(2 \pi)^{n / 2}} \int_{\mathbb{R}^{n}} e^{i \mathbf{X} \cdot \mathbf{t}} F f(\mathbf{x}) d x \overline{g(\mathbf{t})} d t=\int_{\mathbb{R}^{n}}\left(F^{-1} \circ F\right) f(\mathbf{t}) \overline{g(\mathbf{t})} d t \\
= & \int_{\mathbb{R}^{n}} f(\mathbf{t}) \overline{g(\mathbf{t})} d t .
\end{aligned}
$$

The formula with $F^{-1}$ is exactly similar.
Now to verify the corollary, let $\phi_{k} \rightarrow f$ in $L^{2}\left(\mathbb{R}^{n}\right)$ and let $\psi_{k} \rightarrow g$ in $L^{2}\left(\mathbb{R}^{n}\right)$. Then by Lemma 32.3.13

$$
\int_{\mathbb{R}^{n}} F f \overline{F g} d x=\lim _{k \rightarrow \infty} \int_{\mathbb{R}^{n}} F \phi_{k} \overline{F \psi_{k}} d x=\lim _{k \rightarrow \infty} \int_{\mathbb{R}^{n}} \phi_{k} \overline{\psi_{k}} d x=\int_{\mathbb{R}^{n}} f \bar{g} d x
$$

A similar argument holds for $F^{-1}$.
How does one compute $F f$ and $F^{-1} f$ ?
Theorem 32.3.18 For $f \in L^{2}\left(\mathbb{R}^{n}\right)$, let $f_{r}=f \mathscr{X}_{E_{r}}$ where $E_{r}$ is a bounded measurable set with $E_{r} \uparrow \mathbb{R}^{n}$. Then the following limits hold in $L^{2}\left(\mathbb{R}^{n}\right)$.

$$
F f=\lim _{r \rightarrow \infty} F f_{r}, F^{-1} f=\lim _{r \rightarrow \infty} F^{-1} f_{r}
$$

Proof: $\left\|f-f_{r}\right\|_{2} \rightarrow 0$ and so $\left\|F f-F f_{r}\right\|_{2} \rightarrow 0$ and $\left\|F^{-1} f-F^{-1} f_{r}\right\|_{2} \rightarrow 0$ by Plancherel's Theorem.

What are $F f_{r}$ and $F^{-1} f_{r}$ ? Let $\phi \in \mathscr{G}$

$$
\begin{aligned}
\int_{\mathbb{R}^{n}} F f_{r} \phi d x & =\int_{\mathbb{R}^{n}} f_{r} F \phi d x \\
& =(2 \pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} f_{r}(\mathbf{x}) e^{-i \mathbf{x} \cdot \mathbf{y}} \phi(\mathbf{y}) d y d x \\
& =\int_{\mathbb{R}^{n}}\left[(2 \pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{n}} f_{r}(\mathbf{x}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x\right] \phi(\mathbf{y}) d y .
\end{aligned}
$$

Since this holds for all $\phi \in \mathscr{G}$, a dense subset of $L^{2}\left(\mathbb{R}^{n}\right)$, it follows that

$$
F f_{r}(\mathbf{y})=(2 \pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{n}} f_{r}(\mathbf{x}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x
$$

Similarly

$$
F^{-1} f_{r}(\mathbf{y})=(2 \pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{n}} f_{r}(\mathbf{x}) e^{i \mathbf{x} \cdot \mathbf{y}} d x
$$

This shows that to take the Fourier transform of a function in $L^{2}\left(\mathbb{R}^{n}\right)$, it suffices to take the limit as $r \rightarrow \infty$ in $L^{2}\left(\mathbb{R}^{n}\right)$ of $(2 \pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{n}} f_{r}(\mathbf{x}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x$. A similar procedure works for the inverse Fourier transform.

Note this reduces to the earlier definition in case $f \in L^{1}\left(\mathbb{R}^{n}\right)$. Now consider the convolution of a function in $L^{2}$ with one in $L^{1}$.

Theorem 32.3.19 Let $h \in L^{2}\left(\mathbb{R}^{n}\right)$ and let $f \in L^{1}\left(\mathbb{R}^{n}\right)$. Then $h * f \in L^{2}\left(\mathbb{R}^{n}\right)$,

$$
\begin{gathered}
F^{-1}(h * f)=(2 \pi)^{n / 2} F^{-1} h F^{-1} f, \\
F(h * f)=(2 \pi)^{n / 2} F h F f,
\end{gathered}
$$

and

$$
\begin{equation*}
\|h * f\|_{2} \leq\|h\|_{2}\|f\|_{1} . \tag{32.3.11}
\end{equation*}
$$

Proof: An application of Minkowski’s inequality yields

$$
\begin{equation*}
\left(\int_{\mathbb{R}^{n}}\left(\int_{\mathbb{R}^{n}}|h(\mathbf{x}-\mathbf{y})||f(\mathbf{y})| d y\right)^{2} d x\right)^{1 / 2} \leq\|f\|_{1}\|h\|_{2} \tag{32.3.12}
\end{equation*}
$$

Hence $\int|h(\mathbf{x}-\mathbf{y})||f(\mathbf{y})| d y<\infty$ a.e. $\mathbf{x}$ and

$$
\mathbf{x} \rightarrow \int h(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d y
$$

is in $L^{2}\left(\mathbb{R}^{n}\right)$. Let $E_{r} \uparrow \mathbb{R}^{n}, m\left(E_{r}\right)<\infty$. Thus,

$$
h_{r} \equiv \mathscr{X}_{E_{r}} h \in L^{2}\left(\mathbb{R}^{n}\right) \cap L^{1}\left(\mathbb{R}^{n}\right)
$$

and letting $\phi \in \mathscr{G}$,

$$
\begin{aligned}
& \int F\left(h_{r} * f\right)(\phi) d x \\
\equiv & \int\left(h_{r} * f\right)(F \phi) d x \\
= & (2 \pi)^{-n / 2} \iiint h_{r}(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) e^{-i \mathbf{x} \cdot \mathbf{t}} \phi(\mathbf{t}) d t d y d x \\
= & (2 \pi)^{-n / 2} \iint\left(\int h_{r}(\mathbf{x}-\mathbf{y}) e^{-i(\mathbf{x}-\mathbf{y}) \cdot \mathbf{t}} d x\right) f(\mathbf{y}) e^{-i \mathbf{y} \cdot \mathbf{t}} d y \phi(\mathbf{t}) d t \\
= & \int(2 \pi)^{n / 2} F h_{r}(\mathbf{t}) F f(\mathbf{t}) \phi(\mathbf{t}) d t .
\end{aligned}
$$

Since $\phi$ is arbitrary and $\mathscr{G}$ is dense in $L^{2}\left(\mathbb{R}^{n}\right)$,

$$
F\left(h_{r} * f\right)=(2 \pi)^{n / 2} F h_{r} F f .
$$

Now by Minkowski's Inequality, $h_{r} * f \rightarrow h * f$ in $L^{2}\left(\mathbb{R}^{n}\right)$ and also it is clear that $h_{r} \rightarrow h$ in $L^{2}\left(\mathbb{R}^{n}\right)$; so, by Plancherel's theorem, you may take the limit in the above and conclude

$$
F(h * f)=(2 \pi)^{n / 2} F h F f .
$$

The assertion for $F^{-1}$ is similar and 32.3.11 follows from 32.3.12.

### 32.3.4 The Schwartz Class

The problem with $\mathscr{G}$ is that it does not contain $C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$. I have used it in presenting the Fourier transform because the functions in $\mathscr{G}$ have a very specific form which made some technical details work out easier than in any other approach I have seen. The Schwartz class is a larger class of functions which does contain $C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$ and also has the same nice properties as $\mathscr{G}$. The functions in the Schwartz class are infinitely differentiable and they vanish very rapidly as $|\mathbf{x}| \rightarrow \infty$ along with all their partial derivatives. This is the description of these functions, not a specific form involving polynomials times $e^{-\alpha|\mathbf{x}|^{2}}$. To describe this precisely requires some notation.

Definition 32.3.20 $f \in \mathfrak{S}$, the Schwartz class, if $f \in C^{\infty}\left(\mathbb{R}^{n}\right)$ and for all positive integers $N$,

$$
\rho_{N}(f)<\infty
$$

where

$$
\rho_{N}(f)=\sup \left\{\left(1+|\mathbf{x}|^{2}\right)^{N}\left|D^{\alpha} f(\mathbf{x})\right|: \mathbf{x} \in \mathbb{R}^{n},|\alpha| \leq N\right\}
$$

Thus $f \in \mathfrak{S}$ if and only if $f \in C^{\infty}\left(\mathbb{R}^{n}\right)$ and

$$
\begin{equation*}
\sup \left\{\left|\mathbf{x}^{\beta} D^{\alpha} f(\mathbf{x})\right|: \mathbf{x} \in \mathbb{R}^{n}\right\}<\infty \tag{32.3.13}
\end{equation*}
$$

for all multi indices $\alpha$ and $\beta$.
Also note that if $f \in \mathfrak{S}$, then $p(f) \in \mathfrak{S}$ for any polynomial, $p$ with $p(0)=0$ and that

$$
\mathfrak{S} \subseteq L^{p}\left(\mathbb{R}^{n}\right) \cap L^{\infty}\left(\mathbb{R}^{n}\right)
$$

for any $p \geq 1$. To see this assertion about the $p(f)$, it suffices to consider the case of the product of two elements of the Schwartz class. If $f, g \in \mathfrak{S}$, then $D^{\alpha}(f g)$ is a finite sum of derivatives of $f$ times derivatives of $g$. Therefore, $\rho_{N}(f g)<\infty$ for all $N$. You may wonder about examples of things in $\mathfrak{S}$. Clearly any function in $C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$ is in $\mathfrak{S}$. However there are other functions in $\mathfrak{S}$. For example $e^{-|\mathbf{x}|^{2}}$ is in $\mathfrak{S}$ as you can verify for yourself and so is any function from $\mathscr{G}$. Note also that the density of $C_{c}\left(\mathbb{R}^{n}\right)$ in $L^{p}\left(\mathbb{R}^{n}\right)$ shows that $\mathfrak{S}$ is dense in $L^{p}\left(\mathbb{R}^{n}\right)$ for every $p$.

Recall the Fourier transform of a function in $L^{1}\left(\mathbb{R}^{n}\right)$ is given by

$$
F f(\mathbf{t}) \equiv(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \cdot \mathbf{x}} f(\mathbf{x}) d x
$$

Therefore, this gives the Fourier transform for $f \in \mathfrak{S}$. The nice property which $\mathfrak{S}$ has in common with $\mathscr{G}$ is that the Fourier transform and its inverse map $\mathfrak{S}$ one to one onto $\mathfrak{S}$. This means I could have presented the whole of the above theory in terms of $\mathfrak{S}$ rather than in terms of $\mathscr{G}$. However, it is more technical.

Theorem 32.3.21 If $f \in \mathfrak{S}$, then $F f$ and $F^{-1} f$ are also in $\mathfrak{S}$.

Proof: To begin with, let $\alpha=\mathbf{e}_{j}=(0,0, \cdots, 1,0, \cdots, 0)$, the 1 in the $j^{\text {th }}$ slot.

$$
\begin{equation*}
\frac{F^{-1} f\left(\mathbf{t}+h \mathbf{e}_{j}\right)-F^{-1} f(\mathbf{t})}{h}=(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{i \mathbf{t} \cdot \mathbf{x}} f(\mathbf{x})\left(\frac{e^{i h x_{j}}-1}{h}\right) d x \tag{32.3.14}
\end{equation*}
$$

Consider the integrand in 32.3.14.

$$
\begin{aligned}
\left|e^{i \cdot \mathbf{x}} f(\mathbf{x})\left(\frac{e^{i h x_{j}}-1}{h}\right)\right| & =|f(\mathbf{x})|\left|\left(\frac{e^{i(h / 2) x_{j}}-e^{-i(h / 2) x_{j}}}{h}\right)\right| \\
& =|f(\mathbf{x})|\left|\frac{i \sin \left((h / 2) x_{j}\right)}{(h / 2)}\right| \leq|f(\mathbf{x})|\left|x_{j}\right|
\end{aligned}
$$

and this is a function in $L^{1}\left(\mathbb{R}^{n}\right)$ because $f \in \mathfrak{S}$. Therefore by the Dominated Convergence Theorem,

$$
\begin{aligned}
\frac{\partial F^{-1} f(\mathbf{t})}{\partial t_{j}} & =(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{i \mathbf{t} \cdot \mathbf{x}} i x_{j} f(\mathbf{x}) d x \\
& =i(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{i \mathbf{t} \cdot \mathbf{x}} \mathbf{x}^{\mathbf{e}_{j}} f(\mathbf{x}) d x
\end{aligned}
$$

Now $\mathbf{x}^{\mathbf{e}_{j}} f(\mathbf{x}) \in \mathfrak{S}$ and so one can continue in this way and take derivatives indefinitely. Thus $F^{-1} f \in C^{\infty}\left(\mathbb{R}^{n}\right)$ and from the above argument,

$$
D^{\alpha} F^{-1} f(\mathbf{t})=(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{i \mathbf{t} \cdot \mathbf{x}}(i \mathbf{x})^{\alpha} f(\mathbf{x}) d x
$$

To complete showing $F^{-1} f \in \mathfrak{S}$,

$$
\mathbf{t}^{\beta} D^{\alpha} F^{-1} f(\mathbf{t})=(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{i \mathbf{t} \cdot \mathbf{x}} \mathbf{t}^{\beta}(i \mathbf{x})^{a} f(\mathbf{x}) d x
$$

Integrate this integral by parts to get

$$
\begin{equation*}
\mathbf{t}^{\beta} D^{\alpha} F^{-1} f(\mathbf{t})=(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} i^{|\beta|} e^{i \mathbf{t} \cdot \mathbf{x}} D^{\beta}\left((i \mathbf{x})^{a} f(\mathbf{x})\right) d x \tag{32.3.15}
\end{equation*}
$$

Here is how this is done.

$$
\begin{aligned}
\int_{\mathbb{R}} e^{i t_{j} x_{j}} t_{j}^{\beta_{j}}(i \mathbf{x})^{\alpha} f(\mathbf{x}) d x_{j}= & \left.\frac{e^{i t_{j} x_{j}}}{i t_{j}} t_{j}^{\beta_{j}}(i \mathbf{x})^{\alpha} f(\mathbf{x})\right|_{-\infty} ^{\infty}+ \\
& i \int_{\mathbb{R}} e^{i t_{j} x_{j}} t_{j}^{\beta_{j}-1} D^{\mathbf{e}_{j}}\left((i \mathbf{x})^{\alpha} f(\mathbf{x})\right) d x_{j}
\end{aligned}
$$

where the boundary term vanishes because $f \in \mathfrak{S}$. Returning to 32.3 .15 , use the fact that $\left|e^{i a}\right|=1$ to conclude

$$
\left|\mathbf{t}^{\beta} D^{\alpha} F^{-1} f(\mathbf{t})\right| \leq C \int_{\mathbb{R}^{n}}\left|D^{\beta}\left((i \mathbf{x})^{a} f(\mathbf{x})\right)\right| d x<\infty
$$

It follows $F^{-1} f \in \mathfrak{S}$. Similarly $F f \in \mathfrak{S}$ whenever $f \in \mathfrak{S}$.
Of course $\mathfrak{S}$ can be considered a subset of $\mathscr{G}^{*}$ as follows. For $\psi \in \mathfrak{S}$,

$$
\psi(\phi) \equiv \int_{\mathbb{R}^{n}} \psi \phi d x
$$

Theorem 32.3.22 Let $\psi \in \mathfrak{S}$. Then $\left(F \circ F^{-1}\right)(\psi)=\psi$ and $\left(F^{-1} \circ F\right)(\psi)=\psi$ whenever $\psi \in \mathfrak{S}$. Also $F$ and $F^{-1}$ map $\mathfrak{S}$ one to one and onto $\mathfrak{S}$.

Proof: The first claim follows from the fact that $F$ and $F^{-1}$ are inverses of each other on $\mathscr{G}^{*}$ which was established above. For the second, let $\psi \in \mathfrak{S}$. Then $\psi=F\left(F^{-1} \psi\right)$. Thus $F$ maps $\mathfrak{S}$ onto $\mathfrak{S}$. If $F \psi=0$, then do $F^{-1}$ to both sides to conclude $\psi=0$. Thus $F$ is one to one and onto. Similarly, $F^{-1}$ is one to one and onto.

### 32.3.5 Convolution

To begin with it is necessary to discuss the meaning of $\phi f$ where $f \in \mathscr{G}^{*}$ and $\phi \in \mathscr{G}$. What should it mean? First suppose $f \in L^{p}\left(\mathbb{R}^{n}\right)$ or measurable with polynomial growth. Then $\phi f$ also has these properties. Hence, it should be the case that $\phi f(\psi)=\int_{\mathbb{R}^{n}} \phi f \psi d x=$ $\int_{\mathbb{R}^{n}} f(\phi \psi) d x$. This motivates the following definition.

Definition 32.3.23 Let $T \in \mathscr{G}^{*}$ and let $\phi \in \mathscr{G}$. Then $\phi T \equiv T \phi \in \mathscr{G}^{*}$ will be defined by

$$
\phi T(\psi) \equiv T(\phi \psi) .
$$

The next topic is that of convolution. It was just shown that

$$
F(f * \phi)=(2 \pi)^{n / 2} F \phi F f, F^{-1}(f * \phi)=(2 \pi)^{n / 2} F^{-1} \phi F^{-1} f
$$

whenever $f \in L^{2}\left(\mathbb{R}^{n}\right)$ and $\phi \in \mathscr{G}$ so the same definition is retained in the general case because it makes perfect sense and agrees with the earlier definition.

Definition 32.3.24 Let $f \in \mathscr{G}^{*}$ and let $\phi \in \mathscr{G}$. Then define the convolution of $f$ with an element of $\mathscr{G}$ as follows.

$$
f * \phi \equiv(2 \pi)^{n / 2} F^{-1}(F \phi F f) \in \mathscr{G}^{*}
$$

There is an obvious question. With this definition, is it true that

$$
F^{-1}(f * \phi)=(2 \pi)^{n / 2} F^{-1} \phi F^{-1} f
$$

as it was earlier?
Theorem 32.3.25 Let $f \in \mathscr{G}^{*}$ and let $\phi \in \mathscr{G}$.

$$
\begin{gather*}
F(f * \phi)=(2 \pi)^{n / 2} F \phi F f,  \tag{32.3.16}\\
F^{-1}(f * \phi)=(2 \pi)^{n / 2} F^{-1} \phi F^{-1} f . \tag{32.3.17}
\end{gather*}
$$

Proof: Note that 32.3.16 follows from Definition 32.3.24 and both assertions hold for $f \in \mathscr{G}$. Consider 32.3.17. Here is a simple formula involving a pair of functions in $\mathscr{G}$.

$$
\left(\psi * F^{-1} F^{-1} \phi\right)(\mathbf{x})
$$

$$
\begin{aligned}
& =\left(\iiint \psi(\mathbf{x}-\mathbf{y}) e^{i \mathbf{y} \cdot \mathbf{y}_{1}} e^{i \mathbf{y}_{1} \cdot \mathbf{z}} \phi(\mathbf{z}) d z d y_{1} d y\right)(2 \pi)^{n} \\
& =\left(\iiint \psi(\mathbf{x}-\mathbf{y}) e^{-i \mathbf{y} \cdot \tilde{\mathbf{y}}_{1}} e^{-i \tilde{\mathbf{y}}_{1} \cdot \mathbf{z}} \phi(\mathbf{z}) d z d \tilde{y}_{1} d y\right)(2 \pi)^{n} \\
& =(\psi * F F \phi)(\mathbf{x}) .
\end{aligned}
$$

Now for $\psi \in \mathscr{G}$,

$$
\begin{gather*}
(2 \pi)^{n / 2} F\left(F^{-1} \phi F^{-1} f\right)(\psi) \equiv(2 \pi)^{n / 2}\left(F^{-1} \phi F^{-1} f\right)(F \psi) \equiv \\
(2 \pi)^{n / 2} F^{-1} f\left(F^{-1} \phi F \psi\right) \equiv(2 \pi)^{n / 2} f\left(F^{-1}\left(F^{-1} \phi F \psi\right)\right)= \\
f\left((2 \pi)^{n / 2} F^{-1}\left(\left(F F^{-1} F^{-1} \phi\right)(F \psi)\right)\right) \equiv \\
f\left(\psi * F^{-1} F^{-1} \phi\right)=f(\psi * F F \phi) \tag{32.3.18}
\end{gather*}
$$

Also

$$
\begin{gather*}
(2 \pi)^{n / 2} F^{-1}(F \phi F f)(\psi) \equiv(2 \pi)^{n / 2}(F \phi F f)\left(F^{-1} \psi\right) \equiv \\
(2 \pi)^{n / 2} F f\left(F \phi F^{-1} \psi\right) \equiv(2 \pi)^{n / 2} f\left(F\left(F \phi F^{-1} \psi\right)\right)= \\
=f\left(F\left((2 \pi)^{n / 2}\left(F \phi F^{-1} \psi\right)\right)\right) \\
=f\left(F\left((2 \pi)^{n / 2}\left(F^{-1} F F \phi F^{-1} \psi\right)\right)\right)=f\left(F\left(F^{-1}(F F \phi * \psi)\right)\right) \\
f(F F \phi * \psi)=f(\psi * F F \phi) . \tag{32.3.19}
\end{gather*}
$$

The last line follows from the following.

$$
\begin{aligned}
\int F F \phi(\mathbf{x}-\mathbf{y}) \psi(\mathbf{y}) d y & =\int F \phi(\mathbf{x}-\mathbf{y}) F \psi(\mathbf{y}) d y \\
& =\int F \psi(\mathbf{x}-\mathbf{y}) F \phi(\mathbf{y}) d y \\
& =\int \psi(\mathbf{x}-\mathbf{y}) F F \phi(y) d y
\end{aligned}
$$

From 32.3.19 and 32.3.18, since $\psi$ was arbitrary,

$$
(2 \pi)^{n / 2} F\left(F^{-1} \phi F^{-1} f\right)=(2 \pi)^{n / 2} F^{-1}(F \phi F f) \equiv f * \phi
$$

which shows 32.3.17.

### 32.4 Exercises

1. For $f \in L^{1}\left(\mathbb{R}^{n}\right)$, show that if $F^{-1} f \in L^{1}$ or $F f \in L^{1}$, then $f$ equals a continuous bounded function a.e.
2. Suppose $f, g \in L^{1}(\mathbb{R})$ and $F f=F g$. Show $f=g$ a.e.
3. Show that if $f \in L^{1}\left(\mathbb{R}^{n}\right)$, then $\lim _{|\mathbf{x}| \rightarrow \infty} F f(\mathbf{x})=0$.
4. $\uparrow$ Suppose $f * f=f$ or $f * f=0$ and $f \in L^{1}(\mathbb{R})$. Show $f=0$.
5. For this problem define $\int_{a}^{\infty} f(t) d t \equiv \lim _{r \rightarrow \infty} \int_{a}^{r} f(t) d t$. Note this coincides with the Lebesgue integral when $f \in L^{1}(a, \infty)$. Show
(a) $\int_{0}^{\infty} \frac{\sin (u)}{u} d u=\frac{\pi}{2}$
(b) $\lim _{r \rightarrow \infty} \int_{\delta}^{\infty} \frac{\sin (r u)}{u} d u=0$ whenever $\delta>0$.
(c) If $f \in L^{1}(\mathbb{R})$, then $\lim _{r \rightarrow \infty} \int_{\mathbb{R}} \sin (r u) f(u) d u=0$.

Hint: For the first two, use $\frac{1}{u}=\int_{0}^{\infty} e^{-u t} d t$ and apply Fubini's theorem to

$$
\int_{0}^{R} \sin u \int_{\mathbb{R}} e^{-u t} d t d u
$$

For the last part, first establish it for $f \in C_{c}^{\infty}(\mathbb{R})$ and then use the density of this set in $L^{1}(\mathbb{R})$ to obtain the result. This is sometimes called the Riemann Lebesgue lemma.
6. $\uparrow$ Suppose that $g \in L^{1}(\mathbb{R})$ and that at some $x>0, g$ is locally Holder continuous from the right and from the left. This means

$$
\lim _{r \rightarrow 0+} g(x+r) \equiv g(x+)
$$

exists,

$$
\lim _{r \rightarrow 0+} g(x-r) \equiv g(x-)
$$

exists and there exist constants $K, \delta>0$ and $r \in(0,1]$ such that for $|x-y|<\delta$,

$$
|g(x+)-g(y)|<K|x-y|^{r}
$$

for $y>x$ and

$$
|g(x-)-g(y)|<K|x-y|^{r}
$$

for $y<x$. Show that under these conditions,

$$
\begin{gathered}
\lim _{r \rightarrow \infty} \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin (u r)}{u}\left(\frac{g(x-u)+g(x+u)}{2}\right) d u \\
=\frac{g(x+)+g(x-)}{2}
\end{gathered}
$$

7. $\uparrow$ Let $g \in L^{1}(\mathbb{R})$ and suppose $g$ is locally Holder continuous from the right and from the left at $x$. Show that then

$$
\lim _{R \rightarrow \infty} \frac{1}{2 \pi} \int_{-R}^{R} e^{i x t} \int_{-\infty}^{\infty} e^{-i t y} g(y) d y d t=\frac{g(x+)+g(x-)}{2}
$$

This is very interesting. If $g \in L^{2}(\mathbb{R})$, this shows $F^{-1}(F g)(x)=\frac{g(x+)+g(x-)}{2}$, the midpoint of the jump in $g$ at the point, $x$. In particular, if $g \in \mathscr{G}, F^{-1}(F g)=g$. Hint: Show the left side of the above equation reduces to

$$
\frac{2}{\pi} \int_{0}^{\infty} \frac{\sin (u r)}{u}\left(\frac{g(x-u)+g(x+u)}{2}\right) d u
$$

and then use Problem 6 to obtain the result.
8. $\uparrow$ A measurable function $g$ defined on $(0, \infty)$ has exponential growth if $|g(t)| \leq C e^{\eta t}$ for some $\eta$. For $\operatorname{Re}(s)>\eta$, define the Laplace Transform by

$$
L g(s) \equiv \int_{0}^{\infty} e^{-s u} g(u) d u
$$

Assume that $g$ has exponential growth as above and is Holder continuous from the right and from the left at $t$. Pick $\gamma>\eta$. Show that

$$
\lim _{R \rightarrow \infty} \frac{1}{2 \pi} \int_{-R}^{R} e^{\gamma t} e^{i y t} L g(\gamma+i y) d y=\frac{g(t+)+g(t-)}{2}
$$

This formula is sometimes written in the form

$$
\frac{1}{2 \pi i} \int_{\gamma-i \infty}^{\gamma+i \infty} e^{s t} \operatorname{Lg}(s) d s
$$

and is called the complex inversion integral for Laplace transforms. It can be used to find inverse Laplace transforms. Hint:

$$
\begin{gathered}
\frac{1}{2 \pi} \int_{-R}^{R} e^{\gamma t} e^{i y t} L g(\gamma+i y) d y= \\
\frac{1}{2 \pi} \int_{-R}^{R} e^{\gamma t} e^{i y t} \int_{0}^{\infty} e^{-(\gamma+i y) u} g(u) d u d y
\end{gathered}
$$

Now use Fubini's theorem and do the integral from $-R$ to $R$ to get this equal to

$$
\frac{e^{\gamma t}}{\pi} \int_{-\infty}^{\infty} e^{-\gamma u} \bar{g}(u) \frac{\sin (R(t-u))}{t-u} d u
$$

where $\bar{g}$ is the zero extension of $g$ off $[0, \infty)$. Then this equals

$$
\frac{e^{\gamma t}}{\pi} \int_{-\infty}^{\infty} e^{-\gamma(t-u)} \bar{g}(t-u) \frac{\sin (R u)}{u} d u
$$

which equals

$$
\frac{2 e^{\gamma t}}{\pi} \int_{0}^{\infty} \frac{\bar{g}(t-u) e^{-\gamma(t-u)}+\bar{g}(t+u) e^{-\gamma(t+u)}}{2} \frac{\sin (R u)}{u} d u
$$

and then apply the result of Problem 6.
9. Suppose $f \in \mathfrak{S}$. Show $F\left(f_{x_{j}}\right)(\mathbf{t})=i t_{j} F f(\mathbf{t})$.
10. Let $f \in \mathfrak{S}$ and let $k$ be a positive integer.

$$
\|f\|_{k, 2} \equiv\left(\|f\|_{2}^{2}+\sum_{|\alpha| \leq k}\left\|D^{\alpha} f\right\|_{2}^{2}\right)^{1 / 2}
$$

One could also define

$$
\||f|\|_{k, 2} \equiv\left(\int_{R^{n}}|F f(\mathbf{x})|^{2}\left(1+|\mathbf{x}|^{2}\right)^{k} d x\right)^{1 / 2}
$$

Show both $\left\|\left\|\|_{k, 2}\right.\right.$ and $\left.\|\right\|\left\|\|_{k, 2}\right.$ are norms on $\mathfrak{S}$ and that they are equivalent. These are Sobolev space norms. For which values of $k$ does the second norm make sense? How about the first norm?
11. $\uparrow$ Define $H^{k}\left(\mathbb{R}^{n}\right), k \geq 0$ by $f \in L^{2}\left(\mathbb{R}^{n}\right)$ such that

$$
\begin{gathered}
\left(\int|F f(\mathbf{x})|^{2}\left(1+|\mathbf{x}|^{2}\right)^{k} d x\right)^{\frac{1}{2}}<\infty \\
\left\|\left||f| \|_{k, 2} \equiv\left(\int|F f(\mathbf{x})|^{2}\left(1+|\mathbf{x}|^{2}\right)^{k} d x\right)^{\frac{1}{2}}\right.\right.
\end{gathered}
$$

Show $H^{k}\left(\mathbb{R}^{n}\right)$ is a Banach space, and that if $k$ is a positive integer, $H^{k}\left(\mathbb{R}^{n}\right)=\{f \in$ $L^{2}\left(\mathbb{R}^{n}\right)$ : there exists $\left\{u_{j}\right\} \subseteq \mathscr{G}$ with $\left\|u_{j}-f\right\|_{2} \rightarrow 0$ and $\left\{u_{j}\right\}$ is a Cauchy sequence in $\left\|\|_{k, 2}\right.$ of Problem 10$\}$. This is one way to define Sobolev Spaces. Hint: One way to do the second part of this is to define a new measure, $\mu$ by

$$
\mu(E) \equiv \int_{E}\left(1+|\mathbf{x}|^{2}\right)^{k} d x
$$

Then show $\mu$ is a Radon measure and show there exists $\left\{g_{m}\right\}$ such that $g_{m} \in \mathscr{G}$ and $g_{m} \rightarrow F f$ in $L^{2}(\mu)$. Thus $g_{m}=F f_{m}, f_{m} \in \mathscr{G}$ because $F$ maps $\mathscr{G}$ onto $\mathscr{G}$. Then by Problem 10, $\left\{f_{m}\right\}$ is Cauchy in the norm $\left\|\|_{k, 2}\right.$.
12. $\uparrow$ If $2 k>n$, show that if $f \in H^{k}\left(\mathbb{R}^{n}\right)$, then $f$ equals a bounded continuous function a.e. Hint: Show that for $k$ this large, $F f \in L^{1}\left(\mathbb{R}^{n}\right)$, and then use Problem 1. To do this, write

$$
|F f(\mathbf{x})|=|F f(\mathbf{x})|\left(1+|\mathbf{x}|^{2}\right)^{\frac{k}{2}}\left(1+|\mathbf{x}|^{2}\right)^{\frac{-k}{2}}
$$

So

$$
\int|F f(\mathbf{x})| d x=\int|F f(\mathbf{x})|\left(1+|\mathbf{x}|^{2}\right)^{\frac{k}{2}}\left(1+|\mathbf{x}|^{2}\right)^{\frac{-k}{2}} d x
$$

Use the Cauchy Schwarz inequality. This is an example of a Sobolev imbedding Theorem.
13. Let $u \in \mathscr{G}$. Then $F u \in \mathscr{G}$ and so, in particular, it makes sense to form the integral,

$$
\int_{\mathbb{R}} F u\left(\mathbf{x}^{\prime}, x_{n}\right) d x_{n}
$$

where $\left(\mathbf{x}^{\prime}, x_{n}\right)=\mathbf{x} \in \mathbb{R}^{n}$. For $u \in \mathscr{G}$, define $\gamma u\left(\mathbf{x}^{\prime}\right) \equiv u\left(\mathbf{x}^{\prime}, 0\right)$. Find a constant such that $F(\gamma u)\left(\mathbf{x}^{\prime}\right)$ equals this constant times the above integral. Hint: By the dominated convergence theorem

$$
\int_{\mathbb{R}} F u\left(\mathbf{x}^{\prime}, x_{n}\right) d x_{n}=\lim _{\varepsilon \rightarrow 0} \int_{\mathbb{R}} e^{-\left(\varepsilon x_{n}\right)^{2}} F u\left(\mathbf{x}^{\prime}, x_{n}\right) d x_{n} .
$$

Now use the definition of the Fourier transform and Fubini's theorem as required in order to obtain the desired relationship.
14. Recall the Fourier series of a function in $L^{2}(-\pi, \pi)$ converges to the function in $L^{2}(-\pi, \pi)$. Prove a similar theorem with $L^{2}(-\pi, \pi)$ replaced by $L^{2}(-m \pi, m \pi)$ and the functions

$$
\left\{(2 \pi)^{-(1 / 2)} e^{i n x}\right\}_{n \in \mathbb{Z}}
$$

used in the Fourier series replaced with

$$
\left\{(2 m \pi)^{-(1 / 2)} e^{i \frac{n}{m} x}\right\}_{n \in \mathbb{Z}}
$$

Now suppose $f$ is a function in $L^{2}(\mathbb{R})$ satisfying $F f(t)=0$ if $|t|>m \pi$. Show that if this is so, then

$$
f(x)=\frac{1}{\pi} \sum_{n \in \mathbb{Z}} f\left(\frac{-n}{m}\right) \frac{\sin (\pi(m x+n))}{m x+n} .
$$

Here $m$ is a positive integer. This is sometimes called the Shannon sampling theorem.Hint: First note that since $F f \in L^{2}$ and is zero off a finite interval, it follows $F f \in L^{1}$. Also

$$
f(t)=\frac{1}{\sqrt{2 \pi}} \int_{-m \pi}^{m \pi} e^{i t x} F f(x) d x
$$

and you can conclude from this that $f$ has all derivatives and they are all bounded. Thus $f$ is a very nice function. You can replace $F f$ with its Fourier series. Then consider carefully the Fourier coefficient of $F f$. Argue it equals $f\left(\frac{-n}{m}\right)$ or at least an appropriate constant times this. When you get this the rest will fall quickly into place if you use $F f$ is zero off $[-m \pi, m \pi]$.

## Chapter 33

## Fourier Analysis In $\mathbb{R}^{n}$

The purpose of this chapter is to present some of the most important theorems on Fourier analysis in $\mathbb{R}^{n}$. These theorems are the Marcinkiewicz interpolation theorem, the Calderon Zygmund decomposition, and Mihlin's theorem. They are all fundamental results whose proofs depend on the methods of real analysis.

### 33.1 The Marcinkiewicz Interpolation Theorem

Let $(\Omega, \mu, \mathscr{S})$ be a measure space.
Definition 33.1.1 $L^{p}(\Omega)+L^{1}(\Omega)$ will denote the space of measurable functions, $f$, such that $f$ is the sum of a function in $L^{p}(\Omega)$ and $L^{1}(\Omega)$. Also, if $T: L^{p}(\Omega)+L^{1}(\Omega) \rightarrow$ space of measurable functions, $T$ is subadditive if

$$
|T(f+g)(x)| \leq|T f(x)|+|T g(x)|
$$

$T$ is of type $(p, p)$ if there exists a constant independent of $f \in L^{p}(\Omega)$ such that

$$
\|T f\|_{p} \leq A\|f\|_{p}, f \in L^{p}(\Omega)
$$

$T$ is weak type $(p, p)$ if there exists a constant $A$ independent of $f$ such that

$$
\mu([x:|T f(x)|>\alpha]) \leq\left(\frac{A}{\alpha}\|f\|_{p}\right)^{p}, f \in L^{p}(\Omega)
$$

The following lemma involves writing a function as a sum of a functions whose values are small and one whose values are large.
Lemma 33.1.2 If $p \in[1, r]$, then $L^{p}(\Omega) \subseteq L^{1}(\Omega)+L^{r}(\Omega)$.
Proof: Let $\lambda>0$ and let $f \in L^{p}(\Omega)$

$$
f_{1}(x) \equiv\left\{\begin{array}{l}
f(x) \text { if }|f(x)| \leq \lambda \\
0 \text { if }|f(x)|>\lambda
\end{array} \quad, \quad f_{2}(x) \equiv\left\{\begin{array}{l}
f(x) \text { if }|f(x)|>\lambda \\
0 \text { if }|f(x)| \leq \lambda
\end{array}\right.\right.
$$

Thus $f(x)=f_{1}(x)+f_{2}(x)$.

$$
\int\left|f_{1}(x)\right|^{r} d \mu=\int_{[|f| \leq \lambda]}|f(x)|^{r} d \mu \leq \lambda^{r-p} \int_{[|f| \leq \lambda]}|f(x)|^{p} d \mu<\infty
$$

Therefore, $f_{1} \in L^{r}(\Omega)$.

$$
\int\left|f_{2}(x)\right| d \mu=\int_{[|f|>\lambda]}|f(x)| d \mu \leq \mu[|f|>\lambda]^{1 / p^{\prime}}\left(\int|f|^{p} d \mu\right)^{1 / p}<\infty
$$

This proves the lemma since $f=f_{1}+f_{2}, f_{1} \in L^{r}$ and $f_{2} \in L^{1}$.
For $f$ a function having nonnegative real values, $\alpha \rightarrow \mu([f>\alpha])$ is called the distribution function.

Lemma 33.1.3 Let $\phi(0)=0, \phi$ is strictly increasing, and $C^{1}$. Let $f: \Omega \rightarrow[0, \infty)$ be measurable. Then

$$
\begin{equation*}
\int_{\Omega}(\phi \circ f) d \mu=\int_{0}^{\infty} \phi^{\prime}(\alpha) \mu[f>\alpha] d \alpha \tag{33.1.1}
\end{equation*}
$$

Proof: First suppose

$$
f=\sum_{i=1}^{m} a_{i} \mathscr{X}_{E_{i}}
$$

where $a_{i}>0$ and the $a_{i}$ are all distinct nonzero values of $f$, the sets, $E_{i}$ being disjoint. Thus,

$$
\int_{\Omega}(\phi \circ f) d \mu=\sum_{i=1}^{m} \phi\left(a_{i}\right) \mu\left(E_{i}\right)
$$

Suppose without loss of generality $a_{1}<a_{2}<\cdots<a_{m}$. Observe

$$
\alpha \rightarrow \mu([f>\alpha])
$$

is constant on the intervals $\left[0, a_{1}\right),\left[a_{1}, a_{2}\right), \cdots$. For example, on $\left[a_{i}, a_{i+1}\right)$, this function has the value

$$
\sum_{j=i+1}^{m} \mu\left(E_{j}\right)
$$

The function equals zero on $\left[a_{m}, \infty\right)$. Therefore,

$$
\alpha \rightarrow \phi^{\prime}(\alpha) \mu([|f|>\alpha])
$$

is Lebesgue measurable and letting $a_{0}=0$, the second integral in 33.1.1 equals

$$
\begin{aligned}
\int_{0}^{\infty} \phi^{\prime}(\alpha) \mu([f>\alpha]) d \alpha & =\sum_{i=1}^{m} \int_{a_{i-1}}^{a_{i}} \phi^{\prime}(\alpha) \mu([f>\alpha]) d \alpha \\
& =\sum_{i=1}^{m} \sum_{j=i}^{m} \mu\left(E_{j}\right) \int_{a_{i-1}}^{a_{i}} \phi^{\prime}(\alpha) d \alpha \\
& =\sum_{j=1}^{m} \sum_{i=1}^{j} \mu\left(E_{j}\right)\left(\phi\left(a_{i}\right)-\phi\left(a_{i-1}\right)\right) \\
& =\sum_{j=1}^{m} \mu\left(E_{j}\right) \phi\left(a_{j}\right)=\int_{\Omega}(\phi \circ f) d \mu
\end{aligned}
$$

and so this establishes 33.1 .1 in the case when $f$ is a nonnegative simple function. Since every measurable nonnegative function may be written as the pointwise limit of such simple functions, the desired result will follow by the Monotone convergence theorem and the next claim.

Claim: If $f_{n} \uparrow f$, then for each $\alpha>0$,

$$
\mu([f>\alpha])=\lim _{n \rightarrow \infty} \mu\left(\left[f_{n}>\alpha\right]\right)
$$

Proof of the claim: $\left[f_{n}>\alpha\right] \uparrow[f>\alpha]$ because if $f(x)>\alpha$ then for large enough $n$, $f_{n}(x)>\alpha$ and so

$$
\mu\left(\left[f_{n}>\alpha\right]\right) \uparrow \mu([f>\alpha])
$$

This proves the lemma. (Note the importance of the strict inequality in $[f>\alpha]$ in proving the claim.)

The next theorem is the main result in this section. It is called the Marcinkiewicz interpolation theorem.

Theorem 33.1.4 Let $(\Omega, \mu, \mathscr{S})$ be a $\sigma$ finite measure space, $1<r<\infty$, and let

$$
T: L^{1}(\Omega)+L^{r}(\Omega) \rightarrow \text { space of measurable functions }
$$

be subadditive, weak $(r, r)$, and weak $(1,1)$. Then $T$ is of type $(p, p)$ for every $p \in(1, r)$ and

$$
\|T f\|_{p} \leq A_{p}\|f\|_{p}
$$

where the constant $A_{p}$ depends only on $p$ and the constants in the definition of weak $(1,1)$ and weak $(r, r)$.

Proof: Let $\alpha>0$ and let $f_{1}$ and $f_{2}$ be defined as in Lemma 33.1.2,

$$
f_{1}(x) \equiv\left\{\begin{array}{l}
f(x) \text { if }|f(x)| \leq \alpha \\
0 \text { if }|f(x)|>\alpha
\end{array} \quad, f_{2}(x) \equiv\left\{\begin{array}{l}
f(x) \text { if }|f(x)|>\alpha \\
0 \text { if }|f(x)| \leq \alpha
\end{array} .\right.\right.
$$

Thus $f=f_{1}+f_{2}$ where $f_{1} \in L^{r}$ and $f_{2} \in L^{1}$. Since $T$ is subadditive,

$$
[|T f|>\alpha] \subseteq\left[\left|T f_{1}\right|>\alpha / 2\right] \cup\left[\left|T f_{2}\right|>\alpha / 2\right]
$$

Let $p \in(1, r)$. By Lemma 33.1.3,

$$
\begin{aligned}
& \int|T f|^{p} d \mu \leq p \int_{0}^{\infty} \alpha^{p-1} \mu\left(\left[\left|T f_{1}\right|>\alpha / 2\right]\right) d \alpha+ \\
& \quad+p \int_{0}^{\infty} \alpha^{p-1} \mu\left(\left[\left|T f_{2}\right|>\alpha / 2\right]\right) d \alpha
\end{aligned}
$$

Therefore, since $T$ is weak $(1,1)$ and weak $(r, r)$,

$$
\begin{equation*}
\int|T f|^{p} d \mu \leq p \int_{0}^{\infty} \alpha^{p-1}\left(\frac{2 A_{r}}{\alpha}\left\|f_{1}\right\|_{r}\right)^{r} d \alpha+p \int_{0}^{\infty} \alpha^{p-1} \frac{2 A_{1}}{\alpha}\left\|f_{2}\right\|_{1} d \alpha \tag{33.1.2}
\end{equation*}
$$

Therefore, the right side of 33.1.2 equals

$$
\begin{gathered}
p\left(2 A_{r}\right)^{r} \int_{0}^{\infty} \alpha^{p-1-r} \int_{\Omega}\left|f_{1}\right|^{r} d \mu d \alpha+2 A_{1} p \int_{0}^{\infty} \alpha^{p-2} \int_{\Omega}\left|f_{2}\right| d \mu d \alpha= \\
p\left(2 A_{r}\right)^{r} \int_{\Omega} \int_{0}^{\infty} \alpha^{p-1-r}\left|f_{1}\right|^{r} d \alpha d \mu+2 A_{1} p \int_{\Omega} \int_{0}^{\infty} \alpha^{p-2}\left|f_{2}\right| d \alpha d \mu
\end{gathered}
$$

Now $f_{1}(x)=0$ unless $\left|f_{1}(x)\right| \leq \alpha$ and $f_{2}(x)=0$ unless $\left|f_{2}(x)\right|>\alpha$ so this equals

$$
p\left(2 A_{r}\right)^{r} \int_{\Omega}|f(x)|^{r} \int_{|f(x)|}^{\infty} \alpha^{p-1-r} d \alpha d \mu+2 A_{1} p \int_{\Omega}|f(x)| \int_{0}^{|f(x)|} \alpha^{p-2} d \alpha d \mu
$$

which equals

$$
\begin{aligned}
& \frac{2^{r} A_{r}^{r} p}{r-p} \int_{\Omega}|f(x)|^{p} d \mu+\frac{2 p A_{1}}{p-1} \int_{\Omega}|f(x)|^{p} d \mu \\
& \quad \leq \max \left(\frac{2^{r} A_{r}^{r} p}{r-p}, \frac{2 p A_{1}}{p-1}\right)\|f\|_{L^{p}(\Omega)}^{p}
\end{aligned}
$$

and this proves the theorem.

### 33.2 The Calderon Zygmund Decomposition

For a given nonnegative integrable function, $\mathbb{R}^{n}$ can be decomposed into a set where the function is small and a set which is the union of disjoint cubes on which the average of the function is under some control. The measure in this section will always be Lebesgue measure on $\mathbb{R}^{n}$. This theorem depends on the Lebesgue theory of differentiation.

Theorem 33.2.1 Let $f \geq 0, \int f d x<\infty$, and let $\alpha$ be a positive constant. Then there exist sets $F$ and $\Omega$ such that

$$
\begin{gather*}
\mathbb{R}^{n}=F \cup \Omega, F \cap \Omega=\emptyset  \tag{33.2.3}\\
f(x) \leq \alpha \text { a.e. on } F \tag{33.2.4}
\end{gather*}
$$

$\Omega=\cup_{k=1}^{\infty} Q_{k}$ where the interiors of the cubes are disjoint and for each cube, $Q_{k}$,

$$
\begin{equation*}
\alpha<\frac{1}{m\left(Q_{k}\right)} \int_{Q_{k}} f(x) d x \leq 2^{n} \alpha . \tag{33.2.5}
\end{equation*}
$$

Proof: Let $S_{0}$ be a tiling of $\mathbb{R}^{n}$ into cubes having sides of length $M$ where $M$ is chosen large enough that if $Q$ is one of these cubes, then

$$
\begin{equation*}
\frac{1}{m(Q)} \int_{Q} f d m \leq \alpha \tag{33.2.6}
\end{equation*}
$$

Suppose $S_{0}, \cdots, S_{m}$ have been chosen. To get $S_{m+1}$, replace each cube of $S_{m}$ by the $2^{n}$ cubes obtained by bisecting the sides. Then $S_{m+1}$ consists of exactly those cubes of $S_{m}$ for which 33.2.6 holds and let $T_{m+1}$ consist of the bisected cubes from $S_{m}$ for which 33.2.6 does not hold. Now define

$$
\begin{gathered}
F \equiv\left\{\mathbf{x}: \mathbf{x} \text { is contained in some cube from } S_{m} \text { for all } m\right\}, \\
\Omega \equiv \mathbb{R}^{n} \backslash F=\cup_{m=1}^{\infty} \cup\left\{Q: Q \in T_{m}\right\}
\end{gathered}
$$

Note that the cubes from $T_{m}$ have pair wise disjoint interiors and also the interiors of cubes from $T_{m}$ have empty intersections with the interiors of cubes of $T_{k}$ if $k \neq m$.

Let $\mathbf{x}$ be a point of $\Omega$ and let $\mathbf{x}$ be in a cube of $T_{m}$ such that $m$ is the first index for which this happens. Let $Q$ be the cube in $S_{m-1}$ containing $\mathbf{x}$ and let $Q^{*}$ be the cube in the bisection of $Q$ which contains $\mathbf{x}$. Therefore 33.2.6 does not hold for $Q^{*}$. Thus

$$
\alpha<\frac{1}{m\left(Q^{*}\right)} \int_{Q^{*}} f d x \leq \frac{m(Q)}{m\left(Q^{*}\right)} \overbrace{\frac{1}{m(Q)} \int_{Q} f d x}^{\leq \alpha} \leq 2^{n} \alpha
$$

which shows $\Omega$ is the union of cubes having disjoint interiors for which 33.2 .5 holds.
Now a.e. point of $F$ is a Lebesgue point of $f$. Let $\mathbf{x}$ be such a point of $F$ and suppose $\mathbf{x}$ $\in Q_{k}$ for $Q_{k} \in S_{k}$. Let $d_{k} \equiv$ diameter of $Q_{k}$. Thus $d_{k} \rightarrow 0$.

$$
\begin{gathered}
\frac{1}{m\left(Q_{k}\right)} \int_{Q_{k}}|f(\mathbf{y})-f(\mathbf{x})| d y \leq \frac{1}{m\left(Q_{k}\right)} \int_{B\left(\mathbf{x}, d_{k}\right)}|f(\mathbf{y})-f(\mathbf{x})| d y \\
\quad=\frac{m\left(B\left(\mathbf{x}, d_{k}\right)\right)}{m\left(Q_{k}\right)} \frac{1}{m\left(B\left(\mathbf{x}, d_{k}\right)\right)} \int_{B\left(\mathbf{x}, d_{k}\right)}|f(\mathbf{x})-f(\mathbf{y})| d y \\
\quad \leq K_{n} \frac{1}{m\left(B\left(\mathbf{x}, d_{k}\right)\right)} \int_{B\left(\mathbf{x}, d_{k}\right)}|f(\mathbf{x})-f(\mathbf{y})| d y
\end{gathered}
$$

where $K_{n}$ is a constant which depends on $n$ and measures the ratio of the volume of a ball with diamiter $2 d$ and a cube with diameter $d$. The last expression converges to 0 because $\mathbf{x}$ is a Lebesgue point. Hence

$$
f(\mathbf{x})=\lim _{k \rightarrow \infty} \frac{1}{m\left(Q_{k}\right)} \int_{Q_{k}} f(\mathbf{y}) d y \leq \alpha
$$

and this shows $f(\mathbf{x}) \leq \alpha$ a.e. on $F$. This proves the theorem.

### 33.3 Mihlin's Theorem

In this section, the Marcinkiewicz interpolation theorem and Calderon Zygmund decomposition will be used to establish a remarkable theorem of Mihlin, a generalization of Plancherel's theorem to the $L^{p}$ spaces. It is of fundamental importance in the study of elliptic partial differential equations and can also be used to give proofs for the theory of singular integrals. Mihlin's theorem involves a conclusion which is of the form

$$
\begin{equation*}
\left\|F^{-1} \rho * \phi\right\|_{p} \leq A_{p}\|\phi\|_{p} \tag{33.3.7}
\end{equation*}
$$

for $p>1$ and $\phi \in \mathscr{G}$. Thus $F^{-1} \rho *$ extends to a continuous linear map defined on $L^{p}$ because of the density of $\mathscr{G}$. It is proved by showing various weak type estimates and then applying the Marcinkiewicz Interpolation Theorem to get an estimate like the above.

Recall that by Corollary 32.3.19, if $f \in L^{2}\left(\mathbb{R}^{n}\right)$ and if $\phi \in \mathscr{G}$, then $f * \phi \in L^{2}\left(\mathbb{R}^{n}\right)$ and

$$
F(f * \phi)(\mathbf{x})=(2 \pi)^{n / 2} F \phi(\mathbf{x}) F f(\mathbf{x})
$$

The next lemma is essentially a weak $(1,1)$ estimate. The inequality 33.3 .7 is established under the condition, 33.3.8 and then it is shown there exist conditions which are easier to
verify which imply condition 33.3.8. I think the approach used here is due to Hormander [69] and is found in Berg and Lofstrom [16]. For many more references and generalizations, you might look in Triebel [124]. A different proof based on singular integrals is in Stein [122]. Functions, $\rho$ which yield an inequality of the sort in 33.3.7 are called $L^{p}$ multipliers.

Lemma 33.3.1 Suppose $\rho \in L^{\infty}\left(\mathbb{R}^{n}\right) \cap L^{2}\left(\mathbb{R}^{n}\right)$ and suppose also there exists a constant $C_{1}$ such that

$$
\begin{equation*}
\int_{|\mathbf{x}| \geq 2|\mathbf{y}|}\left|F^{-1} \rho(\mathbf{x}-\mathbf{y})-F^{-1} \rho(\mathbf{x})\right| d x \leq C_{1} . \tag{33.3.8}
\end{equation*}
$$

Then there exists a constant A depending only on $C_{1},\|\rho\|_{\infty}$, and $n$ such that

$$
m\left(\left[\mathbf{x}:\left|F^{-1} \rho * \phi(\mathbf{x})\right|>\alpha\right]\right) \leq \frac{A}{\alpha}\|\phi\|_{1}
$$

for all $\phi \in \mathscr{G}$.
Proof: Let $\phi \in \mathscr{G}$ and use the Calderon decomposition to write $\mathbb{R}^{n}=E \cup \Omega$ where $\Omega$ is a union of cubes, $\left\{Q_{i}\right\}$ with disjoint interiors such that

$$
\begin{equation*}
\alpha m\left(Q_{i}\right) \leq \int_{Q_{i}}|\phi(\mathbf{x})| d x \leq 2^{n} \alpha m\left(Q_{i}\right),|\phi(\mathbf{x})| \leq \alpha \text { a.e. on } E . \tag{33.3.9}
\end{equation*}
$$

The proof is accomplished by writing $\phi$ as the sum of a good function and a bad function and establishing a similar weak inequality for these two functions separately. Then this information is used to obtain the desired conclusion.

$$
g(\mathbf{x})=\left\{\begin{array}{l}
\phi(\mathbf{x}) \text { if } \mathbf{x} \in E  \tag{33.3.10}\\
\frac{1}{m\left(Q_{i}\right)} \int_{Q_{i}} \phi(\mathbf{x}) d x \text { if } \mathbf{x} \in Q_{i} \subseteq \Omega
\end{array}, g(\mathbf{x})+b(\mathbf{x})=\phi(\mathbf{x}) .\right.
$$

Thus

$$
\begin{align*}
\int_{Q_{i}} b(\mathbf{x}) d x & =\int_{Q_{i}}(\phi(\mathbf{x})-g(\mathbf{x})) d x=\int_{Q_{i}} \phi(\mathbf{x}) d x-\int_{Q_{i}} \phi(\mathbf{x}) d x=0  \tag{33.3.11}\\
b(\mathbf{x}) & =0 \text { if } \mathbf{x} \notin \Omega \tag{33.3.12}
\end{align*}
$$

## Claim:

$$
\begin{equation*}
\|g\|_{2}^{2} \leq \alpha\left(1+4^{n}\right)\|\phi\|_{1},\|g\|_{1} \leq\|\phi\|_{1} . \tag{33.3.13}
\end{equation*}
$$

## Proof of claim:

$$
\|g\|_{2}^{2}=\|g\|_{L^{2}(E)}^{2}+\|g\|_{L^{2}(\Omega)}^{2} .
$$

Thus

$$
\begin{aligned}
\|g\|_{L^{2}(\Omega)}^{2} & =\sum_{i} \int_{Q_{i}}|g(x)|^{2} d x \\
& \leq \sum_{i} \int_{Q_{i}}\left(\frac{1}{m\left(Q_{i}\right)} \int_{Q_{i}}|\phi(y)| d y\right)^{2} d x \\
& \leq \sum_{i} \int_{Q_{i}}\left(2^{n} \alpha\right)^{2} d x \leq 4^{n} \alpha^{2} \sum_{i} m\left(Q_{i}\right) \\
& \leq 4^{n} \alpha^{2} \frac{1}{\alpha} \sum_{i} \int_{Q_{i}}|\phi(x)| d x \leq 4^{n} \alpha\|\phi\|_{1} .
\end{aligned}
$$

$$
\|g\|_{L^{2}(E)}^{2}=\int_{E}|\phi(x)|^{2} d x \leq \alpha \int_{E}|\phi(x)| d x=\alpha\|\phi\|_{1} .
$$

Now consider the second of the inequalities in 33.3.13.

$$
\begin{aligned}
\|g\|_{1} & =\int_{E}|g(\mathbf{x})| d x+\int_{\Omega}|g(\mathbf{x})| d x \\
& =\int_{E}|\phi(\mathbf{x})| d x+\sum_{i} \int_{Q_{i}}|g| d x \\
& \leq \int_{E}|\phi(\mathbf{x})| d x+\sum_{i} \int_{Q_{i}} \frac{1}{m\left(Q_{i}\right)} \int_{Q_{i}}|\phi(\mathbf{x})| d m(x) d m \\
& =\int_{E}|\phi(\mathbf{x})| d x+\sum_{i} \int_{Q_{i}}|\phi(\mathbf{x})| d m(x)=\|\phi\|_{1}
\end{aligned}
$$

This proves the claim. From the claim, it follows that $b \in L^{2}\left(\mathbb{R}^{n}\right) \cap L^{1}\left(\mathbb{R}^{n}\right)$.
Because of 33.3.13, $g \in L^{1}\left(\mathbb{R}^{n}\right)$ and so $F^{-1} \rho * g \in L^{2}\left(\mathbb{R}^{n}\right)$. (Since $\rho \in L^{2}$, it follows $F^{-1} \rho \in L^{2}$ and so this convolution is indeed in $L^{2}$.) By Plancherel's theorem,

$$
\left\|F^{-1} \rho * g\right\|_{2}=\left\|F\left(F^{-1} \rho * g\right)\right\|_{2}
$$

By Corollary 32.3.19 on Page 1110, the expression on the right equals

$$
(2 \pi)^{n / 2}\|\rho F g\|_{2}
$$

and so

$$
\left\|F^{-1} \rho * g\right\|_{2}=(2 \pi)^{n / 2}\|\rho F g\|_{2} \leq C_{n}\|\rho\|_{\infty}\|g\|_{2}
$$

From this and 33.3.13

$$
\begin{gather*}
m\left(\left[\left|F^{-1} \rho * g\right| \geq \alpha / 2\right]\right) \\
\leq \frac{C_{n}\|\rho\|_{\infty}^{2}}{\alpha^{2}} \alpha\left(1+4^{n}\right)\|\phi\|_{1}=C_{n} \alpha^{-1}\|\phi\|_{1} . \tag{33.3.14}
\end{gather*}
$$

This is what is wanted so far as $g$ is concerned. Next it is required to estimate

$$
m\left(\left[\left|F^{-1} \rho * b\right| \geq \alpha / 2\right]\right)
$$

If $Q$ is one of the cubes whose union is $\Omega$, let $Q^{*}$ be the cube with the same center as $Q$ but whose sides are $2 \sqrt{n}$ times as long.


Let

$$
\Omega^{*} \equiv \cup_{i=1}^{\infty} Q_{i}^{*}
$$

and let

$$
E^{*} \equiv \mathbb{R}^{n} \backslash \Omega^{*}
$$

Thus $E^{*} \subseteq E$. Let $\mathbf{x} \in E^{*}$. Then because of 33.3.11,

$$
\begin{gather*}
\int_{Q_{i}} F^{-1} \rho(\mathbf{x}-\mathbf{y}) b(\mathbf{y}) d y \\
=\int_{Q_{i}}\left[F^{-1} \rho(\mathbf{x}-\mathbf{y})-F^{-1} \rho\left(\mathbf{x}-\mathbf{y}_{i}\right)\right] b(\mathbf{y}) d y \tag{33.3.15}
\end{gather*}
$$

where $\mathbf{y}_{i}$ is the center of $Q_{i}$. Consequently if the sides of $Q_{i}$ have length $2 t / \sqrt{n}$, 33.3.15 implies

$$
\begin{gather*}
\int_{E^{*}}\left|\int_{Q_{i}} F^{-1} \rho(\mathbf{x}-\mathbf{y}) b(\mathbf{y}) d y\right| d x \leq  \tag{33.3.16}\\
\\
\int_{E^{*}} \int_{Q_{i}}\left|F^{-1} \rho(\mathbf{x}-\mathbf{y})-F^{-1} \rho\left(\mathbf{x}-\mathbf{y}_{i}\right)\right||b(\mathbf{y})| d y d x  \tag{33.3.17}\\
=  \tag{33.3.18}\\
\int_{Q_{i}} \int_{E^{*}}\left|F^{-1} \rho(\mathbf{x}-\mathbf{y})-F^{-1} \rho\left(\mathbf{x}-\mathbf{y}_{i}\right)\right| d x|b(\mathbf{y})| d y \\
\leq \\
\int_{Q_{i}} \int_{\left|\mathbf{x}-\mathbf{y}_{i}\right| \geq 2 t}\left|F^{-1} \rho(\mathbf{x}-\mathbf{y})-F^{-1} \rho\left(\mathbf{x}-\mathbf{y}_{i}\right)\right| d x|b(\mathbf{y})| d y
\end{gather*}
$$

since if $\mathbf{x} \in E^{*}$, then $\left|\mathbf{x}-\mathbf{y}_{i}\right| \geq 2 t$. Now for $\mathbf{y} \in Q_{i}$,

$$
\left|\mathbf{y}-\mathbf{y}_{i}\right| \leq\left(\sum_{j=1}^{n}\left(\frac{t}{\sqrt{n}}\right)^{2}\right)^{1 / 2}=t
$$

From 33.3.8 and the change of variables $\mathbf{u}=\mathbf{x}-\mathbf{y}_{i}$ 33.3.16-33.3.18 imply

$$
\begin{equation*}
\int_{E^{*}}\left|\int_{Q_{i}} F^{-1} \rho(\mathbf{x}-\mathbf{y}) b(\mathbf{y}) d y\right| d x \leq C_{1} \int_{Q_{i}}|b(\mathbf{y})| d y \tag{33.3.19}
\end{equation*}
$$

Now from 33.3.19, and the fact that $b=0$ off $\Omega$,

$$
\begin{aligned}
\int_{E^{*}}\left|F^{-1} \rho * b(\mathbf{x})\right| d x & =\int_{E^{*}}\left|\int_{\mathbb{R}^{n}} F^{-1} \rho(\mathbf{x}-\mathbf{y}) b(\mathbf{y}) d y\right| d x \\
& =\int_{E^{*}}\left|\sum_{i=1}^{\infty} \int_{Q_{i}} F^{-1} \rho(\mathbf{x}-\mathbf{y}) b(\mathbf{y}) d y\right| d x \\
& \leq \int_{E^{*}} \sum_{i=1}^{\infty}\left|\int_{Q_{i}} F^{-1} \rho(\mathbf{x}-\mathbf{y}) b(\mathbf{y}) d y\right| d x \\
& =\sum_{i=1}^{\infty} \int_{E^{*}}\left|\int_{Q_{i}} F^{-1} \rho(\mathbf{x}-\mathbf{y}) b(\mathbf{y}) d y\right| d x \\
& \leq \sum_{i=1}^{\infty} C_{1} \int_{Q_{i}}|b(\mathbf{y})| d y=C_{1}| | b \|_{1}
\end{aligned}
$$

Thus, by 33.3.13,

$$
\begin{aligned}
\int_{E^{*}}\left|F^{-1} \rho * b(\mathbf{x})\right| d x & \leq C_{1}\|b\|_{1} \\
& \leq C_{1}\left[\|\phi\|_{1}+\|g\|_{1}\right] \\
& \leq C_{1}\left[\|\phi\|_{1}+\|\phi\|_{1}\right] \\
& \leq 2 C_{1}\|\phi\|_{1}
\end{aligned}
$$

Consequently,

$$
m\left(\left[\left|F^{-1} \rho * b\right| \geq \frac{\alpha}{2}\right] \cap E^{*}\right) \leq \frac{4 C_{1}}{\alpha}\|\phi\|_{1}
$$

From 33.3.10, 33.3.14, and 33.3.9,

$$
\begin{gathered}
m\left[\left|F^{-1} \rho * \phi\right|>\alpha\right] \leq m\left[\left|F^{-1} \rho * g\right| \geq \frac{\alpha}{2}\right]+m\left[\left|F^{-1} \rho * b\right| \geq \frac{\alpha}{2}\right] \\
\leq \frac{C_{n}}{\alpha}\|\phi\|_{1}+m\left(\left[\left|F^{-1} \rho * b\right| \geq \frac{\alpha}{2}\right] \cap E^{*}\right)+m\left(\Omega^{*}\right) \\
\leq \frac{C_{n}}{\alpha}\|\phi\|_{1}+\frac{4 C_{1}}{\alpha}\|\phi\|_{1}+C_{n} m(\Omega) \leq \frac{A}{\alpha}\|\phi\|_{1}
\end{gathered}
$$

because

$$
m(\Omega) \leq \alpha^{-1}\|\phi\|_{1}
$$

by 33.3.9. This proves the lemma.
The next lemma extends this lemma by giving a weak $(2,2)$ estimate and a $(2,2)$ estimate.

Lemma 33.3.2 Suppose $\rho \in L^{\infty}\left(\mathbb{R}^{n}\right) \cap L^{2}\left(\mathbb{R}^{n}\right)$ and suppose also that there exists a constant $C_{1}$ such that

$$
\begin{equation*}
\int_{|\mathbf{x}|>2|\mathbf{y}|}\left|F^{-1} \rho(\mathbf{x}-\mathbf{y})-F^{-1} \rho(\mathbf{x})\right| d x \leq C_{1} \tag{33.3.20}
\end{equation*}
$$

Then $F^{-1} \rho *$ maps $L^{1}\left(\mathbb{R}^{n}\right)+L^{2}\left(\mathbb{R}^{n}\right)$ to measurable functions and there exists a constant $A$ depending only on $C_{1}, n,\|\rho\|_{\infty}$ such that

$$
\begin{gather*}
m\left(\left[\left|F^{-1} \rho * f\right|>\alpha\right]\right) \leq A \frac{\|f\|_{1}}{\alpha} \text { if } f \in L^{1}\left(\mathbb{R}^{n}\right)  \tag{33.3.21}\\
m\left(\left[\left|F^{-1} \rho * f\right|>\alpha\right]\right) \leq\left(A \frac{\|f\|_{2}}{\alpha}\right)^{2} \text { if } f \in L^{2}\left(\mathbb{R}^{n}\right) \tag{33.3.22}
\end{gather*}
$$

Thus, $F^{-1} \rho *$ is weak type $(1,1)$ and weak type $(2,2)$. Also

$$
\begin{equation*}
\left\|F^{-1} \rho * f\right\|_{2} \leq A\|f\|_{2} \text { if } f \in L^{2}\left(\mathbb{R}^{n}\right) \tag{33.3.23}
\end{equation*}
$$

Proof: By Plancherel's theorem $F^{-1} \rho$ is in $L^{2}\left(\mathbb{R}^{n}\right)$. If $f \in L^{1}\left(\mathbb{R}^{n}\right)$, then by Minkowski's inequality,

$$
F^{-1} \rho * f \in L^{2}\left(\mathbb{R}^{n}\right)
$$

Now let $g \in L^{2}\left(\mathbb{R}^{n}\right)$. By Holder's inequality,

$$
\int\left|F^{-1} \rho(\mathbf{x}-\mathbf{y})\right||g(\mathbf{y})| d y \leq\left(\int\left|F^{-1} \rho(\mathbf{x}-\mathbf{y})\right|^{2} d y\right)^{1 / 2}\left(\int|g(\mathbf{y})|^{2} d y\right)^{1 / 2}<\infty
$$

and so the following is well defined a.e.

$$
F^{-1} \rho * g(\mathbf{x}) \equiv \int F^{-1} \rho(\mathbf{x}-\mathbf{y}) g(\mathbf{y}) d y
$$

also,

$$
\begin{aligned}
\left|F^{-1} \rho * g(\mathbf{x})-F^{-1} \rho * g\left(\mathbf{x}^{\prime}\right)\right| & \leq \int\left|F^{-1} \rho(\mathbf{x}-\mathbf{y})-F^{-1} \rho\left(\mathbf{x}^{\prime}-\mathbf{y}\right)\right||g(\mathbf{y})| d y \\
& \leq\left\|F^{-1} \rho-F^{-1} \rho_{\mathbf{x}^{\prime}-\mathbf{x}}\right\|\|g\|_{l^{2}}
\end{aligned}
$$

and by continuity of translation in $L^{2}\left(\mathbb{R}^{n}\right)$, this shows $\mathbf{x} \rightarrow F^{-1} \rho * g(\mathbf{x})$ is continuous. Therefore, $F^{-1} \rho * \operatorname{maps} L^{1}\left(\mathbb{R}^{n}\right)+L^{2}\left(\mathbb{R}^{n}\right)$ to the space of measurable functions. (Continuous functions are measurable.) It is clear that $F^{-1} \rho *$ is subadditive.

If $\phi \in \mathscr{G}$, Plancherel's theorem implies as before,

$$
\begin{gather*}
\left\|F^{-1} \rho * \phi\right\|_{2}=\left\|F\left(F^{-1} \rho * \phi\right)\right\|_{2}= \\
(2 \pi)^{n / 2}\|\rho F \phi\|_{2} \leq(2 \pi)^{n / 2}\|\rho\|_{\infty}\|\phi\|_{2} \tag{33.3.24}
\end{gather*}
$$

Now let $f \in L^{2}\left(\mathbb{R}^{n}\right)$ and let $\phi_{k} \in \mathscr{G}$, with

$$
\left\|\phi_{k}-f\right\|_{2} \rightarrow 0
$$

Then by Holder's inequality,

$$
\int F^{-1} \rho(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d y=\lim _{k \rightarrow \infty} \int F^{-1} \rho(\mathbf{x}-\mathbf{y}) \phi_{k}(\mathbf{y}) d y
$$

and so by Fatou's lemma, Plancherel's theorem, and 33.3.24,

$$
\begin{gathered}
\left\|F^{-1} \rho * f\right\|_{2}=\left(\int\left|\int F^{-1} \rho(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d y\right|^{2} d x\right)^{1 / 2} \leq \\
\leq \lim _{k \rightarrow \infty}\left(\int\left|\int F^{-1} \rho(\mathbf{x}-\mathbf{y}) \phi_{k}(\mathbf{y}) d y\right|^{2} d x\right)^{1 / 2}=\lim _{k \rightarrow \infty} \inf \left\|F^{-1} \rho * \phi_{k}\right\|_{2} \\
\leq\|\rho\|_{\infty}(2 \pi)^{n / 2} \lim _{k \rightarrow \infty} \inf _{k \rightarrow \infty} \phi_{k}\left\|_{2}=\right\| \rho\left\|_{\infty}(2 \pi)^{n / 2}\right\| f \|_{2} .
\end{gathered}
$$

Thus, 33.3.23 holds with $A=\|\rho\|_{\infty}(2 \pi)^{n / 2}$. Consequently,

$$
\begin{aligned}
A\|f\|_{2} \geq & \left(\int_{\left[\left|F^{-1} \rho * f\right|>\alpha\right]}\left|F^{-1} \rho * f(\mathbf{x})\right|^{2} d x\right)^{1 / 2} \\
& \geq \alpha m\left(\left[\left|F^{-1} \rho * f\right|>\alpha\right]\right)^{1 / 2}
\end{aligned}
$$

and so 33.3.22 follows.
It remains to prove 33.3.21 which holds for all $f \in \mathscr{G}$ by Lemma 33.3.1. Let $f \in L^{1}\left(\mathbb{R}^{n}\right)$ and let $\phi_{k} \rightarrow f$ in $L^{1}\left(\mathbb{R}^{n}\right), \phi_{k} \in \mathscr{G}$. Without loss of generality, assume that both $f$ and $F^{-1} \rho$ are Borel measurable. Therefore, by Minkowski's inequality, and Plancherel's theorem,

$$
\begin{aligned}
& \left\|F^{-1} \rho * \phi_{k}-F^{-1} \rho * f\right\|_{2} \\
& \leq\left(\int\left|\int F^{-1} \rho(\mathbf{x}-\mathbf{y})\left(\phi_{k}(\mathbf{y})-f(\mathbf{y})\right) d y\right|^{2} d x\right)^{1 / 2} \\
& \leq\left\|\phi_{k}-f\right\|_{1}\|\rho\|_{2}
\end{aligned}
$$

which shows that $F^{-1} \rho * \phi_{k}$ converges to $F^{-1} \rho * f$ in $L^{2}\left(\mathbb{R}^{n}\right)$. Therefore, there exists a subsequence such that the convergence is pointwise a.e. Then, denoting the subsequence by $k$,

$$
\mathscr{X}_{\left[\left|F^{-1} \rho * f\right|>\alpha\right]}(\mathbf{x}) \leq \lim _{k \rightarrow \infty} \operatorname{Xin}_{\left[\left|F^{-1} \rho * \phi_{k}\right|>\alpha\right]}(\mathbf{x}) \text { a.e. } \mathbf{x} .
$$

Thus by Lemma 33.3.1 and Fatou's lemma, there exists a constant, $A$, depending on $C_{1}, n$, and $\|\rho\|_{\infty}$ such that

$$
\begin{aligned}
m\left(\left[\left|F^{-1} \rho * f\right|>\alpha\right]\right) & \leq \lim _{k \rightarrow \infty} m\left(\left[\left|F^{-1} \rho * \phi_{k}\right|>\alpha\right]\right) \\
& \leq \lim _{k \rightarrow \infty} A \frac{\left\|\phi_{k}\right\|_{1}}{\alpha}=A \frac{\|f\|_{1}}{\alpha}
\end{aligned}
$$

This shows 33.3.21 and proves the lemma.
Theorem 33.3.3 Let $\rho \in L^{2}\left(\mathbb{R}^{n}\right) \cap L^{\infty}\left(\mathbb{R}^{n}\right)$ and suppose

$$
\int_{|\mathbf{x}| \geq 2|\mathbf{y}|}\left|F^{-1} \rho(\mathbf{x}-\mathbf{y})-F^{-1} \rho(\mathbf{x})\right| d x \leq C_{1}
$$

Then for each $p \in(1, \infty)$, there exists a constant, $A_{p}$, depending only on

$$
p, n,\|\rho\|_{\infty}
$$

and $C_{1}$ such that for all $\phi \in \mathscr{G}$,

$$
\left\|F^{-1} \rho * \phi\right\|_{p} \leq A_{p}\|\phi\|_{p}
$$

Proof: From Lemma 33.3.2, $F^{-1} \rho *$ is weak $(1,1)$, weak $(2,2)$, and maps

$$
L^{1}\left(\mathbb{R}^{n}\right)+L^{2}\left(\mathbb{R}^{n}\right)
$$

to measurable functions. Therefore, by the Marcinkiewicz interpolation theorem, there exists a constant $A_{p}$ depending only on $p, C_{1}, n$, and $\|\rho\|_{\infty}$ for $p \in(1,2]$, such that for $f \in L^{p}\left(\mathbb{R}^{n}\right)$, and $p \in(1,2]$,

$$
\left\|F^{-1} \rho * f\right\|_{p} \leq A_{p}\|f\|_{p}
$$

Thus the theorem is proved for these values of $p$. Now suppose $p>2$. Then $p^{\prime}<2$ where

$$
\frac{1}{p}+\frac{1}{p^{\prime}}=1
$$

By Plancherel's theorem and Theorem 32.3.25,

$$
\begin{aligned}
\int F^{-1} \rho * \phi(\mathbf{x}) \psi(\mathbf{x}) d x & =(2 \pi)^{n / 2} \int \rho(\mathbf{x}) F \phi(\mathbf{x}) F \psi(\mathbf{x}) d x \\
& =\int F\left(F^{-1} \rho * \psi\right) F \phi d x \\
& =\int\left(F^{-1} \rho * \psi\right)(\phi) d x
\end{aligned}
$$

Thus by the case for $p \in(1,2)$ and Holder's inequality,

$$
\begin{aligned}
\left|\int F^{-1} \rho * \phi(\mathbf{x}) \psi(\mathbf{x}) d x\right| & =\left|\int\left(F^{-1} \rho * \psi\right)(\phi) d x\right| \\
& \leq\left\|F^{-1} \rho * \psi\right\|_{p^{\prime}}\|\phi\|_{p} \\
& \leq A_{p^{\prime}}\|\psi\|_{p^{\prime}}\|\phi\|_{p} .
\end{aligned}
$$

Letting $L \psi \equiv \int F^{-1} \rho * \phi(\mathbf{x}) \psi(\mathbf{x}) d x$, this shows that $L \in L^{p^{\prime}}\left(\mathbb{R}^{n}\right)^{\prime}$ and also that $\|L\|_{\left(L^{p^{\prime}}\right)^{\prime}} \leq$ $A_{p^{\prime}}\|\phi\|_{p}$ which implies by the Riesz representation theorem that $F^{-1} \rho * \phi$ represents $L$ and

$$
\|L\|_{\left(L^{p^{\prime}}\right)^{\prime}}=\left\|F^{-1} \rho * \phi\right\|_{L^{p}} \leq A_{p^{\prime}}\|\phi\|_{p}
$$

Since $p^{\prime}=p /(p-1)$, this proves the theorem.
It is possible to give verifiable conditions on $\rho$ which imply 33.3.20. The condition on $\rho$ which is presented here is the existence of a constant, $C_{0}$ such that

$$
\begin{gathered}
C_{0} \geq \sup \left\{|\mathbf{x}|^{|\alpha|}\left|D^{\alpha} \rho(\mathbf{x})\right|:|\alpha| \leq L, \mathbf{x} \in \mathbb{R}^{n} \backslash\{\boldsymbol{0}\}\right\}, L>n / 2 \\
\rho \in C^{L}\left(\mathbb{R}^{n} \backslash\{\mathbf{0}\}\right) \text { where } L \text { is an integer. }
\end{gathered}
$$

Here $\alpha$ is a multi-index and $|\alpha|=\sum_{i=1}^{n} \alpha_{i}$. The condition says roughly that $\rho$ is pretty smooth away from $\mathbf{0}$ and all the partial derivatives vanish pretty fast as $|\mathbf{x}| \rightarrow \infty$. Also recall the notation

$$
\mathbf{x}^{\alpha} \equiv x_{1}^{\alpha_{1}} \cdots x_{n}^{\alpha_{n}}
$$

where $\alpha=\left(\alpha_{1} \cdots \alpha_{n}\right)$. For more general conditions, see [69].

Lemma 33.3.4 Let 33.3.25 hold and suppose $\psi \in C_{c}^{\infty}\left(\mathbb{R}^{n} \backslash\{\mathbf{0}\}\right)$. Then for each $\alpha,|\alpha| \leq$ $L$, there exists a constant $C \equiv C(\alpha, n, \psi)$ independent of $k$ such that

$$
\sup _{\mathbf{x} \in \mathbb{R}^{n}}|\mathbf{x}|^{|\alpha|}\left|D^{\alpha}\left(\rho(\mathbf{x}) \psi\left(2^{k} \mathbf{x}\right)\right)\right| \leq C C_{0}
$$

Proof:

$$
\begin{aligned}
& |\mathbf{x}|^{|\alpha|}\left|D^{\alpha}\left(\rho(\mathbf{x}) \psi\left(2^{k} \mathbf{x}\right)\right)\right| \leq|\mathbf{x}|^{|\alpha|} \sum_{\beta+\gamma=\alpha}\left|D^{\beta} \rho(\mathbf{x})\right| 2^{k|\gamma|}\left|D^{\gamma} \psi\left(2^{k} \mathbf{x}\right)\right| \\
& =\sum_{\beta+\gamma=\alpha}|\mathbf{x}|^{|\beta|}\left|D^{\beta} \rho(\mathbf{x})\right|\left|2^{k} \mathbf{x}\right| \\
& \leq C_{0} C(\alpha, n)\left|D^{\gamma} \psi\left(2^{k} \mathbf{x}\right)\right| \\
& \sum_{|\gamma| \leq|\alpha|} \sup \left\{|\mathbf{z}|^{|\gamma|}\left|D^{\gamma} \psi(\mathbf{z})\right|: \mathbf{z} \in \mathbb{R}^{n}\right\}=C_{0} C(\alpha, n, \psi)
\end{aligned}
$$

and this proves the lemma.
Lemma 33.3.5 There exists

$$
\phi \in C_{c}^{\infty}\left(\left[\mathbf{x}: 4^{-1}<|\mathbf{x}|<4\right]\right), \phi(\mathbf{x}) \geq 0
$$

and

$$
\sum_{k=-\infty}^{\infty} \phi\left(2^{k} \mathbf{x}\right)=1
$$

for each $\mathbf{x} \neq \mathbf{0}$.
Proof: Let

$$
\begin{gathered}
\psi \geq 0, \psi=1 \text { on }\left[2^{-1} \leq|\mathbf{x}| \leq 2\right] \\
\operatorname{spt}(\psi) \subseteq\left[4^{-1}<|\mathbf{x}|<4\right]
\end{gathered}
$$

Consider

$$
g(\mathbf{x})=\sum_{k=-\infty}^{\infty} \psi\left(2^{k} \mathbf{x}\right)
$$

Then for each $\mathbf{x}$, only finitely many terms are not equal to 0 . Also, $g(\mathbf{x})>0$ for all $\mathbf{x} \neq \mathbf{0}$. To verify this last claim, note that for some $k$ an integer, $|\mathbf{x}| \in\left[2^{l}, 2^{l+2}\right]$. Therefore, choose $k$ an integer such that $2^{k}|\mathbf{x}| \in\left[2^{-1}, 2\right]$. For example, let $k=-l-1$. This works because $2^{k}|\mathbf{x}| \in$ $\left[2^{l} 2^{k}, 2^{l+2} 2^{k}\right]=\left[2^{l-l-1}, 2^{l+2-l-1}\right]=\left[2^{-1}, 2\right]$. Therefore, for this value of $k, \psi\left(2^{k} \mathbf{x}\right)=1$ so $g(\mathbf{x})>0$.

Now notice that

$$
g\left(2^{r} \mathbf{x}\right)=\sum_{k=-\infty}^{\infty} \psi\left(2^{k} 2^{r} \mathbf{x}\right)=\sum_{k=-\infty}^{\infty} \psi\left(2^{k} \mathbf{x}\right)=g(\mathbf{x})
$$

Let $\phi(\mathbf{x}) \equiv \psi(\mathbf{x}) g(\mathbf{x})^{-1}$. Then

$$
\sum_{k=-\infty}^{\infty} \phi\left(2^{k} \mathbf{x}\right)=\sum_{k=-\infty}^{\infty} \frac{\psi\left(2^{k} \mathbf{x}\right)}{g\left(2^{k} \mathbf{x}\right)}=g(\mathbf{x})^{-1} \sum_{k=-\infty}^{\infty} \psi\left(2^{k} \mathbf{x}\right)=1
$$

for each $\mathbf{x} \neq \mathbf{0}$. This proves the lemma.
Now define

$$
\rho_{m}(\mathbf{x}) \equiv \sum_{k=-m}^{m} \rho(\mathbf{x}) \phi\left(2^{k} \mathbf{x}\right), \gamma_{k}(\mathbf{x}) \equiv \rho(\mathbf{x}) \phi\left(2^{k} \mathbf{x}\right)
$$

Let $t>0$ and let $|\mathbf{y}| \leq t$. Consider the problem of estimating

$$
\begin{equation*}
\int_{|\mathbf{x}| \geq 2 t}\left|F^{-1} \gamma_{k}(\mathbf{x}-\mathbf{y})-F^{-1} \gamma_{k}(\mathbf{x})\right| d x \tag{33.3.26}
\end{equation*}
$$

In the following estimates, $C(a, b, \cdots, d)$ will denote a generic constant depending only on the indicated objects, $a, b, \cdots, d$. For the first estimate, note that since $|\mathbf{y}| \leq t, 33.3 .26$ is no larger than

$$
\begin{aligned}
& 2 \int_{|\mathbf{x}| \geq t}\left|F^{-1} \gamma_{k}(\mathbf{x})\right| d x=2 \int_{|\mathbf{x}| \geq t}\left|F^{-1} \gamma_{k}(\mathbf{x})\right||\mathbf{x}|^{-L}|\mathbf{x}|^{L} d x \\
\leq & 2\left(\int_{|\mathbf{x}| \geq t}|\mathbf{x}|^{-2 L} d x\right)^{1 / 2}\left(\int_{|\mathbf{x}| \geq t}|\mathbf{x}|^{2 L}\left|F^{-1} \gamma_{k}(\mathbf{x})\right|^{2} d x\right)^{1 / 2}
\end{aligned}
$$

Using spherical coordinates and Plancherel's theorem,

$$
\begin{align*}
& \leq C(n, L) t^{n / 2-L}\left(\int|\mathbf{x}|^{2 L}\left|F^{-1} \gamma_{k}(\mathbf{x})\right|^{2} d x\right)^{1 / 2} \\
\leq & C(n, L) t^{n / 2-L}\left(\int \sum_{j=1}^{n}\left|x_{j}\right|^{2 L}\left|F^{-1} \gamma_{k}(\mathbf{x})\right|^{2} d x\right)^{1 / 2} \\
\leq & C(n, L) t^{n / 2-L}\left(\sum_{j=1}^{n} \int\left|F^{-1} D_{j}^{L} \gamma_{k}(\mathbf{x})\right|^{2} d x\right)^{1 / 2}  \tag{33.3.27}\\
= & C(n, L) t^{n / 2-L}\left(\sum_{j=1}^{n} \int_{S_{k}}\left|D_{j}^{L} \gamma_{k}(\mathbf{x})\right|^{2} d x\right)^{1 / 2}
\end{align*}
$$

where

$$
\begin{equation*}
S_{k} \equiv\left[\mathbf{x}: 2^{-2-k}<|\mathbf{x}|<2^{2-k}\right] \tag{33.3.28}
\end{equation*}
$$

a set containing the support of $\gamma_{k}$. Now from the definition of $\gamma_{k}$,

$$
\left|D_{j}^{L} \gamma_{k}(\mathbf{z})\right|=\left|D_{j}^{L}\left(\rho(\mathbf{z}) \phi\left(2^{k} \mathbf{z}\right)\right)\right|
$$

By Lemma 33.3.4, this is no larger than

$$
\begin{equation*}
C(L, n, \phi) C_{0}|\mathbf{z}|^{-L} . \tag{33.3.29}
\end{equation*}
$$

It follows, using polar coordinates, that the last expression in 33.3.27 is no larger than

$$
\begin{equation*}
C\left(n, L, \phi, C_{0}\right) t^{n / 2-L}\left(\int_{S_{k}}|\mathbf{z}|^{-2 L} d z\right)^{1 / 2} \leq C\left(n, L, \phi, C_{0}\right) t^{n / 2-L} \tag{33.3.30}
\end{equation*}
$$

$$
\left(\int_{2^{-2-k}}^{2^{2-k}} \rho^{n-1-2 L} d \rho\right)^{1 / 2} \leq C\left(n, L, \phi, C_{0}\right) t^{n / 2-L} 2^{k(L-n / 2)} .
$$

Now estimate 33.3.26 in another way. The support of $\gamma_{k}$ is in $S_{k}$, a bounded set, and so $F^{-1} \gamma_{k}$ is differentiable. Therefore,

$$
\begin{gather*}
\int_{|\mathbf{x}| \geq 2 t}\left|F^{-1} \gamma_{k}(\mathbf{x}-\mathbf{y})-F^{-1} \gamma_{k}(\mathbf{x})\right| d x= \\
\int_{|\mathbf{x}| \geq 2 t}\left|\int_{0}^{1} \sum_{j=1}^{n} D_{j} F^{-1} \gamma_{k}(\mathbf{x}-s \mathbf{y}) y_{j} d s\right| d x \\
\leq t \int_{|\mathbf{x}| \geq 2 t} \int_{0}^{1} \sum_{j=1}^{n}\left|D_{j} F^{-1} \gamma_{k}(\mathbf{x}-s \mathbf{y})\right| d s d x \\
\leq t \int \sum_{j=1}^{n}\left|D_{j} F^{-1} \gamma_{k}(\mathbf{x})\right| d x \\
\leq \quad t \sum_{j=1}^{n}\left(\int\left(1+\left|2^{-k} \mathbf{x}\right|^{2}\right)^{-L} d x\right)^{1 / 2} \\
\\
\quad\left(\int\left(1+\left|2^{-k} \mathbf{x}\right|^{2}\right)^{L}\left|D_{j} F^{-1} \gamma_{k}(\mathbf{x})\right|^{2} d x\right)^{1 / 2}  \tag{33.3.31}\\
\leq C(n, L) t 2^{k n / 2} \sum_{j=1}^{n}\left(\int\left(1+\left|2^{-k} \mathbf{x}\right|^{2}\right)^{L}\left|D_{j} F^{-1} \gamma_{k}(\mathbf{x})\right|^{2} d x\right)^{1 / 2} .
\end{gather*}
$$

Now consider the $j^{t h}$ term in the last sum in 33.3.31.

$$
\begin{gather*}
\int\left(1+\left|2^{-k} \mathbf{x}\right|^{2}\right)^{L}\left|D_{j} F^{-1} \gamma_{k}(\mathbf{x})\right|^{2} d x \leq \\
C(n, L) \int \sum_{|\alpha| \leq L} 2^{-2 k|\alpha|} \mathbf{x}^{2 \alpha}\left|D_{j} F^{-1} \gamma_{k}(\mathbf{x})\right|^{2} d x  \tag{33.3.32}\\
=C(n, L) \sum_{|\alpha| \leq L} 2^{-2 k|\alpha|} \int \mathbf{x}^{2 \alpha}\left|F^{-1}\left(\pi_{j} \gamma_{k}\right)(\mathbf{x})\right|^{2} d x
\end{gather*}
$$

where $\pi_{j}(\mathbf{z}) \equiv z_{j}$. This last assertion follows from

$$
D_{j} \int e^{-i \mathbf{x} \cdot \mathbf{y}} \gamma_{k}(\mathbf{y}) d y=\int(-i) e^{-i \mathbf{x} \cdot \mathbf{y}} y_{j} \gamma_{k}(\mathbf{y}) d y
$$

Therefore, a similar computation and Plancherel's theorem implies 33.3.32 equals

$$
=C(n, L) \sum_{|\alpha| \leq L} 2^{-2 k|\alpha|} \int\left|F^{-1} D^{\alpha}\left(\pi_{j} \gamma_{k}\right)(\mathbf{x})\right|^{2} d x
$$

$$
\begin{equation*}
=C(n, L) \sum_{|\alpha| \leq L} 2^{-2 k|\alpha|} \int_{S_{k}}\left|D^{\alpha}\left(z_{j} \gamma_{k}(\mathbf{z})\right)\right|^{2} d z \tag{33.3.33}
\end{equation*}
$$

where $S_{k}$ is given in 33.3.28. Now

$$
\begin{aligned}
\left|D^{\alpha}\left(z_{j} \gamma_{k}(\mathbf{z})\right)\right| & =2^{-k}\left|D^{\alpha}\left(\rho(\mathbf{z}) z_{j} 2^{k} \phi\left(2^{k} \mathbf{z}\right)\right)\right| \\
& =2^{-k}\left|D^{\alpha}\left(\rho(\mathbf{z}) \psi_{j}\left(2^{k} \mathbf{z}\right)\right)\right|
\end{aligned}
$$

where $\psi_{j}(\mathbf{z}) \equiv z_{j} \phi(\mathbf{z})$. By Lemma 33.3.4, this is dominated by

$$
2^{-k} C\left(\alpha, n, \phi, j, C_{0}\right)|\mathbf{z}|^{-|\alpha|}
$$

Therefore, 33.3.33 is dominated by

$$
\begin{aligned}
& C\left(L, n, \phi, j, C_{0}\right) \sum_{|\alpha| \leq L} 2^{-2 k|\alpha|} \int_{S_{k}} 2^{-2 k}|\mathbf{z}|^{-2|\alpha|} d z \\
& \leq C\left(L, n, \phi, j, C_{0}\right) \sum_{|\alpha| \leq L} 2^{-2 k|\alpha|} 2^{-2 k}\left(2^{-2-k}\right)^{(-2|\alpha|)}\left(2^{2-k}\right)^{n} \\
& \leq C\left(L, n, \phi, j, C_{0}\right) \sum_{|\alpha| \leq L} 2^{-k n-2 k} \\
& \leq C\left(L, n, \phi, j, C_{0}\right) 2^{-k n} 2^{-2 k}
\end{aligned}
$$

It follows that 33.3 .31 is no larger than

$$
\begin{equation*}
C\left(L, n, \phi, C_{0}\right) t 2^{k n / 2} 2^{-k n / 2} 2^{-k}=C\left(L, n, \phi, C_{0}\right) t 2^{-k} \tag{33.3.34}
\end{equation*}
$$

It follows from 33.3.34 and 33.3.30 that if $|\mathbf{y}| \leq t$,

$$
\begin{aligned}
& \int_{|\mathbf{x}| \geq 2 t}\left|F^{-1} \gamma_{k}(\mathbf{x}-\mathbf{y})-F^{-1} \gamma_{k}(\mathbf{x})\right| d x \leq \\
& C\left(L, n, \phi, C_{0}\right) \min \left(t 2^{-k},\left(2^{-k} t\right)^{n / 2-L}\right)
\end{aligned}
$$

With this inequality, the next lemma which is the desired result can be obtained.
Lemma 33.3.6 There exists a constant depending only on the indicated objects, $C_{1}=$ $C\left(L, n, \phi, C_{0}\right)$ such that when $|\mathbf{y}| \leq t$,

$$
\begin{gather*}
\int_{|\mathbf{x}| \geq 2 t}\left|F^{-1} \rho(\mathbf{x}-\mathbf{y})-F^{-1} \rho(\mathbf{x})\right| d x \leq C_{1} \\
\int_{|\mathbf{x}| \geq 2 t}\left|F^{-1} \rho_{m}(\mathbf{x}-\mathbf{y})-F^{-1} \rho_{m}(\mathbf{x})\right| d x \leq C_{1} \tag{33.3.35}
\end{gather*}
$$

Proof: $F^{-1} \rho=\lim _{m \rightarrow \infty} F^{-1} \rho_{m}$ in $L^{2}\left(\mathbb{R}^{n}\right)$. Let $m_{k} \rightarrow \infty$ be such that convergence is pointwise a.e. Then if $|\mathbf{y}| \leq t$, Fatou's lemma implies

$$
\begin{gather*}
\int_{|\mathbf{x}| \geq 2 t}\left|F^{-1} \rho(\mathbf{x}-\mathbf{y})-F^{-1} \rho(\mathbf{x})\right| d x \leq \\
\quad \lim \inf _{l \rightarrow \infty} \int_{|\mathbf{x}| \geq 2 t}\left|F^{-1} \rho_{m_{l}}(\mathbf{x}-\mathbf{y})-F^{-1} \rho_{m_{l}}(\mathbf{x})\right| d x \\
\leq \lim _{l \rightarrow \infty} \inf _{l=-m_{l}} \sum_{l|x| \geq 2 t}^{m_{l}} \int_{\mid=1}\left|F^{-1} \gamma_{k}(\mathbf{x}-\mathbf{y})-F^{-1} \gamma_{k}(\mathbf{x})\right| d x \\
\leq C\left(L, n, \phi, C_{0}\right) \sum_{k=-\infty}^{\infty} \min \left(t 2^{-k},\left(2^{-k} t\right)^{n / 2-L}\right) . \tag{33.3.36}
\end{gather*}
$$

Now consider the sum in 33.3.36,

$$
\begin{equation*}
\sum_{k=-\infty}^{\infty} \min \left(t 2^{-k},\left(2^{-k} t\right)^{n / 2-L}\right) \tag{33.3.37}
\end{equation*}
$$

$t 2^{j}=\min \left(t 2^{j},\left(2^{j} t\right)^{n / 2-L}\right)$ exactly when $t 2^{j} \leq 1$. This occurs if and only if

$$
j \leq-\ln (t) / \ln (2)
$$

Therefore 33.3 .37 is no larger than

$$
\sum_{j \leq-\ln (t) / \ln (2)} 2^{j} t+\sum_{j \geq-\ln (t) / \ln (2)}\left(2^{j} t\right)^{n / 2-L} .
$$

Letting $a=L-n / 2$, this equals

$$
\begin{aligned}
& t \sum_{k \geq \ln (t) / \ln (2)} 2^{-k}+t^{-\alpha} \sum_{j \geq-\ln (t) / \ln (2)}\left(2^{-a}\right)^{j} \\
\leq & 2 t\left(\frac{1}{2}\right)^{\ln (t) / \ln (2)}+t^{-a}\left(\frac{1}{2^{a}}\right)^{-\ln (t) / \ln (2)} \\
= & 2 t\left(\frac{1}{2}\right)^{\log _{2}(t)}+t^{-a}\left(\frac{1}{2^{a}}\right)^{-\log _{2}(t)} \\
= & 2+1=3 .
\end{aligned}
$$

Similarly, 33.3.35 holds. This proves the lemma.
Now it is possible to prove Mihlin's theorem.
Theorem 33.3.7 (Mihlin's theorem) Suppose $\rho$ satisfies

$$
C_{0} \geq \sup \left\{|\mathbf{x}|^{|\alpha|}\left|D^{\alpha} \rho(\mathbf{x})\right|:|\alpha| \leq L, \mathbf{x} \in \mathbb{R}^{n} \backslash\{\mathbf{0}\}\right\}
$$

where $L$ is an integer greater than $n / 2$ and $\rho \in C^{L}\left(\mathbb{R}^{n} \backslash\{\mathbf{0}\}\right)$. Then for every $p>1$, there exists a constant $A_{p}$ depending only on $p, C_{0}, \phi, n$, and $L$, such that for all $\psi \in \mathscr{G}$,

$$
\left\|F^{-1} \rho * \psi\right\|_{p} \leq A_{p}\|\psi\|_{p}
$$

Proof: Since $\rho_{m}$ satisfies 33.3.35, and is obviously in $L^{2}\left(\mathbb{R}^{n}\right) \cap L^{\infty}\left(\mathbb{R}^{n}\right)$, Theorem 33.3.3 implies there exists a constant $A_{p}$ depending only on $p, n,\left\|\rho_{m}\right\|_{\infty}$, and $C_{1}$ such that for all $\psi \in \mathscr{G}$ and $p \in(1, \infty)$,

$$
\left\|F^{-1} \rho_{m} * \psi\right\|_{p} \leq A_{p}\|\psi\|_{p}
$$

Now $\left\|\rho_{m}\right\|_{\infty} \leq\|\rho\|_{\infty}$ because

$$
\begin{equation*}
\left|\rho_{m}(\mathbf{x})\right| \leq|\rho(\mathbf{x})| \sum_{k=-m}^{m} \phi\left(2^{k} \mathbf{x}\right) \leq|\rho(\mathbf{x})| . \tag{33.3.38}
\end{equation*}
$$

Therefore, since $C_{1}=C_{1}\left(L, n, \phi, C_{0}\right)$ and $C_{0} \geq\|\rho\|_{\infty}$,

$$
\left\|F^{-1} \rho_{m} * \psi\right\|_{p} \leq A_{p}\left(L, n, \phi, C_{0}, p\right)\|\psi\|_{p}
$$

In particular, $A_{p}$ does not depend on $m$. Now, by 33.3.38, the observation that $\rho \in L^{\infty}\left(\mathbb{R}^{n}\right)$, $\lim _{m \rightarrow \infty} \rho_{m}(\mathbf{y})=\rho(\mathbf{y})$ and the dominated convergence theorem, it follows that for $\theta \in \mathscr{G}$.

$$
\begin{gathered}
\left|\left(F^{-1} \rho * \psi\right)(\theta)\right| \equiv\left|(2 \pi)^{n / 2} \int \rho(\mathbf{x}) F \psi(\mathbf{x}) F^{-1} \theta(\mathbf{x}) d x\right| \\
=\lim _{m \rightarrow \infty}\left|\left(F^{-1} \rho_{m} * \psi\right)(\theta)\right| \leq \lim _{m \rightarrow \infty} \sup \left\|F^{-1} \rho_{m} * \psi\right\|\left\|_{p}\right\| \theta \|_{p^{\prime}} \\
\leq A_{p}\left(L, n, \phi, C_{0}, p\right)\|\psi\|_{p}\|\theta\|_{p^{\prime}}
\end{gathered}
$$

Hence $F^{-1} \rho * \psi \in L^{p}\left(\mathbb{R}^{n}\right)$ and $\left\|F^{-1} \rho * \psi\right\|_{p} \leq A_{p}\|\psi\|_{p}$. This proves the theorem.

### 33.4 Singular Integrals

If $K \in L^{1}\left(\mathbb{R}^{n}\right)$ then when $p>1$,

$$
\|K * f\|_{p} \leq\|f\|_{p}
$$

It turns out that some meaning can be assigned to $K * f$ for some functions $K$ which are not in $L^{1}$. This involves assuming a certain form for $K$ and exploiting cancellation. The resulting theory of singular integrals is very useful. To illustrate, an application will be given to the Helmholtz decomposition of vector fields in the next section. Like Mihlin's theorem, the theory presented here rests on Theorem 33.3.3, restated here for convenience.
Theorem 33.4.1 Let $\rho \in L^{2}\left(\mathbb{R}^{n}\right) \cap L^{\infty}\left(\mathbb{R}^{n}\right)$ and suppose

$$
\int_{|\mathbf{x}| \geq 2|\mathbf{y}|}\left|F^{-1} \rho(\mathbf{x}-\mathbf{y})-F^{-1} \rho(\mathbf{x})\right| d x \leq C_{1}
$$

Then for each $p \in(1, \infty)$, there exists a constant, $A_{p}$, depending only on

$$
p, n,\|\rho\|_{\infty}
$$

and $C_{1}$ such that for all $\phi \in \mathscr{G}$,

$$
\left\|F^{-1} \rho * \phi\right\|_{p} \leq A_{p}\|\phi\|_{p}
$$

Lemma 33.4.2 Suppose

$$
\begin{equation*}
K \in L^{2}\left(\mathbb{R}^{n}\right),\|F K\|_{\infty} \leq B<\infty, \tag{33.4.39}
\end{equation*}
$$

and

$$
\int_{|\mathbf{x}|>2|\mathbf{y}|}|K(\mathbf{x}-\mathbf{y})-K(\mathbf{x})| d x \leq B
$$

Then for all $p>1$, there exists a constant, $A(p, n, B)$, depending only on the indicated quantities such that

$$
\|K * f\|_{p} \leq A(p, n, B)\|f\|_{p}
$$

for all $f \in \mathscr{G}$.
Proof: Let $F K=\rho$ so $F^{-1} \rho=K$. Then from 33.4.39 $\rho \in L^{2}\left(\mathbb{R}^{n}\right) \cap L^{\infty}\left(\mathbb{R}^{n}\right)$ and $K=$ $F^{-1} \rho$. By Theorem 33.3.3 listed above,

$$
\|K * f\|_{p}=\left\|F^{-1} \rho * f\right\|_{p} \leq A(p, n, B)\|f\|_{p}
$$

for all $f \in \mathscr{G}$. This proves the lemma.
The next lemma provides a situation in which the above conditions hold.
Lemma 33.4.3 Suppose

$$
\begin{gather*}
|K(\mathbf{x})| \leq B|\mathbf{x}|^{-n},  \tag{33.4.40}\\
\int_{a<|\mathbf{x}|<b} K(\mathbf{x}) d x=0,  \tag{33.4.41}\\
\int_{|\mathbf{x}|>2|\mathbf{y}|}|K(\mathbf{x}-\mathbf{y})-K(\mathbf{x})| d x \leq B . \tag{33.4.42}
\end{gather*}
$$

Define

$$
K_{\varepsilon}(\mathbf{x})=\left\{\begin{array}{l}
K(\mathbf{x}) \text { if }|\mathbf{x}| \geq \varepsilon  \tag{33.4.43}\\
0 \text { if }|\mathbf{x}|<\varepsilon
\end{array}\right.
$$

Then there exists a constant $C(n)$ such that

$$
\begin{equation*}
\int_{|\mathbf{x}|>2|\mathbf{y}|}\left|K_{\mathcal{\varepsilon}}(\mathbf{x}-\mathbf{y})-K_{\mathcal{\varepsilon}}(\mathbf{x})\right| d x \leq C(n) B \tag{33.4.44}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|F K_{\mathcal{\varepsilon}}\right\|_{\infty} \leq C(n) B \tag{33.4.45}
\end{equation*}
$$

Proof: In the argument, $C(n)$ will denote a generic constant depending only on $n$. Consider 33.4.44 first. The integral is broken up according to whether $|\mathbf{x}|,|\mathbf{x}-\mathbf{y}|>\varepsilon$.

| $\|\mathbf{x}\|$ | $>\varepsilon$ | $>\varepsilon$ | $<\varepsilon$ | $<\varepsilon$ |
| :--- | :---: | :---: | :---: | :---: |
| $\|\mathbf{x}-\mathbf{y}\|$ | $>\varepsilon$ | $<\varepsilon$ | $<\varepsilon$ | $>\varepsilon$ |

$$
\begin{align*}
& \int_{|\mathbf{x}| \geq 2|\mathbf{y}|}\left|K_{\mathcal{E}}(\mathbf{x}-\mathbf{y})-K_{\mathcal{E}}(\mathbf{x})\right| d x= \\
& \int_{|\mathbf{x}| \geq 2|\mathbf{y}|,|\mathbf{x}-\mathbf{y}|>\varepsilon,|\mathbf{x}|<\varepsilon}\left|K_{\mathcal{E}}(\mathbf{x}-\mathbf{y})-K_{\mathcal{E}}(\mathbf{x})\right| d x+ \\
& +\int_{|\mathbf{x}| \geq 2|\mathbf{y}|,|\mathbf{x}-\mathbf{y}|<\varepsilon,|\mathbf{x}| \geq \varepsilon}\left|K_{\mathcal{\varepsilon}}(\mathbf{x}-\mathbf{y})-K_{\mathcal{\varepsilon}}(\mathbf{x})\right| d x+  \tag{33.4.46}\\
& \left|\int_{|\mathbf{x}| \geq 2|\mathbf{y}|,|\mathbf{x}-\mathbf{y}|>\varepsilon,|\mathbf{x}|>\varepsilon}\right| K_{\mathcal{E}}(\mathbf{x}-\mathbf{y})-K_{\mathcal{E}}(\mathbf{x}) \mid d x+ \\
& +\int_{|\mathbf{x}| \geq 2|\mathbf{y}|,|\mathbf{x}-\mathbf{y}|<\varepsilon,|\mathbf{x}|<\varepsilon}\left|K_{\mathcal{E}}(\mathbf{x}-\mathbf{y})-K_{\mathcal{E}}(\mathbf{x})\right| d x .
\end{align*}
$$

Now consider the terms in the above expression. The last integral in 33.4.46 equals 0 from the definition of $K_{\varepsilon}$. The third integral on the right is no larger than $B$ by the definition of $K_{\varepsilon}$ and 33.4.42. Consider the second integral on the right. This integral is no larger than

$$
\int_{|\mathbf{x}| \geq 2|\mathbf{y}|,|\mathbf{x}| \geq \varepsilon,|\mathbf{x}-\mathbf{y}|<\varepsilon} B|\mathbf{x}|^{-n} d x .
$$

Now $|\mathbf{x}| \leq|\mathbf{y}|+\varepsilon \leq|\mathbf{x}| / 2+\varepsilon$ and so $|\mathbf{x}|<2 \varepsilon$. Thus this is no larger than

$$
\int_{\varepsilon \leq|\mathbf{x}| \leq 2 \varepsilon} B|\mathbf{x}|^{-n} d x=B \int_{S^{n-1}} \int_{\varepsilon}^{2 \varepsilon} \rho^{n-1} \frac{1}{\rho^{n}} d \rho d \sigma \leq B C(n) \ln 2=C(n) B .
$$

It remains to estimate the first integral on the right in 33.4.46. This integral is bounded by

$$
\int_{|\mathbf{x}| \geq 2|\mathbf{y}|,|\mathbf{x}-\mathbf{y}|>\varepsilon,|\mathbf{x}|<\varepsilon} B|\mathbf{x}-\mathbf{y}|^{-n} d x
$$

In the integral above, $|\mathbf{x}|<\varepsilon$ and so $|\mathbf{x}-\mathbf{y}|-|\mathbf{y}|<\varepsilon$. Therefore, $|\mathbf{x}-\mathbf{y}|<\varepsilon+|\mathbf{y}|<\varepsilon+$ $|\mathbf{x}| / 2<\varepsilon+\varepsilon / 2=(3 / 2) \varepsilon$. Hence $\varepsilon \leq|\mathbf{x}-\mathbf{y}| \leq(3 / 2)|\mathbf{x}-\mathbf{y}|$. Therefore, the above integral is no larger than

$$
\int_{\varepsilon}^{(3 / 2) \varepsilon} B|\mathbf{z}|^{-n} d z=B \int_{S^{n-1}} \int_{\varepsilon}^{(3 / 2) \varepsilon} \rho^{-1} d \rho d \sigma=B C(n) \ln (3 / 2) .
$$

This establishes 33.4.44.
Now it remains to show 33.4.45, a statement about the Fourier transforms of $K_{\varepsilon}$. Fix $\varepsilon$ and let $\mathbf{y} \neq \mathbf{0}$ also be given.

$$
K_{\varepsilon R}(\mathbf{y}) \equiv\left\{\begin{array}{l}
K_{\varepsilon}(\mathbf{y}) \text { if }|\mathbf{y}|<R \\
0 \text { if }|\mathbf{y}| \geq R
\end{array}\right.
$$

where $R>\frac{3 \pi}{|\mathbf{y}|}$. (The 3 here isn't important. It just needs to be larger than 1.) Then

$$
\left|F K_{\varepsilon R}(\mathbf{y})\right| \leq\left|\int_{0<|\mathbf{x}|<3 \pi|\mathbf{y}|^{-1}} K_{\varepsilon}(\mathbf{x}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x\right|+\left|\int_{\beta \pi|\mathbf{y}|^{-1}<|\mathbf{x}| \leq R} K_{\varepsilon}(\mathbf{x}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x\right|
$$

$$
\begin{equation*}
=\mathbf{A}+\mathbf{B} . \tag{33.4.47}
\end{equation*}
$$

Consider A. By 33.4.41

$$
\int_{\varepsilon<|\mathbf{x}|<3 \pi|\mathbf{y}|^{-1}} K_{\varepsilon}(\mathbf{x}) d x=0
$$

and so

$$
\mathbf{A}=\left|\int_{\varepsilon<|\mathbf{x}|<3 \pi|\mathbf{y}|^{-1}} K_{\mathcal{\varepsilon}}(\mathbf{x})\left(e^{-i \mathbf{x} \cdot \mathbf{y}}-1\right) d x\right|
$$

Now

$$
\left|e^{-i \mathbf{x} \cdot \mathbf{y}}-1\right|=|2-2 \cos (\mathbf{x} \cdot \mathbf{y})|^{1 / 2} \leq 2|\mathbf{x} \cdot \mathbf{y}| \leq 2|\mathbf{x}||\mathbf{y}|
$$

so, using polar coordinates, this expression is no larger than

$$
2 B \int_{\varepsilon<|\mathbf{x}|<3 \pi|\mathbf{y}|^{-1}}|\mathbf{x}|^{-n}|\mathbf{x}||\mathbf{y}| d x \leq C(n) B|\mathbf{y}| \int_{\varepsilon}^{3 \pi /|\mathbf{y}|} d \rho \leq B C(n)
$$

Next, consider B. This estimate is based on the trick which follows. Let

$$
\mathbf{z} \equiv \mathbf{y} \pi /|\mathbf{y}|^{2}
$$

so that

$$
|\mathbf{z}|=\pi /|\mathbf{y}|, \mathbf{z} \cdot \mathbf{y}=\pi
$$

Then

$$
\begin{gather*}
\int_{3 \pi|\mathbf{y}|^{-1}<|\mathbf{x}| \leq R} K_{\mathcal{E}}(\mathbf{x}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x=\frac{1}{2} \int_{3 \pi|\mathbf{y}|^{-1}<|\mathbf{x}| \leq R} K_{\mathcal{E}}(\mathbf{x}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x  \tag{33.4.48}\\
-\frac{1}{2} \int_{3 \pi|\mathbf{y}|^{-1}<|\mathbf{x}| \leq R} K_{\varepsilon}(\mathbf{x}) e^{-i(\mathbf{x}+\mathbf{z}) \cdot \mathbf{y}} d x .
\end{gather*}
$$

Here is why. Note in the second of these integrals,

$$
\begin{aligned}
& -\frac{1}{2} \int_{3 \pi|\mathbf{y}|^{-1}<|\mathbf{x}| \leq R} K_{\mathcal{E}}(\mathbf{x}) e^{-i(\mathbf{x}+\mathbf{z}) \cdot \mathbf{y}} d x \\
= & -\frac{1}{2} \int_{3 \pi|\mathbf{y}|^{-1}<|\mathbf{x}| \leq R} K_{\mathcal{E}}(\mathbf{x}) e^{-i \mathbf{x} \cdot \mathbf{y}} e^{-i \mathbf{z} \cdot \mathbf{y}} d x \\
= & -\frac{1}{2} \int_{3 \pi|\mathbf{y}|^{-1}<|\mathbf{x}| \leq R} K_{\mathcal{E}}(\mathbf{x}) e^{-i \mathbf{x} \cdot \mathbf{y}} e^{-i \pi} d x \\
= & \frac{1}{2} \int_{3 \pi|\mathbf{y}|^{-1}<|\mathbf{x}| \leq R} K_{\varepsilon}(\mathbf{x}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x .
\end{aligned}
$$

Then changing the variables in 33.4.48,

$$
\begin{gathered}
\int_{3 \pi|\mathbf{y}|^{-1}<|\mathbf{x}| \leq R} K_{\mathcal{E}}(\mathbf{x}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x \\
=\frac{1}{2} \int_{3 \pi|\mathbf{y}|^{-1}<|\mathbf{x}| \leq R} K_{\mathcal{E}}(\mathbf{x}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x \\
-\frac{1}{2} \int_{3 \pi|\mathbf{y}|^{-1}<|\mathbf{x}-\mathbf{z}| \leq R} K_{\mathcal{E}}(\mathbf{x}-\mathbf{z}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x .
\end{gathered}
$$

Thus

$$
\begin{gather*}
\int K_{\mathcal{E}}(\mathbf{x}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x= \\
+\frac{1}{2} \int_{|\mathbf{x}-\mathbf{z}| \leq 3 \pi|\mathbf{y}|^{-1}} \int_{|\mathbf{x}| \leq R} K_{\mathcal{E}}(\mathbf{x}-\mathbf{z}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x-\frac{1}{2} K_{|\mathbf{x}-\mathbf{y}| \leq 3 \pi|\mathbf{y}|^{-1}} \int_{\mathcal{L}}(\mathbf{x}) e^{-i \mathbf{x} \cdot \mathbf{y} \mid \leq R} d x-\frac{1}{2} \int_{\mathcal{\varepsilon}}(\mathbf{x}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x . \tag{33.4.49}
\end{gather*}
$$

Since $|\mathbf{z}|=\pi /|\mathbf{y}|$, it follows $|\mathbf{z}|=\frac{\pi}{|\mathbf{y}|}<\frac{3 \pi}{|\mathbf{y}|}<R$ and so the following picture describes the situation. In this picture, the radius of each ball equals either $R$ or $3 \pi|\mathbf{y}|^{-1}$ and each integral above is taken over one of the two balls in the picture, either the one centered at $\mathbf{0}$ or the one centered at $\mathbf{z}$.


To begin with, consider the integrals which involve $K_{\mathcal{E}}(\mathbf{x}-\mathbf{z})$.

$$
\begin{gather*}
\int_{|\mathbf{x}-\mathbf{z}| \leq R} K_{\mathcal{E}}(\mathbf{x}-\mathbf{z}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x \\
=\int_{|\mathbf{x}| \leq R} K_{\mathcal{E}}(\mathbf{x}-\mathbf{z}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x \\
-\int_{|\mathbf{x}-\mathbf{z}|>R,|\mathbf{x}|<R} K_{\mathcal{E}}(\mathbf{x}-\mathbf{z}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x  \tag{33.4.50}\\
+\int_{|\mathbf{x}-\mathbf{z}|<R,|\mathbf{x}|>R} K_{\mathcal{E}}(\mathbf{x}-\mathbf{z}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x .
\end{gather*}
$$

Look at the picture. Similarly,

$$
\begin{gather*}
\iint_{\mid \mathbf{x}} K_{\mathcal{E}}(\mathbf{x}-\mathbf{z}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x \\
=\int_{|\mathbf{z}| \leq 3 \pi|\mathbf{y}|^{-1}} K_{\mathcal{E}}(\mathbf{x}-\mathbf{z}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x \\
-\int_{|\mathbf{x}-\mathbf{z}|>3 \pi|\mathbf{y}|^{-1}}^{|\mathbf{y}|^{-1},|\mathbf{x}|<3 \pi|\mathbf{y}|^{-1}} K_{\mathcal{E}}(\mathbf{x}-\mathbf{z}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x+  \tag{33.4.51}\\
\int_{\mathcal{E}}(\mathbf{x}-\mathbf{z}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x
\end{gather*}
$$

The last integral in 33.4.50 is taken over a set that is contained in

$$
B(\mathbf{0}, R+|\mathbf{z}|) \backslash B(\mathbf{0}, R)
$$

illustrated in the following picture as the region between the small ball centered at $\mathbf{0}$ and the big ball which surrounds the two small balls

and so this integral is dominated by

$$
B\left(\frac{1}{(R-|\mathbf{z}|)^{n}}\right) \alpha(n)\left((R+|\mathbf{z}|)^{n}-R^{n}\right)
$$

an expression which converges to 0 as $R \rightarrow \infty$. Similarly, the second integral on the right in 33.4.50 converges to zero as $R \rightarrow \infty$. Now consider the last two integrals in 33.4.51. Letting $3 \pi|\mathbf{y}|^{-1}$ play the role of $R$ and using $|\mathbf{z}|=\pi /|\mathbf{y}|$, these are each dominated by an expression of the form

$$
\begin{gathered}
B\left(\frac{1}{\left(3 \pi|\mathbf{y}|^{-1}-|\mathbf{z}|\right)^{n}}\right) \alpha(n)\left(\left(3 \pi|\mathbf{y}|^{-1}+|\mathbf{z}|\right)^{n}-\left(3 \pi|\mathbf{y}|^{-1}\right)^{n}\right) \\
=B\left(\frac{1}{\left(3 \pi|\mathbf{y}|^{-1}-\pi|\mathbf{y}|^{-1}\right)^{n}}\right) \alpha(n) \\
\left(\left(3 \pi|\mathbf{y}|^{-1}+\pi|\mathbf{y}|^{-1}\right)^{n}-\left(3 \pi|\mathbf{y}|^{-1}\right)^{n}\right)
\end{gathered}
$$

$$
=\alpha(n) B \frac{|\mathbf{y}|^{n}}{(2 \pi)^{n}} \frac{1}{|\mathbf{y}|^{n}}\left((4 \pi)^{n}-(3 \pi)^{n}\right)=C(n) B .
$$

Returning to 33.4.49, the terms involving $\mathbf{x}-\mathbf{y}$ have now been estimated. Thus, collecting the terms which have not yet been estimated along with those that have,

$$
\begin{gathered}
\mathbf{B}=\left|\int_{\beta \pi|\mathbf{y}|^{-1}<|\mathbf{x}| \leq R} K_{\mathcal{E}}(\mathbf{x}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x\right| \\
\leq \frac{1}{2} \int_{|\mathbf{x}|<R} K_{\mathcal{E}}(\mathbf{x}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x-\int_{|\mathbf{x}|<R} K_{\mathcal{E}}(\mathbf{x}-\mathbf{z}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x \\
+\int_{|\mathbf{x}|<3 \pi|\mathbf{y}|^{-1}} K_{\mathcal{E}}(\mathbf{x}-\mathbf{z}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x-\int_{|\mathbf{x}|<3 \pi|\mathbf{y}|^{-1}} K_{\mathcal{E}}(\mathbf{x}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x \mid \\
+C(n) B+g(R)
\end{gathered}
$$

where $g(R) \rightarrow 0$ as $R \rightarrow \infty$. Using $|\mathbf{z}|=\pi /|\mathbf{y}|$ again,

$$
\mathbf{B} \leq \frac{1}{2} \int_{3|\mathbf{z}|<|\mathbf{x}|<R}\left|K_{\mathcal{E}}(\mathbf{x})-K_{\mathcal{E}}(\mathbf{x}-\mathbf{z})\right| d x+C(n) B+g(R) .
$$

But the integral in the above is dominated by $C(n) B$ by 33.4 .44 which was established earlier. Therefore, from 33.4.47,

$$
\left|F K_{\varepsilon R}\right| \leq C(n) B+g(R)
$$

where $g(R) \rightarrow 0$.
Now $K_{\varepsilon R} \rightarrow K_{\varepsilon}$ in $L^{2}\left(\mathbb{R}^{n}\right)$ because

$$
\begin{aligned}
\left\|K_{\varepsilon R}-K_{\mathcal{E}}\right\|_{L^{2}\left(\mathbb{R}^{n}\right)} & \leq B \int_{|\mathbf{x}|>R} \frac{1}{|\mathbf{x}|^{2 n}} d x \\
& =B \int_{S^{n-1}} \int_{R}^{\infty} \frac{1}{\rho^{n+1}} d \rho d \sigma
\end{aligned}
$$

which converges to 0 as $R \rightarrow \infty$ and so $F K_{\varepsilon R} \rightarrow F K_{\varepsilon}$ in $L^{2}\left(\mathbb{R}^{n}\right)$ by Plancherel's theorem. Therefore, by taking a subsequence, still denoted by $R, F K_{\varepsilon R}(\mathbf{y}) \rightarrow F K_{\mathcal{\varepsilon}}(\mathbf{y})$ a.e. which shows

$$
\left|F K_{\varepsilon}(\mathbf{y})\right| \leq C(n) B \text { a.e. }
$$

This proves the lemma.
Corollary 33.4.4 Suppose 33.4.40-33.4.42 hold. Then if $g \in C_{c}^{1}\left(\mathbb{R}^{n}\right), K_{\mathcal{\varepsilon}} * g$ converges uniformly and in $L^{p}\left(\mathbb{R}^{n}\right)$ as $\varepsilon \rightarrow 0$.

Proof:

$$
K_{\mathcal{\varepsilon}} * g(\mathbf{x}) \equiv \int K_{\mathcal{\varepsilon}}(\mathbf{y}) g(\mathbf{x}-\mathbf{y}) d y
$$

Let $0<\eta<\varepsilon$. Then since $g \in C_{c}^{1}\left(\mathbb{R}^{n}\right)$, there exists a constant, $K$ such that $K|\mathbf{u}-\mathbf{v}| \geq$ $|g(\mathbf{u})-g(\mathbf{v})|$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{n}$.

$$
\begin{aligned}
\left|K_{\varepsilon} * g(\mathbf{x})-K_{\eta} * g(\mathbf{x})\right| & \leq B K \int_{\eta<|\mathbf{y}|<\varepsilon} \frac{1}{|\mathbf{y}|^{n}}|\mathbf{y}| d y \\
& =B K \int_{S^{n-1}} \int_{\eta}^{\varepsilon} d \rho d \sigma=C_{n}|\varepsilon-\eta|
\end{aligned}
$$

This proves the corollary.
Theorem 33.4.5 Suppose 33.4.40-33.4.42. Then for $K_{\varepsilon}$ given by 33.4.43 and $p>1$, there exists a constant $A(p, n, B)$ such that for all $f \in L^{p}\left(\mathbb{R}^{n}\right)$,

$$
\begin{equation*}
\left\|K_{\varepsilon} * f\right\|_{p} \leq A(p, n, B)\|f\|_{p} \tag{33.4.52}
\end{equation*}
$$

Also, for each $f \in L^{p}\left(\mathbb{R}^{n}\right)$,

$$
\begin{equation*}
T f \equiv \lim _{\varepsilon \rightarrow 0} K_{\mathcal{E}} * f \tag{33.4.53}
\end{equation*}
$$

exists in $L^{p}\left(\mathbb{R}^{n}\right)$ and for all $f \in L^{p}\left(\mathbb{R}^{n}\right)$,

$$
\begin{equation*}
\|T f\|_{p} \leq A(p, n, B)\|f\|_{p} \tag{33.4.54}
\end{equation*}
$$

Thus $T$ is a linear and continuous map defined on $L^{p}\left(\mathbb{R}^{n}\right)$ for each $p>1$.
Proof: From 33.4.40 it follows $K_{\mathcal{E}} \in L^{p^{\prime}}\left(\mathbb{R}^{n}\right) \cap L^{2}\left(\mathbb{R}^{n}\right)$ where, as usual, $1 / p+1 / p^{\prime}=1$. By continuity of translation in $L^{p^{\prime}}\left(\mathbb{R}^{n}\right), x \rightarrow K_{\mathcal{\varepsilon}} * f(x)$ is a continuous function.By Lemma 33.4.3, $\left\|F K_{\mathcal{E}}\right\|_{\infty} \leq C(n) B$ for all $\varepsilon$. Therefore, by Lemma 33.4.2,

$$
\left\|K_{\varepsilon} * g\right\|_{p} \leq A(p, n, B)\|g\|_{p}
$$

for all $g \in \mathscr{G}$. Now let $f \in L^{p}\left(\mathbb{R}^{n}\right)$ and $g_{k} \rightarrow f$ in $L^{p}\left(\mathbb{R}^{n}\right)$ where $g_{k} \in \mathscr{G}$. Then

$$
\begin{aligned}
\left|K_{\mathcal{\varepsilon}} * f(\mathbf{x})-K_{\mathcal{\varepsilon}} * g_{k}(\mathbf{x})\right| & \leq \int\left|K_{\mathcal{\varepsilon}}(\mathbf{x}-\mathbf{y})\right|\left|g_{k}(\mathbf{y})-f(\mathbf{y})\right| d y \\
& \leq\left\|K_{\mathcal{E}}\right\|_{p^{\prime}}\left\|g_{k}-f\right\|_{p}
\end{aligned}
$$

which shows that $K_{\mathcal{\varepsilon}} * g_{k}(\mathbf{x}) \rightarrow K_{\mathcal{\varepsilon}} * f(\mathbf{x})$ pointwise and so by Fatou's lemma,

$$
\begin{aligned}
\left\|K_{\varepsilon} * f\right\|_{p} & \leq \lim _{k \rightarrow \infty} \inf _{k \rightarrow}\left\|K_{\mathcal{\varepsilon}} * g_{k}\right\|_{p} \leq \lim _{k \rightarrow \infty} \inf _{k \rightarrow \infty}(p, n, B)\left\|g_{k}\right\|_{p} \\
& =A(p, n, B)\|f\|_{p}
\end{aligned}
$$

This verifies 33.4.52.
To verify 33.4 .53 , let $\delta>0$ be given and let

$$
f \in L^{p}\left(\mathbb{R}^{n}\right), g \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right)
$$

$$
\begin{aligned}
& \| K_{\mathcal{\varepsilon}} * f- K_{\eta} * f \|_{p} \leq \\
&\left\|K_{\varepsilon} *(f-g)\right\|_{p}+\left\|K_{\varepsilon} * g-K_{\eta} * g\right\|_{p} \\
&+\left\|K_{\eta} *(f-g)\right\|_{p} \\
& \leq 2 A(p, n, B)\|f-g\|_{p}+\left\|K_{\varepsilon} * g-K_{\eta} * g\right\|_{p}
\end{aligned}
$$

Choose $g$ such that $2 A(p, n, B)\|f-g\|_{p} \leq \delta / 2$. Then if $\varepsilon, \eta$ are small enough, Corollary 33.4.4 implies the last term is also less than $\delta / 2$. Thus, $\lim _{\varepsilon \rightarrow 0} K_{\varepsilon} * f$ exists in $L^{p}\left(\mathbb{R}^{n}\right)$. Let $T f$ be the element of $L^{p}\left(\mathbb{R}^{n}\right)$ to which it converges. Then 33.4.54 follows and $T$ is obviously linear because

$$
\begin{aligned}
T(a f+b g) & =\lim _{\varepsilon \rightarrow 0} K_{\varepsilon} *(a f+b g)=\lim _{\varepsilon \rightarrow 0}\left(a K_{\mathcal{\varepsilon}} * f+b K_{\mathcal{\varepsilon}} * g\right) \\
& =a T f+b T g .
\end{aligned}
$$

This proves the theorem.
When do conditions 33.4.40-33.4.42 hold? It turns out this happens for $K$ given by the following.

$$
\begin{equation*}
K(\mathbf{x}) \equiv \frac{\Omega(\mathbf{x})}{|\mathbf{x}|^{n}} \tag{33.4.55}
\end{equation*}
$$

where

$$
\begin{gather*}
\Omega(\lambda \mathbf{x})=\Omega(\mathbf{x}) \text { for all } \lambda>0  \tag{33.4.56}\\
\Omega \text { is Lipschitz on } S^{n-1} \\
\int_{S^{n-1}} \Omega(\mathbf{x}) d \sigma=0 \tag{33.4.57}
\end{gather*}
$$

Theorem 33.4.6 For $K$ given by 33.4.55-33.4.57, it follows there exists a constant B such that

$$
\begin{gather*}
|K(\mathbf{x})| \leq B|\mathbf{x}|^{-n}  \tag{33.4.58}\\
\int_{a<|\mathbf{x}|<b} K(\mathbf{x}) d x=0  \tag{33.4.59}\\
\int_{|\mathbf{x}|>2|\mathbf{y}|}|K(\mathbf{x}-\mathbf{y})-K(\mathbf{x})| d x \leq B \tag{33.4.60}
\end{gather*}
$$

Consequently, the conclusions of Theorem 33.4.5 hold also.
Proof: 33.4.58 is obvious. To verify 33.4.59,

$$
\begin{aligned}
\int_{a<|\mathbf{x}|<b} K(\mathbf{x}) d x & =\int_{a}^{b} \int_{S^{n-1}} \frac{\Omega(\rho \mathbf{w})}{\rho^{n}} \rho^{n-1} d \sigma d \rho \\
& =\int_{a}^{b} \frac{1}{\rho} \int_{S^{n-1}} \Omega(\mathbf{w}) d \sigma d \rho=0
\end{aligned}
$$

It remains to show 33.4.60.

$$
\begin{align*}
K(\mathbf{x}-\mathbf{y})-K(\mathbf{x})= & |\mathbf{x}-\mathbf{y}|^{-n}\left(\Omega\left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)-\Omega\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right)\right) \\
& +\Omega(\mathbf{x})\left(\frac{1}{|\mathbf{x}-\mathbf{y}|^{n}}-\frac{1}{|\mathbf{x}|^{n}}\right) \tag{33.4.61}
\end{align*}
$$

where 33.4 .56 was used to write $\Omega\left(\frac{\mathbf{z}}{|\mathbf{z}|}\right)=\Omega(\mathbf{z})$. The first group of terms in 33.4.61 is dominated by

$$
|\mathbf{x}-\mathbf{y}|^{-n} \operatorname{Lip}(\Omega)\left|\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}-\frac{\mathbf{x}}{|\mathbf{x}|}\right|
$$

and an estimate is required for $|\mathbf{x}|>2|\mathbf{y}|$. Since $|\mathbf{x}|>2|\mathbf{y}|$,

$$
|\mathbf{x}-\mathbf{y}|^{-n} \leq(|\mathbf{x}|-|\mathbf{y}|)^{-n} \leq \frac{2^{n}}{|\mathbf{x}|^{n}}
$$

Also

$$
\begin{gathered}
\left|\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}-\frac{\mathbf{x}}{|\mathbf{x}|}\right|=\left|\frac{(\mathbf{x}-\mathbf{y})|\mathbf{x}|-\mathbf{x}|\mathbf{x}-\mathbf{y}|}{|\mathbf{x}||\mathbf{x}-\mathbf{y}|}\right| \\
\leq\left|\frac{(\mathbf{x}-\mathbf{y})|\mathbf{x}|-\mathbf{x}|\mathbf{x}-\mathbf{y}|}{|\mathbf{x}|(|\mathbf{x}|-|\mathbf{y}|)}\right| \leq\left|\frac{(\mathbf{x}-\mathbf{y})|\mathbf{x}|-\mathbf{x}|\mathbf{x}-\mathbf{y}|}{|\mathbf{x}|(|\mathbf{x}| / 2)}\right| \\
=\frac{2}{|\mathbf{x}|^{2}}|\mathbf{x}| \mathbf{x}|-\mathbf{y}| \mathbf{x}|-\mathbf{x}| \mathbf{x}-\mathbf{y}| |=\frac{2}{|\mathbf{x}|^{2}}|\mathbf{x}(|\mathbf{x}|-|\mathbf{x}-\mathbf{y}|)-\mathbf{y}| \mathbf{x}| | \\
\leq \frac{2}{|\mathbf{x}|^{2}}|\mathbf{x}|| | \mathbf{x}|-|\mathbf{x}-\mathbf{y}||+|\mathbf{y}||\mathbf{x}| \leq \frac{2}{|\mathbf{x}|^{2}}(|\mathbf{x}||\mathbf{x}-(\mathbf{x}-\mathbf{y})|+|\mathbf{y}||\mathbf{x}|) \\
\leq \frac{4}{|\mathbf{x}|^{2}}|\mathbf{x}||\mathbf{y}|=4 \frac{|\mathbf{y}|}{|\mathbf{x}|}
\end{gathered}
$$

Therefore,

$$
\begin{align*}
\int_{|\mathbf{x}|>2|\mathbf{y}|} & |\mathbf{x}-\mathbf{y}|^{-n}\left|\Omega\left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)-\Omega\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right)\right| d x \\
& \leq 4\left(2^{n}\right) \int_{|\mathbf{x}|>2|\mathbf{y}|} \frac{1}{|\mathbf{x}|^{n}} \frac{|\mathbf{y}|}{|\mathbf{x}|} d x \operatorname{Lip}(\Omega) \\
& =C(n, \operatorname{Lip} \Omega) \int_{|\mathbf{x}|>2|\mathbf{y}|} \frac{|\mathbf{y}|}{|\mathbf{x}|^{n+1}} d x \\
& =C(n, \operatorname{Lip} \Omega) \int_{|\mathbf{u}|>2} \frac{1}{|\mathbf{u}|^{n+1}} d u \tag{33.4.62}
\end{align*}
$$

It remains to consider the second group of terms in 33.4.61 when $|\mathbf{x}|>2|\mathbf{y}|$.

$$
\begin{aligned}
& \left|\frac{1}{|\mathbf{x}-\mathbf{y}|^{n}}-\frac{1}{|\mathbf{x}|^{n}}\right|=\left|\frac{|\mathbf{x}|^{n}-|\mathbf{x}-\mathbf{y}|^{n}}{|\mathbf{x}-\mathbf{y}|^{n}|\mathbf{x}|^{n}}\right| \\
& \left.\quad \leq\left.\frac{2^{n}}{|\mathbf{x}|^{2 n}}| | \mathbf{x}\right|^{n}-|\mathbf{x}-\mathbf{y}|^{n} \right\rvert\, \\
& \leq \frac{2^{n}}{|\mathbf{x}|^{2 n}}|\mathbf{y}|\left[|\mathbf{x}|^{n-1}+|\mathbf{x}|^{n-2}|\mathbf{x}-\mathbf{y}|+\right. \\
& \left.\cdots+|\mathbf{x}||\mathbf{x}-\mathbf{y}|^{n-2}+|\mathbf{x}-\mathbf{y}|^{n-1}\right]
\end{aligned}
$$

$$
\leq \frac{2^{n}|\mathbf{y}| C(n)|\mathbf{x}|^{n-1}}{|\mathbf{x}|^{2 n}}=\frac{C(n) 2^{n}|\mathbf{y}|}{|\mathbf{x}|^{n+1}}
$$

Thus

$$
\begin{gather*}
\int_{|\mathbf{x}|>2|\mathbf{y}|}\left|\Omega(\mathbf{x})\left(\frac{1}{|\mathbf{x}-\mathbf{y}|^{n}}-\frac{1}{|\mathbf{x}|^{n}}\right)\right| d x \\
\leq \quad C(n) \int_{|\mathbf{x}|>2|\mathbf{y}|} \frac{|\mathbf{y}|}{|\mathbf{x}|^{n+1}} d x \\
\leq C(n) \int_{|\mathbf{u}|>2} \frac{1}{|\mathbf{u}|^{n+1}} d u . \tag{33.4.63}
\end{gather*}
$$

From 33.4.62 and 33.4.63,

$$
\int_{|\mathbf{x}|>2|\mathbf{y}|}|K(\mathbf{x}-\mathbf{y})-K(\mathbf{x})| d x \leq C(n, \operatorname{Lip} \Omega) .
$$

This proves the theorem.

### 33.5 Helmholtz Decompositions

It turns out that every vector field which has its components in $L^{p}$ can be written as a sum of a gradient and a vector field which has zero divergence. This is a very remarkable result, especially when applied to vector fields which are only in $L^{p}$. Recall that for $u$ a function of $n$ variables, $\Delta u=\sum_{i=1}^{n} \frac{\partial^{2} u}{\partial x_{i}^{2}}$.
Definition 33.5.1 Define

$$
\Phi(\mathbf{y}) \equiv\left\{\begin{array}{l}
-\frac{1}{a_{1}} \ln |\mathbf{y}|, \text { if } n=2 \\
\frac{1}{(n-2) a_{n-1}}|\mathbf{y}|^{2-n}, \text { if } n>2
\end{array}\right.
$$

where $a_{k}$ denotes the area of the unit sphere, $S^{k}$.
Then it is routine to verify $\Delta \Phi=0$ away from 0 . In fact, if $n>2$,

$$
\begin{equation*}
\Phi_{, i i}(\mathbf{y})=C_{n}\left[\frac{1}{|\mathbf{y}|^{n}}-n \frac{y_{i}^{2}}{|\mathbf{y}|^{n+2}}\right], \Phi_{, i j}(\mathbf{y})=C_{n} \frac{y_{i} y_{j}}{|\mathbf{y}|^{n+2}} \tag{33.5.64}
\end{equation*}
$$

while if $n=2$,

$$
\begin{gathered}
\Phi_{, 22}(\mathbf{y})=C_{2} \frac{y_{1}^{2}-y_{2}^{2}}{\left(y_{1}^{2}+y_{2}^{2}\right)^{2}}, \Phi_{, 11}(\mathbf{y})=C_{2} \frac{y_{2}^{2}-y_{1}^{2}}{\left(y_{1}^{2}+y_{2}^{2}\right)^{2}} \\
\Phi_{, i j}(\mathbf{y})=C_{2} \frac{y_{1} y_{2}}{\left(y_{1}^{2}+y_{2}^{2}\right)^{2}}
\end{gathered}
$$

Also,

$$
\begin{equation*}
\nabla \Phi(\mathbf{y})=\frac{-\mathbf{y}}{a_{n-1}|\mathbf{y}|^{n}} \tag{33.5.65}
\end{equation*}
$$

In the above the subscripts following a comma denote partial derivatives.

Lemma 33.5.2 For $n \geq 2$

$$
\Phi_{, i j}(\mathbf{y})=\frac{\Omega_{i j}(\mathbf{y})}{|\mathbf{y}|^{n}}
$$

where

$$
\begin{gather*}
\Omega_{i j} \text { is Lipschitz continuous on } S^{n-1},  \tag{33.5.66}\\
\Omega_{i j}(\lambda \mathbf{y})=\Omega_{i j}(\mathbf{y}) \tag{33.5.67}
\end{gather*}
$$

for all $\lambda>0$, and

$$
\begin{equation*}
\int_{S^{n-1}} \Omega_{i j}(\mathbf{y}) d \sigma=0 \tag{33.5.68}
\end{equation*}
$$

Proof:
Proof: The case $n=2$ is left to the reader. 33.5.66 and 33.5.67 are obvious from the above descriptions. It remains to verify 33.5 .68. If $n \geq 3$ and $i \neq j$, then this formula is also clear from 33.5.64. Thus consider the case when $n \geq 3$ and $i=j$. By symmetry,

$$
I \equiv \int_{S^{n-1}} 1-n y_{i}^{2} d \sigma=\int_{S^{n-1}} 1-n y_{j}^{2} d \sigma
$$

Hence

$$
\begin{aligned}
n I & =\sum_{i=1}^{n} \int_{S^{n-1}} 1-n y_{i}^{2} d \sigma=\int_{S^{n-1}}\left(n-n \sum_{i} y_{i}^{2}\right) d \sigma \\
& =\int_{S^{n-1}}(n-n) d \sigma=0 .
\end{aligned}
$$

This proves the lemma.
Let $U$ be a bounded open set locally on one side of its boundary having Lipschitz boundary so the divergence theorem holds and let $B=B(\mathbf{0}, R)$ where

$$
B \supseteq U-U \equiv\{\mathbf{x}-\mathbf{y}: \mathbf{x} \in U, \mathbf{y} \in U\}
$$

Let $f \in C_{c}^{\infty}(U)$ and define for $\mathbf{x} \in U$,

$$
u(\mathbf{x}) \equiv \int_{B} \Phi(\mathbf{y}) f(\mathbf{x}-\mathbf{y}) d y=\int_{U} \Phi(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d y
$$

Let $h(\mathbf{y})=f(\mathbf{x}-\mathbf{y})$. Then since $\Phi$ is in $L^{1}(B)$,

$$
\begin{aligned}
& \Delta u(\mathbf{x})=\int_{B} \Phi(\mathbf{y}) \Delta f(\mathbf{x}-\mathbf{y}) d y=\int_{B} \Phi(\mathbf{y}) \Delta h(\mathbf{y}) d y \\
&=\int_{B \backslash B(\mathbf{0}, \varepsilon)} \nabla \cdot(\nabla h(\mathbf{y}) \Phi(\mathbf{y}))-\nabla \Phi(\mathbf{y}) \cdot \nabla h(\mathbf{y}) d y \\
&+\int_{B(\mathbf{0}, \varepsilon)} \Phi(\mathbf{y}) \Delta h(\mathbf{y}) d y
\end{aligned}
$$

The last term converges to 0 as $\varepsilon \rightarrow 0$ because $\Phi$ is in $L^{1}$ and $\Delta h$ is bounded. Since $\operatorname{spt}(h) \subseteq B$, the divergence theorem implies

$$
\begin{equation*}
\Delta u(\mathbf{x})=-\int_{\partial B(\mathbf{0}, \varepsilon)} \Phi(\mathbf{y}) \nabla h(\mathbf{y}) \cdot \mathbf{n} d \sigma-\int_{B \backslash B(\mathbf{0}, \varepsilon)} \nabla \Phi(\mathbf{y}) \cdot \nabla h(\mathbf{y}) d y+e(\varepsilon) \tag{33.5.69}
\end{equation*}
$$

where here and below, $e(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. The first term in 33.5 .69 converges to 0 as $\varepsilon \rightarrow 0$ because

$$
\left|\int_{\partial B(\mathbf{0}, \varepsilon)} \Phi(\mathbf{y}) \nabla h(\mathbf{y}) \cdot \mathbf{n} d \sigma\right| \leq\left\{\begin{array}{l}
C_{n h} \frac{1}{\varepsilon^{n-2}} \varepsilon^{n-1}=C_{n h} \varepsilon \text { if } n>2 \\
C_{h}(\ln \varepsilon) \varepsilon \text { if } n=2
\end{array}\right.
$$

and since $\Delta \Phi(\mathbf{y})=0$,

$$
\nabla \Phi(\mathbf{y}) \cdot \nabla h(\mathbf{y})=\nabla \cdot(\nabla \Phi(\mathbf{y}) h(\mathbf{y}))
$$

Consequently

$$
\Delta u(\mathbf{x})=-\int_{B \backslash B(\mathbf{0}, \varepsilon)} \nabla \cdot(\nabla \Phi(\mathbf{y}) h(\mathbf{y})) d y+e(\varepsilon)
$$

Thus, by the divergence theorem, 33.5.65, and the definition of $h$ above,

$$
\begin{aligned}
\Delta u(\mathbf{x}) & =\int_{\partial B(\mathbf{0}, \varepsilon)} f(\mathbf{x}-\mathbf{y}) \nabla \Phi(\mathbf{y}) \cdot \mathbf{n} d \sigma+e(\varepsilon) \\
& =\int_{\partial B(\mathbf{0}, \varepsilon)} f(\mathbf{x}-\mathbf{y})\left(-\frac{\mathbf{y}}{a_{n-1}|\mathbf{y}|^{n}}\right) \cdot\left(-\frac{\mathbf{y}}{|\mathbf{y}|}\right) d \sigma+e(\varepsilon) \\
& =-\left(\int_{\partial B(\mathbf{0}, \varepsilon)} f(\mathbf{x}-\mathbf{y}) d \sigma(y)\right) \frac{1}{a_{n-1} \varepsilon^{n-1}}+e(\varepsilon)
\end{aligned}
$$

Letting $\varepsilon \rightarrow 0$,

$$
-\Delta u(\mathbf{x})=f(\mathbf{x})
$$

This proves the following lemma.
Lemma 33.5.3 Let $U$ be a bounded open set in $\mathbb{R}^{n}$ with Lipschitz boundary and let $B \supseteq$ $U-U$ where $B=B(\mathbf{0}, R)$. Let $f \in C_{c}^{\infty}(U)$. Then for $\mathbf{x} \in U$,

$$
\int_{B} \Phi(\mathbf{y}) f(\mathbf{x}-\mathbf{y}) d y=\int_{U} \Phi(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d y
$$

and it follows that if $u$ is given by one of the above formulas, then for all $x \in U$,

$$
-\Delta u(\mathbf{x})=f(\mathbf{x})
$$

Theorem 33.5.4 Let $f \in L^{p}(U)$. Then there exists $u \in L^{p}(U)$ whose weak derivatives are also in $L^{p}(U)$ such that in the sense of weak derivatives,

$$
-\Delta u=f
$$

It is given by

$$
\begin{equation*}
u(\mathbf{x})=\int_{B} \Phi(\mathbf{y}) \tilde{f}(\mathbf{x}-\mathbf{y}) d y=\int_{U} \Phi(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d y \tag{33.5.70}
\end{equation*}
$$

where $\tilde{f}$ denotes the zero extension of $f$ off of $U$.

Proof: Let $f \in L^{p}(U)$ and let $f_{k} \in C_{c}^{\infty}(U),\left\|f_{k}-f\right\|_{L^{p}(U)} \rightarrow 0$, and let $u_{k}$ be given by 33.5.70 with $f_{k}$ in place of $f$. Then by Minkowski's inequality,

$$
\begin{aligned}
\left\|u-u_{k}\right\|_{L^{p}(U)} & =\left(\int_{U}\left(\int_{B} \Phi(\mathbf{y})\left|\widetilde{f}(\mathbf{x}-\mathbf{y})-f_{k}(\mathbf{x}-\mathbf{y})\right| d y\right)^{p} d x\right)^{1 / p} \\
& \leq\left(\int_{B}|\Phi(\mathbf{y})|\left(\int_{U}\left|\widetilde{f}(\mathbf{x}-\mathbf{y})-f_{k}(\mathbf{x}-\mathbf{y})\right|^{p} d x\right)^{1 / p} d y\right) \\
& \leq \int_{B}|\Phi(\mathbf{y})| d y| | f-f_{k}\left\|_{L^{p}(U)}=C(B)\right\| f-f_{k} \|_{L^{p}(U)}
\end{aligned}
$$

and so $u_{k} \rightarrow u$ in $L^{p}(U)$. Also

$$
u_{k, i}(\mathbf{x})=\int_{U} \Phi_{, i}(\mathbf{x}-\mathbf{y}) f_{k}(\mathbf{y}) d y=\int_{B} f_{k}(\mathbf{x}-\mathbf{y}) \Phi_{, i}(\mathbf{y}) d y
$$

Now let

$$
\begin{equation*}
w_{i} \equiv \int_{B} \widetilde{f}(\mathbf{x}-\mathbf{y}) \Phi_{, i}(\mathbf{y}) d y \tag{33.5.71}
\end{equation*}
$$

and since $\Phi_{, i} \in L^{1}(B)$, it follows from Minkowski's inequality that

$$
\begin{aligned}
& \left\|u_{k, i}-w_{i}\right\|_{L^{p}(U)} \\
\leq & \left(\int_{U}\left(\int_{B}\left|f_{k}(\mathbf{x}-\mathbf{y})-\widetilde{f}(\mathbf{x}-\mathbf{y})\right|\left|\Phi_{, i}(\mathbf{y})\right| d y\right)^{p} d x\right)^{1 / p} \\
\leq & \int_{B}\left|\Phi_{, i}(\mathbf{y})\right|\left(\int_{U}\left|f_{k}(\mathbf{x}-\mathbf{y})-\widetilde{f}(\mathbf{x}-\mathbf{y})\right|^{p} d x\right)^{1 / p} d y \\
\leq & C(B)\left\|f_{k}-f\right\|_{L^{p}(U)}
\end{aligned}
$$

and so $u_{k, i} \rightarrow w_{i}$ in $L^{p}(U)$.
Now let $\phi \in C_{c}^{\infty}(U)$. Then

$$
\int_{U} w_{i} \phi d x=-\lim _{k \rightarrow \infty} \int_{U} u_{k} \phi_{, i} d x=-\int_{U} u \phi_{, i} d x
$$

Thus $u_{, i}=w_{i} \in L^{p}\left(\mathbb{R}^{n}\right)$ and so if $\phi \in C_{c}^{\infty}(U)$,

$$
\int_{U} f \phi d x=\lim _{k \rightarrow \infty} \int_{U} f_{k} \phi d x=\lim _{k \rightarrow \infty} \int_{U} \nabla u_{k} \cdot \nabla \phi d x=\int_{U} \nabla u \cdot \nabla \phi d x
$$

and so $-\Delta u=f$ as claimed. This proves the theorem.
One could also ask whether the second weak partial derivatives of $u$ are in $L^{p}(U)$. This is where the theory singular integrals is used. Recall from 33.5 .70 and 33.5 .71 along with the argument of the above lemma, that if $u$ is given by 33.5.70, then $u_{, i}$ is given by 33.5.71 which equals

$$
\int_{U} \Phi_{, i}(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d y
$$

Lemma 33.5.5 Let $f \in L^{p}(U)$ and let

$$
w_{i}(\mathbf{x}) \equiv \int_{U} \Phi_{, i}(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d y
$$

Then $w_{i, j} \in L^{p}(U)$ for each $j=1 \cdots n$ and the map $f \rightarrow w_{i, j}$ is continuous and linear on $L^{p}(U)$.

Proof: First let $f \in C_{c}^{\infty}(U)$. For such $f$,

$$
\begin{aligned}
w_{i}(\mathbf{x}) & =\int_{U} \Phi_{, i}(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d y=\int_{\mathbb{R}^{n}} \Phi_{, i}(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d y \\
& =\int_{\mathbb{R}^{n}} \Phi_{, i}(\mathbf{y}) f(\mathbf{x}-\mathbf{y}) d y=\int_{B} \Phi_{, i}(\mathbf{y}) f(\mathbf{x}-\mathbf{y}) d y
\end{aligned}
$$

and

$$
\begin{aligned}
w_{i, j}(\mathbf{x}) & =\int_{B} \Phi_{, i}(\mathbf{y}) f_{, j}(\mathbf{x}-\mathbf{y}) d y \\
& =\int_{B \backslash B(\mathbf{0}, \boldsymbol{\varepsilon})} \Phi_{, i}(\mathbf{y}) f_{, j}(\mathbf{x}-\mathbf{y}) d y+\int_{B(\mathbf{0}, \boldsymbol{\varepsilon})} \Phi_{, i}(\mathbf{y}) f_{, j}(\mathbf{x}-\mathbf{y}) d y
\end{aligned}
$$

The second term converges to 0 because $f_{, j}$ is bounded and by 33.5.65, $\Phi_{, i} \in L_{l o c}^{1}$. Thus

$$
\begin{aligned}
w_{i, j}(\mathbf{x}) & =\int_{B \backslash B(\mathbf{0}, \varepsilon)} \Phi_{, i}(\mathbf{y}) f_{, j}(\mathbf{x}-\mathbf{y}) d y+e(\varepsilon) \\
& =\int_{B \backslash B(\mathbf{0}, \varepsilon)}-\left(\Phi_{, i}(\mathbf{y}) f(\mathbf{x}-\mathbf{y})\right)_{, j}+\Phi_{, i j}(\mathbf{y}) f(\mathbf{x}-\mathbf{y}) d y+e(\varepsilon)
\end{aligned}
$$

where $e(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Using the divergence theorem, this yields

$$
w_{i, j}(\mathbf{x})=\int_{\partial B(\mathbf{0}, \varepsilon)} \Phi_{, i}(\mathbf{y}) f(\mathbf{x}-\mathbf{y}) n_{j} d \sigma+\int_{B \backslash B(\mathbf{0}, \varepsilon)} \Phi_{, i j}(\mathbf{y}) f(\mathbf{x}-\mathbf{y}) d y+e(\varepsilon) .
$$

Consider the first term on the right. This term equals, after letting $\mathbf{y}=\varepsilon \mathbf{z}$,

$$
\begin{aligned}
\varepsilon^{n-1} \int_{\partial B(\mathbf{0}, 1)} \Phi_{, i}(\varepsilon \mathbf{z}) f(\mathbf{x}-\varepsilon \mathbf{z}) n_{j} d \sigma & =C_{n} \varepsilon^{n-1} \int_{\partial B(\mathbf{0}, 1)} \varepsilon^{1-n} z_{i} z_{j} f(\mathbf{x}-\varepsilon \mathbf{z}) d \sigma(z) \\
& =C_{n} \int_{\partial B(\mathbf{0}, 1)} z_{i} z_{j} f(\mathbf{x}-\varepsilon \mathbf{z}) d \sigma(z)
\end{aligned}
$$

and this converges to 0 if $i \neq j$ and it converges to

$$
C_{n} f(\mathbf{x}) \int_{\partial B(\mathbf{0}, 1)} z_{i}^{2} d \sigma(z)
$$

if $i=j$. Thus

$$
w_{i, j}(\mathbf{x})=C_{n} \delta_{i j} f(\mathbf{x})+\int_{B \backslash B(\mathbf{0}, \varepsilon)} \Phi_{, i j}(\mathbf{y}) f(\mathbf{x}-\mathbf{y}) d y+e(\varepsilon)
$$

Letting

$$
\Phi_{i j}^{\varepsilon} \equiv\left\{\begin{array}{l}
0 \text { if }|\mathbf{y}|<\varepsilon \\
\Phi_{, i j}(\mathbf{y}) \text { if }|\mathbf{y}| \geq \varepsilon
\end{array}\right.
$$

it follows

$$
w_{i, j}(\mathbf{x})=C_{n} \delta_{i j} f(\mathbf{x})+\Phi_{i j}^{\varepsilon} * \widetilde{f}(\mathbf{x})+e(\varepsilon)
$$

By the theory of singular integrals, there exists a continuous linear map,

$$
K_{i j} \in \mathscr{L}\left(L^{p}\left(\mathbb{R}^{n}\right), L^{p}\left(\mathbb{R}^{n}\right)\right)
$$

such that

$$
K_{i j} f \equiv \lim _{\varepsilon \rightarrow 0} \Phi_{i j}^{\varepsilon} * f
$$

Therefore, letting $\varepsilon \rightarrow 0$,

$$
w_{i, j}=C_{n} \delta_{i j} f+K_{i j} \widetilde{f}
$$

whenever $f \in C_{c}^{\infty}(U)$.
Now let $f \in L^{p}(U)$, let

$$
\left\|f_{k}-f\right\|_{L^{p}(U)} \rightarrow 0
$$

where $f_{k} \in C_{c}^{\infty}(U)$, and let

$$
w_{i}^{k}(\mathbf{x})=\int_{U} \Phi_{, i}(\mathbf{x}-\mathbf{y}) f_{k}(\mathbf{y}) d y
$$

Then it follows as before that $w_{i}^{k} \rightarrow w_{i}$ in $L^{p}(U)$ and

$$
w_{i, j}^{k}=C_{n} \delta_{i j} f_{k}+K_{i j} \widetilde{f}_{k} .
$$

Now let $\phi \in C_{c}^{\infty}(U)$.

$$
\begin{aligned}
w_{i, j}(\phi) & \equiv-\int_{U} w_{i} \phi_{, j} d x=-\lim _{k \rightarrow \infty} \int_{U} w_{i}^{k} \phi_{, j} d x \\
& =\lim _{k \rightarrow \infty} \int_{U} w_{i, j}^{k} \phi d x=\lim _{k \rightarrow \infty} \int_{U}\left(C_{n} \delta_{i j} \widetilde{f}_{k}+K_{i j} \widetilde{f}_{k}\right) \phi d x \\
& =\int_{U}\left(C_{n} \delta_{i j} \widetilde{f}+K_{i j} \widetilde{f}\right) \phi d x
\end{aligned}
$$

It follows

$$
w_{i, j}=C_{n} \delta_{i j} \tilde{f}+K_{i j} \tilde{f}
$$

and this proves the lemma.
Corollary 33.5.6 In the situation of Theorem 33.5.4, all weak derivatives of $u$ of order 2 are in $L^{p}(U)$ and also $f \rightarrow u_{, i j}$ is a continuous map.

Proof:

$$
u_{, i}(\mathbf{x})=\int_{U} \Phi_{, i}(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d y
$$

and so $u_{, i j} \in L^{p}(U)$ and $f \rightarrow u_{, i j}$ is continuous by Lemma 33.5.5.
With this preparation, it is possible to consider the Helmholtz decomposition. Let $\mathbf{F} \in$ $L^{p}\left(U ; \mathbb{R}^{n}\right)$ and define

$$
\begin{equation*}
\phi(\mathbf{x}) \equiv \int_{U} \nabla \Phi(\mathbf{x}-\mathbf{y}) \cdot \mathbf{F}(\mathbf{y}) d y \tag{33.5.72}
\end{equation*}
$$

Then by Lemma 33.5.5,

$$
\phi_{, j}=C_{n} \widetilde{F}_{j}+\sum_{i} K_{i j} \widetilde{F}_{i} \in L^{p}\left(\mathbb{R}^{n}\right)
$$

and the mapping $\mathbf{F} \rightarrow \nabla \phi$ is continuous from $L^{p}\left(U ; \mathbb{R}^{n}\right)$ to $L^{p}\left(U ; \mathbb{R}^{n}\right)$.
Now suppose $\mathbf{F} \in C_{c}^{\infty}\left(U ; \mathbb{R}^{n}\right)$. Then

$$
\begin{aligned}
\phi(\mathbf{x}) & =\int_{U} \sum_{i=1}^{n}-\frac{\partial}{\partial y^{i}}\left(\Phi(\mathbf{x}-\mathbf{y}) F_{i}(\mathbf{y})\right)+\Phi(\mathbf{x}-\mathbf{y}) \nabla \cdot \mathbf{F}(\mathbf{y}) d y \\
& =\int_{U} \Phi(\mathbf{x}-\mathbf{y}) \nabla \cdot \mathbf{F}(\mathbf{y}) d y
\end{aligned}
$$

and so by Lemma 33.5.3,

$$
\nabla \cdot \nabla \phi=\Delta \phi=-\nabla \cdot \mathbf{F}
$$

This continues to hold in the sense of weak derivatives if $\mathbf{F}$ is only in $L^{p}\left(U ; \mathbb{R}^{n}\right)$ because by Minkowski's inequality and 33.5 .72 the $\operatorname{map} \mathbf{F} \rightarrow \phi$ is continuous. Also note that for $\mathbf{F}$ $\in C_{c}^{\infty}\left(U ; \mathbb{R}^{n}\right)$,

$$
\phi(\mathbf{x})=\int_{B} \Phi(\mathbf{y}) \nabla \cdot \mathbf{F}(\mathbf{x}-\mathbf{y}) d y
$$

Next define $\pi: L^{p}\left(U ; \mathbb{R}^{n}\right) \rightarrow L^{p}\left(U ; \mathbb{R}^{n}\right)$ by

$$
\pi \mathbf{F}=-\nabla \phi, \phi(\mathbf{x})=\int_{U} \nabla \Phi(\mathbf{x}-\mathbf{y}) \cdot \mathbf{F}(\mathbf{y}) d y
$$

It was already shown that $\pi$ is continuous, linear, and $\nabla \cdot \pi \mathbf{F}=\nabla \cdot \mathbf{F}$. It is also true that $\pi$ is a projection. To see this, let $\mathbf{F} \in C_{c}^{\infty}\left(U ; \mathbb{R}^{n}\right)$. Then for $B$ large enough,

$$
\begin{aligned}
\pi^{2} \mathbf{F}(\mathbf{x}) & =-\nabla \int_{B} \Phi(\mathbf{z}) \nabla \cdot \pi \mathbf{F}(\mathbf{x}-\mathbf{z}) d z \\
& =-\nabla \int_{B} \Phi(\mathbf{z}) \nabla \cdot \nabla \int_{B} \Phi(\mathbf{w}) \nabla \cdot \mathbf{F}(\mathbf{x}-\mathbf{z}-\mathbf{w}) d w d z \\
& =-\nabla \int_{B} \Phi(\mathbf{z}) \nabla \cdot \mathbf{F}(\mathbf{x}-\mathbf{z}) d z=\pi \mathbf{F}(\mathbf{x})
\end{aligned}
$$

Since $\pi$ is continuous and $C_{c}^{\infty}\left(U ; \mathbb{R}^{n}\right)$ is dense in $L^{p}\left(U ; \mathbb{R}^{n}\right)$, it follows that $\pi^{2} \mathbf{F}=\pi \mathbf{F}$ for all $\mathbf{F} \in L^{p}\left(U ; \mathbb{R}^{n}\right)$. This proves the following theorem which is the Helmholtz decomposition.

Theorem 33.5.7 There exists a continuous projection

$$
\pi: L^{p}\left(U ; \mathbb{R}^{n}\right) \rightarrow L^{p}\left(U ; \mathbb{R}^{n}\right)
$$

such that $\pi \mathbf{F}$ is a gradient and

$$
\nabla \cdot(\mathbf{F}-\pi \mathbf{F})=0
$$

in the sense of weak derivatives.
Note this theorem shows that any $L^{p}$ vector field is the sum of a gradient and a part which is divergence free. $\mathbf{F}=\mathbf{F}-\pi \mathbf{F}+\pi \mathbf{F}$.

## Chapter 34

## Gelfand Triples And Related Stuff

Let $H$ be a separable real Hilbert space and let $V \subseteq H$ be a separable Banach space which is embedded continuously into $H$ and which is also dense in $H$. Then identifying $H$ and $H^{\prime}$ you can write

$$
V \subseteq H=H^{\prime} \subseteq V^{\prime}
$$

This is called a Gelfand triple. If $V$ is reflexive, you could conclude separability of $V$ from the separability of $H$. However, if $V$ is not reflexive, this might not happen. For example, you could take $V=L^{\infty}(0,1)$ and $H=L^{2}(0,1)$.

Proposition 34.0.1 Suppose $V$ is reflexive and a subset of $H$ a separable Hilbert space with the inclusion map continuous. Suppose also that $V$ is dense in $H$. Then identifying $H$ and $H^{\prime}$, it follows that $H$ is dense in $V^{\prime}$ and $V$ is separable.

Proof: If $H$ is not dense in $V^{\prime}$, then by the Hahn Banach theorem, there exists $\phi^{* *} \in V^{\prime \prime}$ such that $\phi^{* *}(H)=0$ but $\phi^{* *}\left(\phi^{*}\right) \neq 0$ for some $\phi^{*} \in V^{\prime} \backslash \bar{H}$. Since $V$ is reflexive there exists $v \in V$ such that $\phi^{* *}=J v$ for $J$ the standard mapping from $V$ to $V^{\prime \prime}$. Thus

$$
\phi^{* *}(h) \equiv\langle h, v\rangle \equiv(v, h)_{H}=0
$$

for all $h \in H$. Therefore, $v=0$ and so $J v=0=\phi^{* *}$ which contradicts $\phi^{* *}\left(\phi^{*}\right) \neq 0$. Therefore, $H$ is dense in $V^{\prime}$. Now by Theorem 21.1.16 which says separability of the dual space implies separability of the space, it follows $V$ is separable as claimed. This proves the proposition.

From now on, it is assumed $V$ and $V^{\prime}$ are both separable and that $H$ is dense in $V^{\prime}$. This is summarized in the following definition.

Definition 34.0.2 $V, H, V^{\prime}$ will be called a Gelfand triple if $V, V^{\prime}$ are separable, $V \subseteq H$ with the inclusion map continuous, $H=H^{\prime}$, and $H=H^{\prime}$ is dense in $V^{\prime}$.

What about the Borel sets on $V$ and $H$ ?
Proposition 34.0.3 Denote by $\mathscr{B}(X)$ the Borel sets of $X$ where $X$ is any separable Banach space. Then

$$
\mathscr{B}(X)=\sigma\left(X^{\prime}\right)
$$

Here $\sigma\left(X^{\prime}\right)$ is the smallest $\sigma$ algebra such that each $\phi \in X^{\prime}$ is measurable. Also in the context of the above definition, $\mathscr{B}(V)=\sigma\left(i^{*} H^{\prime}\right)$ because $H^{\prime}$ is dense in $V^{\prime}$. Here $i^{*}$ is the restriction to $V$ so that $i^{*} h(v) \equiv h(v) \equiv(h, v)_{H}$ for all $v \in V$ and $\sigma\left(i^{*} H^{\prime}\right)$ denotes the smallest $\sigma$ algebra such that $i^{*} h$ is measurable for each $h \in H^{\prime}$.

Proof: By Lemma 21.1.6 there exists a countable subset of the unit ball in $X^{\prime}$

$$
\left\{\phi_{n}\right\}_{n=1}^{\infty}=D^{\prime}
$$

such that

$$
\|v\|_{X}=\sup \left\{|\phi(v)|: \phi \in D^{\prime}\right\}
$$

Consider a closed ball $\overline{B\left(v_{0}, r\right)}$ in $X$. This equals

$$
\left\{v \in X: \sup _{n}\left|\phi_{n}(v)-\phi_{n}\left(v_{0}\right)\right| \leq r\right\}=\cap_{n=1}^{\infty} \phi_{n}^{-1}\left(\overline{B\left(\phi_{n}\left(v_{0}\right), r\right)}\right)
$$

and this last set is in $\sigma\left(D^{\prime}\right)$. Therefore, every closed ball is in $\sigma\left(D^{\prime}\right)$ which implies every open ball is also in $\sigma\left(D^{\prime}\right)$ since open balls are the countable union of closed balls. Since $X$ is separable, it follows every open set is the countable union of balls and so every open set is in $\sigma\left(D^{\prime}\right)$. It follows $\mathscr{B}(X) \subseteq \sigma\left(D^{\prime}\right) \subseteq \sigma\left(X^{\prime}\right)$. On the other hand, every $\phi \in X^{\prime}$ is continuous and so it is Borel measurable. Hence $\sigma\left(X^{\prime}\right) \subseteq \mathscr{B}(X)$.

Now consider the last claim. From Lemma 21.1.6 and density of $H^{\prime}=H$ in $V^{\prime}$, it can be assumed $D^{\prime} \subseteq H=H^{\prime}$. Therefore, from the first part of the argument

$$
\mathscr{B}(V) \subseteq \sigma\left(D^{\prime}\right) \subseteq \sigma\left(i^{*} H^{\prime}\right)
$$

Also each $i^{*} h$ is continuous on $V$ so in fact, equality holds in the above because $\sigma\left(i^{*} H^{\prime}\right) \subseteq$ $\mathscr{B}(V)$. This proves the proposition.

Next I want to verify that $V$ is in $\mathscr{B}(H)$. This will be true if $V$ is reflexive. More generally, here is an interesting result.

Proposition 34.0.4 Let $X \subseteq Y, X$ dense in $Y$ and suppose $X, Y$ are Banach spaces and that $X$ is reflexive. Then $X \in \mathscr{B}(Y)$.

Proof: Define the functional

$$
\phi(x) \equiv\left\{\begin{array}{l}
\|x\|_{X} \text { if } x \in X \\
\infty \text { if } x \in Y \backslash X
\end{array}\right.
$$

Then $\phi$ is lower semicontinuous on $Y$. Here is why. Suppose $(x, a) \notin \mathrm{epi}(\phi)$ so that $a<$ $\phi(x)$. I need to verify this situation persists for $(x, b)$ near $(x, a)$. If this is not so, there exists $x_{n} \rightarrow x$ and $a_{n} \rightarrow a$ such that $a_{n} \geq \phi\left(x_{n}\right)$. If $\liminf _{n \rightarrow \infty} \phi\left(x_{n}\right)<\infty$, then there exists a subsequence still denoted by $n$ such that $\left\|x_{n}\right\|_{X}$ is bounded. Then by the Eberlein Smulian theorem, there exists a further subsequence such that $x_{n}$ converges weakly in $X$ to some $z$. Now since $X$ is dense in $Y$ it follows $Y^{\prime}$ can be considered a subspace of $X^{\prime}$ and so for $f \in Y^{\prime}$

$$
f\left(x_{n}\right) \rightarrow f(z), f\left(x_{n}\right) \rightarrow f(x)
$$

and so $f(z-x)=0$ for all $f \in Y^{\prime}$ which requires $z=x$. Now $x \rightarrow\|x\|_{X}$ is convex and lower semicontinuous on $X$ so it follows from Corollary 18.2.12

$$
a=\lim \inf _{n \rightarrow \infty} a_{n} \geq \lim \inf _{n \rightarrow \infty} \phi\left(x_{n}\right) \geq \phi(x)>a
$$

which is a contradiction. If $\liminf _{n \rightarrow \infty} \phi\left(x_{n}\right)=\infty$, then

$$
\infty>a=\lim \inf _{n \rightarrow \infty} a_{n}=\infty
$$

another contradiction. Therefore, epi $(\phi)$ is closed and so $\phi$ is lower semicontinuous as claimed. Therefore,

$$
X=Y \backslash\left(\cap_{n=1}^{\infty} \phi^{-1}((n, \infty))\right)
$$

and since $\phi$ is lower semicontinuous, each $\phi^{-1}((n, \infty))$ is open. Hence $X$ is a Borel subset of $Y$. This proves the proposition.

### 34.1 An Unnatural Example

Recall Gelfand triples are of the form

$$
V \subseteq H \subseteq V^{\prime}
$$

where $H$ is a Hilbert space and $V$ is a Banach space contained in $H$ and each of the above inclusions is continuous and each space is dense in the next one. The standard example of a Gelfand triple is $H_{0}^{1}(D) \subseteq L^{2}(D) \subseteq\left(H_{0}^{1}(D)\right)^{\prime}$ with the convention that $L^{2}(D)$ is identified with its dual space. Thus for $f \in L^{2}(D), f$ is considered as something in $\left(H_{0}^{1}(D)\right)^{\prime}$ according to the rule

$$
\langle f, \phi\rangle \equiv(f, \phi)_{L^{2}(D)}
$$

This is a very pleasant thing to contemplate and it is natural and transparent. However, there are other ways to come up with a Gelfand triple which are much more perverse. The following is an example of such a thing along with an application. See [108] and references given there. I think this idea is due to Lions.

First consider the following situation.

$$
X \xrightarrow{\theta} Y
$$

where $\theta$ is continuous, linear and one to one and $X$ is a Banach space. Then $\theta(X) \subseteq Y$ and you could define

$$
\|\theta x\|_{\theta(X)} \equiv\|x\|_{X} .
$$

Then $\theta(X)$ can be considered the same thing as $X$ because $\theta$ preserves distances and all algebraic properties. Thus people write $X \subseteq Y$ to save space. In the above simple example, it is obvious what $\theta$ is. This is because the things in $H_{0}^{1}$ and things in $L^{2}$ are both functions defined on $D$ and we can simply take $\theta$ to be the identity map. However, you might have $H$ be the dual space of something. Thus it consists of bounded linear transformations defined on some Banach space. Then it becomes necessary to specify the manner in which vectors in $V$ can be considered as vectors of $H$.

Let $\infty>p \geq 2$. Then letting $D$ be a bounded open set, $H_{0}^{1}(D)$ embedds continuously into $L^{p^{\prime}}(D)$. That is

$$
\begin{equation*}
\|\phi\|_{L^{p^{\prime}}} \leq C\|\phi\|_{H_{0}^{1}} \tag{34.1.1}
\end{equation*}
$$

Here $\frac{1}{p^{\prime}}+\frac{1}{p}=1$. Also note that an equivalent inner product on $H_{0}^{1}(D)$ is

$$
(f, g)_{H_{0}^{1}} \equiv \int_{D} \nabla f \cdot \nabla g d x
$$

Then with respect to this inner product, the Riesz map is given by $-\Delta$.

$$
-\Delta: H_{0}^{1}(D) \rightarrow\left(H_{0}^{1}(D)\right)^{\prime}
$$

Thus a typical vector of $\left(H_{0}^{1}(D)\right)^{\prime}$ is of the form $-\Delta \phi$ where $\phi \in H_{0}^{1}(D)$ and the following hold.

$$
(\phi, \psi)_{H_{0}^{1}} \equiv\langle-\Delta \phi, \psi\rangle,(-\Delta \phi,-\Delta \psi)_{\left(H_{0}^{1}\right)^{\prime}} \equiv(\phi, \psi)_{H_{0}^{1}}=\langle-\Delta \psi, \phi\rangle
$$

The following is about the Gelfand triple

$$
V=L^{p}(D) \subseteq\left(H_{0}^{1}\right)^{\prime} \subseteq\left(L^{p}(D)\right)^{\prime}
$$

Lemma 34.1.1 It is possible to consider $L^{p}(D) \equiv V$ as a dense subspace of $\left(H_{0}^{1}\right)^{\prime} \equiv H$ as follows. For $f \in L^{p}(D)$ and $\phi \in H_{0}^{1}(D)$,

$$
\langle f, \phi\rangle \equiv \int_{D} f(x) \phi(x) d x
$$

One can also consider $H \equiv\left(H_{0}^{1}\right)^{\prime}$ as a dense subspace of $\left(L^{p}(D)\right)^{\prime} \equiv V^{\prime}$ as follows. For $-\Delta \phi \in H$ and $f \in L^{p}(D)$,

$$
\langle-\Delta \phi, f\rangle \equiv(-\Delta \phi, f)_{H} \equiv\langle f, \phi\rangle
$$

$-\Delta$ maps $H_{0}^{1}(D)$ to $H \equiv\left(H_{0}^{1}\right)^{\prime} \subseteq V^{\prime} .-\Delta$ can be extended to yield a map $-\Delta_{1}$ from $L^{p^{\prime}}(D)$ to $V^{\prime}$.

$$
\begin{aligned}
& H_{0}^{1}(D) \xrightarrow{-\Delta}\left(H_{0}^{1}\right)^{\prime} \\
& L^{p^{\prime}}(D)=V \xrightarrow{-\Delta_{1}} V^{\prime}
\end{aligned}
$$

Proof: First of all, note that by 34.1.1

$$
|\langle f, \phi\rangle| \leq\|f\|_{L^{p}}\|\phi\|_{L^{p^{\prime}}} \leq C\|f\|_{L^{p}}\|\phi\|_{H_{0}^{1}}
$$

and so it is certainly possible to consider $L^{p} \subseteq H \equiv\left(H_{0}^{1}\right)^{\prime}$ as just claimed. Now why can $L^{p}(D)$ be considered dense in $H \equiv\left(H_{0}^{1}\right)^{\prime}$ ? If it isn't dense, then there exists $\psi \in$ $H_{0}^{1}(D), \psi \neq 0$ such that

$$
(-\Delta \psi, f)_{H}=0
$$

for all $f \in L^{p}(D)$. However, the above would say that for all $f \in L^{p}$,

$$
(-\Delta \psi, f)_{H} \equiv\langle f, \psi\rangle \equiv \int_{D} f \psi=0
$$

But $\psi \in L^{p^{\prime}}(D)$ because $H_{0}^{1}(D)$ embedds continuously into $L^{p^{\prime}}(D)$ and so the above holding for all $f \in L^{p}(D)$ implies by the usual Riesz representation theorem that $\psi=0$ contrary to the way $\psi$ was chosen.

Now consider the next claim. For $-\Delta \phi \in H \equiv\left(H_{0}^{1}\right)^{\prime}$ and $f \in L^{p}(D)$ and from the first part

$$
|\langle-\Delta \phi, f\rangle| \equiv\left|(-\Delta \phi, f)_{H}\right| \equiv|\langle f, \phi\rangle| \leq C\|f\|_{L^{p}}\|\phi\|_{H_{0}^{1}(D)}
$$

Thus $-\Delta \phi \in H$ can be considered in $\left(L^{p}(D)\right)^{\prime}$. Why should $H$ be dense in $\left(L^{p}(D)\right)^{\prime}$ ? If it is not dense, then there exists $g^{*} \in\left(L^{p}(D)\right)^{\prime}$ which is not the limit of vectors of $H$. Then
since $L^{p}(D)$ is reflexive, an application of the Hahn Banach theorem shows there exists $f \in L^{p}(D)$ such that

$$
\begin{equation*}
\left\langle g^{*}, f\right\rangle_{\left(L^{p}(D)\right)^{\prime}, L^{p}(D)} \neq 0,\langle-\Delta \phi, f\rangle_{\left(L^{p}(D)\right)^{\prime}, L^{p}(D)}=0 \tag{34.1.2}
\end{equation*}
$$

for all $-\Delta \phi \in H$. However, it was just shown $H$ could be considered a subset of $\left(L^{p}(D)\right)^{\prime}$ in the manner described above. Therefore, the last equation in the above is of the form

$$
0=(-\Delta \phi, f)_{H}=\langle f, \phi\rangle=\int_{D} f \phi d x
$$

and since this holds for all $\phi \in H_{0}^{1}(D)$, it follows by density of $H_{0}^{1}(D)$ in $L^{p^{\prime}}(D)$, that $f=0$ and now this contradicts the inequality in 34.1.2.

Now $\Delta$ is defined on $H_{0}^{1}(D)$ and it delivers something in $\left(H_{0}^{1}\right)^{\prime} \equiv H$. Of course $H_{0}^{1}(D)$ is dense in $L^{p^{\prime}}(D)$. Can $\Delta$ be extended to all of $L^{p^{\prime}}(D)$ ? The answer is yes and it is more of the same given above. For $\phi \in H_{0}^{1}(D),-\Delta \phi \in H \subseteq\left(L^{p}(D)\right)^{\prime}$. Then by the above, for $\phi \in H_{0}^{1}(D)$ and $f \in L^{p}(D)$,

$$
\begin{gathered}
\langle-\Delta \phi, f\rangle \equiv\langle f, \phi\rangle \equiv \int_{D} f \phi d x \\
|\langle-\Delta \phi, f\rangle| \equiv|\langle f, \phi\rangle| \equiv\left|\int_{D} f \phi d s\right| \leq\|\phi\|_{L^{p^{\prime}(D)}}\|f\|_{L^{p}(D)}
\end{gathered}
$$

and so $-\Delta$ is a continuous linear mapping defined on a dense subspace $H_{0}^{1}(D)$ of $L^{p^{\prime}}(D)$ and so this does indeed extend to a continuous linear map defined on all of $L^{p^{\prime}}(D)$ given by the formula

$$
\langle-\Delta g, f\rangle \equiv \int_{D} f g d x
$$

This proves the lemma.
Thus letting $V \equiv L^{p}(D)$, and $H \equiv\left(H_{0}^{1}(D)\right)^{\prime}$, it follows $V \subseteq H \subseteq V^{\prime}$ is a Gelfand triple with the understanding of what it means for one space to be included in another described above. To emphasize the above, for $-\Delta \phi \in H, f \in L^{p}$,

$$
\langle-\Delta \phi, f\rangle \equiv(-\Delta \phi, f)_{H} \equiv\langle f, \phi\rangle \equiv \int_{D} f \phi d x
$$

More generally, for $g \in L^{p^{\prime}}(D),-\Delta g \in\left(L^{p}(D)\right)^{\prime}$ according to the rule

$$
\langle-\Delta g, f\rangle \equiv \int_{D} f g d x
$$

With this example of a Gelfand triple, one can define a "porous medium operator" $A: V \rightarrow V^{\prime}$. Let $\Psi$ be a real valued function defined on $\mathbb{R}$ which satisfies

$$
\begin{equation*}
\Psi \text { is continuous } \tag{34.1.3}
\end{equation*}
$$

$$
\begin{equation*}
(t-s)(\Psi(t)-\Psi(s)) \geq 0 \tag{34.1.4}
\end{equation*}
$$

There exists $p \geq 2, p<\infty$ and $\alpha \in(0, \infty)$ such that for all $s \in \mathbb{R}$

$$
\begin{equation*}
s \Psi(s) \geq \alpha|s|^{p}-c \tag{34.1.5}
\end{equation*}
$$

There exist $c_{3}, c_{4} \in(0, \infty)$ such that for all $s \in \mathbb{R}$

$$
\begin{equation*}
|\Psi(s)| \leq c_{4}+c_{3}|s|^{p-1} \tag{34.1.6}
\end{equation*}
$$

Note that 34.1.6 implies that if $v \in L^{p}(D)$, Then

$$
\int_{D}|\Psi(v)|^{p^{\prime}} d x \leq C \int_{D}\left(1+|v|^{p^{\prime}(p-1)}\right) d x=C \int_{D}\left(1+|v|^{p}\right) d x<\infty
$$

Thus for $v \in L^{p}(D), \Psi(v)$ is something you can do $\Delta$ to and obtain something in $V^{\prime}$. The porous medium operator $A: V \rightarrow V^{\prime}$ is given as follows.

$$
\langle A v, w\rangle_{V^{\prime}, V} \equiv\langle\Delta \Psi(v), w\rangle_{V^{\prime}, V} \equiv-\int_{D} \Psi(v) w d x
$$

What are the properties of $A$ ?

$$
\langle A(u+\lambda v), w\rangle \equiv-\int_{D} \Psi(u+\lambda v) w d x
$$

and this is easily seen to be a continuous function of $\lambda$ Thus $A$ is Hemicontinuous.

$$
\langle A(u)-A(v), u-v\rangle \equiv-\int_{D} \Psi(u)(u-v) d x+\int_{D} \Psi(v)(u-v) d x \leq 0
$$

Thus $-A$ is monotone. Also there is a coercivity estimate which is routine.

$$
\langle A(v), v\rangle \equiv-\int_{D} \Psi(v) v \leq \int_{D} c-\alpha|v|^{p} d x=C-\alpha\|v\|_{V}^{p}
$$

This operator also has a boundedness estimate.

$$
\begin{aligned}
& \|A(v)\|_{V^{\prime}} \equiv \sup _{\|w\|_{V} \leq 1}|\langle A(v), w\rangle| \equiv \sup _{\|w\|_{V} \leq 1}\left|\int_{D} \Psi(v) w\right| \\
& \leq \sup _{\|w\|_{V} \leq 1}\left(\int_{D}\left(c_{4}+c_{3}|v|^{p-1}\right) w d x\right) \\
& \leq\left(\int_{D} C\left(1+|v|^{p}\right) d x\right)^{1 / p^{\prime}} \leq C+C\left(\int_{D}|v|^{p} d x\right)^{1 / p^{\prime}} \\
& =C+C\|v\|_{V}^{p / p^{\prime}}=C+C\|v\|_{V}^{p-1} .
\end{aligned}
$$

Since $\Psi$ is continuous, it will also follow that $A$ is $\mathscr{B}(V)$ measurable. Consider

$$
u \rightarrow\langle A u, w\rangle \equiv-\int_{D} \Psi(u) w d x
$$

for fixed $w \in V$. Suppose $u_{n} \rightarrow u$ in $V$ and fix $w \in L^{\infty}(D) \subseteq V$. Then it follows from an easy argument using the Vitali convergence theorem and the fact that from the estimates above

$$
\Psi\left(u_{n}\right) w
$$

is uniformly integrable that

$$
u \rightarrow-\int_{D} \Psi(u) w d x
$$

is continuous. For general $w \in L^{p}(D)$, let $w_{n} \rightarrow w$ in $L^{p}(D)$ where each $w_{n}$ is in $L^{\infty}(D)$. Then the function

$$
\begin{equation*}
u \rightarrow-\int_{D} \Psi(u) w d x \equiv\langle A u, w\rangle \tag{34.1.7}
\end{equation*}
$$

is the limit of the continuous functions

$$
u \rightarrow-\int_{D} \Psi(u) w_{n} d x
$$

and so the function 34.1.7 is Borel measurable. Now by the Pettis theorem this shows $A: V \rightarrow V^{\prime}$ is $\mathscr{B}(V)$ measurable. This shows $A$ is an example of an operator which satisfies some conditions which will be considered later.

### 34.2 Standard Techniques In Evolution Equations

In this section, several significant theorems are presented. Unless indicated otherwise, the measure will be Lebesgue measure. First here is a lemma.

Lemma 34.2.1 Suppose $g \in L^{1}([a, b] ; X)$ where $X$ is a Banach space. Then if

$$
\int_{a}^{b} g(t) \phi(t) d t=0
$$

for all $\phi \in C_{c}^{\infty}(a, b)$, then $g(t)=0$ a.e.
Proof: Let $S$ be a measurable subset of $(a, b)$ and let $K \subseteq S \subseteq V \subseteq(a, b)$ where $K$ is compact, $V$ is open and $m(V \backslash K)<\varepsilon$. Let $K \prec h \prec V$ as in the proof of the Riesz representation theorem for positive linear functionals. Enlarging $K$ slightly and convolving with a mollifier, it can be assumed $h \in C_{c}^{\infty}(a, b)$. Then

$$
\begin{aligned}
\left|\int_{a}^{b} \mathscr{X}_{S}(t) g(t) d t\right| & =\left|\int_{a}^{b}\left(\mathscr{X}_{S}(t)-h(t)\right) g(t) d t\right| \\
& \leq \int_{a}^{b}\left|\mathscr{X}_{S}(t)-h(t)\right|\|g(t)\| d t \\
& \leq \int_{V \backslash K}\|g(t)\| d t
\end{aligned}
$$

Now let $K_{n} \subseteq S \subseteq V_{n}$ with $m\left(V_{n} \backslash K_{n}\right)<2^{-n}$. Then from the above,

$$
\left|\int_{a}^{b} \mathscr{X}_{S}(t) g(t) d t\right| \leq \int_{a}^{b} \mathscr{X}_{V_{n} \backslash K_{n}}(t)\|g(t)\| d t
$$

and the integrand of the last integral converges to 0 a.e. as $n \rightarrow \infty$ because $\sum_{n} m\left(V_{n} \backslash K_{n}\right)<$ $\infty$. By the dominated convergence theorem, this last integral converges to 0 . Therefore, whenever $S \subseteq(a, b)$,

$$
\int_{a}^{b} \mathscr{X}_{S}(t) g(t) d t=0
$$

Since the endpoints have measure zero, it also follows that for any measurable $S$, the above equation holds.

Now $g \in L^{1}([a, b] ; X)$ and so it is measurable. Therefore, $g([a, b])$ is separable. Let $D$ be a countable dense subset and let $E$ denote the set of linear combinations of the form $\sum_{i} a_{i} d_{i}$ where $a_{i}$ is a rational point of $\mathbb{F}$ and $d_{i} \in D$. Thus $E$ is countable. Denote by $Y$ the closure of $E$ in $X$. Thus $Y$ is a separable closed subspace of $X$ which contains all the values of $g$.

Now let $S_{n} \equiv g^{-1}\left(B\left(y_{n},\left\|y_{n}\right\| / 2\right)\right)$ where $E=\left\{y_{n}\right\}_{n=1}^{\infty}$. Then $\cup_{n} S_{n}=g^{-1}(X \backslash\{0\})$. This follows because if $x \in Y$ and $x \neq 0$, then in $B\left(x, \frac{\|x\|}{4}\right)$ there is a point of $E, y_{n}$. Therefore, $\left\|y_{n}\right\|>\frac{3}{4}\|x\|$ and so $\frac{\left\|y_{n}\right\|}{2}>\frac{3\|x\|}{8}>\frac{\|x\|}{4}$ so $x \in B\left(y_{n},\left\|y_{n}\right\| / 2\right)$. It follows that if each $S_{n}$ has measure zero, then $g(t)=0$ for a.e. $t$. Suppose then that for some $n$, the set, $S_{n}$ has positive measure. Then from what was shown above,

$$
\begin{aligned}
\left\|y_{n}\right\| & =\left\|\frac{1}{m\left(S_{n}\right)} \int_{S_{n}} g(t) d t-y_{n}\right\|=\left\|\frac{1}{m\left(S_{n}\right)} \int_{S_{n}} g(t)-y_{n} d t\right\| \\
& \leq \frac{1}{m\left(S_{n}\right)} \int_{S_{n}}\left\|g(t)-y_{n}\right\| d t \leq \frac{1}{m\left(S_{n}\right)} \int_{S_{n}}\left\|y_{n}\right\| / 2 d t=\left\|y_{n}\right\| / 2
\end{aligned}
$$

and so $y_{n}=0$ which implies $S_{n}=\emptyset$, a contradiction to $m\left(S_{n}\right)>0$. This contradiction shows each $S_{n}$ has measure zero and so as just explained, $g(t)=0$ a.e.

Definition 34.2.2 For $f \in L^{1}(a, b ; X)$, define an extension, $\bar{f}$ defined on

$$
[2 a-b, 2 b-a]=[a-(b-a), b+(b-a)]
$$

as follows.

$$
\bar{f}(t) \equiv\left\{\begin{array}{l}
f(t) \text { if } t \in[a, b] \\
f(2 a-t) \text { if } t \in[2 a-b, a] \\
f(2 b-t) \text { if } t \in[b, 2 b-a]
\end{array}\right.
$$

Definition 34.2.3 Also if $f \in L^{p}(a, b ; X)$ and $h>0$, define for $t \in[a, b], f_{h}(t) \equiv \bar{f}(t-h)$ for all $h<b-a$. Thus the map $f \rightarrow f_{h}$ is continuous and linear on $L^{p}(a, b ; X)$. It is continuous because

$$
\begin{aligned}
\int_{a}^{b}\left\|f_{h}(t)\right\|^{p} d t & =\int_{a}^{a+h}\|f(2 a-t+h)\|^{p} d t+\int_{a}^{b-h}\|f(t)\|^{p} d t \\
& =\int_{a}^{a+h}\|f(t)\|^{p} d t+\int_{a}^{b-h}\|f(t)\|^{p} d t \leq 2\|f\|_{p}^{p}
\end{aligned}
$$

The following lemma is on continuity of translation in $L^{p}(a, b ; X)$.

Lemma 34.2.4 Let $\bar{f}$ be as defined in Definition 34.2.2. Then for $f \in L^{p}(a, b ; X)$ for $p \in$ $[1, \infty)$,

$$
\lim _{\delta \rightarrow 0} \int_{a}^{b}\|\bar{f}(t-\delta)-f(t)\|_{X}^{p} d t=0
$$

Proof: Regarding the measure space as $(a, b)$ with Lebesgue measure, by regularity of the measure, there exists $g \in C_{c}(a, b ; X)$ such that $\|f-g\|_{p}<\varepsilon$. Here the norm is the norm in $L^{p}(a, b ; X)$. Therefore,

$$
\begin{aligned}
\left\|f_{h}-f\right\|_{p} & \leq\left\|f_{h}-g_{h}\right\|_{p}+\left\|g_{h}-g\right\|_{p}+\|g-f\|_{p} \\
& \leq\left(2^{1 / p}+1\right)\|f-g\|_{p}+\left\|g_{h}-g\right\|_{p} \\
& <\left(2^{1 / p}+1\right) \varepsilon+\varepsilon
\end{aligned}
$$

whenever $h$ is sufficiently small. This is because of the uniform continuity of $g$. Therefore, since $\varepsilon>0$ is arbitrary, this proves the lemma.

Definition 34.2.5 Let $f \in L^{1}(a, b ; X)$. Then the distributional derivative in the sense of $X$ valued distributions is given by

$$
f^{\prime}(\phi) \equiv-\int_{a}^{b} f(t) \phi^{\prime}(t) d t
$$

Then $f^{\prime} \in L^{1}(a, b ; X)$ if there exists $h \in L^{1}(a, b ; X)$ such that for all $\phi \in C_{c}^{\infty}(a, b)$,

$$
f^{\prime}(\phi)=\int_{a}^{b} h(t) \phi(t) d t
$$

Then $f^{\prime}$ is defined to equal $h$. Here $f$ and $f^{\prime}$ are considered as vector valued distributions in the same way as was done for scalar valued functions.

Lemma 34.2.6 The above definition is well defined.
Proof: Suppose both $h$ and $g$ work in the definition for $f^{\prime}$. Then for all $\phi \in C_{c}^{\infty}(a, b)$,

$$
\int_{a}^{b}(h(t)-g(t)) \phi(t) d t=0 .
$$

Therefore, by Lemma 34.2.1, $h(t)-g(t)=0$ a.e.
The other thing to notice about this is the following lemma. It follows immediately from the definition.

Lemma 34.2.7 Suppose $f, f^{\prime} \in L^{1}(a, b ; X)$. Then if $[c, d] \subseteq[a, b]$, it follows that $\left(\left.f\right|_{[c, d]}\right)^{\prime}=$ $\left.f^{\prime}\right|_{[c, d]}$. This notation means the restriction to $[c, d]$.

Recall that in the case of scalar valued functions, if you had both $f$ and its weak derivative, $f^{\prime}$ in $L^{1}(a, b)$, then you were able to conclude that $f$ is almost everywhere equal to a continuous function, still denoted by $f$ and

$$
f(t)=f(a)+\int_{a}^{t} f^{\prime}(s) d s
$$

In particular, you can define $f(a)$ to be the initial value of this continuous function. It turns out that an identical theorem holds in this case. To begin with here is the same sort of lemma which was used earlier for the case of scalar valued functions. It says that if $f^{\prime}=0$ where the derivative is taken in the sense of $X$ valued distributions, then $f$ equals a constant.

Lemma 34.2.8 Suppose $f \in L^{1}(a, b ; X)$ and for all $\phi \in C_{c}^{\infty}(a, b)$,

$$
\int_{a}^{b} f(t) \phi^{\prime}(t) d t=0
$$

Then there exists a constant, $a \in X$ such that $f(t)=a$ a.e.
Proof: Let $\phi_{0} \in C_{c}^{\infty}(a, b), \int_{a}^{b} \phi_{0}(x) d x=1$ and define for $\phi \in C_{c}^{\infty}(a, b)$

$$
\psi_{\phi}(x) \equiv \int_{a}^{x}\left[\phi(t)-\left(\int_{a}^{b} \phi(y) d y\right) \phi_{0}(t)\right] d t
$$

Then $\psi_{\phi} \in C_{c}^{\infty}(a, b)$ and $\psi_{\phi}^{\prime}=\phi-\left(\int_{a}^{b} \phi(y) d y\right) \phi_{0}$. Then

$$
\begin{aligned}
\int_{a}^{b} f(t)(\phi(t)) d t & =\int_{a}^{b} f(t)\left(\psi_{\phi}^{\prime}(t)+\left(\int_{a}^{b} \phi(y) d y\right) \phi_{0}(t)\right) d t \\
& =\overbrace{\int_{a}^{b} f(t) \psi_{\phi}^{\prime}(t) d t}^{=0 \text { by assumption }}+\left(\int_{a}^{b} \phi(y) d y\right) \int_{a}^{b} f(t) \phi_{0}(t) d t \\
& =\left(\int_{a}^{b}\left(\int_{a}^{b} f(t) \phi_{0}(t) d t\right) \phi(y) d y\right) .
\end{aligned}
$$

It follows that for all $\phi \in C_{c}^{\infty}(a, b)$,

$$
\int_{a}^{b}\left(f(y)-\left(\int_{a}^{b} f(t) \phi_{0}(t) d t\right)\right) \phi(y) d y=0
$$

and so by Lemma 34.2.1,

$$
f(y)-\left(\int_{a}^{b} f(t) \phi_{0}(t) d t\right)=0 \text { a.e. } y
$$

Theorem 34.2.9 Suppose $f, f^{\prime}$ both are in $L^{1}(a, b ; X)$ where the derivative is taken in the sense of $X$ valued distributions. Then there exists a unique point of $X$, denoted by $f(a)$ such that the following formula holds a.e. $t$.

$$
f(t)=f(a)+\int_{a}^{t} f^{\prime}(s) d s
$$

## Proof:

$$
\int_{a}^{b}\left(f(t)-\int_{a}^{t} f^{\prime}(s) d s\right) \phi^{\prime}(t) d t=\int_{a}^{b} f(t) \phi^{\prime}(t) d t-\int_{a}^{b} \int_{a}^{t} f^{\prime}(s) \phi^{\prime}(t) d s d t
$$

Now consider $\int_{a}^{b} \int_{a}^{t} f^{\prime}(s) \phi^{\prime}(t) d s d t$. Let $\Lambda \in X^{\prime}$. Then it is routine from approximating $f^{\prime}$ with simple functions to verify

$$
\Lambda\left(\int_{a}^{b} \int_{a}^{t} f^{\prime}(s) \phi^{\prime}(t) d s d t\right)=\int_{a}^{b} \int_{a}^{t} \Lambda\left(f^{\prime}(s)\right) \phi^{\prime}(t) d s d t
$$

Now the ordinary Fubini theorem can be applied to obtain

$$
=\int_{a}^{b} \int_{s}^{b} \Lambda\left(f^{\prime}(s)\right) \phi^{\prime}(t) d t d s=\Lambda\left(\int_{a}^{b} \int_{s}^{b} f^{\prime}(s) \phi^{\prime}(t) d t d s\right)
$$

Since $X^{\prime}$ separates the points of $X$, it follows

$$
\int_{a}^{b} \int_{a}^{t} f^{\prime}(s) \phi^{\prime}(t) d s d t=\int_{a}^{b} \int_{s}^{b} f^{\prime}(s) \phi^{\prime}(t) d t d s
$$

Therefore,

$$
\begin{aligned}
& \int_{a}^{b}\left(f(t)-\int_{a}^{t} f^{\prime}(s) d s\right) \phi^{\prime}(t) d t \\
= & \int_{a}^{b} f(t) \phi^{\prime}(t) d t-\int_{a}^{b} \int_{s}^{b} f^{\prime}(s) \phi^{\prime}(t) d t d s \\
= & \int_{a}^{b} f(t) \phi^{\prime}(t) d t-\int_{a}^{b} f^{\prime}(s) \int_{s}^{b} \phi^{\prime}(t) d t d s \\
= & \int_{a}^{b} f(t) \phi^{\prime}(t) d t+\int_{a}^{b} f^{\prime}(s) \phi(s) d s=0 .
\end{aligned}
$$

Therefore, by Lemma 34.2.8, there exists a constant, denoted as $f(a)$ such that

$$
f(t)-\int_{a}^{t} f^{\prime}(s) d s=f(a)
$$

There is also a useful theorem about continuity of pointwise evaluation.
Corollary 34.2.10 Let $f, f^{\prime} \in L^{1}(a, b ; X)$ so that

$$
\begin{equation*}
f(t)=f(0)+\int_{0}^{t} f^{\prime}(s) d s \tag{34.2.8}
\end{equation*}
$$

where in this formula, $t \rightarrow f(t)$ is the continuous representative of $f$. Then there exists $a$ constant $C$ such that for each $t \in[a, b]$,

$$
\|f(t)\|_{X} \leq C\left(\|f\|_{L^{1}(a, b ; X)}+\left\|f^{\prime}\right\|_{L^{1}(a, b ; X)}\right)
$$

Proof: From the integral equation 34.2.8,

$$
\begin{gathered}
f(t)=f(s)+\int_{s}^{t} f^{\prime}(r) d r \\
\|f(t)\|_{X} \leq\|f(s)\|_{X}+\left|\int_{s}^{t}\left\|f^{\prime}(r)\right\|_{X} d r\right| \\
\leq\|f(s)\|_{X}+\int_{a}^{b}\left\|f^{\prime}(r)\right\|_{X} d r
\end{gathered}
$$

and so, integrating both sides with respect to $s$

$$
(b-a)\|f(t)\|_{X} \leq\|f\|_{L^{1}(a, b ; X)}+(b-a)\left\|f^{\prime}\right\|_{L^{1}(a, b ; X)}
$$

and so

$$
\|f(t)\|_{X} \leq\left(\frac{1}{b-a}+1\right)\left(\|f\|_{L^{1}(a, b ; X)}+\left\|f^{\prime}\right\|_{L^{1}(a, b ; X)}\right)
$$

Let $\mathfrak{X}$ be the space of functions $f \in L^{1}(a, b ; X)$ such that their weak derivatives $f^{\prime}$ are also in $L^{1}(a, b ; X)$. Then $\mathfrak{X}$ is a Banach space with norm given by

$$
\|f\|_{\mathfrak{X}} \equiv\|f\|_{L^{1}(a, b ; X)}+\left\|f^{\prime}\right\|_{L^{1}(a, b ; X)}
$$

This is because the map $f \rightarrow f^{\prime}$ is a closed map. If $f_{n} \rightarrow f$ in $L^{1}(a, b ; X)$ and $f_{n}^{\prime} \rightarrow \xi$ in $L^{1}(a, b ; X)$, then for $\phi \in C_{c}^{\infty}(a, b)$,

$$
\int_{a}^{b} \xi \phi d t=\lim _{n \rightarrow \infty} \int_{a}^{b} f_{n}^{\prime} \phi d t=\lim _{n \rightarrow \infty}-\int_{a}^{b} f_{n} \phi^{\prime} d t=-\int_{a}^{b} f \phi^{\prime} d t
$$

showing that $\xi=f^{\prime}$. Thus if you have a Cauchy sequence in $\mathfrak{X},\left\{f_{n}\right\}$, then $f_{n} \rightarrow f$ in $L^{1}(a, b ; X)$ and $f_{n}^{\prime} \rightarrow \xi$ in $L^{1}(a, b ; X)$ for some $\xi$. Hence $f^{\prime}=\xi$.

Then the above corollary says that pointwise evaluation is continuous as a map from $\mathfrak{X}$ to $X$. This is clearly a linear map. Also the formula obtained shows that in fact, this is continuous into $C([a, b] ; X)$.

$$
\|f\|_{C([a, b] ; X)}=\sup _{t \in[a, b]}\|f(t)\|_{X} \leq C\left(\|f\|_{L^{1}(a, b ; X)}+\left\|f^{\prime}\right\|_{L^{1}(a, b ; X)}\right)=C\|f\|_{\mathfrak{X}}
$$

Now let $\theta: \mathfrak{X} \rightarrow C([a, b] ; X)$ be given by $\theta f(t) \equiv f(t)$ where $f(t)=f(0)+\int_{0}^{t} f^{\prime}(s) d s$, $f$ being the continuous representative of $f$. Then $\theta$ is continuous and linear. If $\theta_{t} f \equiv f(t)$ so that it is pointwise evaluation at $t$, then this $\theta_{t}$ is also continuous and linear. Suppose $X$ is also reflexive. It follows that if you have a sequence in $X\left\{f_{n}\right\}$ which is converging weakly to $f \in \mathfrak{X}$, then you would also have $\theta_{t} f_{n}=f_{n}(t) \rightarrow \theta_{t} f \equiv f(t)$ weakly in $X$. If this is not so, then since $X$ is reflexive, there is a subsequence, still denoted as $f_{n}$ such that $f_{n}(t) \rightarrow \xi \neq f(t)$. However, this says that $(f, \boldsymbol{\xi})$ is in the weak closure of the graph of $\theta_{t}$. Since this graph is strongly closed and convex, it is also weakly closed and hence $\xi=\theta_{t} f \equiv f(t)$, a contradiction. This proves the following nice corollary.

Corollary 34.2.11 Suppose $f_{n} \rightarrow f$ weakly in $\mathfrak{X}$ where we assume also that $X$ is reflexive. Then $f_{n}(t) \rightarrow f(t)$ weakly in $X$.

The integration by parts formula is also important.

Corollary 34.2.12 Suppose $f, f^{\prime} \in L^{1}(a, b ; X)$ and suppose $\phi \in C^{1}([a, b])$. Then the following integration by parts formula holds.

$$
\int_{a}^{b} f(t) \phi^{\prime}(t) d t=f(b) \phi(b)-f(a) \phi(a)-\int_{a}^{b} f^{\prime}(t) \phi(t) d t
$$

Proof: From Theorem 34.2.9

$$
\begin{aligned}
& \int_{a}^{b} f(t) \phi^{\prime}(t) d t \\
= & \int_{a}^{b}\left(f(a)+\int_{a}^{t} f^{\prime}(s) d s\right) \phi^{\prime}(t) d t \\
= & f(a)(\phi(b)-\phi(a))+\int_{a}^{b} \int_{a}^{t} f^{\prime}(s) d s \phi^{\prime}(t) d t \\
= & f(a)(\phi(b)-\phi(a))+\int_{a}^{b} f^{\prime}(s) \int_{s}^{b} \phi^{\prime}(t) d t d s \\
= & f(a)(\phi(b)-\phi(a))+\int_{a}^{b} f^{\prime}(s)(\phi(b)-\phi(s)) d s \\
= & f(a)(\phi(b)-\phi(a))-\int_{a}^{b} f^{\prime}(s) \phi(s) d s+(f(b)-f(a)) \phi(b) \\
= & f(b) \phi(b)-f(a) \phi(a)-\int_{a}^{b} f^{\prime}(s) \phi(s) d s .
\end{aligned}
$$

The interchange in order of integration is justified as in the proof of Theorem 34.2.9.
With this integration by parts formula, the following interesting lemma is obtained. This lemma shows why it was appropriate to define $\bar{f}$ as in Definition 34.2.2.

Lemma 34.2.13 Let $\bar{f}$ be given in Definition 34.2.2 and suppose $f, f^{\prime} \in L^{1}(a, b ; X)$. Then $\bar{f}, \bar{f}^{\prime} \in L^{1}(2 a-b, 2 b-a ; X)$ also and

$$
\bar{f}^{\prime}(t) \equiv\left\{\begin{array}{l}
f^{\prime}(t) \text { if } t \in[a, b]  \tag{34.2.9}\\
-f^{\prime}(2 a-t) \text { if } t \in[2 a-b, a] \\
-f^{\prime}(2 b-t) \text { if } t \in[b, 2 b-a]
\end{array}\right.
$$

Proof: It is clear from the definition of $\bar{f}$ that $\bar{f} \in L^{1}(2 a-b, 2 b-a ; X)$ and that in fact

$$
\begin{equation*}
\|\bar{f}\|_{L^{1}(2 a-b, 2 b-a ; X)} \leq 3\|f\|_{L^{1}(a, b ; X)} \tag{34.2.10}
\end{equation*}
$$

Let $\phi \in C_{c}^{\infty}(2 a-b, 2 b-a)$. Then from the integration by parts formula,

$$
\begin{aligned}
& \int_{2 a-b}^{2 b-a} \bar{f}(t) \phi^{\prime}(t) d t \\
= & \int_{a}^{b} f(t) \phi^{\prime}(t) d t+\int_{b}^{2 b-a} f(2 b-t) \phi^{\prime}(t) d t+\int_{2 a-b}^{a} f(2 a-t) \phi^{\prime}(t) d t \\
= & \int_{a}^{b} f(t) \phi^{\prime}(t) d t+\int_{a}^{b} f(u) \phi^{\prime}(2 b-u) d u+\int_{a}^{b} f(u) \phi^{\prime}(2 a-u) d u \\
= & f(b) \phi(b)-f(a) \phi(a)-\int_{a}^{b} f^{\prime}(t) \phi(t) d t-f(b) \phi(b)+f(a) \phi(2 b-a) \\
& +\int_{a}^{b} f^{\prime}(u) \phi(2 b-u) d u-f(b) \phi(2 a-b) \\
& +f(a) \phi(a)+\int_{a}^{b} f^{\prime}(u) \phi(2 a-u) d u \\
= & -\int_{a}^{b} f^{\prime}(t) \phi(t) d t+\int_{a}^{b} f^{\prime}(u) \phi(2 b-u) d u+\int_{a}^{b} f^{\prime}(u) \phi(2 a-u) d u \\
= & -\int_{a}^{b} f^{\prime}(t) \phi(t) d t-\int_{b}^{2 b-a}-f^{\prime}(2 b-t) \phi(t) d t-\int_{2 a-b}^{a}-f^{\prime}(2 a-t) \phi(t) d t \\
= & -\int_{2 a-b}^{2 b-a} \bar{f}^{\prime}(t) \phi(t) d t
\end{aligned}
$$

where $\bar{f}^{\prime}(t)$ is given in 34.2.9.
Definition 34.2.14 Let $V$ be a Banach space and let $H$ be a Hilbert space. (Typically $H=L^{2}(\Omega)$ ) Suppose $V \subseteq H$ is dense in $H$ meaning that the closure in $H$ of $V$ gives $H$. Then it is often the case that $H$ is identified with its dual space, and then because of the density of $V$ in $H$, it is possible to write

$$
V \subseteq H=H^{\prime} \subseteq V^{\prime}
$$

When this is done, $H$ is called a pivot space. Another notation which is often used is $\langle f, g\rangle$ to denote $f(g)$ for $f \in V^{\prime}$ and $g \in V$. This may also be written as $\langle f, g\rangle_{V^{\prime}, V}$. Another term is that $V \subseteq H=H^{\prime} \subseteq V^{\prime}$ is called a Gelfand triple.

The next theorem is an example of a trace theorem. In this theorem, $f \in L^{p}(0, T ; V)$ while $f^{\prime} \in L^{p}\left(0, T ; V^{\prime}\right)$. It makes no sense to consider the initial values of $f$ in $V$ because it is not even continuous with values in $V$. However, because of the derivative of $f$ it will turn out that $f$ is continuous with values in a larger space and so it makes sense to consider initial values of $f$ in this other space. This other space is called a trace space.

Theorem 34.2.15 Let $V$ and $H$ be a Banach space and Hilbert space as described in Definition 34.2.14. Suppose $f \in L^{p}(0, T ; V)$ and $f^{\prime} \in L^{p^{\prime}}\left(0, T ; V^{\prime}\right)$. Then $f$ is a.e. equal to a continuous function mapping $[0, T]$ to $H$. Furthermore, there exists $f(0) \in H$ such that

$$
\begin{equation*}
\frac{1}{2}|f(t)|_{H}^{2}-\frac{1}{2}|f(0)|_{H}^{2}=\int_{0}^{t}\left\langle f^{\prime}(s), f(s)\right\rangle d s \tag{34.2.11}
\end{equation*}
$$

and for all $t \in[0, T]$,

$$
\begin{equation*}
\int_{0}^{t} f^{\prime}(s) d s \in H \tag{34.2.12}
\end{equation*}
$$

and for a.e. $t \in[0, T]$,

$$
\begin{equation*}
f(t)=f(0)+\int_{0}^{t} f^{\prime}(s) d s \text { in } H \tag{34.2.13}
\end{equation*}
$$

Here $f^{\prime}$ is being taken in the sense of $V^{\prime}$ valued distributions and $\frac{1}{p}+\frac{1}{p^{\prime}}=1$ and $p \geq 2$.
Proof: Let $\Psi \in C_{c}^{\infty}(-T, 2 T)$ satisfy $\Psi(t)=1$ if $t \in[-T / 2,3 T / 2]$ and $\Psi(t) \geq 0$. For $t \in \mathbb{R}$, define

$$
\widehat{f}(t) \equiv\left\{\begin{array}{l}
\bar{f}(t) \Psi(t) \text { if } t \in[-T, 2 T] \\
0 \text { if } t \notin[-T, 2 T]
\end{array}\right.
$$

and

$$
\begin{equation*}
f_{n}(t) \equiv \int_{-1 / n}^{1 / n} \widehat{f}(t-s) \phi_{n}(s) d s \tag{34.2.14}
\end{equation*}
$$

where $\phi_{n}$ is a mollifier having support in $(-1 / n, 1 / n)$. Then by Minkowski's inequality

$$
\begin{aligned}
\| f_{n}- & \widehat{f} \|_{L^{p}(\mathbb{R} ; V)}=\left(\int_{\mathbb{R}}\left\|\widehat{f}(t)-\int_{-1 / n}^{1 / n} \widehat{f}(t-s) \phi_{n}(s) d s\right\|_{V}^{p} d t\right)^{1 / p} \\
& =\left(\int_{\mathbb{R}}\left\|\int_{-1 / n}^{1 / n}(\widehat{f}(t)-\widehat{f}(t-s)) \phi_{n}(s) d s\right\|_{V}^{p} d t\right)^{1 / p} \\
& \leq\left(\int_{\mathbb{R}}\left(\int_{-1 / n}^{1 / n}\|\widehat{f}(t)-\widehat{f}(t-s)\|_{V} \phi_{n}(s) d s\right)^{p} d t\right)^{1 / p} \\
& \leq \int_{-1 / n}^{1 / n} \phi_{n}(s)\left(\int_{\mathbb{R}}\|\widehat{f}(t)-\widehat{f}(t-s)\|_{V}^{p} d t\right)^{1 / p} d s \\
& \leq \int_{-1 / n}^{1 / n} \phi_{n}(s) \varepsilon d s=\varepsilon
\end{aligned}
$$

provided $n$ is large enough. This follows from continuity of translation in $L^{p}$ with Lebesgue measure. Since $\varepsilon>0$ is arbitrary, it follows $f_{n} \rightarrow \widehat{f}$ in $L^{p}(\mathbb{R} ; V)$. Similarly, $f_{n} \rightarrow f$ in $L^{2}(\mathbb{R} ; H)$. This follows because $p \geq 2$ and the norm in $V$ and norm in $H$ are related by $|x|_{H} \leq C\|x\|_{V}$ for some constant, $C$. Now

$$
\widehat{f}(t)=\left\{\begin{array}{l}
\Psi(t) f(t) \text { if } t \in[0, T] \\
\Psi(t) f(2 T-t) \text { if } t \in[T, 2 T] \\
\Psi(t) f(-t) \text { if } t \in[0, T] \\
0 \text { if } t \notin[-T, 2 T]
\end{array}\right.
$$

An easy modification of the argument of Lemma 34.2.13 yields

$$
\widehat{f}^{\prime}(t)=\left\{\begin{array}{l}
\Psi^{\prime}(t) f(t)+\Psi(y) f^{\prime}(t) \text { if } t \in[0, T] \\
\Psi^{\prime}(t) f(2 T-t)-\Psi(t) f^{\prime}(2 T-t) \text { if } t \in[T, 2 T] \\
\Psi^{\prime}(t) f(-t)-\Psi(t) f^{\prime}(-t) \text { if } t \in[-T, 0] \\
0 \text { if } t \notin[-T, 2 T]
\end{array}\right.
$$

Recall

$$
\begin{aligned}
f_{n}(t) & =\int_{-1 / n}^{1 / n} \widehat{f}(t-s) \phi_{n}(s) d s=\int_{\mathbb{R}} \widehat{f}(t-s) \phi_{n}(s) d s \\
& =\int_{\mathbb{R}} \widehat{f}(s) \phi_{n}(t-s) d s
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f_{n}^{\prime}(t) & =\int_{\mathbb{R}} \widehat{f}(s) \phi_{n}^{\prime}(t-s) d s=\int_{-T-\frac{1}{n}}^{2 T+\frac{1}{n}} \widehat{f}(s) \phi_{n}^{\prime}(t-s) d s \\
& =\int_{-T-\frac{1}{n}}^{2 T+\frac{1}{n}} \widehat{f}^{\prime}(s) \phi_{n}(t-s) d s=\int_{\mathbb{R}} \widehat{f}^{\prime}(s) \phi_{n}(t-s) d s \\
& =\int_{\mathbb{R}} \widehat{f}^{\prime}(t-s) \phi_{n}(s) d s=\int_{-1 / n}^{1 / n} \widehat{f}^{\prime}(t-s) \phi_{n}(s) d s
\end{aligned}
$$

and it follows from the first line above that $f_{n}^{\prime}$ is continuous with values in $V$ for all $t \in \mathbb{R}$. Also note that both $f_{n}^{\prime}$ and $f_{n}$ equal zero if $t \notin[-T, 2 T]$ whenever $n$ is large enough. Exactly similar reasoning to the above shows that $f_{n}^{\prime} \rightarrow \widehat{f}^{\prime}$ in $L^{p^{\prime}}\left(\mathbb{R} ; V^{\prime}\right)$.

Now let $\phi \in C_{c}^{\infty}(0, T)$.

$$
\begin{align*}
\int_{\mathbb{R}}\left|f_{n}(t)\right|_{H}^{2} \phi^{\prime}(t) d t & =\int_{\mathbb{R}}\left(f_{n}(t), f_{n}(t)\right)_{H} \phi^{\prime}(t) d t  \tag{34.2.15}\\
=-\int_{\mathbb{R}} 2\left(f_{n}^{\prime}(t), f_{n}(t)\right) \phi(t) d t & =-\int_{\mathbb{R}} 2\left\langle f_{n}^{\prime}(t), f_{n}(t)\right\rangle \phi(t) d t
\end{align*}
$$

Now

$$
\begin{aligned}
& \left|\int_{\mathbb{R}}\left\langle f_{n}^{\prime}(t), f_{n}(t)\right\rangle \phi(t) d t-\int_{\mathbb{R}}\left\langle f^{\prime}(t), f(t)\right\rangle \phi(t) d t\right| \\
\leq & \int_{\mathbb{R}}\left(\left|\left\langle f_{n}^{\prime}(t)-f^{\prime}(t), f_{n}(t)\right\rangle\right|+\left|\left\langle f^{\prime}(t), f_{n}(t)-f(t)\right\rangle\right|\right) \phi(t) d t
\end{aligned}
$$

From the first part of this proof which showed that $f_{n} \rightarrow \widehat{f}$ in $L^{p}(\mathbb{R} ; V)$ and $f_{n}^{\prime} \rightarrow \widehat{f}^{\prime}$ in $L^{p^{\prime}}\left(\mathbb{R} ; V^{\prime}\right)$, an application of Holder's inequality shows the above converges to 0 as $n \rightarrow \infty$. Therefore, passing to the limit as $n \rightarrow \infty$ in the 34.2.16,

$$
\int_{\mathbb{R}}|\widehat{f}(t)|_{H}^{2} \phi^{\prime}(t) d t=-\int_{\mathbb{R}} 2\left\langle\widehat{f}^{\prime}(t), \widehat{f}(t)\right\rangle \phi(t) d t
$$

which shows $t \rightarrow|\widehat{f}(t)|_{H}^{2}$ equals a continuous function a.e. and it also has a weak derivative equal to $2\left\langle\widehat{f}^{\prime}, \widehat{f}\right\rangle$.

It remains to verify that $\widehat{f}$ is continuous on $[0, T]$. Of course $\widehat{f}=f$ on this interval. Let
$N$ be large enough that $f_{n}(-T)=0$ for all $n>N$. Then for $m, n>N$ and $t \in[-T, 2 T]$

$$
\begin{aligned}
\left|f_{n}(t)-f_{m}(t)\right|_{H}^{2} & =2 \int_{-T}^{t}\left(f_{n}^{\prime}(s)-f_{m}^{\prime}(s), f_{n}(s)-f_{m}(s)\right) d s \\
& =2 \int_{-T}^{t}\left\langle f_{n}^{\prime}(s)-f_{m}^{\prime}(s), f_{n}(s)-f_{m}(s)\right\rangle_{V^{\prime}, V} d s \\
& \leq 2 \int_{\mathbb{R}}\left\|f_{n}^{\prime}(s)-f_{m}^{\prime}(s)\right\|_{V^{\prime}}\left\|f_{n}(s)-f_{m}(s)\right\|_{V} d s \\
& \leq 2\left\|f_{n}-f_{m}\right\|_{L^{p^{\prime}\left(\mathbb{R} ; V^{\prime}\right)}}\left\|f_{n}-f_{m}\right\|_{L^{p}(\mathbb{R} ; V)}
\end{aligned}
$$

which shows from the above that $\left\{f_{n}\right\}$ is uniformly Cauchy on $[-T, 2 T]$ with values in $H$. Therefore, there exists $g$ a continuous function defined on $[-T, 2 T]$ having values in $H$ such that

$$
\lim _{n \rightarrow \infty} \max \left\{\left|f_{n}(t)-g(t)\right|_{H} ; t \in[-T, 2 T]\right\}=0
$$

However, $g=\widehat{f}$ a.e. because $f_{n}$ converges to $f$ in $L^{p}(0, T ; V)$. Therefore, taking a subsequence, the convergence is a.e. It follows from the fact that $V \subseteq H=H^{\prime} \subseteq V^{\prime}$ and Theorem 34.2.9, there exists $f(0) \in V^{\prime}$ such that for a.e. $t$,

$$
f(t)=f(0)+\int_{0}^{t} f^{\prime}(s) d s \text { in } V^{\prime}
$$

Now $g=f$ a.e. and $g$ is continuous with values in $H$ hence continuous with values in $V^{\prime}$ and so

$$
g(t)=f(0)+\int_{0}^{t} f^{\prime}(s) d s \text { in } V^{\prime}
$$

for all $t$. Since $g$ is continuous with values in $H$ it is continuous with values in $V^{\prime}$. Taking the limit as $t \downarrow 0$ in the above, $g(a)=\lim _{t \rightarrow 0+} g(t)=f(0)$, showing that $f(0) \in H$. Therefore, for a.e. $t$,

$$
f(t)=f(0)+\int_{0}^{t} f^{\prime}(s) d s \text { in } H, \int_{0}^{t} f^{\prime}(s) d s \in H
$$

Note that if $f \in L^{p}(0, T ; V)$ and $f^{\prime} \in L^{p^{\prime}}\left(0, T ; V^{\prime}\right)$, then you can consider the initial value of $f$ and it will be in $H$. What if you start with something in $H$ ? Is it an initial condition for a function $f \in L^{p}(0, T ; V)$ such that $f^{\prime} \in L^{p^{\prime}}\left(0, T ; V^{\prime}\right)$ ? This is worth thinking about. If it is not so, what is the space of initial values? How can you give this space a norm? What are its properties? It turns out that if $V$ is a closed subspace of the Sobolev space, $W^{1, p}(\Omega)$ which contains $W_{0}^{1, p}(\Omega)$ for $p \geq 2$ and $H=L^{2}(\Omega)$ the answer to the above question is yes. Not surprisingly, there are many generalizations of the above ideas.

### 34.3 An Important Formula

It is not necessary to have $p>2$ in order to do the sort of thing just described. First is an approximation theorem which says that a given functionin $L^{p}([0, T] ; E)$ can be approximated by step functions.

Lemma 34.3.1 Let $\Phi:[0, T] \rightarrow E$, be Lebesgue measurable and suppose

$$
\Phi \in K \equiv L^{p}([0, T] ; E), p \geq 1
$$

Then there exists a sequence of nested partitions, $\mathscr{P}_{k} \subseteq \mathscr{P}_{k+1}$,

$$
\mathscr{P}_{k} \equiv\left\{t_{0}^{k}, \cdots, t_{m_{k}}^{k}\right\}
$$

such that the step functions given by

$$
\begin{aligned}
\Phi_{k}^{r}(t) & \equiv \sum_{j=1}^{m_{k}} \Phi\left(t_{j}^{k}\right) \mathscr{X}_{\left[t_{j-1}^{k}, t_{j}^{k}\right)}(t) \\
\Phi_{k}^{l}(t) & \equiv \sum_{j=1}^{m_{k}} \Phi\left(t_{j-1}^{k}\right) \mathscr{X}_{\left[t_{j-1}^{k}, t_{j}^{k}\right)}(t)
\end{aligned}
$$

both converge to $\Phi$ in $K$ as $k \rightarrow \infty$ and

$$
\lim _{k \rightarrow \infty} \max \left\{\left|t_{j}^{k}-t_{j+1}^{k}\right|: j \in\left\{0, \cdots, m_{k}\right\}\right\}=0
$$

In the formulas, define $\Phi(0)=0$. The mesh points $\left\{t_{j}^{k}\right\}_{j=0}^{m_{k}}$ can be chosen to miss a given set of measure zero.

Note that it would make no difference in terms of the conclusion of this lemma if you defined

$$
\Phi_{k}^{l}(t) \equiv \sum_{j=1}^{m_{k}} \Phi\left(t_{j-1}^{k}\right) \mathscr{X}_{\left(t_{j-1}^{k}, t_{j}^{k}\right]}(t)
$$

because the modified function equals the one given above off a countable subset of $[0, T]$, the union of the mesh points.

Proof: For $t \in \mathbb{R}$ let $\gamma_{n}(t) \equiv k / 2^{n}, \delta_{n}(t) \equiv(k+1) / 2^{n}$, where

$$
t \in\left(k / 2^{n},(k+1) / 2^{n}\right]
$$

and $2^{-n}<T / 4$. Also suppose $\Phi$ is defined to equal 0 on $[0, T]^{C} \times \Omega$. There exists a set of measure zero $N$ such that for $\omega \notin N, t \rightarrow\|\Phi(t, \omega)\|$ is in $L^{p}(\mathbb{R})$. Therefore by continuity of translation, as $n \rightarrow \infty$ it follows that for $\omega \notin N$, and $t \in[0, T]$,

$$
\int_{\mathbb{R}}\left\|\Phi\left(\gamma_{n}(t)+s\right)-\Phi(t+s)\right\|_{E}^{p} d s \rightarrow 0
$$

The above is dominated by

$$
\begin{aligned}
& \int_{\mathbb{R}} 2^{p-1}\left(\|\Phi(s)\|^{p}+\|\Phi(s)\|^{p}\right) \mathscr{X}_{[-2 T, 2 T]}(s) d s \\
= & \int_{-2 T}^{2 T} 2^{p-1}\left(\|\Phi(s)\|^{p}+\|\Phi(s)\|^{p}\right) d s<\infty
\end{aligned}
$$

Consider

$$
\int_{-2 T}^{2 T}\left(\int_{\mathbb{R}}\left\|\Phi\left(\gamma_{n}(t)+s\right)-\Phi(t+s)\right\|_{E}^{p} d s\right) d t
$$

By the dominated convergence theorem, this converges to 0 as $n \rightarrow \infty$. Now Fubini. This yields

$$
\int_{\mathbb{R}} \int_{-2 T}^{2 T}\left\|\Phi\left(\gamma_{n}(t)+s\right)-\Phi(t+s)\right\|_{E}^{p} d t d s
$$

Change the variables on the inside.

$$
\int_{\mathbb{R}} \int_{-2 T+s}^{2 T+s}\left\|\Phi\left(\gamma_{n}(t-s)+s\right)-\Phi(t)\right\|_{E}^{p} d t d s
$$

Now by definition, $\Phi(t)$ vanishes if $t \notin[0, T]$, thus the above reduces to

$$
\begin{gathered}
\int_{\mathbb{R}} \int_{0}^{T}\left\|\Phi\left(\gamma_{n}(t-s)+s\right)-\Phi(t)\right\|_{E}^{p} d t d s \\
+\int_{\mathbb{R}} \int_{-2 T+s}^{2 T+s} \mathscr{X}_{[0, T]}^{C}\left\|\Phi\left(\gamma_{n}(t-s)+s\right)\right\|_{E}^{p} d t d s \\
=\quad \int_{\mathbb{R}} \int_{0}^{T}\left\|\Phi\left(\gamma_{n}(t-s)+s\right)-\Phi(t)\right\|_{E}^{p} d t d s \\
+\int_{\mathbb{R}} \int_{-2 T+s}^{2 T+s} \mathscr{X}_{[0, T]^{c}}\left\|\Phi\left(\gamma_{n}(t-s)+s\right)-\Phi(t)\right\|_{E}^{p} d t d s
\end{gathered}
$$

Also by definition, $\gamma_{n}(t-s)+s$ is within $2^{-n}$ of $t$ and so the integrand in the integral on the right equals 0 unless $t \in\left[-2^{-n}-T, T+2^{-n}\right] \subseteq[-2 T, 2 T]$. Thus the above reduces to

$$
\int_{\mathbb{R}} \int_{-2 T}^{2 T}\left\|\Phi\left(\gamma_{n}(t-s)+s\right)-\Phi(t)\right\|_{E}^{p} d t d s
$$

This converges to 0 as $n \rightarrow \infty$ as was shown above. Therefore,

$$
\int_{0}^{T} \int_{0}^{T}\left\|\Phi\left(\gamma_{n}(t-s)+s\right)-\Phi(t)\right\|_{E}^{p} d t d s
$$

also converges to 0 as $n \rightarrow \infty$. The only problem is that $\gamma_{n}(t-s)+s \geq t-2^{-n}$ and so $\gamma_{n}(t-s)+s$ could be less than 0 for $t \in\left[0,2^{-n}\right]$. Since this is an interval whose measure converges to 0 it follows

$$
\int_{0}^{T} \int_{0}^{T}\left\|\Phi\left(\left(\gamma_{n}(t-s)+s\right)^{+}\right)-\Phi(t)\right\|_{E}^{p} d t d s
$$

converges to 0 as $n \rightarrow \infty$. Let

$$
m_{n}(s)=\int_{0}^{T}\left\|\Phi\left(\left(\gamma_{n}(t-s)+s\right)^{+}\right)-\Phi(t)\right\|_{E}^{p} d t
$$

Then letting $\mu$ denote Lebesgue measure,

$$
\mu\left(\left[m_{n}(s)>\lambda\right]\right) \leq \frac{1}{\lambda} \int_{0}^{T} m_{n}(s) d s
$$

It follows there exists a subsequence $n_{k}$ such that

$$
\mu\left(\left[m_{n_{k}}(s)>\frac{1}{k}\right]\right)<2^{-k}
$$

Hence by the Borel Cantelli lemma, there exists a set of measure zero $N$ such that for $s \notin N$,

$$
m_{n_{k}}(s) \leq 1 / k
$$

for all $k$ sufficiently large. Pick such an $s$. Then consider $t \rightarrow \Phi\left(\left(\gamma_{n_{k}}(t-s)+s\right)^{+}\right)$. For $n_{k}, t \rightarrow\left(\gamma_{n_{k}}(t-s)+s\right)^{+}$has jumps at points of the form $0, s+l 2^{-n_{k}}$ where $l$ is an integer. Thus $\mathscr{P}_{n_{k}}$ consists of points of $[0, T]$ which are of this form and these partitions are nested. Define $\Phi_{k}^{l}(0) \equiv 0, \Phi_{k}^{l}(t) \equiv \Phi\left(\left(\gamma_{n_{k}}(t-s)+s\right)^{+}\right)$. Now suppose $N_{1}$ is a set of measure zero. Can $s$ be chosen such that all jumps for all partitions occur off $N_{1}$ ? Let $(a, b)$ be an interval contained in $[0, T]$. Let $S_{j}$ be the points of $(a, b)$ which are translations of the measure zero set $N_{1}$ by $t_{j}^{l}$ for some $j$. Thus $S_{j}$ has measure 0 . Now pick $s \in(a, b) \backslash \cup_{j} S_{j}$. To get the other sequence of step functions, the right step functions, just use a similar argument with $\delta_{n}$ in place of $\gamma_{n}$. Just apply the argument to a subsequence of $n_{k}$ so that the same $s$ can hold for both.

Theorem 34.3.2 Let $V \subseteq H=H^{\prime} \subseteq V^{\prime}$ be a Gelfand triple and suppose $Y \in L^{p^{\prime}}\left(0, T ; V^{\prime}\right) \equiv$ $K^{\prime}$ and

$$
\begin{equation*}
X(t)=X_{0}+\int_{0}^{t} Y(s) d s \text { in } V^{\prime} \tag{34.3.16}
\end{equation*}
$$

where $X_{0} \in H$, and it is known that $X \in L^{p}(0, T, V) \equiv K$ for $p>1$. Then $t \rightarrow X(t)$ is in $C([0, T], H)$ and also

$$
\frac{1}{2}|X(t)|_{H}^{2}=\frac{1}{2}\left|X_{0}\right|_{H}^{2}+\int_{0}^{t}\langle Y(s), X(s)\rangle d s
$$

Proof: By Lemma 34.3.1, there exists a sequence of uniform partitions $\left\{t_{k}^{n}\right\}_{k=0}^{m_{n}}=$ $\mathscr{P}_{n}, \mathscr{P}_{n} \subseteq \mathscr{P}_{n+1}$, of $[0, T]$ such that the step functions

$$
\begin{aligned}
\sum_{k=0}^{m_{n}-1} X\left(t_{k}^{n}\right) \mathscr{X}_{\left(t_{k}^{n}, t_{k+1}^{n}\right]}(t) & \equiv X^{l}(t) \\
\sum_{k=0}^{m_{n}-1} X\left(t_{k+1}^{n}\right) \mathscr{X}_{\left(t_{k}^{n}, t_{k+1}^{n}\right]}(t) & \equiv X^{r}(t)
\end{aligned}
$$

converge to $X$ in $K$ and in $L^{2}([0, T], H)$.

Lemma 34.3.3 Let $s<t$. Then for $X, Y$ satisfying 34.3.16

$$
\begin{equation*}
|X(t)|^{2}=|X(s)|^{2}+2 \int_{s}^{t}\langle Y(u), X(t)\rangle d u-|X(t)-X(s)|^{2} \tag{34.3.17}
\end{equation*}
$$

Proof: It follows from the following computations

$$
\begin{gathered}
X(t)-X(s)=\int_{s}^{t} Y(u) d u \\
-|X(t)-X(s)|^{2}=-|X(t)|^{2}+2(X(t), X(s))-|X(s)|^{2} \\
=-|X(t)|^{2}+2\left(X(t), X(t)-\int_{s}^{t} Y(u) d u\right)-|X(s)|^{2} \\
=-|X(t)|^{2}+2|X(t)|^{2}-2\left\langle\int_{s}^{t} Y(u) d u, X(t)\right\rangle-|X(s)|^{2}
\end{gathered}
$$

Hence

$$
|X(t)|^{2}=|X(s)|^{2}+2 \int_{s}^{t}\langle Y(u), X(t)\rangle d u-|X(t)-X(s)|^{2}
$$

Lemma 34.3.4 In the above situation,

$$
\sup _{t \in[0, T]}|X(t)|_{H} \leq C\left(\|Y\|_{K^{\prime}},\|X\|_{K}\right)
$$

Also, $t \rightarrow X(t)$ is weakly continuous with values in $H$.
Proof: From the above formula applied to the $k^{t h}$ partition of $[0, T]$ described above,

$$
\begin{aligned}
& \left|X\left(t_{m}\right)\right|^{2}-\left|X_{0}\right|^{2}=\sum_{j=0}^{m-1}\left|X\left(t_{j+1}\right)\right|^{2}-\left|X\left(t_{j}\right)\right|^{2} \\
= & \sum_{j=0}^{m-1} 2 \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X\left(t_{j+1}\right)\right\rangle d u-\left|X\left(t_{j+1}\right)-X\left(t_{j}\right)\right|_{H}^{2} \\
= & \sum_{j=0}^{m-1} 2 \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u-\left|X\left(t_{j+1}\right)-X\left(t_{j}\right)\right|_{H}^{2}
\end{aligned}
$$

Thus, discarding the negative terms and denoting by $\mathscr{P}_{k}$ the $k^{t h}$ of these partitions,

$$
\begin{gathered}
\sup _{t_{j} \in \mathscr{P}_{k}}\left|X\left(t_{j}\right)\right|_{H}^{2} \leq\left|X_{0}\right|^{2}+2 \int_{0}^{T}\left|\left\langle Y(u), X_{k}^{r}(u)\right\rangle\right| d u \\
\quad \leq\left|X_{0}\right|^{2}+2 \int_{0}^{T}\|Y(u)\|_{V^{\prime}}\left\|X_{k}^{r}(u)\right\|_{V} d u
\end{gathered}
$$

$$
\leq\left|X_{0}\right|^{2}+2\left(\int_{0}^{T}\|Y(u)\|_{V^{\prime}}^{p^{\prime}} d u\right)^{1 / p^{\prime}}\left(\int_{0}^{T}\left\|X_{k}^{r}(u)\right\|_{V}^{p} d u\right)^{1 / p} \leq C\left(\|Y\|_{K^{\prime}},\|X\|_{K}\right)
$$

because these partitions are chosen such that

$$
\lim _{k \rightarrow \infty}\left(\int_{0}^{T}\left\|X_{k}^{r}(u)\right\|_{V}^{p}\right)^{1 / p}=\left(\int_{0}^{T}\|X(u)\|_{V}^{p}\right)^{1 / p}
$$

and so these are bounded. This has shown that for the dense subset of $[0, T], D \equiv \cup_{k} \mathscr{P}_{k}$,

$$
\sup _{t \in D}|X(t)|<C\left(\|Y\|_{K^{\prime}},\|X\|_{K}\right)
$$

Now let $\left\{g_{k}\right\}_{k=1}^{\infty}$ be linearly independent vectors of $V$ whose span is dense in $V$. This is possible because $V$ is separable. Then let $\left\{e_{j}\right\}_{j=1}^{\infty}$ be an orthonormal basis for $H$ such that $e_{k} \in \operatorname{span}\left(g_{1}, \ldots, g_{k}\right)$ and each $g_{k} \in \operatorname{span}\left(e_{1}, \ldots, e_{k}\right)$. This is done with the Gram Schmidt process. Then it follows that $\operatorname{span}\left(\left\{e_{k}\right\}_{k=1}^{\infty}\right)$ is dense in $V$. I claim

$$
|y|_{H}^{2}=\sum_{j=1}^{\infty}\left|\left\langle y, e_{j}\right\rangle\right|^{2}
$$

This is certainly true if $y \in H$ because

$$
\left\langle y, e_{j}\right\rangle=\left(y, e_{j}\right)_{H}
$$

If $y \notin H$, then the series must diverge since otherwise, you could consider the infinite sum

$$
\sum_{j=1}^{\infty}\left\langle y, e_{j}\right\rangle e_{j} \in H
$$

because

$$
\left|\sum_{j=p}^{q}\left\langle y, e_{j}\right\rangle e_{j}\right|^{2}=\sum_{j=p}^{q}\left|\left\langle y, e_{j}\right\rangle\right|^{2} \rightarrow 0 \text { as } p, q \rightarrow \infty
$$

Letting $z=\sum_{j=1}^{\infty}\left\langle y, e_{j}\right\rangle e_{j}$, it follows that $\left\langle y, e_{j}\right\rangle$ is the $j^{\text {th }}$ Fourier coefficient of $z$ and that

$$
\langle z-y, v\rangle=0
$$

for all $v \in \operatorname{span}\left(\left\{e_{k}\right\}_{k=1}^{\infty}\right)$ which is dense in $V$. Therefore, $z=y$ in $V^{\prime}$ and so $y \in H$.
It follows

$$
|X(t)|^{2}=\sup _{n} \sum_{j=1}^{n}\left|\left\langle X(t), e_{j}\right\rangle\right|^{2}
$$

which is just the sup of continuous functions of $t$. Therefore, $t \rightarrow|X(t)|^{2}$ is lower semicontinuous. It follows that for any $t$, letting $t_{j} \rightarrow t$ for $t_{j} \in D$,

$$
|X(t)|^{2} \leq \lim \inf _{j \rightarrow \infty}\left|X\left(t_{j}\right)\right|^{2} \leq C\left(\|Y\|_{K^{\prime}},\|X\|_{K}\right)
$$

This proves the first claim of the lemma.
Consider now the claim that $t \rightarrow X(t)$ is weakly continuous. Letting $v \in V$,

$$
\lim _{t \rightarrow s}(X(t), v)=\lim _{t \rightarrow s}\langle X(t), v\rangle=\langle X(s), v\rangle=(X(s), v)
$$

Since it was shown that $|X(t)|$ is bounded independent of $t$, and since $V$ is dense in $H$, the claim follows.

Now

$$
\begin{aligned}
-\sum_{j=0}^{m-1}\left|X\left(t_{j+1}\right)-X\left(t_{j}\right)\right|_{H}^{2} & =\left|X\left(t_{m}\right)\right|^{2}-\left|X_{0}\right|^{2}-\sum_{j=0}^{m-1} 2 \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u \\
& =\left|X\left(t_{m}\right)\right|^{2}-\left|X_{0}\right|^{2}-2 \int_{0}^{t_{m}}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u
\end{aligned}
$$

Thus, since the partitions are nested, eventually $\left|X\left(t_{m}\right)\right|^{2}$ is constant for all $k$ large enough and the integral term converges to

$$
\int_{0}^{t_{m}}\langle Y(u), X(u)\rangle d u
$$

It follows that the term on the left does converge to something. It just remains to consider what it does converge to. However, from the equation solved by $X$,

$$
X\left(t_{j+1}\right)-X\left(t_{j}\right)=\int_{t_{j}}^{t_{j+1}} Y(u) d u
$$

Therefore, this term is dominated by an expression of the form

$$
\begin{gathered}
\sum_{j=0}^{m_{k}-1}\left(\int_{t_{j}}^{t_{j+1}} Y(u) d u, X\left(t_{j+1}\right)-X\left(t_{j}\right)\right) \\
=\sum_{j=0}^{m_{k}-1}\left\langle\int_{t_{j}}^{t_{j+1}} Y(u) d u, X\left(t_{j+1}\right)-X\left(t_{j}\right)\right\rangle \\
=\sum_{j=0}^{m_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X\left(t_{j+1}\right)-X\left(t_{j}\right)\right\rangle d u \\
=\sum_{j=0}^{m_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X\left(t_{j+1}\right)\right\rangle-\sum_{j=0}^{m_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X\left(t_{j}\right)\right\rangle \\
=\int_{0}^{T}\left\langle Y(u), X^{r}(u)\right\rangle d u-\int_{0}^{T}\left\langle Y(u), X^{l}(u)\right\rangle d u
\end{gathered}
$$

However, both $X^{r}$ and $X^{l}$ converge to $X$ in $K=L^{p}(0, T, V)$. Therefore, this term must converge to 0 . Passing to a limit, it follows that for all $t \in D$, the desired formula holds. Thus, for such $t$,

$$
|X(t)|^{2}=\left|X_{0}\right|^{2}+2 \int_{0}^{t}\langle Y(u), X(u)\rangle d u
$$

It remains to verify that this holds for all $t$. Let $t \notin D$ and let $t(k) \in \mathscr{P}_{k}$ be the largest point of $\mathscr{P}_{k}$ which is less than $t$. Suppose $t(m) \leq t(k)$ so that $m \leq k$. Then

$$
X(t(m))=X_{0}+\int_{0}^{t(m)} Y(s) d s
$$

a similar formula for $X(t(k))$. Thus for $t>t(m)$,

$$
X(t)-X(t(m))=\int_{t(m)}^{t} Y(s) d s
$$

which is the same sort of thing already looked at except that it starts at $t(m)$ rather than at 0 and $X_{0}=0$. Therefore,

$$
|X(t(k))-X(t(m))|^{2}=2 \int_{t(m)}^{t(k)}\langle Y(s), X(s)-X(t(m))\rangle d s
$$

Thus, for $m \leq k$

$$
\lim _{m, k \rightarrow \infty}|X(t(k))-X(t(m))|^{2}=0
$$

Hence $\{X(t(k))\}_{k=1}^{\infty}$ is a convergent sequence in $H$. Does it converge to $X(t)$ ? Let $\xi(t) \in H$ be what it does converge to. Let $v \in V$. Then

$$
(\xi(t), v)=\lim _{k \rightarrow \infty}(X(t(k)), v)=\lim _{k \rightarrow \infty}\langle X(t(k)), v\rangle=\langle X(t), v\rangle=(X(t), v)
$$

because it is known that $t \rightarrow X(t)$ is continuous into $V^{\prime}$ and it is also known that $X(t) \in H$ and that the $X(t)$ for $t \in[0, T]$ are uniformly bounded. Therefore, since $V$ is dense in $H$, it follows that $\xi(t)=X(t)$.

Now for every $t \in D$, it was shown above that

$$
|X(t)|^{2}=\left|X_{0}\right|^{2}+2 \int_{0}^{t}\langle Y(s), X(s)\rangle d s
$$

Thus, using what was just shown, if $t \notin D$ and $t_{k} \rightarrow t$,

$$
\begin{aligned}
|X(t)|^{2} & =\lim _{k \rightarrow \infty}\left|X\left(t_{k}\right)\right|^{2}=\lim _{k \rightarrow \infty}\left(\left|X_{0}\right|^{2}+2 \int_{0}^{t_{k}}\langle Y(s), X(s)\rangle d s\right) \\
& =\left|X_{0}\right|^{2}+2 \int_{0}^{t}\langle Y(s), X(s)\rangle d s
\end{aligned}
$$

which proves the desired formula. From this it follows right away that $t \rightarrow X(t)$ is continuous into $H$ because it was just shown that $t \rightarrow|X(t)|$ is continuous and $t \rightarrow X(t)$ is weakly continuous. Since Hilbert space is uniformly convex, this implies the $t \rightarrow X(t)$ is continuous. To see this in the special case of Hilbert space,

$$
|X(t)-X(s)|^{2}=|X(t)|^{2}-2(X(s), X(t))+|X(s)|^{2}
$$

Then $\lim _{t \rightarrow s}\left(|X(t)|^{2}-2(X(s), X(t))+|X(s)|^{2}\right)=0$ by weak convergence of $X(t)$ to $X(s)$ and the convergence of $|X(t)|^{2}$ to $|X(s)|^{2}$.

### 34.4 The Implicit Case

The above theorem can be generalized to the case where the formula is of the form

$$
B X(t)=B X_{0}+\int_{0}^{t} Y(s) d s
$$

This involves an operator $B \in \mathscr{L}\left(W, W^{\prime}\right)$ and $B$ satisfies

$$
\langle B x, x\rangle \geq 0,\langle B x, y\rangle=\langle B y, x\rangle
$$

for

$$
V \subseteq W, W^{\prime} \subseteq V^{\prime}
$$

Where $V$ is dense in the Banach space $W$. Before giving the theorem, here is a technical lemma. First is one which is not so technical.

Lemma 34.4.1 Let $V$ be a separable Banach space. Then there exists $\left\{g_{k}\right\}_{k=1}^{\infty}$ which are linearly independent and whose span is dense in $V$.

Proof: Let $\left\{f_{k}\right\}$ be a countable dense subset. Thus their span is dense. Delete $f_{k_{1}}$ such that $k_{1}$ is the first index such that $f_{k}$ is in the span of the other vectors. That is, it is the first which is a finite linear combination of the others. If no such vector exists, then you have what is wanted. Next delete $f_{k_{2}}$ where $k_{2}$ is the next for which $f_{k}$ is a linear combination of the others. Continue. The remaining vectors must be linearly independent. If not, there would be a first which is a linear combination of the others. Say $f_{m}$. But the process would have eliminated it at the $m^{t h}$ step.

Lemma 34.4.2 Suppose $V, W$ are separable Banach spaces such that $V$ is dense in $W$ and $B \in \mathscr{L}\left(W, W^{\prime}\right)$ satisfies

$$
\langle B x, x\rangle \geq 0,\langle B x, y\rangle=\langle B y, x\rangle, B \neq 0 .
$$

Then there exists a countable set $\left\{e_{i}\right\}$ of vectors in $V$ such that

$$
\left\langle B e_{i}, e_{j}\right\rangle=\delta_{i j}
$$

and for each $x \in W$,

$$
\langle B x, x\rangle=\sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2},
$$

and also

$$
B x=\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i},
$$

the series converging in $W^{\prime}$. If $B=B(\omega)$ and $B$ is $\mathscr{F}$ measurable into $\mathscr{L}\left(W, W^{\prime}\right)$ and if the $e_{i}=e_{i}(\omega)$ are as described above, then these $e_{i}$ are measurable into $V$. If $t \rightarrow B(t, \omega)$ is $C^{1}\left([0, T], \mathscr{L}\left(W, W^{\prime}\right)\right)$ and iffor each $w \in W$,

$$
\left\langle B^{\prime}(t, \omega) w, w\right\rangle \leq k_{w, \omega}(t)\langle B(t, \omega) w, w\rangle
$$

Where $k_{w, \omega} \in L^{1}([0, T])$, then the vectors $e_{i}(t)$ can be chosen to also be right continuous functions of $t$.

In the case of dependence on $t$, the extra condition is trivial if $\langle B(t, \omega) x, x\rangle \geq \delta\|w\|_{W}^{2}$ for example. This includes the usual case of evolution equations where $W=H=H^{\prime}=W^{\prime}$. It also includes the case where $B$ does not depend on $t$.

Proof: Let $\left\{g_{k}\right\}_{k=1}^{\infty}$ be linearly independent vectors of $V$ whose span is dense in $V$. This is possible because $V$ is separable. Thus, their span is also dense in $W$. Let $n_{1}$ be the first index such that $\left\langle B g_{n_{1}}, g_{n_{1}}\right\rangle \neq 0$.

Claim: If there is no such index, then $B=0$.
Proof of claim: First note that if there is no such first index, then if $x=\sum_{i=1}^{k} a_{i} g_{i}$

$$
\begin{aligned}
|\langle B x, x\rangle| & =\left|\sum_{i \neq j} a_{i} a_{j}\left\langle B g_{i}, g_{j}\right\rangle\right| \leq \sum_{i \neq j}\left|a_{i}\right|\left|a_{j}\right|\left|\left\langle B g_{i}, g_{j}\right\rangle\right| \\
& \leq \sum_{i \neq j}\left|a_{i}\right|\left|a_{j}\right|\left\langle B g_{i}, g_{i}\right\rangle^{1 / 2}\left\langle B g_{j}, g_{j}\right\rangle^{1 / 2}=0
\end{aligned}
$$

Therefore, if $x$ is given, you could take $x_{k}$ in the span of $\left\{g_{1}, \cdots, g_{k}\right\}$ such that $\left\|x_{k}-x\right\|_{W} \rightarrow$ 0 . Then

$$
|\langle B x, y\rangle|=\lim _{k \rightarrow \infty}\left|\left\langle B x_{k}, y\right\rangle\right| \leq \lim _{k \rightarrow \infty}\left\langle B x_{k}, x_{k}\right\rangle^{1 / 2}\langle B y, y\rangle^{1 / 2}=0
$$

because $\left\langle B x_{k}, x_{k}\right\rangle$ is zero by what was just shown. Hence the conclusion of the lemma is trivially true. Just pick $e_{1}=g_{1}$ and let $\left\{e_{1}\right\}$ be your set of vectors.

Thus assume there is such a first index. Let

$$
e_{1} \equiv \frac{g_{n_{1}}}{\left\langle B g_{n_{1}}, g_{n_{1}}\right\rangle^{1 / 2}}
$$

Then $\left\langle B e_{1}, e_{1}\right\rangle=1$. Now if you have constructed $e_{j}$ for $j \leq k$,

$$
e_{j} \in \operatorname{span}\left(g_{n_{1}}, \cdots, g_{n_{k}}\right),\left\langle B e_{i}, e_{j}\right\rangle=\delta_{i j}
$$

$g_{n_{j+1}}$ being the first in the list $\left\{g_{j}\right\}$ for which

$$
\left\langle B g_{n_{j+1}}-\sum_{i=1}^{j}\left\langle B g_{n_{j+1}}, e_{i}\right\rangle B e_{i}, g_{n_{j+1}}-\sum_{i=1}^{j}\left\langle B g_{n j}, e_{i}\right\rangle e_{i}\right\rangle \neq 0
$$

and

$$
\operatorname{span}\left(g_{n_{1}}, \cdots, g_{n_{k}}\right)=\operatorname{span}\left(e_{1}, \cdots, e_{k}\right)
$$

let $g_{n_{k+1}}$ be such that $g_{n_{k+1}}$ is the first in the list $\left\{g_{n}\right\} n_{k+1}>n_{k}$ such that

$$
\left\langle B g_{n_{k+1}}-\sum_{i=1}^{k}\left\langle B g_{n_{k+1}}, e_{i}\right\rangle B e_{i}, g_{n_{k+1}}-\sum_{i=1}^{k}\left\langle B g_{n_{k+1}}, e_{i}\right\rangle e_{i}\right\rangle \neq 0
$$

Note the difference between this and the Gram Schmidt process. Here you don't necessarily use all of the $g_{k}$ due to the possible degeneracy of $B$.

Claim: If there is no such first $g_{n_{k+1}}$, then $B\left(\operatorname{span}\left(e_{i}, \cdots, e_{k}\right)\right)=B W$ so in this case, $\left\{B e_{i}\right\}_{i=1}^{k}$ is actually a basis for $B W$.

Proof: To see this, note that if $p \in\left(n_{j}, n_{j+1}\right)$, then by assumption,

$$
\left\langle B\left(g_{p}-\sum_{i=1}^{j}\left\langle B g_{p}, e_{i}\right\rangle e_{i}\right), g_{p}-\sum_{i=1}^{j}\left\langle B g_{p}, e_{i}\right\rangle e_{i}\right\rangle=0
$$

Therefore,

$$
B g_{p}=\sum_{i=1}^{j}\left\langle B g_{p}, e_{i}\right\rangle B e_{i}
$$

Also, by assumption, if $p>n_{k}$

$$
\left\langle B\left(g_{p}-\sum_{i=1}^{k}\left\langle B g_{p}, e_{i}\right\rangle e_{i}\right), g_{p}-\sum_{i=1}^{k}\left\langle B g_{p}, e_{i}\right\rangle e_{i}\right\rangle=0
$$

so

$$
B g_{p}=\sum_{i=1}^{k}\left\langle B g_{p}, e_{i}\right\rangle B e_{i}
$$

which shows that span $\left(\left\{B g_{j}\right\}_{j=1}^{\infty}\right) \subseteq \operatorname{span}\left(\left\{B e_{i}\right\}_{i=1}^{k}\right)$. If $\sum_{i=1}^{k} c_{i} B e_{i}=0$, then for $j \leq k$,

$$
0=\sum_{i=1}^{k} c_{i}\left\langle B e_{i}, e_{j}\right\rangle=c_{j}
$$

so $\left\{B e_{i}\right\}_{i=1}^{k}$ is a basis for $\operatorname{span}\left(\left\{B g_{j}\right\}_{j=1}^{\infty}\right)=B\left(\operatorname{span}\left(\left\{g_{j}\right\}_{j=1}^{\infty}\right)\right)$. Hence if $x \in W$, then letting $x_{r} \in \operatorname{span}\left(\left\{g_{j}\right\}_{j=1}^{\infty}\right)$ with $x_{r} \rightarrow x$ in $W$, it follows

$$
B x_{r}=\sum_{i=1}^{k} a_{i} B e_{i}=\sum_{i=1}^{k}\left\langle B x_{r}, e_{i}\right\rangle B e_{i}
$$

Then passing to a limit, you get

$$
B x=\sum_{i=1}^{k}\left\langle B x, e_{i}\right\rangle B e_{i}
$$

Thus $\left\{B e_{i}\right\}_{i=1}^{k}$ is a basis for $B W$. This proves the claim.
If this happens, the process being described stops. You have found what is desired which has only finitely many vectors involved.

If the process does not stop, let

$$
e_{k+1} \equiv \frac{g_{n_{k+1}}-\sum_{i=1}^{k}\left\langle B g_{n_{k+1}}, e_{i}\right\rangle e_{i}}{\left\langle B\left(g_{n_{k+1}}-\sum_{i=1}^{k}\left\langle B g_{n_{k+1}}, e_{i}\right\rangle e_{i}\right), g_{n_{k+1}}-\sum_{i=1}^{k}\left\langle B g_{n_{k+1}}, e_{i}\right\rangle e_{i}\right\rangle^{1 / 2}}
$$

Thus, as in the usual argument for the Gram Schmidt process, $\left\langle B e_{i}, e_{j}\right\rangle=\delta_{i j}$ for $i, j \leq k+1$. This is already known for $i, j \leq k$. Letting $l \leq k$, and using the orthogonality already shown,

$$
\begin{aligned}
\left\langle B e_{k+1}, e_{l}\right\rangle & =C\left\langle B\left(g_{n_{k+1}}-\sum_{i=1}^{k}\left\langle B g_{n_{k+1}}, e_{i}\right\rangle e_{i}\right), e_{l}\right\rangle \\
& =C\left(\left\langle B g_{k+1}, e_{l}\right\rangle-\left\langle B g_{n_{k+1}}, e_{l}\right\rangle\right)=0
\end{aligned}
$$

Consider

$$
\left\langle B g_{p}-B\left(\sum_{i=1}^{k}\left\langle B g_{p}, e_{i}\right\rangle e_{i}\right), g_{p}-\sum_{i=1}^{k}\left\langle B g_{p}, e_{i}\right\rangle e_{i}\right\rangle
$$

If $p \in\left(n_{k}, n_{k+1}\right)$, then the above equals zero which implies

$$
B g_{p}=\sum_{i=1}^{k}\left\langle B g_{p}, e_{i}\right\rangle B e_{i}
$$

On the other hand, suppose $g_{p}=g_{n_{k+1}}$ for some $n_{k+1}$ and so, from the construction, $g_{n_{k+1}}=$ $g_{p} \in \operatorname{span}\left(e_{1}, \cdots, e_{k+1}\right)$ and therefore,

$$
g_{p}=\sum_{j=1}^{k+1} a_{j} e_{j}
$$

which requires easily that

$$
B g_{p}=\sum_{i=1}^{k+1}\left\langle B g_{p}, e_{i}\right\rangle B e_{i},
$$

the above holding for all $k$ large enough. To see this last claim, note that the coefficients of $B g=\sum_{j=1}^{m} a_{j} B e_{j}$ are required to be $a_{j}=\left\langle B g, e_{j}\right\rangle$ and from the construction, $\left\langle B e_{i}, e_{j}\right\rangle=\delta_{i j}$. Thus if the upper limit is increased beyond what is needed, the new terms are all zero. It follows that for any $x \in \operatorname{span}\left(\left\{g_{k}\right\}_{k=1}^{\infty}\right)$, (finite linear combination of vectors in $\left\{g_{k}\right\}_{k=1}^{\infty}$ )

$$
\begin{equation*}
B x=\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i} \tag{34.4.18}
\end{equation*}
$$

because for all $k$ large enough,

$$
B x=\sum_{i=1}^{k}\left\langle B x, e_{i}\right\rangle B e_{i}
$$

Also note that for such $x \in \operatorname{span}\left(\left\{g_{j}\right\}_{j=1}^{\infty}\right)$,

$$
\begin{aligned}
\langle B x, x\rangle & =\left\langle\sum_{i=1}^{k}\left\langle B x, e_{i}\right\rangle B e_{i}, x\right\rangle=\sum_{i=1}^{k}\left\langle B x, e_{i}\right\rangle\left\langle B x, e_{i}\right\rangle \\
& =\sum_{i=1}^{k}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}=\sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}
\end{aligned}
$$

Now for $x$ arbitrary, let $x_{k} \rightarrow x$ in $W$ where $x_{k} \in \operatorname{span}\left(\left\{g_{k}\right\}_{k=1}^{\infty}\right)$. Then by Fatou's lemma,

$$
\begin{align*}
\sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2} & \leq \lim \inf _{k \rightarrow \infty} \sum_{i=1}^{\infty}\left|\left\langle B x_{k}, e_{i}\right\rangle\right|^{2} \\
& =\lim \inf _{k \rightarrow \infty}\left\langle B x_{k}, x_{k}\right\rangle=\langle B x, x\rangle  \tag{34.4.19}\\
& \leq\|B x\|_{W^{\prime}}\|x\|_{W} \leq\|B\|\|x\|_{W}^{2}
\end{align*}
$$

Thus the series on the left converges. Then also, from the above inequality,

$$
\begin{aligned}
& \left|\left\langle\sum_{i=p}^{q}\left\langle B x, e_{i}\right\rangle B e_{i}, y\right\rangle\right| \leq \sum_{i=p}^{q}\left|\left\langle B x, e_{i}\right\rangle\right|\left|\left\langle B e_{i}, y\right\rangle\right| \\
& \leq\left(\sum_{i=p}^{q}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}\right)^{1 / 2}\left(\sum_{i=p}^{q}\left|\left\langle B y, e_{i}\right\rangle\right|^{2}\right)^{1 / 2} \\
& \leq\left(\sum_{i=p}^{q}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}\right)^{1 / 2}\left(\sum_{i=1}^{\infty}\left|\left\langle B y, e_{i}\right\rangle\right|^{2}\right)^{1 / 2}
\end{aligned}
$$

By 34.4.19,

$$
\leq\left(\sum_{i=p}^{q}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}\right)^{1 / 2}\left(\|B\|\|y\|_{W}^{2}\right)^{1 / 2} \leq\left(\sum_{i=p}^{q}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}\right)^{1 / 2}\|B\|^{1 / 2}\|y\|_{W}
$$

It follows that

$$
\begin{equation*}
\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i} \tag{34.4.20}
\end{equation*}
$$

converges in $W^{\prime}$ because it was just shown that

$$
\left\|\sum_{i=p}^{q}\left\langle B x, e_{i}\right\rangle B e_{i}\right\|_{W^{\prime}} \leq\left(\sum_{i=p}^{q}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}\right)^{1 / 2}\|B\|^{1 / 2}
$$

and it was shown above that $\sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}<\infty$, so the partial sums of the series 34.4.20 are a Cauchy sequence in $W^{\prime}$. Also, the above estimate shows that for $\|y\|=1$,

$$
\begin{aligned}
\left|\left\langle\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i}, y\right\rangle\right| & \leq\left(\sum_{i=1}^{\infty}\left|\left\langle B y, e_{i}\right\rangle\right|^{2}\right)^{1 / 2}\left(\sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}\right)^{1 / 2} \\
& \leq\left(\sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}\right)^{1 / 2}\|B\|^{1 / 2}
\end{aligned}
$$

and so

$$
\begin{equation*}
\left\|\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i}\right\|_{W^{\prime}} \leq\left(\sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}\right)^{1 / 2}\|B\|^{1 / 2} \tag{34.4.21}
\end{equation*}
$$

Now for $x$ arbitrary, let $x_{k} \in \operatorname{span}\left(\left\{g_{j}\right\}_{j=1}^{\infty}\right)$ and $x_{k} \rightarrow x$ in $W$. Then for a fixed $k$ large enough,

$$
\left\|B x-\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i}\right\| \leq\left\|B x-B x_{k}\right\|
$$

$$
\begin{gathered}
+\left\|B x_{k}-\sum_{i=1}^{\infty}\left\langle B x_{k}, e_{i}\right\rangle B e_{i}\right\|+\left\|\sum_{i=1}^{\infty}\left\langle B x_{k}, e_{i}\right\rangle B e_{i}-\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i}\right\| \\
\leq \varepsilon+\left\|\sum_{i=1}^{\infty}\left\langle B\left(x_{k}-x\right), e_{i}\right\rangle B e_{i}\right\|
\end{gathered}
$$

the term

$$
\left\|B x_{k}-\sum_{i=1}^{\infty}\left\langle B x_{k}, e_{i}\right\rangle B e_{i}\right\|
$$

equaling 0 by 34.4.18. From 34.4.21 and 34.4.19,

$$
\begin{aligned}
& \leq \varepsilon+\|B\|^{1 / 2}\left(\sum_{i=1}^{\infty}\left|\left\langle B\left(x_{k}-x\right), e_{i}\right\rangle\right|^{2}\right)^{1 / 2} \\
& \leq \varepsilon+\|B\|^{1 / 2}\left\langle B\left(x_{k}-x\right), x_{k}-x\right\rangle^{1 / 2}<2 \varepsilon
\end{aligned}
$$

whenever $k$ is large enough, the second inequality being implied by 34.4.19. Therefore,

$$
B x=\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i}
$$

in $W^{\prime}$. It follows that

$$
\langle B x, x\rangle=\lim _{k \rightarrow \infty}\left\langle\sum_{i=1}^{k}\left\langle B x, e_{i}\right\rangle B e_{i}, x\right\rangle=\lim _{k \rightarrow \infty} \sum_{i=1}^{k}\left|\left\langle B x, e_{i}\right\rangle\right|^{2} \equiv \sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}
$$

Now consider the measurability assertion on the $e_{i}$. Consider first $e_{1}$. Begin by considering $n_{1}(\omega)$

$$
E_{k}^{1} \equiv\left\{\omega:\left\langle B(\omega) g_{k}, g_{k}\right\rangle \neq 0\right\} \cap \cap_{j<k}\left\{\omega:\left\langle B(\omega) g_{j}, g_{j}\right\rangle=0\right\}
$$

As explained above, $B(\omega)=0$, if and only if $E_{k}^{1}=\emptyset$ for all $k$. Also note that these $E_{k}^{1}$ are disjoint and $\mathscr{F}$ measurable. Then

$$
n_{1}(\omega) \equiv\left\{\begin{array}{l}
1 \text { if } \omega \notin \cup_{k} E_{k}^{1}=\emptyset \\
k \text { if } \omega \in E_{k}^{1}
\end{array}\right.
$$

Then $n_{1}(\omega)$ is clearly measurable because it is constant on measurable sets. Then from the algorithm,

$$
e_{1}(\omega) \equiv \mathscr{X}_{\cup_{k} E_{k}^{1}}(\omega) \frac{g_{n_{1}(\omega)}}{\left\langle B g_{n_{1}(\omega)}, g_{n_{1}(\omega)}\right\rangle^{1 / 2}}
$$

Thus $e_{1}(\omega)=0$ if $\omega \notin \cup_{k} E_{k}^{1}$. Also $e_{1}(\omega)$ is measurable because $\omega \rightarrow n_{1}(\omega)$ is measurable. Thus $e_{1}$ has constant values on measurable sets. So suppose $n_{i}(\omega)$ is measurable for $i \leq m$. Then define $E_{p}^{m+1} \equiv$

$$
\left\{\omega:\left\langle B g_{p}-\sum_{i=1}^{m}\left\langle B g_{p}, e_{i}\right\rangle B e_{i}, g_{p}-\sum_{i=1}^{m}\left\langle B g_{p}, e_{i}\right\rangle e_{i}\right\rangle \neq 0\right\} \cap\left\{\omega: n_{m}(\omega)<p\right\}
$$

$$
\cap_{n_{m}(\omega)<r<p}\left\{\omega:\left\langle B g_{r}-\sum_{i=1}^{m}\left\langle B g_{r}, e_{i}\right\rangle B e_{i}, g_{r}-\sum_{i=1}^{m}\left\langle B g_{r}, e_{i}\right\rangle e_{i}\right\rangle=0\right\}
$$

As earlier, these sets $\left\{E_{p}^{m+1}\right\}_{p=1}^{\infty}$ are disjoint and measurable. As before, let $n_{m+1}(\omega)=p$ where $\omega \in E_{p}^{m+1}$. Then from the algorithm, $e_{m+1}(\omega) \equiv$

$$
\mathscr{X}_{\cup_{p} E_{p}^{m+1}}(\omega) \frac{g_{n_{m+1}(\omega)}-\sum_{i=1}^{m}\left\langle B g_{n_{m+1}(\omega)}, e_{i}\right\rangle e_{i}}{D_{m}}
$$

where $D_{m}=$

$$
\left\langle\begin{array}{c}
B\left(g_{n_{m+1}(\omega)}-\sum_{i=1}^{m}\left\langle B g_{n_{m+1}(\omega)}, e_{i}\right\rangle e_{i}\right), \\
g_{n_{m+1}(\omega)}-\sum_{i=1}^{m}\left\langle B g_{n_{m+1}(\omega)}, e_{i}\right\rangle e_{i}
\end{array}\right\rangle^{1 / 2}
$$

Thus the $e_{k}(\omega)$ are all measurable into $W$ thanks to the algorithm. However, they all have values in $V$. Thus if $\phi \in V^{\prime}$, let $\phi_{n} \rightarrow \phi$ in $V^{\prime}$ where $\phi_{n} \in W^{\prime}$.

$$
\left\langle\phi, e_{k}(\omega)\right\rangle_{V^{\prime}, V}=\lim _{n \rightarrow \infty}\left\langle\phi_{n}, e_{k}(\omega)\right\rangle_{V^{\prime}, V}=\lim _{n \rightarrow \infty}\left\langle\phi_{n}, e_{k}(\omega)\right\rangle_{W^{\prime}, W}
$$

which is the limit of measurable functions. By the Pettis theorem, this shows $e_{k}$ is measurable into $V$ also.

To verify the assertion on right continuity, the same kind of argument holds. We suppress the dependence on $\omega$. Consider first $e_{1}$. Begin by considering $n_{1}(t)$

$$
E_{k}^{1} \equiv\left\{t:\left\langle B(t) g_{k}, g_{k}\right\rangle \neq 0\right\} \cap \cap_{j<k}\left\{t:\left\langle B(t) g_{j}, g_{j}\right\rangle=0\right\}
$$

As explained above, $B(t)=0$, if and only if $E_{k}^{1}=\emptyset$ for all $k$. Also note that these $E_{k}^{1}$ are disjoint. Then

$$
n_{1}(t) \equiv\left\{\begin{array}{l}
1 \text { if } t \notin \cup_{k} E_{k}^{1}=\emptyset \\
k \text { if } t \in E_{k}^{1}
\end{array}\right.
$$

If $t \in E_{k}^{1}$, then from the definition, $\left\langle B(t) g_{k}, g_{k}\right\rangle \neq 0$ and $k$ is the first index for which this is nonzero. Let $t_{l} \downarrow t$. Then by continuity, for all $l$ large enough, $\left\langle B\left(t_{l}\right) g_{k}, g_{k}\right\rangle \neq 0$. What of $\left\langle B\left(t_{l}\right) g_{j}, g_{j}\right\rangle$ for $j<k$ ? By assumption,

$$
\left\langle B^{\prime}(t) g_{j}, g_{j}\right\rangle \leq k_{g_{j}}(t)\left\langle B(t) g_{j}, g_{j}\right\rangle
$$

and so, letting $K_{g_{j}}(t)=\int_{0}^{t} k_{g_{j}}(s) d s$,

$$
\begin{gathered}
\frac{d}{d t}\left(e^{-K_{g_{j}}(t)}\left\langle B(t) g_{j}, g_{j}\right\rangle\right) \leq 0 \\
e^{-K_{g_{j}}\left(t_{l}\right)}\left\langle B\left(t_{l}\right) g_{j}, g_{j}\right\rangle \leq e^{-K_{g_{j}}(t)}\left\langle B(t) g_{j}, g_{j}\right\rangle=0
\end{gathered}
$$

Thus one obtains right continuity of $t \rightarrow n_{1}(t)$ and for $E_{k}^{1}$, there is an interval $[t, t+\boldsymbol{\delta}) \subseteq E_{k}^{1}$. From the algorithm,

$$
e_{1}(t) \equiv \mathscr{X}_{\cup_{k} E_{k}^{1}}(t) \frac{g_{n_{1}(t)}}{\left\langle B g_{n_{1}(t)}, g_{n_{1}(t)}\right\rangle^{1 / 2}}
$$

Thus $e_{1}(t)=0$ if $t \notin \cup_{k} E_{k}^{1}$. Also $e_{1}(t)$ is right continuous because $t \rightarrow n_{1}(t)$ is. Thus $e_{1}$ has constant values on a small interval starting at $t$. But what about $t \notin \cup_{k} E_{k}^{1}$ ? Why should it be right continuous there? If you have such a $t$, then as explained above, $B(t)=0$. Then letting $s$ be arbitrary, $s>t$ and $x \in W$,

$$
\left\langle B^{\prime}(s) x, x\right\rangle \leq k_{x}\langle B(s) x, x\rangle
$$

and so as above,

$$
e^{-K_{x}(s)}\langle B(s) x, x\rangle \leq 0
$$

Thus this case reduces to having $B(s) \equiv 0$ for all $s \geq t$ and there is nothing to prove. You have $n_{1}(s)=1$ and $e_{1}(s)=0$ for all $s \geq t$.

Suppose $t \rightarrow n_{i}(t)$ is right continuous for $i \leq m$ and that $e_{i}$ is also. Then define $E_{p}^{m+1} \equiv$

$$
\begin{gathered}
\left\{t:\left\langle B g_{p}-\sum_{i=1}^{m}\left\langle B g_{p}, e_{i}\right\rangle B e_{i}, g_{p}-\sum_{i=1}^{m}\left\langle B g_{p}, e_{i}\right\rangle e_{i}\right\rangle \neq 0\right\} \cap\left\{t: n_{m}(t)<p\right\} \\
\cap_{n_{m}(t)<r<p}\left\{t:\left\langle B g_{r}-\sum_{i=1}^{m}\left\langle B g_{r}, e_{i}\right\rangle B e_{i}, g_{r}-\sum_{i=1}^{m}\left\langle B g_{r}, e_{i}\right\rangle e_{i}\right\rangle=0\right\}
\end{gathered}
$$

As earlier, these sets $\left\{E_{p}^{m+1}\right\}_{p=1}^{\infty}$ are disjoint. As before, let $n_{m+1}(t)=p$ where $t \in E_{p}^{m+1}$. Then by similar reasoning to the above, for small $\boldsymbol{\delta},[t, t+\boldsymbol{\delta}) \in E_{p}^{m+1}$ and $n_{m+1}(s)=p$ for $s \in[t, t+\delta)$. Then from the algorithm, $e_{m+1}(t) \equiv$

$$
\mathscr{X}_{\cup_{p} E_{p}^{m+1}}(t) \frac{g_{n_{m+1}(t)}-\sum_{i=1}^{m}\left\langle B g_{n_{m+1}(t)}, e_{i}(t)\right\rangle e_{i}(t)}{D_{m}}
$$

where $D_{m}=$

$$
\left\langle\begin{array}{c}
B\left(g_{n_{m+1}(t)}-\sum_{i=1}^{m}\left\langle B g_{n_{m+1}(t)}, e_{i}(t)\right\rangle e_{i}(t)\right), \\
g_{n_{m+1}(t)}-\sum_{i=1}^{m}\left\langle B g_{n_{m+1}(t)}, e_{i}(t)\right\rangle e_{i}(t)
\end{array}\right\rangle^{1 / 2}
$$

and so is right continuous. What of $t \notin \cup_{p} E_{p}^{m+1}$ ? In this case, the process has terminated and what is desired has been found.

Then the main result in this section is the following integration by parts theorem.
Theorem 34.4.3 Let $V \subseteq W, W^{\prime} \subseteq V^{\prime}$ be separable Banach spaces, and let $Y \in L^{p^{\prime}}\left(0, T ; V^{\prime}\right)$ and

$$
\begin{equation*}
B u(t)=B u_{0}+\int_{0}^{t} Y(s) d s \text { in } V^{\prime}, u_{0} \in W, B u(t)=B(u(t)) \text { for a.e. } t \tag{34.4.22}
\end{equation*}
$$

As indicated, $B u$ is the name of a function satisfying the above equation which satisfies $B u(t)=B(u(t))$ for a.e. $t$. Thus $Y=(B u)^{\prime}$ as a weak derivative in the sense of $V^{\prime}$ valued distributions. It is known that $u \in L^{p}(0, T, V)$ for $p>1$. Then $t \rightarrow B u(t)$ is continuous into $W^{\prime}$ for $t$ off a set of measure zero $N$ and also there exists a continuous function $t \rightarrow$ $\langle B u, u\rangle(t)$ such that for all $t \notin N,\langle B u, u\rangle(t)=\langle B(u(t)), u(t)\rangle, B u(t)=B(u(t))$, and for all t,

$$
\frac{1}{2}\langle B u, u\rangle(t)=\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle+\int_{0}^{t}\langle Y(s), u(s)\rangle d s
$$

Note that $\langle B u, u\rangle(0)=\left\langle B u_{0}, u_{0}\right\rangle$.

Proof: By Lemma 34.3.1, there exists a sequence of partitions $\left\{t_{k}^{n}\right\}_{k=0}^{m_{n}}=\mathscr{P}_{n}, \mathscr{P}_{n} \subseteq$ $\mathscr{P}_{n+1}$, of $[0, T]$ such that the lengths of the sub intervals converge uniformly to 0 as $n \rightarrow \infty$ and the step functions

$$
\begin{aligned}
\sum_{k=0}^{m_{n}-1} u\left(t_{k}^{n}\right) \mathscr{X}_{\left(t_{k}^{n}, t_{k+1}^{n}\right]}(t) & \equiv u^{l}(t) \\
\sum_{k=0}^{m_{n}-1} u\left(t_{k+1}^{n}\right) \mathscr{X}_{\left(t_{k}^{n}, t_{k+1}^{n}\right]}(t) & \equiv u^{r}(t)
\end{aligned}
$$

converge to $u$ in $L^{p}(0, T ; V) \equiv K$. We assume that all of these partition points have empty intersection with the set of measure zero where $B u(t) \neq B(u(t))$. Thus, at every partition point, $B u\left(t_{k}\right)=B\left(u\left(t_{k}\right)\right)$. As just mentioned, $L^{p}(0, T ; V) \equiv K, L^{p^{\prime}}\left(0, T ; V^{\prime}\right)=K^{\prime}$.

Lemma 34.4.4 Let $s<t$. Then for $u, Y$ satisfying 34.4.22

$$
\begin{gather*}
\langle B u(t), u(t)\rangle=\langle B u(s), u(s)\rangle \\
+2 \int_{s}^{t}\langle Y(r), u(t)\rangle d r-\langle B u(t)-B u(s), u(t)-u(s)\rangle \tag{34.4.23}
\end{gather*}
$$

Proof: It follows from the following computations

$$
B u(t)-B u(s)=\int_{s}^{t} Y(r) d r
$$

and so

$$
\begin{gathered}
2 \int_{s}^{t}\langle Y(r), u(t)\rangle d r-\langle B u(t)-B u(s), u(t)-u(s)\rangle \\
=2\left\langle\int_{s}^{t} Y(r) d r, u(t)\right\rangle-\langle B u(t)-B u(s), u(t)-u(s)\rangle \\
=2\langle B u(t)-B u(s), u(t)\rangle-\langle B u(t)-B u(s), u(t)-u(s)\rangle \\
=\quad 2\langle B u(t), u(t)\rangle-2\langle B u(s), u(t)\rangle-\langle B u(t), u(t)\rangle \\
\quad+2\langle B u(s), u(t)\rangle-\langle B u(s), u(s)\rangle \\
=\langle B u(t), u(t)\rangle-\langle B u(s), u(s)\rangle
\end{gathered}
$$

Thus

$$
\begin{gathered}
\langle B u(t), u(t)\rangle-\langle B u(s), u(s)\rangle \\
=2 \int_{s}^{t}\langle Y(r), u(t)\rangle d r-\langle B u(t)-B u(s), u(t)-u(s)\rangle
\end{gathered}
$$

Note that in case $s=0$, you can simply write $B u(0)=B u_{0}$ and the same argument appears to work.

Lemma 34.4.5 In the above situation,

$$
\sup _{t \in N^{C}}\langle B u(t), u(t)\rangle \leq C\left(\|Y\|_{K^{\prime}},\|u\|_{K}\right)
$$

Also, $t \rightarrow B u(t)$ is weakly continuous with values in $W^{\prime}$ on $N^{C}$ where $N$ is the set of measure zero where $B u(t) \neq B(u(t))$.

Proof: From the above formula of Lemma 34.4.4 applied to the $k^{t h}$ partition of $[0, T]$ described above,

$$
\begin{aligned}
& \left\langle B u\left(t_{m}\right), u\left(t_{m}\right)\right\rangle-\left\langle B u_{0}, u_{0}\right\rangle=\sum_{j=0}^{m-1}\left\langle B u\left(t_{j+1}\right), u\left(t_{j+1}\right)\right\rangle-\left\langle B u\left(t_{j}\right), u\left(t_{j}\right)\right\rangle \\
& =\sum_{j=0}^{m-1} 2 \int_{t_{j}}^{t_{j+1}}\left\langle Y(r), u\left(t_{j+1}\right)\right\rangle d r-\left\langle B\left(u\left(t_{j+1}\right)-u\left(t_{j}\right)\right), u\left(t_{j+1}\right)-u\left(t_{j}\right)\right\rangle \\
& =\sum_{j=0}^{m-1} 2 \int_{t_{j}}^{t_{j+1}}\left\langle Y(r), u_{k}^{r}(r)\right\rangle d r-\left\langle B\left(u\left(t_{j+1}\right)-u\left(t_{j}\right)\right), u\left(t_{j+1}\right)-u\left(t_{j}\right)\right\rangle
\end{aligned}
$$

Thus, discarding the negative terms and denoting by $\mathscr{P}_{k}$ the $k^{t h}$ of these partitions,

$$
\begin{aligned}
& \sup _{t_{j} \in \mathscr{P}_{k}}\left\langle B u\left(t_{j}\right), u\left(t_{j}\right)\right\rangle \leq\left\langle B u_{0}, u_{0}\right\rangle+2 \int_{0}^{T}\left|\left\langle Y(r), u_{k}^{r}(r)\right\rangle\right| d r \\
& \leq\left\langle B u_{0}, u_{0}\right\rangle+2 \int_{0}^{T}\|Y(r)\|_{V^{\prime}}\left\|u_{k}^{r}(r)\right\|_{V} d r \\
& \leq\left\langle B u_{0}, u_{0}\right\rangle+2\left(\int_{0}^{T}\|Y(r)\|_{V^{\prime}}^{p^{\prime}} d r\right)^{1 / p^{\prime}}\left(\int_{0}^{T}\left\|u_{k}^{r}(r)\right\|_{V}^{p} d r\right)^{1 / p} \\
& \leq C\left(\|Y\|_{K^{\prime}},\|u\|_{K}\right)
\end{aligned}
$$

because these partitions are chosen such that

$$
\lim _{k \rightarrow \infty}\left(\int_{0}^{T}\left\|u_{k}^{r}(r)\right\|_{V}^{p}\right)^{1 / p}=\left(\int_{0}^{T}\|u(r)\|_{V}^{p}\right)^{1 / p}
$$

and so these are bounded. This has shown that for the dense subset of $[0, T], D \equiv \cup_{k} \mathscr{P}_{k}$,

$$
\sup _{t \in D}\langle B u(t), u(t)\rangle<C\left(\|Y\|_{K^{\prime}},\|u\|_{K}\right)
$$

From Lemma 34.4.2 above, there exists $\left\{e_{i}\right\} \subseteq V$ such that $\left\langle B e_{i}, e_{j}\right\rangle=\delta_{i j}$ and for $t \notin N$,

$$
\langle B u(t), u(t)\rangle=\sum_{k=1}^{\infty}\left|\left\langle B u(t), e_{i}\right\rangle\right|^{2}=\sup _{m} \sum_{k=1}^{m}\left|\left\langle B u(t), e_{i}\right\rangle\right|^{2}
$$

Thus, if $s_{n} \rightarrow t, s_{n} \in D$, Fatou's lemma implies

$$
\begin{aligned}
\langle B u, u\rangle(t) & =\langle B(u(t)), u(t)\rangle=\sum_{k=1}^{\infty}\left|\left\langle B u(t), e_{i}\right\rangle\right|^{2} \\
& \leq \lim \inf _{n \rightarrow \infty} \sum_{k=1}^{\infty}\left|\left\langle B u\left(s_{n}\right), e_{i}\right\rangle\right|^{2} \leq C\left(\|Y\|_{K^{\prime}},\|u\|_{K}\right)
\end{aligned}
$$

and so

$$
\sup _{t \in N^{C}}\langle B u, u\rangle(t)=\sup _{t \in N^{C}}\langle B(u(t)), u(t)\rangle \leq C\left(\|Y\|_{K^{\prime}},\|u\|_{K}\right)
$$

It only remains to verify the claim about weak continuity.
Consider now the claim that $t \rightarrow B u(t)$ is weakly continuous on $N^{C}$. Letting $v \in V, s \in$ $N^{C}$,

$$
\begin{equation*}
\lim _{t \rightarrow s}\langle B u(t), v\rangle=\langle B u(s), v\rangle=\langle B u(s), v\rangle \tag{34.4.24}
\end{equation*}
$$

The limit follows from the formula 34.4.22 which implies $t \rightarrow B u(t)$ is continuous into $V^{\prime}$. Now for $t \in N^{C}$,

$$
\|B u(t)\|=\sup _{\|v\| \leq 1}|\langle B u(t), v\rangle| \leq\langle B v, v\rangle^{1 / 2}\langle B u(t), u(t)\rangle^{1 / 2}
$$

which was shown to be bounded for $t, s \in N^{C}$. Now let $w \in W$. Then

$$
|\langle B u(t), w\rangle-\langle B u(s), w\rangle| \leq|\langle B u(t)-B u(s), w-v\rangle|+|\langle B u(t)-B u(s), v\rangle|
$$

Then the first term is less than $\varepsilon$ if $v$ is close enough to $w$ and the second converges to 0 so 34.4.24 holds for all $v \in W$ and so this shows the weak continuity on $N^{C}$.

Now pick $t \in D$, the union of all the mesh points. Then for all $k$ large enough, $t \in \mathscr{P}_{k}$. Say $t=t_{m}$. From Lemma 34.4.4,

$$
\begin{gathered}
-\sum_{j=0}^{m-1}\left\langle B\left(u\left(t_{j+1}\right)-u\left(t_{j}\right)\right),\left(u\left(t_{j+1}\right)-u\left(t_{j}\right)\right)\right\rangle= \\
\left\langle B u\left(t_{m}\right), u\left(t_{m}\right)\right\rangle-\left\langle B u_{0}, u_{0}\right\rangle-2 \sum_{j=0}^{m-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(r), u_{k}^{r}(r)\right\rangle d r
\end{gathered}
$$

Thus, $\left\langle B u\left(t_{m}\right), u\left(t_{m}\right)\right\rangle$ is constant for all $k$ large enough and the integral term converges to

$$
\int_{0}^{t_{m}}\langle Y(r), u(r)\rangle d r
$$

It follows that the term on the left does converge to something as $k \rightarrow \infty$. It just remains to consider what it does converge to. However, from the equation solved by $u$,

$$
B u\left(t_{j+1}\right)-B u\left(t_{j}\right)=\int_{t_{j}}^{t_{j+1}} Y(r) d r
$$

Therefore, this term is dominated by an expression of the form

$$
\begin{aligned}
& \quad\left|\sum_{j=0}^{m_{k}-1}\left\langle\int_{t_{j}}^{t_{j+1}} Y(r) d r, u\left(t_{j+1}\right)-u\left(t_{j}\right)\right\rangle\right| \\
& =\left|\sum_{j=0}^{m_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(r), u\left(t_{j+1}\right)-u\left(t_{j}\right)\right\rangle d r\right| \\
& =\left|\sum_{j=0}^{m_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(r), u\left(t_{j+1}\right)\right\rangle-\sum_{j=0}^{m_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(r), u\left(t_{j}\right)\right\rangle\right| \\
& =\left|\int_{0}^{t_{m}}\left\langle Y(r), u^{r}(r)\right\rangle d r-\int_{0}^{t_{m}}\left\langle Y(r), u^{l}(r)\right\rangle d r\right| \\
& \leq \int_{0}^{T}\left|\left\langle Y(r), u^{r}(r)-u^{l}(r)\right\rangle\right| d r
\end{aligned}
$$

However, both $u^{r}$ and $u^{l}$ converge to $u$ in $K=L^{p}(0, T, V)$. Therefore, this term must converge to 0 . Passing to a limit, it follows that for all $t \in D$, the desired formula holds. Thus, for such $t \in D$,

$$
\langle B u(t), u(t)\rangle=\left\langle B u_{0}, u_{0}\right\rangle+2 \int_{0}^{t}\langle Y(r), u(r)\rangle d r
$$

It remains to verify that this holds for all $t \notin N$. Let $t \in N^{C} \backslash D$ and let $t(k) \in \mathscr{P}_{k}$ be the largest point of $\mathscr{P}_{k}$ which is less than $t$. Suppose $t(m) \leq t(k)$ so that $m \leq k$. Then

$$
B u(t(m))=B u_{0}+\int_{0}^{t(m)} Y(s) d s
$$

a similar formula for $u(t(k))$. Thus for $t>t(m)$,

$$
B u(t)-B u(t(m))=\int_{t(m)}^{t} Y(s) d s
$$

which is the same sort of thing already looked at except that it starts at $t(m)$ rather than at 0 and $u_{0}=0$. Therefore,

$$
\begin{aligned}
& \langle B(u(t(k))-u(t(m))), u(t(k))-u(t(m))\rangle \\
= & 2 \int_{t(m)}^{t(k)}\langle Y(s), u(s)-u(t(m))\rangle d s
\end{aligned}
$$

Thus, for $m \leq k$

$$
\begin{equation*}
\lim _{m, k \rightarrow \infty}\langle B(u(t(k))-u(t(m))), u(t(k))-u(t(m))\rangle=0 \tag{34.4.25}
\end{equation*}
$$

Hence $\{B u(t(k))\}_{k=1}^{\infty}$ is a convergent sequence in $W^{\prime}$ because

$$
\begin{aligned}
& |\langle B(u(t(k))-u(t(m))), y\rangle| \\
\leq & \langle B(u(t(k))-u(t(m))), u(t(k))-u(t(m))\rangle^{1 / 2}\langle B y, y\rangle^{1 / 2} \\
\leq & \langle B(u(t(k))-u(t(m))), u(t(k))-u(t(m))\rangle^{1 / 2}\|B\|^{1 / 2}\|y\|_{W}
\end{aligned}
$$

Does it converge to $B u(t)$ ? Let $\xi(t) \in W^{\prime}$ be what it does converge to. Let $v \in V$. Then

$$
\langle\xi(t), v\rangle=\lim _{k \rightarrow \infty}\langle B u(t(k)), v\rangle=\lim _{k \rightarrow \infty}\langle B u(t(k)), v\rangle=\langle B u(t), v\rangle
$$

because it is known that $t \rightarrow B u(t)$ is continuous into $V^{\prime}$. It is also known that for $t \in N^{C}$, $B u(t) \in W^{\prime} \subseteq V^{\prime}$ and that the $B u(t)$ for $t \in N^{C}$ are uniformly bounded in $W^{\prime}$. Therefore, since $V$ is dense in $W$, it follows that $\xi(t)=B u(t)$.

Now for every $t \in D$, it was shown above that

$$
\langle B u(t), u(t)\rangle=\left\langle B u_{0}, u_{0}\right\rangle+2 \int_{0}^{t}\langle Y(r), u(r)\rangle d r
$$

Also it was just shown that $B u(t(k)) \rightarrow B u(t)$ for $t \notin N$. Then for $t \notin N$

$$
\begin{gathered}
|\langle B u(t(k)), u(t(k))\rangle-\langle B u(t), u(t)\rangle| \\
\leq|\langle B u(t(k)), u(t(k))-u(t)\rangle|+|\langle B u(t(k))-B u(t), u(t)\rangle|
\end{gathered}
$$

Then the second term converges to 0 . The first equals

$$
\begin{aligned}
& |\langle B u(t(k))-B u(t), u(t(k))\rangle| \\
\leq & \langle B(u(t(k))-u(t)), u(t(k))-u(t)\rangle^{1 / 2}\langle B u(t(k)), u(t(k))\rangle^{1 / 2}
\end{aligned}
$$

From the above, this is dominated by an expression of the form

$$
\langle B(u(t(k))-u(t)), u(t(k))-u(t)\rangle^{1 / 2} C
$$

Then using the lower semicontinuity of $t \rightarrow\langle B(u(t(k))-u(t)), u(t(k))-u(t)\rangle$ on $N^{C}$ which follows from the above, this is no larger than

$$
\lim \inf _{m \rightarrow \infty}\langle B(u(t(k))-u(t(m))), u(t(k))-u(t(m))\rangle^{1 / 2} C<\varepsilon
$$

provided $k$ is large enough. This follows from 34.4.25. Since $\varepsilon$ is arbitrary, it follows that

$$
\lim _{k \rightarrow \infty}|\langle B u(t(k)), u(t(k))\rangle-\langle B u(t), u(t)\rangle|=0
$$

Then from the formula,

$$
\langle B u(t), u(t)\rangle=\left\langle B u_{0}, u_{0}\right\rangle+2 \int_{0}^{t}\langle Y(r), u(r)\rangle d r
$$

valid for $t \in D$, it follows that the same formula holds for all $t \notin N$. Then define $\langle B u, u\rangle(t)$ to equal $\langle B u(t), u(t)\rangle$ off $N$ and the right side for $t \in N$. Thus $t \rightarrow\langle B u, u\rangle(t)$ is continuous and for all $t \in[0, T]$,

$$
\langle B u, u\rangle(t)=\left\langle B u_{0}, u_{0}\right\rangle+2 \int_{0}^{t}\langle Y(r), u(r)\rangle d r
$$

Also recall that $t \rightarrow B u(t)$ was shown to be weakly continuous into $W^{\prime}$ on $N^{C}$. Then for $t, s \in N^{C}$,

$$
\begin{aligned}
& \langle B(u(t)-u(s)), u(t)-u(s)\rangle \\
= & \langle B u(t), u(t)\rangle-2\langle B u(t), u(s)\rangle+\langle B u(s), u(s)\rangle
\end{aligned}
$$

From this, it follows that $t \rightarrow B u(t)$ is continuous into $W^{\prime}$ on $N^{C}$ because $\lim _{t \rightarrow s}$ of the right side gives 0 and so the same is true of the left. Hence,

$$
\begin{aligned}
|\langle B(u(t)-u(s)), y\rangle| & \leq\langle B y, y\rangle^{1 / 2}\langle B(u(t)-u(s)), u(t)-u(s)\rangle^{1 / 2} \\
& \leq\|B\|^{1 / 2}\langle B(u(t)-u(s)), u(t)-u(s)\rangle^{1 / 2}\|y\|
\end{aligned}
$$

so

$$
\|B(u(t)-u(s))\|_{W^{\prime}} \leq\|B\|^{1 / 2}\langle B(u(t)-u(s)), u(t)-u(s)\rangle^{1 / 2}
$$

which converges to 0 as $t \rightarrow s$.
Consider the case that $t \rightarrow B(u(t))$ has a weak derivative, denoted as $(B u)^{\prime}(t)$ which is in $L^{p^{\prime}}\left(0, T ; V^{\prime}\right)$. Then as shown above, there is a continuous function, denoted as $B u(t)$ which equals $B(u(t))$ for a.e. $t$ and

$$
B u(t)=B u(0)+\int_{0}^{t}(B u)^{\prime}(s) d s
$$

Then the above theorem applies. Then one obtains the following corollary.
Corollary 34.4.6 Let $V \subseteq W, W^{\prime} \subseteq V^{\prime}$ be separable Banach spaces, and $B \in \mathscr{L}\left(W, W^{\prime}\right)$ is nonnegative and self adjoint. Also suppose $t \rightarrow B(u(t))$ has a weak derivative $(B u)^{\prime} \in$ $L^{p^{\prime}}\left(0, T ; V^{\prime}\right)$ for $u \in L^{p}(0, T ; V)$. Then there is a continuous function denoted as $B u(t)$ which equals $B(u(t))$ a.e. $t$. Say for $t \notin N$. Suppose $B u(0)=B u_{0}, u_{0} \in W$. Then

$$
\begin{equation*}
B u(t)=B u_{0}+\int_{0}^{t}(B u)^{\prime}(s) d s \text { in } V^{\prime} \tag{34.4.26}
\end{equation*}
$$

Then $t \rightarrow B u(t)$ is in $C\left(N^{C}, W^{\prime}\right)$ and also for such $t$,

$$
\frac{1}{2}\langle B u(t), u(t)\rangle=\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle+\int_{0}^{t}\left\langle(B u)^{\prime}(s), u(s)\right\rangle d s
$$

There exists a continuous function $t \rightarrow\langle B u, u\rangle(t)$ which equals the right side of the above for all $t$ and equals $\langle B u(t), u(t)\rangle$ off $N$. This also satisfies

$$
\sup _{t \in[0, T]}\langle B u, u\rangle(t) \leq C\left(\left\|(B u)^{\prime}\right\|_{L^{p^{\prime}}\left(0, T, V^{\prime}\right)},\|u\|_{L^{p}(0, T, V)}\right)
$$

where we can take the right side to equal

$$
\left\langle B u_{0}, u_{0}\right\rangle+2\left\|(B u)^{\prime}\right\|_{L^{p^{\prime}}\left(0, T, V^{\prime}\right)}\|u\|_{L^{p}(0, T, V)}
$$

This follows from the above theorem, in particular Lemma 34.4.5.
This also makes it easy to verify continuity of pointwise evaluation of $B u$. Let $L u=$ $(B u)^{\prime}$.

$$
\begin{gather*}
u=D(L) \equiv\left\{u \in L^{p}(0, T ; V): L u \equiv(B u)^{\prime} \in L^{p^{\prime}}\left(0, T, V^{\prime}\right)\right\} \\
\|u\|_{X} \equiv\|u\|_{L^{p}(0, T, V)}+\|L u\|_{L^{p^{\prime}}\left(0, T, V^{\prime}\right)} \tag{34.4.27}
\end{gather*}
$$

Since $L$ is closed, this $X$ is a Banach space.
Then the following theorem is obtained.
Theorem 34.4.7 $\operatorname{Say}(B u)^{\prime} \in L^{p^{\prime}}\left(0, T, V^{\prime}\right)$ so

$$
B u(t)=B u(0)+\int_{0}^{t}(B u)^{\prime}(s) d s \text { in } V^{\prime}
$$

the map $u \rightarrow B u(t)$ is continuous as a map from $X$ to $V^{\prime}$. Also, if $Y$ denotes those $f \in$ $L^{p}([0, T] ; V)$ for which $f^{\prime} \in L^{p}([0, T] ; V)$, so that $f$ has a representative such that $f(t)=$ $f(0)+\int_{0}^{t} f^{\prime}(s) d s$, then if $\|f\|_{Y} \equiv\|f\|_{L^{p}([0, T] ; V)}+\left\|f^{\prime}\right\|_{L^{p}([0, T] ; V)}$, the map $f \rightarrow f(t)$ is continuous.

Proof: First, why is $u \rightarrow B u(0)$ continuous? Say $u, v \in X$ and say $p \geq 2$ first.

$$
B u(t)-B v(t)=B u(0)-B v(0)+\int_{0}^{t}(B u)^{\prime}(s)-(B v)^{\prime}(s) d s
$$

and so,

$$
\|B u(0)-B v(0)\|_{V^{\prime}} \leq\|B u(t)-B v(t)\|_{V^{\prime}}+\int_{0}^{t}\left\|(B u)^{\prime}(s)-(B v)^{\prime}(s)\right\|_{V^{\prime}} d s
$$

then using the triangle inequality,

$$
\begin{aligned}
& \left(\int_{0}^{T}\|B u(0)-B v(0)\|_{V^{\prime}}^{p^{\prime}} d t\right)^{1 / p^{\prime}} \leq\left(\int_{0}^{T}\|B u(t)-B v(t)\|_{V^{\prime}}^{p^{\prime}} d t\right)^{1 / p^{\prime}} \\
& +\left(\int_{0}^{T}\left\|\int_{0}^{t}(B u)^{\prime}(s)-(B v)^{\prime}(s) d s\right\|^{p^{\prime}} d t\right)^{1 / p^{\prime}}
\end{aligned}
$$

and so

$$
\begin{gathered}
\|B u(0)-B v(0)\|_{V^{\prime}} T^{1 / p^{\prime}} \leq \\
\left(\|B\|\|u-v\|_{L^{p^{\prime}}([0, T] ; V)}+T^{1 / p^{\prime}}\left\|(B u)^{\prime}-(B v)^{\prime}\right\|_{L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)}\right) \\
\leq\left(\|B\|\|u-v\|_{L^{p}([0, T] ; V)}+T^{1 / p^{\prime}}\left\|(B u)^{\prime}-(B v)^{\prime}\right\|_{L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)}\right)
\end{gathered}
$$

$$
\leq C(\|B\|, T)\|u-v\|_{X}
$$

Thus $u \rightarrow B u(0)$ is continuous into $V^{\prime}$. If $p<2$, then you do something similar.

$$
\begin{gathered}
\left(\int_{0}^{T}\|B u(0)-B v(0)\|_{V^{\prime}}^{p} d t\right)^{1 / p} \leq\left(\int_{0}^{T}\|B u(t)-B v(t)\|_{V^{\prime}}^{p} d t\right)^{1 / p} \\
+\left(\int_{0}^{T}\left\|\int_{0}^{t}(B u)^{\prime}(s)-(B v)^{\prime}(s) d s\right\|^{p} d t\right)^{1 / p} \\
\leq\|B\|\|u-v\|_{L^{p}([0, T] ; V)}+\int_{0}^{T}\left(\int_{0}^{T}\left\|(B u)^{\prime}(s)-(B v)^{\prime}(s)\right\|^{p} d t\right)^{1 / p} d s \\
\leq\|B\|\|u-v\|_{L^{p}([0, T] ; V)}+T^{1 / p} \int_{0}^{T}\left\|(B u)^{\prime}(s)-(B v)^{\prime}(s)\right\| d s \\
\leq\|B\|\|u-v\|_{L^{p}([0, T] ; V)}+C T^{1 / p}\left\|(B u)^{\prime}-(B v)^{\prime}\right\|_{L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)}
\end{gathered}
$$

Thus

$$
\begin{aligned}
\|B u(0)-B v(0)\|_{V^{\prime}} T^{1 / p} & \leq\|B\|\|u-v\|_{L^{p}([0, T] ; V)}+C(T)\left\|(B u)^{\prime}-(B v)^{\prime}\right\|_{L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)} \\
& \leq C(\|B\|, T)\|u-v\|_{X}
\end{aligned}
$$

However, one could just as easily have done this for an arbitrary $s<T$ by repeating the argument for

$$
B u(t)=B u(s)+\int_{s}^{t}(B u)^{\prime}(r) d r
$$

Thus this mapping is certainly continuous into $V^{\prime}$. The last assertion is similar. You just use $f$ instead of $B u$ and make easy modifications in the argument. It is all happening in one space in the second case.

For $u \in X$ defined above,

$$
B u(t)=B u(0)+\int_{0}^{t}(B u)^{\prime}(s) d s
$$

and also

$$
\frac{1}{2}\langle B u(t), u(t)\rangle=\frac{1}{2}\langle B u, u\rangle(0)+\int_{0}^{t}\left\langle(B u)^{\prime}(s), u(s)\right\rangle d s
$$

This follows from a similar argument given above, (Note we write $\langle B u, u\rangle(0)$ instead of $\left\langle B u_{0}, u_{0}\right\rangle$ since no $u_{0}$ is mentioned. One could also use the above by considering the problem on $[s, t]$ where $s$ is not in the exceptional set where it makes a difference between writing $B u(s)$ and $B(u(s))$. Then you would get the above with 0 replaced with $s$ and then let $s \rightarrow 0$ to finally obtain the above displayed formula. ) and

$$
\sup _{t \in[0, T]}\langle B u, u\rangle(t) \leq C\left(\left\|(B u)^{\prime}\right\|_{L^{p^{\prime}}\left(0, T, V^{\prime}\right)},\|u\|_{L^{p}(0, T, V)}\right)=C\left(\|u\|_{X}\right)
$$

where $X$ was defined in 34.4.27, then

$$
\sup _{t \in[0, T]}\left\langle B \frac{u}{\|u\|_{X}}, \frac{u}{\|u\|_{X}}\right\rangle(t) \leq C(1)=C
$$

and so

$$
\sup _{t \in[0, T]}\langle B u, u\rangle(t) \leq C\|u\|_{X}^{2}
$$

Now define for $u, v \in X$

$$
\langle B u, v\rangle(t) \equiv \frac{1}{2}[\langle B(u+v), u+v\rangle(t)-(\langle B u, u\rangle(t)+\langle B v, v\rangle(t))]
$$

and so for a.e. $t,\langle B u, v\rangle(t)=\langle B(u(t)), v(t)\rangle$ and $t \rightarrow\langle B u, v\rangle(t)$ is continous. Also, there must exist $C$ such that for all $u, v$ and $t \in[0, T]$,

$$
|\langle B u, v\rangle(t)| \leq C\|u\|_{X}\|v\|_{X}
$$

If this is not so, then you could get $u_{n}, v_{n}$ having norm equal to 1 in $X$ such that

$$
\sup _{t \in[0, T]}\left|\left\langle B u_{n}, v_{n}\right\rangle(t)\right|>n
$$

But then, letting $t_{n}$ be a point where $\left|\left\langle B u_{n}, v_{n}\right\rangle\left(t_{n}\right)\right|>n$,

$$
n<\left|\left\langle B u_{n}, v_{n}\right\rangle\left(t_{n}\right)\right| \leq \frac{1}{2}\left[C\left(\left\|u_{n}+v_{n}\right\|_{X}^{2}+\|u\|_{X}^{2}+\|v\|_{X}^{2}\right)\right]=\frac{C}{2}(4+1+1)=3 C
$$

which is clearly a contradiction. It follows that one can define $K: X \rightarrow X^{\prime}$ as follows.

$$
\langle K u, v\rangle \equiv \int_{0}^{T}\langle L u, v\rangle d s+\langle B u, v\rangle(0)
$$

Thus $K$ is linear and continuous. In addition,

$$
\langle K u, u\rangle=\frac{1}{2}[\langle B u, u\rangle(T)+\langle B u, u\rangle(0)]
$$

To see this, Corollary 34.4.6 implies

$$
\frac{1}{2}\langle B u, u\rangle(T)=\frac{1}{2}\langle B u, u\rangle(0)+\int_{0}^{T}\left\langle(B u)^{\prime}(s), u(s)\right\rangle d s
$$

and so

$$
\begin{aligned}
& \frac{1}{2}\langle B u, u\rangle(T)+\langle B u, u\rangle(0) \\
= & \frac{1}{2}\langle B u, u\rangle(0)+\int_{0}^{T}\left\langle(B u)^{\prime}(s), u(s)\right\rangle d s+\langle B u, u\rangle(0)
\end{aligned}
$$

and so, this yields

$$
\int_{0}^{T}\left\langle(B u)^{\prime}(s), u(s)\right\rangle d s+\langle B u, u\rangle(0)=\langle K u, u\rangle=\frac{1}{2}[\langle B u, u\rangle(T)+\langle B u, u\rangle(0)]
$$

as claimed. This proves most of the following.

Proposition 34.4.8 Let

$$
X=\left\{u \in L^{p}(0, T ; V) \equiv \mathscr{V}: L u \equiv(B u)^{\prime} \in L^{p^{\prime}}\left(0, T, V^{\prime}\right)\right\}
$$

where $V$ is a reflexive Banach space. Let a norm on $X$ be given by

$$
\|u\|_{X} \equiv\|u\|_{\mathscr{V}}+\|L u\|_{\mathscr{V}^{\prime}}
$$

Then there is a continuous function $t \rightarrow\langle B u, v\rangle(t)$ such that $\langle B u, v\rangle(t)=\langle B(u(t)), v(t)\rangle$ a.e. $t$ such that

$$
\sup _{t \in[0, T]}|\langle B u, v\rangle(t)| \leq C\|u\|_{X}\|v\|_{X}
$$

and if $K: X \rightarrow X^{\prime}$

$$
\langle K u, v\rangle \equiv \int_{0}^{T}\langle L u, v\rangle d s+\langle B u, v\rangle(0)
$$

Then $K$ is continuous and linear and

$$
\langle K u, u\rangle=\frac{1}{2}[\langle B u, u\rangle(T)+\langle B u, u\rangle(0)]
$$

If $u \in X$ and $B u(0)=0$ then there exists a sequence $\left\{u_{n}\right\}$ such that $\left\|u_{n}-u\right\|_{X} \rightarrow 0$ but $u_{n}(t)=0$ for all t close to 0 .

Proof: It only remains to verify the last assertion. Let $\psi_{n}$ be increasing and piecewise linear such that $\psi_{n}(t)=1$ for $t \geq 2 / n$ and equals 0 on $[0,1 / n]$. Then clearly $\psi_{n} u \rightarrow u$ in $\mathscr{V}$.

$$
\left(B\left(\psi_{n} u\right)\right)^{\prime}=\psi_{n}^{\prime} B u+\psi_{n}(B u)^{\prime}
$$

The second term converges to $(B u)^{\prime}$ in $\mathscr{V}^{\prime}$. It remains to consider the first term.

$$
\begin{gathered}
\int_{0}^{T}\left\|\psi_{n}^{\prime} B u\right\|_{V^{\prime}}^{p^{\prime}} d t \leq \int_{0}^{2 / n} n\left\|\int_{0}^{t}(B u)^{\prime} d s\right\|^{p^{\prime}} d t \\
\leq n \int_{0}^{2 / n} t^{p^{\prime}-1} \int_{0}^{t}\left\|(B u)^{\prime}\right\|_{V^{\prime}}^{p^{\prime}} d s d t \leq \int_{0}^{2 / n}\left\|(B u)^{\prime}\right\|_{V^{\prime}}^{p^{\prime}} d s \frac{1}{p^{\prime}}(2 / n)^{p^{\prime}} n
\end{gathered}
$$

Since $p^{\prime}>1$, this converges to 0 .
Note that, by convolving with a mollifier, we could assume each $u_{n}$ is also smooth. In addition to this, we can draw a similar conclusion at the right endpoint. That is, if $B u(T)=0$ there is a sequence $\left\{u_{n}\right\} \subseteq X$ where $u_{n}(t)=0$ for $t$ near $T$ which converges to $u$ in $X$.

### 34.5 The Implicit Case, $B=B(t)$

The above theorem can be generalized to the case where the formula is of the form

$$
B X(t)=B X_{0}+\int_{0}^{t} Y(s) d s
$$

This involves an operator $B(t) \in \mathscr{L}\left(W, W^{\prime}\right)$ and $B(t)$ satisfies

$$
\langle B(t) x, x\rangle \geq 0,\langle B(t) x, y\rangle=\langle B(t) y, x\rangle
$$

for

$$
V \subseteq W, W^{\prime} \subseteq V^{\prime}
$$

Where we assume $t \rightarrow B(t)$ is in $C^{1}\left([0, T] ; \mathscr{L}\left(W, W^{\prime}\right)\right)$ and $V$ is dense in the Banach space $W$.

Then the main result in this section is the following integration by parts theorem.
Theorem 34.5.1 Let $V \subseteq W, W^{\prime} \subseteq V^{\prime}$ be separable Banach spaces, and let $Y \in L^{p^{\prime}}\left(0, T ; V^{\prime}\right)$ and

$$
\begin{equation*}
B u(t)=B u_{0}+\int_{0}^{t} Y(s) d s \text { in } V^{\prime}, u_{0} \in W, B u(t)=B(t)(u(t)) \text { for a.e. } t \tag{34.5.28}
\end{equation*}
$$

As indicated, $B u$ is the name of a function satisfying the above equation which satisfies $B u(t)=B(t)(u(t))$ for a.e. $t$. Thus $Y=(B u)^{\prime}$ as a weak derivative in the sense of $V^{\prime}$ valued distributions. Suppose that $u \in L^{p}([0, T], V)$ and $(s, t) \rightarrow B^{\prime}(s) u(t)$ is bounded in $V^{\prime}$ in case $p<2$. (If $B(t)$ is constant in this is obvious.) In the case where $p \geq 2$, it is enough to assume $B^{\prime} \in C^{1}\left([0, T] ; \mathscr{L}\left(W, W^{\prime}\right)\right)$. Then $t \rightarrow B u(t)$ is continuous into $W^{\prime}$ for $t$ off a set of measure zero $N$ and also there exists a continuous function $t \rightarrow\langle B u, u\rangle(t)$ such that for all $t \notin N,\langle B u, u\rangle(t)=\langle B(u(t)), u(t)\rangle, B u(t)=B(t)(u(t))$, and for all $t$,

$$
\frac{1}{2}\langle B u, u\rangle(t)+\frac{1}{2} \int_{0}^{t}\left\langle B^{\prime} u, u\right\rangle d s=\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle+\int_{0}^{t}\langle Y(s), u(s)\rangle d s
$$

Proof: By Lemma 34.3.1, there exists a sequence of partitions $\left\{t_{k}^{n}\right\}_{k=0}^{m_{n}}=\mathscr{P}_{n}, \mathscr{P}_{n} \subseteq$ $\mathscr{P}_{n+1}$, of $[0, T]$ such that the lengths of the sub intervals converge uniformly to 0 as $n \rightarrow \infty$ and the step functions

$$
\begin{aligned}
\sum_{k=0}^{m_{n}-1} u\left(t_{k}^{n}\right) \mathscr{X}_{\left(t_{k}^{n}, t_{k+1}^{n}\right]}(t) & \equiv u_{n}^{l}(t) \\
\sum_{k=0}^{m_{n}-1} u\left(t_{k+1}^{n}\right) \mathscr{X}_{\left(t_{k}^{n}, t_{k+1}^{n}\right]}(t) & \equiv u_{n}^{r}(t)
\end{aligned}
$$

converge to $u$ in $L^{p}(0, T ; V) \equiv K$. We assume that all of these partition points have empty intersection with the set of measure zero where $B u(t) \neq B(t)(u(t))$. Thus, at every partition point, $B u\left(t_{k}\right)=B\left(t_{k}\right)\left(u\left(t_{k}\right)\right)$. As just mentioned, $L^{p}(0, T ; V) \equiv K, L^{p^{\prime}}\left(0, T ; V^{\prime}\right)=K^{\prime}$.

Taking a subsequence, we can have

$$
\begin{align*}
& \quad\left\|u_{n}^{l}-u\right\|_{K}+\left\|u_{n}^{r}-u\right\|_{K}+\left\|B u_{n}^{l}-B u\right\|_{K^{\prime}}+\left\|B u_{n}^{r}-B u\right\|_{K^{\prime}} \\
& +\left\|B^{\prime} u_{n}^{r}-B^{\prime} u\right\|_{L^{2}\left([0, T], W^{\prime}\right)}+\left\|B^{\prime} u_{n}^{l}-B^{\prime} u\right\|_{L^{2}\left([0, T], W^{\prime}\right)}<2^{-n} \tag{34.5.29}
\end{align*}
$$

and so, we can assume that $a . e$. convergence also takes place for $B u_{n}^{l}, B u_{n}^{r}, B^{\prime} u_{n}^{l}, B^{\prime} u_{n}^{r}, u_{n}^{r}, u_{n}^{l}$.
Is $B u(0)=B(0) u_{0}$ ? The integral equation gives this it seems. To save notation, $B(0) u_{0}$ will be written as $B u_{0}$. This is not inconsistent because $t \rightarrow B(t) u_{0}$ is continuous and its value at 0 is $B(0) u_{0}$.

Lemma 34.5.2 Let $s<t$. Then for $u, Y$ satisfying 34.5.28

$$
\begin{gather*}
\langle B u(t), u(t)\rangle-\langle B u(s), u(s)\rangle+\langle(B(t)-B(s)) u(s), u(t)\rangle \\
+\langle(B(t)-B(s)) u(s), u(t)-u(s)\rangle=2 \int_{s}^{t}\langle Y(r), u(t)\rangle d r \\
-\langle B(t) u(t)-B(t) u(s), u(t)-u(s)\rangle \tag{34.5.30}
\end{gather*}
$$

Proof: It follows from the following computations

$$
B(t) u(t)-B(s) u(s)=\int_{s}^{t} Y(r) d r
$$

and so

$$
\begin{gathered}
2 \int_{s}^{t}\langle Y(r), u(t)\rangle d r-\langle B(t) u(t)-B(s) u(s), u(t)-u(s)\rangle \\
=2\left\langle\int_{s}^{t} Y(r) d r, u(t)\right\rangle-\langle B(t) u(t)-B(s) u(s), u(t)-u(s)\rangle \\
=2\langle B(t) u(t)-B(s) u(s), u(t)\rangle-\langle B(t) u(t)-B(s) u(s), u(t)-u(s)\rangle \\
=\quad 2\langle B(t) u(t), u(t)\rangle-2\langle B(s) u(s), u(t)\rangle-\langle B(t) u(t), u(t)\rangle \\
+\langle B(t) u(t), u(s)\rangle+\langle B(s) u(s), u(t)\rangle-\langle B(s) u(s), u(s)\rangle \\
=\quad\langle B(t) u(t), u(t)\rangle-\langle B(s) u(s), u(s)\rangle \\
+[\langle B(t) u(t), u(s)\rangle-\langle B(s) u(s), u(t)\rangle] \\
=\quad\langle B(t) u(t), u(t)\rangle-\langle B(s) u(s), u(s)\rangle \\
\\
+\langle(B(t)-B(s)) u(s), u(t)\rangle
\end{gathered}
$$

Thus

$$
\begin{aligned}
& \langle B u(t), u(t)\rangle-\langle B u(s), u(s)\rangle+\langle(B(t)-B(s)) u(s), u(t)\rangle \\
= & 2 \int_{s}^{t}\langle Y(r), u(t)\rangle d r-\langle B(t) u(t)-B(s) u(s), u(t)-u(s)\rangle
\end{aligned}
$$

Now consider the last term. It equals

$$
\begin{gathered}
\langle B(t) u(t)-(B(s)-B(t)+B(t)) u(s), u(t)-u(s)\rangle \\
=\langle B(t) u(t)-((B(s)-B(t)) u(s)+B(t) u(s)), u(t)-u(s)\rangle \\
=\langle B(t) u(t)-B(t) u(s), u(t)-u(s)\rangle+\langle(B(t)-B(s)) u(s), u(t)-u(s)\rangle
\end{gathered}
$$

It follows that

$$
\langle B u(t), u(t)\rangle-\langle B u(s), u(s)\rangle+\langle(B(t)-B(s)) u(s), u(t)\rangle
$$

$$
\begin{gathered}
+\langle(B(t)-B(s)) u(s), u(t)-u(s)\rangle \\
=2 \int_{s}^{t}\langle Y(r), u(t)\rangle d r-\langle B(t) u(t)-B(t) u(s), u(t)-u(s)\rangle
\end{gathered}
$$

Of course this computation is under the assumption that neither $s, t$ are in the exceptional set off which $B(t) u(t)=B u(t)$. In case $s=0$ the same formula holds except you need to replace $u(s)$ with $u_{0}$ and $B u(s)$ with $B(0) u_{0}=B u(0)$.

It is good to emphasize part of the above.

$$
\begin{gathered}
\langle B(t) u(t)-B(t) u(s), u(t)-u(s)\rangle-\langle B(t) u(t)-B(s) u(s), u(t)-u(s)\rangle \\
=\langle(B(s)-B(t)) u(s), u(t)-u(s)\rangle
\end{gathered}
$$

Lemma 34.5.3 Let the partitions $\mathscr{P}_{k}$ be as above such that 34.5.29, $\mathscr{P}_{k}=\left\{t_{j}^{k}\right\}_{j=0}^{m_{k}}$. Then for any $m \leq m_{k}$,

$$
\begin{aligned}
& \sum_{j=0}^{m-1}\left\langle B\left(t_{j+1}^{k}\right) u\left(t_{j+1}^{k}\right)-B\left(t_{j+1}^{k}\right) u\left(t_{j}^{k}\right), u\left(t_{j+1}^{k}\right)-u\left(t_{j}^{k}\right)\right\rangle- \\
& \sum_{j=0}^{m-1}\left\langle B\left(t_{j+1}^{k}\right) u\left(t_{j+1}^{k}\right)-B\left(t_{j}^{k}\right) u\left(t_{j}^{k}\right), u\left(t_{j+1}^{k}\right)-u\left(t_{j}^{k}\right)\right\rangle=\varepsilon^{m}(k)
\end{aligned}
$$

where $\lim _{k \rightarrow \infty} \varepsilon^{m}(k)=0$. Here

$$
\varepsilon^{m}(k)=\sum_{j=0}^{m-1}\left\langle\left(B\left(t_{j}^{k}\right)-B\left(t_{j+1}^{k}\right)\right) u\left(t_{j}^{k}\right), u\left(t_{j+1}^{k}\right)-u\left(t_{j}^{k}\right)\right\rangle
$$

Proof: From the above lemma, the absolute value of the left side is no larger than

$$
\begin{align*}
& \sum_{j=0}^{m-1}\left|\left\langle\left(B\left(t_{j}^{k}\right)-B\left(t_{j+1}^{k}\right)\right) u\left(t_{j}^{k}\right), u\left(t_{j+1}^{k}\right)-u\left(t_{j}^{k}\right)\right\rangle\right| \\
\leq & \sum_{j=0}^{m-1} \int_{t_{j}^{k}}^{t_{j+1}^{k}}\left\|B^{\prime}(\tau) u\left(t_{j}^{k}\right)\right\|_{W^{\prime}} d \tau\left\|u\left(t_{j+1}^{k}\right)-u\left(t_{j}^{k}\right)\right\|_{W} \tag{34.5.31}
\end{align*}
$$

In case $p \geq 2$ then for $C \geq \max _{s}\left\|B^{\prime}(s)\right\|_{\mathscr{L}\left(W, W^{\prime}\right)}$,

$$
\begin{aligned}
& \leq C \sum_{j=0}^{m-1} \int_{t_{j}^{k}}^{t_{j+1}^{k}}\left\|u^{l}(\tau)\right\|_{W}\left\|u_{k}^{r}(\tau)-u_{k}^{l}(\tau)\right\|_{W} d \tau \\
& =C \sum_{j=0}^{m-1} \int_{0}^{t_{m}^{k}} \mathscr{X}_{\left[t_{j}^{k}, t_{j+1}^{k}\right]}(\tau)\left\|u_{k}^{l}(\tau)\right\|_{W}\left\|u_{k}^{r}(\tau)-u_{k}^{l}(\tau)\right\|_{W} d \tau \\
& =C \int_{0}^{t_{m}^{k}} \sum_{j=0}^{m-1} \mathscr{X}_{\left[t_{j}^{k}, t_{j+1}^{k}\right]}(\tau)\left\|u_{k}^{l}(\tau)\right\|_{W}\left\|u_{k}^{r}(\tau)-u_{k}^{l}(\tau)\right\|_{W} d \tau \\
& =C \int_{0}^{t_{m}^{k}}\left\|u^{l}(\tau)\right\|_{W}\left\|u_{k}^{r}(\tau)-u_{k}^{l}(\tau)\right\|_{W} d \tau \\
& \leq C\left\|u_{k}^{l}\right\|_{L^{p}([0, T], V)}\left\|u_{k}^{r}(\tau)-u_{k}^{l}(\tau)\right\|_{L^{p}([0, T], V)} \\
& \leq \hat{C}(2) 2^{-k}
\end{aligned}
$$

by 34.5.29. In case $p<2$, then from assumption and 34.5.31, the absolute value of the left side is no larger than

$$
\begin{aligned}
& \sum_{j=0}^{m-1} C\left(t_{j+1}^{k}-t_{j}^{k}\right)\left\|u\left(t_{j+1}^{k}\right)-u\left(t_{j}^{k}\right)\right\|_{W} \\
= & C \sum_{j=0}^{m-1} \int_{t_{j}^{k}}^{t_{j+1}^{k}} \mathscr{X}_{\left[\left[_{j}^{k}, t_{j+1}^{k}\right]\right.}(s)\left\|u_{k}^{r}(s)-u_{k}^{l}(s)\right\|_{W} \\
= & C \int_{0}^{t_{m}^{k}}\left\|u_{k}^{r}(s)-u_{k}^{l}(s)\right\|_{W}
\end{aligned}
$$

which converges to 0 as $k \rightarrow \infty$ thanks to 34.5.29.
Lemma 34.5.4 In the above situation,

$$
\sup _{t \in N^{C}}\langle B u(t), u(t)\rangle+\int_{0}^{T}\left\langle B^{\prime} u, u\right\rangle d s \leq C\left(\|Y\|_{K^{\prime}},\|u\|_{K}\right)
$$

Also, $t \rightarrow B u(t)$ is weakly continuous with values in $W^{\prime}$ on $N^{C}$ where $N$ is a set of measure zero including the set where $B u(t) \neq B(t)(u(t))$.

Proof: From the above formula of Lemma 34.5.2 applied to the $k^{t h}$ partition of $[0, T]$ described above,

$$
\begin{gathered}
\left\langle B u\left(t_{m}\right), u\left(t_{m}\right)\right\rangle-\left\langle B u_{0}, u_{0}\right\rangle+\sum_{j=0}^{m-1}\left\langle\left(B\left(t_{j+1}\right)-B\left(t_{j}\right)\right) u\left(t_{j}\right), u\left(t_{j+1}\right)\right\rangle \\
+\sum_{j=0}^{m-1}\left\langle\left(B\left(t_{j+1}\right)-B\left(t_{j}\right)\right) u\left(t_{j}\right), u\left(t_{j+1}\right)-u\left(t_{j}\right)\right\rangle
\end{gathered}
$$

$$
\begin{equation*}
=\sum_{j=0}^{m-1} 2 \int_{t_{j}}^{t_{j+1}}\left\langle Y(r), u\left(t_{j+1}\right)\right\rangle d r-\left\langle B\left(t_{j+1}\right) u\left(t_{j+1}\right)-B\left(t_{j+1}\right) u\left(t_{j}\right), u\left(t_{j+1}\right)-u\left(t_{j}\right)\right\rangle \tag{34.5.32}
\end{equation*}
$$

Consider the third term on the left,

$$
\begin{aligned}
& \sum_{j=0}^{m_{n}-1}\left\langle\left(B\left(t_{j+1}^{n}\right)-B\left(t_{j}^{n}\right)\right) u\left(t_{j}^{n}\right), u\left(t_{j+1}^{n}\right)\right\rangle \\
= & \int_{0}^{t_{m_{n}}}\left\langle\sum_{j=0}^{m_{n}-1} \mathscr{X}_{\left(t_{j}^{n}, t_{j+1}^{n}\right]}(t) \frac{B\left(t_{j+1}^{n}\right)-B\left(t_{j}^{n}\right)}{t_{j+1}^{n}-t_{j}^{n}} u_{n}^{l}(t), u_{n}^{r}(t)\right\rangle d t
\end{aligned}
$$

Using a simple approximate identity argument and the assumption that $t \rightarrow B(t)$ is in $C^{1}\left([0, T], \mathscr{L}\left(W, W^{\prime}\right)\right)$,

$$
\sum_{j=0}^{m_{n}-1} \mathscr{X}_{\left(t_{j}^{n}, t_{j+1}^{n}\right]}(t) \frac{B\left(t_{j+1}^{n}\right)-B\left(t_{j}^{n}\right)}{t_{j+1}^{n}-t_{j}^{n}} \rightarrow B^{\prime}(t)
$$

uniformly on $(0, T]$. Then

$$
\sum_{j=0}^{m_{n}-1} \mathscr{X}_{\left(t_{j}^{n}, t_{j+1}^{n}\right]}(t) \frac{B\left(t_{j+1}^{n}\right)-B\left(t_{j}^{n}\right)}{t_{j+1}^{n}-t_{j}^{n}} u_{n}^{l} \rightarrow B^{\prime} u
$$

strongly in $L^{2}\left([0, T], W^{\prime}\right)$ while $u_{n}^{r} \rightarrow u$ strongly in $L^{2}([0, T] ; W)$. It follows that the third term on the left in 34.5 .32 is

$$
\varepsilon(k)+2 \int_{0}^{T}\left\langle B^{\prime} u, u\right\rangle d s, \varepsilon(k) \rightarrow 0 .
$$

whenever $n$ is sufficiently large. Also, $T$ could be replaced with $t_{j}$ for any of the mesh points.

Next consider the term labelled $*$. From Lemma 34.5.3, it is of the form $\varepsilon^{m}(k)$ where $\lim _{k \rightarrow \infty} \varepsilon^{m}(k)=0$. Thus 34.5 .32 reduces to

$$
\begin{align*}
& \left\langle B u\left(t_{m}\right), u\left(t_{m}\right)\right\rangle-\left\langle B u_{0}, u_{0}\right\rangle+\int_{0}^{t_{m}}\left\langle B^{\prime} u, u\right\rangle d s=\sum_{j=0}^{m-1} 2 \int_{t_{j}}^{t_{j+1}}\left\langle Y(r), u_{k}^{r}(r)\right\rangle d r \\
& -\sum_{j=0}^{m-1}\left\langle B\left(t_{j+1}\right) u\left(t_{j+1}\right)-B\left(t_{j+1}\right) u\left(t_{j}\right), u\left(t_{j+1}\right)-u\left(t_{j}\right)\right\rangle+\varepsilon(k) \tag{34.5.33}
\end{align*}
$$

where $t_{m} \in \mathscr{P}_{k}$.
Thus, discarding the negative terms which occur at the end and denoting by $\mathscr{P}_{k}$ the $k^{\text {th }}$ of these partitions,

$$
\sup _{t_{j} \in \mathscr{P}_{k}}\left\langle B u\left(t_{j}\right), u\left(t_{j}\right)\right\rangle+\int_{0}^{T}\left\langle B^{\prime} u, u\right\rangle d s \leq\left\langle B u_{0}, u_{0}\right\rangle+2 \int_{0}^{T}\left|\left\langle Y(r), u_{k}^{r}(r)\right\rangle\right| d r+\varepsilon
$$

$$
\begin{aligned}
& \leq\left\langle B u_{0}, u_{0}\right\rangle+2 \int_{0}^{T}\|Y(r)\|_{V^{\prime}}\left\|u_{k}^{r}(r)\right\|_{V} d r+\varepsilon \\
\leq & \left\langle B u_{0}, u_{0}\right\rangle+2\left(\int_{0}^{T}\|Y(r)\|_{V^{\prime}}^{p^{\prime}} d r\right)^{1 / p^{\prime}}\left(\int_{0}^{T}\left\|u_{k}^{r}(r)\right\|_{V}^{p} d r\right)^{1 / p}+\varepsilon \\
\leq & C\left(\|Y\|_{K^{\prime}},\|u\|_{K}\right)+\varepsilon
\end{aligned}
$$

whenever $k$ is large enough because these partitions are chosen such that

$$
\lim _{k \rightarrow \infty}\left(\int_{0}^{T}\left\|u_{k}^{r}(r)\right\|_{V}^{p}\right)^{1 / p}=\left(\int_{0}^{T}\|u(r)\|_{V}^{p}\right)^{1 / p}
$$

and so these are bounded. This has shown that for the dense subset of $[0, T], D \equiv \cup_{k} \mathscr{P}_{k}$,

$$
\sup _{t \in D}\langle B(t) u(t), u(t)\rangle+\int_{0}^{T}\left\langle B^{\prime} u, u\right\rangle d s<C\left(\|Y\|_{K^{\prime}},\|u\|_{K}\right)+\varepsilon
$$

However, $\varepsilon$ was arbitrary and the partitions are nested. Hence the above holds for all $\varepsilon$ and so

$$
\sup _{t \in D}\langle B(t) u(t), u(t)\rangle+\int_{0}^{T}\left\langle B^{\prime} u, u\right\rangle d s<C\left(\|Y\|_{K^{\prime}},\|u\|_{K}\right)
$$

By 34.5.29 and the integral equation, there is a set of measure zero including all the earlier sets of measure zero $N$ such that for $t \notin N, u_{n}^{l}(t), u_{n}^{r}(t) \rightarrow u(t)$ pointwise in $V$. Also, $B(t) u_{n}^{r}(t) \rightarrow B u(t)$ in $V^{\prime}$. This last can be obtained from the integral equation solved. $t \rightarrow$ $B u(t)$ is continuous into $V^{\prime}$. Then let $t \notin N$. We have $u_{n}^{r}(t) \rightarrow u(t)$ in $V$. Now $B(t) u_{n}^{r}(t)=$ $B(t) u\left(s_{n}\right)$ where $s_{n} \in D$ and $s_{n} \rightarrow t$. Then $B u(t)=B(t) u(t)$ and

$$
\begin{gathered}
\left\|B\left(s_{n}\right) u\left(s_{n}\right)-B(t) u(t)\right\|_{V^{\prime}} \leq\left\|\left(B\left(s_{n}\right)-B(t)\right) u\left(s_{n}\right)\right\|_{V^{\prime}}+\left\|B(t)\left(u\left(s_{n}\right)-u(t)\right)\right\|_{V^{\prime}} \\
\leq C_{t}\left\|B\left(s_{n}\right)-B(t)\right\|+C\left\|u\left(s_{n}\right)-u(t)\right\|_{V}
\end{gathered}
$$

where $C_{t}$ is a constant which comes because $u\left(s_{n}\right) \rightarrow u(t)$ in $V$ and so is bounded. The constant $C$ is just $\max _{t \in[0, T]}\|B(t)\|$. Then, since the two terms on the right converge to 0 as $n \rightarrow \infty$, it follows that as $s_{n} \rightarrow t, B\left(s_{n}\right) u\left(s_{n}\right) \rightarrow B(t) u(t)=B u(t)$ in $V^{\prime}$ while $u\left(s_{n}\right) \rightarrow u(t)$ in $V$. It follows that for $t \notin N$,

$$
\langle B u(t), u(t)\rangle+\int_{0}^{T}\left\langle B^{\prime} u, u\right\rangle d s=\lim _{n \rightarrow \infty}\left\langle B u\left(s_{n}\right), u\left(s_{n}\right)\right\rangle+\int_{0}^{T}\left\langle B^{\prime} u, u\right\rangle d s \leq C\left(\|Y\|_{K^{\prime}},\|u\|_{K}\right)
$$

Hence,

$$
\sup _{t \notin N}\langle B u(t), u(t)\rangle+\int_{0}^{T}\left\langle B^{\prime} u, u\right\rangle d s \leq C\left(\|Y\|_{K^{\prime}},\|u\|_{K}\right)
$$

It only remains to verify the claim about weak continuity.
Consider now the claim that $t \rightarrow B u(t)$ is weakly continuous on $N^{C}$. Letting $v \in V, s \in$ $N^{C}$,

$$
\begin{equation*}
\lim _{t \rightarrow s}\langle B u(t), v\rangle=\langle B u(s), v\rangle=\langle B u(s), v\rangle \tag{34.5.34}
\end{equation*}
$$

The limit follows from the formula 34.5.28 which implies $t \rightarrow B u(t)$ is continuous into $V^{\prime}$. Now for $t \in N^{C}$,

$$
\begin{aligned}
\|B u(t)\|_{W^{\prime}} & =\sup _{\|v\|_{W} \leq 1}|\langle B u(t), v\rangle| \leq\langle B v, v\rangle^{1 / 2}\langle B u(t), u(t)\rangle^{1 / 2} \\
& \leq\left(C\left(\|Y\|_{K^{\prime}},\|u\|_{K}\right)-\int_{0}^{T}\left\langle B^{\prime} u, u\right\rangle d s\right) \\
\sup _{t \notin N}\|B u(t)\|_{W^{\prime}} & \leq\left(C\left(\|Y\|_{K^{\prime}},\|u\|_{K}\right)-\int_{0}^{T}\left\langle B^{\prime} u, u\right\rangle d s\right)
\end{aligned}
$$

Now let $w \in W$. Then

$$
|\langle B u(t), w\rangle-\langle B u(s), w\rangle| \leq|\langle B u(t)-B u(s), w-v\rangle|+\left|\langle B u(t)-B u(s), v\rangle_{V^{\prime}, V}\right|
$$

Then the first term is less than $\varepsilon$ if $v$ is close enough to $w$ and the second converges to 0 by continuity of $t \rightarrow B u(t)$ which comes from the integral equation, so 34.5 .34 holds for all $v \in W$ and so this shows the weak continuity of $t \rightarrow B u(t)$ on $N^{C}$.

Now pick $t \in D$, the union of all the mesh points. Then for all $k$ large enough, $t \in \mathscr{P}_{k}$. Say $t=t_{m}$. From

$$
\begin{gather*}
\left\langle B u\left(t_{m}\right), u\left(t_{m}\right)\right\rangle-\left\langle B u_{0}, u_{0}\right\rangle+\int_{0}^{t_{m}}\left\langle B^{\prime} u, u\right\rangle d s=\sum_{j=0}^{m-1} 2 \int_{t_{j}}^{t_{j+1}}\left\langle Y(r), u^{r}(r)\right\rangle d r+\varepsilon(k) \\
\quad-\sum_{j=0}^{m-1}\left\langle B\left(t_{j+1}\right) u\left(t_{j+1}\right)-B\left(t_{j+1}\right) u\left(t_{j}\right), u\left(t_{j+1}\right)-u\left(t_{j}\right)\right\rangle \tag{34.5.35}
\end{gather*}
$$

where $\varepsilon(k) \rightarrow 0$. By Lemma 34.5.3, you can modify $\varepsilon(k)$ and write this in the form

$$
\begin{gather*}
\left\langle B u\left(t_{m}\right), u\left(t_{m}\right)\right\rangle-\left\langle B u_{0}, u_{0}\right\rangle+\int_{0}^{t_{m}}\left\langle B^{\prime} u, u\right\rangle d s=\sum_{j=0}^{m-1} 2 \int_{t_{j}}^{t_{j+1}}\left\langle Y(r), u^{r}(r)\right\rangle d r+\varepsilon(k) \\
\quad-\sum_{j=0}^{m-1}\left\langle B\left(t_{j+1}\right) u\left(t_{j+1}\right)-B\left(t_{j}\right) u\left(t_{j}\right), u\left(t_{j+1}\right)-u\left(t_{j}\right)\right\rangle \tag{34.5.36}
\end{gather*}
$$

Thus, $\left\langle B u\left(t_{m}\right), u\left(t_{m}\right)\right\rangle$ is constant for all $k$ large enough and the integral term on the right converges as $k \rightarrow \infty$ to

$$
\int_{0}^{t_{m}}\langle Y(r), u(r)\rangle d r
$$

It follows that the last term on the right does converge to something as $k \rightarrow \infty$. It just remains to consider what it does converge to. However, from the equation solved by $u$,

$$
B u\left(t_{j+1}\right)-B u\left(t_{j}\right)=\int_{t_{j}}^{t_{j+1}} Y(r) d r
$$

Therefore, this term is dominated by an expression of the form

$$
\begin{aligned}
&\left|\sum_{j=0}^{m_{k}-1}\left\langle\int_{t_{j}}^{t_{j+1}} Y(r) d r, u\left(t_{j+1}\right)-u\left(t_{j}\right)\right\rangle\right| \\
&=\left|\sum_{j=0}^{m_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(r), u\left(t_{j+1}\right)-u\left(t_{j}\right)\right\rangle d r\right| \\
&=\left|\sum_{j=0}^{m_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(r), u\left(t_{j+1}\right)\right\rangle-\sum_{j=0}^{m_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(r), u\left(t_{j}\right)\right\rangle\right| \\
&=\left|\int_{0}^{t_{m}}\left\langle Y(r), u^{r}(r)\right\rangle d r-\int_{0}^{t_{m}}\left\langle Y(r), u^{l}(r)\right\rangle d r\right| \\
& \leq \int_{0}^{T}\left|\left\langle Y(r), u^{r}(r)-u^{l}(r)\right\rangle\right| d r
\end{aligned}
$$

However, both $u^{r}$ and $u^{l}$ converge to $u$ in $K=L^{p}(0, T, V)$. Therefore, this term must converge to 0 . Passing to a limit, it follows that for all $t \in D$, the desired formula holds. Thus, for such $t \in D$,

$$
\langle B u(t), u(t)\rangle+\int_{0}^{t}\left\langle B^{\prime} u, u\right\rangle d r=\left\langle B u_{0}, u_{0}\right\rangle+2 \int_{0}^{t}\langle Y(r), u(r)\rangle d r
$$

It remains to verify that this holds for all $t \notin N$. Let $t \in N^{C} \backslash D$ and let $t(k) \in \mathscr{P}_{k}$ be the largest point of $\mathscr{P}_{k}$ which is less than $t$. Suppose $t(m) \leq t(k)$ so that $m \leq k$. Then

$$
B u(t(m))=B u_{0}+\int_{0}^{t(m)} Y(s) d s
$$

a similar formula for $u(t(k))$. Thus for $t>t(m)$,

$$
B u(t)-B u(t(m))=\int_{t(m)}^{t} Y(s) d s
$$

which is the same sort of thing already looked at except that it starts at $t(m)$ rather than at 0 and $u_{0}=0$. Therefore,

$$
\begin{aligned}
& \langle B(u(t(k))-u(t(m))), u(t(k))-u(t(m))\rangle \\
& +\int_{t(m)}^{t(k)}\left\langle B^{\prime}(s)(u(s)-u(t(m))), u(s)-u(t(m))\right\rangle \\
& \quad=2 \int_{t(m)}^{t(k)}\langle Y(s), u(s)-u(t(m))\rangle d s
\end{aligned}
$$

Thus, for $m \leq k$

$$
\begin{equation*}
\lim _{m, k \rightarrow \infty}\langle B(u(t(k))-u(t(m))), u(t(k))-u(t(m))\rangle=0 \tag{34.5.37}
\end{equation*}
$$

Hence $\{B u(t(k))\}_{k=1}^{\infty}$ is a convergent sequence in $W^{\prime}$ because

$$
\begin{aligned}
& |\langle B(u(t(k))-u(t(m))), y\rangle| \\
\leq & \langle B(u(t(k))-u(t(m))), u(t(k))-u(t(m))\rangle^{1 / 2}\langle B y, y\rangle^{1 / 2} \\
\leq & \langle B(u(t(k))-u(t(m))), u(t(k))-u(t(m))\rangle^{1 / 2}\|B\|^{1 / 2}\|y\|_{W}
\end{aligned}
$$

Does it converge to $B u(t)$ ? Let $\xi(t) \in W^{\prime}$ be what it does converge to. Let $v \in V$. Then

$$
\langle\xi(t), v\rangle=\lim _{k \rightarrow \infty}\langle B u(t(k)), v\rangle=\lim _{k \rightarrow \infty}\langle B u(t(k)), v\rangle=\langle B u(t), v\rangle
$$

because it is known that $t \rightarrow B u(t)$ is continuous into $V^{\prime}$. It is also known that for $t \in N^{C}$, $B u(t) \in W^{\prime} \subseteq V^{\prime}$ and that the $B u(t)$ for $t \in N^{C}$ are uniformly bounded in $W^{\prime}$. Therefore, since $V$ is dense in $W$, it follows that $\xi(t)=B u(t)$.

Now for every $t \in D$, it was shown above that

$$
\langle B u(t), u(t)\rangle+\int_{0}^{t}\left\langle B^{\prime} u, u\right\rangle d r=\left\langle B u_{0}, u_{0}\right\rangle+2 \int_{0}^{t}\langle Y(r), u(r)\rangle d r
$$

Also it was just shown that $B u(t(k)) \rightarrow B u(t)$ for $t \notin N$. Then for $t \notin N$

$$
\begin{gathered}
|\langle B u(t(k)), u(t(k))\rangle-\langle B u(t), u(t)\rangle| \\
\leq|\langle B(t(k)) u(t(k)), u(t(k))-u(t)\rangle|+|\langle B u(t(k))-B u(t), u(t)\rangle|
\end{gathered}
$$

Then the second term converges to 0 . The first equals

$$
\begin{aligned}
& |\langle B(t(k)) u(t(k))-B(t(k)) u(t), u(t(k))\rangle| \\
\leq & \langle B(t(k))(u(t(k))-u(t)), u(t(k))-u(t)\rangle^{1 / 2}\langle B u(t(k)), u(t(k))\rangle^{1 / 2}
\end{aligned}
$$

From the above, this is dominated by an expression of the form

$$
\langle B(t(k))(u(t(k))-u(t)), u(t(k))-u(t)\rangle^{1 / 2} C
$$

Then from the choice of $N$ and the pointwise convergence of $u_{n}^{r}$ to $u$ off $N$ the above converges to 0 for each $t \notin N$. It follows that

$$
\lim _{k \rightarrow \infty}|\langle B u(t(k)), u(t(k))\rangle-\langle B u(t), u(t)\rangle|=0
$$

Then from the formula,

$$
\langle B u(t), u(t)\rangle=\left\langle B u_{0}, u_{0}\right\rangle+2 \int_{0}^{t}\langle Y(r), u(r)\rangle d r-\int_{0}^{t}\left\langle B^{\prime} u, u\right\rangle d r
$$

valid for $t \in D$, it follows that the same formula holds for all $t \notin N$. Then define $\langle B u, u\rangle(t)$ to equal $\langle B u(t), u(t)\rangle$ off $N$ and the right side for $t \in N$. Thus $t \rightarrow\langle B u, u\rangle(t)$ is continuous and for all $t \in[0, T]$,

$$
\langle B u, u\rangle(t)=\left\langle B u_{0}, u_{0}\right\rangle+2 \int_{0}^{t}\langle Y(r), u(r)\rangle d r-\int_{0}^{t}\left\langle B^{\prime} u, u\right\rangle d r
$$

Also recall that $t \rightarrow B(t) u(t)$ was shown to be weakly continuous into $W^{\prime}$ on $N^{C}$. Is it continuous on $N^{C}$ ? Suppose $t \in N^{C}$ and let $s_{n} \rightarrow t$ where $s_{n} \in D$. Then $u\left(s_{n}\right)=u_{m_{n}}^{r}(t)$ because $s_{n}$ is one of the mesh points. Since $s_{n} \rightarrow t$ one can assume that $m_{n} \rightarrow \infty$. Hence $u\left(s_{n}\right)=u_{m_{n}}^{r}(t) \rightarrow u(t)$ by the pointwise convergence implied by 34.5.29. Then obviously

$$
B\left(s_{n}\right) u\left(s_{n}\right)=B\left(s_{n}\right) u_{m_{n}}^{l}(t) \rightarrow B(t) u(t)
$$

Now suppose you just have $t_{n} \rightarrow t$ where each of $t_{n}, t$ are in $N^{C}$. Does it always follow that $B\left(t_{n}\right) u\left(t_{n}\right) \rightarrow B(t) u(t)$ ? Suppose not. Then there exists such a sequence $t_{n} \rightarrow t$ of points in $N^{C}$ and $\varepsilon>0$ such that

$$
\left\|B\left(t_{n}\right) u\left(t_{n}\right)-B(t) u(t)\right\| \geq \varepsilon
$$

However, from the density of $D$ and what was just shown, there exists $s_{n} \in D$ such that $\left|s_{n}-t_{n}\right|<\frac{1}{2^{n}}$ and

$$
\left\|B\left(s_{n}\right) u\left(s_{n}\right)-B\left(t_{n}\right) u\left(t_{n}\right)\right\|<\frac{1}{2^{n}}
$$

Then

$$
\begin{aligned}
\varepsilon & \leq\left\|B\left(t_{n}\right) u\left(t_{n}\right)-B\left(s_{n}\right) u\left(s_{n}\right)\right\|+\left\|B\left(s_{n}\right) u\left(s_{n}\right)-B(t) u(t)\right\| \\
& <\frac{1}{2^{n}}+\left\|B\left(s_{n}\right) u\left(s_{n}\right)-B(t) u(t)\right\|
\end{aligned}
$$

Since $s_{n} \rightarrow t$, what was just shown implies both terms on the right converge to 0 . This is a contradiction. Thus $t \rightarrow B(t) u(t)$ must be continuous on $N^{C}$ into $W^{\prime}$.

Consider the case that $t \rightarrow B(u(t))$ has a weak derivative, denoted as $(B u)^{\prime}(t)$ which is in $L^{p^{\prime}}\left(0, T ; V^{\prime}\right)$. Then as shown above, there is a continuous function, denoted as $B u(t)$ which equals $B(t)(u(t))$ for a.e. $t$ and

$$
B u(t)=B u(0)+\int_{0}^{t}(B u)^{\prime}(s) d s
$$

Then the above theorem applies. Then one obtains the following corollary.
Corollary 34.5.5 Let $V \subseteq W, W^{\prime} \subseteq V^{\prime}$ be separable Banach spaces, and $B(t) \in \mathscr{L}\left(W, W^{\prime}\right)$ is nonnegative and self adjoint, $B \in C^{1}\left([0, T] ; W^{\prime}\right)$. Also suppose $t \rightarrow B(u(t))$ has a weak derivative $(B u)^{\prime} \in L^{p^{\prime}}\left(0, T ; V^{\prime}\right)$ for $u \in L^{p}([0, T] ; V) \cap L^{2}([0, T] ; W)$. Then there is a continuous function denoted as $B u(t)$ which equals $B(t)(u(t))$ a.e. $t$. Say for $t \notin N$. Suppose $B u(0)=B u_{0}, u_{0} \in W$. Then

$$
\begin{equation*}
B u(t)=B u_{0}+\int_{0}^{t}(B u)^{\prime}(s) d s \text { in } V^{\prime} \tag{34.5.38}
\end{equation*}
$$

Then $t \rightarrow B u(t)$ is in $C\left(N^{C}, W^{\prime}\right)$ and also for such $t$,

$$
\frac{1}{2}\langle B u(t), u(t)\rangle+\frac{1}{2} \int_{0}^{t}\left\langle B^{\prime}(s) u(s), u(s)\right\rangle d s=\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle+\int_{0}^{t}\left\langle(B u)^{\prime}(s), u(s)\right\rangle d s
$$

There exists a continuous function $t \rightarrow\langle B u, u\rangle(t)$ which equals the right side of the above for all $t$ and equals $\langle B(t) u(t), u(t)\rangle$ off $N$. This satisfies

$$
\sup _{t \in[0, T]}\langle B u, u\rangle(t) \leq C\left(\|Y\|_{K^{\prime}},\|u\|_{K}\right)
$$

In particular, this last inequality follows from Lemma 34.5.4 and the assumption that $B^{\prime}$ is bounded.

Note how if everything is nice and smooth, this integration by parts formula is what you would be expected to get. To see this, assume $u$ is smooth and formally work on the right side.

$$
\begin{aligned}
\frac{d}{d t}\langle B u, u\rangle & =\left\langle(B u)^{\prime}, u\right\rangle+\left\langle B u, u^{\prime}\right\rangle \\
& =\left\langle(B u)^{\prime}, u\right\rangle+\left\langle B u^{\prime}, u\right\rangle \\
& =2\left\langle(B u)^{\prime}, u\right\rangle-\left\langle B^{\prime} u, u\right\rangle
\end{aligned}
$$

Thus

$$
\begin{aligned}
& \frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle+\int_{0}^{t}\left\langle(B u)^{\prime}(s), u(s)\right\rangle d s \\
& =\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle+\frac{1}{2}\left[\int_{0}^{t} \frac{d}{d s}\langle B u, u\rangle d s+\int_{0}^{t}\left\langle B^{\prime} u, u\right\rangle d s\right] \\
& =\frac{1}{2}\langle B u(t), u(t)\rangle+\frac{1}{2} \int_{0}^{t}\left\langle B^{\prime} u, u\right\rangle d s
\end{aligned}
$$

which equals the left side.
A related topic is the continuity of pointwise evaluation of $B u$. Let $L u=(B u)^{\prime}$.

$$
\begin{aligned}
u & =D(L) \equiv\left\{u \in L^{p}(0, T ; V): L u \in L^{p^{\prime}}\left(0, T, V^{\prime}\right)\right\} \\
\|u\|_{X} & \equiv\|u\|_{L^{p}(0, T, V)}+\|L u\|_{L^{p^{\prime}}\left(0, T, V^{\prime}\right)}
\end{aligned}
$$

Since $L$ is closed, this $X$ is a Banach space. Then the following theorem is obtained.
Theorem 34.5.6 In the above corollary, the map $u \rightarrow B u(t)$ is continuous as a map from $X$ to $V^{\prime}$. Also if $Y$ denotes those $f \in L^{p}([0, T] ; V)$ for which $f^{\prime} \in L^{p}([0, T] ; V)$, so that $f$ has a representative such that $f(t)=f(0)+\int_{0}^{t} f^{\prime}(s) d s$, then if $\|f\|_{Y} \equiv\|f\|_{L^{p}([0, T] ; V)}+$ $\left\|f^{\prime}\right\|_{L^{p}([0, T] ; V)}$ the map $f \rightarrow f(t)$ is continuous.

Proof: First, why is $u \rightarrow B u(0)$ continuous? Say $u, v \in X$ and say $p \geq 2$ first.

$$
B u(t)-B v(t)=B u(0)-B v(0)+\int_{0}^{t}(B u)^{\prime}(s)-(B v)^{\prime}(s) d s
$$

and so,

$$
\left(\int_{0}^{T}\|B u(0)-B v(0)\|_{V^{\prime}}^{p^{\prime}} d t\right)^{1 / p^{\prime}} \leq\left(\int_{0}^{T}\|B u(t)-B v(t)\|_{V^{\prime}}^{p^{\prime}} d t\right)^{1 / p^{\prime}}
$$

$$
+\left(\int_{0}^{T}\left\|\int_{0}^{t}(B u)^{\prime}(s)-(B v)^{\prime}(s) d s\right\|^{p^{\prime}} d t\right)^{1 / p^{\prime}}
$$

and so

$$
\begin{gathered}
\|B u(0)-B v(0)\|_{V^{\prime}} T^{1 / p^{\prime}} \leq \\
\left(\|B\|\|u-v\|_{L^{p^{\prime}}([0, T] ; V)}+T^{1 / p^{\prime}}\left\|(B u)^{\prime}-(B v)^{\prime}\right\|_{L^{p^{\prime}\left([0, T] ; V^{\prime}\right)}}\right) \\
\leq C(\|B\|, T)\|u-v\|_{X}
\end{gathered}
$$

Thus $u \rightarrow B u(0)$ is continuous into $V^{\prime}$. If $p<2$, then you do something similar.

$$
\begin{gathered}
\left(\int_{0}^{T}\|B u(0)-B v(0)\|_{V^{\prime}}^{p} d t\right)^{1 / p} \leq\left(\int_{0}^{T}\|B u(t)-B v(t)\|_{V^{\prime}}^{p} d t\right)^{1 / p} \\
+\left(\int_{0}^{T}\left\|\int_{0}^{t}(B u)^{\prime}(s)-(B v)^{\prime}(s) d s\right\|^{p} d t\right)^{1 / p} \\
\|B u(0)-B v(0)\|_{V^{\prime}} T^{1 / p} \leq\|B\|\|u-v\|_{L^{p}}+C(T)\left\|(B u)^{\prime}-(B v)^{\prime}\right\|_{L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)} \\
\leq C(\|B\|, T)\|u-v\|_{X}
\end{gathered}
$$

However, one could just as easily have done this for an arbitrary $s<T$ by repeating the argument for

$$
B u(t)=B u(s)+\int_{s}^{t}(B u)^{\prime}(r) d r
$$

Thus this mapping is certainly continuous into $V^{\prime}$. The last assertion is similar.

### 34.6 Another Approach

The above approach is pretty interesting, but there is a quicker way to do it discussed in this section. I am also including the case where the operator $B$ is actually a function of $t$. I have never had a reason to use this level of generality, but it is here if it is of any interest. Also, this is presented in the context of complex Banach spaces. In addition, it is shown that by including $i^{*}$ in various formulas, you don't need to have $V$ dense in $W$. Of course, this is typically not of any interest, but for the sake of generality, it is included. The approach is due to Lions. It is assumed for convenience that $p \geq 2$. This was apparently not needed in the last section. It may be that this approach can also be generalized to not require this.

Let $B(t) \in \mathscr{L}\left(W, W^{\prime}\right)$ satisfy

$$
\begin{gather*}
\langle B(t) u, v\rangle=\overline{\langle B(t) v, u\rangle}, u, v \in W  \tag{34.6.39}\\
\langle B(t) u, u\rangle \geq 0  \tag{34.6.40}\\
B(t)=B(0)+\int_{0}^{t} B^{\prime}(s) d s \tag{34.6.41}
\end{gather*}
$$

where $B^{\prime} \in L^{\infty}\left(0, T ; \mathscr{L}\left(W, W^{\prime}\right)\right)$. Here $W$ is a Banach space such that $V \subseteq W$. Also $\mathscr{V}_{I} \equiv$ $L^{p}(I ; V)$ and $\mathscr{W}_{I} \equiv L^{2}(I ; W)$.

Now let $I=[a, b]$ and $c<a<b<d$. Here and in what follows $\phi_{n}(t)=n \phi(n t)$ where $\phi \geq 0, \phi \in C_{0}^{\infty}(-1,1)$, and $\int \phi d t=1$. The following proposition is known and the essential features of its proof may be found in [92]. We give a proof for the convenience of the reader.

Proposition 34.6.1 Suppose $D(t) \in \mathscr{L}\left(W, W^{\prime}\right)$ and $D(t)=0$ if $t \notin(c, d)$. Suppose also that

$$
D(t)=\int_{c}^{t} D^{\prime}(s) d s, D^{\prime} \in L^{\infty}\left(c, d ; \mathscr{L}\left(W, W^{\prime}\right)\right)
$$

For $u \in \mathscr{W}_{I}$ and $a-n^{-1}>c, b+n^{-1}<d$, define

$$
\begin{equation*}
T_{n} u=\left(D\left(u * \phi_{n}\right)\right)^{\prime}-\left((D u) * \phi_{n}\right)^{\prime} \tag{34.6.42}
\end{equation*}
$$

where we let $u=0$ off I. Then

$$
\begin{equation*}
\left\|T_{n} u\right\|_{\mathscr{W}_{I}^{\prime}} \rightarrow 0 \tag{34.6.43}
\end{equation*}
$$

Proof: First, we show that $\left\|T_{n}\right\|$ is uniformly bounded. Letting $w=0$ off $I$,

$$
\begin{gathered}
\left|\left\langle T_{n} u, w\right\rangle\right|=\left|\int_{\mathbb{R}}\left\langle D^{\prime}(t) \int_{\mathbb{R}} u(s) \phi_{n}(t-s) d s, w(t)\right\rangle d t\right| \\
+\left|\int_{\mathbb{R}}\left\langle\int_{\mathbb{R}}(D(t)-D(s)) u(s) \phi_{n}^{\prime}(t-s) d s, w(t)\right\rangle d t\right| \\
\leq C\|u\|_{W_{I}}\|w\|_{\mathscr{W}_{I}}+\int_{\mathbb{R}} \int_{\mathbb{R}}\|D(t)-D(s)\|\|u(s)\| n^{2}\left|\phi^{\prime}(n(t-s))\right|\|w(t)\| d s d t \\
\leq C\|u\|_{\mathscr{W}_{I}}\|w\|_{\mathscr{W}_{I}}+ \\
\int_{\mathbb{R}} \int_{-1}^{1}\left\|D(t)-D\left(t-\frac{r}{n}\right)\right\|\left\|u\left(t-\frac{r}{n}\right)\right\| n^{2}\left|\phi^{\prime}(r)\right|\|w(t)\| \frac{1}{n} d r d t \\
\leq C\|u\|_{\mathscr{W}_{I}}\|w\|_{\mathscr{W}_{I}}+C \int_{-1}^{1} \int_{\mathbb{R}}\left\|u\left(t-\frac{r}{n}\right)\right\|_{W}\|w(t)\|_{W} d t d r \\
\leq C\|u\|_{\mathscr{W}_{I}}\|w\|_{\mathscr{W}_{I}} .
\end{gathered}
$$

Where $C$ is a positive constant independent of $n$ and $u$. Thus $\left\|T_{n}\right\|$ is bounded independent of $n$.

Next let $u \in C_{0}^{\infty}(I ; V)$, a dense subset of $\mathscr{W}_{I}$. Then a little computation shows

$$
\begin{gathered}
\left|\left\langle T_{n} u, w\right\rangle_{\mathscr{W}}\right| \leq \\
C(\phi) \int_{a}^{b} \int_{-1}^{1}\left\|\left\lvert\, D^{\prime}(t)-D^{\prime}\left(t-\frac{r}{n}\right)\right.\right\|\left\|u\left(t-\frac{r}{n}\right)\right\|\left\|_{W}\right\| w(t) \|_{W} d r d t \\
+\left.C(\phi) \int_{a}^{b} \int_{-1}^{1}\left\|D(t)-D\left(t-\frac{r}{n}\right)\right\|\left\|u^{\prime}\left(t-\frac{r}{n}\right)\right\|\right|_{W}\|w(t)\|_{W} d r d t \\
\equiv \mathbf{A}+\mathbf{B} .
\end{gathered}
$$

Now

$$
\mathbf{B} \leq C\left(\phi, D^{\prime}\right) n^{-1 / 2}\left\|u^{\prime}\right\|_{\mathscr{W}_{I}}\|w\|_{\mathscr{W}_{I}} .
$$

Since $u$ is bounded,

$$
\begin{aligned}
\mathbf{A} & \leq C(\phi, u) \int_{a}^{b} \int_{-1}^{1}\left\|D^{\prime}(t)-D^{\prime}\left(t-\frac{r}{n}\right)\right\|\|w(t)\|_{W} d r d t \\
& \leq C(\phi, u) \int_{a}^{b}\|w(t)\|_{W} n \int_{t-n^{-1}}^{t+n^{-1}}\left\|D^{\prime}(t)-D^{\prime}(s)\right\| d s d t
\end{aligned}
$$

By Holder's inequality, this is no larger than

$$
C(\phi, u)\left(\int_{a}^{b}\left(n \int_{t-n^{-1}}^{t+n^{-1}}\left\|D^{\prime}(t)-D^{\prime}(s)\right\| d s\right)^{2} d t\right)^{1 / 2}\|w\|_{\mathscr{W}_{I}}
$$

If $t$ is a Lebesgue point,

$$
n \int_{t-n^{-1}}^{t+n^{-1}}\left\|D^{\prime}(t)-D^{\prime}(s)\right\| d s \rightarrow 0
$$

and also

$$
n \int_{t-n^{-1}}^{t+n^{-1}}\left\|D^{\prime}(t)-D^{\prime}(s)\right\| d s \leq 4\left\|D^{\prime}\right\|_{\infty}
$$

so the dominated convergence theorem implies

$$
\int_{a}^{b}\left(n \int_{t-n^{-1}}^{t+n^{-1}}\left\|D^{\prime}(t)-D^{\prime}(s)\right\| d s\right)^{2} d t \rightarrow 0
$$

Hence

$$
\begin{gathered}
\left\|T_{n} u\right\|_{\mathscr{W}_{I}^{\prime}} \leq \\
C\left(\phi, u, D^{\prime}\right)\left(n^{-1 / 2}+\left(\int_{a}^{b}\left(n \int_{t-n^{-1}}^{t+n^{-1}}\left\|D^{\prime}(t)-D^{\prime}(s)\right\| d s\right)^{2} d t\right)^{1 / 2}\right)
\end{gathered}
$$

and so $T_{n} u \rightarrow 0$ for all $u$ in the dense subset, $C_{0}^{\infty}(I ; V)$.
We have also the following simple corollary.
Corollary 34.6.2 In the situation of Proposition 34.6.1,

$$
\left\|\left(i^{*} D\left(u * \phi_{n}\right)\right)^{\prime}-\left(\left(i^{*} D u\right) * \phi_{n}\right)^{\prime}\right\|_{\mathscr{Y}_{I}^{\prime}} \rightarrow 0
$$

where $i$ is the inclusion map of $V$ into $W$.
For $f \in L^{1}\left(a, b ; V^{\prime}\right)$ we define $f^{\prime}$ in the sense of $V^{\prime}$ valued distributions as follows. For $\phi \in C_{0}^{\infty}(a, b)$,

$$
f^{\prime}(\phi) \equiv-\int_{a}^{b} f(t) \phi^{\prime}(t) d t
$$

We say $f^{\prime} \in L^{1}\left(a, b ; V^{\prime}\right)$ if there exists $g \in L^{1}\left(a, b ; V^{\prime}\right)$, necessarily unique, such that for all $\phi \in C_{0}^{\infty}(a, b)$,

$$
\int_{a}^{b} g(t) \phi(t) d t=f^{\prime}(\phi)
$$

To save on notation, we let $\mathscr{V} \equiv \mathscr{V}_{[0, T]}$ and $\mathscr{W} \equiv \mathscr{W}_{[0, T]}$. Define

$$
\begin{align*}
D(L) & \equiv\left\{u \in \mathscr{V}:\left(i^{*} B u\right)^{\prime} \in \mathscr{V}^{\prime}\right\}  \tag{34.6.44}\\
L u & \equiv\left(i^{*} B u\right)^{\prime} \text { for } u \in D(L) \tag{34.6.45}
\end{align*}
$$

Note that for $u \in D(L)$, it is automatically the case that $i^{*} B u \in \mathscr{V}^{\prime}$.

Lemma 34.6.3 L is a closed operator.

We define

$$
X \equiv D(L),\|u\|_{X} \equiv\|L u\|_{\mathscr{V}^{\prime}}+\|u\|_{\mathscr{V}} .
$$

Then $X$ is isometric to a closed subspace of a product of reflexive Banach spaces and so $X$ is reflexive by Lemma 17.5.11.

Theorem 34.6.4 Let $p \geq 2$ in what follows. For $u, v \in X$, the following hold.

1. $t \rightarrow\langle B(t) u(t), v(t)\rangle_{W^{\prime}, W}$ equals an absolutely continuous function a.e., denoted by $\langle B u, v\rangle(\cdot)$.
2. $\operatorname{Re}\langle L u(t), u(t)\rangle=\frac{1}{2}\left[\langle B u, u\rangle^{\prime}(t)+\left\langle B^{\prime}(t) u(t), u(t)\right\rangle\right]$ a.e. $t$
3. $|\langle B u, v\rangle(t)| \leq C\|u\|_{X}\|v\|_{X}$ for some $C>0$ and for all $t \in[0, T]$.
4. $t \rightarrow B(t) u(t)$ equals a function in $C\left(0, T ; W^{\prime}\right)$ a.e., denoted by $B u(\cdot)$.
5. $\sup \left\{\|B u(t)\|_{W^{\prime}}, t \in[0, T]\right\} \leq C\|u\|_{X}$ for some $C>0$.

If $K: X \rightarrow X^{\prime}$ is given by

$$
\langle K u, v\rangle_{X^{\prime}, X} \equiv \int_{0}^{T}\langle L u(t), v(t)\rangle d t+\langle B u, v\rangle(0),
$$

then
6. K is linear, continuous and weakly continuous.
7. $\operatorname{Re}\langle K u, u\rangle=\frac{1}{2}[\langle B u, u\rangle(T)+\langle B u, u\rangle(0)]+\frac{1}{2} \int_{0}^{T}\left\langle B^{\prime}(t) u(t), u(t)\right\rangle d t$.
8. If $B u(0)=0$, for $u \in X$, there exists $u_{n} \rightarrow u$ in $X$ such that $u_{n}(t)$ is 0 near 0 . A similar conclusion could be deduced at $T$ if $B u(T)=0$.

Proof: For $h$ a function defined on $[0, T]$, let $h_{1}$ be even, $2 T$ periodic, and $h_{1}(t)=h(t)$ for all $t \in[0, T]$. Let $C(\cdot) \in C_{0}^{\infty}(-T, 2 T), C(t) \in[0,1], C(t)=1$ on $[0, T]$.


Let $\tilde{B}(t)=C(t) B_{1}(t)$ for all $t \in \mathbb{R}$ and define

$$
\tilde{u}(t)=\left\{\begin{array}{l}
u_{1}(t), t \in[-T, 2 T] \\
0, t \notin[-T, 2 T]
\end{array}\right.
$$

Now let $u \in X$. Then

$$
\left(i^{*} \tilde{B} \tilde{u}\right)^{\prime}(t)=\left\{\begin{array}{l}
0, t<-T  \tag{34.6.46}\\
C^{\prime}(t)\left(i^{*} B u\right)(-t)-C(t)\left(i^{*} B u\right)^{\prime}(-t), t \in[-T, 0] \\
\left(i^{*} B u\right)^{\prime}(t), t \in[0, T] \\
C^{\prime}(t)\left(i^{*} B u\right)(2 T-t)-C(t)\left(i^{*} B u\right)^{\prime}(2 T-t), t \in[T, 2 T] \\
0, t>2 T
\end{array}\right.
$$

Thus, if $I \supseteq[-T, 2 T]$, then $\left(i^{*} \tilde{B} \tilde{u}\right)^{\prime} \in \mathscr{V}_{I}^{\prime}$. Defining $u_{n} \equiv \tilde{u} * \phi_{n}$, then for a.e. $t$,

$$
\begin{equation*}
\operatorname{Re}\left\langle\left(i^{*} \tilde{B} u_{n}\right)^{\prime}(t), u_{n}(t)\right\rangle=\frac{1}{2}\left[\left\langle\tilde{B} u_{n}, u_{n}\right\rangle^{\prime}(t)+\left\langle\tilde{B}^{\prime}(t) u_{n}(t), u_{n}(t)\right\rangle\right] . \tag{34.6.47}
\end{equation*}
$$

From 34.6.46 and Proposition 34.6.1, the following holds in $\mathscr{V}_{[-T, 2 T]}^{\prime}$.

$$
\begin{align*}
\lim _{n \rightarrow \infty}\left(i^{*} \tilde{B} u_{n}\right)^{\prime} & =\lim _{n \rightarrow \infty}\left(i^{*} \tilde{B}\left(\tilde{u} * \phi_{n}\right)\right)^{\prime}  \tag{34.6.48}\\
& =\lim _{n \rightarrow \infty}\left(\left(i^{*} \tilde{B} \tilde{u}\right) * \phi_{n}\right)^{\prime} \\
& =\lim _{n \rightarrow \infty}\left(i^{*} \tilde{B} \tilde{u}\right)^{\prime} * \phi_{n} \\
& =\left(i^{*} \tilde{B} \tilde{u}\right)^{\prime}
\end{align*}
$$

Where the second equality follows from Corollary 34.6.2, the third follows from the pointwise a.e. equality of $\left(\left(i^{*} \tilde{B} \tilde{u}\right) * \phi_{n}\right)^{\prime}$ and $\left(i^{*} \tilde{B} \tilde{u}\right)^{\prime} * \phi_{n}$, while the fourth follows from 34.6.46 and standard properties of convolutions.

By choosing a subsequence we can use 34.6 .48 to obtain

$$
\begin{gather*}
u_{n} \rightarrow u \text { a.e. and in } \mathscr{V}  \tag{34.6.49}\\
\left(i^{*} \tilde{B} u_{n}\right)^{\prime} \rightarrow\left(i^{*} B u\right)^{\prime} \text { a.e. and in } \mathscr{V}^{\prime} .
\end{gather*}
$$

From 34.6.49,

$$
\begin{align*}
\operatorname{Re}\left\langle\left(i^{*} \tilde{B} u_{n}\right)^{\prime}(t), u_{n}(t)\right\rangle & \rightarrow \operatorname{Re}\left\langle\left(i^{*} B u\right)^{\prime}(t), u(t)\right\rangle \text { a.e. } t \in[0, T]  \tag{34.6.50}\\
\left\langle B^{\prime}(t) u_{n}(t), u_{n}(t)\right\rangle & \rightarrow\left\langle B^{\prime}(t) u(t), u(t)\right\rangle \text { a.e. } t \in[0, T] \tag{34.6.51}
\end{align*}
$$

If $g \in L^{\infty}(0, T)$,

$$
\begin{gathered}
\lim _{n \rightarrow \infty} \int_{0}^{T} g(t)\left\langle\left(i^{*} \tilde{B} u_{n}\right)^{\prime}(t), u_{n}(t)\right\rangle d t=\lim _{n \rightarrow \infty}\left\langle\left(i^{*} \tilde{B} u_{n}\right)^{\prime}, g u_{n}(t)\right\rangle \\
=\left\langle\left(i^{*} B u\right)^{\prime}, g u\right\rangle=\int_{0}^{T} g(t)\left\langle\left(i^{*} B u\right)^{\prime}(t), u(t)\right\rangle d t
\end{gathered}
$$

Thus we have the following weak convergence:

$$
\operatorname{Re}\left\langle\left(i^{*} \tilde{B} u_{n}\right)^{\prime}, u_{n}\right\rangle \rightharpoonup \operatorname{Re}\left\langle\left(i^{*} B u\right)^{\prime}, u\right\rangle \text { in } L^{1}(0, T)
$$

Similarly,

$$
\left\langle B^{\prime} u_{n}, u_{n}\right\rangle \rightharpoonup\left\langle B^{\prime} u, u\right\rangle \text { in } L^{1}(0, T)
$$

It follows from 34.6.47 that
$\left\langle\tilde{B} u_{n}, u_{n}\right\rangle^{\prime}(\cdot)$ converges a.e. and weakly in $L^{1}(0, T)$.

$$
\left\langle\tilde{B} u_{n}, u_{n}\right\rangle(\cdot) \text { converges a.e. and strongly in } L^{1}(0, T) \text { to }\langle B u, u\rangle(\cdot) .
$$

Therefore, $\left\langle\tilde{B} u_{n}, u_{n}\right\rangle^{\prime}(\cdot)$ converges a.e. and weakly in $L^{1}(0, T)$ to $\langle B u, u\rangle^{\prime}(\cdot)$. Since $\langle\tilde{B} u, u\rangle$ and $\langle\tilde{B} u, u\rangle^{\prime}$ are both in $L^{1}(0, T)$, this proves part 1 in the case where $v=u$. This also establishes formula 2. To get 1 for $u \neq v$, apply what was just shown to

$$
\langle B(t)(u(t)+v(t)), u(t)+v(t)\rangle .
$$

Next let $t \in[0, T]$ and use 34.6 .47 to write

$$
\begin{gather*}
\left\langle\tilde{B} u_{n}, u_{n}\right\rangle(t)= \\
2 \operatorname{Re} \int_{-T}^{t}\left\langle\left(i^{*} \tilde{B} u_{n}\right)^{\prime}(s), u_{n}(s)\right\rangle d s-2 \operatorname{Re} \int_{-T}^{t}\left\langle\tilde{B}^{\prime}(s) u_{n}(s), u_{n}(s)\right\rangle d s \tag{34.6.52}
\end{gather*}
$$

Using 34.6.49, we let $n \rightarrow \infty$ in 34.6.52 and obtain

$$
\begin{equation*}
\langle\tilde{B} u, u\rangle(t)=2 \operatorname{Re} \int_{-T}^{t}\left\langle\left(i^{*} \tilde{B} \tilde{u}\right)^{\prime}(s), \tilde{u}(s)\right\rangle d s-2 \operatorname{Re} \int_{-T}^{t}\left\langle\tilde{B}^{\prime}(s) \tilde{u}(s), \tilde{u}(s)\right\rangle d s . \tag{34.6.53}
\end{equation*}
$$

Hence from 34.6.46,

$$
|\langle\tilde{B} u, u\rangle(t)| \leq C\left[\left\|\left(i^{*} \tilde{B} \tilde{u}\right)^{\prime}\right\|_{\mathscr{Y}_{[-T, 2 T]}^{\prime}}\|\tilde{u}\|_{Y_{[-T, 2 T]}}+\|\tilde{u}\|_{\left.V_{[-T, 2 T]}\right]}^{2}\right]
$$

$$
\begin{equation*}
\leq C\left[\|L u\|_{\mathscr{V}_{[0, T]}^{\prime}}^{2}+\|u\|_{\left.\mathscr{V}_{[0, T]}\right]}^{2}\right] \leq C\|u\|_{X}^{2} . \tag{34.6.54}
\end{equation*}
$$

This verifies 3 in the case $u=v$. To obtain the general case,

$$
|\langle B u, v\rangle(t)| \leq\langle B u, u\rangle^{1 / 2}(t)\langle B v, v\rangle^{1 / 2}(t) \leq C\|u\|_{X}\|v\|_{X}
$$

To verify 4 , use 34.6 .47 to write for $t \in[0, T]$ and $I=[-T, 2 T]$,

$$
\begin{gather*}
\left|\left\langle\tilde{B} u_{n}(t)-\tilde{B} u_{m}(t), u_{n}(t)-u_{m}(t)\right\rangle\right| \\
\leq 2\left|\int_{-T}^{2 T}\left\langle\left(i^{*} \tilde{B}\left(u_{n}-u_{m}\right)\right)^{\prime}(s), u_{n}(s)-u_{m}(s)\right\rangle d s\right| \\
C\left[\left\|\left(i^{*} \tilde{B} u_{n}\right)^{\prime}-\left(i^{*} \tilde{B} u_{m}\right)^{\prime}\right\|_{\mathscr{V}_{I}^{\prime}}^{2 T}\left|\left\langle\tilde{B}^{\prime}(s)\left(u_{n}(s)-u_{m}(s)\right), u_{n}(s)-u_{m}(s)\right\rangle\right| d s \leq\right.  \tag{34.6.55}\\
\left.\mathscr{V}_{I}+\left\|u_{n}-u_{m}\right\|_{\mathscr{W}_{I}}^{2}\right] \equiv E_{n m} .
\end{gather*}
$$

Then from 34.6.48, $\lim _{n, m \rightarrow \infty} E_{n m}=0$ and so, for $t \in[0, T]$,

$$
\left|\left\langle\tilde{B} u_{n}(t)-\tilde{B} u_{m}(t), w\right\rangle\right| \leq E_{n m}^{1 / 2}\langle B(t) w, w\rangle^{1 / 2} \leq C E_{n m}^{1 / 2}\|w\|_{W}
$$

It follows that $\tilde{B} u_{n}(\cdot)$ is uniformly Cauchy in the space of continuous functions $C\left(0, T ; W^{\prime}\right)$ and so it converges to $z \in C\left(0, T ; W^{\prime}\right)$. But $\tilde{B} u_{n}$ converges in $L^{2}\left(0, T ; W^{\prime}\right)$ to $B u(\cdot)$. Therefore $B(t) u(t)=z(t)$ a.e. Letting $B u(\cdot)=z(\cdot)$, this shows 4 . Formula 5 follows from 3 and the following argument.

$$
|\langle B u(t), w\rangle| \leq\langle B u, u\rangle^{1 / 2}(t)\langle B w, w\rangle^{1 / 2} \leq C\|u\|_{X}\|w\|_{W} .
$$

Assertion 6 follows easily from the first five parts. It remains to get 7 .

$$
\begin{aligned}
\operatorname{Re}\langle K u, u\rangle & =\int_{0}^{T} \operatorname{Re}\langle L u, u\rangle d t+\langle B u, u\rangle(0) \\
& =\int_{0}^{T} \frac{1}{2}\left[\langle B u, u\rangle^{\prime}(t)+\left\langle B^{\prime}(t) u(t), u(t)\right\rangle\right] d t+\langle B u, u\rangle(0) \\
& =\frac{1}{2}\langle B u, u\rangle(T)+\frac{1}{2}\langle B u, u\rangle(0)+\frac{1}{2} \int_{0}^{T}\left\langle B^{\prime}(t) u(t), u(t)\right\rangle d t
\end{aligned}
$$

It only remains to verify the last assertion. Let $\psi_{n}$ be increasing and piecewise linear such that $\psi_{n}(t)=1$ for $t \geq 2 / n$ and equals 0 on $[0,1 / n]$. Then clearly $\psi_{n} u \rightarrow u$ in $\mathscr{V}$. Also

$$
\left(B\left(\psi_{n} u\right)\right)^{\prime}=\psi_{n}^{\prime} B u+\psi_{n}(B u)^{\prime}
$$

The latter term converges to $(B u)^{\prime}$ in $\mathscr{V}^{\prime}$. Now consider the first term.

$$
\begin{gathered}
\int_{0}^{T}\left\|\psi_{n}^{\prime} B u\right\|_{V^{\prime}}^{p^{\prime}} d t \leq \int_{0}^{2 / n} n\left\|\int_{0}^{t}(B u)^{\prime} d s\right\|^{p^{\prime}} d t \\
\leq n \int_{0}^{2 / n}{ }_{t} t^{p^{\prime}-1} \int_{0}^{t}\left\|(B u)^{\prime}\right\|_{V^{\prime}}^{p^{\prime}} d s d t \leq \int_{0}^{2 / n}\left\|(B u)^{\prime}\right\|_{V^{\prime}}^{p^{\prime}} d s \frac{1}{p^{\prime}}(2 / n)^{p^{\prime}} n
\end{gathered}
$$

Since $p^{\prime}>1$, this converges to 0 .

Corollary 34.6.5 If $B u(0)=0$ for $u \in X$, then $\langle B u, u\rangle(0)=0$. The converse is also true. An analogous result will hold with 0 replaced with $T$.

Proof: Let $u_{n} \rightarrow u$ in $X$ with $u_{n}(t)=0$ for all $t$ close enough to 0 . For $t$ off a set of measure zero consisting of the union of sets of measure zero corresponding to $u_{n}$ and $u$,

$$
\begin{gathered}
\left\langle B u_{n}, u_{n}\right\rangle(t)=\left\langle B(t) u_{n}(t), u_{n}(t)\right\rangle,\langle B u, u\rangle(t)=\langle B(t) u(t), u(t)\rangle, \\
\left\langle B\left(u-u_{n}\right), u\right\rangle(t)=\left\langle B(t)\left(u(t)-u_{n}(t)\right), u(t)\right\rangle \\
\left\langle B u_{n}, u-u_{n}\right\rangle(t)=\left\langle B(t) u_{n}(t), u(t)-u_{n}(t)\right\rangle
\end{gathered}
$$

Then, considering such $t$,

$$
\begin{aligned}
\langle B(t) u(t), u(t)\rangle-\left\langle B(t) u_{n}(t), u_{n}(t)\right\rangle= & \left\langle B(t)\left(u(t)-u_{n}(t)\right), u(t)\right\rangle \\
& +\left\langle B(t) u_{n}(t), u(t)-u_{n}(t)\right\rangle
\end{aligned}
$$

Hence from Theorem 34.6.4,

$$
\left|\langle B(t) u(t), u(t)\rangle-\left\langle B(t) u_{n}(t), u_{n}(t)\right\rangle\right| \leq C\left\|u-u_{n}\right\|_{X}\left(\|u\|_{X}+\left\|u_{n}\right\|_{X}\right)
$$

Thus if $n$ is sufficiently large,

$$
\left|\langle B(t) u(t), u(t)\rangle-\left\langle B(t) u_{n}(t), u_{n}(t)\right\rangle\right|<\varepsilon
$$

So let $n$ be fixed and this large and now let $t_{k} \rightarrow 0$ to obtain $\left\langle B\left(t_{k}\right) u_{n}\left(t_{k}\right), u_{n}\left(t_{k}\right)\right\rangle=0$ for $k$ large enough. Hence

$$
\langle B u, u\rangle(0)=\lim _{k \rightarrow \infty}\left\langle B\left(t_{k}\right) u\left(t_{k}\right), u\left(t_{k}\right)\right\rangle<\varepsilon
$$

Since $\varepsilon$ is arbitrary, $\langle B u, u\rangle(0)=0$.
Next suppose $\langle B u, u\rangle(0)=0$. Then letting $v \in X$, with $v$ smooth,

$$
\langle B u(0), v(0)\rangle=\langle B u, v\rangle(0)=\langle B u, u\rangle^{1 / 2}(0)\langle B v, v\rangle^{1 / 2}(0)=0
$$

and it follows that $B u(0)=0$.

### 34.7 Some Imbedding Theorems

The next theorem is very useful in getting estimates in partial differential equations. It is called Erling's lemma.

Definition 34.7.1 Let $E, W$ be Banach spaces such that $E \subseteq W$ and the injection map from $E$ into $W$ is continuous. The injection map is said to be compact if every bounded set in $E$ has compact closure in $W$. In other words, if a sequence is bounded in $E$ it has a convergent subsequence converging in $W$. This is also referred to by saying that bounded sets in $E$ are precompact in $W$.

Theorem 34.7.2 Let $E \subseteq W \subseteq X$ where the injection map is continuous from $W$ to $X$ and compact from $E$ to $W$. Then for every $\varepsilon>0$ there exists a constant, $C_{\varepsilon}$ such that for all $u \in E$,

$$
\|u\|_{W} \leq \varepsilon\|u\|_{E}+C_{\varepsilon}\|u\|_{X}
$$

Proof: Suppose not. Then there exists $\varepsilon>0$ and for each $n \in \mathbb{N}, u_{n}$ such that

$$
\left\|u_{n}\right\|_{W}>\varepsilon\left\|u_{n}\right\|_{E}+n\left\|u_{n}\right\|_{X}
$$

Now let $v_{n}=u_{n} /\left\|u_{n}\right\|_{E}$. Therefore, $\left\|v_{n}\right\|_{E}=1$ and

$$
\left\|v_{n}\right\|_{W}>\varepsilon+n\left\|v_{n}\right\|_{X}
$$

It follows there exists a subsequence, still denoted by $v_{n}$ such that $v_{n}$ converges to $v$ in $W$. However, the above inequality shows that $\left\|v_{n}\right\|_{X} \rightarrow 0$. Therefore, $v=0$. But then the above inequality would imply that $\left\|v_{n}\right\|_{W}>\varepsilon$ and passing to the limit yields $0>\varepsilon$, a contradiction.

Definition 34.7.3 Define $C([a, b] ; X)$ the space of functions continuous at every point of $[a, b]$ having values in $X$.

You should verify that this is a Banach space with norm

$$
\|u\|_{\infty, X}=\max \left\{\left\|u_{n_{k}}(t)-u(t)\right\|_{X}: t \in[a, b]\right\} .
$$

The following theorem is an infinite dimensional version of the Ascoli Arzela theorem. It is like a well known result due to Simon [117]. It is an appropriate generalization when you do not have weak derivatives.

Theorem 34.7.4 Let $q>1$ and let $E \subseteq W \subseteq X$ where the injection map is continuous from $W$ to $X$ and compact from $E$ to $W$. Let $S$ be defined by

$$
\left\{u \text { such that }\|u(t)\|_{E} \leq R \text { for all } t \in[a, b], \text { and }\|u(s)-u(t)\|_{X} \leq R|t-s|^{1 / q}\right\}
$$

Thus $S$ is bounded in $L^{\infty}(a, b, E)$ and in addition, the functions are uniformly Holder continuous into $X$. Then $S \subseteq C([a, b] ; W)$ and if $\left\{u_{n}\right\} \subseteq S$, there exists a subsequence, $\left\{u_{n_{k}}\right\}$ which converges to a function $u \in C([a, b] ; W)$ in the following way.

$$
\lim _{k \rightarrow \infty}\left\|u_{n_{k}}-u\right\|_{\infty, W}=0
$$

Proof: First consider the issue of $S$ being a subset of $C([a, b] ; W)$. Let $\varepsilon>0$ be given. Then by Theorem 34.7.2 there exists a constant, $C_{\varepsilon}$ such that for all $u \in W$

$$
\|u\|_{W} \leq \frac{\varepsilon}{6 R}\|u\|_{E}+C_{\varepsilon}\|u\|_{X}
$$

Therefore, for all $u \in S$,

$$
\begin{align*}
\|u(t)-u(s)\|_{W} & \leq \frac{\varepsilon}{6 R}\|u(t)-u(s)\|_{E}+C_{\varepsilon}\|u(t)-u(s)\|_{X} \\
& \leq \frac{\varepsilon}{6 R}\left(\|u(t)\|_{E}+\|u(s)\|_{E}\right)+C_{\varepsilon}\|u(t)-u(s)\|_{X} \\
& \leq \frac{\varepsilon}{3}+C_{\varepsilon} R|t-s|^{1 / q} \tag{34.7.56}
\end{align*}
$$

Since $\varepsilon$ is arbitrary, it follows $u \in C([a, b] ; W)$.
Let $D=\mathbb{Q} \cap[a, b]$ so $D$ is a countable dense subset of $[a, b]$. Let $D=\left\{t_{n}\right\}_{n=1}^{\infty}$. By compactness of the embedding of $E$ into $W$, there exists a subsequence $u_{(n, 1)}$ such that as $n \rightarrow \infty, u_{(n, 1)}\left(t_{1}\right)$ converges to a point in $W$. Now take a subsequence of this, called $(n, 2)$ such that as $n \rightarrow \infty, u_{(n, 2)}\left(t_{2}\right)$ converges to a point in $W$. It follows that $u_{(n, 2)}\left(t_{1}\right)$ also converges to a point of $W$. Continue this way. Now consider the diagonal sequence, $u_{k} \equiv$ $u_{(k, k)}$ This sequence is a subsequence of $u_{(n, l)}$ whenever $k>l$. Therefore, $u_{k}\left(t_{j}\right)$ converges for all $t_{j} \in D$.

Claim: Let $\left\{u_{k}\right\}$ be as just defined, converging at every point of $D \equiv[a, b] \cap \mathbb{Q}$. Then $\left\{u_{k}\right\}$ converges at every point of $[a, b]$.

Proof of claim: Let $\varepsilon>0$ be given. Let $t \in[a, b]$. Pick $t_{m} \in D \cap[a, b]$ such that in 34.7.56 $C_{\varepsilon} R\left|t-t_{m}\right|<\varepsilon / 3$. Theefore it follows that there exists $N$ such that if $l, n>N$, then $\left\|u_{l}\left(t_{m}\right)-u_{n}\left(t_{m}\right)\right\|_{X}<\varepsilon / 3$. It follows that for $l, n>N$,

$$
\begin{aligned}
\left\|u_{l}(t)-u_{n}(t)\right\|_{W} \leq & \left\|u_{l}(t)-u_{l}\left(t_{m}\right)\right\|_{W}+\left\|u_{l}\left(t_{m}\right)-u_{n}\left(t_{m}\right)\right\|_{W} \\
& +\left\|u_{n}\left(t_{m}\right)-u_{n}(t)\right\|_{W} \\
\leq & \frac{2 \varepsilon}{3}+\frac{\varepsilon}{3}+\frac{2 \varepsilon}{3}<2 \varepsilon
\end{aligned}
$$

Since $\varepsilon$ was arbitrary, this shows $\left\{u_{k}(t)\right\}_{k=1}^{\infty}$ is a Cauchy sequence. Since $W$ is complete, this shows this sequence converges.

Now for $t \in[a, b]$, it was just shown that if $\varepsilon>0$ there exists $N_{t}$ such that if $n, m>N_{t}$, then

$$
\left\|u_{n}(t)-u_{m}(t)\right\|_{W}<\frac{\varepsilon}{3}
$$

Now let $s \neq t$. Then

$$
\left\|u_{n}(s)-u_{m}(s)\right\|_{W} \leq\left\|u_{n}(s)-u_{n}(t)\right\|_{W}+\left\|u_{n}(t)-u_{m}(t)\right\|_{W}+\left\|u_{m}(t)-u_{m}(s)\right\|_{W}
$$

From 34.7.56

$$
\left\|u_{n}(s)-u_{m}(s)\right\|_{W} \leq 2\left(\frac{\varepsilon}{3}+C_{\varepsilon} R|t-s|^{1 / q}\right)+\left\|u_{n}(t)-u_{m}(t)\right\|_{W}
$$

and so it follows that if $\delta$ is sufficiently small and $s \in B(t, \delta)$, then when $n, m>N_{t}$

$$
\left\|u_{n}(s)-u_{m}(s)\right\|<\varepsilon .
$$

Since $[a, b]$ is compact, there are finitely many of these balls, $\left\{B\left(t_{i}, \delta\right)\right\}_{i=1}^{p}$, such that for $s \in B\left(t_{i}, \delta\right)$ and $n, m>N_{t_{i}}$, the above inequality holds. Let $N>\max \left\{N_{t_{1}}, \cdots, N_{t_{p}}\right\}$. Then
if $m, n>N$ and $s \in[a, b]$ is arbitrary, it follows the above inequality must hold. Therefore, this has shown the following claim.

Claim: Let $\varepsilon>0$ be given. Then there exists $N$ such that if $m, n>N$, then

$$
\left\|u_{n}-u_{m}\right\|_{\infty, W}<\varepsilon
$$

Now let $u(t)=\lim _{k \rightarrow \infty} u_{k}(t)$.

$$
\begin{equation*}
\|u(t)-u(s)\|_{W} \leq\left\|u(t)-u_{n}(t)\right\|_{W}+\left\|u_{n}(t)-u_{n}(s)\right\|_{W}+\left\|u_{n}(s)-u(s)\right\|_{W} \tag{34.7.57}
\end{equation*}
$$

Let $N$ be in the above claim and fix $n>N$. Then

$$
\left\|u(t)-u_{n}(t)\right\|_{W}=\lim _{m \rightarrow \infty}\left\|u_{m}(t)-u_{n}(t)\right\|_{W} \leq \varepsilon
$$

and similarly, $\left\|u_{n}(s)-u(s)\right\|_{W} \leq \varepsilon$. Then if $|t-s|$ is small enough, 34.7 .56 shows the middle term in 34.7 .57 is also smaller than $\varepsilon$. Therefore, if $|t-s|$ is small enough,

$$
\|u(t)-u(s)\|_{W}<3 \varepsilon
$$

Thus $u$ is continuous. Finally, let $N$ be as in the above claim. Then letting $m, n>N$, it follows that for all $t \in[a, b]$,

$$
\left\|u_{m}(t)-u_{n}(t)\right\|_{W}<\varepsilon
$$

Therefore, letting $m \rightarrow \infty$, it follows that for all $t \in[a, b]$,

$$
\left\|u(t)-u_{n}(t)\right\|_{W} \leq \varepsilon
$$

and so $\left\|u-u_{n}\right\|_{\infty, W} \leq \varepsilon$.
Here is an interesting corollary. Recall that for $E$ a Banach space $C^{0, \alpha}([0, T], E)$ is the space of continuous functions $u$ from $[0, T]$ to $E$ such that

$$
\|u\|_{\alpha, E} \equiv\|u\|_{\infty, E}+\rho_{\alpha, E}(u)<\infty
$$

where here

$$
\rho_{\alpha, E}(u) \equiv \sup _{t \neq s} \frac{\|u(t)-u(s)\|_{E}}{|t-s|^{\alpha}}
$$

Corollary 34.7.5 Let $E \subseteq W \subseteq X$ where the injection map is continuous from $W$ to $X$ and compact from $E$ to $W$. Then if $\gamma>\alpha$, the embedding of $C^{0, \gamma}([0, T], E)$ into $C^{0, \alpha}([0, T], X)$ is compact.

Proof: Let $\phi \in C^{0, \gamma}([0, T], E)$

$$
\begin{gathered}
\frac{\|\phi(t)-\phi(s)\|_{X}}{|t-s|^{\alpha}} \leq\left(\frac{\|\phi(t)-\phi(s)\|_{W}}{|t-s|^{\gamma}}\right)^{\alpha / \gamma}\|\phi(t)-\phi(s)\|_{W}^{1-(\alpha / \gamma)} \\
\leq\left(\frac{\|\phi(t)-\phi(s)\|_{E}}{|t-s|^{\gamma}}\right)^{\alpha / \gamma}\|\phi(t)-\phi(s)\|_{W}^{1-(\alpha / \gamma)} \leq \rho_{\gamma, E}(\phi)\|\phi(t)-\phi(s)\|_{W}^{1-(\alpha / \gamma)}
\end{gathered}
$$

Now suppose $\left\{u_{n}\right\}$ is a bounded sequence in $C^{0, \gamma}([0, T], E)$. By Theorem 34.7.4 above, there is a subsequence still called $\left\{u_{n}\right\}$ which converges in $C^{0}([0, T], W)$. Thus from the above inequality

$$
\begin{aligned}
& \frac{\left\|u_{n}(t)-u_{m}(t)-\left(u_{n}(s)-u_{m}(s)\right)\right\|_{X}}{|t-s|^{\alpha}} \\
\leq & \rho_{\gamma, E}\left(u_{n}-u_{m}\right)\left\|u_{n}(t)-u_{m}(t)-\left(u_{n}(s)-u_{m}(s)\right)\right\|_{W}^{1-(\alpha / \gamma)} \\
\leq & C\left(\left\{u_{n}\right\}\right)\left(2\left\|u_{n}-u_{m}\right\|_{\infty, W}\right)^{1-(\alpha / \gamma)}
\end{aligned}
$$

which converges to 0 as $n, m \rightarrow \infty$. Thus

$$
\rho_{\alpha, X}\left(u_{n}-u_{m}\right) \rightarrow 0 \text { as } n, m \rightarrow \infty
$$

Also $\left\|u_{n}-u_{m}\right\|_{\infty, X} \rightarrow 0$ as $n, m \rightarrow \infty$ so this is a Cauchy sequence in $C^{0, \alpha}([0, T], X)$.
The next theorem is a well known result probably due to Lions, Temam, or Aubin.
Theorem 34.7.6 Let $E \subseteq W \subseteq X$ where the injection map is continuous from $W$ to $X$ and compact from $E$ to $W$. Let $p \geq 1$, let $q>1$, and define

$$
\begin{gathered}
S \equiv\left\{u \in L^{p}([a, b] ; E): \text { for some } C,\|u(t)-u(s)\|_{X} \leq C|t-s|^{1 / q}\right. \\
\text { and } \left.\|u\|_{L^{p}([a, b] ; E)} \leq R\right\} .
\end{gathered}
$$

Thus $S$ is bounded in $L^{p}([a, b] ; E)$ and Holder continuous into $X$. Then $S$ is precompact in $L^{p}([a, b] ; W)$. This means that if $\left\{u_{n}\right\}_{n=1}^{\infty} \subseteq S$, it has a subsequence $\left\{u_{n_{k}}\right\}$ which converges in $L^{p}([a, b] ; W)$.

Proof: By Proposition 7.6 .5 on Page 144 it suffices to show that for each $\eta>0, S$ has an $\eta$ net in $L^{p}([a, b] ; W)$.

If not, there exists $\eta>0$ and a sequence $\left\{u_{n}\right\} \subseteq S$, such that

$$
\begin{equation*}
\left\|u_{n}-u_{m}\right\| \geq \eta \tag{34.7.58}
\end{equation*}
$$

for all $n \neq m$ and the norm refers to $L^{p}([a, b] ; W)$. Let

$$
a=t_{0}<t_{1}<\cdots<t_{k}=b, t_{i}-t_{i-1}=(b-a) / k
$$

Now define

$$
\bar{u}_{n}(t) \equiv \sum_{i=1}^{k} \bar{u}_{n_{i}} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t), \bar{u}_{n_{i}} \equiv \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} u_{n}(s) d s .
$$

The idea is to show that $\bar{u}_{n}$ approximates $u_{n}$ well and then to argue that a subsequence of the $\left\{\bar{u}_{n}\right\}$ is a Cauchy sequence yielding a contradiction to 34.7.58.

Therefore,

$$
u_{n}(t)-\bar{u}_{n}(t)=\sum_{i=1}^{k} u_{n}(t) \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)-\sum_{i=1}^{k} \bar{u}_{n_{i}} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)
$$

$$
\begin{gathered}
=\sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} u_{n}(t) d s \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)-\sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} u_{n}(s) d s \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t) \\
=\sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left(u_{n}(t)-u_{n}(s)\right) d s \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)
\end{gathered}
$$

It follows from Jensen's inequality that

$$
\begin{aligned}
& \left\|u_{n}(t)-\bar{u}_{n}(t)\right\|_{W}^{p} \\
= & \sum_{i=1}^{k}\left\|\frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left(u_{n}(t)-u_{n}(s)\right) d s\right\|_{W}^{p} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t) \\
\leq & \sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left\|u_{n}(t)-u_{n}(s)\right\|_{W}^{p} d s \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)
\end{aligned}
$$

and so

$$
\begin{align*}
& \int_{a}^{b}\left\|\left(u_{n}(t)-\bar{u}_{n}(s)\right)\right\|_{W}^{p} d s \\
\leq & \int_{a}^{b} \sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left\|u_{n}(t)-u_{n}(s)\right\|_{W}^{p} d s \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t) d t \\
= & \sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t_{i}}\left\|u_{n}(t)-u_{n}(s)\right\|_{W}^{p} d s d t \tag{34.7.59}
\end{align*}
$$

From Theorem 34.7.2 if $\varepsilon>0$, there exists $C_{\varepsilon}$ such that

$$
\begin{gathered}
\left\|u_{n}(t)-u_{n}(s)\right\|_{W}^{p} \leq \varepsilon\left\|u_{n}(t)-u_{n}(s)\right\|_{E}^{p}+C_{\varepsilon}\left\|u_{n}(t)-u_{n}(s)\right\|_{X}^{p} \\
\leq 2^{p-1} \varepsilon\left(\left\|u_{n}(t)\right\|^{p}+\left\|u_{n}(s)\right\|^{p}\right)+C_{\varepsilon}|t-s|^{p / q}
\end{gathered}
$$

This is substituted in to 34.7 .59 to obtain

$$
\begin{aligned}
& \int_{a}^{b}\left\|\left(u_{n}(t)-\bar{u}_{n}(s)\right)\right\|_{W}^{p} d s \leq \\
& \sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t_{i}}\left(2^{p-1} \varepsilon\left(\left\|u_{n}(t)\right\|^{p}+\left\|u_{n}(s)\right\|^{p}\right)+C_{\varepsilon}|t-s|^{p / q}\right) d s d t \\
= & \sum_{i=1}^{k} 2^{p} \varepsilon \int_{t_{i-1}}^{t_{i}}\left\|u_{n}(t)\right\|_{W}^{p}+\frac{C_{\varepsilon}}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t_{i}}|t-s|^{p / q} d s d t \\
\leq & 2^{p} \varepsilon \int_{a}^{b}\left\|u_{n}(t)\right\|^{p} d t+C_{\varepsilon} \sum_{i=1}^{k} \frac{1}{\left(t_{i}-t_{i-1}\right)}\left(t_{i}-t_{i-1}\right)^{p / q} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t_{i}} d s d t \\
= & 2^{p} \varepsilon \int_{a}^{b}\left\|u_{n}(t)\right\|^{p} d t+C_{\varepsilon} \sum_{i=1}^{k} \frac{1}{\left(t_{i}-t_{i-1}\right)}\left(t_{i}-t_{i-1}\right)^{p / q}\left(t_{i}-t_{i-1}\right)^{2} \\
\leq & 2^{p} \varepsilon R^{p}+C_{\varepsilon} \sum_{i=1}^{k}\left(t_{i}-t_{i-1}\right)^{1+p / q}=2^{p} \varepsilon R^{p}+C_{\varepsilon} k\left(\frac{b-a}{k}\right)^{1+p / q} .
\end{aligned}
$$

Taking $\varepsilon$ so small that $2^{p} \varepsilon R^{p}<\eta^{p} / 8^{p}$ and then choosing $k$ sufficiently large, it follows

$$
\left\|u_{n}-\bar{u}_{n}\right\|_{L^{p}([a, b] ; W)}<\frac{\eta}{4} .
$$

Thus $k$ is fixed and $\bar{u}_{n}$ at a step function with $k$ steps having values in $E$. Now use compactness of the embedding of $E$ into $W$ to obtain a subsequence such that $\left\{\bar{u}_{n}\right\}$ is Cauchy in $L^{p}(a, b ; W)$ and use this to contradict 34.7.58. The details follow.

Suppose $\bar{u}_{n}(t)=\sum_{i=1}^{k} u_{i}^{n} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)$. Thus

$$
\left\|\bar{u}_{n}(t)\right\|_{E}=\sum_{i=1}^{k}\left\|u_{i}^{n}\right\|_{E} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)
$$

and so

$$
R \geq \int_{a}^{b}\left\|\bar{u}_{n}(t)\right\|_{E}^{p} d t=\frac{T}{k} \sum_{i=1}^{k}\left\|u_{i}^{n}\right\|_{E}^{p}
$$

Therefore, the $\left\{u_{i}^{n}\right\}$ are all bounded. It follows that after taking subsequences $k$ times there exists a subsequence $\left\{u_{n_{k}}\right\}$ such that $u_{n_{k}}$ is a Cauchy sequence in $L^{p}(a, b ; W)$. You simply get a subsequence such that $u_{i}^{n_{k}}$ is a Cauchy sequence in $W$ for each $i$. Then denoting this subsequence by $n$,

$$
\begin{aligned}
\left\|u_{n}-u_{m}\right\|_{L^{p}(a, b ; W)} \leq & \left\|u_{n}-\bar{u}_{n}\right\|_{L^{p}(a, b ; W)} \\
& +\left\|\bar{u}_{n}-\bar{u}_{m}\right\|_{L^{p}(a, b ; W)}+\left\|\bar{u}_{m}-u_{m}\right\|_{L^{p}(a, b ; W)} \\
\leq & \frac{\eta}{4}+\left\|\bar{u}_{n}-\bar{u}_{m}\right\|_{L^{p}(a, b ; W)}+\frac{\eta}{4}<\eta
\end{aligned}
$$

provided $m, n$ are large enough, contradicting 34.7.58.
You can give a different version of the above to include the case where there is, instead of a Holder condition, a bound on $u^{\prime}$ for $u \in S$. It is stated next. See [117].

Corollary 34.7.7 Let $E \subseteq W \subseteq X$ where the injection map is continuous from $W$ to $X$ and compact from $E$ to $W$. Let $p \geq 1$, let $q>1$, and define

$$
\begin{gathered}
S \equiv\left\{u \in L^{p}([a, b] ; E): \text { for some } C,\|u(t)-u(s)\|_{X} \leq C|t-s|^{1 / q}\right. \\
\text { and } \left.\|u\|_{L^{p}([a, b] ; E)} \leq R\right\} .
\end{gathered}
$$

Thus $S$ is bounded in $L^{p}([a, b] ; E)$ and Holder continuous into $X$. Then $S$ is precompact in $L^{p}([a, b] ; W)$. This means that if $\left\{u_{n}\right\}_{n=1}^{\infty} \subseteq S$, it has a subsequence $\left\{u_{n_{k}}\right\}$ which converges in $L^{p}([a, b] ; W)$. The same conclusion can be drawn if it is known instead of the Holder condition that $\left\|u^{\prime}\right\|_{L^{1}([a, b] ; X)}$ is bounded.

Proof: The first part is Theorem 34.7.6. Therefore, we just prove the new stuff which involves a bound on the $L^{1}$ norm of the derivative. By Proposition 7.6.5 on Page 144 it suffices to show $S$ has an $\eta$ net in $L^{p}([a, b] ; W)$ for each $\eta>0$.

If not, there exists $\eta>0$ and a sequence $\left\{u_{n}\right\} \subseteq S$, such that

$$
\begin{equation*}
\left\|u_{n}-u_{m}\right\| \geq \eta \tag{34.7.60}
\end{equation*}
$$

for all $n \neq m$ and the norm refers to $L^{p}([a, b] ; W)$. Let

$$
a=t_{0}<t_{1}<\cdots<t_{k}=b, t_{i}-t_{i-1}=(b-a) / k
$$

Now define

$$
\bar{u}_{n}(t) \equiv \sum_{i=1}^{k} \bar{u}_{n_{i}} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t), \bar{u}_{n_{i}} \equiv \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} u_{n}(s) d s
$$

The idea is to show that $\bar{u}_{n}$ approximates $u_{n}$ well and then to argue that a subsequence of the $\left\{\bar{u}_{n}\right\}$ is a Cauchy sequence yielding a contradiction to 34.7.60.

Therefore,

$$
\begin{gathered}
u_{n}(t)-\bar{u}_{n}(t)=\sum_{i=1}^{k} u_{n}(t) \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)-\sum_{i=1}^{k} \bar{u}_{n_{i}} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t) \\
=\sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} u_{n}(t) d s \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)-\sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} u_{n}(s) d s \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t) \\
=\sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left(u_{n}(t)-u_{n}(s)\right) d s \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)
\end{gathered}
$$

It follows from Jensen's inequality that

$$
\begin{aligned}
& \left\|u_{n}(t)-\bar{u}_{n}(t)\right\|_{W}^{p} \\
= & \sum_{i=1}^{k}\left\|\frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left(u_{n}(t)-u_{n}(s)\right) d s\right\|_{W}^{p} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)
\end{aligned}
$$

And so

$$
\begin{gather*}
\int_{0}^{T}\left\|u_{n}(t)-\bar{u}_{n}(t)\right\|_{W}^{p} d t=\sum_{i=1}^{k} \int_{t_{i-1}}^{t_{i}}\left\|\frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left(u_{n}(t)-u_{n}(s)\right) d s\right\|_{W}^{p} d t \\
\leq \sum_{i=1}^{k} \int_{t_{i-1}}^{t_{i}} \varepsilon\left\|\frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left(u_{n}(t)-u_{n}(s)\right) d s\right\|_{E}^{p} d t \\
\quad+C_{\varepsilon} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_{i}}\left\|\frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left(u_{n}(t)-u_{n}(s)\right) d s\right\|_{X}^{p} d t \tag{34.7.61}
\end{gather*}
$$

Consider the second of these. It equals

$$
C_{\varepsilon} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_{i}}\left\|\frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} \int_{s}^{t} u_{n}^{\prime}(\tau) d \tau d s\right\|_{X}^{p} d t
$$

This is no larger than

$$
\leq C_{\varepsilon} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_{i}}\left(\frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t_{i}}\left\|u_{n}^{\prime}(\tau)\right\|_{X} d \tau d s\right)^{p} d t
$$

$$
\begin{aligned}
& =C_{\varepsilon} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_{i}}\left(\int_{t_{i-1}}^{t_{i}}\left\|u_{n}^{\prime}(\tau)\right\|_{X} d \tau\right)^{p} d t \\
= & C_{\varepsilon} \sum_{i=1}^{k}\left(\left(t_{i}-t_{i-1}\right)^{1 / p} \int_{t_{i-1}}^{t_{i}}\left\|u_{n}^{\prime}(\tau)\right\|_{X} d \tau\right)^{p}
\end{aligned}
$$

Since $p \geq 1$,

$$
\begin{aligned}
& \leq C_{\varepsilon}\left(\sum_{i=1}^{k}\left(t_{i}-t_{i-1}\right)^{1 / p} \int_{t_{i-1}}^{t_{i}}\left\|u_{n}^{\prime}(\tau)\right\|_{X} d \tau\right)^{p} \\
& \leq \frac{C_{\varepsilon}(b-a)}{k}\left(\sum_{i=1}^{k} \int_{t_{i-1}}^{t_{i}}\left\|u_{n}^{\prime}(\tau)\right\|_{X} d \tau\right)^{p} \\
& =\frac{C_{\varepsilon}(b-a)}{k}\left(\left\|u_{n}^{\prime}\right\|_{L^{1}([a, b], X)}\right)^{p}<\frac{\eta^{p}}{10^{p}}
\end{aligned}
$$

if $k$ is chosen large enough. Now consider the first in 34.7.61. By Jensen's inequality

$$
\begin{gathered}
\sum_{i=1}^{k} \int_{t_{i-1}}^{t_{i}} \varepsilon\left\|\frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left(u_{n}(t)-u_{n}(s)\right) d s\right\|_{E}^{p} d t \leq \\
\sum_{i=1}^{k} \int_{t_{i-1}}^{t_{i}} \varepsilon \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left\|u_{n}(t)-u_{n}(s)\right\|_{E}^{p} d s d t \\
\leq \quad \varepsilon 2^{p-1} \sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t_{i}}\left(\left\|u_{n}(t)\right\|^{p}+\left\|u_{n}(s)\right\|^{p}\right) d s d t \\
=\quad 2 \varepsilon 2^{p-1} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_{i}}\left(\left\|u_{n}(t)\right\|^{p}\right) d t=\varepsilon(2)\left(2^{p-1}\right)\left\|u_{n}\right\|_{L^{p}([a, b], E)} \leq M \varepsilon
\end{gathered}
$$

Now pick $\varepsilon$ sufficiently small that $M \varepsilon<\frac{\eta^{p}}{10^{p}}$ and then $k$ large enough that the second term in 34.7 .61 is also less than $\eta^{p} / 10^{p}$. Then it will follow that

$$
\left\|\bar{u}_{n}-u_{n}\right\|_{L^{p}([a, b], W)}<\left(\frac{2 \eta^{p}}{10^{p}}\right)^{1 / p}=2^{1 / p} \frac{\eta}{10} \leq \frac{\eta}{5}
$$

Thus $k$ is fixed and $\bar{u}_{n}$ at a step function with $k$ steps having values in $E$. Now use compactness of the embedding of $E$ into $W$ to obtain a subsequence such that $\left\{\bar{u}_{n}\right\}$ is Cauchy in $L^{p}([a, b] ; W)$ and use this to contradict 34.7.60. The details follow.

Suppose $\bar{u}_{n}(t)=\sum_{i=1}^{k} u_{i}^{n} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)$. Thus

$$
\left\|\bar{u}_{n}(t)\right\|_{E}=\sum_{i=1}^{k}\left\|u_{i}^{n}\right\|_{E} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)
$$

and so

$$
R \geq \int_{a}^{b}\left\|\bar{u}_{n}(t)\right\|_{E}^{p} d t=\frac{T}{k} \sum_{i=1}^{k}\left\|u_{i}^{n}\right\|_{E}^{p}
$$

Therefore, the $\left\{u_{i}^{n}\right\}$ are all bounded. It follows that after taking subsequences $k$ times there exists a subsequence $\left\{u_{n_{k}}\right\}$ such that $u_{n_{k}}$ is a Cauchy sequence in $L^{p}([a, b] ; W)$. You simply get a subsequence such that $u_{i}^{n_{k}}$ is a Cauchy sequence in $W$ for each $i$. Then denoting this subsequence by $n$,

$$
\begin{aligned}
\left\|u_{n}-u_{m}\right\|_{L^{p}(a, b ; W)} \leq & \left\|u_{n}-\bar{u}_{n}\right\|_{L^{p}(a, b ; W)} \\
& +\left\|\bar{u}_{n}-\bar{u}_{m}\right\|_{L^{p}(a, b ; W)}+\left\|\bar{u}_{m}-u_{m}\right\|_{L^{p}(a, b ; W)} \\
\leq & \frac{\eta}{4}+\left\|\bar{u}_{n}-\bar{u}_{m}\right\|_{L^{p}(a, b ; W)}+\frac{\eta}{4}<\eta
\end{aligned}
$$

provided $m, n$ are large enough, contradicting 34.7.60.

### 34.8 Some Evolution Inclusions

Let $H$ be a Hilbert space and let $\mathscr{H}$ denote $L^{2}(0, T ; H)$. Here will be an application to an evolution equation having values in $\mathscr{H}$. It will always be the case that $H=H^{\prime}$ so this is the simplest sort of a Gelfand triple, $V=H=H^{\prime}=V^{\prime}$. First is given a maximal monotone operator.

Definition 34.8.1 Let $D(L) \equiv\left\{u \in \mathscr{H}\right.$ such that $u^{\prime} \in \mathscr{H}$ and $\left.u(0)=u_{0}\right\}$. Then for $u \in$ $D(L), L u \equiv u^{\prime}$.

Note that $L$ is not linear.
Lemma 34.8.2 For $L$ as just defined, $L$ is maximal monotone $L: \mathscr{H} \rightarrow \mathscr{H}$.
Proof: To show it is maximal monotone, it suffices to verify that $L+I$ is onto. This is by Theorem 25.7.13 on Page 881. Thus consider the equation

$$
u^{\prime}+u=f, u(0)=u_{0}
$$

Is there a solution? Of course there is and it equals

$$
u(t)=e^{-t} u_{0}+\int_{0}^{t} e^{-(t-s)} f(s) d s
$$

by the usual application of integrating factors and so forth.
Then with this, the following is from Theorem 25.7.55 on Page 919. This is a well known result found in Brezis [24].

Theorem 34.8.3 Let $u_{0} \in D(\phi)$ where $\phi: H \rightarrow[0, \infty]$ is proper, lower semicontinuous, and convex. Also let $f \in \mathscr{H}$ be given and $u_{0} \in D(\phi)$. Then there exists a solution $u$ to the evolution initial value problem,

$$
u^{\prime}(t)+\partial \phi(u(t)) \ni f(t) \text { a.e.in } H, u(0)=u_{0}
$$

This solution satisfies $u(t) \in D(\partial \phi)$ for a.e. $t$, there exists $z \in \mathscr{H}$ such that $z(t) \in \partial \phi(u(t))$ for a.e. $t$ such that the inclusion is an equation with $\partial \phi(u(t))$ replaced with $z(t)$.

Proof: Define a function $\Phi: \mathscr{H} \rightarrow \mathbb{R}$

$$
\Phi(u) \equiv \int_{0}^{T} \phi(u) d t
$$

There are no measurability issues because $\phi$ is lower semicontinuous and so the composition $\phi(u)$ will be appropriately measurable. Then this is clearly convex. It is proper because $\Phi\left(u_{0}\right)=\phi\left(u_{0}\right) T$ so $u_{0} \in D(\Phi)$. If $u_{n} \rightarrow u$ in $\mathscr{H}$, does it follow that

$$
\lim _{n \rightarrow \infty} \Phi\left(u_{n}\right) \geq \Phi(u)
$$

Suppose not so $\Phi(u)>\liminf _{n \rightarrow \infty} \Phi\left(u_{n}\right)$. Then choosing a subsequence such that

$$
u_{n} \rightarrow u \text { pointwise a.e., }
$$

$$
\begin{aligned}
\Phi(u) & >\lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty} \Phi\left(u_{n}\right) \equiv \lim \inf _{n \rightarrow \infty} \int_{0}^{T} \phi\left(u_{n}\right) d t \\
& \geq \int_{0}^{T} \lim _{n \rightarrow \infty} \phi\left(u_{n}\right) d t=\int_{0}^{T} \phi(u) d t=\Phi(u)
\end{aligned}
$$

which is a contradiction. Thus $\Phi$ is also lower semicontinuous.
The constant function $u \equiv u_{0}$ is in $D(L) \cap D(\Phi)$. To use Theorem 25.7.55 on Page 919, it is required to show that

$$
\Phi\left(J_{\lambda} u\right) \leq \Phi(u)+C \lambda
$$

In this case, the duality map is just the identity map. Hence $J_{\lambda} u$ is the solution to

$$
0=\left(J_{\lambda} u-u\right)+\lambda L\left(J_{\lambda} u\right)
$$

Hence letting $J_{\lambda} u$ be denoted by $u_{\lambda}$, it follows that $u_{\lambda}$ would be the solution to

$$
\lambda u_{\lambda}^{\prime}+u_{\lambda}=u, u_{\lambda}(0)=u_{0}
$$

Using the usual integrating factor procedure, it follows that

$$
u_{\lambda}(t)=e^{-(1 / \lambda) t} u_{0}+\int_{0}^{t} e^{-(1 / \lambda)(t-s)} \frac{1}{\lambda} u(s) d s
$$

Note that

$$
e^{-(1 / \lambda) t}+\int_{0}^{t} \frac{1}{\lambda} e^{-(1 / \lambda)(t-s)} d s=1
$$

Thus by Jensen's inequality,

$$
\phi\left(u_{\lambda}(t)\right) \leq e^{-(1 / \lambda) t} \phi\left(u_{0}\right)+\int_{0}^{t} e^{-(1 / \lambda)(t-s)} \frac{1}{\lambda} \phi(u(s)) d s
$$

Then

$$
\Phi\left(u_{\lambda}\right) \leq \int_{0}^{T} e^{-(1 / \lambda) t} \phi\left(u_{0}\right) d t+\int_{0}^{T} \int_{0}^{t} e^{-(1 / \lambda)(t-s)} \frac{1}{\lambda} \phi(u(s)) d s d t
$$

$$
=\int_{0}^{T} e^{-(1 / \lambda) t} \phi\left(u_{0}\right) d t+\int_{0}^{T} \phi(u(s)) \int_{s}^{T} e^{-(1 / \lambda)(t-s)} \frac{1}{\lambda} d t d s
$$

Now $\int_{s}^{T} e^{-(1 / \lambda)(t-s)} \frac{1}{\lambda} d t=1-e^{\frac{1}{\lambda} s-\frac{1}{\lambda} T}<1$ and since $\phi \geq 0$, this shows that

$$
\Phi\left(u_{\lambda}\right) \leq \phi\left(u_{0}\right) \int_{0}^{\infty} e^{-(1 / \lambda) t} d t+\int_{0}^{T} \phi(u(s)) d t
$$

so

$$
\Phi\left(u_{\lambda}\right) \leq \phi\left(u_{0}\right) \lambda+\Phi(u)
$$

It follows that the conditions of Theorem 25.7.55 on Page 919 are satisfied and so $L+\partial \Phi$ is maximal monotone. Thus if $f \in \mathscr{H}$, there exists $u \in D(L) \cap D(\partial \Phi)$ such that for each $v \in \mathscr{H}$ there exists a solution $u_{v}$ to

$$
u_{v}^{\prime}+z_{v}+u_{v}=f+v, u_{v}(0)=u_{0} \in D(\phi)
$$

where $z_{v} \in \partial \Phi\left(u_{v}\right)$. I will show that $v \rightarrow u_{v}$ has a fixed point. Note that if $z_{i} \in \partial \Phi\left(u_{i}\right)$, then

$$
\int_{t-h}^{t}\left(z_{1}-z_{2}, u_{1}-u_{2}\right) d s \geq 0, \text { any } h \leq t
$$

To see this, you could simply let $u_{2}=u_{1}$ off $[t-h, t]$ and pick $z_{1}=z_{2}$ also on this set. Also, you can conclude that $u \in \partial \Phi$ implies $u(t) \in \partial \phi$ for a.e. $t$. I show this now. Let $[a, b] \in \mathscr{G}(\partial \phi)$. Then as just noted, for $[u, z] \in \mathscr{G}(\partial \Phi)$,

$$
\int_{t-h}^{t}(z-b, u-a) d s \geq 0
$$

Then by the fundamental theorem of calculus, for a.e. $t,(z(t)-b, u(t)-a) \geq 0$ a.e. Letting $\left\{\left[a_{i}, b_{i}\right]\right\}_{i=1}^{\infty}$ be a dense subset of $\mathscr{G}(\partial \phi)$, one can take the union of countably many sets of measure zero, one for each $\left[a_{i}, b_{i}\right]$ and conclude that off this set of measure zero, $\left(z(t)-b_{i}, u(t)-a_{i}\right) \geq 0$ for all $i$. Hence this is also true for all $[a, b] \in \mathscr{G}(\partial \phi)$ and so $z(t) \in \partial \phi(u(t))$ for a.e. $t$.

Then if you have $v_{i}, i=1,2$

$$
L u_{v_{1}}-L u_{v_{2}}+z_{v_{1}}-z_{v_{2}}+u_{v_{1}}-u_{v_{2}}=v_{1}-v_{2}
$$

Then taking inner products with $u_{v_{1}}-u_{v_{2}}$ and integrating up to $t$,

$$
\begin{aligned}
& \frac{1}{2}\left|\left(u_{v_{1}}-u_{v_{2}}\right)(t)\right|_{H}^{2}+\int_{0}^{t}\left(z_{v_{1}}-z_{v_{2}}, u_{v_{1}}-u_{v_{2}}\right) d t+\int_{0}^{t}\left|u_{v_{1}}-u_{v_{2}}\right|^{2} d s \\
\leq & \frac{1}{2} \int_{0}^{t}\left|v_{1}-v_{2}\right|^{2} d s+\frac{1}{2} \int_{0}^{t}\left|u_{v_{1}}-u_{v_{2}}\right|^{2} d s
\end{aligned}
$$

Now by monotonicity of $\phi$ and the above,

$$
\left|\left(u_{v_{1}}-u_{v_{2}}\right)(t)\right|_{H}^{2} \leq \int_{0}^{t}\left|v_{1}-v_{2}\right|^{2} d s
$$

which shows that a high enough power of the mapping $v \rightarrow u_{v}$ is a contraction map on $\mathscr{H}$ and so there exists a unique fixed point $u$. Thus $u_{u}=u$ and so

$$
u^{\prime}+z+u=f+u, u_{v}(0)=u_{0} \in D(\phi), z(t) \in \partial \phi(t) \text { a.e. }
$$

and so

$$
u^{\prime}+z=f \text { in } \mathscr{H}, u(t) \in D(\partial \phi) \text { a.e., } z(t) \in \partial \phi(u(t)) \text { a.e., } u^{\prime} \in \mathscr{H}, \text { and } u(0)=u_{0}
$$

Note that in the above, the initial condition only needs to be in $D(\phi)$, not in the smaller $D(\partial \phi)$, although the solution is in $D(\partial \phi)$ for a.e. $t$. Also note that $f$ has no smoothness. It only is in $\mathscr{H}$. This is really a nice result.

## Chapter 35

## Maximal Monotone Operators, Hilbert Space

### 35.1 Basic Theory

Here is provided a short introduction to some of the most important properties of maximal monotone operators in Hilbert space. The following definition describes them. It is more specialized than the earlier material on maximal monotone operators from a Banach space to its dual and therefore, better results can be obtained. More on this can be read in [24] and [116].

Definition 35.1.1 Let $H$ be a real Hilbert space and let $A: D(A) \rightarrow \mathscr{P}(H)$ have the following properties.

1. For each $y \in H$ there exists $x \in D(A)$ such that $y \in x+A x$.
2. $A$ is monotone. That is, if $z \in A x$ and $w \in$ Ay then

$$
(z-w, x-y) \geq 0
$$

Such an operator is called a maximal monotone operator.
It turns out that whenever $A$ is maximal monotone, so is $\lambda A$ for all $\lambda>0$.
Lemma 35.1.2 Suppose $A$ is maximal monotone. Then so is $\lambda A$. Also $J_{\lambda} \equiv(I+\lambda A)^{-1}$ makes sense for each $\lambda>0$ and is Lipschitz continuous.

Proof: To begin with consider $(I+A)^{-1}$. Suppose

$$
x_{1}, x_{2} \in(I+A)^{-1}(y)
$$

Then $y \in(I+A) x_{i}$ and so $y-x_{i} \in A x_{i}$. By monotonicity

$$
\left(y-x_{1}-\left(y-x_{2}\right), x_{1}-x_{2}\right) \geq 0
$$

and so

$$
0 \geq\left|x_{1}-x_{2}\right|^{2}
$$

which shows $J_{1} \equiv(I+A)^{-1}$ makes sense. In fact this is Lipschitz with Lipschitz constant 1. Here is why. $x \in(I+A) J_{1} x$ and $y \in(I+A) J_{1} y$. Then

$$
x-J_{1} x \in A J_{1} x, y-J_{1} y \in A J_{1} y
$$

and so by monotonicity

$$
0 \leq\left(x-J_{1} x-\left(y-J_{1} y\right), J_{1} x-J_{1} y\right)
$$

which yields

$$
\begin{aligned}
\left|J_{1} x-J_{1} y\right|^{2} & \leq\left(x-y, J_{1} x-J_{1} y\right) \\
& \leq|x-y|\left|J_{1} x-J_{1} y\right|
\end{aligned}
$$

which yields the result.
Next consider the claim that $\lambda A$ is maximal monotone. The monotone part is immediate. The only thing in question is whether $I+\lambda A$ is onto. Let $r \in(-1,1)$ and pick $f \in H$. Consider solving the equation for $u$

$$
\begin{equation*}
(1+r) u+A u \ni(1+r) f \tag{35.1.1}
\end{equation*}
$$

This is equivalent to finding $u$ such that

$$
(I+A) u \ni(1+r) f-r u
$$

or in other words finding $u$ such that

$$
u=J_{1}((1+r) f-r u)
$$

However, if

$$
T u \equiv J_{1}((1+r) f-r u)
$$

then since $|r|<1, T$ is a contraction mapping and so there exists a unique solution to 35.1.1. Thus

$$
u+\frac{1}{1+r} A u \ni f
$$

It follows for any $|r|<1,(1+r)^{-1} A$ is maximal monotone. This takes care of all $\lambda \in$ $\left(\frac{1}{2}, \infty\right)$. Now do the same thing for $(2 / 3) A$ to get the result for all $\lambda \in\left(\left(\frac{2}{3}\right)\left(\frac{1}{2}\right), \infty\right)$. Now apply the same argument to $(2 / 3)^{2} A$ to get the result for all $\lambda \in\left(\left(\frac{2}{3}\right)^{2}\left(\frac{1}{2}\right), \infty\right)$. Next consider the same argument to $(2 / 3)^{3} A$ to get the desired result for all $\lambda \in\left(\left(\frac{2}{3}\right)^{3}\left(\frac{1}{2}\right), \infty\right)$. Continuing this way shows $\lambda A$ is maximal monotone for all $\lambda>0$. Also from the first part of the proof $(I+\lambda A)^{-1}$ is Lipschitz continuous with Lipschitz constant 1 . This proves the lemma.

A maximal monotone operator can be approximated with a Lipschitz continuous operator which is also monotone and has certain salubrious properties. This operator is called the Yosida approximation and as in the case of linear operators it is obtained by formally considering

$$
\frac{A}{1+\lambda A}
$$

If you do the division formally you get the definition for $A_{\lambda}$,

$$
\begin{equation*}
A_{\lambda} x \equiv \frac{1}{\lambda} x-\frac{1}{\lambda} J_{\lambda} x \tag{35.1.2}
\end{equation*}
$$

where $J_{\lambda}=(I+\lambda A)^{-1}$ as above. It is obvious that $A_{\lambda}$ is Lipschitz continuous with Lipschitz constant no more than $2 / \lambda$. Actually you can show $1 / \lambda$ also works but this is not important here.

Lemma 35.1.3 $A_{\lambda} x \in A J_{\lambda} x$ and $\left|A_{\lambda} x\right| \leq|y|$ for all $y \in A x$ whenever $x \in D(A)$. Also $A_{\lambda}$ is monotone.

Proof: Consider the first claim. From the definition,

$$
A_{\lambda} x \equiv \frac{1}{\lambda} x-\frac{1}{\lambda} J_{\lambda} x
$$

Is

$$
\frac{1}{\lambda} x-\frac{1}{\lambda} J_{\lambda} x \in A J_{\lambda} x ?
$$

Is

$$
x-J_{\lambda} x \in \lambda A J_{\lambda} x ?
$$

Is

$$
x \in J_{\lambda} x+\lambda A J_{\lambda} x ?
$$

Is

$$
x \in(I+\lambda A) J_{\lambda} x ?
$$

Certainly so. This is how $J_{\lambda}$ is defined.
Now consider the second claim. Let $y \in A x$ for some $x \in D(A)$. Then by monotonicity and what was just shown

$$
0 \leq\left(A_{\lambda} x-y, J_{\lambda} x-x\right)=-\lambda\left(A_{\lambda} x-y, A_{\lambda} x\right)
$$

and so

$$
\left|A_{\lambda} x\right|^{2} \leq\left(y, A_{\lambda} x\right) \leq|y|\left|A_{\lambda} x\right|
$$

Finally, to show $A_{\lambda}$ is monotone,

$$
\begin{gathered}
\left(A_{\lambda} x-A_{\lambda} y, x-y\right)= \\
\left(\frac{1}{\lambda} x-\frac{1}{\lambda} J_{\lambda} x-\left(\frac{1}{\lambda} y-\frac{1}{\lambda} J_{\lambda} y\right), x-y\right) \\
=\frac{1}{\lambda}|x-y|^{2}-\frac{1}{\lambda}\left(J_{\lambda} x-J_{\lambda} y, x-y\right) \\
\geq \frac{1}{\lambda}|x-y|^{2}-\frac{1}{\lambda}|x-y|\left|J_{\lambda} x-J_{\lambda} y\right| \\
\geq \frac{1}{\lambda}|x-y|^{2}-\frac{1}{\lambda}|x-y|^{2}=0
\end{gathered}
$$

and this proves the lemma.
Proposition 35.1.4 Suppose $D(A)$ is dense in $H$. Then for all $x \in H$,

$$
\left|J_{\lambda} x-x\right| \rightarrow 0
$$

Proof: From the above, if $u \in D(A)$ and $y \in A u$, then

$$
\left|\frac{1}{\lambda} u-\frac{1}{\lambda} J_{\lambda} u\right| \leq|y|
$$

Hence $J_{\lambda} u \rightarrow u$. Now for $x$ arbitrary,

$$
\begin{aligned}
\left|J_{\lambda} x-x\right| & \leq\left|J_{\lambda} x-J_{\lambda} u\right|+\left|J_{\lambda} u-u\right|+|u-x| \\
& <2 \varepsilon+\left|J_{\lambda} u-u\right|
\end{aligned}
$$

where the last term converges to 0 as $\lambda \rightarrow 0$. Since $\varepsilon$ is arbitrary, this shows the proposition.
Thus in the case where $D(A)$ is dense, if you have

$$
x \in \varepsilon A x_{\varepsilon}+x_{\varepsilon}
$$

so that $x_{\varepsilon}=J_{\varepsilon} x$, then $\left|x-x_{\varepsilon}\right| \rightarrow 0$.
The next lemma gives a way to determine whether a pair $[x, y]$ is in the graph of $A$ defined as

$$
\{[x, y]: y \in A x\} \equiv \mathscr{G}(A)
$$

Here I am writing $[\cdot, \cdot]$ rather than $(\cdot, \cdot)$ to avoid confusion with the inner product. It is the conclusion of this lemma which accounts for the use of the term "maximal". It essentially says there is no larger monotone graph which includes the one for $A$.

Lemma 35.1.5 Suppose $\left(y_{1}-y, x_{1}-x\right) \geq 0$ for all $[x, y] \in \mathscr{G}(A)$ where $A$ is maximal monotone. Then $x_{1} \in D(A)$ and $y_{1} \in A x_{1}$. Also if $\left[x_{k}, y_{k}\right] \in \mathscr{G}(A)$ and $x_{k} \rightarrow x, y_{k} \rightharpoonup y$ where the half arrow denotes weak convergence, then $[x, y] \in \mathscr{G}(A)$.

Proof: I want to show $y_{1} \in A x_{1}$ or in other words I want to show

$$
x_{1}+\lambda y_{1} \in x_{1}+\lambda A x_{1}
$$

or in other words

$$
J_{\lambda}\left(x_{1}+\lambda y_{1}\right)=x_{1} .
$$

This is the motivation for the following argument.
From Lemma 35.1.3 $A_{\lambda}\left(x_{1}+\lambda y_{1}\right) \in A J_{\lambda}\left(x_{1}+\lambda y_{1}\right)$ and so by the above assumption

$$
\begin{aligned}
& \quad 0 \leq\left(y_{1}-A_{\lambda}\left(x_{1}+\lambda y_{1}\right), x_{1}-J_{\lambda}\left(x_{1}+\lambda y_{1}\right)\right) \\
& =\left(y_{1}-\left(\frac{1}{\lambda}\left(x_{1}+\lambda y_{1}\right)-\frac{1}{\lambda} J_{\lambda}\left(x_{1}+\lambda y_{1}\right)\right), x_{1}-J_{\lambda}\left(x_{1}+\lambda y_{1}\right)\right) \\
& =\left(\left(-\frac{1}{\lambda} x_{1}+\frac{1}{\lambda} J_{\lambda}\left(x_{1}+\lambda y_{1}\right)\right), x_{1}-J_{\lambda}\left(x_{1}+\lambda y_{1}\right)\right) \\
& =-\frac{1}{\lambda}\left(x_{1}-J_{\lambda}\left(x_{1}+\lambda y_{1}\right), x_{1}-J_{\lambda}\left(x_{1}+\lambda y_{1}\right)\right)
\end{aligned}
$$

which requires

$$
x_{1}=J_{\lambda}\left(x_{1}+\lambda y_{1}\right)
$$

and this says $x_{1} \in D(A)$ because $J_{\lambda}$ maps into $D(A)$. Also it says

$$
x_{1}+A x_{1} \ni x_{1}+\lambda y_{1}
$$

and so $y_{1} \in A x_{1}$.
This makes the last claim pretty easy. Suppose $x_{k} \rightarrow x$ where $x_{k} \in D(A)$ and that $y_{k} \in$ $A x_{k}$ and $y_{k} \rightharpoonup y$. I need to verify $y=A x$ and $x \in D(A)$. Let $[u, v] \in \mathscr{G}(A)$. Then

$$
(y-v, x-u)=\lim _{k \rightarrow \infty}\left(y_{k}-v, x_{k}-u\right) \geq 0
$$

and so, by the first part, $x \in D(A)$ and $y \in A x$. Why does that limit hold? It is because

$$
\begin{gathered}
\left|(y-v, x-u)-\left(y_{k}-v, x_{k}-u\right)\right| \\
\leq\left|(y-v, x-u)-\left(y_{k}-v, x-u\right)\right|+\left|\left(y_{k}-v, x_{k}-x\right)\right|
\end{gathered}
$$

The second term is no larger than

$$
\left|y_{k}-v\right|\left|x_{k}-x\right|
$$

which converges to 0 since $y_{k}$ is weakly convergent, hence bounded. The first term converges to 0 because of the assumption that $y_{k}$ converges weakly to $y$. This proves the lemma.

What about the sum of maximal monotone operators? This might not be maximal monotone but what you can say is the following.

Proposition 35.1.6 Let A be maximal monotone and let B be Lipschitz and monotone. Then $A+B$ is maximal monotone.

Proof: First suppose $B$ has a Lipschitz constant less than 1. The monotonicity is obvious. I need to show that for any $y$ there exists $x \in D(A)$ such that

$$
y \in x+B x+A x
$$

This hapens if and only if

$$
y-B x \in(I+A) x
$$

if and only if $x=(I+A)^{-1}(y-B x)$. Let

$$
T x \equiv(I+A)^{-1}(y-B x)
$$

Then $T$ is clearly a contraction mapping because $(I+A)^{-1}$ is Lipschits with Lipschitz constant 1. Therefore, there exists a unique fixed point and this shows $A+B$ is maximal monotone. Now the same argument applied to $A+B$ shows that $A+2 B$ is maximal monotone. Continuing this way $A+n B$ is maximal monotone. Now for arbitrary $B$ let $n$ be large enough that $n^{-1} B$ has Lipschitz constant less than 1. Then as just explained, $A+n\left(n^{-1} B\right)=A+B$ is maximal monotone. This proves the proposition.

The following is a useful result for determining conditions under which $A+B$ is maximal monotone or more particularly whether a given $y$ is in $(I+A+B)(H)$ where $A, B$ are both maximal monotone.

Theorem 35.1.7 Let $A$ and $B$ be maximal monotone, let

$$
y \in x_{\lambda}+B_{\lambda} x_{\lambda}+A x_{\lambda}
$$

and suppose $B_{\lambda} x_{\lambda}$ is bounded independent of $\lambda$. Then there exists $x \in D(A) \cap D(B)$ such that $y=x+A x+B x$.

Proof: First of all, it follows from Proposition 35.1.6 that there exists a unique $x_{\lambda}$. Note

$$
\begin{aligned}
y-x_{\lambda}-B_{\lambda} x_{\lambda} & \in A x_{\lambda} \\
y-x_{\mu}-B_{\mu} x_{\mu} & \in A x_{\mu}
\end{aligned}
$$

and so by monotonicity of $A$,

$$
\left(x_{\mu}-x_{\lambda}+B_{\mu} x_{\mu}-B_{\lambda} x_{\lambda}, x_{\lambda}-x_{\mu}\right) \geq 0
$$

and so

$$
\begin{align*}
\left|x_{\lambda}-x_{\mu}\right|^{2} & \leq\left(B_{\mu} x_{\mu}-B_{\lambda} x_{\lambda}, x_{\lambda}-x_{\mu}\right) \\
& =-\left(B_{\lambda} x_{\lambda}-B_{\mu} x_{\mu}, x_{\lambda}-x_{\mu}\right) \tag{35.1.3}
\end{align*}
$$

I want to write as many things as possible in terms of the $B_{\lambda}$ and $B_{\mu}$. Denote as $J_{\lambda}(B)$ the operator $(I+\lambda B)^{-1}$. Then

$$
B_{\lambda} x_{\lambda}=\frac{1}{\lambda}\left(x_{\lambda}-J_{\lambda}(B) x_{\lambda}\right)
$$

and so

$$
x_{\lambda}=\lambda B_{\lambda} x_{\lambda}+J_{\lambda}(B) x_{\lambda}
$$

Thus 35.1.3 becomes

$$
\begin{gathered}
\left|x_{\lambda}-x_{\mu}\right|^{2}= \\
-\left(B_{\lambda} x_{\lambda}-B_{\mu} x_{\mu}, \lambda B_{\lambda} x_{\lambda}+J_{\lambda}(B) x_{\lambda}-\left(\mu B_{\mu} x_{\mu}+J_{\mu}(B) x_{\mu}\right)\right) \\
=-\left(B_{\lambda} x_{\lambda}-B_{\mu} x_{\mu}, \lambda B_{\lambda} x_{\lambda}-\mu B_{\mu} x_{\mu}\right) \\
+\left(B_{\mu} x_{\mu}-B_{\lambda} x_{\lambda}, J_{\lambda}(B) x_{\lambda}-J_{\mu}(B) x_{\mu}\right) \\
=-\left(B_{\lambda} x_{\lambda}-B_{\mu} x_{\mu}, \lambda B_{\lambda} x_{\lambda}-\lambda B_{\mu} x_{\mu}\right)-\left(B_{\lambda} x_{\lambda}-B_{\mu} x_{\mu},(\lambda-\mu) B_{\mu} x_{\mu}\right) \\
-\left(B_{\lambda} x_{\lambda}-B_{\mu} x_{\mu}, J_{\lambda}(B) x_{\lambda}-J_{\mu}(B) x_{\mu}\right)
\end{gathered}
$$

Now recall $B_{\mu} x \in B J_{\mu}(B) x$. Then by monotonicity the first and last terms to the right of the equal sign in the above are negative. Therefore,

$$
\left|x_{\lambda}-x_{\mu}\right|^{2} \leq\left|\left(B_{\lambda} x_{\lambda}-B_{\mu} x_{\mu},(\lambda-\mu) B_{\mu} x_{\mu}\right)\right| \leq C|\lambda-\mu|
$$

where $C$ is some constant which comes from the assumption the $B_{\lambda} x_{\lambda}$ are bounded.

Therefore, letting $\lambda$ denote a sequence converging to 0 it follows

$$
\lim _{\lambda \rightarrow 0} x_{\lambda}=x_{1} \in H
$$

for some $x$, the convergence being strong convergence. Also taking a further subsequence and using weak compactness it can be assumed

$$
B_{\lambda} x_{\lambda} \rightharpoonup z_{1}
$$

where this time the convergence is weak. Taking another subsequence, it can also be assumed

$$
\begin{equation*}
y-x_{\lambda}-B_{\lambda} x_{\lambda} \rightharpoonup z_{2} \tag{35.1.4}
\end{equation*}
$$

the convergence being weak convergence. Recall $B_{\lambda} x_{\lambda} \in B J_{\lambda}(B) x_{\lambda}$ and also note that by assumption there is a constant $C$ independent of $\lambda$ such that

$$
C \geq\left|B_{\lambda} x_{\lambda}\right| \geq \frac{1}{\lambda}\left(x_{\lambda}-J_{\lambda}(B) x\right)
$$

which shows

$$
J_{\lambda}(B) x_{\lambda} \rightarrow x_{1}
$$

also. Now it follows from Lemma 35.1.5 that $x_{1} \in D(B)$ and $z_{1} \in B x_{1}$. Recall

$$
y-x_{\lambda}-B_{\lambda} x_{\lambda} \in A x_{\lambda}
$$

and so by the same lemma again,

$$
x_{1} \in D(A), z_{2} \in A x_{1}
$$

By 35.1.4 it follows

$$
y-x_{1}-z_{1}=z_{2} \in A x_{1}
$$

Thus

$$
y=x_{1}+z_{1}+z_{2} \in x_{1}+B x_{1}+A x_{1}
$$

and this proves the theorem.

### 35.2 Evolution Inclusions

One of the interesting things about maximal monotone operators is the concept of evolution inclusions. To facilitate this, here is a little lemma.

Lemma 35.2.1 Let $f:[0, T] \rightarrow \mathbb{R}$ be continuous and suppose

$$
D^{+} f(t) \equiv \lim \sup _{h \rightarrow 0+} \frac{f(t+h)-f(t)}{h}<g(t)
$$

where $g$ is a continuous function. Then

$$
f(t)-f(0) \leq \int_{0}^{t} g(s) d s
$$

Proof: Suppose this is not so. Then let

$$
S \equiv\left\{t \in[0, T]: f(t)-f(0)>\int_{0}^{t} g(s) d s\right\}
$$

and it would follow that $S \neq \emptyset$. Let $a=\inf S$. Then there exists a decreasing sequence $h_{n} \rightarrow 0$ such that

$$
\begin{equation*}
f\left(a+h_{n}\right)-f(0)>\int_{0}^{a+h_{n}} g(s) d s \tag{35.2.5}
\end{equation*}
$$

First suppose $a=0$. Then dividing by $h_{n}$ and taking the limit,

$$
g(0)>D^{+} f(0) \geq g(0)
$$

a contradiction. Therefore, assume $a>0$. Then by continuity

$$
f(a)-f(0) \geq \int_{0}^{a} g(s) d s
$$

If strict inequality holds, then $a \neq \inf S$. It follows

$$
f(a)-f(0)=\int_{0}^{a} g(s) d s
$$

and so from 35.2.5

$$
\frac{f\left(a+h_{n}\right)-f(a)}{h_{n}}>\frac{1}{h_{n}} \int_{a}^{a+h_{n}} g(s) d s
$$

Then doing limsup ${ }_{n \rightarrow \infty}$ to both sides,

$$
g(a)>D^{+} f(a) \geq g(a)
$$

the same sort of contradiction obtained earlier. Thus $S=\emptyset$ and this proves the lemma.
The following is the main result.
Theorem 35.2.2 Let $H$ be a Hilbert space and let $A$ be a maximal monotone operator as described above. Let $f:[0, T] \rightarrow H$ be continuous such that $f^{\prime} \in L^{2}(0, T ; H)$. Then there exists a unique solution to the evolution inclusion

$$
y^{\prime}+A y \ni f, y(0)=y_{0} \in D(A)
$$

Here $y^{\prime}$ exists a.e., $y(t) \in D(A)$ a.e., $y$ is continuous.
Proof: Let $y_{\lambda}$ be the solution to

$$
y_{\lambda}^{\prime}+A_{\lambda} y_{\lambda}=f, y_{\lambda}(0)=y_{0}
$$

I will base the entire proof on estimating the solutions to the corresponding integral equation

$$
\begin{equation*}
y_{\lambda}(t)-y_{0}+\int_{0}^{t} A_{\lambda} y_{\lambda}(s) d s=\int_{0}^{t} f(s) d s \tag{35.2.6}
\end{equation*}
$$

Let $h, k$ be small positive numbers. Then

$$
\begin{equation*}
y_{\lambda}(t+h)-y_{\lambda}(t)+\int_{t}^{t+h} A_{\lambda} y_{\lambda}(s) d s=\int_{t}^{t+h} f(s) d s \tag{35.2.7}
\end{equation*}
$$

Next consider the difference operator

$$
D_{k} g(t) \equiv \frac{g(t+k)-g(t)}{k}
$$

Do this $D_{k}$ to both sides of 35.2 .7 where $k<h$. This gives

$$
\begin{align*}
D_{k}\left(y_{\lambda}(t+h)-\right. & \left.y_{\lambda}(t)\right)+\frac{1}{k}\left(\int_{t+h}^{t+h+k} A_{\lambda} y_{\lambda}(s) d s-\int_{t}^{t+k} A_{\lambda} y_{\lambda}(s) d s\right) \\
& =\frac{1}{k}\left(\int_{t+h}^{t+h+k} f(s) d s-\int_{t}^{t+k} f(s) d s\right) \tag{35.2.8}
\end{align*}
$$

Now multiply both sides by $y_{\lambda}(t+h+k)-y_{\lambda}(t+k)$. Consider the first term. To simplify the ideas consider instead

$$
\begin{align*}
\left(D_{k} g(t), g(t+k)\right) & =\frac{1}{k}\left(|g(t+k)|^{2}-(g(t), g(t+k))\right) \\
& \geq \frac{1}{k}\left(|g(t+k)|^{2}-|g(t)||g(t+h)|\right) \\
& \geq \frac{1}{k}\left(\frac{1}{2}|g(t+k)|^{2}-\frac{1}{2}|g(t)|^{2}\right) \tag{35.2.9}
\end{align*}
$$

Then applying this simple observation to 35.2.8,

$$
\begin{gathered}
\frac{1}{2} \frac{1}{k}\left(\left|y_{\lambda}(t+h+k)-y_{\lambda}(t+k)\right|^{2}-\left|y_{\lambda}(t+h)-y_{\lambda}(t)\right|^{2}\right)+ \\
\left(\frac{1}{k}\left(\int_{t+h}^{t+h+k} A_{\lambda} y_{\lambda}(s) d s-\int_{t}^{t+k} A_{\lambda} y_{\lambda}(s) d s\right), y_{\lambda}(t+h+k)-y_{\lambda}(t+k)\right)+ \\
\leq\left(\frac{1}{k}\left(\int_{t+h}^{t+h+k} f(s) d s-\int_{t}^{t+k} f(s) d s\right), y_{\lambda}(t+h+k)-y_{\lambda}(t+k)\right)
\end{gathered}
$$

Taking limsup ${ }_{k \rightarrow 0}$ of both sides yields

$$
\begin{gathered}
\frac{1}{2} D^{+}\left(\left|y_{\lambda}(t+h)-y_{\lambda}(t)\right|^{2}\right)+\left(A_{\lambda} y_{\lambda}(t+h)-A_{\lambda} y_{\lambda}(t), y_{\lambda}(t+h)-y_{\lambda}(t)\right) \\
\leq\left(f(t+h)-f(t), y_{\lambda}(t+h)-y_{\lambda}(t)\right)
\end{gathered}
$$

Now recall that $A_{\lambda}$ is monotone. Therefore,

$$
D^{+}\left(\left|y_{\lambda}(t+h)-y_{\lambda}(t)\right|^{2}\right) \leq|f(t+h)-f(t)|^{2}+\left|y_{\lambda}(t+h)-y_{\lambda}(t)\right|^{2}
$$

From Lemma 35.2.1 it follows that for all $\varepsilon>0$,

$$
\begin{aligned}
& \left|y_{\lambda}(t+h)-y_{\lambda}(t)\right|^{2}-\left|y_{\lambda}(h)-y_{0}\right|^{2} \\
\leq & \int_{0}^{t}|f(s+h)-f(s)|^{2} d s+\int_{0}^{t}\left|y_{\lambda}(s+h)-y_{\lambda}(s)\right|^{2} d s+\varepsilon t
\end{aligned}
$$

and so since $\varepsilon$ is arbitrary, the term $\varepsilon t$ can be eliminated. By Gronwall's inequality,

$$
\begin{equation*}
\left|y_{\lambda}(t+h)-y_{\lambda}(t)\right|^{2} \leq e^{t}\left(\left|y_{\lambda}(h)-y_{0}\right|^{2}+\int_{0}^{t}|f(s+h)-f(s)|^{2} d s\right) \tag{35.2.10}
\end{equation*}
$$

The last integral equals

$$
\begin{gathered}
\int_{0}^{t}\left|\int_{s}^{s+h} f^{\prime}(r) d r\right|^{2} d s \leq \int_{0}^{t} h \int_{s}^{s+h}\left|f^{\prime}(r)\right|^{2} d r d s \\
=h\left[\int_{0}^{h} \int_{0}^{r}\left|f^{\prime}(r)\right|^{2} d s d r+\int_{h}^{t} \int_{r-h}^{r}\left|f^{\prime}(r)\right|^{2} d s d r+\int_{t}^{t+h} \int_{r-h}^{t}\left|f^{\prime}(r)\right|\right]^{2} d s d r \\
\leq h^{2} \int_{0}^{t+h}\left|f^{\prime}(r)\right|^{2} d r
\end{gathered}
$$

and now it follows that for all $t+h<T$,

$$
\begin{equation*}
\left|\frac{y_{\lambda}(t+h)-y_{\lambda}(t)}{h}\right|^{2} \leq e^{T}\left(\left|\frac{y_{\lambda}(h)-y_{0}}{h}\right|^{2}+\left\|f^{\prime}\right\|_{L^{2}(0, T ; H)}^{2}\right) \tag{35.2.11}
\end{equation*}
$$

Now return to 35.2.7.

$$
\left|\frac{y_{\lambda}(h)-y_{0}}{h}\right| \leq\left|\frac{1}{h} \int_{0}^{h} A_{\lambda} y_{\lambda}(s) d s\right|+\left|\frac{1}{h} \int_{0}^{h} f(s) d s\right|
$$

Then taking limsup ${ }_{h \rightarrow 0}$ of both sides

$$
\limsup \sup _{h \rightarrow 0}\left|\frac{y_{\lambda}(h)-y_{0}}{h}\right| \leq\left|A_{\lambda} y_{0}\right|+|f(0)|
$$

From Lemma 35.1.3, $\left|A_{\lambda} y_{0}\right| \leq|a|$ for all $a \in A y_{0}$. This is where $y_{0} \in D(A)$ is used. Thus from 35.2.11, there exists a constant $C$ independent of $t$ and $h$ and $\lambda$ such that

$$
\left|\frac{y_{\lambda}(t+h)-y_{\lambda}(t)}{h}\right|^{2} \leq C
$$

From the estimate just obtained and 35.2.7, this implies

$$
\begin{equation*}
\frac{y_{\lambda}(t+h)-y_{\lambda}(t)}{h}+\frac{1}{h} \int_{t}^{t+h} A_{\lambda} y_{\lambda}(s) d s=\frac{1}{h} \int_{t}^{t+h} f(s) d s \tag{35.2.12}
\end{equation*}
$$

Now letting $h \rightarrow 0$, it follows that for all $t \in[0, T)$, there exists a constant $C$ independent of $t, \lambda$ such that

$$
\begin{equation*}
\left|A_{\lambda} y_{\lambda}(t)\right| \leq C \tag{35.2.13}
\end{equation*}
$$

This is a very nice estimate. The next task is to show uniform convergence of the $y_{\lambda}$ as $\lambda \rightarrow 0$. From 35.2.7

$$
\begin{gathered}
\left(D_{h}\left(y_{\lambda}(t)-y_{\mu}(t)\right), y_{\lambda}(t+h)-y_{\mu}(t+h)\right)+ \\
\left(\frac{1}{h} \int_{t}^{t+h}\left(A_{\lambda} y_{\lambda}(s)-A_{\mu} y_{\mu}(s)\right) d s, y_{\lambda}(t+h)-y_{\mu}(t+h)\right)=0
\end{gathered}
$$

Then from the argument in 35.2.9,

$$
\begin{gathered}
\frac{1}{h} \frac{1}{2}\left(\left|y_{\lambda}(t+h)-y_{\mu}(t+h)\right|^{2}-\left|y_{\lambda}(t)-y_{\mu}(t)\right|^{2}\right) \\
+\left(\frac{1}{h} \int_{t}^{t+h}\left(A_{\lambda} y_{\lambda}(s)-A_{\mu} y_{\mu}(s)\right) d s, y_{\lambda}(t+h)-y_{\mu}(t+h)\right) \leq 0
\end{gathered}
$$

Now take limsup $\sin _{h \rightarrow 0}$ to obtain

$$
\frac{1}{2} D^{+}\left|y_{\lambda}(t)-y_{\mu}(t)\right|^{2}+\left(A_{\lambda} y_{\lambda}(t)-A_{\mu} y_{\mu}(t), y_{\lambda}(t)-y_{\mu}(t)\right) \leq 0
$$

Using the definition of $A_{\lambda}$ this equals

$$
\begin{gathered}
\frac{1}{2} D^{+}\left|y_{\lambda}(t)-y_{\mu}(t)\right|^{2}+ \\
\left(A_{\lambda} y_{\lambda}(t)-A_{\mu} y_{\mu}(t), \lambda A_{\lambda} y_{\lambda}(t)+J_{\lambda} y_{\lambda}(t)-\left(\mu A_{\mu} y_{\mu}(t)+J_{\mu} y_{\mu}(t)\right)\right) \leq 0
\end{gathered}
$$

Now this last term splits into the following sum

$$
\begin{aligned}
& \left(A_{\lambda} y_{\lambda}(t)-A_{\mu} y_{\mu}(t), \lambda A_{\lambda} y_{\lambda}(t)-\mu A_{\mu} y_{\mu}(t)\right) \\
& +\left(A_{\lambda} y_{\lambda}(t)-A_{\mu} y_{\mu}(t), J_{\lambda} y_{\lambda}(t)-J_{\mu} y_{\mu}(t)\right)
\end{aligned}
$$

By Lemma 35.1.3 the second of these terms is nonnegative. Also from the estimate 35.2.13, the first term converges to 0 uniformly in $t$ as $\lambda, \mu \rightarrow 0$. Then by Lemma 35.2.1 it follows that if $\lambda$ is any sequence converging to $0, y_{\lambda}(t)$ is uniformly Cauchy. Let

$$
y(t) \equiv \lim _{\lambda \rightarrow 0} y_{\lambda}(t) .
$$

Thus $y$ is continuous because it is the uniform limit of continuous functions. Since $A_{\lambda} y_{\lambda}(t)$ is uniformly bounded, it also follows

$$
\begin{equation*}
y(t)=\lim _{\lambda \rightarrow 0} J_{\lambda} y_{\lambda}(t) \text { uniformly in } t \tag{35.2.14}
\end{equation*}
$$

Taking a further subsequence, you can assume

$$
\begin{equation*}
A_{\lambda} y_{\lambda} \rightharpoonup z \text { weak } * \text { in } L^{\infty}(0, T ; H) \tag{35.2.15}
\end{equation*}
$$

Thus $z \in L^{\infty}(0, T ; H)$. Recall $A_{\lambda} y_{\lambda} \in A J_{\lambda} y_{\lambda}$.

Now $A$ can be considered a maximal monotone operator on $L^{2}(0, T ; H)$ according to the rule

$$
A y(t) \equiv A(y(t))
$$

where

$$
D(A) \equiv\left\{f \in L^{2}(0, T ; H): f(t) \in D(A) \text { a.e. } t\right\}
$$

By Lemma 35.1.5 applied to $A$ considered as a maximal monotone operator on $L^{2}(0, T ; H)$ and using 35.2.14 and 35.2.15, it follows $y(t) \in D(A)$ a.e. $t$ and $z(t) \in A y(t)$ a.e. $t$. Then passing to the limit in 35.2.6 yields

$$
\begin{equation*}
y(t)-y_{0}+\int_{0}^{t} z(s) d s=\int_{0}^{t} f(s) d s \tag{35.2.16}
\end{equation*}
$$

Then by fundamental theorem of calculus, $y^{\prime}(t)$ exists a.e. $t$ and

$$
y^{\prime}+z=f, y(0)=y_{0}
$$

where $z(t) \in A y(t)$ a.e.
It remains to verify uniqueness. Suppose $\left[y_{1}, z_{1}\right]$ is another pair which works. Then from 35.2.16,

$$
\begin{aligned}
& y(t)-y_{1}(t)+\int_{0}^{t}\left(z(r)-z_{1}(r)\right) d r=0 \\
& y(s)-y_{1}(s)+\int_{0}^{s}\left(z(r)-z_{1}(r)\right) d r=0
\end{aligned}
$$

Therefore for $s<t$,

$$
y(t)-y_{1}(t)-\left(y(s)-y_{1}(s)\right)=\int_{s}^{t}\left(z(r)-z_{1}(r)\right) d r
$$

and so

$$
\left|\left|y(t)-y_{1}(t)\right|-\left|y(s)-y_{1}(s) \| \leq K\right| s-t\right|
$$

for some $K$ depending on $\|z\|_{L^{\infty}},\left\|z_{1}\right\|_{L^{\infty}}$. Since $y, y_{1}$ are bounded, it follows that $t \rightarrow$ $\left|y(t)-y_{1}(t)\right|^{2}$ is also Lipschitz. Therefore by Corollary 26.4.3, it is the integral of its derivative which exists a.e. So what is this derivative? As before,

$$
\begin{gathered}
\left(D_{h}\left(y(t)-y_{1}(t)\right), y(t+h)-y_{1}(t+h)\right) \\
+\left(\frac{1}{h} \int_{t}^{t+h}\left(z(s)-z_{1}(s)\right) d s, y(t+h)-y_{1}(t+h)\right)=0
\end{gathered}
$$

and so

$$
\begin{aligned}
& \frac{1}{h}\left(\frac{\left|y(t+h)-y_{1}(t+h)\right|^{2}}{2}-\frac{\left|y(t)-y_{1}(t)\right|^{2}}{2}\right) \\
&+\left(\frac{1}{h} \int_{t}^{t+h}\left(z(s)-z_{1}(s)\right) d s, y(t+h)-y_{1}(t+h)\right) \leq 0
\end{aligned}
$$

Then taking $\lim _{h \rightarrow 0}$ it follows that for a.e. $t$ (Lebesgue points of $z-z_{1}$ intersected with the points where $\left|y-y_{1}\right|^{2}$ has a derivative)

$$
\frac{1}{2} \frac{d}{d t}\left|y(t)-y_{1}(t)\right|^{2}+\left(z(t)-z_{1}(t), y(t)-y_{1}(t)\right) \leq 0
$$

Thus for a.e. $t$,

$$
\frac{d}{d t}\left|y(t)-y_{1}(t)\right|^{2} \leq 0
$$

and so

$$
\left|y(t)-y_{1}(t)\right|^{2}-\left|y_{0}-y_{0}\right|^{2}=\int_{0}^{t} \frac{d}{d t}\left|y(s)-y_{1}(s)\right|^{2} d s \leq 0
$$

This proves the theorem.

### 35.3 Subgradients

### 35.3.1 General Results

Definition 35.3.1 Let $X$ be a real locally convex topological vector space. For $x \in X$, $\delta \phi(x) \subseteq X^{\prime}$, possibly $\emptyset$. This subset of $X^{\prime}$ is defined by $y^{*} \in \delta \phi(x)$ means for all $z \in X$,

$$
y^{*}(z-x) \leq \phi(z)-\phi(x)
$$

Also $x \in \delta \phi^{*}\left(y^{*}\right)$ means that for all $z^{*} \in X^{\prime}$,

$$
\left(z^{*}-y^{*}\right)(x) \leq \phi^{*}\left(z^{*}\right)-\phi^{*}\left(y^{*}\right)
$$

We define $\operatorname{dom}(\delta \phi) \equiv\{x: \delta \phi(x) \neq \emptyset\}$.
The subgradient is an attempt to generalize the derivative. For example, a function may have a subgradient but fail to be differentiable at some point. A good example is $f(x)=|x|$. At $x=0$, this function fails to have a derivative but it does have a subgradient. In fact, $\delta f(0)=[-1,1]$.

To begin with consider the question of existence of the subgradient of a convex function. There is a very simple criterion for existence. It is essentially that the subgradient is nonempty at every point of the interior of the domain of $\phi$. First recall Lemma 18.2.15 which says the interior of a convex set is convex and if nonempty, then every point of the convex set can be obtained as the limit of a sequence of points of the interior.

Theorem 35.3.2 Let $\phi: X \rightarrow(-\infty, \infty]$ be convex and suppose for some $u \in \operatorname{dom}(\phi), \phi$ is continuous. Then $\delta \phi(x) \neq \emptyset$ for all $x \in \operatorname{int}(\operatorname{dom}(\phi))$. Thus

$$
\operatorname{dom}(\delta \phi) \supseteq \operatorname{int}(\operatorname{dom}(\phi))
$$

Proof: Let $x_{0} \in \operatorname{int}(\operatorname{dom}(\phi))$ and let

$$
A \equiv\left\{\left(x_{0}, \phi\left(x_{0}\right)\right)\right\}, B \equiv \operatorname{epi}(\phi) \cap X \times \mathbb{R}
$$

Then $A$ and $B$ are both nonempty and convex. Recall epi $(\phi)$ can contain a point like $(x, \infty)$. Since $\phi$ is continuous at $u \in \operatorname{dom}(\phi)$,

$$
(u, \phi(u)+1) \in \operatorname{int}(\operatorname{epi} \phi \cap X \times \mathbb{R}) .
$$

Thus $\operatorname{int}(B) \neq \emptyset$ and also $\operatorname{int}(B) \cap A=\emptyset$. By Lemma $18.2 .15 \operatorname{int}(B)$ is convex and so by Theorem 18.2.14 there exists $x^{*} \in X^{\prime}$ and $\beta \in \mathbb{R}$ such that

$$
\begin{equation*}
\left(x^{*}, \beta\right) \neq(0,0) \tag{35.3.17}
\end{equation*}
$$

and for all $(x, a) \in \operatorname{int} B$,

$$
\begin{equation*}
x^{*}(x)+\beta a>x^{*}\left(x_{0}\right)+\beta \phi\left(x_{0}\right) . \tag{35.3.18}
\end{equation*}
$$

From Lemma 18.2.15, whenever $x \in \operatorname{dom}(\phi)$,

$$
x^{*}(x)+\beta \phi(x) \geq x^{*}\left(x_{0}\right)+\beta \phi\left(x_{0}\right) .
$$

If $\beta=0$, this would mean $x^{*}\left(x-x_{0}\right) \geq 0$ for all $x \in \operatorname{dom}(\phi)$. Since $x_{0} \in \operatorname{int}(\operatorname{dom}(\phi))$, this implies $x^{*}=0$, contradicting 35.3.17. If $\beta<0$, apply 35.3 .18 to the case when $a=$ $\phi\left(x_{0}\right)+1$ and $x=x_{0}$ to obtain a contradiction. It follows $\beta>0$ and so

$$
\phi(x)-\phi\left(x_{0}\right) \geq-\frac{x^{*}}{\beta}\left(x-x_{0}\right)
$$

which says $-x^{*} / \beta \in \delta \phi\left(x_{0}\right)$. This proves the theorem.
Definition 35.3.3 Let $\phi: X \rightarrow(-\infty, \infty]$ be some function, not necessarily convex but satisfying $\phi(y)<\infty$ for some $y \in X$. Define $\phi^{*}: X^{\prime} \rightarrow(-\infty, \infty]$ by

$$
\phi^{*}\left(x^{*}\right) \equiv \sup \left\{x^{*}(y)-\phi(y): y \in X\right\} .
$$

This function, $\phi^{*}$, defined above, is called the conjugate function of $\phi$ or the polar of $\phi$. Note $\phi^{*}\left(x^{*}\right) \neq-\infty$ because $\phi(y)<\infty$ for some $y$.

Theorem 35.3.4 Let $X$ be a real Banach space. Then $\phi^{*}$ is convex and l.s.c.
Proof: Let $\lambda \in[0,1]$. Then

$$
\begin{gathered}
\phi^{*}\left(\lambda x^{*}+(1-\lambda) y^{*}\right)=\sup \left\{\left(\lambda x^{*}+(1-\lambda) y^{*}\right)(y)-\phi(y): y \in X\right\} \\
\sup \left\{\lambda\left(x^{*}(y)-\phi(y)\right)+(1-\lambda)\left(y^{*}(y)-\phi(y)\right): y \in X\right\} \\
\leq \lambda \phi^{*}\left(x^{*}\right)+(1-\lambda) \phi^{*}\left(y^{*}\right)
\end{gathered}
$$

It remains to show the function is l.s.c. Consider $f_{y}\left(x^{*}\right) \equiv x^{*}(y)-\phi(y)$. Then $f_{y}$ is obviously convex. Also to say that $(x, \alpha) \in \operatorname{epi}\left(\phi^{*}\right)$ is to say that $\alpha \geq x^{*}(y)-\phi(y)$ for all $y$. Thus

$$
\operatorname{epi}\left(\phi^{*}\right)=\cap_{y \in X} \operatorname{epi}\left(f_{y}\right)
$$

Therefore, if epi $\left(f_{y}\right)$ is closed, this will prove the theorem. If $\left(x^{*}, a\right) \notin \mathrm{epi}\left(f_{y}\right)$, then $a<$ $x^{*}(y)-\phi(y)$ and, by continuity, for $b$ close enough to $a$ and $y^{*}$ close enough to $x^{*}$ then

$$
b<y^{*}(y)-\phi(y),\left(y^{*}, b\right) \notin \operatorname{epi}\left(f_{y}\right)
$$

Thus epi $\left(f_{y}\right)$ is closed.
Note this theorem holds with no change in the proof if $X$ is only a locally convex topological vector space and $X^{\prime}$ is given the weak $*$ topology.

Definition 35.3.5 We define $\phi^{* *}$ on $X$ by

$$
\phi^{* *}(x) \equiv \sup \left\{x^{*}(x)-\phi^{*}\left(x^{*}\right), x^{*} \in X^{\prime}\right\} .
$$

The following lemma comes from separation theorems. First is a simple observation.
Observation 35.3.6 $f \in(X \times \mathbb{R})^{\prime}$ if and only if there exists $x^{*} \in X^{\prime}$ and $\alpha \in \mathbb{R}$ such that $f(x, \lambda)=x^{*}(x)+\lambda \alpha$. To get $x^{*}$, you can simply define $x^{*}(x) \equiv f(x, 0)$ and to get $\alpha$ you just let $\alpha \lambda \equiv f(0, \lambda)$. Why does such an $\alpha$ exist? You know that $f(0, a \lambda+b \delta)=$ $a f(0, \alpha)+b f(0, \delta)$ and so in fact $\lambda \rightarrow f(0, \lambda)$ satisfies the Cauchy functional equation $g(x+y)=g(x)+g(y)$ and is continuous so there is only one thing it can be and that is $f(0, \lambda)=\alpha \lambda$ for some $\alpha$.

This picture illustrates the conclusion of the following lemma.


Lemma 35.3.7 Let $\phi: X \rightarrow(-\infty, \infty]$ be convex and lower semicontinuous and $\phi(x)<\infty$ for some $x$. (proper). Then if $\beta<\phi\left(x_{0}\right)$ so that $\left(x_{0}, \beta\right)$ is not in epi $(\phi)$, it follows that there exists $\delta>0$ and $z^{*} \in X^{\prime}$ such that for all $y$,

$$
z^{*}\left(y-x_{0}\right)+\beta+\delta<\phi(y), \text { all } y \in X
$$

Proof: Let $C=\operatorname{epi}(\phi) \cap(X \times \mathbb{R})$. Then $C$ is a closed convex nonempty set and it does not contain the point $\left(x_{0}, \beta\right)$. Let $\hat{\beta}>\beta$ be slightly larger so that also $\left(x_{0}, \hat{\beta}\right) \notin C$. Thus there exists $y^{*} \in X^{\prime}$ and $\alpha \in \mathbb{R}$ such that for some $\hat{c}$, and all $y \in X$,

$$
y^{*}\left(x_{0}\right)+\alpha \hat{\beta}>\hat{c}>y^{*}(y)+\alpha \phi(y)
$$

for all $y \in X$. Now you can't have $\alpha \geq 0$ because

$$
\alpha(\hat{\beta}-\phi(y))>y^{*}\left(y-x_{0}\right)
$$

and you can let $y=x_{0}$ to have

$$
\alpha(\overbrace{\hat{\beta}-\phi\left(x_{0}\right)}^{<0})>0
$$

Hence $\alpha<0$ and so, dividing by it yields that for all $y \in X$,

$$
x^{*}\left(x_{0}\right)+\hat{\beta}<c<x^{*}(y)+\phi(y)
$$

where $x^{*}=y^{*} / \alpha, \hat{c} / \alpha \equiv c$. Then

$$
\begin{aligned}
\left(-x^{*}\right)\left(y-x_{0}\right)+\beta+(\hat{\beta}-\beta) & <c-x^{*}(y)<\phi(y) \\
\left(-x^{*}\right)\left(y-x_{0}\right)+\beta+\delta & <\phi(y), \delta \equiv \hat{\beta}-\beta
\end{aligned}
$$

Let $z^{*}=-x^{*}$.
Theorem 35.3.8 $\phi^{* *}(x) \leq \phi(x)$ for all $x$ and if $\phi$ is convex and l.s.c., $\phi^{* *}(x)=\phi(x)$ for all $x \in X$.

Proof:

$$
\begin{aligned}
\phi^{* *}(x) \equiv & \sup \{x^{*}(x)-\overbrace{\sup \left\{x^{*}(y)-\phi(y): y \in X\right\}}^{\phi^{*}\left(x^{*}\right)}: x^{*} \in X^{\prime}\} \\
& \leq \sup \left\{x^{*}(x)-\left(x^{*}(x)-\phi(x)\right)\right\}=\phi(x) .
\end{aligned}
$$

Next suppose $\phi$ is convex and l.s.c. If $\phi^{* *}\left(x_{0}\right)<\phi\left(x_{0}\right)$, then using Lemma 35.3.7, there exists $x_{0}^{*}, \delta>0$ such that for all $y \in X$,

$$
\begin{gathered}
\left(x_{0}^{*}\right)\left(y-x_{0}\right)+\phi^{* *}\left(x_{0}\right)+\delta<\phi(y) \\
x_{0}^{*}(y)-\phi(y)+\delta<x_{0}^{*}\left(x_{0}\right)-\phi^{* *}\left(x_{0}\right)
\end{gathered}
$$

Thus, since this holds for all $y$,

$$
\begin{aligned}
\phi^{*}\left(x_{0}^{*}\right)+\delta & \leq x_{0}^{*}\left(x_{0}\right)-\phi^{* *}\left(x_{0}\right) \\
\phi^{* *}\left(x_{0}\right)+\delta & \leq x_{0}^{*}\left(x_{0}\right)-\phi^{*}\left(x_{0}^{*}\right)
\end{aligned}
$$

Then

$$
\begin{aligned}
\phi^{* *}\left(x_{0}\right) & \equiv \sup \left\{x^{*}\left(x_{0}\right)-\phi^{*}\left(x^{*}\right), x^{*} \in X^{\prime}\right\} \\
& \geq x_{0}^{*}\left(x_{0}\right)-\phi^{*}\left(x_{0}^{*}\right) \geq \phi^{* *}\left(x_{0}\right)+\delta
\end{aligned}
$$

a contradiction.
The following corollary is descriptive of the situation just discussed. It says that to find epi $\left(\phi^{* *}\right)$ it suffices to take the intersection of all closed convex sets which contain epi $(\phi)$.

Corollary 35.3.9 epi $\left(\phi^{* *}\right)$ is the smallest closed convex set containing epi $(\phi)$.
Proof: epi $\left(\phi^{* *}\right) \supseteq$ epi $(\phi)$ from Theorem 35.3.8. Also epi $\left(\phi^{* *}\right)$ is closed by the proof of Theorem 35.3.4. Suppose epi $(\phi) \subseteq K \subseteq \operatorname{epi}\left(\phi^{* *}\right)$ and $K$ is convex and closed. Let

$$
\psi(x) \equiv \min \{a:(x, a) \in K\}
$$

( $\{a:(x, a) \in K\}$ is a closed subset of $(-\infty, \infty]$ so the minimum exists.) $\psi$ is also a convex function with epi $(\psi)=K$. To see $\psi$ is convex, let $\lambda \in[0,1]$. Then, by the convexity of $K$,

$$
\begin{gathered}
\lambda(x, \psi(x))+(1-\lambda)(y, \psi(y)) \\
=(\lambda x+(1-\lambda) y, \lambda \psi(x)+(1-\lambda) \psi(y)) \in K
\end{gathered}
$$

It follows from the definition of $\psi$ that

$$
\psi(\lambda x+(1-\lambda) y) \leq \lambda \psi(x)+(1-\lambda) \psi(y)
$$

Then

$$
\phi^{* *} \leq \psi \leq \phi
$$

and so from the definitions,

$$
\phi^{* * *} \geq \psi^{*} \geq \phi^{*}
$$

which implies from the definitions and Theorem 35.3.8 that

$$
\phi^{* *}=\phi^{* * * *} \leq \psi^{* *}=\psi \leq \phi^{* *}
$$

Therefore, $\psi=\phi^{* *}$ and epi $\left(\phi^{* *}\right)$ is the smallest closed convex set containing epi $(\phi)$ as claimed.

There is an interesting symmetry which relates $\delta \phi, \delta \phi^{*}, \phi$, and $\phi^{*}$.
Theorem 35.3.10 Suppose $\phi$ is convex, l.s.c. (lower semicontinuous or in other words having a closed epigraph), and proper. Then

$$
y^{*} \in \delta \phi(x) \text { if and only if } x \in \delta \phi^{*}\left(y^{*}\right)
$$

where this last expression means

$$
\left(z^{*}-y^{*}\right)(x) \leq \phi^{*}\left(z^{*}\right)-\phi^{*}\left(y^{*}\right)
$$

for all $z^{*}$ and in this case,

$$
y^{*}(x)=\phi^{*}\left(y^{*}\right)+\phi(x)
$$

Proof: If $y^{*} \in \delta \phi(x)$ then $y^{*}(z-x) \leq \phi(z)-\phi(x)$ and so

$$
y^{*}(z)-\phi(z) \leq y^{*}(x)-\phi(x)
$$

for all $z \in X$. Therefore,

$$
\phi^{*}\left(y^{*}\right) \leq y^{*}(x)-\phi(x) \leq \phi^{*}\left(y^{*}\right)
$$

Hence

$$
\begin{equation*}
y^{*}(x)=\phi^{*}\left(y^{*}\right)+\phi(x) . \tag{35.3.19}
\end{equation*}
$$

Now if $z^{*} \in X^{\prime}$ is arbitrary, 35.3 .19 shows

$$
\left(z^{*}-y^{*}\right)(x)=z^{*}(x)-y^{*}(x)=z^{*}(x)-\phi(x)-\phi^{*}\left(y^{*}\right) \leq \phi^{*}\left(z^{*}\right)-\phi^{*}\left(y^{*}\right)
$$

and this shows $x \in \delta \phi^{*}\left(y^{*}\right)$.
Now suppose $x \in \delta \phi^{*}\left(y^{*}\right)$. Then for $z^{*} \in X^{\prime}$,

$$
\left(z^{*}-y^{*}\right)(x) \leq \phi^{*}\left(z^{*}\right)-\phi^{*}\left(y^{*}\right)
$$

so

$$
z^{*}(x)-\phi^{*}\left(z^{*}\right) \leq y^{*}(x)-\phi^{*}\left(y^{*}\right)
$$

and so, taking sup over all $z^{*}$, and using Theorem 35.3.8,

$$
\phi^{* *}(x)=\phi(x) \leq y^{*}(x)-\phi^{*}\left(y^{*}\right) \leq \phi^{* *}(x)
$$

Thus

$$
y^{*}(x)=\phi^{*}\left(y^{*}\right)+\phi^{* *}(x)=\phi^{*}\left(y^{*}\right)+\phi(x) \geq \overbrace{y^{*}(z)-\phi(z)}^{\leq \phi^{*}\left(y^{*}\right)}+\phi(x)
$$

for all $z \in X$ and this implies for all $z \in X$,

$$
\phi(z)-\phi(x) \geq y^{*}(z-x)
$$

so $y^{*} \in \delta \phi(x)$ and this proves the theorem.
Definition 35.3.11 If $X$ is a Banach space define $u \in W^{1, p}([0, T] ; X)$ if there exists $g \in$ $L^{p}([0, T] ; X)$ such that

$$
u(t)=u(0)+\int_{0}^{t} g(s) d s
$$

When this occurs define $u^{\prime}(\cdot) \equiv g(\cdot)$. As usual, $p>1$.
The next Lemma is quite interesting for its own sake but it is also used in the next theorem.

Lemma 35.3.12 Suppose $g \in L^{p}(0, T ; X)$. Then as $h \rightarrow 0$,

$$
\frac{1}{h} \int_{(\cdot)}^{(\cdot)+h} g(s) d s \mathscr{X}_{[0, T-h]}(\cdot) \rightarrow g
$$

in $L^{p}([0, T] ; X)$.
Proof: Let

$$
\widetilde{g}(u) \equiv\left\{\begin{array}{l}
g(u) \text { if } u \in[0, T] \\
0 \text { if } u \notin[0, T]
\end{array}, \phi_{h}(r) \equiv \frac{1}{h} \mathscr{X}_{[-h, 0]}(r)\right.
$$

Thus $\widetilde{g} \in L^{p}(\mathbb{R} ; X)$ and

$$
\widetilde{g} * \phi_{h}(t) \equiv \int_{\mathbb{R}} \widetilde{g}(t-s) \phi_{h}(s) d s
$$

Then

$$
\left\|\widetilde{g} * \phi_{h}-\widetilde{g}\right\|_{L^{p}(\mathbb{R} ; X)} \leq\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\|\widetilde{g}(t)-\widetilde{g}(t-s)\|_{X} \phi_{h}(s) d s\right)^{p} d t\right)^{1 / p}
$$

which by Minkowski's inequality for integrals is no larger than

$$
\begin{gathered}
\leq \int_{\mathbb{R}} \phi_{h}(s)\left(\int_{\mathbb{R}}\|\widetilde{g}(t)-\widetilde{g}(t-s)\|_{X}^{p} d t\right)^{1 / p} d s \\
=\frac{1}{h} \int_{-h}^{0}\left(\int_{\mathbb{R}}\|\widetilde{g}(t)-\widetilde{g}(t-s)\|_{X}^{p} d t\right)^{1 / p} d s<\frac{1}{h} \int_{-h}^{0} \varepsilon d s=\varepsilon
\end{gathered}
$$

whenever $h$ is small enough. This follows from continuity of translation in $L^{p}(\mathbb{R} ; X)$, a consequence of the regularity of the measure. Thus, $\tilde{g} * \phi_{h} \rightarrow \tilde{g}$ in $L^{p}(\mathbb{R} ; X)$. Now

$$
\begin{aligned}
& \widetilde{g} * \phi_{h}(t)-\frac{1}{h} \int_{t}^{t+h} g(s) d s \mathscr{X}_{[0, T-h]}(t) \\
& =\left\{\begin{array}{l}
0 \text { if } t \in[0, T-h] \\
\frac{1}{h} \int_{t}^{t+h} \widetilde{g}(u) d u \text { if } t \notin[0, T-h]
\end{array}\right.
\end{aligned}
$$

and therefore,

$$
\begin{gathered}
\left\|\widetilde{g} * \phi_{h}(\cdot)-\frac{1}{h} \int_{(\cdot)}^{(\cdot)+h} g(s) d s \mathscr{X}_{[0, T-h]}(\cdot)\right\|_{L^{p}(\mathbb{R} ; X)}= \\
\left(\int_{-h}^{0}\left\|\frac{1}{h} \int_{t}^{t+h} \widetilde{g}(u) d u\right\|_{X}^{p} d t\right)^{1 / p}+\left(\int_{T-h}^{T}\left\|\frac{1}{h} \int_{t}^{t+h} \widetilde{g}(u) d u\right\|_{X}^{p} d t\right)^{1 / p} \\
\leq \frac{1}{h}\left(\int_{-h}^{0}\left(\int_{-h}^{h}\|\widetilde{g}(u)\|_{X} d u\right)^{p} d t\right)^{1 / p} \\
+\frac{1}{h}\left(\int_{T-h}^{T}\left(\int_{T-h}^{T+h}\|\widetilde{g}(u)\|_{X} d u\right)^{p} d t\right)^{1 / p}
\end{gathered}
$$

which by Minkowski's inequality for integrals is no larger than

$$
\begin{gathered}
\leq \frac{1}{h} \int_{-h}^{h}\left(\int_{-h}^{0}\|\widetilde{g}(u)\|_{X}^{p} d t\right)^{1 / p} d u+\frac{1}{h} \int_{T-h}^{T+h}\left(\int_{T-h}^{T}\|\widetilde{g}(u)\|_{X}^{p} d t\right)^{1 / p} d u \\
\leq \frac{1}{h} \int_{-h}^{h} \varepsilon d u+\frac{1}{h} \int_{T-h}^{T+h} \varepsilon d u=4 \varepsilon
\end{gathered}
$$

whenever $h$ is small enough because of the fact that $\|\widetilde{g}\|_{X}^{p} \in L^{1}(\mathbb{R} ; X)$. Since $\varepsilon$ is arbitrary, this shows

$$
\left\|\widetilde{g} * \phi_{h}(\cdot)-\frac{1}{h} \int_{(\cdot)}^{(\cdot)+h} g(s) d s \mathscr{X}_{[0, T-h]}(\cdot)\right\|_{L^{p}(\mathbb{R} ; X)} \rightarrow 0
$$

and also, it was shown above that

$$
\left\|\widetilde{g} * \phi_{h}(\cdot)-\tilde{g}\right\|_{L^{p}(\mathbb{R} ; X)} \rightarrow 0
$$

It follows that

$$
\frac{1}{h} \int_{(\cdot)}^{(\cdot)+h} g(s) d s \mathscr{X}_{[0, T-h]}(\cdot) \rightarrow \widetilde{g}
$$

in $L^{p}(\mathbb{R} ; X)$ and consequently in $L^{p}([0, T] ; X)$ as well. But $\widetilde{g}=g$ on $[0, T]$.
The following theorem is a form of the chain rule in which the derivative is replaced by the subgradient.

Theorem 35.3.13 Suppose $u \in W^{1, p}([0, T] ; X), z \in L^{p^{\prime}}\left([0, T] ; X^{\prime}\right)$, and $z(t) \in \delta \phi(u(t))$ a.e $t \in[0, T]$. Then the function, $t \rightarrow \phi(u(t))$ is in $L^{1}(0, T)$ and its weak derivative equals $\left\langle z, u^{\prime}\right\rangle$. In particular,

$$
\phi(u(t))-\phi(u(0))=\int_{0}^{t}\left\langle z(s), u^{\prime}(s)\right\rangle d s
$$

Proof: Modify $u$ on a set of measure zero such that $\delta \phi(u(t)) \neq \emptyset$ for all $t$. Next modify $z$ on a set of measure zero such that for $\widetilde{u}$ and $\widetilde{z}$ the modified functions, $\widetilde{z}(t) \in \delta \phi(\widetilde{u}(t))$ for all $t$. First I claim $t \rightarrow \phi(\widetilde{u}(t))$ is in $L^{1}(0, T)$. Pick $t_{0} \in[0, T]$ and let

$$
\widetilde{z}\left(t_{0}\right) \in \delta \phi\left(\widetilde{u}\left(t_{0}\right)\right)
$$

Then for $t \in[0, T]$,

$$
\begin{gather*}
\left\langle\widetilde{z}\left(t_{0}\right), \widetilde{u}(t)-\widetilde{u}\left(t_{0}\right)\right\rangle+\phi\left(\widetilde{u}\left(t_{0}\right)\right) \leq \\
\phi(\widetilde{u}(t)) \leq\left\langle\widetilde{z}(t), \widetilde{u}(t)-\widetilde{u}\left(t_{0}\right)\right\rangle+\phi\left(\widetilde{u}\left(t_{0}\right)\right) \tag{35.3.20}
\end{gather*}
$$

Then 35.3 .20 shows $t \rightarrow \phi(\widetilde{u}(t))$ is in $L^{1}(0, T)$ since $\widetilde{z} \in L^{p^{\prime}}\left([0, T] ; X^{\prime}\right), \widetilde{u} \in L^{p}([0, T] ; X)$. Also, for $t \in[0, T-h]$,

$$
\begin{gathered}
\left\langle\mathscr{X}_{[0, T-h]}(t) \widetilde{z}(t), \frac{\widetilde{u}(t+h)-\widetilde{u}(t)}{h}\right\rangle \leq \mathscr{X}_{[0, T-h]}(t) \frac{\phi(\widetilde{u}(t+h))-\phi(\widetilde{u}(t))}{h} \\
\leq\left\langle\mathscr{X}_{[0, T-h]}(t) \widetilde{z}(t+h), \frac{\widetilde{u}(t+h)-\widetilde{u}(t)}{h}\right\rangle
\end{gathered}
$$

Now $\mathscr{X}_{[0, T-h]}(\cdot) \widetilde{z}(\cdot+h) \rightarrow z(\cdot)$ in $L^{p^{\prime}}\left(0, T ; X^{\prime}\right)$ by continuity of translation. Also,

$$
\begin{gathered}
\mathscr{X}_{[0, T-h]}(\cdot) \frac{\widetilde{u}(\cdot+h)-\widetilde{u}(\cdot)}{h}=\mathscr{X}_{[0, T-h]}(\cdot) \frac{u(\cdot+h)-u(\cdot)}{h} \\
=\mathscr{X}_{[0, T-h]}(\cdot) \frac{1}{h} \int_{(\cdot)}^{(\cdot)+h} u^{\prime}(s) d s
\end{gathered}
$$

in $L^{p}(0, T ; X)$ and so by Lemma 35.3.12,

$$
\mathscr{X}_{[0, T-h]}(\cdot) \frac{\phi(\widetilde{u}(\cdot+h))-\phi(\widetilde{u}(\cdot))}{h} \rightarrow\left\langle z, u^{\prime}\right\rangle
$$

in $L^{1}(0, T)$.
It follows from the definition of weak derivatives that in the sense of weak derivatives,

$$
\frac{d}{d t}(\phi(u(\cdot)))=\left\langle z, u^{\prime}\right\rangle \in L^{1}(0, T)
$$

Note that by Theorem 26.3.3 this implies that for a.e. $t \in[0, T], \phi(u(t))$ is equal to a continuous function, $\phi \circ u$, and that

$$
(\phi \circ u)(t)-(\phi \circ u)(0)=\int_{0}^{t}\left\langle z(s), u^{\prime}(s)\right\rangle d s
$$

There are other rules of calculus which have a generalization to subgradients. The following theorem is on such a generalization. It generalizes the theorem which states that the derivative of a sum equals the sum of the derivatives.

Theorem 35.3.14 Let $\phi_{1}$ and $\phi_{2}$ be convex, l.s.c. and proper having values in $(-\infty, \infty]$. Then

$$
\begin{equation*}
\delta\left(\lambda \phi_{i}\right)(x)=\lambda \delta \phi_{i}(x), \delta\left(\phi_{1}+\phi_{2}\right)(x) \supseteq \delta \phi_{1}(x)+\delta \phi_{2}(x) \tag{35.3.21}
\end{equation*}
$$

if $\lambda>0$. If there exists $\bar{x} \in \operatorname{dom}\left(\phi_{1}\right) \cap \operatorname{dom}\left(\phi_{2}\right)$ and $\phi_{1}$ is continuous at $\bar{x}$ then for all $x \in X$,

$$
\begin{equation*}
\delta\left(\phi_{1}+\phi_{2}\right)(x)=\delta \phi_{1}(x)+\delta \phi_{2}(x) . \tag{35.3.22}
\end{equation*}
$$

Proof: 35.3 .21 is obvious so we only need to show 35.3.22. Suppose $\bar{x}$ is as described. It is clear 35.3.22 holds whenever $x \notin \operatorname{dom}\left(\phi_{1}\right) \cap \operatorname{dom}\left(\phi_{2}\right)$ since then both sides equal $\emptyset$. Therefore, assume

$$
x \in \operatorname{dom}\left(\phi_{1}\right) \cap \operatorname{dom}\left(\phi_{2}\right)
$$

in what follows. Let $x^{*} \in \delta\left(\phi_{1}+\phi_{2}\right)(x)$. Is $x^{*}$ is the sum of an element of $\delta \phi_{1}(x)$ and $\delta \phi_{2}(x)$ ? Does there exist $x_{1}^{*}$ and $x_{2}^{*}$ such that for every $y$,

$$
\begin{aligned}
x^{*}(y-x) & =x_{1}^{*}(y-x)+x_{2}^{*}(y-x) \\
& \leq \phi_{1}(y)-\phi_{1}(x)+\phi_{2}(y)-\phi_{2}(x) ?
\end{aligned}
$$

If so, then

$$
\phi_{1}(y)-\phi_{1}(x)-x^{*}(y-x) \geq \phi_{2}(x)-\phi_{2}(y)
$$

Define

$$
\begin{aligned}
C_{1} \equiv & \left\{(y, a) \in X \times \mathbb{R}: \phi_{1}(y)-\phi_{1}(x)-x^{*}(y-x) \leq a\right\}, \\
& C_{2} \equiv\left\{(y, a) \in X \times \mathbb{R}: a \leq \phi_{2}(x)-\phi_{2}(y)\right\} .
\end{aligned}
$$

I will show $\operatorname{int}\left(C_{1}\right) \cap C_{2}=\emptyset$ and then by Theorem 18.2.14 there exists an element of $X^{\prime}$ which does something interesting.

Both $C_{1}$ and $C_{2}$ are convex and nonempty. $C_{1}$ is nonempty because it contains

$$
\left(\bar{x}, \phi_{1}(\bar{x})-\phi_{1}(x)-x^{*}(\bar{x}-x)\right)
$$

since

$$
\phi_{1}(\bar{x})-\phi_{1}(x)-x^{*}(\bar{x}-x) \leq \phi_{1}(\bar{x})-\phi_{1}(x)-x^{*}(\bar{x}-x)
$$

$C_{2}$ is also nonempty because it contains $\left(\bar{x}, \phi_{2}(x)-\phi_{2}(\bar{x})\right)$ since

$$
\phi_{2}(x)-\phi_{2}(\bar{x}) \leq \phi_{2}(x)-\phi_{2}(\bar{x})
$$

In addition to this,

$$
\left(\bar{x}, \phi_{1}(\bar{x})-x^{*}(\bar{x}-x)-\phi_{1}(x)+1\right) \in \operatorname{int}\left(C_{1}\right)
$$

due to the assumed continuity of $\phi_{1}$ at $\bar{x}$ and so $\operatorname{int}\left(C_{1}\right) \neq \emptyset$. If $(y, a) \in \operatorname{int}\left(C_{1}\right)$ then

$$
\phi_{1}(y)-x^{*}(y-x)-\phi_{1}(x) \leq a-\varepsilon
$$

whenever $\varepsilon$ is small enough. Therefore, if $(y, a)$ is also in $C_{2}$, the assumption that $x^{*} \in$ $\boldsymbol{\delta}\left(\phi_{1}+\phi_{2}\right)(x)$ implies

$$
a-\varepsilon \geq \phi_{1}(y)-x^{*}(y-x)-\phi_{1}(x) \geq \phi_{2}(x)-\phi_{2}(y) \geq a
$$

a contradiction. Therefore $\operatorname{int}\left(C_{1}\right) \cap C_{2}=\emptyset$ and so by Theorem 18.2.14, there exists $\left(w^{*}, \boldsymbol{\beta}\right) \in X^{\prime} \times \mathbb{R}$ with

$$
\begin{equation*}
\left(w^{*}, \beta\right) \neq(0,0) \tag{35.3.23}
\end{equation*}
$$

and

$$
\begin{equation*}
w^{*}(y)+\beta a \geq w^{*}\left(y_{1}\right)+\beta a_{1} \tag{35.3.24}
\end{equation*}
$$

whenever $(y, a) \in C_{1}$ and $\left(y_{1}, a_{1}\right) \in C_{2}$.
Claim: $\beta>0$.
Proof of claim: If $\beta<0$ let

$$
\begin{aligned}
a & =\phi_{1}(\bar{x})-x^{*}(\bar{x}-x)-\phi_{1}(x)+1 \\
a_{1} & =\phi_{2}(x)-\phi_{2}(\bar{x}), \text { and } y=y_{1}=\bar{x}
\end{aligned}
$$

Then from 35.3.24

$$
\beta\left(\phi_{1}(\bar{x})-x^{*}(\bar{x}-x)-\phi_{1}(x)+1\right) \geq \beta\left(\phi_{2}(x)-\phi_{2}(\bar{x})\right) .
$$

Dividing by $\beta$ yields

$$
\phi_{1}(\bar{x})-x^{*}(\bar{x}-x)-\phi_{1}(x)+1 \leq \phi_{2}(x)-\phi_{2}(\bar{x})
$$

and so

$$
\begin{aligned}
\phi_{1}(\bar{x}) & +\phi_{2}(\bar{x})-\left(\phi_{1}(x)+\phi_{2}(x)\right)+1 \leq x^{*}(\bar{x}-x) \\
& \leq \phi_{1}(\bar{x})+\phi_{2}(\bar{x})-\left(\phi_{1}(x)+\phi_{2}(x)\right),
\end{aligned}
$$

a contradiction. Therefore, $\beta \geq 0$.
Now suppose $\beta=0$. Letting

$$
\begin{gathered}
a=\phi_{1}(\bar{x})-x^{*}(\bar{x}-x)-\phi_{1}(x)+1, \\
(\bar{x}, a) \in \operatorname{int}\left(C_{1}\right)
\end{gathered}
$$

and so there exists an open set $U$ containing 0 and $\eta>0$ such that

$$
\bar{x}+U \times(a-\eta, a+\eta) \subseteq C_{1} .
$$

Therefore, 35.3.24 applied to $(\bar{x}+z, a) \in C_{1}$ and $\left(\bar{x}, \phi_{2}(x)-\phi_{2}(\bar{x})\right) \in C_{2}$ for $z \in U$ yields

$$
w^{*}(\bar{x}+z) \geq w^{*}(\bar{x})
$$

for all $z \in U$. Hence $w^{*}(z)=0$ on $U$ which implies $w^{*}=0$, contradicting 35.3.23. This proves the claim.

Now with the claim, it follows $\beta>0$ and so, letting $z^{*}=w^{*} / \beta, 35.3 .24$ and Lemma 18.2.15 implies

$$
\begin{equation*}
z^{*}(y)+a \geq z^{*}\left(y_{1}\right)+a_{1} \tag{35.3.25}
\end{equation*}
$$

whenever $(y, a) \in C_{1}$ and $\left(y_{1}, a_{1}\right) \in C_{2}$. In particular,

$$
\begin{equation*}
\left(y, \phi_{1}(y)-\phi_{1}(x)-x^{*}(y-x)\right) \in C_{1} \tag{35.3.26}
\end{equation*}
$$

because

$$
\phi_{1}(y)-\phi_{1}(x)-x^{*}(y-x) \leq \phi_{1}(y)-x^{*}(y-x)-\phi_{1}(x)
$$

and

$$
\begin{equation*}
\left(y_{1}, \phi_{2}(x)-\phi_{2}\left(y_{1}\right)\right) \in C_{2} \tag{35.3.27}
\end{equation*}
$$

by similar reasoning so letting $y=x$,

$$
z^{*}(x)+(\overbrace{\phi_{1}(x)-x^{*}(x-x)-\phi_{1}(x)}^{=0}) \geq z^{*}\left(y_{1}\right)+\phi_{2}(x)-\phi_{2}\left(y_{1}\right) .
$$

Therefore,

$$
z^{*}\left(y_{1}-x\right) \leq \phi_{2}\left(y_{1}\right)-\phi_{2}(x)
$$

for all $y_{1}$ and so $z^{*} \in \delta \phi_{2}(x)$. Now let $y_{1}=x$ in 35.3.27 and using 35.3.25 and 35.3.26, it follows

$$
\begin{gathered}
z^{*}(y)+\phi_{1}(y)-x^{*}(y-x)-\phi_{1}(x) \geq z^{*}(x) \\
\phi_{1}(y)-\phi_{1}(x) \geq x^{*}(y-x)-z^{*}(y-x)
\end{gathered}
$$

and so $x^{*}-z^{*} \in \delta \phi_{1}(x)$ so $x^{*}=z^{*}+\left(x^{*}-z^{*}\right) \in \delta \phi_{2}(x)+\delta \phi_{1}(x)$ and this proves the theorem.

Next is a very important example known as the duality map from a Banach space to its dual space. Before doing this, consider a Hilbert space $H$. Define a map $R$ from $H$ to $H^{\prime}$, called the Riesz map, by the rule

$$
R(x)(y) \equiv(y, x)
$$

By the Riesz representation theorem, this map is onto and one to one with the properties

$$
R(x)(x)=\|x\|^{2}, \text { and }\|R x\|^{2}=\|x\|^{2} .
$$

The duality map from a Banach space to its dual is an attempt to generalize this notion of Riesz map to an arbitrary Banach space.

Definition 35.3.15 For $X$ a Banach space define $F: X \rightarrow \mathscr{P}\left(X^{\prime}\right)$ by

$$
F(x) \equiv\left\{x^{*} \in X^{\prime}: x^{*}(x)=\|x\|^{2},\left\|x^{*}\right\| \leq\|x\|\right\}
$$

Lemma 35.3.16 With $F(x)$ defined as above, it follows that

$$
F(x)=\left\{x^{*} \in X^{\prime}: x^{*}(x)=\|x\|^{2},\left\|x^{*}\right\|=\|x\|\right\}
$$

and $F(x)$ is a closed, nonempty, convex subset of $X^{\prime}$.
Proof: If $x^{*}$ is in the set described in 35.3.28,

$$
x^{*}\left(\frac{x}{\|x\|}\right)=\|x\|
$$

and so $\left\|x^{*}\right\| \geq\|x\|$. Therefore

$$
x^{*} \in\left\{x^{*} \in X^{\prime}: x^{*}(x)=\|x\|^{2},\left\|x^{*}\right\|=\|x\|\right\} .
$$

This shows this set and the set of 35.3.28 are equal. It is also clear the set of 35.3.28 is closed and convex. It only remains to show this set is nonempty.

Define $f: \mathbb{R} x \rightarrow \mathbb{R}$ by $f(\alpha x)=\alpha\|x\|^{2}$. Then the norm of $f$ on $\mathbb{R} x$ is $\|x\|$ and $f(x)=$ $\|x\|^{2}$. By the Hahn Banach theorem, $f$ has an extension to all of $X x^{*}$, and this extension is in the set of 35.3 .28 , showing this set is nonempty as required.

The next theorem shows this duality map is the subgradient of $\frac{1}{2}\|x\|^{2}$.
Theorem 35.3.17 For $X$ a real Banach space, let $\phi(x) \equiv \frac{1}{2}\|x\|^{2}$. Then $F(x)=\delta \phi(x)$.
Proof: Let $x^{*} \in F(x)$. Then

$$
\begin{aligned}
\left\langle x^{*}, y-x\right\rangle & =\left\langle x^{*}, y\right\rangle-\left\langle x^{*}, x\right\rangle \\
& \leq\|x\|\|y\|-\|x\|^{2} \leq \frac{1}{2}\|y\|^{2}-\frac{1}{2}\|x\|^{2} .
\end{aligned}
$$

This shows $F(x) \subseteq \delta \phi(x)$.
Now let $x^{*} \in \delta \phi(x)$. Then for all $t \in \mathbb{R}$,

$$
\begin{equation*}
\left\langle x^{*}, t y\right\rangle=\left\langle x^{*},(t y+x)-x\right\rangle \leq \frac{1}{2}\left(\|x+t y\|^{2}-\|x\|^{2}\right) . \tag{35.3.29}
\end{equation*}
$$

Now if $t>0$, divide both sides by $t$. This yields

$$
\begin{aligned}
\left\langle x^{*}, y\right\rangle & \leq \frac{1}{2 t}\left((\|x\|+t\|y\|)^{2}-\|x\|^{2}\right) \\
& =\frac{1}{2 t}\left(2 t\|x\|\|y\|+t^{2}\|y\|^{2}\right)
\end{aligned}
$$

Letting $t \rightarrow 0$,

$$
\begin{equation*}
\left\langle x^{*}, y\right\rangle \leq\|x\|\|y\| . \tag{35.3.30}
\end{equation*}
$$

Next suppose $t=-s$, where $s>0$ in 25.7.66. Then, since when you divide by a negative, you reverse the inequality, for $s>0$

$$
\begin{gather*}
\left\langle x^{*}, y\right\rangle \geq \frac{1}{2 s}\left[\|x\|^{2}-\|x-s y\|^{2}\right] \geq \\
\frac{1}{2 s}\left[\|x-s y\|^{2}-2\|x-s y\|\|s y\|+\|s y\|^{2}-\|x-s y\|\right]^{2}  \tag{35.3.31}\\
=\frac{1}{2 s}\left[-2\|x-s y\|\|s y\|+\|s y\|^{2}\right] \tag{35.3.32}
\end{gather*}
$$

Taking a limit as $s \rightarrow 0$ yields

$$
\begin{equation*}
\left\langle x^{*}, y\right\rangle \geq-\|x\|\|y\| . \tag{35.3.33}
\end{equation*}
$$

It follows from 35.3.33 and 35.3.30 that

$$
\left|\left\langle x^{*}, y\right\rangle\right| \leq\|x\|\|y\|
$$

and that, therefore, $\left\|x^{*}\right\| \leq\|x\|$ and $\left|\left\langle x^{*}, x\right\rangle\right| \leq\|x\|^{2}$. Now return to 35.3.32 and let $y=x$. Then

$$
\begin{aligned}
\left\langle x^{*}, x\right\rangle & \geq \frac{1}{2 s}\left[-2\|x-s x\|\|s x\|+\|s x\|^{2}\right] \\
& =-\|x\|^{2}(1-s)+s\|x\|^{2}
\end{aligned}
$$

Letting $s \rightarrow 1$,

$$
\left\langle x^{*}, x\right\rangle \geq\|x\|^{2}
$$

Since it was already shown that $\left|\left\langle x^{*}, x\right\rangle\right| \leq\|x\|^{2}$, this shows $\left\langle x^{*}, x\right\rangle=\|x\|^{2}$ and also $\left\|x^{*}\right\| \leq$ $\|x\|$. Thus

$$
\left\|x^{*}\right\| \geq\left\langle x^{*} \frac{x}{\|x\|}\right\rangle=\|x\|
$$

so in fact $x^{*} \in F(x)$.
The next result gives conditions under which the subgradient is onto. This means that if $y^{*} \in X^{\prime}$, then there exists $x \in X$ such that $y^{*} \in \delta \phi(x)$.

Theorem 35.3.18 Suppose $X$ is a reflexive Banach space and suppose $\phi: X \rightarrow(-\infty, \infty]$ is convex, proper, l.s.c., and for all $y^{*} \in X^{\prime}, x \rightarrow \phi(x)-y^{*}(x)$ is coercive. Then $\delta \phi$ is onto.

Proof: The function $x \rightarrow \phi(x)-y^{*}(x) \equiv \psi(x)$ is convex, proper, l.s.c., and coercive. Let

$$
\lambda \equiv \inf \left\{\phi(x)-y^{*}(x): x \in X\right\}
$$

and let $\left\{x_{n}\right\}$ be a minimizing sequence satisfying

$$
\lambda=\lim _{n \rightarrow \infty} \phi\left(x_{n}\right)-y^{*}\left(x_{n}\right)
$$

By coercivity,

$$
\lim _{\|x\| \rightarrow \infty} \phi(x)-y^{*}(x)=\infty
$$

and so this minimizing sequence is bounded. By the Eberlein Smulian theorem, Theorem 17.5.12, there is a weakly convergent subsequence $x_{n_{k}} \rightarrow x$. By Theorem $18.2 .11 \phi$ is also weakly lower semicontinuous. Therefore,

$$
\lambda=\phi(x)-y^{*}(x) \leq \lim \inf _{k \rightarrow \infty} \phi\left(x_{n_{k}}\right)-y^{*}\left(x_{n_{k}}\right)=\lambda
$$

so there exists $x$ which minimizes $x \rightarrow \phi(x)-y^{*}(x) \equiv \psi(x)$. Therefore, $0 \in \delta \psi(x)$ because

$$
\psi(y)-\psi(x) \geq 0=0(y-x)
$$

by Theorem 35.3.14, $0 \in \delta \psi(x)=\delta \phi(x)-y^{*}$ and this proves the theorem.
Corollary 35.3.19 Suppose $X$ is a reflexive Banach space and $\phi: X \rightarrow(-\infty, \infty]$ is convex, proper, and l.s.c. Then for each $y^{*} \in X^{\prime}$ there exist $x \in X, x_{1}^{*} \in F(x)$, and $x_{2}^{*} \in \delta \phi(x)$ such that

$$
y^{*}=x_{1}^{*}+x_{2}^{*} .
$$

Proof: Apply Theorem 35.3 .18 to the convex function $\frac{1}{2}\|x\|^{2}+\phi(x)$ and use Theorems 35.3.14 and 35.3.17.

### 35.3.2 Hilbert Space

In this section the subgradients are of a slightly different form and defined on a subset of $H$, a real Hilbert space. In Hilbert space the duality map is just the Riesz map defined earlier by

$$
R x(y) \equiv(y, x)
$$

Definition 35.3.20 $\operatorname{dom}(\partial \phi) \equiv \operatorname{dom}(\delta \phi)$ and for $x \in \operatorname{dom}(\partial \phi)$,

$$
\partial \phi(x) \equiv R^{-1} \delta \phi(x)
$$

Thus $y \in \partial \phi(x)$ if and only if for all $z \in H$,

$$
R y(z-x)=(y, z-x) \leq \phi(z)-\phi(x)
$$

Recall the definition of a maximal monotone operator.
Definition 35.3.21 A mapping $A: D(A) \subseteq H \rightarrow \mathscr{P}(H)$ is called monotone if whenever $y_{i} \in A x_{i}$,

$$
\left(y_{1}-y_{2}, x_{1}-x_{2}\right) \geq 0 .
$$

A monotone map is called maximal monotone if whenever $z \in H$, there exists $x \in D(A)$ and $y \in A(x)$ such that $z=y+x$. Put more simply, $I+A$ maps $D(A)$ onto $H$.

The following lemma states, among other things, that when $\phi$ is a convex, proper, l.s.c. function defined on a Hilbert space, $\partial \phi$ is maximal monotone.

Lemma 35.3.22 If $\phi$ is a convex, proper, l.s.c. function defined on a Hilbert space, then $\partial \phi$ is maximal monotone and $(I+\partial \phi)^{-1}$ is a Lipschitz continuous map from $H$ to dom $(\partial \phi)$ having Lipschitz constant 1.

Proof: Let $y \in H$. Then $R y \in H^{\prime}$ and by Corollary 35.3.19, there exists $x \in \operatorname{dom}(\delta \phi)$ such that $R x+\delta \phi(x) \ni R y$. Multiplying by $R^{-1}$ we see $y \in x+\partial \phi(x)$. This shows $I+\partial \phi$ is onto. If $y_{i} \in \partial \phi\left(x_{i}\right)$, then $R y_{i} \in \delta \phi\left(x_{i}\right)$ and so by the definition of subgradients,

$$
\begin{aligned}
\left(y_{1}-y_{2}, x_{1}-x_{2}\right) & =R\left(y_{1}-y_{2}\right)\left(x_{1}-x_{2}\right) \\
& =R y_{1}\left(x_{1}-x_{2}\right)-R y_{2}\left(x_{1}-x_{2}\right) \\
& \geq \phi\left(x_{1}\right)-\phi\left(x_{2}\right)-\left(\phi\left(x_{1}\right)-\phi\left(x_{2}\right)\right)=0
\end{aligned}
$$

showing $\partial \phi$ is monotone. Now suppose $x_{i} \in(I+\partial \phi)^{-1}(y)$. Then $y-x_{i} \in \partial \phi\left(x_{i}\right)$ and by monotonicity of $\partial \phi$,

$$
-\left|x_{1}-x_{2}\right|^{2}=\left(y-x_{1}-\left(y-x_{2}\right), x_{1}-x_{2}\right) \geq 0
$$

and so $x_{1}=x_{2}$. Thus $(I+\partial \phi)^{-1}$ is well defined. If $x_{i}=(I+\partial \phi)^{-1}\left(y_{i}\right)$, then by the monotonicity of $\partial \phi$,

$$
\left(y_{1}-x_{1}-\left(y_{1}-x_{2}\right), x_{1}-x_{2}\right) \geq 0
$$

and so

$$
\left|y_{1}-y_{2}\right|\left|x_{1}-x_{2}\right| \geq\left|x_{1}-x_{2}\right|^{2}
$$

which shows

$$
\left|(I+\partial \phi)^{-1}\left(y_{1}\right)-(I+\partial \phi)^{-1}\left(y_{2}\right)\right| \leq\left|y_{1}-y_{2}\right| .
$$

This proves the lemma.
Here is another proof.
Lemma 35.3.23 Let $\phi$ be convex, proper and lower semicontinuous on $X$ a reflexive Banach space having strictly convex norm, then for each $\alpha>0$,

$$
I+\alpha \partial \phi
$$

is onto.
Proof: By separation theorems applied to the eipgraph of $\phi$, and since $\phi$ is proper, there exists $w^{*}$ such that

$$
\left(w^{*}, x\right)+b \leq \alpha \phi(x)
$$

for all $x$. Pick $y \in H$. Then consider

$$
\frac{1}{2}|y-x|^{2}+\alpha \phi(x)
$$

This functional of $x$ is bounded below by

$$
\frac{1}{2}|y-x|^{2}+\left(w^{*}, x\right)+b
$$

Thus it is clearly coercive. Hence any minimizing sequence has a weakly convergent subsequence. It follows from lower semicontinuity that there exists $x_{0}$ which minimizes this functional. Hence, if $z \neq x_{0}$,

$$
0 \leq \frac{1}{2}|y-z|^{2}+\alpha \phi(z)-\left(\frac{1}{2}\left|y-x_{0}\right|^{2}+\alpha \phi\left(x_{0}\right)\right)
$$

Then writing $|y-z|^{2}=\left|y-x_{0}\right|^{2}+\left|z-x_{0}\right|^{2}-2\left(y-x_{0}, z-x_{0}\right)$,

$$
\begin{aligned}
=\frac{1}{2}\left|y-x_{0}\right|^{2} & +\frac{1}{2}\left|z-x_{0}\right|^{2}-\left(y-x_{0}, z-x_{0}\right)+\alpha \phi(z)-\frac{1}{2}\left|y-x_{0}\right|^{2}-\alpha \phi\left(x_{0}\right) \\
& =\frac{1}{2}\left|z-x_{0}\right|^{2}-\left(y-x_{0}, z-x_{0}\right)+\alpha \phi(z)-\alpha \phi\left(x_{0}\right)
\end{aligned}
$$

Thus, letting $z$ be replaced with $x_{0}+t\left(z-x_{0}\right)$ for small positive $t$,

$$
\begin{gathered}
t\left(y-x_{0}, z-x_{0}\right) \leq \frac{t^{2}}{2}\left|z-x_{0}\right|^{2}+\alpha \phi\left(x_{0}+t\left(z-x_{0}\right)\right)-\alpha \phi\left(x_{0}\right) \\
\leq \frac{t^{2}}{2}\left|z-x_{0}\right|^{2}+\alpha \phi\left(x_{0}+t\left(z-x_{0}\right)\right)-\alpha \phi\left(x_{0}\right)
\end{gathered}
$$

Using convexity of $\phi$,

$$
\leq \frac{t^{2}}{2}\left|z-x_{0}\right|^{2}+t \alpha \phi(z)-t \alpha \phi\left(x_{0}\right)
$$

Divide by $t$ and let $t \rightarrow 0$ to obtain that

$$
\left(y-x_{0}, z-x_{0}\right) \leq \alpha \phi(z)-\alpha \phi\left(x_{0}\right)
$$

and so

$$
y-x_{0} \in \partial\left(\alpha \phi\left(x_{0}\right)\right)
$$

Thus $y=x_{0}+\alpha \partial \phi\left(x_{0}\right)$ because $\partial(\alpha \phi)=\alpha \partial \phi$.
Thus $\partial \phi$ is maximal monotone.
There is a really amazing theorem, Moreau's theorem. It is in [24], [13] and [116]. It involves approximating a convex function with one which is differentiable.

Theorem 35.3.24 Let $\phi$ be a convex lower semicontinuous proper function defined on $H$. Define

$$
\phi_{\lambda}(x) \equiv \min _{y \in H}\left(\frac{1}{2 \lambda}|x-y|^{2}+\phi(y)\right)
$$

Then the function is well defined, convex, Frechet differentiable, and for all $x \in H$,

$$
\lim _{\lambda \rightarrow 0} \phi_{\lambda}(x)=\phi(x)
$$

$\phi_{\lambda}(x)$ increasing as $\lambda$ decreases. In addition,

$$
\phi_{\lambda}(x)=\frac{1}{2 \lambda}\left|x-J_{\lambda} x\right|^{2}+\phi\left(J_{\lambda}(x)\right)
$$

where $J_{\lambda} x \equiv(I+\lambda \partial \phi)^{-1}(x)$. The Frechet derivative at $x$ equals $A_{\lambda} x$ where

$$
A_{\lambda}=\frac{1}{\lambda}-\frac{1}{\lambda}(I+\lambda \partial \phi)^{-1}=\frac{1}{\lambda}-\frac{1}{\lambda} J_{\lambda}
$$

Also, there is an interesting relation between the domain of $\phi$ and the domain of $\partial \phi$

$$
D(\partial \phi) \subseteq D(\phi) \subseteq \overline{D(\partial \phi)}
$$

Proof: First of all, why does the minimum take place? By the convexity, closed epigraph, and assumption that $\phi$ is proper, separation theorems apply and one can say that there exists $z^{*}$ such that for all $y \in H$,

$$
\begin{equation*}
\frac{1}{2 \lambda}|x-y|^{2}+\phi(y) \geq \frac{1}{2 \lambda}|x-y|^{2}+\left(z^{*}, y\right)+c \tag{35.3.34}
\end{equation*}
$$

It follows easily that a minimizing sequence is bounded and so from lower semicontinuity which implies weak lower semicontinuity, there exists $y_{x}$ such that

$$
\min _{y \in H}\left(\frac{1}{2 \lambda}|x-y|^{2}+\phi(y)\right)=\left(\frac{1}{2 \lambda}\left|x-y_{x}\right|^{2}+\phi\left(y_{x}\right)\right)
$$

Why is $\phi_{\lambda}$ convex? For $\theta \in[0,1]$,

$$
\begin{gathered}
\phi_{\lambda}(\theta x+(1-\theta) z)=\frac{1}{2 \lambda}\left|\theta x+(1-\theta) z-y_{(\theta x+(1-\theta) z)}\right|^{2}+\phi\left(y_{\theta x+(1-\theta) z}\right) \\
\leq \frac{1}{2 \lambda}\left|\theta x+(1-\theta) z-\left(\theta y_{x}+(1-\theta) y_{z}\right)\right|^{2}+\phi\left(\theta y_{x}+(1-\theta) y_{z}\right) \\
\leq \frac{\theta}{2 \lambda}\left|x-y_{x}\right|^{2}+\frac{1-\theta}{2 \lambda}\left|z-y_{z}\right|^{2}+\theta \phi\left(y_{x}\right)+(1-\theta) \phi\left(y_{z}\right) \\
=\theta \phi_{\lambda}(x)+(1-\theta) \phi_{\lambda}(z)
\end{gathered}
$$

So is there a formula for $y_{x}$ ? Since it involves minimization of the functional, it follows as in Lemma 35.3.23 that

$$
\frac{1}{\lambda}\left(x-y_{x}\right) \in \partial \phi\left(y_{x}\right)
$$

Thus

$$
x \in y_{x}+\lambda \partial \phi\left(y_{x}\right)
$$

and so

$$
y_{x}=J_{\lambda} x
$$

Thus

$$
\phi_{\lambda}(x)=\frac{1}{2 \lambda}\left|x-J_{\lambda} x\right|^{2}+\phi\left(J_{\lambda}(x)\right)=\frac{\lambda}{2}\left|A_{\lambda} x\right|^{2}+\phi\left(J_{\lambda} x\right)
$$

Note that $J_{\lambda} x \in D(\partial \phi)$ and so it must also be in $D(\phi)$. Now also

$$
A_{\lambda} x \equiv \frac{x}{\lambda}-\frac{1}{\lambda} J_{\lambda} x \in \partial \phi\left(J_{\lambda} x\right)
$$

This is so if and only if

$$
x \in J_{\lambda} x+\lambda \partial \phi\left(J_{\lambda} x\right)=(I+\lambda \partial \phi)\left(J_{\lambda} x\right)=(I+\lambda \partial \phi)(I+\lambda \partial \phi)^{-1} x
$$

which is clearly true by definition.
Next consider the claim about differentiability.

$$
\phi_{\lambda}(y)-\phi_{\lambda}(x)=\frac{\lambda}{2}\left|A_{\lambda} y\right|^{2}+\phi\left(J_{\lambda} y\right)-\left(\frac{\lambda}{2}\left|A_{\lambda} x\right|^{2}+\phi\left(J_{\lambda} x\right)\right)
$$

$$
\begin{gather*}
=\frac{\lambda}{2}\left(\left|A_{\lambda} y\right|^{2}-\left|A_{\lambda} x\right|^{2}\right)+\phi\left(J_{\lambda} y\right)-\phi\left(J_{\lambda} x\right) \\
\geq \frac{\lambda}{2}\left(\left|A_{\lambda} y\right|^{2}-\left|A_{\lambda} x\right|^{2}\right)+\left(A_{\lambda} x, J_{\lambda} y-J_{\lambda} x\right) \\
=\frac{\lambda}{2}\left(\left|A_{\lambda} y\right|^{2}-\left|A_{\lambda} x\right|^{2}\right)+\left(A_{\lambda} x, y-\lambda A_{\lambda} y-\left(x-\lambda A_{\lambda} x\right)\right) \\
=\frac{\lambda}{2}\left(\left|A_{\lambda} y\right|^{2}-\left|A_{\lambda} x\right|^{2}\right)+\left(A_{\lambda} x, y-x\right)+\lambda\left(A_{\lambda} x, A_{\lambda} x-A_{\lambda} y\right) \\
\geq \frac{\lambda}{2}\left(\left|A_{\lambda} y\right|^{2}-\left|A_{\lambda} x\right|^{2}\right)+\lambda\left|A_{\lambda} x\right|^{2}-\frac{\lambda}{2}\left|A_{\lambda} x\right|^{2}-\frac{\lambda}{2}\left|A_{\lambda} y\right|^{2}+\left(A_{\lambda} x, y-x\right) \\
=\left(A_{\lambda} x, y-x\right)=\left(A_{\lambda} x-A_{\lambda} y, y-x\right)+\left(A_{\lambda} y, y-x\right) \tag{35.3.35}
\end{gather*}
$$

Then it follows that

$$
-\left(A_{\lambda} x-A_{\lambda} y, y-x\right) \geq \phi_{\lambda}(x)-\phi_{\lambda}(y)-\left(A_{\lambda} y, x-y\right)
$$

However, $A_{\lambda}$ is Lipschitz continuous with constant $1 / \lambda$ and so

$$
\begin{equation*}
\frac{1}{\lambda}|x-y|^{2} \geq \phi_{\lambda}(x)-\phi_{\lambda}(y)-\left(A_{\lambda} y, x-y\right) \tag{35.3.36}
\end{equation*}
$$

Then switching $x, y$ in the equation 35.3.36,

$$
\begin{equation*}
\frac{1}{\lambda}|x-y|^{2} \geq \phi_{\lambda}(y)-\phi_{\lambda}(x)-\left(A_{\lambda} x, y-x\right) \tag{35.3.37}
\end{equation*}
$$

But also that term on the end in 35.3.36 equals $\left(A_{\lambda} y, y-x\right) \geq\left(A_{\lambda} x, y-x\right)$ and so it is also the case that

$$
\begin{align*}
\frac{1}{\lambda}|x-y|^{2} & \geq \phi_{\lambda}(x)-\phi_{\lambda}(y)+\left(A_{\lambda} x, y-x\right) \\
& =-\left(\phi_{\lambda}(y)-\phi_{\lambda}(x)-\left(A_{\lambda} x, y-x\right)\right) \tag{35.3.38}
\end{align*}
$$

From 35.3.37 and 35.3.38 it follows that

$$
\frac{1}{\lambda}|x-y|^{2} \geq\left|\phi_{\lambda}(y)-\phi_{\lambda}(x)-\left(A_{\lambda} x, y-x\right)\right|
$$

which shows that $D \phi_{\lambda}(x)=A_{\lambda} x$. This proves the differentiability part.
Next recall that for any maximal monotone operatior $A$, if you have $x \in \overline{D(A)}$, then

$$
\lim _{\lambda \rightarrow 0} J_{\lambda} x=x
$$

Recall why this was so. If $x \in D(A)$, then

$$
x-J_{\lambda} x \in \lambda A x
$$

and so, $\left|x-J_{\lambda} x\right| \rightarrow 0$ as $\lambda \rightarrow 0$. If $x$ is only in $\overline{D(A)}$, it also works because for $y \in D(A)$

$$
\begin{aligned}
\left|x-J_{\lambda} x\right| & \leq|x-y|+\left|y-J_{\lambda} y\right|+\left|J_{\lambda} y-J_{\lambda} x\right| \\
& \leq 2|x-y|+\left|y-J_{\lambda} y\right|
\end{aligned}
$$

If $\varepsilon$ is given, simply pick $|y-x|<\varepsilon / 2$ and then

$$
\left|x-J_{\lambda} x\right| \leq \varepsilon+\left|y-J_{\lambda} y\right|
$$

and the last converges to 0 . Therefore, $J_{\lambda} x \rightarrow x$ on $\overline{D(A)}$.
Returning to the proof of the theorem, if $x \in \overline{D(\partial \phi)}$ then recall that

$$
\phi_{\lambda}(x)=\frac{1}{2 \lambda}\left|x-J_{\lambda} x\right|^{2}+\phi\left(J_{\lambda} x\right)
$$

and so,

$$
\lim \inf _{\lambda \rightarrow 0} \phi_{\lambda}(x) \geq \lim _{\lambda \rightarrow 0} \inf _{\lambda \rightarrow 0} \phi\left(J_{\lambda} x\right) \geq \phi(x) \geq \lim _{\lambda \rightarrow 0} \sup _{\lambda \rightarrow 0}(x)
$$

which shows the desired result in case $x \in \overline{D(\partial \phi)}$. Now consider the case where $x \notin$ $\overline{D(\partial \phi)}$. In this case, there is a positive lower bound $\delta$ to $\left|x-J_{\lambda} x\right|$ because each $J_{\lambda} x \in$ $D(\partial \phi)$. Then from the definition and what was shown above,

$$
\begin{aligned}
\phi_{\lambda}(x)= & \frac{\lambda}{2}\left|A_{\lambda} x\right|^{2}+\phi\left(J_{\lambda} x\right) \geq \frac{\lambda}{2}\left|A_{\lambda} x\right|^{2}+\left(z^{*}, J_{\lambda} x\right)+c \\
& \geq \frac{\lambda}{2}\left|A_{\lambda} x\right|^{2}+\left(z^{*}, J_{\lambda} x-x\right)+\left(z^{*}, x\right)+c \\
\geq & \frac{1}{2}\left|A_{\lambda} x\right|\left|x-J_{\lambda} x\right|-\left|z^{*}\right|\left|J_{\lambda} x-x\right|-\left|z^{*}\right||x|+c \\
\geq & \frac{1}{2}\left(\left|A_{\lambda} x\right|-\left|z^{*}\right|\right) \delta-\left|z^{*}\right||x|+c \\
\geq & \frac{1}{2}\left(\frac{\delta}{\lambda}-\left|z^{*}\right|\right) \delta-\left|z^{*}\right||x|+c
\end{aligned}
$$

Hence $\phi_{\lambda}(x) \rightarrow \infty$ and since $\phi(x) \geq \phi_{\lambda}(x)$ by construction, it follows that $\phi(x)=\infty$. The construction of $\phi_{\lambda}$ also shows that as $\lambda$ decreases, $\phi_{\lambda}(x)$ increases.

Note that the last part of the argument shows that if $x \notin \overline{D(\partial \phi)}$, then $x \notin D(\phi)$. Hence this shows that

$$
D(\partial \phi) \subseteq D(\phi) \subseteq \overline{D(\partial \phi)}
$$

### 35.4 A Perturbation Theorem

In this section is a simple perturbation theorem found in [24] and [116].
Recall that for $B$ a maximal monotone operator, $B_{\lambda}$, the Yosida approximation, is defined by

$$
B_{\lambda} x \equiv \frac{1}{\lambda}\left(x-J_{\lambda} x\right), J_{\lambda} x \equiv(I+\lambda B)^{-1} x .
$$

This follows from Theorem 35.1.7 on Page 1234

Theorem 35.4.1 Let $A$ and $B$ be maximal monotone operators and let $x_{\lambda}$ be the solution to

$$
y \in x_{\lambda}+B_{\lambda} x_{\lambda}+A x_{\lambda} .
$$

Then $y \in x+B x+A x$ for some $x \in D(A) \cap D(B)$ if $B_{\lambda} x_{\lambda}$ is bounded independent of $\lambda$.
The following is the perturbation theorem of this section. See [24] and [116].
Theorem 35.4.2 Let $H$ be a real Hilbert space and let $\Phi$ be non negative, convex, proper, and lower semicontinuous. Suppose also that A is a maximal monotone operator and there exists

$$
\begin{equation*}
\xi \in D(A) \cap D(\Phi) . \tag{35.4.39}
\end{equation*}
$$

Suppose also that for $J_{\lambda} x \equiv(I+\lambda A)^{-1} x$,

$$
\begin{equation*}
\Phi\left(J_{\lambda} x\right) \leq \Phi(x)+C \lambda \tag{35.4.40}
\end{equation*}
$$

Then $A+\partial \Phi$ is maximal monotone.
Proof: Letting $A_{\lambda}$ be the Yosida approximation of $A$,

$$
A_{\lambda} x=\frac{1}{\lambda}\left(x-J_{\lambda} x\right)
$$

and letting $y \in H$, it follows from the Hilbert space version of Proposition 35.1.6 there exists $x_{\lambda} \in H$ such that

$$
y \in x_{\lambda}+A_{\lambda} x_{\lambda}+\partial \Phi\left(x_{\lambda}\right) .
$$

Consequently,

$$
\begin{equation*}
y-x_{\lambda}-A_{\lambda} x_{\lambda} \in \partial \Phi\left(x_{\lambda}\right) \tag{35.4.41}
\end{equation*}
$$

and so

$$
\begin{equation*}
\left(y-x_{\lambda}-A_{\lambda} x_{\lambda}, J_{\lambda} x_{\lambda}-x_{\lambda}\right) \leq \Phi\left(J_{\lambda} x_{\lambda}\right)-\Phi\left(x_{\lambda}\right) \leq C \lambda \tag{35.4.42}
\end{equation*}
$$

which implies

$$
\begin{equation*}
-\left(y-x_{\lambda}-A_{\lambda} x_{\lambda}, A_{\lambda} x_{\lambda}\right)=\left|A_{\lambda} x_{\lambda}\right|^{2}-\left|y-x_{\lambda}\right|\left|A_{\lambda} x_{\lambda}\right| \leq C . \tag{35.4.43}
\end{equation*}
$$

By 35.4.41 and monotonicity of $A_{\lambda}$,

$$
\begin{gathered}
\Phi(\xi)-\Phi\left(x_{\lambda}\right) \geq(\overbrace{y-x_{\lambda}-A_{\lambda} x_{\lambda}}^{\in \partial \Phi\left(x_{\lambda}\right)}, \xi-x_{\lambda}) \\
=\left(y-x_{\lambda}, \xi-x_{\lambda}\right)-\left(A_{\lambda} x_{\lambda}, \xi-x_{\lambda}\right) \\
\geq\left(y-x_{\lambda}, \xi-x_{\lambda}\right)-\left(A_{\lambda} \xi, \xi-x_{\lambda}\right) \\
\geq\left(y-\xi, \xi-x_{\lambda}\right)+\left|\xi-x_{\lambda}\right|^{2}-\left(A_{\lambda} \xi, \xi-x_{\lambda}\right) \\
=\left|\xi-x_{\lambda}\right|^{2}+\left(y-\xi-A_{\lambda} \xi, \xi-x_{\lambda}\right)
\end{gathered}
$$

$$
\geq\left|\xi-x_{\lambda}\right|^{2}-C_{\xi y}\left|\xi-x_{\lambda}\right|
$$

where $C_{\xi y}$ depends on $\xi$ and $y$ but is independent of $\lambda$ because of the assumption that $\xi \in D(A) \cap D(\Phi)$ and Lemma 35.1.3 which gives a bound on $\left|A_{\lambda} \xi\right|$ in terms $|y|$ for $y \in A x$. Therefore, there exist constants, $C_{1}$ and $C_{2}$, depending on $\xi$ and $y$ but not on $\lambda$ such that

$$
\Phi(\xi) \geq \Phi\left(x_{\lambda}\right)+\left|x_{\lambda}\right|^{2}-C_{1}\left|x_{\lambda}\right|-C_{2}
$$

Since $\Phi \geq 0$, this shows that $\left|x_{\lambda}\right|$ is bounded independent of $\lambda$.

$$
2\left(\Phi(\xi)+C_{2}+\frac{C_{1}^{2}}{2}\right) \geq \Phi\left(x_{\lambda}\right)+\left|x_{\lambda}\right|^{2}
$$

This shows $\left|x_{\lambda}\right|$ is bounded independent of $\lambda$. Therefore, by 35.4.43, $\left|A_{\lambda} x_{\lambda}\right|$ is bounded independent of $\lambda$. By Theorem 35.4.1 this shows there exists $x \in D(\partial \Phi) \cap D(A)$ such that

$$
y \in A x+\partial \Phi(x)+x
$$

and so $A+\partial \Phi$ is maximal monotone since $y \in H$ was arbitrary.

### 35.5 An Evolution Inclusion

In this section is a theorem on existence and uniqueness for the initial value problem

$$
x^{\prime}+\partial \phi(x) \ni f, x(0)=x_{0}
$$

Suppose $\phi$ is a mapping from $H$ to $[0, \infty]$ which satisfy the following axioms.

$$
\begin{equation*}
\phi \text { is convex and lower semicontinuous, and proper, } \tag{35.5.44}
\end{equation*}
$$

Lemma 35.5.1 For $x \in L^{2}(0, T ; H), t \rightarrow \phi(x)$ is measurable.
Proof: This follows because $\phi$ is Borel measurable and so $\phi \circ x$ is also measurable. Now define the following function $\Phi$, on the Hilbert space, $L^{2}(0, T ; H)$.

$$
\Phi(x) \equiv\left\{\begin{array}{l}
\int_{0}^{T} \phi(x(t)) d t \text { if } x(t) \in D \text { for a.e. } t  \tag{35.5.45}\\
+\infty \text { otherwise }
\end{array}\right.
$$

Lemma 35.5.2 $\Phi$ is convex, nonnegative, and lower semicontinuous on $L^{2}(0, T ; H)$.
Proof: Since $\phi$ is nonnegative and convex, it follows that $\Phi$ is also nonnegative and convex. It remains to verify lower semicontinuity. Suppose, $x_{n} \rightarrow x$ in $L^{2}(0, T ; H)$ and let

$$
\lambda=\lim _{n \rightarrow \infty} \inf _{n} \Phi\left(x_{n}\right)
$$

Is $\lambda \geq \Phi(x)$ ? Then is suffices to assume $\lambda<\infty$. Suppose not. Then $\lambda<\Phi(x)$. Taking a subsequence, we can have $\lambda=\lim _{n \rightarrow \infty} \Phi\left(x_{n}\right)$ and we can take a further subsequence for which convergence of $x_{n}$ to $x$ is pointwise a.e. Then

$$
\begin{aligned}
\lambda & <\Phi(x) \equiv \int_{0}^{T} \phi(x(t)) d t \leq \int_{0}^{T} \lim _{n \rightarrow \infty} \inf _{n} \phi\left(x_{n}(t)\right) d t \\
& \leq \lim \inf _{n \rightarrow \infty} \int_{0}^{T} \phi\left(x_{n}(t)\right) d t=\lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty} \Phi\left(x_{n}\right)=\lambda
\end{aligned}
$$

which is a contradiction.
Define

$$
\begin{gather*}
D(L) \equiv\left\{x \in L^{2}(0, T ; H):\right. \text { such that } \\
\left.x(t)=x_{0}+\int_{0}^{t} x^{\prime}(s) d s \text { where } x^{\prime} \in L^{2}(0, T ; H)\right\} \tag{35.5.46}
\end{gather*}
$$

and for $x \in D(L)$,

$$
L x \equiv x^{\prime}
$$

Then $L$ is maximal monotone. To see this, consider the equation

$$
\lambda x^{\prime}+x=z, x(0)=x_{0}
$$

It clearly has a solution so $\lambda L+I$ is onto. In fact, the solution is

$$
x=e^{\frac{-t}{\lambda}} x_{0}+\frac{1}{\lambda} e^{\frac{-t}{\lambda}} \int_{0}^{t} e^{\frac{1}{\lambda} s} z(s) d s
$$

Also,

$$
\begin{aligned}
(L x & -L y, x-y)_{L^{2}(0, T ; H)}=\int_{0}^{T}\left(\left(x^{\prime}-y^{\prime}\right), x-y\right)_{H} d t \\
& =\int_{0}^{T}\left(x^{\prime}(t)-y^{\prime}(t), \int_{0}^{t} x^{\prime}(s)-y^{\prime}(s) d s\right) d t \\
& =\frac{1}{2} \int_{0}^{T} \frac{d}{d t}\left(\left|\int_{0}^{t} x^{\prime}(s)-y^{\prime}(s) d s\right|^{2}\right) d t \\
& =\left|\int_{0}^{T} x^{\prime}(s)-y^{\prime}(s) d s\right|_{H}^{2} \geq 0
\end{aligned}
$$

Thus we have the following lemma.
Lemma 35.5.3 L is maximal monotone and if $z \in L^{2}(0, T ; H)$, then $J_{\lambda} z$ is given by

$$
\begin{equation*}
J_{\lambda}[z](t) \equiv(I+\lambda L)^{-1}([z])(t)=e^{\frac{-t}{\lambda}} x_{0}+\frac{1}{\lambda} e^{\frac{-t}{\lambda}} \int_{0}^{t} e^{\frac{1}{\lambda} s} z(s) d s \tag{35.5.47}
\end{equation*}
$$

The main theorem is the following.
Theorem 35.5.4 Let $x_{0} \in D \equiv D(\phi)$. Then $L+\partial \Phi$ is maximal monotone so there exists $a$ unique solution to

$$
\begin{equation*}
L x+x+\partial \Phi(x) \ni f \tag{35.5.48}
\end{equation*}
$$

for every $f \in L^{2}(0, T ; H)$. Thus there exists $x \in L^{2}(0, T ; H)$ such that

$$
x^{\prime} \in L^{2}(0, T ; H), x(0)=x_{0} \in D(\phi)
$$

and

$$
x^{\prime}+x+\partial \Phi(x) \ni f, x(0)=x_{0}
$$

Proof: This is from Theorem 35.4.2. Since $x_{0} \in D$, it follows that $\phi\left(x_{0}\right)<\infty$.
Let $z \in D(\Phi)$, the effective domain of $\Phi$. Then $\int_{0}^{T} \phi(z(t)) d t<\infty$, so by convexity of $\phi$ and 35.5.47,

$$
\begin{equation*}
\phi\left(J_{\lambda} z(t)\right) \leq e^{\frac{-t}{\lambda}} \phi\left(x_{0}\right)+\frac{1}{\lambda} e^{\frac{-t}{\lambda}} \int_{0}^{t} e^{\frac{s}{\lambda}} \phi(z(s)) d s . \tag{35.5.49}
\end{equation*}
$$

Then

$$
\begin{gathered}
\Phi\left(J_{\lambda} z\right)= \\
\int_{0}^{T} \phi\left(J_{\lambda} z(t)\right) d t \leq \phi\left(x_{0}\right) \lambda+\int_{0}^{T} \frac{1}{\lambda} \int_{0}^{t} e^{-(t-s) / \lambda} \phi(z(s)) d s d t \\
\leq \lambda \phi\left(x_{0}\right)+\frac{1}{\lambda} \int_{0}^{T} \phi(z(s)) \int_{s}^{T} e^{-(t-s) / \lambda} d t d s \\
\leq \lambda \phi\left(x_{0}\right)+\left(\int_{0}^{T} \phi(z(s)) d s\right) \frac{1}{\lambda} \int_{0}^{\infty} e^{-t / \lambda} d t \\
=\phi\left(x_{0}\right) \lambda+\int_{0}^{T} \phi(z(s)) d s \\
=\phi\left(x_{0}\right) \lambda+\Phi(z)
\end{gathered}
$$

The conditions of Theorem 35.4.2 are satisfied. This proves $L+\partial \Phi$ is maximal monotone on $L^{2}(0, T ; H)$ and consequently there exists a unique solution to the differential inclusion of the theorem.

Then the main result is the following.
Theorem 35.5.5 Let $f \in L^{2}(0, T ; H)$ and $x_{0} \in D$. Let $\phi$ be as described above, a lower semicontinuous convex proper function defined on $H$. Then there exists a unique solution $x \in L^{2}(0, T ; H), x^{\prime} \in L^{2}(0, T ; H)$, to

$$
x^{\prime}+\partial \Phi(x) \ni f \text { in } L^{2}(0, T ; H), x(0)=x_{0}
$$

This satisfies the pointwise condition

$$
x^{\prime}(t)+\partial \phi(x(t)) \ni f(t) \text { for a.e. } t, x(0)=x_{0}
$$

Proof: From Theorem 35.5.4, there exists a unique solution to

$$
x_{v}^{\prime}+\partial \Phi\left(x_{v}\right)+x_{v} \ni f+v \text { in } L^{2}(0, T ; H), x_{v}(0)=x_{0}
$$

whenever $v \in L^{2}(0, T ; H)$. Then a simple argument based on fundamental theorem of calculus implies that for a.e. $t$,

$$
x_{v}^{\prime}(t)+\partial \phi\left(x_{v}(t)\right)+x_{v}(t) \ni f(t)+v(t)
$$

Then for given $v, u$ one can act on $x_{v}(t)-x_{u}(t)$ and integrate. This yields

$$
\frac{1}{2}\left|x_{v}(t)-x_{u}(t)\right|_{H}^{2}+\int_{0}^{t}\left|x_{v}-x_{u}\right|^{2} d s \leq \int_{0}^{t}|v(s)-u(s)|_{H}^{2} d s
$$

It follows that a sufficiently high power of the mapping $u \rightarrow x_{u}$ is a contraction map on $L^{2}(0, T ; H)$ and so there exists a unique fixed point $v$ in $L^{2}(0, T ; H)$. Thus $x_{v}=v$ and so

$$
v^{\prime}+\partial \Phi(v) \ni f \text { in } L^{2}(0, T ; H), v(0)=x_{0}
$$

### 35.6 A More Complicated Perturbation Theorem

In this section is a simple perturbation theorem which is a small generalization of one found in [24] and [116].

Recall that for $B$ a maximal monotone operator, $B_{\lambda}$, the Yosida approximation, is defined by

$$
B_{\lambda} x \equiv \frac{1}{\lambda}\left(x-J_{\lambda} x\right), J_{\lambda} x \equiv(I+\lambda B)^{-1} x .
$$

This follows from Theorem 35.1.7 on Page 1234
Theorem 35.6.1 Let $A$ and $B$ be maximal monotone operators and let $x_{\lambda}$ be the solution to

$$
y \in x_{\lambda}+B_{\lambda} x_{\lambda}+A x_{\lambda} .
$$

Then $y \in x+B x+A x$ for some $x \in D(A) \cap D(B)$ if $B_{\lambda} x_{\lambda}$ is bounded independent of $\lambda$.
The following is the perturbation theorem of this section. It generalizes a well known result in [24] and [116].

Theorem 35.6.2 Let $H$ be a real Hilbert space and let $\Phi$ be non negative, convex, proper, and lower semicontinuous. Suppose also that A is a maximal monotone operator and there exists

$$
\begin{equation*}
\xi \in D(A) \cap D(\Phi) \tag{35.6.50}
\end{equation*}
$$

Suppose also that for $J_{\lambda} x \equiv(I+\lambda A)^{-1} x$,

$$
\begin{equation*}
\Phi\left(J_{\lambda} x\right) \leq \Phi(x)+C(x) \lambda \tag{35.6.51}
\end{equation*}
$$

where for some constants, $K_{1}, K_{2}$,

$$
\begin{equation*}
K_{2}+K_{1}\left(\Phi(x)+|x|^{2}\right) \geq C(x) \tag{35.6.52}
\end{equation*}
$$

Then $A+\partial \Phi$ is maximal monotone.
Proof: Letting $A_{\lambda}$ be the Yosida approximation of $A$,

$$
A_{\lambda} x=\frac{1}{\lambda}\left(x-J_{\lambda} x\right)
$$

and letting $y \in H$, it follows from the Hilbert space version of Proposition 35.1.6 there exists $x_{\lambda} \in H$ such that

$$
y \in x_{\lambda}+A_{\lambda} x_{\lambda}+\partial \Phi\left(x_{\lambda}\right)
$$

Consequently,

$$
\begin{equation*}
y-x_{\lambda}-A_{\lambda} x_{\lambda} \in \partial \Phi\left(x_{\lambda}\right) \tag{35.6.53}
\end{equation*}
$$

and so

$$
\begin{equation*}
\left(y-x_{\lambda}-A_{\lambda} x_{\lambda}, J_{\lambda} x_{\lambda}-x_{\lambda}\right) \leq \Phi\left(J_{\lambda} x_{\lambda}\right)-\Phi\left(x_{\lambda}\right) \leq C\left(x_{\lambda}\right) \lambda \tag{35.6.54}
\end{equation*}
$$

which implies

$$
\begin{equation*}
-\left(y-x_{\lambda}-A_{\lambda} x_{\lambda}, A_{\lambda} x_{\lambda}\right) \leq C\left(x_{\lambda}\right) . \tag{35.6.55}
\end{equation*}
$$

I claim $\left\{C\left(x_{\lambda}\right)\right\}$ and $\left\{\left|x_{\lambda}\right|\right\}$ are bounded independent of $\lambda$.
By 35.6.53 and monotonicity of $A_{\lambda}$,

$$
\begin{gathered}
\Phi(\xi)-\Phi\left(x_{\lambda}\right) \geq\left(y-x_{\lambda}-A_{\lambda} x_{\lambda}, \xi-x_{\lambda}\right) \\
\geq\left(y-x_{\lambda}, \xi-x_{\lambda}\right)-\left(A_{\lambda} x_{\lambda}, \xi-x_{\lambda}\right) \\
\geq\left(y-x_{\lambda}, \xi-x_{\lambda}\right)-\left(A_{\lambda} \xi, \xi-x_{\lambda}\right) \\
\geq\left(y-\xi, \xi-x_{\lambda}\right)+\left|\xi-x_{\lambda}\right|^{2}-\left(A_{\lambda} \xi, \xi-x_{\lambda}\right) \\
\geq\left|\xi-x_{\lambda}\right|^{2}-C_{\xi y}\left|\xi-x_{\lambda}\right|
\end{gathered}
$$

where $C_{\xi y}$ depends on $\xi$ and $y$ but is independent of $\lambda$ because of the assumption that $\xi \in D(A) \cap D(\Phi)$ and Lemma 35.1.3 which gives a bound on $\left|A_{\lambda} \xi\right|$ in terms $|y|$ for $y \in A x$. Therefore, there exist constants, $C_{1}$ and $C_{2}$, depending on $\xi$ and $y$ but not on $\lambda$ such that

$$
\Phi(\xi) \geq \Phi\left(x_{\lambda}\right)+\left|x_{\lambda}\right|^{2}-C_{1}\left|x_{\lambda}\right|-C_{2}
$$

Since $\Phi \geq 0$,

$$
2\left(\Phi(\xi)+C_{2}+\frac{C_{1}^{2}}{2}\right) \geq \Phi\left(x_{\lambda}\right)+\left|x_{\lambda}\right|^{2}
$$

This shows $\left|x_{\lambda}\right|$ is bounded independent of $\lambda$. Therefore, by 35.6 .52

$$
K_{2}+2 K_{1}\left(\Phi(\xi)+C_{2}+\frac{C_{1}^{2}}{2}\right) \geq K_{2}+K_{1}\left(\Phi\left(x_{\lambda}\right)+\left|x_{\lambda}\right|^{2}\right) \geq C\left(x_{\lambda}\right)
$$

showing that both $\left|x_{\lambda}\right|$ and $C\left(x_{\lambda}\right)$ are bounded independent of $\lambda$. Therefore, from 35.6.55, it follows $A_{\lambda} x_{\lambda}$ is bounded independent of $\lambda$. By Theorem 35.6.1 this shows there exists $x \in D(\partial \Phi) \cap D(A)$ such that

$$
y \in A x+\partial \Phi(x)+x
$$

and so $A+\partial \Phi$ is maximal monotone since $y \in H$ was arbitrary. This proves the theorem.

### 35.7 An Evolution Inclusion

In this section is a theorem on existence and uniqueness for the initial value problem

$$
x^{\prime}+\partial_{2} \phi(t, x) \ni f, x(0)=x_{0}
$$

Suppose $\{\phi(t, \cdot)\}_{t \in[0, T]}$ is a family of functions mapping $H$ to $[0, \infty]$ which satisfy the following axioms.

$$
\begin{align*}
& \phi(t, \cdot) \text { is convex and lower semicontinuous, }  \tag{35.7.56}\\
& D(\phi(t, \cdot))=D, \text { independent of } t \in[0, T] \tag{35.7.57}
\end{align*}
$$

There exists a constant, $K$, such that for all $x \in D$,

$$
\begin{equation*}
|\phi(t, x)-\phi(s, x)| \leq K\left(\phi(r, x)+|x|^{2}+1\right)|t-s| \tag{35.7.58}
\end{equation*}
$$

for all $r \in[0, T]$.

Lemma 35.7.1 Under the conditions, 35.7.56-35.7.58, $\phi: H \times[0, T] \rightarrow[0, \infty]$ is lower semicontinuous.

Proof: Let $\left(x_{n}, t_{n}\right) \rightarrow(x, t)$ and let $\lambda \equiv \liminf _{n \rightarrow \infty} \phi\left(t_{n}, x_{n}\right)$. Is

$$
\phi(t, x) \leq \lambda ?
$$

It suffices to assume $\lambda<\infty$ and by taking a subsequence, $x_{n} \in D$ for all $n$ and

$$
\phi\left(t_{n}, x_{n}\right) \rightarrow \lambda .
$$

Then

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \inf _{n} \phi\left(t_{n}, x_{n}\right)=\lim \inf _{n \rightarrow \infty}\left[\phi\left(t_{n}, x_{n}\right)-\phi\left(t, x_{n}\right)+\phi\left(t, x_{n}\right)\right] . \tag{35.7.59}
\end{equation*}
$$

Now

$$
\begin{gathered}
\lim \sup _{n \rightarrow \infty}\left|\phi\left(t_{n}, x_{n}\right)-\phi\left(t, x_{n}\right)\right| \leq \\
\lim \sup _{n \rightarrow \infty} K\left(\phi\left(t_{n}, x_{n}\right)+\left|x_{n}\right|^{2}+1\right)\left|t_{n}-t\right|=0 .
\end{gathered}
$$

Therefore, from 35.7.59

$$
\lambda=\lim \inf _{n \rightarrow \infty} \phi\left(t_{n}, x_{n}\right)=\lim \inf _{n \rightarrow \infty} \phi\left(t, x_{n}\right) \geq \phi(t, x)
$$

because of the assumption that $\phi(t, \cdot)$ is lower semicontinuous. This proves the lemma.
In all that follows $[x]$ is an element of $L^{2}(0, T ; H)$. Thus $[x]$ is the equivalence class of measurable square integrable functions which equal $x$ a.e. This seems a little fussy but since the existence results are based on surjectivity theorems and the Hilbert space they apply to is $L^{2}(0, T ; H)$, it seems best to emphasize the equivalence classes of functions by using this notation, at least while proving theorems on existence and uniqueness.

Corollary 35.7.2 For $[x] \in L^{2}(0, T ; H), t \rightarrow \phi(t, x(t))$ is measurable.
Proof: This follows because, due to Lemma 35.7.1, $\phi$ is Borel measurable and so $\phi \circ x$ is also measurable.

Now define the following function, $\Phi$, on the Hilbert space, $L^{2}(0, T ; H)$.

$$
\Phi([x]) \equiv\left\{\begin{array}{l}
\int_{0}^{T} \phi(t, x(t)) d t \text { if } x(t) \in D \text { for all } t \text { for some } x(\cdot) \in[x]  \tag{35.7.60}\\
+\infty \text { otherwise }
\end{array}\right.
$$

Note that since the functions $\phi(t, \cdot)$ are proper, the top condition is equivalent to the condition

$$
\int_{0}^{T} \phi(t, x(t)) d t \text { if } x(t) \in D \text { a.e. for all } x(\cdot) \in[x] .
$$

Lemma 35.7.3 $\Phi$ is convex, nonnegative, and lower semicontinuous on $L^{2}(0, T ; H)$.

Proof: Since each $\phi(t, \cdot)$ is nonnegative and convex, it follows that $\Phi$ is also nonnegative and convex. It remains to verify lower semicontinuity. Suppose, $\left[x_{n}\right] \rightarrow[x]$ in $L^{2}(0, T ; H)$ and let

$$
\lambda=\lim \inf _{n \rightarrow \infty} \Phi\left(\left[x_{n}\right]\right) .
$$

Is $\lambda \geq \Phi([x])$ ? It suffices to assume $\lambda<\infty, x_{n}(t) \in D$ for all $t$, and $x_{n}(t) \rightarrow x(t)$ a.e. say for $t \notin N$ where $N$ has measure zero. Let

$$
\widetilde{x}(t)=\left\{\begin{array}{c}
x(t) \text { if } t \notin N \\
x_{1}(t) \text { if } t \in N
\end{array}\right.
$$

Then $[\widetilde{x}]=[x]$ and $\widetilde{x}(t) \in D$ for all $t$. Then by pointwise convergence and Fatou's lemma,

$$
\begin{aligned}
\Phi([x]) & =\Phi([\widetilde{x}])=\int_{0}^{T} \phi(t, \widetilde{x}(t)) d t \leq \int_{0}^{T} \lim _{n \rightarrow \infty} \inf _{n} \phi\left(t, x_{n}(t)\right) d t \\
\leq & \lim \inf _{n \rightarrow \infty} \int_{0}^{T} \phi\left(t, x_{n}(t)\right) d t=\lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty} \Phi\left(\left[x_{n}\right]\right) \equiv \lambda
\end{aligned}
$$

This proves the lemma.
Define

$$
\begin{align*}
D(L) & \equiv\left\{[x] \in L^{2}(0, T ; H): \text { for some } x \in[x]\right. \text { such that } \\
x(t) & \left.=x_{0}+\int_{0}^{t} x^{\prime}(s) d s \text { where }\left[x^{\prime}\right] \in L^{2}(0, T ; H)\right\} \tag{35.7.61}
\end{align*}
$$

and for $[x] \in D(L)$,

$$
L[x] \equiv\left[x^{\prime}\right] .
$$

The following lemma is easily obtained.
Lemma 35.7.4 $L$ is maximal monotone and if $[z] \in L^{2}(0, T ; H)$, then the equivalence class, $\left[J_{\lambda}[z]\right]$ is determined by the function,

$$
\begin{equation*}
J_{\lambda}[z](t) \equiv(I+\lambda L)^{-1}([z])(t)=e^{\frac{-t}{\lambda}} x_{0}+\frac{1}{\lambda} e^{\frac{-t}{\lambda}} \int_{0}^{t} e^{\frac{1}{\lambda} s} z(s) d s \tag{35.7.62}
\end{equation*}
$$

The main theorem is the following.
Theorem 35.7.5 Let $x_{0} \in D$. Then $L+\partial \Phi$ is maximal monotone so there exists a unique solution to

$$
\begin{equation*}
L[x]+[x]+\partial \Phi([x]) \ni[f] \tag{35.7.63}
\end{equation*}
$$

for every $[f] \in L^{2}(0, T ; H)$.
Proof: This is from Theorem 35.6.2. Since $x_{0} \in D$, it follows from 35.7.58 that $\phi\left(t, x_{0}\right)$ is bounded.

Let $[z] \in D(\Phi)$, the effective domain of $\Phi$. Then there exists $z \in[z]$ such that $z(t) \in D$ for all $t$, and $\int_{0}^{T} \phi(t, z(t)) d t<\infty$, so by convexity of $\phi(t, \cdot)$ and 35.7.62,

$$
\begin{equation*}
\phi\left(t, J_{\lambda}[z](t)\right) \leq e^{\frac{-t}{\lambda}} \phi\left(t, x_{0}\right)+\frac{1}{\lambda} e^{\frac{-t}{\lambda}} \int_{0}^{t} e^{\frac{s}{\lambda}} \phi(t, z(s)) d s . \tag{35.7.64}
\end{equation*}
$$

Now the first term in 35.7.64 is bounded so consider the second. The integral in this term is of the form

$$
\begin{equation*}
\int_{0}^{t} e^{\frac{s}{\lambda}} \phi(s, z(s)) d s+\int_{0}^{t} e^{\frac{s}{\lambda}}(\phi(t, z(s))-\phi(s, z(s))) d s \tag{35.7.65}
\end{equation*}
$$

Since $[z] \in D(\Phi), \phi(s, z(s))<\infty$ for all $s$ and also the first integral in 35.7.65 is finite. By 35.7.58, the second term in 35.7.65 is dominated by

$$
C_{\lambda} \int_{0}^{t} K\left(1+\phi(s, z(s))+|z(s)|^{2}\right)|t-s| d s<\infty
$$

This shows $\phi\left(t, J_{\lambda}[z](t)\right)<\infty$ for all $t$ and so $\Phi\left(\left[J_{\lambda}[z]\right]\right)$ is given by the top line of 35.7.60. Therefore, by convexity of $\phi(t, \cdot)$ and Jensen's inequality,

$$
\begin{align*}
& \Phi\left(\left[J_{\lambda}[z]\right]\right)=\int_{0}^{T} \phi\left(t, e^{\frac{-t}{\lambda}} x_{0}+\frac{1}{\lambda} e^{\frac{-t}{\lambda}} \int_{0}^{t} e^{\frac{s}{\lambda}} z(s) d s\right) d t \\
& \leq \int_{0}^{T}\left(e^{\frac{-t}{\lambda}} \phi\left(t, x_{0}\right)+\frac{1}{\lambda} e^{\frac{-t}{\lambda}} \int_{0}^{t} e^{\frac{s}{\lambda}} \phi(t, z(s)) d s\right) d t \\
& =\int_{0}^{T} e^{\frac{-t}{\lambda}} \phi\left(t, x_{0}\right) d t+\int_{0}^{T} \frac{1}{\lambda} e^{\frac{-t}{\lambda}} \int_{0}^{t} e^{\frac{s}{\lambda}} \phi(s, z(s)) d s d t \\
& \quad+\int_{0}^{T} \frac{1}{\lambda} e^{\frac{-t}{\lambda}} \int_{0}^{t} e^{\frac{s}{\lambda}}(\phi(t, z(s))-\phi(s, z(s))) d s d t \tag{35.7.66}
\end{align*}
$$

By 35.7.58, the last term is dominated by

$$
\begin{gather*}
\int_{0}^{T} \int_{0}^{t} \frac{1}{\lambda} e^{\frac{-(t-s)}{\lambda}} K\left(1+\phi(s, z(s))+|z(s)|^{2}\right)|t-s| d s d t= \\
\int_{0}^{T} \int_{s}^{T} e^{\frac{-(t-s)}{\lambda}} \frac{t-s}{\lambda} d t K\left(1+\phi(s, z(s))+|z(s)|^{2}\right) d s \\
\leq C \lambda+C \lambda\left(\Phi([z])+|[z]|^{2}\right) \tag{35.7.67}
\end{gather*}
$$

for some constant, $C$. From 35.7.58, $\phi\left(t, x_{0}\right)$ is bounded and so the first term in 35.7.66 is dominated by an expression of the form $C \lambda$. Now consider the middle term of 35.7.66. Since $\phi$ is nonnegative,

$$
\begin{gather*}
\int_{0}^{T} \frac{1}{\lambda} e^{\frac{-t}{\lambda}} \int_{0}^{t} e^{\frac{s}{\lambda}} \phi(s, z(s)) d s d t=\int_{0}^{T} \int_{s}^{T} \frac{1}{\lambda} e^{\frac{-(t-s)}{\lambda}} d t \phi(s, z(s)) d s \\
\leq \int_{0}^{T} \int_{0}^{\infty} e^{-u} d u \phi(s, z(s)) d s=\Phi([z]) \tag{35.7.68}
\end{gather*}
$$

It follows

$$
\Phi\left(\left[J_{\lambda}[z]\right]\right) \leq \Phi([z])+C \lambda+C \lambda\left(\Phi([z])+|[z]|^{2}\right)
$$

The conditions of Theorem 35.6 .2 are satisfied with $K_{1}=K_{2}=C$. This proves $L+\partial \Phi$ is maximal monotone on $L^{2}(0, T ; H)$ and consequently there exists a unique solution to the differential inclusion of the theorem.

Of course it is desirable to be able to say that $[y] \in \partial \Phi([x])$ if and only if $y(t) \in$ $\partial_{2} \phi(t, x(t))$ for some $x \in[x]$. To obtain this, here are two more assumptions. For all $x \in H$,

$$
\begin{equation*}
t \rightarrow J_{1}(t) x \text { is measurable } \tag{35.7.69}
\end{equation*}
$$

where $J_{1}(t) x$ is the solution, $y$, to $y(t)+\partial_{2} \phi(t, y(t)) \ni x$, and there exists $[\xi] \in L^{2}(0, T ; H)$ such that

$$
\begin{equation*}
\left[J_{1}(\cdot)[\xi]\right] \in L^{2}(0, T ; H) . \tag{35.7.70}
\end{equation*}
$$

Lemma 35.7.6 If 35.7.69 and 35.7.70 hold, and if $[y] \in L^{2}(0, T ; H)$, then $[y] \in \partial \Phi([x])$ if and only if there exists $x \in[x]$ such that $\partial_{2} \phi(t, x(t)) \neq \emptyset$ for all $t$ and $y(t) \in \partial_{2} \phi(t, x(t))$ a.e.

Proof: First suppose $y(t) \in \partial_{2} \phi(t, x(t))$ a.e. and $\partial_{2} \phi(t, x(t)) \neq \emptyset$ for all $t$ where $x \in[x]$. Then for all $[w] \in L^{2}(0, T ; H)$,

$$
\begin{gathered}
([y],[w])_{L^{2}(0, T ; H)} \equiv \int_{0}^{T}(y(t), w(t))_{H} d t \\
\leq \int_{0}^{T} \phi(t, x(t)+w(t)) d t-\int_{0}^{T} \phi(t, x(t)) d t \leq \Phi([x]+[w])-\Phi([x])
\end{gathered}
$$

To prove the converse, define $A: D(\partial \Phi) \rightarrow \mathscr{P}\left(L^{2}(0, T ; H)\right)$ as follows.

$$
\begin{gathered}
{[y] \in A[x] \text { if and only if for some } x \in[x]} \\
\partial_{2} \phi(t, x(t)) \neq \emptyset \text { for all } t \text { and } y(t) \in \partial_{2} \phi(t, x(t)) \text { a.e. } t .
\end{gathered}
$$

It follows $A$ is monotone. I will show $A$ is maximal monotone. From the first part of the proof, the graph of $A$ is contained in the graph of $\partial \Phi$. Since $A$ is maximal, this will imply $A=\partial \Phi$ and prove the lemma.

It remains to show $A$ is maximal monotone. By 35.7.69, for each $x \in H, J_{1}(t) x$ is measurable. Now from 35.7.70, and using the fact that $J_{1}(t)$ is a contraction,

$$
\left|J_{1}(t) x-J_{1}(t) \xi(t)\right| \leq|x-\xi(t)|
$$

and so $\left[J_{1}(\cdot) x\right]$ is in $L^{2}(0, T ; H)$. Now if

$$
s(t)=\sum_{i=1}^{n} \mathscr{X}_{E_{i}}(t) x_{i}
$$

is a simple function,

$$
J_{1}(t) s(t)=\sum_{i=1}^{n} \mathscr{X}_{E_{i}}(t) J_{1}(t) x
$$

and $\left[J_{1}(\cdot) s\right]$ is in $L^{2}(0, T ; H)$. If $[f] \in L^{2}(0, T ; H)$ is arbitrary, take a sequence of simple functions, $s_{n}$ converging to $f$ pointwise and $\left[s_{n}\right] \rightarrow[f]$ in $L^{2}(0, T ; H)$. Then

$$
\left|J_{1}(t) s_{n}(t)-J_{1}(t) f(t)\right| \leq\left|s_{n}(t)-f(t)\right|
$$

and it follows $J_{1}(t) s_{n}(t)$ converges pointwise to $J_{1}(t) f(t)$ showing that $t \rightarrow J_{1}(t) f(t)$ is measurable. Now the equivalence class of functions equal to this one a.e. is in $L^{2}(0, T ; H)$ by Fatou's lemma and the assumption that the simple functions, $s_{n}$ converge in $L^{2}(0, T ; H)$. This shows $A$ is maximal and proves the lemma.

Conditions 35.7.69 and 35.7.70 are just what is needed to obtain the conclusion of Lemma 35.7 .6 but it may not be clear how to verify these conditions easily. The following lemma gives sufficient conditions which are easy to verify which imply 35.7.69 and 35.7.70.

Lemma 35.7.7 Suppose there exists $[\xi] \in L^{2}(0, T ; H)$ such that

$$
J_{1}(t) \xi(t), \phi\left(t, J_{1}(t) \xi(t)\right)
$$

are bounded independent of $t \in[0, T]$ and $t \rightarrow J_{1}(t) \xi(t)$ is measurable. Then the conclusion of Lemma 35.7.6 holds.

Proof: Let $y(t)=J_{1}(t) \xi(t)$. Thus

$$
y(t)+\partial_{2} \phi(t, y(t)) \ni \xi(t) .
$$

Now suppose $x \in H$ and let

$$
\begin{equation*}
x(s)+z(s)=x \tag{35.7.71}
\end{equation*}
$$

where $z(s) \in \partial_{2} \phi(s, x(s))$, so $x(s)=J_{1}(s) x$. Take the inner product of both sides with $x(s)-y(s)$ to obtain

$$
(x(s), x(s)-y(s))_{H}+(z(s), x(s)-y(s))_{H}=(x, x(s)-y(s))_{H}
$$

and therefore,

$$
\begin{gathered}
\frac{1}{2}|x(s)|_{H}^{2}-\frac{1}{2}|y(s)|_{H}^{2} \leq \phi(s, y(s))-\phi(s, x(s)) \\
+|x|_{H}|x(s)|_{H}+|x|_{H}|y(s)|_{H} \leq \frac{1}{4}|x(s)|_{H}^{2}+c|x|_{H}^{2}+\frac{1}{2}|y(s)|_{H}^{2} \\
+\phi(s, y(s))-\phi(s, x(s))
\end{gathered}
$$

Consequently,

$$
\begin{equation*}
\phi(s, x(s))+\frac{1}{4}|x(s)|_{H}^{2} \leq|y(s)|_{H}^{2}+c|x|_{H}^{2}+\phi(s, y(s))<C \tag{35.7.72}
\end{equation*}
$$

a constant depending on $x$. Replacing $s$ with $t$ in 35.7.71 and subtracting yields

$$
x(t)-x(s)+z(t)-z(s)=0
$$

Now taking the inner product of this with $x(t)-x(s)$ it follows from 35.7.58,

$$
\begin{gathered}
|x(s)-x(t)|_{H}^{2}=(z(s)-z(t), x(t)-x(s))_{H} \\
\leq \phi(s, x(t))-\phi(s, x(s))+\phi(t, x(s))-\phi(t, x(t)) \\
\leq\left(K\left(\phi(t, x(t))+|x(t)|_{H}^{2}+1\right)+K\left(\phi(s, x(s))+|x(s)|_{H}^{2}+1\right)\right)|t-s|
\end{gathered}
$$

which shows by 35.7.72 that $x(\cdot)$ is Lipschitz continuous and is therefore measurable which verifies 35.7.69. The assumptions of the lemma include 35.7.70. It follows the conclusion of Lemma 35.7.6 holds.

Remark 35.7.8 Note that if $\phi(t, \cdot)$ has a minimum at $\xi(t)$ and if $t \rightarrow \xi(t)$ and $t \rightarrow$ $\phi(t, \xi(t))$ are bounded and measurable, then

$$
\xi(t)+0=\xi(t)
$$

and $0 \in \partial_{2} \phi(t, \xi(t))$. Therefore, in this case $J_{1}(t) \xi(t)=\xi(t)$ and so the hypotheses of Lemma 35.7.7 hold.

Corollary 35.7.9 Assume 35.7.56-35.7.58 and 35.7.69, 35.7.70. Let $x_{0} \in D$ and let $[f] \in$ $L^{2}(0, T ; H)$. Then there exists a unique function, $x$, satisfying

$$
[x] \text { and }\left[x^{\prime}\right] \text { are in } L^{2}(0, T ; H)
$$

which is a solution to

$$
\begin{equation*}
x^{\prime}+\partial_{2} \phi(t, x) \ni f \text { a.e., } x(0)=x_{0}, x(t)=x_{0}+\int_{0}^{t} x^{\prime}(s) d s \tag{35.7.73}
\end{equation*}
$$

Proof: Let $[v] \in L^{2}(0, T ; H)$ and let $[x]$ be the unique solution to

$$
\begin{equation*}
L[x]+[x]+\partial \Phi([x]) \ni[f]+[v] . \tag{35.7.74}
\end{equation*}
$$

Letting $\left[x_{i}\right]$ be the solution corresponding to 35.7 .74 in which $v$ is replaced with $v_{i}$, and $x_{i} \in\left[x_{i}\right]$ is such that

$$
x_{i}(t)=x_{0}+\int_{0}^{t} x_{i}^{\prime}(s) d s, i=1,2
$$

from Lemma 35.7.6 and 35.7.74 that for each $t \in[0, T]$,

$$
\frac{1}{2}\left|x_{1}(t)-x_{2}(t)\right|_{H}^{2}+\frac{1}{2} \int_{0}^{t}\left|x_{1}-x_{2}\right|_{H}^{2} d s \leq \frac{1}{2} \int_{0}^{t}\left|v_{1}(s)-v_{2}(s)\right|_{H}^{2} d s
$$

and so

$$
\left|x_{1}(t)-x_{2}(t)\right|_{H}^{2} \leq \int_{0}^{t}\left|v_{1}(s)-v_{2}(s)\right|_{H}^{2} d s
$$

Now define a mapping, $\Lambda: L^{2}(0, T ; H) \rightarrow L^{2}(0, T ; H)$ by $\Lambda[v]=[x]$ where $[x]$ is the solution to 35.7.74. Then, if $\left[v_{i}\right]$ is in $L^{2}(0, T ; H)$ and $\left[x_{i}\right]$ is the corresponding solution to 35.7.74,

$$
\left\|\Lambda\left[v_{1}\right]-\Lambda\left[v_{2}\right]\right\|_{L^{2}(0, t ; H)}^{2} \equiv
$$

$$
\int_{0}^{t}\left|x_{1}(s)-x_{2}(s)\right|_{H}^{2} d s \leq \int_{0}^{t} \int_{0}^{s}\left|v_{1}(r)-v_{2}(r)\right|_{H}^{2} d r d s
$$

Iterating this inequality, by replacing $\Lambda$ with $\Lambda^{k}$, it follows that for all $k$ large enough, $\Lambda^{k}$ is a contraction map on $L^{2}(0, T ; H)$. Thus there exists a unique fixed point for $\Lambda,[x]$. Thus

$$
L[x]+[x]+\partial \Phi([x]) \ni[f]+[x] .
$$

Let $x \in[x]$ be such that

$$
x(t)=x_{0}+\int_{0}^{t} x^{\prime}(s) d s
$$

By Lemma 35.7.6,

$$
x^{\prime}+x+\partial_{2} \phi(t, x) \ni f+x
$$

This function, $x(\cdot)$ is the unique solution to 35.7 .73 because if $x_{1}$ is another solution, then $\left[x_{1}\right]=[x]$ and since both functions are continuous, they must coincide. This proves the corollary.

## Part III

## Sobolev Spaces

## Chapter 36

## Weak Derivatives

### 36.1 Weak $*$ Convergence

A very important sort of convergence in applications of functional analysis is the concept of weak or weak $*$ convergence. It is important because it allows you to assert the existence of a convergent subsequence of a given bounded sequence. The only problem is the convergence is very weak so it does not tell you as much as you would like. Nevertheless, it is a very useful concept. The big theorems in the subject are the Eberlein Smulian theorem and the Banach Alaoglu theorem about the weak or weak $*$ compactness of the closed unit balls in either a Banach space or its dual space. These theorems are proved in Yosida [127]. Here I will present a special case which turns out to be by far the most important in applications and it is not hard to get from the Riesz representation theorem for $L^{p}$. First I define weak and weak $*$ convergence.

Definition 36.1.1 Let $X^{\prime}$ be the dual of a Banach space $X$ and let $\left\{x_{n}^{*}\right\}$ be a sequence of elements of $X^{\prime}$. Then $x_{n}^{*}$ converges weak $*$ to $x^{*}$ if and only if for all $x \in X$,

$$
\lim _{n \rightarrow \infty} x_{n}^{*}(x)=x^{*}(x)
$$

A sequence in $X,\left\{x_{n}\right\}$ converges weakly to $x \in X$ if and only if for all $x^{*} \in X^{\prime}$

$$
\lim _{n \rightarrow \infty} x^{*}\left(x_{n}\right)=x^{*}(x)
$$

The main result is contained in the following lemma.
Lemma 36.1.2 Let $X^{\prime}$ be the dual of a Banach space, $X$ and suppose $X$ is separable. Then if $\left\{x_{n}^{*}\right\}$ is a bounded sequence in $X^{\prime}$, there exists a weak $*$ convergent subsequence.

Proof: Let $D$ be a dense countable set in $X$. Then the sequence, $\left\{x_{n}^{*}(x)\right\}$ is bounded for all $x$ and in particular for all $x \in D$. Use the Cantor diagonal process to obtain a subsequence, still denoted by $n$ such that $x_{n}^{*}(d)$ converges for each $d \in D$. Now let $x \in X$ be completely arbitrary. In fact $\left\{x_{n}^{*}(x)\right\}$ is a Cauchy sequence. Let $\varepsilon>0$ be given and pick $d \in D$ such that for all $n$

$$
\left|x_{n}^{*}(x)-x_{n}^{*}(d)\right|<\frac{\varepsilon}{3}
$$

This is possible because $D$ is dense. By the first part of the proof, there exists $N_{\varepsilon}$ such that for all $m, n>N_{\varepsilon}$,

$$
\left|x_{n}^{*}(d)-x_{m}^{*}(d)\right|<\frac{\varepsilon}{3}
$$

Then for such $m, n$,

$$
\begin{aligned}
\left|x_{n}^{*}(x)-x_{m}^{*}(x)\right| & \leq\left|x_{n}^{*}(x)-x_{n}^{*}(d)\right|+\left|x_{n}^{*}(d)-x_{m}^{*}(d)\right| \\
+\left|x_{m}^{*}(d)-x_{m}^{*}(x)\right| & <\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon .
\end{aligned}
$$

Since $\mathcal{\varepsilon}$ is arbitrary, this shows $\left\{x_{n}^{*}(x)\right\}$ is a Cauchy sequence for all $x \in X$.

Now define $f(x) \equiv \lim _{n \rightarrow \infty} x_{n}^{*}(x)$. Since each $x_{n}^{*}$ is linear, it follows $f$ is also linear. In addition to this,

$$
|f(x)|=\lim _{n \rightarrow \infty}\left|x_{n}^{*}(x)\right| \leq K\|x\|
$$

where $K$ is some constant which is larger than all the norms of the $x_{n}^{*}$. Such a constant exists because the sequence, $\left\{x_{n}^{*}\right\}$ was bounded. This proves the lemma.

The lemma implies the following important theorem.
Theorem 36.1.3 Let $\Omega$ be a measurable subset of $\mathbb{R}^{n}$ and let $\left\{f_{k}\right\}$ be a bounded sequence in $L^{p}(\Omega)$ where $1<p \leq \infty$. Then there exists a weak $*$ convergent subsequence.

Proof: Since $L^{p^{\prime}}(\Omega)$ is separable, this follows from the Riesz representation theorem.
Note that from the Riesz representation theorem, it follows that if $p<\infty$, then the sequence converges weakly.

### 36.2 Test Functions And Weak Derivatives

In elementary courses in mathematics, functions are often thought of as things which have a formula associated with them and it is the formula which receives the most attention. For example, in beginning calculus courses the derivative of a function is defined as the limit of a difference quotient. You start with one function which tends to be identified with a formula and, by taking a limit, you get another formula for the derivative. A jump in abstraction occurs as soon as you encounter the derivative of a function of $n$ variables where the derivative is defined as a certain linear transformation which is determined not by a formula but by what it does to vectors. When this is understood, it reduces to the usual idea in one dimension. The idea of weak partial derivatives goes further in the direction of defining something in terms of what it does rather than by a formula, and extra generality is obtained when it is used. In particular, it is possible to differentiate almost anything if the notion of what is meant by the derivative is sufficiently weak. This has the advantage of allowing the consideration of the weak partial derivative of a function without having to agonize over the important question of existence but it has the disadvantage of not being able to say much about the derivative. Nevertheless, it is the idea of weak partial derivatives which makes it possible to use functional analytic techniques in the study of partial differential equations and it is shown in this chapter that the concept of weak derivative is useful for unifying the discussion of some very important theorems. Certain things which shold be true are.

Let $\Omega \subseteq \mathbb{R}^{n}$. A distribution on $\Omega$ is defined to be a linear functional on $C_{c}^{\infty}(\Omega)$, called the space of test functions. The space of all such linear functionals will be denoted by $\mathscr{D}^{*}(\Omega)$. Actually, more is sometimes done here. One imposes a topology on $C_{c}^{\infty}(\Omega)$ making it into a topological vector space, and when this has been done, $\mathscr{D}^{\prime}(\Omega)$ is defined as the dual space of this topological vector space. To see this, consult the book by Yosida [127] or the book by Rudin [114].

Example: The space $L_{l o c}^{1}(\Omega)$ may be considered as a subset of $\mathscr{D}^{*}(\Omega)$ as follows.

$$
f(\phi) \equiv \int_{\Omega} f(\mathbf{x}) \phi(\mathbf{x}) d x
$$

for all $\phi \in C_{c}^{\infty}(\Omega)$. Recall that $f \in L_{l o c}^{1}(\Omega)$ if $f \mathscr{X}_{K} \in L^{1}(\Omega)$ whenever $K$ is compact.
Example: $\delta_{x} \in \mathscr{D}^{*}(\Omega)$ where $\delta_{\mathbf{x}}(\phi) \equiv \phi(\mathbf{x})$.
It will be observed from the above two examples and a little thought that $\mathscr{D}^{*}(\Omega)$ is truly enormous. The derivative of a distribution will be defined in such a way that it agrees with the usual notion of a derivative on those distributions which are also continuously differentiable functions. With this in mind, let $f$ be the restriction to $\Omega$ of a smooth function defined on $\mathbb{R}^{n}$. Then $D_{x_{i}} f$ makes sense and for $\phi \in C_{c}^{\infty}(\Omega)$

$$
D_{x_{i}} f(\phi) \equiv \int_{\Omega} D_{x_{i}} f(\mathbf{x}) \phi(\mathbf{x}) d x=-\int_{\Omega} f D_{x_{i}} \phi d x=-f\left(D_{x_{i}} \phi\right)
$$

This motivates the following definition.
Definition 36.2.1 For $T \in \mathscr{D}^{*}(\Omega)$

$$
D_{x_{i}} T(\phi) \equiv-T\left(D_{x_{i}} \phi\right)
$$

Of course one can continue taking derivatives indefinitely. Thus,

$$
D_{x_{i} x_{j}} T \equiv D_{x_{i}}\left(D_{x_{j}} T\right)
$$

and it is clear that all mixed partial derivatives are equal because this holds for the functions in $C_{c}^{\infty}(\Omega)$. In this weak sense, the derivative of almost anything exists, even functions that may be discontinuous everywhere. However the notion of "derivative" is very weak, hence the name, "weak derivatives".

Example: Let $\Omega=\mathbb{R}$ and let

$$
H(x) \equiv\left\{\begin{array}{l}
1 \text { if } x \geq 0 \\
0 \text { if } x<0
\end{array}\right.
$$

Then

$$
D H(\phi)=-\int H(x) \phi^{\prime}(x) d x=\phi(0)=\delta_{0}(\phi)
$$

Note that in this example, $D H$ is not a function.
What happens when $D f$ is a function?
Theorem 36.2.2 Let $\Omega=(a, b)$ and suppose that $f$ and $D f$ are both in $L^{1}(a, b)$. Then $f$ is equal to a continuous function a.e., still denoted by $f$ and

$$
f(x)=f(a)+\int_{a}^{x} D f(t) d t
$$

In proving Theorem 36.2.2 the following lemma is useful.
Lemma 36.2.3 Let $T \in \mathscr{D}^{*}(a, b)$ and suppose $D T=0$. Then there exists a constant $C$ such that

$$
T(\phi)=\int_{a}^{b} C \phi d x
$$

Proof: $T(D \phi)=0$ for all $\phi \in C_{c}^{\infty}(a, b)$ from the definition of $D T=0$. Let

$$
\phi_{0} \in C_{c}^{\infty}(a, b), \int_{a}^{b} \phi_{0}(x) d x=1
$$

and let

$$
\psi_{\phi}(x)=\int_{a}^{x}\left[\phi(t)-\left(\int_{a}^{b} \phi(y) d y\right) \phi_{0}(t)\right] d t
$$

for $\phi \in C_{c}^{\infty}(a, b)$. Thus $\psi_{\phi} \in C_{c}^{\infty}(a, b)$ and

$$
D \psi_{\phi}=\phi-\left(\int_{a}^{b} \phi(y) d y\right) \phi_{0}
$$

Therefore,

$$
\phi=D \psi_{\phi}+\left(\int_{a}^{b} \phi(y) d y\right) \phi_{0}
$$

and so

$$
T(\phi)=T\left(D \psi_{\phi}\right)+\left(\int_{a}^{b} \phi(y) d y\right) T\left(\phi_{0}\right)=\int_{a}^{b} T\left(\phi_{0}\right) \phi(y) d y
$$

Let $C=T \phi_{0}$. This proves the lemma.
Proof of Theorem 36.2.2 Since $f$ and $D f$ are both in $L^{1}(a, b)$,

$$
D f(\phi)-\int_{a}^{b} D f(x) \phi(x) d x=0
$$

Consider

$$
f(\cdot)-\int_{a}^{(\cdot)} D f(t) d t
$$

and let $\phi \in C_{c}^{\infty}(a, b)$.

$$
\begin{gathered}
D\left(f(\cdot)-\int_{a}^{(\cdot)} D f(t) d t\right)(\phi) \\
\equiv-\int_{a}^{b} f(x) \phi^{\prime}(x) d x+\int_{a}^{b}\left(\int_{a}^{x} D f(t) d t\right) \phi^{\prime}(x) d x \\
=D f(\phi)+\int_{a}^{b} \int_{t}^{b} D f(t) \phi^{\prime}(x) d x d t \\
=D f(\phi)-\int_{a}^{b} D f(t) \phi(t) d t=0
\end{gathered}
$$

By Lemma 36.2.3, there exists a constant, $C$, such that

$$
\left(f(\cdot)-\int_{a}^{(\cdot)} D f(t) d t\right)(\phi)=\int_{a}^{b} C \phi(x) d x
$$

for all $\phi \in C_{c}^{\infty}(a, b)$. Thus

$$
\int_{a}^{b}\left\{\left(f(x)-\int_{a}^{x} D f(t) d t\right)-C\right\} \boldsymbol{\phi}(x) d x=0
$$

for all $\phi \in C_{c}^{\infty}(a, b)$. It follows from Lemma 36.3.3 in the next section that

$$
f(x)-\int_{a}^{x} D f(t) d t-C=0 \text { a.e. } x
$$

Thus let $f(a)=C$ and write

$$
f(x)=f(a)+\int_{a}^{x} D f(t) d t
$$

This proves Theorem 36.2.2.
Theorem 36.2.2 says that

$$
f(x)=f(a)+\int_{a}^{x} D f(t) d t
$$

whenever it makes sense to write $\int_{a}^{x} D f(t) d t$, if $D f$ is interpreted as a weak derivative. Somehow, this is the way it ought to be. It follows from the fundamental theorem of calculus that $f^{\prime}(x)$ exists for a.e. $x$ where the derivative is taken in the sense of a limit of difference quotients and $f^{\prime}(x)=D f(x)$. This raises an interesting question. Suppose $f$ is continuous on $[a, b]$ and $f^{\prime}(x)$ exists in the classical sense for a.e. $x$. Does it follow that

$$
f(x)=f(a)+\int_{a}^{x} f^{\prime}(t) d t ?
$$

The answer is no. To see an example, consider Problem 4 on Page 970 which gives an example of a function which is continuous on $[0,1]$, has a zero derivative for a.e. $x$ but climbs from 0 to 1 on $[0,1]$. Thus this function is not recovered from integrating its classical derivative.

In summary, if the notion of weak derivative is used, one can at least give meaning to the derivative of almost anything, the mixed partial derivatives are always equal, and, in one dimension, one can recover the function from integrating its derivative. None of these claims are true for the classical derivative. Thus weak derivatives are convenient and rule out pathologies.

### 36.3 Weak Derivatives In $L_{l o c}^{p}$

Definition 36.3.1 Let $U$ be an open set in $\mathbb{R}^{n} . f \in L_{l o c}^{p}(U)$ if $f \mathscr{X}_{K} \in L^{p}$ whenever $K$ is a compact subset of $U$.

Definition 36.3.2 For $\alpha=\left(k_{1}, \cdots, k_{n}\right)$ where the $k_{i}$ are nonnegative integers, define

$$
|\alpha| \equiv \sum_{i=1}^{n}\left|k_{x_{i}}\right|, D^{\alpha} f(\mathbf{x}) \equiv \frac{\partial^{|\alpha|} f(\mathbf{x})}{\partial x_{1}^{k_{1}} \partial x_{2}^{k_{2}} \cdots \partial x_{n}^{k_{n}}}
$$

Also define $\phi_{k}$ to be a mollifier if

$$
\operatorname{spt}\left(\phi_{k}\right) \subseteq B\left(\mathbf{0}, \frac{1}{k}\right), \phi_{k} \geq 0
$$

, $\int \phi_{k} d x=1$, and $\phi_{k} \in C_{c}^{\infty}\left(B\left(\mathbf{0}, \frac{1}{k}\right)\right)$. In the case a Greek letter like $\delta$ or $\varepsilon$ is used as a subscript, it will mean $\operatorname{spt}\left(\phi_{\delta}\right) \subseteq B(\mathbf{0}, \boldsymbol{\delta}), \phi_{\delta} \geq 0, \int \phi_{\delta} d x=1$, and $\phi_{\delta} \in C_{c}^{\infty}(B(\mathbf{0}, \boldsymbol{\delta}))$. You can always get a mollifier by letting $\phi \geq 0, \phi \in C_{c}^{\infty}(B(\mathbf{0}, 1)), \int \phi d x=1$, and then defining $\phi_{k}(\mathbf{x}) \equiv k^{n} \phi(k \mathbf{x})$ or in the case of a Greek subscript, $\phi_{\delta}(\mathbf{x})=\frac{1}{\delta^{n}} \phi\left(\frac{\mathbf{x}}{\delta}\right)$.

Consider the case where $u$ and $D^{\alpha} u$ for $|\alpha|=1$ are each in $L_{l o c}^{p}\left(\mathbb{R}^{n}\right)$. The next lemma is the one alluded to in the proof of Theorem 36.2.2.

Lemma 36.3.3 Suppose $f \in L_{l o c}^{1}(U)$ and suppose

$$
\int f \phi d x=0
$$

for all $\phi \in C_{c}^{\infty}(U)$. Then $f(\mathbf{x})=0$ a.e. $\mathbf{x}$.
Proof: Without loss of generality $f$ is real valued. Let

$$
E \equiv\{\mathbf{x}: f(\mathbf{x})>\varepsilon\}
$$

and let

$$
E_{m} \equiv E \cap B(0, m)
$$

Is $m\left(E_{m}\right)=0$ ? If not, there exists an open set, $V$, and a compact set $K$ satisfying

$$
\begin{gathered}
K \subseteq E_{m} \subseteq V \subseteq B(0, m), m(V \backslash K)<4^{-1} m\left(E_{m}\right), \\
\int_{V \backslash K}|f| d x<\varepsilon 4^{-1} m\left(E_{m}\right)
\end{gathered}
$$

Let $H$ and $W$ be open sets satisfying

$$
K \subseteq H \subseteq \bar{H} \subseteq W \subseteq \bar{W} \subseteq V
$$

and let

$$
\bar{H} \prec g \prec W
$$

where the symbol, $\prec$, in the above implies $\operatorname{spt}(g) \subseteq W, g$ has all values in $[0,1]$, and $g(\mathbf{x})=1$ on $\bar{H}$. Then let $\phi_{\delta}$ be a mollifier and let $h \equiv g * \phi_{\delta}$ for $\delta$ small enough that

$$
K \prec h \prec V .
$$

Thus

$$
\begin{aligned}
0 & =\int f h d x=\int_{K} f d x+\int_{V \backslash K} f h d x \\
& \geq \varepsilon m(K)-\varepsilon 4^{-1} m\left(E_{m}\right) \\
& \geq \varepsilon\left(m\left(E_{m}\right)-4^{-1} m\left(E_{m}\right)\right)-\varepsilon 4^{-1} m\left(E_{m}\right) \\
& \geq 2^{-1} \varepsilon m\left(E_{m}\right)
\end{aligned}
$$

Therefore, $m\left(E_{m}\right)=0$, a contradiction. Thus

$$
m(E) \leq \sum_{m=1}^{\infty} m\left(E_{m}\right)=0
$$

and so, since $\varepsilon>0$ is arbitrary,

$$
m(\{\mathbf{x}: f(\mathbf{x})>0\})=0
$$

Similarly $m(\{\mathbf{x}: f(\mathbf{x})<0\})=0$. This proves the lemma.
This lemma allows the following definition.
Definition 36.3.4 Let $U$ be an open subset of $\mathbb{R}^{n}$ and let $u \in L_{l o c}^{1}(U)$. Then $D^{\alpha} u \in L_{l o c}^{1}(U)$ if there exists a function $g \in L_{l o c}^{1}(U)$, necessarily unique by Lemma 36.3.3, such that for all $\phi \in C_{c}^{\infty}(U)$,

$$
\int_{U} g \phi d x=D^{\alpha} u(\phi) \equiv \int_{U}(-1)^{|\alpha|} u\left(D^{\alpha} \phi\right) d x
$$

Then $D^{\alpha} u$ is defined to equal $g$ when this occurs.
Lemma 36.3.5 Let $u \in L_{\text {loc }}^{1}\left(\mathbb{R}^{n}\right)$ and suppose $u_{, i} \in L_{\text {loc }}^{1}\left(\mathbb{R}^{n}\right)$, where the subscript on the $u$ following the comma denotes the $i^{\text {th }}$ weak partial derivative. Then if $\phi_{\varepsilon}$ is a mollifier and $u_{\varepsilon} \equiv u * \phi_{\varepsilon}$, it follows $u_{\varepsilon, i} \equiv u_{, i} * \phi_{\varepsilon}$.

Proof: If $\psi \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$, then

$$
\begin{aligned}
\int u(\mathbf{x}-\mathbf{y}) \psi_{, i}(\mathbf{x}) d x & =\int u(\mathbf{z}) \psi_{, i}(\mathbf{z}+\mathbf{y}) d z \\
& =-\int u_{, i}(\mathbf{z}) \psi(\mathbf{z}+\mathbf{y}) d z \\
& =-\int u_{, i}(\mathbf{x}-\mathbf{y}) \psi(\mathbf{x}) d x
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
u_{\varepsilon, i}(\psi) & =-\int u_{\varepsilon} \psi_{, i}=-\iint u(\mathbf{x}-\mathbf{y}) \phi_{\varepsilon}(\mathbf{y}) \psi_{, i}(\mathbf{x}) d y d x \\
& =-\iint u(\mathbf{x}-\mathbf{y}) \psi_{, i}(\mathbf{x}) \phi_{\varepsilon}(\mathbf{y}) d x d y \\
& =\iint u_{, i}(\mathbf{x}-\mathbf{y}) \psi(\mathbf{x}) \phi_{\varepsilon}(\mathbf{y}) d x d y \\
& =\int u_{, i} * \phi_{\varepsilon}(\mathbf{x}) \psi(\mathbf{x}) d x
\end{aligned}
$$

The technical questions about product measurability in the use of Fubini's theorem may be resolved by picking a Borel measurable representative for $u$. This proves the lemma.

What about the product rule? Does it have some form in the context of weak derivatives?

Lemma 36.3.6 Let $U$ be an open set, $\psi \in C^{\infty}(U)$ and suppose $u, u_{, i} \in L_{\text {loc }}^{p}(U)$. Then $(u \psi)_{, i}$ and $u \psi$ are in $L_{l o c}^{p}(U)$ and

$$
(u \psi)_{, i}=u_{, i} \psi+u \psi_{, i} .
$$

Proof: Let $\phi \in C_{c}^{\infty}(U)$ then

$$
\begin{aligned}
(u \psi)_{, i}(\phi) & \equiv-\int_{U} u \psi \phi_{, i} d x \\
& =-\int_{U} u\left[(\psi \phi)_{, i}-\phi \psi_{, i}\right] d x \\
& =\int_{U}\left(u_{, i} \psi \phi+u \psi_{, i} \phi\right) d x \\
& =\int_{U}\left(u_{, i} \psi+u \psi_{, i}\right) \phi d x
\end{aligned}
$$

This proves the lemma.
Recall the notation for the gradient of a function.

$$
\nabla u(\mathbf{x}) \equiv\left(u_{, 1}(\mathbf{x}) \cdots u_{, n}(\mathbf{x})\right)^{T}
$$

thus

$$
D u(\mathbf{x}) \mathbf{v}=\nabla u(\mathbf{x}) \cdot \mathbf{v}
$$

### 36.4 Morrey's Inequality

The following inequality will be called Morrey's inequality. It relates an expression which is given pointwise to an integral of the $p^{t h}$ power of the derivative.

Lemma 36.4.1 Let $u \in C^{1}\left(\mathbb{R}^{n}\right)$ and $p>n$. Then there exists a constant, $C$, depending only on $n$ such that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n}$,

$$
\begin{gather*}
|u(\mathbf{x})-u(\mathbf{y})| \\
\leq C\left(\int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)}|\nabla u(\mathbf{z})|^{p} d z\right)^{1 / p}\left(|\mathbf{x}-\mathbf{y}|^{(1-n / p)}\right) . \tag{36.4.1}
\end{gather*}
$$

Proof: In the argument $C$ will be a generic constant which depends on $n$. Consider the following picture.


This is a picture of two balls of radius $r$ in $\mathbb{R}^{n}, U$ and $V$ having centers at $\mathbf{x}$ and $\mathbf{y}$ respectively, which intersect in the set, $W$. The center of $U$ is on the boundary of $V$ and the
center of $V$ is on the boundary of $U$ as shown in the picture. There exists a constant, $C$, independent of $r$ depending only on $n$ such that

$$
\frac{m(W)}{m(U)}=\frac{m(W)}{m(V)}=C
$$

You could compute this constant if you desired but it is not important here.
Define the average of a function over a set, $E \subseteq \mathbb{R}^{n}$ as follows.

$$
f_{E} f d x \equiv \frac{1}{m(E)} \int_{E} f d x
$$

Then

$$
\begin{aligned}
|u(\mathbf{x})-u(\mathbf{y})| & =f_{W}|u(\mathbf{x})-u(\mathbf{y})| d z \\
& \leq f_{W}|u(\mathbf{x})-u(\mathbf{z})| d z+f_{W}|u(\mathbf{z})-u(\mathbf{y})| d z \\
& =\frac{C}{m(U)}\left[\int_{W}|u(\mathbf{x})-u(\mathbf{z})| d z+\int_{W}|u(\mathbf{z})-u(\mathbf{y})| d z\right] \\
& \leq C\left[f_{U}|u(\mathbf{x})-u(\mathbf{z})| d z+f_{V}|u(\mathbf{y})-u(\mathbf{z})| d z\right]
\end{aligned}
$$

Now consider these two terms. Using spherical coordinates and letting $U_{0}$ denote the ball of the same radius as $U$ but with center at $\mathbf{0}$,

$$
\begin{aligned}
& f_{U}|u(\mathbf{x})-u(\mathbf{z})| d z \\
= & \frac{1}{m\left(U_{0}\right)} \int_{U_{0}}|u(\mathbf{x})-u(\mathbf{z}+\mathbf{x})| d z \\
= & \frac{1}{m\left(U_{0}\right)} \int_{0}^{r} \rho^{n-1} \int_{S^{n-1}}|u(\mathbf{x})-u(\rho \mathbf{w}+\mathbf{x})| d \sigma(w) d \rho \\
\leq & \frac{1}{m\left(U_{0}\right)} \int_{0}^{r} \rho^{n-1} \int_{S^{n-1}} \int_{0}^{\rho}|\nabla u(\mathbf{x}+t \mathbf{w}) \cdot \mathbf{w}| d t d \sigma d \rho \\
\leq & \frac{1}{m\left(U_{0}\right)} \int_{0}^{r} \rho^{n-1} \int_{S^{n-1}} \int_{0}^{\rho}|\nabla u(\mathbf{x}+t \mathbf{w})| d t d \sigma d \rho \\
\leq & C \frac{1}{r} \int_{0}^{r} \int_{S^{n-1}} \int_{0}^{r}|\nabla u(\mathbf{x}+t \mathbf{w})| d t d \sigma d \rho \\
= & C \frac{1}{r} \int_{0}^{r} \int_{S^{n-1}} \int_{0}^{r} \frac{|\nabla u(\mathbf{x}+t \mathbf{w})|}{t^{n-1}} t^{n-1} d t d \sigma d \rho
\end{aligned}
$$

$$
\begin{aligned}
& =C \int_{S^{n-1}} \int_{0}^{r} \frac{|\nabla u(\mathbf{x}+t \mathbf{w})|}{t^{n-1}} t^{n-1} d t d \sigma \\
& =C \int_{U_{0}} \frac{|\nabla u(\mathbf{x}+\mathbf{z})|}{|\mathbf{z}|^{n-1}} d z \\
& \leq C\left(\int_{U_{0}}|\nabla u(\mathbf{x}+\mathbf{z})|^{p} d z\right)^{1 / p}\left(\int_{U}|\mathbf{z}|^{p^{\prime}-n p^{\prime}}\right)^{1 / p^{\prime}} \\
& =C\left(\int_{U}|\nabla u(\mathbf{z})|^{p} d z\right)^{1 / p}\left(\int_{S^{n-1}} \int_{0}^{r} \rho^{p^{\prime}-n p^{\prime}} \rho^{n-1} d \rho d \sigma\right)^{(p-1) / p} \\
& =C\left(\int_{U}|\nabla u(\mathbf{z})|^{p} d z\right)^{1 / p}\left(\int_{S^{n-1}} \int_{0}^{r} \frac{1}{\left.\rho^{\frac{n-1}{p-1}} d \rho d \sigma\right)^{(p-1) / p}}\right. \\
& =C\left(\frac{p-1}{p-n}\right)^{(p-1) / p}\left(\int_{U}|\nabla u(\mathbf{z})|^{p} d z\right)^{1 / p} r^{1-\frac{n}{p}} \\
& =C\left(\frac{p-1}{p-n}\right)^{(p-1) / p}\left(\int_{U}|\nabla u(\mathbf{z})|^{p} d z\right)^{1 / p}|\mathbf{x}-\mathbf{y}|^{1-\frac{n}{p}}
\end{aligned}
$$

Similarly,

$$
f_{V}|u(\mathbf{y})-u(\mathbf{z})| d z \leq C\left(\frac{p-1}{p-n}\right)^{(p-1) / p}\left(\int_{V}|\nabla u(\mathbf{z})|^{p} d z\right)^{1 / p}|\mathbf{x}-\mathbf{y}|^{1-\frac{n}{p}}
$$

Therefore,

$$
|u(\mathbf{x})-u(\mathbf{y})| \leq C\left(\frac{p-1}{p-n}\right)^{(p-1) / p}\left(\int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)}|\nabla u(\mathbf{z})|^{p} d z\right)^{1 / p}|\mathbf{x}-\mathbf{y}|^{1-\frac{n}{p}}
$$

because $B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|) \supseteq V \cup U$. This proves the lemma.
The following corollary is also interesting
Corollary 36.4.2 Suppose $u \in C^{1}\left(\mathbb{R}^{n}\right)$. Then

$$
\begin{gather*}
|u(\mathbf{y})-u(\mathbf{x})-\nabla u(\mathbf{x}) \cdot(\mathbf{y}-\mathbf{x})| \\
\leq C\left(\frac{1}{m(B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|))} \int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)}|\nabla u(\mathbf{z})-\nabla u(\mathbf{x})|^{p} d z\right)^{1 / p}|\mathbf{x}-\mathbf{y}| . \tag{36.4.2}
\end{gather*}
$$

Proof: This follows easily from letting $g(\mathbf{y}) \equiv u(\mathbf{y})-u(\mathbf{x})-\nabla u(\mathbf{x}) \cdot(\mathbf{y}-\mathbf{x})$. Then $g \in C^{1}\left(\mathbb{R}^{n}\right), g(\mathbf{x})=0$, and $\nabla g(\mathbf{z})=\nabla u(\mathbf{z})-\nabla u(\mathbf{x})$. From Lemma 36.4.1,

$$
\begin{aligned}
& |u(\mathbf{y})-u(\mathbf{x})-\nabla u(\mathbf{x}) \cdot(\mathbf{y}-\mathbf{x})| \\
= & |g(\mathbf{y})|=|g(\mathbf{y})-g(\mathbf{x})| \\
\leq & C\left(\int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)}|\nabla u(\mathbf{z})-\nabla u(\mathbf{x})|^{p} d z\right)^{1 / p}|\mathbf{x}-\mathbf{y}|^{1-\frac{n}{p}} \\
= & C\left(\frac{1}{m(B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|))} \int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)}|\nabla u(\mathbf{z})-\nabla u(\mathbf{x})|^{p} d z\right)^{1 / p}|\mathbf{x}-\mathbf{y}| .
\end{aligned}
$$

This proves the corollary.
It may be interesting at this point to recall the definition of differentiability on Page 117. If you knew the above inequality held for $\nabla u$ having components in $L_{l o c}^{1}\left(\mathbb{R}^{n}\right)$, then at Lebesgue points of $\nabla u$, the above would imply $D u(\mathbf{x})$ exists. This is exactly the approach taken below.

### 36.5 Rademacher's Theorem

The inequality of Corollary 36.4 .2 can be extended to the case where $u$ and $u_{, i}$ are in $L_{l o c}^{p}\left(\mathbb{R}^{n}\right)$ for $p>n$. This leads to an elegant proof of the differentiability a.e. of a Lipschitz continuous function as well as a more general theorem.

Theorem 36.5.1 Suppose $u$ and all its weak partial derivatives, $u_{, i}$ are in $L_{\text {loc }}^{p}\left(\mathbb{R}^{n}\right)$. Then there exists a set of measure zero, $E$ such that if $\mathbf{x}, \mathbf{y} \notin E$ then inequalities 36.4.2 and 36.4.1 are both valid. Furthermore, u equals a continuous function a.e.

Proof: Let $u \in L_{l o c}^{p}\left(\mathbb{R}^{n}\right)$ and $\psi_{k} \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right), \psi_{k} \geq 0$, and $\psi_{k}(\mathbf{z})=1$ for all $\mathbf{z} \in B(\mathbf{0}, k)$. Then it is routine to verify that

$$
u \psi_{k},\left(u \psi_{k}\right)_{, i} \in L^{p}\left(\mathbb{R}^{n}\right)
$$

Here is why:

$$
\begin{aligned}
\left(u \psi_{k}\right)_{, i}(\phi) & \equiv-\int_{\mathbb{R}^{n}} u \psi_{k} \phi_{, i} d x \\
& =-\int_{\mathbb{R}^{n}} u \psi_{k} \phi_{, i} d x-\int_{\mathbb{R}^{n}} u \psi_{k, i} \phi d x+\int_{\mathbb{R}^{n}} u \psi_{k, i} \phi d x \\
& =-\int_{\mathbb{R}^{n}} u\left(\psi_{k} \phi\right)_{, i} d x+\int_{\mathbb{R}^{n}} u \psi_{k, i} \phi d x \\
& =\int_{\mathbb{R}^{n}}\left(u_{, i} \psi_{k}+u \psi_{k, i}\right) \phi d x
\end{aligned}
$$

which shows

$$
\left(u \psi_{k}\right)_{, i}=u_{, i} \psi_{k}+u \psi_{k, i}
$$

as expected.
Let $\phi_{\varepsilon}$ be a mollifier and consider

$$
\left(u \psi_{k}\right)_{\varepsilon} \equiv u \psi_{k} * \phi_{\varepsilon}
$$

By Lemma 36.3.5 on Page 1281,

$$
\left(u \psi_{k}\right)_{\varepsilon, i}=\left(u \psi_{k}\right)_{, i} * \phi_{\varepsilon}
$$

Therefore

$$
\begin{equation*}
\left(u \psi_{k}\right)_{\mathcal{\varepsilon}, i} \rightarrow\left(u \psi_{k}\right)_{, i} \text { in } L^{p}\left(\mathbb{R}^{n}\right) \tag{36.5.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(u \psi_{k}\right)_{\varepsilon} \rightarrow u \psi_{k} \operatorname{in} L^{p}\left(\mathbb{R}^{n}\right) \tag{36.5.4}
\end{equation*}
$$

as $\varepsilon \rightarrow 0$. By 36.5.4, there exists a subsequence $\varepsilon \rightarrow 0$ such that for $|\mathbf{z}|<k$ and for each $i=1,2, \cdots, n$

$$
\begin{gather*}
\left(u \psi_{k}\right)_{\varepsilon, i}(\mathbf{z}) \rightarrow\left(u \psi_{k}\right)_{, i}(\mathbf{z})=u_{, i}(\mathbf{z}) \text { a.e. } \\
\quad\left(u \psi_{k}\right)_{\varepsilon}(\mathbf{z}) \rightarrow u \psi_{k}(\mathbf{z})=u(\mathbf{z}) \text { a.e. } \tag{36.5.5}
\end{gather*}
$$

Denoting the exceptional set by $E_{k}$, let

$$
\mathbf{x}, \mathbf{y} \notin \cup_{k=1}^{\infty} E_{k} \equiv E
$$

and let $k$ be so large that

$$
B(\mathbf{0}, k) \supseteq B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|) .
$$

Then by 36.4.1 and for $\mathbf{x}, \mathbf{y} \notin E$,

$$
\begin{gathered}
\left|\left(u \psi_{k}\right)_{\varepsilon}(\mathbf{x})-\left(u \psi_{k}\right)_{\varepsilon}(\mathbf{y})\right| \\
\leq C\left(\int_{B(\mathbf{x}, 2|\mathbf{y}-\mathbf{x}|)}\left|\nabla\left(u \psi_{k}\right)_{\varepsilon}\right|^{p} d z\right)^{1 / p}|\mathbf{x}-\mathbf{y}|^{(1-n / p)}
\end{gathered}
$$

where $C$ depends only on $n$. Similarly, by 36.4.2,

$$
\begin{gathered}
\left|\left(u \psi_{k}\right)_{\varepsilon}(\mathbf{x})-\left(u \psi_{k}\right)_{\varepsilon}(\mathbf{y})-\nabla\left(u \psi_{k}\right)_{\mathcal{\varepsilon}}(\mathbf{x}) \cdot(\mathbf{y}-\mathbf{x})\right| \leq \\
C\left(\frac{1}{m(B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|))} \int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)}\left|\nabla\left(u \psi_{k}\right)_{\mathcal{\varepsilon}}(\mathbf{z})-\nabla\left(u \psi_{k}\right)_{\mathcal{\varepsilon}}(\mathbf{x})\right|^{p} d z\right)^{1 / p}|\mathbf{x}-\mathbf{y}| .
\end{gathered}
$$

Now by 36.5 .5 and 36.5 .3 passing to the limit as $\varepsilon \rightarrow 0$ yields

$$
\begin{equation*}
|u(\mathbf{x})-u(\mathbf{y})| \leq C\left(\int_{B(\mathbf{x}, 2|\mathbf{y}-\mathbf{x}|)}|\nabla u|^{p} d z\right)^{1 / p}|\mathbf{x}-\mathbf{y}|^{(1-n / p)} \tag{36.5.6}
\end{equation*}
$$

and

$$
\begin{gather*}
|u(\mathbf{y})-u(\mathbf{x})-\nabla u(\mathbf{x}) \cdot(\mathbf{y}-\mathbf{x})| \\
\leq C\left(\frac{1}{m(B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|))} \int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)}|\nabla u(\mathbf{z})-\nabla u(\mathbf{x})|^{p} d z\right)^{1 / p}|\mathbf{x}-\mathbf{y}| . \tag{36.5.7}
\end{gather*}
$$

Redefining $u$ on the set of mesure zero, $E$ yields 36.5 .6 for all $\mathbf{x}, \mathbf{y}$. This proves the theorem.
Corollary 36.5.2 Let $u, u_{, i} \in L_{l o c}^{p}\left(\mathbb{R}^{n}\right)$ for $i=1, \cdots, n$ and $p>n$. Then the representative of $u$ described in Theorem 36.5.1 is differentiable a.e.

Proof: From Theorem 36.5.1

$$
\begin{gather*}
|u(\mathbf{y})-u(\mathbf{x})-\nabla u(\mathbf{x}) \cdot(\mathbf{y}-\mathbf{x})| \\
\leq C\left(\frac{1}{m(B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|))} \int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)}|\nabla u(\mathbf{z})-\nabla u(\mathbf{x})|^{p} d z\right)^{1 / p}|\mathbf{x}-\mathbf{y}| \tag{36.5.8}
\end{gather*}
$$

and at every Lebesgue point, $\mathbf{x}$ of $\nabla u$

$$
\lim _{\mathbf{y} \rightarrow \mathbf{x}}\left(\frac{1}{m(B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|))} \int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)}|\nabla u(\mathbf{z})-\nabla u(\mathbf{x})|^{p} d z\right)^{1 / p}=0
$$

and so at each of these points,

$$
\lim _{\mathbf{y} \rightarrow \mathbf{x}} \frac{|u(\mathbf{y})-u(\mathbf{x})-\nabla u(\mathbf{x}) \cdot(\mathbf{y}-\mathbf{x})|}{|\mathbf{x}-\mathbf{y}|}=0
$$

which says that $u$ is differentiable at $\mathbf{x}$ and $D u(\mathbf{x})(\mathbf{v})=\nabla u(\mathbf{x}) \cdot(\mathbf{v})$. See Page 117. This proves the corollary.

Definition 36.5.3 Now suppose $u$ is Lipschitz on $\mathbb{R}^{n}$,

$$
|u(\mathbf{x})-u(\mathbf{y})| \leq K|\mathbf{x}-\mathbf{y}|
$$

for some constant $K$. Define $\operatorname{Lip}(u)$ as the smallest value of $K$ that works in this inequality.
The following corollary is known as Rademacher's theorem. It states that every Lipschitz function is differentiable a.e.

Corollary 36.5.4 If $u$ is Lipschitz continuous then $u$ is differentiable a.e. and $\left\|u_{, i}\right\|_{\infty} \leq$ $\operatorname{Lip}(u)$.

Proof: This is done by showing that Lipschitz continuous functions have weak derivatives in $L^{\infty}\left(\mathbb{R}^{n}\right)$ and then using the previous results. Let

$$
D_{\mathbf{e}_{i}}^{h} u(\mathbf{x}) \equiv h^{-1}\left[u\left(\mathbf{x}+h \mathbf{e}_{i}\right)-u(\mathbf{x})\right] .
$$

Then $D_{\mathbf{e}_{i}}^{h} u$ is bounded in $L^{\infty}\left(\mathbb{R}^{n}\right)$ and

$$
\left\|D_{\mathbf{e}_{i}}^{h} u\right\|_{\infty} \leq \operatorname{Lip}(u)
$$

It follows that $D_{\mathbf{e}_{i}}^{h} u$ is contained in a ball in $L^{\infty}\left(\mathbb{R}^{n}\right)$, the dual space of $L^{1}\left(\mathbb{R}^{n}\right)$. By Theorem 36.1.3 on Page 1276, there is a subsequence $h \rightarrow 0$ such that

$$
D_{\mathbf{e}_{i}}^{h} u \rightharpoonup w,\|w\|_{\infty} \leq \operatorname{Lip}(u)
$$

where the convergence takes place in the weak $*$ topology of $L^{\infty}\left(\mathbb{R}^{n}\right)$. Let $\phi \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$. Then

$$
\begin{gathered}
\int w \phi d x=\lim _{h \rightarrow 0} \int D_{\mathbf{e}_{i}}^{h} u \phi d x \\
=\lim _{h \rightarrow 0} \int u(\mathbf{x}) \frac{\left(\phi\left(\mathbf{x}-h \mathbf{e}_{i}\right)-\phi(\mathbf{x})\right)}{h} d x \\
=-\int u(\mathbf{x}) \phi_{, i}(\mathbf{x}) d x
\end{gathered}
$$

Thus $w=u_{, i}$ and $u_{, i} \in L^{\infty}\left(\mathbb{R}^{n}\right)$ for each $i$. Hence $u, u_{, i} \in L_{l o c}^{p}\left(\mathbb{R}^{n}\right)$ for all $p>n$ and so $u$ is differentiable a.e. by Corollary 36.5.2. This proves the corollary.

### 36.6 Change Of Variables Formula Lipschitz Maps

With Rademacher's theorem, one can give a general change of variables formula involving Lipschitz maps. First here is an elementary estimate.

Lemma 36.6.1 Suppose $V$ is an $n-1$ dimensional subspace of $\mathbb{R}^{n}$ and $K$ is a compact subset of $V$. Then letting

$$
K_{\varepsilon} \equiv \cup_{\mathbf{x} \in K} B(\mathbf{x}, \boldsymbol{\varepsilon})=K+B(\mathbf{0}, \varepsilon),
$$

it follows that

$$
m_{n}\left(K_{\varepsilon}\right) \leq 2^{n} \varepsilon(\operatorname{diam}(K)+\varepsilon)^{n-1}
$$

Proof: Let an orthonormal basis for $V$ be $\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{n-1}\right\}$ and let

$$
\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{n-1}, \mathbf{v}_{n}\right\}
$$

be an orthonormal basis for $\mathbb{R}^{n}$. Now define a linear transformation, $Q$ by $Q \mathbf{v}_{i}=\mathbf{e}_{i}$. Thus $Q Q^{*}=Q^{*} Q=I$ and $Q$ preserves all distances because

$$
\left|Q \sum_{i} a_{i} \mathbf{e}_{i}\right|^{2}=\left|\sum_{i} a_{i} \mathbf{v}_{i}\right|^{2}=\sum_{i}\left|a_{i}\right|^{2}=\left|\sum_{i} a_{i} \mathbf{e}_{i}\right|^{2}
$$

Letting $\mathbf{k}_{0} \in K$, it follows $K \subseteq B\left(\mathbf{k}_{0}, \operatorname{diam}(K)\right)$ and so,

$$
Q K \subseteq B^{n-1}\left(Q \mathbf{k}_{0}, \operatorname{diam}(Q K)\right)=B^{n-1}\left(Q \mathbf{k}_{0}, \operatorname{diam}(K)\right)
$$

where $B^{n-1}$ refers to the ball taken with respect to the usual norm in $\mathbb{R}^{n-1}$. Every point of $K_{\varepsilon}$ is within $\varepsilon$ of some point of $K$ and so it follows that every point of $Q K_{\varepsilon}$ is within $\varepsilon$ of some point of $Q K$. Therefore,

$$
Q K_{\varepsilon} \subseteq B^{n-1}\left(Q \mathbf{k}_{0}, \operatorname{diam}(Q K)+\varepsilon\right) \times(-\varepsilon, \varepsilon)
$$

To see this, let $\mathbf{x} \in Q K_{\varepsilon}$. Then there exists $\mathbf{k} \in Q K$ such that $|\mathbf{k}-\mathbf{x}|<\varepsilon$. Therefore, $\left|\left(x_{1}, \cdots, x_{n-1}\right)-\left(k_{1}, \cdots, k_{n-1}\right)\right|<\varepsilon$ and $\left|x_{n}-k_{n}\right|<\varepsilon$ and so $\mathbf{x}$ is contained in the set on the right in the above inclusion because $k_{n}=0$. However, the measure of the set on the right is smaller than

$$
[2(\operatorname{diam}(Q K)+\varepsilon)]^{n-1}(2 \varepsilon)=2^{n}[(\operatorname{diam}(K)+\varepsilon)]^{n-1} \varepsilon
$$

This proves the lemma.
Next is the definition of a point of density. This is sort of like an interior point but not as good.

Definition 36.6.2 Let $E$ be a Lebesgue measurable set. $\mathbf{x} \in E$ is a point of density if

$$
\lim _{r \rightarrow 0} \frac{m(E \cap B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))}=1
$$

You see that if $\mathbf{x}$ were an interior point of $E$, then this limit will equal 1. However, it is sometimes the case that the limit equals 1 even when $\mathbf{x}$ is not an interior point. In fact, these points of density make sense even for sets that have empty interior.

Lemma 36.6.3 Let $E$ be a Lebesgue measurable set. Then there exists a set of measure zero, $N$, such that if $\mathbf{x} \in E \backslash N$, then $\mathbf{x}$ is a point of density of $E$.

Proof: Consider the function, $f(\mathbf{x})=\mathscr{X}_{E}(\mathbf{x})$. This function is in $L_{l o c}^{1}\left(\mathbb{R}^{n}\right)$. Let $N^{C}$ denote the Lebesgue points of $f$. Then for $\mathbf{x} \in E \backslash N$,

$$
\begin{aligned}
1 & =\mathscr{X}_{E}(\mathbf{x})=\lim _{r \rightarrow 0} \frac{1}{m_{n}(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} \mathscr{X}_{E}(\mathbf{y}) d m_{n} \\
& =\lim _{r \rightarrow 0} \frac{m_{n}(B(\mathbf{x}, r) \cap E)}{m_{n}(B(\mathbf{x}, r))}
\end{aligned}
$$

In this section, $\Omega$ will be a Lebesgue measurable set in $\mathbb{R}^{n}$ and $\mathbf{h}: \Omega \rightarrow \mathbb{R}^{n}$ will be Lipschitz. Recall the following definition and theorems. See Page 13.4.2 for the proofs and more discussion.

Definition 36.6.4 Let $\mathscr{F}$ be a collection of balls that cover a set, $E$, which have the property that if $\mathbf{x} \in E$ and $\varepsilon>0$, then there exists $B \in \mathscr{F}$, diameter of $B<\varepsilon$ and $\mathbf{x} \in B$. Such a collection covers $E$ in the sense of Vitali.

Theorem 36.6.5 Let $E \subseteq \mathbb{R}^{n}$ and suppose $\overline{m_{n}}(E)<\infty$ where $\overline{m_{n}}$ is the outer measure determined by $m_{n}, n$ dimensional Lebesgue measure, and let $\mathscr{F}$, be a collection of closed balls of bounded radii such that $\mathscr{F}$ covers $E$ in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathscr{F},\left\{B_{j}\right\}_{j=1}^{\infty}$, such that $\overline{m_{n}}\left(E \backslash \cup_{j=1}^{\infty} B_{j}\right)=0$.

Now this theorem implies a simple lemma which is what will be used.
Lemma 36.6.6 Let $V$ be an open set in $\mathbb{R}^{r}, m_{r}(V)<\infty$. Then there exists a sequence of disjoint open balls $\left\{B_{i}\right\}$ having radii less than $\delta$ and a set of measure $0, T$, such that

$$
V=\left(\cup_{i=1}^{\infty} B_{i}\right) \cup T
$$

As in the proof of the change of variables theorem given earlier, the first step is to show that $\mathbf{h}$ maps Lebesgue measurable sets to Lebesgue measurable sets. In showing this the key result is the next lemma which states that $\mathbf{h}$ maps sets of measure zero to sets of measure zero.

Lemma 36.6.7 If $m_{n}(T)=0$ then $m_{n}(\mathbf{h}(T))=0$.
Proof: Let $V$ be an open set containing $T$ whose measure is less than $\varepsilon$. Now using the Vitali covering theorem, there exists a sequence of disjoint balls $\left\{B_{i}\right\}, B_{i}=B\left(\mathbf{x}_{i}, r_{i}\right)$ which are contained in $V$ such that the sequence of enlarged balls, $\left\{\widehat{B}_{i}\right\}$, having the same center but 5 times the radius, covers $T$. Then

$$
m_{n}(\mathbf{h}(T)) \leq m_{n}\left(\mathbf{h}\left(\cup_{i=1}^{\infty} \widehat{B}_{i}\right)\right)
$$

$$
\begin{aligned}
& \leq \sum_{i=1}^{\infty} m_{n}\left(\mathbf{h}\left(\widehat{B}_{i}\right)\right) \\
& \leq \sum_{i=1}^{\infty} \alpha(n)(\operatorname{Lip}(\mathbf{h}))^{n} 5^{n} r_{i}^{n}=5^{n}(\operatorname{Lip}(\mathbf{h}))^{n} \sum_{i=1}^{\infty} m_{n}\left(B_{i}\right) \\
& \leq \quad(\operatorname{Lip}(\mathbf{h}))^{n} 5^{n} m_{n}(V) \leq \varepsilon(\operatorname{Lip}(\mathbf{h}))^{n} 5^{n} .
\end{aligned}
$$

Since $\varepsilon$ is arbitrary, this proves the lemma.
With the conclusion of this lemma, the next lemma is fairly easy to obtain.
Lemma 36.6.8 If A is Lebesgue measurable, then $\mathbf{h}(A)$ is $m_{n}$ measurable. Furthermore,

$$
\begin{equation*}
m_{n}(\mathbf{h}(A)) \leq(\operatorname{Lip}(\mathbf{h}))^{n} m_{n}(A) . \tag{36.6.9}
\end{equation*}
$$

Proof: Let $A_{k}=A \cap B(\mathbf{0}, k), k \in \mathbb{N}$. Let $V \supseteq A_{k}$ and let $m_{n}(V)<\infty$. By Lemma 36.6.6, there is a sequence of disjoint balls $\left\{B_{i}\right\}$ and a set of measure $0, T$, such that

$$
V=\cup_{i=1}^{\infty} B_{i} \cup T, B_{i}=B\left(x_{i}, r_{i}\right) .
$$

By Lemma 36.6.7,

$$
\begin{gathered}
\overline{m_{n}}\left(\mathbf{h}\left(A_{k}\right)\right) \leq \overline{m_{n}}(\mathbf{h}(V)) \\
\leq \overline{m_{n}}\left(\mathbf{h}\left(\cup_{i=1}^{\infty} B_{i}\right)\right)+\overline{m_{n}}(\mathbf{h}(T))=\overline{m_{n}}\left(\mathbf{h}\left(\cup_{i=1}^{\infty} B_{i}\right)\right) \\
\leq \sum_{i=1}^{\infty} \overline{m_{n}}\left(\mathbf{h}\left(B_{i}\right)\right) \leq \sum_{i=1}^{\infty} \overline{m_{n}}\left(B\left(\mathbf{h}\left(x_{i}\right), \operatorname{Lip}(\mathbf{h}) r_{i}\right)\right) \\
\leq \sum_{i=1}^{\infty} \alpha(n)\left(\operatorname{Lip}(\mathbf{h}) r_{i}\right)^{n}=\operatorname{Lip}(\mathbf{h})^{n} \sum_{i=1}^{\infty} m_{n}\left(B_{i}\right)=\operatorname{Lip}(\mathbf{h})^{n} m_{n}(V)
\end{gathered}
$$

Therefore,

$$
\overline{m_{n}}\left(\mathbf{h}\left(A_{k}\right)\right) \leq \operatorname{Lip}(\mathbf{h})^{n} m_{n}(V) .
$$

Since $V$ is an arbitrary open set containing $A_{k}$, it follows from regularity of Lebesgue measure that

$$
\begin{equation*}
\overline{m_{n}}\left(\mathbf{h}\left(A_{k}\right)\right) \leq \operatorname{Lip}(\mathbf{h})^{n} m_{n}\left(A_{k}\right) . \tag{36.6.10}
\end{equation*}
$$

Now let $k \rightarrow \infty$ to obtain 36.6.9. This proves the formula. It remains to show $\mathbf{h}(A)$ is measurable.

By inner regularity of Lebesgue measure, there exists a set, $F$, which is the countable union of compact sets and a set $T$ with $m_{n}(T)=0$ such that

$$
F \cup T=A_{k} .
$$

Then $\mathbf{h}(F) \subseteq \mathbf{h}\left(A_{k}\right) \subseteq \mathbf{h}(F) \cup \mathbf{h}(T)$. By continuity of $\mathbf{h}, \mathbf{h}(F)$ is a countable union of compact sets and so it is Borel. By 36.6.10 with $T$ in place of $A_{k}$,

$$
\overline{m_{n}}(\mathbf{h}(T))=0
$$

and so $\mathbf{h}(T)$ is $m_{n}$ measurable. Therefore, $\mathbf{h}\left(A_{k}\right)$ is $m_{n}$ measurable because $m_{n}$ is a complete measure and this exhibits $\mathbf{h}\left(A_{k}\right)$ between two $m_{n}$ measurable sets whose difference has measure 0 . Now

$$
\mathbf{h}(A)=\cup_{k=1}^{\infty} \mathbf{h}\left(A_{k}\right)
$$

so $\mathbf{h}(A)$ is also $m_{n}$ measurable and this proves the lemma.
The following lemma, depending on the Brouwer fixed point theorem and found in Rudin [113], will be important for the following arguments. The idea is that if a continuous function mapping a ball in $\mathbb{R}^{k}$ to $\mathbb{R}^{k}$ doesn't move any point very much, then the image of the ball must contain a slightly smaller ball.

Lemma 36.6.9 Let $B=B(\mathbf{0}, r)$, a ball in $\mathbb{R}^{k}$ and let $\mathbf{F}: \bar{B} \rightarrow \mathbb{R}^{k}$ be continuous and suppose for some $\varepsilon<1$,

$$
|\mathbf{F}(\mathbf{v})-\mathbf{v}|<\varepsilon r
$$

for all $\mathbf{v} \in \bar{B}$. Then

$$
\mathbf{F}(\bar{B}) \supseteq \overline{B(\mathbf{0}, r(1-\varepsilon))}
$$

Proof: Suppose $\mathbf{a} \in \overline{B(\mathbf{0}, r(1-\boldsymbol{\varepsilon}))} \backslash \mathbf{F}(\bar{B})$ and let

$$
\mathbf{G}(\mathbf{v}) \equiv \frac{r(\mathbf{a}-\mathbf{F}(\mathbf{v}))}{|\mathbf{a}-\mathbf{F}(\mathbf{v})|}
$$

Then by the Brouwer fixed point theorem, $\mathbf{G}(\mathbf{v})=\mathbf{v}$ for some $\mathbf{v} \in \bar{B}$. Using the formula for $\mathbf{G}$, it follows $|\mathbf{v}|=r$. Taking the inner product with $\mathbf{v}$,

$$
\begin{aligned}
(\mathbf{G}(\mathbf{v}), \mathbf{v}) & =|\mathbf{v}|^{2}=r^{2}=\frac{r}{|\mathbf{a}-\mathbf{F}(\mathbf{v})|}(\mathbf{a}-\mathbf{F}(\mathbf{v}), \mathbf{v}) \\
& =\frac{r}{|\mathbf{a}-\mathbf{F}(\mathbf{v})|}(\mathbf{a}-\mathbf{v}+\mathbf{v}-\mathbf{F}(\mathbf{v}), \mathbf{v}) \\
& =\frac{r}{|\mathbf{a}-\mathbf{F}(\mathbf{v})|}[(\mathbf{a}-\mathbf{v}, \mathbf{v})+(\mathbf{v}-\mathbf{F}(\mathbf{v}), \mathbf{v})] \\
& =\frac{r}{|\mathbf{a}-\mathbf{F}(\mathbf{v})|}\left[(\mathbf{a}, \mathbf{v})-|\mathbf{v}|^{2}+(\mathbf{v}-\mathbf{F}(\mathbf{v}), \mathbf{v})\right] \\
& \leq \frac{r}{|\mathbf{a}-\mathbf{F}(\mathbf{v})|}\left[r^{2}(1-\varepsilon)-r^{2}+r^{2} \varepsilon\right]=0
\end{aligned}
$$

a contradiction. Therefore, $\overline{B(\mathbf{0}, r(1-\varepsilon))} \backslash \mathbf{F}(\bar{B})=\emptyset$ and this proves the lemma.
Now let $\Omega$ be a Lebesgue measurable set and suppose $\mathbf{h}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ is Lipschitz continuous and one to one on $\Omega$. Let

$$
\begin{gather*}
N \equiv\{\mathbf{x} \in \Omega: D \mathbf{h}(\mathbf{x}) \text { does not exist }\}  \tag{36.6.11}\\
S \equiv\left\{\mathbf{x} \in \Omega \backslash N: D \mathbf{h}(\mathbf{x})^{-1} \text { does not exist }\right\} \tag{36.6.12}
\end{gather*}
$$

Lemma 36.6.10 Let $\mathbf{x} \in \Omega \backslash(S \cup N)$. Then if $\varepsilon \in(0,1)$ the following hold for all $r$ small enough.

$$
\begin{equation*}
m_{n}(\mathbf{h}(\overline{B(\mathbf{x}, r)})) \geq m_{n}(D \mathbf{h}(\mathbf{x}) B(\mathbf{0}, r(1-\varepsilon))) \tag{36.6.13}
\end{equation*}
$$

$$
\begin{gather*}
\mathbf{h}(B(\mathbf{x}, r)) \subseteq \mathbf{h}(\mathbf{x})+D \mathbf{h}(\mathbf{x}) B(\mathbf{0}, r(1+\varepsilon))  \tag{36.6.14}\\
m_{n}(\mathbf{h}(\overline{B(\mathbf{x}, r)})) \leq m_{n}(D \mathbf{h}(\mathbf{x}) B(\mathbf{0}, r(1+\varepsilon))) \tag{36.6.15}
\end{gather*}
$$

If $\mathbf{x} \in \Omega \backslash(S \cup N)$ is also a point of density of $\Omega$, then

$$
\begin{equation*}
\lim _{r \rightarrow 0} \frac{m_{n}(\mathbf{h}(B(\mathbf{x}, r) \cap \Omega))}{m_{n}(\mathbf{h}(B(\mathbf{x}, r)))}=1 \tag{36.6.16}
\end{equation*}
$$

If $\mathbf{x} \in \Omega \backslash N$, then

$$
\begin{equation*}
|\operatorname{det} D \mathbf{h}(\mathbf{x})|=\lim _{r \rightarrow 0} \frac{m_{n}(\mathbf{h}(B(\mathbf{x}, r)))}{m_{n}(B(\mathbf{x}, r))} \text { a.e. } \tag{36.6.17}
\end{equation*}
$$

Proof: Since $D \mathbf{h}(\mathbf{x})^{-1}$ exists,

$$
\begin{align*}
\mathbf{h}(\mathbf{x}+\mathbf{v}) & =\mathbf{h}(\mathbf{x})+D \mathbf{h}(\mathbf{x}) \mathbf{v}+o(|\mathbf{v}|)  \tag{36.6.18}\\
& =\mathbf{h}(\mathbf{x})+D \mathbf{h}(\mathbf{x})(\mathbf{v}+\overbrace{D \mathbf{h}(\mathbf{x})^{-1} o(|\mathbf{v}|)}^{=o(|\mathbf{v}|)}) \tag{36.6.19}
\end{align*}
$$

Consequently, when $r$ is small enough, 36.6 .14 holds. Therefore, 36.6 .15 holds. From 36.6.19, and the assumption that $D \mathbf{h}(\mathbf{x})^{-1}$ exists,

$$
\begin{equation*}
D \mathbf{h}(\mathbf{x})^{-1} \mathbf{h}(\mathbf{x}+\mathbf{v})-D \mathbf{h}(\mathbf{x})^{-1} \mathbf{h}(\mathbf{x})-\mathbf{v}=o(|\mathbf{v}|) \tag{36.6.20}
\end{equation*}
$$

Letting

$$
\mathbf{F}(\mathbf{v})=D \mathbf{h}(\mathbf{x})^{-1} \mathbf{h}(\mathbf{x}+\mathbf{v})-D \mathbf{h}(\mathbf{x})^{-1} \mathbf{h}(\mathbf{x})
$$

apply Lemma 36.6.9 in 36.6.20 to conclude that for $r$ small enough, whenever $|\mathbf{v}|<r$,

$$
D \mathbf{h}(\mathbf{x})^{-1} \mathbf{h}(\mathbf{x}+\mathbf{v})-D \mathbf{h}(\mathbf{x})^{-1} \mathbf{h}(\mathbf{x}) \supseteq B(\mathbf{0},(1-\varepsilon) r) .
$$

Therefore,

$$
\mathbf{h}(\overline{B(\mathbf{x}, r)}) \supseteq \mathbf{h}(\mathbf{x})+D \mathbf{h}(\mathbf{x}) B(\mathbf{0},(1-\varepsilon) r)
$$

which implies

$$
m_{n}(\mathbf{h}(\overline{B(\mathbf{x}, r)})) \geq m_{n}(D \mathbf{h}(\mathbf{x}) B(\mathbf{0}, r(1-\boldsymbol{\varepsilon})))
$$

which shows 36.6.13.
Now suppose that $\mathbf{x}$ is a point of density of $\Omega$ as well as being a point where $D \mathbf{h}(\mathbf{x})^{-1}$ and $D \mathbf{h}(\mathbf{x})$ exist. Then whenever $r$ is small enough,

$$
1-\varepsilon<\frac{m_{n}(\mathbf{h}(B(\mathbf{x}, r) \cap \Omega))}{m_{n}(\mathbf{h}(B(\mathbf{x}, r)))} \leq 1
$$

and so

$$
\begin{aligned}
1-\varepsilon & <\frac{m_{n}\left(\mathbf{h}\left(B(\mathbf{x}, r) \cap \Omega^{C}\right)\right)}{m_{n}(\mathbf{h}(B(\mathbf{x}, r)))}+\frac{m_{n}(\mathbf{h}(B(\mathbf{x}, r) \cap \Omega))}{m_{n}(\mathbf{h}(B(\mathbf{x}, r)))} \\
& \leq \frac{m_{n}\left(\mathbf{h}\left(B(\mathbf{x}, r) \cap \Omega^{C}\right)\right)}{m_{n}(\mathbf{h}(B(\mathbf{x}, r)))}+1
\end{aligned}
$$

which implies

$$
\begin{equation*}
m_{n}(B(\mathbf{x}, r) \backslash \Omega)<\varepsilon \alpha(n) r^{n} . \tag{36.6.21}
\end{equation*}
$$

Then for such $r$,

$$
\begin{gathered}
1 \geq \frac{m_{n}(\mathbf{h}(B(\mathbf{x}, r) \cap \Omega))}{m_{n}(\mathbf{h}(B(\mathbf{x}, r)))} \\
\geq \frac{m_{n}(\mathbf{h}(B(\mathbf{x}, r)))-m_{n}(\mathbf{h}(B(\mathbf{x}, r) \backslash \Omega))}{m_{n}(\mathbf{h}(B(\mathbf{x}, r)))} .
\end{gathered}
$$

From Lemma 36.6.8, 36.6.21, and 36.6.13, this is no larger than

$$
1-\frac{\operatorname{Lip}(\mathbf{h})^{n} \varepsilon \alpha(n) r^{n}}{m_{n}(D \mathbf{h}(\mathbf{x}) B(\mathbf{0}, r(1-\varepsilon)))}
$$

By the theorem on the change of variables for a linear map, this expression equals

$$
1-\frac{\operatorname{Lip}(\mathbf{h})^{n} \varepsilon \alpha(n) r^{n}}{|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| r^{n} \alpha(n)(1-\varepsilon)^{n}} \equiv 1-g(\varepsilon)
$$

where $\lim _{\varepsilon \rightarrow 0} g(\varepsilon)=0$. Then for all $r$ small enough,

$$
1 \geq \frac{m_{n}(\mathbf{h}(B(\mathbf{x}, r) \cap \Omega))}{m_{n}(\mathbf{h}(B(\mathbf{x}, r)))} \geq 1-g(\varepsilon)
$$

which shows 36.6 .16 since $\varepsilon$ is arbitrary. It remains to verify 36.6.17.
In case $\mathbf{x} \in S$, for small $|\mathbf{v}|$,

$$
\mathbf{h}(\mathbf{x}+\mathbf{v})=\mathbf{h}(\mathbf{x})+D \mathbf{h}(\mathbf{x}) \mathbf{v}+o(|\mathbf{v}|)
$$

where $|o(|\mathbf{v}|)|<\varepsilon|\mathbf{v}|$. Therefore, for small enough $r$,

$$
\mathbf{h}(B(\mathbf{x}, r))-\mathbf{h}(\mathbf{x}) \subseteq K+B(\mathbf{0}, r \varepsilon)
$$

where $K$ is a compact subset of an $n-1$ dimensional subspace contained in $D \mathbf{h}(\mathbf{x})\left(\mathbb{R}^{n}\right)$ which has diameter no more than $2\|D \mathbf{h}(\mathbf{x})\| r$. By Lemma 36.6.1 on Page 1288,

$$
\begin{aligned}
m_{n}(\mathbf{h}(B(\mathbf{x}, r))) & =m_{n}(\mathbf{h}(B(\mathbf{x}, r))-\mathbf{h}(\mathbf{x})) \\
& \leq 2^{n} \varepsilon r(2\|D \mathbf{h}(\mathbf{x})\| r+r \varepsilon)^{n-1}
\end{aligned}
$$

and so, in this case, letting $r$ be small enough,

$$
\frac{m_{n}(\mathbf{h}(B(\mathbf{x}, r)))}{m_{n}(B(\mathbf{x}, r))} \leq \frac{2^{n} \varepsilon r(2\|D \mathbf{h}(\mathbf{x})\| r+r \varepsilon)^{n-1}}{\alpha(n) r^{n}} \leq C \varepsilon
$$

Since $\varepsilon$ is arbitrary, the limit as $r \rightarrow 0$ of this quotient equals 0 .
If $\mathbf{x} \notin S$, use 36.6.13-36.6.15 along with the change of variables formula for linear maps. This proves the Lemma.

Since $\mathbf{h}$ is one to one, there exists a measure, $\mu$, defined by

$$
\mu(E) \equiv m_{n}(\mathbf{h}(E))
$$

on the Lebesgue measurable subsets of $\Omega$. By Lemma $36.6 .8 \mu \ll m_{n}$ and so by the Radon Nikodym theorem, there exists a nonnegative function, $J(\mathbf{x})$ in $L_{l o c}^{1}\left(\mathbb{R}^{n}\right)$ such that whenever $E$ is Lebesgue measurable,

$$
\begin{equation*}
\mu(E)=m_{n}(\mathbf{h}(E \cap \Omega))=\int_{E \cap \Omega} J(\mathbf{x}) d m_{n} \tag{36.6.22}
\end{equation*}
$$

Extend $J$ to equal zero off $\Omega$.
Lemma 36.6.11 The function, $J(\mathbf{x})$ equals $|\operatorname{det} D \mathbf{h}(\mathbf{x})|$ a.e.
Proof: Define

$$
\begin{aligned}
Q \equiv & \{\mathbf{x} \in \Omega: \mathbf{x} \text { is not a point of density of } \Omega\} \cup N \cup \\
& \{\mathbf{x} \in \Omega: \mathbf{x} \text { is not a Lebesgue point of } J\}
\end{aligned}
$$

Then $Q$ is a set of measure zero and if $\mathbf{x} \notin Q$, then by 36.6.17, and 36.6.16,

$$
\begin{aligned}
& |\operatorname{det} D \mathbf{h}(\mathbf{x})| \\
= & \lim _{r \rightarrow 0} \frac{m_{n}(\mathbf{h}(B(\mathbf{x}, r)))}{m_{n}(B(\mathbf{x}, r))} \\
= & \lim _{r \rightarrow 0} \frac{m_{n}(\mathbf{h}(B(\mathbf{x}, r)))}{m_{n}(\mathbf{h}(B(\mathbf{x}, r) \cap \Omega))} \frac{m_{n}(\mathbf{h}(B(\mathbf{x}, r) \cap \Omega))}{m_{n}(B(\mathbf{x}, r))} \\
= & \lim _{r \rightarrow 0} \frac{1}{m_{n}(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r) \cap \Omega} J(\mathbf{y}) d m_{n} \\
= & \lim _{r \rightarrow 0} \frac{1}{m_{n}(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} J(\mathbf{y}) d m_{n}=J(\mathbf{x}) .
\end{aligned}
$$

the last equality because $J$ was extended to be zero off $\Omega$. This proves the lemma.
Here is the change of variables formula for Lipschitz mappings. It is a special case of the area formula.

Theorem 36.6.12 Let $\Omega$ be a Lebesgue measurable set, let $f \geq 0$ be Lebesgue measurable. Then for $\mathbf{h}$ a Lipschitz mapping defined on $\mathbb{R}^{n}$ which is one to one on $\Omega$,

$$
\begin{equation*}
\int_{\mathbf{h}(\Omega)} f(\mathbf{y}) d m_{n}=\int_{\Omega} f(\mathbf{h}(\mathbf{x}))|\operatorname{det} D \mathbf{h}(\mathbf{x})| d m_{n} \tag{36.6.23}
\end{equation*}
$$

Proof: Let $F$ be a Borel set. It follows that $\mathbf{h}^{-1}(F)$ is a Lebesgue measurable set. Therefore, by 36.6.22,

$$
\begin{align*}
& m_{n}\left(\mathbf{h}\left(\mathbf{h}^{-1}(F) \cap \Omega\right)\right)  \tag{36.6.24}\\
= & \int_{\mathbf{h}(\Omega)} \mathscr{X}_{F}(\mathbf{y}) d m_{n}=\int_{\Omega} \mathscr{X}_{\mathbf{h}^{-1}(F)}(\mathbf{x}) J(\mathbf{x}) d m_{n} \\
= & \int_{\Omega} \mathscr{X}_{F}(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) d m_{n} .
\end{align*}
$$

What if $F$ is only Lebesgue measurable? Note there are no measurability problems with the above expression because $\mathbf{x} \rightarrow \mathscr{X}_{F}(\mathbf{h}(\mathbf{x}))$ is Borel measurable due to the assumption that $\mathbf{h}$ is continuous while $J$ is given to be Lebesgue measurable. However, if $F$ is Lebesgue measurable, not necessarily Borel measurable, then it is no longer clear that $\mathbf{x} \rightarrow \mathscr{X}_{F}(\mathbf{h}(\mathbf{x}))$ is measurable. In fact this is not always even true. However, $\mathbf{x} \rightarrow \mathscr{X}_{F}(\mathbf{h}(\mathbf{x})) J(\mathbf{x})$ is measurable and 36.6.24 holds.

Let $F$ be Lebesgue measurable. Then by inner regularity, $F=H \cup N$ where $N$ has measure zero, $H$ is the countable union of compact sets so it is a Borel set, and $H \cap N=\emptyset$. Therefore, letting $N^{\prime}$ denote a Borel set of measure zero which contains $N$,

$$
\begin{aligned}
& b(\mathbf{x}) \equiv \mathscr{X}_{H}(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) \leq \mathscr{X}_{F}(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) \\
= & \mathscr{X}_{H}(\mathbf{h}(\mathbf{x})) J(\mathbf{x})+\mathscr{X}_{N}(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) \\
\leq & \mathscr{X}_{H}(\mathbf{h}(\mathbf{x})) J(\mathbf{x})+\mathscr{X}_{N^{\prime}}(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) \equiv u(\mathbf{x})
\end{aligned}
$$

Now since $N^{\prime}$ is Borel,

$$
\begin{aligned}
& \int_{\Omega}(u(\mathbf{x})-b(\mathbf{x})) d m_{n}=\int_{\Omega} \mathscr{X}_{N^{\prime}}(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) d m_{n} \\
& =m_{n}\left(\mathbf{h}\left(\mathbf{h}^{-1}\left(N^{\prime}\right) \cap \Omega\right)\right)=m_{n}\left(N^{\prime} \cap \mathbf{h}(\Omega)\right)=0
\end{aligned}
$$

and this shows $\mathscr{X}_{H}(\mathbf{h}(\mathbf{x})) J(\mathbf{x})=\mathscr{X}_{F}(\mathbf{h}(\mathbf{x})) J(\mathbf{x})$ except on a set of measure zero. By completeness of Lebesgue measure, it follows $\mathbf{x} \rightarrow \mathscr{X}_{F}(\mathbf{h}(\mathbf{x})) J(\mathbf{x})$ is Lebesgue measurable and also since $\mathbf{h}$ maps sets of measure zero to sets of measure zero,

$$
\begin{aligned}
\int_{\Omega} \mathscr{X}_{F}(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) d m_{n} & =\int_{\Omega} \mathscr{X}_{H}(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) d m_{n} \\
& =\int_{\mathbf{h}(\Omega)} \mathscr{X}_{H}(\mathbf{y}) d m_{n} \\
& =\int_{\mathbf{h}(\Omega)} \mathscr{X}_{F}(\mathbf{y}) d m_{n}
\end{aligned}
$$

It follows that if $s$ is any nonnegative Lebesgue measurable simple function,

$$
\begin{equation*}
\int_{\Omega} s(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) d m_{n}=\int_{\mathbf{h}(\Omega)} s(\mathbf{y}) d m_{n} \tag{36.6.25}
\end{equation*}
$$

and now, if $f \geq 0$ is Lebesgue measurable, let $s_{k}$ be an increasing sequence of Lebesgue measurable simple functions converging pointwise to $f$. Then since 36.6 .25 holds for $s_{k}$, the monotone convergence theorem applies and yields 36.6.23. This proves the theorem.

It turns out that a Lipschitz function defined on some subset of $\mathbb{R}^{n}$ always has a Lipschitz extension to all of $\mathbb{R}^{n}$. The next theorem gives a proof of this. For more on this sort of theorem see Federer [50]. He gives a better but harder theorem than what follows.

Theorem 36.6.13 If $\mathbf{h}: \Omega \rightarrow \mathbb{R}^{m}$ is Lipschitz, then there exists $\overline{\mathbf{h}}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ which extends h and is also Lipschitz.

Proof: It suffices to assume $m=1$ because if this is shown, it may be applied to the components of $\mathbf{h}$ to get the desired result. Suppose

$$
\begin{equation*}
|h(\mathbf{x})-h(\mathbf{y})| \leq K|\mathbf{x}-\mathbf{y}| \tag{36.6.26}
\end{equation*}
$$

Define

$$
\begin{equation*}
\bar{h}(\mathbf{x}) \equiv \inf \{h(\mathbf{w})+K|\mathbf{x}-\mathbf{w}|: \mathbf{w} \in \Omega\} \tag{36.6.27}
\end{equation*}
$$

If $\mathbf{x} \in \Omega$, then for all $\mathbf{w} \in \Omega$,

$$
h(\mathbf{w})+K|\mathbf{x}-\mathbf{w}| \geq h(\mathbf{x})
$$

by 36.6.26. This shows $h(\mathbf{x}) \leqq \bar{h}(\mathbf{x})$. But also you could take $\mathbf{w}=\mathbf{x}$ in 36.6.27 which yields $\bar{h}(\mathbf{x}) \leq h(\mathbf{x})$. Therefore $\bar{h}(\mathbf{x})=h(\mathbf{x})$ if $\mathbf{x} \in \Omega$.

Now suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n}$ and consider $|\bar{h}(\mathbf{x})-\bar{h}(\mathbf{y})|$. Without loss of generality assume $\bar{h}(\mathbf{x}) \geq \bar{h}(\mathbf{y})$. (If not, repeat the following argument with $\mathbf{x}$ and $\mathbf{y}$ interchanged.) Pick $\mathbf{w} \in \Omega$ such that

$$
h(\mathbf{w})+K|\mathbf{y}-\mathbf{w}|-\varepsilon<\bar{h}(\mathbf{y})
$$

Then

$$
\begin{aligned}
|\bar{h}(\mathbf{x})-\bar{h}(\mathbf{y})|=\bar{h}(\mathbf{x})-\bar{h}(\mathbf{y}) & \leq h(\mathbf{w})+K|\mathbf{x}-\mathbf{w}|- \\
{[h(\mathbf{w})+K|\mathbf{y}-\mathbf{w}|-\varepsilon] } & \leq K|\mathbf{x}-\mathbf{y}|+\varepsilon .
\end{aligned}
$$

Since $\varepsilon$ is arbitrary,

$$
|\bar{h}(\mathbf{x})-\bar{h}(\mathbf{y})| \leq K|\mathbf{x}-\mathbf{y}|
$$

and this proves the theorem.
This yields a simple corollary to Theorem 36.6.12.
Corollary 36.6.14 Let $\mathbf{h}: \Omega \rightarrow \mathbb{R}^{n}$ be Lipschitz continuous and one to one where $\Omega$ is a Lebesgue measurable set. Then if $f \geq 0$ is Lebesgue measurable,

$$
\begin{equation*}
\int_{\mathbf{h}(\Omega)} f(\mathbf{y}) d m_{n}=\int_{\Omega} f(\mathbf{h}(\mathbf{x}))|\operatorname{det} D \overline{\mathbf{h}}(\mathbf{x})| d m_{n} \tag{36.6.28}
\end{equation*}
$$

where $\overline{\mathbf{h}}$ denotes a Lipschitz extension of $\mathbf{h}$.

## Chapter 37

## Integration On Manifolds

You can do integration on various manifolds by using the Hausdorff measure of an appropriate dimension. However, it is possible to discuss this through the use of the Riesz representation theorem and some of the machinery for accomplishing this is interesting for its own sake so I will present this alternate point of view.

### 37.1 Partitions Of Unity

This material has already been mostly discussed starting on Page 1023. However, that was a long time ago and it seems like it might be good to go over it again and so, for the sake of convenience, here it is again.

Definition 37.1.1 Let $\mathfrak{C}$ be a set whose elements are subsets of $\mathbb{R}^{n} .{ }^{1}$ Then $\mathfrak{C}$ is said to be locally finite if for every $\mathbf{x} \in \mathbb{R}^{n}$, there exists an open set, $U_{\mathbf{x}}$ containing $\mathbf{x}$ such that $U_{\mathbf{x}}$ has nonempty intersection with only finitely many sets of $\mathfrak{C}$.

Lemma 37.1.2 Let $\mathfrak{C}$ be a set whose elements are open subsets of $\mathbb{R}^{n}$ and suppose $\cup \mathfrak{C} \supseteq H$, a closed set. Then there exists a countable list of open sets, $\left\{U_{i}\right\}_{i=1}^{\infty}$ such that each $U_{i}$ is bounded, each $U_{i}$ is a subset of some set of $\mathfrak{C}$, and $\cup_{i=1}^{\infty} U_{i} \supseteq H$.

Proof: Let $W_{k} \equiv B(\mathbf{0}, k), W_{0}=W_{-1}=\emptyset$. For each $\mathbf{x} \in H \cap \overline{W_{k}}$ there exists an open set, $U_{\mathbf{x}}$ such that $U_{\mathbf{x}}$ is a subset of some set of $\mathfrak{C}$ and $U_{\mathbf{x}} \subseteq W_{k+1} \backslash \overline{W_{k-1}}$. Then since $H \cap \overline{W_{k}}$ is compact, there exist finitely many of these sets, $\left\{U_{i}^{k}\right\}_{i=1}^{m(k)}$ whose union contains $H \cap \overline{W_{k}}$. If $H \cap \overline{W_{k}}=\emptyset$, let $m(k)=0$ and there are no such sets obtained.The desired countable list of open sets is $\cup_{k=1}^{\infty}\left\{U_{i}^{k}\right\}_{i=1}^{m(k)}$. Each open set in this list is bounded. Furthermore, if $\mathbf{x} \in \mathbb{R}^{n}$, then $\mathbf{x} \in W_{k}$ where $k$ is the first positive integer with $\mathbf{x} \in W_{k}$. Then $W_{k} \backslash \overline{W_{k-1}}$ is an open set containing $\mathbf{x}$ and this open set can have nonempty intersection only with with a set of $\left\{U_{i}^{k}\right\}_{i=1}^{m(k)} \cup\left\{U_{i}^{k-1}\right\}_{i=1}^{m(k-1)}$, a finite list of sets. Therefore, $\cup_{k=1}^{\infty}\left\{U_{i}^{k}\right\}_{i=1}^{m(k)}$ is locally finite.

The set, $\left\{U_{i}\right\}_{i=1}^{\infty}$ is said to be a locally finite cover of $H$. The following lemma gives some important reasons why a locally finite list of sets is so significant. First of all consider the rational numbers, $\left\{r_{i}\right\}_{i=1}^{\infty}$ each rational number is a closed set.

$$
\mathbb{Q}=\left\{r_{i}\right\}_{i=1}^{\infty}=\cup_{i=1}^{\infty} \overline{\left\{r_{i}\right\}} \neq \overline{\cup_{i=1}^{\infty}\left\{r_{i}\right\}}=\mathbb{R}
$$

The set of rational numbers is definitely not locally finite.
Lemma 37.1.3 Let $\mathfrak{C}$ be locally finite. Then

$$
\overline{\cup \mathfrak{C}}=\cup\{\bar{H}: H \in \mathfrak{C}\}
$$

Next suppose the elements of $\mathfrak{C}$ are open sets and that for each $U \in \mathfrak{C}$, there exists a differentiable function, $\psi_{U}$ having $\operatorname{spt}\left(\psi_{U}\right) \subseteq U$. Then you can define the following finite sum for each $\mathbf{x} \in \mathbb{R}^{n}$

$$
f(\mathbf{x}) \equiv \sum\left\{\psi_{U}(\mathbf{x}): \mathbf{x} \in U \in \mathfrak{C}\right\}
$$

[^25]Furthermore, $f$ is also a differentiable function ${ }^{2}$ and

$$
D f(\mathbf{x})=\sum\left\{D \psi_{U}(\mathbf{x}): \mathbf{x} \in U \in \mathfrak{C}\right\} .
$$

Proof: Let $\mathbf{p}$ be a limit point of $\cup \mathfrak{C}$ and let $W$ be an open set which intersects only finitely many sets of $\mathfrak{C}$. Then $\mathbf{p}$ must be a limit point of one of these sets. It follows $\mathbf{p} \in \cup\{\bar{H}: H \in \mathfrak{C}\}$ and so $\overline{\cup C} \subseteq \cup\{\bar{H}: H \in \mathfrak{C}\}$. The inclusion in the other direction is obvious.

Now consider the second assertion. Letting $\mathbf{x} \in \mathbb{R}^{n}$, there exists an open set, $W$ intersecting only finitely many open sets of $\mathfrak{C}, U_{1}, U_{2}, \cdots, U_{m}$. Then for all $\mathbf{y} \in W$,

$$
f(\mathbf{y})=\sum_{i=1}^{m} \psi_{U_{i}}(\mathbf{y})
$$

and so the desired result is obvious. It merely says that a finite sum of differentiable functions is differentiable. Recall the following definition.

Definition 37.1.4 Let $K$ be a closed subset of an open set, $U . K \prec f \prec U$ if $f$ is continuous, has values in $[0,1]$, equals 1 on $K$, and has compact support contained in $U$.

Lemma 37.1.5 Let $U$ be a bounded open set and let $K$ be a closed subset of $U$. Then there exist an open set, $W$, such that $W \subseteq \bar{W} \subseteq U$ and a function, $f \in C_{c}^{\infty}(U)$ such that $K \prec f \prec U$.

Proof: The set, $K$ is compact so is at a positive distance from $U^{C}$. Let

$$
W \equiv\left\{\mathbf{x}: \operatorname{dist}(\mathbf{x}, K)<3^{-1} \operatorname{dist}\left(K, U^{C}\right)\right\}
$$

Also let

$$
W_{1} \equiv\left\{\mathbf{x}: \operatorname{dist}(\mathbf{x}, K)<2^{-1} \operatorname{dist}\left(K, U^{C}\right)\right\}
$$

Then it is clear

$$
K \subseteq W \subseteq \bar{W} \subseteq W_{1} \subseteq \overline{W_{1}} \subseteq U
$$

Now consider the function,

$$
h(\mathbf{x}) \equiv \frac{\operatorname{dist}\left(\mathbf{x}, W_{1}^{C}\right)}{\operatorname{dist}\left(\mathbf{x}, W_{1}^{C}\right)+\operatorname{dist}(\mathbf{x}, \bar{W})}
$$

Since $\bar{W}$ is compact it is at a positive distance from $W_{1}^{C}$ and so $h$ is a well defined continuous function which has compact support contained in $\bar{W}_{1}$, equals 1 on $W$, and has values in $[0,1]$. Now let $\phi_{k}$ be a mollifier. Letting

$$
k^{-1}<\min \left(\operatorname{dist}\left(K, W^{C}\right), 2^{-1} \operatorname{dist}\left(\bar{W}_{1}, U^{C}\right)\right)
$$

it follows that for such $k$, the function, $h * \phi_{k} \in C_{c}^{\infty}(U)$, has values in $[0,1]$, and equals 1 on $K$. Let $f=h * \phi_{k}$.

The above lemma is used repeatedly in the following.

[^26]Lemma 37.1.6 Let $K$ be a closed set and let $\left\{V_{i}\right\}_{i=1}^{\infty}$ be a locally finite list of bounded open sets whose union contains $K$. Then there exist functions, $\psi_{i} \in C_{c}^{\infty}\left(V_{i}\right)$ such that for all $\mathbf{x} \in K$,

$$
1=\sum_{i=1}^{\infty} \psi_{i}(\mathbf{x})
$$

and the function $f(\mathbf{x})$ given by

$$
f(\mathbf{x})=\sum_{i=1}^{\infty} \psi_{i}(\mathbf{x})
$$

is in $C^{\infty}\left(\mathbb{R}^{n}\right)$.
Proof: Let $K_{1}=K \backslash \cup_{i=2}^{\infty} V_{i}$. Thus $K_{1}$ is compact because $K_{1} \subseteq V_{1}$. Let

$$
K_{1} \subseteq W_{1} \subseteq \bar{W}_{1} \subseteq V_{1}
$$

Thus $W_{1}, V_{2}, \cdots, V_{n}$ covers $K$ and $\bar{W}_{1} \subseteq V_{1}$. Suppose $W_{1}, \cdots, W_{r}$ have been defined such that $\overline{W_{i}} \subseteq V_{i}$ for each $i$, and $W_{1}, \cdots, W_{r}, V_{r+1}, \cdots, V_{n}$ covers $K$. Then let

$$
K_{r+1} \equiv K \backslash\left(\left(\cup_{i=r+2}^{\infty} V_{i}\right) \cup\left(\cup_{j=1}^{r} W_{j}\right)\right)
$$

It follows $K_{r+1}$ is compact because $K_{r+1} \subseteq V_{r+1}$. Let $W_{r+1}$ satisfy

$$
K_{r+1} \subseteq W_{r+1} \subseteq \bar{W}_{r+1} \subseteq V_{r+1}
$$

Continuing this way defines a sequence of open sets, $\left\{W_{i}\right\}_{i=1}^{\infty}$ with the property

$$
\overline{W_{i}} \subseteq V_{i}, K \subseteq \cup_{i=1}^{\infty} W_{i}
$$

Note $\left\{W_{i}\right\}_{i=1}^{\infty}$ is locally finite because the original list, $\left\{V_{i}\right\}_{i=1}^{\infty}$ was locally finite. Now let $U_{i}$ be open sets which satisfy

$$
\bar{W}_{i} \subseteq U_{i} \subseteq \bar{U}_{i} \subseteq V_{i}
$$

Similarly, $\left\{U_{i}\right\}_{i=1}^{\infty}$ is locally finite.


Since the set, $\left\{W_{i}\right\}_{i=1}^{\infty}$ is locally finite, it follows $\overline{\cup_{i=1}^{\infty} W_{i}}=\cup_{i=1}^{\infty} \overline{W_{i}}$ and so it is possible to define $\phi_{i}$ and $\gamma$, infinitely differentiable functions having compact support such that

$$
\bar{U}_{i} \prec \phi_{i} \prec V_{i}, \cup_{i=1}^{\infty} \bar{W}_{i} \prec \gamma \prec \cup_{i=1}^{\infty} U_{i} .
$$

Now define

$$
\psi_{i}(\mathbf{x})=\left\{\begin{array}{l}
\gamma(\mathbf{x}) \phi_{i}(\mathbf{x}) / \sum_{j=1}^{\infty} \phi_{j}(\mathbf{x}) \text { if } \sum_{j=1}^{\infty} \phi_{j}(\mathbf{x}) \neq 0 \\
0 \text { if } \sum_{j=1}^{\infty} \phi_{j}(\mathbf{x})=0
\end{array}\right.
$$

If $\mathbf{x}$ is such that $\sum_{j=1}^{\infty} \phi_{j}(\mathbf{x})=0$, then $\mathbf{x} \notin \cup_{i=1}^{\infty} \overline{U_{i}}$ because $\phi_{i}$ equals one on $\overline{U_{i}}$. Consequently $\gamma(\mathbf{y})=0$ for all $\mathbf{y}$ near $\mathbf{x}$ thanks to the fact that $\cup_{i=1}^{\infty} \overline{U_{i}}$ is closed and so $\psi_{i}(\mathbf{y})=0$ for all $\mathbf{y}$ near $\mathbf{x}$. Hence $\psi_{i}$ is infinitely differentiable at such $\mathbf{x}$. If $\sum_{j=1}^{\infty} \phi_{j}(\mathbf{x}) \neq 0$, this situation persists near $\mathbf{x}$ because each $\phi_{j}$ is continuous and so $\psi_{i}$ is infinitely differentiable at such points also thanks to Lemma 37.1.3. Therefore $\psi_{i}$ is infinitely differentiable. If $\mathbf{x} \in K$, then $\gamma(\mathbf{x})=1$ and so $\sum_{j=1}^{\infty} \psi_{j}(\mathbf{x})=1$. Clearly $0 \leq \psi_{i}(\mathbf{x}) \leq 1$ and $\operatorname{spt}\left(\psi_{j}\right) \subseteq V_{j}$. This proves the theorem.

The method of proof of this lemma easily implies the following useful corollary.
Corollary 37.1.7 If $H$ is a compact subset of $V_{i}$ for some $V_{i}$ there exists a partition of unity such that $\psi_{i}(x)=1$ for all $x \in H$ in addition to the conclusion of Lemma 37.1.6.

Proof: Keep $V_{i}$ the same but replace $V_{j}$ with $\widetilde{V}_{j} \equiv V_{j} \backslash H$. Now in the proof above, applied to this modified collection of open sets, if $j \neq i, \phi_{j}(x)=0$ whenever $x \in H$. Therefore, $\psi_{i}(x)=1$ on $H$.

Theorem 37.1.8 Let $H$ be any closed set and let $\mathfrak{C}$ be any open cover of $H$. Then there exist functions $\left\{\psi_{i}\right\}_{i=1}^{\infty}$ such that $\operatorname{spt}\left(\psi_{i}\right)$ is contained in some set of $\mathfrak{C}$ and $\psi_{i}$ is infinitely differentiable having values in $[0,1]$ such that on $H, \sum_{i=1}^{\infty} \psi_{i}(\mathbf{x})=1$. Furthermore, the function, $f(\mathbf{x}) \equiv \sum_{i=1}^{\infty} \psi_{i}(\mathbf{x})$ is infinitely differentiable on $\mathbb{R}^{n}$. Also, $\operatorname{spt}\left(\psi_{i}\right) \subseteq U_{i}$ where $U_{i}$ is a bounded open set with the property that $\left\{U_{i}\right\}_{i=1}^{\infty}$ is locally finite and each $U_{i}$ is contained in some set of $\mathfrak{C}$.

Proof: By Lemma 37.1.2 there exists an open cover of $H$ composed of bounded open sets, $U_{i}$ such that each $U_{i}$ is a subset of some set of $\mathfrak{C}$ and the collection, $\left\{U_{i}\right\}_{i=1}^{\infty}$ is locally finite. Then the result follows from Lemma 37.1.6 and Lemma 37.1.3.

Corollary 37.1.9 Let $H$ be any closed set and let $\left\{V_{i}\right\}_{i=1}^{m}$ be a finite open cover of H. Then there exist functions $\left\{\phi_{i}\right\}_{i=1}^{m}$ such that $\operatorname{spt}\left(\phi_{i}\right) \subseteq V_{i}$ and $\phi_{i}$ is infinitely differentiable having values in $[0,1]$ such that on $H, \sum_{i=1}^{m} \phi_{i}(\mathbf{x})=1$.

Proof: By Theorem 37.1.8 there exists a set of functions, $\left\{\psi_{i}\right\}_{i=1}^{\infty}$ having the properties listed in this theorem relative to the open covering, $\left\{V_{i}\right\}_{i=1}^{m}$. Let $\phi_{1}(\mathbf{x})$ equal the sum of all $\psi_{j}(\mathbf{x})$ such that $\operatorname{spt}\left(\psi_{j}\right) \subseteq V_{1}$. Next let $\phi_{2}(\mathbf{x})$ equal the sum of all $\psi_{j}(\mathbf{x})$ which have not already been included and for which $\operatorname{spt}\left(\psi_{j}\right) \subseteq V_{2}$. Continue in this manner. Since the open sets, $\left\{U_{i}\right\}_{i=1}^{\infty}$ mentioned in Theorem 37.1.8 are locally finite, it follows from Lemma 37.1.3 that each $\phi_{i}$ is infinitely differentiable having support in $V_{i}$. This proves the corollary.

### 37.2 Integration On Manifolds

Manifolds are things which locally appear to be $\mathbb{R}^{n}$ for some $n$. The extent to which they have such a local appearance varies according to various analytical characteristics which the manifold possesses.

Definition 37.2.1 Let $U \subseteq \mathbb{R}^{n}$ be an open set and let $\mathbf{h}: U \rightarrow \mathbb{R}^{m}$. Then for $r \in[0,1)$, $\mathbf{h} \in C^{k, r}(U)$ for $k$ a nonnegative integer means that $D^{\alpha} \mathbf{h}$ exists for all $|\alpha| \leq k$ and each $D^{\alpha} \mathbf{h}$ is Holder continuous with exponent $r$. That is

$$
\left|D^{\alpha} \mathbf{h}(\mathbf{x})-D^{\alpha} \mathbf{h}(\mathbf{y})\right| \leq K|\mathbf{x}-\mathbf{y}|^{r} .
$$

Also $\mathbf{h} \in C^{k, r}(\bar{U})$ if it is the restriction of a function of $C^{k, r}\left(\mathbb{R}^{n}\right)$ to $U$.
Definition 37.2.2 Let $\Gamma$ be a closed subset of $\mathbb{R}^{p}$ where $p \geq n$. Suppose $\Gamma=\cup_{i=1}^{\infty} \Gamma_{i}$ where $\Gamma_{i}=\Gamma \cap W_{i}$ for $W_{i}$ a bounded open set. Suppose also $\left\{W_{i}\right\}_{i=1}^{\infty}$ is locally finite. This means every bounded open set intersects only finitely many. Also suppose there are open bounded sets, $U_{i}$ having Lipschitz boundaries and functions $\mathbf{h}_{i}: U_{i} \rightarrow \Gamma_{i}$ which are one to one, onto, and in $C^{m, 1}\left(U_{i}\right)$. Suppose also there exist functions, $\mathbf{g}_{i}: W_{i} \rightarrow U_{i}$ such that $\mathbf{g}_{i}$ is $C^{m, 1}\left(W_{i}\right)$, and $\mathbf{g}_{i} \circ \mathbf{h}_{i}=\mathrm{id}$ on $U_{i}$ while $\mathbf{h}_{i} \circ \mathbf{g}_{i}=\mathrm{id}$ on $\Gamma_{i}$. The collection of sets, $\Gamma_{j}$ and mappings, $\mathbf{g}_{j},\left\{\left(\Gamma_{j}, \mathbf{g}_{j}\right)\right\}$ is called an atlas and an individual entry in the atlas is called a chart. Thus $\left(\Gamma_{j}, \mathbf{g}_{j}\right)$ is a chart. Then $\Gamma$ as just described is called a $C^{m, 1}$ manifold. The number, $m$ is just a nonnegative integer. When $m=0$ this would be called a Lipschitz manifold, the least smooth of the manifolds discussed here.

For example, take $p=n+1$ and let

$$
\mathbf{h}_{i}(\mathbf{u})=\left(u_{1}, \cdots, u_{i}, \phi_{i}(\mathbf{u}), u_{i+1}, \cdots, u_{n}\right)^{T}
$$

for $\mathbf{u}=\left(u_{1}, \cdots, u_{i}, u_{i+1}, \cdots, u_{n}\right)^{T} \in U_{i}$ for $\phi_{i} \in C^{m, 1}\left(U_{i}\right)$ and $\mathbf{g}_{i}: U_{i} \times \mathbb{R} \rightarrow U_{i}$ given by

$$
\mathbf{g}_{i}\left(u_{1}, \cdots, u_{i}, y, u_{i+1}, \cdots, u_{n}\right) \equiv \mathbf{u}
$$

for $i=1,2, \cdots, p$. Then for $\mathbf{u} \in U_{i}$, the definition gives

$$
\mathbf{g}_{i} \circ \mathbf{h}_{i}(\mathbf{u})=\mathbf{g}_{i}\left(u_{1}, \cdots, u_{i}, \phi_{i}(\mathbf{u}), u_{i+1}, \cdots, u_{n}\right)=\mathbf{u}
$$

and for $\Gamma_{i} \equiv \mathbf{h}_{i}\left(U_{i}\right)$ and $\left(u_{1}, \cdots, u_{i}, \phi_{i}(\mathbf{u}), u_{i+1}, \cdots, u_{n}\right)^{T} \in \Gamma_{i}$,

$$
\begin{aligned}
& \mathbf{h}_{i} \circ \mathbf{g}_{i}\left(u_{1}, \cdots, u_{i}, \phi_{i}(\mathbf{u}), u_{i+1}, \cdots, u_{n}\right) \\
= & \mathbf{h}_{i}(\mathbf{u})=\left(u_{1}, \cdots, u_{i}, \phi_{i}(\mathbf{u}), u_{i+1}, \cdots, u_{n}\right)^{T}
\end{aligned}
$$

This example can be used to describe the boundary of a bounded open set and since $\phi_{i} \in$ $C^{m, 1}\left(U_{i}\right)$, such an open set is said to have a $C^{m, 1}$ boundary. Note also that in this example, $U_{i}$ could be taken to be $\mathbb{R}^{n}$ or if $U_{i}$ is given, both $\mathbf{h}_{i}$ and and $\mathbf{g}_{i}$ can be taken as restrictions of functions defined on all of $\mathbb{R}^{n}$ and $\mathbb{R}^{p}$ respectively.

The symbol, $I$ will refer to an increasing list of $n$ indices taken from $\{1, \cdots, p\}$. Denote by $\Lambda(p, n)$ the set of all such increasing lists of $n$ indices.

Let

$$
J_{i}(\mathbf{u}) \equiv\left[\sum_{I \in \Lambda(p, n)}\left(\frac{\partial\left(x^{i_{1}} \cdots x^{i_{n}}\right)}{\partial\left(u^{1} \cdots u^{n}\right)}\right)^{2}\right]^{1 / 2}
$$

where here the sum is taken over all possible increasing lists of $n$ indices, $I$, from $\{1, \cdots, p\}$ and $\mathbf{x}=\mathbf{h}_{i} \mathbf{u}$. Thus there are $\binom{p}{n}$ terms in the sum. In this formula, $\frac{\partial\left(x^{i} \ldots x^{i n}\right)}{\partial\left(u^{1} \ldots u^{n}\right)}$ is defined to be the determinant of the following matrix.

$$
\left(\begin{array}{ccc}
\frac{\partial x^{i_{1}}}{\partial u_{1}} & \cdots & \frac{\partial x^{i_{1}}}{\partial u_{n}} \\
\vdots & & \vdots \\
\frac{\partial x^{i_{n}}}{\partial u_{1}} & \cdots & \frac{\partial i^{i_{n}}}{\partial u_{n}}
\end{array}\right) .
$$

Note that if $p=n$ there is only one term in the sum, the absolute value of the determinant of $D \mathbf{x}(\mathbf{u})$. Define a positive linear functional, $\Lambda$ on $C_{c}(\Gamma)$ as follows: First let $\left\{\psi_{i}\right\}$ be a $C^{\infty}$ partition of unity subordinate to the open sets, $\left\{W_{i}\right\}$. Thus $\psi_{i} \in C_{c}^{\infty}\left(W_{i}\right)$ and $\sum_{i} \psi_{i}(\mathbf{x})=1$ for all $\mathbf{x} \in \Gamma$. Then

$$
\begin{equation*}
\Lambda f \equiv \sum_{i=1}^{\infty} \int_{\mathbf{g}_{i} \Gamma_{i}} f \psi_{i}\left(\mathbf{h}_{i}(\mathbf{u})\right) J_{i}(\mathbf{u}) d u \tag{37.2.1}
\end{equation*}
$$

Is this well defined?
Lemma 37.2.3 The functional defined in 37.2. 1 does not depend on the choice of atlas or the partition of unity.

Proof: In 37.2.1, let $\left\{\psi_{i}\right\}$ be a $C^{\infty}$ partition of unity which is associated with the atlas $\left(\Gamma_{i}, \mathbf{g}_{i}\right)$ and let $\left\{\eta_{i}\right\}$ be a $C^{\infty}$ partition of unity associated in the same manner with the atlas $\left(\Gamma_{i}^{\prime}, \mathbf{g}_{i}^{\prime}\right)$. In the following argument, the local finiteness of the $\Gamma_{i}$ implies that all sums are finite. Using the change of variables formula with $\mathbf{u}=\left(\mathbf{g}_{i} \circ \mathbf{h}_{j}^{\prime}\right) \mathbf{v}$

$$
\begin{gather*}
\sum_{i=1}^{\infty} \int_{\mathbf{g}_{i} \Gamma_{i}} \psi_{i} f\left(\mathbf{h}_{i}(\mathbf{u})\right) J_{i}(\mathbf{u}) d u=  \tag{37.2.2}\\
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{\mathbf{g}_{i} \Gamma_{i}} \eta_{j} \psi_{i} f\left(\mathbf{h}_{i}(\mathbf{u})\right) J_{i}(\mathbf{u}) d u=\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{\mathbf{g}_{j}^{\prime}\left(\Gamma_{i} \cap \Gamma_{j}^{\prime}\right)} . \\
\eta_{j}\left(\mathbf{h}_{j}^{\prime}(\mathbf{v})\right) \psi_{i}\left(\mathbf{h}_{j}^{\prime}(\mathbf{v})\right) f\left(\mathbf{h}_{j}^{\prime}(\mathbf{v})\right) J_{i}(\mathbf{u})\left|\frac{\partial\left(u^{1} \cdots u^{n}\right)}{\partial\left(v^{1} \cdots v^{n}\right)}\right| d v \\
=\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{\mathbf{g}_{j}^{\prime}\left(\Gamma_{i} \cap \Gamma_{j}^{\prime}\right)} \eta_{j}\left(\mathbf{h}_{j}^{\prime}(\mathbf{v})\right) \psi_{i}\left(\mathbf{h}_{j}^{\prime}(\mathbf{v})\right) f\left(\mathbf{h}_{j}^{\prime}(\mathbf{v})\right) J_{j}(\mathbf{v}) d v . \tag{37.2.3}
\end{gather*}
$$

Thus

$$
\begin{gathered}
\text { the definition of } \Lambda f \text { using }\left(\Gamma_{i}, \mathbf{g}_{i}\right) \equiv \\
\sum_{i=1}^{\infty} \int_{\mathbf{g}_{i} \Gamma_{i}} \psi_{i} f\left(\mathbf{h}_{i}(\mathbf{u})\right) J_{i}(\mathbf{u}) d u= \\
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{\mathbf{g}_{j}^{\prime}\left(\Gamma_{i} \cap \Gamma_{j}^{\prime}\right)} \eta_{j}\left(\mathbf{h}_{j}^{\prime}(\mathbf{v})\right) \psi_{i}\left(\mathbf{h}_{j}^{\prime}(\mathbf{v})\right) f\left(\mathbf{h}_{j}^{\prime}(\mathbf{v})\right) J_{j}(\mathbf{v}) d v
\end{gathered}
$$

$$
=\sum_{j=1}^{\infty} \int_{\mathbf{g}_{j}^{\prime}\left(\Gamma_{j}^{\prime}\right)} \eta_{j}\left(\mathbf{h}_{j}^{\prime}(\mathbf{v})\right) f\left(\mathbf{h}_{j}^{\prime}(\mathbf{v})\right) J_{j}(\mathbf{v}) d v
$$

the definition of $\Lambda f$ using $\left(V_{i}, \mathbf{g}_{i}^{\prime}\right)$.
This proves the lemma.
This lemma and the Riesz representation theorem for positive linear functionals implies the part of the following theorem which says the functional is well defined.

Theorem 37.2.4 Let $\Gamma$ be a $C^{m, 1}$ manifold. Then there exists a unique Radon measure, $\mu$, defined on $\Gamma$ such that whenever $f$ is a continuous function having compact support which is defined on $\Gamma$ and $\left(\Gamma_{i}, \mathbf{g}_{i}\right)$ denotes an atlas and $\left\{\psi_{i}\right\}$ a partition of unity subordinate to this atlas,

$$
\begin{equation*}
\Lambda f=\int_{\Gamma} f d \mu=\sum_{i=1}^{\infty} \int_{\mathbf{g}_{i} \Gamma_{i}} \psi_{i} f\left(\mathbf{h}_{i}(\mathbf{u})\right) J_{i}(\mathbf{u}) d u \tag{37.2.4}
\end{equation*}
$$

Also, a subset, $A$, of $\Gamma$ is $\mu$ measurable if and only if for all $r, \mathbf{g}_{r}\left(\Gamma_{r} \cap A\right)$ is $v_{r}$ measurable where $v_{r}$ is the measure defined by

$$
v_{r}\left(\mathbf{g}_{r}\left(\Gamma_{r} \cap A\right)\right) \equiv \int_{\mathbf{g}_{r}\left(\Gamma_{r} \cap A\right)} J_{r}(\mathbf{u}) d u
$$

Proof: To begin, here is a claim.
Claim : A set, $S \subseteq \Gamma_{i}$, has $\mu$ measure zero if and only if $\mathbf{g}_{i} S$ has measure zero in $\mathbf{g}_{i} \Gamma_{i}$ with respect to the measure, $v_{i}$.

Proof of the claim: Let $\varepsilon>0$ be given. By outer regularity, there exists a set, $V \subseteq \Gamma_{i}$, open $^{3}$ in $\Gamma$ such that $\mu(V)<\varepsilon$ and $S \subseteq V \subseteq \Gamma_{i}$. Then $\mathbf{g}_{i} V$ is open in $\mathbb{R}^{n}$ and contains $\mathbf{g}_{i} S$. Letting $h \prec \mathbf{g}_{i} V$ and $h_{1}(\mathbf{x}) \equiv h\left(\mathbf{g}_{i}(\mathbf{x})\right)$ for $\mathbf{x} \in \Gamma_{i}$ it follows $h_{1} \prec V$. By Corollary 37.1.7 on Page 1300 there exists a partition of unity such that $\operatorname{spt}\left(h_{1}\right) \subseteq\left\{\mathbf{x} \in \mathbb{R}^{p}: \psi_{i}(\mathbf{x})=1\right\}$. Thus $\psi_{j} h_{1}\left(\mathbf{h}_{j}(u)\right)=0$ unless $j=i$ when this reduces to $h_{1}\left(\mathbf{h}_{i}(u)\right)$. It follows

$$
\begin{aligned}
\varepsilon & \geq \mu(V) \geq \int_{V} h_{1} d \mu=\int_{\Gamma} h_{1} d \mu \\
& =\sum_{j=1}^{\infty} \int_{\mathbf{g}_{j} \Gamma_{j}} \psi_{j} h_{1}\left(\mathbf{h}_{j}(\mathbf{u})\right) J_{j}(\mathbf{u}) d u \\
& =\int_{\mathbf{g}_{i} \Gamma_{i}} h_{1}\left(\mathbf{h}_{i}(\mathbf{u})\right) J_{i}(\mathbf{u}) d u=\int_{\mathbf{g}_{i} \Gamma_{i}} h(\mathbf{u}) J_{i}(\mathbf{u}) d u \\
& =\int_{\mathbf{g}_{i} V} h(\mathbf{u}) J_{i}(\mathbf{u}) d u
\end{aligned}
$$

Now this holds for all $h \prec \mathbf{g}_{i} V$ and so

$$
\int_{\mathbf{g}_{i} V} J_{i}(\mathbf{u}) d u \leq \varepsilon
$$

[^27]Since $\varepsilon$ is arbitrary, this shows $\mathbf{g}_{i} V$ has measure no more than $\varepsilon$ with respect to the measure, $v_{i}$. Since $\varepsilon$ is arbitrary, $\mathbf{g}_{i} S$ has measure zero.

Consider the converse. Suppose $\mathbf{g}_{i} S$ has $v_{i}$ measure zero. Then there exists an open set, $O \subseteq \mathbf{g}_{i} \Gamma_{i}$ such that $O \supseteq \mathbf{g}_{i} S$ and

$$
\int_{O} J_{i}(\mathbf{u}) d u<\varepsilon
$$

Thus $\mathbf{h}_{i}(O)$ is open in $\Gamma$ and contains $S$. Let $h \prec \mathbf{h}_{i}(O)$ be such that

$$
\begin{equation*}
\int_{\Gamma} h d \mu+\varepsilon>\mu\left(\mathbf{h}_{i}(O)\right) \geq \mu(S) \tag{37.2.5}
\end{equation*}
$$

As in the first part, Corollary 37.1 .7 on Page 1300 implies there exists a partition of unity such that $h(\mathbf{x})=0$ off the set,

$$
\left\{\mathbf{x} \in \mathbb{R}^{p}: \psi_{i}(\mathbf{x})=1\right\}
$$

and so as in this part of the argument,

$$
\begin{align*}
\int_{\Gamma} h d \mu & \equiv \sum_{j=1}^{\infty} \int_{\mathbf{g}_{j} U_{j}} \psi_{j} h\left(\mathbf{h}_{j}(\mathbf{u})\right) J_{j}(\mathbf{u}) d u \\
& =\int_{\mathbf{g}_{i} \Gamma_{i}} h\left(\mathbf{h}_{i}(\mathbf{u})\right) J_{i}(\mathbf{u}) d u \\
& =\int_{O \mathbf{g}_{i} \Gamma_{i}} h\left(\mathbf{h}_{i}(\mathbf{u})\right) J_{i}(\mathbf{u}) d u \\
& \leq \int_{O} J_{i}(\mathbf{u}) d u<\varepsilon \tag{37.2.6}
\end{align*}
$$

and so from 37.2 .5 and $37.2 .6 \mu(S) \leq 2 \varepsilon$. Since $\varepsilon$ is arbitrary, this proves the claim.
For the last part of the theorem, it suffices to let $A \subseteq \Gamma_{r}$ because otherwise, the above argument would apply to $A \cap \Gamma_{r}$. Thus let $A \subseteq \Gamma_{r}$ be $\mu$ measurable. By the regularity of the measure, there exists an $F_{\sigma}$ set, $F$ and a $G_{\delta}$ set, $G$ such that $\Gamma_{r} \supseteq G \supseteq A \supseteq F$ and $\mu(G \backslash F)=0$. (Recall a $G_{\delta}$ set is a countable intersection of open sets and an $F_{\sigma}$ set is a countable union of closed sets.) Then since $\overline{\Gamma_{r}}$ is compact, it follows each of the closed sets whose union equals $F$ is a compact set. Thus if $F=\cup_{k=1}^{\infty} F_{k}, \mathbf{g}_{r}\left(F_{k}\right)$ is also a compact set and so $\mathbf{g}_{r}(F)=\cup_{k=1}^{\infty} \mathbf{g}_{r}\left(F_{k}\right)$ is a Borel set. Similarly, $\mathbf{g}_{r}(G)$ is also a Borel set. Now by the claim,

$$
\int_{\mathbf{g}_{r}(G \backslash F)} J_{r}(\mathbf{u}) d u=0
$$

Since $\mathbf{g}_{r}$ is one to one,

$$
\mathbf{g}_{r} G \backslash \mathbf{g}_{r} F=\mathbf{g}_{r}(G \backslash F)
$$

and so

$$
\mathbf{g}_{r}(F) \subseteq \mathbf{g}_{r}(A) \subseteq \mathbf{g}_{r}(G)
$$

where $\mathbf{g}_{r}(G) \backslash \mathbf{g}_{r}(F)$ has measure zero. By completeness of the measure, $v_{r}, \mathbf{g}_{r}(A)$ is measurable. It follows that if $A \subseteq \Gamma$ is $\mu$ measurable, then $\mathbf{g}_{r}\left(\Gamma_{r} \cap A\right)$ is $v_{r}$ measurable for all $r$. The converse is entirely similar. This proves the theorem.

Corollary 37.2.5 Let $f \in L^{1}(\Gamma ; \mu)$ and suppose $f(\mathbf{x})=0$ for all $\mathbf{x} \notin \Gamma_{r}$ where $\left(\Gamma_{r}, \mathbf{g}_{r}\right)$ is a chart. Then

$$
\begin{equation*}
\int_{\Gamma} f d \mu=\int_{\Gamma_{r}} f d \mu=\int_{\mathbf{g}_{r} \Gamma_{r}} f\left(\mathbf{h}_{r}(\mathbf{u})\right) J_{r}(\mathbf{u}) d u \tag{37.2.7}
\end{equation*}
$$

Furthermore, if $\left\{\left(\Gamma_{i}, \mathbf{g}_{i}\right)\right\}$ is an atlas and $\left\{\psi_{i}\right\}$ is a partition of unity as described earlier, then for any $f \in L^{1}(\Gamma, \mu)$,

$$
\begin{equation*}
\int_{\Gamma} f d \mu=\sum_{r=1}^{\infty} \int_{\mathbf{g}_{r} \Gamma_{r}} \psi_{r} f\left(\mathbf{h}_{r}(\mathbf{u})\right) J_{r}(\mathbf{u}) d u \tag{37.2.8}
\end{equation*}
$$

Proof: Let $f \in L^{1}(\Gamma, \mu)$ with $f=0$ off $\Gamma_{r}$. Without loss of generality assume $f \geq 0$ because if the formulas can be established for this case, the same formulas are obtained for an arbitrary complex valued function by splitting it up into positive and negative parts of the real and imaginary parts in the usual way. Also, let $K \subseteq \Gamma_{r}$ a compact set. Since $\mu$ is a Radon measure there exists a sequence of continuous functions, $\left\{f_{k}\right\}, f_{k} \in C_{c}\left(\Gamma_{r}\right)$, which converges to $f$ in $L^{1}(\Gamma, \mu)$ and for $\mu$ a.e. $\mathbf{x}$. Take the partition of unity, $\left\{\psi_{i}\right\}$ to be such that

$$
K \subseteq\left\{\mathbf{x}: \psi_{r}(\mathbf{x})=1\right\}
$$

Therefore, the sequence $\left\{f_{k}\left(\mathbf{h}_{r}(\cdot)\right)\right\}$ is a Cauchy sequence in the sense that

$$
\lim _{k, l \rightarrow \infty} \int_{\mathbf{g}_{r}(K)}\left|f_{k}\left(\mathbf{h}_{r}(\mathbf{u})\right)-f_{l}\left(\mathbf{h}_{r}(\mathbf{u})\right)\right| J_{r}(\mathbf{u}) d u=0
$$

It follows there exists $g$ such that

$$
\int_{\mathbf{g}_{r}(K)}\left|f_{k}\left(\mathbf{h}_{r}(\mathbf{u})\right)-g(\mathbf{u})\right| J_{r}(\mathbf{u}) d u \rightarrow 0
$$

and

$$
g \in L^{1}\left(\mathbf{g}_{r} K ; v_{r}\right)
$$

By the pointwise convergence and the claim used in the proof of Theorem 37.2.4,

$$
g(\mathbf{u})=f\left(\mathbf{h}_{r}(\mathbf{u})\right)
$$

for $\mu$ a.e. $\mathbf{h}_{r}(\mathbf{u}) \in K$. Therefore,

$$
\begin{align*}
\int_{K} f d \mu & =\lim _{k \rightarrow \infty} \int_{K} f_{k} d \mu=\lim _{k \rightarrow \infty} \int_{\mathbf{g}_{r}(K)} f_{k}\left(\mathbf{h}_{r}(\mathbf{u})\right) J_{r}(\mathbf{u}) d u \\
& =\int_{\mathbf{g}_{r}(K)} g(\mathbf{u}) J_{r}(\mathbf{u}) d u=\int_{\mathbf{g}_{r}(K)} f\left(\mathbf{h}_{r}(\mathbf{u})\right) J_{r}(\mathbf{u}) d u \tag{37.2.9}
\end{align*}
$$

Now let $\cdots K_{j} \subseteq K_{j+1} \cdots$ and $\cup_{j=1}^{\infty} K_{j}=\Gamma_{r}$ where $K_{j}$ is compact for all $j$. Replace $K$ in 37.2.9 with $K_{j}$ and take a limit as $j \rightarrow \infty$. By the monotone convergence theorem,

$$
\int_{\Gamma_{r}} f d \mu=\int_{\mathbf{g}_{r}\left(\Gamma_{r}\right)} f\left(\mathbf{h}_{r}(\mathbf{u})\right) J_{r}(\mathbf{u}) d u
$$

This establishes 37.2.7.
To establish 37.2.8, let $f \in L^{1}(\Gamma, \mu)$ and let $\left\{\left(\Gamma_{i}, \mathbf{g}_{i}\right)\right\}$ be an atlas and $\left\{\psi_{i}\right\}$ be a partition of unity. Then $f \psi_{i} \in L^{1}(\Gamma, \mu)$ and is zero off $\Gamma_{i}$. Therefore, from what was just shown,

$$
\begin{aligned}
\int_{\Gamma} f d \mu & =\sum_{i=1}^{\infty} \int_{\Gamma_{i}} f \psi_{i} d \mu \\
& =\sum_{r=1}^{\infty} \int_{\mathbf{g}_{r}\left(\Gamma_{r}\right)} \psi_{r} f\left(\mathbf{h}_{r}(\mathbf{u})\right) J_{r}(\mathbf{u}) d u
\end{aligned}
$$

### 37.3 Comparison With $\mathscr{H}^{n}$

The above gives a measure on a manifold, $\Gamma$. I will now show that the measure obtained is nothing more than $\mathscr{H}^{n}$, the $n$ dimensional Hausdorff measure. Recall $\Lambda(p, n)$ was the set of all increasing lists of $n$ indices taken from $\{1,2, \cdots, p\}$

Recall

$$
J_{i}(\mathbf{u}) \equiv\left[\sum_{I \in \Lambda(p, n)}\left(\frac{\partial\left(x^{i_{1}} \cdots x^{i_{n}}\right)}{\partial\left(u^{1} \cdots u^{n}\right)}\right)^{2}\right]^{1 / 2}
$$

where here the sum is taken over all possible increasing lists of $n$ indices, $I$, from $\{1, \cdots, p\}$ and $\mathbf{x}=\mathbf{h}_{i} \mathbf{u}$ and the functional was given as

$$
\begin{equation*}
\Lambda f \equiv \sum_{i=1}^{\infty} \int_{\mathbf{g}_{i} \Gamma_{i}} f \psi_{i}\left(\mathbf{h}_{i}(\mathbf{u})\right) J_{i}(\mathbf{u}) d u \tag{37.3.10}
\end{equation*}
$$

where the $\left\{\psi_{i}\right\}_{i=1}^{\infty}$ was a partition of unity subordinate to the open sets, $\left\{W_{i}\right\}_{i=1}^{\infty}$ as described above. I will show

$$
J_{i}(\mathbf{u})=\operatorname{det}\left(D \mathbf{h}(\mathbf{u})^{*} D \mathbf{h}(\mathbf{u})\right)^{1 / 2}
$$

and then use the area formula. The key result is really a special case of the Binet Cauchy theorem and this special case is presented in the next lemma.

Lemma 37.3.1 Let $A=\left(a_{i j}\right)$ be a real $p \times n$ matrix in which $p \geq n$. For $I \in \Lambda(p, n)$ denote by $A_{I}$ the $n \times n$ matrix obtained by deleting from $A$ all rows except for those corresponding to an element of I. Then

$$
\sum_{I \in \Lambda(p, n)} \operatorname{det}\left(A_{I}\right)^{2}=\operatorname{det}\left(A^{*} A\right)
$$

Proof: For $\left(j_{1}, \cdots, j_{n}\right) \in \Lambda(p, n)$, define $\theta\left(j_{k}\right) \equiv k$. Then for

$$
\left\{k_{1}, \cdots, k_{n}\right\}=\left\{j_{1}, \cdots, j_{n}\right\}
$$

define

$$
\operatorname{sgn}\left(k_{1}, \cdots, k_{n}\right) \equiv \operatorname{sgn}\left(\theta\left(k_{1}\right), \cdots, \theta\left(k_{n}\right)\right)
$$

Then from the definition of the determinant and matrix multiplication,

$$
\begin{gathered}
\operatorname{det}\left(A^{*} A\right)=\sum_{i_{1}, \cdots, i_{n}} \operatorname{sgn}\left(i_{1}, \cdots, i_{n}\right) \sum_{k_{1}=1}^{p} a_{k_{1} i_{1}} a_{k_{1} 1} \sum_{k_{2}=1}^{p} a_{k_{2} i_{2}} a_{k_{2} 2} \\
\cdots \sum_{k_{n}=1}^{p} a_{k_{n} i_{n}} a_{k_{n} n} \\
=\sum_{J \in \Lambda(p, n)} \sum_{\left\{k_{1}, \cdots, k_{n}\right\}=J i_{1}, \cdots, i_{n}} \operatorname{sgn}\left(i_{1}, \cdots, i_{n}\right) a_{k_{1} i_{1}} a_{k_{1} 1} a_{k_{2} i_{2}} a_{k_{2} 2} \cdots a_{k_{n} i_{n}} a_{k_{n} n} \\
=\sum_{J \in \Lambda(p, n)} \sum_{\left\{k_{1}, \cdots, k_{n}\right\}=J i_{1}, \cdots, i_{n}} \operatorname{sgn}\left(i_{1}, \cdots, i_{n}\right) a_{k_{1} i_{1}} a_{k_{2} i_{2}} \cdots a_{k_{n} i_{n}} \cdot a_{k_{1} 1} a_{k_{2} 2} \cdots a_{k_{n} n} \\
=\sum_{J \in \Lambda(p, n)\left\{k_{1}, \cdots, k_{n}\right\}=J} \operatorname{sgn}\left(k_{1}, \cdots, k_{n}\right) \operatorname{det}\left(A_{J}\right) a_{k_{1} 1} a_{k_{2} 2} \cdots a_{k_{n} n} \\
=\sum_{J \in \Lambda(p, n)} \operatorname{det}\left(A_{J}\right) \operatorname{det}\left(A_{J}\right)
\end{gathered}
$$

and this proves the lemma.
It follows from this lemma that

$$
J_{i}(\mathbf{u})=\operatorname{det}\left(D \mathbf{h}(\mathbf{u})^{*} D \mathbf{h}(\mathbf{u})\right)^{1 / 2}
$$

From 37.3.10 and the area formula, the functional equals

$$
\begin{aligned}
\Lambda f & \equiv \sum_{i=1}^{\infty} \int_{\mathbf{g}_{i} \Gamma_{i}} f \psi_{i}\left(\mathbf{h}_{i}(\mathbf{u})\right) J_{i}(\mathbf{u}) d u \\
& =\sum_{i=1}^{\infty} \int_{\Gamma_{i}} f \psi_{i}(\mathbf{y}) d \mathscr{H}^{n}=\int_{\Gamma} f(\mathbf{y}) d \mathscr{H}^{n}
\end{aligned}
$$

Now $\mathscr{H}^{n}$ is a Borel measure defined on $\Gamma$ which is finite on all compact subsets of $\Gamma$. This finiteness follows from the above formula. If $K$ is a compact subset of $\Gamma$, then there exists an open set, $W$ whose closure is compact and a continuous function with compact support, $f$ such that $K \prec f \prec W$. Then $\mathscr{H}^{n}(K) \leq \int_{\Gamma} f(\mathbf{y}) d \mathscr{H}^{n}<\infty$ because of the above formula.

Lemma 37.3.2 $\mu=\mathscr{H}^{n}$ on every $\mu$ measurable set.
Proof: The Riesz representation theorem shows that

$$
\int_{\Gamma} f d \mu=\int_{\Gamma} f d \mathscr{H}^{n}
$$

for every continuous function having compact support. Therefore, since every open set is the countable union of compact sets, it follows $\mu=\mathscr{H}^{n}$ on all open sets. Since compact sets can be obtained as the countable intersection of open sets, these two measures are also equal on all compact sets. It follows they are also equal on all countable unions of compact sets. Suppose now that $E$ is a $\mu$ measurable set of finite measure. Then there exist sets,
$F, G$ such that $G$ is the countable intersection of open sets each of which has finite measure and $F$ is the countable union of compact sets such that $\mu(G \backslash F)=0$ and $F \subseteq E \subseteq G$. Thus $\mathscr{H}^{n}(G \backslash F)=0$,

$$
\mathscr{H}^{n}(G)=\mu(G)=\mu(F)=\mathscr{H}^{n}(F)
$$

By completeness of $\mathscr{H}^{n}$ it follows $E$ is $\mathscr{H}^{n}$ measurable and $\mathscr{H}^{n}(E)=\mu(E)$. If $E$ is not of finite measure, consider $E_{r} \equiv E \cap B(\mathbf{0}, r)$. This is contained in the compact set $\Gamma \cap \overline{B(\mathbf{0}, r)}$ and so $\mu\left(E_{r}\right)$ if finite. Thus from what was just shown, $\mathscr{H}^{n}\left(E_{r}\right)=\mu\left(E_{r}\right)$ and so, taking $r \rightarrow \infty \mathscr{H}^{n}(E)=\mu(E)$.

This shows you can simply use $\mathscr{H}^{n}$ for the measure on $\Gamma$.

## Chapter 38

## Basic Theory Of Sobolev Spaces

Definition 38.0.1 Let $U$ be an open set of $\mathbb{R}^{n}$. Define $X^{m, p}(U)$ as the set of all functions in $L^{p}(U)$ whose weak partial derivatives up to order $m$ are also in $L^{p}(U)$ where $1 \leq p$. The norm ${ }^{1}$ in this space is given by

$$
\|u\|_{m, p} \equiv\left(\int_{U} \sum_{|\alpha| \leq m}\left|D^{\alpha} u\right|^{p} d x\right)^{1 / p}
$$

where $\alpha=\left(\alpha_{1}, \cdots, \alpha_{n}\right) \in \mathbb{N}^{n}$ and $|\alpha| \equiv \sum \alpha_{i}$.Here $D^{\mathbf{0}} u \equiv u . C^{\infty}(\bar{U})$ is defined to be the set of functions which are restrictions to $U$ of a function in $C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$. Thus $C^{\infty}(\bar{U}) \subseteq W^{m, p}(U)$. The Sobolev space, $W^{m, p}(U)$ is defined to be the closure of $C^{\infty}(\bar{U})$ in $X^{m, p}(U)$ with respect to the above norm. Denote this norm by $\|u\|_{W^{m, p}(U)},\|u\|_{X^{m, p}(U)}$, or $\|u\|_{m, p, U}$ when it is important to identify the open set, $U$.

Also the following notation will be used pretty consistently.
Definition 38.0.2 Let u be a function defined on $U$. Define

$$
\widetilde{u}(\mathbf{x}) \equiv\left\{\begin{array}{c}
u(\mathbf{x}) \text { if } \mathbf{x} \in U \\
0 \text { if } \mathbf{x} \notin U
\end{array}\right.
$$

Theorem 38.0.3 Both $X^{m, p}(U)$ and $W^{m, p}(U)$ are separable reflexive Banach spaces provided $p>1$.

Proof: Define $\Lambda: X^{m, p}(U) \rightarrow L^{p}(U)^{w}$ where $w$ equals the number of multi indices, $\alpha$, such that $|\alpha| \leq m$ as follows. Letting $\left\{\alpha_{i}\right\}_{i=1}^{w}$ be the set of all multi indices with $\alpha_{1}=\mathbf{0}$,

$$
\Lambda(u) \equiv\left(D^{\alpha_{1}} u, D^{\alpha_{2}} u, \cdots, D^{\alpha_{w}} u\right)=\left(u, D^{\alpha_{2}} u, \cdots, D^{\alpha_{w}} u\right)
$$

Then $\Lambda$ is one to one because one of the multi indices is $\mathbf{0}$. Also

$$
\Lambda\left(X^{m, p}(U)\right)
$$

is a closed subspace of $L^{p}(U)^{w}$. To see this, suppose

$$
\left(u_{k}, D^{\alpha_{2}} u_{k}, \cdots, D^{\alpha_{w}} u_{k}\right) \rightarrow\left(f_{1}, f_{2}, \cdots, f_{w}\right)
$$

in $L^{p}(U)^{w}$. Then $u_{k} \rightarrow f_{1}$ in $L^{p}(U)$ and $D^{\alpha_{j}} u_{k} \rightarrow f_{j}$ in $L^{p}(U)$. Therefore, letting $\phi \in$ $C_{c}^{\infty}(U)$ and letting $k \rightarrow \infty$,

$$
\begin{array}{cc}
\int_{U}\left(D^{\alpha_{j}} u_{k}\right) \phi d x & =(-1)^{|\alpha|} \int_{U} u_{k} D^{\alpha_{j}} \phi d x \\
\downarrow & \downarrow \\
\int_{U} f_{j} \phi d x & (-1)^{|\alpha|} \int_{U} f_{1} D^{\alpha_{j}} \phi d x \quad \equiv D^{\alpha_{j}}\left(f_{1}\right)(\phi)
\end{array}
$$

[^28]It follows $D^{\alpha_{j}}\left(f_{1}\right)=f_{j}$ and so $\Lambda\left(X^{m, p}(U)\right)$ is closed as claimed. This is clearly also a subspace of $L^{p}(U)^{w}$ and so it follows that $\Lambda\left(X^{m, p}(U)\right)$ is a reflexive Banach space. This is because $L^{p}(U)^{w}$, being the product of reflexive Banach spaces, is reflexive and any closed subspace of a reflexive Banach space is reflexive. Now $\Lambda$ is an isometry of $X^{m, p}(U)$ and $\Lambda\left(X^{m, p}(U)\right)$ which shows that $X^{m, p}(U)$ is a reflexive Banach space. Finally, $W^{m, p}(U)$ is a closed subspace of the reflexive Banach space, $X^{m, p}(U)$ and so it is also reflexive. To see $X^{m, p}(U)$ is separable, note that $L^{p}(U)^{w}$ is separable because it is the finite product of the separable hence completely separable metric space, $L^{p}(U)$ and $\Lambda\left(X^{m, p}(U)\right)$ is a subset of $L^{p}(U)^{w}$. Therefore, $\Lambda\left(X^{m, p}(U)\right)$ is separable and since $\Lambda$ is an isometry, it follows $X^{m, p}(U)$ is separable also. Now $W^{m, p}(U)$ must also be separable because it is a subset of $X^{m, p}(U)$.

The following theorem is obvious but is worth noting because it says that if a function has a weak derivative in $L^{p}(U)$ on a large open set, $U$ then the restriction of this weak derivative is also the weak derivative for any smaller open set.

Theorem 38.0.4 Suppose $U$ is an open set and $U_{0} \subseteq U$ is another open set. Suppose also $D^{\alpha} u \in L^{p}(U)$. Then for all $\psi \in C_{c}^{\infty}\left(U_{0}\right)$,

$$
\int_{U_{0}}\left(D^{\alpha} u\right) \psi d x=(-1)^{|\alpha|} \int_{U_{0}} u\left(D^{\alpha} \psi\right)
$$

The following theorem is a fundamental approximation result for functions in $X^{m, p}(U)$.

Theorem 38.0.5 Let $U$ be an open set and let $U_{0}$ be an open subset of $U$ with the property that $\operatorname{dist}\left(\overline{U_{0}}, U^{C}\right)>0$. Then if $u \in X^{m, p}(U)$ and $\widetilde{u}$ denotes the zero extention of $u$ off $U$,

$$
\lim _{l \rightarrow \infty}\left\|\widetilde{u} * \phi_{l}-u\right\|_{X^{m, p}\left(U_{0}\right)}=0
$$

Proof: Always assume $l$ is large enough that $1 / l<\operatorname{dist}\left(\overline{U_{0}}, U^{C}\right)$. Thus for $\mathbf{x} \in U_{0}$,

$$
\begin{equation*}
\widetilde{u} * \phi_{l}(\mathbf{x})=\int_{B\left(\mathbf{0}, \frac{1}{l}\right)} u(\mathbf{x}-\mathbf{y}) \phi_{l}(\mathbf{y}) d y \tag{38.0.1}
\end{equation*}
$$

The theorem is proved if it can be shown that $D^{\alpha}\left(\widetilde{u} * \phi_{l}\right) \rightarrow D^{\alpha} u$ in $L^{p}\left(U_{0}\right)$. Let $\psi \in$ $C_{c}^{\infty}\left(U_{0}\right)$

$$
\begin{aligned}
D^{\alpha}\left(\widetilde{u} * \phi_{l}\right)(\psi) & \equiv(-1)^{|\alpha|} \int_{U_{0}}\left(\widetilde{u} * \phi_{l}\right)\left(D^{\alpha} \psi\right) d x \\
& =(-1)^{|\alpha|} \int_{U_{0}} \int \widetilde{u}(\mathbf{y}) \phi_{l}(\mathbf{x}-\mathbf{y})\left(D^{\alpha} \psi\right)(\mathbf{x}) d y d x \\
& =(-1)^{|\alpha|} \int_{U} u(\mathbf{y}) \int_{U_{0}} \phi_{l}(\mathbf{x}-\mathbf{y})\left(D^{\alpha} \psi\right)(\mathbf{x}) d x d y
\end{aligned}
$$

Also,

$$
\begin{aligned}
\left(\widetilde{D^{\alpha} u} * \phi_{l}\right)(\psi) & \equiv \int_{U_{0}}\left(\int \widetilde{D^{\alpha} u}(\mathbf{y}) \phi_{l}(\mathbf{x}-\mathbf{y}) d y\right) \psi(\mathbf{x}) d x \\
& =\int_{U_{0}}\left(\int_{U} D^{\alpha} u(\mathbf{y}) \phi_{l}(\mathbf{x}-\mathbf{y}) d y\right) \psi(\mathbf{x}) d x \\
& =\int_{U_{0}}\left(\int_{U} u(\mathbf{y})\left(D^{\alpha} \phi_{l}\right)(\mathbf{x}-\mathbf{y}) d y\right) \psi(\mathbf{x}) d x \\
& =\int_{U} u(\mathbf{y}) \int_{U_{0}}\left(D^{\alpha} \phi_{l}\right)(\mathbf{x}-\mathbf{y}) \psi(\mathbf{x}) d x d y \\
& =(-1)^{|\alpha|} \int_{U} u(\mathbf{y}) \int_{U_{0}} \phi_{l}(\mathbf{x}-\mathbf{y})\left(D^{\alpha} \psi\right)(\mathbf{x}) d x d y
\end{aligned}
$$

It follows that $D^{\alpha}\left(\widetilde{u} * \phi_{l}\right)=\left(\widetilde{D^{\alpha} u} * \phi_{l}\right)$ as weak derivatives defined on $C_{c}^{\infty}\left(U_{0}\right)$. Therefore,

$$
\begin{aligned}
\left\|D^{\alpha}\left(\widetilde{u} * \phi_{l}\right)-D^{\alpha} u\right\|_{L^{p}\left(U_{0}\right)} & =\left\|\widetilde{D^{\alpha} u} * \phi_{l}-D^{\alpha} u\right\|_{L^{p}\left(U_{0}\right)} \\
& \leq \| \widetilde{D^{\alpha} u * \phi_{l}-\widetilde{D^{\alpha} u} \|_{L^{p}\left(\mathbb{R}^{n}\right)} \rightarrow 0} .
\end{aligned}
$$

This proves the theorem.
As part of the proof of the theorem, the following corollary was established.
Corollary 38.0.6 Let $U_{0}$ and $U$ be as in the above theorem. Then for all l large enough and $\phi_{l}$ a mollifier,

$$
\begin{equation*}
D^{\alpha}\left(\widetilde{u} * \phi_{l}\right)=\left(\widetilde{D^{\alpha} u} * \phi_{l}\right) \tag{38.0.2}
\end{equation*}
$$

as distributions on $C_{c}^{\infty}\left(U_{0}\right)$.
Definition 38.0.7 Let $U$ be an open set. $C^{\infty}(U)$ denotes the set of functions which are defined and infinitely differentiable on $U$.

Note that $f(x)=\frac{1}{x}$ is a function in $C^{\infty}(0,1)$. However, it is not equal to the restriction to $(0,1)$ of some function which is in $C_{c}^{\infty}(\mathbb{R})$. This illustrates the distinction between $C^{\infty}(U)$ and $C^{\infty}(\bar{U})$. The set, $C^{\infty}(\bar{U})$ is a subset of $C^{\infty}(U)$. The following theorem is known as the Meyer Serrin theorem.

Theorem 38.0.8 (Meyer Serrin) Let $U$ be an open subset of $\mathbb{R}^{n}$. Then if $\delta>0$ and $u \in$ $X^{m, p}(U)$, there exists $J \in C^{\infty}(U)$ such that $\|J-u\|_{m, p, U}<\delta$.

Proof: Let $\cdots U_{k} \subseteq \overline{U_{k}} \subseteq U_{k+1} \cdots$ be a sequence of open subsets of $U$ whose union equals $U$ such that $\overline{U_{k}}$ is compact for all $k$. Also let $U_{-3}=U_{-2}=U_{-1}=U_{0}=\emptyset$. Now define $V_{k} \equiv U_{k+1} \backslash \overline{U_{k-1}}$. Thus $\left\{V_{k}\right\}_{k=1}^{\infty}$ is an open cover of $U$. Note the open cover is locally finite and therefore, there exists a partition of unity subordinate to this open cover,
$\left\{\eta_{k}\right\}_{k=1}^{\infty}$ such that each $\operatorname{spt}\left(\eta_{k}\right) \in C_{c}\left(V_{k}\right)$. Let $\psi_{m}$ denote the sum of all the $\eta_{k}$ which are non zero at some point of $V_{m}$. Thus

$$
\begin{equation*}
\operatorname{spt}\left(\psi_{m}\right) \subseteq U_{m+2} \backslash \overline{U_{m-2}}, \psi_{m} \in C_{c}^{\infty}(U), \sum_{m=1}^{\infty} \psi_{m}(\mathbf{x})=1 \tag{38.0.3}
\end{equation*}
$$

for all $\mathbf{x} \in U$, and $\psi_{m} u \in W^{m, p}\left(U_{m+2}\right)$.
Now let $\phi_{l}$ be a mollifier and consider

$$
\begin{equation*}
J \equiv \sum_{m=0}^{\infty} u \psi_{m} * \phi_{l_{m}} \tag{38.0.4}
\end{equation*}
$$

where $l_{m}$ is chosen large enough that the following two conditions hold:

$$
\begin{gather*}
\operatorname{spt}\left(u \psi_{m} * \phi_{l_{m}}\right) \subseteq U_{m+3} \backslash \overline{U_{m-3}},  \tag{38.0.5}\\
\left\|\left(u \psi_{m}\right) * \phi_{l_{m}}-u \psi_{m}\right\|_{m, p, U_{m+3}}=\left\|\left(u \psi_{m}\right) * \phi_{l_{m}}-u \psi_{m}\right\|_{m, p, U}<\frac{\delta}{2^{m+5}}, \tag{38.0.6}
\end{gather*}
$$

where 38.0.6 is obtained from Theorem 38.0.5. Because of 38.0 .3 only finitely many terms of the series in 38.0.4 are nonzero and therefore, $J \in C^{\infty}(U)$. Now let $N>10$, some large value.

$$
\begin{aligned}
\|J-u\|_{m, p, U_{N-3}} & =\left\|\sum_{k=0}^{N}\left(u \psi_{k} * \phi_{l_{k}}-u \psi_{k}\right)\right\|_{m, p, U_{N-3}} \\
& \leq \sum_{k=0}^{N}\left\|u \psi_{k} * \phi_{l_{k}}-u \psi_{k}\right\|_{m, p, U_{N-3}} \\
& \leq \sum_{k=0}^{N} \frac{\delta}{2^{m+5}}<\delta .
\end{aligned}
$$

Now apply the monotone convergence theorem to conclude that $\|J-u\|_{m, p, U} \leq \delta$. This proves the theorem.

Note that $J=0$ on $\partial U$. Later on, you will see that this is pathological.
In the study of partial differential equations it is the space $W^{m, p}(U)$ which is of the most use, not the space $X^{m, p}(U)$. This is because of the density of $C^{\infty}(\bar{U})$. Nevertheless, for reasonable open sets, $U$, the two spaces coincide.

Definition 38.0.9 An open set, $U \subseteq \mathbb{R}^{n}$ is said to satisfy the segment condition if for all $\mathbf{z} \in \bar{U}$, there exists an open set $U_{\mathbf{z}}$ containing $\mathbf{z}$ and a vector $\mathbf{a}$ such that

$$
\bar{U} \cap \bar{U}_{\mathbf{z}}+t \mathbf{a} \subseteq U
$$

for all $t \in(0,1)$.


You can imagine open sets which do not satisfy the segment condition. For example, a pair of circles which are tangent at their boundaries. The condition in the above definition breaks down at their point of tangency.

Here is a simple lemma which will be used in the proof of the following theorem.
Lemma 38.0.10 If $u \in W^{m, p}(U)$ and $\psi \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$, then $u \psi \in W^{m, p}(U)$.
Proof: Let $|\alpha| \leq m$ and let $\phi \in C_{c}^{\infty}(U)$. Then

$$
\begin{aligned}
\left(D_{x_{i}}(u \psi)\right)(\phi) & \equiv-\int_{U} u \psi \phi_{, x_{i}} d x \\
& =-\int_{U} u\left((\psi \phi)_{x_{i}}-\phi \psi_{, x_{i}}\right) d x \\
& =\left(D_{x_{i}} u\right)(\psi \phi)+\int_{U} u \psi_{, x_{i}} \phi d x \\
& =\int_{U}\left(\psi D_{x_{i}} u+u \psi_{, x_{i}}\right) \phi d x
\end{aligned}
$$

Therefore, $D_{x_{i}}(u \psi)=\psi D_{x_{i}} u+u \psi_{x_{i}} \in L^{p}(U)$. In other words, the product rule holds. Now considering the terms in the last expression, you can do the same argument with each of these as long as they all have derivatives in $L^{p}(U)$. Therefore, continuing this process the lemma is proved.

Theorem 38.0.11 Let $U$ be an open set and suppose there exists a locally finite covering ${ }^{2}$ of $\bar{U}$ which is of the form $\left\{U_{i}\right\}_{i=1}^{\infty}$ such that each $U_{i}$ is a bounded open set which satisfies the conditions of Definition 38.0.9. Thus there exist vectors, $\mathbf{a}_{i}$ such that for all $t \in(0,1)$,

$$
\overline{U_{i}} \cap U+t \mathbf{a}_{i} \subseteq U
$$

Then $C^{\infty}(\bar{U})$ is dense in $X^{m, p}(U)$ and so $W^{m, p}(U)=X^{m, p}(U)$.

[^29]Proof: Let $\left\{\psi_{i}\right\}_{i=1}^{\infty}$ be a partition of unity subordinate to the given open cover with $\psi_{i} \in C_{c}^{\infty}\left(U_{i}\right)$ and let $u \in X^{m, p}(U)$. Thus

$$
u=\sum_{k=1}^{\infty} \psi_{k} u .
$$

Consider $U_{k}$ for some $k$. Let $\mathbf{a}_{k}$ be the special vector associated with $U_{k}$ such that

$$
\begin{equation*}
t \mathbf{a}_{k}+\bar{U} \cap \overline{U_{k}} \subseteq U \tag{38.0.7}
\end{equation*}
$$

for all $t \in(0,1)$ and consider only $t$ small enough that

$$
\begin{equation*}
\operatorname{spt}\left(\psi_{k}\right)-t \mathbf{a}_{k} \subseteq U_{k} \tag{38.0.8}
\end{equation*}
$$

Pick $l(t)>1 / t$ which is also large enough that

$$
\begin{equation*}
t \mathbf{a}_{k}+\bar{U} \cap \overline{U_{k}}+B\left(\mathbf{0}, \frac{1}{l(t)}\right) \subseteq U, \overline{\operatorname{spt}\left(\psi_{k}\right)+B\left(\mathbf{0}, \frac{1}{l\left(t_{k}\right)}\right)-t \mathbf{a}_{k}} \subseteq U_{k} \tag{38.0.9}
\end{equation*}
$$

This can be done because $t \mathbf{a}_{k}+\bar{U} \cap \overline{U_{k}}$ is a compact subset of $U$ and so has positive distance to $U^{C}$ and $\operatorname{spt}\left(\psi_{k}\right)-t \mathbf{a}_{k}$ is a compact subset of $U_{k}$ having positive distance to $U_{k}^{C}$. Let $t_{k}$ be such a value for $t$ and for $\phi_{l}$ a mollifier, define

$$
\begin{equation*}
v_{t_{k}}(\mathbf{x}) \equiv \int_{\mathbb{R}^{n}} \widetilde{u}\left(\mathbf{x}+t_{k} \mathbf{a}_{k}-\mathbf{y}\right) \psi_{k}\left(\mathbf{x}+t_{k} \mathbf{a}_{k}-\mathbf{y}\right) \phi_{l\left(t_{k}\right)}(\mathbf{y}) d y \tag{38.0.10}
\end{equation*}
$$

where as usual, $\widetilde{u}$ is the zero extention of $u$ off $U$. For $v_{t_{k}}(\mathbf{x}) \neq 0$, it is necessary that $\mathbf{x}+t_{k} \mathbf{a}_{k}-\mathbf{y} \in \operatorname{spt}\left(\psi_{k}\right)$ for some $\mathbf{y} \in B\left(\mathbf{0}, \frac{1}{l\left(t_{k}\right)}\right)$. Therefore, using 38.0.9, for $v_{t_{k}}(\mathbf{x}) \neq 0$, it is necessary that

$$
\begin{gathered}
\mathbf{x} \in \mathbf{y}-t_{k} \mathbf{a}_{k}+U \cap \operatorname{spt}\left(\psi_{k}\right) \subseteq B\left(\mathbf{0}, \frac{1}{l\left(t_{k}\right)}\right)+\operatorname{spt}\left(\psi_{k}\right)-t_{k} \mathbf{a}_{k} \\
\subseteq \overline{B\left(\mathbf{0}, \frac{1}{l\left(t_{k}\right)}\right)+\operatorname{spt}\left(\psi_{k}\right)-t_{k} \mathbf{a}_{k}} \subseteq U_{k}
\end{gathered}
$$

showing that $v_{t_{k}}$ has compact support in $U_{k}$. Now change variables in 38.0.10 to obtain

$$
\begin{equation*}
v_{t_{k}}(\mathbf{x}) \equiv \int_{\mathbb{R}^{n}} \widetilde{u}(\mathbf{y}) \psi_{k}(\mathbf{y}) \phi_{l\left(t_{k}\right)}\left(\mathbf{x}+t_{k} \mathbf{a}_{k}-\mathbf{y}\right) d y \tag{38.0.11}
\end{equation*}
$$

For $\mathbf{x} \in U \cap U_{k}$, the above equals zero unless

$$
\mathbf{y}-t_{k} \mathbf{a}_{k}-\mathbf{x} \in B\left(\mathbf{0}, \frac{1}{l\left(t_{k}\right)}\right)
$$

which implies by 38.0.9 that

$$
\mathbf{y} \in t_{k} \mathbf{a}_{k}+U \cap U_{k}+B\left(\mathbf{0}, \frac{1}{l\left(t_{k}\right)}\right) \subseteq U
$$

Therefore, for such $\mathbf{x} \in U \cap U_{k}, 38.0 .11$ reduces to

$$
\begin{aligned}
v_{t_{k}}(\mathbf{x}) & =\int_{\mathbb{R}^{n}} u(\mathbf{y}) \psi_{k}(\mathbf{y}) \phi_{l\left(t_{k}\right)}\left(\mathbf{x}+t_{k} \mathbf{a}_{k}-\mathbf{y}\right) d y \\
& =\int_{U} u(\mathbf{y}) \psi_{k}(\mathbf{y}) \phi_{l\left(t_{k}\right)}\left(\mathbf{x}+t_{k} \mathbf{a}_{k}-\mathbf{y}\right) d y
\end{aligned}
$$

It follows that for $|\alpha| \leq m$, and $\mathbf{x} \in U \cap U_{k}$

$$
\begin{align*}
D^{\alpha} v_{t_{k}}(\mathbf{x}) & =\int_{U} u(\mathbf{y}) \psi_{k}(\mathbf{y}) D^{\alpha} \phi_{l\left(t_{k}\right)}\left(\mathbf{x}+t_{k} \mathbf{a}_{k}-\mathbf{y}\right) d y \\
& =\int_{U} D^{\alpha}\left(u \psi_{k}\right)(\mathbf{y}) \phi_{l\left(t_{k}\right)}\left(\mathbf{x}+t_{k} \mathbf{a}_{k}-\mathbf{y}\right) d y \\
& =\int_{\mathbb{R}^{n}} \widetilde{\left.D^{\alpha\left(u \psi_{k}\right.}\right)}(\mathbf{y}) \phi_{l\left(t_{k}\right)}\left(\mathbf{x}+t_{k} \mathbf{a}_{k}-\mathbf{y}\right) d y \\
& =\int_{\mathbb{R}^{n}} \widetilde{\left.D^{\alpha\left(u \psi_{k}\right.}\right)}\left(\mathbf{x}+t_{k} \mathbf{a}_{k}-\mathbf{y}\right) \phi_{l\left(t_{k}\right)}(\mathbf{y}) d y \tag{38.0.12}
\end{align*}
$$

Actually, this formula holds for all $\mathbf{x} \in U$. If $\mathbf{x} \in U$ but $\mathbf{x} \notin U_{k}$, then the left side of the above formula equals zero because, as noted above, $\operatorname{spt}\left(v_{t_{k}}\right) \subseteq U_{k}$. The integrand of the right side equals zero unless

$$
\mathbf{x} \in B\left(\mathbf{0}, \frac{1}{l\left(t_{k}\right)}\right)+\operatorname{spt}\left(\psi_{k}\right)-t_{k} \mathbf{a}_{k} \subseteq U_{k}
$$

by 38.0.9 and here $\mathbf{x} \notin U_{k}$.
Next an estimate is obtained for $\left\|D^{\alpha} v_{t_{k}}-D^{\alpha}\left(u \psi_{k}\right)\right\|_{L^{p}(U)}$. By 38.0.12,

$$
\begin{gathered}
\| D^{\alpha} v_{t_{k}}-\left.D^{\alpha}\left(u \psi_{k}\right)\right|_{L^{p}(U)} \leq \\
\left(\int _ { U } \left(\int_{\mathbb{R}^{n}}\left|\widetilde{\left.D^{\alpha\left(u \psi_{k}\right.}\right)}\left(\mathbf{x}+t_{k} \mathbf{a}_{k}-\mathbf{y}\right)-\widetilde{\left.D^{\alpha\left(u \psi_{k}\right.}\right)}(\mathbf{x})\right| \widetilde{\left.\left.\phi_{l\left(t_{k}\right)}(\mathbf{y}) d y\right)^{p} d x\right)^{1 / p}}\right.\right. \\
\leq \int_{\mathbb{R}^{n}} \phi_{l\left(t_{k}\right)}(\mathbf{y})\left(\int_{U}\left|\widetilde{\left.D^{\alpha\left(u \psi_{k}\right.}\right)}\left(\mathbf{x}+t_{k} \mathbf{a}_{k}-\mathbf{y}\right)-\widetilde{\left.D^{\alpha\left(u \psi_{k}\right.}\right)}(\mathbf{x})\right|^{p} d x\right)^{1 / p} d y \\
\leq \frac{\varepsilon}{2^{k}}
\end{gathered}
$$

whenever $t_{k}$ is taken small enough. Pick $t_{k}$ this small and let $w_{k} \equiv v_{t_{k}}$. Thus

$$
\left\|D^{\alpha} w_{k}-D^{\alpha}\left(u \psi_{k}\right)\right\|_{L^{p}(U)} \leq \frac{\varepsilon}{2^{k}}
$$

and $w_{k} \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$. Now let

$$
J(\mathbf{x}) \equiv \sum_{k=1}^{\infty} w_{k}
$$

Since the $U_{k}$ are locally finite and $\operatorname{spt}\left(w_{k}\right) \subseteq U_{k}$ for each $k$, it follows

$$
D^{\alpha} J=\sum_{k=0}^{\infty} D^{\alpha} w_{k}
$$

and the sum is always finite. Similarly,

$$
D^{\alpha} \sum_{k=1}^{\infty}\left(\psi_{k} u\right)=\sum_{k=1}^{\infty} D^{\alpha}\left(\psi_{k} u\right)
$$

and the sum is always finite. Therefore,

$$
\begin{aligned}
\left\|D^{\alpha} J-D^{\alpha} u\right\|_{L^{p}(U)} & =\left\|\sum_{k=1}^{\infty} D^{\alpha} w_{k}-D^{\alpha}\left(\psi_{k} u\right)\right\|_{L^{p}(U)} \\
& \leq \sum_{k=1}^{\infty}\left\|D^{\alpha} w_{k}-D^{\alpha}\left(\psi_{k} u\right)\right\|_{L^{p}(U)} \leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k}}=\varepsilon
\end{aligned}
$$

By choosing $t_{k}$ small enough, such an inequality can be obtained for

$$
\left\|D^{\beta} J-D^{\beta} u\right\|_{L^{p}(U)}
$$

for each multi index, $\beta$ such that $|\beta| \leq m$. Therefore, there exists

$$
J \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right)
$$

such that

$$
\|J-u\|_{W^{m, p}(U)} \leq \varepsilon K
$$

where $K$ equals the number of multi indices no larger than $m$. Since $\varepsilon$ is arbitrary, this proves the theorem.

Corollary 38.0.12 Let $U$ be an open set which has the segment property. Then $W^{m, p}(U)=$ $X^{m, p}(U)$.

Proof: Start with an open covering of $\bar{U}$ whose sets satisfy the segment condition and obtain a locally finite refinement consisting of bounded sets which are of the sort in the above theorem.

Now consider a situation where $\mathbf{h}: U \rightarrow V$ where $U$ and $V$ are two open sets in $\mathbb{R}^{n}$ and $D^{\alpha} \mathbf{h}$ exists and is continuous and bounded if $|\alpha|<m-1$ and $D^{\alpha} \mathbf{h}$ is Lipschitz if $|\alpha|=m-1$.

Definition 38.0.13 Whenever $\mathbf{h}: U \rightarrow V$, define $\mathbf{h}^{*}$ mapping the functions which are defined on $V$ to the functions which are defined on $U$ as follows.

$$
\mathbf{h}^{*} f(\mathbf{x}) \equiv f(\mathbf{h}(\mathbf{x}))
$$

$\mathbf{h}: U \rightarrow V$ is bilipschitz if $\mathbf{h}$ is one to one, onto and Lipschitz and $\mathbf{h}^{-1}$ is also one to one, onto and Lipschitz.

Theorem 38.0.14 Let $\mathbf{h}: U \rightarrow V$ be one to one and onto where $U$ and $V$ are two open sets. Also suppose that $D^{\alpha} \mathbf{h}$ and $D^{\alpha}\left(\mathbf{h}^{-1}\right)$ exist and are Lipschitz continuous if $|\alpha| \leq m-1$ for $m$ a positive integer. Then

$$
\mathbf{h}^{*}: W^{m, p}(V) \rightarrow W^{m, p}(U)
$$

is continuous, linear, one to one, and has an inverse with the same properties, the inverse being $\left(\mathbf{h}^{-1}\right)^{*}$.

Proof: It is clear that $\mathbf{h}^{*}$ is linear. It is required to show it is one to one and continuous. First suppose $\mathbf{h}^{*} f=0$. Then

$$
0=\int_{V}|f(\mathbf{h}(\mathbf{x}))|^{p} d x
$$

and so $f(\mathbf{h}(\mathbf{x}))=0$ for a.e. $\mathbf{x} \in U$. Since $\mathbf{h}$ is Lipschitz, it takes sets of measure zero to sets of measure zero. Therefore, $f(\mathbf{y})=0$ a.e. This shows $\mathbf{h}^{*}$ is one to one.

By the Meyer Serrin theorem, Theorem 38.0.8, it suffices to verify that $\mathbf{h}^{*}$ is continuous on functions in $C^{\infty}(V)$. Let $f$ be such a function. Then using the chain rule and product $\operatorname{rule},\left(\mathbf{h}^{*} f\right)_{, i}(\mathbf{x})=f_{, k}(\mathbf{h}(\mathbf{x})) h_{k, i}(\mathbf{x})$,

$$
\begin{aligned}
\left(\mathbf{h}^{*} f\right)_{, i j}(\mathbf{x}) & =\left(f_{, k}(\mathbf{h}(\mathbf{x})) h_{k, i}(\mathbf{x})\right)_{, j} \\
& =f_{, k l}(\mathbf{h}(\mathbf{x})) h_{l, j}(\mathbf{x}) h_{k, i}(\mathbf{x})+f_{, k}(\mathbf{h}(\mathbf{x})) h_{k, i j}(\mathbf{x})
\end{aligned}
$$

etc. In general, for $|\alpha| \leq m-1$, succsessive applications of the product rule and chain rule yield that $D^{\alpha}\left(\mathbf{h}^{*} f\right)(\mathbf{x})$ has the form

$$
D^{\alpha}\left(\mathbf{h}^{*} f\right)(\mathbf{x})=\sum_{|\beta| \leq|\alpha|} \mathbf{h}^{*}\left(D^{\beta} f\right)(\mathbf{x}) g_{\beta}(\mathbf{x})
$$

where $g_{\beta}$ is a bounded Lipschitz function with Lipschitz constant dependent on $\mathbf{h}$ and its derivatives. It only remains to take one more derivative of the functions, $D^{\alpha} f$ for $|\alpha|=$ $m-1$. This can be done again but this time you have to use Rademacher's theorem which assures you that the derivative of a Lipschitz function exists a.e. in order to take the partial derivative of the $g_{\beta}(\mathbf{x})$. When this is done, the above formula remains valid for all $|\alpha| \leq m$. Therefore, using the change of variables formula for multiple integrals, Corollary 36.6.14 on Page 1296,

$$
\begin{aligned}
\int_{U}\left|D^{\alpha}\left(\mathbf{h}^{*} f\right)(\mathbf{x})\right|^{p} d x & \leq C_{m, p, \mathbf{h}} \sum_{|\beta| \leq m} \int_{U}\left|\mathbf{h}^{*}\left(D^{\beta} f\right)(\mathbf{x})\right|^{p} d x \\
& =C_{m, p, \mathbf{h}} \sum_{|\beta| \leq m} \int_{U}\left|\left(D^{\beta} f\right)(\mathbf{h}(\mathbf{x}))\right|^{p} d x \\
& =C_{m, p, \mathbf{h}} \sum_{|\beta| \leq m} \int_{V}\left|\left(D^{\beta} f\right)(\mathbf{y})\right|^{p}\left|\operatorname{det} D \mathbf{h}^{-1}(\mathbf{y})\right| d y \\
& \leq C_{m, p, \mathbf{h}, \mathbf{h}^{-1}} \|\left. f\right|_{m, p, V}
\end{aligned}
$$

This shows $\mathbf{h}^{*}$ is continuous on $C^{\infty}(V) \cap W^{m, p}(U)$ and since this set is dense, this proves $\mathbf{h}^{*}$ is continuous. The same argument applies to $\left(\mathbf{h}^{-1}\right)^{*}$ and now the definitions of $\mathbf{h}^{*}$ and $\left(\mathbf{h}^{-1}\right)^{*}$ show these are inverses.

### 38.1 Embedding Theorems For $W^{m, p}\left(\mathbb{R}^{n}\right)$

Recall Theorem 36.5.1 which is listed here for convenience.

Theorem 38.1.1 Suppose $u, u_{, i} \in L_{\text {loc }}^{p}\left(\mathbb{R}^{n}\right)$ for $i=1, \cdots, n$ and $p>n$. Then $u$ has a representative, still denoted by $u$, such that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n}$,

$$
\begin{equation*}
|u(\mathbf{x})-u(\mathbf{y})| \leq C\left(\int_{B(\mathbf{x}, 2|\mathbf{y}-\mathbf{x}|)}|\nabla u|^{p} d z\right)^{1 / p}|\mathbf{x}-\mathbf{y}|^{(1-n / p)} \tag{38.1.13}
\end{equation*}
$$

This amazing result shows that every $u \in W^{m, p}\left(\mathbb{R}^{n}\right)$ has a representative which is continuous provided $p>n$.

Using the above inequality, one can give an important embedding theorem.
Definition 38.1.2 Let $X, Y$ be two Banach spaces and let $f: X \rightarrow Y$ be a function. Then $f$ is a compact map if whenever $S$ is a bounded set in $X$, it follows that $f(S)$ is precompact in $Y$.

Theorem 38.1.3 Let $U$ be a bounded open set and for $u$ a function defined on $\mathbb{R}^{n}$, let $r_{U} u(\mathbf{x}) \equiv u(\mathbf{x})$ for $\mathbf{x} \in \bar{U}$. Then if $p>n, r_{U}: W^{1, p}\left(\mathbb{R}^{n}\right) \rightarrow C(\bar{U})$ is continuous and compact.

Proof: First suppose $u_{k} \rightarrow 0$ in $W^{1, p}\left(\mathbb{R}^{n}\right)$. Then if $r_{U} u_{k}$ does not converge to 0 , it follows there exists a sequence, still denoted by $k$ and $\varepsilon>0$ such that $u_{k} \rightarrow 0$ in $W^{1, p}\left(\mathbb{R}^{n}\right)$ but $\left\|r_{U} u_{k}\right\|_{\infty} \geq \varepsilon$. Selecting a further subsequence which is still denoted by $k$, you can also assume $u_{k}(\mathbf{x}) \rightarrow 0$ a.e. Pick such an $\mathbf{x}_{0} \in U$ where this convergence takes place. Then from 38.1.13, for all $\mathbf{x} \in \bar{U}$,

$$
\left|u_{k}(\mathbf{x})\right| \leq\left|u_{k}\left(\mathbf{x}_{0}\right)\right|+C\left\|u_{k}\right\|_{1, p, \mathbb{R}^{n}} \operatorname{diam}(U)
$$

showing that $u_{k}$ converges uniformly to 0 on $\bar{U}$ contrary to $\left\|r_{U} u_{k}\right\|_{\infty} \geq \varepsilon$. Therefore, $r_{U}$ is continuous as claimed.

Next let $S$ be a bounded subset of $W^{1, p}\left(\mathbb{R}^{n}\right)$ with $\|u\|_{1, p}<M$ for all $u \in S$. Then for $u \in S$

$$
r^{p} m_{n}([|u|>r] \cap U) \leq \int_{[|u|>r] \cap U}|u|^{p} d m_{n} \leq M^{p}
$$

and so

$$
m_{n}([|u|>r] \cap U) \leq \frac{M^{p}}{r^{p}}
$$

Now choosing $r$ large enough, $M^{p} / r^{p}<m_{n}(U)$ and so, for such $r$, there exists $\mathbf{x}_{u} \in U$ such that $\left|u\left(\mathbf{x}_{u}\right)\right| \leq r$. Therefore from 38.1.13, whenever $\mathbf{x} \in U$,

$$
\begin{aligned}
|u(\mathbf{x})| & \leq\left|u\left(\mathbf{x}_{u}\right)\right|+C M \operatorname{diam}(U)^{1-n / p} \\
& \leq r+C M \operatorname{diam}(U)^{1-n / p}
\end{aligned}
$$

showing that $\left\{r_{U} u: u \in S\right\}$ is uniformly bounded. But also, for $\mathbf{x}, \mathbf{y} \in \bar{U}, 38.1 .13$ implies

$$
|u(\mathbf{x})-u(\mathbf{y})| \leq C M|\mathbf{x}-\mathbf{y}|^{1-\frac{n}{p}}
$$

showing that $\left\{r_{U} u: u \in S\right\}$ is equicontinuous. By the Ascoli Arzela theorem, it follows $r_{U}(S)$ is precompact and so $r_{U}$ is compact.

Definition 38.1.4 Let $\alpha \in(0,1]$ and $K$ a compact subset of $\mathbb{R}^{n}$

$$
C^{\alpha}(K) \equiv\left\{f \in C(K): \rho_{\alpha}(f)+\|f\| \equiv\|f\|_{\alpha}<\infty\right\}
$$

where

$$
\|f\| \equiv\|f\|_{\infty} \equiv \sup \{|f(\mathbf{x})|: \mathbf{x} \in K\}
$$

and

$$
\rho_{\alpha}(f) \equiv \sup \left\{\frac{|f(\mathbf{x})-f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{\alpha}}: \mathbf{x}, \mathbf{y} \in K, \mathbf{x} \neq \mathbf{y}\right\}
$$

Then $\left(C^{\alpha}(K),\|\cdot\|_{\alpha}\right)$ is a complete normed linear space called a Holder space.
The verification that this is a complete normed linear space is routine and is left for you. More generally, one considers the following class of Holder spaces.

Definition 38.1.5 Let $K$ be a compact subset of $\mathbb{R}^{n}$ and let $\lambda \in(0,1]$. $C^{m, \lambda}(K)$ denotes the set of functions, $u$ which are restrictions of functions defined on $\mathbb{R}^{n}$ to Ksuch that for $|\alpha| \leq m$,

$$
D^{\alpha} u \in C(K)
$$

and if $|\alpha|=m$,

$$
D^{\alpha} u \in C^{\lambda}(K) .
$$

Thus $C^{0, \lambda}(K)=C^{\lambda}(K)$. The norm of a function in $C^{m, \lambda}(K)$ is given by

$$
\|u\|_{m, \lambda} \equiv \sup _{|\alpha|=m} \rho_{\lambda}\left(D^{\alpha} u\right)+\sum_{|\alpha| \leq m}\left\|D^{\alpha} u\right\|_{\infty}
$$

Lemma 38.1.6 Let $m$ be a positive integer, $K$ a compact subset of $\mathbb{R}^{n}$, and let $0<\beta<\lambda \leq$ 1. Then the identity map from $C^{m, \lambda}(K)$ into $C^{m, \beta}(K)$ is compact.

Proof: First note that the containment is obvious because for any function, $f$, if

$$
\rho_{\lambda}(f) \equiv \sup \left\{\frac{|f(\mathbf{x})-f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{\lambda}}: \mathbf{x}, \mathbf{y} \in K, \mathbf{x} \neq \mathbf{y}\right\}<\infty
$$

Then

$$
\begin{aligned}
\rho_{\beta}(f) & \equiv \sup \left\{\frac{|f(\mathbf{x})-f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{\beta}}: \mathbf{x}, \mathbf{y} \in K, \mathbf{x} \neq \mathbf{y}\right\} \\
& =\sup \left\{\frac{|f(\mathbf{x})-f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{\lambda}}|\mathbf{x}-\mathbf{y}|^{\lambda-\beta}: \mathbf{x}, \mathbf{y} \in K, \mathbf{x} \neq \mathbf{y}\right\} \\
& \leq \sup \left\{\frac{|f(\mathbf{x})-f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{\lambda}} \operatorname{diam}(K)^{\lambda-\beta}: \mathbf{x}, \mathbf{y} \in K, \mathbf{x} \neq \mathbf{y}\right\}<\infty .
\end{aligned}
$$

Suppose the identity map, id, is not compact. Then there exists $\varepsilon>0$ and a sequence, $\left\{f_{k}\right\}_{k=1}^{\infty} \subseteq C^{m, \lambda}(K)$ such that $\left\|f_{k}\right\|_{m, \lambda}<M$ for all $k$ but $\left\|f_{k}-f_{l}\right\|_{\beta} \geq \varepsilon$ whenever $k \neq l$. By
the Ascoli Arzela theorem, there exists a subsequence of this, still denoted by $f_{k}$ such that $\sum_{|\alpha| \leq m}\left\|D^{\alpha}\left(f_{l}-f_{k}\right)\right\|_{\infty}<\delta$ where $\delta$ satisfies

$$
\begin{equation*}
0<\delta<\min \left(\frac{\varepsilon}{2},\left(\frac{\varepsilon}{8}\right)\left(\frac{\varepsilon}{8 M}\right)^{\beta /(\lambda-\beta)}\right) \tag{38.1.14}
\end{equation*}
$$

Therefore, $\sup _{|\alpha|=m} \rho_{\beta}\left(D^{\alpha}\left(f_{k}-f_{l}\right)\right) \geq \varepsilon-\delta$ for all $k \neq l$. It follows that there exist pairs of points and a multi index, $\alpha$ with $|\alpha|=m,\left\{\mathbf{x}_{k l}, \mathbf{y}_{k l}, \alpha\right\}$ such that

$$
\begin{equation*}
\frac{\varepsilon-\delta}{2}<\frac{\left|\left(D^{\alpha} f_{k}-D^{\alpha} f_{l}\right)\left(\mathbf{x}_{k l}\right)-\left(\left(D^{\alpha} f_{k}-D^{\alpha} f_{l}\right)\left(\mathbf{y}_{k l}\right)\right)\right|}{\left|\mathbf{x}_{k l}-\mathbf{y}_{k l}\right|^{\beta}} \leq 2 M\left|\mathbf{x}_{k l}-\mathbf{y}_{k l}\right|^{\lambda-\beta} \tag{38.1.15}
\end{equation*}
$$

and so considering the ends of the above inequality,

$$
\left(\frac{\varepsilon-\delta}{4 M}\right)^{1 /(\lambda-\beta)}<\left|\mathbf{x}_{k l}-\mathbf{y}_{k l}\right|
$$

Now also, since $\sum_{|\alpha| \leq m}\left\|D^{\alpha}\left(f_{l}-f_{k}\right)\right\|_{\infty}<\boldsymbol{\delta}$, it follows from the first inequality in 38.1.15 that

$$
\frac{\varepsilon-\delta}{2}<\frac{2 \delta}{\left(\frac{\varepsilon-\delta}{4 M}\right)^{\beta /(\lambda-\beta)}}
$$

Since $\delta<\varepsilon / 2$, this implies

$$
\frac{\varepsilon}{4}<\frac{2 \delta}{\left(\frac{\varepsilon}{8 M}\right)^{\beta /(\lambda-\beta)}}
$$

and so

$$
\left(\frac{\varepsilon}{8}\right)\left(\frac{\varepsilon}{8 M}\right)^{\beta /(\lambda-\beta)}<\delta
$$

contrary to 38.1 .14 . This proves the lemma.
Corollary 38.1.7 Let $p>n, U$ and $r_{U}$ be as in Theorem 38.1.3 and let $m$ be a nonnegative integer. Then $r_{U}: W^{m+1, p}\left(\mathbb{R}^{n}\right) \rightarrow C^{m, \lambda}(\bar{U})$ is continuous as a map into $C^{m, \lambda}(\bar{U})$ for all $\lambda \in\left[0,1-\frac{n}{p}\right]$ and $r_{U}$ is compact if $\lambda<1-\frac{n}{p}$.

Proof: Suppose $u_{k} \rightarrow 0$ in $W^{m+1, p}\left(\mathbb{R}^{n}\right)$. Then from 38.1.13, if $\lambda \leq 1-\frac{n}{p}$ and $|\alpha|=m$

$$
\rho_{\lambda}\left(D^{\alpha} u_{k}\right) \leq C\left\|D^{\alpha} u_{k}\right\|_{1, p} \operatorname{diam}(U)^{1-\frac{n}{p}-\lambda} .
$$

Therefore, $\rho_{\lambda}\left(D^{\alpha} u_{k}\right) \rightarrow 0$. From Theorem 38.1.3 it follows that for $|\alpha| \leq m$,

$$
\left\|D^{\alpha} u_{k}\right\|_{\infty} \rightarrow 0
$$

and so $\left\|u_{k}\right\|_{m, \lambda} \rightarrow 0$. This proves the claim about continuity. The claim about compactness for $\lambda<1-\frac{n}{p}$ follows from Lemma 38.1.6 and this.
(Bounded in $W^{m, p}\left(\mathbb{R}^{n}\right) \xrightarrow{r_{U}}$ Bounded in $C^{m, 1-\frac{n}{p}}(\bar{U}) \xrightarrow{\text { id }}$ Compact in $C^{m, \lambda}(\bar{U})$.)
It is just as important to consider the case where $p<n$. To do this case the following lemma due to Gagliardo [53] will be of interest. See also [1].

Lemma 38.1.8 Suppose $n \geq 2$ and $w_{j}$ does not depend on the $j^{\text {th }}$ component of $\mathbf{x}, x_{j}$. Then

$$
\int_{\mathbb{R}^{n}} \prod_{j=1}^{n}\left|w_{j}(\mathbf{x})\right| d m_{n} \leq \prod_{i=1}^{n}\left(\int_{\mathbb{R}^{n-1}}\left|w_{j}(\mathbf{x})\right|^{n-1} d m_{n-1}\right)^{1 /(n-1)}
$$

In this inequality, assume all the functions are continuous so there can be no measurability questions.

Proof: First note that for $n=2$ the inequality reduces to the statement

$$
\iint\left|w_{1}\left(x_{2}\right)\right|\left|w_{2}\left(x_{1}\right)\right| d x_{1} d x_{2} \leq \int\left|w_{1}\left(x_{2}\right)\right| d x_{2} \int\left|w_{2}\left(x_{1}\right)\right| d x_{1}
$$

which is obviously true. Suppose then that the inequality is valid for some $n$. Using Fubini's theorem, Holder's inequality, and the induction hypothesis,

$$
\begin{aligned}
& \int_{\mathbb{R}^{n+1}} \prod_{j=1}^{n+1}\left|w_{j}(\mathbf{x})\right| d m_{n+1} \\
= & \int_{\mathbb{R}} \int_{\mathbb{R}^{n}}\left|w_{n+1}(\mathbf{x})\right| \prod_{j=1}^{n}\left|w_{j}(\mathbf{x})\right| d m_{n} d x_{n+1} \\
= & \int_{\mathbb{R}^{n}}\left|w_{n+1}(\mathbf{x})\right| \int_{\mathbb{R}} \prod_{j=1}^{n}\left|w_{j}(\mathbf{x})\right| d x_{n+1} d m_{n} \\
= & \int_{\mathbb{R}^{n}}\left|w_{n+1}(\mathbf{x})\right|\left(\prod_{j=1}^{n} \int_{\mathbb{R}}\left|w_{j}(\mathbf{x})\right|^{n} d x_{n+1}\right)^{1 / n} d m_{n} \\
& =\int_{\mathbb{R}^{n}}\left|w_{n+1}(\mathbf{x})\right| \prod_{j=1}^{n}\left(\int_{\mathbb{R}}\left|w_{j}(\mathbf{x})\right|^{n} d x_{n+1}\right)^{1 / n} d m_{n} \\
\leq & \left(\int_{\mathbb{R}^{n}}\left|w_{n+1}(\mathbf{x})\right|^{n} d m_{n}\right)^{1 / n} \cdot \\
& \left(\int_{\mathbb{R}^{n}}\left(\prod_{j=1}^{n}\left(\int_{\mathbb{R}}\left|w_{j}(\mathbf{x})\right|^{n} d x_{n+1}\right)^{1 / n}\right)^{n /(n-1)} d m_{n}\right)^{(n-1) / n} \\
= & \left(\int_{\mathbb{R}^{n}}\left|w_{n+1}(\mathbf{x})\right|^{n} d m_{n}\right)^{1 / n} \cdot \\
& \left(\int_{\mathbb{R}^{n}} \prod_{j=1}^{n}\left(\int_{\mathbb{R}}\left|w_{j}(\mathbf{x})\right|^{n} d x_{n+1}\right)^{1 /(n-1)} d m_{n}\right)^{(n-1) / n}
\end{aligned}
$$

$$
\begin{aligned}
\leq & \left(\int_{\mathbb{R}^{n}}\left|w_{n+1}(\mathbf{x})\right|^{n} d m_{n}\right)^{1 / n} \\
& \left(\prod_{j=1}^{n}\left(\int_{\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}}\left|w_{j}(\mathbf{x})\right|^{n} d x_{n+1}\right) d m_{n-1}\right)^{1 /(n-1)}\right)^{(n-1) / n} \\
= & \left(\int_{\mathbb{R}^{n}}\left|w_{n+1}(\mathbf{x})\right|^{n} d m_{n}\right)^{1 / n} \prod_{j=1}^{n}\left(\int_{\mathbb{R}^{n}}\left|w_{j}(\mathbf{x})\right|^{n} d m_{n}\right)^{1 / n} \\
= & \prod_{j=1}^{n+1}\left(\int_{\mathbb{R}^{n}}\left|w_{j}(\mathbf{x})\right|^{n} d m_{n}\right)^{1 / n}
\end{aligned}
$$

This proves the lemma.
Lemma 38.1.9 If $\phi \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$ and $n \geq 1$, then

$$
\|\phi\|_{n /(n-1)} \leq \frac{1}{\sqrt[n]{n}} \sum_{j=1}^{n}\left\|\frac{\partial \phi}{\partial x_{j}}\right\|_{1}
$$

Proof: The case where $n=1$ is obvious if $n /(n-1)$ is interpreted as $\infty$. Assume then that $n>1$ and note that for $a_{i} \geq 0$,

$$
n \prod_{i=1}^{n} a_{i} \leq\left(\sum_{j=1}^{n} a_{i}\right)^{n}
$$

In fact, the term on the left is one of many terms of the expression on the right. Therefore, taking $n^{\text {th }}$ roots

$$
\prod_{i=1}^{n} a_{i}^{1 / n} \leq \frac{1}{\sqrt[n]{n}} \sum_{j=1}^{n} a_{i}
$$

Then observe that for each $j=1,2, \cdots, n$,

$$
|\phi(\mathbf{x})| \leq \int_{-\infty}^{\infty}\left|\phi_{, j}(\mathbf{x})\right| d x_{j}
$$

so

$$
\begin{aligned}
& \|\phi\|_{n /(n-1)}^{n /(n-1)} \equiv \int_{\mathbb{R}^{n}}|\phi(\mathbf{x})|^{n /(n-1)} d m_{n} \\
\leq & \int_{\mathbb{R}^{n}} \prod_{j=1}^{n}\left(\int_{-\infty}^{\infty}\left|\phi_{, j}(\mathbf{x})\right| d x_{j}\right)^{1 /(n-1)} d m_{n}
\end{aligned}
$$

and from Lemma 38.1.8 this is dominated by

$$
\leq \prod_{j=1}^{n}\left(\int_{\mathbb{R}^{n}}\left|\phi_{, j}(\mathbf{x})\right| d m_{n}\right)^{1 /(n-1)}
$$

Hence $\prod_{i=1}^{n} a_{i}^{1 / n} \leq \frac{1}{\sqrt[n]{n}} \sum_{j=1}^{n} a_{i}$

$$
\begin{aligned}
\|\phi\|_{n /(n-1)} & \leq \prod_{j=1}^{n}\left(\int_{\mathbb{R}^{n}}\left|\phi_{, j}(\mathbf{x})\right| d m_{n}\right)^{1 / n} \\
& \leq \frac{1}{\sqrt[n]{n}} \sum_{j=1}^{n} \int_{\mathbb{R}^{n}}\left|\phi_{, j}(\mathbf{x})\right| d m_{n} \\
& =\frac{1}{\sqrt[n]{n}} \sum_{j=1}^{n}\left\|\phi_{, i}\right\|_{1}
\end{aligned}
$$

and this proves the lemma.
The above lemma is due to Gagliardo and Nirenberg.
With this lemma, it is possible to prove a major embedding theorem which follows.
Theorem 38.1.10 Let $1 \leq p<n$ and $\frac{1}{q}=\frac{1}{p}-\frac{1}{n}$. Then if $f \in W^{1, p}\left(\mathbb{R}^{n}\right)$,

$$
\|f\|_{q} \leq \frac{1}{\sqrt[n]{n}} \frac{(n-1) p}{n-p}\|f\|_{1, p, \mathbb{R}^{n}}
$$

Proof: From the definition of $W^{1, p}\left(\mathbb{R}^{n}\right), C_{c}^{1}\left(\mathbb{R}^{n}\right)$ is dense in $W^{1, p}$. Here $C_{c}^{1}\left(\mathbb{R}^{n}\right)$ is the space of continuous functions having continuous derivatives which have compact support. The desired inequality will be established for such $\phi$ and then the density of this set in $W^{1, p}\left(\mathbb{R}^{n}\right)$ will be exploited to obtain the inequality for all $f \in W^{1, p}\left(\mathbb{R}^{n}\right)$. First note that the case where $p=1$ follows immediately from the above lemma and so it is only necessary to consider the case where $p>1$.

Let $\phi \in C_{c}^{1}\left(\mathbb{R}^{n}\right)$ and consider $|\phi|^{r}$ where $r>1$. Then a short computation shows $|\phi|^{r} \in$ $C_{c}^{1}\left(\mathbb{R}^{n}\right)$ and

$$
\left||\phi|_{, i}^{r}\right|=r|\phi|^{r-1}\left|\phi_{, i}\right| .
$$

Therefore, from Lemma 38.1.9,

$$
\begin{aligned}
& \left(\int|\phi|^{\frac{r n}{n-1}} d m_{n}\right)^{(n-1) / n} \\
\leq & \frac{r}{\sqrt[n]{n}} \sum_{i=1}^{n} \int|\phi|^{r-1}\left|\phi_{, i}\right| d m_{n} \\
\leq & \frac{r}{\sqrt[n]{n}} \sum_{i=1}^{n}\left(\int\left|\phi_{, i}\right|^{p}\right)^{1 / p}\left(\int\left(|\phi|^{r-1}\right)^{p /(p-1)} d m_{n}\right)^{(p-1) / p} .
\end{aligned}
$$

Now choose $r$ such that

$$
\frac{(r-1) p}{p-1}=\frac{r n}{n-1}
$$

That is, let $r=\frac{p(n-1)}{n-p}>1$ and so $\frac{r n}{n-1}=\frac{n p}{n-p}$. Then this reduces to

$$
\left(\int|\phi|^{\frac{n p}{n-p}} d m_{n}\right)^{(n-1) / n} \leq \frac{r}{\sqrt[n]{n}} \sum_{i=1}^{n}\left(\int\left|\phi_{, i}\right|^{p}\right)^{1 / p}\left(\int|\phi|^{\frac{n p}{n-p}} d m_{n}\right)^{(p-1) / p}
$$

Also, $\frac{n-1}{n}-\frac{p-1}{p}=\frac{n-p}{n p}$ and so, dividing both sides by the last term yields

$$
\left(\int|\phi|^{\frac{n p}{n-p}} d m_{n}\right)^{\frac{n-p}{n p}} \leq \frac{r}{\sqrt[n]{n}} \sum_{i=1}^{n}\left(\int\left|\phi_{, i}\right|^{p}\right)^{1 / p} \leq \frac{r}{\sqrt[n]{n}}\|\phi\|_{1, p, \mathbb{R}^{n}}
$$

Letting $q=\frac{n p}{n-p}$, it follows $\frac{1}{q}=\frac{n-p}{n p}=\frac{1}{p}-\frac{1}{n}$ and

$$
\|\phi\|_{q} \leq \frac{r}{\sqrt[n]{n}}\|\phi\|_{1, p, \mathbb{R}^{n}}
$$

Now let $f \in W^{m, p}\left(\mathbb{R}^{n}\right)$ and let $\left\|\phi_{k}-f\right\|_{1, p, \mathbb{R}^{n}} \rightarrow 0$ as $k \rightarrow \infty$. Taking another subsequence, if necessary, you can also assume $\phi_{k}(\mathbf{x}) \rightarrow f(\mathbf{x})$ a.e. Therefore, by Fatou's lemma,

$$
\begin{aligned}
\|f\|_{q} & \leq \lim \inf _{k \rightarrow \infty}\left(\int_{\mathbb{R}^{n}}\left|\phi_{k}(\mathbf{x})\right|^{q} d m_{n}\right)^{1 / q} \\
& \leq \lim _{k \rightarrow \infty} \frac{r}{\sqrt[n]{n}}\left\|\phi_{k}\right\|_{1, p, \mathbb{R}^{n}}=\|f\|_{1, p, \mathbb{R}^{n}}
\end{aligned}
$$

This proves the theorem.
Corollary 38.1.11 Suppose $m p<n$. Then $W^{m, p}\left(\mathbb{R}^{n}\right) \subseteq L^{q}\left(\mathbb{R}^{n}\right)$ where $q=\frac{n p}{n-m p}$ and the identity map, id : $W^{m, p}\left(\mathbb{R}^{n}\right) \rightarrow L^{q}\left(\mathbb{R}^{n}\right)$ is continuous.

Proof: This is true if $m=1$ according to Theorem 38.1.10. Suppose it is true for $m-1$ where $m>1$. If $u \in W^{m, p}\left(\mathbb{R}^{n}\right)$ and $|\alpha| \leq 1$, then $D^{\alpha} u \in W^{m-1, p}\left(\mathbb{R}^{n}\right)$ so by induction, for all such $\alpha$,

$$
D^{\alpha} u \in L^{\frac{n p}{n-(m-1) p}}\left(\mathbb{R}^{n}\right)
$$

Thus $u \in W^{1, q_{1}}\left(\mathbb{R}^{n}\right)$ where

$$
q_{1}=\frac{n p}{n-(m-1) p}
$$

By Theorem 38.1.10, it follows that $u \in L^{q}\left(\mathbb{R}^{n}\right)$ where

$$
\frac{1}{q}=\frac{n-(m-1) p}{n p}-\frac{1}{n}=\frac{n-m p}{n p}
$$

This proves the corollary.
There is another similar corollary of the same sort which is interesting and useful.
Corollary 38.1.12 Suppose $m \geq 1$ and $j$ is a nonnegative integer satisfying $j p<n$. Then

$$
W^{m+j, p}\left(\mathbb{R}^{n}\right) \subseteq W^{m, q}\left(\mathbb{R}^{n}\right)
$$

for

$$
\begin{equation*}
q \equiv \frac{n p}{n-j p} \tag{38.1.16}
\end{equation*}
$$

and the identity map is continuous.

Proof: If $|\alpha| \leq m$, then $D^{\alpha} u \in W^{j, p}\left(\mathbb{R}^{n}\right)$ and so by Corollary 38.1.11, $D^{\alpha} u \in L^{q}\left(\mathbb{R}^{n}\right)$ where $q$ is given above. This means $u \in W^{m, q}\left(\mathbb{R}^{n}\right)$.

The above corollaries imply yet another interesting corollary which involves embeddings in the Holder spaces.

Corollary 38.1.13 Suppose $j p<n<(j+1) p$ and let $m$ be a positive integer. Let $U$ be any bounded open set in $\mathbb{R}^{n}$. Then letting $r_{U}$ denote the restriction to $\bar{U}, r_{U}: W^{m+j, p}\left(\mathbb{R}^{n}\right) \rightarrow$ $C^{m-1, \lambda}(\bar{U})$ is continuous for every $\lambda \leq \lambda_{0} \equiv(j+1)-\frac{n}{p}$ and if $\lambda<(j+1)-\frac{n}{p}$, then $r_{U}$ is compact.

Proof: From Corollary 38.1.12 $W^{m+j, p}\left(\mathbb{R}^{n}\right) \subseteq W^{m, q}\left(\mathbb{R}^{n}\right)$ where $q$ is given by 38.1.16. Therefore,

$$
\frac{n p}{n-j p}>n
$$

and so by Corollary 38.1.7, $W^{m, q}\left(\mathbb{R}^{n}\right) \subseteq C^{m-1, \lambda}(\bar{U})$ for all $\lambda$ satisfying

$$
0<\lambda<1-\frac{(n-j p) n}{n p}=\frac{p(j+1)-n}{p}=(j+1)-\frac{n}{p} .
$$

The assertion about compactness follows from the compactness of the embedding of

$$
C^{m-1, \lambda_{0}}(\bar{U})
$$

into $C^{m-1, \lambda}(\bar{U})$ for $\lambda<\lambda_{0}$. See Lemma 38.1.6.
There are other embeddings of this sort available. You should see Adams [1] for a more complete listing of these. Next are some theorems about compact embeddings. This requires some consideration of which subsets of $L^{p}(U)$ are compact. The main theorem is the following. See [1].

Theorem 38.1.14 Let $K$ be a bounded subset of $L^{p}(U)$ and suppose that for all $\varepsilon>0$, there exist a $\delta>0$ such that if $|\mathbf{h}|<\delta$, then

$$
\begin{equation*}
\int_{\mathbb{R}^{n}}|\widetilde{u}(\mathbf{x}+\mathbf{h})-\widetilde{u}(\mathbf{x})|^{p} d x<\varepsilon^{p} \tag{38.1.17}
\end{equation*}
$$

Suppose also that for each $\varepsilon>0$ there exists an open set, $G \subseteq U$ such that $\bar{G}$ is compact and for all $u \in K$,

$$
\begin{equation*}
\int_{U \backslash \bar{G}}|u(\mathbf{x})|^{p} d x<\varepsilon^{p} \tag{38.1.18}
\end{equation*}
$$

Then $K$ is precompact in $L^{p}\left(\mathbb{R}^{n}\right)$.
Proof: To save fussing first consider the case where $U=\mathbb{R}^{n}$ so that $\widetilde{u}=u$. Suppose the two conditions hold and let $\phi_{k}$ be a mollifier of the form $\phi_{k}(\mathbf{x})=k^{n} \phi(k \mathbf{x})$ where $\operatorname{spt}(\phi) \subseteq B(\mathbf{0}, 1)$. Consider

$$
K_{k} \equiv\left\{u * \phi_{k}: u \in K\right\}
$$

and verify the conditions for the Ascoli Arzela theorem for these functions defined on $\bar{G}$. Say $\|u\|_{p} \leq M$ for all $u \in K$.

First of all, for $u \in K$ and $\mathbf{x} \in \mathbb{R}^{n}$,

$$
\begin{aligned}
\left|u * \phi_{k}(\mathbf{x})\right|^{p} & \leq\left(\int\left|u(\mathbf{x}-\mathbf{y}) \phi_{k}(\mathbf{y})\right| d y\right)^{p} \\
& =\left(\int\left|u(\mathbf{y}) \phi_{k}(\mathbf{x}-\mathbf{y})\right| d y\right)^{p} \\
& \leq \int|u(\mathbf{y})|^{p} \phi_{k}(\mathbf{x}-\mathbf{y}) d y \\
& \leq\left(\sup _{\mathbf{z} \in \mathbb{R}^{n}} \phi_{k}(\mathbf{z})\right) \int|u(\mathbf{y})| d y \leq M\left(\sup _{\mathbf{z} \in \mathbb{R}^{n}} \phi_{k}(\mathbf{z})\right)
\end{aligned}
$$

showing the functions in $K_{k}$ are uniformly bounded.
Next suppose $\mathbf{x}, \mathbf{x}_{1} \in K_{k}$ and consider

$$
\begin{aligned}
& \left|u * \phi_{k}(\mathbf{x})-u * \phi_{k}\left(\mathbf{x}_{1}\right)\right| \\
\leq & \int\left|u(\mathbf{x}-\mathbf{y})-u\left(\mathbf{x}_{1}-\mathbf{y}\right)\right| \phi_{k}(\mathbf{y}) d y \\
\leq & \left(\int\left|u(\mathbf{x}-\mathbf{y})-u\left(\mathbf{x}_{1}-\mathbf{y}\right)\right|^{p} d y\right)^{1 / p}\left(\int \phi_{k}(\mathbf{y})^{q} d y\right)^{q}
\end{aligned}
$$

which by assumption 38.1 .17 is small independent of the choice of $u$ whenever $\left|\mathbf{x}-\mathbf{x}_{1}\right|$ is small enough. Note that $k$ is fixed in the above. Therefore, the set, $K_{k}$ is precompact in $C(\bar{G})$ thanks to the Ascoli Arzela theorem. Next consider how well $u \in K$ is approximated by $u * \phi_{k}$ in $L^{p}\left(\mathbb{R}^{n}\right)$. By Minkowski’s inequality,

$$
\begin{aligned}
& \left(\int\left|u(\mathbf{x})-u * \phi_{k}(\mathbf{x})\right|^{p} d x\right)^{1 / p} \\
\leq & \left(\int\left(\int|u(\mathbf{x})-u(\mathbf{x}-\mathbf{y})| \phi_{k}(\mathbf{y}) d y\right)^{p} d x\right)^{1 / p} \\
\leq & \int_{B\left(\mathbf{0}, \frac{1}{k}\right)} \phi_{k}(\mathbf{y})\left(\int|u(\mathbf{x})-u(\mathbf{x}-\mathbf{y})|^{p} d x\right)^{1 / p} d y
\end{aligned}
$$

Now let $\eta>0$ be given. From 38.1.17 there exists $k$ large enough that for all $u \in K$,

$$
\int_{B\left(\mathbf{0}, \frac{1}{k}\right)} \phi_{k}(\mathbf{y})\left(\int|u(\mathbf{x})-u(\mathbf{x}-\mathbf{y})|^{p} d x\right)^{1 / p} d y \leq \int_{B\left(\mathbf{0}, \frac{1}{k}\right)} \phi_{k}(\mathbf{y}) \eta d y=\eta
$$

Now let $\varepsilon>0$ be given and let $\delta$ and $G$ correspond to $\varepsilon$ as given in the hypotheses and let $1 / k<\delta$ and also $k$ is large enough that for all $u \in K$,

$$
\left\|u-u * \phi_{k}\right\|_{p}<\varepsilon
$$

as in the above inequality. By the Ascoli Arzela theorem there exists an

$$
\left(\frac{\varepsilon}{m(\bar{G}+B(\mathbf{0}, 1))}\right)^{1 / p}
$$

net for $K_{k}$ in $C(\bar{G})$. That is, there exist $\left\{u_{i}\right\}_{i=1}^{m} \subseteq K$ such that for any $u \in K$,

$$
\left\|u * \phi_{k}-u_{j} * \phi_{k}\right\|_{\infty}<\left(\frac{\varepsilon}{m(\bar{G}+B(\mathbf{0}, 1))}\right)^{1 / p}
$$

for some $j$. Letting $u \in K$ be given, let $u_{j} \in\left\{u_{i}\right\}_{i=1}^{m} \subseteq K$ be such that the above inequality holds. Then

$$
\begin{aligned}
&\left\|u-u_{j}\right\|_{p} \leq\left\|u-u * \phi_{k}\right\|_{p}+\left\|u * \phi_{k}-u_{j} * \phi_{k}\right\|_{p}+\left\|u_{j} * \phi_{k}-u_{j}\right\|_{p} \\
&<2 \varepsilon+\left\|u * \phi_{k}-u_{j} * \phi_{k}\right\|_{p} \\
& \leq 2 \varepsilon+\left(\int_{\bar{G}+B(\mathbf{0}, 1)}\left|u * \phi_{k}-u_{j} * \phi_{k}\right|^{p} d x\right)^{1 / p} \\
&+\left(\int_{\mathbb{R}^{n} \backslash(\bar{G}+B(\mathbf{0}, 1))}\left|u * \phi_{k}-u_{j} * \phi_{k}\right|^{p} d x\right)^{1 / p} \\
& \leq 2 \varepsilon+\varepsilon^{1 / p} \\
&+\left(\int_{\mathbb{R}^{n} \backslash(\bar{G}+B(\mathbf{0}, 1))}\left(\int\left|u(\mathbf{x}-\mathbf{y})-u_{j}(\mathbf{x}-\mathbf{y})\right| \phi_{k}(\mathbf{y}) d y\right)^{p} d x\right)^{1 / p} \\
& \leq 2 \varepsilon+\varepsilon^{1 / p} \\
&+\int \phi_{k}(\mathbf{y})\left(\int_{\mathbb{R}^{n} \backslash(\bar{G}+B(\mathbf{0}, 1))}\left(|u(\mathbf{x}-\mathbf{y})|+\left|u_{j}(\mathbf{x}-\mathbf{y})\right|\right)^{p} d x\right)^{1 / p} d y \\
& \leq 2 \varepsilon+\varepsilon^{1 / p}+\int \phi_{k}(\mathbf{y})\left(\int_{\mathbb{R}^{n} \backslash \bar{G}}\left(|u(\mathbf{x})|+\left|u_{j}(\mathbf{x})\right|\right)^{p} d x\right)^{1 / p} d y \\
& \leq 2 \varepsilon+\varepsilon^{1 / p}+2^{p-1} \int \phi_{k}(\mathbf{y})\left(\int_{\mathbb{R}^{n} \backslash \bar{G}}\left(|u(\mathbf{x})|^{p}+\left|u_{j}(\mathbf{x})\right|^{p}\right) d x\right)^{1 / p} d y \\
& \leq 2 \varepsilon+\varepsilon^{1 / p}+2^{p-1} 2^{1 / p} \varepsilon
\end{aligned}
$$

and since $\varepsilon>0$ is arbitrary, this shows that $K$ is totally bounded and is therefore precompact.

Now for an arbitrary open set, $\underset{\sim}{U}$ and $K$ given in the hypotheses of the theorem, let $\widetilde{K} \equiv\{\widetilde{u}: u \in K\}$ and observe that $\widetilde{K}$ is precompact in $L^{p}\left(\mathbb{R}^{n}\right)$. But this is the same as saying that $K$ is precompact in $L^{p}(U)$. This proves the theorem.

Actually the converse of the above theorem is also true [1] but this will not be needed so I have left it as an exercise for anyone interested.

Lemma 38.1.15 Let $u \in W^{1,1}(U)$ for $U$ an open set and let $\phi \in C_{c}^{\infty}(U)$. Then there exists a constant,

$$
C\left(\phi,\|u\|_{1,1, U}\right)
$$

depending only on the indicated quantities such that whenever $\mathbf{v} \in \mathbb{R}^{n}$ with

$$
|\mathbf{v}|<\operatorname{dist}\left(\operatorname{spt}(\phi), U^{C}\right)
$$

it follows that

$$
\int_{\mathbb{R}^{n}}|\widetilde{\phi u}(\mathbf{x}+\mathbf{v})-\widetilde{\phi u}(\mathbf{x})| d x \leq C\left(\phi,\|u\|_{1,1, U}\right)|\mathbf{v}| .
$$

Proof: First suppose $u \in C^{\infty}(\bar{U})$. Then for any $\mathbf{x} \in \operatorname{spt}(\phi) \cup(\operatorname{spt}(\phi)-\mathbf{v}) \equiv G_{\mathbf{v}}$, the chain rule implies

$$
\begin{aligned}
|\phi u(\mathbf{x}+\mathbf{v})-\phi u(\mathbf{x})| & \leq \int_{0}^{1} \sum_{i=1}^{n}\left|(\phi u)_{, i}(\mathbf{x}+t \mathbf{v}) v_{i}\right| d t \\
& \leq \int_{0}^{1} \sum_{i=1}^{n}\left|\left(\phi_{, i} u+u_{, i} \phi\right)(\mathbf{x}+t \mathbf{v})\right| d t|\mathbf{v}|
\end{aligned}
$$

Therefore, for such $u$,

$$
\begin{aligned}
& \int_{\mathbb{R}^{n}}|\widetilde{\phi u}(\mathbf{x}+\mathbf{v})-\widetilde{\phi u}(\mathbf{x})| d x \\
= & \int_{G_{\mathbf{v}}}|\phi u(\mathbf{x}+\mathbf{v})-\phi u(\mathbf{x})| d x \\
\leq & \int_{G_{\mathbf{v}}} \int_{0}^{1} \sum_{i=1}^{n}\left|\left(\phi_{, i} u+u_{, i} \phi\right)(\mathbf{x}+t \mathbf{v})\right| d t d x|\mathbf{v}| \\
\leq & \int_{0}^{1} \int_{G_{\mathbf{v}}} \sum_{i=1}^{n}\left|\left(\phi_{, i} u+u_{, i} \phi\right)(\mathbf{x}+t \mathbf{v})\right| d x d t|\mathbf{v}| \\
\leq & C\left(\phi,\|u\|_{1,1, U}\right)|\mathbf{v}|
\end{aligned}
$$

where $C$ is a continuous function of $\|u\|_{1,1, U}$. Now for general $u \in W^{1,1}(U)$, let $u_{k} \rightarrow u$ in $W^{1,1}(U)$ where $u_{k} \in C^{\infty}(\bar{U})$. Then for $|\mathbf{v}|<\operatorname{dist}\left(\operatorname{spt}(\phi), U^{C}\right)$,

$$
\begin{aligned}
& \int_{\mathbb{R}^{n}}|\widetilde{\phi u}(\mathbf{x}+\mathbf{v})-\widetilde{\phi u}(\mathbf{x})| d x \\
= & \int_{G_{\mathbf{v}}}|\phi u(\mathbf{x}+\mathbf{v})-\phi u(\mathbf{x})| d x \\
= & \lim _{k \rightarrow \infty} \int_{G_{\mathbf{v}}}\left|\phi u_{k}(\mathbf{x}+\mathbf{v})-\phi u_{k}(\mathbf{x})\right| d x \\
\leq & \lim _{k \rightarrow \infty} C\left(\phi,\left\|u_{k}\right\|_{1,1, U}\right)|\mathbf{v}| \\
= & C\left(\phi,\|u\|_{1,1, U}\right)|\mathbf{v}| .
\end{aligned}
$$

This proves the lemma.

Lemma 38.1.16 Let $U$ be a bounded open set and define for $p>1$

$$
\begin{equation*}
S \equiv\left\{u \in W^{1,1}(U) \cap L^{p}(U):\|u\|_{1,1, U}+\|u\|_{L^{p}(U)} \leq M\right\} \tag{38.1.19}
\end{equation*}
$$

and let $\phi \in C_{c}^{\infty}(U)$ and

$$
\begin{equation*}
S_{1} \equiv\{u \phi: u \in S\} \tag{38.1.20}
\end{equation*}
$$

Then $S_{1}$ is precompact in $L^{q}(U)$ where $1 \leq q<p$.
Proof: This depends on Theorem 38.1.14. The second condition is satisfied by taking $G \equiv \operatorname{spt}(\phi)$. Thus, for $w \in S_{1}$,

$$
\int_{U \backslash \bar{G}}|w(\mathbf{x})|^{q} d x=0<\varepsilon^{p} .
$$

It remains to satisfy the first condition. It is necessary to verify there exists $\delta>0$ such that if $|\mathbf{v}|<\delta$, then

$$
\begin{equation*}
\int_{\mathbb{R}^{n}}|\widetilde{\phi} u(\mathbf{x}+\mathbf{v})-\widetilde{\phi u}(\mathbf{x})|^{q} d x<\varepsilon^{p} \tag{38.1.21}
\end{equation*}
$$

Let $\operatorname{spt}(\phi) \cup(\operatorname{spt}(\phi)-\mathbf{v}) \equiv G_{\mathbf{v}}$. Now if $h$ is any measurable function, and if $\theta \in(0,1)$ is chosen small enough that $\theta q<1$,

$$
\begin{align*}
\int_{G_{\mathbf{v}}}|h|^{q} d x & =\int_{G_{\mathbf{V}}}|h|^{\theta q}|h|^{(1-\theta) q} d x \\
& \leq\left(\int_{G_{\mathbf{v}}}|h| d x\right)^{\theta q}\left(\int_{G_{\mathbf{v}}}\left(|h|^{(1-\theta) q}\right)^{\frac{1}{1-\theta q}}\right)^{1-\theta q} \\
& =\left(\int_{G_{\mathbf{v}}}|h| d x\right)^{\theta q}\left(\int_{G_{\mathbf{v}}}|h|^{\frac{(1-\theta) q}{1-\theta q}}\right)^{1-\theta q} \tag{38.1.22}
\end{align*}
$$

Now let $\theta$ also be small enough that there exists $r>1$ such that

$$
r \frac{(1-\theta) q}{1-\theta q}=p
$$

and use Holder's inequality in the last factor of the right side of 38.1.22. Then 38.1.22 is dominated by

$$
\begin{aligned}
& \left(\int_{G_{\mathbf{v}}}|h| d x\right)^{\theta q}\left(\int_{G_{\mathbf{v}}}|h|^{p}\right)^{\frac{1-\theta q}{r}}\left(\int_{G_{\mathbf{v}}} 1 d x\right)^{1 / r^{\prime}} \\
= & C\left(\|h\|_{L^{p}\left(G_{\mathbf{v}}\right)}, m_{n}\left(G_{\mathbf{v}}\right)\right)\left(\int_{G_{\mathbf{v}}}|h| d x\right)^{\theta q}
\end{aligned}
$$

Therefore, for $u \in S$,

$$
\int_{\mathbb{R}^{n}}|\widetilde{\phi u}(\mathbf{x}+\mathbf{v})-\widetilde{\phi} u(\mathbf{x})|^{q} d x=\int_{G_{\mathbf{v}}}|\phi u(\mathbf{x}+\mathbf{v})-\phi u(\mathbf{x})|^{q} d x \leq
$$

$$
\begin{gather*}
C\left(\|\phi u(\cdot+\mathbf{v})-\phi u(\cdot)\|_{L^{p}\left(G_{\mathbf{v}}\right)}, m_{n}\left(G_{\mathbf{v}}\right)\right)\left(\int_{G_{\mathbf{v}}}|\phi u(\mathbf{x}+\mathbf{v})-\phi u(\mathbf{x})| d x\right)^{\theta q} \\
\leq C\left(2\|\phi u(\cdot)\|_{L^{p}(U)}, m_{n}(U)\right)\left(\int_{G_{\mathbf{v}}}|\phi u(\mathbf{x}+\mathbf{v})-\phi u(\mathbf{x})| d x\right)^{\theta q} \\
\leq C\left(\phi, M, m_{n}(U)\right)\left(\int_{G_{\mathbf{v}}}|\phi u(\mathbf{x}+\mathbf{v})-\phi u(\mathbf{x})| d x\right)^{\theta q} \\
\quad=C\left(\phi, M, m_{n}(U)\right)\left(\int_{\mathbb{R}^{n}}|\widetilde{\phi} u(\mathbf{x}+\mathbf{v})-\widetilde{\phi u}(\mathbf{x})| d x\right)^{\theta q} \tag{38.1.23}
\end{gather*}
$$

Now by Lemma 38.1.15,

$$
\begin{equation*}
\int_{\mathbb{R}^{n}}|\widetilde{\phi u}(\mathbf{x}+\mathbf{v})-\widetilde{\phi u}(\mathbf{x})| d x \leq C\left(\phi,\|u\|_{1,1, U}\right)|\mathbf{v}| \tag{38.1.24}
\end{equation*}
$$

and so from 38.1.23 and 38.1.24, and adjusting the constants

$$
\begin{aligned}
\int_{\mathbb{R}^{n}}|\widetilde{\phi u}(\mathbf{x}+\mathbf{v})-\widetilde{\phi u}(\mathbf{x})|^{q} d x & \leq C\left(\phi, M, m_{n}(U)\right)\left(C\left(\phi,\|u\|_{1,1, U}\right)|\mathbf{v}|\right)^{\theta q} \\
& =C\left(\phi, M, m_{n}(U)\right)|\mathbf{v}|^{\theta q}
\end{aligned}
$$

which verifies 38.1 .21 whenever $|\mathbf{v}|$ is sufficiently small. This proves the lemma because the conditions of Theorem 38.1.14 are satisfied.

Theorem 38.1.17 Let $U$ be a bounded open set and define for $p>1$

$$
\begin{equation*}
S \equiv\left\{u \in W^{1,1}(U) \cap L^{p}(U):\|u\|_{1,1, U}+\|u\|_{L^{p}(U)} \leq M\right\} \tag{38.1.25}
\end{equation*}
$$

Then $S$ is precompact in $L^{q}(U)$ where $1 \leq q<p$.
Proof: If suffices to show that every sequence, $\left\{u_{k}\right\}_{k=1}^{\infty} \subseteq S$ has a subsequence which converges in $L^{q}(U)$. Let $\left\{K_{m}\right\}_{m=1}^{\infty}$ denote a sequence of compact subsets of $U$ with the property that $K_{m} \subseteq K_{m+1}$ for all $m$ and $\cup_{m=1}^{\infty} K_{m}=U$. Now let $\phi_{m} \in C_{c}^{\infty}(U)$ such that $\phi_{m}(\mathbf{x}) \in[0,1]$ and $\phi_{m}(\mathbf{x})=1$ for all $\mathbf{x} \in K_{m}$. Let $S_{m} \equiv\left\{\phi_{m} u: u \in S\right\}$. By Lemma 38.1.16 there exists a subsequence of $\left\{u_{k}\right\}_{k=1}^{\infty}$, denoted here by $\left\{u_{1, k}\right\}_{k=1}^{\infty}$ such that $\left\{\phi_{1} u_{1, k}\right\}_{k=1}^{\infty}$ converges in $L^{q}(U)$. Now $S_{2}$ is also precompact in $L^{q}(U)$ and so there exists a subsequence of $\left\{u_{1, k}\right\}_{k=1}^{\infty}$, denoted by $\left\{u_{2, k}\right\}_{k=1}^{\infty}$ such that $\left\{\phi_{2} u_{2, k}\right\}_{k=1}^{\infty}$ converges in $L^{2}(U)$. Thus it is also the case that $\left\{\phi_{1} u_{2, k}\right\}_{k=1}^{\infty}$ converges in $L^{q}(U)$. Continue taking subsequences in this manner such that for all $l \leq m,\left\{\phi_{l} u_{m, k}\right\}_{k=1}^{\infty}$ converges in $L^{q}(U)$. Let $\left\{w_{m}\right\}_{m=1}^{\infty}=\left\{u_{m, m}\right\}_{m=1}^{\infty}$ so that $\left\{w_{k}\right\}_{k=m}^{\infty}$ is a subsequence of $\left\{u_{m, k}\right\}_{k=1}^{\infty}$. Then it follows for all $k,\left\{\phi_{k} w_{m}\right\}_{m=1}^{\infty}$ must converge in $L^{q}(U)$. For $u \in S$,

$$
\begin{aligned}
\left\|u-\phi_{k} u\right\|_{L^{q}(U)}^{q} & =\int_{U}|u|^{q}\left(1-\phi_{k}\right)^{q} d x \\
& \leq\left(\int_{U}|u|^{p} d x\right)^{q / p}\left(\int_{U}\left(1-\phi_{k}\right)^{q r} d x\right)^{1 / r} \\
& \leq M\left(\int_{U}\left(1-\phi_{k}\right)^{q r} d x\right)^{1 / r}
\end{aligned}
$$

where $q / p+1 / r=1$. Now $\phi_{l}(\mathbf{x}) \rightarrow \mathscr{X}_{U}(\mathbf{x})$ and so the integrand in the last integral converges to 0 by the dominated convergence theorem. Therefore, $k$ may be chosen large enough that for all $u \in S$,

$$
\left\|u-\phi_{k} u\right\|_{L^{q}(U)}^{q} \leq\left(\frac{\varepsilon}{3}\right)^{q} .
$$

Fix such a value of $k$. Then

$$
\begin{gathered}
\left\|w_{q}-w_{p}\right\|_{L^{q}(U)} \leq \\
\left\|w_{q}-\phi_{k} w_{q}\right\|_{L^{q}(U)}+\left\|\phi_{k} w_{q}-\phi_{k} w_{p}\right\|_{L^{q}(U)}+\left\|w_{p}-\phi_{k} w_{p}\right\|_{L^{q}(U)} \\
\leq \frac{2 \varepsilon}{3}+\left\|\phi_{k} w_{q}-\phi_{k} w_{p}\right\|_{L^{q}(U)}
\end{gathered}
$$

But $\left\{\phi_{k} w_{m}\right\}_{m=1}^{\infty}$ converges in $L^{q}(U)$ and so the last term in the above is less than $\varepsilon / 3$ whenever $p, q$ are large enough. Thus $\left\{w_{m}\right\}_{m=1}^{\infty}$ is a Cauchy sequence and must therefore converge in $L^{q}(U)$. This proves the theorem.

### 38.2 An Extension Theorem

Definition 38.2.1 An open subset, $U$, of $\mathbb{R}^{n}$ has a Lipschitz boundary if it satisfies the following conditions. For each $p \in \partial U \equiv \bar{U} \backslash U$, there exists an open set, $Q$, containing $p$, an open interval $(a, b)$, a bounded open box $B \subseteq \mathbb{R}^{n-1}$, and an orthogonal transformation $R$ such that

$$
\begin{gather*}
R Q=B \times(a, b),  \tag{38.2.26}\\
R(Q \cap U)=\left\{\mathbf{y} \in \mathbb{R}^{n}: \widehat{\mathbf{y}} \in B, a<y_{n}<g(\widehat{\mathbf{y}})\right\} \tag{38.2.27}
\end{gather*}
$$

where $g$ is Lipschitz continuous on $\bar{B}, a<\min \{g(\mathbf{x}): \mathbf{x} \in \bar{B}\}$, and

$$
\widehat{\mathbf{y}} \equiv\left(y_{1}, \cdots, y_{n-1}\right) .
$$

Letting $W=Q \cap U$ the following picture describes the situation.


The following lemma is important.
Lemma 38.2.2 If $U$ is an open subset of $\mathbb{R}^{n}$ which has a Lipschitz boundary, then it satisfies the segment condition and so $X^{m, p}(U)=W^{m, p}(U)$.

Proof: For $\mathbf{x} \in \partial U$, simply look at a single open set, $Q_{\mathbf{x}}$ described in the above which contains $\mathbf{x}$. Then consider an open set whose intersection with $U$ is of the form

$$
R^{T}\left(\left\{\mathbf{y}: \widehat{\mathbf{y}} \in B, g(\widehat{\mathbf{y}})-\varepsilon<y_{n}<g(\widehat{\mathbf{y}})\right\}\right)
$$

and a vector of the form $\varepsilon R^{T}\left(-\mathbf{e}_{n}\right)$ where $\varepsilon$ is chosen smaller than $\min \{g(\mathbf{x}): \mathbf{x} \in \bar{B}\}-a$. There is nothing to prove for points of $U$.

One way to extend many of the above theorems to more general open sets than $\mathbb{R}^{n}$ is through the use of an appropriate extension theorem. In this section, a fairly general one will be presented.

Lemma 38.2.3 Let $B \times(a, b)$ be as described in Definition 38.2.1 and let

$$
V^{-} \equiv\left\{\left(\widehat{\mathbf{y}}, y_{n}\right): y_{n}<g(\widehat{\mathbf{y}})\right\}, V^{+} \equiv\left\{\left(\widehat{\mathbf{y}}, y_{n}\right): y_{n}>g(\widehat{\mathbf{y}})\right\}
$$

for $g$ a Lipschitz function of the sort described in this definition. Suppose $u^{+}$and $u^{-}$are Lipschitz functions defined on $\overline{V^{+}}$and $\overline{V^{-}}$respectively and suppose that $u^{+}(\widehat{\mathbf{y}}, g(\widehat{\mathbf{y}}))=$ $u^{-}(\widehat{\mathbf{y}}, g(\widehat{\mathbf{y}}))$ for all $\widehat{\mathbf{y}} \in B$. Let

$$
u\left(\widehat{\mathbf{y}}, y_{n}\right) \equiv \begin{cases}u^{+}\left(\widehat{\mathbf{y}}, y_{n}\right) & \text { if }\left(\widehat{\mathbf{y}}, y_{n}\right) \in V^{+} \\ u^{-}\left(\widehat{\mathbf{y}}, y_{n}\right) & \text { if }\left(\widehat{\mathbf{y}}, y_{n}\right) \in V^{-}\end{cases}
$$

and suppose $\operatorname{spt}(u) \subseteq B \times(a, b)$. Then extending $u$ to be 0 off of $B \times(a, b)$, $u$ is continuous and the weak partial derivatives, $u_{, i}$, are all in $L^{\infty}\left(\mathbb{R}^{n}\right) \cap L^{p}\left(\mathbb{R}^{n}\right)$ for all $p>1$ and $u_{, i}=$ $\left(u^{+}\right)_{, i}$ on $V^{+}$and $u_{, i}=\left(u^{-}\right)_{, i}$ on $V^{-}$.

Proof: Consider the following picture which is descriptive of the situation.


Note first that $u$ is Lipschitz continuous. To see this, consider $\left|u\left(\mathbf{y}_{1}\right)-u\left(\mathbf{y}_{2}\right)\right|$ where $\left(\widehat{\mathbf{y}}_{i}, y_{n}^{i}\right)=\mathbf{y}_{i}$. There are various cases to consider depending on whether $y_{n}^{i}$ is above $g\left(\widehat{\mathbf{y}}_{i}\right)$. Suppose $y_{n}^{1}<g\left(\widehat{\mathbf{y}}_{1}\right)$ and $y_{n}^{2}>g\left(\widehat{\mathbf{y}}_{2}\right)$. Then letting $K \geq \max \left(\operatorname{Lip}\left(u^{+}\right), \operatorname{Lip}\left(u^{-}\right), \operatorname{Lip}(g)\right)$,

$$
\left|u\left(\widehat{\mathbf{y}}_{1}, y_{n}^{1}\right)-u\left(\widehat{\mathbf{y}}_{2}, y_{n}^{2}\right)\right| \leq
$$

$$
\begin{aligned}
& \left|u\left(\widehat{\mathbf{y}}_{1}, y_{n}^{1}\right)-u\left(\widehat{\mathbf{y}}_{2}, y_{n}^{1}\right)\right|+\left|u\left(\widehat{\mathbf{y}}_{2}, y_{n}^{1}\right)-u\left(\widehat{\mathbf{y}}_{2}, g\left(\widehat{\mathbf{y}}_{2}\right)\right)\right| \\
& +\left|u\left(\widehat{\mathbf{y}}_{2}, g\left(\widehat{\mathbf{y}}_{2}\right)\right)-u\left(\widehat{\mathbf{y}}_{2}, y_{n}^{2}\right)\right| \\
\leq & K\left|\widehat{\mathbf{y}}_{1}-\widehat{\mathbf{y}}_{2}\right|+K\left[\left|g\left(\widehat{\mathbf{y}}_{2}\right)-g\left(\widehat{\mathbf{y}}_{1}\right)\right|+g\left(\widehat{\mathbf{y}}_{1}\right)-y_{n}^{1}+y_{n}^{2}-g\left(\widehat{\mathbf{y}}_{2}\right)\right] \\
\leq & \left(2 K+K^{2}\right)\left|\widehat{\mathbf{y}}_{1}-\widehat{\mathbf{y}}_{2}\right|+K\left|y_{n}^{1}-y_{n}^{2}\right| \\
= & \left(2 K+K^{2}\right)\left(\left|\widehat{\mathbf{y}}_{1}-\widehat{\mathbf{y}}_{2}\right|+\left|y_{n}^{1}-y_{n}^{2}\right|\right) \leq\left(2 K+K^{2}\right) \sqrt{2}\left|\mathbf{y}_{1}-\mathbf{y}_{2}\right|
\end{aligned}
$$

The other cases are similar. Thus $u$ is a Lipschitz continuous function which has compact support. By Corollary 36.5 . 4 on Page 1287 it follows that $u_{, i} \in L^{\infty}\left(\mathbb{R}^{n}\right) \cap L^{p}\left(\mathbb{R}^{n}\right)$ for all $p>1$. It remains to verify $u_{, i}=\left(u^{+}\right)_{, i}$ on $V^{+}$and $u_{, i}=\left(u^{-}\right)_{, i}$ on $V^{-}$. The last claim is obvious from the definition of weak derivatives.

Lemma 38.2.4 In the situation of Lemma 38.2.3 let $u \in C^{1}\left(\overline{V^{-}}\right) \cap C_{c}^{1}(B \times(a, b))^{3}$ and define

$$
w\left(\widehat{\mathbf{y}}, y_{n}\right) \equiv\left\{\begin{array}{l}
u\left(\widehat{\mathbf{y}}, y_{n}\right) \text { if } \widehat{\mathbf{y}} \in B \text { and } y_{n} \leq g(\widehat{\mathbf{y}}) \\
u\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-y_{n}\right), \text { if } \widehat{\mathbf{y}} \in B \text { and } y_{n}>g(\widehat{\mathbf{y}}) \\
0 \text { if } \widehat{\mathbf{y}} \notin B .
\end{array}\right.
$$

Then $w \in W^{1, p}\left(\mathbb{R}^{n}\right)$ and there exists a constant, $C$ depending only on $\operatorname{Lip}(g)$ and dimension such that

$$
\|w\|_{W^{1, p}\left(\mathbb{R}^{n}\right)} \leq C\|u\|_{W^{1, p}\left(V^{-}\right)}
$$

Denote $w$ by $E_{0} u$. Thus $E_{0}(u)(\mathbf{y})=u(\mathbf{y})$ for all $\mathbf{y} \in V^{-}$but $E_{0} u=w$ is defined on all of $\mathbb{R}^{n}$. Also, $E_{0}$ is a linear mapping.

Proof: As in the previous lemma, $w$ is Lipschitz continuous and has compact support so it is clear $w \in W^{1, p}\left(\mathbb{R}^{n}\right)$. The main task is to find $w_{i}$ for $\widehat{\mathbf{y}} \in B$ and $y_{n}>g(\widehat{\mathbf{y}})$ and then to extract an estimate of the right sort. Denote by $U$ the set of points of $\mathbb{R}^{n}$ with the property that $\left(\widehat{\mathbf{y}}, y_{n}\right) \in U$ if and only if $\widehat{\mathbf{y}} \notin B$ or $\widehat{\mathbf{y}} \in B$ and $y_{n}>g(\widehat{\mathbf{y}})$. Then letting $\phi \in C_{c}^{\infty}(U)$, suppose first that $i<n$. Then

$$
\begin{gather*}
\int_{U} w\left(\widehat{\mathbf{y}}, y_{n}\right) \phi_{, i}(\mathbf{y}) d y \\
\equiv \lim _{h \rightarrow 0} \int_{U} \phi(\mathbf{y}) \frac{u\left(\widehat{\mathbf{y}}-h \mathbf{e}_{i}^{n-1}, 2 g\left(\widehat{\mathbf{y}}-h \mathbf{e}_{i}^{n-1}\right)-y_{n}\right)-u\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-y_{n}\right)}{h} d y  \tag{38.2.28}\\
=\lim _{h \rightarrow 0}\left\{\frac { - 1 } { h } \int _ { U } \phi ( \mathbf { y } ) \left[D_{1} u\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-y_{n}\right)\left(h \mathbf{e}_{i}^{n-1}\right)\right.\right. \\
\left.+2 D_{2} u\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-y_{n}\right)\left(g\left(\widehat{\mathbf{y}}-h \mathbf{e}_{i}^{n-1}\right)-g(\widehat{\mathbf{y}})\right)\right] d y \\
\left.+\frac{-1}{h} \int_{U} \phi(\mathbf{y})\left[o\left(g\left(\widehat{\mathbf{y}}-h \mathbf{e}_{i}^{n-1}\right)-g(\widehat{\mathbf{y}})\right)+o(h)\right] d y\right\}
\end{gather*}
$$

[^30]where $\mathbf{e}_{i}^{n-1}$ is the unit vector in $\mathbb{R}^{n-1}$ having all zeros except for a 1 in the $i^{t h}$ position. Now by Rademacher's theorem, $D g(\widehat{\mathbf{y}})$ exists for a.e. $\widehat{\mathbf{y}}$ and so except for a set of measure zero, the expression, $o\left(g\left(\widehat{\mathbf{y}}-h \mathbf{e}_{i}^{n-1}\right)-g(\widehat{\mathbf{y}})\right)$ is $o(h)$ and also for $\widehat{\mathbf{y}}$ not in the exceptional set,
$$
g\left(\widehat{\mathbf{y}}-h \mathbf{e}_{i}^{n-1}\right)-g(\widehat{\mathbf{y}})=-h D g(\widehat{\mathbf{y}}) \mathbf{e}_{i}^{n-1}+o(h)
$$

Therefore, since the integrand in 38.2.28 has compact support and because of the Lipschitz continuity of all the functions, the dominated convergence theorem may be applied to obtain

$$
\begin{gathered}
\int_{U} w\left(\widehat{\mathbf{y}}, y_{n}\right) \phi_{, i}(\mathbf{y}) d y= \\
\int_{U} \phi(\mathbf{y})\left[-D_{1} u\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-y_{n}\right)\left(\mathbf{e}_{i}^{n-1}\right)+2 D_{2} u\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-y_{n}\right)\left(D g(\widehat{\mathbf{y}}) \mathbf{e}_{i}^{n-1}\right)\right] d y \\
=\int_{U} \phi(\mathbf{y})\left[-\frac{\partial u}{\partial y_{i}}\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-y_{n}\right)+2 \frac{\partial u}{\partial y_{n}}\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-y_{n}\right) \frac{\partial g(\widehat{\mathbf{y}})}{\partial y_{i}}\right] d y
\end{gathered}
$$

and so

$$
\begin{equation*}
w_{, i}(\mathbf{y})=\frac{\partial u}{\partial y_{i}}\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-y_{n}\right)-2 \frac{\partial u}{\partial y_{n}}\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-y_{n}\right) \frac{\partial g(\widehat{\mathbf{y}})}{\partial y_{i}} \tag{38.2.29}
\end{equation*}
$$

whenever $i<n$ which is what you would expect from a formal application of the chain rule. Next suppose $i=n$.

$$
\begin{gathered}
\int_{U} w\left(\widehat{\mathbf{y}}, y_{n}\right) \phi_{, n}(\mathbf{y}) d y \\
=\lim _{h \rightarrow 0}-\int_{U} \frac{u\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-\left(y_{n}+h\right)\right)-u\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-y_{n}\right)}{h} \phi(\mathbf{y}) d y \\
=\lim _{h \rightarrow 0} \int_{U} \frac{D_{2} u\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-y_{n}\right) h+o(h)}{h} \phi(\mathbf{y}) d y \\
=\int_{U} \frac{\partial u}{\partial y_{n}}\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-y_{n}\right) \phi(\mathbf{y}) d y
\end{gathered}
$$

showing that

$$
\begin{equation*}
w_{, n}(\mathbf{y})=\frac{-\partial u}{\partial y_{n}}\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-y_{n}\right) \tag{38.2.30}
\end{equation*}
$$

which is also expected.
From the definnition, for $\mathbf{y} \in \mathbb{R}^{n} \backslash U \equiv\left\{\left(\widehat{\mathbf{y}}, y_{n}\right): y_{n} \leq g(\widehat{\mathbf{y}})\right\}$ it follows $w_{, i}=u_{, i}$ and on $U, w_{, i}$ is given by 38.2.29 and 38.2.30. Consider $\left\|w_{, i}\right\|_{L^{p}(U)}^{p}$ for $i<n$. From 38.2.29

$$
\begin{aligned}
&\left\|w_{, i}\right\|_{L^{p}(U)}^{p}=\int_{U} \mid \frac{\partial u}{\partial y_{i}}\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-y_{n}\right)-\left.2 \frac{\partial u}{\partial y_{n}}\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-y_{n}\right) \frac{\partial g(\widehat{\mathbf{y}})}{\partial y_{i}}\right|^{p} d y \\
& \quad \leq 2^{p-1} \int_{U}\left|\frac{\partial u}{\partial y_{i}}\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-y_{n}\right)\right|^{p} \\
&+2^{p}\left|\frac{\partial u}{\partial y_{n}}\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-y_{n}\right)\right|^{p} \operatorname{Lip}(g)^{p} d y
\end{aligned}
$$

$$
\begin{gathered}
\leq 4^{p}\left(1+\operatorname{Lip}(g)^{p}\right) \int_{U}\left|\frac{\partial u}{\partial y_{i}}\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-y_{n}\right)\right|^{p} \\
+\left|\frac{\partial u}{\partial y_{n}}\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-y_{n}\right)\right|^{p} d y \\
=4^{p}\left(1+\operatorname{Lip}(g)^{p}\right) \int_{B} \int_{g(\widehat{\mathbf{y}})}^{\infty}\left|\frac{\partial u}{\partial y_{i}}\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-y_{n}\right)\right|^{p} \\
+\left|\frac{\partial u}{\partial y_{n}}\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-y_{n}\right)\right|^{p} d y_{n} d \widehat{y} \\
=4^{p}\left(1+\operatorname{Lip}(g)^{p}\right) \int_{B} \int_{-\infty}^{g(\widehat{\mathbf{y}})}\left|\frac{\partial u}{\partial y_{i}}\left(\widehat{\mathbf{y}}, z_{n}\right)\right|^{p}+\left|\frac{\partial u}{\partial y_{n}}\left(\widehat{\mathbf{y}}, z_{n}\right)\right|^{p} d z_{n} d \widehat{y} \\
=4^{p}\left(1+\operatorname{Lip}(g)^{p}\right) \int_{B} \int_{a}^{g(\hat{\mathbf{y}})}\left|\frac{\partial u}{\partial y_{i}}\left(\widehat{\mathbf{y}}, z_{n}\right)\right|^{p} \\
+\left|\frac{\partial u}{\partial y_{n}}\left(\widehat{\mathbf{y}}, z_{n}\right)\right|^{p} d z_{n} d \widehat{y} \leq 4^{p}\left(1+\operatorname{Lip}(g)^{p}\right)\|u\|_{1, p, V^{-}}^{p}
\end{gathered}
$$

Now by similar reasoning,

$$
\begin{aligned}
\left\|w_{, n}\right\|_{L^{p}(U)}^{p} & =\int_{U}\left|\frac{-\partial u}{\partial y_{n}}\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-y_{n}\right)\right|^{p} d y \\
& =\int_{B} \int_{g(\widehat{\mathbf{y}})}^{\infty}\left|\frac{-\partial u}{\partial y_{n}}\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-y_{n}\right)\right|^{p} d y_{n} d \widehat{y} \\
& =\int_{B} \int_{a}^{g(\widehat{\mathbf{y}})}\left|\frac{-\partial u}{\partial y_{n}}\left(\widehat{\mathbf{y}}, z_{n}\right)\right|^{p} d z_{n} d \widehat{y}=\left\|u_{, n}\right\|_{1, p, V^{-}}^{p} .
\end{aligned}
$$

It follows

$$
\begin{aligned}
\|w\|_{1, p, \mathbb{R}^{n}}^{p} & =\|w\|_{1, p, U}^{p}+\|u\|_{1, p, V^{-}}^{p} \\
& \leq 4^{p} n\left(1+\operatorname{Lip}(g)^{p}\right)\|u\|_{1, p, V^{-}}^{p}+\|u\|_{1, p, V^{-}}^{p}
\end{aligned}
$$

and so

$$
\|w\|_{1, p, \mathbb{R}^{n}}^{p} \leq 4^{p} n\left(2+\operatorname{Lip}(g)^{p}\right)\|u\|_{1, p, V^{-}}^{p}
$$

which implies

$$
\|w\|_{1, p, \mathbb{R}^{n}} \leq 4 n^{1 / p}\left(2+\operatorname{Lip}(g)^{p}\right)^{1 / p}\|u\|_{1, p, V^{-}}
$$

It is obvious that $E_{0}$ is a continuous linear mapping. This proves the lemma.
Now recall Definition 38.2.1, listed here for convenience.
Definition 38.2.5 An open subset, $U$, of $\mathbb{R}^{n}$ has a Lipschitz boundary if it satisfies the following conditions. For each $p \in \partial U \equiv \bar{U} \backslash U$, there exists an open set, $Q$, containing $p$, an open interval $(a, b)$, a bounded open box $B \subseteq \mathbb{R}^{n-1}$, and an orthogonal transformation $R$ such that

$$
\begin{equation*}
R Q=B \times(a, b), \tag{38.2.31}
\end{equation*}
$$

$$
\begin{equation*}
R(Q \cap U)=\left\{\mathbf{y} \in \mathbb{R}^{n}: \widehat{\mathbf{y}} \in B, a<y_{n}<g(\widehat{\mathbf{y}})\right\} \tag{38.2.32}
\end{equation*}
$$

where $g$ is Lipschitz continuous on $\bar{B}, a<\min \{g(\mathbf{x}): \mathbf{x} \in \bar{B}\}$, and

$$
\widehat{\mathbf{y}} \equiv\left(y_{1}, \cdots, y_{n-1}\right) .
$$

Letting $W=Q \cap U$ the following picture describes the situation.


Lemma 38.2.6 In the situation of Definition 38.2.1 let $u \in C^{1}(\bar{U}) \cap C_{c}^{1}(Q)$ and define

$$
E u \equiv R^{*} E_{0}\left(R^{T}\right)^{*} u
$$

where $\left(R^{T}\right)^{*}$ maps $W^{1, p}(U \cap Q)$ to $W^{1, p}(R(W))$. Then $E$ is linear and satisfies

$$
\|E u\|_{W^{1, p}\left(\mathbb{R}^{n}\right)} \leq C\|u\|_{W^{1, p}(Q \cap U)}, E u(\mathbf{x})=u(\mathbf{x}) \text { for } \mathbf{x} \in Q \cap U .
$$

where $C$ depends only on the dimension and $\operatorname{Lip}(g)$.
Proof: This follows from Theorem 38.0.14 and Lemma 38.2.4.
The following theorem is a general extension theorem for Sobolev spaces.
Theorem 38.2.7 Let $U$ be a bounded open set which has Lipschitz boundary. Then for each $p \geq 1$, there exists $E \in \mathscr{L}\left(W^{1, p}(U), W^{1, p}\left(\mathbb{R}^{n}\right)\right)$ such that $E u(\mathbf{x})=u(\mathbf{x})$ a.e. $\mathbf{x} \in U$.

Proof: Let $\partial U \subseteq \cup_{i=1}^{p} Q_{i}$ Where the $Q_{i}$ are as described in Definition 38.2.5. Also let $R_{i}$ be the orthogonal trasformation and $g_{i}$ the Lipschitz functions associated with $Q_{i}$ as in this definition. Now let $Q_{0} \subseteq \overline{Q_{0}} \subseteq U$ be such that $\bar{U} \subseteq \cup_{i=0}^{p} Q_{i}$, and let $\psi_{i} \in C_{c}^{\infty}\left(Q_{i}\right)$ with $\psi_{i}(\mathbf{x}) \in[0,1]$ and $\sum_{i=0}^{p} \psi_{i}(\mathbf{x})=1$ on $\bar{U}$. For $u \in C^{\infty}(\bar{U})$, let $E^{0}\left(\psi_{0} u\right) \equiv \psi_{0} u$ on $Q_{0}$ and 0 off $Q_{0}$. Thus

$$
\left\|E^{0}\left(\psi_{0} u\right)\right\|_{1, p, \mathbb{R}^{n}}=\left\|\psi_{0} u\right\|_{1, p, U}
$$

For $i \geq 1$, let

$$
E^{i}\left(\psi_{i} u\right) \equiv R_{i}^{*} E_{0}\left(R^{T}\right)^{*}\left(\psi_{i} u\right)
$$

Thus, by Lemma 38.2.6

$$
\left\|E^{1}\left(\psi_{i} u\right)\right\|_{1, p, \mathbb{R}^{n}} \leq C\left\|\psi_{i} u\right\|_{1, p, Q_{i} \cap U}
$$

where the constant depends on $\operatorname{Lip}\left(g_{i}\right)$ but is independent of $u \in C^{\infty}(\bar{U})$. Now define $E$ as follows.

$$
E u \equiv \sum_{i=0}^{p} E^{i}\left(\psi_{i} u\right)
$$

Thus for $u \in C^{\infty}(\bar{U})$, it follows $E u(\mathbf{x})=u(\mathbf{x})$ for all $\mathbf{x} \in U$. Also,

$$
\begin{align*}
\|E u\|_{1, p, \mathbb{R}^{n}} & \leq \sum_{i=0}^{p}\left\|E^{i}\left(\psi_{i} u\right)\right\| \leq \sum_{i=0}^{p} C_{i}\left\|\psi_{i} u\right\|_{1, p, Q_{i} \cap U} \\
& =\sum_{i=0}^{p} C_{i}\left\|\psi_{i} u\right\|_{1, p, U} \leq \sum_{i=0}^{p} C_{i}\|u\|_{1, p, U} \\
& \leq(p+1) \sum_{i=0}^{p} C_{i}\|u\|_{1, p, U} \equiv C\|u\|_{1, p, U} \tag{38.2.33}
\end{align*}
$$

where $C$ depends on the $\psi_{i}$ and the $g_{i}$ but is independent of $u \in C^{\infty}(\bar{U})$. Therefore, by density of $C^{\infty}(\bar{U})$ in $W^{1, p}(U), E$ has a unique continuous extension to $W^{1, p}(U)$ still denoted by $E$ satisfying the inequality determined by the ends of 38.2.33. It remains to verify that $E u(\mathbf{x})=u(\mathbf{x})$ a.e. for $\mathbf{x} \in U$.

Let $u_{k} \rightarrow u$ in $W^{1, p}(U)$ where $u_{k} \in C^{\infty}(\bar{U})$. Therefore, by 38.2.33, $E u_{k} \rightarrow E u$ in $W^{1, p}\left(\mathbb{R}^{n}\right)$. Since $E u_{k}(\mathbf{x})=u_{k}(\mathbf{x})$ for each $k$,

$$
\begin{aligned}
\|u-E u\|_{L^{p}(U)} & =\lim _{k \rightarrow \infty}\left\|u_{k}-E u_{k}\right\|_{L^{p}(U)} \\
& =\lim _{k \rightarrow \infty}\left\|E u_{k}-E u_{k}\right\|_{L^{p}(U)}=0
\end{aligned}
$$

which shows $u(\mathbf{x})=E u(\mathbf{x})$ for a.e. $\mathbf{x} \in U$ as claimed. This proves the theorem.
Definition 38.2.8 Let $U$ be an open set. Then $W_{0}^{m, p}(U)$ is the closure of the set, $C_{c}^{\infty}(U)$ in $W^{m, p}(U)$.

Corollary 38.2.9 Let $U$ be a bounded open set which has Lipschitz boundary and let $W$ be an open set containing $\bar{U}$. Then for each $p \geq 1$, there exists $E_{W} \in \mathscr{L}\left(W^{1, p}(U), W_{0}^{1, p}(W)\right)$ such that $E_{W} u(\mathbf{x})=u(\mathbf{x})$ a.e. $\mathbf{x} \in U$.

Proof: Let $\psi \in C_{c}^{\infty}(W)$ and $\psi=1$ on $U$. Then let $E_{W} u \equiv \psi E u$ where $E$ is the extension operator of Theorem 38.2.7.

Extension operators of the above sort exist for many open sets, $U$, not just for bounded ones. In particular, the above discussion would apply to an open set, $U$, not necessarily bounded, if you relax the condition that the $Q_{i}$ must be bounded but require the existence of a finite partition of unity $\left\{\psi_{i}\right\}_{i=1}^{p}$ having the property that $\psi_{i}$ and $\psi_{i, j}$ are uniformly bounded for all $i, j$. The proof would be identical to the above. My main interest is in bounded open sets so the above theorem will suffice. Such an extension operator will be referred to as a $(1, p)$ extension operator.

### 38.3 General Embedding Theorems

With the extension theorem it is possible to give a useful theory of embeddings.
Theorem 38.3.1 Let $1 \leq p<n$ and $\frac{1}{q}=\frac{1}{p}-\frac{1}{n}$ and let $U$ be any open set for which there exists a $(1, p)$ extension operator. Then if $u \in W^{1, p}(U)$, there exists a constant independent of $u$ such that

$$
\|u\|_{L^{q}(U)} \leq C\|u\|_{1, p, U}
$$

If $U$ is bounded and $r<q$, then id : $W^{1, p}(U) \rightarrow L^{r}(U)$ is also compact.
Proof: Let $E$ be the $(1, p)$ extension operator. Then by Theorem 38.1.10 on Page 1323

$$
\begin{aligned}
\|u\|_{L^{q}(U)} & \leq\|E u\|_{L^{q}\left(\mathbb{R}^{n}\right)} \leq \frac{1}{\sqrt[n]{n}} \frac{(n-1) p}{(n-p)}\|E u\|_{1, p, \mathbb{R}^{n}} \\
& \leq C\|u\|_{1, p, U}
\end{aligned}
$$

It remains to prove the assertion about compactness. If $S \subseteq W^{1, p}(U)$ is bounded then

$$
\sup _{u \in S}\left\{\|u\|_{1,1, U}+\|u\|_{L^{q}(U)}\right\}<\infty
$$

and so by Theorem 38.1.17 on Page 1330, it follows $S$ is precompact in $L^{r}(U)$.This proves the theorem.

Corollary 38.3.2 Suppose $m p<n$ and $U$ is an open set satisfying the segment condition which has a $(1, p)$ extension operator for all $p$. Then $\mathrm{id} \in \mathscr{L}\left(W^{m, p}(U), L^{q}(U)\right)$ where $q=\frac{n p}{n-m p}$.

Proof: This is true if $m=1$ according to Theorem 38.3.1. Suppose it is true for $m-1$ where $m>1$. If $u \in W^{m, p}(U)$ and $|\alpha| \leq 1$, then $D^{\alpha} u \in W^{m-1, p}(U)$ so by induction, for all such $\alpha$,

$$
D^{\alpha} u \in L^{\frac{n p}{n-(m-1) p}}(U)
$$

Thus, since $U$ has the segment condition, $u \in W^{1, q_{1}}(U)$ where

$$
q_{1}=\frac{n p}{n-(m-1) p}
$$

By Theorem 38.3.1, it follows $u \in L^{q}\left(\mathbb{R}^{n}\right)$ where

$$
\frac{1}{q}=\frac{n-(m-1) p}{n p}-\frac{1}{n}=\frac{n-m p}{n p}
$$

This proves the corollary.
There is another similar corollary of the same sort which is interesting and useful.

Corollary 38.3.3 Suppose $m \geq 1$ and $j$ is a nonnegative integer satisfying $j p<n$. Also suppose $U$ has $a(1, p)$ extension operator for all $p \geq 1$ and satisfies the segment condition. Then

$$
\mathrm{id} \in \mathscr{L}\left(W^{m+j, p}(U), W^{m, q}(U)\right)
$$

where

$$
\begin{equation*}
q \equiv \frac{n p}{n-j p} \tag{38.3.34}
\end{equation*}
$$

If, in addition to the above, $U$ is bounded and $1 \leq r<q$, then

$$
\mathrm{id} \in \mathscr{L}\left(W^{m+j, p}(U), W^{m, r}(U)\right)
$$

and is compact.
Proof: If $|\alpha| \leq m$, then $D^{\alpha} u \in W^{j, p}(U)$ and so by Corollary 38.3.2, $D^{\alpha} u \in L^{q}(U)$ where $q$ is given above. Since $U$ has the segment property, this means $u \in W^{m, q}(U)$. It remains to verify the assertion about compactness of id.

Let $S$ be bounded in $W^{m+j, p}(U)$. Then $S$ is bounded in $W^{m, q}(U)$ by the first part. Now let $\left\{u_{k}\right\}_{k=1}^{\infty}$ be any sequence in $S$. The corollary will be proved if it is shown that any such sequence has a convergent subsequence in $W^{m, r}(U)$. Let $\left\{\alpha_{1}, \alpha_{2}, \cdots, \alpha_{h}\right\}$ denote the indices satisfying $|\alpha| \leq m$. Then for each of these indices, $\alpha$,

$$
\sup _{u \in S}\left\{\left\|D^{\alpha} u\right\|_{1,1, U}+\left\|D^{\alpha} u\right\|_{L^{q}(U)}\right\}<\infty
$$

and so for each such $\alpha$, satisfying $|\alpha| \leq m$, it follows from Lemma 38.1.16 on Page 1328 that $\left\{D^{\alpha} u: u \in S\right\}$ is precompact in $L^{r}(U)$. Therefore, there exists a subsequence, still denoted by $u_{k}$ such that $D^{\alpha_{1}} u_{k}$ converges in $L^{r}(U)$. Applying the same lemma, there exists a subsequence of this subsequence such that both $D^{\alpha_{1}} u_{k}$ and $D^{\alpha_{2}} u_{k}$ converge in $L^{r}(U)$. Continue taking subsequences until you obtain a subsequence, $\left\{u_{k}\right\}_{k=1}^{\infty}$ for which $\left\{D^{\alpha} u_{k}\right\}_{k=1}^{\infty}$ converges in $L^{r}(U)$ for all $|\alpha| \leq m$. But this must be a convergent subsequence in $W^{m, r}(U)$ and this proves the corollary.

Theorem 38.3.4 Let $U$ be a bounded open set having $a(1, p)$ extension operator and let $p>n$. Then id : $W^{1, p}(U) \rightarrow C(\bar{U})$ is continuous and compact.

Proof: Theorem 38.1.3 on Page 38.1.3 implies $r_{U}: W^{1, p}\left(\mathbb{R}^{n}\right) \rightarrow C(\bar{U})$ is continuous and compact. Thus

$$
\|u\|_{\infty, U}=\|E u\|_{\infty, U} \leq C\|E u\|_{1, p, \mathbb{R}^{n}} \leq C\|u\|_{1, p, U} .
$$

This proves continuity. If $S$ is a bounded set in $W^{1, p}(U)$, then define $S_{1} \equiv\{E u: u \in S\}$. Then $S_{1}$ is a bounded set in $W^{1, p}\left(\mathbb{R}^{n}\right)$ and so by Theorem 38.1.3 the set of restrictions to $U$, is precompact. However, the restrictions to $U$ are just the functions of $S$. Therefore, id is compact as well as continuous.

Corollary 38.3.5 Let $p>n$, let $U$ be a bounded open set having $a(1, p)$ extension operator which also satisfies the segment condition, and let $m$ be a nonnegative integer. Then id : $W^{m+1, p}(U) \rightarrow C^{m, \lambda}(\bar{U})$ is continuous for all $\lambda \in\left[0,1-\frac{n}{p}\right]$ and id is compact if $\lambda<1-\frac{n}{p}$.

Proof: Let $u_{k} \rightarrow 0$ in $W^{m+1, p}(U)$. Then it follows that for each $|\alpha| \leq m, D^{\alpha} u_{k} \rightarrow 0$ in $W^{1, p}(U)$. Therefore,

$$
E\left(D^{\alpha} u_{k}\right) \rightarrow 0 \text { in } W^{1, p}\left(\mathbb{R}^{n}\right)
$$

Then from Morrey's inequality, 38.1.13 on Page 1318, if $\lambda \leq 1-\frac{n}{p}$ and $|\alpha|=m$

$$
\rho_{\lambda}\left(E\left(D^{\alpha} u_{k}\right)\right) \leq C\left\|E\left(D^{\alpha} u_{k}\right)\right\|_{1, p, \mathbb{R}^{n}} \operatorname{diam}(U)^{1-\frac{n}{p}-\lambda} .
$$

Therefore, $\rho_{\lambda}\left(E\left(D^{\alpha} u_{k}\right)\right)=\rho_{\lambda}\left(D^{\alpha} u_{k}\right) \rightarrow 0$. From Theorem 38.3.4 it follows that for $|\alpha| \leq$ $m,\left\|D^{\alpha} u_{k}\right\|_{\infty} \rightarrow 0$ and so $\left\|u_{k}\right\|_{m, \lambda} \rightarrow 0$. This proves the claim about continuity. The claim about compactness for $\lambda<1-\frac{n}{p}$ follows from Lemma 38.1.6 on Page 1319 and this. (Bounded in $W^{m, p}(U) \xrightarrow{\text { id }}$ Bounded in $C^{m, 1-\frac{n}{p}}(\bar{U}) \xrightarrow{\text { id }}$ Compact in $\left.C^{m, \lambda}(\bar{U}).\right)$

Theorem 38.3.6 Suppose $j p<n<(j+1) p$ and let $m$ be a positive integer. Let $U$ be any bounded open set in $\mathbb{R}^{n}$ which has a $(1, p)$ extension operator for each $p \geq 1$ and the segment property. Then $\mathrm{id} \in \mathscr{L}\left(W^{m+j, p}(U), C^{m-1, \lambda}(\bar{U})\right)$ for every $\lambda \leq \lambda_{0} \equiv(j+1)-\frac{n}{p}$ and if $\lambda<(j+1)-\frac{n}{p}$, id is compact.

Proof: From Corollary 38.3.3 $W^{m+j, p}(U) \subseteq W^{m, q}(U)$ where $q$ is given by 38.3.34. Therefore,

$$
\frac{n p}{n-j p}>n
$$

and so by Corollary 38.3.5, $W^{m, q}(U) \subseteq C^{m-1, \lambda}(\bar{U})$ for all $\lambda$ satisfying

$$
0<\lambda<1-\frac{(n-j p) n}{n p}=\frac{p(j+1)-n}{p}=(j+1)-\frac{n}{p} .
$$

The assertion about compactness follows from the compactness of the embedding of

$$
C^{m-1, \lambda_{0}}(\bar{U})
$$

into $C^{m-1, \lambda}(\bar{U})$ for $\lambda<\lambda_{0}$, Lemma 38.1.6 on Page 1319.

### 38.4 More Extension Theorems

The theorem about the existence of a $(1, p)$ extension is all that is needed to obtain general embedding theorems for Sobolev spaces. However, a more general theory is needed in order to tie the theory of Sobolev spaces presented thus far to a very appealing description using Fourier transforms. First the problem of extending $W^{k, p}(H)$ to $W^{k, p}\left(\mathbb{R}^{n}\right)$ is considered for $\mathrm{H}^{-}$a half space

$$
\begin{equation*}
H^{-} \equiv\left\{\mathbf{y} \in \mathbb{R}^{n}: y_{n}<0\right\} \tag{38.4.35}
\end{equation*}
$$

I am following Adams [1].

Lemma 38.4.1 Let $H^{-}$be a half space as in 38.4.35. Let $H^{+}$be the half space in which $y_{n}<0$ is replaced with $y_{n}>0$. Also let $\left(\mathbf{y}^{\prime}, y_{n}\right)=\mathbf{y}$

$$
u\left(\mathbf{y}^{\prime}, y_{n}\right) \equiv\left\{\begin{array}{ll}
u^{+}\left(\mathbf{y}^{\prime}, y_{n}\right) & \text { if } \mathbf{y} \in H^{+} \\
u^{-}\left(\mathbf{y}^{\prime}, y_{n}\right) & \text { if } \mathbf{y} \in H^{-}
\end{array},\right.
$$

suppose $u^{+} \in C^{\infty}\left(\overline{H^{+}}\right)$and $u^{-} \in C^{\infty}\left(\overline{H^{-}}\right)$, and that for $l \leq k-1$,

$$
D^{l \mathbf{e}_{n}} \mathbf{u}^{+}\left(\mathbf{y}^{\prime}, 0\right)=D^{l \mathbf{e}_{n}} \mathbf{u}^{-}\left(\mathbf{y}^{\prime}, 0\right)
$$

Then $u \in W^{k, p}\left(\mathbb{R}^{n}\right)$. Furthermore,

$$
D^{\alpha} u\left(\mathbf{y}^{\prime}, y_{n}\right) \equiv\left\{\begin{array}{l}
D^{\alpha} u^{+}\left(\mathbf{y}^{\prime}, y_{n}\right) \text { if } \mathbf{y} \in H^{+} \\
D^{\alpha} u^{-}\left(\mathbf{y}^{\prime}, y_{n}\right) \text { if } \mathbf{y} \in H^{-}
\end{array}\right.
$$

Proof: Consider the following for $\phi \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$ and $|\alpha| \leq k$.

$$
(-1)^{|\alpha|}\left(\int_{\mathbb{R}^{n-1}} \int_{0}^{\infty} u^{+} D^{\alpha} \phi d y_{n} d y^{\prime}+\int_{\mathbb{R}^{n-1}} \int_{-\infty}^{0} u^{-} D^{\alpha} \phi d y_{n} d y^{\prime}\right)
$$

Integrating by parts, this yields

$$
\begin{aligned}
& (-1)^{|\alpha|}(-1)^{|\beta|}\left(\int_{\mathbb{R}^{n-1}} \int_{0}^{\infty} D^{\beta} u^{+} D^{\alpha_{n} \mathbf{e}_{n}} \phi d y_{n} d y^{\prime}\right. \\
& \left.+\int_{\mathbb{R}^{n-1}} \int_{-\infty}^{0} D^{\beta} u^{-} D^{\alpha_{n} \mathbf{e}_{n}} \phi d y_{n} d y^{\prime}\right)
\end{aligned}
$$

where $\beta \equiv\left(\alpha_{1}, \alpha_{2}, \cdots \alpha_{n-1}, 0\right)$. Do integration by parts on the inside integral and by assumption, the boundary terms will cancel and the whole thing reduces to

$$
\begin{aligned}
& (-1)^{|\alpha|}(-1)^{|\beta|}(-1)^{\alpha_{n}}\left(\int_{\mathbb{R}^{n-1}} \int_{0}^{\infty} D^{\alpha} u^{+} \phi d y_{n} d y^{\prime}\right. \\
& \left.+\int_{\mathbb{R}^{n-1}} \int_{-\infty}^{0} D^{\alpha} u^{-} \phi d y_{n} d y^{\prime}\right) \\
= & \left(\int_{\mathbb{R}^{n-1}} \int_{0}^{\infty} D^{\alpha} u^{+} \phi d y_{n} d y^{\prime}+\int_{\mathbb{R}^{n-1}} \int_{-\infty}^{0} D^{\alpha} u^{-} \phi d y_{n} d y^{\prime}\right)
\end{aligned}
$$

which proves the lemma.
Lemma 38.4.2 Let $H^{-}$be the half space in 38.4.35 and let $u \in C^{\infty}\left(\overline{H^{-}}\right)$. Then there exists a mapping,

$$
E: C^{\infty}\left(\overline{H^{-}}\right) \rightarrow W^{k, p}\left(\mathbb{R}^{n}\right)
$$

and a constant, $C$ which is independent of $u \in C^{\infty}\left(\overline{H^{-}}\right)$such that $E$ is linear and for all $l \leq k$,

$$
\begin{equation*}
\|E u\|_{l, p, \mathbb{R}^{n}} \leq C\|u\|_{l, p, H^{-}} \tag{38.4.36}
\end{equation*}
$$

Proof: Define

$$
E u\left(\mathbf{x}^{\prime}, x_{n}\right) \equiv\left\{\begin{array}{l}
u\left(\mathbf{x}^{\prime}, x_{n}\right) \text { if } x_{n}<0 \\
\sum_{j=1}^{k} \lambda_{j} u\left(\mathbf{x}^{\prime},-j x_{n}\right) \text { if } x_{n} \geq 0
\end{array}\right.
$$

where the $\lambda_{j}$ are chosen in such a way that for $l \leq k-1$,

$$
D^{l \mathbf{e}_{n}} u\left(\mathbf{x}^{\prime}, 0\right)-D^{l \mathbf{e}_{n}}\left(\sum_{j=1}^{k} \lambda_{j} u\right)\left(\mathbf{x}^{\prime}, 0\right)=0
$$

so that Lemma 38.4.1 may be applied. Do there exist such $\lambda_{j}$ ? It is necessary to have the following hold for each $r=0,1, \cdots, k-1$.

$$
\sum_{j=1}^{k}(-j)^{r} \lambda_{j} D^{r \mathbf{e}_{n}} u\left(\mathbf{x}^{\prime}, 0\right)=D^{r \mathbf{e}_{n}} u\left(\mathbf{x}^{\prime}, 0\right)
$$

This is satisfied if

$$
\sum_{j=1}^{k}(-j)^{r} \lambda_{j}=1
$$

for $r=0,1, \cdots, k-1$. This is a system of $k$ equations for the $k$ variables, the $\lambda_{j}$. The matrix of coefficients is of the form

$$
\left(\begin{array}{ccccc}
1 & 1 & 1 & \cdots & 1 \\
-1 & -2 & -3 & \cdots & -k \\
1 & 4 & 9 & \cdots & k^{2} \\
\vdots & \vdots & \vdots & & \vdots \\
(-1)^{k} & (-2)^{k} & (-3)^{k} & \cdots & (-k)^{k}
\end{array}\right)
$$

This matrix has an inverse because its determinant is nonzero.
Now from Lemma 38.4.1, it follows from the above description of $E$ that for $|\alpha| \leq k$,

$$
D^{\alpha}(E u)\left(\mathbf{x}^{\prime}, x_{n}\right) \equiv\left\{\begin{array}{l}
D^{\alpha} u\left(\mathbf{x}^{\prime}, x_{n}\right) \text { if } x_{n}<0 \\
\sum_{j=1}^{k} \lambda_{j}(-j)^{\alpha_{n}}\left(D^{\alpha} u\right)\left(\mathbf{x}^{\prime},-j x_{n}\right) \text { if } x_{n} \geq 0
\end{array}\right.
$$

It follows that $E$ is linear and there exists a constant, $C$ independent of $u$ such that 38.4.36 holds. This proves the lemma.

Corollary 38.4.3 Let $H^{-}$be the half space of 38.4.35. There exists $E$ with the property that $E: W^{l, p}\left(H^{-}\right) \rightarrow W^{l, p}\left(\mathbb{R}^{n}\right)$ and is linear and continuous for each $l \leq k$.

Proof: This immediate from the density of $C_{c}^{\infty}\left(\overline{H^{-}}\right)$in $W^{k, p}\left(\overline{H^{-}}\right)$and Lemma 38.4.2.
There is nothing sacred about a half space or this particular half space. It is clear that everything works as well for a half space of the form

$$
H_{k}^{-} \equiv\left\{\mathbf{x}: x_{k}<0\right\}
$$

Thus the half space featured in the above discussion is $H_{n}^{-}$.

Corollary 38.4.4 Let $\left\{k_{1}, \cdots, k_{r}\right\} \subseteq\{1, \cdots, n\}$ where the $k_{i}$ are distinct and let

$$
\begin{equation*}
H_{k_{1} \cdots k_{r}}^{-} \equiv H_{k_{1}}^{-} \cap H_{k_{2}}^{-} \cap \cdots \cap H_{k_{r}}^{-} . \tag{38.4.37}
\end{equation*}
$$

Then there exists $E: W^{k, p}\left(H_{k_{1} \cdots k_{r}}^{-}\right) \rightarrow W^{k, p}\left(\mathbb{R}^{n}\right)$ such that $E$ is linear and continuous.
Proof: Follow the above argument with minor modifications to first extend from $H_{k_{1} \cdots k_{r}}^{-}$ to $H_{k_{1} \cdots k_{r-1}}^{-}$and then from from $H_{k_{1} \cdots k_{r-1}}^{-}$to $H_{k_{1} \cdots k_{r-2}}^{-}$etc.

This easily implies the ability to extend off bounded open sets which near their boundaries look locally like an intersection of half spaces.

Theorem 38.4.5 Let $U$ be a bounded open set and suppose $U_{0}, U_{1}, \cdots, U_{m}$ are open sets with the property that $\bar{U} \subseteq \cup_{k=0}^{m} U_{k}, \overline{U_{0}} \subseteq U$, and $\partial U \subseteq \cup_{k=1}^{m} U_{k}$. Suppose also there exist one to one and onto functions, $\mathbf{h}_{k}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}, \mathbf{h}_{k}\left(U_{k} \cap U\right)=W_{k}$ where $W_{k}$ equals the intersection of a bounded open set with a finite intersection of half spaces, $H_{k_{1} \cdots k_{r}}^{-}$, as in 38.4 .37 such that $\mathbf{h}_{k}\left(\partial U \cap U_{k}\right) \subseteq \partial H_{k_{1} \cdots k_{r}}^{-}$. Suppose also that for all $|\alpha| \leq k-1$,

$$
D^{\alpha} \mathbf{h}_{k} \text { and } D^{\alpha} \mathbf{h}_{k}^{-1}
$$

exist and are Lipschitz continuous. Then letting $W$ be an open set which contains $\bar{U}$, there exists $E: W^{k, p}(U) \rightarrow W^{k, p}(W)$ such that $E$ is a linear continuous map from $W^{l, p}(U)$ to $W^{l, p}(W)$ for each $l \leq k$.

Proof: Let $\psi_{j} \in C_{c}^{\infty}\left(U_{j}\right), \psi_{j}(\mathbf{x}) \in[0,1]$ for all $\mathbf{x} \in \mathbb{R}^{n}$, and $\sum_{j=0}^{m} \psi_{j}(\mathbf{x})=1$ on $\bar{U}$. This is a $C^{\infty}$ partition of unity on $\bar{U}$. By Theorem 38.0.14 $\left(\mathbf{h}_{j}^{-1}\right)^{*} u \psi_{j} \in W^{k, p}\left(W_{j}\right)$. By the assumption that $\mathbf{h}_{j}\left(\partial U \cap U_{j}\right) \subseteq \partial H_{k_{1} \cdots k_{r}}^{-}$, the zero extension of $\left(\mathbf{h}_{j}^{-1}\right)^{*} u \psi_{j}$ to the rest of $H_{k_{1} \cdots k_{r}}^{-}$results in an element of $W^{k, p}\left(H_{k_{1} \cdots k_{r}}^{-}\right)$. Apply Corollary 38.4.4 to conclude there exists $E_{j}: W^{k, p}\left(H_{k_{1} \cdots k_{r}}^{-}\right) \rightarrow W^{k, p}\left(\mathbb{R}^{n}\right)$ which is continuous and linear. Abusing notation slightly, by using $\left(\mathbf{h}_{j}^{-1}\right)^{*} u \psi_{j}$ as the above zero extension, it follows $E_{j}\left(\left(\mathbf{h}_{j}^{-1}\right)^{*} u \psi_{j}\right) \in$ $W^{k, p}\left(\mathbb{R}^{n}\right)$. Now let $\eta$ be a function in $C_{c}^{\infty}(\mathbf{h}(W))$ such that $\eta(\mathbf{y})=1$ on $\mathbf{h}(\bar{U})$. Then Define

$$
E u \equiv \sum_{j=0}^{m} \mathbf{h}_{j}^{*} \eta E_{j}\left(\left(\mathbf{h}_{j}^{-1}\right)^{*}\left(u \psi_{j}\right)\right)
$$

Clearly $E u(\mathbf{x})=u(\mathbf{x})$ if $\mathbf{x} \in U$. It is also clear that $E$ is linear. It only remains to verify $E$ is continuous. In what follows, $C_{j}$ will denote a constant which is independent of $u$ which may change from line to line. By Theorem 38.0.14,

$$
\begin{aligned}
\|E u\|_{k, p, W} & \leq \sum_{j=0}^{m}\left\|\mathbf{h}_{j}^{*} \eta E_{j}\left(\left(\mathbf{h}_{j}^{-1}\right)^{*}\left(u \psi_{j}\right)\right)\right\|_{k, p, W} \\
& \leq \sum_{j=0}^{m} C_{j}\left\|\eta E_{j}\left(\left(\mathbf{h}_{j}^{-1}\right)^{*}\left(u \psi_{j}\right)\right)\right\|_{k, p, \mathbf{h}(W)}
\end{aligned}
$$

$$
\begin{gathered}
=\sum_{j=0}^{m} C_{j}\left\|\eta E_{j}\left(\left(\mathbf{h}_{j}^{-1}\right)^{*}\left(u \psi_{j}\right)\right)\right\|_{k, p, \mathbb{R}^{n}} \leq \sum_{j=0}^{m} C_{j}\left\|E_{j}\left(\left(\mathbf{h}_{j}^{-1}\right)^{*}\left(u \psi_{j}\right)\right)\right\|_{k, p, \mathbb{R}^{n}} \\
\leq \sum_{j=0}^{m} C_{j}\left\|\left(\mathbf{h}_{j}^{-1}\right)^{*}\left(u \psi_{j}\right)\right\|_{k, p, \mathbf{h}_{j}\left(U \cap U_{j}\right)} \leq \sum_{j=0}^{m} C_{j}\left\|u \psi_{j}\right\|_{k, p, U \cap U_{k}} \\
\leq \sum_{j=0}^{m} C_{j}\|u\|_{k, p, U \cap U_{k}} \leq\left(\sum_{j=0}^{m} C_{j}\right)\|u\|_{k, p, U} .
\end{gathered}
$$

Similarly $E: W^{l, p}(U) \rightarrow W^{l, p}(U)$ for $l \leq k$. This proves the theorem.
Definition 38.4.6 When $E$ is a linear continuous map from $W^{l, p}(U)$ to $W^{l, p}\left(\mathbb{R}^{n}\right)$ for each $l \leq k$. it is called a strong ( $k, p$ ) extension map.

There is also a very easy sort of extension theorem for the space, $W_{0}^{m, p}(U)$ which does not require any assumptions on the boundary of $U$ other than $m_{n}(\partial U)=0$. First here is the definition of $W_{0}^{m, p}(U)$.

Definition 38.4.7 Denote by $W_{0}^{m, p}(U)$ the closure of $C_{c}^{\infty}(U)$ in $W^{m, p}(U)$.
Theorem 38.4.8 For $u \in W_{0}^{m, p}(U)$, define

$$
E u(\mathbf{x}) \equiv\left\{\begin{array}{l}
u(\mathbf{x}) \text { if } \mathbf{x} \in U \\
0 \text { if } \mathbf{x} \notin U
\end{array}\right.
$$

Then $E$ is a strong ( $k, p$ ) extension map.
Proof: Letting $l \leq m$, it is clear that for $|\alpha| \leq l$,

$$
D^{\alpha} E u=\left\{\begin{array}{l}
D^{\alpha} u \text { for } \mathbf{x} \in U \\
0 \text { for } \mathbf{x} \notin U
\end{array}\right.
$$

This follows because, since $m_{n}(\partial U)=0$ it suffices to consider $\phi \in C_{c}^{\infty}(U)$ and

$$
\phi \in C_{c}^{\infty}\left(\bar{U}^{C}\right) .
$$

Therefore, $\|E u\|_{l, p, \mathbb{R}^{n}}=\|u\|_{l, p, U}$.
There are many other extension theorems and if you are interested in pursuing this further, consult Adams [1]. One of the most famous which is discussed in this reference is due to Calderon and depends on the theory of singular integrals.

## Chapter 39

## Sobolev Spaces Based On $L^{2}$

### 39.1 Fourier Transform Techniques

Much insight can be obtained easily through the use of Fourier transform methods. This technique will be developed in this chapter. When this is done, it is necessary to use Sobolev spaces of the form $W^{k, 2}(U)$, those Sobolev spaces which are based on $L^{2}(U)$. It is true there are generalizations which use Fourier transform methods in the context of $L^{p}$ but the spaces so considered are called Bessel potential spaces. They are not really Sobolev spaces. Furthermore, it is Mihlin's theorem rather than the Plancherel theorem which is the main tool of the analysis. This is a hard theorem.

It is convenient to consider the Schwartz class of functions, $\mathfrak{S}$. These are functions which have infinitely many derivatives and vanish quickly together with their derivatives as $|\mathbf{x}| \rightarrow \infty$. In particular, $C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$ is contained in $\mathfrak{S}$ which is not true of the functions, $\mathscr{G}$ used earlier in defining the Fourier transforms which are a suspace of $\mathfrak{S}$. Recall the following definition.

Definition 39.1.1 $f \in \mathfrak{S}$, the Schwartz class, if $f \in C^{\infty}\left(\mathbb{R}^{n}\right)$ and for all positive integers $N$,

$$
\rho_{N}(f)<\infty
$$

where

$$
\rho_{N}(f)=\sup \left\{\left(1+|\mathbf{x}|^{2}\right)^{N}\left|D^{\alpha} f(\mathbf{x})\right|: \mathbf{x} \in \mathbb{R}^{n},|\alpha| \leq N\right\}
$$

Thus $f \in \mathfrak{S}$ if and only if $f \in C^{\infty}\left(\mathbb{R}^{n}\right)$ and

$$
\begin{equation*}
\sup \left\{\left|\mathbf{x}^{\beta} D^{\alpha} f(\mathbf{x})\right|: \mathbf{x} \in \mathbb{R}^{n}\right\}<\infty \tag{39.1.1}
\end{equation*}
$$

for all multi indices $\alpha$ and $\beta$.
Thus all partial derivatives of a function in $\mathfrak{S}$ are in $L^{p}\left(\mathbb{R}^{n}\right)$ for all $p \geq 1$. Therefore, for $f \in \mathfrak{S}$, the Fourier and inverse Fourier transforms are given in the usual way,

$$
F f(\mathbf{t})=\left(\frac{1}{2 \pi}\right)^{n / 2} \int_{\mathbb{R}^{n}} f(\mathbf{x}) e^{-i \mathbf{t} \cdot \mathbf{x}} d x, F^{-1} f(\mathbf{t})=\left(\frac{1}{2 \pi}\right)^{n / 2} \int_{\mathbb{R}^{n}} f(\mathbf{x}) e^{i \mathbf{t} \cdot \mathbf{x}} d x
$$

Also recall that the Fourier transform and its inverse are one to one and onto maps from $\mathfrak{S}$ to $\mathfrak{S}$.

To tie the Fourier transform technique in with what has been done so far, it is necessary to make the following assumption on the set, $U$. This assumption is made so that it is possible to consider elements of $W^{k, 2}(U)$ as restrictions of elements of $W^{k, 2}\left(\mathbb{R}^{n}\right)$.

Assumption 39.1.2 Assume $U$ satisfies the segment condition and that for any $m$ of interest, there exists $E \in \mathscr{L}\left(W^{m, p}(U), W^{m, p}\left(\mathbb{R}^{n}\right)\right)$ such that for each

$$
k \leq m, E \in \mathscr{L}\left(W^{k, p}(U), W^{k, p}\left(\mathbb{R}^{n}\right)\right)
$$

That is, there exists a stong ( $m, p$ ) extension operator.

Lemma 39.1.3 The Schwartz class, $\mathfrak{S}$, is dense in $W^{m, p}\left(\mathbb{R}^{n}\right)$.
Proof: The set, $\mathbb{R}^{n}$ satisfies the segment condition and so $C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$ is dense in $W^{m, p}\left(\mathbb{R}^{n}\right)$. However, $C_{c}^{\infty}\left(\mathbb{R}^{n}\right) \subseteq \mathfrak{S}$. This proves the lemma.

Recall now Plancherel's theorem which states that $\|f\|_{0,2, \mathbb{R}^{n}}=\|F f\|_{0,2, \mathbb{R}^{n}}$ whenever $f \in L^{2}\left(\mathbb{R}^{n}\right)$. Also it is routine to verify from the definition of the Fourier transform that for $u \in \mathfrak{S}$,

$$
F \partial_{k} u=i x_{k} F u .
$$

From this it follows that

$$
\left\|D^{\alpha} u\right\|_{0,2, \mathbb{R}^{n}}=\left\|\mathbf{x}^{\alpha} F u\right\|_{0,2, \mathbb{R}^{n}}
$$

Here $\mathbf{x}^{\alpha}$ denotes the function $\mathbf{x} \rightarrow \mathbf{x}^{\alpha}$. Therefore,

$$
\|u\|_{m, 2, \mathbb{R}^{n}}=\left(\int_{\mathbb{R}^{n}} \sum_{|\alpha| \leq m} x_{1}^{2 \alpha_{1}} \cdots x_{n}^{2 \alpha_{n}}|F u(\mathbf{x})|^{2} d x\right)^{1 / 2}
$$

Also, it is not hard to verify that

$$
\sum_{|\alpha| \leq m} x_{1}^{2 \alpha_{1}} \cdots x_{n}^{2 \alpha_{n}} \leq\left(1+\sum_{j=1}^{n} x_{j}^{2}\right)^{m} \leq C(n, m) \sum_{|\alpha| \leq m} x_{1}^{2 \alpha_{1}} \cdots x_{n}^{2 \alpha_{n}}
$$

where $C(n, m)$ is the largest of the multinomial coefficients obtained in the expansion,

$$
\left(1+\sum_{j=1}^{n} x_{j}^{2}\right)^{m}
$$

Therefore, for all $u \in \mathfrak{S}$,

$$
\begin{equation*}
\|u\|_{m, 2, \mathbb{R}^{n}} \leq\left(\int_{\mathbb{R}^{n}}\left(1+|\mathbf{x}|^{2}\right)^{m}|F u(\mathbf{x})|^{2} d x\right)^{1 / 2} \leq C(n, m)\|u\|_{m, 2, \mathbb{R}^{n}} \tag{39.1.2}
\end{equation*}
$$

This motivates the following definition.
Definition 39.1.4 Let $H^{m}\left(\mathbb{R}^{n}\right) \equiv$

$$
\begin{equation*}
\left\{u \in L^{2}\left(\mathbb{R}^{n}\right):\|u\|_{H^{m}\left(\mathbb{R}^{n}\right)} \equiv\left(\int_{\mathbb{R}^{n}}\left(1+|\mathbf{x}|^{2}\right)^{m}|F u(\mathbf{x})|^{2} d x\right)^{1 / 2}<\infty\right\} \tag{39.1.3}
\end{equation*}
$$

Lemma 39.1.5 $\mathfrak{S}$ is dense in $H^{m}\left(\mathbb{R}^{n}\right)$ and $H^{m}\left(\mathbb{R}^{n}\right)=W^{2, m}\left(\mathbb{R}^{n}\right)$. Furthermore, the norms are equivalent.

Proof: First it is shown that $\mathfrak{S}$ is dense in $H^{m}\left(\mathbb{R}^{n}\right)$. Let $u \in H^{m}\left(\mathbb{R}^{n}\right)$. Let $\mu(E) \equiv$ $\int_{E}\left(1+|\mathbf{x}|^{2}\right)^{m} d x$. Thus $\mu$ is a regular measure and $u \in H^{m}\left(\mathbb{R}^{n}\right)$ just means that $F u \in$ $L^{2}(\mu)$, the space of functions which are in $L^{2}\left(\mathbb{R}^{n}\right)$ with respect to this measure, $\mu$. Therefore, from the regularity of the measure, $\mu$, there exists $u_{k} \in C_{c}\left(\mathbb{R}^{n}\right)$ such that

$$
\left\|u_{k}-F u\right\|_{L^{2}(\mu)} \rightarrow 0
$$

Now let $\psi_{\varepsilon}$ be a mollifier and pick $\varepsilon_{k}$ small enough that

$$
\left\|u_{k} * \psi_{\varepsilon_{k}}-u_{k}\right\|_{L^{2}(\mu)}<\frac{1}{2^{k}} .
$$

Then $u_{k} * \psi_{\varepsilon_{k}} \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right) \subseteq \mathfrak{S}$. Therefore, there exists $w_{k} \in \mathscr{G}$ such that $F w_{k}=u_{k} * \psi_{\varepsilon_{k}}$. It follows

$$
\left\|F w_{k}-F u\right\|_{L^{2}(\mu)} \leq\left\|F w_{k}-u_{k}\right\|_{L^{2}(\mu)}+\left\|u_{k}-F u\right\|_{L^{2}(\mu)}
$$

and these last two terms converge to 0 as $k \rightarrow \infty$. Therefore, $w_{k} \rightarrow u$ in $H^{m}\left(\mathbb{R}^{n}\right)$ and this proves the first part of this lemma.

Now let $u \in H^{m}\left(\mathbb{R}^{n}\right)$. By what was just shown, there exists a sequence, $u_{k} \rightarrow u$ in $H^{m}\left(\mathbb{R}^{n}\right)$ where $u_{k} \in \mathfrak{S}$. It follows from 39.1.2 that

$$
\left\|u_{k}-u_{l}\right\|_{H^{m}} \geq\left\|u_{k}-u_{l}\right\|_{m, 2, \mathbb{R}^{n}}
$$

and so $\left\{u_{k}\right\}$ is a Cauchy sequence in $W^{m, 2}\left(\mathbb{R}^{n}\right)$. Therefore, there exists $w \in W^{m, 2}\left(\mathbb{R}^{n}\right)$ such that

$$
\left\|u_{k}-w\right\|_{m, 2, \mathbb{R}^{n}} \rightarrow 0
$$

But this implies

$$
0=\lim _{k \rightarrow \infty}\left\|u_{k}-w\right\|_{0,2, \mathbb{R}^{n}}=\lim _{k \rightarrow \infty}\left\|u_{k}-u\right\|_{0,2, \mathbb{R}^{n}}
$$

showing $u=w$ which verifies $H^{m}\left(\mathbb{R}^{n}\right) \subseteq W^{2, m}\left(\mathbb{R}^{n}\right)$. The opposite inclusion is proved the same way, using density of $\mathfrak{S}$ and the fact that the norms in both spaces are larger than the norms in $L^{2}\left(\mathbb{R}^{n}\right)$. The equivalence of the norms follows from the density of $\mathfrak{S}$ and the equivalence of the norms on $\mathfrak{S}$. This proves the lemma.

The conclusion of this lemma with the density of $\mathfrak{S}$ and 39.1.2 implies you can use either norm, $\|u\|_{H^{m}\left(\mathbb{R}^{n}\right)}$ or $\|u\|_{m, 2, \mathbb{R}^{n}}$ when working with these Sobolev spaces.

What of open sets satisfying Assumption 39.1.2? How does $W^{m, 2}(U)$ relate to the Fourier transform?

Definition 39.1.6 Let $U$ be an open set in $\mathbb{R}^{n}$. Then

$$
\begin{equation*}
H^{m}(U) \equiv\left\{u: u=\left.v\right|_{U} \text { for some } v \in H^{m}\left(\mathbb{R}^{n}\right)\right\} \tag{39.1.4}
\end{equation*}
$$

Here the notation, $\left.v\right|_{U}$ means $v$ restricted to $U$. Define the norm in this space by

$$
\begin{equation*}
\|u\|_{H^{m}(U)} \equiv \inf \left\{\|v\|_{H^{m}\left(\mathbb{R}^{n}\right)}:\left.v\right|_{U}=u\right\} \tag{39.1.5}
\end{equation*}
$$

Lemma 39.1.7 $H^{m}(U)$ is a Banach space.
Proof: First it is necessary to verify that the given norm really is a norm. Suppose then that $u=0$. Is $\|u\|_{H^{m}(U)}=0$ ? Of course it is. Just take $v \equiv 0$. Then $\left.v\right|_{U}=u$ and $\|v\|_{H^{m}}=0$. Next suppose $\|u\|_{H^{m}(U)}=0$. Does it follow that $u=0$ ? Letting $\varepsilon>0$ be given, there exists $v \in H^{m}\left(\mathbb{R}^{n}\right)$ such that $\left.v\right|_{U}=u$ and $\|v\|_{H^{m}\left(\mathbb{R}^{n}\right)}<\varepsilon$. Therefore,

$$
\|u\|_{0, U} \leq\|v\|_{0, \mathbb{R}^{n}} \leq\|v\|_{H^{m}(U)}<\varepsilon
$$

Since $\varepsilon>0$ is arbitrary, it follows $u=0$ a.e. Next suppose $u_{i} \in H^{m}(U)$ for $i=1,2$. There exists $v_{i} \in H^{m}\left(\mathbb{R}^{n}\right)$ such that

$$
\left\|v_{i}\right\|_{H^{m}\left(\mathbb{R}^{n}\right)}<\left\|u_{i}\right\|_{H^{m}(U)}+\varepsilon
$$

Therefore,

$$
\begin{aligned}
\left\|u_{1}+u_{2}\right\|_{H^{m}(U)} & \leq\left\|v_{1}+v_{2}\right\|_{H^{m}\left(\mathbb{R}^{n}\right)} \leq\left\|v_{1}\right\|_{H^{m}\left(\mathbb{R}^{n}\right)}+\left\|v_{2}\right\|_{H^{m}\left(\mathbb{R}^{n}\right)} \\
& \leq\left\|u_{1}\right\|_{H^{m}(U)}+\left\|u_{2}\right\|_{H^{m}(U)}+2 \varepsilon
\end{aligned}
$$

and since $\varepsilon>0$ is arbitrary, this shows the triangle inequality.
The interesting question is the one about completeness. Suppose then $\left\{u_{k}\right\}$ is a Cauchy sequence in $H^{m}(U)$. There exists $N_{k}$ such that if $k, l \geq N_{k}$, it follows $\left\|u_{k}-u_{l}\right\|_{H^{m}(U)}<\frac{1}{2^{k}}$ and the numbers, $N_{k}$ can be taken to be strictly increasing in $k$. Thus for

$$
l \geq N_{k},| | u_{l}-u_{N_{k}} \|_{H^{m}(U)}<1 / 2^{l}
$$

Therefore, there exists $w_{l} \in H^{m}\left(\mathbb{R}^{n}\right)$ such that

$$
\left.w_{l}\right|_{U}=u_{l}-u_{N_{k}},\left\|w_{l}\right\|_{H^{m}\left(\mathbb{R}^{n}\right)}<\frac{1}{2^{l}}
$$

Also let $\left.v_{N_{k}}\right|_{U}=u_{N_{k}}$ with $v_{N_{k}} \in H^{m}\left(\mathbb{R}^{n}\right)$ and

$$
\left|\left|v_{N_{k}}\right|\right|_{H^{m}\left(\mathbb{R}^{n}\right)}<\| u_{N_{k}}| |_{H^{m}(U)}+\frac{1}{2^{k}}
$$

Now for $l>N_{k}$, define $v_{l}$ by $v_{l}-v_{N_{k}}=w_{N_{k}}$ so that $\left|\mid v_{l}-v_{N_{k}} \|_{H^{m}\left(\mathbb{R}^{n}\right)}<1 / 2^{k}\right.$. In particular,

$$
\left\|v_{N_{k+1}}-v_{N_{k}}\right\|_{H^{m}\left(\mathbb{R}^{n}\right)}<1 / 2^{k}
$$

which shows that $\left\{v_{N_{k}}\right\}_{k=1}^{\infty}$ is a Cauchy sequence. Consequently it must converge to $v \in$ $H^{m}\left(\mathbb{R}^{n}\right)$. Let $u=\left.v\right|_{U}$. Then

$$
\left\|u-\left.u_{N_{k}}\right|_{H^{m}(U)} \leq\right\| v-\left.v_{N_{k}}\right|_{H^{m}\left(\mathbb{R}^{n}\right)}
$$

which shows the subsequence, $\left\{u_{N_{k}}\right\}_{k}$ converges to $u$. Since $\left\{u_{k}\right\}$ is a Cauchy sequence, it follows it too must converge to $u$. This proves the lemma.

The main result is next.
Theorem 39.1.8 Suppose $U$ satisfies Assumption 39.1.2. Then for $m$ a nonnegative integer, $H^{m}(U)=W^{m, 2}(U)$ and the two norms are equivalent.

Proof: Let $u \in H^{m}(U)$. Then there exists $v \in H^{m}\left(\mathbb{R}^{n}\right)$ such that $\left.v\right|_{U}=u$. Hence $v \in W^{k, 2}\left(\mathbb{R}^{n}\right)$ and so all its weak derivatives up to order $m$ are in $L^{2}\left(\mathbb{R}^{n}\right)$. Therefore, the restrictions of these weak derivitves are in $L^{2}(U)$. Since $U$ satisfies the segment condition, it follows $u \in W^{m, 2}(U)$ which shows $H^{m}(U) \subseteq W^{m, 2}(U)$.

Next take $u \in W^{m, 2}(U)$. Then $E u \in W^{m, 2}\left(\mathbb{R}^{n}\right)=H^{m}\left(\mathbb{R}^{n}\right)$ and this shows $u \in H^{m}(U)$. This has shown the two spaces are the same. It remains to verify their norms are equivalent. Let $u \in H^{m}(U)$ and let $\left.v\right|_{U}=u$ where $v \in H^{m}\left(\mathbb{R}^{n}\right)$ and

$$
\|u\|_{H^{m}(U)}+\varepsilon>\|v\|_{H^{m}\left(\mathbb{R}^{n}\right)} .
$$

Then recalling that $\|\cdot\|_{H^{m}\left(\mathbb{R}^{n}\right)}$ and $\|\cdot\|_{m, 2, \mathbb{R}^{n}}$ are equivalent norms for $H^{m}\left(\mathbb{R}^{n}\right)$, there exists a constant, $C$ such that

$$
\|u\|_{H^{m}(U)}+\varepsilon>\|v\|_{H^{m}\left(\mathbb{R}^{n}\right)} \geq C\|v\|_{m, 2, \mathbb{R}^{n}} \geq C\|u\|_{m, 2, U}
$$

Now consider the two Banach spaces,

$$
\left(H^{m}(U),\|\cdot\|_{H^{m}(U)}\right),\left(W^{m, 2}(U),\|\cdot\|_{m, 2, U}\right)
$$

The above inequality shows since $\varepsilon>0$ is arbitrary that

$$
\text { id }:\left(H^{m}(U),\|\cdot\|_{H^{m}(U)}\right) \rightarrow\left(W^{m, 2}(U),\|\cdot\|_{m, 2, U}\right)
$$

is continuous. By the open mapping theorem, it follows id is continuous in the other direction. Thus there exists a constant, $K$ such that $\|u\|_{H^{m}(U)} \leq K\|u\|_{k, 2, U}$. Hence the two norms are equivalent as claimed.

Specializing Corollary 38.3.3 and Theorem 38.3.6 starting on Page 1339 to the case of $p=2$ while also assuming more on $U$ yields the following embedding theorems.

Theorem 39.1.9 Suppose $m \geq 0$ and $j$ is a nonnegative integer satisfying $2 j<n$. Also let $U$ bean open set which satisfies Assumption 39.1.2. Then id $\in \mathscr{L}\left(H^{m+j}(U), W^{m, q}(U)\right)$ where

$$
\begin{equation*}
q \equiv \frac{2 n}{n-2 j} \tag{39.1.6}
\end{equation*}
$$

If, in addition to the above, $U$ is bounded and $1 \leq r<q$, then

$$
\mathrm{id} \in \mathscr{L}\left(H^{m+j}(U), W^{m, r}(U)\right)
$$

and is compact.
Theorem 39.1.10 Suppose for $j$ a nonnegative integer, $2 j<n<2(j+1)$ and let $m$ be a positive integer. Let $U$ be any bounded open set in $\mathbb{R}^{n}$ which satisfies Assumption 39.1.2. Then $\operatorname{id} \in \mathscr{L}\left(H^{m+j}(U), C^{m-1, \lambda}(\bar{U})\right)$ for every $\lambda \leq \lambda_{0} \equiv(j+1)-\frac{n}{2}$ and if $\lambda<(j+1)-$ $\frac{n}{2}$, id is compact.

### 39.2 Fractional Order Spaces

What has been gained by all this? The main thing is that $H^{m+s}(U)$ makes sense for any $s \in(0,1)$ and $m$ an integer. You simply replace $m$ with $m+s$ in the above for $s \in(0,1)$. This gives what is meant by $H^{m+s}\left(\mathbb{R}^{n}\right)$

Definition 39.2.1 For $m$ an integer and $s \in(0,1)$, let $H^{m+s}\left(\mathbb{R}^{n}\right) \equiv$

$$
\begin{equation*}
\left\{u \in L^{2}\left(\mathbb{R}^{n}\right):\|u\|_{H^{m+s}\left(\mathbb{R}^{n}\right)} \equiv\left(\int_{\mathbb{R}^{n}}\left(1+|\mathbf{x}|^{2}\right)^{m+s}|F u(\mathbf{x})|^{2} d x\right)^{1 / 2}<\infty\right\} \tag{39.2.7}
\end{equation*}
$$

You could also simply refer to $H^{t}\left(\mathbb{R}^{n}\right)$ where $t$ is a real number replacing the $m+s$ in the above formula with $t$ but I want to emphasize the notion that $t=m+s$ where $m$ is a nonnegative integer. Therefore, I will often write $m+s$. Let $U$ be an open set in $\mathbb{R}^{n}$. Then

$$
\begin{equation*}
H^{m+s}(U) \equiv\left\{u: u=\left.v\right|_{U} \text { for some } v \in H^{m+s}\left(\mathbb{R}^{n}\right)\right\} \tag{39.2.8}
\end{equation*}
$$

Define the norm in this space by

$$
\begin{equation*}
\|u\|_{H^{m+s}(U)} \equiv \inf \left\{\|v\|_{H^{m+s}\left(\mathbb{R}^{n}\right)}:\left.v\right|_{U}=u\right\} \tag{39.2.9}
\end{equation*}
$$

Lemma 39.2.2 $H^{m+s}(U)$ is a Banach space.
Proof: Just repeat the proof of Lemma 39.1.7.
The theorem about density of $\mathfrak{S}$ also remains true in $H^{m+s}\left(\mathbb{R}^{n}\right)$. Just repeat the proof of that part of Lemma 39.1.5 replacing the integer, $m$, with the symbol, $m+s$.

Lemma 39.2.3 $\mathfrak{S}$ is dense in $H^{m+s}\left(\mathbb{R}^{n}\right)$.
In fact, more can be said.
Corollary 39.2.4 Let $U$ be an open set and let $\left.\mathfrak{S}\right|_{U}$ denote the restrictions of functions of $\mathfrak{S}$ to $U$. Then $\left.\mathfrak{S}\right|_{U}$ is dense in $H^{t}(U)$.

Proof: Let $u \in H^{t}(U)$ and let $v \in H^{t}\left(\mathbb{R}^{n}\right)$ such that $\left.v\right|_{U}=u$ a.e. Then since $\mathfrak{S}$ is dense in $H^{t}\left(\mathbb{R}^{n}\right)$, there exists $w \in \mathfrak{S}$ such that

$$
\|w-v\|_{H^{t}\left(\mathbb{R}^{n}\right)}<\varepsilon .
$$

It follows that

$$
\begin{aligned}
\|u-w\|_{H^{t}(U)} & \leq\|u-v\|_{H^{t}(U)}+\|v-w\|_{H^{t}(U)} \\
& \leq 0+\|v-w\|_{H^{t}\left(\mathbb{R}^{n}\right)}<\varepsilon
\end{aligned}
$$

These fractional order spaces are important when trying to understand the trace on the boundary. The Fourier transform description also makes it very easy to establish interesting inequalities such as interpolation inequalities.

Lemma 39.2.5 Let $0 \leq r<s<t$. Then if $u \in H^{t}\left(\mathbb{R}^{n}\right)$,

$$
\|u\|_{H^{s}\left(\mathbb{R}^{n}\right)} \leq\|u\|_{H^{r}\left(\mathbb{R}^{n}\right)}^{\theta}\|u\|_{H^{t}\left(\mathbb{R}^{n}\right)}^{1-\theta}
$$

where $\theta$ is a positve number such that $\theta r+(1-\theta) t=s$.

Proof: This follows from Holder's inequality applied to the measure $\mu$ given by

$$
\mu(E)=\int_{E}|F u|^{2} d x
$$

Thus

$$
\begin{aligned}
& \int\left(1+|\mathbf{x}|^{2}\right)^{s}|F u|^{2} d x \\
= & \int\left(1+|\mathbf{x}|^{2}\right)^{r \theta}\left(1+|\mathbf{x}|^{2}\right)^{(1-\theta) t}|F u|^{2} d x \\
\leq & \left(\int\left(1+|\mathbf{x}|^{2}\right)^{r}|F u|^{2} d x\right)^{\theta}\left(\int\left(1+|\mathbf{x}|^{2}\right)^{(1-\theta) t}|F u|^{2} d x\right)^{1-\theta} \\
= & \|u\|_{H^{r}\left(\mathbb{R}^{n}\right)}^{2 \theta}| | u \|_{H^{t}\left(\mathbb{R}^{n}\right)}^{2(1-\theta)} .
\end{aligned}
$$

Taking square roots yields the desired inequality.
Corollary 39.2.6 Let $U$ be an open set satisfying Assumption 39.1.2 and let $p<q$ where $p, q$ are two nonnegative integers. Also let $t \in(p, q)$. Then exists a constant, $C$ independent of $u \in H^{q}(U)$ such that for all $u \in H^{q}(U)$,

$$
\|u\|_{H^{t}(U)} \leq C\|u\|_{H^{p}(U)}^{\theta}\|u\|_{H^{q}(U)}^{1-\theta}
$$

where $\theta$ is such that $t=\theta p+(1-\theta) q$.
Proof: Let $E \in \mathscr{L}\left(H^{q}(U), H^{q}\left(\mathbb{R}^{n}\right)\right)$ such that for all positive integers, $l$ less than or equal to $q, E \in \mathscr{L}\left(H^{l}(U), H^{l}\left(\mathbb{R}^{n}\right)\right)$. Then $\left.E u\right|_{U}=u$ and $E u \in H^{t}\left(\mathbb{R}^{n}\right)$. Therefore, by Lemma 39.2.5,

$$
\begin{aligned}
\|u\|_{H^{t}(U)} & \leq\|E u\|_{H^{t}\left(\mathbb{R}^{n}\right)} \leq\|E u\|_{H^{p}\left(\mathbb{R}^{n}\right)}^{\theta}\|E u\|_{H^{q}\left(\mathbb{R}^{n}\right)}^{1-\theta} \\
& \leq C\|u\|_{H^{p}(U)}^{\theta}\|u\|_{H^{q}(U)}^{1-\theta} .
\end{aligned}
$$

Now recall the very important Theorem 38.0.14 on Page 1316 which is listed here for convenience.

Theorem 39.2.7 Let $\mathbf{h}: U \rightarrow V$ be one to one and onto where $U$ and $V$ are two open sets. Also suppose that $D^{\alpha} \mathbf{h}$ and $D^{\alpha}\left(\mathbf{h}^{-1}\right)$ exist and are Lipschitz continuous if $|\alpha| \leq m-1$ for $m$ a positive integer. Then

$$
\mathbf{h}^{*}: W^{m, p}(V) \rightarrow W^{m, p}(U)
$$

is continuous, linear, one to one, and has an inverse with the same properties, the inverse being $\left(\mathbf{h}^{-1}\right)^{*}$.

### 39.3 An Intrinsic Norm

Is there something like this for the fractional order spaces? Yes there is. However, in order to prove it, it is convenient to use an equivalent norm for $H^{m+s}\left(\mathbb{R}^{n}\right)$ which does not depend explicitly on the Fourier transform. The following theorem is similar to one in [68]. It describes the norm in $H^{m+s}\left(\mathbb{R}^{n}\right)$ in terms which are free of the Fourier transform. This is also called an intrinsic norm [1].

Theorem 39.3.1 Let $s \in(0,1)$ and let $m$ be a nonnegative integer. Then an equivalent norm for $H^{m+s}\left(\mathbb{R}^{n}\right)$ is

$$
\left\|\|u\|_{m+s}^{2} \equiv\right\| u \|_{m, 2, \mathbb{R}^{n}}^{2}+\sum_{|\alpha|=m} \iint\left|D^{\alpha} u(\mathbf{x})-D^{\alpha} u(\mathbf{y})\right|^{2}|\mathbf{x}-\mathbf{y}|^{-n-2 s} d x d y
$$

Also if $|\beta| \leq m$, there are constants, $m(s)$ and $M(s)$ such that

$$
\begin{gather*}
m(s) \int|F u(\mathbf{z})|^{2}\left|\mathbf{z}^{\beta}\right|^{2}|\mathbf{z}|^{2 s} d z \leq \iint\left|D^{\beta} u(\mathbf{x})-D^{\beta} u(\mathbf{y})\right|^{2}|\mathbf{x}-\mathbf{y}|^{-n-2 s} d x d y \\
\leq M(s) \int|F u(\mathbf{z})|^{2}\left|\mathbf{z}^{\beta}\right|^{2}|\mathbf{z}|^{2 s} d z \tag{39.3.10}
\end{gather*}
$$

Proof: Let $u \in \mathfrak{S}$ which is dense in $H^{m+s}\left(\mathbb{R}^{n}\right)$. The Fourier transform of the function, $\mathbf{y} \rightarrow D^{\alpha} u(\mathbf{x}+\mathbf{y})-D^{\alpha} u(\mathbf{y})$ equals

$$
\left(e^{i \mathbf{x} \cdot \mathbf{z}}-1\right) F D^{\alpha} u(\mathbf{z})
$$

Now by Fubini's theorem and Plancherel's theorem along with the above, taking $|\alpha|=m$,

$$
\begin{align*}
& \iint\left|D^{\alpha} u(\mathbf{x})-D^{\alpha} u(\mathbf{y})\right|^{2}|\mathbf{x}-\mathbf{y}|^{-n-2 s} d x d y \\
= & \iint\left|D^{\alpha} u(\mathbf{y}+\mathbf{t})-D^{\alpha} u(\mathbf{y})\right|^{2}|\mathbf{t}|^{-n-2 s} d t d y \\
= & \int|\mathbf{t}|^{-n-2 s} \int\left|D^{\alpha} u(\mathbf{y}+\mathbf{t})-D^{\alpha} u(\mathbf{y})\right|^{2} d y d t \\
= & \int|\mathbf{t}|^{-n-2 s} \int\left|\left(e^{i \mathbf{t} \cdot \mathbf{z}}-1\right) F D^{\alpha} u(\mathbf{z})\right|^{2} d z d t \\
= & \int\left|F D^{\alpha} u(\mathbf{z})\right|^{2}\left(\int|\mathbf{t}|^{-n-2 s}\left|\left(e^{i \mathbf{t} \cdot \mathbf{z}}-1\right)\right|^{2} d t\right) d z \tag{39.3.11}
\end{align*}
$$

Consider the inside integral, the one taken with respect to $\mathbf{t}$.

$$
G(\mathbf{z}) \equiv\left(\int|\mathbf{t}|^{-n-2 s}\left|\left(e^{i \mathbf{t} \cdot \mathbf{z}}-1\right)\right|^{2} d t\right)
$$

The essential thing to notice about this function of $\mathbf{z}$ is that it is a positive real number whenever $\mathbf{z} \neq \mathbf{0}$. This is because for small $|\mathbf{t}|$, the integrand is dominated by $C|\mathbf{t}|^{-n+2(1-s)}$. Changing to polar coordinates, you see that

$$
\int_{[|\mathbf{t}| \leq 1]}|\mathbf{t}|^{-n-2 s}\left|\left(e^{i \mathbf{t} \cdot \mathbf{z}}-1\right)\right|^{2} d t<\infty
$$

Next, for $|\mathbf{t}|>1$, the integrand is bounded by $4|\mathbf{t}|^{-n-2 s}$, and changing to polar coordinates shows

$$
\int_{[|\mathbf{t}|>1]}|\mathbf{t}|^{-n-2 s}\left|\left(e^{i \mathbf{t} \cdot \mathbf{z}}-1\right)\right|^{2} d t \leq 4 \int_{[|\mathbf{t}|>1]}|\mathbf{t}|^{-n-2 s} d t<\infty .
$$

Now for $\alpha>0$,

$$
\begin{aligned}
G(\alpha \mathbf{z}) & =\int|\mathbf{t}|^{-n-2 s}\left|\left(e^{i \mathbf{t} \cdot \alpha \mathbf{z}}-1\right)\right|^{2} d t \\
& =\int|\mathbf{t}|^{-n-2 s}\left|\left(e^{i \alpha \mathbf{x} \cdot \mathbf{z}}-1\right)\right|^{2} d t \\
& =\int\left|\frac{\mathbf{r}}{\alpha}\right|^{-n-2 s}\left|\left(e^{i \mathbf{r} \cdot \mathbf{z}}-1\right)\right|^{2} \frac{1}{\alpha^{n}} d r \\
& =\alpha^{2 s} \int|\mathbf{r}|^{-n-2 s}\left|\left(e^{i \mathbf{r} \cdot \mathbf{z}}-1\right)\right|^{2} d r=\alpha^{2 s} G(\mathbf{z})
\end{aligned}
$$

Also $G$ is continuous and strictly positive. Letting

$$
0<m(s)=\min \{G(\mathbf{w}):|\mathbf{w}|=1\}
$$

and

$$
M(s)=\max \{G(\mathbf{w}):|\mathbf{w}|=1\}
$$

it follows from this, and letting $\alpha=|\mathbf{z}|, \mathbf{w} \equiv \mathbf{z} /|\mathbf{z}|$, that

$$
G(\mathbf{z}) \in\left(m(s)|\mathbf{z}|^{2 s}, M(s)|\mathbf{z}|^{2 s}\right)
$$

More can be said but this will suffice. Also observe that for $s \in(0,1)$ and $b>0$,

$$
(1+b)^{s} \leq 1+b^{s}, 2^{1-s}(1+b)^{s} \geq 1+b^{s}
$$

In what follows, $C(s)$ will denote a constant which depends on the indicated quantities which may be different on different lines of the argument. Then from 39.3.11,

$$
\begin{aligned}
& \iint\left|D^{\alpha} u(\mathbf{x})-D^{\alpha} u(\mathbf{y})\right|^{2}|\mathbf{x}-\mathbf{y}|^{-n-2 s} d x d y \\
& \quad \leq M(s) \int\left|F D^{\alpha} u(\mathbf{z})\right|^{2}|\mathbf{z}|^{2 s} d z \\
& \quad=M(s) \int|F u(\mathbf{z})|^{2}\left|\mathbf{z}^{\alpha}\right|^{2}|\mathbf{z}|^{2 s} d z
\end{aligned}
$$

No reference was made to $|\alpha|=m$ and so this establishes the top half of 39.3.10. Therefore,

$$
\begin{aligned}
\|u\| \|_{m+s}^{2} & \equiv\|u\|_{m, 2, \mathbb{R}^{n}}^{2}+\sum_{|\alpha|=m} \iint\left|D^{\alpha} u(\mathbf{x})-D^{\alpha} u(\mathbf{y})\right|^{2}|\mathbf{x}-\mathbf{y}|^{-n-2 s} d x d y \\
& \leq C \int\left(1+|\mathbf{z}|^{2}\right)^{m}|F u(\mathbf{z})|^{2} d z+M(s) \int|F u(\mathbf{z})|^{2} \sum_{|\alpha|=m}\left|\mathbf{z}^{\alpha}\right|^{2}|\mathbf{z}|^{2 s} d z
\end{aligned}
$$

Recall that

$$
\begin{equation*}
\sum_{|\alpha| \leq m} z_{1}^{2 \alpha_{1}} \cdots z_{n}^{2 \alpha_{n}} \leq\left(1+\sum_{j=1}^{n} z_{j}^{2}\right)^{m} \leq C(n, m) \sum_{|\alpha| \leq m} z_{1}^{2 \alpha_{1}} \cdots z_{n}^{2 \alpha_{n}} \tag{39.3.12}
\end{equation*}
$$

Therefore, where $C(n, m)$ is the largest of the multinomial coefficients obtained in the expansion,

$$
\left(1+\sum_{j=1}^{n} z_{j}^{2}\right)^{m}
$$

Therefore,

$$
\begin{aligned}
& \left\|\left||u \||_{m+s}^{2}\right.\right. \\
\leq & C \int\left(1+|\mathbf{z}|^{2}\right)^{m}|F u(\mathbf{z})|^{2} d z+M(s) \int|F u(\mathbf{z})|^{2} \sum_{|\alpha|=m}\left|\mathbf{z}^{\alpha}\right|^{2}|\mathbf{z}|^{2 s} d z \\
\leq & C \int\left(1+|\mathbf{z}|^{2}\right)^{m+s}|F u(\mathbf{z})|^{2} d z+M(s) \int|F u(\mathbf{z})|^{2}\left(1+|\mathbf{z}|^{2}\right)^{m}|\mathbf{z}|^{2 s} d z \\
\leq & C \int\left(1+|\mathbf{z}|^{2}\right)^{m+s}|F u(\mathbf{z})|^{2} d z=C\|u\|_{H^{m+s}\left(\mathbb{R}^{n}\right)}
\end{aligned}
$$

It remains to show the other inequality. From 39.3.11,

$$
\begin{aligned}
& \iint\left|D^{\alpha} u(\mathbf{x})-D^{\alpha} u(\mathbf{y})\right|^{2}|\mathbf{x}-\mathbf{y}|^{-n-2 s} d x d y \\
& \quad \geq m(s) \int\left|F D^{\alpha} u(\mathbf{z})\right|^{2}|\mathbf{z}|^{2 s} d z \\
& \quad=m(s) \int|F u(\mathbf{z})|^{2}\left|\mathbf{z}^{\alpha}\right|^{2}|\mathbf{z}|^{2 s} d z
\end{aligned}
$$

No reference was made to $|\alpha|=m$ and so this establishes the bottom half of 39.3.10. Therefore, from 39.3.12,

$$
\begin{aligned}
& \|\|u\|\|_{m+s}^{2} \\
\geq & C \int\left(1+|\mathbf{z}|^{2}\right)^{m}|F u(\mathbf{z})|^{2} d z+m(s) \int|F u(\mathbf{z})|^{2} \sum_{|\alpha|=m}\left|\mathbf{z}^{\alpha}\right|^{2}|\mathbf{z}|^{2 s} d z \\
\geq & C \int\left(1+|\mathbf{z}|^{2}\right)^{m}|F u(\mathbf{z})|^{2} d z+C \int|F u(\mathbf{z})|^{2}\left(1+|\mathbf{z}|^{2}\right)^{m}|\mathbf{z}|^{2 s} d z \\
= & C \int\left(1+|\mathbf{z}|^{2}\right)^{m}\left(1+|\mathbf{z}|^{2 s}\right)|F u(\mathbf{z})|^{2} d z \\
\geq & C \int\left(1+|\mathbf{z}|^{2}\right)^{m}\left(1+|\mathbf{z}|^{2}\right)^{s}|F u(\mathbf{z})|^{2} d z \\
= & C \int\left(1+|\mathbf{z}|^{2}\right)^{m+s}|F u(\mathbf{z})|^{2} d z=\|u\|_{H^{m+s}\left(\mathbb{R}^{n}\right)} .
\end{aligned}
$$

This proves the theorem.
With the above intrinsic norm, it becomes possible to prove the following version of Theorem 39.2.7.

Lemma 39.3.2 Let $\mathbf{h}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be one to one and onto. Also suppose that $D^{\alpha} \mathbf{h}$ and $D^{\alpha}\left(\mathbf{h}^{-1}\right)$ exist and are Lipschitz continuous if $|\alpha| \leq m$ for $m$ a positive integer. Then

$$
\mathbf{h}^{*}: H^{m+s}\left(\mathbb{R}^{n}\right) \rightarrow H^{m+s}\left(\mathbb{R}^{n}\right)
$$

is continuous, linear, one to one, and has an inverse with the same properties, the inverse being $\left(\mathbf{h}^{-1}\right)^{*}$.

Proof: Let $u \in \mathfrak{S}$. From Theorem 39.2.7 and the equivalence of the norms in $W^{m, 2}\left(\mathbb{R}^{n}\right)$ and $H^{m}\left(\mathbb{R}^{n}\right)$,

$$
\begin{gather*}
\left\|\mathbf{h}^{*} u\right\|_{H^{m}\left(\mathbb{R}^{n}\right)}^{2}+\iint \sum_{|\alpha|=m}\left|D^{\alpha} \mathbf{h}^{*} u(\mathbf{x})-D^{\alpha} \mathbf{h}^{*} u(\mathbf{y})\right|^{2}|\mathbf{x}-\mathbf{y}|^{-n-2 s} d x d y \\
\leq C\|u\|_{H^{m}\left(\mathbb{R}^{n}\right)}^{2}+\iint \sum_{|\alpha|=m}\left|D^{\alpha} \mathbf{h}^{*} u(\mathbf{x})-D^{\alpha} \mathbf{h}^{*} u(\mathbf{y})\right|^{2}|\mathbf{x}-\mathbf{y}|^{-n-2 s} d x d y \\
=C\|u\|_{H^{m}\left(\mathbb{R}^{n}\right)}^{2}+\iint \sum_{|\alpha|=m} \mid \sum_{|\beta(\alpha)| \leq m} \mathbf{h}^{*}\left(D^{\beta(\alpha)} u\right) g_{\beta(\alpha)}(\mathbf{x}) \\
\quad-\left.\mathbf{h}^{*}\left(D^{\beta(\alpha)} u\right) g_{\beta(\alpha)}(\mathbf{y})\right|^{2}|\mathbf{x}-\mathbf{y}|^{-n-2 s} d x d y  \tag{39.3.13}\\
\leq C\|u\|_{H^{m}\left(\mathbb{R}^{n}\right)}^{2}+C \iint \sum_{|\alpha|=m} \sum_{|\beta(\alpha)| \leq m} \mid \mathbf{h}^{*}\left(D^{\beta(\alpha)} u\right) g_{\beta(\alpha)}(\mathbf{x}) \\
\quad-\left.\mathbf{h}^{*}\left(D^{\beta(\alpha)} u\right) g_{\beta(\alpha)}(\mathbf{y})\right|^{2}|\mathbf{x}-\mathbf{y}|^{-n-2 s} d x d y
\end{gather*}
$$

A single term in the last sum corresponding to a given $\alpha$ is then of the form,

$$
\begin{align*}
& \iint\left|\mathbf{h}^{*}\left(D^{\beta} u\right) g_{\beta}(\mathbf{x})-\mathbf{h}^{*}\left(D^{\beta} u\right) g_{\beta}(\mathbf{y})\right|^{2}|\mathbf{x}-\mathbf{y}|^{-n-2 s} d x d y  \tag{39.3.14}\\
\leq & {\left[\iint\left|\mathbf{h}^{*}\left(D^{\beta} u\right)(\mathbf{x}) g_{\beta}(\mathbf{x})-\mathbf{h}^{*}\left(D^{\beta} u\right)(\mathbf{y}) g_{\beta}(\mathbf{x})\right|^{2}|\mathbf{x}-\mathbf{y}|^{-n-2 s} d x d y+\right.} \\
& \left.\iint\left|\mathbf{h}^{*}\left(D^{\beta} u\right)(\mathbf{y}) g_{\beta}(\mathbf{x})-\mathbf{h}^{*}\left(D^{\beta} u\right)(\mathbf{y}) g_{\beta}(\mathbf{y})\right|^{2}|\mathbf{x}-\mathbf{y}|^{-n-2 s} d x d y\right] \\
\leq & {\left[C(\mathbf{h}) \iint\left|\mathbf{h}^{*}\left(D^{\beta} u\right)(\mathbf{x})-\mathbf{h}^{*}\left(D^{\beta} u\right)(\mathbf{y})\right|^{2}|\mathbf{x}-\mathbf{y}|^{-n-2 s} d x d y+\right.} \\
& \left.\iint\left|\mathbf{h}^{*}\left(D^{\beta} u\right)(\mathbf{y})\right|^{2}\left|g_{\beta}(\mathbf{x})-g_{\beta}(\mathbf{y})\right|^{2}|\mathbf{x}-\mathbf{y}|^{-n-2 s} d x d y\right]
\end{align*}
$$

Changing variables, and then using the names of the old variables to simplify the notation,

$$
\begin{aligned}
\leq & {\left[C\left(\mathbf{h}, \mathbf{h}^{-1}\right) \iint\left|\left(D^{\beta} u\right)(\mathbf{x})-\left(D^{\beta} u\right)(\mathbf{y})\right|^{2}|\mathbf{x}-\mathbf{y}|^{-n-2 s} d x d y+\right.} \\
& \left.\iint\left|\mathbf{h}^{*}\left(D^{\beta} u\right)(\mathbf{y})\right|^{2}\left|g_{\beta}(\mathbf{x})-g_{\beta}(\mathbf{y})\right|^{2}|\mathbf{x}-\mathbf{y}|^{-n-2 s} d x d y\right]
\end{aligned}
$$

By 39.3.10,

$$
\begin{aligned}
\leq & C(\mathbf{h}) \int|F(u)(\mathbf{z})|^{2}\left|\mathbf{z}^{\beta}\right|^{2}|\mathbf{z}|^{2 s} d z \\
& +\iint\left|\mathbf{h}^{*}\left(D^{\beta} u\right)(\mathbf{y})\right|^{2}\left|g_{\beta}(\mathbf{x})-g_{\beta}(\mathbf{y})\right|^{2}|\mathbf{x}-\mathbf{y}|^{-n-2 s} d x d y
\end{aligned}
$$

In the second term, let $\mathbf{t}=\mathbf{x}-\mathbf{y}$. Then this term is of the form

$$
\begin{align*}
& \int\left|\mathbf{h}^{*}\left(D^{\beta} u\right)(\mathbf{y})\right|^{2} \int\left|g_{\beta}(\mathbf{y}+\mathbf{t})-g_{\beta}(\mathbf{y})\right|^{2}|\mathbf{t}|^{-n-2 s} d t d y  \tag{39.3.15}\\
\leq & C \int\left|\mathbf{h}^{*}\left(D^{\beta} u\right)(\mathbf{y})\right|^{2} d y \leq C| | u \|_{H^{m}\left(\mathbb{R}^{n}\right)}^{2} \tag{39.3.16}
\end{align*}
$$

because the inside integral equals a constant which depends on the Lipschitz constants and bounds of the function, $g_{\beta}$ and these things depend only on $\mathbf{h}$. The reason this integral is finite is that for $|\mathbf{t}| \leq 1$,

$$
\left|g_{\beta}(\mathbf{y}+\mathbf{t})-g_{\beta}(\mathbf{y})\right|^{2}|\mathbf{t}|^{-n-2 s} \leq K|\mathbf{t}|^{2}|\mathbf{t}|^{-n-2 s}
$$

and using polar coordinates, you see

$$
\int_{[|\mathbf{t}| \leq 1]}\left|g_{\beta}(\mathbf{y}+\mathbf{t})-g_{\beta}(\mathbf{y})\right|^{2}|\mathbf{t}|^{-n-2 s} d t<\infty
$$

Now for $|\mathbf{t}|>1$, the integrand in 39.3 .15 is dominated by $4|\mathbf{t}|^{-n-2 s}$ and using polar coordinates, this yields

$$
\int_{[|\mathbf{t}|>1]}\left|g_{\beta}(\mathbf{y}+\mathbf{t})-g_{\beta}(\mathbf{y})\right|^{2}|\mathbf{t}|^{-n-2 s} d t \leq 4 \int_{[|\mathbf{t}|>1]}|\mathbf{t}|^{-n-2 s} d t<\infty
$$

It follows 39.3.14 is dominated by an expression of the form

$$
C(\mathbf{h}) \int|F(u)(\mathbf{z})|^{2}\left|\mathbf{z}^{\beta}\right|^{2}|\mathbf{z}|^{2 s} d z+C\|u\|_{H^{m}\left(\mathbb{R}^{n}\right)}^{2}
$$

and so the sum in 39.3.13 is dominated by

$$
\begin{aligned}
& C(m, \mathbf{h}) \int|F(u)(\mathbf{z})|^{2}|\mathbf{z}|^{2 s} \sum_{|\beta| \leq m}\left|\mathbf{z}^{\beta}\right|^{2} d z+C\|u\|_{H^{m}\left(\mathbb{R}^{n}\right)}^{2} \\
\leq & C(m, \mathbf{h}) \int|F(u)(\mathbf{z})|^{2}\left(1+|\mathbf{z}|^{2}\right)^{s}\left(1+|\mathbf{z}|^{2}\right)^{m} d z+C\|u\|_{H^{m}\left(\mathbb{R}^{n}\right)}^{2} \\
\leq & C\|u\|_{H^{m+s}\left(\mathbb{R}^{n}\right)}^{2} .
\end{aligned}
$$

This proves the theorem because the assertion about $\mathbf{h}^{-1}$ is obvious. Just replace $\mathbf{h}$ with $\mathbf{h}^{-1}$ in the above argument.

Next consider the case where $U$ is an open set.
Lemma 39.3.3 Let $\mathbf{h}(U) \subseteq V$ where $U$ and $V$ are open subsets of $\mathbb{R}^{n}$ and suppose that $\mathbf{h}, \mathbf{h}^{-1}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ are both functions in $C^{m, 1}\left(\mathbb{R}^{n}\right)$. Recall this means $D^{\alpha} \mathbf{h}$ and $D^{\alpha} \mathbf{h}^{-1}$ exist and are Lipschitz continuous for all $|\alpha| \leq m$. Then $\mathbf{h}^{*} \in \mathscr{L}\left(H^{m+s}(V), H^{m+s}(U)\right)$.

Proof: Let $u \in H^{m+s}(V)$ and let $v \in H^{m+s}\left(\mathbb{R}^{n}\right)$ such that $\left.v\right|_{V}=u$. Then from the above, $\mathbf{h}^{*} v \in H^{m+s}\left(\mathbb{R}^{n}\right)$ and so $\mathbf{h}^{*} u \in H^{m+s}(U)$ because $\mathbf{h}^{*} u=\left.\mathbf{h}^{*} v\right|_{U}$. Then by Lemma 39.3.2,

$$
\left\|\mathbf{h}^{*} u\right\|_{H^{m+s}(U)} \leq\left\|\mathbf{h}^{*} v\right\|_{H^{m+s}\left(\mathbb{R}^{n}\right)} \leq C\|v\|_{H^{m+s}\left(\mathbb{R}^{n}\right)}
$$

Since this is true for all $v \in H^{m+s}\left(\mathbb{R}^{n}\right)$, it follows that

$$
\left\|\mathbf{h}^{*} u\right\|_{H^{m+s}(U)} \leq C\|u\|_{H^{m+s}(V)} .
$$

With harder work, you don't need to have $\mathbf{h}, \mathbf{h}^{-1}$ defined on all of $\mathbb{R}^{n}$ but I don't feel like including the details so this lemma will suffice.

Another interesting application of the intrinsic norm is the following.
Lemma 39.3.4 Let $\phi \in C^{m, 1}\left(\mathbb{R}^{n}\right)$ and suppose $\operatorname{spt}(\phi)$ is compact. Then there exists a constant, $C_{\phi}$ such that whenever $u \in H^{m+s}\left(\mathbb{R}^{n}\right)$,

$$
\|\phi u\|_{H^{m+s}\left(\mathbb{R}^{n}\right)} \leq C_{\phi}\|u\|_{H^{m+s}\left(\mathbb{R}^{n}\right)}
$$

Proof: It is a routine exercise in the product rule to verify that

$$
\|\phi u\|_{H^{m}\left(\mathbb{R}^{n}\right)} \leq C_{\phi}\|u\|_{H^{m}\left(\mathbb{R}^{n}\right)}
$$

It only remains to consider the term involving the integral. A typical term is

$$
\iint\left|D^{\alpha} \phi u(\mathbf{x})-D^{\alpha} \phi u(\mathbf{y})\right|^{2}|\mathbf{x}-\mathbf{y}|^{-n-2 s} d x d y
$$

This is a finite sum of terms of the form

$$
\iint\left|D^{\gamma} \phi(\mathbf{x}) D^{\beta} u(\mathbf{x})-D^{\gamma} \phi(\mathbf{y}) D^{\beta} u(\mathbf{y})\right|^{2}|\mathbf{x}-\mathbf{y}|^{-n-2 s} d x d y
$$

where $|\gamma|$ and $|\beta| \leq m$.

$$
\begin{aligned}
\leq & 2 \iint\left|D^{\gamma} \phi(\mathbf{x})\right|^{2}\left|D^{\beta} u(\mathbf{x})-D^{\beta} u(\mathbf{y})\right|^{2}|\mathbf{x}-\mathbf{y}|^{-n-2 s} d x d y \\
& +2 \iint\left|D^{\beta} u(\mathbf{y})\right|^{2}\left|D^{\gamma} \phi(\mathbf{x})-D^{\gamma} \phi(\mathbf{y})\right|^{2}|\mathbf{x}-\mathbf{y}|^{-n-2 s} d x d y
\end{aligned}
$$

By 39.3.10 and the Lipschitz continuity of all the derivatives of $\phi$, this is dominated by

$$
\begin{aligned}
& C M(s) \int|F u(\mathbf{z})|^{2}\left|\mathbf{z}^{\beta}\right|^{2}|\mathbf{z}|^{2 s} d z \\
& +K \iint\left|D^{\beta} u(\mathbf{y})\right|^{2}|\mathbf{x}-\mathbf{y}|^{2}|\mathbf{x}-\mathbf{y}|^{-n-2 s} d x d y \\
= & C M(s) \int|F u(\mathbf{z})|^{2}\left|\mathbf{z}^{\beta}\right|^{2}|\mathbf{z}|^{2 s} d z \\
& +K \int\left|D^{\beta} u(\mathbf{y})\right|^{2} \int|\mathbf{t}|^{-n+2(1-s)} d t d y \\
\leq & C(s)\left(\int|F u(\mathbf{z})|^{2}\left|\mathbf{z}^{\beta}\right|^{2}|\mathbf{z}|^{2 s} d z+K \int\left|D^{\beta} u(\mathbf{y})\right|^{2} d y\right) \\
\leq & C(s) \int\left(1+|\mathbf{y}|^{2}\right)^{m+s}|F u(\mathbf{y})|^{2} d y .
\end{aligned}
$$

Since there are only finitely many such terms, this proves the lemma.

Corollary 39.3.5 Let $t=m+s$ for $s \in[0,1)$ and let $U, V$ be open sets. Let $\phi \in C_{c}^{m, 1}(V)$. This means $\operatorname{spt}(\phi) \subseteq V$ and $\phi \in C^{m, 1}\left(\mathbb{R}^{n}\right)$. Then if $u \in H^{t}(U)$ then $u \phi \in H^{t}(U \cap V)$ and $\|u \phi\|_{H^{t}(U \cap V)} \leq C_{\phi}\|u\|_{H^{t}(U)}$.

Proof: Let $\left.v\right|_{U}=u$ a.e. where $v \in H^{t}\left(\mathbb{R}^{n}\right)$. Then by Lemma 39.3.4, $\phi v \in H^{t}\left(\mathbb{R}^{n}\right)$ and $\left.\phi v\right|_{U \cap V}=\phi u$ a.e. Therefore, $\phi u \in H^{t}(U \cap V)$ and

$$
\|\phi u\|_{H^{t}(U \cap V)} \leq\|\phi v\|_{H^{t}\left(\mathbb{R}^{n}\right)} \leq C_{\phi}\|v\|_{H^{t}\left(\mathbb{R}^{n}\right)}
$$

Taking the infimum for all such $v$ whose restrictions equal $u$, this yields

$$
\|\phi u\|_{H^{t}(U \cap V)} \leq C_{\phi}\|u\|_{H^{t}(U)}
$$

This proves the corollary.

### 39.4 Embedding Theorems

The Fourier transform description of Sobolev spaces makes possible fairly easy proofs of various embedding theorems.

Definition 39.4.1 Let $C_{b}^{m}\left(\mathbb{R}^{n}\right)$ denote the functions which are m times continuously differentiable and for which

$$
\sup _{|\alpha| \leq m x \in \mathbb{R}^{n}} \sup \left|D^{\alpha} u(\mathbf{x})\right| \equiv\|u\|_{C_{b}^{m}\left(\mathbb{R}^{n}\right)}<\infty
$$

For $U$ an open set, $C^{m}(\bar{U})$ denotes the functions which are restrictions of $C_{b}^{m}\left(\mathbb{R}^{n}\right)$ to $U$.
It is clear this is a Banach space, the proof being a simple exercise in the use of the fundamental theorem of calculus along with standard results about uniform convergence.

Lemma 39.4.2 Let $u \in \mathfrak{S}$ and let $\frac{n}{2}+m<t$. Then there exists $C$ independent of $u$ such that

$$
\|u\|_{C_{b}^{m}\left(\mathbb{R}^{n}\right)} \leq C\|u\|_{H^{t}\left(\mathbb{R}^{n}\right)}
$$

Proof: Using the fact that the Fourier transform maps $\mathfrak{S}$ to $\mathfrak{S}$ and the definition of the Fourier transform,

$$
\begin{aligned}
\left|D^{\alpha} u(\mathbf{x})\right| & \leq C\left\|F D^{\alpha} u\right\|_{L^{1}\left(\mathbb{R}^{n}\right)} \\
& =C \int\left|\mathbf{x}^{\alpha}\right||F u(\mathbf{x})| d x \\
& \leq C \int\left(1+|\mathbf{x}|^{2}\right)^{|\alpha| / 2}|F u(\mathbf{x})| d x \\
& \leq C \int\left(1+|\mathbf{x}|^{2}\right)^{m / 2}\left(1+|\mathbf{x}|^{2}\right)^{-t / 2}\left(1+|\mathbf{x}|^{2}\right)^{t / 2}|F u(\mathbf{x})| d x \\
& \leq C\left(\int\left(1+|\mathbf{x}|^{2}\right)^{m-t} d x\right)^{1 / 2}\left(\int\left(1+|\mathbf{x}|^{2}\right)^{t}|F u(\mathbf{x})|^{t}\right)^{1 / 2} \\
& \leq C\|u\|_{H^{t}\left(\mathbb{R}^{n}\right)}
\end{aligned}
$$

because for the given values of $t$ and $m$ the first integral is finite. This follows from a use of polar coordinates. Taking sup over all $\mathbf{x} \in \mathbb{R}^{n}$ and $|\alpha| \leq m$, this proves the lemma.

Corollary 39.4.3 Let $u \in H^{t}\left(\mathbb{R}^{n}\right)$ where $t>m+\frac{n}{2}$. Then $u$ is a.e. equal to a function of $C_{b}^{m}\left(\mathbb{R}^{n}\right)$ still denoted by $u$. Furthermore, there exists a constant, $C$ independent of $u$ such that

$$
\|u\|_{C_{b}^{m}\left(\mathbb{R}^{n}\right)} \leq C\|u\|_{H^{t}\left(\mathbb{R}^{n}\right)}
$$

Proof: This follows from the above lemma. Let $\left\{u_{k}\right\}$ be a sequence of functions of $\mathfrak{S}$ which converges to $u$ in $H^{t}$ and a.e. Then by the inequality of the above lemma, this sequence is also Cauchy in $C_{b}^{m}\left(\mathbb{R}^{n}\right)$ and taking the limit,

$$
\|u\|_{C_{b}^{m}\left(\mathbb{R}^{n}\right)}=\lim _{k \rightarrow \infty}\left\|u_{k}\right\|_{C_{b}^{m}\left(\mathbb{R}^{n}\right)} \leq C \lim _{k \rightarrow \infty}\left\|u_{k}\right\|_{H^{t}\left(\mathbb{R}^{n}\right)}=C\|u\|_{H^{t}\left(\mathbb{R}^{n}\right)}
$$

What about open sets, $U$ ?
Corollary 39.4.4 Let $t>m+\frac{n}{2}$ and let $U$ be an open set with $u \in H^{t}(U)$. Then $u$ is a.e. equal to a function of $C^{m}(\bar{U})$ still denoted by $u$. Furthermore, there exists a constant, $C$ independent of $u$ such that

$$
\|u\|_{C^{m}(\bar{U})} \leq C\|u\|_{H^{t}(U)}
$$

Proof: Let $u \in H^{t}(U)$ and let $v \in H^{t}\left(\mathbb{R}^{n}\right)$ such that $\left.v\right|_{U}=u$. Then

$$
\|u\|_{C^{m}(\bar{U})} \leq\|v\|_{C_{b}^{m}\left(\mathbb{R}^{n}\right)} \leq C\|v\|_{H^{t}\left(\mathbb{R}^{n}\right)}
$$

Now taking the inf for all such $v$ yields

$$
\|u\|_{C^{m}(\bar{U})} \leq C\|u\|_{H^{t}(U)}
$$

### 39.5 The Trace On The Boundary Of A Half Space

It is important to consider the restriction of functions in a Sobolev space onto a smaller dimensional set such as the boundary of an open set.

Definition 39.5.1 For $u \in \mathfrak{S}$, define $\gamma и$ a function defined on $\mathbb{R}^{n-1}$ by $\gamma u\left(\mathbf{x}^{\prime}\right) \equiv u\left(\mathbf{x}^{\prime}, 0\right)$ where $\mathbf{x}^{\prime} \in \mathbb{R}^{n-1}$ is defined by $\mathbf{x}=\left(\mathbf{x}^{\prime}, x_{n}\right)$.

The following elementary lemma featuring trig. substitutions is the basis for the proof of some of the arguments which follow.

Lemma 39.5.2 Consider the integral,

$$
\int_{\mathbb{R}}\left(a^{2}+x^{2}\right)^{-t} d x
$$

for $a>0$ and $t>1 / 2$. Then this integral is no more than $C_{t} a^{-2 t+1}$ where $C_{t}$ is some constant which depends on $t$.

Proof: If $t>1 / 2$ the integrand is in $L^{1}(\mathbb{R})$. This is easily seen because it is of the form $\frac{1}{\left(a^{2}+x^{2}\right)^{t}}$. Now change the variable letting $x=a u$ and the result is obtained.

Lemma 39.5.3 Let $u \in \mathfrak{S}$. Then there exists a constant, $C_{n}$, depending on $n$ but independent of $u \in \mathfrak{S}$ such that

$$
F \gamma u\left(\mathbf{x}^{\prime}\right)=C_{n} \int_{\mathbb{R}} F u\left(\mathbf{x}^{\prime}, x_{n}\right) d x_{n}
$$

Proof: Using the dominated convergence theorem,

$$
\begin{aligned}
& \int_{\mathbb{R}} F u\left(\mathbf{x}^{\prime}, x_{n}\right) d x_{n} \equiv \lim _{\varepsilon \rightarrow 0} \int_{\mathbb{R}} e^{-\left(\varepsilon x_{n}\right)^{2}} F u\left(\mathbf{x}^{\prime}, x_{n}\right) d x_{n} \\
\equiv & \lim _{\varepsilon \rightarrow 0} \int_{\mathbb{R}} e^{-\left(\varepsilon x_{n}\right)^{2}}\left(\frac{1}{2 \pi}\right)^{n / 2} \int_{\mathbb{R}^{n}} e^{-i\left(\mathbf{x}^{\prime} \cdot \mathbf{y}^{\prime}+x_{n} y_{n}\right)} u\left(\mathbf{y}^{\prime}, y_{n}\right) d y^{\prime} d y_{n} d x_{n} \\
= & \lim _{\varepsilon \rightarrow 0}\left(\frac{1}{2 \pi}\right)^{n / 2} \int_{\mathbb{R}^{n}} u\left(\mathbf{y}^{\prime}, y_{n}\right) e^{-i \mathbf{x}^{\prime} \cdot \mathbf{y}^{\prime}} \int_{\mathbb{R}} e^{-\left(\varepsilon x_{n}\right)^{2}} e^{-i x_{n} y_{n}} d x_{n} d y^{\prime} d y_{n} .
\end{aligned}
$$

Now $-\left(\varepsilon x_{n}\right)^{2}-i x_{n} y_{n}=-\varepsilon^{2}\left(x_{n}+\frac{i y_{n}}{2}\right)^{2}-\varepsilon^{2} \frac{y_{n}^{2}}{4}$ and so the above reduces to

$$
\begin{aligned}
& \lim _{\varepsilon \rightarrow 0}\left(\frac{1}{2 \pi}\right)^{n / 2} \int_{\mathbb{R}^{n}} u\left(\mathbf{y}^{\prime}, y_{n}\right) e^{-i \mathbf{x}^{\prime} \cdot \mathbf{y}^{\prime}} \int_{\mathbb{R}} e^{-\varepsilon^{2}\left(x_{n}+\frac{i y_{n}}{2}\right)^{2}-\varepsilon^{2} \frac{y_{n}^{2}}{4}} d x_{n} d y^{\prime} d y_{n} \\
= & \lim _{\varepsilon \rightarrow 0} K_{n} \int_{\mathbb{R}^{n}} u\left(\mathbf{y}^{\prime}, y_{n}\right) e^{-i \mathbf{x}^{\prime} \cdot \mathbf{y}^{\prime}} e^{-\varepsilon^{2} \frac{y_{n}^{2}}{4}} \int_{\mathbb{R}} e^{-\varepsilon^{2}\left(x_{n}+\frac{i y_{n}}{2}\right)^{2}} d x_{n} d y^{\prime} d y_{n} \\
= & \lim _{\varepsilon \rightarrow 0} K_{n} \int_{\mathbb{R}^{n}} u\left(\mathbf{y}^{\prime}, y_{n}\right) e^{-i \mathbf{x}^{\prime} \cdot y^{\prime}} e^{-\varepsilon^{2} \frac{y_{n}^{2}}{4}} \frac{1}{\varepsilon} d y^{\prime} d y_{n}
\end{aligned}
$$

which is an expression of the form

$$
\begin{aligned}
\lim _{\varepsilon \rightarrow 0} K_{n} \int_{\mathbb{R}} \frac{1}{\varepsilon} e^{-\varepsilon^{2} \frac{y_{n}^{2}}{4}} \int_{\mathbb{R}^{n-1}} u\left(\mathbf{y}^{\prime}, y_{n}\right) e^{-i \mathbf{x}^{\prime} \cdot \mathbf{y}^{\prime}} d y^{\prime} d y_{n} & =K_{n} \int_{\mathbb{R}^{n}} u\left(\mathbf{y}^{\prime}, 0\right) e^{-i \mathbf{x}^{\prime} \cdot \mathbf{y}^{\prime}} d y^{\prime} \\
& =K_{n} F \gamma u\left(\mathbf{x}^{\prime}\right)
\end{aligned}
$$

and this proves the lemma with $C_{n} \equiv K_{n}^{-1}$.
Earlier $H^{t}\left(\mathbb{R}^{n}\right)$ was defined and then for $U$ an open subset of $\mathbb{R}^{n}, H^{t}(U)$ was defined to be the space of restrictions of functions of $H^{t}\left(\mathbb{R}^{n}\right)$ to $U$ and a norm was given which made $H^{t}(U)$ into a Banach space. The next task is to consider $\mathbb{R}^{n-1} \times\{0\}$, a smaller dimensional subspace of $\mathbb{R}^{n}$ and examine the functions defined on this set, denoted by $\mathbb{R}^{n-1}$ for short which are restrictions of functions in $H^{t}\left(\mathbb{R}^{n}\right)$. You note this is somewhat different because heuristically, the dimension of the domain of the function is changing. An open set in $\mathbb{R}^{n}$ is considered an $n$ dimensional thing but $\mathbb{R}^{n-1}$ is only $n-1$ dimensional. I realize this is vague because the standard definition of dimension requires a vector space and an open set is not a vector space. However, think in terms of fatness. An open set is fat in $n$ directions whereas $\mathbb{R}^{n-1}$ is only fat in $n-1$ directions. Therefore, something interesting is likely to happen.

Let $\mathfrak{S}$ denote the Schwartz class of functions on $\mathbb{R}^{n}$ and $\mathfrak{S}^{\prime}$ the Schwartz class of functions on $\mathbb{R}^{n-1}$. Also, $\mathbf{y}^{\prime} \in \mathbb{R}^{n-1}$ while $\mathbf{y} \in \mathbb{R}^{n}$. Let $u \in \mathfrak{S}$. Then from Lemma 39.5.3
and $s>0$,

$$
\begin{gathered}
\int_{\mathbb{R}^{n-1}}\left(1+\left|\mathbf{y}^{\prime}\right|^{2}\right)^{s}\left|F \gamma u\left(\mathbf{y}^{\prime}\right)\right|^{2} d y^{\prime} \\
=C_{n} \int_{\mathbb{R}^{n-1}}\left(1+\left|\mathbf{y}^{\prime}\right|^{2}\right)^{s}\left|\int_{\mathbb{R}} F u\left(\mathbf{y}^{\prime}, y_{n}\right) d y_{n}\right|^{2} d y^{\prime} \\
=C_{n} \int_{\mathbb{R}^{n-1}}\left(1+\left|\mathbf{y}^{\prime}\right|^{2}\right)^{s}\left|\int_{\mathbb{R}} F u\left(\mathbf{y}^{\prime}, y_{n}\right)\left(1+|\mathbf{y}|^{2}\right)^{t / 2}\left(1+|\mathbf{y}|^{2}\right)^{-t / 2} d y_{n}\right|^{2} d y^{\prime}
\end{gathered}
$$

Then by the Cauchy Schwarz inequality,

$$
\begin{equation*}
\leq C_{n} \int_{\mathbb{R}^{n-1}}\left(1+\left|\mathbf{y}^{\prime}\right|^{2}\right)^{s} \int_{\mathbb{R}}\left|F u\left(\mathbf{y}^{\prime}, y_{n}\right)\right|^{2}\left(1+|\mathbf{y}|^{2}\right)^{t} d y_{n} \int_{\mathbb{R}}\left(1+|\mathbf{y}|^{2}\right)^{-t} d y_{n} d y^{\prime} \tag{39.5.17}
\end{equation*}
$$

Consider

$$
\int_{\mathbb{R}}\left(1+|\mathbf{y}|^{2}\right)^{-t} d y_{n}=\int_{\mathbb{R}}\left(1+\left|\mathbf{y}^{\prime}\right|^{2}+y_{n}^{2}\right)^{-t} d y_{n}
$$

by Lemma 39.5.2 and taking $a=\left(1+\left|\mathbf{y}^{\prime}\right|^{2}\right)^{1 / 2}$, this equals

$$
C_{t}\left(\left(1+\left|\mathbf{y}^{\prime}\right|^{2}\right)^{1 / 2}\right)^{-2 t+1}=C_{t}\left(1+\left|\mathbf{y}^{\prime}\right|^{2}\right)^{(-2 t+1) / 2}
$$

Now using this in 39.5.17,

$$
\begin{aligned}
& \int_{\mathbb{R}^{n-1}}\left(1+\left|\mathbf{y}^{\prime}\right|^{2}\right)^{s}\left|F \gamma u\left(\mathbf{y}^{\prime}\right)\right|^{2} d y^{\prime} \\
\leq & C_{n, t} \int_{\mathbb{R}^{n-1}}\left(1+\left|\mathbf{y}^{\prime}\right|^{2}\right)^{s} \int_{\mathbb{R}}\left|F u\left(\mathbf{y}^{\prime}, y_{n}\right)\right|^{2}\left(1+|\mathbf{y}|^{2}\right)^{t} d y_{n} \\
& \left(1+\left|\mathbf{y}^{\prime}\right|^{2}\right)^{(-2 t+1) / 2} d y^{\prime} \\
= & C_{n, t} \int_{\mathbb{R}^{n-1}}\left(1+\left|\mathbf{y}^{\prime}\right|^{2}\right)^{s+(-2 t+1) / 2} \int_{\mathbb{R}}\left|F u\left(\mathbf{y}^{\prime}, y_{n}\right)\right|^{2}\left(1+|\mathbf{y}|^{2}\right)^{t} d y_{n} d y^{\prime} .
\end{aligned}
$$

What is the correct choice of $t$ so that the above reduces to $\|u\|_{H^{t}\left(\mathbb{R}^{n}\right)}^{2}$ ? It is clearly the one for which

$$
s+(-2 t+1) / 2=0
$$

which occurs when $t=s+\frac{1}{2}$. Then for this choice of $t$, the following inequality is obtained for any $u \in \mathfrak{S}$.

$$
\begin{equation*}
\|\gamma u\|_{H^{t-1 / 2}\left(\mathbb{R}^{n-1}\right)} \leq C_{n, t}\|u\|_{H^{t}\left(\mathbb{R}^{n}\right)} \tag{39.5.18}
\end{equation*}
$$

This has proved part of the following theorem.
Theorem 39.5.4 For each $t>1 / 2$ there exists a unique mapping

$$
\gamma \in \mathscr{L}\left(H^{t}\left(\mathbb{R}^{n}\right), H^{t-1 / 2}\left(\mathbb{R}^{n-1}\right)\right)
$$

which has the property that for $u \in \mathfrak{S}, \gamma u\left(\mathbf{x}^{\prime}\right)=u\left(\mathbf{x}^{\prime}, 0\right)$. In addition to this, $\gamma$ is onto. In fact, there exists a continuous map, $\zeta \in \mathscr{L}\left(H^{t-1 / 2}\left(\mathbb{R}^{n-1}\right), H^{t}\left(\mathbb{R}^{n}\right)\right)$ such that $\gamma \circ \zeta=\mathrm{id}$.

Proof: It only remains to verify that $\gamma$ is onto and that the continuous map, $\zeta$ exists. Now define

$$
\phi(\mathbf{y}) \equiv \phi\left(\mathbf{y}^{\prime}, y_{n}\right) \equiv \frac{\left(1+\left|\mathbf{y}^{\prime}\right|^{2}\right)^{t-1 / 2}}{\left(1+|\mathbf{y}|^{2}\right)^{t}}
$$

Then for $u \in \mathfrak{S}^{\prime}$, let

$$
\begin{gather*}
\zeta u(\mathbf{x}) \equiv C F^{-1}(\phi F u)(\mathbf{x})= \\
C \int_{\mathbb{R}^{n}} e^{i \mathbf{y} \cdot \mathbf{x}} \frac{\left(1+\left|\mathbf{y}^{\prime}\right|^{2}\right)^{t-1 / 2}}{\left(1+|\mathbf{y}|^{2}\right)^{t}} F u\left(\mathbf{y}^{\prime}\right) d y \tag{39.5.19}
\end{gather*}
$$

Here the inside Fourier transform is taken with respect to $\mathbb{R}^{n-1}$ because $u$ is only defined on $\mathbb{R}^{n-1}$ and $C$ will be chosen in such a way that $\gamma \circ \zeta=\mathrm{id}$. First the existence of $C$ such that $\gamma \circ \zeta=$ id will be shown. Since $u \in \mathfrak{S}^{\prime}$ it follows

$$
\mathbf{y} \rightarrow \frac{\left(1+\left|\mathbf{y}^{\prime}\right|^{2}\right)^{t-1 / 2}}{\left(1+|\mathbf{y}|^{2}\right)^{t}} F u\left(\mathbf{y}^{\prime}\right)
$$

is in $\mathfrak{S}$. Hence the inverse Fourier transform of this function is also in $\mathfrak{S}$ and so for $u \in \mathfrak{S}^{\prime}$, it follows $\zeta u \in \mathfrak{S}$. Therefore, to check $\gamma \circ \zeta=$ id it suffices to plug in $x_{n}=0$. From Lemma 39.5.2 this yields

$$
\begin{gathered}
\gamma(\zeta u)\left(\mathbf{x}^{\prime}, 0\right) \\
=C \int_{\mathbb{R}^{n}} e^{i \mathbf{y}^{\prime} \cdot \mathbf{x}^{\prime}} \frac{\left(1+\left|\mathbf{y}^{\prime}\right|^{2}\right)^{t-1 / 2}}{\left(1+|\mathbf{y}|^{2}\right)^{t}} F u\left(\mathbf{y}^{\prime}\right) d y \\
=C \int_{\mathbb{R}^{n-1}}\left(1+\left|\mathbf{y}^{\prime}\right|^{2}\right)^{t-1 / 2} e^{i \mathbf{y}^{\prime} \cdot \mathbf{x}^{\prime}} F u\left(\mathbf{y}^{\prime}\right) \int_{\mathbb{R}} \frac{1}{\left(1+|\mathbf{y}|^{2}\right)^{t}} d y_{n} d y^{\prime} \\
=C C_{t} \int_{\mathbb{R}^{n-1}}\left(1+\left|\mathbf{y}^{\prime}\right|^{2}\right)^{t-1 / 2} e^{i \mathbf{y}^{\prime} \cdot \mathbf{x}^{\prime}} F u\left(\mathbf{y}^{\prime}\right)\left(1+\left|\mathbf{y}^{\prime}\right|^{2}\right)^{\frac{-2 t+1}{2}} d y^{\prime} \\
=C C_{t} \int_{\mathbb{R}^{n-1}} e^{i \mathbf{y}^{\prime} \cdot \mathbf{x}^{\prime}} F u\left(\mathbf{y}^{\prime}\right) d y^{\prime}=C C_{t}(2 \pi)^{n / 2} F^{-1}(F u)\left(\mathbf{x}^{\prime}\right)
\end{gathered}
$$

and so the correct value of $C$ is $\left(C_{t}(2 \pi)^{n / 2}\right)^{-1}$ to obtain $\gamma \circ \zeta=\mathrm{id}$. It only remains to verify that $\zeta$ is continuous. From 39.5.19, and Lemma 39.5.2,

$$
\begin{aligned}
& \|\zeta u\|_{H^{t}\left(\mathbb{R}^{n}\right)}^{2} \\
= & \int_{\mathbb{R}^{n}}\left(1+|\mathbf{x}|^{2}\right)^{t}|F \zeta u(\mathbf{x})|^{2} d x \\
= & C^{2} \int_{\mathbb{R}^{n}}\left(1+|\mathbf{x}|^{2}\right)^{t}\left|F\left(F^{-1}(\phi F u)(\mathbf{x})\right)\right|^{2} d x
\end{aligned}
$$

$$
\begin{gathered}
=C^{2} \int_{\mathbb{R}^{n}}\left(1+|\mathbf{x}|^{2}\right)^{t}\left|\phi(\mathbf{x}) F u\left(\mathbf{x}^{\prime}\right)\right|^{2} d x \\
=C^{2} \int_{\mathbb{R}^{n}}\left(1+|\mathbf{x}|^{2}\right)^{t}\left|\frac{\left(1+\left|\mathbf{x}^{\prime}\right|^{2}\right)^{t-1 / 2}}{\left(1+\mid \mathbf{x}^{2}\right)^{t}} F u\left(\mathbf{x}^{\prime}\right)\right|^{2} d x \\
=C^{2} \int_{\mathbb{R}^{n}}\left(1+|\mathbf{x}|^{2}\right)^{-t}\left|\left(1+\left|\mathbf{x}^{\prime}\right|^{2}\right)^{t-1 / 2} F u\left(\mathbf{x}^{\prime}\right)\right|^{2} d x \\
=C^{2} \int_{\mathbb{R}^{n-1}}\left(1+\left|\mathbf{x}^{\prime}\right|^{2}\right)^{2 t-1}\left|F u\left(\mathbf{x}^{\prime}\right)\right|^{2} \int_{\mathbb{R}}\left(1+|\mathbf{x}|^{2}\right)^{-t} d x_{n} d x^{\prime} \\
=C^{2} C_{t} \int_{\mathbb{R}^{n-1}}\left(1+\left|\mathbf{x}^{\prime}\right|^{2}\right)^{2 t-1}\left|F u\left(\mathbf{x}^{\prime}\right)\right|^{2}\left(1+\left|\mathbf{y}^{\prime}\right|^{2}\right)^{\frac{-2 t+1}{2}} d x^{\prime} \\
=C^{2} C_{t} \int_{\mathbb{R}^{n-1}}\left(1+\left|\mathbf{x}^{\prime}\right|^{2}\right)^{t-1 / 2}\left|F u\left(\mathbf{x}^{\prime}\right)\right|^{2} d x^{\prime}=\left.C^{2} C_{t}| | u\right|_{H^{t-1 / 2}\left(\mathbb{R}^{n-1}\right)} ^{2} .
\end{gathered}
$$

This proves the theorem because $\mathfrak{S}$ is dense in $\mathbb{R}^{n}$.
Actually, the assertion that $\gamma u\left(\mathbf{x}^{\prime}\right)=u\left(\mathbf{x}^{\prime}, 0\right)$ holds for more functions, $u$ than just $u \in$ $\mathfrak{S}$. I will make no effort to obtain the most general description of such functions but the following is a useful lemma which will be needed when the trace on the boundary of an open set is considered.

Lemma 39.5.5 Suppose $u$ is continuous and $u \in H^{1}\left(\mathbb{R}^{n}\right)$. Then there exists a set of $m_{1}$ measure zero, $N$ such that if $x_{n} \notin N$, then for every $\phi \in L^{2}\left(\mathbb{R}^{n-1}\right)$

$$
(\gamma u, \phi)_{H}+\int_{0}^{x_{n}}\left(u_{, n}(\cdot, t), \phi\right)_{H} d t=\left(u\left(\cdot, x_{n}\right), \phi\right)_{H}
$$

where here

$$
(f, g)_{H} \equiv \int_{\mathbb{R}^{n-1}} f \bar{g} d x^{\prime}
$$

just the inner product in $L^{2}\left(\mathbb{R}^{n-1}\right)$. Furthermore,

$$
u(\cdot, 0)=\gamma u \text { a.e. } \mathbf{x}^{\prime}
$$

Proof: Let $\left\{u_{k}\right\}$ be a sequence of functions from $\mathfrak{S}$ which converges to $u$ in $H^{1}\left(\mathbb{R}^{n}\right)$ and let $\left\{\phi_{k}\right\}$ denote a countable dense subset of $L^{2}\left(\mathbb{R}^{n-1}\right)$. Then

$$
\begin{equation*}
\left(\gamma u_{k}, \phi_{j}\right)_{H}+\int_{0}^{x_{n}}\left(u_{k, n}(\cdot, t), \phi_{j}\right)_{H} d t=\left(u_{k}\left(\cdot, x_{n}\right), \phi_{j}\right)_{H} \tag{39.5.20}
\end{equation*}
$$

Now

$$
\begin{aligned}
& \left(\int_{0}^{\infty}\left|\left(u_{k}\left(\cdot, x_{n}\right), \phi_{j}\right)_{H}-\left(u\left(\cdot, x_{n}\right), \phi_{j}\right)_{H}\right|^{2} d x_{n}\right)^{1 / 2} \\
= & \left(\int_{0}^{\infty}\left|\left(u_{k}\left(\cdot, x_{n}\right)-u\left(\cdot, x_{n}\right), \phi_{j}\right)_{H}\right|^{2} d x_{n}\right)^{1 / 2} \\
\leq & \left(\int_{0}^{\infty}\left|u_{k}\left(\cdot, x_{n}\right)-u\left(\cdot, x_{n}\right)\right|_{H}^{2}\left|\phi_{j}\right|_{H}^{2} d x_{n}\right)^{1 / 2} \\
= & \left|\phi_{j}\right|_{H}^{2}\left(\int_{0}^{\infty}\left|u_{k}\left(\cdot, x_{n}\right)-u\left(\cdot, x_{n}\right)\right|_{H}^{2} d x_{n}\right)^{1 / 2} \\
= & \left|\phi_{j}\right|_{H}^{2}\left(\int_{0}^{\infty} \int_{\mathbb{R}^{n-1}}\left|u_{k}\left(\mathbf{x}^{\prime}, x_{n}\right)-u\left(\mathbf{x}^{\prime}, x_{n}\right)\right|^{2} d x^{\prime} d x_{n}\right)^{1 / 2}
\end{aligned}
$$

which converges to zero. Therefore, there exists a set of measure zero, $N_{j}$ and a subsequence, still denoted by $k$ such that if $x_{n} \notin N_{j}$, then

$$
\left(u_{k}\left(\cdot, x_{n}\right), \phi_{j}\right)_{H} \rightarrow\left(u\left(\cdot, x_{n}\right), \phi_{j}\right)_{H} .
$$

Now by Theorem 39.5.4, $\gamma u_{k} \rightarrow \gamma u$ in $H=L^{2}\left(\mathbb{R}^{n-1}\right)$. It only remains to consider the term of 39.5.20 which involves an integral.

$$
\begin{aligned}
& \left|\int_{0}^{x_{n}}\left(u_{k, n}(\cdot, t), \phi_{j}\right)_{H} d t-\int_{0}^{x_{n}}\left(u_{, n}(\cdot, t), \phi_{j}\right)_{H} d t\right| \\
\leq & \int_{0}^{x_{n}}\left|\left(u_{k, n}(\cdot, t)-u_{, n}(\cdot, t), \phi_{j}\right)_{H}\right| d t \\
\leq & \int_{0}^{x_{n}}\left|u_{k, n}(\cdot, t)-u_{, n}(\cdot, t)\right|_{H}\left|\phi_{j}\right|_{H} d t \\
\leq & \left(\int_{0}^{x_{n}}\left|u_{k, n}(\cdot, t)-u_{, n}(\cdot, t)\right|_{H}^{2} d t\right)^{1 / 2}\left(\int_{0}^{x_{n}}\left|\phi_{j}\right|_{H}^{2} d t\right)^{1 / 2} \\
= & x_{n}^{1 / 2}\left|\phi_{j}\right|_{H}\left(\int_{0}^{x_{n}} \int_{\mathbb{R}^{n-1}}\left|u_{k, n}\left(\mathbf{x}^{\prime}, t\right)-u_{, n}\left(\mathbf{x}^{\prime}, t\right)\right|^{2} d x^{\prime}\right)^{1 / 2} d t
\end{aligned}
$$

and this converges to zero as $k \rightarrow \infty$. Therefore, using a diagonal sequence argument, there exists a subsequence, still denoted by $k$ and a set of measure zero, $N \equiv \cup_{j=1}^{\infty} N_{j}$ such that for $\mathbf{x}^{\prime} \notin N$, you can pass to the limit in 39.5.20 and obtain that for all $\phi_{j}$,

$$
\left(\gamma u, \phi_{j}\right)_{H}+\int_{0}^{x_{n}}\left(u_{, n}(\cdot, t), \phi_{j}\right)_{H} d t=\left(u\left(\cdot, x_{n}\right), \phi_{j}\right)_{H}
$$

By density of $\left\{\phi_{j}\right\}$, this equality holds for all $\phi \in L^{2}\left(\mathbb{R}^{n-1}\right)$. In particular, the equality holds for every $\phi \in C_{c}\left(\mathbb{R}^{n-1}\right)$. Since $u$ is uniformly continuous on the compact set, $\operatorname{spt}(\phi) \times[0,1]$, there exists a sequence, $\left(x_{n}\right)_{k} \rightarrow 0$ such that the above equality holds for
$x_{n}$ replaced with $\left(x_{n}\right)_{k}$ and $\phi$ in place of $\phi_{j}$. Now taking $k \rightarrow \infty$, this uniform continuity implies

$$
(\gamma u, \phi)_{H}=(u(\cdot, 0), \phi)_{H}
$$

This implies since $C_{c}\left(\mathbb{R}^{n-1}\right)$ is dense in $L^{2}\left(\mathbb{R}^{n-1}\right)$ that $\gamma u=u(\cdot, 0)$ a.e. and this proves the lemma.

Lemma 39.5.6 Suppose $U$ is an open subset of $\mathbb{R}^{n}$ of the form

$$
U \equiv\left\{\mathbf{u} \in \mathbb{R}^{n}: \mathbf{u}^{\prime} \in U^{\prime} \text { and } 0<u_{n}<\phi\left(\mathbf{u}^{\prime}\right)\right\}
$$

where $U^{\prime}$ is an open subset of $\mathbb{R}^{n-1}$ and $\phi\left(\mathbf{u}^{\prime}\right)$ is a positive function such that $\phi\left(\mathbf{u}^{\prime}\right) \leq \infty$ and

$$
\inf \left\{\phi\left(\mathbf{u}^{\prime}\right): \mathbf{u}^{\prime} \in U^{\prime}\right\}=\delta>0
$$

Suppose $v \in H^{t}\left(\mathbb{R}^{n}\right)$ such that $v=0$ a.e. on $U$. Then $\gamma v=0 m_{n-1}$ a.e. point of $U^{\prime}$. Also, if $v \in H^{t}\left(\mathbb{R}^{n}\right)$ and $\phi \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$, then $\gamma v \gamma \phi=\gamma(\phi v)$.

Proof: First consider the second claim. Let $v \in H^{t}\left(\mathbb{R}^{n}\right)$ and let $v_{k} \rightarrow v$ in $H^{t}\left(\mathbb{R}^{n}\right)$ where $v_{k} \in \mathfrak{S}$. Then from Lemma 39.3.4 and Theorem 39.5.4

$$
\|\gamma(\phi v)-\gamma \phi \gamma v\|_{H^{t-1 / 2}\left(\mathbb{R}^{n-1}\right)}=\lim _{k \rightarrow \infty}\left\|\gamma\left(\phi v_{k}\right)-\gamma \phi \gamma v_{k}\right\|_{H^{t-1 / 2}\left(\mathbb{R}^{n-1}\right)}=0
$$

because each term in the sequence equals zero due to the observation that for $v_{k} \in \mathfrak{S}$ and $\phi \in C_{c}^{\infty}(U), \gamma\left(\phi v_{k}\right)=\gamma v_{k} \gamma \phi$.

Now suppose $v=0$ a.e. on $U$. Define for $0<r<\delta, v_{r}(\mathbf{x}) \equiv v\left(\mathbf{x}^{\prime}, x_{n}+r\right)$.
Claim: If $u \in H^{t}\left(\mathbb{R}^{n}\right)$, then

$$
\lim _{r \rightarrow 0}\left\|v_{r}-v\right\|_{H^{t}\left(\mathbb{R}^{n}\right)}=0
$$

Proof of claim: First of all, let $v \in \mathfrak{S}$. Then $v \in H^{m}\left(\mathbb{R}^{n}\right)$ for all $m$ and so by Lemma 39.2.5,

$$
\left\|v_{r}-v\right\|_{H^{t}\left(\mathbb{R}^{n}\right)} \leq\left\|v_{r}-v\right\|_{H^{m}\left(\mathbb{R}^{n}\right)}^{\theta}\left\|v_{r}-v\right\|_{H^{m+1}\left(\mathbb{R}^{n}\right)}^{1-\theta}
$$

where $t \in[m, m+1]$. It follows from continuity of translation in $L^{p}\left(\mathbb{R}^{n}\right)$ that

$$
\lim _{r \rightarrow 0}\left\|v_{r}-v\right\|_{H^{m}\left(\mathbb{R}^{n}\right)}^{\theta}\left\|v_{r}-v\right\|_{H^{m+1}\left(\mathbb{R}^{n}\right)}^{1-\theta}=0
$$

and so the claim is proved if $v \in \mathfrak{S}$. Now suppose $u \in H^{t}\left(\mathbb{R}^{n}\right)$ is arbitrary. By density of $\mathfrak{S}$ in $H^{t}\left(\mathbb{R}^{n}\right)$, there exists $v \in \mathfrak{S}$ such that

$$
\|u-v\|_{H^{t}\left(\mathbb{R}^{n}\right)}<\varepsilon / 3 .
$$

Therefore,

$$
\begin{aligned}
\left\|u_{r}-u\right\|_{H^{t}\left(\mathbb{R}^{n}\right)} & \leq\left\|u_{r}-v_{r}\right\|_{H^{t}\left(\mathbb{R}^{n}\right)}+\left\|v_{r}-v\right\|_{H^{t}\left(\mathbb{R}^{n}\right)}+\|v-u\|_{H^{t}\left(\mathbb{R}^{n}\right)} \\
& =2 \varepsilon / 3+\left\|v_{r}-v\right\|_{H^{t}\left(\mathbb{R}^{n}\right)}
\end{aligned}
$$

Now using what was just shown, it follows that for $r$ small enough, $\left\|u_{r}-u\right\|_{H^{t}\left(\mathbb{R}^{n}\right)}<\varepsilon$ and this proves the claim.

Now suppose $v \in H^{t}\left(\mathbb{R}^{n}\right)$. By the claim,

$$
\left\|v_{r}-v\right\|_{H^{t}\left(\mathbb{R}^{n}\right)} \rightarrow 0
$$

and so by continuity of $\gamma$,

$$
\begin{equation*}
\gamma v_{r} \rightarrow \gamma v \text { in } H^{t-1 / 2}\left(\mathbb{R}^{n-1}\right) \tag{39.5.21}
\end{equation*}
$$

Note $v_{r}=0$ a.e. on

$$
U_{r} \equiv\left\{\mathbf{u} \in \mathbb{R}^{n}: \mathbf{u}^{\prime} \in U^{\prime} \text { and }-r<u_{n}<\phi\left(\mathbf{u}^{\prime}\right)-r\right\}
$$

Let $\phi \in C_{c}^{\infty}\left(U_{r}\right)$ and consider $\phi v_{r}$. Then it follows $\phi v_{r}=0$ a.e. on $\mathbb{R}^{n}$. Let $w \equiv 0$. Then $w \in \mathfrak{S}$ and so $\gamma w=0=\gamma\left(\phi v_{r}\right)=\gamma \phi \gamma v_{r}$ in $H^{t-1 / 2}\left(\mathbb{R}^{n-1}\right)$. It follows that for $m_{n-1}$ a.e. $\mathbf{x}^{\prime} \in[\phi \neq 0] \cap \mathbb{R}^{n-1}, \gamma v_{r}\left(\mathbf{x}^{\prime}\right)=0$. Now let $U^{\prime}=\cup_{k=1}^{\infty} K_{k}$ where the $K_{k}$ are compact sets such that $K_{k} \subseteq K_{k+1}$ and let $\phi_{k} \in C_{c}^{\infty}(U)$ such that $\phi_{k}$ has values in $[0,1]$ and $\phi_{k}\left(\mathbf{x}^{\prime}\right)=1$ if $\mathbf{x}^{\prime} \in K_{k}$. Then from what was just shown, $\gamma v_{r}=0$ for a.e. point of $K_{k}$. Therefore, $\gamma v_{r}=0$ for $m_{n-1}$ a.e. point in $U^{\prime}$. Therefore, since each $\gamma v_{r}=0$, it follows from 39.5.21 that $\gamma \nu=0$ also. This proves the lemma.

Theorem 39.5.7 Let $t>1 / 2$ and let $U$ be of the form

$$
\left\{\mathbf{u} \in \mathbb{R}^{n}: \mathbf{u}^{\prime} \in U^{\prime} \text { and } 0<u_{n}<\phi\left(\mathbf{u}^{\prime}\right)\right\}
$$

where $U^{\prime}$ is an open subset of $\mathbb{R}^{n-1}$ and $\phi\left(\mathbf{u}^{\prime}\right)$ is a positive function such that $\phi\left(\mathbf{u}^{\prime}\right) \leq \infty$ and

$$
\inf \left\{\phi\left(\mathbf{u}^{\prime}\right): \mathbf{u}^{\prime} \in U^{\prime}\right\}=\delta>0
$$

Then there exists a unique

$$
\gamma \in \mathscr{L}\left(H^{t}(U), H^{t-1 / 2}\left(U^{\prime}\right)\right)
$$

which has the property that if $u=\left.v\right|_{U}$ where $v$ is continuous and also a function of $H^{1}\left(\mathbb{R}^{n}\right)$, then $\gamma u\left(\mathbf{x}^{\prime}\right)=u\left(\mathbf{x}^{\prime}, 0\right)$ for a.e. $\mathbf{x}^{\prime} \in U^{\prime}$.

Proof: Let $u \in H^{t}(U)$. Then $u=\left.v\right|_{U}$ for some $v \in H^{t}\left(\mathbb{R}^{n}\right)$. Define

$$
\left.\gamma u \equiv \gamma v\right|_{U^{\prime}}
$$

Is this well defined? The answer is yes because if $\left.v_{i}\right|_{U}=u$ a.e., then $\gamma\left(v_{1}-v_{2}\right)=0$ a.e. on $U^{\prime}$ which implies $\gamma \nu_{1}=\gamma \nu_{2}$ a.e. and so the two different versions of $\gamma u$ differ only on a set of measure zero.

If $u=\left.v\right|_{U}$ where $v$ is continuous and also a function of $H^{1}\left(\mathbb{R}^{n}\right)$, then for a.e. $\mathbf{x}^{\prime} \in \mathbb{R}^{n-1}$, it follows from Lemma 39.5.5 on Page 1363 that $\gamma v\left(\mathbf{x}^{\prime}\right)=v\left(\mathbf{x}^{\prime}, 0\right)$. Hence, it follows that for a.e. $\mathbf{x}^{\prime} \in U^{\prime}, \gamma u\left(\mathbf{x}^{\prime}\right) \equiv u\left(\mathbf{x}^{\prime}, 0\right)$.

In particular, $\gamma$ is determined by $\gamma u\left(\mathbf{x}^{\prime}\right)=u\left(\mathbf{x}^{\prime}, 0\right)$ on $\left.\mathfrak{S}\right|_{U}$ and the density of $\left.\mathfrak{S}\right|_{U}$ and continuity of $\gamma$ shows $\gamma$ is unique.

It only remains to show $\gamma$ is continuous. Let $u \in H^{t}(U)$. Thus there exists $v \in H^{t}\left(\mathbb{R}^{n}\right)$ such that $u=\left.v\right|_{U}$. Then

$$
\|\gamma u\|_{H^{t-1 / 2}\left(U^{\prime}\right)} \leq\|\gamma v\|_{H^{t-1 / 2}\left(\mathbb{R}^{n-1}\right)} \leq C\|v\|_{H^{t}\left(\mathbb{R}^{n}\right)}
$$

for $C$ independent of $v$. Then taking the inf for all such $v \in H^{t}\left(\mathbb{R}^{n}\right)$ which are equal to $u$ a.e. on $U$, it follows

$$
\|\gamma u\|_{H^{t-1 / 2}\left(U^{\prime}\right)} \leq C\|v\|_{H^{t}\left(\mathbb{R}^{n}\right)}
$$

and this proves $\gamma$ is continuous.

### 39.6 Sobolev Spaces On Manifolds

### 39.6.1 General Theory

The type of manifold, $\Gamma$ for which Sobolev spaces will be defined on is:
Definition 39.6.1 1. $\Gamma$ is a closed subset of $\mathbb{R}^{p}$ where $p \geq n$.
2. $\Gamma=\cup_{i=1}^{\infty} \Gamma_{i}$ where $\Gamma_{i}=\Gamma \cap W_{i}$ for $W_{i}$ a bounded open set.
3. $\left\{W_{i}\right\}_{i=1}^{\infty}$ is locally finite.
4. There are open bounded sets, $U_{i}$ and functions $\mathbf{h}_{i}: U_{i} \rightarrow \Gamma_{i}$ which are one to one, onto, and in $C^{m, 1}\left(U_{i}\right)$. There exists a constant, $C$, such that $C \geq \operatorname{Lip} \mathbf{h}_{r}$ for all $r$.
5. There exist functions, $\mathbf{g}_{i}: W_{i} \rightarrow U_{i}$ such that $\mathbf{g}_{i}$ is $C^{m, 1}\left(W_{i}\right)$, and $\mathbf{g}_{i} \circ \mathbf{h}_{i}=\mathrm{id}$ on $U_{i}$ while $\mathbf{h}_{i} \circ \mathbf{g}_{i}=\mathrm{id}$ on $\Gamma_{i}$.
This will be referred to as a $C^{m, 1}$ manifold.
Lemma 39.6.2 Let $\mathbf{g}_{i}, \mathbf{h}_{i}, U_{i}, W_{i}$, and $\Gamma_{i}$ be as defined above. Then

$$
\mathbf{g}_{i} \circ \mathbf{h}_{k}: U_{k} \cap \mathbf{h}_{k}^{-1}\left(\Gamma_{i}\right) \rightarrow U_{i} \cap \mathbf{h}_{i}^{-1}\left(\Gamma_{k}\right)
$$

is $C^{m, 1}$. Furthermore, the inverse of this map is $\mathbf{g}_{k} \circ \mathbf{h}_{i}$.
Proof: First it is well to show it does indeed map the given open sets. Let $\mathbf{x} \in U_{k} \cap$ $\mathbf{h}_{k}^{-1}\left(\Gamma_{i}\right)$. Then $\mathbf{h}_{k}(\mathbf{x}) \in \Gamma_{k} \cap \Gamma_{i}$ and so $\mathbf{g}_{i}\left(\mathbf{h}_{k}(\mathbf{x})\right) \in U_{i}$ because $\mathbf{h}_{k}(\mathbf{x}) \in \Gamma_{i}$. Now since $\mathbf{h}_{k}(\mathbf{x}) \in \Gamma_{k}, \mathbf{g}_{i}\left(\mathbf{h}_{k}(\mathbf{x})\right) \in \mathbf{h}_{i}^{-1}\left(\Gamma_{k}\right)$ also and this proves the mappings do what they should in terms of mapping the two open sets. That $\mathbf{g}_{i} \circ \mathbf{h}_{k}$ is $C^{m, 1}$ follows immediately from the chain rule and the assumptions that the functions $\mathbf{g}_{i}$ and $\mathbf{h}_{k}$ are $C^{m, 1}$. The claim about the inverse follows immediately from the definitions of the functions.

Let $\left\{\psi_{i}\right\}_{i=1}^{\infty}$ be a partition of unity subordinate to the open cover $\left\{W_{i}\right\}$ satisfying $\psi_{i} \in$ $C_{c}^{\infty}\left(W_{i}\right)$. Then the following definition provides a norm for $H^{m+s}(\Gamma)$.

Definition 39.6.3 Let $s \in(0,1)$ and $m$ is a nonnegative integer. Also let $\mu$ denote the surface measure for $\Gamma$ defined in the last section. A $\mu$ measurable function, $u$ is in $H^{m+s}(\Gamma)$ if whenever $\left\{W_{i}, \psi_{i}, \Gamma_{i}, U_{i}, \mathbf{h}_{i}, \mathbf{g}_{i}\right\}_{i=1}^{\infty}$ is described above, $\mathbf{h}_{i}^{*}\left(u \psi_{i}\right) \in H^{m+s}\left(U_{i}\right)$ and

$$
\|u\|_{H^{m+s}(\Gamma)} \equiv\left(\sum_{i=1}^{\infty}\left\|\mathbf{h}_{i}^{*}\left(u \psi_{i}\right)\right\|_{H^{m+s}\left(U_{i}\right)}^{2}\right)^{1 / 2}<\infty
$$

Are there functions which are in $H^{m+s}(\Gamma)$ ? The answer is yes. Just take the restriction to $\Gamma$ of any function, $u \in C_{c}^{\infty}\left(\mathbb{R}^{m}\right)$. Then each $\mathbf{h}_{i}^{*}\left(u \psi_{i}\right) \in H^{m+s}\left(U_{i}\right)$ and the sum is finite because $\operatorname{spt} u$ has nonempty intersection with only finitely many $W_{i}$.

It is not at all obvious this norm is well defined. What if $\left\{W_{i}^{\prime}, \psi_{i}^{\prime}, \Gamma_{i}^{\prime}, U_{i}, \mathbf{h}_{i}^{\prime}, \mathbf{g}_{i}^{\prime}\right\}_{i=1}^{\infty}$ is as described above? Would the two norms be equivalent? If they aren't, then this is not a good way to define $H^{m+s}(\Gamma)$ because it would depend on the choice of partition of unity and functions, $\mathbf{h}_{i}$ and choice of the open sets, $U_{i}$. To begin with pick a particular choice for $\left\{W_{i}, \psi_{i}, \Gamma_{i}, U_{i}, \mathbf{h}_{i}, \mathbf{g}_{i}\right\}_{i=1}^{\infty}$.

Lemma 39.6.4 $H^{m+s}(\Gamma)$ as just described, is a Banach space.
Proof: Let $\left\{u_{j}\right\}_{j=1}^{\infty}$ be a Cauchy sequence in $H^{m+s}(\Gamma)$. Then $\left\{\mathbf{h}_{i}^{*}\left(u_{j} \psi_{i}\right)\right\}_{j=1}^{\infty}$ is a Cauchy sequence in $H^{m+s}\left(U_{i}\right)$ for each $i$. Therefore, for each $i$, there exists $w_{i} \in H^{m+s}\left(U_{i}\right)$ such that

$$
\begin{equation*}
\lim _{j \rightarrow \infty} \mathbf{h}_{i}^{*}\left(u_{j} \psi_{i}\right)=w_{i} \text { in } H^{m+s}\left(U_{i}\right) \tag{39.6.22}
\end{equation*}
$$

It is required to show there exists $u \in H^{m+s}(\Gamma)$ such that $w_{i}=\mathbf{h}_{i}^{*}\left(u \psi_{i}\right)$ for each $i$.
Now from Corollary 37.2 .5 it follows easily by approximating with simple functions that for ever nonnegative $\mu$ measurable function, $f$,

$$
\int_{\Gamma} f d \mu=\sum_{r=1}^{\infty} \int_{\mathbf{g}_{r} \Gamma_{r}} \psi_{r} f\left(\mathbf{h}_{r}(\mathbf{u})\right) J_{r}(\mathbf{u}) d u
$$

Therefore,

$$
\begin{aligned}
\int_{\Gamma}\left|u_{j}-u_{k}\right|^{2} d \mu & =\sum_{r=1}^{\infty} \int_{\mathbf{g}_{r} \Gamma_{r}} \psi_{r}\left|u_{j}-u_{k}\right|^{2}\left(\mathbf{h}_{r}(\mathbf{u})\right) J_{r}(\mathbf{u}) d u \\
& \leq C \sum_{r=1}^{\infty} \int_{\mathbf{g}_{r} \Gamma_{r}} \psi_{r}\left|u_{j}-u_{k}\right|^{2}\left(\mathbf{h}_{r}(\mathbf{u})\right) d u \\
& =C \sum_{r=1}^{\infty}\left\|\mathbf{h}_{r}^{*}\left(\psi_{r}\left|u_{j}-u_{k}\right|\right)\right\|_{0,2, U_{r}}^{2} \\
& \leq C \| u_{j}-u_{k}| |_{H^{m+s}(\Gamma)}
\end{aligned}
$$

and it follows there exists $u \in L^{2}(\Gamma)$ such that

$$
\left\|u_{j}-u\right\|_{0,2, \Gamma} \rightarrow 0
$$

and a subsequence, still denoted by $u_{j}$ such that $u_{j}(\mathbf{x}) \rightarrow u(\mathbf{x})$ for $\mu$ a.e. $\mathbf{x} \in \Gamma$. It is required to show that $u \in H^{m+s}(\Gamma)$ such that $w_{i}=\mathbf{h}_{i}^{*}\left(u \psi_{i}\right)$ for each $i$. First of all, $u$ is measurable because it is the limit of measurable functions. The pointwise convergence just established and the fact that sets of measure zero on $\Gamma_{i}$ correspond to sets of measure zero on $U_{i}$ which was discussed in the claim found in the proof of Theorem 37.2.4 on Page 1303 shows that

$$
\mathbf{h}_{i}^{*}\left(u_{j} \psi_{i}\right)(\mathbf{x}) \rightarrow \mathbf{h}_{i}^{*}\left(u \psi_{i}\right)(\mathbf{x})
$$

a.e. x. Therefore,

$$
\mathbf{h}_{i}^{*}\left(u \psi_{i}\right)=w_{i}
$$

and this shows that $\mathbf{h}_{i}^{*}\left(u \psi_{i}\right) \in H^{m+s}\left(U_{i}\right)$. It remains to verify that $u \in H^{m+s}(\Gamma)$. This follows from Fatou's lemma. From 39.6.22,

$$
\left\|\mathbf{h}_{i}^{*}\left(u_{j} \psi_{i}\right)\right\|_{H^{m+s}\left(U_{i}\right)}^{2} \rightarrow\left\|\mathbf{h}_{i}^{*}\left(u \psi_{i}\right)\right\|_{H^{m+s}\left(U_{i}\right)}^{2}
$$

and so

$$
\begin{aligned}
\sum_{i=1}^{\infty}\left\|\mathbf{h}_{i}^{*}\left(u \psi_{i}\right)\right\|_{H^{m+s}\left(U_{i}\right)}^{2} & \leq \lim \inf _{j \rightarrow \infty} \sum_{i=1}^{\infty}\left\|\mathbf{h}_{i}^{*}\left(u_{j} \psi_{i}\right)\right\|_{H^{m+s}\left(U_{i}\right)}^{2} \\
& =\lim \inf _{j \rightarrow \infty}\left\|u_{j}\right\|_{H^{m+s}(\Gamma)}^{2}<\infty
\end{aligned}
$$

This proves the lemma.
In fact any two such norms are equivalent. This follows from the open mapping theorem. Suppose $\|\cdot\|_{1}$ and $\|\cdot\|_{2}$ are two such norms and consider the norm $\|\cdot\|_{3} \equiv$ $\max \left(\|\cdot\|_{1},\|\cdot\|_{2}\right)$. Then $\left(H^{m+s}(\Gamma),\|\cdot\|_{3}\right)$ is also a Banach space and the identity map from this Banach space to $\left(H^{m+s}(\Gamma),\|\cdot\|_{i}\right)$ for $i=1,2$ is continuous. Therefore, by the open mapping theorem, there exist constants, $C, C^{\prime}$ such that for all $u \in H^{m+s}(\Gamma)$,

$$
\|u\|_{1} \leq\|u\|_{3} \leq C\|u\|_{2} \leq C\|u\|_{3} \leq C C^{\prime}\|u\|_{1}
$$

Therefore,

$$
\|u\|_{1} \leq C\|u\|_{2},\|u\|_{2} \leq C^{\prime}\|u\|_{1} .
$$

This proves the following theorem.
Theorem 39.6.5 Let $\Gamma$ be described above. Defining $H^{t}(\Gamma)$ as in Definition 39.6.3, any two norms like those given in this definition are equivalent.

Suppose $\left(\Gamma, W_{i}, U_{i}, \Gamma_{i}, \mathbf{h}_{i}, \mathbf{g}_{i}\right)$ are as defined above where $\mathbf{h}_{i}, \mathbf{g}_{i}$ are $C^{m, 1}$ functions. Take $W$, an open set in $\mathbb{R}^{p}$ and define $\Gamma^{\prime} \equiv W \cap \Gamma$. Then letting

$$
W_{i}^{\prime} \equiv W \cap W_{i}, \Gamma_{i}^{\prime} \equiv W_{i}^{\prime} \cap \Gamma
$$

and

$$
U_{i}^{\prime} \equiv \mathbf{g}_{i}\left(\Gamma_{i}^{\prime}\right)=\mathbf{h}_{i}^{-1}\left(W_{i}^{\prime} \cap \Gamma\right)
$$

it follows that $U_{i}^{\prime}$ is an open set because $\mathbf{h}_{i}$ is continuous and $\left(\Gamma^{\prime}, W_{i}^{\prime}, U_{i}^{\prime}, \Gamma_{i}^{\prime}, \mathbf{h}_{i}^{\prime}, \mathbf{g}_{i}^{\prime}\right)$ is also a $C^{m, 1}$ manifold if you define $\mathbf{h}_{i}^{\prime}$ to be the restriction of $\mathbf{h}_{i}$ to $U_{i}^{\prime}$ and $\mathbf{g}_{i}^{\prime}$ to be the restriction of $\mathbf{g}_{i}$ to $W_{i}^{\prime}$.

As a case of this, consider a $C^{m, 1}$ manifold, $\Gamma$ where $\left(\Gamma, W_{i}, U_{i}, \Gamma_{i}, \mathbf{h}_{i}, \mathbf{g}_{i}\right)$ are as described in Definition 39.6.1 and the submanifold consisting of $\Gamma_{i}$. The next lemma shows there is a simple way to define a norm on $H^{t}\left(\Gamma_{i}\right)$ which does not depend on dragging in a partition of unity.

Lemma 39.6.6 Suppose $\Gamma$ is a $C^{m, 1}$ manifold and $\left(\Gamma, W_{i}, U_{i}, \Gamma_{i}, \mathbf{h}_{i}, \mathbf{g}_{i}\right)$ are as described in Definition 39.6.1. Then for $t \in[m, m+s)$, it follows that if $u \in H^{t}(\Gamma)$, then $u \in H^{t}\left(\Gamma_{k}\right)$ and the restriction map is continuous. Also an equivalent norm for $H^{t}\left(\Gamma_{k}\right)$ is given by

$$
\left\|\|u\|_{t} \equiv\right\| \mathbf{h}_{k}^{*} u \|_{H^{t}\left(U_{k}\right)} .
$$

Proof: Let $u \in H^{t}(\Gamma)$ and let $\left(\Gamma_{k}, W_{i}^{\prime}, U_{i}^{\prime}, \Gamma_{i}^{\prime}, \mathbf{h}_{i}^{\prime}, \mathbf{g}_{i}^{\prime}\right)$ be the sets and functions which define what is meant by $\Gamma_{k}$ being a $C^{m, 1}$ manifold as described in Definition 39.6.1. Also let $\left(\Gamma, W_{i}, U_{i}, \Gamma_{i}, \mathbf{h}_{i}, \mathbf{g}_{i}\right)$ be pertain to $\Gamma$ in the same way and let $\left\{\phi_{j}\right\}$ be a $C^{\infty}$ partition of unity for the $\left\{W_{j}\right\}$. Since the $\left\{W_{i}^{\prime}\right\}$ are locally finite, only finitely many can intersect $\Gamma_{k}$, say $\left\{W_{1}^{\prime}, \cdots, W_{s}^{\prime}\right\}$. Also only finitely many of the $W_{i}$ can intersect $\Gamma_{k}$, say $\left\{W_{1}, \cdots, W_{q}\right\}$. Then letting $\left\{\psi_{i}^{\prime}\right\}$ be a $C^{\infty}$ partition of unity subordinate to the $\left\{W_{i}^{\prime}\right\}$.

$$
\begin{gathered}
\sum_{i=1}^{\infty}\left\|\mathbf{h}_{i}^{\prime *}\left(u \psi_{i}^{\prime}\right)\right\|_{H^{t}\left(U_{i}^{\prime}\right)}=\sum_{i=1}^{s}\left\|\mathbf{h}_{i}^{\prime *}\left(\sum_{j=1}^{q} \phi_{j} u \psi_{i}^{\prime}\right)\right\|_{H^{t}\left(U_{i}^{\prime}\right)} \\
\leq \sum_{i=1}^{s} \sum_{j=1}^{q}\left\|\mathbf{h}_{i}^{\prime *} \phi_{j} u \psi_{i}^{\prime}\right\|_{H^{t}\left(U_{i}^{\prime}\right)}=\sum_{j=1}^{q} \sum_{i=1}^{s}\left\|\mathbf{h}_{i}^{* *} \phi_{j} u \psi_{i}^{\prime}\right\| \|_{H^{t}\left(U_{i}^{\prime}\right)} \\
=\sum_{j=1}^{q} \sum_{i=1}^{s}\left\|\left(\mathbf{g}_{j} \circ \mathbf{h}_{i}^{\prime}\right)^{*} \mathbf{h}_{j}^{*} \phi_{j} u \psi_{i}^{\prime}\right\|_{H^{t}\left(U_{i}^{\prime}\right)}
\end{gathered}
$$

By Lemma 39.3.3 on page 1356, there exists a single constant, $C$ such that the above is dominated by $C \sum_{j=1}^{q} \sum_{i=1}^{s}\left\|\mathbf{h}_{j}^{*} \phi_{j} u \psi_{i}^{\prime}\right\|_{H^{t}\left(U_{j}\right)}$. Now by Corollary 39.3.5 on Page 1358, this is no larger than

$$
C \sum_{j=1}^{q} \sum_{i=1}^{s} C_{\psi_{i}^{\prime}}\left\|\mathbf{h}_{j}^{*} \phi_{j} u\right\|_{H^{t}\left(U_{j}\right)} \leq C \sum_{j=1}^{q} \sum_{i=1}^{s}\left\|\mathbf{h}_{j}^{*} \phi_{j} u\right\|_{H^{t}\left(U_{j}\right)} \leq C \sum_{j=1}^{q}\left\|\mathbf{h}_{j}^{*} \phi_{j} u\right\|_{H^{t}\left(U_{j}\right)}<\infty
$$

This shows that $u$ restricted to $\Gamma_{k}$ is in $H^{t}\left(\Gamma_{k}\right)$. It also shows that the restriction map of $H^{t}(\Gamma)$ to $H^{t}\left(\Gamma_{k}\right)$ is continuous.

Now consider the norm $\||\cdot|\|_{t}$. For $u \in H^{t}\left(\Gamma_{k}\right)$, let $\left(\Gamma_{k}, W_{i}^{\prime}, U_{i}^{\prime}, \Gamma_{i}^{\prime}, \mathbf{h}_{i}^{\prime}, \mathbf{g}_{i}^{\prime}\right)$ be sets and functions which define an atlas for $\Gamma_{k}$. Since the $\left\{W_{i}^{\prime}\right\}$ are locally finite, only finitely many can have nonempty intersection with $\Gamma_{k}$, say $\left\{W_{1}, \cdots, W_{s}\right\}$. Thus $i \leq s$ for some finite $s$. The problem is to compare $\mid\|\cdot\| \|_{t}$ with $\|\cdot\|_{H^{t}\left(\Gamma_{k}\right)}$. As above, let $\left\{\psi_{i}^{\prime}\right\}$ denote a $C^{\infty}$ partition of unity subordinate to the $\left\{W_{j}^{\prime}\right\}$. Then

$$
\begin{gathered}
\|\mid u\|\left\|_{t} \equiv\right\| \mathbf{h}_{k}^{*} u\left\|_{H^{t}\left(U_{k}\right)}=\right\| \mathbf{h}_{k}^{*} \sum_{j=1}^{s} \psi_{j}^{\prime} u\left\|_{H^{t}\left(U_{k}\right)} \leq \sum_{j=1}^{s}\right\| \mathbf{h}_{k}^{*}\left(\psi_{j}^{\prime} u\right) \|_{H^{t}\left(U_{k}\right)} \\
=\sum_{j=1}^{s}\left\|\left(\mathbf{g}_{j}^{\prime} \circ \mathbf{h}_{k}\right)^{*} \mathbf{h}_{j}^{\prime *}\left(\psi_{j}^{\prime} u\right)\right\|_{H^{t}\left(U_{k}\right)} \leq C \sum_{j=1}^{s}\left\|\mathbf{h}_{j}^{\prime *}\left(\psi_{j}^{\prime} u\right)\right\|_{H^{t}\left(U_{j}^{\prime}\right)}
\end{gathered}
$$

$$
\leq C\left(\sum_{j=1}^{s}\left\|\mathbf{h}_{j}^{* *}\left(\psi_{j}^{\prime} u\right)\right\|_{H^{t}\left(U_{j}^{\prime}\right)}^{2}\right)^{1 / 2}=\|u\|_{H^{t}\left(\Gamma_{k}\right)}
$$

where Lemma 39.3.3 on page 1356 was used in the last step. Now also, from Lemma 39.3.3 on page 1356

$$
\begin{aligned}
& \|u\|_{H^{t}\left(\Gamma_{k}\right)}=\left(\sum_{j=1}^{s}\left\|\mathbf{h}_{j}^{\prime *}\left(\psi_{j}^{\prime} u\right)\right\|_{H^{t}\left(U_{j}^{\prime}\right)}^{2}\right)^{1 / 2}=\left(\sum_{j=1}^{s}\left\|\left(\mathbf{g}_{k} \circ \mathbf{h}_{j}^{\prime}\right)^{*} \mathbf{h}_{k}^{*}\left(\psi_{j}^{\prime} u\right)\right\|_{H^{t}\left(U_{j}^{\prime}\right)}^{2}\right)^{1 / 2} \\
& \leq C\left(\sum_{j=1}^{s}\left\|\mathbf{h}_{k}^{*}\left(\psi_{j}^{\prime} u\right)\right\|_{H^{t}\left(U_{k}\right)}^{2}\right)^{1 / 2} \leq C\left(\sum_{j=1}^{s}\left\|\mathbf{h}_{k}^{*} u\right\|_{H^{t}\left(U_{k}\right)}^{2}\right)^{1 / 2}=C_{S}\left\|\mathbf{h}_{k}^{*} u\right\|_{H^{t}\left(U_{k}\right)}=\|u\| \|_{t} .
\end{aligned}
$$

This proves the lemma.

### 39.6.2 The Trace On The Boundary

Definition 39.6.7 A bounded open subset, $\Omega$, of $\mathbb{R}^{n}$ has a $C^{m, 1}$ boundary if it satisfies the following conditions. For each $p \in \Gamma \equiv \bar{\Omega} \backslash \Omega$, there exists an open set, $W$, containing $p$, an open interval $(0, b)$, a bounded open box $U^{\prime} \subseteq \mathbb{R}^{n-1}$, and an affine orthogonal transformation, $R_{W}$ consisting of a distance preserving linear transformation followed by a translation such that

$$
\begin{gather*}
R_{W} W=U^{\prime} \times(0, b)  \tag{39.6.23}\\
R_{W}(W \cap \Omega)=\left\{\mathbf{u} \in \mathbb{R}^{n}: \mathbf{u}^{\prime} \in U^{\prime}, 0<u_{n}<\phi_{W}\left(\mathbf{u}^{\prime}\right)\right\} \equiv U_{W} \tag{39.6.24}
\end{gather*}
$$

where $\phi_{W} \in C^{m, 1}\left(\overline{U^{\prime}}\right)$ meaning $\phi_{W}$ is the restriction to $U^{\prime}$ of a function, still denoted by $\phi_{W}$ which is in $C^{m, 1}\left(\mathbb{R}^{n-1}\right)$ and

$$
\inf \left\{\phi_{W}\left(\mathbf{u}^{\prime}\right): \mathbf{u}^{\prime} \in U^{\prime}\right\}>0
$$

The following picture depicts the situation.


For the situation described in the above definition, let $\mathbf{h}_{W}: U^{\prime} \rightarrow \Gamma \cap W$ be defined by

$$
\mathbf{h}_{W}\left(\mathbf{u}^{\prime}\right) \equiv R_{W}^{-1}\left(\mathbf{u}^{\prime}, \phi_{W}\left(\mathbf{u}^{\prime}\right)\right), \mathbf{g}_{W}(\mathbf{x}) \equiv\left(R_{W} \mathbf{x}\right)^{\prime}, \mathbf{H}_{W}(\mathbf{u}) \equiv R_{W}^{-1}\left(\mathbf{u}^{\prime}, \phi_{W}\left(\mathbf{u}^{\prime}\right)-u_{n}\right)
$$

where $\mathbf{x}^{\prime} \equiv\left(x_{1}, \cdots, x_{n-1}\right)$ for $\mathbf{x}=\left(x_{1}, \cdots, x_{n}\right)$. Thus $\mathbf{g}_{W} \circ \mathbf{h}_{W}=\mathrm{id}$ on $U^{\prime}$ and $\mathbf{h}_{W} \circ \mathbf{g}_{W}=\mathrm{id}$ on $\Gamma \cap W$. Also note that $\mathbf{H}_{W}$ is defined on all of $\mathbb{R}^{n}$ is $C^{m, 1}$, and has an inverse with the
same properties. To see this, let $\mathbf{G}_{W}(\mathbf{u})=\left(\mathbf{u}^{\prime}, \phi_{W}\left(\mathbf{u}^{\prime}\right)-u_{n}\right)$. Then $\mathbf{H}_{W}=R_{W}^{-1} \circ \mathbf{G}_{W}$ and $\mathbf{G}_{W}^{-1}=\left(\mathbf{u}^{\prime}, \phi_{W}\left(\mathbf{u}^{\prime}\right)-u_{n}\right)$ and so $\mathbf{H}_{W}^{-1}=\mathbf{G}_{W}^{-1} \circ R_{W}$. Note also that as indicated in the picture,

$$
R_{W}(W \cap \Omega)=\left\{\mathbf{u} \in \mathbb{R}^{n}: \mathbf{u}^{\prime} \in U^{\prime} \text { and } 0<u_{n}<\phi_{W}\left(\mathbf{u}^{\prime}\right)\right\}
$$

Since $\Gamma=\partial \Omega$ is compact, there exist finitely many of these open sets, $W$, denoted by $\left\{W_{i}\right\}_{i=1}^{q}$ such that $\Gamma \subseteq \cup_{i=1}^{q} W_{i}$. Let the corresponding sets, $U^{\prime}$ be denoted by $U_{i}^{\prime}$ and let the functions, $\phi$ be denoted by $\phi_{i}$. Also let $\mathbf{h}_{i}=\mathbf{h}_{W_{i}}$ etc. Now let $\left\{\psi_{i}\right\}_{i=1}^{q}$ be a $C^{\infty}$ partition of unity subordinate to the $\left\{W_{i}\right\}_{i=1}^{q}$. If $u \in H^{t}(\Omega)$, then by Corollary 39.3.5 on Page 1358 it follows that $u \psi_{i} \in H^{t}\left(W_{i} \cap \Omega\right)$. Now

$$
\mathbf{H}_{i}: U_{i} \equiv\left\{\mathbf{u} \in \mathbb{R}^{n}: \mathbf{u}^{\prime} \in U_{i}^{\prime}, 0<u_{n}<\phi_{i}\left(\mathbf{u}^{\prime}\right)\right\} \rightarrow W_{i} \cap \Omega
$$

and $\mathbf{H}_{i}$ and its inverse are defined on $\mathbb{R}^{n}$ and are in $C^{m, 1}\left(\mathbb{R}^{n}\right)$. Therefore, by Lemma 39.3.3 on Page 1356,

$$
\mathbf{H}_{i}^{*} \in \mathscr{L}\left(H^{t}\left(W_{i} \cap \Omega\right), H^{t}\left(U_{i}\right)\right)
$$

Provide $t=m+s$ where $s>0$.
Now it is possible to define the trace on $\Gamma \equiv \partial \Omega$. For $u \in H^{t}(\Omega)$,

$$
\begin{equation*}
\gamma u \equiv \sum_{i=1}^{q} \mathbf{g}_{i}^{*}\left(\gamma \mathbf{H}_{i}^{*}\left(u \psi_{i}\right)\right) . \tag{39.6.25}
\end{equation*}
$$

I must show it satisfies what it should. Recall the definition of what it means for a function to be in $H^{t-1 / 2}(\Gamma)$ where $t=m+s$.

Definition 39.6.8 Let $s \in(0,1)$ and $m$ is a nonnegative integer. Also let $\mu$ denote the surface measure for $\Gamma$. A $\mu$ measurable function, $u$ is in $H^{m+s}(\Gamma)$ if whenever

$$
\left\{W_{i}, \psi_{i}, \Gamma_{i}, U_{i}, \mathbf{h}_{i}, \mathbf{g}_{i}\right\}_{i=1}^{\infty}
$$

is described above, $\mathbf{h}_{i}^{*}\left(u \psi_{i}\right) \in H^{m+s}\left(U_{i}\right)$ and

$$
\|u\|_{H^{m+s}(\Gamma)} \equiv\left(\sum_{i=1}^{\infty}\left\|\mathbf{h}_{i}^{*}\left(u \psi_{i}\right)\right\|_{H^{m+s}\left(U_{i}\right)}^{2}\right)^{1 / 2}<\infty
$$

Recall that all these norms which are obtained from various partitions of unity and functions, $\mathbf{h}_{i}$ and $\mathbf{g}_{i}$, are equivalent. Here there are only finitely many $W_{i}$ so the sum is a finite sum. The theorem is the following.

Theorem 39.6.9 Let $\Omega$ be a bounded open set having $C^{m, 1}$ boundary as discussed above in Definition 39.6.7. Then for $t \leq m+1$, there exists a unique

$$
\gamma \in \mathscr{L}\left(H^{t}(\Omega), H^{t-1 / 2}(\Gamma)\right)
$$

which has the property that for $\mu$ the measure on the boundary,

$$
\begin{equation*}
\gamma u(\mathbf{x})=u(\mathbf{x}) \text { for } \mu \text { a.e. } \mathbf{x} \in \Gamma \text { whenever }\left.u \in \mathfrak{S}\right|_{\Omega} \tag{39.6.26}
\end{equation*}
$$

Proof: First consider the claim that $\gamma \in \mathscr{L}\left(H^{t}(\Omega), H^{t-1 / 2}(\Gamma)\right)$. This involves first showing that for $u \in H^{t}(\Omega), \gamma u \in H^{t-1 / 2}(\Gamma)$. To do this, use the above definition.

$$
\begin{align*}
\mathbf{h}_{j}^{*}\left(\psi_{j}(\gamma u)\right) & =\sum_{i=1}^{q} \mathbf{h}_{j}^{*}\left(\psi_{j} \mathbf{g}_{i}^{*}\left(\gamma \mathbf{H}_{i}^{*}\left(u \psi_{i}\right)\right)\right)=\sum_{i=1}^{q}\left(\mathbf{h}_{j}^{*} \psi_{j}\right)\left(\mathbf{h}_{j}^{*}\left(\mathbf{g}_{i}^{*}\left(\gamma \mathbf{H}_{i}^{*}\left(u \psi_{i}\right)\right)\right)\right) \\
& =\sum_{i=1}^{q}\left(\mathbf{h}_{j}^{*} \psi_{j}\right)\left(\mathbf{g}_{i} \circ \mathbf{h}_{j}\right)^{*}\left(\gamma \mathbf{H}_{i}^{*}\left(u \psi_{i}\right)\right) \tag{39.6.27}
\end{align*}
$$

First note that $\gamma \mathbf{H}_{i}^{*}\left(u \psi_{i}\right) \in H^{t-1 / 2}\left(U_{i}^{\prime}\right)$. Now $\mathbf{g}_{i} \circ \mathbf{h}_{j}$ and its inverse, $\mathbf{g}_{j} \circ \mathbf{h}_{i}$ are both functions in $C^{m, 1}\left(\mathbb{R}^{n-1}\right)$ and

$$
\mathbf{g}_{i} \circ \mathbf{h}_{j}: U_{j}^{\prime} \rightarrow U_{i}^{\prime}
$$

Therefore, by Lemma 39.3.3 on Page 1356,

$$
\left(\mathbf{g}_{i} \circ \mathbf{h}_{j}\right)^{*}\left(\gamma \mathbf{H}_{i}^{*}\left(u \psi_{i}\right)\right) \in H^{t-1 / 2}\left(U_{j}^{\prime}\right)
$$

and $\left\|\left(\mathbf{g}_{i} \circ \mathbf{h}_{j}\right)^{*}\left(\gamma \mathbf{H}_{i}^{*}\left(u \psi_{i}\right)\right)\right\|_{H^{t-1 / 2}\left(U_{j}^{\prime}\right)} \leq C_{i j}\left\|\gamma \mathbf{H}_{i}^{*}\left(u \psi_{i}\right)\right\|_{H^{t-1 / 2}\left(U_{i}^{\prime}\right)}$. Also it follows that $\mathbf{h}_{j}^{*} \psi_{j} \in C^{m, 1}\left(U_{j}^{\prime}\right)$ and has compact support in $U_{j}^{\prime}$ and so by Corollary 39.3.5 on Page 1358

$$
\left(\mathbf{h}_{j}^{*} \psi_{j}\right)\left(\mathbf{g}_{i} \circ \mathbf{h}_{j}\right)^{*}\left(\gamma \mathbf{H}_{i}^{*}\left(u \psi_{i}\right)\right) \in H^{t-1 / 2}\left(U_{j}^{\prime}\right)
$$

and

$$
\begin{align*}
& \left\|\left(\mathbf{h}_{j}^{*} \psi_{j}\right)\left(\mathbf{g}_{i} \circ \mathbf{h}_{j}\right)^{*}\left(\gamma \mathbf{H}_{i}^{*}\left(u \psi_{i}\right)\right)\right\|_{H^{t-1 / 2}\left(U_{j}^{\prime}\right)} \\
& \quad \leq C_{i j}\left\|\left(\mathbf{g}_{i} \circ \mathbf{h}_{j}\right)^{*}\left(\gamma \mathbf{H}_{i}^{*}\left(u \psi_{i}\right)\right)\right\|_{H^{t-1 / 2}\left(U_{j}^{\prime}\right)}  \tag{39.6.28}\\
& \quad \leq C_{i j}\left\|\gamma \mathbf{H}_{i}^{*}\left(u \psi_{i}\right)\right\|_{H^{t-1 / 2}\left(U_{i}^{\prime}\right)} \tag{39.6.29}
\end{align*}
$$

This shows $\gamma u \in H^{t-1 / 2}(\Gamma)$ because each $\mathbf{h}_{j}^{*}\left(\psi_{j}(\gamma u)\right) \in H^{t-1 / 2}\left(U_{j}^{\prime}\right)$. Also from 39.6.29 and 39.6.27

$$
\begin{gathered}
\|\gamma u\|_{H^{t-1 / 2}(\Gamma)}^{2} \leq \sum_{j=1}^{q}\left\|\mathbf{h}_{j}^{*}\left(\psi_{j}(\gamma u)\right)\right\|_{H^{t-1 / 2}\left(U_{j}^{\prime}\right)}^{2} \\
=\sum_{j=1}^{q}\left\|\mathbf{h}_{j}^{*}\left(\psi_{j}(\gamma u)\right)\right\|_{H^{t-1 / 2}\left(U_{j}^{\prime}\right)}^{2}=\sum_{j=1}^{q}\left\|\sum_{i=1}^{q}\left(\mathbf{h}_{j}^{*} \psi_{j}\right)\left(\mathbf{g}_{i} \circ \mathbf{h}_{j}\right)^{*}\left(\gamma \mathbf{H}_{i}^{*}\left(u \psi_{i}\right)\right)\right\|_{H^{t-1 / 2}\left(U_{j}^{\prime}\right)}^{2} \\
\leq C_{q} \sum_{j=1}^{q} \sum_{i=1}^{q}\left\|\left(\mathbf{h}_{j}^{*} \psi_{j}\right)\left(\mathbf{g}_{i} \circ \mathbf{h}_{j}\right)^{*}\left(\gamma \mathbf{H}_{i}^{*}\left(u \psi_{i}\right)\right)\right\|_{H^{t-1 / 2}\left(U_{j}^{\prime}\right)}^{2} \\
\leq C_{q} \sum_{j=1}^{q} \sum_{i=1}^{q} C_{i j}\left\|\left(\gamma \mathbf{H}_{i}^{*}\left(u \psi_{i}\right)\right)\right\|_{H^{t-1 / 2}\left(U_{i}^{\prime}\right)}^{2} \leq C_{q} \sum_{i=1}^{q}\left\|\left(\gamma \mathbf{H}_{i}^{*}\left(u \psi_{i}\right)\right)\right\|_{H^{t-1 / 2}\left(U_{i}^{\prime}\right)}^{2}
\end{gathered}
$$

$$
\leq C_{q} \sum_{i=1}^{q}\left\|\mathbf{H}_{i}^{*}\left(u \psi_{i}\right)\right\|_{H^{t}\left(R_{i}\left(W_{i} \cap \Omega\right)\right)}^{2} \leq C_{q} \sum_{i=1}^{q}\left\|u \psi_{i}\right\|_{H^{t}\left(W_{i} \cap \Omega\right)}^{2} \leq C_{q}\|u\|_{H^{t}(\Omega)}^{2}
$$

Does $\gamma$ satisfy 39.6.26? Let $\mathbf{x} \in \Gamma$ and $\left.u \in \mathfrak{S}\right|_{\Omega}$. Let

$$
I_{\mathbf{x}} \equiv\left\{i \in\{1,2, \cdots, q\}: \mathbf{x}=\mathbf{h}_{i}\left(\mathbf{u}_{i}^{\prime}\right) \text { for some } \mathbf{u}_{i}^{\prime} \in U_{i}^{\prime}\right\}
$$

Then

$$
\gamma u(\mathbf{x})=\sum_{i \in I_{\mathbf{x}}}\left(\gamma \mathbf{H}_{i}^{*}\left(u \psi_{i}\right)\right)\left(\mathbf{g}_{i}(\mathbf{x})\right)=\sum_{i \in I_{\mathbf{x}}}\left(\gamma \mathbf{H}_{i}^{*}\left(u \psi_{i}\right)\right)\left(\mathbf{g}_{i}\left(\mathbf{h}_{i}\left(\mathbf{u}_{i}^{\prime}\right)\right)\right)=\sum_{i \in I_{\mathbf{x}}}\left(\gamma \mathbf{H}_{i}^{*}\left(u \psi_{i}\right)\right)\left(\mathbf{u}_{i}^{\prime}\right) .
$$

Now because $\mathbf{H}_{i}$ is Lipschitz continuous and $u \psi \in \mathfrak{S}$, it follows that $\mathbf{H}_{i}^{*}\left(u \psi_{i}\right) \in H^{1}\left(\mathbb{R}^{n}\right)$ and is continuous and so by Theorem 39.5.7 on Page 1366 for a.e. $\mathbf{u}_{i}^{\prime}$,

$$
=\sum_{i \in I_{\mathbf{x}}} \mathbf{H}_{i}^{*}\left(u \psi_{i}\right)\left(\mathbf{u}_{i}^{\prime}, 0\right)=\sum_{i \in I_{\mathbf{x}}} \mathbf{h}_{i}^{*}\left(u \psi_{i}\right)\left(\mathbf{u}_{i}^{\prime}\right)=\sum_{i \in I_{\mathbf{x}}}\left(u \psi_{i}\right)\left(\mathbf{h}_{i}\left(\mathbf{u}_{i}^{\prime}\right)\right)=u(\mathbf{x}) \text { for } \mu \text { a.e.x. }
$$

This verifies 39.6.26 and completes the proof of the theorem.

## Chapter 40

## Weak Solutions

### 40.1 The Lax Milgram Theorem

The Lax Milgram theorem is a fundamental result which is useful for obtaining weak solutions to many types of partial differential equations. It is really a general theorem in functional analysis.

Definition 40.1.1 Let $A \in \mathscr{L}\left(V, V^{\prime}\right)$ where $V$ is a Hilbert space. Then $A$ is said to be coercive if

$$
A(v)(v) \geq \boldsymbol{\delta}\|v\|^{2}
$$

for some $\delta>0$.
Theorem 40.1.2 (Lax Milgram) Let $A \in \mathscr{L}\left(V, V^{\prime}\right)$ be coercive. Then A maps one to one and onto.

Proof: The proof that $A$ is onto involves showing $A(V)$ is both dense and closed.
Consider first the claim that $A(V)$ is closed. Let $A x_{n} \rightarrow y^{*} \in V^{\prime}$. Then

$$
\delta\left\|x_{n}-x_{m}\right\|_{V}^{2} \leq\left\|A x_{n}-A x_{m}\right\|_{V^{\prime}}\left\|x_{n}-x_{m}\right\|_{V} .
$$

Therefore, $\left\{x_{n}\right\}$ is a Cauchy sequence in $V$. It follows $x_{n} \rightarrow x \in V$ and since $A$ is continuous, $A x_{n} \rightarrow A x$. This shows $A(V)$ is closed.

Now let $R: V \rightarrow V^{\prime}$ denote the Riesz map defined by $R x(y)=(y, x)$. Recall that the Riesz map is one to one, onto, and preserves norms. Therefore, $R^{-1}(A(V))$ is a closed subspace of $V$. If there $R^{-1}(A(V)) \neq V$, then $\left(R^{-1}(A(V))\right)^{\perp} \neq\{0\}$. Let $x \in\left(R^{-1}(A(V))\right)^{\perp}$ and $x \neq 0$. Then in particular,

$$
0=\left(x, R^{-1} A x\right)=R\left(R^{-1}(A(x))\right)(x)=A(x)(x) \geq \delta\|x\|_{V}^{2}
$$

a contradiction to $x \neq 0$. Therefore, $R^{-1}(A(V))=V$ and so $A(V)=R(V)=V^{\prime}$.
Since $A(V)$ is both closed and dense, $A(V)=V^{\prime}$. This shows $A$ is onto.
If $A x=A y$, then $0=A(x-y)(x-y) \geq \delta\|x-y\|_{V}^{2}$, and this shows $A$ is one to one. This proves the theorem.

Here is a simple example which illustrates the use of the above theorem. In the example the repeated index summation convention is being used. That is, you sum over the repeated indices.

Example 40.1.3 Let $U$ be an open subset of $\mathbb{R}^{n}$ and let $V$ be a closed subspace of $H^{1}(U)$. Let $\alpha^{i j} \in L^{\infty}(U)$ for $i, j=1,2, \cdots, n$. Now define $A: V \rightarrow V^{\prime}$ by

$$
A(u)(v) \equiv \int_{U}\left(\alpha^{i j}(\mathbf{x}) u_{, i}(\mathbf{x}) v_{, j}(\mathbf{x})+u(\mathbf{x}) v(\mathbf{x})\right) d x
$$

Suppose also that

$$
\alpha^{i j} v_{i} v_{j} \geq \delta|\mathbf{v}|^{2}
$$

whenever $\mathbf{v} \in \mathbb{R}^{n}$. Then A maps $V$ to $V^{\prime}$ one to one and onto.

Here is why. It is obvious that $A$ is in $\mathscr{L}\left(V, V^{\prime}\right)$. It only remains to verify that it is coercive.

$$
\begin{aligned}
A(u)(u) & \equiv \int_{U}\left(\alpha^{i j}(\mathbf{x}) u_{, i}(\mathbf{x}) u_{, j}(\mathbf{x})+u(\mathbf{x}) u(\mathbf{x})\right) d x \\
& \geq \int_{U} \delta|\nabla u(\mathbf{x})|^{2}+|u(\mathbf{x})|^{2} d x \\
& \geq \delta\|u\|_{H^{1}(U)}^{2}
\end{aligned}
$$

This proves coercivity and verifies the claim.
What has been obtained in the above example? This depends on how you choose $V$. In Example 40.1.3 suppose $U$ is a bounded open set with $C^{0,1}$ boundary and $V=H_{0}^{1}(U)$ where

$$
H_{0}^{1}(U) \equiv\left\{u \in H^{1}(U): \gamma u=0\right\}
$$

Also suppose $f \in L^{2}(U)$. Then you can consider $F \in V^{\prime}$ by defining

$$
F(v) \equiv \int_{U} f(\mathbf{x}) v(\mathbf{x}) d x
$$

According to the Lax Milgram theorem and the verification of its conditions in Example 40.1.3, there exists a unique solution to the problem of finding $u \in H_{0}^{1}(U)$ such that for all $v \in H_{0}^{1}(U)$,

$$
\begin{equation*}
\int_{U}\left(\alpha^{i j}(\mathbf{x}) u_{, i}(\mathbf{x}) v_{, j}(\mathbf{x})+u(\mathbf{x}) v(\mathbf{x})\right) d x=\int_{U} f(\mathbf{x}) v(\mathbf{x}) d x \tag{40.1.1}
\end{equation*}
$$

In particular, this holds for all $v \in C_{c}^{\infty}(U)$. Thus for all such $v$,

$$
\int_{U}\left(-\left(\alpha^{i j}(\mathbf{x}) u_{, i}(\mathbf{x})\right)_{, j}+u(\mathbf{x})-f(\mathbf{x})\right) v(\mathbf{x}) d x=0
$$

Therefore, in terms of weak derivatives,

$$
-\left(\alpha^{i j} u_{, i}\right)_{, j}+u=f
$$

and since $u \in H_{0}^{1}(U)$, it must be the case that $\gamma u=0$ on $\partial U$. This is why the solution to 40.1.1 is referred to as a weak solution to the boundary value problem

$$
-\left(\alpha^{i j}(\mathbf{x}) u_{, i}(\mathbf{x})\right)_{, j}+u(\mathbf{x})=f(\mathbf{x}), u=0 \text { on } \partial U
$$

Of course you then begin to ask the important question whether $u$ really has two derivatives. It is not immediately clear that just because $-\left(\alpha^{i j}(\mathbf{x}) u_{, i}(\mathbf{x})\right)_{, j} \in L^{2}(U)$ it follows that the second derivatives of $u$ exist. Actually this will often be true and is discussed somewhat in the next section.

Next suppose you choose $V=H^{1}(U)$ and let $g \in H^{1 / 2}(\partial U)$. Define $F \in V^{\prime}$ by

$$
F(v) \equiv \int_{U} f(\mathbf{x}) v(\mathbf{x}) d x+\int_{\partial U} g(\mathbf{x}) \gamma v(\mathbf{x}) d \mu
$$

Everything works the same way and you get the existence of a unique $u \in H^{1}(U)$ such that for all $v \in H^{1}(U)$,

$$
\begin{equation*}
\int_{U}\left(\alpha^{i j}(\mathbf{x}) u_{, i}(\mathbf{x}) v_{, j}(\mathbf{x})+u(\mathbf{x}) v(\mathbf{x})\right) d x=\int_{U} f(\mathbf{x}) v(\mathbf{x}) d x+\int_{\partial U} g(\mathbf{x}) \gamma v(\mathbf{x}) d \mu \tag{40.1.2}
\end{equation*}
$$

is satisfied. If you pretend $u$ has all second order derivatives in $L^{2}(U)$ and apply the divergence theorem, you find that you have obtained a weak solution to

$$
-\left(\alpha^{i j} u_{, i}\right)_{, j}+u=f, \alpha^{i j} u_{, i} n_{j}=g \text { on } \partial U
$$

where $n_{j}$ is the $j^{\text {th }}$ component of $\mathbf{n}$, the unit outer normal. Therefore, $u$ is a weak solution to the above boundary value problem.

The conclusion is that the Lax Milgram theorem gives a way to obtain existence and uniqueness of weak solutions to various boundary value problems. The following theorem is often very useful in establishing coercivity. To prove this theorem, here is a definition.

Definition 40.1.4 Let $U$ be an open set and $\delta>0$. Then

$$
U_{\delta} \equiv\left\{\mathbf{x} \in U: \operatorname{dist}\left(\mathbf{x}, U^{C}\right)>\delta\right\}
$$

Theorem 40.1.5 Let $U$ be a connected bounded open set having $C^{0,1}$ boundary such that for some sequence, $\eta_{k} \downarrow 0$,

$$
\begin{equation*}
U=\cup_{k=1}^{\infty} U_{\eta_{k}} \tag{40.1.3}
\end{equation*}
$$

and $U_{\eta_{k}}$ is a connected open set. Suppose $\Gamma \subseteq \partial U$ has positive surface measure and that

$$
V \equiv\left\{u \in H^{1}(U): \gamma u=0 \text { a.e. on } \Gamma\right\} .
$$

Then the norm |||•||| given by

$$
\||u|\| \equiv\left(\int_{U}|\nabla u|^{2} d x\right)^{1 / 2}
$$

is equivalent to the usual norm on $V$.
Proof: First it is necessary to verify this is actually a norm. It clearly satisfies all the usual axioms of a norm except for the condition that $\|\|u\|\|=0$ if and only if $u=0$. Suppose then that $\|\|u\|\|=0$. Let $\delta_{0}=\eta_{k}$ for one of those $\eta_{k}$ mentioned above and define

$$
u_{\delta}(\mathbf{x}) \equiv \int_{B(\mathbf{0}, \delta)} u(\mathbf{x}-\mathbf{y}) \phi_{\delta}(\mathbf{y}) d y
$$

where $\phi_{\delta}$ is a mollifier having support in $B(\mathbf{0}, \boldsymbol{\delta})$. Then changing the variables, it follows that for $\mathbf{x} \in U_{\delta_{0}}$

$$
u_{\delta}(\mathbf{x})=\int_{B(\mathbf{x}, \delta)} u(\mathbf{t}) \phi_{\delta}(\mathbf{x}-\mathbf{t}) d t=\int_{U} u(\mathbf{t}) \phi_{\delta}(\mathbf{x}-\mathbf{t}) d t
$$

and so $u_{\delta} \in C^{\infty}\left(U_{\delta_{0}}\right)$ and

$$
\nabla u_{\delta}(\mathbf{x})=\int_{U} u(\mathbf{t}) \nabla \phi_{\delta}(\mathbf{x}-\mathbf{t}) d t=\int_{B(\mathbf{0}, \delta)} \nabla u(\mathbf{x}-\mathbf{y}) \phi_{\delta}(\mathbf{y}) d y=0
$$

Therefore, $u_{\delta}$ equals a constant on $U_{\delta_{0}}$ because $U_{\delta_{0}}$ is a connected open set and $u_{\delta}$ is a smooth function defined on this set which has its gradient equal to $\mathbf{0}$. By Minkowski's inequality,

$$
\begin{aligned}
& \left(\int_{U_{\delta_{0}}}\left|u(\mathbf{x})-u_{\delta}(\mathbf{x})\right|^{2} d x\right)^{1 / 2} \\
\leq & \int_{B(\mathbf{0}, \delta)} \phi_{\delta}(\mathbf{y})\left(\int_{U_{\delta_{0}}}|u(\mathbf{x})-u(\mathbf{x}-\mathbf{y})|^{2} d x\right)^{1 / 2} d y
\end{aligned}
$$

and this converges to 0 as $\delta \rightarrow 0$ by continuity of translation in $L^{2}$. It follows there exists a sequence of constants, $c_{\delta} \equiv u_{\delta}(\mathbf{x})$ such that $\left\{c_{\delta}\right\}$ converges to $u$ in $L^{2}\left(U_{\delta_{0}}\right)$. Consequently, a subsequence, still denoted by $u_{\delta}$, converges to $u$ a.e. By Eggoroff's theorem there exists a set, $N_{k}$ having measure no more than $3^{-k} m_{n}\left(U_{\delta_{0}}\right)$ such that $u_{\delta}$ converges to $u$ uniformly on $N_{k}^{C}$. Thus $u$ is constant on $N_{k}^{C}$. Now $\sum_{k} m_{n}\left(N_{k}\right) \leq \frac{1}{2} m_{n}\left(U_{\delta_{0}}\right)$ and so there exists $\mathbf{x}_{0} \in$ $U_{\delta_{0}} \backslash \cup_{k=1}^{\infty} N_{k}$. Therefore, if $\mathbf{x} \notin N_{k}$ it follows $u(\mathbf{x})=u\left(\mathbf{x}_{0}\right)$ and so, if $u(\mathbf{x}) \neq u\left(\mathbf{x}_{0}\right)$ it must be the case that $\mathbf{x} \in \cap_{k=1}^{\infty} N_{k}$, a set of measure zero. This shows that $u$ equals a constant a.e. on $U_{\delta_{0}}=U_{\eta_{k}}$. Since $k$ is arbitrary, 40.1.3 shows $u$ is a.e. equal to a constant on $U$. Therefore, $u$ equals the restriction of a function of $\mathfrak{S}$ to $U$ and so $\gamma u$ equals this constant in $L^{2}(\partial \Omega)$. Since the surface measure of $\Gamma$ is positive, the constant must equal zero. Therefore, $\|\|\cdot|\||$ is a norm.

It remains to verify that it is equivalent to the usual norm. It is clear that $\|\|u\|\| \leq\|u\|_{1,2}$. What about the other direction? Suppose it is not true that for some constant, $K,\|u\|_{1,2} \leq$ $K\|\|u\|\|$. Then for every $k \in \mathbb{N}$, there exists $u_{k} \in V$ such that

$$
\left\|u_{k}\right\|_{1,2}>k \mid\left\|u_{k}\right\| \|
$$

Replacing $u_{k}$ with $u_{k} /\left\|u_{k}\right\|_{1,2}$, it can be assumed that $\left\|u_{k}\right\|_{1,2}=1$ for all $k$. Therefore, using the compactness of the embedding of $H^{1}(U)$ into $L^{2}(U)$, there exists a subsequence, still denoted by $u_{k}$ such that

$$
\begin{align*}
u_{k} & \rightarrow u \text { weakly in } V,  \tag{40.1.4}\\
u_{k} & \rightarrow u \text { strongly in } L^{2}(U),  \tag{40.1.5}\\
\left\|\left\|u_{k} \mid\right\|\right. & \rightarrow 0,  \tag{40.1.6}\\
u_{k} & \rightarrow u \text { weakly in }(V,|\|\cdot|\||) . \tag{40.1.7}
\end{align*}
$$

From 40.1.6 and 40.1.7, it follows $u=0$. Therefore, $\left|u_{k}\right|_{L^{2}(U)} \rightarrow 0$. This with 40.1.6 contradicts the fact that $\left\|u_{k}\right\|_{1,2}=1$ and this proves the equivalence of the two norms.

The proof of the above theorem yields the following interesting corollary.

Corollary 40.1.6 Let $U$ be a connected open set with the property that for some sequence, $\eta_{k} \downarrow 0$,

$$
U=\cup_{k=1}^{\infty} U_{\eta_{k}}
$$

for $U_{\eta_{k}}$ a connected open set and suppose $u \in W^{1, p}(U)$ and $\nabla u=0$ a.e. Then $u$ equals $a$ constant a.e.

Example 40.1.7 Let $U$ be a bounded open connected subset of $\mathbb{R}^{n}$ and let $V$ be a closed subspace of $H^{1}(U)$ defined by

$$
V \equiv\left\{u \in H^{1}(U): \gamma u=0 \text { on } \Gamma\right\}
$$

where the surface measure of $\Gamma$ is positive.
Let $\alpha^{i j} \in L^{\infty}(U)$ for $i, j=1,2, \cdots, n$ and define $A: V \rightarrow V^{\prime}$ by

$$
A(u)(v) \equiv \int_{U} \alpha^{i j}(\mathbf{x}) u_{, i}(\mathbf{x}) v_{, j}(\mathbf{x}) d x
$$

for

$$
\alpha^{i j} v_{i} v_{j} \geq \delta|\mathbf{v}|^{2}
$$

whenever $\mathbf{v} \in \mathbb{R}^{n}$. Then A maps $V$ to $V^{\prime}$ one to one and onto.
This follows from Theorem 40.1.5 using the equivalent norm defined there. Define $F \in V^{\prime}$ by

$$
\int_{U} f(\mathbf{x}) v(\mathbf{x}) d x+\int_{\partial U \backslash \Gamma} g(\mathbf{x}) \gamma v(\mathbf{x}) d x
$$

for $f \in L^{2}(U)$ and $g \in H^{1 / 2}(\partial U)$. Then the equation,

$$
A u=F \text { in } V^{\prime}
$$

which is equivalent to $u \in V$ and for all $v \in V$,

$$
\int_{U} \alpha^{i j}(\mathbf{x}) u_{, i}(\mathbf{x}) v_{, j}(\mathbf{x}) d x=\int_{U} f(\mathbf{x}) v(\mathbf{x}) d x+\int_{\partial U \backslash \Gamma} g(\mathbf{x}) \gamma v(\mathbf{x}) d \mu
$$

is a weak solution for the boundary value problem,

$$
-\left(\alpha^{i j} u_{, i}\right)_{, j}=f \text { in } U, \alpha^{i j} u_{, i} n_{j}=g \text { on } \partial U \backslash \Gamma, u=0 \text { on } \Gamma
$$

as you can verify by using the divergence theorem formally.

### 40.2 An Application Of The Mountain Pass Theorem

Recall the mountain pass theorem 24.1.3.
Theorem 40.2.1 Let $H$ be a Hilbert space and let $I: H \rightarrow \mathbb{R}$ be a $C^{1}$ functional having $I^{\prime}$ Lipschitz continuous and such that I satisfies the Palais Smale condition. Suppose I $(0)=0$
and $I(u) \geq a>0$ for all $\|u\|=r$. Suppose also that there exists $v,\|v\|>r$ such that $I(v) \leq 0$. Then define

$$
\Gamma \equiv\{g \in C([0,1] ; H): g(0)=0, g(1)=v\}
$$

Let

$$
c \equiv \inf _{g \in \Gamma 0 \leq t \leq 1} \max _{0} I(g(t))
$$

Then $c$ is a critical value of I meaning that there exists $u$ such that $I(u)=c$ and $I^{\prime}(u)=0$. In particular, there is $u \neq 0$ such that $I^{\prime}(u)=0$.

This nice example is in Evans [49]. Let the Hilbert space be $H_{0}^{1}(U)$ where $U$ is a bounded open set. To avoid cases, assume $U$ is in $\mathbb{R}^{3}$ or higher. The main results will work in general but it would involve cases. Consider the functional

$$
\frac{1}{2}\|u\|_{H_{0}^{1}}^{2}-\int_{U} F(u) d x \equiv I_{1}(u)-I_{2}(u)
$$

where $F^{\prime}(u)=f(u), f(0)=0$. Here it is assumed that

$$
\begin{equation*}
|f(u)| \leq C\left(1+|u|^{p}\right),\left|f^{\prime}(u)\right| \leq C\left(1+|u|^{p-1}\right), 1<p<\frac{n+2}{n-2} \tag{40.2.8}
\end{equation*}
$$

Also suppose that

$$
\begin{equation*}
0 \leq F(u) \leq \gamma f(u) u \text { where } 0<\gamma<1 / 2 \tag{40.2.9}
\end{equation*}
$$

and finally that

$$
\begin{equation*}
\alpha|u|^{p+1} \leq F(u) \leq A|u|^{p+1}, \alpha, A>0 \tag{40.2.10}
\end{equation*}
$$

Let $R: H_{0}^{1}(U) \rightarrow H^{-1}(U)$ be the Riesz map.

## Showing Functional is $C^{1,1}$

Then it is not hard to verify that $\left(I_{1}(u), v\right)=(u, v)$ and so it is clearly the case that $I^{\prime}(u)$ exists and is a continuous function of $u$. In addition to this, it is Lipschitz.

Next consider $I_{2}$.

$$
\begin{aligned}
& I_{2}(u+v)-I_{2}(u)=\int_{U} F(u+v)-F(u) d x \\
& \quad=\int_{U} f(u) v+\frac{1}{2} f^{\prime}(\hat{u}) v^{2} d x, \hat{u} \in[u, u+v]
\end{aligned}
$$

Now $H_{0}^{1}(U)$ embeds continuously into $L^{2 n /(n-2)}(U)$. Because of the estimate for $f(u)$, we can regard $f(u)$ as being in $H^{-1}(U)$ as follows.

$$
\begin{aligned}
\left|\int_{U} f(u) v d x\right| & \leq\left(\int_{U}|f(u)|^{2 n /(n+2)} d x\right)^{(n+2) / 2 n}\left(\int_{U}|v|^{2 n /(n-2)}\right)^{(n-2) / 2 n} \\
& \leq\left(\int_{U} C\left(1+|u|^{2 n /(n-2)}\right) d x\right)^{(n+2) / 2 n}\|v\|_{H_{0}^{1}}
\end{aligned}
$$

where $C$ will be adusted as needed here and elsewhere. Thus, writing in terms of the inner product on $H_{0}^{1}$,

$$
\left(I_{2}^{\prime}(u), v\right)_{H_{0}^{1}}=\left(R^{-1} f(u), v\right)_{H_{0}^{1}}
$$

This is so if the $\frac{1}{2} f^{\prime}(\hat{u}) v^{2}$ term is as it should be. We need to verify that

$$
\frac{\int_{U}\left|\frac{1}{2} f^{\prime}(\hat{u}) v^{2}\right| d x}{\|v\|_{H_{0}^{1}}} \rightarrow 0
$$

However, we can use the estimate and write that this is no larger than

$$
\begin{equation*}
\frac{\int_{U} C\left(1+|v|^{p-1}+|u|^{p-1}\right)\left|v^{2}\right| d x}{\left(\int_{U}|v|^{2 n /(n-2)} d x\right)^{(n-2) / 2 n}} \tag{40.2.11}
\end{equation*}
$$

Then consider the term involving $|u|$.

$$
\int_{U}|u|^{p-1}|v|^{2} d x \leq\left(\int_{U}|u|^{p+1}\right)^{\frac{p-1}{p+1}}\left(\int_{U}|v|^{p+1}\right)^{2 /(p+1)}
$$

Now $p+1 \leq 2 \frac{n}{n-2}$ and so the first factor is finite. As to the second, it equals

$$
\left(\left(\int_{U}|v|^{2 n /(n-2)}\right)^{\frac{1}{2 n}(n-2)}\right)^{2}
$$

and so this term from 40.2 .11 is $o(v)$ on $H_{0}^{1}(U)$. The term involving $|v|^{p-1}$ is obviously $o(v)$. Consider the constant term.

$$
\int_{U}|v|^{2} d x \leq\left(\left(\int_{U}|v|^{2 n /(n-2)} d x\right)^{(n-2) / 2 n}\right)^{2}
$$

so it is also all right. Thus the derivative is as claimed. Is this derivative Lipschitz on bounded sets?

$$
f(\hat{u})-f(u)=\int_{0}^{1} f^{\prime}(u+t(\hat{u}-u))(\hat{u}-u) d t
$$

Thus in $H^{-1}$ and using the estimates,

$$
\begin{aligned}
& \left|\int_{U}(f(\hat{u})-f(u)) v d x\right| \leq \int_{U} \int_{0}^{1} C\left(1+|u+t(\hat{u}-u)|^{p-1}\right)|\hat{u}-u| d t d x \\
& =\int_{0}^{1} \int_{U} C\left(1+|u+t(\hat{u}-u)|^{p-1}\right)|\hat{u}-u| d x d t \\
& \leq C\left(\int_{U}\left(1+|\hat{u}|^{p-1}+|u|^{p-1}\right)^{2 n /(n+2)}\right)^{\frac{n+2}{2 n}}\left(\int_{U}|\hat{u}-u|^{2 n /(n-2)}\right)^{(n-2) / 2 n} \\
& \leq C\left(\int_{U}\left(1+|\hat{u}|^{p-1}+|u|^{p-1}\right)^{2 n /(n+2)}\right)^{\frac{n+2}{2 n}}\|\hat{u}-u\|_{H_{0}^{1}(U)}
\end{aligned}
$$

Now $(p-1) \frac{2 n}{n+2} \leq\left(\frac{n+2}{n-2}-1\right) \frac{2 n}{n+2}=8 \frac{n}{n^{2}-4} \leq \frac{2 n}{n-2}$ and so the derivative is Lipschitz on bounded sets of $H_{0}^{1}(U)$.

## Palais Smale Conditions

Here we verify the Palais Smale conditions. Suppose then that $I\left(u_{k}\right)$ is bounded and $I^{\prime}\left(u_{k}\right) \rightarrow 0$ in $H_{0}^{1}(U)$. Then

$$
\begin{equation*}
\left|\frac{1}{2}\left\|u_{k}\right\|_{H_{0}^{1}}^{2}-\int_{U} F\left(u_{k}\right) d x\right| \leq C \tag{40.2.12}
\end{equation*}
$$

Since $I^{\prime}\left(u_{k}\right) \rightarrow 0$,

$$
\begin{equation*}
u_{k}-R^{-1} f\left(u_{k}\right) \rightarrow 0 \text { in } H_{0}^{1}(U) \tag{40.2.13}
\end{equation*}
$$

Take inner product of the second term with $u_{k}$.

$$
\left(R^{-1} f\left(u_{k}\right), u_{k}\right) \equiv\left\langle f\left(u_{k}\right), u_{k}\right\rangle_{H^{-1}, H_{0}^{1}}=\int_{U} f\left(u_{k}\right) u_{k} d x
$$

Then by assumption, for $\varepsilon>0$, and all $k$ large enough,

$$
\left|\left(I^{\prime}\left(u_{k}\right), u_{k}\right)\right| \leq\left|\left\|u_{k}\right\|_{H_{0}^{1}(U)}^{2}-\int_{U} f\left(u_{k}\right) u_{k} d x\right| \leq \varepsilon\left\|u_{k}\right\|_{H_{0}^{1}(U)}
$$

Then also for large $k$, letting $\varepsilon=1$,

$$
\left|\int_{U} f\left(u_{k}\right) u_{k} d x\right| \leq\left\|u_{k}\right\|_{H_{0}^{1}(U)}^{2}+\left\|u_{k}\right\|_{H_{0}^{1}(U)}
$$

Now from the estimates assumed and 40.2.12,

$$
\begin{aligned}
\frac{1}{2}\left\|u_{k}\right\|_{H_{0}^{1}}^{2} & \leq C+\int_{U} F\left(u_{k}\right) d x \leq C+\gamma \int_{U} f\left(u_{k}\right) u_{k} d x \\
& \leq C+\gamma\left(\left\|u_{k}\right\|_{H_{0}^{1}(U)}^{2}+\left\|u_{k}\right\|_{H_{0}^{1}(U)}\right)
\end{aligned}
$$

and since $\gamma<1 / 2$,

$$
\left(\frac{1}{2}-\gamma\right)\left\|u_{k}\right\|_{H_{0}^{1}}^{2} \leq C+\left\|u_{k}\right\|_{H_{0}^{1}(U)}
$$

and so $\left\|u_{k}\right\|_{H_{0}^{1}(U)}$ is bounded. Hence it has a subsequence still denoted as $u_{k}$ which converges weakly in $H_{0}^{1}(U)$ to $u \in H_{0}^{1}(U)$. Since $p<\frac{n+2}{n-2}$, it follows that

$$
p+1<\frac{n+2}{n-2}+1=\frac{2 n}{n-2}
$$

and so by compactness of the embedding, it follows that $u_{k} \rightarrow u$ strongly in $L^{p+1}(U)$. We can assume convergence also takes place pointwise by taking a suitable subsequence.

Now $|f(u) v| \leq C\left(1+|u|^{p}\right)|v|$. Therefore, adjusting the constants,

$$
\begin{aligned}
|f(u) v| & \leq C\left(1+|u|^{p}\right)|v| \leq C\left(1+|u|^{p+1}\right)^{p /(p+1)}|v| \\
\left|\int_{U} f(u) v d x\right| & \leq C\left(\int_{U}\left(1+|u|^{p+1}\right)\right)^{p /(p+1)}\left(\int_{U}|v|^{p+1}\right)^{1 /(p+1)} \\
& \leq C\left(\int_{U}\left(1+|u|^{p+1}\right)\right)^{p /(p+1)}\|v\|_{H_{0}^{1}(U)}
\end{aligned}
$$

and so

$$
\|f(u)\|_{H^{-1}(U)} \leq\|f(u)\|_{L^{p+1}(U)} \leq C\left(\int_{U}\left(1+|u|^{p+1}\right)\right)^{p /(p+1)}
$$

It follows that

$$
f\left(u_{k}\right) \rightarrow f(u) \text { pointwise }
$$

and also

$$
\left|f\left(u_{k}\right)-f(u)\right|^{p+1} \leq C_{p}\left(\left|f\left(u_{k}\right)\right|^{p+1}+|f(u)|^{p+1}\right)
$$

where

$$
\lim _{k \rightarrow \infty} \int_{U}\left(\left|f\left(u_{k}\right)\right|^{p+1}+|f(u)|^{p+1}\right) d x=\int_{U} 2|f(u)|^{p+1} d x
$$

then by the dominated convergence theorem or more precisely Corollary 11.4.10,

$$
\lim _{k \rightarrow \infty}\left(\int_{U}\left|f\left(u_{k}\right)-f(u)\right|^{p+1}\right)^{1 /(p+1)}=0
$$

It follows that $f\left(u_{k}\right) \rightarrow f(u)$ in $H^{-1}(U)$. Hence $R^{-1} f\left(u_{k}\right) \rightarrow R^{-1} f(u)$ in $H_{0}^{1}(U)$ and so from 40.2.13, $u_{k} \rightarrow u$ strongly in $H_{0}^{1}(U)$ also. Thus $\left\{u_{k}\right\}$ is precompact. This verifies the Palais Smale conditions.

## mountain pass conditions

It is clear that $I(0)=0$. It remains to verify that for some $r>0, I(u) \geq a>0$ whenever $\|u\|_{H_{0}^{1}(U)}=r$ and for some $v$ with $\|v\|>r, I(v)=0$. Now consider $r u$ where $\|u\|=1$.

$$
I(r u)=\frac{1}{2} r^{2}-\int_{U} F(r u) d x
$$

From the assumed estimates and Sobolev embedding,

$$
\begin{aligned}
I(r u) & \geq \frac{1}{2} r^{2}-\int_{U} A|u|^{p+1} r^{p+1} d x \geq \frac{1}{2} r^{2}-C A r^{p+1}\|u\|_{H_{0}^{1}(U)}^{p+1} \\
& =\frac{1}{2} r^{2}-C A r^{p+1}
\end{aligned}
$$

Now this is independent of $u$ such that $\|u\|=1$. Then the derivative of the right side is

$$
r-(p+1) C A r^{p}
$$

where $p>1$. Thus this is positive for a while and then when $r$ is larger, it becomes negative. Thus there is $r_{0}>0$ where $\frac{1}{2} r_{0}^{2}-C A r_{0}^{p+1} \equiv a>0$. Hence when $\|u\|=r_{0}$, you have $I(u) \geq$ $a>0$. This is part of the mountain pass conditions. Now consider the other part. Letting $\|u\|=1$ be fixed, the estimates imply

$$
I(r u) \leq \frac{1}{2} r^{2}-\int_{U} \alpha|u|^{p+1} r^{p+1} d x \leq \frac{1}{2} r^{2}-r^{p+1} C
$$

Hence, for $r$ large enough, the right side becomes negative because $p+1>2$. Therefore, $r \rightarrow I(r u)$ is positive for small $r$ and is eventually negative as $r$ gets larger. hence there is some value of $r$ where this equals 0 . Then $v=r u$. This verifies the conditions for the mountain pass theorem.

## conclusions

It follows from the mountain pass theorem that there is some $u \neq 0$ such that $I^{\prime}(u)=0$. From the above computations,

$$
u-R^{-1} f(u)=0
$$

Now $R=-\Delta$ the Laplacian. In terms of weak derivatives,

$$
\langle-\Delta u, v\rangle_{H^{-1}, H_{0}^{1}}=\int_{U} \nabla u \cdot \nabla v d x=(u, v)_{H_{0}^{1}(U)}
$$

and so in terms of weak derivatives,

$$
-\Delta u=f(u) \text { in } H^{-1}(U), u \in H_{0}^{1}(U) \text { so } u=0 \text { on } \partial U
$$

This proves the following theorem.
Theorem 40.2.2 Suppose the conditions 40.2.8-40.2.10 hold. Then there exists a nonzero $u \in H_{0}^{1}(U)$ such that

$$
-\Delta u=f(u)
$$

One can verify that an example of such a function $f(u)$ is

$$
f(u)=|u|^{p-2} u
$$

This is very exciting to a large number of people because it gives an interesting example of non uniqueness of a boundary value problem. It is clear that $u=0$ works.

## Chapter 41

## Korn's Inequality

A fundamental inequality used in elasticity to obtain coercivity and then apply the Lax Milgram theorem or some other theorem is Korn's inequality. The proof given here of this fundamental result follows [101] and [46].

### 41.1 A Fundamental Inequality

The proof of Korn's inequality depends on a fundamental inequality involving negative Sobolev space norms. The theorem to be proved is the following.

Theorem 41.1.1 Let $f \in L^{2}(\Omega)$ where $\Omega$ is a bounded Lipschitz domain. Then there exist constants, $C_{1}$ and $C_{2}$ such that

$$
C_{1}\|f\|_{0,2, \Omega} \leq\left(\|f\|_{-1,2, \Omega}+\sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_{i}}\right\|_{-1,2, \Omega}\right) \leq C_{2}\|f\|_{0,2, \Omega}
$$

where here $\|\cdot\|_{0,2, \Omega}$ represents the $L^{2}$ norm and $\|\cdot\|_{-1,2, \Omega}$ represents the norm in the dual space of $H_{0}^{1}(\Omega)$, denoted by $H^{-1}(\Omega)$.

Similar conventions will apply for any domain in place of $\Omega$. The proof of this theorem will proceed through the use of several lemmas.

Lemma 41.1.2 Let $U^{-}$denote the set,

$$
\left\{\left(\mathbf{x}, x_{n}\right) \in \mathbb{R}^{n}: x_{n}<g(\mathbf{x})\right\}
$$

where $g: \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ is Lipschitz and denote by $U^{+}$the set

$$
\left\{\left(\mathbf{x}, x_{n}\right) \in \mathbb{R}^{n}: x_{n}>g(\mathbf{x})\right\}
$$

Let $f \in L^{2}\left(U^{-}\right)$and extend $f$ to all of $\mathbb{R}^{n}$ in the following way.

$$
f\left(\mathbf{x}, x_{n}\right) \equiv-3 f\left(\mathbf{x}, 2 g(\mathbf{x})-x_{n}\right)+4 f\left(\mathbf{x}, 3 g(\mathbf{x})-2 x_{n}\right)
$$

Then there is a constant, $C_{g}$, depending on $g$ such that

$$
\|f\|_{-1,2, \mathbb{R}^{n}}+\sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_{i}}\right\|_{-1,2, \mathbb{R}^{n}} \leq C_{g}\left(\|f\|_{-1,2, U^{-}}+\sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_{i}}\right\| \|_{-1,2, U^{-}}\right)
$$

Proof: Let $\phi \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$. Then,

$$
\begin{align*}
\int_{\mathbb{R}^{n}} f \frac{\partial \phi}{\partial x_{n}} d x=\int_{U^{+}} \frac{\partial \phi}{\partial x_{n}}[ & \left.-3 f\left(\mathbf{x}, 2 g(\mathbf{x})-x_{n}\right)+4 f\left(\mathbf{x}, 3 g(\mathbf{x})-2 x_{n}\right)\right] d x \\
& +\int_{U^{-}} f \frac{\partial \phi}{\partial x_{n}} d x \tag{41.1.1}
\end{align*}
$$

Consider the first integral on the right in 41.1.1. Changing the variables, letting

$$
y_{n}=2 g(\mathbf{x})-x_{n}
$$

in the first term of the integrand and $3 g(\mathbf{x})-2 x_{n}$ in the next, it equals

$$
\begin{aligned}
& -3 \int_{U^{-}} \frac{\partial \phi}{\partial x_{n}}\left(\mathbf{x}, 2 g(\mathbf{x})-y_{n}\right) f\left(\mathbf{x}, y_{n}\right) d y_{n} d x \\
+ & 2 \int_{U^{-}} \frac{\partial \phi}{\partial x_{n}}\left(\mathbf{x}, \frac{3}{2} g(\mathbf{x})-\frac{y_{n}}{2}\right) f\left(\mathbf{x}, y_{n}\right) d y_{n} d x
\end{aligned}
$$

For $\left(\mathbf{x}, y_{n}\right) \in U^{-}$, and defining

$$
\psi\left(\mathbf{x}, y_{n}\right) \equiv \phi\left(\mathbf{x}, y_{n}\right)+3 \phi\left(\mathbf{x}, 2 g(\mathbf{x})-y_{n}\right)-4 \phi\left(\mathbf{x}, \frac{3}{2} g(\mathbf{x})-\frac{y_{n}}{2}\right)
$$

it follows $\psi=0$ when $y_{n}=g(\mathbf{x})$ and so

$$
\int_{\mathbb{R}^{n}} f \frac{\partial \phi}{\partial x_{n}} d x=\int_{U^{-}} \frac{\partial \psi}{\partial y_{n}} f\left(\mathbf{x}, y_{n}\right) d x d y_{n}
$$

Now from the definition of $\psi$ given above,

$$
\|\psi\|_{1,2, U^{-}} \leq C_{g}\|\phi\|_{1,2, U^{-}} \leq C_{g}\|\phi\|_{1,2, \mathbb{R}^{n}}
$$

and so

$$
\begin{gather*}
\|\left.\frac{\partial f}{\partial x_{n}}\right|_{-1,2, \mathbb{R}^{n}} \equiv \\
\sup \left\{\int_{\mathbb{R}^{n}} f \frac{\partial \phi}{\partial x_{n}} d x: \phi \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right),\|\phi\|_{1,2, \mathbb{R}^{n}} \leq 1\right\} \leq \\
\sup \left\{\left|\int_{U^{-}} f \frac{\partial \psi}{\partial x_{n}} d x d y_{n}\right|: \psi \in H_{0}^{1}\left(U^{-}\right),\|\psi\|_{1,2, U^{-}} \leq C_{g}\right\} \\
=C_{g}| | \frac{\partial f}{\partial x_{n}} \|\left.\right|_{-1,2, U^{-}} \tag{41.1.2}
\end{gather*}
$$

It remains to establish a similar inequality for the case where the derivatives are taken with respect to $x_{i}$ for $i<n$. Let $\phi \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$. Then

$$
\begin{gathered}
\int_{\mathbb{R}^{n}} f \frac{\partial \phi}{\partial x_{i}} d x=\int_{U^{-}} f \frac{\partial \phi}{\partial x_{i}} d x \\
\int_{U^{+}} \frac{\partial \phi}{\partial x_{i}}\left[-3 f\left(\mathbf{x}, g(\mathbf{x})-x_{n}\right)+4 f\left(\mathbf{x}, 3 g(\mathbf{x})-2 x_{n}\right)\right] d x .
\end{gathered}
$$

Changing the variables as before, this last integral equals

$$
-3 \int_{U^{-}} D_{i} \phi\left(\mathbf{x}, 2 g(\mathbf{x})-y_{n}\right) f\left(\mathbf{x}, y_{n}\right) d y_{n} d x
$$

$$
\begin{equation*}
+2 \int_{U^{-}} D_{i} \phi\left(\mathbf{x}, \frac{3}{2} g(\mathbf{x})-\frac{y_{n}}{2}\right) f\left(\mathbf{x}, y_{n}\right) d y_{n} d x \tag{41.1.3}
\end{equation*}
$$

Now let

$$
\psi_{1}\left(\mathbf{x}, y_{n}\right) \equiv \phi\left(\mathbf{x}, 2 g(\mathbf{x})-y_{n}\right), \psi_{2}\left(\mathbf{x}, y_{n}\right) \equiv \phi\left(\mathbf{x}, \frac{3}{2} g(\mathbf{x})-\frac{y_{n}}{2}\right)
$$

Then

$$
\begin{gathered}
\frac{\partial \psi_{1}}{\partial x_{i}}=D_{i} \phi\left(\mathbf{x}, 2 g(\mathbf{x})-y_{n}\right)+D_{n} \phi\left(\mathbf{x}, 2 g(\mathbf{x})-y_{n}\right) 2 D_{i} g(\mathbf{x}), \\
\frac{\partial \psi_{2}}{\partial x_{i}}=D_{i} \phi\left(\mathbf{x}, \frac{3}{2} g(\mathbf{x})-\frac{y_{n}}{2}\right)+D_{n} \phi\left(\mathbf{x}, \frac{3}{2} g(\mathbf{x})-\frac{y_{n}}{2}\right) \frac{3}{2} D_{i} g(\mathbf{x}) .
\end{gathered}
$$

Also

$$
\begin{gathered}
\frac{\partial \psi_{1}}{\partial y_{n}}\left(\mathbf{x}, y_{n}\right)=-D_{n} \phi\left(\mathbf{x}, 2 g(\mathbf{x})-y_{n}\right) \\
\frac{\partial \psi_{2}}{\partial y_{n}}\left(\mathbf{x}, y_{n}\right)=\left(\frac{-1}{2}\right) D_{n} \phi\left(\mathbf{x}, \frac{3}{2} g(\mathbf{x})-\frac{y_{n}}{2}\right)
\end{gathered}
$$

Therefore,

$$
\begin{gathered}
\frac{\partial \psi_{1}}{\partial x_{i}}\left(\mathbf{x}, y_{n}\right)=D_{i} \phi\left(\mathbf{x}, 2 g(\mathbf{x})-y_{n}\right)-2 \frac{\partial \psi_{1}}{\partial y_{n}}\left(\mathbf{x}, y_{n}\right) D_{i} g(\mathbf{x}), \\
\frac{\partial \psi_{2}}{\partial x_{i}}\left(\mathbf{x}, y_{n}\right)=D_{i} \phi\left(\mathbf{x}, \frac{3}{2} g(\mathbf{x})-\frac{y_{n}}{2}\right)-3 \frac{\partial \psi_{2}}{\partial y_{n}}\left(\mathbf{x}, y_{n}\right) D_{i} g(\mathbf{x}) .
\end{gathered}
$$

Using this in 41.1.3, the integrals in this expression equal

$$
\begin{gathered}
-3 \int_{U^{-}}\left[\frac{\partial \psi_{1}}{\partial x_{i}}\left(\mathbf{x}, y_{n}\right)+2 \frac{\partial \psi_{1}}{\partial y_{n}}\left(\mathbf{x}, y_{n}\right) D_{i} g(\mathbf{x})\right] f\left(\mathbf{x}, y_{n}\right) d y_{n} d x+ \\
2 \int_{U^{-}}\left[\frac{\partial \psi_{2}}{\partial x_{i}}\left(\mathbf{x}, y_{n}\right)+3 \frac{\partial \psi_{2}}{\partial y_{n}}\left(\mathbf{x}, y_{n}\right) D_{i} g(\mathbf{x})\right] f\left(\mathbf{x}, y_{n}\right) d y_{n} d x \\
=\int_{U^{-}}\left[-3 \frac{\partial \psi_{1}(\mathbf{x}, y)}{\partial x_{i}}+2 \frac{\partial \psi_{2}\left(\mathbf{x}, y_{n}\right)}{\partial x_{i}}\right] f\left(\mathbf{x}, y_{n}\right) d y_{n} d x
\end{gathered}
$$

Therefore,

$$
\int_{\mathbb{R}^{n}} \frac{\partial \phi}{\partial x_{i}} f d x=\int_{U^{-}}\left[\frac{\partial \phi}{\partial x_{i}}-3 \frac{\partial \psi_{1}}{\partial x_{i}}+2 \frac{\partial \psi_{2}}{\partial x_{i}}\right] f d x d y_{n}
$$

and also

$$
\begin{aligned}
& \phi(\mathbf{x}, g(\mathbf{x}))-3 \psi_{1}(\mathbf{x}, g(\mathbf{x}))+2 \psi_{2}(\mathbf{x}, g(\mathbf{x}))= \\
& \phi(\mathbf{x}, g(\mathbf{x}))-3 \phi(\mathbf{x}, g(\mathbf{x}))+2 \phi(\mathbf{x}, g(\mathbf{x}))=0
\end{aligned}
$$

and so $\phi-3 \psi_{1}+2 \psi_{2} \in H_{0}^{1}\left(U^{-}\right)$. It also follows from the definition of the functions, $\psi_{i}$ and the assumption that $g$ is Lipschitz, that

$$
\begin{equation*}
\left\|\psi_{i}\right\|_{1,2, U^{-}} \leq C_{g}\|\phi\|_{1,2, U^{-}} \leq C_{g}\|\phi\|_{1,2, \mathbb{R}^{n}} \tag{41.1.4}
\end{equation*}
$$

Therefore,

$$
\begin{gathered}
\| \frac{\partial f}{\partial x_{i}}| |_{-1,2, \mathbb{R}^{n}} \equiv \sup \left\{\left|\int_{\mathbb{R}^{n}} f \frac{\partial \phi}{\partial x_{i}} d x\right|:\|\phi\|_{1,2, \mathbb{R}^{n}} \leq 1\right\} \\
=\sup \left\{\left|\int_{U^{-}} f\left[\frac{\partial \phi}{\partial x_{i}}-3 \frac{\partial \psi_{1}}{\partial x_{i}}+2 \frac{\partial \psi_{2}}{\partial x_{i}}\right] d x\right|:\|\phi\|_{1,2, \mathbb{R}^{n}} \leq 1\right\} \\
\leq C_{g}| | \frac{\partial f}{\partial x_{i}}| |_{-1,2, U^{-}}
\end{gathered}
$$

where $C_{g}$ is a constant which depends on $g$. This inequality along with 41.1 .2 yields

$$
\sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_{i}}\right\|_{-1,2, \mathbb{R}^{n}} \leq C_{g}\left(\sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_{i}}\right\|_{-1,2, U^{-}}\right)
$$

The inequality,

$$
\|f\|_{-1,2, \mathbb{R}^{n}} \leq C_{g}\|f\|_{-1,2, U^{-}}
$$

follows from 41.1.4 and the equation,

$$
\begin{aligned}
\int_{\mathbb{R}^{n}} f \phi d x= & \int_{U^{-}} f \phi d x-3 \int_{U^{-}} f\left(\mathbf{x}, y_{n}\right) \psi_{1}\left(\mathbf{x}, y_{n}\right) d x d y_{n} \\
& +2 \int_{U^{-}} f\left(\mathbf{x}, y_{n}\right) \psi_{2}\left(\mathbf{x}, y_{n}\right) d x d y_{n}
\end{aligned}
$$

which results in the same way as before by changing variables using the definition of $f$ off $U^{-}$. This proves the lemma.

The next lemma is a simple application of Fourier transforms.
Lemma 41.1.3 If $f \in L^{2}\left(\mathbb{R}^{n}\right)$, then the following formula holds.

$$
C_{n}\|f\|_{0,2, \mathbb{R}^{n}}=\sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_{i}}\right\|_{-1,2, \mathbb{R}^{n}}+\|f\|_{-1,2, \mathbb{R}^{n}}
$$

Proof: For $\phi \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$

$$
\|\phi\|_{1,2, \mathbb{R}^{n}} \equiv\left(\int_{\mathbb{R}^{n}}\left(1+|\mathbf{t}|^{2}\right)|F \phi|^{2} d t\right)^{1 / 2}
$$

is an equivalent norm to the usual Sobolev space norm for $H_{0}^{1}\left(\mathbb{R}^{n}\right)$ and is used in the following argument which depends on Plancherel's theorem and the fact that $F\left(\frac{\partial \phi}{\partial x_{i}}\right)=$ $t_{i} F(\phi)$.

$$
\begin{aligned}
& \| \frac{\partial f}{\partial x_{i}}| |_{-1,2, \mathbb{R}^{n}} \equiv \sup \left\{\left|\int_{\mathbb{R}^{n}} \frac{\partial \phi}{\partial x_{i}} \bar{f} d x\right|:\|\phi\|_{1,2} \leq 1\right\} \\
& \quad=C_{n} \sup \left\{\left|\int_{\mathbb{R}^{n}} t_{i}(F \phi) \overline{(F f)} d t\right|:\|\phi\|_{1,2} \leq 1\right\}
\end{aligned}
$$

$$
\begin{gather*}
=C_{n} \sup \left\{\left|\int_{\mathbb{R}^{n}} \frac{t_{i}(F \phi)\left(1+|\mathbf{t}|^{2}\right)^{1 / 2}}{\left(1+|\mathbf{t}|^{2}\right)^{1 / 2}} \overline{(F f)} d t\right|:\|\phi\|_{1,2} \leq 1\right\} \\
=C_{n}\left(\int \frac{|F f|^{2} t_{i}^{2}}{\left(1+|\mathbf{t}|^{2}\right)} d t\right)^{1 / 2} \tag{41.1.5}
\end{gather*}
$$

Also,

$$
\begin{gathered}
\|f\|_{-1,2} \equiv \sup \left\{\left|\int_{\mathbb{R}^{n}} \phi \bar{f} d x\right|:\|\phi\|_{1,2} \leq 1\right\} \\
=C_{n} \sup \left\{\left|\int_{\mathbb{R}^{n}}(F \phi)(\overline{F f}) d x\right|:\|\phi\|_{1,2} \leq 1\right\} \\
=C_{n} \sup \left\{\left|\int_{\mathbb{R}^{n}} \frac{F \phi\left(1+|\mathbf{t}|^{2}\right)^{1 / 2}}{\left(1+|\mathbf{t}|^{2}\right)^{1 / 2}} \overline{(F f)} d t\right|:\|\phi\|_{1,2} \leq 1\right\} \\
=C_{n}\left(\int_{\mathbb{R}^{n}} \frac{|F f|^{2}}{\left(1+|\mathbf{t}|^{2}\right)} d t\right)^{1 / 2}
\end{gathered}
$$

This along with 41.1.5 yields the conclusion of the lemma because

$$
\sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_{i}}\right\|_{-1,2}^{2}+\|f\|_{-1,2}^{2}=C_{n} \int_{\mathbb{R}^{n}}|F f|^{2} d x=C_{n}\|f\|_{0,2}^{2}
$$

Now consider Theorem 41.1.1. First note that by Lemma 41.1.2 and $U^{-}$defined there, Lemma 41.1.3 implies that for $f$ extended as in Lemma 41.1.2,

$$
\begin{align*}
\|f\|_{0,2, U^{-}} & \leq\|f\|_{0,2, \mathbb{R}^{n}}=C_{n}\left(\|f\|_{-1,2, \mathbb{R}^{n}}+\sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_{i}}\right\|_{-1,2, \mathbb{R}^{n}}\right) \\
& \leq C_{g n}\left(\|f\|_{-1,2, U^{-}}+\sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_{i}}\right\|_{-1,2, U^{-}}\right) . \tag{41.1.6}
\end{align*}
$$

Let $\Omega$ be a bounded open set having Lipschitz boundary which lies locally on one side of its boundary. Let $\left\{Q_{i}\right\}_{i=0}^{p}$ be cubes of the sort used in the proof of the divergence theorem such that $\overline{Q_{0}} \subseteq \Omega$ and the other cubes cover the boundary of $\Omega$. Let $\left\{\psi_{i}\right\}$ be a $C^{\infty}$ partition of unity with $\operatorname{spt}\left(\psi_{i}\right) \subseteq Q_{i}$ and let $f \in L^{2}(\Omega)$. Then for $\phi \in C_{c}^{\infty}(\Omega)$ and $\psi$ one of these functions in the partition of unity,

$$
\left|\left|\frac{\partial(f \psi)}{\partial x_{i}} \|_{-1,2, \Omega} \leq \sup _{\|\phi\|_{1,2} \leq 1}\right| \int_{\Omega} f \frac{\partial}{\partial x_{i}}(\psi \phi) d x\right|+\sup _{\|\phi\|_{1,2} \leq 1}\left|\int_{\Omega} f \phi \frac{\partial \psi}{\partial x_{i}} d x\right|
$$

Now if $\|\phi\|_{1,2} \leq 1$, then for a suitable constant, $C_{\psi}$,

$$
\|\psi \phi\|_{1,2} \leq C_{\psi}\|\phi\|_{1,2} \leq C_{\psi},\left\|\phi \frac{\partial \psi}{\partial x_{i}}\right\|_{1,2} \leq C_{\psi}
$$

Therefore,

$$
\begin{gather*}
\left\|\frac{\partial(f \psi)}{\partial x_{i}}\right\| \|_{-1,2, \Omega} \leq \sup _{\|\eta\|_{1,2} \leq C_{\psi}}\left|\int_{\Omega} f \frac{\partial \eta}{\partial x_{i}} d x\right|+\sup _{\|\eta\|_{1,2} \leq C_{\psi}}\left|\int_{\Omega} f \eta d x\right| \\
\leq C_{\psi}\left(\left\|\frac{\partial f}{\partial x_{i}}\right\|_{-1,2, \Omega}+\|f\|_{-1,2, \Omega}\right) \tag{41.1.7}
\end{gather*}
$$

Now using 41.1.7 and 41.1.6

$$
\begin{gathered}
\left\|f \psi_{j}\right\|_{0,2, \Omega} \leq C_{g}\left(\left\|f \psi_{j}\right\|_{-1,2, \Omega}+\sum_{i=1}^{n}\left\|\frac{\partial\left(f \psi_{j}\right)}{\partial x_{i}}\right\|_{-1,2, \Omega}\right) \\
\leq C_{\psi_{j}} C_{g}\left(\|f\|_{-1,2, \Omega}+\sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_{i}}\right\|_{-1,2, \Omega}\right)
\end{gathered}
$$

Therefore, letting $C=\sum_{j=1}^{p} C_{\psi_{j}} C_{g}$,

$$
\begin{equation*}
\|f\|_{0,2, \Omega} \leq \sum_{j=1}^{p}\left\|f \psi_{j}\right\|_{0,2, \Omega} \leq C\left(\|f\|_{-1,2, \Omega}+\sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_{i}}\right\|_{-1,2, \Omega}\right) \tag{41.1.8}
\end{equation*}
$$

This proves the hard half of the inequality of Theorem 41.1.1.
To complete the proof, let $\bar{f}$ denote the zero extension of $f$ off $\Omega$. Then

$$
\begin{gathered}
\|f\|_{-1,2, \Omega}+\left.\sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_{i}}\right\|\right|_{-1,2, \Omega} \leq\|\bar{f}\|_{-1,2, \mathbb{R}^{n}}+\sum_{i=1}^{n}\left\|\frac{\partial \bar{f}}{\partial x_{i}}\right\| \|_{-1,2, \mathbb{R}^{n}} \\
\leq C_{n}\|\bar{f}\|_{0,2, \mathbb{R}^{n}}=C_{n}\|f\|_{0,2, \Omega}
\end{gathered}
$$

This along with 41.1.8 proves Theorem 41.1.1.

### 41.2 Korn's Inequality

The inequality in this section is known as Korn's second inequality. It is also known as coercivity of strains. For $\mathbf{u}$ a vector valued function in $\mathbb{R}^{n}$, define

$$
\varepsilon_{i j}(\mathbf{u}) \equiv \frac{1}{2}\left(u_{i, j}+u_{j, i}\right)
$$

This is known as the strain or small strain. Korn's inequality says that the norm given by,

$$
\begin{equation*}
\|\mid \mathbf{u}\| \equiv\left(\sum_{i=1}^{n}\left\|u_{i}\right\|_{0,2, \Omega}^{2}+\sum_{i=1}^{n} \sum_{j=1}^{n}\left\|\varepsilon_{i j}(\mathbf{u})\right\|_{0,2, \Omega}^{2}\right)^{1 / 2} \tag{41.2.9}
\end{equation*}
$$

is equivalent to the norm,

$$
\begin{equation*}
\|\mathbf{u}\| \equiv\left(\sum_{i=1}^{n}\left\|u_{i}\right\|_{0,2, \Omega}^{2}+\sum_{i=1}^{n} \sum_{j=1}^{n}\left\|\frac{\partial u_{i}}{\partial x_{j}}\right\|_{0,2, \Omega}^{2}\right)^{1 / 2} \tag{41.2.10}
\end{equation*}
$$

It is very significant because it is the strain as just defined which occurs in many of the physical models proposed in continuum mechanics. The inequality is far from obvious because the strains only involve certain combinations of partial derivatives.

Theorem 41.2.1 (Korn's second inequality) Let $\Omega$ be any domain for which the conclusion of Theorem 41.1.1 holds. Then the two norms in 41.2 .9 and 41.2.10 are equivalent.

Proof: Let $\mathbf{u}$ be such that $u_{i} \in H^{1}(\Omega)$ for each $i=1, \cdots, n$. Note that

$$
\frac{\partial^{2} u_{i}}{\partial x_{j}, \partial x_{k}}=\frac{\partial}{\partial x_{j}}\left(\varepsilon_{i k}(\mathbf{u})\right)+\frac{\partial}{\partial x_{k}}\left(\varepsilon_{i j}(\mathbf{u})\right)-\frac{\partial}{\partial x_{i}}\left(\varepsilon_{j k}(\mathbf{u})\right) .
$$

Therefore, by Theorem 41.1.1,

$$
\begin{aligned}
& \left\|\frac{\partial u_{i}}{\partial x_{j}}\right\|_{0,2, \Omega} \leq C\left[\left\|\frac{\partial u_{i}}{\partial x_{j}}\right\|_{-1,2, \Omega}+\sum_{k=1}^{n}\left\|\frac{\partial^{2} u_{i}}{\partial x_{j}, \partial x_{k}}\right\|_{-1,2, \Omega}\right] \\
& \quad \leq C\left[\left\|\frac{\partial u_{i}}{\partial x_{j}}\right\|_{-1,2, \Omega}+\sum_{r, s, p}\left\|\frac{\partial \varepsilon_{r s}(\mathbf{u})}{\partial x_{p}}\right\|_{-1,2, \Omega}\right] \\
& \quad \leq C\left[\left\|\frac{\partial u_{i}}{\partial x_{j}}\right\|_{-1,2, \Omega}+\sum_{r, s}\left\|\varepsilon_{r s}(\mathbf{u})\right\|_{0,2, \Omega}\right]
\end{aligned}
$$

But also by this theorem,

$$
\left\|u_{i}\right\|_{-1,2, \Omega}+\sum_{p}\left\|\frac{\partial u_{i}}{\partial x_{p}}\right\|_{-1,2, \Omega} \leq C\left\|u_{i}\right\|_{0,2, \Omega}
$$

and so

$$
\left\|\frac{\partial u_{i}}{\partial x_{j}}\right\|_{0,2, \Omega} \leq C\left[\left\|u_{i}\right\|_{0,2, \Omega}+\sum_{r, s}\left\|\varepsilon_{r s}(\mathbf{u})\right\|_{0,2, \Omega}\right]
$$

This proves the theorem.
Note that $\Omega$ did not need to be bounded. It suffices to be able to conclude the result of Theorem 41.1.1 which would hold whenever the boundary of $\Omega$ can be covered with finitely many boxes of the sort to which Lemma 41.1.2 can be applied.

## Chapter 42

## Elliptic Regularity

### 42.1 The Case Of A Half Space

Regularity theorems are concerned with obtaining more regularity given a weak solution. This extra regularity is essential in order to obtain error estimates for various problems. In this section a regularity is given for weak solutions to various elliptic boundary value problems. To save on notation, I will use the repeated index summation convention. Thus you sum over repeated indices. Consider the following picture.


Here $V$ is an open set,

$$
U \equiv\left\{\mathbf{y} \in V: y_{n}<0\right\}, \Gamma \equiv\left\{\mathbf{y} \in V: y_{n}=0\right\}
$$

and $U_{1}$ is an open set as shown for which $U_{1} \subseteq V \cap U$. Assume also that $V$ is bounded. Suppose

$$
\begin{gather*}
f \in L^{2}(U), \\
\alpha^{r s} \in C^{0,1}(\bar{U}),  \tag{42.1.1}\\
\alpha^{r s}(\mathbf{y}) v_{r} v_{s} \geq \delta|\mathbf{v}|^{2}, \delta>0 . \tag{42.1.2}
\end{gather*}
$$

The following technical lemma gives the essential ideas.
Lemma 42.1.1 Suppose

$$
\begin{align*}
w & \in H^{1}(U),  \tag{42.1.3}\\
\alpha^{r s} & \in C^{0,1}(\bar{U}),  \tag{42.1.4}\\
h_{s} & \in H^{1}(U),  \tag{42.1.5}\\
f & \in L^{2}(U) . \tag{42.1.6}
\end{align*}
$$

and

$$
\begin{equation*}
\int_{U} \alpha^{r s}(\mathbf{y}) \frac{\partial w}{\partial y^{r}} \frac{\partial z}{\partial y^{s}} d y+\int_{U} h_{s}(\mathbf{y}) \frac{\partial z}{\partial y^{s}} d y=\int_{U} f z d y \tag{42.1.7}
\end{equation*}
$$

for all $z \in H^{1}(U)$ having the property that $\operatorname{spt}(z) \subseteq V$. Then $w \in H^{2}\left(U_{1}\right)$ and for some constant $C$, independent of $f, w$, and $g$, the following estimate holds.

$$
\begin{equation*}
\|w\|_{H^{2}\left(U_{1}\right)}^{2} \leq C\left(\|w\|_{H^{1}(U)}^{2}+\|f\|_{L^{2}(U)}^{2}+\sum_{s}\left\|h_{s}\right\|_{H^{1}(U)}^{2}\right) \tag{42.1.8}
\end{equation*}
$$

Proof: Define for small real $h$,

$$
D_{k}^{h} l(\mathbf{y}) \equiv \frac{1}{h}\left(l\left(\mathbf{y}+h \mathbf{e}_{k}\right)-l(\mathbf{y})\right) .
$$

Let $U_{1} \subseteq \overline{U_{1}} \subseteq W \subseteq \bar{W} \subseteq V$ and let $\eta \in C_{c}^{\infty}(W)$ with $\eta(\mathbf{y}) \in[0,1]$, and $\eta=1$ on $\overline{U_{1}}$ as shown in the following picture.


For $h$ small $\left(3 h<\operatorname{dist}\left(\bar{W}, V^{C}\right)\right)$, let

$$
\begin{align*}
z(\mathbf{y}) \equiv & \frac{1}{h}\left\{\eta^{2}\left(\mathbf{y}-h \mathbf{e}_{k}\right)\left[\frac{w(\mathbf{y})-w\left(\mathbf{y}-h \mathbf{e}_{k}\right)}{h}\right]\right. \\
& \left.-\eta^{2}(\mathbf{y})\left[\frac{w\left(\mathbf{y}+h \mathbf{e}_{k}\right)-w(\mathbf{y})}{h}\right]\right\}  \tag{42.1.9}\\
\equiv & -D_{k}^{-h}\left(\eta^{2} D_{k}^{h} w\right) \tag{42.1.10}
\end{align*}
$$

where here $k<n$. Thus $z$ can be used in equation 42.1.7. Begin by estimating the left side of 42.1.7.

$$
\begin{aligned}
& \int_{U} \alpha^{r s}(\mathbf{y}) \frac{\partial w}{\partial y^{r}} \frac{\partial z}{\partial y^{s}} d y \\
= & \frac{1}{h} \int_{U} \alpha^{r s}\left(\mathbf{y}+h \mathbf{e}_{k}\right) \frac{\partial w}{\partial y^{r}}\left(\mathbf{y}+h \mathbf{e}_{k}\right) \frac{\partial\left(\eta^{2} D_{k}^{h} w\right)}{\partial y^{s}} d y \\
& -\frac{1}{h} \int_{U} \alpha^{r s}(\mathbf{y}) \frac{\partial w}{\partial y^{r}} \frac{\partial\left(\eta^{2} D_{k}^{h} w\right)}{\partial y^{s}} d y
\end{aligned}
$$

$$
\begin{gather*}
=\int_{U} \alpha^{r s}\left(\mathbf{y}+h \mathbf{e}_{k}\right) \frac{\partial\left(D_{k}^{h} w\right)}{\partial y^{r}} \frac{\partial\left(\eta^{2} D_{k}^{h} w\right)}{\partial y^{s}} d y+ \\
\frac{1}{h} \int_{U}\left(\alpha^{r s}\left(\mathbf{y}+h \mathbf{e}_{k}\right)-\alpha^{r s}(\mathbf{y})\right) \frac{\partial w}{\partial y^{r}} \frac{\partial\left(\eta^{2} D_{k}^{h} w\right)}{\partial y^{s}} d y \tag{42.1.11}
\end{gather*}
$$

Now

$$
\begin{equation*}
\frac{\partial\left(\eta^{2} D_{k}^{h} w\right)}{\partial y^{s}}=2 \eta \frac{\partial \eta}{\partial y^{s}} D_{k}^{h} w+\eta^{2} \frac{\partial\left(D_{k}^{h} w\right)}{\partial y^{s}} \tag{42.1.12}
\end{equation*}
$$

therefore,

$$
\begin{align*}
&= \int_{U} \eta^{2} \alpha^{r s}\left(\mathbf{y}+h \mathbf{e}_{k}\right) \frac{\partial\left(D_{k}^{h} w\right)}{\partial y^{r}} \frac{\partial\left(D_{k}^{h} w\right)}{\partial y^{s}} d y \\
&+\left\{\int_{W \cap U} \alpha^{r s}\left(\mathbf{y}+h \mathbf{e}_{k}\right) \frac{\partial\left(D_{k}^{h} w\right)}{\partial y^{r}} 2 \eta \frac{\partial \eta}{\partial y^{s}} D_{k}^{h} w d y\right. \\
&\left.+\frac{1}{h} \int_{W \cap U}\left(\alpha^{r s}\left(\mathbf{y}+h \mathbf{e}_{k}\right)-\alpha^{r s}(\mathbf{y})\right) \frac{\partial w}{\partial y^{r}} \frac{\partial\left(\eta^{2} D_{k}^{h} w\right)}{\partial y^{s}} d y\right\} \equiv A .+\{B .\} . \tag{42.1.13}
\end{align*}
$$

Now consider these two terms. From 42.1.2,

$$
\begin{equation*}
A . \geq \delta \int_{U} \eta^{2}\left|\nabla D_{k}^{h} w\right|^{2} d y \tag{42.1.14}
\end{equation*}
$$

Using the Lipschitz continuity of $\alpha^{r s}$ and 42.1.12,

$$
\begin{gather*}
B . \leq C(\eta, \operatorname{Lip}(\alpha), \alpha)\left\{\left\|D_{k}^{h} w\right\|_{L^{2}(W \cap U)}\left\|\eta \nabla D_{k}^{h} w\right\|_{L^{2}\left(W \cap U ; \mathbb{R}^{n}\right)}+\right. \\
\|\eta \nabla w\|_{L^{2}\left(W \cap U ; \mathbb{R}^{n}\right)}\left\|\eta \nabla D_{k}^{h} w\right\|_{L^{2}\left(W \cap U ; \mathbb{R}^{n}\right)} \\
\left.+\|\eta \nabla w\|_{L^{2}\left(W \cap U ; \mathbb{R}^{n}\right)}\left\|D_{k}^{h} w\right\|_{L^{2}(W \cap U)}\right\} .  \tag{42.1.15}\\
\leq C(\eta, \operatorname{Lip}(\alpha), \alpha) C_{\varepsilon}\left(\left\|D_{k}^{h} w\right\|_{L^{2}(W \cap U)}^{2}+\|\eta \nabla w\|_{L^{2}\left(W \cap U ; \mathbb{R}^{n}\right)}^{2}\right)+ \\
\varepsilon C(\eta, \operatorname{Lip}(\alpha), \alpha)\left(\left\|\eta \nabla D_{k}^{h} w\right\|_{L^{2}\left(W \cap U ; \mathbb{R}^{n}\right)}^{2}+\left\|D_{k}^{h} w\right\|_{L^{2}(W \cap U)}^{2}\right) . \tag{42.1.16}
\end{gather*}
$$

Now

$$
\begin{equation*}
\left\|D_{k}^{h} w\right\|_{L^{2}(W)} \leq\|\nabla w\|_{L^{2}\left(U ; \mathbb{R}^{n}\right)}^{2} . \tag{42.1.17}
\end{equation*}
$$

To see this, observe that if $w$ is smooth, then

$$
\begin{aligned}
& \left(\int_{W}\left|\frac{w\left(\mathbf{y}+h \mathbf{e}_{k}\right)-w(\mathbf{y})}{h}\right|^{2} d y\right)^{1 / 2} \\
\leq & \left(\int_{W}\left|\frac{1}{h} \int_{0}^{h} \nabla w\left(\mathbf{y}+t \mathbf{e}_{k}\right) \cdot \mathbf{e}_{k} d t\right|^{2} d y\right)^{1 / 2}
\end{aligned}
$$

$$
\leq\left(\int_{0}^{h}\left(\int_{W}\left|\nabla w\left(\mathbf{y}+t \mathbf{e}_{k}\right) \cdot \mathbf{e}_{k}\right|^{2} d y\right)^{1 / 2} \frac{d t}{h}\right) \leq\|\nabla w\|_{L^{2}\left(U ; \mathbb{R}^{n}\right)}
$$

so by density of such functions in $H^{1}(U), 42.1 .17$ holds. Therefore, changing $\varepsilon$, yields

$$
\begin{equation*}
B . \leq C_{\varepsilon}(\eta, \operatorname{Lip}(\alpha), \alpha)\|\nabla w\|_{L^{2}\left(U ; \mathbb{R}^{n}\right)}^{2}+\varepsilon\left\|\eta \nabla D_{k}^{h} w\right\|_{L^{2}\left(W \cap U ; \mathbb{R}^{n}\right)}^{2} \tag{42.1.18}
\end{equation*}
$$

With 42.1.14 and 42.1.18 established, consider the other terms of 42.1.7.

$$
\begin{align*}
& \left|\int_{U} f z d y\right| \\
& \leq\left|\int_{U} f\left(-D_{k}^{-h} \eta^{2} D_{k}^{h} w\right) d y\right| \\
& \leq\left(\int_{U}|f|^{2} d y\right)^{1 / 2}\left(\int_{U}\left|D_{k}^{-h}\left(\eta^{2} D_{k}^{h} w\right)\right|^{2} d y\right)^{1 / 2} \\
& \leq\|f\|_{L^{2}(U)}\left\|\nabla\left(\eta^{2} D_{k}^{h} w\right)\right\|_{L^{2}\left(U ; \mathbb{R}^{n}\right)} \\
& \leq\|f\|_{L^{2}(U)}\left(\left\|2 \eta \nabla \eta D_{k}^{h} w\right\|_{L^{2}\left(U ; \mathbb{R}^{n}\right)}+\left\|\eta^{2} \nabla D_{k}^{h} w\right\|_{L^{2}\left(U ; \mathbb{R}^{n}\right)}\right) \\
& \leq C\|f\|_{L^{2}(U)}| | \nabla w\left\|_{L^{2}\left(U ; \mathbb{R}^{n}\right)}+\right\| f\left\|_{L^{2}(U)} \mid\right\| \eta \nabla D_{k}^{h} w \|_{L^{2}\left(U ; \mathbb{R}^{n}\right)} \\
& \leq C_{\varepsilon}\left(\|f\|_{L^{2}(U)}^{2}+\|\nabla w\|_{L^{2}\left(U ; \mathbb{R}^{n}\right)}^{2}\right)+\varepsilon\left\|\eta \nabla D_{k}^{h} w\right\|_{L^{2}\left(U ; \mathbb{R}^{n}\right)}^{2}  \tag{42.1.19}\\
& \left|\int_{U} h_{s}(\mathbf{y}) \frac{\partial z}{\partial y^{s}} d y\right| \\
& \leq\left|\int_{U} h_{s}(\mathbf{y}) \frac{\partial\left(-D_{k}^{-h}\left(\eta^{2} D_{k}^{h} w\right)\right)}{\partial y^{s}} d y\right| \\
& \leq\left|\int_{U} D_{k}^{h} h_{s}(\mathbf{y}) \frac{\partial\left(\left(\eta^{2} D_{k}^{h} w\right)\right)}{\partial y^{s}}\right| \\
& \leq \int_{U}\left|D_{k}^{h} h_{s} 2 \eta \frac{\partial \eta}{\partial y^{s}} D_{k}^{h} w\right| d y+\int_{U}\left|\left(\eta D_{k}^{h} h_{s}\right)\left(\eta \frac{\partial\left(D_{k}^{h} w\right)}{\partial y^{s}}\right)\right| d y \\
& \leq C \sum_{s}\left\|h_{S}\right\|_{H^{1}(U)}\left(\|w\|_{H^{1}(U)}+\left\|\eta \nabla D_{k}^{h} w\right\|_{L^{2}\left(U ; \mathbb{R}^{n}\right)}\right) \\
& \leq C_{\varepsilon} \sum_{s}\left\|h_{s}\right\|_{H^{1}(U)}^{2}+\|w\|_{H^{1}(U)}^{2}+\varepsilon\left\|\eta \nabla D_{k}^{h} w\right\|_{L^{2}\left(U ; \mathbb{R}^{n}\right)}^{2} . \tag{42.1.20}
\end{align*}
$$

The following inequalities in 42.1.14,42.1.18, 42.1.19and 42.1.20 are summarized here.

$$
A . \geq \delta \int_{U} \eta^{2}\left|\nabla D_{k}^{h} w\right|^{2} d y
$$

$$
\begin{gathered}
B . \leq C_{\varepsilon}(\eta, \operatorname{Lip}(\alpha), \alpha)\|\nabla w\|_{L^{2}\left(U ; \mathbb{R}^{n}\right)}^{2}+\varepsilon\left\|\eta \nabla D_{k}^{h} w\right\|_{L^{2}\left(W \cap U ; \mathbb{R}^{n}\right)}^{2} \\
\left|\int_{U} f z d y\right| \leq C_{\varepsilon}\left(\|f\|_{L^{2}(U)}^{2}+\|\nabla w\|_{L^{2}\left(U ; \mathbb{R}^{n}\right)}^{2}\right)+\varepsilon\left\|\eta \nabla D_{k}^{h} w\right\|_{L^{2}\left(U ; \mathbb{R}^{n}\right)}^{2} \\
\left|\int_{U} h_{s}(\mathbf{y}) \frac{\partial z}{\partial y^{s}} d y\right| \leq
\end{gathered} \begin{gathered}
\leq \sum_{s}\left\|h_{s}\right\|_{H^{1}(U)}^{2} \\
+\|w\|_{H^{1}(U)}^{2}+\varepsilon\left\|\eta \nabla D_{k}^{h} w\right\|_{L^{2}\left(U ; \mathbb{R}^{n}\right)}^{2}
\end{gathered}
$$

Therefore,

$$
\begin{aligned}
& \delta\left\|\eta \nabla D_{k}^{h} w\right\|_{L^{2}\left(U ; \mathbb{R}^{n}\right)}^{2} \\
\leq & C_{\varepsilon}(\eta, \operatorname{Lip}(\alpha), \alpha)\|\nabla w\|_{L^{2}\left(U ; \mathbb{R}^{n}\right)}^{2}+\varepsilon\left\|\eta \nabla D_{k}^{h} w\right\|_{L^{2}\left(U ; \mathbb{R}^{n}\right)}^{2} \\
& +C_{\varepsilon} \sum_{s}\left\|h_{S}\right\|_{H^{1}(U)}^{2}+\|w\|_{H^{1}(U)}^{2}+\varepsilon\left\|\eta \nabla D_{k}^{h} w\right\|_{L^{2}\left(U ; \mathbb{R}^{n}\right)}^{2} \\
+ & C_{\varepsilon}\left(\|f\|_{L^{2}(U)}^{2}+\|\nabla w\|_{L^{2}\left(U ; \mathbb{R}^{n}\right)}^{2}\right)+\varepsilon\left\|\eta \nabla D_{k}^{h} w\right\|_{L^{2}\left(U ; \mathbb{R}^{n}\right)}^{2}
\end{aligned}
$$

Letting $\varepsilon$ be small enough and adjusting constants yields

$$
\begin{gathered}
\left\|\nabla D_{k}^{h} w\right\|_{L^{2}\left(U_{1} ; \mathbb{R}^{n}\right)}^{2} \leq\left\|\eta \nabla D_{k}^{h} w\right\|_{L^{2}\left(U ; \mathbb{R}^{n}\right)}^{2} \leq \\
C\left(\|w\|_{H^{1}(U)}^{2}+\|f\|_{L^{2}(U)}^{2}+C_{\varepsilon} \sum_{s}\left\|h_{s}\right\|_{H^{1}(U)}^{2}\right)
\end{gathered}
$$

where the constant, $C$, depends on $\eta, \operatorname{Lip}(\alpha), \alpha, \delta$. Since this holds for all $h$ small enough, it follows $\frac{\partial w}{\partial y^{k}} \in H^{1}\left(U_{1}\right)$ and

$$
\begin{gather*}
\left\|\nabla \frac{\partial w}{\partial y^{k}}\right\|_{L^{2}\left(U_{1} ; \mathbb{R}^{n}\right)}^{2} \leq \\
C\left(\|w\|_{H^{1}(U)}^{2}+\|f\|_{L^{2}(U)}^{2}+C_{\varepsilon} \sum_{s}\left\|h_{s}\right\|_{H^{1}(U)}^{2}\right) \tag{42.1.21}
\end{gather*}
$$

for each $k<n$. It remains to estimate $\left\|\frac{\partial^{2} w}{\partial y_{n}^{2}}\right\|_{L^{2}\left(U_{1}\right)}^{2}$. To do this return to 42.1 .7 which must hold for all $z \in C_{c}^{\infty}\left(U_{1}\right)$. Therefore, using 42.1.7 it follows that for all $z \in C_{c}^{\infty}\left(U_{1}\right)$,

$$
\int_{U} \alpha^{r s}(\mathbf{y}) \frac{\partial w}{\partial y^{r}} \frac{\partial z}{\partial y^{s}} d y=-\int_{U} \frac{\partial h_{s}}{\partial y^{s}} z d y+\int_{U} f z d y
$$

Now from the Lipschitz assumption on $\alpha^{r s}$, it follows

$$
\begin{aligned}
F \equiv & \sum_{r, s \leq n-1} \frac{\partial}{\partial y^{s}}\left(\alpha^{r s} \frac{\partial w}{\partial y^{r}}\right) \\
& +\sum_{\substack{s \leq n-1}} \frac{\partial}{\partial y^{s}}\left(\alpha^{n s} \frac{\partial w}{\partial y^{n}}\right)-\sum_{s} \frac{\partial h_{s}}{\partial y^{s}}+f \\
\in & L^{2}\left(U_{1}\right)
\end{aligned}
$$

and

$$
\begin{equation*}
\|F\|_{L^{2}\left(U_{1}\right)} \leq C\left(\|w\|_{H^{1}(U)}^{2}+\|f\|_{L^{2}(U)}^{2}+C_{\varepsilon} \sum_{s}\left\|h_{s}\right\|_{H^{1}(U)}^{2}\right) . \tag{42.1.22}
\end{equation*}
$$

Therefore, from density of $C_{c}^{\infty}\left(U_{1}\right)$ in $L^{2}\left(U_{1}\right)$,

$$
-\frac{\partial}{\partial y^{n}}\left(\alpha^{n n}(\mathbf{y}) \frac{\partial w}{\partial y^{n}}\right)=F, \text { no sum on } n
$$

and so

$$
-\frac{\partial \alpha^{n n}}{\partial y^{n}} \frac{\partial w}{\partial y^{n}}-\alpha^{n n} \frac{\partial^{2} w}{\partial\left(y^{n}\right)^{2}}=F
$$

By 42.1.2 $\alpha^{n n}(\mathbf{y}) \geq \delta$ and so it follows from 42.1.22 that there exists a constant, $C$ depending on $\delta$ such that

$$
\left|\frac{\partial^{2} w}{\partial\left(y^{n}\right)^{2}}\right|_{L^{2}\left(U_{1}\right)} \leq C\left(|F|_{L^{2}\left(U_{1}\right)}+\|w\|_{H^{1}(U)}\right)
$$

which with 42.1 .21 and 42.1 .22 implies the existence of a constant, $C$ depending on $\delta$ such that

$$
\|w\|_{H^{2}\left(U_{1}\right)}^{2} \leq C\left(\|w\|_{H^{1}(U)}^{2}+\|f\|_{L^{2}(U)}^{2}+C_{\varepsilon} \sum_{s}\left\|h_{S}\right\|_{H^{1}(U)}^{2}\right),
$$

proving the lemma.
What if more regularity is known for $f, h_{s}, \alpha^{r s}$ and $w$ ? Could more be said about the regularity of the solution? The answer is yes and is the content of the next corollary.

First here is some notation. For $\alpha$ a multi-index with $|\alpha|=k-1, \alpha=\left(\alpha_{1}, \cdots, \alpha_{n}\right)$ define

$$
D_{\alpha}^{h} l(\mathbf{y}) \equiv \prod_{k=1}^{n}\left(D_{k}^{h}\right)^{\alpha_{k}} l(\mathbf{y})
$$

Also, for $\alpha$ and $\tau$ multi indices, $\tau<\alpha$ means $\tau_{i}<\alpha_{i}$ for each $i$.
Corollary 42.1.2 Suppose in the context of Lemma 42.1.1 the following for $k \geq 1$.

$$
\begin{aligned}
w & \in H^{k}(U), \\
\alpha^{r s} & \in C^{k-1,1}(\bar{U}), \\
h_{s} & \in H^{k}(U), \\
f & \in H^{k-1}(U),
\end{aligned}
$$

and

$$
\begin{equation*}
\int_{U} \alpha^{r s}(\mathbf{y}) \frac{\partial w}{\partial y^{r}} \frac{\partial z}{\partial y^{s}} d y+\int_{U} h_{s}(\mathbf{y}) \frac{\partial z}{\partial y^{s}} d y=\int_{U} f z d y \tag{42.1.23}
\end{equation*}
$$

for all $z \in H^{1}(U)$ or $H_{0}^{1}(U)$ such that $\operatorname{spt}(z) \subseteq V$. Then there exists $C$ independent of $w$ such that

$$
\begin{equation*}
\|w\|_{H^{k+1}\left(U_{1}\right)} \leq C\left(\|f\|_{H^{k-1}(U)}+\sum_{s}\left\|h_{S}\right\|_{H^{k}(U)}+\|w\|_{H^{k}(U)}\right) . \tag{42.1.24}
\end{equation*}
$$

Proof: The proof involves the following claim which is proved using the conclusion of Lemma 42.1.1 on Page 1393.

Claim : If $\alpha=\left(\alpha^{\prime}, 0\right)$ where $\left|\alpha^{\prime}\right| \leq k-1$, then there exists a constant independent of $w$ such that

$$
\begin{equation*}
\left\|D^{\alpha} w\right\|_{H^{2}\left(U_{1}\right)} \leq C\left(\|f\|_{H^{k-1}(U)}+\sum_{s}\left\|h_{s}\right\|_{H^{k}(U)}+\|w\|_{H^{k}(U)}\right) . \tag{42.1.25}
\end{equation*}
$$

Proof of claim: First note that if $|\alpha|=0$, then 42.1.25 follows from Lemma 42.1.1 on Page 1393. Now suppose the conclusion of the claim holds for all $|\alpha| \leq j-1$ where $j<k$. Let $|\alpha|=j$ and $\alpha=\left(\alpha^{\prime}, 0\right)$. Then for $z \in H^{1}(U)$ having compact support in $V$, it follows that for $h$ small enough,

$$
D_{\alpha}^{-h} z \in H^{1}(U), \operatorname{spt}\left(D_{\alpha}^{h} z\right) \subseteq V
$$

Therefore, you can replace $z$ in 42.1.23 with $D_{\alpha}^{-h} z$. Now note that you can apply the following manipulation.

$$
\int_{U} p(\mathbf{y}) D_{\alpha}^{-h} z(\mathbf{y}) d y=\int_{U} D_{\alpha}^{h} p(\mathbf{y}) z(\mathbf{y}) d y
$$

and obtain

$$
\begin{equation*}
\int_{U}\left(D_{\alpha}^{h}\left(\alpha^{r s} \frac{\partial w}{\partial y^{r}}\right) \frac{\partial z}{\partial y^{s}}+D_{\alpha}^{h}\left(h_{s}\right) \frac{\partial z}{\partial y^{s}}\right) d y=\int_{U}\left(\left(D_{\alpha}^{h} f\right) z\right) d y \tag{42.1.26}
\end{equation*}
$$

Letting $h \rightarrow 0$, this gives

$$
\int_{U}\left(D^{\alpha}\left(\alpha^{r s} \frac{\partial w}{\partial y^{r}}\right) \frac{\partial z}{\partial y^{s}}+D^{\alpha}\left(h_{s}\right) \frac{\partial z}{\partial y^{s}}\right) d y=\int_{U}\left(\left(D^{\alpha} f\right) z\right) d y
$$

Now

$$
D^{\alpha}\left(\alpha^{r s} \frac{\partial w}{\partial y^{r}}\right)=\alpha^{r s} \frac{\partial\left(D^{\alpha} w\right)}{\partial y^{r}}+\sum_{\tau<\alpha} C(\tau) D^{\alpha-\tau}\left(\alpha^{r s}\right) \frac{\partial\left(D^{\tau} w\right)}{\partial y^{r}}
$$

where $C(\tau)$ is some coefficient. Therefore, from 42.1.26,

$$
\int_{U} \alpha^{r s} \frac{\partial\left(D^{\alpha} w\right)}{\partial y^{r}} \frac{\partial z}{\partial y^{s}} d y+\int_{U}\left(\sum_{\tau<\alpha} C(\tau) D^{\alpha-\tau}\left(\alpha^{r s}\right) \frac{\partial\left(D^{\tau} w\right)}{\partial y^{r}}+D^{\alpha}\left(h_{s}\right)\right) \frac{\partial z}{\partial y^{s}} d y
$$

$$
\begin{equation*}
=\int_{U}\left(D^{\alpha} f\right) z d y \tag{42.1.27}
\end{equation*}
$$

Let $\widehat{U}_{1}$ be as indicated in the following picture


Now apply the induction hypothesis to $\widehat{U}_{1}$ in order to write

$$
\begin{gathered}
\left\|\frac{\partial\left(D^{\tau} w\right)}{\partial y^{r}}\right\|_{H^{1}\left(\widehat{U}_{1}\right)} \leq\left\|D^{\tau} w\right\|_{H^{2}\left(\widehat{U}_{1}\right)} \\
\leq C\left(\|f\|_{H^{k-1}(U)}+\sum_{s}\left\|h_{s}\right\|_{H^{k}(U)}+\|w\|_{H^{k}(U)}\right)
\end{gathered}
$$

Since $\alpha^{r s} \in C^{k-1,1}(\bar{U})$, it follows that each term from the sum in 42.1.27 satisfies an inequality of the form

$$
\begin{gathered}
\left\|C(\tau) D^{\alpha-\tau}\left(\alpha^{r s}\right) \frac{\partial\left(D^{\tau} w\right)}{\partial y^{r}}\right\|_{H^{1}\left(\widehat{U}_{1}\right)} \leq \\
C\left(\|f\|_{H^{k-1}(U)}+\sum_{s}\left\|h_{S}\right\|_{H^{k}(U)}+\|w\|_{H^{k}(U)}\right)
\end{gathered}
$$

and consequently,

$$
\begin{gather*}
\left\|\sum_{\tau<\alpha} C(\tau) D^{\alpha-\tau}\left(\alpha^{r s}\right) \frac{\partial\left(D^{\tau} w\right)}{\partial y^{r}}+D^{\alpha}\left(h_{s}\right)\right\|_{H^{1}\left(\widehat{U}_{1}\right)} \leq \\
C\left(\|f\|_{H^{k-1}(U)}+\sum_{s}\left\|h_{S}\right\|_{H^{k}(U)}+\|w\|_{H^{k}(U)}\right) . \tag{42.1.28}
\end{gather*}
$$

Now consider 42.1.27. The equation remains true if you replace $U$ with $\widehat{U}_{1}$ and require that $\operatorname{spt}(z) \subseteq \widehat{U}_{1}$. Therefore, by Lemma 42.1.1 on Page 1393 there exists a constant, $C$ independent of $w$ such that

$$
\left\|D^{\alpha} w\right\|_{H^{2}\left(U_{1}\right)} \leq C\left(\left\|D^{\alpha} f\right\|_{L^{2}\left(\widehat{U}_{1}\right)}+\left\|D^{\alpha} w\right\|_{H^{1}\left(\widehat{U}_{1}\right)}+\right.
$$

$$
\left.+\sum_{s}\left\|\sum_{\tau<\alpha} C(\tau) D^{\alpha-\tau}\left(\alpha^{r s}\right) \frac{\partial\left(D^{\tau} w\right)}{\partial y^{r}}+D^{\alpha}\left(h_{s}\right)\right\|_{H^{1}\left(\widehat{U}_{1}\right)}\right)
$$

and by 42.1.28, this implies

$$
\left\|D^{\alpha} w\right\|_{H^{2}\left(U_{1}\right)} \leq C\left(\|f\|_{H^{k-1}(U)}+\|w\|_{H^{k}(U)}+\sum_{s}\left\|h_{S}\right\|_{H^{k}(U)}\right)
$$

which proves the Claim.
To establish 42.1 .24 it only remains to verify that if $|\alpha| \leq k+1$, then

$$
\begin{equation*}
\left\|D^{\alpha} w\right\|_{L^{2}\left(U_{1}\right)} \leq C\left(\|f\|_{H^{k-1}(U)}+\|w\|_{H^{k}(U)}+\sum_{s}\left\|h_{S}\right\|_{H^{k}(U)}\right) . \tag{42.1.29}
\end{equation*}
$$

If $|\alpha|<k+1$, there is nothing to show because it is given that $w \in H^{k}(U)$. Therefore, assume $|\alpha|=k+1$. If $\alpha_{n}$ equals 0 the conclusion follows from the claim because in this case, you can subtract 1 from a pair of positive $\alpha_{i}$ and obtain a new multi index, $\beta$ such that $|\beta|=k-1$ and $\beta_{n}=0$ and then from the claim,

$$
\left\|D^{\alpha} w\right\|_{L^{2}\left(U_{1}\right)} \leq\left\|D^{\beta}\right\|_{H^{2}\left(U_{1}\right)} \leq C\left(\|f\|_{H^{k-1}(U)}+\|w\|_{H^{k}(U)}+\sum_{s}\left\|h_{S}\right\|_{H^{k}(U)}\right)
$$

If $\alpha_{n}=1$, then subtract 1 from some positive $\alpha_{i}$ and consider

$$
\beta=\left(\alpha_{1}, \cdots, \alpha_{i}-1, \alpha_{i+1}, \cdots, \alpha_{n-1}, 0\right)
$$

Then from the claim,

$$
\left\|D^{\alpha} w\right\|_{L^{2}\left(U_{1}\right)} \leq\left\|D^{\beta} w\right\|_{H^{2}\left(U_{1}\right)} \leq C\left(\|f\|_{H^{k-1}(U)}+\|w\|_{H^{k}(U)}+\sum_{s}\left\|h_{s}\right\|_{H^{k}(U)}\right)
$$

Suppose 42.1.29 holds for $\alpha_{n} \leq j-1$ where $j-1 \geq 1$ and consider $\alpha$ for which $|\alpha|=k+1$ and $\alpha_{n}=j$. Let

$$
\beta \equiv\left(\alpha_{1}, \cdots, \alpha_{n-1}, \alpha_{n}-2\right)
$$

Thus $D^{\alpha}=D^{\beta} D_{n}^{2}$. Restricting 42.1.23 to $z \in C_{c}^{\infty}\left(U_{1}\right)$ and using the density of this set of functions in $L^{2}\left(U_{1}\right)$, it follows that

$$
-\frac{\partial}{\partial y^{s}}\left(\alpha^{r s}(\mathbf{y}) \frac{\partial w}{\partial y^{r}}\right)-\frac{\partial h_{s}}{\partial y^{s}}=f
$$

Therefore, from the product rule,

$$
\frac{\partial \alpha^{r s}}{\partial y^{s}} \frac{\partial w}{\partial y^{r}}+\alpha^{r s} \frac{\partial^{2} w}{\partial y^{s} \partial y^{r}}+\frac{\partial h_{s}}{\partial y^{s}}=-f
$$

and so

$$
\begin{aligned}
\alpha^{n n} D_{n}^{2} w= & -\left(\frac{\partial \alpha^{r s}}{\partial y^{s}} \frac{\partial w}{\partial y^{r}}+\sum_{r \leq n-1} \sum_{s \leq n-1} \alpha^{r s} \frac{\partial^{2} w}{\partial y^{s} \partial y^{r}}+\right. \\
& \left.\sum_{s} \alpha^{n s} \frac{\partial^{2} w}{\partial y^{s} \partial y^{n}}+\sum_{r} \alpha^{r n} \frac{\partial^{2} w}{\partial y^{n} \partial y^{r}}+\frac{\partial h_{s}}{\partial y^{s}}+f\right) .
\end{aligned}
$$

As noted earlier, the condition, 42.1.2 implies $\alpha^{n n}(\mathbf{y}) \geq \delta>0$ and so

$$
\begin{aligned}
D_{n}^{2} w= & -\frac{1}{\alpha^{n n}}\left(\frac{\partial \alpha^{r s}}{\partial y^{s}} \frac{\partial w}{\partial y^{r}}+\sum_{r \leq n-1} \sum_{s \leq n-1} \alpha^{r s} \frac{\partial^{2} w}{\partial y^{s} \partial y^{r}}+\right. \\
& \left.\sum_{s} \alpha^{n s} \frac{\partial^{2} w}{\partial y^{s} \partial y^{n}}+\sum_{r} \alpha^{r n} \frac{\partial^{2} w}{\partial y^{n} \partial y^{r}}+\frac{\partial h_{s}}{\partial y^{s}}+f\right) .
\end{aligned}
$$

It follows from $D^{\alpha}=D^{\beta} D_{n}^{2}$ that

$$
\begin{aligned}
D^{\alpha} w= & D^{\beta}\left[-\frac{1}{\alpha^{n n}}\left(\frac{\partial \alpha^{r s}}{\partial y^{s}} \frac{\partial w}{\partial y^{r}}+\sum_{r \leq n-1} \sum_{s \leq n-1} \alpha^{r s} \frac{\partial^{2} w}{\partial y^{s} \partial y^{r}}+\right.\right. \\
& \left.\left.\sum_{s} \alpha^{n s} \frac{\partial^{2} w}{\partial y^{s} \partial y^{n}}+\sum_{r} \alpha^{r n} \frac{\partial^{2} w}{\partial y^{n} \partial y^{r}}+\frac{\partial h_{s}}{\partial y^{s}}+f\right)\right]
\end{aligned}
$$

Now you note that terms like $D^{\beta}\left(\frac{\partial^{2} w}{\partial y^{5} \partial y^{n}}\right)$ have $\alpha_{n}=j-1$ and so, from the induction hypothesis along with the assumptions on the given functions,

$$
\left\|D^{\alpha} w\right\|_{L^{2}\left(U_{1}\right)} \leq C\left(\|f\|_{H^{k-1}(U)}+\|w\|_{H^{k}(U)}+\sum_{s}\left\|h_{S}\right\|_{H^{k}(U)}\right)
$$

This proves the corollary.

### 42.2 The Case Of Bounded Open Sets

The main interest in all this is in the application to bounded open sets. Recall the following definition.

Definition 42.2.1 A bounded open subset, $\Omega$, of $\mathbb{R}^{n}$ has a $C^{m, 1}$ boundary if it satisfies the following conditions. For each $p \in \Gamma \equiv \bar{\Omega} \backslash \Omega$, there exists an open set, $W$, containing $p$, an open interval $(0, b)$, a bounded open box $U^{\prime} \subseteq \mathbb{R}^{n-1}$, and an affine orthogonal transformation, $R_{W}$ consisting of a distance preserving linear transformation followed by a translation such that

$$
\begin{gather*}
R_{W} W=U^{\prime} \times(0, b),  \tag{42.2.30}\\
R_{W}(W \cap \Omega)=\left\{\mathbf{u} \in \mathbb{R}^{n}: \mathbf{u}^{\prime} \in U^{\prime}, 0<u_{n}<\phi_{W}\left(\mathbf{u}^{\prime}\right)\right\} \tag{42.2.31}
\end{gather*}
$$

where $\phi_{W} \in C^{m, 1}\left(\overline{U^{\prime}}\right)$ meaning $\phi_{W}$ is the restriction to $U^{\prime}$ of a function, still denoted by $\phi_{W}$ which is in $C^{m, 1}\left(\mathbb{R}^{n-1}\right)$ and

$$
\inf \left\{\phi_{W}\left(\mathbf{u}^{\prime}\right): \mathbf{u}^{\prime} \in U^{\prime}\right\}>0
$$

The following picture depicts the situation.


For the situation described in the above definition, let $\mathbf{h}_{W}: U^{\prime} \rightarrow \Gamma \cap W$ be defined by

$$
\mathbf{h}_{W}\left(\mathbf{u}^{\prime}\right) \equiv R_{W}^{-1}\left(\mathbf{u}^{\prime}, \phi_{W}\left(\mathbf{u}^{\prime}\right)\right), \mathbf{g}_{W}(\mathbf{x}) \equiv\left(R_{W} \mathbf{x}\right)^{\prime}, \mathbf{H}_{W}(\mathbf{u}) \equiv R_{W}^{-1}\left(\mathbf{u}^{\prime}, \phi_{W}\left(\mathbf{u}^{\prime}\right)-u_{n}\right) .
$$

where $\mathbf{x}^{\prime} \equiv\left(x_{1}, \cdots, x_{n-1}\right)$ for $\mathbf{x}=\left(x_{1}, \cdots, x_{n}\right)$. Thus $\mathbf{g}_{W} \circ \mathbf{h}_{W}=\mathrm{id}$ on $U^{\prime}$ and $\mathbf{h}_{W} \circ \mathbf{g}_{W}=\mathrm{id}$ on $\Gamma \cap W$. Also note that $\mathbf{H}_{W}$ is defined on all of $\mathbb{R}^{n}$ is $C^{m, 1}$, and has an inverse with the same properties. To see this, let $\mathbf{G}_{W}(\mathbf{u})=\left(\mathbf{u}^{\prime}, \phi_{W}\left(\mathbf{u}^{\prime}\right)-u_{n}\right)$. Then $\mathbf{H}_{W}=R_{W}^{-1} \circ \mathbf{G}_{W}$ and $\mathbf{G}_{W}^{-1}=\left(\mathbf{u}^{\prime}, \phi_{W}\left(\mathbf{u}^{\prime}\right)-u_{n}\right)$ and so $\mathbf{H}_{W}^{-1}=\mathbf{G}_{W}^{-1} \circ R_{W}$. Note also that as indicated in the picture,

$$
R_{W}(W \cap \Omega)=\left\{\mathbf{u} \in \mathbb{R}^{n}: \mathbf{u}^{\prime} \in U^{\prime} \text { and } 0<u_{n}<\phi_{W}\left(\mathbf{u}^{\prime}\right)\right\}
$$

Since $\Gamma=\partial \Omega$ is compact, there exist finitely many of these open sets, $W$, denoted by $\left\{W_{i}\right\}_{i=1}^{q}$ such that $\Gamma \subseteq \cup_{i=1}^{q} W_{i}$. Let the corresponding sets, $U^{\prime}$ be denoted by $U_{i}^{\prime}$ and let the functions, $\phi$ be denoted by $\phi_{i}$. Also let $\mathbf{h}_{i}=\mathbf{h}_{W_{i}}, G_{W_{i}}=G_{i}$ etc. Now let

$$
\mathbf{\square}_{i}: \mathbf{G}_{i} R_{i}(\Omega \cap W) \equiv V_{i} \rightarrow \Omega \cap W_{i}
$$

be defined by

$$
\mathbf{m}_{i}(\mathbf{y}) \equiv R_{i}^{-1} \circ \mathbf{G}_{i}^{-1}(\mathbf{y})
$$

Thus $\varpi_{i}, \varpi_{i}^{-1} \in C^{m, 1}\left(\mathbb{R}^{n}\right)$. The following picture might be helpful.


Therefore, by Lemma 39.3.3 on Page 1356, it follows that for $t \in[m, m+1)$,

$$
\mathbf{m}_{i}^{*} \in \mathscr{L}\left(H^{t}\left(W_{i} \cap \Omega\right), H^{t}\left(V_{i}\right)\right)
$$

Assume

$$
\begin{equation*}
a^{i j}(\mathbf{x}) v_{i} v_{j} \geq \delta|\mathbf{v}|^{2} \tag{42.2.32}
\end{equation*}
$$

Lemma 42.2.2 Let $W$ be one of the sets described in the above definition and let $m \geq 1$. Let $W_{1} \subseteq \overline{W_{1}} \subseteq W$ where $W_{1}$ is an open set. Suppose also that

$$
\begin{aligned}
u & \in H^{1}(\Omega), \\
\alpha^{r s} & \in C^{0,1}(\bar{\Omega}), \\
f & \in L^{2}(\Omega), \\
h_{k} & \in H^{1}(\Omega),
\end{aligned}
$$

and that for all $v \in H^{1}(\Omega \cap W)$ such that $\operatorname{spt}(v) \subseteq \Omega \cap W$,

$$
\begin{equation*}
\int_{\Omega} a^{i j}(\mathbf{x}) u_{, i}(\mathbf{x}) v_{, j}(\mathbf{x}) d x+\int_{\Omega} h_{k}(\mathbf{x}) v_{, k}(\mathbf{x}) d x=\int_{\Omega} f(\mathbf{x}) v(\mathbf{x}) d x \tag{42.2.33}
\end{equation*}
$$

Then there exists a constant, $C$, independent of $f, u$, and $g$ such that

$$
\begin{equation*}
\|u\|_{H^{2}\left(\Omega \cap W_{1}\right)}^{2} \leq C\left(\|f\|_{L^{2}(\Omega)}^{2}+\|u\|_{H^{1}(\Omega)}^{2}+\sum_{k}\left\|h_{k}\right\|_{H^{1}(\Omega)}^{2}\right) . \tag{42.2.34}
\end{equation*}
$$

Proof: Let

$$
E \equiv\left\{v \in H^{1}(\Omega \cap W): \operatorname{spt}(v) \subseteq W\right\}
$$

$u$ restricted to $W \cap \Omega$ is in $H^{1}(\Omega \cap W)$ and

$$
\begin{equation*}
\int_{\Omega \cap W} a^{i j}(\mathbf{x}) u_{, i} v_{, j} d x+\int_{\Omega} h_{k}(\mathbf{x}) v_{, k}(\mathbf{x}) d x=\int_{\Omega} f(\mathbf{x}) v(\mathbf{x}) d x \text { for all } v \in E . \tag{42.2.35}
\end{equation*}
$$

Now let $\square_{i}(\mathbf{y})=\mathbf{x}$. For this particular $W$, denote $\Phi_{i}$ more simply by $\Phi, U_{i} \equiv \Phi_{i}\left(\Omega \cap W_{i}\right)$ by $U$, and $V_{i}$ by $V$. Denoting the coordinates of $V$ by $\mathbf{y}$, and letting $u(\mathbf{x}) \equiv w(\mathbf{y})$ and $v(\mathbf{x}) \equiv$ $z(\mathbf{y})$, it follows that in terms of the new coordinates, 42.2.35 takes the form

$$
\begin{aligned}
& \int_{U} a^{i j}(\Phi(\mathbf{y})) \frac{\partial w}{\partial y^{r}} \frac{\partial y^{r}}{\partial x^{i}} \frac{\partial z}{\partial y^{s}} \frac{\partial y^{s}}{\partial x^{j}}|\operatorname{det} D \Phi(\mathbf{y})| d y \\
& +\int_{U} h_{k}(\Phi(\mathbf{y})) \frac{\partial z}{\partial y^{l}} \frac{\partial y^{l}}{\partial x^{k}}|\operatorname{det} D \Phi(\mathbf{y})| d x \\
= & \int_{U} f(\Phi(\mathbf{y})) z(\mathbf{y})|\operatorname{det} D \Phi(\mathbf{y})| d y
\end{aligned}
$$

Let

$$
\begin{align*}
\alpha^{r s}(\mathbf{y}) & \equiv a^{i j}(\Phi(\mathbf{y})) \frac{\partial y^{r}}{\partial x^{i}} \frac{\partial y^{s}}{\partial x^{j}}|\operatorname{det} D \Phi(\mathbf{y})|  \tag{42.2.36}\\
\widetilde{h}_{l}(\mathbf{y}) & \equiv h_{k}(\Phi(\mathbf{y})) \frac{\partial y^{l}}{\partial x^{k}}|\operatorname{det} D \Phi(\mathbf{y})| \tag{42.2.37}
\end{align*}
$$

and

$$
\begin{equation*}
\widetilde{f}(\mathbf{y}) \equiv \Phi^{*} f|\operatorname{det} D \Phi|(\mathbf{y}) \equiv f(\Phi(\mathbf{y}))|\operatorname{det} D \Phi(\mathbf{y})| \tag{42.2.38}
\end{equation*}
$$

Now the function on the right in 42.2 .36 is in $C^{0,1}(\bar{U})$. This is because of the assumption that $m \geq 1$ in the statement of the lemma. This function is therefore a finite product of bounded functions in $C^{0,1}(\bar{U})$.

The function $\widetilde{h_{l}}$ defined in 42.2 .37 is in $H^{1}(U)$ and

$$
\left\|\widetilde{h}_{l}\right\|_{H^{1}(U)} \leq C \sum_{k}\left\|h_{k}\right\|_{H^{1}(\Omega \cap W)}
$$

again because $m \geq 1$.
Finally, the right side of 42.2 .38 is a function in $L^{2}(U)$ by Lemma 39.3.3 on Page 1356 and the observation that $|\operatorname{det} D \Phi(\cdot)| \in C^{0,1}(\bar{U})$ which follows from the assumption of the lemma that $m \geq 1$ so $\Phi \in C^{1,1}\left(\mathbb{R}^{n}\right)$. Also

$$
\|\widetilde{f}\|_{L^{2}(U)} \leq C\|f\|_{L^{2}(\Omega \cap W)}
$$

Therefore, 42.2 .35 is of the form

$$
\begin{equation*}
\int_{U} \alpha^{r s}(\mathbf{y}) w_{, r} z, s d y+\int_{U} \tilde{h}_{l} z_{, l} d y=\int_{U} \tilde{f} z d y \tag{42.2.39}
\end{equation*}
$$

for all $z$ in $H^{1}(U)$ having support in $V$.
Claim: There exists $r>0$ independent of $\mathbf{y} \in \bar{U}$ such that for all $\mathbf{y} \in \bar{U}$,

$$
\alpha^{r s}(\mathbf{y}) v_{r} v_{s} \geq r|\mathbf{v}|^{2}
$$

Proof of the claim: If this is not so, there exist vectors, $\mathbf{v}^{n},\left|\mathbf{v}^{n}\right|=1$, and $\mathbf{y}_{n} \in \bar{U}$ such that $\alpha^{r s}\left(\mathbf{y}_{n}\right) v_{r}^{n} v_{s}^{n} \leq \frac{1}{n}$. Taking a subsequence, there exists $\mathbf{y} \in \bar{U}$ and $|\mathbf{v}|=1$ such that $\alpha^{r s}(\mathbf{y}) v_{r} v_{s}=0$ contradicting 42.2.32.

Therefore, by Lemma 42.1.1, there exists a constant, $C$, independent of $f, g$, and $w$ such that

$$
\|w\|_{H^{2}\left(\Phi^{-1}\left(W_{1} \cap \Omega\right)\right)}^{2} \leq C\left(\|\widetilde{f}\|_{L^{2}(U)}^{2}+\|w\|_{H^{1}(U)}^{2}+\sum_{l}\left\|\widetilde{h}_{l}\right\|_{H^{1}(U)}^{2}\right)
$$

Therefore,

$$
\begin{aligned}
\|u\|_{H^{2}\left(W_{1} \cap \Omega\right)}^{2} & \leq C\left(\|f\|_{L^{2}(W \cap \Omega)}^{2}+\|w\|_{H^{1}(W \cap \Omega)}^{2}+\sum_{k}\left\|h_{k}\right\|_{H^{1}(W \cap \Omega)}^{2}\right) \\
& \leq C\left(\|f\|_{L^{2}(\Omega)}^{2}+\|w\|_{H^{1}(\Omega)}^{2}+\sum_{k}\left\|h_{k}\right\|_{H^{1}(\Omega)}^{2}\right)
\end{aligned}
$$

which proves the lemma.
With this lemma here is the main result.
Theorem 42.2.3 Let $\Omega$ be a bounded open set with $C^{1,1}$ boundary as in Definition 42.2.1, let $f \in L^{2}(\Omega), h_{k} \in H^{1}(\Omega)$, and suppose that for all $\mathbf{x} \in \bar{\Omega}$,

$$
a^{i j}(\mathbf{x}) v_{i} v_{j} \geq \delta|\mathbf{v}|^{2}
$$

Suppose also that $u \in H^{1}(\Omega)$ and

$$
\int_{\Omega} a^{i j}(\mathbf{x}) u_{, i}(\mathbf{x}) v_{, j}(\mathbf{x}) d x+\int_{\Omega} h_{k}(\mathbf{x}) v_{, k}(\mathbf{x}) d x=\int_{\Omega} f(\mathbf{x}) v(\mathbf{x}) d x
$$

for all $v \in H^{1}(\Omega)$. Then $u \in H^{2}(\Omega)$ and for some $C$ independent of $f, g$, and $u$,

$$
\|u\|_{H^{2}(\Omega)}^{2} \leq C\left(\|f\|_{L^{2}(\Omega)}^{2}+\|u\|_{H^{1}(\Omega)}^{2}+\sum_{k}\left\|h_{k}\right\|_{H^{1}(\Omega)}^{2}\right)
$$

Proof: Let the $W_{i}$ for $i=1, \cdots, l$ be as described in Definition 42.2.1. Thus $\partial \Omega \subseteq$ $\cup_{j=1}^{l} W_{j}$. Then let $C_{1} \equiv \partial \Omega \backslash \cup_{i=2}^{l} W_{i}$, a closed subset of $W_{1}$. Let $D_{1}$ be an open set satisfying

$$
C_{1} \subseteq D_{1} \subseteq \overline{D_{1}} \subseteq W_{1}
$$

Then $D_{1}, W_{2}, \cdots, W_{l} \operatorname{cover} \partial \Omega$. Let $C_{2}=\partial \Omega \backslash\left(D_{1} \cup\left(\cup_{i=3}^{l} W_{i}\right)\right)$. Then $C_{2}$ is a closed subset of $W_{2}$. Choose an open set, $D_{2}$ such that

$$
C_{2} \subseteq D_{2} \subseteq \overline{D_{2}} \subseteq W_{2}
$$

Thus $D_{1}, D_{2}, W_{3} \cdots, W_{l}$ covers $\partial \Omega$. Continue in this way to get $\overline{D_{i}} \subseteq W_{i}$, and $\partial \Omega \subseteq \cup_{i=1}^{l} D_{i}$, and $D_{i}$ is an open set. Now let

$$
D_{0} \equiv \Omega \backslash \cup_{i=1}^{l} \overline{D_{i}}
$$

Also, let $\overline{D_{i}} \subseteq V_{i} \subseteq \overline{V_{i}} \subseteq W_{i}$. Therefore, $D_{0}, V_{1}, \cdots, V_{l}$ covers $\Omega$. Then the same estimation process used above yields

$$
\|u\|_{H^{2}\left(D_{0}\right)} \leq C\left(\|f\|_{L^{2}(\Omega)}^{2}+\|u\|_{H^{1}(\Omega)}^{2}+\sum_{k}\left\|h_{k}\right\|_{H^{1}(\Omega)}^{2}\right)
$$

From Lemma 42.2.2

$$
\|u\|_{H^{2}\left(V_{i} \cap \Omega\right)} \leq C\left(\|f\|_{L^{2}(\Omega)}^{2}+\|u\|_{H^{1}(\Omega)}^{2}+\sum_{k}\left\|h_{k}\right\|_{H^{1}(\Omega)}^{2}\right)
$$

also. This proves the theorem since

$$
\|u\|_{H^{2}(\Omega)} \leq \sum_{i=1}^{l}\|u\|_{H^{2}\left(V_{i} \cap \Omega\right)}+\|u\|_{H^{2}\left(D_{0}\right)} .
$$

What about the Dirichlet problem? The same differencing procedure as above yields the following.

Theorem 42.2.4 Let $\Omega$ be a bounded open set with $C^{1,1}$ boundarybrownianmotiontheorem as in Definition 42.2.1, let $f \in L^{2}(\Omega), h_{k} \in H^{1}(\Omega)$, and suppose that for all $\mathbf{x} \in \bar{\Omega}$,

$$
a^{i j}(\mathbf{x}) v_{i} v_{j} \geq \delta|\mathbf{v}|^{2}
$$

Suppose also that $u \in H_{0}^{1}(\Omega)$ and

$$
\int_{\Omega} a^{i j}(\mathbf{x}) u_{, i}(\mathbf{x}) v_{, j}(\mathbf{x}) d x+\int_{\Omega} h_{k}(\mathbf{x}) v_{, k}(\mathbf{x}) d x=\int_{\Omega} f(\mathbf{x}) v(\mathbf{x}) d x
$$

for all $v \in H_{0}^{1}(\Omega)$. Then $u \in H^{2}(\Omega)$ and for some $C$ independent of $f, g$, and $u$,

$$
\|u\|_{H^{2}(\Omega)}^{2} \leq C\left(\|f\|_{L^{2}(\Omega)}^{2}+\|u\|_{H^{1}(\Omega)}^{2}+\sum_{k}\left\|h_{k}\right\|_{H^{1}(\Omega)}^{2}\right)
$$

What about higher regularity?

Lemma 42.2.5 Let $W$ be one of the sets described in Definition 42.2.1 and let $m \geq k$. Let $W_{1} \subseteq \overline{W_{1}} \subseteq W$ where $W_{1}$ is an open set. Suppose also that

$$
\begin{aligned}
u & \in H^{k}(\Omega), \\
\alpha^{r s} & \in C^{k-1,1}(\bar{\Omega}), \\
f & \in H^{k-1}(\Omega), \\
h_{s} & \in H^{k}(\Omega),
\end{aligned}
$$

and that for all $v \in H^{1}(\Omega \cap W)$ such that $\operatorname{spt}(v) \subseteq \Omega \cap W$,

$$
\begin{equation*}
\int_{\Omega} a^{i j}(\mathbf{x}) u_{, i}(\mathbf{x}) v_{, j}(\mathbf{x}) d x+\int_{\Omega} h_{s}(\mathbf{x}) v_{, s}(\mathbf{x}) d x=\int_{\Omega} f(\mathbf{x}) v(\mathbf{x}) d x \tag{42.2.40}
\end{equation*}
$$

Then there exists a constant, $C$, independent of $f, u$, and $g$ such that

$$
\begin{equation*}
\|u\|_{H^{k+1}\left(\Omega \cap W_{1}\right)}^{2} \leq C\left(\|f\|_{H^{k-1}(\Omega)}^{2}+\|u\|_{H^{k}(\Omega)}^{2}+\sum_{s}\left\|h_{s}\right\|_{H^{k}(\Omega)}^{2}\right) . \tag{42.2.41}
\end{equation*}
$$

Proof: Let

$$
E \equiv\left\{v \in H^{k}(\Omega \cap W): \operatorname{spt}(v) \subseteq W\right\}
$$

$u$ restricted to $W \cap \Omega$ is in $H^{k}(\Omega \cap W)$ and

$$
\begin{gather*}
\int_{\Omega \cap W} a^{i j}(\mathbf{x}) u_{, i} v, j d x+\int_{\Omega} h_{s}(\mathbf{x}) v_{, s}(\mathbf{x}) d x \\
=\int_{\Omega} f(\mathbf{x}) v(\mathbf{x}) d x \text { for all } v \in E . \tag{42.2.42}
\end{gather*}
$$

Now let $\mathbf{\Xi}_{i}(\mathbf{y})=\mathbf{x}$. For this particular $W$, denote $\Phi_{i}$ more simply by $\Phi, U_{i} \equiv \Phi_{i}\left(\Omega \cap W_{i}\right)$ by $U$, and $V_{i}$ by $V$. Denoting the coordinates of $V$ by $\mathbf{y}$, and letting $u(\mathbf{x}) \equiv w(\mathbf{y})$ and $v(\mathbf{x}) \equiv$ $z(\mathbf{y})$, it follows that in terms of the new coordinates, 42.2.35 takes the form

$$
\begin{aligned}
& \int_{U} a^{i j}(\Phi(\mathbf{y})) \frac{\partial w}{\partial y^{r}} \frac{\partial y^{r}}{\partial x^{i}} \frac{\partial z}{\partial y^{s}} \frac{\partial y^{s}}{\partial x^{j}}|\operatorname{det} D \Phi(\mathbf{y})| d y \\
& +\int_{U} h_{k}(\Phi(\mathbf{y})) \frac{\partial z}{\partial y^{l}} \frac{\partial y^{l}}{\partial x^{k}}|\operatorname{det} D \Phi(\mathbf{y})| d x \\
= & \int_{U} f(\Phi(\mathbf{y})) z(\mathbf{y})|\operatorname{det} D \Phi(\mathbf{y})| d y
\end{aligned}
$$

Let

$$
\begin{gather*}
\alpha^{r s}(\mathbf{y}) \equiv a^{i j}(\Phi(\mathbf{y})) \frac{\partial y^{r}}{\partial x^{i}} \frac{\partial y^{s}}{\partial x^{j}}|\operatorname{det} D \Phi(\mathbf{y})|  \tag{42.2.43}\\
\widetilde{h}_{l}(\mathbf{y}) \equiv h_{k}(\Phi(\mathbf{y})) \frac{\partial y^{l}}{\partial x^{k}}|\operatorname{det} D \Phi(\mathbf{y})| \tag{42.2.44}
\end{gather*}
$$

and

$$
\begin{equation*}
\widetilde{f}(\mathbf{y}) \equiv \Phi^{*} f|\operatorname{det} D \Phi|(\mathbf{y}) \equiv f(\Phi(\mathbf{y}))|\operatorname{det} D \Phi(\mathbf{y})| \tag{42.2.45}
\end{equation*}
$$

Now the function on the right in 42.2 .43 is in $C^{k, 1}(\bar{U})$. This is because of the assumption that $m \geq k$ in the statement of the lemma. This function is therefore a finite product of bounded functions in $C^{k, 1}(\bar{U})$.

The function $\widetilde{h}_{l}$ defined in 42.2 .44 is in $H^{k}(U)$ and

$$
\left\|\widetilde{h}_{l}\right\|_{H^{k}(U)} \leq C \sum_{S}\left\|h_{s}\right\|_{H^{k}(\Omega \cap W)}
$$

again because $m \geq k$.
Finally, the right side of 42.2 .45 is a function in $H^{k-1}(U)$ by Lemma 39.3.3 on Page 1356 and the observation that $|\operatorname{det} D \Phi(\cdot)| \in C^{k-1,1}(\bar{U})$ which follows from the assumption of the lemma that $m \geq k$ so $\Phi \in C^{k-1,1}\left(\mathbb{R}^{n}\right)$. Also

$$
\|\widetilde{f}\|_{H^{k-1}(U)} \leq C\|f\|_{H^{k-1}(\Omega \cap W)}
$$

Therefore, 42.2.42 is of the form

$$
\begin{equation*}
\int_{U} \alpha^{r s}(\mathbf{y}) w_{, r} z, s d y+\int_{U} \widetilde{h}_{l} z_{, l} d y=\int_{U} \tilde{f} z d y \tag{42.2.46}
\end{equation*}
$$

for all $z$ in $H^{1}(U)$ having support in $V$.
Claim: There exists $r>0$ independent of $\mathbf{y} \in \bar{U}$ such that for all $\mathbf{y} \in \bar{U}$,

$$
\alpha^{r s}(\mathbf{y}) v_{r} v_{s} \geq r|\mathbf{v}|^{2}
$$

Proof of the claim: If this is not so, there exist vectors, $\mathbf{v}^{n},\left|\mathbf{v}^{n}\right|=1$, and $\mathbf{y}_{n} \in \bar{U}$ such that $\alpha^{r s}\left(\mathbf{y}_{n}\right) v_{r}^{n} v_{s}^{n} \leq \frac{1}{n}$. Taking a subsequence, there exists $\mathbf{y} \in \bar{U}$ and $|\mathbf{v}|=1$ such that $\alpha^{r s}(\mathbf{y}) v_{r} v_{s}=0$ contradicting 42.2.32.

Therefore, by Corollary 42.1.2, there exists a constant, $C$, independent of $f, g$, and $w$ such that

$$
\|w\|_{H^{k+1}\left(\Phi^{-1}\left(W_{1} \cap \Omega\right)\right)}^{2} \leq C\left(\|\widetilde{f}\|_{H^{k-1}(U)}^{2}+\|w\|_{H^{k}(U)}^{2}+\sum_{l}\left\|\widetilde{h}_{l}\right\|_{H^{k}(U)}^{2}\right)
$$

Therefore,

$$
\begin{aligned}
\|u\|_{H^{k+1}\left(W_{1} \cap \Omega\right)}^{2} & \leq C\left(\|f\|_{H^{k-1}(W \cap \Omega)}^{2}+\|w\|_{H^{k}(W \cap \Omega)}^{2}+\sum_{s}\left\|h_{S}\right\|_{H^{k}(W \cap \Omega)}^{2}\right) \\
& \leq C\left(\|f\|_{H^{k-1}(\Omega)}^{2}+\|w\|_{H^{k}(\Omega)}^{2}+\sum_{s}\left\|h_{s}\right\|_{H^{k}(\Omega)}^{2}\right)
\end{aligned}
$$

which proves the lemma.
Now here is a theorem which generalizes the one above in the case where more regularity is known.

Theorem 42.2.6 Let $\Omega$ be a bounded open set with $C^{k, 1}$ boundary as in Definition 42.2.1, let $f \in H^{k-1}(\Omega), h_{s} \in H^{k}(\Omega)$, and suppose that for all $\mathbf{x} \in \bar{\Omega}$,

$$
a^{i j}(\mathbf{x}) v_{i} v_{j} \geq \delta|\mathbf{v}|^{2}
$$

Suppose also that $u \in H^{k}(\Omega)$ and

$$
\int_{\Omega} a^{i j}(\mathbf{x}) u_{, i}(\mathbf{x}) v_{, j}(\mathbf{x}) d x+\int_{\Omega} h_{k}(\mathbf{x}) v_{, k}(\mathbf{x}) d x=\int_{\Omega} f(\mathbf{x}) v(\mathbf{x}) d x
$$

for all $v \in H^{k}(\Omega)$. Then $u \in H^{k+1}(\Omega)$ and for some $C$ independent of $f, g$, and $u$,

$$
\|u\|_{H^{k+1}(\Omega)}^{2} \leq C\left(\|f\|_{H^{k-1}(\Omega)}^{2}+\|u\|_{H^{k}(\Omega)}^{2}+\sum_{s}\left\|h_{s}\right\|_{H^{k}(\Omega)}^{2}\right)
$$

Proof: Let the $W_{i}$ for $i=1, \cdots, l$ be as described in Definition 42.2.1. Thus $\partial \Omega \subseteq$ $\cup_{j=1}^{l} W_{j}$. Then let $C_{1} \equiv \partial \Omega \backslash \cup_{i=2}^{l} W_{i}$, a closed subset of $W_{1}$. Let $D_{1}$ be an open set satisfying

$$
C_{1} \subseteq D_{1} \subseteq \overline{D_{1}} \subseteq W_{1}
$$

Then $D_{1}, W_{2}, \cdots, W_{l} \operatorname{cover} \partial \Omega$. Let $C_{2}=\partial \Omega \backslash\left(D_{1} \cup\left(\cup_{i=3}^{l} W_{i}\right)\right)$. Then $C_{2}$ is a closed subset of $W_{2}$. Choose an open set, $D_{2}$ such that

$$
C_{2} \subseteq D_{2} \subseteq \overline{D_{2}} \subseteq W_{2}
$$

Thus $D_{1}, D_{2}, W_{3} \cdots, W_{l}$ covers $\partial \Omega$. Continue in this way to get $\overline{D_{i}} \subseteq W_{i}$, and $\partial \Omega \subseteq \cup_{i=1}^{l} D_{i}$, and $D_{i}$ is an open set. Now let

$$
D_{0} \equiv \Omega \backslash \cup_{i=1}^{l} \overline{D_{i}}
$$

Also, let $\overline{D_{i}} \subseteq V_{i} \subseteq \overline{V_{i}} \subseteq W_{i}$. Therefore, $D_{0}, V_{1}, \cdots, V_{l}$ covers $\Omega$. Then the same estimation process used above yields

$$
\|u\|_{H^{k+1}\left(D_{0}\right)} \leq C\left(\|f\|_{H^{k-1}(\Omega)}^{2}+\|u\|_{H^{k}(\Omega)}^{2}+\sum_{k}\left\|h_{k}\right\|_{H^{k}(\Omega)}^{2}\right)
$$

From Lemma 42.2.5

$$
\|u\|_{H^{k+1}\left(V_{i} \cap \Omega\right)} \leq C\left(\|f\|_{H^{k-1}(\Omega)}^{2}+\|u\|_{H^{k}(\Omega)}^{2}+\sum_{k}\left\|h_{k}\right\|_{H^{k}(\Omega)}^{2}\right)
$$

also. This proves the theorem since

$$
\|u\|_{H^{k+1}(\Omega)} \leq \sum_{i=1}^{l}\|u\|_{H^{k+1}\left(V_{i} \cap \Omega\right)}+\|u\|_{H^{k+1}\left(D_{0}\right)}
$$

## Chapter 43

## Interpolation In Banach Space

### 43.1 Some Standard Techniques In Evolution Equations

### 43.1.1 Weak Vector Valued Derivatives

In this section, several significant theorems are presented. Unless indicated otherwise, the measure will be Lebesgue measure. First here is a lemma.

Lemma 43.1.1 Suppose $g \in L^{1}([a, b] ; X)$ where $X$ is a Banach space. Then if

$$
\int_{a}^{b} g(t) \phi(t) d t=0
$$

for all $\phi \in C_{c}^{\infty}(a, b)$, then $g(t)=0$ a.e.
Proof: Let $E$ be a measurable subset of $(a, b)$ and let $K \subseteq E \subseteq V \subseteq(a, b)$ where $K$ is compact, $V$ is open and $m(V \backslash K)<\varepsilon$. Let $K \prec h \prec V$ as in the proof of the Riesz representation theorem for positive linear functionals. Enlarging $K$ slightly and convolving with a mollifier, it can be assumed $h \in C_{c}^{\infty}(a, b)$. Then

$$
\begin{aligned}
\left|\int_{a}^{b} \mathscr{X}_{E}(t) g(t) d t\right| & =\left|\int_{a}^{b}\left(\mathscr{X}_{E}(t)-h(t)\right) g(t) d t\right| \\
& \leq \int_{a}^{b}\left|\mathscr{X}_{E}(t)-h(t)\right|\|g(t)\| d t \\
& \leq \int_{V \backslash K}\|g(t)\| d t
\end{aligned}
$$

Now let $K_{n} \subseteq E \subseteq V_{n}$ with $m\left(V_{n} \backslash K_{n}\right)<2^{-n}$. Then from the above,

$$
\left|\int_{a}^{b} \mathscr{X}_{E}(t) g(t) d t\right| \leq \int_{a}^{b} \mathscr{X}_{V_{n} \backslash K_{n}}(t)\|g(t)\| d t
$$

and the integrand of the last integral converges to 0 a.e. as $n \rightarrow \infty$ because $\sum_{n} m\left(V_{n} \backslash K_{n}\right)<$ $\infty$. By the dominated convergence theorem, this last integral converges to 0 . Therefore, whenever $E \subseteq(a, b)$,

$$
\int_{a}^{b} \mathscr{X}_{E}(t) g(t) d t=0
$$

Since the endpoints have measure zero, it also follows that for any measurable $E$, the above equation holds.

Now $g \in L^{1}([a, b] ; X)$ and so it is measurable. Therefore, $g([a, b])$ is separable. Let $D$ be a countable dense subset and let $E$ denote the set of linear combinations of the form $\sum_{i} a_{i} d_{i}$ where $a_{i}$ is a rational point of $\mathbb{F}$ and $d_{i} \in D$. Thus $E$ is countable. Denote by $Y$ the closure of $E$ in $X$. Thus $Y$ is a separable closed subspace of $X$ which contains all the values of $g$.

Now let $S_{n} \equiv g^{-1}\left(B\left(y_{n},\left\|y_{n}\right\| / 2\right)\right)$ where $E=\left\{y_{n}\right\}_{n=1}^{\infty}$. Then, $\cup_{n} S_{n}=g^{-1}(X \backslash\{0\})$. This follows because if $x \in Y$ and $x \neq 0$, then in $B\left(x, \frac{\|x\|}{4}\right)$ there is a point of $E, y_{n}$. Therefore, $\left\|y_{n}\right\|>\frac{3}{4}\|x\|$ and so $\frac{\left\|y_{n}\right\|}{2}>\frac{3\|x\|}{8}>\frac{\|x\|}{4}$ so $x \in B\left(y_{n},\left\|y_{n}\right\| / 2\right)$. It follows that if each $S_{n}$ has measure zero, then $g(t)=0$ for a.e. $t$. Suppose then that for some $n$, the set, $S_{n}$ has positive mesure. Then from what was shown above,

$$
\begin{aligned}
\left\|y_{n}\right\| & =\left\|\frac{1}{m\left(S_{n}\right)} \int_{S_{n}} g(t) d t-y_{n}\right\|=\left\|\frac{1}{m\left(S_{n}\right)} \int_{S_{n}} g(t)-y_{n} d t\right\| \\
& \leq \frac{1}{m\left(S_{n}\right)} \int_{S_{n}}\left\|g(t)-y_{n}\right\| d t \leq \frac{1}{m\left(S_{n}\right)} \int_{S_{n}}\left\|y_{n}\right\| / 2 d t=\left\|y_{n}\right\| / 2
\end{aligned}
$$

and so $y_{n}=0$ which implies $S_{n}=\emptyset$, a contradiction to $m\left(S_{n}\right)>0$. This contradiction shows each $S_{n}$ has measure zero and so as just explained, $g(t)=0$ a.e.

Definition 43.1.2 For $f \in L^{1}(a, b ; X)$, define an extension, $\bar{f}$ defined on

$$
[2 a-b, 2 b-a]=[a-(b-a), b+(b-a)]
$$

as follows.

$$
\bar{f}(t) \equiv\left\{\begin{array}{l}
f(t) \text { if } t \in[a, b] \\
f(2 a-t) \text { if } t \in[2 a-b, a] \\
f(2 b-t) \text { if } t \in[b, 2 b-a]
\end{array}\right.
$$

Definition 43.1.3 Also if $f \in L^{p}(a, b ; X)$ and $h>0$, define for $t \in[a, b], f_{h}(t) \equiv \bar{f}(t-h)$ for all $h<b-a$. Thus the map $f \rightarrow f_{h}$ is continuous and linear on $L^{p}(a, b ; X)$. It is continuous because

$$
\begin{aligned}
\int_{a}^{b}\left\|f_{h}(t)\right\|^{p} d t & =\int_{a}^{a+h}\|f(2 a-t+h)\|^{p} d t+\int_{a}^{b-h}\|f(t)\|^{p} d t \\
& =\int_{a}^{a+h}\|f(t)\|^{p} d t+\int_{a}^{b-h}\|f(t)\|^{p} d t \leq 2\|f\|_{p}^{p}
\end{aligned}
$$

The following lemma is on continuity of translation in $L^{p}(a, b ; X)$.
Lemma 43.1.4 Let $\bar{f}$ be as defined in Definition 69.2.2. Then for $f \in L^{p}(a, b ; X)$ for $p \in$ $[1, \infty)$,

$$
\lim _{\delta \rightarrow 0} \int_{a}^{b}\|\bar{f}(t-\delta)-f(t)\|_{X}^{p} d t=0
$$

Proof: Regarding the measure space as $(a, b)$ with Lebesgue measure, by Lemma 21.5.9 there exists $g \in C_{c}(a, b ; X)$ such that $\|f-g\|_{p}<\varepsilon$. Here the norm is the norm in $L^{p}(a, b ; X)$. Therefore,

$$
\begin{aligned}
\left\|f_{h}-f\right\|_{p} & \leq\left\|f_{h}-g_{h}\right\|_{p}+\left\|g_{h}-g\right\|_{p}+\|g-f\|_{p} \\
& \leq\left(2^{1 / p}+1\right)\|f-g\|_{p}+\left\|g_{h}-g\right\|_{p} \\
& <\left(2^{1 / p}+1\right) \varepsilon+\varepsilon
\end{aligned}
$$

whenever $h$ is sufficiently small. This is because of the uniform continuity of $g$. Therefore, since $\varepsilon>0$ is arbitrary, this proves the lemma.

Definition 43.1.5 Let $f \in L^{1}(a, b ; X)$. Then the distributional derivative in the sense of $X$ valued distributions is given by

$$
f^{\prime}(\phi) \equiv-\int_{a}^{b} f(t) \phi^{\prime}(t) d t
$$

Then $f^{\prime} \in L^{1}(a, b ; X)$ if there exists $h \in L^{1}(a, b ; X)$ such that for all $\phi \in C_{c}^{\infty}(a, b)$,

$$
f^{\prime}(\phi)=\int_{a}^{b} h(t) \phi(t) d t
$$

Then $f^{\prime}$ is defined to equal $h$. Here $f$ and $f^{\prime}$ are considered as vector valued distributions in the same way as was done for scalar valued functions.

Lemma 43.1.6 The above definition is well defined.
Proof: Suppose both $h$ and $g$ work in the definition. Then for all $\phi \in C_{c}^{\infty}(a, b)$,

$$
\int_{a}^{b}(h(t)-g(t)) \phi(t) d t=0
$$

Therefore, by Lemma 43.1.1, $h(t)-g(t)=0$ a.e.
The other thing to notice about this is the following lemma. It follows immediately from the definition.

Lemma 43.1.7 Suppose $f, f^{\prime} \in L^{1}(a, b ; X)$. Then if $[c, d] \subseteq[a, b]$, it follows that $\left(\left.f\right|_{[c, d]}\right)^{\prime}=$ $\left.f^{\prime}\right|_{[c, d]}$. This notation means the restriction to $[c, d]$.

Recall that in the case of scalar valued functions, if you had both $f$ and its weak derivative, $f^{\prime}$ in $L^{1}(a, b)$, then you were able to conclude that $f$ is almost everywhere equal to a continuous function, still denoted by $f$ and

$$
f(t)=f(a)+\int_{a}^{t} f^{\prime}(s) d s
$$

In particular, you can define $f(a)$ to be the initial value of this continuous function. It turns out that an identical theorem holds in this case. To begin with here is the same sort of lemma which was used earlier for the case of scalar valued functions. It says that if $f^{\prime}=0$ where the derivative is taken in the sense of $X$ valued distributions, then $f$ equals a constant.

Lemma 43.1.8 Suppose $f \in L^{1}(a, b ; X)$ and for all $\phi \in C_{c}^{\infty}(a, b)$,

$$
\int_{a}^{b} f(t) \phi^{\prime}(t) d t=0
$$

Then there exists a constant, $a \in X$ such that $f(t)=a$ a.e.

Proof: Let $\phi_{0} \in C_{c}^{\infty}(a, b), \int_{a}^{b} \phi_{0}(x) d x=1$ and define for $\phi \in C_{c}^{\infty}(a, b)$

$$
\psi_{\phi}(x) \equiv \int_{a}^{x}\left[\phi(t)-\left(\int_{a}^{b} \phi(y) d y\right) \phi_{0}(t)\right] d t
$$

Then $\psi_{\phi} \in C_{c}^{\infty}(a, b)$ and $\psi_{\phi}^{\prime}=\phi-\left(\int_{a}^{b} \phi(y) d y\right) \phi_{0}$. Then

$$
\begin{aligned}
\int_{a}^{b} f(t)(\phi(t)) d t & =\int_{a}^{b} f(t)\left(\psi_{\phi}^{\prime}(t)+\left(\int_{a}^{b} \phi(y) d y\right) \phi_{0}(t)\right) d t \\
& =\overbrace{\int_{a}^{b} f(t) \psi_{\phi}^{\prime}(t) d t+\left(\int_{a}^{b} \phi(y) d y\right) \int_{a}^{b} f(t) \phi_{0}(t) d t}^{=0 \text { by assumption }} \\
& =\left(\int_{a}^{b}\left(\int_{a}^{b} f(t) \phi_{0}(t) d t\right) \phi(y) d y\right) .
\end{aligned}
$$

It follows that for all $\phi \in C_{c}^{\infty}(a, b)$,

$$
\int_{a}^{b}\left(f(y)-\left(\int_{a}^{b} f(t) \phi_{0}(t) d t\right)\right) \phi(y) d y=0
$$

and so by Lemma 43.1.1,

$$
f(y)-\left(\int_{a}^{b} f(t) \phi_{0}(t) d t\right)=0 \text { a.e. } y
$$

Theorem 43.1.9 Suppose $f, f^{\prime}$ both are in $L^{1}(a, b ; X)$ where the derivative is taken in the sense of $X$ valued distributions. Then there exists a unique point of $X$, denoted by $f(a)$ such that the following formula holds a.e. $t$.

$$
f(t)=f(a)+\int_{a}^{t} f^{\prime}(s) d s
$$

Proof:

$$
\int_{a}^{b}\left(f(t)-\int_{a}^{t} f^{\prime}(s) d s\right) \phi^{\prime}(t) d t=\int_{a}^{b} f(t) \phi^{\prime}(t) d t-\int_{a}^{b} \int_{a}^{t} f^{\prime}(s) \phi^{\prime}(t) d s d t
$$

Now consider $\int_{a}^{b} \int_{a}^{t} f^{\prime}(s) \phi^{\prime}(t) d s d t$. Let $\Lambda \in X^{\prime}$. Then it is routine from approximating $f^{\prime}$ with simple functions to verify

$$
\Lambda\left(\int_{a}^{b} \int_{a}^{t} f^{\prime}(s) \phi^{\prime}(t) d s d t\right)=\int_{a}^{b} \int_{a}^{t} \Lambda\left(f^{\prime}(s)\right) \phi^{\prime}(t) d s d t
$$

Now the ordinary Fubini theorem can be applied to obtain

$$
\begin{aligned}
& =\int_{a}^{b} \int_{s}^{b} \Lambda\left(f^{\prime}(s)\right) \phi^{\prime}(t) d t d s \\
& =\Lambda\left(\int_{a}^{b} \int_{s}^{b} f^{\prime}(s) \phi^{\prime}(t) d t d s\right)
\end{aligned}
$$

Since $X^{\prime}$ separates the points of $X$, it follows

$$
\int_{a}^{b} \int_{a}^{t} f^{\prime}(s) \phi^{\prime}(t) d s d t=\int_{a}^{b} \int_{s}^{b} f^{\prime}(s) \phi^{\prime}(t) d t d s
$$

Therefore,

$$
\begin{aligned}
& \int_{a}^{b}\left(f(t)-\int_{a}^{t} f^{\prime}(s) d s\right) \phi^{\prime}(t) d t \\
= & \int_{a}^{b} f(t) \phi^{\prime}(t) d t-\int_{a}^{b} \int_{s}^{b} f^{\prime}(s) \phi^{\prime}(t) d t d s \\
= & \int_{a}^{b} f(t) \phi^{\prime}(t) d t-\int_{a}^{b} f^{\prime}(s) \int_{s}^{b} \phi^{\prime}(t) d t d s \\
= & \int_{a}^{b} f(t) \phi^{\prime}(t) d t+\int_{a}^{b} f^{\prime}(s) \phi(s) d s=0
\end{aligned}
$$

Therefore, by Lemma 43.1.8, there exists a constant, denoted as $f(a)$ such that

$$
f(t)-\int_{a}^{t} f^{\prime}(s) d s=f(a)
$$

The integration by parts formula is also important.
Corollary 43.1.10 Suppose $f, f^{\prime} \in L^{1}(a, b ; X)$ and suppose $\phi \in C^{1}([a, b])$. Then the following integration by parts formula holds.

$$
\int_{a}^{b} f(t) \phi^{\prime}(t) d t=f(b) \phi(b)-f(a) \phi(a)-\int_{a}^{b} f^{\prime}(t) \phi(t) d t
$$

Proof: From Theorem 43.1.9

$$
\begin{aligned}
& \int_{a}^{b} f(t) \phi^{\prime}(t) d t \\
= & \int_{a}^{b}\left(f(a)+\int_{a}^{t} f^{\prime}(s) d s\right) \phi^{\prime}(t) d t \\
= & f(a)(\phi(b)-\phi(a))+\int_{a}^{b} \int_{a}^{t} f^{\prime}(s) d s \phi^{\prime}(t) d t \\
= & f(a)(\phi(b)-\phi(a))+\int_{a}^{b} f^{\prime}(s) \int_{s}^{b} \phi^{\prime}(t) d t d s \\
= & f(a)(\phi(b)-\phi(a))+\int_{a}^{b} f^{\prime}(s)(\phi(b)-\phi(s)) d s \\
= & f(a)(\phi(b)-\phi(a))-\int_{a}^{b} f^{\prime}(s) \phi(s) d s+(f(b)-f(a)) \phi(b) \\
= & f(b) \phi(b)-f(a) \phi(a)-\int_{a}^{b} f^{\prime}(s) \phi(s) d s .
\end{aligned}
$$

The interchange in order of integration is justified as in the proof of Theorem 43.1.9.
There is an interesting theorem which is easy to present at this point.

Definition 43.1.11 Let

$$
H^{1}(0, T, X)
$$

denote the functions $f \in L^{2}(0, T, X)$ whose weak derivative $f^{\prime}$ is also in $L^{2}(0, T, X)$.
Proposition 43.1.12 Let $f \in H^{1}(0, T, X)$. Then $f \in C^{0,(1 / 2)}([0, T], X)$ and the inclusion map is continuous.

Proof: First note that

$$
f(t)-f(s)=\int_{s}^{t} f^{\prime}(r) d r
$$

and so

$$
\|f(t)-f(s)\|_{X} \leq \int_{s}^{t}\left\|f^{\prime}(r)\right\|_{X} d r \leq\|f\|_{H^{1}}|t-s|^{1 / 2}
$$

It follows that

$$
\sup _{0 \leq s<t \leq T} \frac{\|f(t)-f(s)\|}{|t-s|^{1 / 2}} \leq\|f\|_{H^{1}}
$$

Also

$$
f(t)=f(0)+\int_{0}^{t} f^{\prime}(s) d s
$$

so

$$
\|f(t)\| \leq\|f(0)\|+\int_{0}^{t}\left|f^{\prime}(s)\right| d s \leq\|f(0)\|+T^{1 / 2}\|f\|_{H^{1}}
$$

Now consider $\|f(0)\|$. Then integrating by parts yields

$$
\int_{0}^{T}(T-t) f^{\prime}(t) d t=\left.(T-t) f(t)\right|_{0} ^{T}+\int_{0}^{t} f(t) d t
$$

and so

$$
T\|f(0)\| \leq \int_{0}^{T}\|f(t)\| d t+T \int_{0}^{T}\left\|f^{\prime}(t)\right\| d t \leq C(T)\|f\|_{H^{1}}
$$

Hence

$$
\sup _{t \in[0, T]}\|f(t)\| \leq C(T)\|f\|_{H^{1}}
$$

Therefore, this has shown that

$$
\|f\|_{C^{0,(1 / 2)}([0, T], X)} \equiv \sup _{t \in[0, T]}\|f(t)\|+\sup _{0 \leq s<t \leq T} \frac{\|f(t)-f(s)\|}{|t-s|^{1 / 2}} \leq C(T)\|f\|_{H^{1}}
$$

You could imagine that other interesting versions of this are available with similar proof for the case where the function and its weak derivative are in $L^{p}(0, T, X)$ for $p>1$.

With this integration by parts formula, the following interesting lemma is obtained. This lemma shows why it was appropriate to define $\bar{f}$ as in Definition 43.1.2.

Lemma 43.1.13 Let $\bar{f}$ be given in Definition 43.1.2 and suppose $f, f^{\prime} \in L^{1}(a, b ; X)$. Then $\bar{f}, \bar{f}^{\prime} \in L^{1}(2 a-b, 2 b-a ; X)$ also and

$$
\bar{f}^{\prime}(t) \equiv\left\{\begin{array}{l}
f^{\prime}(t) \text { if } t \in[a, b]  \tag{43.1.1}\\
-f^{\prime}(2 a-t) \text { if } t \in[2 a-b, a] \\
-f^{\prime}(2 b-t) \text { if } t \in[b, 2 b-a]
\end{array}\right.
$$

Proof: It is clear from the definition of $\bar{f}$ that $\bar{f} \in L^{1}(2 a-b, 2 b-a ; X)$ and that in fact

$$
\begin{equation*}
\|\bar{f}\|_{L^{1}(2 a-b, 2 b-a ; X)} \leq 3\|f\|_{L^{1}(a, b ; X)} \tag{43.1.2}
\end{equation*}
$$

Let $\phi \in C_{c}^{\infty}(2 a-b, 2 b-a)$. Then from the integration by parts formula,

$$
\begin{aligned}
& \int_{2 a-b}^{2 b-a} \bar{f}(t) \phi^{\prime}(t) d t \\
= & \int_{a}^{b} f(t) \phi^{\prime}(t) d t+\int_{b}^{2 b-a} f(2 b-t) \phi^{\prime}(t) d t+\int_{2 a-b}^{a} f(2 a-t) \phi^{\prime}(t) d t \\
= & \int_{a}^{b} f(t) \phi^{\prime}(t) d t+\int_{a}^{b} f(u) \phi^{\prime}(2 b-u) d u+\int_{a}^{b} f(u) \phi^{\prime}(2 a-u) d u \\
= & f(b) \phi(b)-f(a) \phi(a)-\int_{a}^{b} f^{\prime}(t) \phi(t) d t-f(b) \phi(b)+f(a) \phi(2 b-a) \\
& +\int_{a}^{b} f^{\prime}(u) \phi(2 b-u) d u-f(b) \phi(2 a-b) \\
& +f(a) \phi(a)+\int_{a}^{b} f^{\prime}(u) \phi(2 a-u) d u \\
= & -\int_{a}^{b} f^{\prime}(t) \phi(t) d t+\int_{a}^{b} f^{\prime}(u) \phi(2 b-u) d u+\int_{a}^{b} f^{\prime}(u) \phi(2 a-u) d u \\
= & -\int_{a}^{b} f^{\prime}(t) \phi(t) d t-\int_{b}^{2 b-a}-f^{\prime}(2 b-t) \phi(t) d t-\int_{2 a-b}^{a}-f^{\prime}(2 a-t) \phi(t) d t \\
= & -\int_{2 a-b}^{2 b-a} \bar{f}^{\prime}(t) \phi(t) d t
\end{aligned}
$$

where $\bar{f}^{\prime}(t)$ is given in 43.1.1.
Definition 43.1.14 Let $V$ be a Banach space and let $H$ be a Hilbert space. (Typically $H=L^{2}(\Omega)$ ) Suppose $V \subseteq H$ is dense in $H$ meaning that the closure in $H$ of $V$ gives $H$. Then it is often the case that $H$ is identified with its dual space, and then because of the density of $V$ in $H$, it is possible to write

$$
V \subseteq H=H^{\prime} \subseteq V^{\prime}
$$

When this is done, $H$ is called a pivot space. Another notation which is often used is $\langle f, g\rangle$ to denote $f(g)$ for $f \in V^{\prime}$ and $g \in V$. This may also be written as $\langle f, g\rangle_{V^{\prime}, V}$. Another term is that $V \subseteq H=H^{\prime} \subseteq V^{\prime}$ is called a Gelfand triple.

The next theorem is an example of a trace theorem. In this theorem, $f \in L^{p}(0, T ; V)$ while $f^{\prime} \in L^{p}\left(0, T ; V^{\prime}\right)$. It makes no sense to consider the initial values of $f$ in $V$ because it is not even continuous with values in $V$. However, because of the derivative of $f$ it will turn out that $f$ is continuous with values in a larger space and so it makes sense to consider initial values of $f$ in this other space. This other space is called a trace space.

Theorem 43.1.15 Let $V$ and $H$ be a Banach space and Hilbert space as described in Definition 43.1.14. Suppose $f \in L^{p}(0, T ; V)$ and $f^{\prime} \in L^{p^{\prime}}\left(0, T ; V^{\prime}\right)$. Then $f$ is a.e. equal to $a$ continuous function mapping $[0, T]$ to $H$. Furthermore, there exists $f(0) \in H$ such that

$$
\begin{equation*}
\frac{1}{2}|f(t)|_{H}^{2}-\frac{1}{2}|f(0)|_{H}^{2}=\int_{0}^{t}\left\langle f^{\prime}(s), f(s)\right\rangle d s \tag{43.1.3}
\end{equation*}
$$

and for all $t \in[0, T]$,

$$
\begin{equation*}
\int_{0}^{t} f^{\prime}(s) d s \in H \tag{43.1.4}
\end{equation*}
$$

and for a.e. $t \in[0, T]$,

$$
\begin{equation*}
f(t)=f(0)+\int_{0}^{t} f^{\prime}(s) d s \text { in } H \tag{43.1.5}
\end{equation*}
$$

Here $f^{\prime}$ is being taken in the sense of $V^{\prime}$ valued distributions and $\frac{1}{p}+\frac{1}{p^{\prime}}=1$ and $p \geq 2$.
Proof: Let $\Psi \in C_{c}^{\infty}(-T, 2 T)$ satisfy $\Psi(t)=1$ if $t \in[-T / 2,3 T / 2]$ and $\Psi(t) \geq 0$. For $t \in \mathbb{R}$, define

$$
\widehat{f}(t) \equiv\left\{\begin{array}{l}
\bar{f}(t) \Psi(t) \text { if } t \in[-T, 2 T] \\
0 \text { if } t \notin[-T, 2 T]
\end{array}\right.
$$

and

$$
\begin{equation*}
f_{n}(t) \equiv \int_{-1 / n}^{1 / n} \widehat{f}(t-s) \phi_{n}(s) d s \tag{43.1.6}
\end{equation*}
$$

where $\phi_{n}$ is a mollifier having support in $(-1 / n, 1 / n)$. Then by Minkowski's inequality

$$
\begin{aligned}
\| f_{n} & -\widehat{f} \|_{L^{p}(\mathbb{R} ; V)}=\left(\int_{\mathbb{R}}\left\|\widehat{f}(t)-\int_{-1 / n}^{1 / n} \widehat{f}(t-s) \phi_{n}(s) d s\right\|_{V}^{p} d t\right)^{1 / p} \\
& =\left(\int_{\mathbb{R}}\left\|\int_{-1 / n}^{1 / n}(\widehat{f}(t)-\widehat{f}(t-s)) \phi_{n}(s) d s\right\|_{V}^{p} d t\right)^{1 / p} \\
& \leq\left(\int_{\mathbb{R}}\left(\int_{-1 / n}^{1 / n}\|\widehat{f}(t)-\widehat{f}(t-s)\|_{V} \phi_{n}(s) d s\right)^{p} d t\right)^{1 / p} \\
& \leq \int_{-1 / n}^{1 / n} \phi_{n}(s)\left(\int_{\mathbb{R}}\|\widehat{f}(t)-\widehat{f}(t-s)\|_{V}^{p} d t\right)^{1 / p} d s \\
& \leq \int_{-1 / n}^{1 / n} \phi_{n}(s) \varepsilon d s=\varepsilon
\end{aligned}
$$

provided $n$ is large enough. This follows from continuity of translation in $L^{p}$ with Lebesgue measure. Since $\varepsilon>0$ is arbitrary, it follows $f_{n} \rightarrow \widehat{f}$ in $L^{p}(\mathbb{R} ; V)$. Similarly, $f_{n} \rightarrow f$ in $L^{2}(\mathbb{R} ; H)$. This follows because $p \geq 2$ and the norm in $V$ and norm in $H$ are related by $|x|_{H} \leq C\|x\|_{V}$ for some constant, $C$. Now

$$
\widehat{f}(t)=\left\{\begin{array}{l}
\Psi(t) f(t) \text { if } t \in[0, T] \\
\Psi(t) f(2 T-t) \text { if } t \in[T, 2 T] \\
\Psi(t) f(-t) \text { if } t \in[0, T] \\
0 \text { if } t \notin[-T, 2 T]
\end{array}\right.
$$

An easy modification of the argument of Lemma 43.1.13 yields

$$
\widehat{f}^{\prime}(t)=\left\{\begin{array}{l}
\Psi^{\prime}(t) f(t)+\Psi(y) f^{\prime}(t) \text { if } t \in[0, T] \\
\Psi^{\prime}(t) f(2 T-t)-\Psi(t) f^{\prime}(2 T-t) \text { if } t \in[T, 2 T] \\
\Psi^{\prime}(t) f(-t)-\Psi(t) f^{\prime}(-t) \text { if } t \in[-T, 0] \\
0 \text { if } t \notin[-T, 2 T]
\end{array}\right.
$$

Recall

$$
\begin{aligned}
f_{n}(t) & =\int_{-1 / n}^{1 / n} \widehat{f}(t-s) \phi_{n}(s) d s=\int_{\mathbb{R}} \widehat{f}(t-s) \phi_{n}(s) d s \\
& =\int_{\mathbb{R}} \widehat{f}(s) \phi_{n}(t-s) d s
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f_{n}^{\prime}(t) & =\int_{\mathbb{R}} \widehat{f}(s) \phi_{n}^{\prime}(t-s) d s=\int_{-T-\frac{1}{n}}^{2 T+\frac{1}{n}} \widehat{f}(s) \phi_{n}^{\prime}(t-s) d s \\
& =\int_{-T-\frac{1}{n}}^{2 T+\frac{1}{n}} \widehat{f}^{\prime}(s) \phi_{n}(t-s) d s=\int_{\mathbb{R}} \widehat{f}^{\prime}(s) \phi_{n}(t-s) d s \\
& =\int_{\mathbb{R}} \widehat{f}^{\prime}(t-s) \phi_{n}(s) d s=\int_{-1 / n}^{1 / n} \widehat{f}^{\prime}(t-s) \phi_{n}(s) d s
\end{aligned}
$$

and it follows from the first line above that $f_{n}^{\prime}$ is continuous with values in $V$ for all $t \in \mathbb{R}$. Also note that both $f_{n}^{\prime}$ and $f_{n}$ equal zero if $t \notin[-T, 2 T]$ whenever $n$ is large enough. Exactly similar reasoning to the above shows that $f_{n}^{\prime} \rightarrow \widehat{f}^{\prime}$ in $L^{p^{\prime}}\left(\mathbb{R} ; V^{\prime}\right)$.

Now let $\phi \in C_{c}^{\infty}(0, T)$.

$$
\begin{align*}
\int_{\mathbb{R}}\left|f_{n}(t)\right|_{H}^{2} \phi^{\prime}(t) d t & =\int_{\mathbb{R}}\left(f_{n}(t), f_{n}(t)\right)_{H} \phi^{\prime}(t) d t  \tag{43.1.7}\\
=-\int_{\mathbb{R}} 2\left(f_{n}^{\prime}(t), f_{n}(t)\right) \phi(t) d t & =-\int_{\mathbb{R}} 2\left\langle f_{n}^{\prime}(t), f_{n}(t)\right\rangle \phi(t) d t
\end{align*}
$$

Now

$$
\begin{aligned}
& \left|\int_{\mathbb{R}}\left\langle f_{n}^{\prime}(t), f_{n}(t)\right\rangle \phi(t) d t-\int_{\mathbb{R}}\left\langle f^{\prime}(t), f(t)\right\rangle \phi(t) d t\right| \\
\leq & \int_{\mathbb{R}}\left(\left|\left\langle f_{n}^{\prime}(t)-f^{\prime}(t), f_{n}(t)\right\rangle\right|+\left|\left\langle f^{\prime}(t), f_{n}(t)-f(t)\right\rangle\right|\right) \phi(t) d t
\end{aligned}
$$

From the first part of this proof which showed that $f_{n} \rightarrow \widehat{f}$ in $L^{p}(\mathbb{R} ; V)$ and $f_{n}^{\prime} \rightarrow \widehat{f^{\prime}}$ in $L^{p^{\prime}}\left(\mathbb{R} ; V^{\prime}\right)$, an application of Holder's inequality shows the above converges to 0 as $n \rightarrow \infty$. Therefore, passing to the limit as $n \rightarrow \infty$ in the 43.1.8,

$$
\int_{\mathbb{R}}|\widehat{f}(t)|_{H}^{2} \phi^{\prime}(t) d t=-\int_{\mathbb{R}} 2\left\langle\widehat{f}^{\prime}(t), \widehat{f}(t)\right\rangle \phi(t) d t
$$

which shows $t \rightarrow|\widehat{f}(t)|_{H}^{2}$ equals a continuous function a.e. and it also has a weak derivative equal to $2\left\langle\widehat{f}^{\prime}, \widehat{f}\right\rangle$.

It remains to verify that $\widehat{f}$ is continuous on $[0, T]$. Of course $\widehat{f}=f$ on this interval. Let $N$ be large enough that $f_{n}(-T)=0$ for all $n>N$. Then for $m, n>N$ and $t \in[-T, 2 T]$

$$
\begin{aligned}
\left|f_{n}(t)-f_{m}(t)\right|_{H}^{2} & =2 \int_{-T}^{t}\left(f_{n}^{\prime}(s)-f_{m}^{\prime}(s), f_{n}(s)-f_{m}(s)\right) d s \\
& =2 \int_{-T}^{t}\left\langle f_{n}^{\prime}(s)-f_{m}^{\prime}(s), f_{n}(s)-f_{m}(s)\right\rangle_{V^{\prime}, V} d s \\
& \leq 2 \int_{\mathbb{R}}\left\|f_{n}^{\prime}(s)-f_{m}^{\prime}(s)\right\|_{V^{\prime}}\left\|f_{n}(s)-f_{m}(s)\right\|_{V} d s \\
& \leq 2\left\|f_{n}-f_{m}\right\|_{L^{p^{\prime}\left(\mathbb{R} ; V^{\prime}\right)}}\left\|f_{n}-f_{m}\right\|_{L^{p}(\mathbb{R} ; V)}
\end{aligned}
$$

which shows from the above that $\left\{f_{n}\right\}$ is uniformly Cauchy on $[-T, 2 T]$ with values in $H$. Therefore, there exists $g$ a continuous function defined on $[-T, 2 T]$ having values in $H$ such that

$$
\lim _{n \rightarrow \infty} \max \left\{\left|f_{n}(t)-g(t)\right|_{H} ; t \in[-T, 2 T]\right\}=0
$$

However, $g=\widehat{f}$ a.e. because $f_{n}$ converges to $f$ in $L^{p}(0, T ; V)$. Therefore, taking a subsequence, the convergence is a.e. It follows from the fact that $V \subseteq H=H^{\prime} \subseteq V^{\prime}$ and Theorem 43.1.9, there exists $f(0) \in V^{\prime}$ such that for a.e. $t$,

$$
f(t)=f(0)+\int_{0}^{t} f^{\prime}(s) d s \text { in } V^{\prime}
$$

Now $g=f$ a.e. and $g$ is continuous with values in $H$ hence continuous with values in $V^{\prime}$ and so

$$
g(t)=f(0)+\int_{0}^{t} f^{\prime}(s) d s \text { in } V^{\prime}
$$

for all $t$. Since $g$ is continuous with values in $H$ it is continuous with values in $V^{\prime}$. Taking the limit as $t \downarrow 0$ in the above, $g(a)=\lim _{t \rightarrow 0+} g(t)=f(0)$, showing that $f(0) \in H$. Therefore, for a.e. $t$,

$$
f(t)=f(0)+\int_{0}^{t} f^{\prime}(s) d s \text { in } H, \int_{0}^{t} f^{\prime}(s) d s \in H .
$$

Note that if $f \in L^{p}(0, T ; V)$ and $f^{\prime} \in L^{p^{\prime}}\left(0, T ; V^{\prime}\right)$, then you can consider the initial value of $f$ and it will be in $H$. What if you start with something in $H$ ? Is it an initial condition for a function $f \in L^{p}(0, T ; V)$ such that $f^{\prime} \in L^{p^{\prime}}\left(0, T ; V^{\prime}\right)$ ? This is worth thinking
about. If it is not so, what is the space of initial values? How can you give this space a norm? What are its properties? It turns out that if $V$ is a closed subspace of the Sobolev space, $W^{1, p}(\Omega)$ which contains $W_{0}^{1, p}(\Omega)$ for $p \geq 2$ and $H=L^{2}(\Omega)$ the answer to the above question is yes. Not surprisingly, there are many generalizations of the above ideas.

### 43.2 An Important Formula

It is not necessary to have $p>2$ in order to do the sort of thing just described. Here is a major result which will have a much more difficult stochastic version presented later. First is a simple version of an approximation theorem of Doob.

Lemma 43.2.1 Let $Y:[0, T] \rightarrow E$, be $\mathscr{B}([0, T])$ measurable and suppose

$$
Y \in L^{p}(0, T ; E) \equiv K, p \geq 1
$$

Then there exists a sequence of nested partitions, $\mathscr{P}_{k} \subseteq \mathscr{P}_{k+1}$,

$$
\mathscr{P}_{k} \equiv\left\{t_{0}^{k}, \cdots, t_{m_{k}}^{k}\right\}
$$

such that the step functions given by

$$
\begin{aligned}
Y_{k}^{r}(t) & \equiv \sum_{j=1}^{m_{k}} Y\left(t_{j}^{k}\right) \mathscr{X}_{\left[t_{j-1}^{k}, t_{j}^{k}\right)}(t) \\
Y_{k}^{l}(t) & \equiv \sum_{j=1}^{m_{k}} Y\left(t_{j-1}^{k}\right) \mathscr{X}_{\left(t_{j-1}^{k}, t_{j}^{k}\right]}(t)
\end{aligned}
$$

both converge to $Y$ in $K$ as $k \rightarrow \infty$ and

$$
\lim _{k \rightarrow \infty} \max \left\{\left|t_{j}^{k}-t_{j+1}^{k}\right|: j \in\left\{0, \cdots, m_{k}\right\}\right\}=0
$$

Also, each $Y\left(t_{j}^{k}\right), Y\left(t_{j-1}^{k}\right)$ is in $E$. One can also assume that $Y(0)=0$. The mesh points $\left\{t_{j}^{k}\right\}_{j=0}^{m_{k}}$ can be chosen to miss a given set of measure zero. In addition to this, we can assume that

$$
\left|t_{j}^{k}-t_{j-1}^{k}\right|=2^{-n_{k}}
$$

except for the case where $j=1$ or $j=m_{n_{k}}$ when this might not be so. In the case of the last subinterval defined by the partition, we can assume

$$
\left|t_{m}^{k}-t_{m-1}^{k}\right|=\left|T-t_{m-1}^{k}\right| \geq 2^{-\left(n_{k}+1\right)}
$$

Proof: For $t \in \mathbb{R}$ let $\gamma_{n}(t) \equiv k / 2^{n}, \delta_{n}(t) \equiv(k+1) / 2^{n}$, where $t \in\left(k / 2^{n},(k+1) / 2^{n}\right]$, and $2^{-n}<T / 4$. Also suppose $Y$ is defined to equal 0 on $[0, T]^{C}$. Then $t \rightarrow\|Y(t)\|$ is in $L^{p}(\mathbb{R})$. Therefore by continuity of translation, as $n \rightarrow \infty$ it follows that for $t \in[0, T]$,

$$
\int_{\mathbb{R}}\left\|Y\left(\gamma_{n}(t)+s\right)-Y(t+s)\right\|_{E}^{p} d s \rightarrow 0
$$

The above is dominated by

$$
\begin{aligned}
& \int_{\mathbb{R}} 2^{p-1}\left(\|Y(s)\|^{p}+\|Y(s)\|^{p}\right) \mathscr{X}_{[-2 T, 2 T]}(s) d s \\
= & \int_{-2 T}^{2 T} 2^{p-1}\left(\|Y(s)\|^{p}+\|Y(s)\|^{p}\right) d s<\infty
\end{aligned}
$$

Therefore,

$$
\lim _{n \rightarrow \infty} \int_{-2 T}^{2 T}\left(\int_{\mathbb{R}}\left\|Y\left(\gamma_{n}(t)+s\right)-Y(t+s)\right\|_{E}^{p} d s\right) d t=0
$$

by the dominated convergence theorem. Now Fubini. The above equals

$$
\int_{\mathbb{R}} \int_{-2 T}^{2 T}\left\|Y\left(\gamma_{n}(t)+s\right)-Y(t+s)\right\|_{E}^{p} d t d s
$$

Change the variables on the inside.

$$
\int_{\mathbb{R}} \int_{-2 T+s}^{2 T+s}\left\|Y\left(\gamma_{n}(t-s)+s\right)-Y(t)\right\|_{E}^{p} d t d s
$$

Since $\gamma_{n}(t-s)+s$ is within $2^{-n}$ of $t$ and $Y(t)$ vanishes if $t \notin[0, T]$, this reduces to

$$
\int_{\mathbb{R}} \int_{-2 T}^{2 T}\left\|Y\left(\gamma_{n}(t-s)+s\right)-Y(t)\right\|_{E}^{p} d t d s
$$

This converges to 0 as $n \rightarrow \infty$ as was shown above. Therefore,

$$
\int_{0}^{T} \int_{0}^{T}\left\|Y\left(\gamma_{n}(t-s)+s\right)-Y(t)\right\|_{E}^{p} d t d s
$$

also converges to 0 as $n \rightarrow \infty$. The only problem is that $\gamma_{n}(t-s)+s \geq t-2^{-n}$ and so $\gamma_{n}(t-s)+s$ could be less than 0 for $t \in\left[0,2^{-n}\right]$. Since this is an interval whose measure converges to 0 it follows

$$
\int_{0}^{T} \int_{0}^{T}\left\|Y\left(\left(\gamma_{n}(t-s)+s\right)^{+}\right)-Y(t)\right\|_{E}^{p} d t d s
$$

converges to 0 as $n \rightarrow \infty$. Let

$$
m_{n}(s)=\int_{0}^{T}\left\|Y\left(\left(\gamma_{n}(t-s)+s\right)^{+}\right)-Y(t)\right\|_{E}^{p} d t
$$

Then

$$
P\left(\left[m_{n}(s)>\lambda\right]\right) \leq \frac{1}{\lambda} \int_{0}^{T} m_{n}(s) d s
$$

It follows there exists a subsequence $n_{k}$ such that

$$
P\left(\left[m_{n_{k}}(s)>\frac{1}{k}\right]\right)<2^{-k}
$$

Hence by the Borel Cantelli lemma, there exists a set of measure zero $N$ such that for $s \notin N$,

$$
m_{n_{k}}(s) \leq 1 / k
$$

for all $k$ sufficiently large. Picking $s_{k} \notin N$,

$$
Y_{k}^{l}(t) \equiv Y\left(\left(\gamma_{n_{k}}\left(t-s_{k}\right)+s_{k}\right)^{+}\right)
$$

Then $t \rightarrow Y\left(\left(\gamma_{n_{k}}\left(t-s_{k}\right)+s_{k}\right)^{+}\right)$is a step function of the sort described above. Of course you can always simply define $Y_{k}^{l}(0) \equiv 0$. This is because the interval affected has length which converges to 0 as $k \rightarrow \infty$. The jumps in $t \rightarrow \gamma_{n_{k}}\left(t-s_{k}\right)$ determine the mesh points of the partition. By picking $s_{k}$ appropriately, you can have each of these mesh points miss a given set of measure zero except for the first and last point. This is because when you slide $s_{k}$ it just moves the mesh points of $\mathscr{P}_{k}$ except for the first point and last point. Let $N_{1}$ be a set of measure zero and let $(a, b) \subseteq[0, T]$. Now let $s$ move through $(a, b)$ and denote by $A_{j}$ the corresponding set of points obtained by the $j^{t h}$ mesh point. Thus $A_{j}$ has positive measure and so it is not contained in $N_{1}$. Let $S_{j}$ be the points of $(a, b)$ which correspond to $A_{j} \cap N_{1}$. Thus $S_{j}$ has measure 0 . Just pick $s_{k} \in(a, b) \backslash \cup_{j} S_{j}$. You can also choose $s_{k}$ such that

$$
T-s_{k}-\gamma_{n_{k}}\left(T-s_{k}\right)>2^{-\left(n_{k}+1\right)}
$$

which will cause the last condition mentioned above to hold.
To get the other sequence of step functions, just use a similar argument with $\boldsymbol{\delta}_{n}$ in place of $\gamma_{n}$.

Theorem 43.2.2 Let $V \subseteq H=H^{\prime} \subseteq V^{\prime}$ be a Gelfand triple and suppose $Y \in L^{p^{\prime}}\left(0, T ; V^{\prime}\right) \equiv$ $K^{\prime}$ and

$$
\begin{equation*}
X(t)=X_{0}+\int_{0}^{t} Y(s) d s \text { in } V^{\prime} \tag{43.2.8}
\end{equation*}
$$

where $X_{0} \in H$, and it is known that $X \in L^{p}(0, T, V) \equiv K$ for $p>1$. Then $t \rightarrow X(t)$ is in $C([0, T], H)$ and also

$$
\frac{1}{2}|X(t)|_{H}^{2}=\frac{1}{2}\left|X_{0}\right|_{H}^{2}+\int_{0}^{t}\langle Y(s), X(s)\rangle d s
$$

Proof: By Lemma 43.2.1, there exists a sequence of uniform partitions $\left\{t_{k}^{n}\right\}_{k=0}^{m_{n}}=$ $\mathscr{P}_{n}, \mathscr{P}_{n} \subseteq \mathscr{P}_{n+1}$, of $[0, T]$ such that the step functions

$$
\begin{aligned}
\sum_{k=0}^{m_{n}-1} X\left(t_{k}^{n}\right) \mathscr{X}_{\left(t_{k}^{n}, t_{k+1}^{n}\right]}(t) & \equiv X^{l}(t) \\
\sum_{k=0}^{m_{n}-1} X\left(t_{k+1}^{n}\right) \mathscr{X}_{\left(t_{k}^{n}, t_{k+1}^{n}\right]}(t) & \equiv X^{r}(t)
\end{aligned}
$$

converge to $X$ in $K$ and in $L^{2}([0, T], H)$.

Lemma 43.2.3 Let $s<t$. Then for $X, Y$ satisfying 43.2.8

$$
\begin{equation*}
|X(t)|^{2}=|X(s)|^{2}+2 \int_{s}^{t}\langle Y(u), X(t)\rangle d u-|X(t)-X(s)|^{2} \tag{43.2.9}
\end{equation*}
$$

Proof: It follows from the following computations

$$
\begin{gathered}
X(t)-X(s)=\int_{s}^{t} Y(u) d u \\
-|X(t)-X(s)|^{2}=-|X(t)|^{2}+2(X(t), X(s))-|X(s)|^{2} \\
=-|X(t)|^{2}+2\left(X(t), X(t)-\int_{s}^{t} Y(u) d u\right)-|X(s)|^{2} \\
=-|X(t)|^{2}+2|X(t)|^{2}-2\left\langle\int_{s}^{t} Y(u) d u, X(t)\right\rangle-|X(s)|^{2}
\end{gathered}
$$

Hence

$$
|X(t)|^{2}=|X(s)|^{2}+2 \int_{s}^{t}\langle Y(u), X(t)\rangle d u-|X(t)-X(s)|^{2}
$$

Lemma 43.2.4 In the above situation,

$$
\sup _{t \in[0, T]}|X(t)|_{H} \leq C\left(\|Y\|_{K^{\prime}},\|X\|_{K}\right)
$$

Also, $t \rightarrow X(t)$ is weakly continuous with values in $H$.
Proof: From the above formula applied to the $k^{t h}$ partition of $[0, T]$ described above,

$$
\begin{aligned}
& \left|X\left(t_{m}\right)\right|^{2}-\left|X_{0}\right|^{2}=\sum_{j=0}^{m-1}\left|X\left(t_{j+1}\right)\right|^{2}-\left|X\left(t_{j}\right)\right|^{2} \\
= & \sum_{j=0}^{m-1} 2 \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X\left(t_{j+1}\right)\right\rangle d u-\left|X\left(t_{j+1}\right)-X\left(t_{j}\right)\right|_{H}^{2} \\
= & \sum_{j=0}^{m-1} 2 \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u-\left|X\left(t_{j+1}\right)-X\left(t_{j}\right)\right|_{H}^{2}
\end{aligned}
$$

Thus, discarding the negative terms and denoting by $\mathscr{P}_{k}$ the $k^{t h}$ of these partitions,

$$
\begin{gathered}
\sup _{t_{j} \in \mathscr{P}_{k}}\left|X\left(t_{j}\right)\right|_{H}^{2} \leq\left|X_{0}\right|^{2}+2 \int_{0}^{T}\left|\left\langle Y(u), X_{k}^{r}(u)\right\rangle\right| d u \\
\quad \leq\left|X_{0}\right|^{2}+2 \int_{0}^{T}\|Y(u)\|_{V^{\prime}}\left\|X_{k}^{r}(u)\right\|_{V} d u
\end{gathered}
$$

$$
\leq\left|X_{0}\right|^{2}+2\left(\int_{0}^{T}\|Y(u)\|_{V^{\prime}}^{p^{\prime}} d u\right)^{1 / p^{\prime}}\left(\int_{0}^{T}\left\|X_{k}^{r}(u)\right\|_{V}^{p} d u\right)^{1 / p} \leq C\left(\|Y\|_{K^{\prime}},\|X\|_{K}\right)
$$

because these partitions are chosen such that

$$
\lim _{k \rightarrow \infty}\left(\int_{0}^{T}\left\|X_{k}^{r}(u)\right\|_{V}^{p}\right)^{1 / p}=\left(\int_{0}^{T}\|X(u)\|_{V}^{p}\right)^{1 / p}
$$

and so these are bounded. This has shown that for the dense subset of $[0, T], D \equiv \cup_{k} \mathscr{P}_{k}$,

$$
\sup _{t \in D}|X(t)|<C\left(\|Y\|_{K^{\prime}},\|X\|_{K}\right)
$$

Now let $\left\{g_{k}\right\}_{k=1}^{\infty}$ be linearly independent vectors of $V$ whose span is dense in $V$. This is possible because $V$ is separable. Then let $\left\{e_{j}\right\}_{j=1}^{\infty}$ be an orthonormal basis for $H$ such that $e_{k} \in \operatorname{span}\left(g_{1}, \ldots, g_{k}\right)$ and each $g_{k} \in \operatorname{span}\left(e_{1}, \ldots, e_{k}\right)$. This is done with the Gram Schmidt process. Then it follows that $\operatorname{span}\left(\left\{e_{k}\right\}_{k=1}^{\infty}\right)$ is dense in $V$. I claim

$$
|y|_{H}^{2}=\sum_{j=1}^{\infty}\left|\left\langle y, e_{j}\right\rangle\right|^{2}
$$

This is certainly true if $y \in H$ because

$$
\left\langle y, e_{j}\right\rangle=\left(y, e_{j}\right)_{H}
$$

If $y \notin H$, then the series must diverge since otherwise, you could consider the infinite sum

$$
\sum_{j=1}^{\infty}\left\langle y, e_{j}\right\rangle e_{j} \in H
$$

because

$$
\left|\sum_{j=p}^{q}\left\langle y, e_{j}\right\rangle e_{j}\right|^{2}=\sum_{j=p}^{q}\left|\left\langle y, e_{j}\right\rangle\right|^{2} \rightarrow 0 \text { as } p, q \rightarrow \infty
$$

Letting $z=\sum_{j=1}^{\infty}\left\langle y, e_{j}\right\rangle e_{j}$, it follows that $\left\langle y, e_{j}\right\rangle$ is the $j^{\text {th }}$ Fourier coefficient of $z$ and that

$$
\langle z-y, v\rangle=0
$$

for all $v \in \operatorname{span}\left(\left\{e_{k}\right\}_{k=1}^{\infty}\right)$ which is dense in $V$. Therefore, $z=y$ in $V^{\prime}$ and so $y \in H$.
It follows

$$
|X(t)|^{2}=\sup _{n} \sum_{j=1}^{n}\left|\left\langle X(t), e_{j}\right\rangle\right|^{2}
$$

which is just the sup of continuous functions of $t$. Therefore, $t \rightarrow|X(t)|^{2}$ is lower semicontinuous. It follows that for any $t$, letting $t_{j} \rightarrow t$ for $t_{j} \in D$,

$$
|X(t)|^{2} \leq \lim \inf _{j \rightarrow \infty}\left|X\left(t_{j}\right)\right|^{2} \leq C\left(\|Y\|_{K^{\prime}},\|X\|_{K}\right)
$$

This proves the first claim of the lemma.
Consider now the claim that $t \rightarrow X(t)$ is weakly continuous. Letting $v \in V$,

$$
\lim _{t \rightarrow s}(X(t), v)=\lim _{t \rightarrow s}\langle X(t), v\rangle=\langle X(s), v\rangle=(X(s), v)
$$

Since it was shown that $|X(t)|$ is bounded independent of $t$, and since $V$ is dense in $H$, the claim follows.

Now

$$
\begin{aligned}
-\sum_{j=0}^{m-1}\left|X\left(t_{j+1}\right)-X\left(t_{j}\right)\right|_{H}^{2} & =\left|X\left(t_{m}\right)\right|^{2}-\left|X_{0}\right|^{2}-\sum_{j=0}^{m-1} 2 \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u \\
& =\left|X\left(t_{m}\right)\right|^{2}-\left|X_{0}\right|^{2}-2 \int_{0}^{t_{m}}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u
\end{aligned}
$$

Thus, since the partitions are nested, eventually $\left|X\left(t_{m}\right)\right|^{2}$ is constant for all $k$ large enough and the integral term converges to

$$
\int_{0}^{t_{m}}\langle Y(u), X(u)\rangle d u
$$

It follows that the term on the left does converge to something. It just remains to consider what it does converge to. However, from the equation solved by $X$,

$$
X\left(t_{j+1}\right)-X\left(t_{j}\right)=\int_{t_{j}}^{t_{j+1}} Y(u) d u
$$

Therefore, this term is dominated by an expression of the form

$$
\begin{gathered}
\sum_{j=0}^{m_{k}-1}\left(\int_{t_{j}}^{t_{j+1}} Y(u) d u, X\left(t_{j+1}\right)-X\left(t_{j}\right)\right) \\
=\sum_{j=0}^{m_{k}-1}\left\langle\int_{t_{j}}^{t_{j+1}} Y(u) d u, X\left(t_{j+1}\right)-X\left(t_{j}\right)\right\rangle \\
=\sum_{j=0}^{m_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X\left(t_{j+1}\right)-X\left(t_{j}\right)\right\rangle d u \\
=\sum_{j=0}^{m_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X\left(t_{j+1}\right)\right\rangle-\sum_{j=0}^{m_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X\left(t_{j}\right)\right\rangle \\
=\int_{0}^{T}\left\langle Y(u), X^{r}(u)\right\rangle d u-\int_{0}^{T}\left\langle Y(u), X^{l}(u)\right\rangle d u
\end{gathered}
$$

However, both $X^{r}$ and $X^{l}$ converge to $X$ in $K=L^{p}(0, T, V)$. Therefore, this term must converge to 0 . Passing to a limit, it follows that for all $t \in D$, the desired formula holds. Thus, for such $t$,

$$
|X(t)|^{2}=\left|X_{0}\right|^{2}+2 \int_{0}^{t}\langle Y(u), X(u)\rangle d u
$$

It remains to verify that this holds for all $t$. Let $t \notin D$ and let $t(k) \in \mathscr{P}_{k}$ be the largest point of $\mathscr{P}_{k}$ which is less than $t$. Suppose $t(m) \leq t(k)$ so that $m \leq k$. Then

$$
X(t(m))=X_{0}+\int_{0}^{t(m)} Y(s) d s
$$

a similar formula for $X(t(k))$. Thus for $t>t(m)$,

$$
X(t)-X(t(m))=\int_{t(m)}^{t} Y(s) d s
$$

which is the same sort of thing already looked at except that it starts at $t(m)$ rather than at 0 and $X_{0}=0$. Therefore,

$$
|X(t(k))-X(t(m))|^{2}=2 \int_{t(m)}^{t(k)}\langle Y(s), X(s)-X(t(m))\rangle d s
$$

Thus, for $m \leq k$

$$
\lim _{m, k \rightarrow \infty}|X(t(k))-X(t(m))|^{2}=0
$$

Hence $\{X(t(k))\}_{k=1}^{\infty}$ is a convergent sequence in $H$. Does it converge to $X(t)$ ? Let $\xi(t) \in H$ be what it does converge to. Let $v \in V$. Then

$$
(\xi(t), v)=\lim _{k \rightarrow \infty}(X(t(k)), v)=\lim _{k \rightarrow \infty}\langle X(t(k)), v\rangle=\langle X(t), v\rangle=(X(t), v)
$$

because it is known that $t \rightarrow X(t)$ is continuous into $V^{\prime}$ and it is also known that $X(t) \in H$ and that the $X(t)$ for $t \in[0, T]$ are uniformly bounded. Therefore, since $V$ is dense in $H$, it follows that $\xi(t)=X(t)$.

Now for every $t \in D$, it was shown above that

$$
|X(t)|^{2}=\left|X_{0}\right|^{2}+2 \int_{0}^{t}\langle Y(s), X(s)\rangle d s
$$

Thus, using what was just shown, if $t \notin D$ and $t_{k} \rightarrow t$,

$$
\begin{aligned}
|X(t)|^{2} & =\lim _{k \rightarrow \infty}\left|X\left(t_{k}\right)\right|^{2}=\lim _{k \rightarrow \infty}\left(\left|X_{0}\right|^{2}+2 \int_{0}^{t_{k}}\langle Y(s), X(s)\rangle d s\right) \\
& =\left|X_{0}\right|^{2}+2 \int_{0}^{t}\langle Y(s), X(s)\rangle d s
\end{aligned}
$$

which proves the desired formula. From this it follows right away that $t \rightarrow X(t)$ is continuous into $H$ because it was just shown that $t \rightarrow|X(t)|$ is continuous and $t \rightarrow X(t)$ is weakly continuous. Since Hilbert space is uniformly convex, this implies the $t \rightarrow X(t)$ is continuous. To see this in the special cas of Hilbert space,

$$
|X(t)-X(s)|^{2}=|X(t)|^{2}-2(X(s), X(t))+|X(s)|^{2}
$$

Then $\lim _{t \rightarrow s}\left(|X(t)|^{2}-2(X(s), X(t))+|X(s)|^{2}\right)=0$ by weak convergence of $X(t)$ to $X(s)$ and the convergence of $|X(t)|^{2}$ to $|X(s)|^{2}$.

### 43.3 The Implicit Case

The above theorem can be generalized to the case where the formula is of the form

$$
B X(t)=B X_{0}+\int_{0}^{t} Y(s) d s
$$

This involves an operator $B \in \mathscr{L}\left(W, W^{\prime}\right)$ and $B$ satisfies

$$
\langle B x, x\rangle \geq 0,\langle B x, y\rangle=\langle B y, x\rangle
$$

for

$$
V \subseteq W, W^{\prime} \subseteq V^{\prime}
$$

Where $V$ is dense in the Hilbert space $W$. Before giving the theorem, here is a technical lemma.

Lemma 43.3.1 Suppose $V, W$ are separable Banach spaces, $W$ also a Hilbert space such that $V$ is dense in $W$ and $B \in \mathscr{L}\left(W, W^{\prime}\right)$ satisfies

$$
\langle B x, x\rangle \geq 0,\langle B x, y\rangle=\langle B y, x\rangle, B \neq 0 .
$$

Then there exists a countable set $\left\{e_{i}\right\}$ of vectors in $V$ such that

$$
\left\langle B e_{i}, e_{j}\right\rangle=\delta_{i j}
$$

and for each $x \in W$,

$$
\langle B x, x\rangle=\sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2},
$$

and also

$$
B x=\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i}
$$

the series converging in $W^{\prime}$.
Proof: Let $\left\{g_{k}\right\}_{k=1}^{\infty}$ be linearly independent vectors of $V$ whose span is dense in $V$. This is possible because $V$ is separable. Thus, their span is also dense in $W$. Let $n_{1}$ be the first index such that $\left\langle B g_{n_{1}}, g_{n_{1}}\right\rangle \neq 0$.

Claim: If there is no such index, then $B=0$.
Proof of claim: First note that if there is no such first index, then if $x=\sum_{i=1}^{k} a_{i} g_{i}$

$$
\begin{aligned}
|\langle B x, x\rangle| & =\left|\sum_{i \neq j} a_{i} a_{j}\left\langle B g_{i}, g_{j}\right\rangle\right| \leq \sum_{i \neq j}\left|a_{i}\right|\left|a_{j}\right|\left|\left\langle B g_{i}, g_{j}\right\rangle\right| \\
& \leq \sum_{i \neq j}\left|a_{i}\right|\left|a_{j}\right|\left\langle B g_{i}, g_{i}\right\rangle^{1 / 2}\left\langle B g_{j}, g_{j}\right\rangle^{1 / 2}=0
\end{aligned}
$$

Therefore, if $x$ is given, you could take $x_{k}$ in the span of $\left\{g_{1}, \cdots, g_{k}\right\}$ such that $\left\|x_{k}-x\right\|_{W} \rightarrow$ 0 . Then

$$
|\langle B x, y\rangle|=\lim _{k \rightarrow \infty}\left|\left\langle B x_{k}, y\right\rangle\right| \leq \lim _{k \rightarrow \infty}\left\langle B x_{k}, x_{k}\right\rangle^{1 / 2}\langle B y, y\rangle^{1 / 2}=0
$$

because $\left\langle B x_{k}, x_{k}\right\rangle$ is zero by what was just shown.
Thus assume there is such a first index. Let

$$
e_{1} \equiv \frac{g_{n_{1}}}{\left\langle B g_{n_{1}}, g_{n_{1}}\right\rangle^{1 / 2}}
$$

Then $\left\langle B e_{1}, e_{1}\right\rangle=1$. Now if you have constructed $e_{j}$ for $j \leq k$,

$$
e_{j} \in \operatorname{span}\left(g_{n_{1}}, \cdots, g_{n_{k}}\right),\left\langle B e_{i}, e_{j}\right\rangle=\delta_{i j},
$$

$g_{n_{j+1}}$ being the first for which

$$
\left\langle B g_{n_{j+1}}-\sum_{i=1}^{j}\left\langle B g_{n_{j+1}}, e_{i}\right\rangle B e_{i}, g_{n_{j+1}}-\sum_{i=1}^{j}\left\langle B g_{n j}, e_{i}\right\rangle e_{i}\right\rangle \neq 0,
$$

and

$$
\operatorname{span}\left(g_{n_{1}}, \cdots, g_{n_{k}}\right)=\operatorname{span}\left(e_{1}, \cdots, e_{k}\right)
$$

let $g_{n_{k+1}}$ be such that $g_{n_{k+1}}$ is the first in the list $\left\{g_{n_{k}}\right\}$ such that

$$
\left\langle B g_{n_{k+1}}-\sum_{i=1}^{k}\left\langle B g_{n_{k+1}}, e_{i}\right\rangle B e_{i}, g_{n_{k+1}}-\sum_{i=1}^{k}\left\langle B g_{n_{k+1}}, e_{i}\right\rangle e_{i}\right\rangle \neq 0
$$

Note the difference between this and the Gram Schmidt process. Here you don't necessarily use all of the $g_{k}$ due to the possible degeneracy of $B$.

Claim: If there is no such first $g_{n_{k+1}}$, then $B\left(\operatorname{span}\left(e_{i}, \cdots, e_{k}\right)\right)=B W$ so in this case, $\left\{B e_{i}\right\}_{i=1}^{k}$ is actually a basis for $B W$.

Proof: Let $x \in W$. Let $x_{r} \in \operatorname{span}\left(g_{1}, \cdots, g_{r}\right), r>n_{k}$ such that $\lim _{r \rightarrow \infty} x_{r}=x$ in $W$. Then

$$
\begin{equation*}
x_{r}=\sum_{i=1}^{k} c_{i}^{r} e_{i}+\sum_{i \notin\left\{n_{1}, \cdots, n_{k}\right\}}^{r} d_{i}^{r} g_{i} \equiv y_{r}+z_{r} \tag{43.3.10}
\end{equation*}
$$

If $l \notin\left\{n_{1}, \cdots, n_{k}\right\}$, then by the construction and the above assumption, for some $j \leq k$

$$
\begin{equation*}
\left\langle B g_{l}-\sum_{i=1}^{j}\left\langle B g_{l}, e_{i}\right\rangle B e_{i}, g_{l}-\sum_{i=1}^{j}\left\langle B g_{l}, e_{i}\right\rangle e_{i}\right\rangle=0 \tag{43.3.11}
\end{equation*}
$$

If $l<n_{k}$, this follows from the construction. If the above is nonzero all $j \leq k$, then $l$ would have been chosen but it wasn't. Thus

$$
B g_{l}=\sum_{i=1}^{j}\left\langle B g_{l}, e_{i}\right\rangle B e_{i}
$$

If $l>n_{k}$, then by assumption, 43.3.11 holds for $j=k$. Thus, in any case, it follows that for each $l \notin\left\{n_{1}, \cdots, n_{k}\right\}$,

$$
B g_{l} \in B\left(\operatorname{span}\left(e_{i}, \cdots, e_{k}\right)\right)
$$

Now it follows from 43.3.10 that

$$
\begin{aligned}
B x_{r} & =\sum_{i=1}^{k} c_{i}^{r} B e_{i}+\sum_{i \notin\left\{n_{1}, \cdots, n_{k}\right\}}^{r} d_{i}^{r} B g_{i} \\
& =\sum_{i=1}^{k} c_{i}^{r} B e_{i}+\sum_{i \notin\left\{n_{1}, \cdots, n_{k}\right\}}^{r} d_{i}^{r} \sum_{j=1}^{k} c_{j}^{i} B e_{j}
\end{aligned}
$$

and so $B x_{r} \in B\left(\operatorname{span}\left(e_{i}, \cdots, e_{k}\right)\right)$. Then

$$
B x=\lim _{r \rightarrow \infty} B x_{r}=\lim _{r \rightarrow \infty} B y_{r}
$$

where $y_{r} \in \operatorname{span}\left(e_{i}, \cdots, e_{k}\right)$. Say

$$
B x_{r}=\sum_{i=1}^{k} a_{i}^{r} B e_{i}
$$

It follows easily that $\left\langle B x_{r}, e_{j}\right\rangle=a_{j}^{r}$. (Act on $e_{j}$ by both sides and use $\left\langle B e_{i}, e_{j}\right\rangle=\delta_{i j}$.) Now since $x_{r}$ is bounded, it follows that these $a_{j}^{r}$ are also bounded. Hence, defining $y_{r} \equiv \sum_{i=1}^{k} a_{i}^{r} e_{i}$, it follows that $y_{r}$ is bounded in $\operatorname{span}\left(e_{i}, \cdots, e_{k}\right)$ and so, there exists a subsequence, still denoted by $r$ such that $y_{r} \rightarrow y \in \operatorname{span}\left(e_{i}, \cdots, e_{k}\right)$. Therefore, $B x=$ $\lim _{r \rightarrow \infty} B y_{r}=B y$. In other words, $B W=B\left(\operatorname{span}\left(e_{i}, \cdots, e_{k}\right)\right)$ as claimed. This proves the claim.

If this happens, the process being described stops. You have found what is desired which has only finitely many vectors involved.

As long as the process does not stop, let

$$
e_{k+1} \equiv \frac{g_{n_{k+1}}-\sum_{i=1}^{k}\left\langle B g_{n_{k+1}}, e_{i}\right\rangle e_{i}}{\left\langle B\left(g_{n_{k+1}}-\sum_{i=1}^{k}\left\langle B g_{n_{k+1}}, e_{i}\right\rangle e_{i}\right), g_{n_{k+1}}-\sum_{i=1}^{k}\left\langle B g_{n_{k+1}}, e_{i}\right\rangle e_{i}\right\rangle^{1 / 2}}
$$

Thus, as in the usual argument for the Gram Schmidt process, $\left\langle B e_{i}, e_{j}\right\rangle=\delta_{i j}$ for $i, j \leq k+1$. This is already known for $i, j \leq k$. Letting $l \leq k$, and using the orthogonality already shown,

$$
\begin{aligned}
\left\langle B e_{k+1}, e_{l}\right\rangle & =C\left\langle B\left(g_{n_{k+1}}-\sum_{i=1}^{k}\left\langle B g_{n_{k+1}}, e_{i}\right\rangle e_{i}\right), e_{l}\right\rangle \\
& =C\left(\left\langle B g_{k+1}, e_{l}\right\rangle-\left\langle B g_{n_{k+1}}, e_{l}\right\rangle\right)=0
\end{aligned}
$$

Consider

$$
\left\langle B g_{p}-B\left(\sum_{i=1}^{k}\left\langle B g_{p}, e_{i}\right\rangle e_{i}\right), g_{p}-\sum_{i=1}^{k}\left\langle B g_{p}, e_{i}\right\rangle e_{i}\right\rangle
$$

Either this equals 0 because $p$ is never one of the $n_{k}$ or eventually it equals 0 for some $k$ because $g_{p}=g_{n_{k}}$ for some $n_{k}$ and so, from the construction, $g_{n_{k}}=g_{p} \in \operatorname{span}\left(e_{1}, \cdots, e_{k}\right)$ and therefore,

$$
g_{p}=\sum_{j=1}^{k} a_{j} e_{j}
$$

which requires easily that

$$
B g_{p}=\sum_{i=1}^{k}\left\langle B g_{p}, e_{i}\right\rangle B e_{i}
$$

the above holding for all $k$ large enough. It follows that for any $x \in \operatorname{span}\left(\left\{g_{k}\right\}_{k=1}^{\infty}\right)$, (finite linear combination of vectors in $\left\{g_{k}\right\}_{k=1}^{\infty}$ )

$$
\begin{equation*}
B x=\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i} \tag{43.3.12}
\end{equation*}
$$

because for all $k$ large enough,

$$
B x=\sum_{i=1}^{k}\left\langle B x, e_{i}\right\rangle B e_{i}
$$

Also note that for such $x \in \operatorname{span}\left(\left\{g_{k}\right\}_{k=1}^{\infty}\right)$,

$$
\begin{aligned}
\langle B x, x\rangle & =\left\langle\sum_{i=1}^{k}\left\langle B x, e_{i}\right\rangle B e_{i}, x\right\rangle=\sum_{i=1}^{k}\left\langle B x, e_{i}\right\rangle\left\langle B x, e_{i}\right\rangle \\
& =\sum_{i=1}^{k}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}=\sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}
\end{aligned}
$$

Now for $x$ arbitrary, let $x_{k} \rightarrow x$ in $W$ where $x_{k} \in \operatorname{span}\left(\left\{g_{k}\right\}_{k=1}^{\infty}\right)$. Then by Fatou's lemma,

$$
\begin{align*}
\sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2} & \leq \lim _{k \rightarrow \infty} \sum_{i=1}^{\infty}\left|\left\langle B x_{k}, e_{i}\right\rangle\right|^{2} \\
& =\liminf _{k \rightarrow \infty}\left\langle B x_{k}, x_{k}\right\rangle=\langle B x, x\rangle  \tag{43.3.13}\\
& \leq\|B x\|_{W^{\prime}}\|x\|_{W} \leq\|B\|\|x\|_{W}^{2}
\end{align*}
$$

Thus the series on the left converges. Then also, from the above inequality,

$$
\begin{aligned}
& \left|\left\langle\sum_{i=p}^{q}\left\langle B x, e_{i}\right\rangle B e_{i}, y\right\rangle\right| \leq \sum_{i=p}^{q}\left|\left\langle B x, e_{i}\right\rangle\right|\left|\left\langle B e_{i}, y\right\rangle\right| \\
& \leq\left(\sum_{i=p}^{q}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}\right)^{1 / 2}\left(\sum_{i=p}^{q}\left|\left\langle B y, e_{i}\right\rangle\right|^{2}\right)^{1 / 2} \\
& \leq\left(\sum_{i=p}^{q}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}\right)^{1 / 2}\left(\sum_{i=1}^{\infty}\left|\left\langle B y, e_{i}\right\rangle\right|^{2}\right)^{1 / 2}
\end{aligned}
$$

By 43.3.13,

$$
\begin{aligned}
& \leq\left(\sum_{i=p}^{q}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}\right)^{1 / 2}\left(\|B\|\|y\|_{W}^{2}\right)^{1 / 2} \\
& \leq\left(\sum_{i=p}^{q}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}\right)^{1 / 2}\|B\|^{1 / 2}\|y\|_{W}
\end{aligned}
$$

It follows that

$$
\begin{equation*}
\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i} \tag{43.3.14}
\end{equation*}
$$

converges in $W^{\prime}$ because it was just shown that

$$
\left\|\sum_{i=p}^{q}\left\langle B x, e_{i}\right\rangle B e_{i}\right\|_{W^{\prime}} \leq\left(\sum_{i=p}^{q}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}\right)^{1 / 2}\|B\|^{1 / 2}
$$

and it was shown above that $\sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}<\infty$, so the partial sums of the series 43.3.14 are a Cauchy sequence in $W^{\prime}$. Also, the above estimate shows that for $\|y\|=1$,

$$
\begin{aligned}
\left|\left\langle\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i}, y\right\rangle\right| & \leq\left(\sum_{i=1}^{\infty}\left|\left\langle B y, e_{i}\right\rangle\right|^{2}\right)^{1 / 2}\left(\sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}\right)^{1 / 2} \\
& \leq\left(\sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}\right)^{1 / 2}\|B\|^{1 / 2}
\end{aligned}
$$

and so

$$
\begin{equation*}
\left\|\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i}\right\|_{W^{\prime}} \leq\left(\sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}\right)^{1 / 2}\|B\|^{1 / 2} \tag{43.3.15}
\end{equation*}
$$

Now for $x$ arbitrary, let $x_{k} \in \operatorname{span}\left(\left\{g_{j}\right\}_{j=1}^{\infty}\right)$ and $x_{k} \rightarrow x$ in $W$. Then for a fixed $k$ large enough,

$$
\begin{gathered}
\left\|B x-\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i}\right\| \leq\left\|B x-B x_{k}\right\| \\
+\left\|B x_{k}-\sum_{i=1}^{\infty}\left\langle B x_{k}, e_{i}\right\rangle B e_{i}\right\|+\left\|\sum_{i=1}^{\infty}\left\langle B x_{k}, e_{i}\right\rangle B e_{i}-\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i}\right\| \\
\leq \varepsilon+\left\|\sum_{i=1}^{\infty}\left\langle B\left(x_{k}-x\right), e_{i}\right\rangle B e_{i}\right\|
\end{gathered}
$$

the term

$$
\left\|B x_{k}-\sum_{i=1}^{\infty}\left\langle B x_{k}, e_{i}\right\rangle B e_{i}\right\|
$$

equaling 0 by 43.3.12. From 43.3.15 and 43.3.13,

$$
\begin{aligned}
& \leq \varepsilon+\|B\|^{1 / 2}\left(\sum_{i=1}^{\infty}\left|\left\langle B\left(x_{k}-x\right), e_{i}\right\rangle\right|^{2}\right)^{1 / 2} \\
& \leq \varepsilon+\|B\|^{1 / 2}\left\langle B\left(x_{k}-x\right), x_{k}-x\right\rangle^{1 / 2}<2 \varepsilon
\end{aligned}
$$

whenever $k$ is large enough. Therefore,

$$
B x=\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i}
$$

in $W^{\prime}$. It follows that

$$
\langle B x, x\rangle=\lim _{k \rightarrow \infty}\left\langle\sum_{i=1}^{k}\left\langle B x, e_{i}\right\rangle B e_{i}, x\right\rangle=\lim _{k \rightarrow \infty} \sum_{i=1}^{k}\left|\left\langle B x, e_{i}\right\rangle\right|^{2} \equiv \sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}
$$

Theorem 43.3.2 Let $V \subseteq W, W^{\prime} \subseteq V^{\prime}$ be separable Banach spaces, $W$ a separable Hilbert space, and let $Y \in L^{p^{\prime}}\left(0, T ; V^{\prime}\right) \equiv K^{\prime}$ and

$$
\begin{equation*}
B X(t)=B X_{0}+\int_{0}^{t} Y(s) d s \operatorname{in} V^{\prime} \tag{43.3.16}
\end{equation*}
$$

where $X_{0} \in W$, and it is known that $X \in L^{p}(0, T, V) \equiv K$ for $p>1$. Also assume $X \in$ $L^{2}(0, T, W)$. Then $t \rightarrow B X(t)$ is in $C\left([0, T], W^{\prime}\right)$ and also

$$
\frac{1}{2}\langle B X(t), X(t)\rangle=\frac{1}{2}\left\langle B X_{0}, X_{0}\right\rangle+\int_{0}^{t}\langle Y(s), X(s)\rangle d s
$$

Proof: By Lemma 43.2.1, there exists a sequence of uniform partitions $\left\{t_{k}^{n}\right\}_{k=0}^{m_{n}}=$ $\mathscr{P}_{n}, \mathscr{P}_{n} \subseteq \mathscr{P}_{n+1}$, of $[0, T]$ such that the step functions

$$
\begin{aligned}
\sum_{k=0}^{m_{n}-1} X\left(t_{k}^{n}\right) \mathscr{X}_{\left(t_{k}^{n}, t_{k+1}^{n}\right]}(t) & \equiv X^{l}(t) \\
\sum_{k=0}^{m_{n}-1} X\left(t_{k+1}^{n}\right) \mathscr{X}_{\left(t_{k}^{n}, t_{k+1}^{n}\right]}(t) & \equiv X^{r}(t)
\end{aligned}
$$

converge to $X$ in $K$ and also $B X^{l}, B X^{r} \rightarrow B X$ in $L^{2}\left([0, T], W^{\prime}\right)$.
Lemma 43.3.3 Let $s<t$. Then for $X, Y$ satisfying 43.3.16

$$
\begin{gather*}
\langle B X(t), X(t)\rangle=\langle B X(s), X(s)\rangle \\
+2 \int_{s}^{t}\langle Y(u), X(t)\rangle d u-\langle B(X(t)-X(s)),(X(t)-X(s))\rangle \tag{43.3.17}
\end{gather*}
$$

Proof: It follows from the following computations

$$
B(X(t)-X(s))=\int_{s}^{t} Y(u) d u
$$

and so

$$
\begin{gathered}
2 \int_{s}^{t}\langle Y(u), X(t)\rangle d u-\langle B(X(t)-X(s)),(X(t)-X(s))\rangle \\
=2\langle B(X(t)-X(s)), X(t)\rangle-\langle B(X(t)-X(s)),(X(t)-X(s))\rangle \\
=2\langle B X(t), X(t)\rangle-2\langle B X(s), X(t)\rangle-\langle B X(t), X(t)\rangle \\
+2\langle B X(s), X(t)\rangle-\langle B X(s), X(s)\rangle \\
=\langle B X(t), X(t)\rangle-\langle B X(s), X(s)\rangle
\end{gathered}
$$

Thus

$$
\begin{gathered}
\langle B X(t), X(t)\rangle-\langle B X(s), X(s)\rangle \\
=2 \int_{s}^{t}\langle Y(u), X(t)\rangle d u-\langle B(X(t)-X(s)),(X(t)-X(s))\rangle
\end{gathered}
$$

Lemma 43.3.4 In the above situation,

$$
\sup _{t \in[0, T]}\langle B X(t), X(t)\rangle \leq C\left(\|Y\|_{K^{\prime}},\|X\|_{K}\right)
$$

Also, $t \rightarrow B X(t)$ is weakly continuous with values in $W^{\prime}$.
Proof: From the above formula applied to the $k^{t h}$ partition of $[0, T]$ described above,

$$
\begin{aligned}
& \left\langle B X\left(t_{m}\right), X\left(t_{m}\right)\right\rangle-\left\langle B X_{0}, X_{0}\right\rangle=\sum_{j=0}^{m-1}\left\langle B X\left(t_{j+1}\right), X\left(t_{j+1}\right)\right\rangle-\left\langle B X\left(t_{j}\right), X\left(t_{j}\right)\right\rangle \\
& =\sum_{j=0}^{m-1} 2 \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X\left(t_{j+1}\right)\right\rangle d u-\left\langle B\left(X\left(t_{j+1}\right)-X\left(t_{j}\right)\right), X\left(t_{j+1}\right)-X\left(t_{j}\right)\right\rangle \\
& =\sum_{j=0}^{m-1} 2 \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u-\left\langle B\left(X\left(t_{j+1}\right)-X\left(t_{j}\right)\right), X\left(t_{j+1}\right)-X\left(t_{j}\right)\right\rangle
\end{aligned}
$$

Thus, discarding the negative terms and denoting by $\mathscr{P}_{k}$ the $k^{t h}$ of these partitions,

$$
\begin{gathered}
\sup _{t_{j} \in \mathscr{P}_{k}}\left\langle B X\left(t_{j}\right), X\left(t_{j}\right)\right\rangle \leq\left\langle B X_{0}, X_{0}\right\rangle+2 \int_{0}^{T}\left|\left\langle Y(u), X_{k}^{r}(u)\right\rangle\right| d u \\
\leq\left\langle B X_{0}, X_{0}\right\rangle+2 \int_{0}^{T}\|Y(u)\|_{V^{\prime}}\left\|X_{k}^{r}(u)\right\|_{V} d u
\end{gathered}
$$

$$
\begin{aligned}
& \leq\left\langle B X_{0}, X_{0}\right\rangle+2\left(\int_{0}^{T}\|Y(u)\|_{V^{\prime}}^{p^{\prime}} d u\right)^{1 / p^{\prime}}\left(\int_{0}^{T}\left\|X_{k}^{r}(u)\right\|_{V}^{p} d u\right)^{1 / p} \\
& \leq C\left(\|Y\|_{K^{\prime}},\|X\|_{K}\right)
\end{aligned}
$$

because these partitions are chosen such that

$$
\lim _{k \rightarrow \infty}\left(\int_{0}^{T}\left\|X_{k}^{r}(u)\right\|_{V}^{p}\right)^{1 / p}=\left(\int_{0}^{T}\|X(u)\|_{V}^{p}\right)^{1 / p}
$$

and so these are bounded. This has shown that for the dense subset of $[0, T], D \equiv \cup_{k} \mathscr{P}_{k}$,

$$
\sup _{t \in D}\langle B X(t), X(t)\rangle<C\left(\|Y\|_{K^{\prime}},\|X\|_{K}\right)
$$

From Lemma 43.3.1 above, there exists $\left\{e_{i}\right\} \subseteq V$ such that $\left\langle B e_{i}, e_{j}\right\rangle=\delta_{i j}$ and

$$
\langle B X(t), X(t)\rangle=\sum_{k=1}^{\infty}\left|\left\langle B X(t), e_{i}\right\rangle\right|^{2}=\sup _{m} \sum_{k=1}^{m}\left|\left\langle B X(t), e_{i}\right\rangle\right|^{2}
$$

Since each $e_{i} \in V$, and since $t \rightarrow B X(t)$ is continuous into $V^{\prime}$ thanks to the formula 43.3.16, it follows that $t \rightarrow \sum_{k=1}^{m}\left|\left\langle B X(t), e_{i}\right\rangle\right|$ is continuous and so $t \rightarrow\langle B X(t), X(t)\rangle$ is the sup of continuous functions. Therefore, this function of $t$ is lower semicontinuous. Since $D$ is dense in $[0, T]$, it follows that for all $t$,

$$
\langle B X(t), X(t)\rangle \leq C\left(\|Y\|_{K^{\prime}},\|X\|_{K}\right)
$$

It only remains to verify the claim about weak continuity.
Consider now the claim that $t \rightarrow B X(t)$ is weakly continuous. Letting $v \in V$,

$$
\begin{equation*}
\lim _{t \rightarrow s}\langle B X(t), v\rangle=\langle B X(s), v\rangle=\langle B X(s), v\rangle \tag{43.3.18}
\end{equation*}
$$

The limit follows from the formula 43.3.16 which implies $t \rightarrow B X(t)$ is continuous into $V^{\prime}$. Now

$$
\|B X(t)\|=\sup _{\|v\| \leq 1}|\langle B X(t), v\rangle| \leq\langle B v, v\rangle^{1 / 2}\langle B X(t), X(t)\rangle^{1 / 2}
$$

which was shown to be bounded for $t \in[0, T]$. Now let $w \in W$. Then

$$
|\langle B X(t), w\rangle-\langle B X(s), w\rangle| \leq|\langle B X(t)-B X(s), w-v\rangle|+|\langle B X(t)-B X(s), v\rangle|
$$

Then the first term is less than $\varepsilon$ if $v$ is close enough to $w$ and the second converges to 0 so 43.3.18 holds for all $v \in W$ and so this shows the weak continuity.

Now pick $t \in D$, the union of all the mesh points. Then for all $k$ large enough, $t \in \mathscr{P}_{k}$. Say $t=t_{m}$. From Lemma 43.3.3,

$$
-\sum_{j=0}^{m-1}\left\langle B\left(X\left(t_{j+1}\right)-X\left(t_{j}\right)\right),\left(X\left(t_{j+1}\right)-X\left(t_{j}\right)\right)\right\rangle=
$$

$$
\left\langle B X\left(t_{m}\right), X\left(t_{m}\right)\right\rangle-\left\langle B X_{0}, X_{0}\right\rangle-2 \sum_{j=0}^{m-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u
$$

Thus, $\left\langle B X\left(t_{m}\right), X\left(t_{m}\right)\right\rangle$ is constant for all $k$ large enough and the integral term converges to

$$
\int_{0}^{t_{m}}\langle Y(u), X(u)\rangle d u
$$

It follows that the term on the left does converge to something as $k \rightarrow \infty$. It just remains to consider what it does converge to. However, from the equation solved by $X$,

$$
B X\left(t_{j+1}\right)-B X\left(t_{j}\right)=\int_{t_{j}}^{t_{j+1}} Y(u) d u
$$

Therefore, this term is dominated by an expression of the form

$$
\begin{gathered}
\sum_{j=0}^{m_{k}-1}\left(\int_{t_{j}}^{t_{j+1}} Y(u) d u, X\left(t_{j+1}\right)-X\left(t_{j}\right)\right) \\
=\sum_{j=0}^{m_{k}-1}\left\langle\int_{t_{j}}^{t_{j+1}} Y(u) d u, X\left(t_{j+1}\right)-X\left(t_{j}\right)\right\rangle \\
=\sum_{j=0}^{m_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X\left(t_{j+1}\right)-X\left(t_{j}\right)\right\rangle d u \\
=\sum_{j=0}^{m_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X\left(t_{j+1}\right)\right\rangle-\sum_{j=0}^{m_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X\left(t_{j}\right)\right\rangle \\
=\int_{0}^{T}\left\langle Y(u), X^{r}(u)\right\rangle d u-\int_{0}^{T}\left\langle Y(u), X^{l}(u)\right\rangle d u
\end{gathered}
$$

However, both $X^{r}$ and $X^{l}$ converge to $X$ in $K=L^{p}(0, T, V)$. Therefore, this term must converge to 0 . Passing to a limit, it follows that for all $t \in D$, the desired formula holds. Thus, for such $t \in D$,

$$
\langle B X(t), X(t)\rangle=\left\langle B X_{0}, X_{0}\right\rangle+2 \int_{0}^{t}\langle Y(u), X(u)\rangle d u
$$

It remains to verify that this holds for all $t$. Let $t \notin D$ and let $t(k) \in \mathscr{P}_{k}$ be the largest point of $\mathscr{P}_{k}$ which is less than $t$. Suppose $t(m) \leq t(k)$ so that $m \leq k$. Then

$$
B X(t(m))=B X_{0}+\int_{0}^{t(m)} Y(s) d s
$$

a similar formula for $X(t(k))$. Thus for $t>t(m)$,

$$
B X(t)-B X(t(m))=\int_{t(m)}^{t} Y(s) d s
$$

which is the same sort of thing already looked at except that it starts at $t(m)$ rather than at 0 and $X_{0}=0$. Therefore,

$$
\begin{aligned}
& \langle B(X(t(k))-X(t(m))), X(t(k))-X(t(m))\rangle \\
= & 2 \int_{t(m)}^{t(k)}\langle Y(s), X(s)-X(t(m))\rangle d s
\end{aligned}
$$

Thus, for $m \leq k$

$$
\begin{equation*}
\lim _{m, k \rightarrow \infty}\langle B(X(t(k))-X(t(m))), X(t(k))-X(t(m))\rangle=0 \tag{43.3.19}
\end{equation*}
$$

Hence $\{B X(t(k))\}_{k=1}^{\infty}$ is a convergent sequence in $W^{\prime}$ because

$$
\begin{aligned}
& |\langle B(X(t(k))-X(t(m))), y\rangle| \\
\leq & \langle B(X(t(k))-X(t(m))), X(t(k))-X(t(m))\rangle^{1 / 2}\langle B y, y\rangle^{1 / 2} \\
\leq & \langle B(X(t(k))-X(t(m))), X(t(k))-X(t(m))\rangle^{1 / 2}\|B\|^{1 / 2}\|y\|_{W}
\end{aligned}
$$

Does it converge to $B X(t)$ ? Let $\xi(t) \in W^{\prime}$ be what it does converge to. Let $v \in V$. Then

$$
\langle\xi(t), v\rangle=\lim _{k \rightarrow \infty}\langle B X(t(k)), v\rangle=\lim _{k \rightarrow \infty}\langle B X(t(k)), v\rangle=\langle B X(t), v\rangle
$$

because it is known that $t \rightarrow B X(t)$ is continuous into $V^{\prime}$. It is also known that $B X(t) \in$ $W^{\prime} \subseteq V^{\prime}$ and that the $B X(t)$ for $t \in[0, T]$ are uniformly bounded in $W^{\prime}$. Therefore, since $V$ is dense in $W$, it follows that $\xi(t)=B X(t)$.

Now for every $t \in D$, it was shown above that

$$
\langle B X(t), X(t)\rangle=\left\langle B X_{0}, X_{0}\right\rangle+2 \int_{0}^{t}\langle Y(u), X(u)\rangle d u
$$

Also it was just shown that $B X(t(k)) \rightarrow B X(t)$. Then

$$
\begin{gathered}
|\langle B X(t(k)), X(t(k))\rangle-\langle B X(t), X(t)\rangle| \\
\leq|\langle B X(t(k)), X(t(k))-X(t)\rangle|+|\langle B X(t(k))-B X(t), X(t)\rangle|
\end{gathered}
$$

Then the second term converges to 0 . The first equals

$$
\begin{aligned}
& |\langle B X(t(k))-B X(t), X(t(k))\rangle| \\
\leq & \langle B(X(t(k))-X(t)), X(t(k))-X(t)\rangle^{1 / 2}\langle B X(t(k)), X(t(k))\rangle^{1 / 2}
\end{aligned}
$$

From the above, this is dominated by an expression of the form

$$
\langle B(X(t(k))-X(t)), X(t(k))-X(t)\rangle^{1 / 2} C
$$

Then using the lower semicontinuity of $t \rightarrow\langle B(X(t(k))-X(t)), X(t(k))-X(t)\rangle$ which follows from the above, this is no larger than

$$
\lim \inf _{m \rightarrow \infty}\langle B(X(t(k))-X(t(m))), X(t(k))-X(t(m))\rangle^{1 / 2} C<\varepsilon
$$

provided $k$ is large enough. This follows from 43.3.19. Since $\varepsilon$ is arbitrary, it follows that

$$
\lim _{k \rightarrow \infty}|\langle B X(t(k)), X(t(k))\rangle-\langle B X(t), X(t)\rangle|=0
$$

Then from the formula,

$$
\langle B X(t), X(t)\rangle=\left\langle B X_{0}, X_{0}\right\rangle+2 \int_{0}^{t}\langle Y(u), X(u)\rangle d u
$$

valid for $t \in D$, it follows that the same formula holds for all $t$. This formula implies $t \rightarrow\langle B X(t), X(t)\rangle$ is continuous. Also recall that $t \rightarrow B X(t)$ was shown to be weakly continuous into $W^{\prime}$. Then

$$
\langle B(X(t)-X(s)), X(t)-X(s)\rangle=\langle B X(t), X(t)\rangle-2\langle B X(t), X(s)\rangle+\langle B X(s), X(s)\rangle
$$

From this, it follows that $t \rightarrow B X(t)$ is continuous into $W^{\prime}$ because $\lim _{t \rightarrow s}$ of the right side gives 0 and so the same is true of the left. Hence,

$$
\begin{aligned}
& |\langle B(X(t)-X(s)), y\rangle| \\
\leq & \langle B y, y\rangle^{1 / 2}\langle B(X(t)-X(s)), X(t)-X(s)\rangle^{1 / 2} \\
\leq & \|B\|^{1 / 2}\langle B(X(t)-X(s)), X(t)-X(s)\rangle^{1 / 2}\|y\|
\end{aligned}
$$

so

$$
\|B(X(t)-X(s))\|_{W^{\prime}} \leq\|B\|^{1 / 2}\langle B(X(t)-X(s)), X(t)-X(s)\rangle^{1 / 2}
$$

which converges to 0 as $t \rightarrow s$.

### 43.4 Some Implicit Inclusions

Let $B \in \mathscr{L}\left(W, W^{\prime}\right)$ and $B$ satisfies $\langle B x, x\rangle \geq 0,\langle B x, y\rangle=\langle B y, x\rangle$ for $V \subseteq W, W^{\prime} \subseteq V^{\prime}$. Where $V$ is dense in the Hilbert space $W$. Now let

$$
\begin{equation*}
D(L) \equiv\left\{u \in \mathscr{V}:(B u)^{\prime} \in \mathscr{V}^{\prime}, B u(0)=0\right\}, L u \equiv(B u)^{\prime} \tag{43.4.20}
\end{equation*}
$$

Then clearly $D(L)$ is dense in $\mathscr{V}$. Here $\mathscr{V} \equiv L^{p}([0, T], V)$ where $p \geq 2$ for simplicity. Now let

$$
\begin{equation*}
D(T) \equiv\left\{u \in \mathscr{V}: u^{\prime} \in \mathscr{V} \text { and } u(T)=0\right\}, T u \equiv-B\left(u^{\prime}\right) \tag{43.4.21}
\end{equation*}
$$

The idea is to show that $L=T^{*}$ and that $T$ is monotone. Then this will imply using Proposition 25.8.2 that $L^{*}$ is monotone. This is done by showing that $\mathscr{G}\left(L^{*}\right)=\overline{\mathscr{G}(T)}$.

Lemma 43.4. $T$ is monotone, $T^{*}=L$ and $L^{*}, L$ are both monotone.
Proof: First, why is $T$ monotone?

$$
\begin{aligned}
\int_{0}^{T}\left\langle-B u^{\prime}, u\right\rangle d t & =\int_{0}^{T}-\left\langle B u, u^{\prime}\right\rangle d t=-\left.\langle B u, u\rangle\right|_{0} ^{T}+\int_{0}^{T}\left\langle(B u)^{\prime}, u\right\rangle d t \\
& =\langle B u(0), u(0)\rangle+\int_{0}^{T}\left\langle B u^{\prime}, u\right\rangle d t
\end{aligned}
$$

and so

$$
2\langle T u, u\rangle=2 \int_{0}^{T}\left\langle-B u^{\prime}, u\right\rangle d t=\langle B u(0), u(0)\rangle \geq 0
$$

Next, why is $T^{*}=L$ ? Let $u \in D(L)$. Then for $v \in D(T)$,

$$
\begin{aligned}
\langle T v, u\rangle & =\int_{0}^{T}\left\langle-B v^{\prime}, u\right\rangle d t=\int_{0}^{T}\left\langle-B u, v^{\prime}\right\rangle d t=\left.\langle-B u, v\rangle\right|_{0} ^{T}+\int_{0}^{T}\left\langle(B u)^{\prime}, v\right\rangle d s \\
& =\langle L u, v\rangle
\end{aligned}
$$

Hence $u \in D\left(T^{*}\right)$ and $T^{*} u=L u$ since $D(T)$ is dense. Thus $D(L) \subseteq D\left(T^{*}\right)$ and on $D(L)$, these two are equal. Next suppose $u \in D\left(T^{*}\right)$. Then for all $v \in D(T),\langle T v, u\rangle \leq C\|v\|_{\mathscr{V}}$. Thus, by density and the Riesz representation theorem, there exists a unique $g^{*} \in \mathscr{V}^{\prime}$ such that

$$
\langle T v, u\rangle=\int_{0}^{T}\left\langle g^{*}, v\right\rangle d t=\int_{0}^{T}\left\langle-B v^{\prime}, u\right\rangle d t=-\int_{0}^{T}\left\langle B u, v^{\prime}\right\rangle d t
$$

In particular, it follows from the definition of weak $V^{\prime}$ valued distributions that $g^{*}=(B u)^{\prime}$. Simply specialize to letting $v(t)=v \phi(t)$ where $\phi \in C_{c}^{\infty}(0, T)$. Thus in particular $(B u)^{\prime} \in \mathscr{V}^{\prime}$ and the above reduces to

$$
\begin{gathered}
\langle T v, u\rangle=\int_{0}^{T}\left\langle(B u)^{\prime}, v\right\rangle d t \\
\langle T v, u\rangle= \\
=\int_{0}^{T}\left\langle-B v^{\prime}, u\right\rangle d t=\int_{0}^{T}\left\langle-B u, v^{\prime}\right\rangle d t=\left.\langle-B u, v\rangle\right|_{0} ^{T}+\int_{0}^{T}\left\langle(B u)^{\prime}, v\right\rangle d t \\
=
\end{gathered}
$$

Thus also $(B u)(0)=0$. Hence $D\left(T^{*}\right) \subseteq D(L)$ and this shows that $L=T^{*}$ as claimed.
Why is $L$ monotone? From the material on weak derivatives,

$$
B u(t)=\int_{0}^{t}(B u)^{\prime}(s) d s
$$

and now use Theorem 43.3.2 to obtain

$$
0 \leq\langle B u(t), u(t)\rangle=2 \int_{0}^{t}\left\langle(B u)^{\prime}, u\right\rangle d s
$$

In particular, this holds for $t=T$ and so $\langle L u, u\rangle \geq 0$.
Why is $L^{*}$ monotone? This follows from Proposition 25.8.2 and the fact that $L=T^{*}$ shown above.

$$
\mathscr{G}\left(L^{*}\right)=(\tau \mathscr{G}(L))^{\perp}
$$

Consider $(\tau S)^{\perp}$. To say that $\left(x, y^{*}\right) \in(\tau S)^{\perp}$ is to say that if $\left(a, b^{*}\right) \in S$, then

$$
\left\langle\left(x, y^{*}\right),\left(-b^{*}, a\right)\right\rangle=0
$$

or in other words, $\left\langle x, b^{*}\right\rangle=\left\langle y^{*}, a\right\rangle$. To say that $\left(x, y^{*}\right) \in \tau\left(S^{\perp}\right)$ is to say that $\left(x, y^{*}\right)=$ $\left(-c, d^{*}\right)$ where

$$
\left\langle\left(d^{*}, c\right),\left(a, b^{*}\right)\right\rangle=0
$$

for all $\left(a, b^{*}\right) \in S$. That is, $\left\langle\left(y^{*},-x\right),\left(a, b^{*}\right)\right\rangle=0$ for all $\left(a, b^{*}\right) \in S$. In other words $\left\langle y^{*}, a\right\rangle=$ $\left\langle x, b^{*}\right\rangle$ for all $\left(a, b^{*}\right) \in S$. Thus $(\tau S)^{\perp}=\tau\left(S^{\perp}\right)$. Now $\tau \tau(M)=M$ if $M$ is a subspace. and $\left(M^{\perp}\right)^{\perp}=\bar{M}$ if $M$ is a subspace. Hence

$$
\begin{aligned}
\mathscr{G}\left(L^{*}\right) & =(\tau \mathscr{G}(L))^{\perp}=\tau\left(\mathscr{G}(L)^{\perp}\right) \\
& =\tau\left(\left(\tau(\mathscr{G}(T))^{\perp}\right)^{\perp}\right)=\tau \tau\left(\mathscr{G}(T)^{\perp}\right)^{\perp}=\overline{\mathscr{G}(T)}
\end{aligned}
$$

Now it follows that, since $T$ is monotone, it follows that $L^{*}$ is also monotone.
Note that as part of this argument, we have proved that for $T$ a densely defined linear operator, $\mathscr{G}\left(T^{* *}\right)=\overline{\mathscr{G}(T)}$.

Now recall Theorem 25.8.8 on Page 923 which is listed next.

Theorem 43.4.2 Let $L: D(L) \subseteq V \rightarrow V^{\prime}$ where $D(L)$ is dense, $L$ is monotone, $L$ is closed, and $L^{*}$ is monotone, L a linear map. Let $T: V \rightarrow \mathscr{P}\left(V^{\prime}\right)$ be L pseudomonotone, bounded, coercive. Then $L+T$ is onto. Here $V$ is a reflexive Banach space such that the norms for $V$ and $V^{\prime}$ are strictly convex.

To apply this theorem, let $B$ be as above and $V \rightarrow \mathscr{V} \equiv L^{p}([0, T], V)$. Letting $u_{0} \in V$, let

$$
T(u) \equiv A\left(u+u_{0}\right)
$$

where $A: \mathscr{V} \rightarrow \mathscr{P}\left(\mathscr{V}^{\prime}\right)$. Suppose that $T$ just defined is set valued pseudomonotone and coercive. Let $L u=(B u)^{\prime}$ as described above in 43.4.20. Then from Theorem 43.4.2 and if $f \in \mathscr{V}^{\prime}$, there exists a solution $u$ to

$$
L u+A\left(u+u_{0}\right) \ni f
$$

Thus there exists $\xi \in A\left(u+u_{0}\right)$ such that $L u+\xi=f$ in $\mathscr{V}^{\prime}$. Then letting $w=u+u_{0}$, it follows that $\xi \in A(w)$ and $L\left(w-w_{0}\right)+\xi=f$. Thus,

$$
(B w)^{\prime}+\xi=f,(B w)(0)=B w_{0}
$$

Written in terms of $A,(B w)^{\prime}+A(w) \ni f$ in $\mathscr{V}^{\prime},(B w)(0)=B u_{0}$. This proves the following theorem about the existence of solutions to implicit evolution inclusions.

Theorem 43.4.3 Suppose $u \rightarrow A\left(u+u_{0}\right)$ is set valued pseudomonotone and coercive for $u_{0} \in V$. Also let

$$
V \subseteq W, W^{\prime} \subseteq V^{\prime}
$$

where $W$ is a Hilbert space, $V$ is a reflexive Banach space dense in $W$. Suppose $B: W \rightarrow W^{\prime}$ is self adjoint and nonnegative. Then there exists a solution $w \in \mathscr{V}$ to the implicit evolution equation

$$
(B w)^{\prime}+A(w) \ni f \text { in } \mathscr{V}^{\prime},(B w)(0)=B u_{0}
$$

### 43.5 Some Imbedding Theorems

The next theorem is very useful in getting estimates in partial differential equations. It is called Erling's lemma.

Definition 43.5.1 Let $E, W$ be Banach spaces such that $E \subseteq W$ and the injection map from $E$ into $W$ is continuous. The injection map is said to be compact if every bounded set in $E$ has compact closure in $W$. In other words, if a sequence is bounded in $E$ it has a convergent subsequence converging in $W$. This is also referred to by saying that bounded sets in $E$ are precompact in $W$.

Theorem 43.5.2 Let $E \subseteq W \subseteq X$ where the injection map is continuous from $W$ to $X$ and compact from $E$ to $W$. Then for every $\varepsilon>0$ there exists a constant, $C_{\varepsilon}$ such that for all $u \in E$,

$$
\|u\|_{W} \leq \varepsilon\|u\|_{E}+C_{\varepsilon}\|u\|_{X}
$$

Proof: Suppose not. Then there exists $\varepsilon>0$ and for each $n \in \mathbb{N}, u_{n}$ such that

$$
\left\|u_{n}\right\|_{W}>\varepsilon\left\|u_{n}\right\|_{E}+n\left\|u_{n}\right\|_{X}
$$

Now let $v_{n}=u_{n} /\left\|u_{n}\right\|_{E}$. Therefore, $\left\|v_{n}\right\|_{E}=1$ and

$$
\left\|v_{n}\right\|_{W}>\varepsilon+n\left\|v_{n}\right\|_{X}
$$

It follows there exists a subsequence, still denoted by $v_{n}$ such that $v_{n}$ converges to $v$ in $W$. However, the above inequality shows that $\left\|v_{n}\right\|_{X} \rightarrow 0$. Therefore, $v=0$. But then the above inequality would imply that $\left\|v_{n}\right\|>\varepsilon$ and passing to the limit yields $0>\varepsilon$, a contradiction.

Definition 43.5.3 Define $C([a, b] ; X)$ the space of functions continuous at every point of $[a, b]$ having values in $X$.

You should verify that this is a Banach space with norm

$$
\|u\|_{\infty, X}=\max \left\{\left\|u_{n_{k}}(t)-u(t)\right\|_{X}: t \in[a, b]\right\} .
$$

The following theorem is an infinite dimensional version of the Ascoli Arzela theorem. [117].

Theorem 43.5.4 Let $q>1$ and let $E \subseteq W \subseteq X$ where the injection map is continuous from $W$ to $X$ and compact from $E$ to $W$. Let $S$ be defined by

$$
\left\{u \text { such that }\|u(t)\|_{E}+\left\|u^{\prime}\right\|_{L^{q}([a, b] ; X)} \leq R \text { for all } t \in[a, b]\right\}
$$

Then $S \subseteq C([a, b] ; W)$ and if $\left\{u_{n}\right\} \subseteq S$, there exists a subsequence, $\left\{u_{n_{k}}\right\}$ which converges to a function $u \in C([a, b] ; W)$ in the following way.

$$
\lim _{k \rightarrow \infty}\left\|u_{n_{k}}-u\right\|_{\infty, W}=0
$$

Proof: First consider the issue of $S$ being a subset of $C([a, b] ; W)$. By Theorem 43.1.9 on Page 1414 the following holds in $X$ for $u \in S$.

$$
u(t)-u(s)=\int_{s}^{t} u^{\prime}(r) d r
$$

Thus $S \subseteq C([a, b] ; X)$. Let $\varepsilon>0$ be given. Then by Theorem 43.5.2 there exists a constant, $C_{\varepsilon}$ such that for all $u \in W$

$$
\|u\|_{W} \leq \frac{\varepsilon}{4 R}\|u\|_{E}+C_{\varepsilon}\|u\|_{X}
$$

Therefore, for all $u \in S$,

$$
\begin{align*}
\|u(t)-u(s)\|_{W} & \leq \frac{\varepsilon}{6 R}\|u(t)-u(s)\|_{E}+C_{\varepsilon}\|u(t)-u(s)\|_{X} \\
& \leq \frac{\varepsilon}{3}+C_{\varepsilon}\left\|\int_{s}^{t} u^{\prime}(r) d r\right\|_{X} \\
& \leq \frac{\varepsilon}{3}+C_{\varepsilon} \int_{s}^{t}\left\|u^{\prime}(r)\right\|_{X} d r \leq \frac{\varepsilon}{3}+C_{\varepsilon} R|t-s|^{1 / q} \tag{43.5.22}
\end{align*}
$$

Since $\varepsilon$ is arbitrary, it follows $u \in C([a, b] ; W)$.
Let $D=\mathbb{Q} \cap[a, b]$ so $D$ is a countable dense subset of $[a, b]$. Let $D=\left\{t_{n}\right\}_{n=1}^{\infty}$. By compactness of the embedding of $E$ into $W$, there exists a subsequence $u_{(n, 1)}$ such that as $n \rightarrow \infty, u_{(n, 1)}\left(t_{1}\right)$ converges to a point in $W$. Now take a subsequence of this, called $(n, 2)$ such that as $n \rightarrow \infty, u_{(n, 2)}\left(t_{2}\right)$ converges to a point in $W$. It follows that $u_{(n, 2)}\left(t_{1}\right)$ also converges to a point of $W$. Continue this way. Now consider the diagonal sequence, $u_{k} \equiv$ $u_{(k, k)}$ This sequence is a subsequence of $u_{(n, l)}$ whenever $k>l$. Therefore, $u_{k}\left(t_{j}\right)$ converges for all $t_{j} \in D$.

Claim: Let $\left\{u_{k}\right\}$ be as just defined, converging at every point of $D \equiv \mathbb{Q} \cap[a, b]$. Then $\left\{u_{k}\right\}$ converges at every point of $[a, b]$.

Proof of claim: Let $\varepsilon>0$ be given. Let $t \in[a, b]$. Pick $t_{m} \in D \cap[a, b]$ such that in 43.5.22 $C_{\varepsilon} R\left|t-t_{m}\right|<\varepsilon / 3$. Then there exists $N$ such that if $l, n>N$, then

$$
\left\|u_{l}\left(t_{m}\right)-u_{n}\left(t_{m}\right)\right\|_{X}<\varepsilon / 3
$$

It follows that for $l, n>N$,

$$
\begin{aligned}
\left\|u_{l}(t)-u_{n}(t)\right\|_{X} \leq & \left\|u_{l}(t)-u_{l}\left(t_{m}\right)\right\|+\left\|u_{l}\left(t_{m}\right)-u_{n}\left(t_{m}\right)\right\| \\
& +\left\|u_{n}\left(t_{m}\right)-u_{n}(t)\right\| \\
\leq & \frac{2 \varepsilon}{3}+\frac{\varepsilon}{3}+\frac{2 \varepsilon}{3}<2 \varepsilon
\end{aligned}
$$

Since $\varepsilon$ was arbitrary, this shows $\left\{u_{k}(t)\right\}_{k=1}^{\infty}$ is a Cauchy sequence. Since $W$ is complete, this shows this sequence converges.

Now for $t \in[a, b]$, it was just shown that if $\varepsilon>0$ there exists $N_{t}$ such that if $n, m>N_{t}$, then

$$
\left\|u_{n}(t)-u_{m}(t)\right\|<\frac{\varepsilon}{3}
$$

Now let $s \neq t$. Then

$$
\left\|u_{n}(s)-u_{m}(s)\right\| \leq\left\|u_{n}(s)-u_{n}(t)\right\|+\left\|u_{n}(t)-u_{m}(t)\right\|+\left\|u_{m}(t)-u_{m}(s)\right\|
$$

From 43.5.22

$$
\left\|u_{n}(s)-u_{m}(s)\right\| \leq 2\left(\frac{\varepsilon}{3}+C_{\varepsilon} R|t-s|^{1 / q}\right)+\left\|u_{n}(t)-u_{m}(t)\right\|
$$

and so it follows that if $\delta$ is sufficiently small and $s \in B(t, \boldsymbol{\delta})$, then when $n, m>N_{t}$

$$
\left\|u_{n}(s)-u_{m}(s)\right\|<\varepsilon .
$$

Since $[a, b]$ is compact, there are finitely many of these balls, $\left\{B\left(t_{i}, \delta\right)\right\}_{i=1}^{p}$, such that for $s \in B\left(t_{i}, \boldsymbol{\delta}\right)$ and $n, m>N_{t_{i}}$, the above inequality holds. Let $N>\max \left\{N_{t_{1}}, \cdots, N_{t_{p}}\right\}$. Then if $m, n>N$ and $s \in[a, b]$ is arbitrary, it follows the above inequality must hold. Therefore, this has shown the following claim.

Claim: Let $\varepsilon>0$ be given. There exists $N$ such that if $m, n>N$, then $\left\|u_{n}-u_{m}\right\|_{\infty, W}<\varepsilon$.
Now let $u(t)=\lim _{k \rightarrow \infty} u_{k}(t)$.

$$
\begin{equation*}
\|u(t)-u(s)\|_{W} \leq\left\|u(t)-u_{n}(t)\right\|_{W}+\left\|u_{n}(t)-u_{n}(s)\right\|_{W}+\left\|u_{n}(s)-u(s)\right\|_{W} \tag{43.5.23}
\end{equation*}
$$

Let $N$ be in the above claim and fix $n>N$. Then

$$
\left\|u(t)-u_{n}(t)\right\|_{W}=\lim _{m \rightarrow \infty}\left\|u_{m}(t)-u_{n}(t)\right\|_{W} \leq \varepsilon
$$

and similarly, $\left\|u_{n}(s)-u(s)\right\|_{W} \leq \varepsilon$. Then if $|t-s|$ is small enough, 43.5.22 shows the middle term in 43.5.23 is also smaller than $\varepsilon$. Therefore, if $|t-s|$ is small enough,

$$
\|u(t)-u(s)\|_{W}<3 \varepsilon
$$

Thus $u$ is continuous. Finally, let $N$ be as in the above claim. Then letting $m, n>N$, it follows that for all $t \in[a, b],\left\|u_{m}(t)-u_{n}(t)\right\|<\varepsilon$. Therefore, letting $m \rightarrow \infty$, it follows that for all $t \in[a, b],\left\|u(t)-u_{n}(t)\right\| \leq \varepsilon$. and so $\left\|u-u_{n}\right\|_{\infty, W} \leq \varepsilon$. Since $\varepsilon$ is arbitrary, this proves the theorem.

The next theorem is another such imbedding theorem found in [91]. It is often used in partial differential equations.

Theorem 43.5.5 Let $E \subseteq W \subseteq X$ where the injection map is continuous from $W$ to $X$ and compact from $E$ to $W$. Let $p \geq 1$, let $q>1$, and define

$$
\begin{aligned}
& S \equiv\left\{u \in L^{p}([a, b] ; E): u^{\prime} \in L^{q}([a, b] ; X)\right. \\
& \text { and } \left.\|u\|_{L^{p}([a, b] ; E)}+\left\|u^{\prime}\right\|_{L^{q}([a, b] ; X)} \leq R\right\} .
\end{aligned}
$$

Then $S$ is precompact in $L^{p}([a, b] ; W)$. This means that if $\left\{u_{n}\right\}_{n=1}^{\infty} \subseteq S$, it has a subsequence $\left\{u_{n_{k}}\right\}$ which converges in $L^{p}([a, b] ; W)$.

Proof: By Proposition 7.6 .5 on Page 144 it suffices to show $S$ has an $\eta$ net in the complete metric space $L^{p}([a, b] ; W)$ for each $\eta>0$.

If not, there exists $\eta>0$ and a sequence $\left\{u_{n}\right\} \subseteq S$, such that

$$
\begin{equation*}
\left\|u_{n}-u_{m}\right\| \geq \eta \tag{43.5.24}
\end{equation*}
$$

for all $n \neq m$ and the norm refers to $L^{p}([a, b] ; W)$. Let

$$
a=t_{0}<t_{1}<\cdots<t_{k}=b, t_{i}-t_{i-1}=(b-a) / k
$$

Now define

$$
\bar{u}_{n}(t) \equiv \sum_{i=1}^{k} \bar{u}_{n_{i}} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t), \bar{u}_{n_{i}} \equiv \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} u_{n}(s) d s
$$

The idea is to show that $\bar{u}_{n}$ approximates $u_{n}$ well and then to argue that a subsequence of the $\left\{\bar{u}_{n}\right\}$ is a Cauchy sequence yielding a contradiction to 43.5.24.

Therefore,

$$
u_{n}(t)-\bar{u}_{n}(t)=\sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left(u_{n}(t)-\bar{u}_{n}(s)\right) d s \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t) .
$$

It follows from Jensen's inequality that

$$
\begin{aligned}
& \left\|u_{n}(t)-\bar{u}_{n}(t)\right\|_{W}^{p} \\
= & \sum_{i=1}^{k}\left\|\frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left(u_{n}(t)-u_{n}(s)\right) d s\right\|_{W}^{p} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t) \\
\leq & \sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left\|u_{n}(t)-u_{n}(s)\right\|_{W}^{p} d s \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)
\end{aligned}
$$

and so

$$
\begin{align*}
& \int_{a}^{b}\left\|\left(u_{n}(t)-\bar{u}_{n}(s)\right)\right\|_{W}^{p} d s \\
\leq & \int_{a}^{b} \sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left\|u_{n}(t)-u_{n}(s)\right\|_{W}^{p} d s \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t) d t \\
= & \sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t_{i}}\left\|u_{n}(t)-u_{n}(s)\right\|_{W}^{p} d s d t \tag{43.5.25}
\end{align*}
$$

From Theorems 43.5 .2 and 43.1 .9 , if $\varepsilon>0$, there exists $C_{\varepsilon}$ such that

$$
\left\|u_{n}(t)-u_{n}(s)\right\|_{W}^{p} \leq \varepsilon\left\|u_{n}(t)-u_{n}(s)\right\|_{E}^{p}+C_{\varepsilon}\left\|u_{n}(t)-u_{n}(s)\right\|_{X}^{p}
$$

$$
\begin{aligned}
\leq & 2^{p-1} \varepsilon\left(\left\|u_{n}(t)\right\|^{p}+\left\|u_{n}(s)\right\|^{p}\right)+C_{\varepsilon}\left\|\int_{s}^{t} u_{n}^{\prime}(r) d r\right\|_{X}^{p} \\
\leq & 2^{p-1} \varepsilon\left(\left\|u_{n}(t)\right\|^{p}+\left\|u_{n}(s)\right\|^{p}\right)+C_{\varepsilon}\left(\int_{s}^{t}\left\|u_{n}^{\prime}(r)\right\|_{X} d r\right)^{p} \\
\leq & 2^{p-1} \varepsilon\left(\left\|u_{n}(t)\right\|^{p}+\left\|u_{n}(s)\right\|^{p}\right) \\
& +C_{\varepsilon}\left(\left(\int_{s}^{t}\left\|u_{n}^{\prime}(r)\right\|_{X}^{q} d r\right)^{1 / q}|t-s|^{1 / q^{\prime}}\right)^{p} \\
= & 2^{p-1} \varepsilon\left(\left\|u_{n}(t)\right\|^{p}+\left\|u_{n}(s)\right\|^{p}\right)+C_{\varepsilon} R^{p / q}|t-s|^{p / q^{\prime}}
\end{aligned}
$$

This is substituted in to 43.5 .25 to obtain

$$
\begin{gathered}
\int_{a}^{b}\left\|\left(u_{n}(t)-\bar{u}_{n}(s)\right)\right\|_{W}^{p} d s \leq \\
\\
\sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t_{i}}\left(2^{p-1} \varepsilon\left(\left\|u_{n}(t)\right\|^{p}+\left\|u_{n}(s)\right\|^{p}\right)\right. \\
\left.\quad+C_{\varepsilon} R^{p / q}|t-s|^{p / q^{\prime}}\right) d s d t \\
=\quad \sum_{i=1}^{k} 2^{p} \varepsilon \int_{t_{i-1}}^{t_{i}}\left\|u_{n}(t)\right\|_{W}^{p}+C_{\varepsilon} R^{p / q} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t_{i}}|t-s|^{p / q^{\prime}} d s d t \\
=\quad 2^{p} \varepsilon \int_{a}^{b}\left\|u_{n}(t)\right\|^{p} d t+C_{\varepsilon} R^{p / q} \sum_{i=1}^{k} \frac{1}{\left(t_{i}-t_{i-1}\right)}\left(t_{i}-t_{i-1}\right)^{p / q^{\prime}} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t_{i}} d s d t \\
=\quad 2^{p} \varepsilon \int_{a}^{b}\left\|u_{n}(t)\right\|^{p} d t+C_{\varepsilon} R^{p / q} \sum_{i=1}^{k} \frac{1}{\left(t_{i}-t_{i-1}\right)}\left(t_{i}-t_{i-1}\right)^{p / q^{\prime}}\left(t_{i}-t_{i-1}\right)^{2} \\
\leq \quad 2^{p} \varepsilon R^{p}+C_{\varepsilon} R^{p / q} \sum_{i=1}^{k}\left(t_{i}-t_{i-1}\right)^{1+p / q^{\prime}}=2^{p} \varepsilon R^{p}+C_{\varepsilon} R^{p / q} k\left(\frac{b-a}{k}\right)^{1+p / q^{\prime}} .
\end{gathered}
$$

Taking $\varepsilon$ so small that $2^{p} \varepsilon R^{p}<\eta^{p} / 8^{p}$ and then choosing $k$ sufficiently large, it follows

$$
\left\|u_{n}-\bar{u}_{n}\right\|_{L^{p}([a, b] ; W)}<\frac{\eta}{4} .
$$

Now use compactness of the embedding of $E$ into $W$ to obtain a subsequence such that $\left\{\bar{u}_{n}\right\}$ is Cauchy in $L^{p}(a, b ; W)$ and use this to contradict 43.5.24. The details follow.

Suppose $\bar{u}_{n}(t)=\sum_{i=1}^{k} u_{i}^{n} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)$. Thus

$$
\left\|\bar{u}_{n}(t)\right\|_{E}=\sum_{i=1}^{k}\left\|u_{i}^{n}\right\|_{E} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)
$$

and so

$$
R \geq \int_{a}^{b}\left\|\bar{u}_{n}(t)\right\|_{E}^{p} d t=\frac{T}{k} \sum_{i=1}^{k}\left\|u_{i}^{n}\right\|_{E}^{p}
$$

Therefore, the $\left\{u_{i}^{n}\right\}$ are all bounded. It follows that after taking subsequences $k$ times there exists a subsequence $\left\{u_{n_{k}}\right\}$ such that $u_{n_{k}}$ is a Cauchy sequence in $L^{p}(a, b ; W)$. You simply get a subsequence such that $u_{i}^{n_{k}}$ is a Cauchy sequence in $W$ for each $i$. Then denoting this subsequence by $n$,

$$
\begin{aligned}
\left\|u_{n}-u_{m}\right\|_{L^{p}(a, b ; W)} \leq & \left\|u_{n}-\bar{u}_{n}\right\|_{L^{p}(a, b ; W)} \\
& +\left\|\bar{u}_{n}-\bar{u}_{m}\right\|_{L^{p}(a, b ; W)}+\left\|\bar{u}_{m}-u_{m}\right\|_{L^{p}(a, b ; W)} \\
\leq & \frac{\eta}{4}+\left\|\bar{u}_{n}-\bar{u}_{m}\right\|_{L^{p}(a, b ; W)}+\frac{\eta}{4}<\eta
\end{aligned}
$$

provided $m, n$ are large enough, contradicting 43.5.24. This proves the theorem.

### 43.6 The $K$ Method

This considers the problem of interpolating Banach spaces. The idea is to build a Banach space between two others in a systematic way, thus constructing a new Banach space from old ones. The first method of defining intermediate Banach spaces is called the $K$ method. For more on this topic as well as the other topics on interpolation see [16] which is what I am following. See also [124]. There is far more on these subjects in these books than what I am presenting here! My goal is to present only enough to give an introduction to the topic and to use it in presenting more theory of Sobolev spaces.

In what follows a topological vector space is a vector space in which vector addition and scalar multiplication are continuous. That is $: ~: \mathbb{F} \times X \rightarrow X$ is continuous and $+: X \times X \rightarrow X$ is also continuous.

A common example of a topological vector space is the dual space, $X^{\prime}$ of a Banach space, $X$ with the weak $*$ topology. For $S \subseteq X$ a finite set, define

$$
B_{S}\left(x^{*}, r\right) \equiv\left\{y^{*} \in X^{\prime}:\left|y^{*}(x)-x^{*}(x)\right|<r \text { for all } x \in S\right\}
$$

Then the $B_{S}\left(x^{*}, r\right)$ for $S$ a finite subset of $X$ and $r>0$ form a basis for the topology on $X^{\prime}$ called the weak $*$ topology. You can check that the vector space operations are continuous.

Definition 43.6.1 Let $A_{0}$ and $A_{1}$ be two Banach spaces with norms $\|\cdot\|_{0}$ and $\|\cdot\|_{1}$ respectively, also written as $\|\cdot\|_{A_{0}}$ and $\|\cdot\|_{A_{1}}$ and let $X$ be a topological vector space such that $A_{i} \subseteq X$ for $i=1,2$, and the identity map from $A_{i}$ to $X$ is continuous. For each $t>0$, define a norm on $A_{0}+A_{1}$ by

$$
K(t, a) \equiv\|a\|_{t} \equiv \inf \left\{\left\|a_{0}\right\|_{0}+t\left\|a_{1}\right\|_{1}: a_{0}+a_{1}=a\right\}
$$

This is short for $K\left(t, a, A_{0}, A_{1}\right)$. Thus $K\left(t, a, A_{1}, A_{0}\right)$ will mean

$$
K\left(t, a, A_{1}, A_{0}\right) \equiv \inf \left\{\left\|a_{1}\right\|_{A_{1}}+t\left\|a_{0}\right\|_{A_{0}}: a_{0}+a_{1}=a\right\}
$$

but the default is $K\left(t, a, A_{0}, A_{1}\right)$ if $K(t, a)$ is written.
The following lemma is an interesting exercise.

Lemma 43.6.2 $\left(A_{0}+A_{1}, K(t, \cdot)\right)$ is a Banach space and all the norms, $K(t, \cdot)$ are equivalent.

Proof: First, why is $K(t, \cdot)$ a norm? It is clear that $K(t, a) \geq 0$ and that if $a=0$ then $K(t, a)=0$. Is this the only way this can happen? Suppose $K(t, a)=0$. Then there exist $a_{0 n} \in A_{0}$ and $a_{1 n} \in A_{1}$ such that $\left\|a_{0 n}\right\|_{0} \rightarrow 0,\left\|a_{1 n}\right\|_{1} \rightarrow 0$, and $a=a_{0 n}+a_{1 n}$. Since the embedding of $A_{i}$ into $X$ is continuous and since $X$ is a topological vector space ${ }^{1}$, it follows

$$
a=a_{0 n}+a_{1 n} \rightarrow 0
$$

and so $a=0$.
Let $\alpha$ be a nonzero scalar. Then

$$
\begin{aligned}
K(t, \alpha a) & =\inf \left\{\left\|a_{0}\right\|_{0}+t \mid a_{1} \|_{1}: a_{0}+a_{1}=\alpha a\right\} \\
& =\inf \left\{|\alpha|\left\|\frac{a_{0}}{\alpha}\right\|\left\|_{0}+t|\alpha|\right\| \frac{a_{1}}{\alpha} \|_{1}: \frac{a_{0}}{\alpha}+\frac{a_{1}}{\alpha}=a\right\} \\
& =|\alpha| \inf \left\{\left\|\frac{a_{0}}{\alpha}\right\|\left\|_{0}+t\right\| \frac{a_{1}}{\alpha}\| \|_{1}: \frac{a_{0}}{\alpha}+\frac{a_{1}}{\alpha}=a\right\} \\
& =|\alpha| \inf \left\{\left\|a_{0}\right\|_{0}+t\left\|a_{1}\right\|_{1}: a_{0}+a_{1}=a\right\}=|\alpha| K(t, a) .
\end{aligned}
$$

It remains to verify the triangle inequality. Let $\varepsilon>0$ be given. Then there exist $a_{0}, a_{1}, b_{0}$, and $b_{1}$ in $A_{0}, A_{1}, A_{0}$, and $A_{1}$ respectively such that $a_{0}+a_{1}=a, b_{0}+b_{1}=b$ and

$$
\begin{aligned}
\varepsilon+K(t, a)+K(t, b) & >\left\|a_{0}\right\|_{0}+t\left\|a_{1}\right\|_{1}+\left\|b_{0}\right\|_{0}+t\left\|b_{1}\right\|_{1} \\
& \geq\left\|a_{0}+b_{0}\right\|_{0}+t\left\|b_{1}+a_{1}\right\|_{1} \geq K(t, a+b)
\end{aligned}
$$

This has shown that $K(t, \cdot)$ is at least a norm. Are all these norms equivalent? If $0<s<t$ then it is clear that $K(t, a) \geq K(s, a)$. To show there exists a constant, $C$ such that $C K(s, a) \geq K(t, a)$ for all $a$,

$$
\begin{aligned}
\stackrel{t}{s} K(s, a) & \equiv \frac{t}{s} \inf \left\{\left\|a_{0}\right\|_{0}+s\left\|a_{1}\right\|_{1}: a_{0}+a_{1}=a\right\} \\
& =\inf \left\{\frac{t}{s}\left\|a_{0}\right\|_{0}+s-\frac{t}{s}\left\|a_{1}\right\|_{1}: a_{0}+a_{1}=a\right\} \\
& =\inf \left\{\frac{t}{s}\left\|a_{0}\right\|_{0}+t\left\|a_{1}\right\|_{1}: a_{0}+a_{1}=a\right\} \\
& \geq \inf \left\{\left\|a_{0}\right\|_{0}+t\left\|a_{1}\right\|_{1}: a_{0}+a_{1}=a\right\}=K(t, a)
\end{aligned}
$$

Therefore, the two norms are equivalent as hoped.
Finally, it is required to verify that $\left(A_{0}+A_{1}, K(t, \cdot)\right)$ is a Banach space. Since all these norms are equivalent, it suffices to only consider the norm, $K(1, \cdot)$. Let $\left\{a_{0 n}+a_{1 n}\right\}_{n=1}^{\infty}$ be a Cauchy sequence in $A_{0}+A_{1}$. Then for $m, n$ large enough,

$$
K\left(1, a_{0 n}+a_{1 n}-\left(a_{0 m}+a_{1 m}\right)\right)<\varepsilon .
$$

[^31]It follows there exist $x_{n} \in A_{0}$ and $y_{n} \in A_{1}$ such that $x_{n}+y_{n}=0$ for every $n$ and whenever $m, n$ are large enough,

$$
\left\|a_{0 n}+x_{n}-\left(a_{0 m}+x_{m}\right)\right\|_{0}+\left\|a_{1 n}+y_{n}-\left(a_{1 m}+y_{m}\right)\right\|_{1}<\varepsilon
$$

Hence $\left\{a_{1 n}+y_{n}\right\}$ is a Cauchy sequence in $A_{1}$ and $\left\{a_{0 n}+x_{n}\right\}$ is a Cauchy sequence in $A_{0}$. Let

$$
\begin{aligned}
a_{0 n}+x_{n} & \rightarrow a_{0} \in A_{0} \\
a_{1 n}+y_{n} & \rightarrow a_{1} \in A_{1} .
\end{aligned}
$$

Then

$$
\begin{aligned}
K\left(1, a_{0 n}+a_{1 n}-\left(a_{0}+a_{1}\right)\right) & =K\left(1, a_{0 n}+x_{n}+a_{1 n}+y_{n}-\left(a_{0}+a_{1}\right)\right) \\
& \leq\left\|a_{0 n}+x_{n}-a_{0}\right\|_{0}+\left\|a_{1 n}+y_{n}-a_{1}\right\|_{1}
\end{aligned}
$$

which converges to 0 . Thus $A_{0}+A_{1}$ is a Banach space as claimed.
With this, there exists a method for constructing a Banach space which lies between $A_{0} \cap A_{1}$ and $A_{0}+A_{1}$.

Definition 43.6.3 Let $1 \leq q<\infty, 0<\theta<1$. Define $\left(A_{0}, A_{1}\right)_{\theta, q}$ to be those elements of $A_{0}+A_{1}, a$, such that

$$
\|a\|_{\theta, q} \equiv\left[\int_{0}^{\infty}\left(t^{-\theta} K\left(t, a, A_{0}, A_{1}\right)\right)^{q} \frac{d t}{t}\right]^{1 / q}<\infty
$$

Theorem 43.6.4 $\left(A_{0}, A_{1}\right)_{\theta, q}$ is a normed linear space satisfying

$$
\begin{equation*}
A_{0} \cap A_{1} \subseteq\left(A_{0}, A_{1}\right)_{\theta, q} \subseteq A_{0}+A_{1} \tag{43.6.26}
\end{equation*}
$$

with the inclusion maps continuous, and

$$
\begin{equation*}
\left(\left(A_{0}, A_{1}\right)_{\theta, q},\|\cdot\|_{\theta, q}\right) \text { is a Banach space. } \tag{43.6.27}
\end{equation*}
$$

If $a \in A_{0} \cap A_{1}$, then

$$
\begin{equation*}
\|a\|_{\theta, q} \leq\left(\frac{1}{q \theta(1-\theta)}\right)^{1 / q}\|a\|_{1}^{\theta}\|a\|_{0}^{1-\theta} \tag{43.6.28}
\end{equation*}
$$

If $A_{0} \subseteq A_{1}$ with $\|\cdot\|_{0} \geq\|\cdot\|_{1}$, then

$$
A_{0} \cap A_{1}=A_{0} \subseteq\left(A_{0}, A_{1}\right)_{\theta, q} \subseteq A_{1}=A_{0}+A_{1} .
$$

Also, if bounded sets in $A_{0}$ have compact closures in $A_{1}$ then the same is true if $A_{1}$ is replaced with $\left(A_{0}, A_{1}\right)_{\theta, q}$. Finally, if

$$
\begin{equation*}
T \in \mathscr{L}\left(A_{0}, B_{0}\right), T \in \mathscr{L}\left(A_{1}, B_{1}\right) \tag{43.6.29}
\end{equation*}
$$

and $T$ is a linear map from $A_{0}+A_{1}$ to $B_{0}+B_{1}$ where the $A_{i}$ and $B_{i}$ are Banach spaces with the properties described above, then it follows

$$
\begin{equation*}
T \in \mathscr{L}\left(\left(A_{0}, A_{1}\right)_{\theta, q},\left(B_{0}, B_{1}\right)_{\theta, q}\right) \tag{43.6.30}
\end{equation*}
$$

and if $M$ is its norm, and $M_{0}$ and $M_{1}$ are the norms of $T$ as a map in $\mathscr{L}\left(A_{0}, B_{0}\right)$ and $\mathscr{L}\left(A_{1}, B_{1}\right)$ respectively, then

$$
\begin{equation*}
M \leq M_{0}^{1-\theta} M_{1}^{\theta} \tag{43.6.31}
\end{equation*}
$$

Proof: Suppose first $a \in A_{0} \cap A_{1}$. Then

$$
\begin{align*}
\|a\|_{\theta, q}^{q} & \equiv \int_{0}^{r}\left(t^{-\theta} K(t, a)\right)^{q} \frac{d t}{t}+\int_{r}^{\infty}\left(t^{-\theta} K(t, a)\right)^{q} \frac{d t}{t}  \tag{43.6.32}\\
& \leq \int_{0}^{r}\left(t^{-\theta}\|a\|_{1} t\right)^{q} \frac{d t}{t}+\int_{r}^{\infty}\left(t^{-\theta}\|a\|_{0}\right)^{q} \frac{d t}{t} \\
& =\|a\|_{1}^{q} \int_{0}^{r} t^{q(1-\theta)-1} d t+\|a\|_{0}^{q} \int_{r}^{\infty} t^{-1-\theta q} d t \\
& =\|a\|_{1}^{q} \frac{r^{q-q \theta}}{q-q \theta}+\|a\|_{0}^{q} \frac{r^{-\theta q}}{\theta q}<\infty \tag{43.6.33}
\end{align*}
$$

Which shows the first inclusion of 43.6.26. The above holds for all $r>0$ and in particular for the value of $r$ which minimizes the expression on the right in 43.6.33, $r=\|a\|_{0} /\|a\|_{1}$. Therefore, doing some calculus,

$$
\|a\|_{\theta, q}^{q} \leq \frac{1}{\theta q(1+\theta)}\|a\|_{0}^{q(1-\theta)}\|a\|_{1}^{q \theta}
$$

which shows 43.6.28. This also verifies that the first inclusion map is continuous in 43.6.26 because if $a_{n} \rightarrow 0$ in $A_{0} \cap A_{1}$, then $a_{n} \rightarrow 0$ in $A_{0}$ and in $A_{1}$ and so the above shows $a_{n} \rightarrow 0$ in $\left(A_{0}, A_{1}\right)_{\theta, q}$.

Now consider the second inclusion in 43.6.26. The inclusion is obvious since $\left(A_{0}, A_{1}\right)_{\theta, q}$ is given to be a subset of $A_{0}+A_{1}$ defined by

$$
\left(\int_{0}^{\infty}\left(t^{-\theta} K(t, a)\right)^{q} \frac{d t}{t}\right)^{1 / q}<\infty
$$

It remains to verify the inclusion map is continuous. Suppose $a_{n} \rightarrow 0$ in $\left(A_{0}, A_{1}\right)_{\theta, q}$. Since $a_{n} \rightarrow 0$ in $\left(A_{0}, A_{1}\right)_{\theta, q}$, it follows the function, $t \rightarrow t^{-\theta} K\left(t, a_{n}\right)$ converges to zero in $L^{q}(0, \infty)$ with respect to the measure, $d t / t$. Therefore, taking another subsequence, still denoted as $a_{n}$, you can assume this function converges to 0 a.e. Pick such a $t$ where this convergence takes place. Then $K\left(t, a_{n}\right) \rightarrow 0$ as $n \rightarrow \infty$ and so $a_{n} \rightarrow 0$ in $A_{0}+A_{1}$. (Recall all these norms $K(t, \cdot)$ are equivalent.) This shows that if $a_{n} \rightarrow 0$ in $\left(A_{0}, A_{1}\right)_{\theta, q}$, then there exists a subsequence $\left\{a_{n_{k}}\right\}$ such that $a_{n_{k}} \rightarrow 0$ in $A_{0}+A_{1}$. It follows that if $a_{n} \rightarrow 0$ in $\left(A_{0}, A_{1}\right)_{\theta, q}$, then $a_{n} \rightarrow 0$ in $A_{0}+A_{1}$. This proves the continuity of the embedding.

What about 43.6.27? Suppose $\left\{a_{n}\right\}$ is a Cauchy sequence in $\left(A_{0}, A_{1}\right)_{\theta, q}$. Then from what was just shown this is a Cauchy sequence in $A_{0}+A_{1}$ and so there exists $a \in A_{0}+A_{1}$
such that $a_{n} \rightarrow a$ in $A_{0}+A_{1}$ because $A_{0}+A_{1}$ is a Banach space. Thus, $K\left(t, a_{n}\right) \rightarrow K(t, a)$ for all $t>0$. (Recall all these norms $K(t, \cdot)$ are equivalent.) Therefore, by Fatou's lemma,

$$
\begin{aligned}
\left(\int_{0}^{\infty}\left(t^{-\theta} K(t, a)\right)^{q} \frac{d t}{t}\right)^{1 / q} & \leq \lim _{n \rightarrow \infty}\left(\int_{0}^{\infty}\left(t^{-\theta} K\left(t, a_{n}\right)\right)^{q} \frac{d t}{t}\right)^{1 / q} \\
& \leq \max \left\{\left\|a_{n}\right\|_{\theta, q}: n \in \mathbb{N}\right\}<\infty
\end{aligned}
$$

and so $a \in\left(A_{0}, A_{1}\right)_{\theta, q}$. Now

$$
\begin{aligned}
\left\|a-a_{n}\right\|_{\theta, q} & \leq \lim _{m \rightarrow \infty}\left(\int_{0}^{\infty}\left(t^{-\theta} K\left(t, a_{n}-a_{m}\right)\right)^{q} \frac{d t}{t}\right)^{1 / q} \\
& =\lim \inf _{m \rightarrow \infty}\left\|a_{n}-a_{m}\right\|_{\theta, q}<\varepsilon
\end{aligned}
$$

whenever $n$ is large enough. Thus $\left(A_{0}, A_{1}\right)_{\theta, q}$ is complete as claimed.
Next suppose $A_{0} \subseteq A_{1}$ and the inclusion map is compact. In this case, $A_{0} \cap A_{1}=A_{0}$ and so it has been shown above that $A_{0} \subseteq\left(A_{0}, A_{1}\right)_{\theta, q}$. It remains to show that every bounded subset, $S$, contained in $A_{0}$ has an $\eta$ net in $\left(A_{0}, A_{1}\right)_{\theta, q}$. Recall the inequality, 43.6.28

$$
\begin{aligned}
\|a\|_{\theta, q} & \leq\left(\frac{1}{q \theta(1-\theta)}\right)^{1 / q}\|a\|_{1}^{\theta}\|a\|_{0}^{1-\theta} \\
& =\frac{C}{\varepsilon}\|a\|_{1}^{\theta} \varepsilon\|a\|_{0}^{1-\theta}
\end{aligned}
$$

Now this implies

$$
\|a\|_{\theta, q} \leq\left(\frac{C}{\varepsilon}\right)^{1 / \theta} \theta\|a\|_{1}+\varepsilon^{1 /(1-\theta)}(1-\theta)\|a\|_{0}
$$

By compactness of the embedding of $A_{0}$ into $A_{1}$, it follows there exists an $\varepsilon^{(1+\theta) / \theta}$ net for $S$ in $A_{1},\left\{a_{1}, \cdots, a_{p}\right\}$. Then for $a \in S$, there exists $k$ such that $\left\|a-a_{k}\right\|_{1}<\varepsilon^{(1+\theta) / \theta}$. It follows

$$
\begin{aligned}
\left\|a-a_{k}\right\|_{\theta, q} & \leq\left(\frac{C}{\varepsilon}\right)^{1 / \theta} \theta\left\|a-a_{k}\right\|_{1}+\varepsilon^{1 /(1-\theta)}(1-\theta)\left\|a-a_{k}\right\|_{0} \\
& \leq\left(\frac{C}{\varepsilon}\right)^{1 / \theta} \theta \varepsilon^{(1+\theta) / \theta}+\varepsilon^{1 /(1-\theta)}(1-\theta) 2 M \\
& =C^{1 / \theta} \theta \varepsilon+\varepsilon^{1 /(1-\theta)}(1-\theta) 2 M
\end{aligned}
$$

where $M$ is large enough that $\|a\|_{0} \leq M$ for all $a \in S$. Since $\varepsilon$ is arbitrary, this shows the existence of a $\eta$ net and proves the compactness of the embedding into $\left(A_{0}, A_{1}\right)_{\theta, q}$.

It remains to verify the assertions 43.6.29-43.6.31. Let $T \in \mathscr{L}\left(A_{0}, B_{0}\right), T \in \mathscr{L}\left(A_{1}, B_{1}\right)$ with $T$ a linear map from $A_{0}+A_{1}$ to $B_{0}+B_{1}$. Let $a \in\left(A_{0}, A_{1}\right)_{\theta, q} \subseteq A_{0}+A_{1}$ and consider
$T a \in B_{0}+B_{1}$. Denote by $K(t, \cdot)$ the norm described above for both $A_{0}+A_{1}$ and $B_{0}+B_{1}$ since this will cause no confusion. Then

$$
\begin{equation*}
\|T a\|_{\theta, q} \equiv\left(\int_{0}^{\infty}\left(t^{-\theta} K(t, T a)\right)^{q} \frac{d t}{t}\right)^{1 / q} \tag{43.6.34}
\end{equation*}
$$

Now let $a_{0}+a_{1}=a$ and so $T a_{0}+T a_{1}=T a$

$$
\begin{aligned}
K(t, T a) & \leq\left\|T a_{0}\right\|_{0}+t\left\|T a_{1}\right\|_{1} \leq M_{0}\left\|a_{0}\right\|_{0}+M_{1} t\left\|a_{1}\right\|_{1} \\
& \leq M_{0}\left(\left\|a_{0}\right\|_{0}+t\left(\frac{M_{1}}{M_{0}}\right)\left\|a_{1}\right\|_{1}\right)
\end{aligned}
$$

and so, taking inf for all $a_{0}+a_{1}=a$, yields

$$
K(t, T a) \leq M_{0} K\left(t\left(\frac{M_{1}}{M_{0}}\right), a\right)
$$

It follows from 43.6.34 that

$$
\begin{aligned}
\|T a\|_{\theta, q} & \equiv\left(\int_{0}^{\infty}\left(t^{-\theta} K(t, T a)\right)^{q} \frac{d t}{t}\right)^{1 / q} \\
& \leq\left(\int_{0}^{\infty}\left(t^{-\theta} M_{0} K\left(t\left(\frac{M_{1}}{M_{0}}\right), a\right)\right)^{q} \frac{d t}{t}\right)^{1 / q} \\
& =M_{0}\left(\int_{0}^{\infty}\left(t^{-\theta} K\left(t\left(\frac{M_{1}}{M_{0}}\right), a\right)\right)^{q} \frac{d t}{t}\right)^{1 / q} \\
& =M_{0}\left(\int_{0}^{\infty}\left(\left(\frac{M_{0}}{M_{1}} s\right)^{-\theta} K(s, a)\right)^{q} \frac{d s}{s}\right)^{1 / q} \\
& =M_{1}^{\theta} M_{0}^{(1-\theta)}\left(\int_{0}^{\infty}\left(s^{-\theta} K(s, a)\right)^{q} \frac{d s}{s}\right)^{1 / q}=M_{1}^{\theta} M_{0}^{(1-\theta)}\|a\|_{\theta, q}
\end{aligned}
$$

This shows $T \in \mathscr{L}\left(\left(A_{0}, A_{1}\right)_{\theta, q},\left(B_{0}, B_{1}\right)_{\theta, q}\right)$ and if $M$ is the norm of $T, M \leq M_{0}^{1-\theta} M_{1}^{\theta}$ as claimed. This proves the theorem.

### 43.7 The $J$ Method

There is another method known as the $J$ method. Instead of

$$
K(t, a) \equiv \inf \left\{\left\|a_{0}\right\|_{A_{0}}+t\left\|a_{1}\right\|_{A_{1}}: a_{0}+a_{1}=a\right\}
$$

for $a \in A_{0}+A_{1}$, this method considers $a \in A_{0} \cap A_{1}$ and $J(t, a)$ defined below gives a norm on $A_{0} \cap A_{1}$.

Definition 43.7.1 For $A_{0}$ and $A_{1}$ Banach spaces as described above, and $a \in A_{0} \cap A_{1}$,

$$
\begin{equation*}
J(t, a) \equiv \max \left(\|a\|_{A_{0}}, t\|a\|_{A_{1}}\right) \tag{43.7.35}
\end{equation*}
$$

this is short for $J\left(t, a, A_{0}, A_{1}\right)$. Thus

$$
J\left(t, a, A_{1}, A_{0}\right) \equiv \max \left(\|a\|_{A_{1}}, t\|a\|_{A_{0}}\right)
$$

but unless indicated otherwise, $A_{0}$ will come first. Now for $\theta \in(0,1)$ and $q \geq 1$, define a space, $\left(A_{0}, A_{1}\right)_{\theta, q, J}$ as follows. The space, $\left(A_{0}, A_{1}\right)_{\theta, q, J}$ will consist of those elements, $a$, of $A_{0}+A_{1}$ which can be written in the form

$$
\begin{equation*}
a=\int_{0}^{\infty} u(t) \frac{d t}{t} \equiv \lim _{\varepsilon \rightarrow 0+} \int_{\varepsilon}^{1} u(t) \frac{d t}{t}+\lim _{r \rightarrow \infty} \int_{1}^{r} u(t) \frac{d t}{t} \tag{43.7.36}
\end{equation*}
$$

the limits taking place in $A_{0}+A_{1}$ with the norm

$$
K(1, a) \equiv \inf _{a=a_{0}+a_{1}}\left(\left\|a_{0}\right\|_{A_{0}}+\left\|a_{1}\right\|_{A_{1}}\right)
$$

where $u(t)$ is strongly measurable with values in $A_{0} \cap A_{1}$ and bounded on every compact subset of $(0, \infty)$ such that

$$
\begin{equation*}
\left(\int_{0}^{\infty}\left(t^{-\theta} J\left(t, u(t), A_{0}, A_{1}\right)\right)^{q} \frac{d t}{t}\right)^{1 / q}<\infty . \tag{43.7.37}
\end{equation*}
$$

For such $a \in A_{0}+A_{1}$, define

$$
\begin{equation*}
\|a\|_{\theta, q, J} \equiv \inf _{u}\left\{\left(\int_{0}^{\infty}\left(t^{-\theta} J\left(t, u(t), A_{0}, A_{1}\right)\right)^{q} \frac{d t}{t}\right)^{1 / q}\right\} \tag{43.7.38}
\end{equation*}
$$

where the infimum is taken over all u satisfying 43.7.36 and 43.7.37.
Note that a norm on $A_{0} \times A_{1}$ would be

$$
\left\|\left(a_{0}, a_{1}\right)\right\| \equiv \max \left(\left\|a_{0}\right\|_{A_{0}}, t\left\|a_{1}\right\|_{A_{1}}\right)
$$

and so $J(t, \cdot)$ is the restriction of this norm to the subspace of $A_{0} \times A_{1}$ defined by

$$
\left\{(a, a): a \in A_{0} \cap A_{1}\right\}
$$

Also for each $t>0 J(t, \cdot)$ is a norm on $A_{0} \cap A_{1}$ and furthermore, any two of these norms are equivalent. In fact, for $0<t<s$,

$$
\begin{aligned}
J(t, a) & =\max \left(\|a\|_{A_{0}}, t\|a\|_{A_{1}}\right) \\
& \geq \max \left(\|a\|_{A_{0}}, s\|a\|_{A_{1}}\right) \\
& =J(s, a) \\
& \geq \max \left(\frac{s}{t}\|a\|_{A_{0}}, s\|a\|_{A_{1}}\right) \\
& =\frac{s}{t} \max \left(\|a\|_{A_{0}}, t\|a\|_{A_{1}}\right) \\
& \geq{ }_{-}^{s} J(t, a)
\end{aligned}
$$

The following lemma is significant and follows immediately from the above definition.

Lemma 43.7.2 Suppose $a \in\left(A_{0}, A_{1}\right)_{\theta, q, J}$ and $a=\int_{0}^{\infty} u(t) \frac{d t}{t}$ where $u$ is described above. Then letting $r>1$,

$$
u_{r}(t) \equiv\left\{\begin{array}{l}
u(t) \text { if } t \in\left(\frac{1}{r}, r\right) \\
0 \text { otherwise }
\end{array}\right.
$$

it follows that

$$
\int_{0}^{\infty} u_{r}(t) \frac{d t}{t} \in A_{0} \cap A_{1}
$$

Proof: The integral equals $\int_{1 / r}^{r} u(t) \frac{d t}{t} \cdot \int_{1 / r}^{r} \frac{1}{t} d t=2 \ln r<\infty$. Now $u_{r}$ is measurable in $A_{0} \cap A_{1}$ and bounded. Therefore, there exists a sequence of measurable simple functions, $\left\{s_{n}\right\}$ having values in $A_{0} \cap A_{1}$ which converges pointwise and uniformly to $u_{r}$. It can also be assumed $J\left(r, s_{n}(t)\right) \leq J\left(r, u_{r}(t)\right)$ for all $t \in[1 / r, r]$. Therefore,

$$
\lim _{n, m \rightarrow \infty} \int_{1 / r}^{r} J\left(r, s_{m}-s_{n}\right) \frac{d t}{t}=0 .
$$

It follows from the definition of the Bochner integral that

$$
\lim _{n \rightarrow \infty} \int_{1 / r}^{r} s_{n} \frac{d t}{t}=\int_{1 / r}^{r} u_{r} \frac{d t}{t} \in A_{0} \cap A_{1} .
$$

This proves the lemma.
The remarkable thing is that the two spaces, $\left(A_{0}, A_{1}\right)_{\theta, q}$ and $\left(A_{0}, A_{1}\right)_{\theta, q, J}$ coincide and have equivalent norms. The following important lemma, called the fundamental lemma of interpolation theory in [16] is used to prove this. This lemma is really incredible.

Lemma 43.7.3 Suppose for $a \in A_{0}+A_{1}, \lim _{t \rightarrow 0+} K(t, a)=0$ and $\lim _{t \rightarrow \infty} \frac{K(t, a)}{t}=0$. Then for any $\varepsilon>0$, there is a representation,

$$
\begin{equation*}
a=\sum_{i=-\infty}^{\infty} u_{i}=\lim _{n, m \rightarrow \infty} \sum_{i=-m}^{n} u_{i}, u_{i} \in A_{0} \cap A_{1} \tag{43.7.39}
\end{equation*}
$$

the convergences taking place in $A_{0}+A_{1}$, such that

$$
\begin{equation*}
J\left(2^{i}, u_{i}\right) \leq 3(1+\varepsilon) K\left(2^{i}, a\right) \tag{43.7.40}
\end{equation*}
$$

Proof: For each $i$, there exist $a_{0, i} \in A_{0}$ and $a_{1, i} \in A_{1}$ such that

$$
a=a_{0, i}+a_{1, i}
$$

and

$$
\begin{equation*}
(1+\varepsilon) K\left(2^{i}, a\right) \geq\left\|a_{0, i}\right\|_{A_{0}}+2^{i}\left\|a_{1, i}\right\|_{A_{1}} . \tag{43.7.41}
\end{equation*}
$$

This follows directly from the definition of $K(t, a)$. From the assumed limit conditions on $K(t, a)$,

$$
\begin{equation*}
\lim _{i \rightarrow \infty}\left\|a_{1, i}\right\|_{A_{1}}=0, \lim _{i \rightarrow-\infty}\left\|a_{0, i}\right\|_{A_{0}}=0 \tag{43.7.42}
\end{equation*}
$$

Then let $u_{i} \equiv a_{0, i}-a_{0, i-1}=a_{1, i-1}-a_{1, i}$. The reason these are equal is $a=a_{0, i}+a_{1, i}=$ $a_{0, i-1}+a_{1, i-1}$. Then

$$
\sum_{i=-m}^{n} u_{i}=a_{0, n}-a_{0,-(m+1)}=a_{1,-(m+1)}-a_{1, n}
$$

It follows $a-\sum_{i=-m}^{n} u_{i}=a-\left(a_{0, n}-a_{0,-(m+1)}\right)=a_{0,-(m+1)}+a_{1, n}$, and both terms converge to zero as $m$ and $n$ converge to $\infty$ by 43.7.42. Therefore,

$$
K\left(1, a-\sum_{i=-m}^{n} u_{i}\right) \leq\left\|a_{0,-(m+1)}\right\|+\left\|a_{1, n}\right\|
$$

and so this shows $a=\sum_{i=-\infty}^{\infty} u_{i}$ which is one of the claims of the lemma. Also

$$
\begin{gathered}
J\left(2^{i}, u_{i}\right) \equiv \max \left(\left\|u_{i}\right\|_{A_{0}}, 2^{i}\left\|u_{i}\right\|_{A_{1}}\right) \leq\left\|u_{i}\right\|_{A_{0}}+2^{i}\left\|u_{i}\right\|_{A_{1}} \\
\leq\left\|a_{0, i}\right\|_{A_{0}}+2^{i}\left\|a_{1, i}\right\|_{A_{1}}+\overbrace{\left\|a_{0, i-1}\right\|_{A_{0}}+2^{i}\left\|a_{1, i-1}\right\|_{A_{1}}}^{\leq 2\left(\left\|a_{0, i-1}\right\|_{A_{0}}+2^{i-1}\left\|a_{1, i-1}\right\|_{A_{1}}\right)} \\
\leq(1+\varepsilon) K\left(2^{i}, a\right)+2(1+\varepsilon) K\left(2^{i-1}, a\right) \leq 3(1+\varepsilon) K\left(2^{i}, a\right)
\end{gathered}
$$

because $t \rightarrow K(t, a)$ is nondecreasing. This proves the lemma.
Lemma 43.7.4 If $a \in A_{0} \cap A_{1}$, then $K(t, a) \leq \min \left(1, \frac{t}{s}\right) J(s, a)$.
Proof: If $s \geq t$, then $\min \left(1, \frac{t}{s}\right)=\frac{t}{s}$ and so

$$
\begin{aligned}
\min \left(1, \frac{t}{s}\right) J(s, a) & =\frac{t}{s} \max \left(\|a\|_{A_{0}}, s\|a\|_{A_{1}}\right) \geq\left(\frac{t}{s}\right) s\|a\|_{A_{1}} \\
& =t\|a\|_{A_{1}} \geq K(t, a)
\end{aligned}
$$

Now in case $s<t$, then $\min \left(1, \frac{t}{s}\right)=1$ and so

$$
\begin{aligned}
\min \left(1, \frac{t}{s}\right) J(s, a) & =\max \left(\|a\|_{A_{0}}, s\|a\|_{A_{1}}\right) \geq\|a\|_{A_{0}} \\
& \geq K(t, a)
\end{aligned}
$$

This proves the lemma.
Theorem 43.7.5 Let $A_{0}, A_{1}, K$ and $J$ be as described above. Then for all $q \geq 1$ and $\theta \in$ $(0,1)$,

$$
\left(A_{0}, A_{1}\right)_{\theta, q}=\left(A_{0}, A_{1}\right)_{\theta, q, J}
$$

and furthermore, the norms are equivalent.

Proof: Begin with $a \in\left(A_{0}, A_{1}\right)_{\theta, q}$. Thus

$$
\begin{equation*}
\|a\|_{\theta, q}^{q}=\int_{0}^{\infty}\left(t^{-\theta} K(t, a)\right)^{q} \frac{d t}{t}<\infty \tag{43.7.43}
\end{equation*}
$$

and it is necessary to produce $u(t)$ as described above,

$$
a=\int_{0}^{\infty} u(t) \frac{d t}{t} \text { where } \int_{0}^{\infty}\left(t^{-\theta} J(t, u(t))\right)^{q} \frac{d t}{t}<\infty .
$$

From 43.7.43, $\lim _{t \rightarrow 0+} K(t, a)=0$ since $t \rightarrow K(t, a)$ is nondecreasing and so if its limit is positive, the integrand would have a non integrable singularity like $t^{-\theta q-1}$. Next consider what happens to $\frac{K(t, a)}{t}$ as $t \rightarrow \infty$.

Claim: $t \rightarrow \frac{K(t, a)}{t}$ is decreasing.
Proof of the claim: Choose $a_{0} \in A_{0}$ and $a_{1} \in A_{1}$ such that $a_{0}+a_{1}=a$ and

$$
K(t, a)+\varepsilon t>\left\|a_{0}\right\|_{A_{0}}+t\left\|a_{1}\right\|_{A_{1}}
$$

let $s>t$. Then

$$
\frac{K(t, a)+t \varepsilon}{t} \geq \frac{\left\|a_{0}\right\|_{A_{0}}+t\left\|a_{1}\right\|_{A_{1}}}{t} \geq \frac{\left\|a_{0}\right\|_{A_{0}}+s\left\|a_{1}\right\|_{A_{1}}}{s} \geq \frac{K(s, a)}{s}
$$

Since $\varepsilon$ is arbitrary, this proves the claim.
Let $r \equiv \lim _{t \rightarrow \infty} \frac{K(t, a)}{t}$. Is $r=0$ ? Suppose to the contrary that $r>0$. Then the integrand of 43.7.43, is at least as large as

$$
\begin{gathered}
t^{-\theta q} K(t, a)^{q-1} \frac{K(t, a)}{t} \geq t^{-\theta q} K(t, a)^{q-1} r \\
\geq t^{-\theta q}(t r)^{q-1} r \geq r^{q} t^{q(1-\theta)-1}
\end{gathered}
$$

whose integral is infinite. Therefore, $r=0$.
Lemma 43.7.3, implies there exist $u_{i} \in A_{0} \cap A_{1}$ such that $a=\sum_{i=-\infty}^{\infty} u_{i}$, the convergence taking place in $A_{0}+A_{1}$ with the inequality of that Lemma holding,

$$
J\left(2^{i}, u_{i}\right) \leq 3(1+\varepsilon) K\left(2^{i}, a\right)
$$

For $i$ an integer and $t \in\left[2^{i-1}, 2^{i}\right)$, let

$$
u(t) \equiv u_{i} / \ln 2
$$

Then

$$
\begin{equation*}
a=\sum_{i=-\infty}^{\infty} u_{i}=\int_{0}^{\infty} u(t) \frac{d t}{t} \tag{43.7.44}
\end{equation*}
$$

Now

$$
\begin{aligned}
\|a\|_{\theta, q, J}^{q} & \leq \int_{0}^{\infty}\left(t^{-\theta} J(t, u(t))\right)^{q} \frac{d t}{t} \\
& =\sum_{i=-\infty}^{\infty} \int_{2^{i-1}}^{2^{i}}\left(t^{-\theta} J\left(t, \frac{u_{i}}{\ln 2}\right)\right)^{q} \frac{d t}{t} \\
& \leq\left(\frac{1}{\ln 2}\right)^{q} \sum_{i=-\infty}^{\infty} \int_{2^{i-1}}^{2^{i}}\left(t^{-\theta} J\left(2^{i}, u_{i}\right)\right)^{q} \frac{d t}{t} \\
& \leq\left(\frac{1}{\ln 2}\right)^{q} \sum_{i=-\infty}^{\infty} \int_{2^{i-1}}^{2^{i}}\left(t^{-\theta} 3(1+\varepsilon) K\left(2^{i}, a\right)\right)^{q} \frac{d t}{t}
\end{aligned}
$$

Using the above claim, $\frac{K\left(2^{i}, a\right)}{2^{i}} \leq \frac{K\left(2^{i-1}, a\right)}{2^{i-1}}$ and so $K\left(2^{i}, a\right) \leq 2 K\left(2^{i-1}, a\right)$. Therefore, the above is no larger than

$$
\begin{align*}
& \leq 2\left(\frac{1}{\ln 2}\right)^{q} \sum_{i=-\infty}^{\infty} \int_{2^{i-1}}^{2^{i}}\left(t^{-\theta} 3(1+\varepsilon) K\left(2^{i-1}, a\right)\right)^{q} \frac{d t}{t} \\
& \leq 2\left(\frac{1}{\ln 2}\right)^{q} \sum_{i=-\infty}^{\infty} \int_{2^{i-1}}^{2^{i}}\left(t^{-\theta} 3(1+\varepsilon) K(t, a)\right)^{q} \frac{d t}{t} \\
& =2\left(\frac{3(1+\varepsilon)}{\ln 2}\right)^{q} \int_{0}^{\infty}\left(t^{-\theta} K(t, a)\right)^{q} \frac{d t}{t} \equiv 2\left(\frac{3(1+\varepsilon)}{\ln 2}\right)^{q}\|a\|_{\theta, q}^{q} \tag{43.7.45}
\end{align*}
$$

This has shown that if $a \in\left(A_{0}, A_{1}\right)_{\theta, q}$, then by 43.7.44 and 43.7.45, $a \in\left(A_{0}, A_{1}\right)_{\theta, q, J}$ and

$$
\begin{equation*}
\|a\|_{\theta, q, J}^{q} \leq 2\left(\frac{3(1+\varepsilon)}{\ln 2}\right)^{q}\|a\|_{\theta, q}^{q} \tag{43.7.46}
\end{equation*}
$$

It remains to prove the other inclusion and norm inequality, both of which are much easier to obtain. Thus, let $a \in\left(A_{0}, A_{1}\right)_{\theta, q, J}$ with

$$
\begin{equation*}
a=\int_{0}^{\infty} u(t) \frac{d t}{t} \tag{43.7.47}
\end{equation*}
$$

where $u$ is a strongly measurable function having values in $A_{0} \cap A_{1}$ and for which

$$
\begin{gather*}
\int_{0}^{\infty}\left(t^{-\theta} J(t, u(t))\right)^{q} d t<\infty  \tag{43.7.48}\\
K(t, a)=K\left(t, \int_{0}^{\infty} u(s) \frac{d s}{s}\right) \leq \int_{0}^{\infty} K(t, u(s)) \frac{d s}{s} \tag{43.7.49}
\end{gather*}
$$

Now by Lemma 43.7.4, this is dominated by an expression of the form

$$
\begin{equation*}
\leq \int_{0}^{\infty} \min \left(1, \frac{t}{s}\right) J(s, u(s)) \frac{d s}{s}=\int_{0}^{\infty} \min \left(1, \frac{1}{s}\right) J(t s, u(t s)) \frac{d s}{s} \tag{43.7.50}
\end{equation*}
$$

where the equation follows from a change of variable. From Minkowski's inequality and 43.7.50,

$$
\begin{aligned}
\|a\|_{\theta, q} & \equiv\left(\int_{0}^{\infty}\left(t^{-\theta} K(t, a)\right)^{q} \frac{d t}{t}\right)^{1 / q} \\
& \leq\left(\int_{0}^{\infty}\left(t^{-\theta} \int_{0}^{\infty} \min \left(1, \frac{1}{s}\right) J(t s, u(t s)) \frac{d s}{s}\right)^{q} \frac{d t}{t}\right)^{1 / q} \\
\leq & \int_{0}^{\infty}\left(\int_{0}^{\infty}\left(t^{-\theta} \min \left(1, \frac{1}{s}\right) J(t s, u(t s))\right)^{q} \frac{d t}{t}\right)^{1 / q} \frac{d s}{s}
\end{aligned}
$$

Now change the variable in the inside integral to obtain, letting $t=\tau s$,

$$
\begin{aligned}
& \leq \int_{0}^{\infty} \min \left(1, \frac{1}{s}\right)\left(\int_{0}^{\infty}\left(t^{-\theta} J(t s, u(t s))\right)^{q} \frac{d t}{t}\right)^{1 / q} \frac{d s}{s} \\
& =\int_{0}^{\infty} \min \left(1, \frac{1}{s}\right) s^{\theta} \frac{d s}{s}\left(\int_{0}^{\infty}\left(\tau^{-\theta} J(\tau, u(\tau))\right)^{q} \frac{d \tau}{\tau}\right)^{1 / q} \\
& =\left(\frac{1}{(1-\theta) \theta}\right)\left(\int_{0}^{\infty}\left(\tau^{-\theta} J(\tau, u(\tau))\right)^{q} \frac{d \tau}{\tau}\right)^{1 / q} .
\end{aligned}
$$

This has shown that

$$
\|a\|_{\theta, q} \leq\left(\frac{1}{(1-\theta) \theta}\right)\left(\int_{0}^{\infty}\left(\tau^{-\theta} J(\tau, u(\tau))\right)^{q} \frac{d \tau}{\tau}\right)^{1 / q}<\infty
$$

for all $u$ satisfying 43.7.47 and 43.7.48. Therefore, taking the infimum it follows $a \in$ $\left(A_{0}, A_{1}\right)_{\theta, q}$ and

$$
\|a\|_{\theta, q} \leq\left(\frac{1}{(1-\theta) \theta}\right)\|a\|_{\theta, q, J}
$$

This proves the theorem.

### 43.8 Duality And Interpolation

In this section it will be assumed that $A_{0} \cap A_{1}$ is dense in $A_{i}$ for $i=0,1$. This is done so that $A_{i}^{\prime} \subseteq\left(A_{0} \cap A_{1}\right)^{\prime}$ and the inclusion map is continuous. Thus it makes sense to add something in $A_{0}^{\prime}$ to something in $A_{1}^{\prime}$.

What is the dual space of $\left(A_{0}, A_{1}\right)_{\theta, q}$ ? The answer is based on the following lemma,
[16]. Remember that

$$
J(t, a)=\max \left(\|a\|_{A_{0}}, t\|a\|_{A_{1}}\right)
$$

and this is a norm on $A_{0} \cap A_{1}$ and

$$
K(t, a)=\inf \left\{\left\|a_{0}\right\|_{A_{0}}+t\left\|a_{1}\right\|_{A_{1}}: a=a_{0}+a_{1}\right\}
$$

As mentioned above, $A_{0}^{\prime}+A_{1}^{\prime} \subseteq\left(A_{0} \cap A_{1}\right)^{\prime}$. In fact these two are equal. This is the first part of the following lemma.

Lemma 43.8.1 Suppose $A_{0} \cap A_{1}$ is dense in $A_{i}, i=0,1$. Then

$$
\begin{equation*}
A_{0}^{\prime}+A_{1}^{\prime}=\left(A_{0} \cap A_{1}\right)^{\prime} \tag{43.8.51}
\end{equation*}
$$

and for $a^{\prime} \in A_{0}^{\prime}+A_{1}^{\prime}=\left(A_{0} \cap A_{1}\right)^{\prime}$,

$$
\begin{equation*}
K\left(t, a^{\prime}\right)=\sup _{a \in A_{0} \cap A_{1}} \frac{\left|a^{\prime}(a)\right|}{J\left(t^{-1}, a\right)} \tag{43.8.52}
\end{equation*}
$$

Thus $K(t, \cdot)$ is an equivalent norm to the usual operator norm on $\left(A_{0} \cap A_{1}\right)^{\prime}$ taken with respect to $J\left(t^{-1}, \cdot\right)$. If, in addition to this, $A_{i}$ is reflexive, then for $a^{\prime} \in A_{0}^{\prime} \cap A_{1}^{\prime}$, and a $\in$ $A_{0} \cap A_{1}$,

$$
\begin{equation*}
J\left(t, a^{\prime}\right) K\left(t^{-1}, a\right) \geq\left|a^{\prime}(a)\right| \tag{43.8.53}
\end{equation*}
$$

Proof: First consider the claim that $A_{0}^{\prime}+A_{1}^{\prime}=\left(A_{0} \cap A_{1}\right)^{\prime}$. As noted above, $\subseteq$ is clear. Define a norm on $A_{0} \times A_{1}$ as follows.

$$
\begin{equation*}
\left\|\left(a_{0}, a_{1}\right)\right\|_{A_{0} \times A_{1}} \equiv \max \left(\left\|a_{0}\right\|_{A_{0}}, t^{-1}\left\|a_{1}\right\|_{A_{1}}\right) \tag{43.8.54}
\end{equation*}
$$

Let $a^{\prime} \in\left(A_{0} \cap A_{1}\right)^{\prime}$. Let

$$
E \equiv\left\{(a, a): a \in A_{0} \cap A_{1}\right\}
$$

with the norm $J\left(t^{-1}, a\right) \equiv \max \left(\|a\|_{A_{0}}, t^{-1}\|a\|_{A_{1}}\right)$. Now define $\lambda$ on $E$, the subspace of $A_{0} \times A_{1}$ by

$$
\lambda((a, a)) \equiv a^{\prime}(a)
$$

Thus $\lambda$ is a continuous linear map on $E$ and in fact,

$$
|\lambda((a, a))|=\left|a^{\prime}(a)\right| \leq \| a^{\prime}| | J\left(t^{-1}, a\right)
$$

By the Hahn Banach theorem there exists an extension of $\lambda$ to all of $A_{0} \times A_{1}$. This extension is of the form $\left(a_{0}^{\prime}, a_{1}^{\prime}\right) \in A_{0}^{\prime} \times A_{1}^{\prime}$. Thus

$$
\left(a_{0}^{\prime}, a_{1}^{\prime}\right)((a, a))=a_{0}^{\prime}(a)+a_{1}^{\prime}(a)=a^{\prime}(a)
$$

and therefore, $a_{0}^{\prime}+a_{1}^{\prime}=a^{\prime}$ provided $a_{0}^{\prime}+a_{1}^{\prime}$ is continuous. But

$$
\begin{aligned}
\left|\left(a_{0}^{\prime}+a_{1}^{\prime}\right)(a)\right| & =\left|a_{0}^{\prime}(a)+a_{1}^{\prime}(a)\right| \leq\left|a_{0}^{\prime}(a)\right|+\left|a_{1}^{\prime}(a)\right| \\
& \leq\left\|a_{0}^{\prime}\right\|\|a\|_{A_{0}}+\left\|a_{1}^{\prime}\right\|\|a\|_{A_{1}} \\
& \leq\left\|a_{0}^{\prime}\right\|\|a\|_{A_{0}}+t\left\|a_{1}^{\prime}\right\| t^{-1}\|a\|_{A_{1}} \\
& \leq\left(\left\|a_{0}^{\prime}\right\|+t\left\|a_{1}^{\prime}\right\|\right) J\left(t^{-1}, a\right)
\end{aligned}
$$

which shows that $a_{0}^{\prime}+a_{1}^{\prime}$ is continuous and in fact

$$
\left\|a_{0}^{\prime}+a_{1}^{\prime}\right\|_{\left(A_{0} \cap A_{1}\right)^{\prime}} \leq\left(\left\|a_{0}^{\prime}\right\|+t\left\|a_{1}^{\prime}\right\|\right) .
$$

This proves the first part of the lemma.
Claim: With this definition of the norm in 43.8.54, the operator norm of $\left(a_{0}^{\prime}, a_{1}^{\prime}\right) \in$ $\left(A_{0} \times A_{1}\right)^{\prime}=A_{0}^{\prime} \times A_{1}^{\prime}$ is

$$
\begin{equation*}
\left\|\left(a_{0}^{\prime}, a_{1}^{\prime}\right)\right\|_{\left(A_{0} \times A_{1}\right)^{\prime}}=\left\|a_{0}^{\prime}\right\|_{A_{0}^{\prime}}+t\left\|a_{1}^{\prime}\right\|_{A_{1}^{\prime}} \tag{43.8.55}
\end{equation*}
$$

Proof of the claim: $\left|\left(a_{0}^{\prime}, a_{1}^{\prime}\right)\left(a_{0}, a_{1}\right)\right| \leq\left\|a_{0}^{\prime}\right\|\left\|a_{0}\right\|+\left\|a_{1}^{\prime}\right\|\left\|a_{1}\right\|$. Now suppose that $\left\|a_{0}\right\|=\max \left(\left\|a_{0}\right\|, t^{-1}\left\|a_{1}\right\|\right)$. Then this is no larger than

$$
\left(\left\|a_{0}^{\prime}\right\|+t\left\|a_{1}^{\prime}\right\|\right)\left\|a_{0}\right\|=\left(\left\|a_{0}^{\prime}\right\|+t\left\|a_{1}^{\prime}\right\|\right) \max \left(\left\|a_{0}\right\|, t^{-1}\left\|a_{1}\right\|\right) .
$$

The other case is that $t^{-1}\left\|a_{1}\right\|=\max \left(\left\|a_{0}\right\|, t^{-1}\left\|a_{1}\right\|\right)$. In this case,

$$
\begin{aligned}
\left|\left(a_{0}^{\prime}, a_{1}^{\prime}\right)\left(a_{0}, a_{1}\right)\right| & \leq\left\|a_{0}^{\prime}\right\|\left\|a_{0}\right\|+\left\|a_{1}^{\prime}\right\|\left\|a_{1}\right\| \\
& \leq\left\|a_{0}^{\prime}\right\| t^{-1}\left\|a_{1}\right\|+\left\|a_{1}^{\prime}\right\|\left\|a_{1}\right\| \\
& =\left(\left\|a_{0}^{\prime}\right\|+t\left\|a_{1}^{\prime}\right\|\right) t^{-1}\left\|a_{1}\right\| \\
& =\left(\left\|a_{0}^{\prime}\right\|+t\left\|a_{1}^{\prime}\right\|\right) \max \left(\left\|a_{0}\right\|, t^{-1}\left\|a_{1}\right\|\right) .
\end{aligned}
$$

This shows $\left\|\left(a_{0}^{\prime}, a_{1}^{\prime}\right)\right\|_{\left(A_{0} \times A_{1}\right)^{\prime}} \leq\left(\left\|a_{0}^{\prime}\right\|+t\left\|a_{1}^{\prime}\right\|\right)$. Is equality achieved? Let $a_{0 n}$ and $a_{1 n}$ be points of $A_{0}$ and $A_{1}$ respectively such that $\left\|a_{0 n}\right\|,\left\|a_{1 n}\right\| \leq 1$ and $\lim _{n \rightarrow \infty} a_{i}^{\prime}\left(a_{i n}\right)=\left\|a_{i}^{\prime}\right\|$. Then

$$
\left(a_{0}^{\prime}, a_{1}^{\prime}\right)\left(a_{0 n}, t a_{1 n}\right) \rightarrow\left\|a_{0}^{\prime}\right\|+t\left\|a_{1}^{\prime}\right\|
$$

and also, $\left\|\left(a_{0 n}, t a_{1 n}\right)\right\|_{A_{0} \times A_{1}}=\max \left(\left\|a_{0 n}\right\|, t^{-1} t\left\|a_{1 n}\right\|_{A_{1}}\right) \leq 1$. Therefore, equality is indeed achieved and this proves the claim.

Consider 43.8.52. Take $a^{\prime} \in A_{0}^{\prime}+A_{1}^{\prime}=\left(A_{0} \cap A_{1}\right)^{\prime}$ and let

$$
E \equiv\left\{(a, a) \in A_{0} \times A_{1}: a \in A_{0} \cap A_{1}\right\} .
$$

Now define a linear map, $\lambda$ on $E$ as before.

$$
\lambda((a, a)) \equiv a^{\prime}(a) .
$$

If $a^{\prime}=\widetilde{a}_{0}+\widetilde{a}_{1}^{\prime}$,

$$
\begin{aligned}
|\lambda((a, a))| & \leq\left\|\widetilde{a}_{0}\right\|_{A_{0}^{\prime}}\|a\|_{A_{0}}+\left\|\widetilde{a}_{1}\right\|\left\|_{A_{1}^{\prime}}\right\| a \|_{A_{1}} \\
& =\left\|\widetilde{a}_{0}\right\|_{A_{0}^{\prime}}\|a\|_{A_{0}}+t\left\|\widetilde{a}_{1}\right\|_{A_{1}^{\prime}} t^{-1}\|a\|_{A_{1}} \\
& \leq\left(\left\|\widetilde{a}_{0}\right\|+t\left\|\widetilde{a}_{1}\right\|\right)\|(a, a)\|_{A_{0} \times A_{1}}
\end{aligned}
$$

so $\lambda$ is continuous on the subspace, $E$ of $A_{0} \times A_{1}$ and

$$
\begin{equation*}
\|\lambda\|_{E^{\prime}} \leq\left\|\widetilde{a}_{0}\right\|+t\left\|\widetilde{a}_{1}\right\| . \tag{43.8.56}
\end{equation*}
$$

By the Hahn Banach theorem, there exists an extension of $\lambda$ defined on all of $A_{0} \times A_{1}$ with the same norm. Thus, from 43.8.55, there exists $\left(a_{0}^{\prime}, a_{1}^{\prime}\right) \in\left(A_{0} \times A_{1}\right)^{\prime}$ which is an extension of $\lambda$ such that

$$
\left\|\left(a_{0}^{\prime}, a_{1}^{\prime}\right)\right\|_{\left(A_{0} \times A_{1}\right)^{\prime}}=\left\|a_{0}^{\prime}\right\|_{A_{0}^{\prime}}+t\left\|a_{1}^{\prime}\right\|_{A_{1}^{\prime}}=\|\lambda\|_{E^{\prime}}
$$

and for all $a \in A_{0} \cap A_{1}$,

$$
a_{0}^{\prime}(a)+a_{1}^{\prime}(a)=\lambda((a, a))=a^{\prime}(a)
$$

It follows that $a_{0}^{\prime}+a_{1}^{\prime}=a^{\prime}$ in $\left(A_{0} \cap A_{1}\right)^{\prime}$. Therefore, from 43.8.56,

$$
\begin{align*}
\|\lambda\|_{E^{\prime}} & \leq \inf \left\{\left\|\widetilde{a}_{0}\right\|_{A_{0}}+t\left\|\widetilde{a}_{1}^{\prime}\right\|_{A_{1}}: a^{\prime}=\widetilde{a}_{0}+\widetilde{a}_{1}\right\} \equiv K\left(t, a^{\prime}\right)  \tag{43.8.57}\\
& \leq\left\|a_{0}^{\prime}\right\|_{A_{0}^{\prime}}+t\left\|a_{1}^{\prime}\right\|_{A_{1}^{\prime}}=\|\lambda\|_{E^{\prime}} \equiv \sup _{a \in A_{0} \cap A_{1}} \frac{\left|a^{\prime}(a)\right|}{J\left(t^{-1}, a\right)} \tag{43.8.58}
\end{align*}
$$

because on $E, J\left(t^{-1}, a\right)=\|(a, a)\|_{A_{0} \times A_{1}}$ which proves 43.8.52.
To obtain 43.8 .53 in the case that $A_{i}$ is reflexive, apply 43.8 .52 to the case where $A_{i}^{\prime \prime}$ plays the role of $A_{i}$ in 43.8 .52 . Thus, for $a^{\prime \prime} \in A_{0}^{\prime \prime}+A_{1}^{\prime \prime}$,

$$
K\left(t, a^{\prime \prime}\right)=\sup _{a^{\prime} \in A_{0}^{\prime} \cap A_{1}^{\prime}} \frac{\left|a^{\prime \prime}\left(a^{\prime}\right)\right|}{J\left(t^{-1}, a^{\prime}\right)}
$$

Now $a^{\prime \prime}=a_{1}^{\prime \prime}+a_{0}^{\prime \prime}=\eta_{1} a_{1}+\eta_{0} a_{0}$ where $\eta_{i}$ is the map from $A_{i}$ to $A_{i}^{\prime \prime}$ which is onto and preserves norms, given by $\eta a\left(a^{\prime}\right) \equiv a^{\prime}(a)$. Therefore, letting $a_{1}+a_{0}=a$

$$
\begin{aligned}
K(t, a) & =K\left(t, a^{\prime \prime}\right)=\sup _{a^{\prime} \in A_{0}^{\prime} \cap A_{1}^{\prime}} \frac{\left|a^{\prime \prime}\left(a^{\prime}\right)\right|}{J\left(t^{-1}, a^{\prime}\right)} \\
& =\sup _{a^{\prime} \in A_{0}^{\prime} \cap A_{1}^{\prime}} \frac{\left|\left(\eta_{1} a_{1}+\eta_{0} a_{0}\right)\left(a^{\prime}\right)\right|}{J\left(t^{-1}, a^{\prime}\right)}=\sup _{a^{\prime} \in A_{0}^{\prime} \cap A_{1}^{\prime}} \frac{\left|\left(a^{\prime}\left(a_{1}+a_{0}\right)\right)\right|}{J\left(t^{-1}, a^{\prime}\right)}
\end{aligned}
$$

and so

$$
K(t, a)=\sup _{a^{\prime} \in A_{0}^{\prime} \cap A_{1}^{\prime}} \frac{\left|a^{\prime}(a)\right|}{J\left(t^{-1}, a^{\prime}\right)}
$$

Changing $t \rightarrow t^{-1}$,

$$
K\left(t^{-1}, a\right) J\left(t, a^{\prime}\right) \geq\left|a^{\prime}(a)\right|
$$

which proves the lemma.
Consider $\left(A_{0}, A_{1}\right)_{\theta, q}^{\prime}$.
Definition 43.8.2 Let $q \geq 1$. Then $\lambda^{\theta, q}$ will denote the sequences, $\left\{\alpha_{i}\right\}_{i=-\infty}^{\infty}$ such that

$$
\sum_{i=-\infty}^{\infty}\left(\left|\alpha_{i}\right| 2^{-i \theta}\right)^{q}<\infty
$$

For $\alpha \in \lambda^{\theta, q}$,

$$
\|\alpha\|_{\lambda^{\theta, q}} \equiv\left(\sum_{i=-\infty}^{\infty}\left(\left|\alpha_{i}\right| 2^{-i \theta}\right)^{q}\right)^{1 / q}
$$

Thus $\alpha \in \lambda^{\theta, q}$ means $\left\{\alpha_{i} 2^{-i \theta}\right\} \in l_{q}$.

Lemma 43.8.3 Let $f(t) \geq 0$, and let $f(t)=\alpha_{i}$ for $t \in\left[2^{i}, 2^{i+1}\right)$ where $\alpha \in \lambda^{\theta, q}$. Then there exists a constant, $C$, such that

$$
\begin{equation*}
\left\|t^{-\theta} f\right\|_{L^{q}\left(0, \infty ; \frac{d t}{t}\right)} \leq C\|\alpha\|_{\lambda^{\theta, q}} \tag{43.8.59}
\end{equation*}
$$

Also, if whenever $\alpha \in \lambda^{\theta, q}$, and $\alpha_{i} \geq 0$ for all $i$,

$$
\begin{equation*}
\sum_{i} f\left(2^{i}\right) 2^{-i} \alpha_{i} \leq C\|\alpha\|_{\lambda^{\theta, q}} \tag{43.8.60}
\end{equation*}
$$

then

$$
\begin{equation*}
\left\|\left\{f\left(2^{i}\right)\right\}_{i=-\infty}^{\infty} \mid\right\|_{\lambda^{1-\theta, q^{\prime}}} \leq C \tag{43.8.61}
\end{equation*}
$$

Proof: Consider 43.8.59.

$$
\begin{gathered}
\int_{0}^{\infty}\left(t^{-\theta} f(t)\right)^{q} \frac{d t}{t}=\sum_{i} \int_{2^{i}}^{2^{i+1}} t^{-\theta q} \alpha_{i}^{q} \frac{d t}{t} \\
\leq \sum_{i} \int_{2^{i}}^{2^{i+1}}\left(2^{-i \theta} \alpha_{i}\right)^{q} \frac{d t}{t}=\ln 2 \sum_{i}\left(2^{-i \theta} \alpha_{i}\right)^{q}=\ln 2\|\alpha\|_{\lambda^{\theta, q}}^{q} .
\end{gathered}
$$

43.8.61 is next. By 43.8.60, whenever $\alpha \in \lambda^{\theta, q}$,

$$
\left|\sum_{i}\left(f\left(2^{i}\right) 2^{-(1-\theta) i}\right) 2^{-\theta i} \alpha_{i}\right| \leq C| |\left\{2^{-\theta i}\left|\alpha_{i}\right|\right\} \|\left.\right|_{l_{q}}
$$

It follows from the Riesz representation theorem that $\left\{f\left(2^{i}\right) 2^{-(1-\theta) i}\right\}$ is in $l_{q^{\prime}}$ and

$$
\left\|\left\{f\left(2^{i}\right) 2^{-(1-\theta) i}\right\}\right\|_{l_{q^{\prime}}}=\left\|\left\{f\left(2^{i}\right)\right\}\right\|_{\lambda^{1-\theta, q^{\prime}}} \leq C .
$$

This proves the lemma.
The dual space of $\left(A_{0}, A_{1}\right)_{\theta, q, J}$ is discussed next.
Lemma 43.8.4 Let $\theta \in(0,1)$ and let $q \geq 1$. Then,

$$
\left(A_{0}, A_{1}\right)_{\theta, q, J}^{\prime} \subseteq\left(A_{1}^{\prime}, A_{0}^{\prime}\right)_{1-\theta, q^{\prime}}
$$

and the inclusion map is continuous.
Proof: Let $a^{\prime} \in\left(A_{0}, A_{1}\right)_{\theta, q, J}^{\prime}$. Now

$$
A_{0} \cap A_{1} \subseteq\left(A_{0}, A_{1}\right)_{\theta, q, J}
$$

and if

$$
a \in\left(A_{0}, A_{1}\right)_{\theta, q, J}
$$

then $a$ has a representation of the form

$$
a=\int_{0}^{\infty} u(t) \frac{d t}{t}
$$

where

$$
\int_{0}^{\infty}\left(t^{-\theta} J(t, u(t))\right)^{q} \frac{d t}{t}<\infty
$$

where

$$
J(t, u(t))=\max \left(\|u(t)\|_{A_{0}}, t\|u(t)\|_{A_{1}}\right)
$$

for $u(t) \in A_{0} \cap A_{1}$. Now let

$$
u_{r}(t) \equiv\left\{\begin{array}{l}
u(t) \text { if } t \in\left(\frac{1}{r}, r\right) \\
0 \text { otherwise }
\end{array}\right.
$$

Then $\int_{0}^{\infty}\left(t^{-\theta} J\left(t, u_{r}(t)\right)\right)^{q} \frac{d t}{t}<\infty$ and

$$
a_{r} \equiv \int_{0}^{\infty} u_{r}(t) \frac{d t}{t} \in A_{0} \cap A_{1}
$$

by Lemma 43.7.2. Also

$$
\begin{gathered}
\left\|a-a_{r}\right\|_{\theta, q, J}^{q} \leq \int_{0}^{\frac{1}{r}}\left(t^{-\theta} J(t, u(t))\right)^{q} \frac{d t}{t}+ \\
\int_{r}^{\infty}\left(t^{-\theta} J(t, u(t))\right)^{q} \frac{d t}{t}
\end{gathered}
$$

which is small whenever $r$ is large enough thanks to the dominated convergence theorem. Therefore, $A_{0} \cap A_{1}$ is dense in $\left(A_{0}, A_{1}\right)_{\theta, q, J}$ and so

$$
\left(A_{0}, A_{1}\right)_{\theta, q, J}^{\prime} \subseteq\left(A_{0} \cap A_{1}\right)^{\prime}=A_{0}^{\prime}+A_{1}^{\prime},
$$

the equality following from Lemma 43.8.1.
It follows $a^{\prime} \in A_{0}^{\prime}+A_{1}^{\prime}$ and so, by Lemma 43.8.1, there exists $b_{i} \in A_{0} \cap A_{1}$ such that

$$
K\left(2^{-i}, a^{\prime}, A_{0}^{\prime}, A_{1}^{\prime}\right)-\varepsilon \min \left(1,2^{-i}\right) \leq \frac{a^{\prime}\left(b_{i}\right)}{J\left(2^{i}, b_{i}, A_{0}, A_{1}\right)}
$$

Now let $\alpha \in \lambda^{\theta, q}$ with $\alpha_{i} \geq 0$ for all $i$ and let

$$
\begin{equation*}
a_{\infty} \equiv \sum_{i} J\left(2^{i}, b_{i}, A_{0}, A_{1}\right)^{-1} b_{i} \alpha_{i} \tag{43.8.62}
\end{equation*}
$$

Consider first whether $a_{\infty}$ makes sense before proceeding further.

$$
a_{\infty} \equiv \sum_{i} \frac{b_{i} 2^{i \theta}}{\max \left(\left\|b_{i}\right\|_{A_{0}}, 2^{i}\left\|b_{i}\right\|_{A_{1}}\right)} 2^{-i \theta} \alpha_{i}
$$

Now

$$
\left\|\frac{b_{i} 2^{i \theta}}{\max \left(\left\|b_{i}\right\|_{A_{0}}, 2^{i}\left\|b_{i}\right\|_{A_{1}}\right)}\right\|_{A_{0}+A_{1}} \leq\left\{\begin{array}{l}
2^{i \theta} \text { if } i<0  \tag{43.8.63}\\
2^{-i(1-\theta)} \text { if } i \geq 0
\end{array} .\right.
$$

This is fairly routine to verify. Consider the case where $i \geq 0$. Then

$$
\left\|\frac{b_{i} 2^{i \theta}}{\max \left(\left\|b_{i}\right\|_{A_{0}}, 2^{i}\left\|b_{i}\right\|_{A_{1}}\right)}\right\|\left\|_{A_{0}+A_{1}} \leq\right\| \frac{b_{i} 2^{i \theta}}{2^{i}\left\|b_{i}\right\|_{A_{1}}} \|_{A_{0}+A_{1}} \leq 2^{-i(1-\theta)}
$$

because $\left\|b_{i}\right\|_{A_{1}} \geq\left\|b_{i}\right\|_{A_{0}+A_{1}}$. Therefore,

$$
\begin{gathered}
\sum_{i=0}^{M}\left\|\frac{b_{i} 2^{i \theta}}{\max \left(\left\|b_{i}\right\|_{A_{0}}, 2^{i}\left\|b_{i}\right\|_{A_{1}}\right)} 2^{-i \theta} \alpha_{i}\right\|_{A_{0}+A_{1}} \leq \\
\sum_{i=0}^{M} 2^{-i(1-\theta)} 2^{-i \theta} \alpha_{i} \leq\left(\sum_{i=0}^{\infty} 2^{-i(1-\theta) q^{\prime}}\right)^{1 / q^{\prime}}\left(\sum_{i=0}^{\infty} 2^{-i q \theta} \alpha_{i}^{q}\right)^{1 / q}<\infty
\end{gathered}
$$

and similarly,

$$
\sum_{i=-\infty}^{0}\left\|\frac{b_{i} 2^{i \theta}}{\max \left(\left\|b_{i}\right\|_{A_{0}}, 2^{i}\left\|b_{i}\right\|_{A_{1}}\right)} 2^{-i \theta} \alpha_{i}\right\|_{A_{0}+A_{1}}
$$

converges. Therefore, $a_{\infty}$ makes sense in $A_{0}+A_{1}$ and also from 43.8.63, we see that

$$
\left\{\frac{\left\|b_{i}\right\|_{A_{0}+A_{1}} 2^{i \theta}}{J\left(2^{i}, b_{i}\right)}\right\} \in \lambda^{(1-\theta) q^{\prime}}
$$

Now let

$$
u(t) \equiv \frac{\alpha_{i} b_{i}}{J\left(2^{i}, b_{i}\right) \ln 2} \text { on }\left[2^{i-1}, 2^{i}\right)
$$

Then

$$
\begin{aligned}
\int_{0}^{\infty} u(t) \frac{d t}{t} & =\sum_{i} \int_{2^{i-1}}^{2^{i}} \frac{\alpha_{i} b_{i}}{J\left(2^{i}, b_{i}\right) \ln 2} \frac{d t}{t} \\
& =\sum_{i} \frac{\alpha_{i} b_{i}}{J\left(2^{i}, b_{i}\right)}=a_{\infty}
\end{aligned}
$$

Also

$$
\begin{gathered}
\int_{0}^{\infty}\left(t^{-\theta} J(t, u(t))\right)^{q} \frac{d t}{t} \leq \sum_{i} \int_{2^{i-1}}^{2^{i}}\left(2^{(1-i) \theta} J\left(2^{i}, u\left(2^{i-1}\right)\right)\right) \frac{d t}{t} \\
\leq \sum_{i}\left[2^{-(i-1) \theta} J\left(2^{i}, u\left(2^{i-1}\right)\right)\right]^{q} \ln 2
\end{gathered}
$$

$$
\begin{gather*}
=\sum_{i}\left[2^{-(i-1) \theta} \frac{J\left(2^{i}, b_{i}\right) \alpha_{i}}{J\left(2^{i}, b_{i}\right) \ln 2}\right]^{q} \ln 2 \\
=C \sum_{i}\left(2^{-i \theta}\left|\alpha_{i}\right|\right)^{q}<\infty \tag{43.8.64}
\end{gather*}
$$

and so $\left\|a_{\infty}\right\|_{\theta, q, J}<\infty$. Now for $a^{\prime}$ as above, $a^{\prime} \in\left(A_{0}, A_{1}\right)_{\theta, q, J}^{\prime} \subseteq\left(A_{0}+A_{1}\right)^{\prime}$, and so since the sum for $a_{\infty}$ converges in $A_{0}+A_{1}$, we have

$$
a^{\prime}\left(a_{\infty}\right)=\sum_{i} J\left(2^{i}, b_{i}\right)^{-1} \alpha_{i} a^{\prime}\left(b_{i}\right)
$$

Therefore,

$$
\begin{align*}
a^{\prime}\left(a_{\infty}\right) & \geq \sum_{i}\left[K\left(2^{-i}, a^{\prime}\right)-\varepsilon \min \left(1,2^{-i}\right)\right] \alpha_{i} \\
& =\sum_{i} K\left(2^{-i}, a^{\prime}\right) \alpha_{i}-\sum_{i} \varepsilon \min \left(1,2^{-i}\right) \alpha_{i} \\
& =\sum_{i} K\left(2^{-i}, a^{\prime}\right) \alpha_{i}-O(\varepsilon) \tag{43.8.65}
\end{align*}
$$

The reason for this is that $\alpha \in \lambda^{\theta, q}$ so $\left\{\alpha_{i} 2^{-i \theta}\right\} \in l_{q}$. Therefore,

$$
\begin{aligned}
& \sum_{i} \varepsilon \min \left(1,2^{-i}\right) \alpha_{i}=\varepsilon\left\{\sum_{i=0}^{\infty} 2^{-i} \alpha_{i}+\sum_{i=-\infty}^{-1} \alpha_{i}\right\} \\
& =\varepsilon\left\{\sum_{i=0}^{\infty} 2^{-i \theta} 2^{(\theta-1) i} \alpha_{i}+\sum_{i=-\infty}^{-1} \alpha_{i} 2^{-i \theta} 2^{i \theta}\right\} \\
& \leq \varepsilon\left\{\left(\sum_{i}\left|\alpha_{i} 2^{-i \theta}\right|^{q}\right)^{1 / q}\left(\sum_{i=0}^{\infty}\left(2^{(\theta-1) i}\right)^{q^{\prime}}\right)^{1 / q^{\prime}}\right. \\
& \left.+\left(\sum_{i}\left|\alpha_{i} 2^{-i \theta}\right|^{q}\right)^{1 / q}\left(\sum_{i=0}^{\infty}\left(2^{\theta i}\right)^{q^{\prime}}\right)^{1 / q^{\prime}}\right\}<C \varepsilon
\end{aligned}
$$

Also $\left|a^{\prime}\left(a_{\infty}\right)\right| \leq\left\|a^{\prime}\right\|_{\left(A_{0}, A_{1}\right)_{\theta, q, J}^{\prime}}\left\|a_{\infty}\right\|_{\left(A_{0}, A_{1}\right)_{\theta, q, J}}$. Now from the definition of $K$,

$$
K\left(2^{-i}, a^{\prime}, A_{0}^{\prime}, A_{1}^{\prime}\right)=2^{-i} K\left(2^{i}, a^{\prime}, A_{1}^{\prime}, A_{0}^{\prime}\right)
$$

and so from 43.8.65

$$
\begin{aligned}
\sum_{i} 2^{-i} K\left(2^{i}, a^{\prime}, A_{1}^{\prime}, A_{0}^{\prime}\right) \alpha_{i}-O(\varepsilon) & \leq a^{\prime}\left(a_{\infty}\right) \\
& \leq\left\|a^{\prime}\right\|_{\left(A_{0}, A_{1}\right)_{\theta, q, J}^{\prime}} C_{\theta}\|\alpha\|_{\lambda^{\theta, q}}
\end{aligned}
$$

Since $\varepsilon$ is arbitrary, it follows that whenever, $\alpha \in \lambda^{\theta, q}, \alpha_{i} \geq 0$,

$$
\sum_{i} 2^{-i} K\left(2^{i}, a^{\prime}, A_{1}^{\prime}, A_{0}^{\prime}\right) \alpha_{i} \leq\left\|a^{\prime}\right\|_{\left(A_{0}, A_{1}\right)_{\theta, q, J}^{\prime}} C_{\theta}\|\alpha\|_{\lambda^{\theta, q}}
$$

By Lemma 43.8.3, $\left\{K\left(2^{i}, a^{\prime}, A_{1}^{\prime}, A_{0}^{\prime}\right)\right\} \in \lambda^{1-\theta, q^{\prime}}$ and

$$
\left\|\left\{K\left(2^{i}, a^{\prime}, A_{1}^{\prime}, A_{0}^{\prime}\right)\right\}\right\|_{\lambda^{1-\theta, q^{\prime}}} \leq\left\|a^{\prime}\right\|_{\left(A_{0}, A_{1}\right)_{\theta, q, J}^{\prime}} C_{\theta}
$$

Therefore,

$$
\begin{aligned}
& \left(\frac{1}{\ln 2} \int_{0}^{\infty}\left(K\left(t, a^{\prime}, A_{1}^{\prime}, A_{0}^{\prime}\right) t^{-(1-\theta)}\right)^{q^{\prime}} \frac{d t}{t}\right)^{1 / q^{\prime}} \\
= & \left(\sum_{i} \frac{1}{\ln 2} \int_{2^{i}}^{2^{i+1}}\left(K\left(t, a^{\prime}, A_{1}^{\prime}, A_{0}^{\prime}\right) t^{-(1-\theta)}\right)^{q^{\prime}} \frac{d t}{t}\right)^{1 / q^{\prime}} \\
\leq & \left(\sum_{i}\left(2^{-i(1-\theta)} K\left(2^{i}, a^{\prime}, A_{1}^{\prime}, A_{0}^{\prime}\right)\right)^{q^{\prime}}\right)^{1 / q^{\prime}} \\
\leq & \left\|a^{\prime}\right\|_{\left(A_{0}, A_{1}\right)_{\theta, q, J}^{\prime}} C_{\theta}
\end{aligned}
$$

Thus

$$
\left\|a^{\prime}\right\|_{\left(A_{1}^{\prime}, A_{0}^{\prime}\right)_{1-\theta, q^{\prime}}} \equiv\left\|t^{-(1-\theta)} K\left(t, a^{\prime}, A_{1}^{\prime}, A_{0}^{\prime}\right)\right\|_{L^{q^{\prime}}\left(0, \infty, \frac{d t}{t}\right)} \leq C\left\|a^{\prime}\right\|_{\left(A_{0}, A_{1}\right)_{\theta, q, J}^{\prime}}
$$

which shows that $\left(A_{0}, A_{1}\right)_{\theta, q, J}^{\prime} \subseteq\left(A_{1}^{\prime}, A_{0}^{\prime}\right)_{1-\theta, q^{\prime}}$ with the inclusion map continuous. This proves the lemma.

Lemma 43.8.5 If $A_{i}$ is reflexive for $i=0,1$ and if $A_{0} \cap A_{1}$ is dense in $A_{i}$, then

$$
\left(A_{1}^{\prime}, A_{0}^{\prime}\right)_{1-\theta, q^{\prime}, J} \subseteq\left(A_{0}, A_{1}\right)_{\theta, q}^{\prime}
$$

and the inclusion map is continuous.
Proof: Let $a^{\prime} \in\left(A_{1}^{\prime}, A_{0}^{\prime}\right)_{1-\theta, q^{\prime}, J}$. Thus, there exists $u^{*}$ bounded on compact subsets of $(0, \infty)$ and measurable with values in $A_{0} \cap A_{1}$ and

$$
\begin{gather*}
a^{\prime}=\int_{0}^{\infty} u^{*}(t) \frac{d t}{t}  \tag{43.8.66}\\
\int_{0}^{\infty}\left(t^{-(1-\theta)} J\left(t, u^{*}(t)\right)\right)^{q^{\prime}} \frac{d t}{t}<\infty .
\end{gather*}
$$

Then

$$
a^{\prime}=\sum_{i=-\infty}^{\infty} \int_{2^{i}}^{2^{i+1}} u^{*}(t) \frac{d t}{t} \equiv \sum_{i=-\infty}^{\infty} a_{i}^{\prime}
$$

where $a_{i}^{\prime} \in A_{1}^{\prime} \cap A_{0}^{\prime}$, the convergence taking place in $A_{1}^{\prime}+A_{0}^{\prime}$. Now let $a \in A_{0} \cap A_{1}$. From Lemma 43.8.1

$$
\begin{aligned}
&\left|a^{\prime}(a)\right| \leq \sum_{i=-\infty}^{\infty}\left|a_{i}^{\prime}(a)\right| \\
& \leq \sum_{i=-\infty}^{\infty} J\left(2^{-i}, a_{i}^{\prime}, A_{0}^{\prime}, A_{1}^{\prime}\right) K\left(2^{i}, a, A_{0}, A_{1}\right) \\
&= \sum_{i=-\infty}^{\infty} 2^{-i} J\left(2^{i}, a_{i}^{\prime}, A_{1}^{\prime}, A_{0}^{\prime}\right) K\left(2^{i}, a, A_{0}, A_{1}\right) \\
& \leq\left(\sum_{i}\left(2^{-(1-\theta) i} J\left(2^{i}, a_{i}^{\prime}, A_{1}^{\prime}, A_{0}^{\prime}\right)\right)^{q^{\prime}}\right)^{1 / q^{\prime}} . \\
& \leq\left(\sum_{i}\left(2^{-\theta i} K\left(2^{i}, a, A_{0}, A_{1}\right)\right)^{q}\right)^{1 / q} \\
& \leq\left[\int_{0}^{\infty}\left(t^{-(1-\theta)} J\left(t, u^{*}(t), A_{1}^{\prime}, A_{0}^{\prime}\right)\right)^{q^{\prime}} \frac{d t}{t}\right]^{1 / q^{\prime}} . \\
& {\left[\int_{0}^{\infty}\left(t^{-\theta} K\left(t, a, A_{0}, A_{1}\right)\right)^{q} \frac{d t}{t}\right]^{1 / q} . }
\end{aligned}
$$

In going from the sums to the integrals, express the first sum as a sum of integrals on $\left[2^{i}, 2^{i+1}\right)$ and the second sum as a sum of integrals on $\left(2^{i-1}, 2^{i}\right]$.

Taking the infimum over all $u^{*}$ representing $a^{\prime}$,

$$
\left|a^{\prime}(a)\right| \leq C\left\|a^{\prime}\right\|_{\left(A_{1}^{\prime}, A_{0}^{\prime}\right)_{1-\theta, q^{\prime}, J}}\|a\|_{\theta, q} .
$$

It follows $a^{\prime} \in\left(A_{0}, A_{1}\right)_{\theta, q}^{\prime}$ and $\left\|a^{\prime}\right\|_{\left(A_{0}, A_{1}\right)_{\theta, q}^{\prime}} \leq C\left\|a^{\prime}\right\|_{\left(A_{1}^{\prime}, A_{0}^{\prime}\right)_{1-\theta, q^{\prime}, J}}$ which proves the lemma.
With these two lemmas the main result follows.
Theorem 43.8.6 Suppose $A_{0} \cap A_{1}$ is dense in $A_{i}$ and $A_{i}$ is reflexive. Then

$$
\left(A_{1}^{\prime}, A_{0}^{\prime}\right)_{1-\theta, q^{\prime}}=\left(A_{0}, A_{1}\right)_{\theta, q}^{\prime}
$$

and the norms are equivalent.
Proof: By Theorem 43.7.5, and the last two lemmas,

$$
\begin{aligned}
\left(A_{0}, A_{1}\right)_{\theta, q}^{\prime} & =\left(A_{0}, A_{1}\right)_{\theta, q, J}^{\prime} \subseteq\left(A_{1}^{\prime}, A_{0}^{\prime}\right)_{1-\theta, q^{\prime}} \\
& =\left(A_{1}^{\prime}, A_{0}^{\prime}\right)_{1-\theta, q^{\prime}, J} \subseteq\left(A_{0}, A_{1}\right)_{\theta, q}^{\prime}
\end{aligned}
$$

This proves the theorem.

## Chapter 44

## Trace Spaces

### 44.1 Definition And Basic Theory Of Trace Spaces

Another approach to these sorts of problems is to use trace spaces. This allows the consideration of fractional order Sobolev spaces. In so far as the subject of Sobolev spaces is concerned, I will present this material in a manner which is essentially independent of the previous material on interpolation spaces.

As in the case of interpolation spaces, suppose $A_{0}$ and $A_{1}$ are two Banach spaces which are continuously embedded in some topological vector space, $X$.

Definition 44.1.1 Define a norm on $A_{0}+A_{1}$ as follows.

$$
\begin{equation*}
\|a\|_{A_{0}+A_{1}} \equiv \inf \left\{\left\|a_{0}\right\|_{A_{0}}+\left\|a_{1}\right\|_{A_{1}}: a_{0}+a_{1}=a\right\} \tag{44.1.1}
\end{equation*}
$$

Lemma 44.1.2 $A_{0}+A_{1}$ with the norm just described is a Banach space.
Proof: This was already explained in the treatment of the $K$ method of interpolation. It is just $K(1, a)$.

Definition 44.1.3 Take $f^{\prime}$ in the sense of distributions for any

$$
f \in L_{l o c}^{1}\left(0, \infty ; A_{0}+A_{1}\right)
$$

as follows.

$$
f^{\prime}(\phi) \equiv \int_{0}^{\infty}-f(t) \phi^{\prime}(t) d t
$$

whenever $\phi \in C_{c}^{\infty}(0, \infty)$. Define a Banach space, $W\left(A_{0}, A_{1}, p, \theta\right)=W$ where $p \geq 1, \theta \in$ $(0,1)$. Let

$$
\begin{equation*}
\|f\|_{W} \equiv \max \left(\left\|t^{\theta} f\right\|_{L^{p}\left(0, \infty, \frac{d t}{t} ; A_{0}\right)},\left\|t^{\theta} f^{\prime}\right\|_{L^{p}\left(0, \infty, \frac{d t}{t} ; A_{1}\right)}\right) \tag{44.1.2}
\end{equation*}
$$

and let $W$ consist of $f \in L_{l o c}^{1}\left(0, \infty ; A_{0}+A_{1}\right)$ such that $\|f\|_{W}<\infty$.
Note that to be in $W, f(t) \in A_{0}$ and $f^{\prime}(t) \in A_{1}$.
Lemma 44.1.4 If $f \in W$, then

$$
\operatorname{Trace}(f) \equiv f(0) \equiv \lim _{t \rightarrow 0} f(t)
$$

exists in $A_{0}+A_{1}$. Also $Z \equiv\{f \in W: f(0)=0\}$ is a closed subspace of $W$. In addition to this, for every $f \in W$ and $\varepsilon>0$ there exists a $g \in W$ such that $\|f-g\|_{W}<\varepsilon$ and $g \in C^{\infty}\left(0, \infty ; A_{0}\right)$ while $g^{\prime} \in C^{\infty}\left(0, \infty ; A_{1}\right)$.

Proof: Let $0<s<t$. Let $v+\frac{1}{p}=\theta$. Then for a generic $g$,

$$
\int_{0}^{\infty}\left\|\tau^{v} g(\tau)\right\|^{p} d \tau=\int_{0}^{\infty}\left\|\tau^{\theta} g(\tau)\right\|^{p} \frac{d \tau}{\tau}
$$

so that $t^{v} f^{\prime} \in L^{p}\left(0, \infty ; A_{1}\right)$, the measure in this case being usual Lebesgue measure. Then

$$
f(t)-f(s)=\int_{s}^{t} f^{\prime}(\tau) d \tau=\int_{s}^{t} \tau^{v} f^{\prime}(\tau) \tau^{-v} d \tau
$$

For $\frac{1}{p}+\frac{1}{p^{\prime}}=1, v p^{\prime}=\left(\theta-\frac{1}{p}\right) p^{\prime}<1$ because $\theta<1=\frac{1}{p^{\prime}}+\frac{1}{p}$. Therefore,

$$
\begin{align*}
& \|f(t)-f(s)\|_{A_{0}+A_{1}} \\
\leq & \int_{s}^{t}\left\|f^{\prime}(\tau)\right\|_{A_{0}+A_{1}} d \tau \\
\leq & \int_{s}^{t}\left\|f^{\prime}(\tau)\right\|_{A_{1}} d \tau=\int_{s}^{t}\left\|\tau^{v} f^{\prime}(\tau)\right\|_{A_{1}} \tau^{-v} d \tau \\
\leq & \left(\int_{s}^{t}\left\|\tau^{v} f^{\prime}(\tau)\right\|_{A_{1}}^{p} d \tau\right)^{1 / p}\left(\int_{s}^{t} \tau^{-v p^{\prime}} d \tau\right)^{1 / p^{\prime}} \\
\leq & \|f\|_{W}\left(\frac{t^{1-v p^{\prime}}}{1-v p^{\prime}}-\frac{s^{1-v p^{\prime}}}{1-v p^{\prime}}\right)  \tag{44.1.3}\\
\leq & \|f\|_{W} \frac{t^{1-v p^{\prime}}}{1-v p^{\prime}} .
\end{align*}
$$

which converges to 0 as $t \rightarrow 0$. This shows that $\lim _{t \rightarrow 0+} f(t)$ exists in $A_{0}+A_{1}$.
Clearly $Z$ is a subspace. Let $f_{n} \rightarrow f$ in $W$ and suppose $f_{n} \in Z$. Then since $f \in W, 44.1 .3$ implies $f$ is continuous. Using 44.1.3 and replacing $f$ with $f_{n}-f_{m}$ and then taking a limit as $s \rightarrow 0$,

$$
\left\|f_{n}(t)-f_{m}(t)\right\|_{A_{0}+A_{1}} \leq\left\|f_{n}-f_{m}\right\|_{W} C_{v} t^{1-v p^{\prime}}
$$

Taking a subsequence, it can be assumed $f_{n}(t)$ converges to $f(t)$ a.e. But the above inequality shows that $f_{n}(t)$ is a Cauchy sequence in $C\left([0, \beta] ; A_{0}+A_{1}\right)$ for all $\beta<\infty$. Therefore, $f_{n}(t) \rightarrow f(t)$ for all $t$. Also,

$$
\left\|f_{n}(t)\right\|_{A_{0}+A_{1}} \leq C_{v}\left\|f_{n}\right\|_{W} t^{1-v p^{\prime}} \leq K t^{1-v p^{\prime}}
$$

for some $K$ depending on $\max \left\{\left|\mid f_{n} \|: n \geq 1\right\}\right.$ and so

$$
\|f(t)\|_{A_{0}+A_{1}} \leq K t^{1-v p^{\prime}}
$$

which implies $f(0)=0$. Thus $Z$ is closed.
Consider the last claim. For a generic $t^{\theta} g \in L^{p}\left(0, \infty, \frac{d t}{t} ; A\right)$, changing variables $t=e^{\tau}$,

$$
\int_{0}^{\infty} t^{\theta p}|g(t)|^{p} \frac{d t}{t}=\int_{-\infty}^{\infty} e^{\tau \theta p}\left|g\left(e^{\tau}\right)\right|^{p} d \tau
$$

Let $\widetilde{g}(\tau) \equiv g\left(e^{\tau}\right)$. Thus $\tau \rightarrow e^{\tau \theta} \widetilde{g}(\tau)$ is $L^{p}(\mathbb{R} ; A)$ and $\widetilde{g} \in L_{l o c}^{1}(\mathbb{R})$. Now let $\psi_{\delta}$ be a mollifier and consider

$$
\begin{aligned}
e^{\theta \tau} \widetilde{g}_{\delta}(\tau) & \equiv \int_{-\infty}^{\infty} e^{\theta \sigma} \widetilde{g}(\sigma) \psi_{\delta}(\tau-\sigma) d \sigma \\
& =\int_{-\infty}^{\infty} e^{\theta(\tau-\sigma)} \widetilde{g}(\tau-\sigma) \psi_{\delta}(\sigma) d \sigma
\end{aligned}
$$

so that

$$
\begin{aligned}
\widetilde{g}_{\delta}(\tau) & =\int_{-\infty}^{\infty} e^{-\theta \sigma} \widetilde{g}(\tau-\sigma) \psi_{\delta}(\sigma) d \sigma \\
& =\int_{-\infty}^{\infty} e^{-\theta(\tau-\sigma)} \widetilde{g}(\sigma) \psi_{\delta}(\tau-\sigma) d \sigma
\end{aligned}
$$

Thus $\widetilde{g}_{\delta} \in C^{\infty}(\mathbb{R} ; A)$ and using Minkowski's inequality,

$$
\begin{gather*}
\left(\int_{-\infty}^{\infty}\left\|e^{\theta \tau} \widetilde{g}_{\delta}(\tau)-e^{\theta \tau} \widetilde{g}(\tau)\right\|^{p} d \tau\right)^{1 / p}=  \tag{44.1.4}\\
\left(\int_{-\infty}^{\infty}\left\|\int_{-\infty}^{\infty}\left(e^{\theta(\tau-\sigma)} \widetilde{g}(\tau-\sigma)-e^{\theta \tau} \widetilde{g}(\tau)\right) \psi_{\delta}(\sigma) d \sigma\right\|^{p} d \tau\right)^{1 / p} \\
\leq \int_{-\delta}^{\delta} \psi_{\delta}(\sigma)\left(\int_{-\infty}^{\infty}\left\|e^{\theta(\tau-\sigma)} \widetilde{g}(\tau-\sigma)-e^{\theta \tau} \widetilde{g}(\tau)\right\|^{p} d \tau\right)^{1 / p} d \sigma \\
\leq \varepsilon \int_{-\infty}^{\infty} \psi_{\delta}(\sigma) d s=\varepsilon
\end{gather*}
$$

provided $\delta$ is small enough due to continuity of translation in $L^{p}$. Thus changing variables in 44.1.4, letting $\tau=\ln (t)$ and $g_{\delta}(t) \equiv \widetilde{g}_{\delta}(\ln (t))$, it follows $g_{\delta} \in C^{\infty}(0, \infty ; A)$ and this integral equals

$$
\left(\int_{0}^{\infty} t^{\theta p}\left\|g_{\delta}(t)-g(t)\right\|^{p} \frac{d t}{t}\right)^{1 / p}
$$

This result applied to $f$ and $f^{\prime}$ with $A=A_{0}$ and then $A=A_{1}$ shows the last claim. This proves the lemma.

Definition 44.1.5 Let $W$ be a Banach space and let $Z$ be a closed subspace. Then the quotient space, denoted by $W / Z$ consists of the set of equivalence classes $[x]$ where the equivalence relation is defined by $x \backsim y$ means $x-y \in Z$. Then $W / Z$ is a vector space if the operations are defined by $\alpha[x] \equiv[\alpha x]$ and $[x]+[y] \equiv[x+y]$ and these vector space operations are well defined. The norm on the quotient space is defined as $\|[x]\| \equiv \inf \{\|x+z\|: z \in Z\}$.

The verification of the algebraic claims made in the above definition is left to the reader. It is routine. What is not as routine is the following lemma. However, it is similar to some topics in the presentation of the $K$ method of interpolation.

Lemma 44.1.6 Let $W$ be a Banach space and let $Z$ be a closed subspace of $W$. Then $W / Z$ with the norm described above is a Banach space.

Proof: That $W / Z$ is a vector space is left to the reader. Why is $\|\cdot\|$ a norm? Suppose $\alpha \neq 0$. Then

$$
\begin{aligned}
\|\alpha[x]\| & =\|[\alpha x]\| \equiv \inf \{\|\alpha x+z\|: z \in Z\} \\
& =\inf \{\|\alpha x+\alpha z\|: z \in Z\} \\
& =|\alpha| \inf \{\|x+z\|: z \in Z\}=|\alpha|\|[x]\| .
\end{aligned}
$$

Now let $\|[x]\| \geq\left\|x+z_{1}\right\|-\varepsilon$ and let $\|[y]\| \geq\left\|y+z_{2}\right\|-\varepsilon$ where $z_{i} \in Z$. Then

$$
\begin{aligned}
\|[x]+[y]\| & \equiv\|[x+y]\| \leq\left\|x+y+z_{1}+z_{2}\right\| \\
& \leq\left\|x+z_{1}\right\|+\left\|y+z_{2}\right\| \leq\|[x]\|+\|[y]\|+2 \varepsilon
\end{aligned}
$$

Since $\varepsilon$ is arbitrary, this shows the triangle inequality. Clearly, $\|[x]\| \geq 0$. It remains to show that the only way $\|[x]\|=0$ is for $x \in Z$. Suppose then that $\|[x]\|=0$. This means there exist $z_{n} \in Z$ such that $\left\|x+z_{n}\right\| \rightarrow 0$. Therefore, $-x$ is a limit of a sequence of points of $Z$ and since $Z$ is closed, this requires $-x \in Z$. Hence $x \in Z$ also because $Z$ is a subspace. This shows $\|\cdot\|$ is a norm on $W / Z$. It remains to verify that $W / Z$ is a Banach space.

Suppose $\left\{\left[x_{n}\right]\right\}$ is a Cauchy sequence in $W / Z$ and suppose $\left\|\left[x_{n}\right]-\left[x_{n+1}\right]\right\|<\frac{1}{2^{n+1}}$. Let $x_{1}^{\prime}=x_{1}$. If $x_{n}^{\prime}$ has been chosen let $x_{n+1}^{\prime}=x_{n+1}+z_{n+1}$ where $z_{n+1} \in Z$ be such that

$$
\begin{aligned}
\left\|x_{n+1}^{\prime}-x_{n}^{\prime}\right\| & \leq\left\|\left[x_{n+1}-x_{n}\right]\right\|+\frac{1}{2^{(n+1)}} \\
& =\left\|\left[x_{n+1}\right]-\left[x_{n}\right]\right\|+\frac{1}{2^{(n+1)}}<\frac{1}{2^{n}}
\end{aligned}
$$

It follows $\left\{x_{n}^{\prime}\right\}$ is a Cauchy sequence in $W$ and so it must converge to some $x \in W$. Now

$$
\left\|[x]-\left[x_{n}\right]\right\|=\left\|\left[x-x_{n}\right]\right\|=\left\|\left[x-x_{n}^{\prime}\right]\right\| \leq\left\|x-x_{n}^{\prime}\right\|
$$

which converges to 0 . Now if $\left\{\left[x_{n}\right]\right\}$ is just a Cauchy sequence, there exists a subsequence satisfying $\left\|\left[x_{n_{k}}\right]-\left[x_{n_{k+1}}\right]\right\|<\frac{1}{2^{k+1}}$ and so from the first part, the subsequence converges to some $[x] \in W / Z$ and so the original Cauchy sequence also converges. therefore, $W / Z$ is a Banach space as claimed.

Definition 44.1.7 Define $T\left(A_{0}, A_{1}, p, \theta\right)=T$, to consist of

$$
\left\{a \in A_{0}+A_{1}: a=\lim _{t \rightarrow 0+} f(t) \text { for some } f \in W\left(A_{0}, A_{1}, p, \theta\right)\right\}
$$

the limit taking place in $A_{0}+A_{1}$. Let $\gamma f$ be defined for $f \in W$ by $\gamma f \equiv \lim _{t \rightarrow 0+} f(t)$. Thus $T=\gamma(W)$. As above $Z \equiv\{f \in W: \gamma f=0\}=\operatorname{ker}(\gamma)$.

Lemma 44.1.8 $T$ is a Banach space with norm given by

$$
\begin{equation*}
\|a\|_{T} \equiv \inf \left\{\|f\|_{W}: f(0)=a\right\} \tag{44.1.5}
\end{equation*}
$$

Proof: Define a mapping, $\psi: W / Z \rightarrow T$ by

$$
\psi([f]) \equiv \gamma f
$$

Then $\psi$ is one to one and onto. Also

$$
\|[f]\| \equiv \inf \{\|f+g\|: g \in Z\}=\inf \left\{\|h\|_{W}: \gamma h=\gamma f\right\}=\|\gamma(f)\|_{T}
$$

Therefore, the Banach space, $W / Z$ and $T$ are isometric and so $T$ must be a Banach space since $W / Z$ is.

The following is an important interpolation inequality.
Theorem 44.1.9 If $a \in T$, then

$$
\begin{equation*}
\|a\|_{T}=\inf \left\{\left\|t^{\theta} f\right\|_{L^{p}\left(0, \infty, \frac{d t}{t} ; A_{0}\right)}^{1-\theta}\left\|t^{\theta} f^{\prime}\right\|_{L^{p}\left(0, \infty, \frac{d t}{t} ; A_{1}\right)}^{\theta}\right\} \tag{44.1.6}
\end{equation*}
$$

where the infimum is taken over all $f \in W$ such that $a=f(0)$. Also, if $a \in A_{0} \cap A_{1}$, then $a \in T$ and

$$
\begin{equation*}
\|a\|_{T} \leq K\|a\|_{A_{1}}^{1-\theta}\|a\|_{A_{0}}^{\theta} \tag{44.1.7}
\end{equation*}
$$

for some constant K. Also

$$
\begin{equation*}
A_{0} \cap A_{1} \subseteq T\left(A_{0}, A_{1}, p, \theta\right) \subseteq A_{0}+A_{1} \tag{44.1.8}
\end{equation*}
$$

and the inclusion maps are continuous.
Proof: First suppose $f(0)=a$ where $f \in W$. Then letting $f_{\lambda}(t) \equiv f(\lambda t)$, it follows that $f_{\lambda}(0)=a$ also and so

$$
\begin{aligned}
\|a\|_{T} & \leq \max \left(\left\|t^{\theta} f_{\lambda}\right\|_{L^{p}\left(0, \infty, \frac{d t}{t} ; A_{0}\right)},\left\|t^{\theta}\left(f_{\lambda}\right)^{\prime}\right\|_{L^{p}\left(0, \infty, \frac{d t}{t} ; A_{1}\right)}\right) \\
& =\max \left(\lambda^{-\theta}\left\|t^{\theta} f\right\|_{L^{p}\left(0, \infty, \frac{d t}{t} ; A_{0}\right)}, \lambda^{1-\theta}\left\|t^{\theta} f^{\prime}\right\|_{L^{p}\left(0, \infty, \frac{d t}{t} ; A_{1}\right)}\right) \\
& \equiv \max \left(\lambda^{-\theta} R, \lambda^{1-\theta} S\right)
\end{aligned}
$$

Now choose $\lambda=R / S$ to obtain

$$
\|a\|_{T} \leq R^{1-\theta} S^{\theta}=\left\|t^{\theta} f\right\|_{L^{p}\left(0, \infty, \frac{d t}{t} ; A_{0}\right)}^{1-\theta}\left\|t^{\theta} f^{\prime}\right\|_{L^{p}\left(0, \infty, \frac{d t}{t} ; A_{1}\right)}^{\theta}
$$

Thus

$$
\|a\|_{T} \leq \inf \left\{\left\|t^{\theta} f\right\|_{L^{p}\left(0, \infty, \frac{d t}{t} ; A_{0}\right)}^{1-\theta}\left\|t^{\theta} f^{\prime}\right\|_{L^{p}\left(0, \infty, \frac{d t}{t} ; A_{1}\right)}^{\theta}\right\}
$$

Next choose $f \in W$ such that $f(0)=a$ and $\|f\|_{W} \approx\|a\|_{T}$. More precisely, pick $f \in W$ such that $f(0)=a$ and $\|a\|_{T}>-\varepsilon+\|f\|_{W}$. Also let

$$
R \equiv\left\|t^{\theta} f\right\|_{L^{p}\left(0, \infty, \frac{d t}{t} ; A_{0}\right)}, S \equiv\left\|t^{\theta} f^{\prime}\right\|_{L^{p}\left(0, \infty, \frac{d t}{t} ; A_{1}\right)}
$$

Then as before,

$$
\begin{equation*}
\left\|t^{\theta} f_{\lambda}\right\|_{L^{p}\left(0, \infty, \frac{d t}{t} ; A_{0}\right)}=\lambda^{-\theta} R,\left\|t^{\theta}\left(f_{\lambda}\right)^{\prime}\right\|_{L^{p}\left(0, \infty, \frac{d t}{t} ; A_{1}\right)}=\lambda^{1-\theta} S \tag{44.1.9}
\end{equation*}
$$

so that $\|f\|_{W}=\max (R, S)$. Then, changing the variables, letting $\lambda=R / S$,

$$
\begin{equation*}
\left\|t^{\theta} f_{\lambda}\right\|_{L^{p}\left(0, \infty, \frac{d t}{t} ; A_{0}\right)}=\left\|t^{\theta}\left(f_{\lambda}\right)^{\prime}\right\|_{L^{p}\left(0, \infty, \frac{d t}{t} ; A_{1}\right)}=R^{1-\theta} S^{\theta} \tag{44.1.10}
\end{equation*}
$$

Since $f_{\lambda}(0)=a, f_{\lambda} \in W$, and it is always the case that for positive $R, S$,

$$
R^{1-\theta} S^{\theta} \leq \max (R, S)
$$

this shows that

$$
\begin{aligned}
\|a\|_{T} & \leq \max \left(\left\|t^{\theta} f_{\lambda}\right\|_{L^{p}\left(0, \infty, \frac{d t}{t} ; A_{0}\right)},\left\|t^{\theta}\left(f_{\lambda}\right)^{\prime}\right\|_{L^{p}\left(0, \infty, \frac{d t}{t} ; A_{1}\right)}\right) \\
& =R^{1-\theta} S^{\theta} \leq \max (R, S)=\|f\|_{W}<\|a\|_{T}+\varepsilon
\end{aligned}
$$

the first inequality holding because $\|a\|_{T}$ is the infimum of such things on the right. This shows 44.1.6.

It remains to verify 44.1.7. To do this, let $\psi \in C^{\infty}([0, \infty))$, with $\psi(0)=1$ and $\psi(t)=0$ for all $t>1$. Then consider the special $f \in W$ which is given by $f(t) \equiv a \psi(t)$ where $a \in A_{0} \cap A_{1}$. Thus $f \in W$ and $f(0)=a$ so $a \in T\left(A_{0}, A_{1}, p, \theta\right)$. From the first part, there exists a constant, $K$ such that

$$
\begin{aligned}
\|a\|_{T} & \leq\left\|t^{\theta} f\right\|_{L^{p}\left(0, \infty, \frac{d t}{t} ; A_{0}\right)}^{1-\theta}\left\|t^{\theta} f^{\prime}\right\|_{L^{p}\left(0, \infty, \frac{d t}{t} ; A_{1}\right)}^{\theta} \\
& \leq K\|a\|_{A_{0}}^{1-\theta}\|a\|_{A_{1}}^{\theta}
\end{aligned}
$$

This shows 44.1.7 and the first inclusion in 44.1.8. From the inequality just obtained,

$$
\begin{aligned}
\|a\|_{T} & \leq K\left((1-\theta)\|a\|_{A_{0}}+\theta\|a\|_{A_{1}}\right) \\
& \leq K\|a\|_{A_{0} \cap A_{1}}
\end{aligned}
$$

This shows the first inclusion map of 44.1.8 is continuous.
Now take $a \in T$. Let $f \in W$ be such that $a=f(0)$ and

$$
\|a\|_{T}+\varepsilon>\|f\|_{W} \geq\|a\|_{T}
$$

By 44.1.3,

$$
\|a-f(t)\|_{A_{0}+A_{1}} \leq C_{v} t^{1-v p^{\prime}}\|f\|_{W}
$$

where $\frac{1}{p}+v=\theta$, and so

$$
\|a\|_{A_{0}+A_{1}} \leq\|f(t)\|_{A_{0}+A_{1}}+C_{v} t^{1-v p^{\prime}}\|f\|_{W}
$$

Now $\|f(t)\|_{A_{0}+A_{1}} \leq\|f(t)\|_{A_{0}}$.

$$
\begin{aligned}
\|a\|_{A_{0}+A_{1}} & \leq t^{v}\|f(t)\|_{A_{0}+A_{1}} t^{-v}+C_{v} t^{1-v p^{\prime}}\|f\|_{W} \\
& \leq t^{v}\|f(t)\|_{A_{0}} t^{-v}+C_{v} t^{1-v p^{\prime}}\|f\|_{W}
\end{aligned}
$$

Therefore, recalling that $v p^{\prime}<1$, and integrating both sides from 0 to 1 ,

$$
\|a\|_{A_{0}+A_{1}} \leq C_{V}\|f\|_{W} \leq C_{V}\left(\|a\|_{T}+\varepsilon\right)
$$

To see this,

$$
\begin{aligned}
\int_{0}^{1} t^{v}\|f(t)\|_{A_{0}} t^{-v} d t & \leq\left(\int_{0}^{1}\left(t^{v}\|f(t)\|_{A_{0}}\right)^{p} d t\right)^{1 / p}\left(\int_{0}^{1} t^{-v p^{\prime}} d t\right)^{1 / p^{\prime}} \\
& \leq C\|f\|_{W}
\end{aligned}
$$

Since $\varepsilon>0$ is arbitrary, this verifies the second inclusion and continuity of the inclusion map completing the proof of the theorem.

The interpolation inequality, 44.1 .7 is very significant. The next result concerns bounded linear transformations.

Theorem 44.1.10 Now suppose $A_{0}, A_{1}$ and $B_{0}, B_{1}$ are pairs of Banach spaces such that $A_{i}$ embeds continuously into a topological vector space, $X$ and $B_{i}$ embeds continuously into a topological vector space, $Y$. Suppose also that $L \in \mathscr{L}\left(A_{0}, B_{0}\right)$ and $L \in \mathscr{L}\left(A_{1}, B_{1}\right)$ where the operator norm of $L$ in these spaces is $K_{i}, i=0,1$. Then

$$
\begin{equation*}
L \in \mathscr{L}\left(A_{0}+A_{1}, B_{0}+B_{1}\right) \tag{44.1.11}
\end{equation*}
$$

with

$$
\begin{equation*}
\|L a\|_{B_{0}+B_{1}} \leq \max \left(K_{0}, K_{1}\right)\|a\|_{A_{0}+A_{1}} \tag{44.1.12}
\end{equation*}
$$

and

$$
\begin{equation*}
L \in \mathscr{L}\left(T\left(A_{0}, A_{1}, p, \theta\right), T\left(B_{0}, B_{1}, p, \theta\right)\right) \tag{44.1.13}
\end{equation*}
$$

and for $K$ the operator norm,

$$
\begin{equation*}
K \leq K_{0}^{1-\theta} K_{1}^{\theta} \tag{44.1.14}
\end{equation*}
$$

Proof: To verify 44.1.11, let $a \in A_{0}+A_{1}$ and pick $a_{0} \in A_{0}$ and $a_{1} \in A_{1}$ such that

$$
\|a\|_{A_{0}+A_{1}}+\varepsilon>\left\|a_{0}\right\|_{A_{0}}+\left\|a_{1}\right\|_{A_{1}}
$$

Then

$$
\begin{aligned}
& \|L(a)\|_{B_{0}+B_{1}}=\left\|L a_{0}+L a_{1}\right\|_{B_{0}+B_{1}} \leq\left\|L a_{0}\right\|_{B_{0}}+\left\|L a_{1}\right\|_{B_{1}} \\
& \leq K_{0}\left\|a_{0}\right\|_{A_{0}}+K_{1}\left\|a_{1}\right\|_{A_{1}} \leq \max \left(K_{0}, K_{1}\right)\left(\|a\|_{A_{0}+A_{1}}+\varepsilon\right) .
\end{aligned}
$$

This establishes 44.1.12. Now consider the other assertions.

Let $a \in T\left(A_{0}, A_{1}, p, \theta\right)$ and pick $f \in W\left(A_{0}, A_{1}, p, \theta\right)$ such that $\gamma f=a$ and

$$
\|a\|_{T\left(A_{0}, A_{1}, p, \theta\right)}+\varepsilon>\left\|t^{\theta} f\right\|_{L^{p}\left(0, \infty, \frac{d t}{t}, A_{0}\right)}^{1-\theta}\left\|t^{\theta} f^{\prime}\right\|_{L^{p}\left(0, \infty, \frac{d t}{t}, A_{1}\right)}^{\theta}
$$

Then consider $L f$. Since $L$ is continuous on $A_{0}+A_{1}$,

$$
L f(0)=L a
$$

and $L f \in W\left(B_{0}, B_{1}, p, \theta\right)$. Therefore, by Theorem 44.1.9,

$$
\begin{aligned}
\|L a\|_{T\left(B_{0}, B_{1}, p, \theta\right)} & \leq\left\|t^{\theta} L f\right\|_{L^{p}\left(0, \infty, \frac{d t}{t}, B_{0}\right)}^{1-\theta}\left\|t^{\theta} L f^{\prime}\right\|_{L^{p}\left(0, \infty, \frac{d t}{t}, B_{1}\right)}^{\theta} \\
& \leq K_{0}^{1-\theta} K_{1}^{\theta}\left\|t^{\theta} f\right\|_{L^{p}\left(0, \infty, \frac{d t}{t}, A_{0}\right)}^{1-\theta}\left\|t^{\theta} f^{\prime}\right\|_{L^{p}\left(0, \infty, \frac{d t}{t}, A_{1}\right)}^{\theta} \\
& \leq K_{0}^{1-\theta} K_{1}^{\theta}\left(\|a\|_{T\left(A_{0}, A_{1}, p, \theta\right)}+\varepsilon\right)
\end{aligned}
$$

and since $\varepsilon>0$ is arbitrary, this proves the theorem.

### 44.2 Trace And Interpolation Spaces

Trace spaces are equivalent to interpolation spaces. In showing this, a more general sort of trace space than that presented earlier will be used.

Definition 44.2.1 Define for $m$ a positive integer, $V^{m}=V^{m}\left(A_{0}, A_{1}, p, \theta\right)$ to be the set of functions, u such that

$$
\begin{equation*}
t \rightarrow t^{\theta} u(t) \in L^{p}\left(0, \infty, \frac{d t}{t} ; A_{0}\right) \tag{44.2.15}
\end{equation*}
$$

and

$$
\begin{equation*}
t \rightarrow t^{\theta+m-1} u^{(m)}(t) \in L^{p}\left(0, \infty, \frac{d t}{t} ; A_{1}\right) \tag{44.2.16}
\end{equation*}
$$

$V^{m}$ is a Banach space with the norm

$$
\|u\|_{V^{m}} \equiv \max \left(\left\|t^{\theta} u(t)\right\|_{L^{p}\left(0, \infty, \frac{d t}{t} ; A_{0}\right)},\left\|t^{\theta+m-1} u^{(m)}(t)\right\|_{L^{p}\left(0, \infty, \frac{d t}{t} ; A_{1}\right)}\right)
$$

Thus $V^{m}$ equals $W$ in the case when $m=1$. More generally, as in [16] different exponents are used for the two $L^{p}$ spaces, $p_{0}$ in place of $p$ for the space corresponding to $A_{0}$ and $p_{1}$ in place of $p$ for the space corresponding to $A_{1}$.

Definition 44.2.2 Denote by $T^{m}\left(A_{0}, A_{1}, p, \theta\right)$ the set of all $a \in A_{0}+A_{1}$ such that for some $u \in V^{m}$,

$$
\begin{equation*}
a=\lim _{t \rightarrow 0+} u(t) \equiv \operatorname{trace}(u) \tag{44.2.17}
\end{equation*}
$$

the limit holding in $A_{0}+A_{1}$. For the norm

$$
\begin{equation*}
\|a\|_{T^{m}} \equiv \inf \left\{\|u\|_{V^{m}}: \operatorname{trace}(u)=a\right\} \tag{44.2.18}
\end{equation*}
$$

The case when $m=1$ was discussed in Section 44.1. Note it is not known at this point whether $\lim _{t \rightarrow 0+} u(t)$ even exists for every $u \in V^{m}$. Of course, if $m=1$ this was shown earlier but it has not been shown for $m>1$. The following theorem is absolutely amazing. Note the lack of dependence on $m$ of the right side!

Theorem 44.2.3 The following hold.

$$
\begin{equation*}
T^{m}\left(A_{0}, A_{1}, p, \theta\right)=\left(A_{0}, A_{1}\right)_{\theta, p, J}=\left(A_{0}, A_{1}\right)_{\theta, p} \tag{44.2.19}
\end{equation*}
$$

Proof: It is enough to show the first equality because of Theorem 43.7 .5 which identifies $\left(A_{0}, A_{1}\right)_{\theta, p, J}$ and $\left(A_{0}, A_{1}\right)_{\theta, p}$. Let $a \in T^{m}$. Then there exists $u \in V^{m}$ such that

$$
a=\lim _{t \rightarrow 0+} u(t) \text { in } A_{0}+A_{1} .
$$

The first task is to modify this $u(t)$ to get a better one which is more usable in order to show $a \in\left(A_{0}, A_{1}\right)_{\theta, p, J}$. Remember, it is required to find $w(t) \in A_{0} \cap A_{1}$ for all $t \in(0, \infty)$ and $a=\int_{0}^{\infty} w(t) \frac{d t}{t}$, a representation which is not known at this time. To get such a thing, let

$$
\begin{equation*}
\phi \in C_{c}^{\infty}(0, \infty), \operatorname{spt}(\phi) \subseteq[\alpha, \beta] \tag{44.2.20}
\end{equation*}
$$

with $\phi \geq 0$ and

$$
\begin{equation*}
\int_{0}^{\infty} \phi(t) \frac{d t}{t}=1 \tag{44.2.21}
\end{equation*}
$$

Then define

$$
\begin{equation*}
\widetilde{u}(t) \equiv \int_{0}^{\infty} \phi\left(\frac{t}{\tau}\right) u(\tau) \frac{d \tau}{\tau}=\int_{0}^{\infty} \phi(s) u\left(\frac{t}{s}\right) \frac{d s}{s} \tag{44.2.22}
\end{equation*}
$$

Claim: $\lim _{t \rightarrow 0+} \widetilde{u}(t)=a$ and $\lim _{t \rightarrow \infty} \widetilde{u}^{(k)}(t)=0$ in $A_{0}+A_{1}$ for all $k \leq m$.
Proof of the claim: From 44.2 .22 and 44.2 .21 it follows that for $\|\cdot\|$ referring to $\|\cdot\|_{A_{0}+A_{1}}$,

$$
\begin{aligned}
\|\widetilde{u}(t)-a\| & \leq \int_{0}^{\infty}\left\|u\left(\frac{t}{s}\right)-a\right\| \phi(s) \frac{d s}{s} \\
& =\int_{0}^{\infty}\|u(\tau)-a\| \phi\left(\frac{t}{\tau}\right) \frac{d \tau}{\tau} \\
& =\int_{t / \beta}^{t / \alpha}\|u(\tau)-a\| \phi\left(\frac{t}{\tau}\right) \frac{d \tau}{\tau} \\
& \leq \int_{t / \beta}^{t / \alpha} \varepsilon \phi\left(\frac{t}{\tau}\right) \frac{d \tau}{\tau}=\varepsilon \int_{\alpha}^{\beta} \phi(s) \frac{d s}{s}=\varepsilon
\end{aligned}
$$

whenever $t$ is small enough due to the convergence of $u(t)$ to $a$ in $A_{0}+A_{1}$.
Now consider what occurs when $t \rightarrow \infty$. For $\|\cdot\|$ referring to the norm in $A_{0}$,

$$
\widetilde{u}^{(k)}(t)=\int_{0}^{\infty} \phi^{(k)}\left(\frac{t}{\tau}\right) \frac{1}{\tau^{k}} u(\tau) \frac{d \tau}{\tau}
$$

and so

$$
\begin{gathered}
\left\|\widetilde{u}^{(k)}(t)\right\|_{A_{0}} \leq C_{k} \int_{t / \beta}^{t / \alpha}\|u(\tau)\|_{A_{0}} \frac{d \tau}{\tau} \\
\leq C\left(\int_{t / \beta}^{t / \alpha} \frac{d \tau}{\tau}\right)^{1 / p^{\prime}}\left(\int_{t / \beta}^{t / \alpha}\|u(\tau)\|_{A_{0}}^{p} \frac{d \tau}{\tau}\right)^{1 / p}
\end{gathered}
$$

Now $\left(\frac{\beta}{t}\right)^{\theta} \tau^{\theta} \geq 1$ for $\tau \geq t / \beta$ and so the above expression

$$
\leq C\left(\ln \frac{\beta}{\alpha}\right)^{1 / p^{\prime}}\left(\frac{\beta}{t}\right)^{\theta}\left(\int_{t / \beta}^{\infty}\left(\tau^{\theta}\|u(\tau)\|_{A_{0}}\right)^{p} \frac{d \tau}{\tau}\right)^{1 / p}
$$

and so $\lim _{t \rightarrow \infty}\left\|\widetilde{u}^{(k)}(t)\right\|_{A_{0}}=0$ and therefore, this also holds in $A_{0}+A_{1}$. This proves the claim.

Thus $\widetilde{u}$ has the same properties as $u$ in terms of having $a$ as its trace. $\widetilde{u}$ is used to build the desired $w$, representing $a$ as an integral. Define

$$
\begin{gather*}
v(t) \equiv \frac{(-1)^{m} t^{m}}{(m-1)!} \widetilde{u}^{(m)}(t)=\frac{(-1)^{m}}{(m-1)!} \int_{0}^{\infty} \frac{t^{m}}{\tau^{m}} \phi^{(m)}\left(\frac{t}{\tau}\right) u(\tau) \frac{d \tau}{\tau} \\
=\frac{(-1)^{m}}{(m-1)!} \int_{0}^{\infty} s^{m} \phi^{(m)}(s) u\left(\frac{t}{s}\right) \frac{d s}{s} \tag{44.2.23}
\end{gather*}
$$

Then from the claim, and integration by parts in the last step,

$$
\begin{equation*}
\int_{0}^{\infty} v\left(\frac{1}{t}\right) \frac{d t}{t}=\int_{0}^{\infty} v(t) \frac{d t}{t}=\frac{(-1)^{m}}{(m-1)!} \int_{0}^{\infty} t^{m-1} \widetilde{u}^{(m)}(t) d t=a \tag{44.2.24}
\end{equation*}
$$

Thus $v\left(\frac{1}{t}\right)$ represents $a$ in the way desired for $\left(A_{0}, A_{1}\right)_{\theta, p, J}$ if it is also true that $v\left(\frac{1}{t}\right) \in$ $A_{0} \cap A_{1}$ and $t \rightarrow t^{-\theta} v\left(\frac{1}{t}\right)$ is in $L^{p}\left(0, \infty, \frac{d t}{t} ; A_{0}\right)$ and $t \rightarrow t^{1-\theta_{v}}\left(\frac{1}{t}\right)$ is in $L^{p}\left(0, \infty, \frac{d t}{t} ; A_{1}\right)$. First consider whether $v(t) \in A_{0} \cap A_{1} . v(t) \in A_{0}$ for each $t$ from 44.2.23 and the assumption that $u \in L^{p}\left(0, \infty, \frac{d t}{t} ; A_{0}\right)$. To verify $v(t) \in A_{1}$, integrate by parts in 44.2.23 to obtain

$$
\begin{gather*}
v(t)=\frac{(-1)^{m}}{(m-1)!} \int_{0}^{\infty} \phi^{(m)}(s)\left(s^{m-1} u\left(\frac{t}{s}\right)\right) d s  \tag{44.2.25}\\
=\frac{1}{(m-1)!} \int_{0}^{\infty} \phi(s) \frac{d^{m}}{d s^{m}}\left(s^{m-1} u\left(\frac{t}{s}\right)\right) d s \\
=\frac{(-1)^{m}}{(m-1)!} \int_{0}^{\infty} \phi(s) \frac{t^{m}}{s^{m+1}} u^{(m)}\left(\frac{t}{s}\right) d s \in A_{1}
\end{gather*}
$$

The last step may look very mysterious. If so, consider the case where $m=2$.

$$
\begin{aligned}
& \phi(s)\left(s u\left(\frac{t}{s}\right)\right)^{\prime \prime} \\
= & \phi(s)\left(-\frac{t}{s} u^{\prime}\left(\frac{t}{s}\right)+u\left(\frac{t}{s}\right)\right)^{\prime} \\
= & \phi(s)\left(\left(-\frac{t}{s}\right) u^{\prime \prime}\left(\frac{t}{s}\right)\left(-\frac{t}{s^{2}}\right)+\frac{t}{s^{2}} u^{\prime}\left(\frac{t}{s}\right)-\frac{t}{s^{2}} u^{\prime}\left(\frac{t}{s}\right)\right) \\
= & \phi(s) \frac{t^{2}}{s^{3}} u^{\prime \prime}\left(\frac{t}{s}\right) .
\end{aligned}
$$

You can see the same pattern will take place for other values of $m$.
Now

$$
\begin{gather*}
\|a\|_{\theta, p, J} \leq\left(\int_{0}^{\infty}\left(t^{-\theta} J\left(t, v\left(\frac{1}{t}\right)\right)\right)^{p} \frac{d t}{t}\right)^{1 / p} \\
\leq C_{p}\left\{\int_{0}^{\infty}\left[\left(t^{-\theta}\left\|v\left(\frac{1}{t}\right)\right\|_{A_{0}}\right)+\left(t^{1-\theta}\left\|v\left(\frac{1}{t}\right)\right\|_{A_{1}}\right)\right]^{p} \frac{d t}{t}\right\}^{1 / p} \\
\leq C_{p}\left\{\left(\int_{0}^{\infty}\left(t^{-\theta}\left\|v\left(\frac{1}{t}\right)\right\|_{A_{0}}\right)^{p} \frac{d t}{t}\right)^{1 / p}\right. \\
\left.+\left(\int_{0}^{\infty}\left(t^{1-\theta}\left\|v\left(\frac{1}{t}\right)\right\|_{A_{1}}\right)^{p} \frac{d t}{t}\right)^{1 / p}\right\} \tag{44.2.26}
\end{gather*}
$$

The first term equals

$$
\begin{gather*}
\left(\int_{0}^{\infty}\left(t^{-\theta}\left\|v\left(\frac{1}{t}\right)\right\|_{A_{0}}\right)^{p} \frac{d t}{t}\right)^{1 / p} \\
=\left(\int_{0}^{\infty}\left(t^{\theta}\|v(t)\|_{A_{0}}\right)^{p} \frac{d t}{t}\right)^{1 / p} \\
=\left(\int_{0}^{\infty}\left(t^{\theta}\left\|\int_{0}^{\infty} s^{m} \phi^{(m)}(s) u\left(\frac{t}{s}\right) \frac{d s}{s}\right\| \|_{A_{0}}\right)^{p} \frac{d t}{t}\right)^{1 / p} \\
\leq \int_{0}^{\infty}\left(\int_{0}^{\infty}\left(t^{\theta} s^{m}\left|\phi^{(m)}(s)\right|\left\|u\left(\frac{t}{s}\right)\right\|_{A_{0}}\right)^{p} \frac{d t}{t}\right)^{1 / p} \frac{d s}{s} \\
\leq \int_{0}^{\infty} s^{m}\left|\phi^{(m)}(s)\right|\left(\int_{0}^{\infty}\left(t^{\theta}\left\|u\left(\frac{t}{s}\right)\right\|_{A_{0}}\right)^{p} \frac{d t}{t}\right)^{1 / p} \frac{d s}{s} \\
=\int_{0}^{\infty} s^{\theta+m}\left|\phi^{(m)}(s)\right| \frac{d s}{s}\left(\int_{0}^{\infty}\left(\tau^{\theta}\|u(\tau)\|_{A_{0}}\right)^{p} \frac{d \tau}{\tau}\right)^{1 / p} \\
=C\left(\int_{0}^{\infty}\left(\tau^{\theta}\|u(\tau)\|_{A_{0}}\right)^{p} \frac{d \tau}{\tau}\right)^{1 / p} \cdot \tag{44.2.27}
\end{gather*}
$$

The second term equals

$$
\begin{aligned}
& \left(\int_{0}^{\infty}\left(t^{1-\theta}\left\|v\left(\frac{1}{t}\right)\right\|_{A_{1}}\right)^{p} \frac{d t}{t}\right)^{1 / p}=\left(\int_{0}^{\infty}\left(t^{\theta-1}\|v(t)\|_{A_{1}}\right)^{p} \frac{d t}{t}\right)^{1 / p} \\
& \quad=\left(\int_{0}^{\infty}\left(t^{\theta-1}\left\|\frac{1}{(m-1)!} \int_{0}^{\infty} \phi(s) \frac{t^{m}}{s^{m}} u^{(m)}\left(\frac{t}{s}\right) \frac{d s}{s}\right\|_{A_{1}}\right)^{p} \frac{d t}{t}\right)^{1 / p}
\end{aligned}
$$

$$
\begin{gather*}
\leq \int_{0}^{\infty}\left(\int_{0}^{\infty}\left(\left(\frac{t^{\theta+m-1}}{s^{m}}\right)|\phi(s)|\left\|u^{(m)}\left(\frac{t}{s}\right)\right\|_{A_{1}}\right)^{p} \frac{d t}{t}\right)^{1 / p} \frac{d s}{s} \\
\leq \int_{0}^{\infty} \frac{|\phi(s)|}{s^{m}}\left(\int_{0}^{\infty}\left(t^{\theta+m-1}\left\|u^{(m)}\left(\frac{t}{s}\right)\right\|_{A_{1}}\right)^{p} \frac{d t}{t}\right)^{1 / p} \frac{d s}{s} \\
=\int_{0}^{\infty} \frac{|\phi(s)|}{s^{m}} s^{\theta+m-1}\left(\int_{0}^{\infty}\left(\tau^{\theta+m-1}\left\|u^{(m)}(\tau)\right\|_{A_{1}}\right)^{p} \frac{d \tau}{\tau}\right)^{1 / p} \frac{d s}{s} \\
=C\left(\int_{0}^{\infty}\left(\tau^{\theta+m-1}\left\|u^{(m)}(\tau)\right\|_{A_{1}}\right)^{p} \frac{d \tau}{\tau}\right)^{1 / p} \tag{44.2.28}
\end{gather*}
$$

Now from the estimates on the two terms in 44.2.26 found in 44.2.27 and 44.2.28, and the simple estimate,

$$
2 \max (\alpha, \beta) \geq \alpha+\beta
$$

it follows

$$
\begin{align*}
& \|a\|_{\theta, p, J}  \tag{44.2.29}\\
\leq & C \max \left(\left(\int_{0}^{\infty}\left(\tau^{\theta}\|u(\tau)\|_{A_{0}}\right)^{p} \frac{d \tau}{\tau}\right)^{1 / p}\right.  \tag{44.2.30}\\
& \left.,\left(\int_{0}^{\infty}\left(\tau^{\theta+m-1}\left\|u^{(m)}(\tau)\right\|_{A_{1}}\right)^{p} \frac{d \tau}{\tau}\right)^{1 / p}\right) \tag{44.2.31}
\end{align*}
$$

which shows that after taking the infimum over all $u$ whose trace is $a$, it follows $a \in$ $\left(A_{0}, A_{1}\right)_{\theta, p, J}$.

$$
\begin{equation*}
\|a\|_{\theta, p, J} \leq C\|a\|_{T^{m}} \tag{44.2.32}
\end{equation*}
$$

Thus $T^{m}\left(A_{0}, A_{1}, \theta, p\right) \subseteq\left(A_{0}, A_{1}\right)_{\theta, p, J}$.
Is $\left(A_{0}, A_{1}\right)_{\theta, p, J} \subseteq T^{m}\left(A_{0}, A_{1}, \theta, p\right)$ ? Let $a \in\left(A_{0}, A_{1}\right)_{\theta, p, J}$. There exists $u$ having values in $A_{0} \cap A_{1}$ and such that

$$
a=\int_{0}^{\infty} u(t) \frac{d t}{t}=\int_{0}^{\infty} u\left(\frac{1}{t}\right) \frac{d t}{t}
$$

in $A_{0}+A_{1}$ such that

$$
\int_{0}^{\infty}\left(t^{-\theta} J(t, u(t))\right)^{p} d t<\infty, \text { where } J(t, a)=\max \left(\|a\|_{A_{0}}, t\|a\|_{A_{1}}\right)
$$

Then let

$$
\begin{gather*}
w(t) \equiv \int_{t}^{\infty}\left(1-\frac{t}{\tau}\right)^{m-1} u\left(\frac{1}{\tau}\right) \frac{d \tau}{\tau}=  \tag{44.2.33}\\
\int_{0}^{1 / t}(1-s t)^{m-1} u(s) \frac{d s}{s}=\int_{0}^{1}(1-\tau)^{m-1} u\left(\frac{\tau}{t}\right) \frac{d \tau}{\tau} \tag{44.2.34}
\end{gather*}
$$

It is routine to verify from 44.2.33 that

$$
\begin{equation*}
w^{(m)}(t)=(m-1)!(-1)^{m} \frac{u\left(\frac{1}{t}\right)}{t^{m}} \tag{44.2.35}
\end{equation*}
$$

For example, consider the case where $m=2$.

$$
\begin{aligned}
\left(\int_{t}^{\infty}\left(1-\frac{t}{\tau}\right) u\left(\frac{1}{\tau}\right) \frac{d \tau}{\tau}\right)^{\prime \prime} & =\left(0+\int_{t}^{\infty}\left(-\frac{1}{\tau}\right) u\left(\frac{1}{\tau}\right) \frac{d \tau}{\tau}\right)^{\prime} \\
& =\frac{1}{t^{2}} u\left(\frac{1}{t}\right)
\end{aligned}
$$

Also from 44.2.33, it follows that trace $(w)=a$. It remains to verify $w \in V^{m}$. From 44.2.35,

$$
\begin{gather*}
\left(\int_{0}^{\infty}\left(t^{\theta+m-1}\left\|w^{(m)}(t)\right\|_{A_{1}}\right)^{p} \frac{d t}{t}\right)^{1 / p}= \\
C_{m}\left(\int_{0}^{\infty}\left(t^{\theta-1}\left\|u\left(\frac{1}{t}\right)\right\| \|_{A_{1}}\right)^{p} \frac{d t}{t}\right)^{1 / p}=C_{m}\left(\int_{0}^{\infty}\left(t^{1-\theta}\|u(t)\|_{A_{1}}\right)^{p} \frac{d t}{t}\right)^{1 / p} \\
\leq C_{m}\left(\int_{0}^{\infty}\left(t^{-\theta} J(t, u(t))\right)^{p} \frac{d t}{t}\right)^{1 / p}<\infty \tag{44.2.36}
\end{gather*}
$$

It remains to consider $\left(\int_{0}^{\infty}\left(t^{\theta}\|w(t)\|_{A_{0}}\right)^{p} \frac{d t}{t}\right)^{1 / p}$. From 44.2.34,

$$
\begin{aligned}
&\left(\int_{0}^{\infty}\left(t^{\theta}\|w(t)\|_{A_{0}}\right)^{p} \frac{d t}{t}\right)^{1 / p} \\
&=\left(\int_{0}^{\infty}\left(t^{\theta}\left\|\int_{0}^{1}(1-\tau)^{m-1} u\left(\frac{\tau}{t}\right) \frac{d \tau}{\tau}\right\|_{A_{0}}\right)^{p} \frac{d t}{t}\right)^{1 / p} \\
&=\left(\int_{0}^{\infty}\left(t^{-\theta}\left\|\int_{0}^{1}(1-\tau)^{m-1} u(\tau t) \frac{d \tau}{\tau}\right\|_{A_{0}}\right)^{p} \frac{d t}{t}\right)^{1 / p} \\
& \leq \int_{0}^{1}\left(\int_{0}^{\infty}\left(t^{-\theta}(1-\tau)^{m-1}\|u(\tau t)\|_{A_{0}}\right)^{p} \frac{d t}{t}\right)^{1 / p} \frac{d \tau}{\tau} \\
&= \int_{0}^{1} \tau^{\theta}(1-\tau)^{m-1}\left(\int_{0}^{\infty}\left(s^{-\theta}\|u(s)\|_{A_{0}}\right)^{p} \frac{d s}{s}\right)^{1 / p} \frac{d \tau}{\tau} \\
&=\left(\int_{0}^{1} \tau^{\theta-1}(1-\tau)^{m-1} d \tau\right)\left(\int_{0}^{\infty}\left(s^{-\theta}\|u(s)\|_{A_{0}}\right)^{p} \frac{d s}{s}\right)^{1 / p} \\
& \quad \leq C\left(\int_{0}^{\infty}\left(s^{-\theta}\|u(s)\|_{A_{0}}\right)^{p} \frac{d s}{s}\right)^{1 / p}
\end{aligned}
$$

$$
\begin{equation*}
\leq C\left(\int_{0}^{\infty}\left(t^{-\theta} J(t, u(t))\right)^{p} d t\right)^{1 / p}<\infty \tag{44.2.37}
\end{equation*}
$$

It follows that

$$
\begin{gathered}
\|w\|_{V^{m}} \equiv \\
\max \left(\left(\int_{0}^{\infty}\left(t^{\theta}\|w(t)\|_{A_{0}}\right)^{p} \frac{d t}{t}\right)^{1 / p},\left(\int_{0}^{\infty}\left(t^{\theta+m-1}\left\|w^{(m)}(t)\right\|_{A_{1}}\right)^{p} \frac{d t}{t}\right)^{1 / p}\right) \\
\leq C\left(\int_{0}^{\infty}\left(t^{-\theta} J(t, u(t))\right)^{p} d t\right)^{1 / p}<\infty
\end{gathered}
$$

which shows that $a \in T^{m}\left(A_{0}, A_{1}, \theta, p\right)$. Taking the infimum,

$$
\|a\|_{T^{m}} \leq C\|a\|_{\theta, p, J}
$$

This together with 44.2 .32 proves the theorem.
By Theorem 44.2.3 and Theorem 43.8.6, we obtain the following important corollary describing the dual space of a trace space.

Corollary 44.2.4 Let $A_{0} \cap A_{1}$ be dense in $A_{i}$ for $i=0,1$ and suppose that $A_{i}$ is reflexive for $i=0,1$. Then for $\infty>p \geq 1$,

$$
T^{m}\left(A_{0}, A_{1}, \theta, p\right)^{\prime}=T^{m}\left(A_{1}^{\prime}, A_{0}^{\prime}, 1-\theta, p^{\prime}\right)
$$

## Chapter 45

## Traces Of Sobolev Spaces

### 45.1 Traces Of Sobolev Spaces, Half Space

In this section consider the trace of $W^{m, p}\left(\mathbb{R}_{+}^{n}\right)$ onto a Sobolev space of functions defined on $\mathbb{R}^{n-1}$. This latter Sobolev space will be defined in terms of the following theory in such a way that the trace map is continuous. The trace map is continuous as a map from $W^{m, p}\left(\mathbb{R}_{+}^{n}\right)$ to $W^{m-1, p}\left(\mathbb{R}^{n-1}\right)$ but here I will give a better conclusion using the above theory.

Definition 45.1.1 Let $\theta \in(0,1)$ and let $\Omega$ be an open subset of $\mathbb{R}^{m}$. We define

$$
W^{\theta, p}(\Omega) \equiv T\left(W^{1, p}(\Omega), L^{p}(\Omega), p, 1-\theta\right)
$$

Thus, from the above general theory, $W^{1, p}(\Omega) \hookrightarrow W^{\theta, p}(\Omega) \hookrightarrow L^{p}(\Omega)=L^{p}(\Omega)+$ $W^{1, p}(\Omega)$. Now we consider the trace map for Sobolev space.

Lemma 45.1.2 Let $\phi \in C^{\infty}\left(\overline{\mathbb{R}_{+}^{n}}\right)$. Then $\gamma \phi\left(\mathbf{x}^{\prime}\right) \equiv \phi\left(\mathbf{x}^{\prime}, 0\right)$. Then $\gamma: C^{\infty}\left(\overline{\mathbb{R}_{+}^{n}}\right) \rightarrow L^{p}\left(\mathbb{R}^{n-1}\right)$ is continuous as a map from $W^{1, p}\left(\mathbb{R}_{+}^{n}\right)$ to $L^{p}\left(\mathbb{R}^{n-1}\right)$.

Proof: We know

$$
\phi\left(\mathbf{x}^{\prime}, x_{n}\right)=\gamma \phi\left(\mathbf{x}^{\prime}\right)+\int_{0}^{x_{n}} \frac{\partial \phi\left(\mathbf{x}^{\prime}, t\right)}{\partial t} d t
$$

Then by Jensen's inequality,

$$
\begin{aligned}
& \int_{\mathbb{R}^{n-1}}\left|\gamma \phi\left(\mathbf{x}^{\prime}\right)\right|^{p} d x^{\prime} \\
= & \int_{0}^{1} \int_{\mathbb{R}^{n-1}}\left|\gamma \phi\left(\mathbf{x}^{\prime}\right)\right|^{p} d x^{\prime} d x_{n} \\
\leq & C \int_{0}^{1} \int_{\mathbb{R}^{n-1}}\left|\phi\left(\mathbf{x}^{\prime}, x_{n}\right)\right|^{p} d x^{\prime} d x_{n} \\
& +C \int_{0}^{1} \int_{\mathbb{R}^{n-1}}\left|\int_{0}^{x_{n}} \frac{\partial \phi\left(\mathbf{x}^{\prime}, t\right)}{\partial t} d t\right|^{p} d x^{\prime} d x_{n} \\
\leq & C\|\phi\|_{0, p, \mathbb{R}_{+}^{n}}^{p}+C \int_{0}^{1} x_{n}^{p-1} \int_{\mathbb{R}^{n-1}} \int_{0}^{x_{n}}\left|\frac{\partial \phi\left(\mathbf{x}^{\prime}, t\right)}{\partial t}\right|^{p} d t d x^{\prime} d x_{n} \\
\leq & C\|\phi\|_{0, p, \mathbb{R}_{+}^{n}}^{p}+C \int_{0}^{1} x_{n}^{p-1} \int_{\mathbb{R}^{n-1}} \int_{0}^{\infty}\left|\frac{\partial \phi\left(\mathbf{x}^{\prime}, t\right)}{\partial t}\right|^{p} d t d x^{\prime} d x_{n} \\
\leq & C\|\phi\|_{0, p, \mathbb{R}_{+}^{n}}^{p}+\frac{C}{p} \int_{\mathbb{R}^{n-1}} \int_{0}^{\infty}\left|\frac{\partial \phi\left(\mathbf{x}^{\prime}, t\right)}{\partial t}\right|^{p} d t d x^{\prime} \\
\leq & C\|\phi\|_{1, p, \mathbb{R}_{+}^{n}}^{p}
\end{aligned}
$$

This proves the lemma.

Definition 45.1.3 We define the trace,

$$
\gamma: W^{1, p}\left(\mathbb{R}_{+}^{n}\right) \rightarrow L^{p}\left(\mathbb{R}^{n-1}\right)
$$

as follows. $\gamma \phi\left(\mathbf{x}^{\prime}\right) \equiv \phi\left(\mathbf{x}^{\prime}, 0\right)$ whenever $\phi \in C^{\infty}\left(\overline{\mathbb{R}_{+}^{n}}\right)$. For $u \in W^{1, p}\left(\overline{\mathbb{R}_{+}^{n}}\right)$, we define $\gamma u \equiv$ $\lim _{k \rightarrow \infty} \gamma \phi_{k}$ in $L^{p}\left(\mathbb{R}^{n-1}\right)$ where $\phi_{k} \rightarrow u$ in $W^{1, p}\left(\overline{\mathbb{R}_{+}^{n}}\right)$. Then the above lemma shows this is well defined.

Also from this lemma we obtain a constant, $C$ such that

$$
\|\phi\|_{0, p, \mathbb{R}^{n-1}} \leq C\|\phi\|_{1, p, \mathbb{R}_{+}^{n}}
$$

and the same constant holds for all $u \in W^{1, p}\left(\mathbb{R}_{+}^{n}\right)$.
From the definition of the norm in the trace space, if $f \in C^{\infty}\left(\overline{\mathbb{R}_{+}^{n}}\right)$, and letting $\theta=$ $1-\frac{1}{p}$, it follows from the definition

$$
\begin{aligned}
& \|\gamma f\|_{1-\frac{1}{p}, p, \mathbb{R}^{n-1}} \\
\leq & \max \left(\left(\int_{0}^{\infty}\left(t^{1 / p}\|f(t)\|_{1, p, \mathbb{R}^{n-1}}\right)^{p} \frac{d t}{t}\right)^{1 / p}\right. \\
& \left.,\left(\int_{0}^{\infty}\left(t^{1 / p}\left\|f^{\prime}(t)\right\|_{0, p, \mathbb{R}^{n-1}}\right)^{p} \frac{d t}{t}\right)^{1 / p}\right) \\
\leq & C\|f\|_{1, p, \mathbb{R}_{+}^{n}}
\end{aligned}
$$

Thus, if $f \in W^{1, p}\left(\mathbb{R}_{+}^{n}\right)$, define $\gamma f \in W^{1-\frac{1}{p}, p}\left(\mathbb{R}^{n-1}\right)$ according to the rule,

$$
\gamma f=\lim _{k \rightarrow \infty} \gamma \phi_{k}
$$

where $\phi_{k} \rightarrow f$ in $W^{1, p}\left(\mathbb{R}_{+}^{n}\right)$ and $\phi_{k} \in C^{\infty}\left(\overline{\mathbb{R}_{+}^{n}}\right)$. This shows the continuity part of the following lemma.

Lemma 45.1.4 The trace map, $\gamma$, is a continuous map from $W^{1, p}\left(\mathbb{R}_{+}^{n}\right)$ onto

$$
W^{1-\frac{1}{p}, p}\left(\mathbb{R}^{n-1}\right)
$$

Furthermore, for $f \in W^{1, p}\left(\mathbb{R}_{+}^{n}\right)$,

$$
\gamma f=f(0)=\lim _{t \rightarrow 0+} f(t)
$$

the limit taking place in $L^{p}\left(\mathbb{R}^{n-1}\right)$.
Proof: It remains to verify $\gamma$ is onto along with the displayed equation. But by definition, things in $W^{1-\frac{1}{p}, p}\left(\mathbb{R}^{n-1}\right)$ are $\lim _{t \rightarrow 0+} f(t)$ where $f \in L^{p}\left(0, \infty ; W^{1, p}\left(\mathbb{R}^{n-1}\right)\right)$, and $f^{\prime} \in L^{p}\left(0, \infty ; L^{p}\left(\mathbb{R}^{n-1}\right)\right)$, the limit taking place in

$$
W^{1, p}\left(\mathbb{R}^{n-1}\right)+L^{p}\left(\mathbb{R}^{n-1}\right)=L^{p}\left(\mathbb{R}^{n-1}\right)
$$

and

$$
\left(\int_{0}^{\infty}\|f(t)\|_{1, p, \mathbb{R}^{n-1}}^{p} d t\right)^{1 / p}+\left(\int_{0}^{\infty}\left\|f^{\prime}(t)\right\|_{0, p}^{p} d t\right)^{1 / p}<\infty
$$

Then taking a measurable representative, we see $f \in W^{1, p}\left(\mathbb{R}_{+}^{n}\right)$ and $f_{, x_{n}}=f^{\prime}$. Also, as an equation in $L^{p}\left(\mathbb{R}^{n-1}\right)$, the following holds for all $t>0$.

$$
f(\cdot, t)=f(0)+\int_{0}^{t} f_{, x_{n}}(\cdot, s) d s
$$

But also, for a.e. $\mathbf{x}^{\prime}$, the following equation holds for a.e. $t>0$.

$$
\begin{equation*}
f\left(\mathbf{x}^{\prime}, t\right)=\gamma f\left(\mathbf{x}^{\prime}\right)+\int_{0}^{t} f_{, x_{n}}\left(\mathbf{x}^{\prime}, s\right) d s \tag{45.1.1}
\end{equation*}
$$

showing that

$$
\gamma f=f(0) \in W^{1-\frac{1}{p}, p}\left(\mathbb{R}^{n-1}\right) \equiv T\left(W^{1, p}(\Omega), L^{p}(\Omega), p, \frac{1}{p}\right)
$$

To see that 45.1 .1 holds, approximate $f$ with a sequence from $C^{\infty}\left(\overline{\mathbb{R}_{+}^{n}}\right)$ and finally obtain an equation of the form

$$
\int_{\mathbb{R}^{n-1}} \int_{0}^{\infty}\left[f\left(\mathbf{x}^{\prime}, t\right)-\gamma f\left(\mathbf{x}^{\prime}\right)-\int_{0}^{t} f_{, x_{n}}\left(\mathbf{x}^{\prime}, s\right) d s\right] \psi\left(\mathbf{x}^{\prime}, t\right) d t d x^{\prime}=0
$$

which holds for all $\psi \in C_{c}^{\infty}\left(\mathbb{R}_{+}^{n}\right)$. This proves the lemma.
Thus taking the trace on the boundary loses exactly $\frac{1}{p}$ derivatives.

### 45.2 A Right Inverse For The Trace For A Half Space

It is also important to show there is a continuous linear function,

$$
R: W^{1-\frac{1}{p}, p}\left(\mathbb{R}^{n-1}\right) \rightarrow W^{1, p}\left(\mathbb{R}_{+}^{n}\right)
$$

which has the property that $\gamma(R g)=g$. Define this function as follows.

$$
\begin{equation*}
\operatorname{Rg}\left(\mathbf{x}^{\prime}, x_{n}\right) \equiv \int_{\mathbb{R}^{n-1}} g\left(\mathbf{y}^{\prime}\right) \phi\left(\frac{\mathbf{x}^{\prime}-\mathbf{y}^{\prime}}{x_{n}}\right) \frac{1}{x_{n}^{n-1}} d \mathbf{y}^{\prime} \tag{45.2.2}
\end{equation*}
$$

where $\phi$ is a mollifier having support in $B(\mathbf{0}, 1)$.
Lemma 45.2.1 Let $R$ be defined in 45.2.2. Then $R g \in W^{1, p}\left(\mathbb{R}_{+}^{n}\right)$ and is a continuous linear map from $W^{1-\frac{1}{p}, p}\left(\mathbb{R}^{n-1}\right)$ to $W^{1, p}\left(\mathbb{R}_{+}^{n}\right)$ with the property that $\gamma R g=g$.

Proof: Let $f \in W^{1, p}\left(\mathbb{R}_{+}^{n}\right)$ be such that $\gamma f=g$. Let $\psi\left(x_{n}\right) \equiv\left(1-x_{n}\right)_{+}$and assume $f$ is Borel measurable by taking a Borel measurable representative. Then for a.e. $\mathbf{x}^{\prime}$ we have the following formula holding for a.e. $x_{n}$.

$$
\begin{aligned}
& \operatorname{Rg}\left(\mathbf{x}^{\prime}, x_{n}\right) \\
= & \int_{\mathbb{R}^{n-1}}\left[\psi\left(x_{n}\right) f\left(\mathbf{y}^{\prime}, \psi\left(x_{n}\right)\right)-\int_{0}^{\psi\left(x_{n}\right)}(\psi f)_{, n}\left(\mathbf{y}^{\prime}, t\right) d t\right] \phi\left(\frac{\mathbf{x}^{\prime}-\mathbf{y}^{\prime}}{x_{n}}\right) x_{n}^{1-n} d y^{\prime} .
\end{aligned}
$$

Using the repeated index summation convention to save space, we obtain that in terms of weak derivatives,

$$
\begin{gathered}
R g_{, n}\left(\mathbf{x}^{\prime}, x_{n}\right) \\
=\int_{\mathbb{R}^{n-1}}\left[\psi\left(x_{n}\right) f\left(\mathbf{y}^{\prime}, \psi\left(x_{n}\right)\right)-\int_{0}^{\psi\left(x_{n}\right)}(\psi f)_{, n}\left(\mathbf{y}^{\prime}, t\right) d t\right] \\
{\left[\phi_{, k}\left(\frac{\mathbf{x}^{\prime}-\mathbf{y}^{\prime}}{x_{n}}\right)\left(\frac{y_{k}-x_{k}}{x_{n}^{n}}\right)+\phi\left(\frac{\mathbf{x}^{\prime}-\mathbf{y}^{\prime}}{x_{n}}\right) \frac{(1-n)}{x_{n}^{n}}\right] d y^{\prime}} \\
=\int_{\mathbb{R}^{n-1}}\left[\psi\left(x_{n}\right) f\left(\mathbf{x}^{\prime}-x_{n} \mathbf{z}^{\prime}, \psi\left(x_{n}\right)\right)-\int_{0}^{\psi\left(x_{n}\right)}(\psi f)_{, n}\left(\mathbf{x}^{\prime}-x_{n} \mathbf{z}^{\prime}, t\right) d t\right] . \\
{\left[\phi_{, k}\left(\mathbf{z}^{\prime}\right)\left(\frac{y_{k}-x_{k}}{x_{n}^{n}}\right) z_{k}+\phi\left(\mathbf{z}^{\prime}\right) \frac{(1-n)}{x_{n}^{n}}\right] x_{n}^{n} d z^{\prime}}
\end{gathered}
$$

and so

$$
\begin{aligned}
\left|R g_{, n}\left(\mathbf{x}^{\prime}, x_{n}\right)\right| \leq & C(\phi) \mid \int_{B(\mathbf{0}, 1)}\left[\psi\left(x_{n}\right) f\left(\mathbf{x}^{\prime}-x_{n} \mathbf{z}^{\prime}, \psi\left(x_{n}\right)\right)\right. \\
& \left.-\int_{0}^{\psi\left(x_{n}\right)}(\psi f)_{, n}\left(\mathbf{x}^{\prime}-x_{n} \mathbf{z}^{\prime}, t\right) d t\right] \mid \\
\leq & \frac{C(\phi)}{x_{n}^{n-1}}\left\{\int_{B\left(\mathbf{0}, x_{n}\right)}\left|\psi\left(x_{n}\right) f\left(\mathbf{x}^{\prime}+\mathbf{y}^{\prime}, \psi\left(x_{n}\right)\right)\right| d y^{\prime}\right. \\
& \left.+\int_{B\left(\mathbf{0}, x_{n}\right)} \int_{0}^{\psi\left(x_{n}\right)}\left|(\psi f)_{, n}\left(\mathbf{x}^{\prime}+\mathbf{y}^{\prime}, t\right)\right| d t d y^{\prime}\right\}
\end{aligned}
$$

Therefore,

$$
\begin{gather*}
\left(\int_{0}^{\infty} \int_{\mathbb{R}^{n-1}}\left|R g_{, n}\left(\mathbf{x}^{\prime}, x_{n}\right)\right|^{p} d x^{\prime} d x_{n}\right)^{1 / p} \leq \\
C(\phi)\left(\int_{0}^{\infty} \int_{\mathbb{R}^{n-1}}\left(\frac{1}{x_{n}^{n-1}} \int_{B\left(\mathbf{0}, x_{n}\right)}\left|\psi\left(x_{n}\right) f\left(\mathbf{x}^{\prime}+\mathbf{y}^{\prime}, \psi\left(x_{n}\right)\right)\right| d y^{\prime}\right)^{p} d x^{\prime} d x_{n}\right)^{1 / p} \\
+C(\phi)\left(\int_{0}^{\infty} \int_{\mathbb{R}^{n-1}}\left(\frac{1}{x_{n}^{n-1}} \int_{B\left(\mathbf{0}, x_{n}\right)} \int_{0}^{\psi\left(x_{n}\right)}\left|(\psi f)_{, n}\left(\mathbf{x}^{\prime}+\mathbf{y}^{\prime}, t\right)\right| d t d y^{\prime}\right)^{p} d x^{\prime} d x_{n}\right)^{1 / p} \tag{45.2.3}
\end{gather*}
$$

Consider the first term on the right. We change variables, letting $\mathbf{y}^{\prime}=\mathbf{z}^{\prime} x_{n}$. Then this term becomes

$$
\begin{aligned}
& C(\phi)\left(\int_{0}^{1} \int_{\mathbb{R}^{n-1}}\left(\int_{B(\mathbf{0}, 1)}\left|\psi\left(x_{n}\right) f\left(\mathbf{x}^{\prime}+x_{n} \mathbf{z}^{\prime}, \psi\left(x_{n}\right)\right)\right| d z^{\prime}\right)^{p} d x^{\prime} d x_{n}\right)^{1 / p} \\
& \leq C(\phi) \int_{B(\mathbf{0}, 1)}\left(\int_{0}^{1} \int_{\mathbb{R}^{n-1}}\left|\psi\left(x_{n}\right) f\left(\mathbf{x}^{\prime}+x_{n} \mathbf{z}^{\prime}, \psi\left(x_{n}\right)\right)\right|^{p} d x^{\prime} d x_{n}\right)^{1 / p} d z^{\prime}
\end{aligned}
$$

Now we change variables, letting $t=\psi\left(x_{n}\right)$. This yields

$$
\begin{equation*}
=C(\phi) \int_{B(\mathbf{0}, 1)}\left(\int_{0}^{1} \int_{\mathbb{R}^{n-1}}\left|t f\left(\mathbf{x}^{\prime}+x_{n} \mathbf{z}^{\prime}, t\right)\right|^{p} d x^{\prime} d t\right)^{1 / p} d z^{\prime} \leq C(\phi)\|f\|_{0, p, \mathbb{R}_{+}^{n}} \tag{45.2.4}
\end{equation*}
$$

Now we consider the second term on the right in 45.2.3. Using the same arguments which were used on the first term involving Minkowski's inequality and changing the variables, we obtain the second term

$$
\begin{align*}
& \leq C(\phi) \int_{B(\mathbf{0}, 1)} \int_{0}^{1}\left(\int_{0}^{1} \int_{\mathbb{R}^{n-1}}\left|(\psi f)_{, n}\left(\mathbf{x}^{\prime}+x_{n} \mathbf{z}^{\prime}, t\right)\right|^{p} d \mathbf{x}^{\prime} d x_{n}\right)^{1 / p} d t d y^{\prime} \\
& \leq C(\phi)\|f\|_{1, p, \mathbb{R}_{+}^{n}} \tag{45.2.5}
\end{align*}
$$

It is somewhat easier to verify that

$$
\left\|R g_{, j}\right\|_{0, p, \mathbb{R}_{+}^{n}} \leq C(\phi)\|f\|_{1, p, \mathbb{R}_{+}^{n}}
$$

Therefore, we have shown that whenever $\gamma f=f(0)=g$,

$$
\|R g\|_{1, p, \mathbb{R}_{+}^{n}} \leq C(\phi)\|f\|_{1, p, \mathbb{R}_{+}^{n}}
$$

Taking the infimum over all such $f$ and using the definition of the norm in

$$
W^{1-\frac{1}{p}, p}\left(\mathbb{R}^{n-1}\right)
$$

it follows

$$
\|R g\|_{1, p, \mathbb{R}_{+}^{n}} \leq C(\phi)\|g\|_{1-\frac{1}{p}, p, \mathbb{R}^{n-1}}
$$

showing that this map, $R$, is continuous as claimed. It is obvious that

$$
\lim _{x_{n} \rightarrow 0} \operatorname{Rg}\left(x_{n}\right)=g
$$

the convergence taking place in $L^{p}\left(\mathbb{R}^{n-1}\right)$ because of general results about convolution with mollifiers. This proves the lemma.

### 45.3 Intrinsic Norms

The above presentation is very abstract, involving the trace of a function in

$$
W\left(A_{0}, A_{1}, p, \theta\right)
$$

and a norm which was the infimum of norms of functions in $W$ which have trace equal to the given function. It is very useful to have a description of the norm in these fractional order spaces which is defined in terms of the function itself rather than functions which have the given function as trace. This leads to something called an intrinsic norm. I am following Adams [1].

The following interesting lemma is called Young's inequality. It holds more generally than stated.

Lemma 45.3.1 Let $g=f * h$ where $f \in L^{1}(\mathbb{R}), h \in L^{p}(\mathbb{R})$, and $f, h$ are all Borel measurable, $p \geq 1$. Then $g \in L^{p}(\mathbb{R})$ and

$$
\|g\|_{L^{p}(\mathbb{R})} \leq\|f\|_{L^{1}(\mathbb{R})}\|h\|_{L^{p}(\mathbb{R})}
$$

Proof: First of all it is good to show $g$ is well defined. Using Minkowski’s inequality

$$
\begin{aligned}
& \left(\int\left(\int|h(t-s) f(s)| d s\right)^{p} d t\right)^{1 / p} \\
& \leq \int\left(\int|h(t-s)|^{p}|f(s)|^{p} d t\right)^{1 / p} d s \\
& =\int|f(s)|\left(\int|h(t-s)|^{p} d t\right)^{1 / p} d s \\
& =\|f\|_{L^{1}}\|h\|_{L^{p}}
\end{aligned}
$$

Therefore, for a.e. $t$,

$$
\int|h(t-s) f(s)| d s=\int|h(s) f(t-s)| d s<\infty
$$

and so for all such $t$ the convolution $f * h(t)$ makes sense. The above also shows

$$
\|g\|_{L^{p}} \equiv\left(\int\left|\int f(t-s) h(s) d s\right|^{p} d t\right)^{1 / p} \leq\|f\|_{L^{1}}\|h\|_{L^{p}}
$$

and this proves the lemma.
The following is a very interesting inequality of Hardy Littlewood and Pólya.
Lemma 45.3.2 Let $f$ be a real valued function defined a.e. on $[0, \infty)$ and let $\alpha \in(-\infty, 1)$ and

$$
\begin{equation*}
g(t)=\frac{1}{t} \int_{0}^{t} f(\xi) d \xi \tag{45.3.6}
\end{equation*}
$$

For $1 \leq p<\infty$

$$
\begin{equation*}
\int_{0}^{\infty} t^{\alpha p}|g(t)|^{p} \frac{d t}{t} \leq \frac{1}{(1-\alpha)^{p}} \int_{0}^{\infty} t^{\alpha p}|f(t)|^{p} \frac{d t}{t} \tag{45.3.7}
\end{equation*}
$$

Proof: First it can be assumed the right side of 45.3 .7 is finite since otherwise there is nothing to show. Changing the variables letting $t=e^{\tau}$, the above inequality takes the form

$$
\int_{-\infty}^{\infty} e^{\tau p \alpha}\left|g\left(e^{\tau}\right)\right|^{p} d \tau \leq \frac{1}{(1-\alpha)^{p}} \int_{-\infty}^{\infty} e^{\tau p \alpha}\left|f\left(e^{\tau}\right)\right|^{p} d \tau
$$

Now from the definition of $g$ it follows

$$
\begin{aligned}
g\left(e^{\tau}\right) & =e^{-\tau} \int_{-\infty}^{e^{\tau}} f(\xi) d \xi \\
& =e^{-\tau} \int_{-\infty}^{\tau} f\left(e^{\sigma}\right) e^{\sigma} d \sigma
\end{aligned}
$$

and so the left side equals

$$
\begin{gathered}
\int_{-\infty}^{\infty} e^{\tau p(\alpha-1)}\left|\int_{-\infty}^{\tau} f\left(e^{\sigma}\right) e^{\sigma} d \sigma\right|^{p} d \tau \\
=\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\tau} e^{\tau \alpha} f\left(e^{\sigma}\right) e^{-(\tau-\sigma)} d \sigma\right|^{p} d \tau \\
=\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\tau} e^{(\tau-\sigma) \alpha} e^{-(\tau-\sigma)} e^{\sigma \alpha} f\left(e^{\sigma}\right) d \sigma\right|^{p} d \tau \\
=\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} \mathscr{X}_{(-\infty, 0)}(\tau-\sigma) e^{(\tau-\sigma)(\alpha-1)} e^{\sigma \alpha} f\left(e^{\sigma}\right) d \sigma\right|^{p} d \tau
\end{gathered}
$$

and by Lemma 45.3.1,

$$
\begin{aligned}
& \leq\left(\int_{-\infty}^{0} e^{(\alpha-1) u} d u\right)^{p} \int_{-\infty}^{\infty} e^{p \sigma \alpha}\left|f\left(e^{\sigma}\right)\right|^{p} d \sigma \\
& =\left(\frac{1}{1-\alpha}\right)^{p} \int_{-\infty}^{\infty} e^{p \sigma \alpha}\left|f\left(e^{\sigma}\right)\right|^{p} d \sigma
\end{aligned}
$$

which was to be shown. This proves the lemma.
Next consider the case where $G(t), t>0$ is a continuous semigroup on $A_{1}$ and $A_{0} \equiv$ $D(\Lambda)$ where $\Lambda$ is the generator of this semigroup. Recall that from Proposition 19.14.5 on Page $577 \Lambda$ is a closed densely defined operator and so $A_{0}$ is a Banach space if the norm is given by

$$
\|u\|_{A_{0}} \equiv\|u\|_{A_{1}}+\|\Lambda u\|_{A_{1}}
$$

Also assume $\|G(t)\|$ is uniformly bounded for $t \in[0, \infty)$. I have in mind the case where $A_{1}=L^{p}\left(\mathbb{R}^{n}\right)$ and $G(t) u(\mathbf{x})=u\left(\mathbf{x}+t \mathbf{e}_{i}\right)$ but it is notationally easier to discuss this in the general case. First here is a simple lemma.

Lemma 45.3.3 Let $A_{0}=D(\Lambda)$ as just described. Then for $u \in A_{1}$

$$
\|u\|_{A_{1}+A_{0}}=\|u\|_{A_{1}}
$$

Proof: $D(\Lambda) \subseteq A_{1}$. Now let $u \in A_{1}$.

$$
\|u\|_{A_{0}+A_{1}} \equiv \inf \left\{\left\|u_{0}\right\|_{A_{1}}+\left\|\Lambda u_{0}\right\|_{A_{1}}+\left\|u_{1}\right\|_{A_{1}}: u=u_{0}+u_{1}\right\}
$$

To make this as small as possible you should clearly take $u_{1}=u$ because

$$
\begin{aligned}
\left\|u_{0}\right\|_{A_{1}}+\left\|\Lambda u_{0}\right\|_{A_{1}}+\left\|u_{1}\right\|_{A_{1}} & \geq\left\|u_{0}+u_{1}\right\|_{A_{1}}+\left\|\Lambda u_{0}\right\| \\
& =\|u\|_{A_{1}}+\left\|\Lambda u_{0}\right\|
\end{aligned}
$$

Therefore, the result of the lemma follows.

Lemma 45.3.4 Let $\Lambda$ be the generator of $G(t)$ and let $t \rightarrow g(t)$ be in $C^{1}\left(0, \infty ; A_{1}\right)$. Then there exists a unique solution to the initial value problem

$$
y^{\prime}-\Lambda y=g, y(0)=y_{0} \in D(\Lambda)
$$

and it is given by

$$
\begin{equation*}
y(t)=G(t) y_{0}+\int_{0}^{t} G(t-s) g(s) d s . \tag{45.3.8}
\end{equation*}
$$

This solution is continuous having continuous derivative and has values in $D(\Lambda)$.
Proof: First I show the following claim.
Claim: $\int_{0}^{t} G(t-s) g(s) d s \in D(\Lambda)$ and

$$
\Lambda\left(\int_{0}^{t} G(t-s) g(s) d s\right)=G(t) g(0)-g(t)+\int_{0}^{t} G(t-s) g^{\prime}(s) d s
$$

Proof of the claim:

$$
\begin{aligned}
& \frac{1}{h}\left(G(h) \int_{0}^{t} G(t-s) g(s) d s-\int_{0}^{t} G(t-s) g(s) d s\right) \\
= & \frac{1}{h}\left(\int_{0}^{t} G(t-s+h) g(s) d s-\int_{0}^{t} G(t-s) g(s) d s\right) \\
= & \frac{1}{h}\left(\int_{-h}^{t-h} G(t-s) g(s+h) d s-\int_{0}^{t} G(t-s) g(s) d s\right) \\
= & \frac{1}{h} \int_{-h}^{0} G(t-s) g(s+h) d s+\int_{0}^{t-h} G(t-s) \frac{g(s+h)-g(s)}{h} \\
& -\frac{1}{h} \int_{t-h}^{t} G(t-s) g(s) d s
\end{aligned}
$$

Using the estimate in Theorem 19.14.3 on Page 577 and the dominated convergence theorem the limit as $h \rightarrow 0$ of the above equals

$$
G(t) g(0)-g(t)+\int_{0}^{t} G(t-s) g^{\prime}(s) d s
$$

which proves the claim.
Since $y_{0} \in D(\Lambda)$,

$$
\begin{align*}
G(t) \Lambda y_{0} & =G(t) \lim _{h \rightarrow 0} \frac{G(h) y_{0}-y_{0}}{h} \\
& =\lim _{h \rightarrow 0} \frac{G(t+h)-G(t)}{h} y_{0} \\
& =\lim _{h \rightarrow 0} \frac{G(h) G(t) y_{0}-G(t) y_{0}}{h} \tag{45.3.9}
\end{align*}
$$

Since this limit exists, the last limit in the above exists and equals

$$
\begin{equation*}
\Lambda G(t) y_{0} \tag{45.3.10}
\end{equation*}
$$

and $G(t) y_{0} \in D(\Lambda)$. Now consider 45.3.8.

$$
\begin{gathered}
\frac{y(t+h)-y(t)}{h}=\frac{G(t+h)-G(t)}{h} y_{0}+ \\
\frac{1}{h}\left(\int_{0}^{t+h} G(t-s+h) g(s) d s-\int_{0}^{t} G(t-s) g(s) d s\right) \\
=\frac{G(t+h)-G(t)}{h} y_{0}+\frac{1}{h} \int_{t}^{t+h} G(t-s+h) g(s) d s \\
+\frac{1}{h}\left(G(h) \int_{0}^{t} G(t-s) g(s) d s-\int_{0}^{t} G(t-s) g(s) d s\right)
\end{gathered}
$$

From the claim and 45.3.9, 45.3.10 the limit of the right side is

$$
\begin{aligned}
& \Lambda G(t) y_{0}+g(t)+\Lambda\left(\int_{0}^{t} G(t-s) g(s) d s\right) \\
= & \Lambda\left(G(t) y_{0}+\int_{0}^{t} G(t-s) g(s) d s\right)+g(t)
\end{aligned}
$$

Hence

$$
y^{\prime}(t)=\Lambda y(t)+g(t)
$$

and from the formula, $y^{\prime}$ is continuous since by the claim and 45.3.10 it also equals

$$
G(t) \Lambda y_{0}+g(t)+G(t) g(0)-g(t)+\int_{0}^{t} G(t-s) g^{\prime}(s) d s
$$

which is continuous. The claim and 45.3 .10 also shows $y(t) \in D(\Lambda)$. This proves the existence part of the lemma.

It remains to prove the uniqueness part. It suffices to show that if

$$
y^{\prime}-\Lambda y=0, y(0)=0
$$

and $y$ is $C^{1}$ having values in $D(\Lambda)$, then $y=0$. Suppose then that $y$ is this way. Letting $0<s<t$,

$$
\begin{gathered}
\frac{d}{d s}(G(t-s) y(s)) \\
\equiv \lim _{h \rightarrow 0} G(t-s-h) \frac{y(s+h)-y(s)}{h} \\
-\frac{G(t-s) y(s)-G(t-s-h) y(s)}{h}
\end{gathered}
$$

provided the limit exists. Since $y^{\prime}$ exists and $y(s) \in D(\Lambda)$, this equals

$$
G(t-s) y^{\prime}(s)-G(t-s) \Lambda y(s)=0
$$

Let $y^{*} \in A_{1}^{\prime}$. This has shown that on the open interval $(0, t), s \rightarrow y^{*}(G(t-s) y(s))$ has a derivative equal to 0 . Also from continuity of $G$ and $y$, this function is continuous on $[0, t]$. Therefore, it is constant on $[0, t]$ by the mean value theorem. At $s=0$, this function equals 0 . Therefore, it equals 0 on $[0, t]$. Thus for fixed $s>0$ and letting $t>s, y^{*}(G(t-s) y(s))=0$. Now let $t$ decrease toward $s$. Then $y^{*}(y(s))=0$ and since $y^{*}$ was arbitrary, it follows $y(s)=0$. This proves uniqueness.

Definition 45.3.5 Let $G(t)$ be a uniformly bounded continuous semigroup defined on $A_{1}$ and let $\Lambda$ be its generator. Let the norm on $D(\Lambda)$ be given by

$$
\|u\|_{D(\Lambda)} \equiv\|u\|_{A_{1}}+\|\Lambda u\|_{A_{1}}
$$

so that by Lemma 45.3 .3 the norm on $A_{1}+D(\Lambda)$ is just $\|\cdot\|_{A_{1}}$. Let

$$
T_{0} \equiv\left\{u \in A_{1}:\|u\|_{A_{1}}^{p}+\int_{0}^{\infty} t^{\theta p}\left\|\frac{G(t) u-u}{t}\right\|_{A_{1}}^{p} \frac{d t}{t} \equiv\|u\|_{T_{0}}^{p}<\infty\right\}
$$

Theorem 45.3.6 $T_{0}=T\left(D(\Lambda), A_{1}, p, \theta\right) \equiv T$ and the two norms are equivalent.
Proof: Take $u \in T\left(D(\Lambda), A_{1}, p, \theta\right)$. I will show $\|u\|_{T_{0}} \leq C(\theta, p)\|u\|_{T}$. By the definition of the norm in $T$, there exists $f \in W\left(D(\Lambda), A_{1}, p, \theta\right)$ such that

$$
\|u\|_{T}^{p}+\delta>\|f\|_{W}^{p}, f(0)=u
$$

Now by Lemma 44.1.4 there exists $g_{r} \in W$ such that $\left\|g_{r}-f\right\|_{W}<r, g_{r} \in C^{\infty}(0, \infty ; D(\Lambda))$ and $g_{r}^{\prime} \in C^{\infty}\left(0, \infty ; A_{1}\right)$. Thus for each $\varepsilon>0, g_{r}(\varepsilon) \in D(\Lambda)$ although possibly $g_{r}(0) \notin D(\Lambda)$. Then letting $h_{r}(t)$ be defined by

$$
g_{r}^{\prime}(t)-\Lambda g_{r}(t)=h_{r}(t)
$$

it follows $h_{r} \in C^{1}\left(0, \infty ; A_{1}\right)$ and applying Lemma 45.3.4 on $[\varepsilon, \infty)$ it follows

$$
\begin{equation*}
g_{r}(t)=G(t-\varepsilon) g_{r}(\varepsilon)+\int_{\varepsilon}^{t} G(t-s) h_{r}(s) d s \tag{45.3.11}
\end{equation*}
$$

By Lemma 44.1.4 again, $g_{r}(\varepsilon)$ converges to $g_{r}(0)$ in $A_{1}$. Thus

$$
\int_{\mathcal{\varepsilon}}^{t}\left\|G(t-s) h_{r}(s)\right\|_{A_{1}} d s \leq C
$$

for some constant independent of $\varepsilon$. Thus $s \rightarrow G(t-s) h_{r}(s)$ is in $L^{1}\left(0, t ; A_{1}\right)$ and it is possible to pass to the limit in 45.3.11 as $\varepsilon \rightarrow 0$ to conclude

$$
g_{r}(t)=G(t) g_{r}(0)+\int_{0}^{t} G(t-s) h_{r}(s) d s
$$

Now

$$
\frac{G(t) g_{r}(0)-g_{r}(0)}{t}=\frac{1}{t} \int_{0}^{t} g_{r}^{\prime}(s) d s-\frac{1}{t} \int_{0}^{t} G(t-s) h_{r}(s) d s
$$

and so using the assumption that $G(t)$ is uniformly bounded,

$$
\begin{gathered}
\left\|\frac{G(t) g_{r}(0)-g_{r}(0)}{t}\right\| \leq \frac{1}{t} \int_{0}^{t}\left\|g_{r}^{\prime}\right\|_{A_{1}}+M\left\|h_{r}\right\|_{A_{1}} \\
\leq \frac{1}{t} \int_{0}^{t}\left\|g_{r}^{\prime}\right\|(M+1)+M\left\|\Lambda g_{r}\right\| d s \\
\leq \frac{M+1}{t} \int_{0}^{t}\left\|g_{r}^{\prime}\right\|_{A_{1}}+\left\|g_{r}\right\|_{D(\Lambda)} d s
\end{gathered}
$$

Therefore, from Lemma 45.3.2

$$
\begin{gathered}
\int_{0}^{\infty} t^{p \theta-p}\left\|G(t) g_{r}(0)-g_{r}(0)\right\|_{A_{1}}^{p} \frac{d t}{t} \\
=\int_{0}^{\infty} t^{p \theta}\left\|\frac{G(t) g_{r}(0)-g_{r}(0)}{t}\right\|_{A_{1}}^{p} \frac{d t}{t} \\
\leq \int_{0}^{\infty} t^{p \theta}\left|\frac{M+1}{t} \int_{0}^{t}\left\|g_{r}^{\prime}\right\|_{A_{1}}+\left\|g_{r}\right\|_{D(\Lambda)} d s\right|^{p} d t / t \\
\leq(M+1)^{p} 2^{p-1}\left(\frac{1}{1-\theta}\right)^{p} \int_{0}^{\infty} t^{p \theta}\left(\left\|g_{r}^{\prime}\right\|_{A_{1}}^{p}+\left\|g_{r}\right\|_{D(\Lambda)}^{p}\right)
\end{gathered}
$$

Now since $g_{r} \rightarrow f$ in $W$, it follows from Lemma 44.1.8 that $g_{r}(0) \rightarrow u$ in $T$ and hence by Theorem 44.1.9 this also in $A_{1}$. Therefore, using Fatou's lemma in the above along with the convergence of $g_{r}$ to $f$,

$$
\begin{aligned}
& \int_{0}^{\infty} t^{p \theta-p}\|G(t) u-u\|_{A_{1}}^{p} \frac{d t}{t} \\
\leq & (M+1)^{p} 2^{p-1}\left(\frac{1}{1-\theta}\right)^{p} \int_{0}^{\infty} t^{p \theta}\left(\left\|f^{\prime}\right\|_{A_{1}}^{p}+\|f\|_{D(\Lambda)}^{p}\right) \\
\leq & (M+1)^{p} 2^{p-1}\left(\frac{1}{1-\theta}\right)^{p}\left(\|u\|_{T}^{p}+\delta\right)
\end{aligned}
$$

Since $u \in T$, Theorem 44.1.9 implies $u \in A_{1}$ and $\|u\|_{A_{1}} \leq C\|u\|_{T}$. Therefore, since $\delta$ was arbitrary, this has shown that $u \in T_{0}$ and

$$
\|u\|_{T_{0}} \leq C(\theta, p)\|u\|_{T}
$$

This shows $T \subseteq T_{0}$ with continuous inclusion.
Now it is necessary to take $u \in T_{0}$ and show it is in $T$. Since $u \in T_{0}$

$$
\infty>\|u\|_{A_{1}}^{p}+\int_{0}^{\infty} t^{\theta p}\left\|\frac{G(t) u-u}{t}\right\|^{p} \frac{d t}{t} \equiv\|u\|_{T_{0}}^{p}
$$

Let $\phi$ be a nonnegative decreasing infinitely differentiable function such that $\phi(0)=1$ and $\phi(t)=0$ for all $t>1$. Then define

$$
f(t) \equiv \phi(t) \frac{1}{t} \int_{0}^{t} G(\tau) u d \tau
$$

It is easy to see that $f(t) \in D(\Lambda)$. In fact, changing variables as needed,

$$
\begin{aligned}
& \frac{1}{h}\left(G(h) \int_{0}^{t} G(\tau) u d \tau-\int_{0}^{t} G(\tau) u d \tau\right) \\
& =\frac{1}{h} \int_{h}^{t+h} G(\tau) u d \tau-\frac{1}{h} \int_{0}^{t} G(\tau) u d \tau \\
& =\frac{1}{h} \int_{t}^{t+h} G(\tau) u d \tau-\frac{1}{h} \int_{0}^{h} G(\tau) u d \tau
\end{aligned}
$$

which converges to $G(t) u-u$ and so

$$
\begin{equation*}
\Lambda \int_{0}^{t} G(\tau) u d \tau=G(t) u-u \tag{45.3.12}
\end{equation*}
$$

Thus

$$
\begin{aligned}
\int_{0}^{\infty} t^{p \theta}\|\Lambda f\|_{A_{1}}^{p} \frac{d t}{t} & \leq \int_{0}^{\infty} t^{p \theta}\left\|\frac{G(t) u-u}{t}\right\|_{A_{1}}^{p} \frac{d t}{t} \\
& \leq\|u\|_{T_{0}}^{p}
\end{aligned}
$$

Next it is necessary to consider

$$
\begin{gathered}
\int_{0}^{\infty} t^{p \theta}\left\|f^{\prime}\right\|_{A_{1}}^{p} \frac{d t}{t} \\
f^{\prime}(t)=\phi^{\prime}(t) \frac{1}{t} \int_{0}^{t} G(\tau) u d \tau+ \\
\phi(t)\left(-\frac{1}{t^{2}} \int_{0}^{t} G(\tau) u d \tau+\frac{1}{t} G(t) u\right) \\
=\phi^{\prime}(t) \frac{1}{t} \int_{0}^{t} G(\tau) u d \tau+\phi(t)\left(\frac{1}{t^{2}} \int_{0}^{t}(G(t) u-G(\tau) u) d \tau\right)
\end{gathered}
$$

and so there is a constant $C$ depending on $\phi$ and the uniform bound on $\|G(t)\|$ such that

$$
\begin{aligned}
\left\|f^{\prime}(t)\right\|_{A_{1}} & \leq C \mathscr{X}_{[0,1]}(t)\left(\|u\|_{A_{1}}+\frac{1}{t^{2}} \int_{0}^{t}\|G(t-\tau) u-u\| d \tau\right) \\
& =C \mathscr{X}_{[0,1]}(t)\left(\|u\|_{A_{1}}+\frac{1}{t^{2}} \int_{0}^{t}\|G(\tau) u-u\| d \tau\right)
\end{aligned}
$$

Now

$$
\int_{0}^{\infty} \mathscr{X}_{[0,1]}(t) t^{p \theta}\|u\|_{A_{1}}^{p} d t / t \leq C(p, \theta)\|u\|_{T_{0}}^{p}
$$

and using Lemma 45.3.2,

$$
\begin{gathered}
\int_{0}^{\infty} \mathscr{X}_{[0,1]}(t)\left|\frac{1}{t^{2}} \int_{0}^{t}\|G(\tau) u-u\| d \tau\right|^{p} t^{p \theta} \frac{d t}{t} \\
\leq \int_{0}^{\infty}\left|\frac{1}{t} \int_{0}^{t}\|G(\tau) u-u\| d \tau\right|^{p} t^{p(\theta-1)} \frac{d t}{t} \\
\leq \frac{1}{(1-(\theta-1))^{p}} \int_{0}^{\infty}\|G(\tau) u-u\|^{p} t^{p(\theta-1)} \frac{d t}{t} \\
=\frac{1}{(2-\theta)^{p}} \int_{0}^{\infty}\left\|\frac{G(\tau) u-u}{t}\right\|^{p} t^{p \theta} \frac{d t}{t} \leq C(\theta, p)\|u\|_{T_{0}}^{p}
\end{gathered}
$$

This proves the theorem.
Of course the case of most interest here is where $A_{1}=L^{p}\left(\mathbb{R}^{n}\right)$ and

$$
G(t) u(\mathbf{x}) \equiv u\left(\mathbf{x}+t \mathbf{e}_{i}\right)
$$

Thus $\Lambda u=\partial u / \partial x_{i}$, the weak derivative. The trace space $T\left(D(\Lambda), L^{p}\left(\mathbb{R}^{n}\right), p, 1-\theta\right)$ then is a space of functions in $L^{p}\left(\mathbb{R}^{n}\right)$ which have a fractional order partial derivative with respect to $x_{i}$.

Recall from Definition 45.1.1 that for $\theta \in(0,1)$,

$$
W^{\theta, p}\left(\mathbb{R}^{n}\right) \equiv T\left(W^{1, p}\left(\mathbb{R}^{n}\right), L^{p}\left(\mathbb{R}^{n}\right), p, 1-\theta\right)
$$

Let $f \in W\left(W^{1, p}\left(\mathbb{R}^{n}\right), L^{p}\left(\mathbb{R}^{n}\right), p, 1-\theta\right)$. Then

$$
\|f\|_{W} \equiv \max \left(\int_{0}^{\infty} t^{(1-\theta) p}\|f(t)\|_{W^{1, p}}^{p} \frac{d t}{t}, \int_{0}^{\infty} t^{(1-\theta) p}\left\|f^{\prime}(t)\right\|_{L^{p}}^{p} \frac{d t}{t}\right)
$$

Letting $G_{i}(t) u(\mathbf{x}) \equiv u\left(\mathbf{x}+t \mathbf{e}_{i}\right)$ and $\Lambda_{i}$ its generator,

$$
W^{1, p}(\Omega)=\cap_{i=1}^{n} D \Lambda_{i} \cap L^{p}\left(\mathbb{R}^{n}\right)
$$

with the norm given by

$$
\|u\|^{p}=\|u\|_{L^{p}}^{p}+\sum_{i=1}^{n}\left\|\Lambda_{i} u\right\|_{L^{p}}^{p}
$$

which is equivalent to the norm

$$
\|u\|^{p}=\sum_{i=1}^{n}\|u\|_{D\left(\Lambda_{i}\right)}^{p} .
$$

Then by considering each of the $G_{i}$ and repeating the above argument in Theorem 45.3.6, it follows an equivalent intrinsic norm is

$$
\|u\|_{W^{\theta, p}\left(\mathbb{R}^{n}\right)}^{p}=\|u\|_{L^{p}\left(\mathbb{R}^{n}\right)}^{p}+\sum_{i=1}^{n} \int_{0}^{\infty} t^{(1-\theta) p}\left\|\frac{G_{i}(t) u-u}{t}\right\|_{L^{p}}^{p} \frac{d t}{t}
$$

$$
\begin{equation*}
=\|u\|_{L^{p}\left(\mathbb{R}^{n}\right)}^{p}+\sum_{i=1}^{n} \int_{0}^{\infty} t^{(1-\theta) p}\left\|\frac{u\left(\cdot+t \mathbf{e}_{i}\right)-u(\cdot)}{t}\right\|_{L^{p}}^{p} \frac{d t}{t} \tag{45.3.13}
\end{equation*}
$$

and $u \in W^{\theta, p}\left(\mathbb{R}^{n}\right)$ when this norm is finite. The only new detail is that in showing that for $u \in T_{0}$ it follows it is in $T$, you use the function

$$
f(t) \equiv \phi(t) \frac{1}{t^{n}} \int_{0}^{t} \cdots \int_{0}^{t} G_{1}\left(\tau_{1}\right) G_{2}\left(\tau_{2}\right) \cdots G_{n}\left(\tau_{n}\right) u d \tau_{1} \cdots d \tau_{n}
$$

and the fact that these semigroups commute. To get this started, note that

$$
g(t) \equiv \int_{0}^{t} \cdots \int_{0}^{t} G_{1}\left(\tau_{1}\right) G_{2}\left(\tau_{2}\right) \cdots G_{n}\left(\tau_{n}\right) u d \tau_{1} \cdots d \tau_{n} \in D\left(\Lambda_{i}\right)
$$

for each $i$. This follows from writing it as

$$
\int_{0}^{t} G_{i}\left(\tau_{i}\right)\left(w_{i}\right) d \tau_{i}
$$

for $w_{i} \in L^{p}$ coming from the other integrals and then repeating the earlier argument to get

$$
\Lambda_{i} g(t)=G_{i}(t) w_{i}-w_{i}
$$

and then

$$
\begin{gathered}
\int_{0}^{\infty} t^{p(1-\theta)}\left\|\Lambda_{i} f\right\|_{L^{p}}^{p} \frac{d t}{t} \\
\leq \int_{0}^{\infty} t^{p(1-\theta)}\left\|\frac{G_{i}(t) w_{i}-w_{i}}{t}\right\|_{L^{p}}^{p} \frac{d t}{t} \\
\leq C \int_{0}^{\infty} t^{p(1-\theta)}\left\|\frac{G_{i}(t) u-u}{t}\right\|_{L^{p}}^{p} \frac{d t}{t} \leq C\|u\|_{T_{0}}^{p}
\end{gathered}
$$

Thus all is well as far as $f$ is concerned and the proof will work as it did earlier in Theorem 45.3.6. What about $f^{\prime}$ ? As before, the only term which is problematic is

$$
\phi(t)\left(\frac{1}{t^{n}} \int_{0}^{t} \cdots \int_{0}^{t} G_{1}\left(\tau_{1}\right) G_{2}\left(\tau_{2}\right) \cdots G_{n}\left(\tau_{n}\right) u d \tau_{1} \cdots d \tau_{n}\right)^{\prime}
$$

After enough massaging, it becomes

$$
\sum_{i=1}^{n} \prod_{j \neq i} \frac{1}{t} \int_{0}^{t} G_{j}\left(\tau_{j}\right) d \tau_{j} \frac{1}{t^{2}} \int_{0}^{t}\left(G_{i}(t) u-G_{i}\left(\tau_{i}\right) u\right) d \tau_{i}
$$

where the operator $\sum_{i=1}^{n} \prod_{j \neq i} \frac{1}{t} \int_{0}^{t} G_{j}\left(\tau_{j}\right) d \tau_{j}$ is bounded. Thus similar arguments to those of Theorem 45.3 .6 will work, the only difference being a sum.

Theorem 45.3.7 An equivalent norm for $W^{\theta, p}\left(\mathbb{R}^{n}\right)$ for $\theta \in(0,1)$ is

$$
\begin{gather*}
\|u\|_{W^{\theta, p}\left(\mathbb{R}^{n}\right)}^{p}= \\
\|u\|_{L^{p}\left(\mathbb{R}^{n}\right)}^{p}+\sum_{i=1}^{n} \int_{0}^{\infty} t^{(1-\theta) p}\left\|\frac{G_{i}(t) u-u}{t}\right\|_{L^{p}}^{p} \frac{d t}{t} \\
=\|u\|_{L^{p}\left(\mathbb{R}^{n}\right)}^{p}+\sum_{i=1}^{n} \int_{0}^{\infty} t^{(1-\theta) p}\left\|\frac{u\left(\cdot+t \mathbf{e}_{i}\right)-u(\cdot)}{t}\right\|_{L^{p}}^{p} \frac{d t}{t} \tag{45.3.14}
\end{gather*}
$$

Note it is obvious from 45.3 .13 that a Lipschitz map takes $W^{\theta, p}\left(\mathbb{R}^{n}\right)$ to $W^{\theta, p}\left(\mathbb{R}^{n}\right)$ and is continuous.

The above description in Theorem 45.3.7 also makes possible the following corollary.
Corollary 45.3.8 $W^{\theta, p}\left(\mathbb{R}^{n}\right)$ is reflexive.
Proof: Let $u \in W^{\theta, p}\left(\mathbb{R}^{n}\right)$. For each $i=1,2, \cdots, n$, define for $t>0$,

$$
\Delta_{i} u(t)(\mathbf{x}) \equiv \frac{u\left(\mathbf{x}+t \mathbf{e}_{i}\right)-u(\mathbf{x})}{t}
$$

Then by Theorem 45.3.7,

$$
\Delta_{i} u \in L^{p}\left((0, \infty) ; L^{p}\left(\mathbb{R}^{n}\right), \mu\right) \equiv Y
$$

where

$$
\mu(E) \equiv \int_{E} t^{(1-\theta) p} t^{-1} d t
$$

Clearly the measure space is $\sigma$ finite and so $Y$ is reflexive by Corollary 21.8.9 on Page 687. Also $\Delta_{i}$ is a closed operator whose domain is $W^{\theta, p}\left(\mathbb{R}^{n}\right)$. To see this, suppose $u_{n} \in W^{\theta, p}\left(\mathbb{R}^{n}\right)$ and $u_{n} \rightarrow u$ in $L^{p}\left(\mathbb{R}^{n}\right)$ while $\Delta_{i} u_{n} \rightarrow g$ in $Y$. Then in particular $\left\|\Delta_{i} u_{n}\right\|_{Y}$ is bounded. Now by Fatou's lemma,

$$
\begin{gathered}
\int_{0}^{\infty} t^{(1-\theta) p}\left\|\frac{u\left(\cdot+t \mathbf{e}_{i}\right)-u(\cdot)}{t}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}^{p} \frac{d t}{t} \leq \\
\lim \inf _{n \rightarrow \infty} \int_{0}^{\infty} t^{(1-\theta) p}\left\|\frac{u_{n}\left(\cdot+t \mathbf{e}_{i}\right)-u_{n}(\cdot)}{t}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}^{p} \frac{d t}{t}<\infty .
\end{gathered}
$$

Letting $\vec{\Delta} \equiv\left(\Delta_{1}, \Delta_{2}, \cdots, \Delta_{n}\right)$, it follows from similar reasoning that $\vec{\Delta}$ is a closed operator mapping $W^{\theta, p}\left(\mathbb{R}^{n}\right)$ to $Y^{n}$. Therefore

$$
(\mathrm{id}, \vec{\Delta})\left(W^{\theta, p}\left(\mathbb{R}^{n}\right)\right) \subseteq L^{p}\left(\mathbb{R}^{n}\right) \times Y^{n}
$$

and is a closed subspace of the reflexive space $L^{p}\left(\mathbb{R}^{n}\right) \times Y^{n}$. With the norm in $L^{p}\left(\mathbb{R}^{n}\right) \times Y^{n}$ given as the sum of the norms of the components, it follows the mapping $(\mathrm{id}, \vec{\Delta})$ is a
norm preserving isomorphism between $W^{\theta, p}\left(\mathbb{R}^{n}\right)$ and this closed subspace of $L^{p}\left(\mathbb{R}^{n}\right) \times Y^{n}$. Since $L^{p}\left(\mathbb{R}^{n}\right)$ and $Y$ is reflexive, their product is reflexive. By Lemma 21.2.7 on Page 655 it follows $(\mathrm{id}, \vec{\Delta})\left(W^{\theta, p}\left(\mathbb{R}^{n}\right)\right)$ and hence $W^{\theta, p}\left(\mathbb{R}^{n}\right)$ is reflexive. This proves the theorem.

One can generalize this to find an intrinsic norm for $W^{\theta, p}(\Omega)$. The version given above will not do because it requires the function to be defined on all of $\mathbb{R}^{n}$ in order to make sense of the shift operators $G_{i}$. However, you can give a different version of this intrinsic norm which will make sense for $\Omega \neq \mathbb{R}^{n}$.

Lemma 45.3.9 Let $t \neq 0$ be a number. Then there is a constant $C(n, \theta, p)$ depending on the indicated quantities such that

$$
\int_{\mathbb{R}^{n-1}} \frac{1}{\left(t^{2}+|\mathbf{s}|^{2}\right)^{\frac{1}{2}(n+p \theta)}} d s=\frac{C(n, \theta, p)}{|t|^{1+p \theta}}
$$

Proof: Change the integral to polar coordinates. Thus the integral equals

$$
\int_{S^{n-1}} \int_{0}^{\infty} \frac{\rho^{n-2}}{\left(t^{2}+\rho^{2}\right)^{\frac{1}{2}(n+p \theta)}} d \rho d \sigma
$$

Now change the variables, $\rho=|t| u$. Then the above integral becomes

$$
\begin{gathered}
C_{n} \int_{0}^{\infty} \frac{|t|^{n-2} u^{n-2}|t|}{|t|^{n+p \theta}\left(1+u^{2}\right)^{\frac{1}{2}(n+p \theta)}} d u \\
=C_{n} \frac{1}{|t|^{1+p \theta}} \int_{0}^{\infty} \frac{u^{n-2}}{\left(1+u^{2}\right)^{\frac{1}{2}(n+p \theta)}} d u \equiv \frac{C(n, \theta, p)}{|t|^{1+p \theta}} .
\end{gathered}
$$

This proves the lemma.
Now let $u \in W^{\theta, p}\left(\mathbb{R}^{n}\right)$. This means the norm of $\|u\|_{W^{\theta, p}}^{p}$ can be taken as

$$
\begin{aligned}
& \|u\|_{L^{p}}^{p}+\sum_{i=1}^{n} \int_{0}^{\infty}|t|^{(1-\theta) p} \int_{\mathbb{R}^{n}}\left|\frac{u\left(\mathbf{x}+t \mathbf{e}_{i}\right)-u(\mathbf{x})}{t}\right|^{p} d x \frac{d t}{|t|} \\
= & \|u\|_{L^{p}}^{p}+\frac{1}{2} \sum_{i=1}^{n} \int_{-\infty}^{\infty}|t|^{(1-\theta) p} \int_{\mathbb{R}^{n}}\left|\frac{u\left(\mathbf{x}+t \mathbf{e}_{i}\right)-u(\mathbf{x})}{t}\right|^{p} d x \frac{d t}{|t|}
\end{aligned}
$$

That integral over $\mathbb{R}^{n}$ can be massaged and one obtains the above equal to

$$
\begin{gathered}
\|u\|_{L^{p}}^{p}+ \\
\left.\frac{1}{2} \sum_{i=1}^{n} \int_{-\infty}^{\infty} \int_{\mathbb{R}^{n-i}} \int_{\mathbb{R}^{i}} \frac{1}{t^{1+p \theta}} \right\rvert\, u\left(x_{1}, \cdots, x_{i}+t, y_{i+1}, \cdots, y_{n}\right) \\
-\left.u\left(x_{1}, \cdots, x_{i}, y_{i+1}, \cdots, y_{n}\right)\right|^{p} d x_{i} d y_{n-i} d t
\end{gathered}
$$

where $d x_{i}$ refers to the first $i$ entries and $d y_{n-i}$ refers to the remaining entries. From Lemma 45.3.9, the complicated expression above equals

$$
\begin{aligned}
& \frac{1}{C(n, \theta, p)} \frac{1}{2} \sum_{i=1}^{n} \int_{-\infty}^{\infty} \int_{\mathbb{R}^{n-i}} \int_{\mathbb{R}^{i}} \int_{\mathbb{R}^{n-1}} \frac{1}{\left(t^{2}+|\mathbf{s}|^{2}\right)^{\frac{1}{2}(n+p \theta)}} \\
& \left|u\left(x_{1}, \cdots, x_{i-1}, x_{i}+t, y_{i+1}, \cdots, y_{n}\right)-u\left(x_{1}, \cdots, x_{i}, y_{i+1}, \cdots, y_{n}\right)\right|^{p} \\
& d s d x_{i} d y_{n-i} d t
\end{aligned}
$$

Now Fubini this to get

$$
\begin{aligned}
& \frac{1}{C(n, \theta, p)} \frac{1}{2} \sum_{i=1}^{n} \int_{\mathbb{R}^{n-i}} \int_{\mathbb{R}^{i}} \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} \frac{1}{\left(t^{2}+|\mathbf{s}|^{2}\right)^{\frac{1}{2}(n+p \theta)}} \\
& \left|u\left(x_{1}, \cdots, x_{i-1}, x_{i}+t, y_{i+1}, \cdots, y_{n}\right)-u\left(x_{1}, \cdots, x_{i}, y_{i+1}, \cdots, y_{n}\right)\right|^{p} \\
& d t d s d x_{i} d y_{n-i}
\end{aligned}
$$

Changing the variable in the inside integral to $t=y_{i}-x_{i}$, this equals

$$
\begin{aligned}
& \frac{1}{C(n, \theta, p)} \frac{1}{2} \sum_{i=1}^{n} \int_{\mathbb{R}^{n-i}} \int_{\mathbb{R}^{i}} \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} \frac{1}{\left(\left(y_{i}-x_{i}\right)^{2}+|\mathbf{s}|^{2}\right)^{\frac{1}{2}(n+p \theta)}} \\
& \left|u\left(x_{1}, \cdots, x_{i-1}, y_{i}, y_{i+1}, \cdots, y_{n}\right)-u\left(x_{1}, \cdots, x_{i}, y_{i+1}, \cdots, y_{n}\right)\right|^{p} \\
& d y_{i} d s d x_{i} d y_{n-i}
\end{aligned}
$$

Next let

$$
\begin{aligned}
& \left(s_{1}, \cdots, s_{n-1}\right) \\
\equiv & \left(y_{1}-x_{1}, \cdots, y_{i-1}-x_{i-1}, x_{i+1}-y_{i+1}, \cdots, x_{n}-y_{n}\right)
\end{aligned}
$$

where the new variables of integration in the integral corresponding to $d s$ are $y_{1}, \cdots, y_{i-1}$ and $x_{i+1}, \cdots, x_{n}$. Then changing the variables, the above reduces to

$$
\begin{aligned}
& \frac{1}{C(n, \theta, p)} \frac{1}{2} \sum_{i=1}^{n} \int_{\mathbb{R}^{n-i}} \int_{\mathbb{R}^{i}} \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} \frac{1}{|\mathbf{x}-\mathbf{y}|^{(n+p \theta)}} \\
& \left|u\left(x_{1}, \cdots, x_{i-1}, y_{i}, y_{i+1}, \cdots, y_{n}\right)-u\left(x_{1}, \cdots, x_{i}, y_{i+1}, \cdots, y_{n}\right)\right|^{p} \\
& d y_{i} d y_{1} \cdots d y_{i-1} d x_{i+1} \cdots d x_{n} d x_{1} \cdots d x_{i} d y_{i+1} \cdots d y_{n}
\end{aligned}
$$

Then if you Fubini again, it reduces to the expression

$$
\begin{gathered}
\frac{1}{C(n, \theta, p)} \frac{1}{2} \sum_{i=1}^{n} \\
\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \frac{\left|u\left(x_{1}, \cdots, x_{i-1}, y_{i}, y_{i+1}, \cdots, y_{n}\right)-u\left(x_{1}, \cdots, x_{i}, y_{i+1}, \cdots, y_{n}\right)\right|^{p}}{|\mathbf{x}-\mathbf{y}|^{(n+p \theta)}} d x d y
\end{gathered}
$$

Now taking the sum inside and adjusting the constants yields

$$
\geq C(n, \theta, p) \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \frac{|u(\mathbf{y})-u(\mathbf{x})|^{p}}{|\mathbf{x}-\mathbf{y}|^{(n+p \theta)}} d x d y
$$

Thus there exists a constant $C(n, \theta, p)$ such that

$$
\|u\|_{W^{\theta, p}\left(\mathbb{R}^{n}\right)} \geq C(n, \theta, p)\left(\|u\|_{L^{p}}^{p}+\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \frac{|u(\mathbf{y})-u(\mathbf{x})|^{p}}{|\mathbf{x}-\mathbf{y}|^{(n+p \theta)}} d x d y\right)^{1 / p}
$$

Next start with the right side of the above. It suffices to consider only the complicated term. First note that for a vector,

$$
\left(\sum_{i=1}^{n} a_{i}^{2}\right)^{p / 2} \geq\left|a_{i}\right|^{p}
$$

and so

$$
\sum_{i=1}^{n}\left|a_{i}\right|^{p} \leq n\left(\sum_{i=1}^{n} a_{i}^{2}\right)^{p / 2}=n|\mathbf{a}|^{p}
$$

from which it follows

$$
\frac{1}{n} \sum_{i=1}^{n}\left|a_{i}\right|^{p} \leq|\mathbf{a}|^{p}
$$

Then it follows

$$
\begin{gather*}
\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \frac{|u(\mathbf{y})-u(\mathbf{x})|^{p}}{|\mathbf{x}-\mathbf{y}|^{(n+p \theta)}} d x d y \geq \frac{1}{n} \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \\
\frac{\sum_{i=1}^{n}\left|u\left(x_{1}, \cdots, x_{i-1}, y_{i}, y_{i+1}, \cdots, y_{n}\right)-u\left(x_{1}, \cdots, x_{i}, y_{i+1}, \cdots, y_{n}\right)\right|^{p}}{|\mathbf{x}-\mathbf{y}|^{(n+p \theta)}} d x d y \tag{45.3.15}
\end{gather*}
$$

Consider the $i^{t h}$ term. By Fubini's theorem it equals

$$
\begin{aligned}
& \frac{1}{n} \int \cdots \int \frac{1}{\left(\left(y_{i}-x_{i}\right)^{2}+\sum_{j \neq i}\left(y_{j}-x_{j}\right)^{2}\right)^{\frac{1}{2}(n+p \theta)}} \\
& \left|u\left(x_{1}, \cdots, x_{i-1}, y_{i}, y_{i+1}, \cdots, y_{n}\right)-u\left(x_{1}, \cdots, x_{i}, y_{i+1}, \cdots, y_{n}\right)\right|^{p} \\
& d y_{i} d x_{1} \cdots d x_{i-1} d y_{i+1} \cdots d y_{n} d y_{1} \cdots d y_{i-1} d x_{i} \cdots d x_{n}
\end{aligned}
$$

Let $t=y_{i}-x_{i}$. Then it reduces to

$$
\begin{aligned}
& \frac{1}{n} \int \cdots \int \frac{1}{\left(t^{2}+\sum_{j \neq i}\left(y_{j}-x_{j}\right)^{2}\right)^{\frac{1}{2}(n+p \theta)}} \\
& \left|u\left(x_{1}, \cdots, x_{i-1}, x_{i}+t, y_{i+1}, \cdots, y_{n}\right)-u\left(x_{1}, \cdots, x_{i}, y_{i+1}, \cdots, y_{n}\right)\right|^{p} \\
& d t d x_{1} \cdots d x_{i-1} d y_{i+1} \cdots d y_{n} d y_{1} \cdots d y_{i-1} d x_{i} \cdots d x_{n}
\end{aligned}
$$

Now let

$$
\begin{gathered}
\left(s_{1}, \cdots, s_{n-1}\right)= \\
\left(x_{1}-y_{1}, \cdots, x_{i-1}-y_{i-1}, y_{i+1}-x_{i+1}, \cdots, y_{n}-x_{n}\right)
\end{gathered}
$$

on the next $n-1$ iterated integrals. Then using Fubini's theorem again and changing the variables, it equals

$$
\begin{aligned}
& \frac{1}{n} \int \cdots \int \frac{1}{\left(t^{2}+|\mathbf{s}|^{2}\right)^{\frac{1}{2}(n+p \theta)}} \\
& \mid u\left(s_{1}+y_{1}, \cdots, y_{i-1}+s_{i-1}, x_{i}+t, x_{i+1}+s_{i}, \cdots, x_{n}+s_{n-1}\right) \\
& -\left.u\left(s_{1}+y_{1}, \cdots, y_{i-1}+s_{i-1}, x_{i}, x_{i+1}+s_{i}, \cdots, x_{n}+s_{n-1}\right)\right|^{p} \\
& d y_{1} \cdots d y_{i-1} d x_{i} \cdots d x_{n} d s_{1} \cdots d s_{i-1} d s_{i} \cdots d s_{n-1} d t
\end{aligned}
$$

By translation invariance of the measure, the inside integrals corresponding to

$$
d y_{1} \cdots d y_{i-1} d x_{i} \cdots d x_{n}
$$

simplify and the expression can be written as

$$
\begin{aligned}
& \frac{1}{n} \int \cdots \int \frac{1}{\left(t^{2}+|\mathbf{s}|^{2}\right)^{\frac{1}{2}(n+p \theta)}} \\
& \left|u\left(x_{1}, \cdots, x_{i-1}, x_{i}+t, x_{i+1}, \cdots, x_{n}\right)-u\left(x_{1}, \cdots, x_{i-1}, x_{i}, x_{i+1}, \cdots, x_{n}\right)\right|^{p} \\
& d x_{1} \cdots d x_{i-1} d x_{i} \cdots d x_{n} d s_{1} \cdots d s_{i-1} d s_{i} \cdots d s_{n-1} d t
\end{aligned}
$$

where I just renamed the variables. Use Fubini's theorem again to get

$$
\begin{aligned}
& \frac{1}{n} \int \cdots \int \frac{1}{\left(t^{2}+|\mathbf{s}|^{2}\right)^{\frac{1}{2}(n+p \theta)}} \\
& \left|u\left(x_{1}, \cdots, x_{i-1}, x_{i}+t, x_{i+1}, \cdots, x_{n}\right)-u\left(x_{1}, \cdots, x_{i-1}, x_{i}, x_{i+1}, \cdots, x_{n}\right)\right|^{p} \\
& d s_{1} \cdots d s_{i-1} d s_{i} \cdots d s_{n-1} d x_{1} \cdots d x_{i-1} d x_{i} \cdots d x_{n} d t
\end{aligned}
$$

Now from Lemma 45.3.9, the inside $n-1$ integrals corresponding to

$$
d s_{1} \cdots d s_{i-1} d s_{i} \cdots d s_{n-1}
$$

can be replaced with

$$
\frac{C(n, \theta, p)}{|t|^{1+p \theta}}
$$

and this yields

$$
\begin{aligned}
& C(n, \theta, p) \int_{\mathbb{R}} \frac{1}{|t|^{p \theta}} \int_{\mathbb{R}^{n}}\left|u\left(\mathbf{x}+t \mathbf{e}_{i}\right)-u(\mathbf{x})\right|^{p} d x \frac{d t}{t} \\
= & \frac{1}{2} C(n, \theta, p) \int_{0}^{\infty} t^{p(1-\theta)} \frac{\| u\left(\cdot+t \mathbf{e}_{i}\right)-\left.u(\cdot)\right|_{L^{p}\left(\mathbb{R}^{n}\right)} ^{p}}{t^{p}} \frac{d t}{t}
\end{aligned}
$$

Applying this to each term of the sum in 45.3.15 and adjusting the constant, it follows

$$
\begin{gathered}
\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \frac{|u(\mathbf{y})-u(\mathbf{x})|^{p}}{|\mathbf{x}-\mathbf{y}|^{(n+p \theta)}} d x d y \geq \\
C(n, \theta, p) \sum_{i=1}^{n} \int_{0}^{\infty} t^{p(1-\theta)} \frac{\| u\left(\cdot+t \mathbf{e}_{i}\right)-\left.u(\cdot)\right|_{L^{p}\left(\mathbb{R}^{n}\right)} ^{p}}{t^{p}} \frac{d t}{t}
\end{gathered}
$$

Therefore,

$$
\begin{aligned}
& \|u\|_{W^{\theta, p}\left(\mathbb{R}^{n}\right)} \\
\geq & C(n, \theta, p)\left(\|u\|_{L^{p}\left(\mathbb{R}^{n}\right)}^{p}+\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \frac{|u(\mathbf{y})-u(\mathbf{x})|^{p}}{|\mathbf{x}-\mathbf{y}|^{(n+p \theta)}} d x d y\right)^{1 / p}
\end{aligned}
$$

This has proved most of the following theorem about the intrinsic norm.
Theorem 45.3.10 An equivalent norm for $W^{\theta, p}\left(\mathbb{R}^{n}\right)$ is

$$
\begin{gathered}
\|u\|= \\
\left(\|u\|_{L^{p}\left(\mathbb{R}^{n}\right)}^{p}+\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \frac{|u(\mathbf{y})-u(\mathbf{x})|^{p}}{|\mathbf{x}-\mathbf{y}|^{(n+p \theta)}} d x d y\right)^{1 / p}
\end{gathered}
$$

Also for any open subset of $\mathbb{R}^{n}$

$$
\|u\|=
$$

$$
\begin{equation*}
\left(\|u\|_{L^{p}(\Omega)}^{p}+\int_{\Omega} \int_{\Omega} \frac{|u(\mathbf{y})-u(\mathbf{x})|^{p}}{|\mathbf{x}-\mathbf{y}|^{(n+p \theta)}} d x d y\right)^{1 / p} \tag{45.3.16}
\end{equation*}
$$

is a norm.
Proof: It only remains to verify this is a norm. Recall the $l_{p}$ norm on $\mathbb{R}^{2}$ given by

$$
|(x, y)|_{l_{p}} \equiv\left(|x|^{p}+|y|^{p}\right)^{1 / p}
$$

For $u, v \in W^{\theta, p}$ denote by $\rho(u)$ the expression

$$
\left(\int_{\Omega} \int_{\Omega} \frac{|u(\mathbf{y})-u(\mathbf{x})|^{p}}{|\mathbf{x}-\mathbf{y}|^{(n+p \theta)}} d x d y\right)^{1 / p}
$$

a similar definition holding for $v$. Then it follows from the usual Minkowski inequality that $\rho(u+v) \leq \rho(u)+\rho(v)$. Then from 45.3.16

$$
\begin{gathered}
\|u+v\|=\left(\|u+v\|_{L^{p}}^{p}+\rho(u+v)^{p}\right)^{1 / p} \\
\leq\left(\left(\|u\|_{L^{p}}+\|v\|_{L^{p}}\right)^{p}+(\rho(u)+\rho(v))^{p}\right)^{1 / p}
\end{gathered}
$$

$$
\begin{gathered}
=\left|\left(\|u\|_{L^{p}}, \rho(u)\right)+\left(\|v\|_{L^{p}}, \rho(v)\right)\right|_{l_{p}} \\
\leq\left|\left(\|u\|_{L^{p}}, \rho(u)\right)\right|_{l_{p}}+\left|\left(\|v\|_{L^{p}}, \rho(v)\right)\right|_{l_{p}} \\
=\left(\|u\|_{L^{p}}^{p}+\rho(u)^{p}\right)^{1 / p}+\left(\|v\|_{L^{p}}^{p}+\rho(v)^{p}\right)^{1 / p} \\
=\|u\|+\|v\|
\end{gathered}
$$

The other properties of a norm are obvious. This proves the theorem.
As pointed out in the above theorem, this is a norm in 45.3.16. One could define a set of functions for which this norm is finite. In the case where $\Omega=\mathbb{R}^{n}$ the conclusion of Theorem 45.3.10 is that this space of functions is the same as $W^{\theta, p}\left(\mathbb{R}^{n}\right)$ and the norms are equivalent. Does this happen for other open subsets of $\mathbb{R}^{n}$ ?
Definition 45.3.11 Denote by $\widetilde{W^{\theta, p}(U)}$ the functions in $L^{p}(U)$ for which the norm of Theorem 45.3.10 is finite. Here $\theta \in(0,1)$.

Proposition 45.3.12 Let $U$ be a bounded open set which has Lipschitz boundary and $\theta \in$ $(0,1)$. Then for each $p \geq 1$, there exists $E \in \mathscr{L}\left(\widehat{W^{\theta, p}(U)}, W^{\theta, p}\left(\mathbb{R}^{n}\right)\right)$ such that $E u(\mathbf{x})=$ $u(\mathbf{x})$ a.e. $\mathbf{x} \in U$.

Proof: In proving this, I will use the equivalent norm of Theorem 45.3.10 as the norm of $W^{\theta, p}\left(\mathbb{R}^{n}\right)$ Consider the following picture.


The wavy line signifies a part of the boundary of $U$ and $\operatorname{spt}(u)$ is contained in the circle as shown. It is drawn as a circle but this is not important. Denote by $U^{+}$the region above the part of the boundary which is shown. Also let the boundary be given by $x_{n}=g(\widehat{\mathbf{x}})$ for $\widehat{\mathbf{x}} \in B \equiv B\left(\widehat{\mathbf{y}}_{0}, r\right) \subseteq \mathbb{R}^{n-1}$. Of course $u$ is only defined on $U$ so actually the support of $u$ is contained in the intersection of the circle with $\bar{U}$. Let the Lipschitz constant for $g$ be very small and denote it by $K$. In fact, assume $8 K^{2}<1$. I will first show how to extend when this condition holds and then I will remove it with a simple trick. Define

$$
E u\left(\widehat{\mathbf{x}}, x_{n}\right) \equiv\left\{\begin{array}{l}
u\left(\widehat{\mathbf{x}}, x_{n}\right) \text { if } x_{n} \leq g(\widehat{\mathbf{x}}) \\
u\left(\widehat{\mathbf{x}}, 2 g(\widehat{\mathbf{x}})-x_{n}\right) \text { if } x_{n}>g(\widehat{\mathbf{x}}) \\
0 \text { if } \widehat{\mathbf{x}} \notin B
\end{array}\right.
$$

I will write $U$ instead of $U \cap B \times(a, b)$ to save space but this does not matter because $u$ is assumed to be zero outside the indicated region. Then

$$
\begin{align*}
& \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \frac{\left|E u\left(\widehat{\mathbf{x}}, x_{n}\right)-E u\left(\widehat{\mathbf{y}}, y_{n}\right)\right|^{p}}{| | \widehat{\mathbf{x}}-\left.\widehat{\mathbf{y}}\right|^{2}+\left.\left(x_{n}-y_{n}\right)^{2}\right|^{(1 / 2)(n+p \theta)}} d x d y \\
& =\int_{U} \int_{U} \frac{\left|u\left(\widehat{\mathbf{x}}, x_{n}\right)-u\left(\widehat{\mathbf{y}}, y_{n}\right)\right|^{p}}{| | \widehat{\mathbf{x}}-\left.\widehat{\mathbf{y}}\right|^{2}+\left.\left(x_{n}-y_{n}\right)^{2}\right|^{(1 / 2)(n+p \theta)}} d x d y+ \\
& \int_{U^{+}} \int_{U} \frac{\left|u\left(\widehat{\mathbf{x}}, x_{n}\right)-E u\left(\widehat{\mathbf{y}}, y_{n}\right)\right|^{p}}{| | \widehat{\mathbf{x}}-\left.\widehat{\mathbf{y}}\right|^{2}+\left.\left(x_{n}-y_{n}\right)^{2}\right|^{(1 / 2)(n+p \theta)}} d x d y+  \tag{45.3.17}\\
& \int_{U} \int_{U^{+}} \frac{\left|E u\left(\widehat{\mathbf{x}}, x_{n}\right)-u\left(\widehat{\mathbf{y}}, y_{n}\right)\right|^{p}}{| | \widehat{\mathbf{x}}-\left.\widehat{\mathbf{y}}\right|^{2}+\left.\left(x_{n}-y_{n}\right)^{2}\right|^{(1 / 2)(n+p \theta)}} d x d y+  \tag{45.3.18}\\
& \int_{U^{+}} \int_{U^{+}} \frac{\left|E u\left(\widehat{\mathbf{x}}, x_{n}\right)-E u\left(\widehat{\mathbf{y}}, y_{n}\right)\right|^{p}}{| | \widehat{\mathbf{x}}-\left.\widehat{\mathbf{y}}\right|^{2}+\left.\left(x_{n}-y_{n}\right)^{2}\right|^{(1 / 2)(n+p \theta)}} d x d y \tag{45.3.19}
\end{align*}
$$

Consider the second of the integrals on the right of the equal sign. Using Fubini's theorem, it equals

$$
\begin{aligned}
& \int_{U} \int_{U^{+}} \frac{\left|u\left(\widehat{\mathbf{x}}, x_{n}\right)-u\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-y_{n}\right)\right|^{p}}{|\widehat{\mathbf{x}}-\widehat{\mathbf{y}}|^{2}+\left.\left(x_{n}-y_{n}\right)^{2}\right|^{(1 / 2)(n+p \theta)}} d y d x \\
= & \int_{U} \int_{B} \int_{g(\widehat{\mathbf{y}})}^{\infty} \frac{\left|u\left(\widehat{\mathbf{x}}, x_{n}\right)-u\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-y_{n}\right)\right|^{p}}{| | \widehat{\mathbf{x}}-\left.\widehat{\mathbf{y}}\right|^{2}+\left.\left(x_{n}-y_{n}\right)^{2}\right|^{(1 / 2)(n+p \theta)}} d y_{n} d \widehat{y} d x \\
= & \int_{U} \int_{B} \int_{-\infty}^{g(\widehat{\mathbf{y}})} \frac{\left|u\left(\widehat{\mathbf{x}}, x_{n}\right)-u\left(\widehat{\mathbf{y}}, z_{n}\right)\right|^{p}}{| | \widehat{\mathbf{x}}-\left.\widehat{\mathbf{y}}\right|^{2}+\left.\left(x_{n}-\left(2 g(\widehat{\mathbf{y}})-z_{n}\right)\right)^{2}\right|^{(1 / 2)(n+p \theta)}} d z_{n} d \widehat{y} d x
\end{aligned}
$$

I need to estimate $\left|x_{n}-z_{n}\right|$.

$$
\begin{aligned}
\left|x_{n}-z_{n}\right| & \leq\left|x_{n}-g(\widehat{\mathbf{x}})\right|+|g(\widehat{\mathbf{x}})-g(\widehat{\mathbf{y}})|+\left|g(\widehat{\mathbf{y}})-z_{n}\right| \\
& \leq g(\widehat{\mathbf{x}})-x_{n}+K|\widehat{\mathbf{x}}-\widehat{\mathbf{y}}|+y_{n}-g(\widehat{\mathbf{y}}) \\
& \leq\left|y_{n}-x_{n}\right|+2 K|\widehat{\mathbf{x}}-\widehat{\mathbf{y}}|
\end{aligned}
$$

and so

$$
\begin{aligned}
\left|x_{n}-z_{n}\right|^{2} & \leq 8 K^{2}|\widehat{\mathbf{x}}-\widehat{\mathbf{y}}|^{2}+2\left|y_{n}-x_{n}\right|^{2} \\
& \leq|\widehat{\mathbf{x}}-\widehat{\mathbf{y}}|^{2}+2\left|y_{n}-x_{n}\right|^{2}
\end{aligned}
$$

Thus,

$$
\left(y_{n}-x_{n}\right)^{2} \geq \frac{1}{2}\left|x_{n}-z_{n}\right|^{2}-\frac{1}{2}|\widehat{\mathbf{x}}-\widehat{\mathbf{y}}|^{2}
$$

and so, the above change of variables results in an expression which is dominated by

$$
\int_{U} \int_{U} \frac{\left|u\left(\widehat{\mathbf{x}}, x_{n}\right)-u\left(\widehat{\mathbf{y}}, z_{n}\right)\right|^{p}}{\left|\frac{1}{2}\right| \widehat{\mathbf{x}}-\left.\widehat{\mathbf{y}}\right|^{2}+\left.\frac{1}{2}\left(x_{n}-z_{n}\right)^{2}\right|^{(1 / 2)(n+p \theta)}} d y d x
$$

where $y$ refers to $\left(\widehat{\mathbf{y}}, z_{n}\right)$ in the above formula. Hence there is a constant $C(n . \theta)$ such that 45.3.17 is dominated by $C(n . \theta)\|u\|_{W^{p, p}(U)}^{p}$. A similar inequality holds for the third term. Finally consider 45.3 .19 .This equals

$$
\int_{U^{+}} \int_{U^{+}} \frac{\left|u\left(\widehat{\mathbf{x}}, 2 g(\widehat{\mathbf{x}})-x_{n}\right)-u\left(\widehat{\mathbf{y}}, 2 g(\widehat{\mathbf{y}})-y_{n}\right)\right|^{p}}{| | \widehat{\mathbf{x}}-\left.\widehat{\mathbf{y}}\right|^{2}+\left.\left(x_{n}-y_{n}\right)^{2}\right|^{(1 / 2)(n+p \theta)}} d x d y
$$

Changing variables, $x_{n}^{\prime}=2 g(\widehat{\mathbf{x}})-x_{n}, y_{n}^{\prime}=2 g(\widehat{\mathbf{y}})-y_{n}$, it equals

$$
\begin{equation*}
\int_{U^{+}} \int_{U^{+}} \frac{\left|u\left(\widehat{\mathbf{x}}, x_{n}^{\prime}\right)-u\left(\widehat{\mathbf{y}}, y_{n}^{\prime}\right)\right|^{p}}{| | \widehat{\mathbf{x}}-\left.\widehat{\mathbf{y}}\right|^{2}+\left.\left(x_{n}-y_{n}\right)^{2}\right|^{(1 / 2)(n+p \theta)}} d x^{\prime} d y^{\prime} \tag{45.3.20}
\end{equation*}
$$

each of $x_{n}, y_{n}$ being a function of $x_{n}^{\prime}, y_{n}^{\prime}$ where an estimate needs to be obtained on $\left|x_{n}^{\prime}-y_{n}^{\prime}\right|$ in terms of $\left|x_{n}-y_{n}\right|$.

$$
\begin{aligned}
&\left(x_{n}^{\prime}-y_{n}^{\prime}\right)^{2}=\left(2(g(\widehat{\mathbf{x}})-g(\widehat{\mathbf{y}}))+y_{n}-x_{n}\right)^{2} \\
&=\left(y_{n}-x_{n}\right)^{2}+4(g(\widehat{\mathbf{x}})-g(\widehat{\mathbf{y}}))\left(y_{n}-x_{n}\right) \\
&+4(g(\widehat{\mathbf{x}})-g(\widehat{\mathbf{y}}))^{2} \\
& \leq\left(y_{n}-x_{n}\right)^{2}+2(g(\widehat{\mathbf{x}})-g(\widehat{\mathbf{y}}))^{2} \\
&+2\left(y_{n}-x_{n}\right)^{2}+4(g(\widehat{\mathbf{x}})-g(\widehat{\mathbf{y}}))^{2}
\end{aligned}
$$

and so

$$
\left(x_{n}^{\prime}-y_{n}^{\prime}\right)^{2} \leq 3\left(y_{n}-x_{n}\right)^{2}+6 K^{2}|\widehat{\mathbf{x}}-\widehat{\mathbf{y}}|^{2}
$$

which implies

$$
\left(y_{n}-x_{n}\right)^{2} \geq \frac{1}{3}\left(x_{n}^{\prime}-y_{n}^{\prime}\right)^{2}-2 K^{2}|\widehat{\mathbf{x}}-\widehat{\mathbf{y}}|^{2}
$$

Then substituting this in to 45.3.20, a short computation shows 45.3 .19 is dominated by an expression of the form $C(n, \theta)\|u\|_{W^{\theta, p(U)}}^{p}$ and this proves the existence of an extension operator provided the Lipschitz constant is small enough. It is clear $E$ is linear where $E$ is defined above.

Now this assumption on the smallness of $K$ needs to be removed. For $\left(\widehat{\mathbf{x}}, x_{n}\right) \in U$ define

$$
U^{\prime} \equiv\left\{\widehat{\mathbf{x}^{\prime}}=\lambda\left(\widehat{\mathbf{x}}-\widehat{\mathbf{b}}_{0}\right): \widehat{\mathbf{x}} \in U\right\}
$$

here this is centering at $\mathbf{0}$ and stretching $B$ since $\lambda$ will be large. Let $\mathbf{h}$ be the name of this mapping. Thus

$$
\begin{aligned}
\mathbf{h}(\widehat{\mathbf{x}}) & \equiv \lambda\left(\widehat{\mathbf{x}}-\widehat{\mathbf{b}}_{0}\right) \\
\mathbf{k}(\widehat{\mathbf{x}}) & \equiv \mathbf{h}^{-1}(\widehat{\mathbf{x}})=\frac{1}{\lambda} \widehat{\mathbf{x}}+\widehat{\mathbf{b}}_{0}
\end{aligned}
$$

These mappings are defined on all of $\mathbb{R}^{n}$. Now let $u^{\prime}$ be defined on $U^{\prime}$ as follows.

$$
u^{\prime}\left(\widehat{\mathbf{x}}, x_{n}\right) \equiv \mathbf{k}^{*} u\left(\widehat{\mathbf{x}}, x_{n}\right)
$$

Also let

$$
g^{\prime}(\widehat{\mathbf{x}}) \equiv \mathbf{k}^{*} g(\widehat{\mathbf{x}})
$$

Thus $g^{\prime}(\widehat{\mathbf{x}}) \equiv g\left(\frac{1}{\lambda} \widehat{\mathbf{x}}+\widehat{\mathbf{b}}_{0}\right)=g(\mathbf{k}(\widehat{\mathbf{x}}))$. Then choosing $\lambda$ large enough the Lipschitz condition for $g^{\prime}$ is as small as desired. Always assume $\lambda$ has been chosen this large and also $\lambda \geq 1$. Furthermore, $g^{\prime}\left(\widehat{\mathbf{x}^{\prime}}\right)=x_{n}^{\prime}$ describes the boundary in the same way as $x_{n}=g(\widehat{\mathbf{x}})$. Now I need to consider whether $u^{\prime} \in \widetilde{W^{\theta, p}\left(U^{\prime}\right)}$. Consider

$$
\begin{aligned}
& \int_{U^{\prime}} \int_{U^{\prime}} \frac{\left|u^{\prime}\left(\widehat{\mathbf{x}^{\prime}}, x_{n}\right)-u^{\prime}\left(\widehat{\mathbf{y}^{\prime}}, y_{n}\right)\right|^{p}}{\left(\left|\widehat{\mathbf{x}^{\prime}}-\widehat{\mathbf{y}^{\prime}}\right|^{2}+\left(x_{n}-y_{n}\right)^{2}\right)^{p+n \theta}} d x^{\prime} d y^{\prime} \\
= & \int_{U^{\prime}} \int_{U^{\prime}} \frac{\left|\mathbf{k}^{*} u\left(\widehat{\mathbf{x}^{\prime}}, x_{n}\right)-\mathbf{k}^{*} u\left(\widehat{\mathbf{y}^{\prime}}, y_{n}\right)\right|^{p}}{\left(\left|\widehat{\mathbf{x}^{\prime}}-\widehat{\mathbf{y}^{\prime}}\right|^{2}+\left(x_{n}-y_{n}\right)^{2}\right)^{p+n \theta}} d x^{\prime} d y^{\prime}
\end{aligned}
$$

Then change the variables $\widehat{\mathbf{x}^{\prime}}=\lambda\left(\widehat{\mathbf{x}}-\widehat{\mathbf{b}}_{0}\right)=\mathbf{h}(\widehat{\mathbf{x}})$ with a similar change for $\widehat{\mathbf{y}^{\prime}}$, the above expression equals

$$
\left(\lambda^{n-1}\right)^{2} \int_{U} \int_{U} \frac{\left|u\left(\widehat{\mathbf{x}}, x_{n}\right)-u\left(\widehat{\mathbf{y}}, y_{n}\right)\right|^{p}}{\left(\lambda^{2}|\widehat{\mathbf{x}}-\widehat{\mathbf{y}}|^{2}+\left(x_{n}-y_{n}\right)^{2}\right)^{p+n \theta}} d x d y
$$

Thus

$$
\begin{equation*}
\left\|u^{\prime}\right\|\left\|_{W^{\theta, p}\left(U^{\prime}\right)} \leq \lambda^{n-1}\right\| u \|_{W^{\theta, p}(U)}<\infty \tag{45.3.21}
\end{equation*}
$$

and $\mathbf{k}^{*}: \widetilde{W^{\theta, p}(U)}$ to $\widetilde{W^{\theta, p}\left(U^{\prime}\right)}$ is continuous and linear. Similar reasoning shows that $\mathbf{h}^{*}$ is continuous and linear mapping $\widetilde{W^{\theta, p}\left(\mathbb{R}^{n}\right)}$ to $\widetilde{W^{\theta, p}\left(\mathbb{R}^{n}\right)}$. By the first part of the argument there exists a continuous linear map

$$
E^{\prime}: \widetilde{W^{\theta, p}\left(U^{\prime}\right)} \rightarrow \widetilde{W^{\theta, p}\left(\mathbb{R}^{n}\right)}
$$

Now define

$$
E u \equiv \mathbf{h}^{*} E^{\prime}\left(\mathbf{k}^{*} u\right)
$$

Say $x_{n} \leq g(\widehat{\mathbf{x}})$. Then $\left(\widehat{\mathbf{x}}, x_{n}\right) \in U$ and so $\left(\mathbf{h}(\widehat{\mathbf{x}}), x_{n}\right) \in U^{\prime}$. Thus

$$
\begin{align*}
E u\left(\widehat{\mathbf{x}}, x_{n}\right) & \equiv \mathbf{h}^{*} E^{\prime}\left(\mathbf{k}^{*} u\right)\left(\widehat{\mathbf{x}}, x_{n}\right) \\
& \equiv E^{\prime}\left(\mathbf{k}^{*} u\right)\left(\mathbf{h}(\widehat{\mathbf{x}}), x_{n}\right) \tag{45.3.22}
\end{align*}
$$

Now $g(\widehat{\mathbf{x}}) \geq x_{n}$ and so $g^{\prime}(\mathbf{h}(\widehat{\mathbf{x}})) \geq x_{n}$ because

$$
g^{\prime}(\mathbf{h}(\widehat{\mathbf{x}})) \equiv \mathbf{k}^{*} g(\mathbf{h}(\widehat{\mathbf{x}}))=g(\widehat{\mathbf{x}})
$$

and so 45.3.22 equals

$$
=\left(\mathbf{k}^{*} u\right)\left(\mathbf{h}(\widehat{\mathbf{x}}), x_{n}\right)=u\left(\widehat{\mathbf{x}}, x_{n}\right)
$$

Thus $E$ leaves $u$ unchanged at points $\left(\widehat{\mathbf{x}}, x_{n}\right)$ where $x_{n} \leq g(\widehat{\mathbf{x}})$. Also

$$
\begin{aligned}
& \|E u\|_{W^{\theta, p}\left(\mathbb{R}^{n}\right)}=\left\|\mathbf{h}^{*} E^{\prime} \mathbf{k}^{*} u\right\|_{W^{\theta, p}\left(\mathbb{R}^{n}\right)} \leq\left\|\mathbf{h}^{*}\right\|\left\|E^{\prime} \mathbf{k}^{*} u\right\|_{W^{\theta, p}\left(\mathbb{R}^{n}\right)} \\
& \quad \leq\left\|\mathbf{h}^{*}\right\|\left\|E^{\prime}\right\|\left\|\mathbf{k}^{*}\right\|\|u\|_{W^{\theta, p}(U)}=C\|u\|_{\widetilde{W^{\theta, p}(U)}}
\end{aligned}
$$

To complete the proof, cover $U$ with finitely many sets of this sort oriented with respect to one of the coordinate axes as this one was along with an open set whose closure is contained in $U$ and then use a smooth partition of unity to localize the function to the situation of the sort just discussed and one whose support is contained in $U$. Extend that one to equal zero off its support and treat the others as above. This proves the proposition.

Recall Theorem 38.2.7 which gives an extension operator which maps from the space $W^{1, p}(U)$ to $W^{1, p}\left(\mathbb{R}^{n}\right)$ also denoted by $E$. Now it is not hard to see that $\widetilde{W^{\theta, p}(U)}=$ $W^{\theta, p}(U)$ and the two norms are equivalent.

Theorem 45.3.13 Let $U$ be a bounded open set which has Lipschitz boundary and $\theta \in$ $(0,1)$. Then $\widetilde{W^{\theta, p}(U)}=W^{\theta, p}(U)$ and the two norms are equivalent.

Proof: Let $u \in \widetilde{W^{\theta, p}(U)}$. Letting $E$ be the extension operator of Lemma 45.3.12, there is a constant $C$ such that

$$
C\|u\|_{W^{\theta, p}(U)} \geq\|E u\|_{W^{\theta, p}\left(\mathbb{R}^{n}\right)}=\|E u\|_{W^{\theta, p}\left(\mathbb{R}^{n}\right)} \geq\|u\|_{W^{\theta, p}(U)}
$$

Thus if $u \in \widetilde{W^{\theta, p}(U)}$, then $u \in W^{\theta, p}(U)$ and the inclusion map is continuous.
Next suppose $u \in W^{\theta, p}(U)$ and let $\gamma f=u$ for

$$
f \in W\left(W^{1, p}(U), L^{p}(U), p, 1-\theta\right) \equiv W_{U}
$$

Then from Theorem 38.2.7 and the definition of the norm in $W_{U}$

$$
E f \in W\left(W^{1, p}\left(\mathbb{R}^{n}\right), L^{p}\left(\mathbb{R}^{n}\right), p, 1-\theta\right) \equiv W
$$

where this $E$ pertains to extending $W^{1, p}(U)$.

$$
C_{1}\|f\|_{W_{U}} \geq\|E f\|_{W} \geq\|E u\|_{W^{\theta, p}\left(\mathbb{R}^{n}\right)} \geq\|u\|_{W^{\theta, p}(U)}
$$

Since this is true for every $f \in W_{U}$, it follows $u \in \widehat{W^{\theta, p}(U)}$ and

$$
\|u\|_{W^{\theta, p}(U)} \leq C_{1}\|u\|_{W^{\theta, p}(U)}
$$

This proves the theorem.
Corollary 45.3.14 Let $U$ be a bounded open set with Lipschitz boundary. Then $W^{\theta, p}(U)$ is reflexive.

Proof: From Proposition 45.3.12 and Theorem 45.3.13, there exists an extension operator $E: W^{\theta, p}(U) \rightarrow W^{\theta, p}\left(\mathbb{R}^{n}\right)$ which is continuous. This operator is one to one and continuous. Furthermore, $\|E u\|_{W^{\theta, p}\left(\mathbb{R}^{n}\right)} \geq\|u\|_{W^{\theta, p}(U)}$ and so $E\left(W^{\theta, p}(U)\right)$ is closed. Therefore, by Corollary 21.2.8 on Page 656 and the fact $W^{\theta, p}\left(\mathbb{R}^{n}\right)$ is reflexive which was shown in Corollary 45.3.8, it follows $W^{\theta, p}(U)$ is reflexive. This proves the corollary.

There may be other sets $U$ for which the intrinsic norm is an equivalent norm for $W^{\theta, p}(U)$ but this much will suffice. It should be routine to verify that this works for $U$ a half space for example and the extension argument should be much easier than that presented above. More generally, the assumption that $U$ was bounded in the above extension argument of Proposition 45.3 .12 was never needed except for giving finitely many of those special sets covering the boundary. If you just assumed this at the outset instead of an assumption the set is bounded, the same sort of extension would work.

### 45.4 Fractional Order Sobolev Spaces

Now it is time to define fractional order Sobolev spaces between $W^{m, p}$ and $W^{m+1, p}$.
Definition 45.4.1 Let $m$ be a nonnegative integer and let $s=m+\sigma$ where $\sigma \in(0,1)$. Then $W^{s, p}(\Omega)$ will consist of those elements of $W^{m, p}(\Omega)$ for which $D^{\alpha} u \in W^{\sigma, p}(\Omega)$ for all $|\alpha|=m$. The norm is given by the following.

$$
\|u\|_{s, p, \Omega} \equiv\left(\|u\|_{m, p, \Omega}^{p}+\sum_{|\alpha|=m}\left\|D^{\alpha} u\right\|_{\sigma, p, \Omega}^{p}\right)^{1 / p}
$$

Corollary 45.4.2 The space, $W^{s, p}(\Omega)$ is a reflexive Banach space whenever $p>1$.
Proof: From the theory of interpolation spaces, $W^{\sigma, p}(\Omega)$ is reflexive. This is because it is an iterpolation space for the two reflexive spaces, $L^{p}(\Omega)$ and $W^{1, p}(\Omega)$. (Alternatively, you could use Corollary 45.3 .14 in the case where $\Omega$ is a bounded open set with Lipschitz boundary or you could use Corollary 45.3 .8 in case $\Omega=\mathbb{R}^{n}$. In addition, the same ideas would work if $\Omega$ were any space for which there was a continuous extension map from $W^{\sigma, p}(\Omega)$ to $W^{\sigma, p}\left(\mathbb{R}^{n}\right)$.) Now the formula for the norm of an element in $W^{s, p}(\Omega)$ shows this space is isometric to a closed subspace of $W^{m, p}(\Omega) \times W^{\sigma, p}(\Omega)^{k}$ for suitable $k$. Therefore, from Corollary 21.2.8 on Page 656, $W^{s, p}(\Omega)$ is also reflexive.

Theorem 45.4.3 The trace map, $\gamma: W^{m, p}\left(\mathbb{R}_{+}^{n}\right) \rightarrow W^{m-\frac{1}{p}, p}\left(\mathbb{R}^{n-1}\right)$ is continuous.
Proof: Let $f \in \mathfrak{S}$, the Schwartz class. Let $\sigma=1-\frac{1}{p}$ so that $m-\left(\frac{1}{p}\right)=m-1+\sigma$. Then from the definition and using $f \in \mathfrak{S}$,

$$
\begin{aligned}
\|\gamma f\|_{m-\frac{1}{p}, p, \mathbb{R}^{n-1}} & =\left(\|\gamma f\|_{m-1, p, \mathbb{R}^{n-1}}^{p}+\sum_{|\alpha|=m-1}\left\|D^{\alpha} \gamma f\right\|_{1-\frac{1}{p}, p, \mathbb{R}^{n-1}}^{p}\right)^{1 / p} \\
& =\left(\|\gamma f\|_{m-1, p, \mathbb{R}^{n-1}}^{p}+\sum_{|\alpha|=m-1}\left\|\gamma D^{\alpha} f\right\|_{1-\frac{1}{p}, p, \mathbb{R}^{n-1}}^{p}\right)^{1 / p}
\end{aligned}
$$

and from Lemma 45.1.4, and the fact that the trace is continuous as a map from $W^{m, p}\left(\mathbb{R}_{+}^{n}\right)$ to $W^{m-1, p}\left(\mathbb{R}^{n-1}\right)$,

$$
\begin{aligned}
\|\gamma f\|_{m-\frac{1}{p}, p, \mathbb{R}^{n-1}} & \leq\left(C_{1}\|f\|_{m, p, \mathbb{R}_{+}^{n}}^{p}+C_{2} \sum_{|\alpha|=m-1}\left\|D^{\alpha} f\right\|_{1, p, \mathbb{R}^{n}}\right)^{1 / p} \\
& \leq C\|f\|_{m, p, \mathbb{R}^{n}+p} .
\end{aligned}
$$

Then using density of $\mathfrak{S}$ this implies the desired result.
With the definition of $W^{s, p}(\Omega)$ for $s$ not an integer, here is a generalization of an earlier theorem.

Theorem 45.4.4 Let $\mathbf{h}: U \rightarrow V$ where $U$ and $V$ are two open sets and suppose $\mathbf{h}$ is bilipschitz and that $D^{\alpha} \mathbf{h}$ and $D^{\alpha} \mathbf{h}^{-1}$ exist and are Lipschitz continuous if $|\alpha| \leq m$ where $m=0,1, \cdots$ and $s=m+\sigma$ where $\sigma \in(0,1)$. Then

$$
\mathbf{h}^{*}: W^{s, p}(V) \rightarrow W^{s, p}(U)
$$

is continuous, linear, one to one, and has an inverse with the same properties, the inverse being $\left(\mathbf{h}^{-1}\right)^{*}$.

Proof: In case $m=0$, the conclusion of the theorem is immediate from the general theory of trace spaces. Therefore, assume $m \geq 1$. It follows from the definition that

$$
\left\|\mathbf{h}^{*} u\right\|_{m+\sigma, p, U} \equiv\left[\left\|\mathbf{h}^{*} u\right\|_{m, p, U}^{p}+\sum_{|\alpha|=m}\left\|D^{\alpha}\left(\mathbf{h}^{*} u\right)\right\|_{\sigma, p, U}^{p}\right]^{1 / p}
$$

Consider the case when $m=1$. Then it is routine to verify that

$$
D_{j} \mathbf{h}^{*} u(\mathbf{x})=u_{, k}(\mathbf{h}(\mathbf{x})) h_{k, j}(\mathbf{x}) .
$$

Let $L_{k}: W^{1, p}(V) \rightarrow W^{1, p}(U)$ be defined by

$$
L_{k} v=\mathbf{h}^{*}(v) h_{k, j}
$$

Then $L_{k}$ is continuous as a map from $W^{1, p}(V)$ to $W^{1, p}(U)$ and as a map from $L^{p}(V)$ to $L^{p}(U)$ and therefore, it follows that $L_{k}$ is continuous as a map from $W^{\sigma, p}(V)$ to $W^{\sigma, p}(U)$. Therefore,

$$
\left\|L_{k}(v)\right\|_{\sigma, p, U} \leq C_{k}\|v\|_{\sigma, p, U}
$$

and so

$$
\begin{aligned}
\left\|D_{j}\left(\mathbf{h}^{*} u\right)\right\|_{\sigma, p, U} & \leq \sum_{k}\left\|L_{k}\left(u_{, k}\right)\right\|_{\sigma, p, U} \\
& \leq \sum_{k} C_{k}\left\|D_{k} u\right\|_{\sigma, p, V} \\
& \leq C\left(\sum_{k}\left\|D_{k} u\right\|_{\sigma, p, V}^{p}\right)^{1 / p}
\end{aligned}
$$

Therefore, it follows that

$$
\begin{aligned}
\left\|\mathbf{h}^{*} u\right\|_{1+\sigma, p, U} & \leq\left[\left\|\mathbf{h}^{*} u\right\|_{1, p, U}^{p}+\sum_{j} C^{p} \sum_{k}\left\|D_{k} u\right\|_{\sigma, p, V}^{p}\right]^{1 / p} \\
& \leq C\left[\|u\|_{1, p, V}^{p}+\sum_{k}\left\|D_{k} u\right\|_{\sigma, p, V}^{p}\right]^{1 / p}=C\|u\|_{1+\sigma, p, V}
\end{aligned}
$$

The general case is similar except for the use of a more complicated linear operator in place of $L_{k}$. This proves the theorem.

It is interesting to prove this theorem using Theorem 45.3.13 and the intrinsic norm.
Now we prove an important interpolation inequality for Sobolev spaces.
Theorem 45.4.5 Let $\Omega$ be an open set in $\mathbb{R}^{n}$ which has the segment property and let $f \in$ $W^{m+1, p}(\Omega)$ and $\sigma \in(0,1)$. Then for some constant, $C$, independent of $f$,

$$
\|f\|_{m+\sigma, p, \Omega} \leq C\|f\|_{m+1, p, \Omega}^{1-\sigma}\|f\|_{m, p, \Omega}^{\sigma}
$$

Also, if $L \in \mathscr{L}\left(W^{m, p}(\Omega), W^{m, p}(\Omega)\right)$ for all $m=0,1, \cdots$, and $L \circ D^{\alpha}=D^{\alpha} \circ L$ on $C^{\infty}(\bar{\Omega})$, then $L \in \mathscr{L}\left(W^{m+\sigma, p}(\Omega), W^{m+\sigma, p}(\Omega)\right)$ for any $m=0,1, \cdots$.

Proof: Recall from above, $W^{1-\theta, p}(\Omega) \equiv T\left(W^{1, p}(\Omega), L^{p}(\Omega), p, \theta\right)$. Therefore, from Theorem 44.1.9, if $f \in W^{1, p}(\Omega)$,

$$
\|f\|_{1-\theta, p, \Omega} \leq K\|f\|_{1, p, \Omega}^{\theta}\|f\|_{0, p, \Omega}^{1-\theta}
$$

Therefore,

$$
\begin{aligned}
\|f\|_{m+\sigma, p, \Omega} & \leq\left(\|f\|_{m, p, \Omega}^{p}+\sum_{|\alpha|=m} K\left(\left\|D^{\alpha} f\right\|_{1, p, \Omega}^{1-\sigma}\left\|D^{\alpha} f\right\|_{0, p, \Omega}^{\sigma}\right)^{p}\right)^{1 / p} \\
& \leq C\left[\|f\|_{m, p, \Omega}^{p}+\left(\|f\|_{m+1, p, \Omega}^{1-\sigma}\|f\|_{m, p, \Omega}^{\sigma}\right)^{p}\right]^{1 / p} \\
& \leq C\left[\left(\|f\|_{m+1, p, \Omega}^{1-\sigma}\|f\|_{m, p, \Omega}^{\sigma}\right)^{p}+\left(\|f\|_{m+1, p, \Omega}^{1-\sigma}\|f\|_{m, p, \Omega}^{\sigma}\right)^{p}\right]^{1 / p} \\
& \leq C\|f\|_{m+1, p, \Omega}^{1-\sigma}\|f\|_{m, p, \Omega}^{\sigma}
\end{aligned}
$$

This proves the first part. Now consider the second. Let $\phi \in C^{\infty}(\bar{\Omega})$

$$
\begin{gather*}
\|L \phi\|_{m+\sigma, p, \Omega}=\left(\|L \phi\|_{m, p, \Omega}^{p}+\sum_{|\alpha|=m}\left\|D^{\alpha} L \phi\right\|_{\sigma, p, \Omega}^{p}\right)^{1 / p} \\
=\left(\|L \phi\|_{m, p, \Omega}^{p}+\sum_{|\alpha|=m}\left\|L D^{\alpha} \phi\right\|_{T\left(W^{\left.1, p, L^{p}, p, 1-\sigma\right)}\right.}^{p}\right)^{1 / p} \\
=\left(\|L \phi\|_{m, p, \Omega}^{p}+\sum_{|\alpha|=m}\left[\inf \left(\left\|t^{1-\sigma} L f_{\alpha}\right\|_{1}^{\sigma}\left\|t^{1-\sigma} L f_{\alpha}^{\prime}\right\|_{2}^{1-\sigma}\right)\right]^{p}\right)^{1 / p} \tag{45.4.23}
\end{gather*}
$$

where

$$
\begin{gathered}
\inf \left(\left\|t^{1-\sigma} L f_{\alpha}\right\|_{1}^{\sigma}\left\|t^{1-\sigma} L f_{\alpha}^{\prime}\right\|_{2}^{1-\sigma}\right)= \\
\inf \left(\left\|t^{1-\sigma} L f_{\alpha}\right\|_{L^{p}\left(0, \infty ; \frac{d t}{t} ; W^{1, p}(\Omega)\right)}^{\sigma}\left\|t^{1-\sigma} L f_{\alpha}^{\prime}\right\|_{L^{p}\left(0, \infty ; \frac{d t}{t} ; L^{p}(\Omega)\right)}^{1-\sigma}\right)
\end{gathered}
$$

$f_{\alpha}(0) \equiv \lim _{t \rightarrow 0} f_{\alpha}(t)=D^{\alpha} \phi$ in $W^{1, p}(\Omega)+L^{p}(\Omega)$, and the infimum is taken over all such functions. Therefore, from 45.4.23, and letting $\|L\|_{1}$ denote the operator norm of $L$ in $W^{1, p}(\Omega)$ and $\|L\|_{2}$ denote the operator norm of $L$ in $L^{p}(\Omega)$,

$$
\begin{aligned}
& \|L \phi\|_{m+\sigma, p, \Omega} \\
\leq & \left(\|L \phi\|_{m, p, \Omega}^{p}+\sum_{|\alpha|=m}\left[\inf \left(\|L\|_{1}^{\sigma}\|L\|_{2}^{1-\sigma}\left\|t^{1-\sigma} f_{\alpha}\right\|_{1}^{\sigma}\left\|t^{1-\sigma} f_{\alpha}^{\prime}\right\|_{2}^{1-\sigma}\right)\right]^{p}\right)^{1 / p} \\
\leq & \left(\|L \phi\|_{m, p, \Omega}^{p}+\left(\|L\|_{1}^{\sigma}\|L\|_{2}^{1-\sigma}\right)^{p} \sum_{|\alpha|=m}\left[\inf \left(\left\|t^{1-\sigma} f_{\alpha}\right\|_{1}^{\sigma}\left\|t^{1-\sigma} f_{\alpha}^{\prime}\right\|_{2}^{1-\sigma}\right)\right]^{p}\right)^{1 / p} \\
\leq & C\left(\|\phi\|_{m, p, \Omega}^{p}+\sum_{|\alpha|=m}\left[\left\|D^{\alpha} \phi\right\|_{\sigma, p, \Omega}\right]^{p}\right)^{1 / p}=C\|\phi\|_{m+\sigma, p, \Omega}
\end{aligned}
$$

Since $C^{\infty}(\bar{\Omega})$ is dense in all the Sobolev spaces, this inequality establishes the desired result.

Definition 45.4.6 Define for $s \geq 0, W^{-s, p^{\prime}}\left(\mathbb{R}^{n}\right)$ to be the dual space of

$$
W^{s, p}\left(\mathbb{R}^{n}\right)
$$

Here $\frac{1}{p}+\frac{1}{p^{\prime}}=1$.
Note that in the case of $m=0$ this is consistent with the Riesz representation theorem for the $L^{p}$ spaces.

## Chapter 46

## Sobolev Spaces On Manifolds

### 46.1 Basic Definitions

Consider the following situation. There exists a set, $\Gamma \subseteq \mathbb{R}^{m}$ where $m>n$, mappings, $\mathbf{h}_{i}: U_{i} \rightarrow \Gamma_{i}=\Gamma \cap W_{i}$ for $W_{i}$ an open set in $\mathbb{R}^{m}$ with $\Gamma \subseteq \cup_{i=1}^{l} W_{i}$ and $U_{i}$ is an open subset of $\mathbb{R}^{n}$ which $\mathbf{h}_{i}$ one to one and onto. Assume $\mathbf{h}_{i}$ is of the form

$$
\begin{equation*}
\mathbf{h}_{i}(\mathbf{x})=\mathbf{H}_{i}(\mathbf{x}, \mathbf{0}) \tag{46.1.1}
\end{equation*}
$$

where for some open set, $O_{i}, \mathbf{H}_{i}: U_{i} \times O_{i} \rightarrow W_{i}$ is bilipschitz having bilipschitz inverse such that for $\mathbf{G}=\mathbf{H}_{i}$ or $\mathbf{H}_{i}^{-1}, D^{\alpha} \mathbf{G}$ is Lipschitz for $|\alpha| \leq k$.

For example, let $m=n+1$ and let

$$
\mathbf{H}_{i}(\mathbf{x}, y)=\binom{\mathbf{x}}{\phi(\mathbf{x})+y}
$$

where $\phi$ is a Lipschitz function having $D^{\alpha} \phi$ Lipschitz for all $|\alpha| \leq k$. This is an example of the sort of thing just described if $\mathbf{x} \in U_{i} \subseteq \mathbb{R}^{n}$ and $O_{i}=\mathbb{R}$, because it is obvious the inverse of $\mathbf{H}_{i}$ is given by

$$
\mathbf{H}_{i}^{-1}(\mathbf{x}, y)=\binom{\mathbf{x}}{y-\phi(\mathbf{x})}
$$

Also let $\left\{\psi_{i}\right\}_{i=1}^{l}$ be a partition of unity subordinate to the open cover $\left\{W_{i}\right\}$ satisfying $\psi_{i} \in$ $C_{c}^{\infty}\left(W_{i}\right)$. Then the definition of $W^{s, p}(\Gamma)$ follows.

Definition 46.1.1 $u \in W^{s, p}(\Gamma)$ if whenever $\left\{W_{i}, \psi_{i}, \Gamma_{i}, U_{i}, \mathbf{h}_{i}, \mathbf{H}_{i}\right\}_{i=1}^{l}$ is described above with $\mathbf{h}_{i} \in C^{k, 1}, \mathbf{h}_{i}^{*}\left(u \psi_{i}\right) \in W^{s, p}\left(U_{i}\right)$. The norm is given by

$$
\|u\|_{s, p, \Gamma} \equiv \sum_{i=1}^{l}\left\|\mathbf{h}_{i}^{*}\left(u \psi_{i}\right)\right\|_{s, p, U_{i}}
$$

It is not at all obvious this norm is well defined. What if

$$
\left\{W_{i}^{\prime}, \phi_{i}, \Gamma_{i}, V_{i}, \mathbf{g}_{i}, \mathbf{G}_{i}\right\}_{i=1}^{r}
$$

is as described above. Would the two norms be equivalent? To begin with consider the following lemma which involves a particular choice for $\left\{W_{i}, \psi_{i}, \Gamma_{i}, U_{i}, \mathbf{h}_{i}, \mathbf{H}_{i}\right\}_{i=1}^{l}$.

Lemma 46.1.2 $W^{s, p}(\Gamma)$ as just described, is a Banach space. If $p>1$ then it is reflexive.
Proof: Let $L: W^{s, p}(\Gamma) \rightarrow \prod_{i=1}^{l} W^{s, p}\left(U_{i}\right)$ be defined by $(L u)_{i} \equiv \mathbf{h}_{i}^{*}\left(u \psi_{i}\right)$. Let $\left\{u_{j}\right\}_{j=1}^{\infty}$ be a Cauchy sequence in $W^{s, p}(\Gamma)$. Then $\left\{\mathbf{h}_{i}^{*}\left(u_{j} \psi_{i}\right)\right\}_{j=1}^{\infty}$ is a Cauchy sequence in $W^{s, p}\left(U_{i}\right)$ for each $i$. Therefore, for each $i$, there exists $w_{i} \in W^{s, p}\left(U_{i}\right)$ such that

$$
\lim _{j \rightarrow \infty} \mathbf{h}_{i}^{*}\left(u_{j} \psi_{i}\right)=w_{i} \text { in } W^{s, p}\left(U_{i}\right)
$$

But also, there exists a subsequence, still denoted by $j$ such that for each $i$

$$
\left\{\mathbf{h}_{i}^{*}\left(u_{j} \psi_{i}\right)(\mathbf{x})\right\}_{j=1}^{\infty}
$$

is a Cauchy sequence for a.e. $\mathbf{x}$. Since $\mathbf{h}_{i}$ is given to be Lipschitz, it maps sets of measure 0 to sets of $n$ dimensional Hausdorff measure zero. Therefore,

$$
\left\{u_{j} \psi_{i}(\mathbf{y})\right\}_{j=1}^{\infty}
$$

is a Cauchy sequence for $\mu$ a.e. $\mathbf{y} \in W_{i} \cap \Gamma$ where $\mu$ denotes the $n$ dimensional Hausdorff measure. It follows that for $\mu$ a.e. $\mathbf{y},\left\{u_{j}(\mathbf{y})\right\}_{j=1}^{\infty}$ is a Cauchy sequence and so it converges to a function denoted as $u(\mathbf{y})$.

$$
u_{j}(\mathbf{y}) \rightarrow u(\mathbf{y}) \mu \text { a.e. }
$$

Therefore, $w_{i}(\mathbf{x})=\mathbf{h}_{i}^{*}\left(u \psi_{i}\right)(\mathbf{x})$ a.e. and this shows $\mathbf{h}_{i}^{*}\left(u \psi_{i}\right) \in W^{s, p}\left(U_{i}\right)$. Thus $u \in W^{s, p}(\Gamma)$ showing completeness. It is clear $\|\cdot\|_{s, p, \Gamma}$ is a norm. Thus $L$ is an isometry of $W^{s, p}(\Gamma)$ and a closed subspace of $\prod_{i=1}^{l} W^{s, p}\left(U_{i}\right)$. By Corollary 45.4.2, $W^{s, p}\left(U_{i}\right)$ is reflexive which implies the product is reflexive. Closed subspaces of reflexive spaces are reflexive by Lemma 21.2.7 on Page 655 and so $W^{s, p}(\Gamma)$ is also reflexive. This proves the lemma.

I now show that any two such norms are equivalent.
Suppose $\left\{W_{j}^{\prime}, \phi_{j}, \Gamma_{j}, V_{j}, \mathbf{g}_{j}, \mathbf{G}_{j}\right\}_{j=1}^{r}$ and $\left\{W_{i}, \Psi_{i}, \Gamma_{i}, U_{i}, \mathbf{h}_{i}, \mathbf{H}_{i}\right\}_{i=1}^{l}$ both satisfy the conditions described above. Let $\|\cdot\|_{s, p, \Gamma}^{1}$ denote the norm defined by

$$
\begin{gather*}
\|u\|_{s, p, \Gamma}^{1} \equiv \sum_{j=1}^{r}\left\|\mathbf{g}_{j}^{*}\left(u \phi_{j}\right)\right\|_{s, p, V_{j}} \\
\leq \sum_{j=1}^{r}\left\|\mathbf{g}_{j}^{*}\left(\sum_{i=1}^{l} u \phi_{j} \psi_{i}\right)\right\|\left\|_{s, p, V_{j}} \leq \sum_{j, i}\right\| \mathbf{g}_{j}^{*}\left(u \phi_{j} \psi_{i}\right) \|_{s, p, V_{j}} \\
=\sum_{j, i}\left\|\mathbf{g}_{j}^{*}\left(u \phi_{j} \psi_{i}\right)\right\|_{s, p, \mathbf{g}_{j}^{-1}\left(W_{i} \cap W_{j}^{\prime}\right)} \tag{46.1.2}
\end{gather*}
$$

Now define a new norm $\|u\|_{s, p, \Gamma}^{1, \mathbf{g}}$ by the formula 46.1.2. This norm is determined by

$$
\left\{W_{j}^{\prime} \cap W_{i}, \psi_{i} \phi_{j}, \Gamma_{j} \cap \Gamma_{i}, V_{j}, \mathbf{g}_{i, j}, \mathbf{G}_{i, j}\right\}
$$

where $\mathbf{g}_{i, j}=\mathbf{g}_{j}$. Thus the identity map

$$
\operatorname{id}:\left(W^{s, p}(\Gamma),\|\cdot\|_{s, p, \Gamma}^{1, \mathbf{g}}\right) \rightarrow\left(W^{s, p}(\Gamma),\|\cdot\| \|_{s, p, \Gamma}^{1}\right)
$$

is continuous. It follows the two norms, $\|\cdot\|_{s, p, \Gamma}^{1, \mathbf{g}}$ and $\|\cdot\|_{s, p, \Gamma}^{1}$, are equivalent by the open mapping theorem. In a similar way, the norms, $\|\cdot\|_{s, p, \Gamma}^{2, \mathbf{h}}$ and $\|\cdot\|_{s, p, \Gamma}^{2}$ are equivalent where

$$
\|u\|_{s, p, \Gamma}^{2} \equiv \sum_{j=1}^{l}\left\|\mathbf{h}_{i}^{*}\left(u \psi_{i}\right)\right\|_{s, p, U_{i}}
$$

and

$$
\|u\|_{s, p, \Gamma}^{2, \mathbf{h}} \equiv \sum_{j, i}\left\|\mathbf{h}_{i}^{*}\left(u \phi_{j} \psi_{i}\right)\right\|_{s, p, U_{i}}=\sum_{j, i}\left\|\mathbf{h}_{i}^{*}\left(u \phi_{j} \psi_{i}\right)\right\|_{s, p, \mathbf{h}_{i}^{-1}\left(w_{i} \cap W_{j}^{\prime}\right)}
$$

But from the assumptions on $\mathbf{h}$ and $\mathbf{g}$, in particular the assumption that these are restrictions of functions which are defined on open subsets of $\mathbb{R}^{m}$ which have Lipschitz derivatives up to order $k$ along with their inverses, Theorem 45.4.4 implies, there exist constants $C_{i}$, independent of $u$ such that

$$
\left\|\mathbf{h}_{i}^{*}\left(u \phi_{j} \psi_{i}\right)\right\|_{s, p, \mathbf{h}_{i}^{-1}\left(W_{i} \cap W_{j}^{\prime}\right)} \leq C_{1}\left\|\mathbf{g}_{j}^{*}\left(u \phi_{j} \psi_{i}\right)\right\|_{s, p, \mathbf{g}_{j}^{-1}\left(W_{i} \cap W_{j}^{\prime}\right)}
$$

and

$$
\left\|\mathbf{g}_{j}^{*}\left(u \phi_{j} \psi_{i}\right)\right\|_{s, p, \mathbf{g}_{j}^{-1}\left(W_{i} \cap W_{j}^{\prime}\right)} \leq C_{2}\left\|\mathbf{h}_{i}^{*}\left(u \phi_{j} \psi_{i}\right)\right\|_{s, p, \mathbf{h}_{i}^{-1}\left(W_{i} \cap W_{j}^{\prime}\right)}
$$

Therefore, the two norms, $\|\cdot\|_{s, p, \Gamma}^{1, \mathbf{g}}$ and $\|\cdot\|_{s, p, \Gamma}^{2, \mathbf{h}}$ are equivalent. It follows that the norms, $\|\cdot\|_{s, p, \Gamma}^{2}$ and $\|\cdot\|_{s, p, \Gamma}^{1}$ are equivalent. This proves the following theorem.

Theorem 46.1.3 Let $\Gamma$ be described above. Then any two norms for $W^{s, p}(\Gamma)$ as in Definition 39.6.3 are equivalent.

### 46.2 The Trace On The Boundary Of An Open Set

Next is a generalization of earlier theorems about the loss of $\frac{1}{p}$ derivatives on the boundary.

## Definition 46.2.1 Define

$$
\mathbb{R}_{k}^{n-1} \equiv\left\{\mathbf{x} \in \mathbb{R}^{n}: x_{k}=0\right\}, \widehat{\mathbf{x}}_{k} \equiv\left(x_{1}, \cdots, x_{k-1}, 0, x_{k+1}, \cdots, x_{n}\right)
$$

An open set, $\Omega$ is $C^{m, 1}$ if there exist open sets, $W_{i}, i=0,1, \cdots, l$ such that

$$
\Omega=\cup_{i=0}^{l} W_{i}
$$

with $\overline{W_{0}} \subseteq \Omega$, open sets $U_{i} \subseteq \mathbb{R}_{k}^{n-1}$ for some $k$, and open intervals, $\left(a_{i}, b_{i}\right)$ containing 0 such that for $i \geq 1$,

$$
\begin{gathered}
\partial \Omega \cap W_{i}=\left\{\widehat{\mathbf{x}}_{k}+\phi_{i}\left(\widehat{\mathbf{x}}_{k}\right) \mathbf{e}_{k}: \widehat{\mathbf{x}}_{k} \in U_{i}\right\} \\
\Omega \cap W_{i}=\left\{\widehat{\mathbf{x}}_{k}+\left(\phi_{i}\left(\widehat{\mathbf{x}}_{k}\right)+x_{k}\right) \mathbf{e}_{k}:\left(\widehat{\mathbf{x}}_{k}, x_{k}\right) \in U_{i} \times I_{i}\right\},
\end{gathered}
$$

where $\phi_{i}$ is Lipschitz with partial derivatives up to order m also Lipschitz. Here $I_{i}=\left(a_{i}, 0\right)$ or $\left(0, b_{i}\right)$. The case of $\left(a_{i}, 0\right)$ is shown in the picture.


Assume $\Omega$ is $C^{m-1,1}$. Define

$$
\mathbf{h}_{i}\left(\widehat{\mathbf{x}}_{k}\right)=\widehat{\mathbf{x}}_{k}+\phi_{i}\left(\widehat{\mathbf{x}}_{k}\right) \mathbf{e}_{k}, \mathbf{H}_{i}(\mathbf{x}) \equiv \widehat{\mathbf{x}}_{k}+\left(\phi_{i}\left(\widehat{\mathbf{x}}_{k}\right)+x_{k}\right) \mathbf{e}_{k},
$$

and let $\psi_{i} \in C_{c}^{\infty}\left(W_{i}\right)$ with $\sum_{i=0}^{l} \psi_{i}(\mathbf{x})=1$ on $\bar{\Omega}$. Thus

$$
\left\{W_{i}, \psi_{i}, \partial \Omega \cap W_{i}, U_{i}, \mathbf{h}_{i}, \mathbf{H}_{i}\right\}_{i=1}^{l}
$$

satisfies all the conditions for defining $W^{s, p}(\partial \Omega)$ for $s \leq m$. Let $u \in C^{\infty}(\bar{\Omega})$ and let $\mathbf{h}_{i}$ be as just described. The trace, denoted by $\gamma$ is that operator which evaluates functions in $C^{\infty}(\bar{\Omega})$ on $\partial \Omega$. Thus for $u \in C^{\infty}(\bar{\Omega})$, and $\mathbf{y} \in \partial \Omega$,

$$
u(\mathbf{y})=\sum_{i=1}^{l}\left(u \psi_{i}\right)(\mathbf{y})
$$

and so using the notation to suppress the reference to $\mathbf{y}$,

$$
\gamma u=\sum_{i=1}^{l} \gamma\left(u \psi_{i}\right)
$$

It is necessary to show this is a continuous map. Letting $u \in W^{m, p}(\Omega)$, it follows from Theorem 45.4.3, and Theorem 38.0.14,

$$
\begin{gathered}
\|\gamma u\|_{m-\frac{1}{p}, p, \partial \Omega}=\sum_{i=1}^{l}\left\|\mathbf{h}_{i}^{*}\left(\gamma\left(\psi_{i} u\right)\right)\right\|_{m-\frac{1}{p}, p, U_{i}} \\
=\sum_{i=1}^{l}\left\|\mathbf{h}_{i}^{*} \gamma\left(\psi_{i} u\right)\right\|_{m-\frac{1}{p}, p, \mathbb{R}_{k}^{n-1}} \leq C \sum_{i=1}^{l}\left\|\mathbf{H}_{i}^{*}\left(\psi_{i} u\right)\right\|_{m, p, \mathbb{R}_{+}^{n}} \\
\leq C \sum_{i=1}^{l}\left\|\mathbf{H}_{i}^{*}\left(\psi_{i} u\right)\right\|_{m, p, U_{i} \times\left(a_{i}, 0\right)} \leq C \sum_{i=1}^{l}\left\|\left(\psi_{i} u\right)\right\|_{m, p, W_{i} \cap \Omega}
\end{gathered}
$$

$$
\leq C \sum_{i=1}^{l}\left\|\left(\psi_{i} u\right)\right\|_{m, p, \Omega} \leq C \sum_{i=1}^{l}\|u\|_{m, p, \Omega} \leq C l\|u\|_{m, p, \Omega}
$$

Now use the density of $C^{\infty}(\bar{\Omega})$ in $W^{m, p}(\Omega)$ to see that $\gamma$ extends to a continuous linear map defined on $W^{m, p}(\Omega)$ still called $\gamma$ such that for all $u \in W^{m, p}(\Omega)$,

$$
\begin{equation*}
\|\gamma u\|_{m-\frac{1}{p}, p, \partial \Omega} \leq C l\|u\|_{m, p, \Omega} . \tag{46.2.3}
\end{equation*}
$$

Also, it can be shown that $\gamma$ maps $W^{m, p}(\Omega)$ onto $W^{m-\frac{1}{p}}(\partial \Omega)$. Let $g \in W^{m-\frac{1}{p}}(\partial \Omega)$. By definition, this means

$$
\mathbf{h}_{i}^{*}\left(\psi_{i} g\right) \in W^{m-\frac{1}{p}}\left(U_{i}\right), \text { each } i
$$

and so, using a cutoff function, there exists $w_{i} \in W^{m, p}\left(U_{i} \times I_{i}\right)$ such that

$$
\gamma w_{i}=\mathbf{h}_{i}^{*}\left(\psi_{i} g\right)=\mathbf{h}_{i}^{*}\left(\gamma \psi_{i} g\right)
$$

Thus $\left(\mathbf{H}_{i}^{-1}\right)^{*} w_{i} \in W^{m p}\left(\Omega \cap W_{i}\right)$. Let

$$
w \equiv \sum_{i=1}^{l} \psi_{i}\left(\mathbf{H}_{i}^{-1}\right)^{*} w_{i} \in W^{m p}(\Omega)
$$

then

$$
\begin{aligned}
\gamma w & =\sum_{i} \gamma \psi_{j} \gamma\left(\mathbf{H}_{i}^{-1}\right)^{*} w_{i}=\sum_{i} \gamma \psi_{j}\left(\mathbf{H}_{i}^{-1}\right)^{*} \gamma w_{i} \\
& =\sum_{i} \gamma \psi_{j}\left(\mathbf{H}_{i}^{-1}\right)^{*} \mathbf{h}_{i}^{*}\left(\gamma \psi_{i} g\right)=g
\end{aligned}
$$

In addition to this, in the case where $m=1$, Lemma 45.2.1 implies there exists a linear map, $R$, from $W^{1-\frac{1}{p}, p}(\partial \Omega)$ to $W^{1, p}(\Omega)$ which has the property that $\gamma R g=g$ for every $g \in W^{1-\frac{1}{p}, p}(\partial \Omega)$. I show this now. Letting $g \in W^{1-\frac{1}{p}, p}(\partial \Omega)$,

$$
g=\sum_{i=1}^{l} \psi_{i} g .
$$

Then also,

$$
\mathbf{h}_{i}^{*}\left(\psi_{i} g\right) \in W^{1-\frac{1}{p}, p}\left(\mathbb{R}^{n-1}\right)
$$

if extended to equal 0 off $U_{i}$. From Lemma 45.2.1, there exists an extension of this to $W^{1, p}\left(\mathbb{R}_{+}^{n}\right), R \mathbf{h}_{i}^{*}\left(\psi_{i} g\right)$. Without loss of generality, assume that

$$
R \mathbf{h}_{i}^{*}\left(\psi_{i} g\right) \in W^{1, p}\left(U_{i} \times\left(a_{i}, 0\right)\right) .
$$

If not so, multiply by a suitable cut off function in the definition of $R$. Then the extension is

$$
R g=\sum_{i=1}^{l}\left(\mathbf{H}_{i}^{-1}\right)^{*} R \mathbf{h}_{i}^{*}\left(\psi_{i} g\right) .
$$

This works because from the definition of $\gamma$ on $C^{\infty}(\bar{\Omega})$ and continuity of the map established above, $\gamma$ and $\left(\mathbf{H}_{i}^{-1}\right)^{*}$ commute and so

$$
\begin{aligned}
\gamma R g & \equiv \sum_{i=1}^{l} \gamma\left(\mathbf{H}_{i}^{-1}\right)^{*} R \mathbf{h}_{i}^{*}\left(\psi_{i} g\right) \\
& =\sum_{i=1}^{l}\left(\mathbf{H}_{i}^{-1}\right)^{*} \gamma R \mathbf{h}_{i}^{*}\left(\psi_{i} g\right) \\
& =\sum_{i=1}^{l}\left(\mathbf{H}_{i}^{-1}\right)^{*} \mathbf{h}_{i}^{*}\left(\psi_{i} g\right)=g .
\end{aligned}
$$

This proves the following theorem about the trace.
Theorem 46.2.2 Let $\Omega \in C^{m-1,1}$. Then there exists a constant, $C$ independent of $u \in$ $W^{m, p}(\Omega)$ and a continuous linear map, $\gamma: W^{m, p}(\Omega) \rightarrow W^{m-\frac{1}{p}, p}(\partial \Omega)$ such that 46.2.3 holds. This map satisfies $\gamma u(\mathbf{x})=u(\mathbf{x})$ for all $u \in C^{\infty}(\bar{\Omega})$ and $\gamma$ is onto. In the case where $m=1$, there exists a continuous linear map, $R: W^{1-\frac{1}{p}, p}(\partial \Omega) \rightarrow W^{1, p}(\Omega)$ which has the property that $\gamma R g=g$ for all $g \in W^{1-\frac{1}{p}, p}(\partial \Omega)$.

Of course more can be proved but this is all to be presented here.

## Part IV

## Multifunctions

## Chapter 47

## The Yankov von Neumann Aumann theorem

The Yankov von Neumann Aumann theorem deals with the projection of a product measurable set. It is a very difficult but interesting theorem. The material of this chapter is taken from [29], [30], [10], and [70]. We use the standard notation that for $\mathscr{S}$ and $\mathscr{F} \sigma$ algebras, $\mathscr{S} \times \mathscr{F}$ is the $\sigma$ algebra generated by the measurable rectangles, the product measure $\sigma$ algebra. The next result is fairly easy and the proof is left for the reader.
Lemma 47.0.1 Let $(X, d)$ be a metric space. Then if $d_{1}(x, y)=\frac{d(x, y)}{1+d(x, y)}$, it follows that $d_{1}$ is a metric on $X$ and the basis of open balls taken with respect to $d_{1}$ yields the same topology as the basis of open balls taken with respect to $d$.

Theorem 47.0.2 Let $\left(X_{i}, d_{i}\right)$ denote a complete metric space and let $X \equiv \prod_{i=1}^{\infty} X_{i}$. Then $X$ is also a complete metric space with the metric

$$
\rho(\mathbf{x}, \mathbf{y}) \equiv \sum_{i=1}^{\infty} 2^{-i} \frac{d_{i}\left(x_{i}, y_{i}\right)}{1+d_{i}\left(x_{i}, y_{i}\right)} .
$$

Also, if $X_{i}$ is separable for each $i$ then so is $X$.
Proof: It is clear from the above lemma that $\rho$ is a metric on $X$. We need to verify $X$ is complete with this metric. Let $\left\{\mathbf{x}^{n}\right\}$ be a Cauchy sequence in $X$. Then it is clear from the definition that $\left\{x_{i}^{n}\right\}$ is a Cauchy sequence for each $i$ and converges to $x_{i} \in X_{i}$. Therefore, letting $\varepsilon>0$ be given, we choose $N$ such that

$$
\sum_{k=N}^{\infty} 2^{-k}<\frac{\varepsilon}{2}
$$

we choose $M$ large enough that for $n>M$,

$$
2^{-i} \frac{d_{i}\left(x_{i}^{n}, x_{i}\right)}{1+d_{i}\left(x_{i}^{n}, x_{i}\right)}<\frac{\varepsilon}{2(N+1)}
$$

for all $i=1,2, \cdots, N$. Then letting $\mathbf{x}=\left\{x_{i}\right\}$,

$$
\rho\left(\mathbf{x}, \mathbf{x}^{n}\right) \leq \frac{\varepsilon N}{2(N+1)}+\sum_{k=N}^{\infty} 2^{-k}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon
$$

We need to verify that $X$ is separable. Let $D_{i}$ denote a countable dense set in $X_{i}, D_{i} \equiv$ $\left\{r_{k}^{i}\right\}_{k=1}^{\infty}$. Then let

$$
\mathscr{D}_{k} \equiv D_{1} \times \cdots \times D_{k} \times\left\{r_{1}^{k+1}\right\} \times\left\{r_{1}^{k+2}\right\} \times \cdots
$$

Thus $\mathscr{D}_{k}$ is a countable subset of $X$. Let $\mathscr{D} \equiv \cup_{k=1}^{\infty} \mathscr{D}_{k}$. Then $\mathscr{D}$ is countable and we can see $\mathscr{D}$ is dense in $X$ as follows. The projection of $\mathscr{D}_{k}$ onto the first $k$ entries is dense in
$\prod_{i=1}^{k} X_{i}$ and for $k$ large enough the remaining component's contribution to the metric, $\rho$ is very small. Therefore, obtaining $\mathbf{d} \in \mathscr{D}$ close to $\mathbf{x} \in X$ may be accomplished by finding $\mathbf{d} \in \mathscr{D}$ such that $\mathbf{d}$ is close to $\mathbf{x}$ in the first $k$ components for $k$ large enough. Note that we do not use $\prod_{k=1}^{\infty} D_{k}$ !

Definition 47.0.3 A complete separable metric space is called a polish space.
Theorem 47.0.4 Let $X$ be a polish space. Then there exists $f: \mathbb{N}^{\mathbb{N}} \rightarrow X$ which is onto and continuous. Here $\mathbb{N}^{\mathbb{N}} \equiv \prod_{i=1}^{\infty} \mathbb{N}$ and a metric is given according to the above theorem. Thus for $\mathbf{n}, \mathbf{m} \in \mathbb{N}^{\mathbb{N}}$,

$$
\rho(\mathbf{n}, \mathbf{m}) \equiv \sum_{i=1}^{\infty} 2^{-i} \frac{\left|n_{i}-m_{i}\right|}{1+\left|n_{i}-m_{i}\right|}
$$

Proof: Since $X$ is polish, there exists a countable covering of $X$ by closed sets having diameters no larger than $2^{-1},\{B(i)\}_{i=1}^{\infty}$. Each of these closed sets is also a polish space and so there exists a countable covering of $B(i)$ by a countable collection of closed sets, $\{B(i, j)\}_{j=1}^{\infty}$ each having diameter no larger than $2^{-2}$ where $B(i, j) \subseteq B(i) \neq \emptyset$ for all $j$. Continue this way. Thus

$$
B\left(n_{1}, n_{2}, \cdots, n_{m}\right)=\cup_{i=1}^{\infty} B\left(n_{1}, n_{2}, \cdots, n_{m}, i\right)
$$

and each of $B\left(n_{1}, n_{2}, \cdots, n_{m}, i\right)$ is a closed set contained in $B\left(n_{1}, n_{2}, \cdots, n_{m}\right)$ whose diameter is at most half of the diameter of $B\left(n_{1}, n_{2}, \cdots, n_{m}\right)$. Now we define our mapping from $\mathbb{N}^{\mathbb{N}}$ to $X$. If $\mathbf{n}=\left\{n_{k}\right\}_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$, we let $f(\mathbf{n}) \equiv \cap_{m=1}^{\infty} B\left(n_{1}, n_{2}, \cdots, n_{m}\right)$. Since the diameters of these sets converge to 0 , there exists a unique point in this countable intersection and this is $f(\mathbf{n})$.

We need to verify $f$ is continuous. Let $\mathbf{n} \in \mathbb{N}^{\mathbb{N}}$ be given and suppose $\mathbf{m}$ is very close to $\mathbf{n}$. The only way this can occur is for $n_{k}$ to coincide with $m_{k}$ for many $k$. Therefore, both $f(\mathbf{n})$ and $f(\mathbf{m})$ must be contained in $B\left(n_{1}, n_{2}, \cdots, n_{m}\right)$ for some fairly large $m$. This implies, from the above construction that $f(\mathbf{m})$ is as close to $f(\mathbf{n})$ as $2^{-m}$, proving $f$ is continuous. To see that $f$ is onto, note that from the construction, if $x \in X$, then $x \in$ $B\left(n_{1}, n_{2}, \cdots, n_{m}\right)$ for some choice of $n_{1}, \cdots, n_{m}$ for each $m$. Note nothing is said about $f$ being one to one. It probably is not one to one.
Definition 47.0.5 We call a topological space $X$ a Suslin space if $X$ is a Hausdorff space and there exists a polish space, $Z$ and a continuous function $f$ which maps $Z$ onto $X$.

$$
Z \underset{\text { continuous }}{\substack{\text { onto }}} X
$$

These Suslin spaces are also called analytic sets in some contexts but we will use the term Suslin space in referring to them.
Corollary 47.0.6 $X$ is a Suslin space, if and only if there exists a continuous mapping from $\mathbb{N}^{\mathbb{N}}$ onto $X$.

Proof: We know there exists a polish space $Z$ and a continuous function, $h: Z \rightarrow X$ which is onto. By the above theorem there exists a continuous map, $g: \mathbb{N}^{\mathbb{N}} \rightarrow Z$ which is onto. Then $h \circ g$ is a continuous map from $\mathbb{N}^{\mathbb{N}}$ onto $X$. The "if" part of this theorem is accomplished by noting that $\mathbb{N}^{\mathbb{N}}$ is a polish space.

Lemma 47.0.7 Let $X$ be a Suslin space and suppose $X_{i}$ is a subspace of $X$ which is also a Suslin space. Then $\cup_{i=1}^{\infty} X_{i}$ and $\cap_{i=1}^{\infty} X_{i}$ are also Suslin spaces. Also every Borel set in $X$ is a Suslin space.

Proof: Let $f_{i}: Z_{i} \rightarrow X_{i}$ where $Z_{i}$ is a polish space and $f_{i}$ is continuous and onto. Without loss of generality we may assume the spaces $Z_{i}$ are disjoint because if not, we could replace $Z_{i}$ with $Z_{i} \times\{i\}$. Now we define a metric, $\rho$, for $Z \equiv \cup_{i=1}^{\infty} Z_{i}$ as follows.

$$
\begin{aligned}
& \rho(x, y) \equiv 1 \text { if } x \in Z_{i}, y \in Z_{k}, i \neq k \\
& \rho(x, y) \equiv \frac{d_{i}(x, y)}{1+d_{i}(x, y)} \text { if } x, y \in Z_{i} .
\end{aligned}
$$

Here $d_{i}$ is the metric on $Z_{i}$. It is easy to verify $\rho$ is a metric and that $(Z, \rho)$ is a polish space. Now we define $f: Z \rightarrow \cup_{i=1}^{\infty} X_{i}$ as follows. For $x \in Z_{i}, f(x) \equiv f_{i}(x)$. This is well defined because the $Z_{i}$ are disjoint. If $y$ is very close to $x$ it must be that $x$ and $y$ are in the same $Z_{i}$ otherwise this could not happen. Therefore, continuity of $f$ follows from continuity of $f_{i}$. This shows countable unions of Suslin subspaces of a Suslin space are Suslin spaces.

If $H \subseteq X$ is a closed subset, then, letting $f: Z \rightarrow X$ be onto and continuous, it follows $f: f^{-1}(H) \rightarrow H$ is onto and continuous. Since $f^{-1}(H)$ is closed, it follows $f^{-1}(H)$ is a polish space. Therefore, $H$ is a Suslin space.

Now we show countable intersections of Suslin spaces are Suslin. It is clear that $\theta$ : $\prod_{i=1}^{\infty} Z_{i} \rightarrow \prod_{i=1}^{\infty} X_{i}$ given by $\theta(\mathbf{z}) \equiv \mathbf{x}=\left\{x_{i}\right\}$ where $x_{i}=f_{i}\left(z_{i}\right)$ is continuous and onto, this with respect to the usual product topology. Note that $\prod_{i=1}^{\infty} Z_{i}$ is a polish space because of the assumption that each $Z_{i}$ is and the above considerations. Therefore, $\prod_{i=1}^{\infty} X_{i}$ is a Suslin space. Now let $P \equiv\left\{\mathbf{y} \in \prod_{i=1}^{\infty} f_{i}\left(Z_{i}\right): y_{i}=y_{j}\right.$ for all $\left.i, j\right\}$ (This is how you get it on the intersection. I guess this must be the case where each $X_{i} \subseteq X$ ). Then $P$ is a closed subspace of a Suslin space and so it is Suslin. Then we define $h: P \rightarrow \cap_{i=1}^{\infty} X_{i}$ by $h(\mathbf{y}) \equiv f_{i}\left(y_{i}\right)$. This shows $\cap_{i=1}^{\infty} X_{i}$ is Suslin because $h$ is continuous and onto. ( $h \circ \theta: \theta^{-1}(P) \rightarrow \cap_{i=1}^{\infty} X_{i}$ is continuous and $\theta^{-1}(P)$ being a closed subset of a polish space is polish.)

Next let $U$ be an open subset of $X$. Then $f^{-1}(U)$, being an open subset of a polish space, can be obtained as an increasing limit of closed sets, $K_{n}$. Therefore, $U=\cup_{n=1}^{\infty} f\left(K_{n}\right)$. Each $f\left(K_{n}\right)$ is a Suslin space because it is the continuous image of a polish space, $K_{n}$. Therefore, by the first part of the lemma, $U$ is a Suslin space. Now let

$$
\mathscr{F} \equiv\left\{E \subseteq X: \text { both } E^{C} \text { and } E \text { are Suslin }\right\} .
$$

We see that $\mathscr{F}$ is closed with respect to taking complements. The first part of this lemma shows $\mathscr{F}$ is closed with respect to countable unions. Therefore, $\mathscr{F}$ is a $\sigma$ algebra and so, since it contains the open sets, must contain the Borel sets.

It turns out that Suslin spaces tend to be measurable sets. In order to develop this idea, we need a technical lemma.

Lemma 47.0.8 Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and denote by $\mu^{*}$ the outer measure generated by $\mu$. Thus

$$
\mu^{*}(S) \equiv \inf \{\mu(E): E \supseteq S, E \in \mathscr{F}\}
$$

Then $\mu^{*}$ is regular, meaning that for every $S$, there exists $E \in \mathscr{F}$ such that $E \supseteq S$ and $\mu(E)=\mu^{*}(S)$. If $S_{n} \uparrow S$, it follows that $\mu^{*}\left(S_{n}\right) \uparrow \mu^{*}(S)$. Also if $\mu(\Omega)<\infty$, then a set, $E$ is measurable if and only if

$$
\mu^{*}(\Omega) \geq \mu^{*}(E)+\mu^{*}(\Omega \backslash E)
$$

Proof: First we verify that $\mu^{*}$ is regular. If $\mu^{*}(S)=\infty$, let $E=\Omega$. Then $\mu^{*}(S)=\mu(E)$ and $E \supseteq S$. On the other hand, if $\mu^{*}(S)<\infty$, then we can obtain $E_{n} \in \mathscr{F}$ such that $\mu^{*}(S)+$ $\frac{1}{n} \geq \mu\left(E_{n}\right)$ and $E_{n} \supseteq S$. Now let $F_{n}=\cap_{i=1}^{n} E_{i}$. Then $F_{n} \supseteq S$ and so $\mu^{*}(S)+\frac{1}{n} \geq \mu\left(F_{n}\right) \geq$ $\mu^{*}(S)$. Therefore, letting $F=\cap_{k=1}^{\infty} F_{k} \in \mathscr{F}$ it follows $\mu(F)=\lim _{n \rightarrow \infty} \mu\left(F_{n}\right)=\mu^{*}(S)$.

Let $E_{n} \supseteq S_{n}$ be such that $E_{n} \in \mathscr{F}$ and $\mu\left(E_{n}\right)=\mu^{*}\left(S_{n}\right)$. Also let $E_{\infty} \supseteq S$ such that $\mu\left(E_{\infty}\right)=\mu^{*}(S)$ and $E_{\infty} \in \mathscr{F}$. Now consider $B_{n} \equiv \cup_{k=1}^{n} E_{k}$. We claim

$$
\begin{equation*}
\mu\left(B_{n}\right)=\mu\left(S_{n}\right) \tag{47.0.1}
\end{equation*}
$$

Here is why:

$$
\mu\left(E_{1} \backslash E_{2}\right)=\mu\left(E_{1}\right)-\mu\left(E_{1} \cap E_{2}\right)=\mu^{*}\left(S_{1}\right)-\mu^{*}\left(S_{1}\right)=0
$$

Therefore,

$$
\mu\left(B_{2}\right)=\mu\left(E_{1} \cup E_{2}\right)=\mu\left(E_{1} \backslash E_{2}\right)+\mu\left(E_{2}\right)=\mu\left(E_{2}\right)=\mu^{*}\left(S_{2}\right)
$$

Continuing in this way we see that 47.0 .1 holds. Now let $B_{n} \cap E_{\infty} \equiv C_{n}$. Then $C_{n} \uparrow C \equiv$ $\cup_{k=1}^{\infty} C_{n} \in \mathscr{F}$ and $\mu\left(C_{n}\right)=\mu^{*}\left(S_{n}\right)$. Since $S_{n} \uparrow S$ and each $C_{n} \supseteq S_{n}$, it follows $C \supseteq S$ and therefore,

$$
\mu^{*}(S) \leq \mu(C)=\lim _{n \rightarrow \infty} \mu\left(C_{n}\right)=\lim _{n \rightarrow \infty} \mu^{*}\left(S_{n}\right) \leq \mu^{*}(S)
$$

Now we verify the second claim of the lemma. It is clear the formula holds whenever $E$ is measurable. Suppose now that the formula holds. Let $S$ be an arbitrary set. We need to verify that

$$
\mu^{*}(S) \geq \mu^{*}(S \cap E)+\mu^{*}(S \backslash E)
$$

Let $F \supseteq S, F \in \mathscr{F}$, and $\mu(F)=\mu^{*}(S)$. Then since $\mu^{*}$ is subadditive,

$$
\begin{equation*}
\mu^{*}(\Omega \backslash F) \leq \mu^{*}(E \backslash F)+\mu^{*}\left(\Omega \cap E^{C} \cap F^{C}\right) \tag{47.0.2}
\end{equation*}
$$

Since $F$ is measurable,

$$
\begin{equation*}
\mu^{*}(E)=\mu^{*}(E \cap F)+\mu^{*}(E \backslash F) \tag{47.0.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\mu^{*}(\Omega \backslash E)=\mu^{*}(F \backslash E)+\mu^{*}\left(\Omega \cap E^{C} \cap F^{C}\right) \tag{47.0.4}
\end{equation*}
$$

and by the hypothesis,

$$
\begin{equation*}
\mu^{*}(\Omega) \geq \mu^{*}(E)+\mu^{*}(\Omega \backslash E) \tag{47.0.5}
\end{equation*}
$$

Therefore,

$$
\begin{aligned}
\mu(\Omega) & \geq \mu^{*}(E)+\mu^{*}(\Omega \backslash E) \\
& =\mu^{*}(E \cap F)+\mu^{*}(E \backslash F)+\mu^{*}(\Omega \backslash E) \\
& =\mu^{*}(E \cap F)+\mu^{*}(E \backslash F)+\mu^{*}(F \backslash E)+\mu^{*}\left(\Omega \cap E^{C} \cap F^{C}\right) \\
& \geq \mu^{*}(\Omega \backslash F)+\mu^{*}(F \backslash E)+\mu^{*}(E \cap F) \\
& \geq \mu^{*}(\Omega \backslash F)+\mu^{*}(F)=\mu(\Omega)
\end{aligned}
$$

showing that all the inequalities must be equal signs. Hence, referring to the top and fourth lines above,

$$
\mu(\Omega)=\mu^{*}(\Omega \backslash F)+\mu^{*}(F \backslash E)+\mu^{*}(E \cap F)
$$

Subtracting $\mu^{*}(\Omega \backslash F)=\mu(\Omega \backslash F)$ from both sides gives

$$
\mu^{*}(S)=\mu(F)=\mu^{*}(F \backslash E)+\mu^{*}(E \cap F) \geq \mu^{*}(S \backslash E)+\mu^{*}(E \cap S)
$$

This proves the lemma.
The next theorem is a major result. It states that the Suslin subsets are measurable under appropriate conditions. This is sort of interesting because something being a Suslin subset has to do with topology and this topological condition implies that the set is measurable.

Theorem 47.0.9 Let $\Omega$ be a metric space and let $(\Omega, \mathscr{F}, \mu)$ be a complete Borel measure space with $\mu(\Omega)<\infty$. Denote by $\mu^{*}$ the outer measure generated by $\mu$. Then if $A$ is a Suslin subset of $\Omega$, it follows that $A$ is $\mu^{*}$ measurable. Since the original measure space is complete, it follows that the completion produces nothing new and so in fact A is in $\mathscr{F}$. See Proposition 12.1.5.

Proof: We need to verify that

$$
\mu^{*}(\Omega) \geq \mu^{*}(A)+\mu^{*}(\Omega \backslash A)
$$

We know from Corollary 47.0.6, there exists a continuous map, $f: \mathbb{N}^{\mathbb{N}} \rightarrow A$ which is onto. Let

$$
E(k) \equiv\left\{\mathbf{n} \in \mathbb{N}^{\mathbb{N}}: n_{1} \leq k\right\}
$$

Then $E(k) \uparrow \mathbb{N}^{\mathbb{N}}$ and so from Lemma 47.0.8 we know $\mu^{*}(f(E(k))) \uparrow \mu^{*}(A)$. Therefore, there exists $m_{1}$ such that

$$
\mu^{*}\left(f\left(E\left(m_{1}\right)\right)\right)>\mu^{*}(A)-\frac{\varepsilon}{2} .
$$

Now $E(k)$ is clearly not compact but it is trying to be as far as the first component is concerned. Now we let

$$
E\left(m_{1}, k\right) \equiv\left\{\mathbf{n} \in \mathbb{N}^{\mathbb{N}}: n_{1} \leq m_{1} \text { and } n_{2} \leq k\right\}
$$

Thus $E\left(m_{1}, k\right) \uparrow E\left(m_{1}\right)$ and so we can pick $m_{2}$ such that

$$
\mu^{*}\left(f\left(E\left(m_{1}, m_{2}\right)\right)\right)>\mu^{*}\left(f\left(E\left(m_{1}\right)\right)\right)-\frac{\varepsilon}{2^{2}}
$$

We continue in this way obtaining a decreasing list of sets, $f\left(E\left(m_{1}, m_{2}, \cdots, m_{k-1}, m_{k}\right)\right)$, such that

$$
\mu^{*}\left(f\left(E\left(m_{1}, m_{2}, \cdots, m_{k-1}, m_{k}\right)\right)\right)>\mu^{*}\left(f\left(E\left(m_{1}, m_{2}, \cdots, m_{k-1}\right)\right)\right)-\frac{\varepsilon}{2^{k}}
$$

Therefore,

$$
\mu^{*}\left(f\left(E\left(m_{1}, m_{2}, \cdots, m_{k-1}, m_{k}\right)\right)\right)-\mu^{*}(A)>\sum_{l=1}^{k}-\left(\frac{\varepsilon}{2^{l}}\right)>-\varepsilon
$$

Now define a closed set,

$$
C \equiv \cap_{k=1}^{\infty} \overline{f\left(E\left(m_{1}, m_{2}, \cdots, m_{k-1}, m_{k}\right)\right)}
$$

The sets $f\left(E\left(m_{1}, m_{2}, \cdots, m_{k-1}, m_{k}\right)\right)$ are decreasing as $k \rightarrow \infty$ and so

$$
\mu^{*}(C)=\lim _{k \rightarrow \infty} \mu^{*}\left(\overline{f\left(E\left(m_{1}, m_{2}, \cdots, m_{k-1}, m_{k}\right)\right)}\right) \geq \mu^{*}(A)-\varepsilon .
$$

We wish to verify that $C \subseteq A$. If we can do this we will be done because $C$, being a closed set, is measurable and so

$$
\mu^{*}(\Omega)=\mu^{*}(C)+\mu^{*}(\Omega \backslash C) \geq \mu^{*}(A)-\varepsilon+\mu^{*}(\Omega \backslash A)
$$

Since $\varepsilon$ is arbitrary, this will conclude the proof. Therefore, we only need to verify that $C \subseteq A$.

What we know is that each $f\left(E\left(m_{1}, m_{2}, \cdots, m_{k-1}, m_{k}\right)\right)$ is contained in $A$. We do not know their closures are contained in $A$. We let $\mathbf{m} \equiv\left\{m_{i}\right\}_{i=1}^{\infty}$ where the $m_{i}$ are defined above. Then letting

$$
K \equiv\left\{\mathbf{n} \in \mathbb{N}^{\mathbb{N}}: n_{i} \leq m_{i} \text { for all } i\right\}
$$

we see that $K$ is a closed, hence complete subset of $\mathbb{N}^{\mathbb{N}}$ which is also totally bounded due to the definition of the distance. Therefore, $K$ is compact and so $f(K)$ is also compact, hence closed due to the assumption that $\Omega$ is a Hausdorff space and we know that $f(K) \subseteq A$. We verify that $C=f(K)$. We know $f(K) \subseteq C$. Suppose therefore, $p \in C$. From the definition of $C$, we know there exists $\mathbf{r}^{k} \in E\left(m_{1}, m_{2}, \cdots, m_{k-1}, m_{k}\right)$ such that $d\left(f\left(\mathbf{r}^{k}\right), p\right)<\frac{1}{k}$. Denote by $\widetilde{\mathbf{r}^{k}}$ the element of $\mathbb{N}^{\mathbb{N}}$ which consists of modifying $\mathbf{r}^{k}$ by taking all components after the $k^{t h}$ equal to one. Thus $\widetilde{\mathbf{r}^{k}} \in K$. Now $\left\{\widetilde{\mathbf{r}^{k}}\right\}$ is in a compact set and so taking a subsequence we can have $\widetilde{\mathbf{r}^{k}} \rightarrow \mathbf{r} \in K$. But from the metric on $\mathbb{N}^{\mathbb{N}}$, it follows that $\rho\left(\widetilde{\mathbf{r}^{k}}, \mathbf{r}^{k}\right)<\frac{1}{2^{k-2}}$. Therefore, $\mathbf{r}^{k} \rightarrow \mathbf{r}$ also and so $f\left(\mathbf{r}^{k}\right) \rightarrow f(\mathbf{r})=p$. Therefore, $p \in f(K)$ and this proves the theorem.

Note we could have proved this under weaker assumptions. If we had assumed only that every point has a countable basis (first axiom of countability) and $\Omega$ is Hausdorff, the same argument would work. We will need the following definition.

Definition 47.0.10 Let $\mathscr{F}$ be a $\sigma$ algebra of sets from $\Omega$ and let $\mu$ denote a finite measure defined on $\mathscr{F}$. We let $\mathscr{F} \mu$ denote the completion of $\mathscr{F}$ with respect to $\mu$. Thus we let $\mu^{*}$ be the outer measure determined by $\mu$ and $\mathscr{F}_{\mu}$ will be the $\sigma$ algebra of $\mu^{*}$ measurable subsets of $\Omega$. We also define $\widehat{\mathscr{F}}$ by

$$
\widehat{\mathscr{F}} \equiv \cap\{\mathscr{F} \mu: \mu \text { is a finite measure defined on } \mathscr{F}\}
$$

Also, if $X$ is a topological space, we will denote by $B(X)$ the Borel sets of $X$.
With this notation, we can give the following simple corollary of Theorem 47.0.9. This is really quite amazing.

Corollary 47.0.11 Let $\Omega$ be a compact metric space and let $A$ be a Suslin subset of $\Omega$. Then $A \in \widehat{B(\Omega)}$.

Proof: Let $\mu$ be a finite measure defined on $B(\Omega)$. By Theorem 47.0.9 $A \in B(\Omega)_{\mu}$. Since this is true for every finite measure, $\mu$, it follows $A \in \widehat{B(\Omega)}$ as claimed. This proves the corollary.

We give another technical lemma about the completion of measure spaces.
Lemma 47.0.12 Let $\mu$ be a finite measure on a $\sigma$ algebra, $\Sigma$. Then $A \in \Sigma_{\mu}$ if and only if there exists $A_{1} \in \Sigma$ and $N_{1}$ such that $A=A_{1} \cup N_{1}$ where there exists $N \in \Sigma$ such that $\mu(N)=0$ and $N_{1} \subseteq N$.

Proof: Suppose first $A=A_{1} \cup N_{1}$ where these sets are as described. Let $S \in \mathscr{P}(\Omega)$ and let $\mu^{*}$ denote the outer measure determined by $\mu$. Then since $A_{1} \in \Sigma \subseteq \Sigma_{\mu}$

$$
\begin{aligned}
\mu^{*}(S) & \leq \mu^{*}(S \backslash A)+\mu^{*}(S \cap A) \\
& \leq \mu^{*}\left(S \backslash A_{1}\right)+\mu^{*}\left(S \cap A_{1}\right)+\mu^{*}\left(N_{1}\right) \\
& =\mu^{*}\left(S \backslash A_{1}\right)+\mu^{*}\left(S \cap A_{1}\right)=\mu^{*}(S)
\end{aligned}
$$

showing that $A \in \Sigma_{\mu}$.
Now suppose $A \in \Sigma_{\mu}$. Then there exists $B_{1} \supseteq A$ such that $\mu^{*}\left(B_{1}\right)=\mu^{*}(A)$, and $B_{1} \in \Sigma$. Also there exists $A_{1}^{C} \in \Sigma$ with $A_{1}^{C} \supseteq A^{C}$ and $\mu\left(A_{1}^{C}\right)=\mu^{*}\left(A^{C}\right)$. Then $A_{1} \subseteq A \subseteq B_{1}$

$$
A \subseteq A_{1} \cup\left(B_{1} \backslash A_{1}\right)
$$

Now

$$
\mu\left(A_{1}\right)+\mu^{*}\left(A^{C}\right)=\mu\left(A_{1}\right)+\mu\left(A_{1}^{C}\right)=\mu(\Omega)
$$

and so

$$
\begin{aligned}
\mu\left(B_{1} \backslash A_{1}\right) & =\mu^{*}\left(B_{1} \backslash A_{1}\right) \\
& =\mu^{*}\left(B_{1} \backslash A\right)+\mu^{*}\left(A \backslash A_{1}\right) \\
& =\mu^{*}\left(B_{1}\right)-\mu^{*}(A)+\mu^{*}(A)-\mu^{*}\left(A_{1}\right) \\
& =\mu^{*}(A)-\left(\mu(\Omega)-\mu^{*}\left(A^{C}\right)\right)=0
\end{aligned}
$$

because $A \in \Sigma_{\mu}$ implying $A=A_{1} \cup \overbrace{\left(B_{1} \backslash A_{1}\right) \cap A}^{N_{1}}$ and $N_{1} \subseteq N \equiv\left(B_{1} \backslash A_{1}\right) \in \Sigma$ with $\mu(N)=$ 0 . This proves the lemma.

Next we need another definition.
Definition 47.0.13 We say $(\Omega, \Sigma)$, where $\Sigma$ is a $\sigma$ algebra of subsets of $\Omega$, is separable if there exists a sequence $\left\{A_{n}\right\}_{n=1}^{\infty} \subseteq \Sigma$ such that $\sigma\left(\left\{A_{n}\right\}\right)=\Sigma$ and if $w \neq w^{\prime}$, then there exists $A \in \Sigma$ such that $\mathscr{X}_{A}(\omega) \neq \mathscr{X}_{A}\left(\omega^{\prime}\right)$. This last condition is referred to by saying $\left\{A_{n}\right\}$ separates the points of $\Omega$. Given two measure spaces, $(\Omega, \Sigma)$ and $\left(\Omega^{\prime}, \Sigma^{\prime}\right)$, we say they are isomorphic if there exists a function, $f: \Omega \rightarrow \Omega^{\prime}$ which is one to one and $f(E) \in \Sigma^{\prime}$ whenever $E \in \Sigma$ and $f^{-1}(F) \in \Sigma$ whenever $F \in \Sigma^{\prime}$.

The interesting thing about separable measure spaces is that they are isomorphic to a very simple sort of measure space in which topology plays a significant role.

Lemma 47.0.14 Let $(\Omega, \Sigma)$ be separable. Then there exists $E \in\{0,1\}^{\mathbb{N}}$ such that $(\Omega, \Sigma)$ and $(E, B(E))$ are isomorphic.

Proof: First we show $\left\{A_{n}\right\}$ separates the points. Here $\sigma\left(\left\{A_{n}\right\}\right)=\Sigma$. We already know $\Sigma$ separates the points but now we show the smaller set does so also. If this is not so, there exists $\omega, \omega_{1} \in \Omega$ such that for all $n, \mathscr{X}_{A_{n}}(\omega)=\mathscr{X}_{A_{n}}\left(\omega_{1}\right)$. Then let

$$
\mathscr{F} \equiv\left\{F \in \Sigma: \mathscr{X}_{F}(\omega)=\mathscr{X}_{F}\left(\omega_{1}\right)\right\}
$$

Thus $A_{n} \in \mathscr{F}$ for all $n$. It is also clear that $\mathscr{F}$ is a $\sigma$ algebra and so $\mathscr{F}=\Sigma$ contradicting the assumption that $\Sigma$ separates points. Now we define a function from $\Omega$ to $\{0,1\}^{\mathbb{N}}$ as follows.

$$
f(\omega) \equiv\left\{\mathscr{X}_{A_{n}}(\omega)\right\}_{n=1}^{\infty}
$$

We also let $E \equiv f(\Omega)$. Since the $\left\{A_{n}\right\}$ separate the points, we see that $f$ is one to one. A subbasis for the topology of $\{0,1\}^{\mathbb{N}}$ consists of sets of the form $\prod_{i=1}^{\infty} H_{i}$ where $H_{i}=\{0,1\}$ for all $i$ except one, when $i=j$ and $H_{j}$ equals either $\{0\}$ or $\{1\}$. Therefore,

$$
f^{-1}(\text { subbasic open set }) \in \Sigma
$$

because if $H_{j}$ is the exceptional set then this equals $A_{j}$ if $H_{j}=\{1\}$ and $A_{j}^{C}$ if $H_{j}=\{0\}$. Intersections of these subbasic sets with $E$ gives a countable subbasis for $E$ and so the inverse image of all sets in a countable subbasis for $E$ are in $\Sigma$, showing that $f^{-1}$ (open set) $\in \Sigma$. Now we consider $f\left(A_{n}\right)$.

$$
f\left(A_{n}\right) \equiv\left\{\left\{\lambda_{k}\right\}_{k=1}^{\infty}: \lambda_{n}=1\right\} \cap E,
$$

an open set in $E$. Hence $f\left(A_{n}\right) \in B(E)$. Now letting

$$
\mathscr{F} \equiv\{G \subseteq \Omega: f(G) \in B(E)\},
$$

we see that $\mathscr{F}$ is a $\sigma$ algebra which contains $\left\{A_{n}\right\}_{n=1}^{\infty}$ and so $\mathscr{F} \supseteq \sigma\left(\left\{A_{n}\right\}\right) \equiv \Sigma$. Thus $f(F) \in B(E)$ for all $A \in \Sigma$. This proves the lemma.

Lemma 47.0.15 Let $\phi:\left(\Omega_{1}, \Sigma_{1}\right) \rightarrow\left(\Omega_{2}, \Sigma_{2}\right)$ where $\phi^{-1}(U) \in \Sigma_{1}$ for all $U \in \Sigma_{2}$. Then if $F \in \widehat{\Sigma}_{2}$, it follows $\phi^{-1}(F) \in \widehat{\Sigma}_{1}$.

Proof: Let $\mu$ be a finite measure on $\Sigma_{1}$ and define a measure $\phi(\mu)$ on $\Sigma_{2}$ by the rule

$$
\phi(\mu)(F) \equiv \mu\left(\phi^{-1}(F)\right)
$$

Now let $A \in \Sigma_{2 \phi(\mu)}$. Then by Lemma 47.0.12, $A=A_{1} \cup N_{1}$ where there exists $N \in \Sigma_{2}$ with $\phi(\mu)(N)=0$ and $A_{1} \in \Sigma_{2}$. Therefore, from the definition of $\phi(\mu)$, we have $\mu\left(\phi^{-1}(N)\right)=$ 0 and therefore, $\phi^{-1}(A)=\phi^{-1}\left(A_{1}\right) \cup \phi^{-1}\left(N_{1}\right)$ where

$$
\phi^{-1}\left(N_{1}\right) \subseteq \phi^{-1}(N) \in \Sigma_{1}
$$

and $\mu\left(\phi^{-1}(N)\right)=0$. Therefore, $\phi^{-1}(A) \in \Sigma_{1 \mu}$ and so if $F \in \widehat{\Sigma}_{2}$, then
$F \in \cap\left\{\Sigma_{2 v}: v\right.$ is a finite measure on $\left.\Sigma_{2}\right\} \subseteq \cap\left\{\Sigma_{2 \phi(\mu)}: \mu\right.$ is a finite measure on $\left.\Sigma_{1}\right\}$,
and so $\phi^{-1}(F) \in \Sigma_{1 \mu}$. Since $\mu$ is arbitrary, this shows $\phi^{-1}(F) \in \widehat{\Sigma}_{1}$.
The next lemma is a special case of the Yankov von Neumann Aumann projection theorem. It contains the main idea of the proof of the more general theorem.

Definition 47.0.16 Let $(\Omega, \Sigma)$ be a measurable space and let $G \in \Sigma \times B(X)$ the product measurable sets resulting from the Borel sets $B(X)$ for $X$ a Suslan space. Then

$$
\operatorname{proj}_{\Omega}(G) \equiv\{\omega \in \Omega: \text { there exists } x \in X \text { with }(\omega, x) \in G\}
$$

What if you had $G=\cup\{(\omega \times A(\omega)): \omega \in \Omega\}$ where $A(\omega)$ is in $Y$ some polish space and $A(\omega)$ is, for example, a closed nonempty set? Then

$$
\operatorname{proj}_{\Omega}(G)=\{\omega \in \Omega: A(\omega) \cap X \neq \emptyset\}
$$

Of course, you might ask whether this particular $G$ is in $\Sigma \times B(X)$. For fixed $\omega$ the $\omega$ section is closed which is Borel. To say that $\operatorname{proj}_{\Omega}(G) \in \Sigma$ would be to say that each $y$ section is in $\Sigma$ so this $G$ is in $\Sigma \times B(X)$ exactly when $\operatorname{proj}_{\Omega}(G) \in \Sigma$. Later this will be defined as saying that $A$ is a measurable multifunction. It will be measurable when $X$ is open, strongly measurable when $X$ is closed.

Lemma 47.0.17 Let $(\Omega, \Sigma)$ be separable and let $X$ be a Suslin space. Let $G \in \Sigma \times B(X)$. (Recall $\Sigma \times B(X)$ is the $\sigma$ algebra of product measurable sets, the smallest $\sigma$ algebra containing the measurable rectangles.) Then

$$
\operatorname{proj}_{\Omega}(G) \in \widehat{\Sigma}
$$

Proof: Let $f:(\Omega, \Sigma) \rightarrow(E, B(E))$ be the isomorphism of Lemma 47.0.14. We have the following claim.

Claim: $f \times \mathrm{id}_{X}$ maps $\Sigma \times B(X)$ to $B(E) \times B(X)$.
Proof of the claim: First of all, assume $A \times B$ is a measurable rectangle where $A \in \Sigma$ and $B \in B(X)$. Then by the assumption that $f$ is an isomorphism, $f(A) \in B(E)$ and so

$$
f \times \operatorname{id}_{X}(A \times B) \in B(E) \times B(X) .
$$

Now let

$$
\mathscr{F} \equiv\left\{P \in \Sigma \times B(X): f \times \operatorname{id}_{X}(P) \in B(E) \times B(X)\right\} .
$$

Then we see that $\mathscr{F}$ is a $\sigma$ algebra and contains the elementary sets. ( $\mathscr{F}$ is closed with respect to complements because $f$ is one to one.) Therefore, $\mathscr{F}=\Sigma \times B(X)$ and this proves the claim.

Therefore, since $G \in \Sigma \times B(X)$, we see

$$
f \times \operatorname{id}_{X}(G) \in B(E) \times B(X) \subseteq B(E \times X) .
$$

The set inclusion follows from the observation that if $A \in B(E)$ and $B \in B(X)$ then $A \times B$ is in $B(E \times X)$ and the collection of sets in $B(E) \times B(X)$ which are in $B(E \times X)$ is a $\sigma$ algebra.

Therefore, there exists $D$, a Borel set in $E \times X$ such that $f \times \mathrm{id}_{X}(G)=D \cap(E \times X)$. Now from this it follows from Lemma 47.0.7 that $D$ is a Suslin space. Letting $Y$ be $\{0,1\}^{\mathbb{N}}$, it follows that $\operatorname{proj}_{Y}(D)$ is a Suslin space in $Y$. By Corollary 47.0.11, we see that $\operatorname{proj}_{Y}(D) \in$ $\widehat{B(Y)}$. Now

$$
\begin{gathered}
\operatorname{proj}_{\Omega}(G)=\{\omega \in \Omega \text { : there exists } x \in X \text { with }(\omega, x) \in G\} \\
=\left\{\omega \in \Omega: \text { there exists } x \in X \text { with }(f(\omega), x) \in f \times \operatorname{id}_{X}(G)\right\} \\
=f^{-1}(\{y \in Y: \text { there exists } x \in X \text { with }(y, x) \in D\}) \\
=f^{-1}\left(\operatorname{proj}_{Y}(D)\right)
\end{gathered}
$$

Now $\operatorname{proj}_{Y}(D) \in \widehat{B(Y)}$ and so Lemma 47.0.15 shows $f^{-1}\left(\operatorname{proj}_{Y}(D)\right) \in \widehat{\Sigma}$. This proves the lemma.

Now we are ready to prove the Yankov von Neumann Aumann projection theorem. First we must present another technical lemma.

Lemma 47.0.18 Let $X$ be a Hausdorff space and let $G \in \Sigma \times B(X)$ where $\Sigma$ is a $\sigma$ algebra of sets of $\Omega$. Then there exists $\Sigma_{0} \subseteq \Sigma$ a countably generated $\sigma$ algebra such that $G \in$ $\Sigma_{0} \times B(X)$.

Proof: First suppose $G$ is a measurable rectangle, $G=A \times B$ where $A \in \Sigma$ and $B \in$ $B(X)$. Letting $\Sigma_{0}$ be the finite $\sigma$ algebra, $\left\{\emptyset, A, A^{C}, \Omega\right\}$, we see that $G \in \Sigma_{0} \times B(X)$. Similarly, if $G$ equals an elementary set, then the conclusion of the lemma holds for $G$. Let

$$
\mathscr{F} \equiv\left\{H \in \Sigma \times B(X): H \in \Sigma_{0} \times B(X)\right\}
$$

for some countably generated $\sigma$ algebra, $\Sigma_{0}$. We just saw that $\mathscr{F}$ contains the elementary sets. If $H \in \mathscr{F}$, then $H^{C} \in \Sigma_{0} \times B(X)$ for the same $\Sigma_{0}$ and so $\mathscr{F}$ is closed with respect to complements. Now suppose $H_{n} \in \mathscr{F}$. Then for each $n$, there exists a countably generated $\sigma$ algebra, $\Sigma_{0 n}$ such that $H_{n} \in \Sigma_{0 n} \times B(X)$. Then $\cup_{n=1}^{\infty} H_{n} \in \sigma\left(\left\{\Sigma_{0 n} \times B(X)\right\}\right)$. We will be done when we show

$$
\sigma\left(\left\{\Sigma_{0 n} \times B(X)\right\}_{n=1}^{\infty}\right) \subseteq \sigma\left(\left\{\Sigma_{0 n}\right\}_{n=1}^{\infty}\right) \times B(X)
$$

because it is clear that $\sigma\left(\left\{\Sigma_{0 n}\right\}_{n=1}^{\infty}\right)$ is countably generated. We see that

$$
\sigma\left(\left\{\Sigma_{0 n} \times B(X)\right\}_{n=1}^{\infty}\right)
$$

is generated by sets of the form $A \times B$ where $A \in \Sigma_{0 n}$ and $B \in B(X)$. But each such set is also contained in $\sigma\left(\left\{\Sigma_{0 n}\right\}_{n=1}^{\infty}\right) \times B(X)$ and so the desired inclusion is obtained. Therefore, $\mathscr{F}$ is a $\sigma$ algebra and so since $\mathscr{F}$ was shown to contain the measurable rectangles, this verifies $\mathscr{F}=\Sigma \times B(X)$ and this proves the lemma.

Theorem 47.0.19 Let $(\Omega, \Sigma)$ be a measure space and let $G \in \widehat{\Sigma} \times B(X)$ where $X$ is a Suslin space. Then

$$
\operatorname{proj}_{\Omega}(G) \in \widehat{\Sigma}
$$

Proof: By the previous lemma, $G \in \Sigma_{0} \times B(X)$ where $\Sigma_{0}$ is countably generated. If $\left(\Omega, \Sigma_{0}\right)$ were separable, we could then apply Lemma 47.0.17 and be done. Unfortunately, we don't know $\Sigma_{0}$ separates the points of $\Omega$. Therefore, we define an equivalence class on the points of $\Omega$ as follows. We say $\omega \backsim \omega_{1}$ if and only if $\mathscr{X}_{A}(\omega)=\mathscr{X}_{A}\left(\omega_{1}\right)$ for all $A \in \Sigma_{0}$. Now the nice thing to notice about this equivalence relation is that if $\omega \in A \in \Sigma_{0}$, and if $\omega \backsim \omega_{1}$, then $1=\mathscr{X}_{A}(\omega)=\mathscr{X}_{A}\left(\omega_{1}\right)$ implying $\omega_{1} \in A$ also. Therefore, every set of $\Sigma_{0}$ is the union of equivalence classes. It follows that for $A \in \Sigma_{0}$, and $\pi$ the map given by $\pi \omega \equiv[\omega]$ where $[\omega]$ is the equivalence class determined by $\omega$,

$$
\pi(A) \cap \pi(\Omega \backslash A)=\emptyset
$$

Suppose now that $H_{n} \in \Sigma_{0} \times B(X)$. If $([\omega], x) \in \cap_{n=1}^{\infty} \pi \times \mathrm{id}_{X}\left(H_{n}\right)$, then for each $n$,

$$
([w], x)=\left(\pi w_{n}, x\right)
$$

for some $\left(\omega_{n}, x\right) \in H_{n}$. But this implies $\omega \backsim \omega_{n}$ and so from the above observation that the sets of $\Sigma_{0}$ are unions of equivalence classes, it follows that $(\omega, x) \in H_{n}$. Therefore, $(\omega, x) \in \cap_{n=1}^{\infty} H_{n}$ and so $([\omega], x)=\pi \times \operatorname{id}_{X}(\omega, x)$ where $(\omega, x) \in \cap_{n=1}^{\infty} H_{n}$. This shows that

$$
\pi \times \operatorname{id}_{X}\left(\cap_{n=1}^{\infty} H_{n}\right) \supseteq \cap_{n=1}^{\infty} \pi \times \operatorname{id}_{X}\left(H_{n}\right)
$$

In fact these two sets are equal because the other inclusion is obvious. We will denote by $\Omega_{1}$ the set of equivalence classes and $\Sigma_{1}$ will be the subsets, $S_{1}$, of $\Omega_{1}$ such that $S_{1}=$ $\left\{[\omega]: \omega \in S \in \Sigma_{0}\right\}$. Then $\left(\Omega_{1}, \Sigma_{1}\right)$ is clearly a measure space which is separable. Let

$$
\mathscr{F} \equiv\left\{H \in \Sigma_{0} \times B(X): \pi \times \operatorname{id}_{X}(H), \pi \times \operatorname{id}_{X}\left(H^{C}\right) \in \Sigma_{1} \times B(X)\right\}
$$

We see that the measurable rectangles, $A \times B$ where $A \in \Sigma_{0}$ and $B \in B(X)$ are in $\mathscr{F}$, that from the above observation on countable intersections, $\mathscr{F}$ is closed with respect to countable unions and closed with respect to complements. Therefore, $\mathscr{F}$ is a $\sigma$ algebra and so $\mathscr{F}=\Sigma_{0} \times B(X)$. By Lemma 47.0.14 $\left(\Omega_{1}, \Sigma_{1}\right)$ is isomorphic to $(E, B(E))$ where $E$ is a subspace of $\{0,1\}^{\mathbb{N}}$. Denoting the isomorphism by $h$, it follows as in Lemma 47.0.17 that $h \times \mathrm{id}_{X}$ maps $\Sigma_{1} \times B(X)$ to $B(E) \times B(X)$. Therefore, we see $f \equiv h \circ \pi$ is a mapping from $\Omega$ to $E$ which has the property that $f \times \operatorname{id}_{X}$ maps $\Sigma_{0} \times B(X)$ to $B(E) \times B(X)$. Now from the proof of Lemma 47.0 .17 starting with the claim, we see that $G \in \widehat{\Sigma}_{0}$. However, if $\mu$ is a finite measure on $\widehat{\Sigma}$, then $(\widehat{\Sigma})_{\mu}=\Sigma_{\mu}$ and so $\widehat{\Sigma}_{0} \subseteq \widehat{(\widehat{\Sigma})} \subseteq \widehat{\Sigma}$. This proves the theorem.

## Chapter 48

## Multifunctions and Their Measurability

### 48.1 The General Case

Let $X$ be a separable complete metric space and let $(\Omega, \mathscr{C}, \mu)$ be a set, a $\sigma$ algebra of subsets of $\Omega$, and a measure $\mu$ such that this is a complete $\sigma$ finite measure space. Also let $\Gamma: \Omega \rightarrow \mathscr{P}_{F}(X)$, the closed subsets of $X$.

Definition 48.1.1 We define $\Gamma^{-}(S) \equiv\{\omega \in \Omega: \Gamma(\omega) \cap S \neq \emptyset\}$
We will consider a theory of measurability of set valued functions. The following theorem is the main result in the subject. In this theorem the third condition is what we will refer to as measurable.

Theorem 48.1.2 The following are equivalent in case of a complete $\sigma$ finite measure space. However 3 and 4 are equivalent for any measurable space consisting only of a set $\Omega$ and $a \sigma$ algebra $\mathscr{C}$.

1. For all B a Borel set in $X, \Gamma^{-}(B) \in \mathscr{C}$.
2. For all $F$ closed in $X, \Gamma^{-}(F) \in \mathscr{C}$
3. For all $U$ open in $X, \Gamma^{-}(U) \in \mathscr{C}$
4. There exists a sequence, $\left\{\sigma_{n}\right\}$ of measurable functions satisfying $\sigma_{n}(\omega) \in \Gamma(\omega)$ such that for all $\omega \in \Omega$,

$$
\Gamma(\omega)=\overline{\left\{\sigma_{n}(\omega): n \in \mathbb{N}\right\}}
$$

These functions are called measurable selections.
5. For all $x \in X, \omega \rightarrow \operatorname{dist}(x, \Gamma(\omega))$ is a measurable real valued function.
6. $\mathscr{G}(\Gamma) \equiv\{(\omega, x): x \in \Gamma(\omega)\} \subseteq \mathscr{C} \times B(X)$.

Proof: It is obvious that 1.) $\Rightarrow 2$.). To see that 2.) $\Rightarrow$ 3.) note that $\Gamma^{-}\left(\cup_{i=1}^{\infty} F_{i}\right)=$ $\cup_{i=1}^{\infty} \Gamma^{-}\left(F_{i}\right)$. Since any open set in $X$ can be obtained as a countable union of closed sets, this implies 2.) $\Rightarrow 3$.).

Now we verify that 3.) $\Rightarrow 4$.). For convenience, drop the assumption that $\Gamma(\omega)$ is closed in this part of the argument. It will just be set valued and satisfy the measurability condition. A measurable selection will be obtained in $\overline{\Gamma(\omega)}$. Let $\left\{x_{n}\right\}_{n=1}^{\infty}$ be a countable dense subset of $X$. For $\omega \in \Omega$, let $\psi_{1}(\omega)=x_{n}$ where $n$ is the smallest integer such that $\Gamma(\omega) \cap B\left(x_{n}, 1\right) \neq \emptyset$. Therefore, $\psi_{1}(\omega)$ has countably many values, $x_{n_{1}}, x_{n_{2}}, \cdots$ where $n_{1}<$ $n_{2}<\cdots$. Now

$$
\begin{gathered}
\left\{\omega: \psi_{1}=x_{n}\right\}= \\
\left\{\omega: \Gamma(\omega) \cap B\left(x_{n}, 1\right) \neq \emptyset\right\} \cap\left[\Omega \backslash \cup_{k<n}\left\{\omega: \Gamma(\omega) \cap B\left(x_{k}, 1\right) \neq \emptyset\right\}\right] \in \mathscr{C} .
\end{gathered}
$$

Thus we see that $\psi_{1}$ is measurable and $\operatorname{dist}\left(\psi_{1}(\omega), \Gamma(\omega)\right)<1$. Let

$$
\Omega_{n} \equiv\left\{\omega \in \Omega: \psi_{1}(\omega)=x_{n}\right\}
$$

Then $\Omega_{n} \in \mathscr{C}$ and $\Omega_{n} \cap \Omega_{m}=\emptyset$ for $n \neq m$ and $\cup_{n=1}^{\infty} \Omega_{n}=\Omega$. Let

$$
D_{n} \equiv\left\{x_{k}: x_{k} \in B\left(x_{n}, 1\right)\right\}
$$

Now for each $n$, and $\omega \in \Omega_{n}$, let $\psi_{2}(\omega)=x_{k}$ where $k$ is the smallest index such that $x_{k} \in D_{n}$ and $B\left(x_{k}, \frac{1}{2}\right) \cap \Gamma(\omega) \neq \emptyset$. Thus dist $\left(\psi_{2}(\omega), \Gamma(\omega)\right)<\frac{1}{2}$ and

$$
d\left(\psi_{2}(\omega), \psi_{1}(\omega)\right)<1
$$

Continue this way obtaining $\psi_{k}$ a measurable function such that

$$
\operatorname{dist}\left(\psi_{k}(\omega), \Gamma(\omega)\right)<\frac{1}{2^{k-1}}, d\left(\psi_{k}(\omega), \psi_{k+1}(\omega)\right)<\frac{1}{2^{k-2}}
$$

Then for each $\omega,\left\{\psi_{k}(\omega)\right\}$ is a Cauchy sequence converging to a point, $\sigma(\omega) \in \overline{\Gamma(\omega)}$. This has shown that if $\Gamma$ is measurable, there exists a measurable selection, $\sigma(\omega) \in \overline{\Gamma(\omega)}$. Of course, if $\Gamma(\omega)$ is closed, then $\sigma(\omega) \in \Gamma(\omega)$. Note that this had nothing to do with the measure. It remains to show there exists a sequence of these measurable selections $\sigma_{n}$ such that the conclusion of 4.) holds. To do this we define for $\Gamma(\omega)$ closed and measurable,

$$
\Gamma_{n i}(\omega) \equiv\left\{\begin{array}{l}
\Gamma(\omega) \cap B\left(x_{n}, 2^{-i}\right) \text { if } \Gamma(\omega) \cap B\left(x_{n}, 2^{-i}\right) \neq \emptyset \\
\Gamma(\omega) \text { otherwise }
\end{array}\right.
$$

Thus in the case of nonempty intersecton in the above cases,

$$
\Gamma(\omega) \cap \overline{B\left(x_{n}, 2^{-(i+1)}\right)} \subseteq \overline{\Gamma_{n i}(\omega)} \subseteq \Gamma(\omega) \cap \overline{B\left(x_{n}, 2^{-i}\right)}
$$

First we show that $\Gamma_{n i}$ is measurable. Let $U$ be open. Then

$$
\begin{gathered}
\left\{\omega: \Gamma_{n i}(\omega) \cap U \neq \emptyset\right\}=\left\{\omega: \Gamma(\omega) \cap B\left(x_{n}, 2^{-i}\right) \cap U \neq \emptyset\right\} \cup \\
{\left[\left\{\omega: \Gamma(\omega) \cap B\left(x_{n}, 2^{-i}\right)=\emptyset\right\} \cap\{\omega: \Gamma(\omega) \cap U \neq \emptyset\}\right]} \\
=\left\{\omega: \Gamma(\omega) \cap B\left(x_{n}, 2^{-i}\right) \cap U \neq \emptyset\right\} \cup \\
{\left[\left(\Omega \backslash\left\{\omega: \Gamma(\omega) \cap B\left(x_{n}, 2^{-i}\right) \neq \emptyset\right\}\right) \cap\{\omega: \Gamma(\omega) \cap U \neq \emptyset\}\right],}
\end{gathered}
$$

a measurable set. By what was just shown, there exists $\sigma_{n i}$, a measurable function such that $\sigma_{n i}(\omega) \in \overline{\Gamma_{n i}(\omega)} \subseteq \Gamma(\omega)$ for all $\omega \in \Omega$. If $x \in \Gamma(\omega)$, then

$$
x \in \overline{B\left(x_{n}, 2^{-(i+2)}\right)}
$$

whenever $x_{n}$ is close enough to $x$. Thus both $x, \sigma_{n(i+2)}(\omega)$ are in $B\left(x_{n}, 2^{-(i+2)}\right)$ and so $\left|\sigma_{n(i+1)}(\omega)-x\right|<2^{-i}$. It follows that condition 4.) holds. Note that this had nothing to do with the measure.

Now we verify that 4.) $\Rightarrow$ 3.). Suppose there exist measurable selections $\sigma_{n}(\omega) \in$ $\Gamma(\omega)$ satisfying condition 4 .). Let $U$ be open. Then

$$
\{\omega: \Gamma(\omega) \cap U \neq \emptyset\}=\cup_{n=1}^{\infty} \sigma_{n}^{-1}(U) \in \mathscr{C} .
$$

Now we verify that 4.) $\Rightarrow$ 5.). Let $F(\omega) \equiv \operatorname{dist}(x, \Gamma(\omega))$. Then letting $U$ be an open set in $[0, \infty), F(\omega) \in U$ if and only if $d\left(x, \sigma_{n}(\omega)\right) \in U$ for some $\sigma_{n}(\omega)$. Let $h_{n}(\omega) \equiv$ $d\left(x, \sigma_{n}(\omega)\right)$. Then $h_{n}$ is measurable and $F^{-1}(U)=\cup_{n=1}^{\infty} h_{n}^{-1}(U) \in \mathscr{C}$. This shows that for all $x \in X, \omega \rightarrow \operatorname{dist}(x, \Gamma(\omega))$ is measurable and this proves 5 .).

Now we verify that 5.$) \Rightarrow 4$.). We know $\operatorname{dist}(x, \Gamma(\cdot))$ is measurable and we show $\{\omega: \Gamma(\omega) \cap U \neq \emptyset\} \in \mathscr{C}$ whenever $U$ is open and then use 3.) $\Rightarrow 4$.). Since $X$ is separable, there exists $B\left(x_{i}, r_{i}\right)$ such that $U=\cup_{i=1}^{\infty} B\left(x_{i}, r_{i}\right)$. Then

$$
\begin{aligned}
\{\omega: \Gamma(\omega) \cap U \neq \emptyset\} & =\cup_{i=1}^{\infty}\left\{\omega: \Gamma(\omega) \cap B\left(x_{i}, r_{i}\right) \neq \emptyset\right\} \\
& =\cup_{i=1}^{\infty}\left\{\omega: \operatorname{dist}\left(x_{i}, \Gamma(\omega)\right)<r_{i}\right\} \in \mathscr{C} .
\end{aligned}
$$

Therefore, 5.) $\Rightarrow 4$.) as claimed.
Now we must prove 5.) $\Rightarrow 6$.). We note that $\omega \rightarrow \operatorname{dist}(x, \Gamma(\omega))$ is measurable and $x \rightarrow \operatorname{dist}(x, \Gamma(\omega))$ is continuous. Also, the graph of $\Gamma, \mathscr{G}(\Gamma)$ is given by

$$
\mathscr{G}(\Gamma)=\{(\omega, x): \operatorname{dist}(x, \Gamma(\omega))=0\} .
$$

We wish to show that $(\omega, x) \rightarrow \operatorname{dist}(x, \Gamma(\omega))$ is product measurable because then $\mathscr{G}(\Gamma)$, being the inverse image of $\{0\}$ will be product measurable. Let $\left\{x_{k}\right\}$ be a countable dense set in $X$ and let

$$
\phi_{k}(\omega, x) \equiv \operatorname{dist}\left(x_{n}, \Gamma(\omega)\right)
$$

where $n$ is the first index such that $x \in B\left(x_{n}, 2^{-k}\right)$. Then $\phi_{k}(\omega, x) \rightarrow \operatorname{dist}(x, \Gamma(\omega))$ due to the continuity of $x \rightarrow \operatorname{dist}(x, \Gamma(\omega))$ and so we must argue that $\phi_{k}$ is product measurable. On

$$
E_{n} \equiv \Omega \times\left(B\left(x_{n}, 2^{-k}\right) \backslash \cup_{m<n} B\left(x_{m}, 2^{-k}\right)\right)
$$

$\phi_{k}(\omega, x)=\operatorname{dist}\left(x_{n}, \Gamma(\omega)\right)$. Thus, on this set, $\phi_{k}$ equals a measurable function of $\omega$ and does not depend on $x$ on $E_{n}$. It follows that there are measurable simple $\mathscr{C}$ measurable functions, $s_{m}(\omega)$ which increase pointwise to $\operatorname{dist}\left(x_{n}, \Gamma(\omega)\right)$ on $E_{n}$. Thus $s_{m}(\omega) \mathscr{X}_{E_{n}}(x)$ increases to $\phi_{k}(\omega, x)$ on $E_{n}$ showing that $\phi_{k} \mathscr{X}_{E_{n}}$ is product measurable with respect to $\mathscr{C} \times \sigma(\tau)$ since $E_{n}$ is a measurable rectangle with respect to $\mathscr{C}$ and $\sigma(\tau)$. Therefore, $\phi_{k}$ is product measurable and so $(\omega, x) \rightarrow \operatorname{dist}(x, \Gamma(\omega))$ is also product measurable.

It remains to prove 6 .) $\Rightarrow 1$.). This follows from Theorem 47.0.19.

$$
\begin{gathered}
\Gamma^{-}(B) \equiv\{\omega: \Gamma(\omega) \cap B \neq \emptyset\} \\
=\operatorname{proj}_{\Omega}(\mathscr{G}(\Gamma) \cap(\Omega \cap B))
\end{gathered}
$$

But from Theorem 47.0.19, $\operatorname{proj}_{\Omega}(\mathscr{G}(\Gamma) \cap(\Omega \cap B)) \in \hat{C} \subseteq \mathscr{C}_{\mu}=\mathscr{C}$.
The last part results from $(\Omega, \mathscr{C}, \mu)$ being a complete measure space. Note that without this assumption we could not draw the conclusion desired. This required consideration of the measure. The following theorem is like part of the above but without an assumption that $\Gamma(\omega)$ is closed.

Theorem 48.1.3 The following are equivalent for any measurable space consisting only of a set $\Omega$ and a $\sigma$ algebra $\mathscr{C}$. Here nothing is known about $\Gamma(\omega)$ other than that is $a$ nonempty set.

1. For all $U$ open in $X, \Gamma^{-}(U) \in \mathscr{C}$

$$
\Gamma^{-}(U) \equiv\{\omega: \Gamma(\omega) \cap U \neq \emptyset\}
$$

2. There exists a sequence, $\left\{\sigma_{n}\right\}$ of measurable functions satisfying $\sigma_{n}(\omega) \in \Gamma(\omega)$ such that for all $\omega \in \Omega$,

$$
\overline{\Gamma(\omega)}=\overline{\left\{\sigma_{n}(\omega): n \in \mathbb{N}\right\}}
$$

These functions are called measurable selections.
Proof: First 1.) $\Rightarrow 2$.). A measurable selection will be obtained in $\overline{\Gamma(\omega)}$. Let $\left\{x_{n}\right\}_{n=1}^{\infty}$ be a countable dense subset of $X$. For $\omega \in \Omega$, let $\psi_{1}(\omega)=x_{n}$ where $n$ is the smallest integer such that $\Gamma(\omega) \cap B\left(x_{n}, 1\right) \neq \emptyset$. Therefore, $\psi_{1}(\omega)$ has countably many values, $x_{n_{1}}, x_{n_{2}}, \cdots$ where $n_{1}<n_{2}<\cdots$. Now

$$
\begin{gathered}
\left\{\omega: \psi_{1}=x_{n}\right\}= \\
\left\{\omega: \Gamma(\omega) \cap B\left(x_{n}, 1\right) \neq \emptyset\right\} \cap\left[\Omega \backslash \cup_{k<n}\left\{\omega: \Gamma(\omega) \cap B\left(x_{k}, 1\right) \neq \emptyset\right\}\right] \in \mathscr{C} .
\end{gathered}
$$

Thus we see that $\psi_{1}$ is measurable and $\operatorname{dist}\left(\psi_{1}(\omega), \Gamma(\omega)\right)<1$. Let

$$
\Omega_{n} \equiv\left\{\omega \in \Omega: \psi_{1}(\omega)=x_{n}\right\}
$$

Then $\Omega_{n} \in \mathscr{C}$ and $\Omega_{n} \cap \Omega_{m}=\emptyset$ for $n \neq m$ and $\cup_{n=1}^{\infty} \Omega_{n}=\Omega$. Let

$$
D_{n} \equiv\left\{x_{k}: x_{k} \in B\left(x_{n}, 1\right)\right\}
$$

Now for each $n$, and $\omega \in \Omega_{n}$, let $\psi_{2}(\omega)=x_{k}$ where $k$ is the smallest index such that $x_{k} \in D_{n}$ and $B\left(x_{k}, \frac{1}{2}\right) \cap \Gamma(\omega) \neq \emptyset$. Thus dist $\left(\psi_{2}(\omega), \Gamma(\omega)\right)<\frac{1}{2}$ and

$$
d\left(\psi_{2}(\omega), \psi_{1}(\omega)\right)<1
$$

Continue this way obtaining $\psi_{k}$ a measurable function such that

$$
\operatorname{dist}\left(\psi_{k}(\omega), \Gamma(\omega)\right)<\frac{1}{2^{k-1}}, d\left(\psi_{k}(\omega), \psi_{k+1}(\omega)\right)<\frac{1}{2^{k-2}}
$$

Then for each $\omega,\left\{\psi_{k}(\omega)\right\}$ is a Cauchy sequence converging to a point, $\sigma(\omega) \in \overline{\Gamma(\omega)}$. This has shown that if $\Gamma$ is measurable, there exists a measurable selection, $\sigma(\omega) \in \overline{\Gamma(\omega)}$. Of course, if $\Gamma(\omega)$ is closed, then $\sigma(\omega) \in \Gamma(\omega)$. Note that this had nothing to do with a measure.

It remains to show there exists a sequence of these measurable selections $\sigma_{n}$ such that the conclusion of 2.) holds. To do this we define

$$
\Gamma_{n i}(\omega) \equiv\left\{\begin{array}{l}
\Gamma(\omega) \cap B\left(x_{n}, 2^{-i}\right) \text { if } \Gamma(\omega) \cap B\left(x_{n}, 2^{-i}\right) \neq \emptyset \\
\Gamma(\omega) \text { otherwise }
\end{array}\right.
$$

First we show that $\Gamma_{n i}$ is measurable. Let $U$ be open. Then

$$
\left\{\omega: \Gamma_{n i}(\omega) \cap U \neq \emptyset\right\}=\left\{\omega: \Gamma(\omega) \cap B\left(x_{n}, 2^{-i}\right) \cap U \neq \emptyset\right\} \cup
$$

$$
\begin{gathered}
{\left[\left\{\omega: \Gamma(\omega) \cap B\left(x_{n}, 2^{-i}\right)=\emptyset\right\} \cap\{\omega: \Gamma(\omega) \cap U \neq \emptyset\}\right]} \\
=\left\{\omega: \Gamma(\omega) \cap B\left(x_{n}, 2^{-i}\right) \cap U \neq \emptyset\right\} \cup \\
{\left[\left(\Omega \backslash\left\{\omega: \Gamma(\omega) \cap B\left(x_{n}, 2^{-i}\right) \neq \emptyset\right\}\right) \cap\{\omega: \Gamma(\omega) \cap U \neq \emptyset\}\right],}
\end{gathered}
$$

a measurable set. By what was just shown, there exists $\sigma_{n i}$, a measurable function such that $\sigma_{n i}(\omega) \in \overline{\Gamma_{n i}(\omega)} \subseteq \overline{\Gamma(\omega)}$ for all $\omega \in \Omega$. If $x \in \overline{\Gamma(\omega)}$, then

$$
x \in B\left(x_{n}, 2^{-(i+2)}\right)
$$

whenever $x_{n}$ is close enough to $x$. Thus both $x, \sigma_{n(i+2)}(\omega)$ are in $B\left(x_{n}, 2^{-(i+2)}\right)$ and so $\left|\sigma_{n(i+1)}(\omega)-x\right|<2^{-i}$. It follows that condition 2.) holds. Note that this had nothing to do with a measure.

Now consider why 2.$) \Rightarrow$ 1.). We have $\left\{\sigma_{n}(\omega)\right\} \subseteq \Gamma(\omega)$ and $\sigma_{n}$ is measurable and $\overline{\cup_{n} \sigma_{n}(\omega)}$ equals $\overline{\Gamma(\omega)}$. Why is $\Gamma$ a measurable multifunction? Let $U$ be an open set

$$
\begin{aligned}
\Gamma^{-}(U) & \equiv\{\omega: \Gamma(\omega) \cap U \neq \emptyset\} \\
& =\{\omega: \overline{\Gamma(\omega)} \cap U \neq \emptyset\} \\
& =\cup_{n} \sigma_{n}^{-1}(U) \in \mathscr{C}
\end{aligned}
$$

For much more on multifunctions, you should see the book by Hu and Papageorgiou . The above proof follows the presentation in this book.

### 48.1.1 A Special Case Which Is Easier

The above is a pretty long and difficult argument to show that $\Gamma^{-}(U) \in \mathscr{C}$ for all $U$ open is equivalent to $\Gamma^{-}(F)$ for all $F$ closed. However, there is a special case for which this is much easier to show. Suppose $\Gamma(\omega)$ is not just closed but is also compact. Then as above, if $\Gamma^{-}(F) \in \mathscr{C}$ for all $F$ closed, then $\Gamma^{-}(U)=\cup_{n} \Gamma^{-}\left(F_{n}\right)$ where $F_{n}$ is an increasing sequence of closed sets whose union is $U$. This follows from the observation that

$$
\Gamma(\omega) \cap U=\cup_{n} \Gamma(\omega) \cap F_{n}
$$

and so to say the set on the left is nonempty is to say that at least one of the sets on the right is nonempty. Thus if $\Gamma^{-}(F) \in \mathscr{C}$ for all $F$ closed, then $\Gamma^{-}(U) \in \mathscr{C}$ for all $U$ open. This requires no special considerations.

Now suppose $\Gamma(\omega)$ is compact for every $\omega$ and that $\Gamma^{-}(U) \in \mathscr{C}$ for every $U$ open. Then let $F$ be a closed set and let $\left\{U_{n}\right\}$ be a decreasing sequence of open sets whose intersection equals $F$ such that also, for all $n, U_{n} \supseteq \overline{U_{n+1}}$. Then

$$
\Gamma(\omega) \cap F=\cap_{n} \Gamma(\omega) \cap U_{n}=\cap_{n} \Gamma(\omega) \cap \overline{U_{n}}
$$

Now because of compactness, the set on the left is nonempty if and only if each set on the right is also nonempty. Thus $\Gamma^{-}(F)=\cap_{n} \Gamma^{-}\left(U_{n}\right) \in \mathscr{C}$. Thus in this special case, it is much easier to see that these two conditions for measurability are equivalent. Note that there is no condition on measures or completeness or any such thing. This proves the following proposition.

Proposition 48.1.4 Let $X$ be a Polish space and let $\Gamma: X \rightarrow \mathscr{P}(X)$ have compact values. Then $\Gamma$ is measurable if and only if it is strongly measurable, the latter being the statement that $\Gamma^{-}(C)$ is measurable whenever $C$ is closed.

Let $\Gamma$ be strongly measurable. Let $\mathscr{G}$ be the sets $G$ such that $\Gamma^{-}(G)$ and $\Gamma^{-}\left(G^{C}\right)$ are both in $\mathscr{C}$. Then clearly $\mathscr{G}$ is closed with respect to complements. If $G \in \mathscr{G}$ is $G^{C}$ ? Is $\Gamma^{-}\left(G^{C}\right)$ and $\Gamma^{-}\left(\left(G^{C}\right)^{C}\right)$ in $\mathscr{C}$ ? I guess this is just the definition of what it means to be in $\mathscr{G}$. Also if you have $\left\{G_{i}\right\} \subseteq \mathscr{G}$, Then

$$
\Gamma^{-}\left(\cup_{i} G_{i}\right)=\cup_{i} \Gamma^{-}\left(G_{i}\right) \in \mathscr{C}
$$

and so $\mathscr{G}$ is closed with respect to countable unions. Hence $\mathscr{G}$ must contain the Borel sets because the strong measurability implies that the closed sets and hence open sets are in $\mathscr{G}$. Thus $\Gamma^{-}(G) \in \mathscr{C}$ whenever $G$ is Borel.

### 48.1.2 Other Measurability Considerations

Here are some general considerations about measurable multifunctions.
Lemma 48.1.5 Suppose $f: K(\omega) \times \Omega \rightarrow X, K \subseteq X$. Here $X$ is Polish space, separable complete metric space, and $(\Omega, \mathscr{F})$ is a measurable space. Also $\omega \rightarrow K(\omega)$ is a measurable multifunction as in Theorem 48.1.3. Also suppose $\omega \rightarrow f(x, \omega)$ is measurable and $x \rightarrow$ $f(x, \omega)$ is continuous. Also suppose that $\mathscr{K}(\omega) \equiv f(K(\omega), \omega)$. Then you can conclude that $\omega \rightarrow \mathscr{K}(\omega)$ is a measurable multifunction. If $\mathscr{K}(\omega)$ is compact, then it is also strongly measurable.

Proof: Let $\left\{x_{n}(\omega)\right\}$ be a countable dense subset of $K(\omega)$, each $x_{n}$ measurable. Then if $U$ is open,

$$
\{\omega: \mathscr{K}(\omega) \cap U \neq \emptyset\}=\cup_{n=1}^{\infty} f\left(x_{n}(\cdot), \cdot\right)^{-1}(U)
$$

and each of the sets in the union is measurable. The latter claim follows from the continuity of $f(\cdot, \omega)$. If $x(\omega)$ is measurable, then we can express it as the limit of simple functions $s_{n}$ for which $\omega \rightarrow f\left(s_{n}(\omega), \omega\right)$ is clearly measurable. Then $f(x(\omega), \omega)$ is the limit of $f\left(s_{n}(\omega), \omega\right)$. The reason for the equality is as follows. It is clear that the right side is contained in the left. Now if $\mathscr{K}(\omega) \cap U \neq \emptyset$, then by definition, $f(x, \omega) \in U$ for some $x \in K(\omega)$ but then by continuity, $f\left(x_{n}(\omega), \omega\right) \in U$ also for some $x_{n}(\omega)$ close to $x$. Thus the two sets are actually equal. Thus $\omega \rightarrow \mathscr{K}(\omega)$ is measurable. If $\mathscr{K}(\omega)$ has compact values it will be strongly measurable.

This lemma gives an easy example of a measurable multifunction having compact values. In fact this is the one of most interest in what follows. However, we also have the following general result. It gives the existence of a measurable $\varepsilon$ net. This is formulated in Banach space because it is convenient to add. It could also be formulated in Polish space with a little more difficulty. One just defines things a little differently.

Proposition 48.1.6 Let $\omega \rightarrow \mathscr{K}(\omega)$ be a measurable multifunction where $\mathscr{K}(\omega)$ is a pre compact set. Recall this means its closure is compact. Also, it must have an $\varepsilon$ net for
each $\varepsilon>0$. Then for each $\varepsilon>0$, there exists $N(\omega)$ and measurable functions $y_{j}, j=$ $1,2, \cdots, N(\omega), y_{j}(\omega) \in \mathscr{K}(\omega)$, such that

$$
\cup_{j=1}^{N} B\left(y_{j}(\omega), \varepsilon\right) \supseteq \mathscr{K}(\omega)
$$

for each $\omega$. Also $\omega \rightarrow N(\omega)$ is measurable.
Proof: Suppose that $\omega \rightarrow \mathscr{K}(\omega)$ is a measurable multifunction having compact values in $X$ a Banach space. Let $\left\{\sigma_{n}(\omega)\right\}$ be the measurable selections such that for each $\omega,\left\{\sigma_{n}(\omega)\right\}_{n=1}^{\infty}$ is dense in $\mathscr{K}(\omega)$. Let $y_{1}(\omega) \equiv \sigma_{1}(\omega)$. Now let $2(\omega)$ be the first index after 1 such that $\left\|\sigma_{2(\omega)}(\omega)-\sigma_{1}(\omega)\right\|>\frac{\varepsilon}{2}$. Thus $2(\omega)=k$ on the measurable set

$$
\left\{\omega \in \Omega:\left\|\sigma_{k}(\omega)-\sigma_{1}(\omega)\right\|>\frac{\varepsilon}{2}\right\} \cap\left\{\omega \in \Omega: \cap_{j=1}^{k-1}\left\|\sigma_{j}(\omega)-\sigma_{1}(\omega)\right\| \leq \frac{\varepsilon}{2}\right\}
$$

Suppose $1(\omega), 2(\omega), \cdots,(m-1)(\omega)$ have been chosen such that this is a strictly increasing sequence for each $\omega$, each is a measurable function, and for $i, j \leq m-1$,

$$
\left\|\sigma_{i(\omega)}(\omega)-\sigma_{j(\omega)}(\omega)\right\|>\frac{\varepsilon}{2}
$$

Each $\omega \rightarrow \sigma_{i(\omega)}(\omega)$ is measurable because it equals

$$
\sum_{k=1}^{\infty} \mathscr{X}_{[i(\omega)=k]}(\omega) \sigma_{k}(\omega)
$$

Then $m(\omega)$ will be the first index larger than $(m-1)(\omega)$ such that

$$
\left\|\sigma_{m(\omega)}(\omega)-\sigma_{j(\omega)}(\omega)\right\|>\frac{\varepsilon}{2}
$$

for all $j(\omega)<m(\omega)$. Thus $\omega \rightarrow m(\omega)$ is also measurable because it equals $k$ on the measurable set

$$
\begin{gathered}
\left(\cap\left\{\omega:\left\|\sigma_{k}(\omega)-\sigma_{j(\omega)}(\omega)\right\|>\frac{\varepsilon}{2}, j \leq m-1\right\}\right) \cap\{\omega:(m-1)(\omega)<k\} \\
\cap\left(\cup\left\{\omega:\left\|\sigma_{k-1}(\omega)-\sigma_{j(\omega)}(\omega)\right\| \leq \frac{\varepsilon}{2}, j \leq m-1\right\}\right)
\end{gathered}
$$

The top line says it does what is wanted and the second says it is the first after $(m-1)(\omega)$ which does so. Since $\mathscr{K}(\omega)$ is a pre compact set, it follows that the above measurable set will be empty for all $m(\omega)$ sufficiently large called $N(\omega)$, also a measurable function, and so the process ends. Let $y_{i}(\omega) \equiv \sigma_{i(\omega)}(\omega)$. Then this gives the desired measurable $\varepsilon$ net. The fact that

$$
\cup_{i=1}^{N(\omega)} B\left(y_{i}(\omega), \varepsilon\right) \supseteq \mathscr{K}(\omega)
$$

follows because if there exists $z \in \mathscr{K}(\omega) \backslash\left(\cup_{i=1}^{N(\omega)} B\left(y_{i}(\omega), \varepsilon\right)\right)$, then $B\left(z, \frac{\varepsilon}{2}\right)$ would have empty intersection with all of the balls $B\left(y_{i}(\omega), \frac{\varepsilon}{3}\right)$ and by density of the $\sigma_{i}(\omega)$ in $\mathscr{K}(\omega)$, there would be some $\sigma_{l}(\omega)$ contained in $B\left(z, \frac{\varepsilon}{3}\right)$ for arbitrarily large $l$ and so the process would not have ended as shown above.

### 48.2 Existence of Measurable Fixed Points

### 48.2.1 Simplices And Labeling

First define an $n$ simplex, denoted by $\left[\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right]$, to be the convex hull of the $n+1$ points, $\left\{\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right\}$ where $\left\{\mathbf{x}_{i}-\mathbf{x}_{0}\right\}_{i=1}^{n}$ are independent. Thus

$$
\left[\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right] \equiv\left\{\sum_{i=0}^{n} t_{i} \mathbf{x}_{i}: \sum_{i=0}^{n} t_{i}=1, t_{i} \geq 0\right\}
$$

Since $\left\{\mathbf{x}_{i}-\mathbf{x}_{0}\right\}_{i=1}^{n}$ is independent, the $t_{i}$ are uniquely determined. If two of them are

$$
\sum_{i=0}^{n} t_{i} \mathbf{x}_{i}=\sum_{i=0}^{n} s_{i} \mathbf{x}_{i}
$$

Then

$$
\sum_{i=0}^{n} t_{i}\left(\mathbf{x}_{i}-\mathbf{x}_{0}\right)=\sum_{i=0}^{n} s_{i}\left(\mathbf{x}_{i}-\mathbf{x}_{0}\right)
$$

so $t_{i}=s_{i}$ for $i \geq 1$. Since the $s_{i}$ and $t_{i}$ sum to 1 , it follows that also $s_{0}=t_{0}$. If $n \leq 2$, the simplex is a triangle, line segment, or point. If $n \leq 3$, it is a tetrahedron, triangle, line segment or point. To say that $\left\{\mathbf{x}_{i}-\mathbf{x}_{0}\right\}_{i=1}^{n}$ are independent is to say that $\left\{\mathbf{x}_{i}-\mathbf{x}_{r}\right\}_{i \neq r}$ are independent for each fixed $r$. Indeed, if $\mathbf{x}_{i}-\mathbf{x}_{r}=\sum_{j \neq i, r} c_{j}\left(\mathbf{x}_{j}-\mathbf{x}_{r}\right)$, then you would have

$$
\mathbf{x}_{i}-\mathbf{x}_{0}+\mathbf{x}_{0}-\mathbf{x}_{r}=\sum_{j \neq i, r} c_{j}\left(\mathbf{x}_{j}-\mathbf{x}_{0}\right)+\left(\sum_{j \neq i, r} c_{j}\right) \mathbf{x}_{0}
$$

and it follows that $\mathbf{x}_{i}-\mathbf{x}_{0}$ is a linear combination of the $\mathbf{x}_{j}-\mathbf{x}_{0}$ for $j \neq i$, contrary to assumption.

A simplex $S$ can be triangulated. This means it is the union of smaller sub-simplices such that if $S_{1}, S_{2}$ are two simplices in the triangulation, with

$$
S_{1} \equiv\left[\mathbf{z}_{0}^{1}, \cdots, \mathbf{z}_{m}^{1}\right], S_{2} \equiv\left[\mathbf{z}_{0}^{2}, \cdots, \mathbf{z}_{p}^{2}\right]
$$

then

$$
S_{1} \cap S_{2}=\left[\mathbf{x}_{k_{0}}, \cdots, \mathbf{x}_{k_{r}}\right]
$$

where $\left[\mathbf{x}_{k_{0}}, \cdots, \mathbf{x}_{k_{r}}\right]$ is in the triangulation and

$$
\left\{\mathbf{x}_{k_{0}}, \cdots, \mathbf{x}_{k_{r}}\right\}=\left\{\mathbf{z}_{0}^{1}, \cdots, \mathbf{z}_{m}^{1}\right\} \cap\left\{\mathbf{z}_{0}^{2}, \cdots, \mathbf{z}_{p}^{2}\right\}
$$

or else the two simplices do not intersect. Does there exist a triangulation in which all sub-simplices have diameter less than $\varepsilon$ ? This is obvious if $n \leq 2$. Supposing it to be true for $n-1$, is it also so for $n$ ? The barycenter $\mathbf{b}$ of a simplex $\left[\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right]$ is just $\frac{1}{1+n} \sum_{i} \mathbf{x}_{i}$. This point is not in the convex hull of any of the faces, those simplices of the form $\left[\mathbf{x}_{0}, \cdots, \hat{\mathbf{x}}_{k}, \cdots, \mathbf{x}_{n}\right]$ where the hat indicates $\mathbf{x}_{k}$ has been left out. Thus $\left[\mathbf{x}_{0}, \cdots, \mathbf{b}, \cdots, \mathbf{x}_{n}\right]$ is a $n$ simplex also. Now in general, if you have an $n$ simplex $\left[\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right]$, its diameter is
the maximum of $\left|\mathbf{x}_{k}-\mathbf{x}_{l}\right|$ for all $k \neq l$. Consider $\left|\mathbf{b}-\mathbf{x}_{j}\right|$. It equals $\left|\sum_{i=0}^{n} \frac{1}{n+1}\left(\mathbf{x}_{i}-\mathbf{x}_{j}\right)\right|=$ $\left|\sum_{i \neq j} \frac{1}{n+1}\left(\mathbf{x}_{i}-\mathbf{x}_{j}\right)\right| \leq \frac{n}{n+1} \operatorname{diam}(S)$. Next consider the $k^{t h}$ face of $S\left[\mathbf{x}_{0}, \cdots, \hat{\mathbf{x}}_{k}, \cdots, \mathbf{x}_{n}\right]$. By induction, it has a triangulation into simplices which each have diameter no more than $\frac{n}{n+1} \operatorname{diam}(S)$. Let these $n-1$ simplices be denoted by $\left\{S_{1}^{k}, \cdots, S_{m_{k}}^{k}\right\}$. Then the simplices $\left\{\left[S_{i}^{k}, \mathbf{b}\right]\right\}_{i=1, k=1}^{m_{k}, n+1}$ are a triangulation of $S$ such that $\operatorname{diam}\left(\left[S_{i}^{k}, \mathbf{b}\right]\right) \leq \frac{n}{n+1} \operatorname{diam}(S)$. Do for $\left[S_{i}^{k}, \mathbf{b}\right]$ what was just done for $S$ obtaining a triangulation of $S$ as the union of what is obtained such that each simplex has diameter no more than $\left(\frac{n}{n+1}\right)^{2} \operatorname{diam}(S)$. Continuing this way shows the existence of the desired triangulation.

### 48.2.2 Labeling Vertices

Next is a way to label the vertices. Let $p_{0}, \cdots, p_{n}$ be the first $n+1$ prime numbers. All vertices of a simplex $S=\left[\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right]$ having $\left\{\mathbf{x}_{k}-\mathbf{x}_{0}\right\}_{k=1}^{n}$ independent will be labeled with one of these primes. In particular, the vertex $\mathbf{x}_{k}$ will be labeled as $p_{k}$ if the simplex is $\left[\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right]$. The value of a simplex will be the product of its labels. Triangulate this $S$. Consider a 1 simplex coming from the original simplex $\left[\mathbf{x}_{k_{1}}, \mathbf{x}_{k_{2}}\right]$, label one end as $p_{k_{1}}$ and the other as $p_{k_{2}}$. Then label all other vertices of this triangulation which occur on $\left[\mathbf{x}_{k_{1}}, \mathbf{x}_{k_{2}}\right]$ either $p_{k_{1}}$ or $p_{k_{2}}$. Then obviously there will be an odd number of simplices in this triangulation having value $p_{k_{1}} p_{k_{2}}$, that is a $p_{k_{1}}$ at one end and a $p_{k_{2}}$ at the other. Suppose that the labeling has been done for all vertices of the triangulation which are on $\left[\mathbf{x}_{j_{1}}, \ldots \mathbf{x}_{j_{k+1}}\right]$,

$$
\left\{\mathbf{x}_{j_{1}}, \ldots \mathbf{x}_{j_{k+1}}\right\} \subseteq\left\{\mathbf{x}_{0}, \ldots \mathbf{x}_{n}\right\}
$$

any $k$ simplex for $k \leq n-1$, and there is an odd number of simplices from the triangulation having value equal to $\prod_{i=1}^{k+1} p_{j_{i}}$. Consider $\hat{S} \equiv\left[\mathbf{x}_{j_{1}}, \ldots \mathbf{x}_{j_{k+1}}, \mathbf{x}_{j_{k+2}}\right]$. Then by induction, there is an odd number of $k$ simplices on the $s^{t h}$ face $\left[\mathbf{x}_{j_{1}}, \ldots, \hat{\mathbf{x}}_{j_{s}}, \cdots, \mathbf{x}_{j_{k+1}}\right.$ ] having value $\prod_{i \neq s} p_{j_{i}}$. In particular the face $\left[\mathbf{x}_{j_{1}}, \ldots, \mathbf{x}_{j_{k+1}}, \hat{\mathbf{x}}_{j_{k+2}}\right]$ has an odd number of simplices with value $\prod_{i \leq k+1} p_{j_{i}}$. Now no simplex in any other face of $\hat{S}$ can have this value by uniqueness of prime factorization. Lable the "interior" vertices, those $\mathbf{u}$ having all $s_{i}>0$ in $\mathbf{u}=\sum_{i=1}^{k+2} s_{i} \mathbf{x}_{j_{i}}$, (These have not yet been labeled.) with any of the $p_{j_{1}}, \cdots, p_{j_{k+2}}$. Pick a simplex on the face $\left[\mathbf{x}_{j_{1}}, \ldots, \mathbf{x}_{j_{k+1}}, \hat{\mathbf{x}}_{j_{k+2}}\right]$ which has value $\prod_{i \leq k+1} p_{j_{i}}$ and cross this simplex into $\hat{S}$. Continue crossing simplices having value $\prod_{i \leq k+1} p_{j_{i}}$ which have not been crossed till the process ends. It must end because there are an odd number of these simplices having value $\prod_{i \leq k+1} p_{j_{i}}$. If the process leads to the outside of $\hat{S}$, then one can always enter it again because there are an odd number of simplices with value $\prod_{i \leq k+1} p_{j_{i}}$ available and you will have used up an even number. When the process ends, the value of the simplex must be $\prod_{i=1}^{k+2} p_{j_{i}}$ because it will have the additional label $p_{j_{k+2}}$ on a vertex since if not, there will be another way out of the simplex. This identifies a simplex in the triangulation with value $\prod_{i=1}^{k+2} p_{j_{i}}$. Then repeat the process with $\prod_{i \leq k+1} p_{j_{i}}$ valued simplices on $\left[\mathbf{x}_{j_{1}}, \ldots, \mathbf{x}_{j_{k+1}}, \hat{\mathbf{x}}_{j_{k+2}}\right]$ which have not been crossed. Repeating the process, entering from the outside, cannot deliver a $\prod_{i=1}^{k+2} p_{j_{i}}$ valued simplex encountered earlier. This is because you cross faces labeled $\prod_{i \leq k+1} p_{j_{i}}$. If the remaining vertex is labeled $p_{j_{i}}$ where $i \neq k+2$, then this yields exactly one other face to cross. There are two, the one with the first vertex $p_{j_{i}}$ and the next one with the new vertex labeled $p_{j_{i}}$ substituted for the first vertex having
this label. Thus there is either one route in to a simplex or two. Thus, starting at a simplex labeled $\prod_{i \leq k+1} p_{j_{i}}$ one can cross faces having this value till one is led to the $\prod_{i \leq k+1} p_{j_{i}}$ valued simplex on the selected face of $\hat{S}$. In other words, the process is one to one in selecting a $\prod_{i \leq k+1} p_{j_{i}}$ vertex from crossing such a vertex on the selected face of $\hat{S}$. Continue doing this, crossing a $\prod_{i \leq k+1} p_{j_{i}}$ simplex on the face of $\hat{S}$ which has not been crossed previously. This identifies an odd number of simplices having value $\prod_{i=1}^{k+2} p_{j_{i}}$. These are the ones which are "accessible" from the outside using this process. If there are any which are not accessible from outside, applying the same process starting inside one of these, leads to exactly one other inaccessible simplex with value $\prod_{i=1}^{k+2} p_{j_{i}}$. Hence these inaccessible simplices occur in pairs and so there are an odd number of simplices in the triangulation having value $\prod_{i=1}^{k+2} p_{j_{i}}$. We refer to this procedure of labeling as Sperner's lemma. The system of labeling is well defined thanks to the assumption that $\left\{\mathbf{x}_{k}-\mathbf{x}_{0}\right\}_{k=1}^{n}$ is independent which implies that $\left\{\mathbf{x}_{k}-\mathbf{x}_{i}\right\}_{k \neq i}$ is also linearly independent. The following is a description of the system of labeling the vertices.

Lemma 48.2.1 Let $\left[\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right]$ be an $n$ simplex with $\left\{\mathbf{x}_{k}-\mathbf{x}_{0}\right\}_{k=1}^{n}$ independent, and let the first $n+1$ primes be $p_{0}, p_{1}, \cdots, p_{n}$. Label $\mathbf{x}_{k}$ as $p_{k}$ and consider a triangulation of this simplex. Labeling the vertices of this triangulation which occur on $\left[\mathbf{x}_{k_{1}}, \cdots, \mathbf{x}_{k_{s}}\right]$ with any of $p_{k_{1}}, \cdots, p_{k_{s}}$, beginning with all 1 simplices $\left[\mathbf{x}_{k_{1}}, \mathbf{x}_{k_{2}}\right]$ and then 2 simplices and so forth, there are an odd number of simplices $\left[\mathbf{y}_{k_{1}}, \cdots, \mathbf{y}_{k_{s}}\right]$ of the triangulation contained in $\left[\mathbf{x}_{k_{1}}, \cdots, \mathbf{x}_{k_{s}}\right]$ which have value $p_{k_{1}}, \cdots, p_{k_{s}}$. This for $s=1,2, \cdots, n$.

## Another way To Explain The Labeling

We now give a brief discussion of the system of labeling for Sperner's lemma from the point of view of counting numbers of faces rather than obtaining them with an algorithm. Let $p_{0}, \cdots, p_{n}$ be the first $n+1$ prime numbers. All vertices of a simplex $S=\left[\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right]$ having $\left\{\mathbf{x}_{k}-\mathbf{x}_{0}\right\}_{k=1}^{n}$ independent will be labeled with one of these primes. In particular, the vertex $\mathbf{x}_{k}$ will be labeled as $p_{k}$. The value of a simplex will be the product of its labels. Triangulate this $S$. Consider a 1 simplex coming from the original simplex $\left[\mathbf{x}_{k_{1}}, \mathbf{x}_{k_{2}}\right]$, label one end as $p_{k_{1}}$ and the other as $p_{k_{2}}$. Then label all other vertices of this triangulation which occur on $\left[\mathbf{x}_{k_{1}}, \mathbf{x}_{k_{2}}\right]$ either $p_{k_{1}}$ or $p_{k_{2}}$. Then obviously there will be an odd number of simplices in this triangulation having value $p_{k_{1}} p_{k_{2}}$, that is a $p_{k_{1}}$ at one end and a $p_{k_{2}}$ at the other. Suppose that the labeling has been done for all vertices of the triangulation which are on $\left[\mathbf{x}_{j_{1}}, \ldots \mathbf{x}_{j_{k+1}}\right]$,

$$
\left\{\mathbf{x}_{j_{1}}, \ldots \mathbf{x}_{j_{k+1}}\right\} \subseteq\left\{\mathbf{x}_{0}, \ldots \mathbf{x}_{n}\right\}
$$

any $k$ simplex for $k \leq n-1$, and there is an odd number of simplices from the triangulation having value equal to $\prod_{i=1}^{k+1} p_{j_{i}}$. Consider $\hat{S} \equiv\left[\mathbf{x}_{j_{1}}, \ldots \mathbf{x}_{j_{k+1}}, \mathbf{x}_{j_{k+2}}\right]$. Then by induction, there is an odd number of $k$ simplices on the $s^{t h}$ face

$$
\left[\mathbf{x}_{j_{1}}, \ldots, \hat{\mathbf{x}}_{j_{s}}, \cdots, \mathbf{x}_{j_{k+1}}\right]
$$

having value $\prod_{i \neq s} p_{j_{i}}$. In particular the face $\left[\mathbf{x}_{j_{1}}, \ldots, \mathbf{x}_{j_{k+1}}, \hat{\mathbf{x}}_{j_{k+2}}\right]$ has an odd number of simplices with value $\prod_{i=1}^{k+1} p_{j_{i}}:=\hat{P}_{k}$. We want to argue that some simplex in the triangulation which is contained in $\hat{S}$ has value $\hat{P}_{k+1}:=\prod_{i=1}^{k+2} p_{j_{i}}$. Let $Q$ be the number of $k+1$
simplices from the triangulation contained in $\hat{S}$ which have two faces with value $\hat{P}_{k}$ (A $k+1$ simplex has either 1 or $2 \hat{P}_{k}$ faces.) and let $R$ be the number of $k+1$ simplices from the triangulation contained in $\hat{S}$ which have exactly one $\hat{P}_{k}$ face. These are the ones we want because they have value $\hat{P}_{k+1}$. Thus the number of faces having value $\hat{P}_{k}$ which is described here is $2 Q+R$. All interior $\hat{P}_{k}$ faces being counted twice by this number. Now we count the total number of $\hat{P}_{k}$ faces another way. There are $P$ of them on the face $\left[\mathbf{x}_{j_{1}}, \ldots, \mathbf{x}_{j_{k+1}}, \hat{\mathbf{x}}_{j_{k+2}}\right]$ and by induction, $P$ is odd. Then there are $O$ of them which are not on this face. These faces got counted twice. Therefore,

$$
2 Q+R=P+2 O
$$

and so, since $P$ is odd, so is $R$. Thus there is an odd number of $\hat{P}_{k+1}$ simplices in $\hat{S}$.
We refer to this procedure of labeling as Sperner's lemma. The system of labeling is well defined thanks to the assumption that $\left\{\mathbf{x}_{k}-\mathbf{x}_{0}\right\}_{k=1}^{n}$ is independent which implies that $\left\{\mathbf{x}_{k}-\mathbf{x}_{i}\right\}_{k \neq i}$ is also linearly independent. Sperner's lemma is now a consequence of this discussion.

### 48.2.3 Measurability Of Brouwer Fixed Points

First, here is a nice measurable selection theorem.
Lemma 48.2.2 Let $U$ be a separable reflexive Banach space. Suppose there is a sequence $\left\{u_{j}(\omega)\right\}_{j=1}^{\infty}$ in $U$, where each $\omega \rightarrow u_{j}(\omega)$ is measurable and for each $\omega$, $\sup _{i}\left\|u_{i}(\omega)\right\|<$ $\infty$. Then, there exists $u(\omega) \in U$ such that $\omega \rightarrow u(\omega)$ is measurable, and a subsequence $n(\omega)$, that depends on $\omega$, such that the weak limit

$$
\lim _{n(\omega) \rightarrow \infty} u_{n(\omega)}(\omega)=u(\omega)
$$

holds.
Proof: Let $\left\{z_{i}\right\}_{i=1}^{\infty}$ be a countable dense subset of $U^{\prime}$. Let $\mathbf{h}: U \rightarrow \prod_{i=1}^{\infty} \mathbb{R}$ be defined by

$$
\mathbf{h}(u)=\prod_{i=1}^{\infty}\left\langle z_{i}, u\right\rangle .
$$

Let $X=\prod_{i=1}^{\infty} \mathbb{R}$ with the product topology. Then, this is a Polish space with the metric defined as $d(\mathbf{x}, \mathbf{y})=\sum_{i=1}^{\infty} \frac{\left|x_{i}-y_{i}\right|}{1+\left|x_{i}-y_{i}\right|} 2^{-i}$. By compactness, for a fixed $\omega$,the $\mathbf{h}\left(u_{n}(\omega)\right)$ are contained in a compact subset of $X$. Next, define

$$
\Gamma_{n}(\omega)=\overline{\cup_{k \geq n} \mathbf{h}\left(u_{k}(\omega)\right)}
$$

which is a nonempty compact subset of $X$. Moreover, $\Gamma_{n}(\omega)$ is a measurable multifunction into $X$.

Next, we claim that $\omega \rightarrow \Gamma_{n}(\omega)$ is a measurable multifunction.
The proof of the claim is as follows. It is necessary to show that $\Gamma_{n}^{-}(O)$ defined as $\left\{\omega: \Gamma_{n}(\omega) \cap O \neq \emptyset\right\}$ is measurable whenever $O$ is open. It suffices to verify this for $O$ a
basic open set in the topology of $X$. Thus let $O=\prod_{i=1}^{\infty} O_{i}$ where each $O_{i}$ is a proper open subset of $\mathbb{R}$ only for $i \in\left\{j_{1}, \cdots, j_{m}\right\}$. Then,

$$
\Gamma_{n}^{-}(O)=\cup_{k \geq n} \cap_{r=1}^{m}\left\{\omega:\left\langle z_{j_{r}}, u_{k}(\omega)\right\rangle \in O_{j_{r}}\right\},
$$

which is a measurable set since $u_{k}$ is measurable.
Then, it follows that $\omega \rightarrow \Gamma_{n}(\omega)$ is strongly measurable because it has compact values in $X$, thanks to Tychonoff's theorem. Thus $\Gamma_{n}^{-}(H)=\left\{\omega: H \cap \Gamma_{n}(\omega) \neq \emptyset\right\}$ is measurable whenever $H$ is a closed set. Now, let $\Gamma(\omega)$ be defined as $\cap_{n} \Gamma_{n}(\omega)$ and then for $H$ closed,

$$
\Gamma^{-}(H)=\cap_{n} \Gamma_{n}^{-}(H)
$$

and each set in the intersection is measurable, so this shows that $\omega \rightarrow \Gamma(\omega)$ is also measurable. Therefore, it has a measurable selection $\mathbf{g}(\omega)$. It follows from the definition of $\Gamma(\omega)$ that there exists a subsequence $n(\omega)$ such that

$$
\mathbf{g}(\omega)=\lim _{n(\omega) \rightarrow \infty} \mathbf{h}\left(u_{n(\omega)}(\omega)\right) \quad \text { in } X
$$

In terms of components, we have

$$
g_{i}(\omega)=\lim _{n(\omega) \rightarrow \infty}\left\langle z_{i}, u_{n(\omega)}(\omega)\right\rangle
$$

Furthermore, there is a further subsequence, still denoted with $n(\omega)$, such that $u_{n(\omega)}(\omega) \rightarrow$ $u(\omega)$ weakly. This means that for each $i$,

$$
g_{i}(\omega)=\lim _{n(\omega) \rightarrow \infty}\left\langle z_{i}, u_{n(\omega)}(\omega)\right\rangle=\left\langle z_{i}, u(\omega)\right\rangle
$$

Thus, for each $z_{i}$ in a dense set, $\omega \rightarrow\left\langle z_{i}, u(\omega)\right\rangle$ is measurable. Since the $z_{i}$ are dense, this implies $\omega \rightarrow\langle z, u(\omega)\rangle$ is measurable for every $z \in U^{\prime}$ and so by the Pettis theorem, $\omega \rightarrow u(\omega)$ is measurable.

There is an easy version of this which follows from the same arguments.
Corollary 48.2.3 Let $K(\omega)$ be a compact subset of a separable metric space $X$ and suppose $\left\{u_{j}(\omega)\right\}_{j=1}^{\infty} \subseteq K(\omega)$ with each $\omega \rightarrow u_{j}(\omega)$ measurable into $X$. Then there exists $u(\omega) \in K(\omega)$ such that $\omega \rightarrow u(\omega)$ is measurable into $X$ and a subsequence $n(\omega)$ depending on $\omega$ such that $\lim _{n(\omega) \rightarrow \infty} u_{n(\omega)}(\omega)=u(\omega)$.

Proof: Define

$$
\Gamma_{n}(\omega)=\overline{\bigcup_{k \geq n} u_{k}(\omega)}
$$

This is a nonempty compact subset of $K(\omega) \subseteq X$. I claim that $\omega \rightarrow \Gamma_{n}(\omega)$ is a measurable multifunction into $X$. It is necessary to show that $\Gamma_{n}^{-}(O)$ defined as $\left\{\omega: \Gamma_{n}(\omega) \cap O \neq \emptyset\right\}$ is measurable whenever $O$ is open in $X$. For $\omega \in \Gamma_{n}^{-}(O)$ it means that some $u_{k}(\omega) \in O, k \geq n$. Thus $\Gamma_{n}^{-}(O)=\cup_{k \geq n} u_{k}^{-1}(O)$ and this is measurable by the assumption that each $u_{k}$ is. Since $\Gamma_{n}^{-}(\omega)$ is compact, it is also strongly measurable by Proposition 48.1.4, meaning that $\Gamma^{-}(H)$ is measurable whenever $H$ is closed. Now, let $\Gamma(\omega)$ be defined as

$$
\Gamma(\omega) \equiv \cap_{n} \Gamma_{n}(\omega)
$$

and then for $H$ closed,

$$
\Gamma^{-}(H)=\cap_{n} \Gamma_{n}^{-}(H)
$$

and each set in the intersection is measurable, so this shows that $\omega \rightarrow \Gamma(\omega)$ is also (strongly) measurable. Therefore, it has a measurable selection $u(\omega)$. It follows from the definition of $\Gamma(\omega)$ that there exists a subsequence $n(\omega)$ such that $u(\omega)=\lim _{n(\omega) \rightarrow \infty} u_{n(\omega)}(\omega)$.

Now we consider the case of fixed points for simplices.
Suppose $\mathbf{f}(\cdot, \omega): S \rightarrow S$ for $S$ a simplex. Then from the Brouwer fixed point theorem, there is a fixed point $\mathbf{x}(\omega)$ provided $\mathbf{f}(\cdot, \omega)$ is continuous. Can it be arranged to have $\omega \rightarrow \mathbf{x}(\omega)$ also measurable? In fact, it can, and this is shown here. In other words, if $P(\omega)$ are the fixed points of $f(\cdot, \omega)$, there exists a measurable selection in $P(\omega)$.
$S \equiv\left[\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right]$ is a simplex in $\mathbb{R}^{n}$. Assume $\left\{\mathbf{x}_{i}-\mathbf{x}_{0}\right\}_{i=1}^{n}$ are linearly independent. Thus a typical point of $S$ is of the form

$$
\sum_{i=0}^{n} t_{i} \mathbf{x}_{i}
$$

where the $t_{i}$ are uniquely determined and the map $\mathbf{x} \rightarrow \mathbf{t}$ is continuous from $S$ to the compact set $\left\{\mathbf{t} \in \mathbb{R}^{n+1}: \sum t_{i}=1, t_{i} \geq 0\right\}$.

To see this, suppose $\mathbf{x}^{k} \rightarrow \mathbf{x}$ in $S$. Let $\mathbf{x}^{k} \equiv \sum_{i=0}^{n} t_{i}^{k} \mathbf{x}_{i}$ with $\mathbf{x}$ defined similarly with $t_{i}^{k}$ replaced with $t_{i}, \mathbf{x} \equiv \sum_{i=0}^{n} t_{i} \mathbf{x}_{i}$. Then

$$
\mathbf{x}^{k}-\mathbf{x}_{0}=\sum_{i=0}^{n} t_{i}^{k} \mathbf{x}_{i}-\sum_{i=0}^{n} t_{i}^{k} \mathbf{x}_{0}=\sum_{i=1}^{n} t_{i}^{k}\left(\mathbf{x}_{i}-\mathbf{x}_{0}\right)
$$

Thus

$$
\mathbf{x}^{k}-\mathbf{x}_{0}=\sum_{i=1}^{n} t_{i}^{k}\left(\mathbf{x}_{i}-\mathbf{x}_{0}\right), \mathbf{x}-\mathbf{x}_{0}=\sum_{i=1}^{n} t_{i}\left(\mathbf{x}_{i}-\mathbf{x}_{0}\right)
$$

Say $t_{i}^{k}$ fails to converge to $t_{i}$ for all $i \geq 1$. Then there exists a subsequence, still denoted with superscript $k$ such that for each $i=1, \cdots, n$, it follows that $t_{i}^{k} \rightarrow s_{i}$ where $s_{i} \geq 0$ and some $s_{i} \neq t_{i}$. But then, taking a limit, it follows that

$$
\mathbf{x}-\mathbf{x}_{0}=\sum_{i=1}^{n} s_{i}\left(\mathbf{x}_{i}-\mathbf{x}_{0}\right)=\sum_{i=1}^{n} t_{i}\left(\mathbf{x}_{i}-\mathbf{x}_{0}\right)
$$

which contradicts independence of the $\mathbf{x}_{i}-\mathbf{x}_{0}$. It follows that for all $i \geq 1, t_{i}^{k} \rightarrow t_{i}$. Since they all sum to 1 , this implies that also $t_{0}^{k} \rightarrow t_{0}$. Thus the claim about continuity is verified.

Let $\mathbf{f}(\cdot, \omega): S \rightarrow S$ be continuous such that $\omega \rightarrow \mathbf{f}(\mathbf{x}, \omega)$ is measurable. When doing $\mathbf{f}(\cdot, \omega)$ to a point $\sum_{i=0}^{n} t_{i} \mathbf{x}_{i}$, one obtains another point of $S$ denoted as $\sum_{i=0}^{n} s_{i}(\omega) \mathbf{x}_{i}$. The coefficients $s_{i}$ must be measurable functions. This is because

$$
\omega \rightarrow \mathbf{f}(\mathbf{x}, \omega)=\sum_{i=0}^{n} s_{i}(\omega) \mathbf{x}_{i}
$$

and the left side is measurable so it follows the right is also. Now as noted above, the map which takes a point of $S$ to its coefficients is continuous and so each $s_{i}$ is measurable as a function of $\omega$. Note that if $\mathbf{x}$ is replaced with $\mathbf{x}(\omega)$, with $\omega \rightarrow \mathbf{x}(\omega)$ measurable, the same
conclusion can be drawn about the $s_{i}(\omega)$. This is because, thanks to the continuity of $\mathbf{f}$ in its first argument, the function on the left in the above is measurable.

Label $\mathbf{x}_{j}$ as $p_{j}$ where $p_{0}, \cdots, p_{n}$ are the first $n+1$ prime numbers. Thus the vertices of $S$ have been labeled. Next triangulate $S$ so that all simplices have diameter less than $\varepsilon$. If $\left[\mathbf{y}_{0}, \cdots, \mathbf{y}_{n}\right]$ is one of these small vertices, each is of the form $\sum_{i=0}^{n} t_{i} \mathbf{x}_{i}$ where $t_{i} \geq 0$ and $\sum_{i} t_{i}=1$. Define $r_{k}(\omega) \equiv s_{k}(\omega) / t_{k}$ if $t_{k}>0$ and $\infty$ if $t_{k}=0$. Thus this is a measurable function.

$$
E_{k} \equiv \cap_{j \neq k}\left[\omega: r_{k}(\omega) \leq r_{j}(\omega)\right], F_{0} \equiv E_{0}, F_{1} \equiv E_{1} \backslash E_{0}, \cdots, F_{n} \equiv E_{n} \backslash \cup_{i=0}^{n-1} E_{i}
$$

Then $p\left(\mathbf{y}_{i}, \omega\right)$ will be the label placed on $\mathbf{y}_{i}$. It equals

$$
p\left(\mathbf{y}_{i}, \omega\right) \equiv \sum_{k=0}^{n} p_{k} \mathscr{X}_{F_{k}}(\omega)
$$

obviously a measurable function. Note also that this new method of labeling does not contradict the original labels placed on the vertices $\mathbf{x}_{i}$. This is because for $\mathbf{x}_{i}, t_{i}=1$ and all other $t_{j}=0$ so the only ratio that is finite will be $s_{i} / t_{i}$. All others are $\infty$ by definition. As for the vertices which are on the $k^{t h}$ face $\left[\mathbf{x}_{0}, \cdots, \hat{\mathbf{x}}_{k}, \cdots, \mathbf{x}_{n}\right]$, these will be labeled from the list $\left\{p_{0}, \cdots, \hat{p}_{k}, \cdots, p_{n}\right\}$ because $t_{k}=0$ for each of these and so $r_{k}(\omega)=\infty$. By the Sperner's lemma procedure described above, there are an odd number of simplices having value $\prod_{i \neq k} p_{i}$ on the $k^{\text {th }}$ face and an odd number of simplices in the triangulation of $S$ for which the product of the labels on their vertices equals $p_{0} p_{1} \cdots p_{n} \equiv P_{n}$. We call this the value of the simplex. Thus if $\left[\mathbf{y}_{0}, \cdots, \mathbf{y}_{n}\right]$ is one of these simplices, and $p\left(\mathbf{y}_{i}, \omega\right)$ is the label for $\mathbf{y}_{i}$, a measurable function of $\omega$,

$$
\prod_{i=0}^{n} p\left(\mathbf{y}_{i}, \omega\right)=\prod_{i=0}^{n} p_{i} \equiv P_{n}
$$

For $\omega \in F_{k}$, what is $r_{k}(\omega)$ ? Could it be larger than 1 ? $r_{k}(\omega)$ is certainly finite because at least some $t_{j} \neq 0$ since they sum to 1 . Thus, if $r_{k}(\omega)>1$, you would have $s_{k}(\omega)>t_{k}$. The $s_{j}$ sum to 1 and so some $s_{j}(\omega)<t_{j}$ since otherwise, the sum of the $t_{j}$ equalling 1 would require the sum of the $s_{j}$ to be larger than 1 . Hence $r_{k}(\omega)$ was not really the smallest so $\omega \notin F_{k}$. Thus $r_{k}(\omega) \leq 1$. Hence $s_{k}(\omega) \leq t_{k}$.

Let $\mathscr{S}$ denote those simplices whose value is $P_{n}$ for some $\omega$. In other words, if $\left\{\mathbf{y}_{0}, \cdots, \mathbf{y}_{n}\right\}$ are the vertices of one of these simplices, and

$$
\mathbf{y}_{s}=\sum_{i=0}^{n} t_{i}^{S} \mathbf{x}_{i}
$$

then for some $\omega, r_{k_{s}}(\omega) \leq r_{j}(\omega)$ for all $j \neq k_{s}$ and $\left\{k_{0}, \cdots, k_{n}\right\}=\{0, \cdots, n\}$. There are finitely many of these simplices, so $\mathscr{S} \equiv\left\{S_{1}, \cdots, S_{m}\right\}$. Let $F_{1} \subseteq \Omega$ be defined by

$$
F_{1} \equiv\left\{\omega: \prod_{i=0}^{n} p\left(\mathbf{y}_{i}, \omega\right)=P_{n}\right\}:\left[\mathbf{y}_{0}, \cdots, \mathbf{y}_{n}\right]=S_{1}
$$

If $F_{1}=\Omega$, then stop. If not, let

$$
F_{2} \equiv\left\{\omega \notin F_{1}: \prod_{i=0}^{n} p\left(\mathbf{y}_{i}, \omega\right)=P_{n}\right\}:\left[\mathbf{y}_{0}, \cdots, \mathbf{y}_{n}\right]=S_{2}
$$

Continue this way obtaining disjoint measurable sets $F_{j}$ whose union is all of $\Omega$. The union is $\Omega$ because every $\omega$ is associated with at least one of the $S_{i}$. Now for $\omega \in F_{k}$ and $\left[\mathbf{y}_{0}, \cdots, \mathbf{y}_{n}\right]=S_{k}$, it follows that $\prod_{i=0}^{n} p\left(\mathbf{y}_{i}, \omega\right)=P_{n}$. For $\omega \in F_{k}$, let $\mathbf{b}(\omega)$ denote the barycenter of $S_{k}$. Thus $\omega \rightarrow \mathbf{b}(\omega)$ is a measurable function, being constant on a measurable set. Thus we let $\mathbf{b}(\omega)=\sum_{i=1}^{m} \mathscr{X}_{F_{i}}(\omega) \mathbf{b}_{i}$ where $\mathbf{b}_{i}$ is the barycenter of $S_{i}$.

Now do this for a sequence $\varepsilon_{k} \rightarrow 0$ where $\mathbf{b}_{k}(\omega)$ is a barycenter as above. By Lemma 48.2.2 there exists $\mathbf{x}(\omega)$ such that $\omega \rightarrow \mathbf{x}(\omega)$ is measurable and a sequence

$$
\left\{\mathbf{b}_{k(\omega)}\right\}_{k(\omega)=1}^{\infty}, \lim _{k(\omega) \rightarrow \infty} \mathbf{b}_{k(\omega)}(\omega)=\mathbf{x}(\omega)
$$

This $\mathbf{x}(\omega)$ is also a fixed point.
Consider this last claim. $\mathbf{x}(\omega)=\sum_{i=0}^{n} t_{i}(\omega) \mathbf{x}_{i}$ and after applying $\mathbf{f}(\cdot, \omega)$, the result is $\sum_{i=0}^{n} s_{i}(\omega) \mathbf{x}_{i}$. Then $\mathbf{b}_{k(\omega)} \in \sigma_{k}(\omega)$ where $\sigma_{k}(\omega)$ is a simplex having vertices

$$
\left\{\mathbf{y}_{0}^{k}(\omega), \cdots, \mathbf{y}_{n}^{k}(\omega)\right\}
$$

and the value of $\left[\mathbf{y}_{0}^{k}(\omega), \cdots, \mathbf{y}_{n}^{k}(\omega)\right]$ is $P_{n}$. Re ordering these if necessary, we can assume that the label for $\mathbf{y}_{i}^{k}(\omega)=p_{i}$ which implies that, as noted above,

$$
\frac{s_{i}(\omega)}{t_{i}(\omega)} \leq 1, s_{i}(\omega) \leq t_{i}(\omega)
$$

the $i^{\text {th }}$ coordinate of $\mathbf{f}\left(\mathbf{y}_{i}^{k}(\omega), \omega\right)$ with respect to the original vertices of $S$ decreases and each $i$ is represented for $i=\{0,1, \cdots, n\}$. Thus

$$
\mathbf{y}_{i}^{k}(\omega) \rightarrow \mathbf{x}(\omega)
$$

and so the $i^{t h}$ coordinate of $\mathbf{y}_{i}^{k}(\omega), t_{i}^{k}(\omega)$ must converge to $t_{i}(\omega)$. Hence if the $i^{t h}$ coordinate of $\mathbf{f}\left(\mathbf{y}_{i}^{k}(\omega), \omega\right)$ is denoted by $s_{i}^{k}(\omega)$,

$$
s_{i}^{k}(\omega) \leq t_{i}^{k}(\omega)
$$

By continuity of $\mathbf{f}$, it follows that $s_{i}^{k}(\omega) \rightarrow s_{i}(\omega)$. Thus the above inequality is preserved on taking $k \rightarrow \infty$ and so

$$
0 \leq s_{i}(\omega) \leq t_{i}(\omega)
$$

this for each $i$. But these $s_{i}$ add to 1 as do the $t_{i}$ and so in fact, $s_{i}(\omega)=t_{i}(\omega)$ for each $i$ and so $\mathbf{f}(\mathbf{x}(\omega), \omega)=\mathbf{x}(\omega)$. This proves the following theorem which gives the existence of a measurable fixed point.

Theorem 48.2.4 Let $S$ be a simplex $\left[\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right]$ such that $\left\{\mathbf{x}_{i}-\mathbf{x}_{0}\right\}_{i=1}^{n}$ are independent. Also let $\mathbf{f}(\cdot, \omega): S \rightarrow S$ be continuous for each $\omega$ and $\omega \rightarrow \mathbf{f}(\mathbf{x}, \omega)$ is measurable, meaning inverse images of sets open in $S$ are in $\mathscr{F}$ where $(\Omega, \mathscr{F})$ is a measurable space. Then there exists $\mathbf{x}(\omega) \in S$ such that $\omega \rightarrow \mathbf{x}(\omega)$ is measurable and $\mathbf{f}(\mathbf{x}(\omega), \omega)=\mathbf{x}(\omega)$.

Corollary 48.2.5 Let $K$ be a closed convex bounded subset of $\mathbb{R}^{n}$. Let $\mathbf{f}(\cdot, \omega): K \rightarrow K$ be continuous for each $\omega$ and $\omega \rightarrow \mathbf{f}(\mathbf{x}, \omega)$ is measurable, meaning inverse images of sets open in $K$ are in $\mathscr{F}$ where $(\Omega, \mathscr{F})$ is a measurable space. Then there exists $\mathbf{x}(\omega) \in K$ such that $\omega \rightarrow \mathbf{x}(\omega)$ is measurable and $\mathbf{f}(\mathbf{x}(\omega), \omega)=\mathbf{x}(\omega)$.

Proof: Let $S$ be a large simplex containing $K$ and let $P$ be the projection map onto $K$. Consider $\mathbf{g}(\mathbf{x}, \omega) \equiv \mathbf{f}(P(\mathbf{x}), \omega)$. Then $\mathbf{g}$ satisfies the necessary conditions for Theorem 48.2.4 and so there exists $\mathbf{x}(\omega) \in S$ such that $\omega \rightarrow \mathbf{x}(\omega)$ is measurable and $\mathbf{g}(\mathbf{x}(\omega), \omega)=$ $\mathbf{x}(\omega)$. But this says $\mathbf{x}(\omega) \in K$ and so $\mathbf{g}(\mathbf{x}(\omega), \omega)=\mathbf{f}(\mathbf{x}(\omega), \omega)$.

## Much shorter proof

The above gives a proof of a measurable fixed point as part of a proof of the Brouwer fixed point theorem directly but it is a lot easier if you simply begin with the existence of the Brouwer fixed point and show it is measurable. We sent the above to be considered for publication and the referee pointed this out. I totally missed it because I had forgotten about the Kuratowski selection theorem. The functions $f: \Omega \times E \rightarrow \mathbb{R}$ in which $f(\cdot, \omega)$ is continuous are called Caratheodory functions.

Kuratowski [82] which is presented next. It is Theorem 11.1.11 in this book. I will give an alternate proof which comes from the measurable selection result of Corollary 48.2.3.

Theorem 48.2.6 Let $E$ be a compact metric space and let $(\Omega, \mathscr{F})$ be a measure space. Suppose $\psi: E \times \Omega \rightarrow \mathbb{R}$ has the property that $x \rightarrow \psi(x, \omega)$ is continuous and $\omega \rightarrow \psi(x, \omega)$ is measurable. Then there exists a measurable function, $f$ having values in $E$ such that

$$
\psi(f(\omega), \omega)=\max _{x \in E} \psi(x, \omega)
$$

Furthermore, $\omega \rightarrow \psi(f(\omega), \omega)$ is measurable.
Proof: Let $C=\left\{e_{i}\right\}_{i=1}^{\infty}$ be a countable dense subset of $E$. For example, take the union of $1 / 2^{n}$ nets for all $n$. Let $C_{n} \equiv\left\{e_{1}, \ldots, e_{n}\right\}$. Let $\omega \rightarrow f_{n}(\omega)$ be measurable and satisfy

$$
\psi\left(f_{n}(\omega), \omega\right)=\sup _{x \in C_{n}} \psi(x, \omega)
$$

This is easily done as follows. Let $B_{k} \equiv\left\{\omega: \psi\left(e_{k}, \omega\right) \geq \psi\left(e_{j}, \omega\right)\right.$ for all $\left.j \neq k\right\}$. Then let $A_{1} \equiv B_{1}$ and if $A_{1}, \ldots, A_{k}$ have been chosen, let $A_{k+1} \equiv B_{k+1} \backslash\left(\cup_{j=1}^{k} B_{k}\right)$. Thus each $A_{k}$ is measurable and you let $f_{n}(\omega) \equiv e_{k}$ for $\omega \in A_{k}$. Using Corollary 48.2.3, there is measurable $f(\omega)$ and a subsequence $n(\omega)$ such that $f_{n(\omega)}(\omega) \rightarrow f(\omega)$. Then by continuity, $\psi(f(\omega), \omega)=\lim _{n(\omega) \rightarrow \infty} \psi\left(f_{n(\omega)}(\omega), \omega\right)$ and this is an increasing sequence in this limit. Hence $\psi(f(\omega), \omega) \geq \sup _{x \in C_{n}} \psi(x, \omega)$ for each $n$ and so $\psi(f(\omega), \omega) \geq \sup _{x \in C} \psi(x, \omega)=$ $\sup _{x \in E} \psi(x, \omega)$. Since $f$ is measurable, it is the limit of a sequence $\left\{f_{n}(\omega)\right\}$ such that $f_{n}$ has finitely many values occuring on measurable sets, Theorem 11.3.10. Hence, by continuity, $\psi(f(\omega), \omega)=\lim _{n \rightarrow \infty} \psi\left(f_{n}(\omega), \omega\right)$ and since $\omega \rightarrow \psi\left(f_{n}(\omega), \omega\right)$ is measurable, so is $\psi(f(\omega), \omega)$.

One can generalize fairly easily. It is the same argument but carrying around more $\omega$.

Theorem 48.2.7 Let $E(\omega)$ be a compact metric space in a separable metric space $(X, d)$ and that $\omega \rightarrow E(\omega)$ is a measurable multifunction where $(\Omega, \mathscr{F})$ be a measure space. Suppose $\psi_{\omega}: E(\omega) \times \Omega \rightarrow \mathbb{R}$ has the property that $x \rightarrow \psi_{\omega}(x, \omega)$ is continuous and $\omega \rightarrow$ $\psi_{\omega}(x(\omega), \omega)$ is measurable if $x(\omega) \in E(\omega)$ and $\omega \rightarrow x(\omega)$ is measurable. Then there exists a measurable function, $f$ with $f(\omega) \in E(\omega)$ such that

$$
\psi_{\omega}(f(\omega), \omega)=\max _{x \in E(\omega)} \psi_{\omega}(x, \omega)
$$

Furthermore, $\omega \rightarrow \psi_{\omega}(f(\omega), \omega)$ is measurable.
Proof: Let $C(\omega)=\left\{e_{i}(\omega)\right\}_{i=1}^{\infty}$ be a countable dense subset of $E(\omega)$ with each $e_{i}(\omega)$ measurable. Since $\omega \rightarrow E(\omega)$ is measurable, such a countable dense subset exists. Let $C_{n}(\omega) \equiv\left\{e_{1}(\omega), \ldots, e_{n}(\omega)\right\}$. Let $\omega \rightarrow f_{n}(\omega)$ be measurable and satisfy

$$
\psi_{\omega}\left(f_{n}(\omega), \omega\right)=\sup _{x \in C_{n}} \psi(x, \omega)
$$

This is easily done as follows. Let

$$
B_{k} \equiv\left\{\omega: \psi_{\omega}\left(e_{k}(\omega), \omega\right) \geq \psi_{\omega}\left(e_{j}(\omega), \omega\right) \text { for all } j \neq k\right\}
$$

Then let $A_{1} \equiv B_{1}$ and if $A_{1}, \ldots, A_{k}$ have been chosen, let $A_{k+1} \equiv B_{k+1} \backslash\left(\cup_{j=1}^{k} B_{k}\right)$. Thus each $A_{k}$ is measurable and you let $f_{n}(\omega) \equiv e_{k}(\omega)$ for $\omega \in A_{k}$, so $f_{n}(\omega) \in E(\omega)$. Using Corollary 48.2.3, there is measurable $f(\omega)$ and a subsequence $n(\omega)$ such that $f_{n(\omega)}(\omega) \rightarrow f(\omega)$. Then by continuity,

$$
\psi_{\omega}(f(\omega), \omega)=\lim _{n(\omega) \rightarrow \infty} \psi_{\omega}\left(f_{n(\omega)}(\omega), \omega\right)
$$

and this is an increasing sequence in this limit. Hence

$$
\psi_{\omega}(f(\omega), \omega) \geq \sup _{x \in C_{n}(\omega)} \psi_{\omega}(x, \omega)
$$

for each $n$ and so

$$
\psi_{\omega}(f(\omega), \omega) \geq \sup _{x \in C(\omega)} \psi_{\omega}(x, \omega)=\sup _{x \in E(\omega)} \psi_{\omega}(x, \omega)
$$

Since $f$ is measurable, it is the limit of a sequence $\left\{f_{n}(\omega)\right\}$ such that $f_{n}$ has finitely many values on measurable sets, Theorem 11.3.10. We can assume each value of $f_{n}(\omega)$ is in $E(\omega)$. Indeed, repeat that theorem's proof applied to $C(\omega)$, letting $f_{n}(\omega)$ be the $e_{k}(\omega)$ in $C_{n}(\omega)$ closest to $f(\omega)$ as done there where this happens on a measurable set where in fact $e_{k}(\omega)$ is maximally close to $f(\omega)$ out of all $e_{i}(\omega), i \leq n$. Hence, by continuity, $\psi_{\omega}(f(\omega), \omega)=\lim _{n \rightarrow \infty} \psi_{\omega}\left(f_{n}(\omega), \omega\right)$ and since $\omega \rightarrow \psi_{\omega}\left(f_{n}(\omega), \omega\right)$ is measurable, so is $\psi_{\omega}(f(\omega), \omega)$.

Note the following: If you have the simpler situation where $\psi(x, \omega)$ defined on $X \times \Omega$ with $x \rightarrow \psi(x, \omega)$ continuous and $\omega \rightarrow \psi(x, \omega)$ measurable but $E(\omega)$ a compact measurable multifunction as above, then the conditions will hold because you would have
$\omega \rightarrow \psi(x(\omega), \omega)$ is measurable if $x(\omega)$ is. Indeed, $x(\omega)$ is the limit of a sequence $\left\{x_{n}(\omega)\right\}$ such that $x_{n}$ has finitely many values on measurable sets, Theorem 11.3.10. Hence, by continuity, $\psi(x(\omega), \omega)=\lim _{n \rightarrow \infty} \psi\left(x_{n}(\omega), \omega\right)$ and since $\omega \rightarrow \psi\left(x_{n}(\omega), \omega\right)$ is measurable, so is $\psi(x(\omega), \omega)$.

Now with the marvelous Kuratowski theorem, one gets the following interesting result on measurability of Brouwer fixed points.

Theorem 48.2.8 Let $K$ be a closed convex bounded subset of $\mathbb{R}^{n}$. Let $\mathbf{f}(\cdot, \omega): K \rightarrow K$ be continuous for each $\omega$ and $\omega \rightarrow \mathbf{f}(\mathbf{x}, \omega)$ is measurable, meaning inverse images of sets open in $K$ are in $\mathscr{F}$ where $(\Omega, \mathscr{F})$ is a measurable space. Then there exists $\mathbf{x}(\omega) \in K$ such that $\omega \rightarrow \mathbf{x}(\omega)$ is measurable and $\mathbf{f}(\mathbf{x}(\omega), \omega)=\mathbf{x}(\omega)$.

Proof: Simply consider $E=K$ and $\psi(\mathbf{x}, \omega) \equiv-|\mathbf{x}-\mathbf{f}(\mathbf{x}, \omega)|$. It has a maximum $\mathbf{x}(\omega)$ for each $\omega$ thanks to continuity of $\mathbf{f}(\cdot, \omega)$. Thanks to the Brouwer fixed point theorem, this $\mathbf{x}(\omega)$ must be a fixed point. By the above Kuratowski theorem, one of these $\mathbf{x}(\omega)$ is measurable. Obviously, by continuity of $\mathbf{f}(\cdot, \omega), \omega \rightarrow \mathbf{f}(\mathbf{x}(\omega), \omega)$ is measurable.

You can also let $K$ be replaced with $K(\omega)$ where $\omega \rightarrow K(\omega)$ is measurable and each $K(\omega)$ is closed, bounded, and convex.

Corollary 48.2.9 Let $K(\omega)$ be a closed convex bounded subset of $\mathbb{R}^{n}$ and let $\omega \rightarrow K(\omega)$ be a measurable multifunction for $\omega \in \Omega$ with $(\Omega, \mathscr{F})$ a measurable space. Let $\mathbf{f}_{\omega}(\cdot, \omega)$ : $K(\omega) \rightarrow K(\omega)$ be continuous, $\omega \rightarrow \mathbf{f}_{\omega}(\mathbf{x}(\omega), \omega)$ is measurable whenever $\omega \rightarrow \mathbf{x}(\omega)$ is measurable and $\mathbf{x}(\omega) \in K(\omega)$. Then there exists a measurable fixed point

$$
\mathbf{x}(\omega), \mathbf{f}_{\omega}(\mathbf{x}(\omega), \omega)=\mathbf{x}(\omega)
$$

Proof: Consider $\psi_{\omega}(\mathbf{x}, \omega) \equiv-\left|\mathbf{f}_{\omega}(\mathbf{x}(\omega), \omega)-\mathbf{x}(\omega)\right|$. By the Brower fixed point theorem, the maximum for fixed $\omega$ is 0 . Therefore, there exists such a measurable $\omega \rightarrow \mathbf{x}(\omega)$, a fixed point, from Theorem 48.2.7.

You can show that for $K(\omega)$ a closed convex bounded subset of $\mathbb{R}^{n}$ which is also a measurable multifunction, then the projection map $\omega \rightarrow P(\omega) \mathbf{x}$ is measurable. Suppose $\mathbf{f}$ : $\mathbb{R}^{n} \times \Omega \rightarrow \mathbb{R}^{n}$ and you know that $\mathbf{x} \rightarrow \mathbf{f}(\mathbf{x}, \omega)$ is continuous. Consider $\mathbf{f}(P(\omega) \mathbf{x}, \omega)$. Since $P(\omega)$ is a continuous map on $\mathbb{R}^{n}, \mathbf{x} \rightarrow \mathbf{f}(P(\omega) \mathbf{x}, \omega)$ is continuous. If $\mathbf{x}(\omega)$ is measurable with values in $K(\omega)$ so it is a pointwise limit of $\mathbf{x}_{n}(\omega)$ having finitely many values on measurable sets, then one can assume all these values of $\mathbf{x}_{n}(\omega)$ are in $K(\omega)$ since you could consider $P(\omega) \mathbf{x}_{n}(\omega) \rightarrow P(\omega) \mathbf{x}(\omega)=\mathbf{x}(\omega)$ and the measurability of $\omega \rightarrow P(\omega) \mathbf{x}$ implies $\omega \rightarrow P(\omega) \mathbf{x}_{n}(\omega)$ is measurable. Thus

$$
\mathbf{f}(\mathbf{x}(\omega), \omega)=\lim _{n \rightarrow \infty} \mathbf{f}\left(\mathbf{x}_{n}(\omega), \omega\right)
$$

and by assumption this last is measurable because it is a measurable function on each of several measurable sets. Thus from Corollary 48.2.9, there is a measurable fixed point $\mathbf{x}(\omega) \in K(\omega)$ for $\mathbf{f}$ so

$$
\mathbf{f}(P(\omega) \mathbf{x}(\omega), \omega)=\mathbf{f}(\mathbf{x}(\omega), \omega)=\mathbf{x}(\omega)
$$

This shows the following corollary.

Corollary 48.2.10 Let $K(\omega)$ be a closed convex bounded subset of $\mathbb{R}^{n}$ and let $\omega \rightarrow K(\omega)$ be a measurable multifunction for $\omega \in \Omega$ with $(\Omega, \mathscr{F})$ a measurable space. Let $\mathbf{f}(\cdot, \omega)$ : $K(\omega) \rightarrow K(\omega)$ and $\mathbf{x} \rightarrow \mathbf{f}(\mathbf{x}, \omega)$ continuous on $\mathbb{R}^{n}$. Suppose also $\omega \rightarrow \mathbf{f}(\mathbf{x}, \omega)$ is measurable for each fixed $\mathbf{x}$. Then there exists a measurable fixed point $\mathbf{x}(\omega), \mathbf{f}(\mathbf{x}(\omega), \omega)=$ $\mathbf{x}(\omega), \mathbf{x}(\omega) \in K(\omega)$.

### 48.2.4 Measurability Of Schauder Fixed Points

Now we consider the Schauder fixed point theorem. Let $\omega \rightarrow K(\omega)$ be a measurable multifunction having closed convex values. Here $K(\omega) \subseteq X$ a separable Banach space. Also assume

$$
\begin{gathered}
f(\cdot, \omega) \text { is continuous, } f(\cdot, \omega): K(\omega) \rightarrow K(\omega), \\
\omega \rightarrow f(x, \omega) \text { is measurable }
\end{gathered}
$$

Next we have the following approximation result.
Lemma 48.2.11 Let $f(\cdot, \omega)$ be as above and $f(K(\omega), \omega) \subseteq K(\omega)$ for $K(\omega)$ convex and closed and $\omega \rightarrow K(\omega)$ a measurable mutifunction. Suppose also that $\overline{f(K(\omega), \omega)}$ is a compact set. For each $r>0$ and $\omega$, there exists a finite set of points

$$
\left\{y_{1}(\omega), \cdots, y_{n(\omega)}(\omega)\right\} \subseteq \overline{f(K(\omega), \omega)}, \omega \rightarrow y_{i}(\omega) \text { measurable }
$$

and continuous functions $\psi_{i}(\cdot, \omega)$ defined on $\overline{f(K(\omega), \omega)}$ such that for $y \in f(K(\omega), \omega)$,

$$
\begin{gather*}
\sum_{i=1}^{n(\omega)} \psi_{i}(y, \omega)=1,  \tag{48.2.1}\\
\psi_{i}(y, \omega)=0 \text { if } y \notin B\left(y_{i}(\omega), r\right), \psi_{i}(y, \omega)>0 \text { if } y \in B\left(y_{i}(\omega), r\right)
\end{gather*}
$$

If

$$
\begin{equation*}
f_{r}(x, \omega) \equiv \sum_{i=1}^{n(\omega)} y_{i}(\omega) \psi_{i}(f(x, \omega), \omega) \tag{48.2.2}
\end{equation*}
$$

then whenever $x \in K(\omega)$,

$$
\left\|f(x, \omega)-f_{r}(x, \omega)\right\| \leq r .
$$

Proof: Using the compactness of $\overline{f(K(\omega), \omega)}$, Proposition 48.1.6 says there exist measurable functions $y_{i}(\omega)$

$$
\left\{y_{1}(\omega), \cdots, y_{n(\omega)}(\omega)\right\} \subseteq \overline{f(K(\omega), \omega)} \subseteq K(\omega)
$$

such that

$$
\left\{B\left(y_{i}(\omega), r\right)\right\}_{i=1}^{n}
$$

covers $\overline{f(K, \omega)}$. Let

$$
\phi_{i}(y, \omega) \equiv\left(r-\left\|y-y_{i}(\omega)\right\|\right)^{+}
$$

Thus $\phi_{i}$ is continuous in $y$ and measurable in $\omega$ for fixed $y$. Also $\phi_{i}(y, \omega)>0$ if $y \in$ $B\left(y_{i}(\omega), r\right)$ and $\phi_{i}(y, \omega)=0$ if $y \notin B\left(y_{i}(\omega), r\right)$. For $y \in \overline{f(K, \omega)}$, let

$$
\psi_{i}(y, \omega) \equiv \phi_{i}(y, \omega)\left(\sum_{j=1}^{n(\omega)} \phi_{j}(y, \omega)\right)^{-1}
$$

From the formula, $\omega \rightarrow \psi_{i}(y, \omega)$ is measurable. Also 48.2.1 is satisfied. Indeed the denominator is not zero because $y$ is in one of the $B\left(y_{i}(\omega), r\right)$. Thus it is obvious that the sum of these equals 1 on $f(K, \omega)$. Now let $f_{r}$ be given by 48.2 .2 for $x \in K(\omega)$. For such $x$,

$$
f(x, \omega)-f_{r}(x, \omega)=\sum_{i=1}^{n}\left(f(x, \omega)-y_{i}(\omega)\right) \psi_{i}(f(x, \omega), \omega)
$$

Thus

$$
\begin{aligned}
& f(x, \omega)- f_{r}(x, \omega)=\sum_{\left\{i: f(x) \in B\left(y_{i}(\omega), r\right)\right\}}\left(f(x, \omega)-y_{i}(\omega)\right) \psi_{i}(f(x, \omega), \omega) \\
&+\sum_{\left\{i: f(x, \omega) \notin B\left(y_{i}(\omega), r\right)\right\}}\left(f(x, \omega)-y_{i}(\omega)\right) \psi_{i}(f(x, \omega), \omega) \\
&=\sum_{\left\{i: f(x, \omega)-y_{i}(\omega) \in B(0, r)\right\}}\left(f(x, \omega)-y_{i}(\omega)\right) \psi_{i}(f(x, \omega), \omega)= \\
&\left\{\sum_{\left\{i: f(x, \omega)-y_{i}(\omega) \in B(0, r)\right\}}\left(f(x, \omega)-y_{i}(\omega)\right) \psi_{i}(f(x, \omega), \omega) \in B(0, r)\right.
\end{aligned}
$$

because $0 \in B(0, r), B(0, r)$ is convex, and 48.2.1. $f(x, \omega)-f_{r}(x, \omega)$ is a convex combination of vectors in $B(0, r)$.

We think of $f_{r}(\cdot, \omega)$ as an approximation to $f(\cdot, \omega)$. In fact it is uniformly within $r$ of $f(\cdot, \omega)$ on $K(\omega)$. The next lemma shows that this $f_{r}(\cdot, \omega)$ has a fixed point. This is the main result and comes from the Brouwer fixed point theorem in $\mathbb{R}^{n}$. It is an approximate fixed point.

Lemma 48.2.12 Let $\overline{f(K(\omega), \omega)}$ be compact. For each $r>0$, there exists $x_{r}(\omega) \in$ convex hull of $\overline{f(K(\omega), \omega)} \subseteq K(\omega)$ such that

$$
f_{r}\left(x_{r}(\omega), \omega\right)=x_{r}(\omega),\left\|f_{r}(x, \omega)-f(x, \omega)\right\|<r \text { for all } x \in K(\omega)
$$

and $\omega \rightarrow x_{r}(\omega)$ is measurable.
Proof: The upper limit in the sum of the above lemma $n(\omega)$ is a measurable function. One can partition the measure space according to the value of $n(\omega)$. This gives a countable set of disjoint measurable subsets $\left\{\Omega_{n}\right\}_{n=1}^{\infty}$ in the partition such that on the measurable set $\Omega_{n}, n(\omega)=n$. Specializing to the measurable space consisting of $\Omega_{n}$, we will assume here that $n(\omega)=n$ and show that there exists a measurable fixed point $x_{r}(\omega) \in K(\omega)$ for $\omega \in \Omega_{n}$. Then the result follows by letting $x_{r}(\omega)$ be that which has been obtained on $\Omega_{n}$.

Thus, from now on, simply denote as $n$ the upper limit and let $\omega \in \Omega_{n}$. If $f_{r}\left(x_{r}, \omega\right)=x_{r}$ and

$$
x_{r}=\sum_{i=1}^{n} a_{i} y_{i}(\omega)
$$

for $\sum_{i=1}^{n} a_{i}=1$ and the $y_{i}$ described in the above lemma, we need

$$
\begin{aligned}
f_{r}\left(x_{r}, \omega\right) & \equiv \sum_{i=1}^{n} y_{i}(\omega) \psi_{i}\left(f\left(x_{r}, \omega\right), \omega\right) \\
& =\sum_{j=1}^{n} y_{j}(\omega) \psi_{j}\left(f\left(\sum_{i=1}^{n} a_{i} y_{i}, \omega\right), \omega\right)=\sum_{j=1}^{n} a_{j} y_{j}(\omega)=x_{r}
\end{aligned}
$$

Also, if this is satisfied, then we have the desired fixed point.
This will be satisfied if for each $j=1, \cdots, n$,

$$
\begin{equation*}
a_{j}=\psi_{j}\left(f\left(\sum_{i=1}^{n} a_{i} y_{i}, \omega\right), \omega\right) \tag{48.2.3}
\end{equation*}
$$

so, let

$$
\Sigma_{n-1} \equiv\left\{\mathbf{a} \in \mathbb{R}^{n}: \sum_{i=1}^{n} a_{i}=1, a_{i} \geq 0\right\}
$$

and let $h(\cdot, \omega): \Sigma_{n-1} \rightarrow \Sigma_{n-1}$ be given by

$$
h(\mathbf{a}, \omega)_{j} \equiv \psi_{j}\left(f\left(\sum_{i=1}^{n} a_{i} y_{i}, \omega\right), \omega\right)
$$

Can we obtain a fixed point $\mathbf{a}(\omega)$ such that $\omega \rightarrow \mathbf{a}(\omega)$ is measurable? Since $h(\cdot, \omega)$ is a continuous function of $\mathbf{a}$ and $\omega \rightarrow h(\mathbf{x}, \omega)$ is measurable, such a measurable fixed point exists thanks to Theorem 48.2 .4 or the much easier Theorem 48.2.8 above. Then $x_{r}(\omega)=$ $\sum_{i=1}^{n} a_{i}(\omega) y_{i}(\omega)$ so $x_{r}$ is measurable.

The following is the Schauder fixed point theorem for measurable fixed points.
Theorem 48.2.13 Let $\omega \rightarrow K(\omega)$ be a measurable multifunction which has convex and closed values in a separable Banach space. Let $f(\cdot, \omega): K(\omega) \rightarrow K(\omega)$ be continuous and $\omega \rightarrow f(x, \omega)$ is measurable and $\overline{f(K(\omega), \omega)}$ is compact. Then $f(\cdot, \omega)$ has a fixed point $x(\omega)$ such that $\omega \rightarrow x(\omega)$ is measurable.

Proof: Recall that $f\left(x_{r}(\underline{\omega}), \omega\right)-f_{r}\left(x_{r}(\omega), \omega\right) \in B(0, r)$ and $f_{r}\left(x_{r}(\omega), \omega\right)=x_{r}(\omega)$ with $x_{r}(\omega) \in$ convex hull of $\overline{f(K(\omega), \omega)} \subseteq K(\omega)$. Here $x_{r}$ is measurable. By Lemma 48.2.2 there is a measurable function $x(\omega)$ which equals the weak $\lim _{r(\omega) \rightarrow 0} x_{r(\omega)}(\omega)$. However, since $\overline{f(K(\omega), \omega)}$ is compact, there is a subsequence still denoted with $r(\omega)$ such that $f\left(x_{r(\omega)}, \omega\right)$ converges strongly to some $x \in \overline{f(K(\omega), \omega)}$. It follows that

$$
f_{r(\omega)}\left(x_{r(\omega)}(\omega), \omega\right)
$$

also converges to $x$ strongly. But this equals $x_{r(\omega)}(\omega)$ which shows that $x_{r(\omega)}(\omega)$ converges strongly to the measurable $x(\omega)$. Therefore,

$$
\begin{aligned}
f(x(\omega), \omega) & =\lim _{r(\omega) \rightarrow 0} f\left(x_{r(\omega)}(\omega), \omega\right)=\lim _{r(\omega) \rightarrow 0} f_{r(\omega)}\left(x_{r(\omega)}(\omega), \omega\right) \\
& =\lim _{r(\omega) \rightarrow 0} x_{r(\omega)}(\omega)=x(\omega)
\end{aligned}
$$

As a special case of the above, here is a corollary which generalizes the earlier result on the Brouwer fixed point theorem.

Corollary 48.2.14 Let $(\Omega, \mathscr{F})$ be a measurable space and let $K(\omega)$ be a convex and compact set in $\mathbb{R}^{n}$ and $\omega \rightarrow K(\omega)$ is a measurable multifunction. Also let $\mathbf{f}(\cdot, \omega): K(\omega) \rightarrow$ $K(\omega)$ be continuous and for fixed $\mathbf{x} \in \mathbb{R}^{n}, \omega \rightarrow \mathbf{f}(\mathbf{x}, \omega)$ is measurable. Then there exists a fixed point $\mathbf{x}(\omega)$ for $\mathbf{f}(\cdot, \omega)$ such that $\omega \rightarrow \mathbf{x}(\omega)$ is measurable.

Note that in all of these considerations, there is no loss of generality in assuming $\mathbf{f}(\cdot, \omega)$ is defined on the whole space $X$ thanks to the theorem which says that a continuous function defined on a convex closed set can be extended to a continuous function defined on the whole space.

In the case of a single set, the following corollary is also obtained.
Corollary 48.2.15 Let $X$ be a Banach space and let $K$ be a compact convex subset. Let $f: K \times \Omega \rightarrow K$ satisfy

$$
\begin{aligned}
x & \rightarrow f(x, \omega) \text { is continous } \\
\omega & \rightarrow f(x, \omega) \text { is measurable }
\end{aligned}
$$

Then $f(\cdot, \omega)$ has a fixed point $x(\omega)$ such that $\omega \rightarrow x(\omega)$ is measurable.
Proof: The set $K$ has a countable dense subset $\left\{k_{i}\right\}$. You could consider $Y$ as the closure in $X$ of the span of these $k_{i}$. Thus $Y$ is a separable Banach space which contains $K$. Now apply the above result.

If $X$ is only a normed linear space, you could just consider its completion and apply the above result. Since $K$ is compact, it is automatically complete with respect to the norm on $X$.

What of the Schaefer fixed point theorem? Is there a measurable version of it? A map $h: X \rightarrow X$ for $X$ a Banach space is a compact map if it is continuous and it takes bounded sets to precompact sets. If you have such a compact map and it maps a closed ball to a closed ball, then it must have a fixed point by the Schauder theorem. If you have $h=f(\cdot, \omega)$ where $\omega \rightarrow f(x, \omega)$ is measurable, $x \rightarrow f(x, \omega)$ compact, then if $f(\cdot, \omega)$ maps a closed ball to a closed ball, it must have a measurable fixed point $x(\omega)$ by the above. Now the following is a version of the Schaefer fixed point theorem which can be used to get measurable fixed points.
Theorem 48.2.16 Let $f(\cdot, \omega): X \rightarrow X$ be a compact map (takes bounded sets to precompact sets and continuous) where $X$ is a Banach space. Also suppose that

$$
\sup \{z \in f(B(0, r), \omega)\} \leq C(r)
$$

independent of $\omega$. Here $\omega \rightarrow f(x, \omega)$ is measurable for each $x \in X$. Then either

1. There is a measurable fixed point $x(\omega)$ for $t f(\cdot, \omega)$ for all $t \in[0,1]$ or
2. For every $r>0$, there exists $\omega$ and $t \in(0,1)$ such that if $x$ satisfies $x=t f(x, \omega)$, then $\|x\|>r$.

Proof: Suppose that alternative 2 does not hold and yet alternative 1 also fails to hold. Since alternative 2 does not hold, there exists $M_{0}$ such that for all $\omega$, and for all $t \in(0,1)$, if $x=t f(x, \omega)$, then $\|x(\omega)\| \leq M_{0}$. If alternative 1 fails, then there is some $t$ with no measurable fixed point $x(\omega)$ for $t f(\cdot, \omega)$. So let $M>M_{0}$. By the measurable Schauder fixed point theorem, Theorem 48.2.13, there is measurable $x_{M}(\omega)$ such that

$$
x_{M}(\omega)=t\left(r_{M} f\left(x_{M}(\omega), \omega\right)\right), r_{M} y=y \text { if }\|y\| \leq M, r_{M} y=\frac{M y}{\|y\|} \text { if }\|y\|>M
$$

Thus $r_{M}$ is continuous and so $r_{M} f(\cdot, \omega)$ is continuous and compact. We must have

$$
\left\|f\left(x_{M}(\hat{\omega}), \hat{\omega}\right)\right\|>M
$$

for some $\hat{\omega}$ and $r_{M} f\left(x_{M}(\hat{\omega}), \omega\right)=\frac{M f\left(x_{M}(\hat{\omega}), \hat{\omega}\right)}{\left\|f\left(x_{M}(\hat{\omega}), \hat{\omega}\right)\right\|}$ since if $\left\|f\left(x_{M}(\omega), \omega\right)\right\| \leq M$ for all $\omega$, then

$$
r_{M} f\left(x_{M}(\omega), \omega\right)=f\left(x_{M}(\omega), \omega\right)
$$

and there would be a measurable fixed point for this $t$. But then, for this $\hat{\omega}$

$$
x_{M}(\hat{\omega})=t\left(r_{M} f\left(x_{M}(\hat{\omega}), \hat{\omega}\right)\right)=t \frac{M f\left(x_{M}(\hat{\omega}), \hat{\omega}\right)}{\left\|f\left(x_{M}(\hat{\omega}), \hat{\omega}\right)\right\|}=\hat{t} f\left(x_{M}(\hat{\omega}), \hat{\omega}\right)
$$

From the hypotheses that 2 does not hold, $\left\|x_{M}(\hat{\omega})\right\| \leq M_{0}$. Thus $\left\|f\left(x_{M}(\hat{\omega}), \hat{\omega}\right)\right\|>M$. But this requires that $C\left(M_{0}\right)>M$ for all $M$ which is clearly impossible. Hence there is a measurable fixed point for $t f(\cdot, \omega)$ for all $t \in[0,1]$.

We will use this very interesting Shaefer theorem to give an easy to use criterion for showing the existence of measurable solutions to ordinary differential equations.

Lemma 48.2.17 Let $X \equiv C\left([0, T] ; \mathbb{R}^{n}\right)$ and let $\mathbf{f}(\cdot, \cdot, \omega):[0, T] \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ where $\omega \in \Omega$ for $(\Omega, \mathscr{F})$ a measurable space and let $(t, \mathbf{x}) \rightarrow \mathbf{f}(t, \mathbf{x}, \omega)$ be continuous on $[0, T] \times \mathbb{R}^{n}$. Also suppose a uniform estimate of the form

$$
\begin{equation*}
\sup _{t \in[0, T],|\mathbf{x}| \leq r}|\mathbf{f}(t, \mathbf{x}, \omega)| \leq C(r) \text { independent of } \omega \tag{48.2.4}
\end{equation*}
$$

If $\mathbf{f}$ is $\mathscr{B}\left([0, T] \times \mathbb{R}^{n}\right) \times \mathscr{F}$ measurable and $\omega \rightarrow \mathbf{x}_{0}(\omega)$ is $\mathscr{F}$ measurable, then for $\mathbf{y} \in X$, define $F(\mathbf{y}, \omega) \in X$ by

$$
F(\mathbf{y}, \omega)(t) \equiv \int_{0}^{t} \mathbf{f}\left(s, \mathbf{y}(s)+\mathbf{x}_{0}(\omega), \omega\right) d s
$$

Then

$$
\omega \rightarrow F(\mathbf{y}, \omega)
$$

is measurable into $X$.

Proof: The space $X$ is separable and so by the Riesz representation theorem and the Pettis theorem, it suffices to verify that for $\mathbf{y} \in X$,

$$
\omega \rightarrow \int_{0}^{T} \int_{0}^{t} \mathbf{f}\left(s, \mathbf{y}(s)+\mathbf{x}_{0}(\omega), \omega\right) d s \cdot \sigma(t) d \mu(t)
$$

is measurable whenever $\sigma \in L^{\infty}([0, T], \mu)$ and $\mu$ is a finite Radon measure. By product measurability, there are simple functions $\mathbf{s}_{n}(t, \mathbf{x}, \omega)$ converging to $\mathbf{f}(t, \mathbf{x}, \omega)$ pointwise and we can have $\left|\mathbf{s}_{n}(t, \mathbf{x}, \omega)\right| \leq|\mathbf{f}(t, \mathbf{x}, \omega)|$ where

$$
\mathbf{s}_{n}(t, \mathbf{x}, \omega)=\sum_{i=1}^{m_{n}} c_{i} \mathscr{X}_{E_{i}^{n}}(t, \mathbf{x}, \omega), E_{i}^{n} \in \mathscr{B}\left([0, T] \times \mathbb{R}^{n}\right) \times \mathscr{F} .
$$

Therefore, for fixed $\omega,(t, \mathbf{x}) \rightarrow s_{n}\left(t, \mathbf{x}+\mathbf{x}_{0}(\omega), \omega\right)$ is Borel measurable and so

$$
t \rightarrow s_{n}\left(t, \mathbf{y}(t)+\mathbf{x}_{0}(\omega), \omega\right)
$$

is also Borel measurable so the above integral with $\mathbf{f}$ replaced with $\mathbf{s}_{n}$ surely makes sense. Then for $\mathbf{y} \in X$, you would have from dominated convergence theorem and assumed estimate 48.2.4,

$$
\begin{aligned}
& \int_{0}^{T} \int_{0}^{t} \mathbf{s}_{n}\left(s, \mathbf{y}(s)+\mathbf{x}_{0}(\omega), \omega\right) d s \cdot \sigma(t) d \mu(t) \\
\rightarrow & \int_{0}^{T} \int_{0}^{t} \mathbf{f}\left(s, \mathbf{y}(s)+\mathbf{x}_{0}(\omega), \omega\right) d s \cdot \sigma(t) d \mu(t)
\end{aligned}
$$

and so the issue devolves to whether

$$
\begin{equation*}
\omega \rightarrow \int_{0}^{T} \int_{0}^{t} \mathbf{s}_{n}\left(s, \mathbf{y}(s)+\mathbf{x}_{0}(\omega), \omega\right) d s \cdot \sigma(t) d \mu(t) \tag{48.2.5}
\end{equation*}
$$

is $\mathscr{F}$ measurable. Let $\mathscr{P}$ be the rectangles $B \times F$ where $B$ is Borel in $[0, T] \times \mathbb{R}^{n}$ and $F \in \mathscr{F}$. Let $\mathscr{G} \equiv$

$$
\left\{E \in \sigma(\mathscr{P}): \omega \rightarrow \int_{0}^{T} \int_{0}^{t} \mathscr{X}_{E}\left(s, \mathbf{y}(s)+\mathbf{x}_{0}(\omega), \omega\right) d s \cdot \sigma(t) d \mu(t) \text { is } \mathscr{F} \text { measurable }\right\}
$$

the above condition holding for all $\sigma d \mu$. Obviously $\mathscr{G} \supseteq \mathscr{P}$. Indeed,

$$
\omega \rightarrow \mathscr{X}_{B}\left(s, \mathbf{y}(s)+\mathbf{x}_{0}(\omega)\right) \mathscr{X}_{F}(\omega)=\mathscr{X}_{B \times F}\left(s, \mathbf{y}(s)+\mathbf{x}_{0}(\omega), \omega\right)
$$

is measurable because $B$ is Borel and composition of Borel functions with a measurable function is measurable. It is also clear that $\mathscr{G}$ is closed with respect to countable disjoint unions and complements. This follows from the monotone convergence theorem in the case of disjoint unions and from the observation that

$$
\mathscr{X}_{E}\left(s, \mathbf{y}(s)+\mathbf{x}_{0}(\omega), \omega\right)+\mathscr{X}_{E^{C}}\left(s, \mathbf{y}(s)+\mathbf{x}_{0}(\omega), \omega\right)=1
$$

in the case of complements. Hence, by Dynkin's lemma, $\mathscr{G}=\sigma(\mathscr{P})=\mathscr{B}\left([0, T] \times \mathbb{R}^{n}\right) \times$ $\mathscr{F}$. On consideration of components of $\mathbf{s}_{n}$, it follows that 48.2.5 is indeed measurable and this establishes the needed result.

There may be other conditions which will imply $\omega \rightarrow F(\mathbf{y}, \omega)$ is measurable into $X$ but an assumption of product measurability as above is fairly attractive. In particular, one could likely relax the estimate .

Now with this lemma, here is a very useable theorem related to measurable solutions to ordinary differential equations.

Theorem 48.2.18 Let $\mathbf{f}(\cdot, \cdot, \omega):[0, T] \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be continuous and suppose

$$
\begin{equation*}
\omega \rightarrow F(\mathbf{y}, \omega) \tag{48.2.6}
\end{equation*}
$$

is measurable into $C\left([0, T] ; \mathbb{R}^{n}\right) \equiv X$ for

$$
F(\mathbf{y}, \omega)(t) \equiv \int_{0}^{t} \mathbf{f}\left(s, \mathbf{y}(s)+\mathbf{x}_{0}(\omega), \omega\right) d s
$$

Also suppose that

$$
\sup _{t \in[0, T],|\mathbf{x}| \leq r}|\mathbf{f}(t, \mathbf{x}, \omega)| \leq C(r) \text { independent of } \omega
$$

and suppose there exists $L>0$ such that for all $\omega$ and $\lambda \in(0,1)$, if

$$
\begin{equation*}
\mathbf{x}^{\prime}=\lambda \mathbf{f}(t, \mathbf{x}, \omega), \mathbf{x}(0, \omega)=\mathbf{x}_{0}(\omega), t \in[0, T] \tag{48.2.7}
\end{equation*}
$$

where $\mathbf{x}_{0}$ is bounded and measurable, then for all $t \in[0, T]$, then it follows that $\|\mathbf{x}\|<L$, the norm in $X \equiv C\left([0, T] ; \mathbb{R}^{n}\right)$. Then there exists a solution to

$$
\begin{equation*}
\mathbf{x}^{\prime}=\mathbf{f}(t, \mathbf{x}, \omega), \mathbf{x}(0, \omega)=\mathbf{x}_{0}(\omega) \tag{48.2.8}
\end{equation*}
$$

for $t \in[0, T]$ where $\omega \rightarrow \mathbf{x}(\cdot, \omega)$ is measurable into $X$. Thus $(t, \omega) \rightarrow \mathbf{x}(t, \omega)$ is product measurable.

Proof: Let $F(\cdot, \omega): X \rightarrow X$ where $X$ described above.

$$
F(\mathbf{y}, \omega)(t) \equiv \int_{0}^{t} \mathbf{f}\left(s, \mathbf{y}(s)+\mathbf{x}_{0}, \omega\right) d s
$$

$F$ is clearly continuous in the first variable and is assumed measurable in the second.
Let $B$ be a bounded set in $X$. Then by assumption $\left|\mathbf{f}\left(s, \mathbf{y}(s)+\mathbf{x}_{0}, \omega\right)\right|$ is bounded for $s \in[0, T]$ if $\mathbf{y} \in B$. Say $\left|\mathbf{f}\left(s, \mathbf{y}(s)+\mathbf{x}_{0}, \omega\right)\right| \leq C_{B}$. Hence $F(B, \omega)$ is bounded in $X$. Also, for $\mathbf{y} \in B, s<t$,

$$
|F(\mathbf{y}, \omega)(t)-F(\mathbf{y}, \omega)(s)| \leq\left|\int_{s}^{t} \mathbf{f}\left(s, \mathbf{y}(s)+\mathbf{x}_{0}, \omega\right) d s\right| \leq C_{B}|t-s|
$$

and so $F(B, \omega)$ is pre-compact by the Ascoli Arzela theorem. By the Schaefer fixed point theorem, there are two alternatives. Either there exist $\omega, \lambda$ resulting in arbitrarily large solutions $\mathbf{y}$ to

$$
\lambda F(\mathbf{y}, \omega)=\mathbf{y}
$$

or there is a fixed point for $\lambda F$ for all $\lambda \in[0,1]$. In the first case, there would be unbounded $\mathbf{y}_{\lambda, \omega}$ solving

$$
\mathbf{y}_{\lambda, \omega}(t)=\lambda \int_{0}^{t} \mathbf{f}\left(s, \mathbf{y}_{\lambda, \omega}(s)+\mathbf{x}_{0}, \omega\right) d s
$$

Then let $\mathbf{x}_{\lambda, \omega}(s) \equiv \mathbf{y}_{\lambda, \omega}(s)+\mathbf{x}_{0}$ and you get arbitrarily large $\left\|\mathbf{x}_{\lambda, \omega}\right\|$ for various $\omega$ and $\lambda \in(0,1)$. The above implies

$$
\mathbf{x}_{\lambda}(t)-\mathbf{x}_{0}=\lambda \int_{0}^{t} \mathbf{f}\left(s, \mathbf{x}_{\lambda}(s), \omega\right) d s
$$

so $\mathbf{x}_{\lambda}^{\prime}=\lambda \mathbf{f}\left(t, \mathbf{x}_{\lambda}, \omega\right), \mathbf{x}_{\lambda}(0, \omega)=\mathbf{x}_{0}(\omega)$ and these would be unbounded for $\lambda \in(0,1)$ contrary to the assumption that there exists an estimate for these valid for all $\lambda \in(0,1)$. Hence the first alternative must hold and hence there is $\mathbf{y}(\omega) \in X$ such that $\omega \rightarrow \mathbf{y}(\omega)$ measurable and

$$
F(\mathbf{y}(\omega), \omega)=\mathbf{y}(\omega)
$$

Then letting $\mathbf{x}(s, \omega) \equiv \mathbf{y}(s, \omega)+\mathbf{x}_{0}(\omega)$, it follows that

$$
\mathbf{x}(t, \omega)-\mathbf{x}_{0}(\omega)=\int_{0}^{t} \mathbf{f}(s, \mathbf{x}(s, \omega), \omega) d s
$$

and so $\mathbf{x}(\cdot, \omega)$ is a solution to the differential equation on $[0, T]$ which is measurable into $X$. In particular, one obtains that $(t, \omega) \rightarrow \mathbf{x}(t, \omega)$ is $\mathscr{B}(0, T) \times \mathscr{F}$ measurable where $\mathscr{F}$ is the $\sigma$ algebra of measurable sets.

### 48.3 A Set Valued Browder Lemma With Measurability

A simple application is a measurable version of the Browder lemma which is also valid for upper semicontinuous set valued maps. In what follows, we do not assume that $A(\cdot, \omega)$ is a set valued measurable multifunction, only that it has a measurable selection which is a weaker assumption. First is a general result on upper set valued maps $(u, \omega) \rightarrow A(u, \omega)$ where $u \rightarrow A(u, \omega)$ is upper semicontinuous and $\omega \rightarrow A(u, \omega)$ has a measurable selection.

Theorem 48.3.1 Let $V$ be a reflexive separable Banach space. Suppose $\omega \rightarrow A(u, \omega)$ has a measurable selection in $V^{\prime}$, for each $u \in V$ and $\omega \in \Omega$ the set $A(u, \omega)$ is closed and convex in $V^{\prime}$ and $u \rightarrow A(u, \omega)$ is bounded. Also, suppose $u \rightarrow A(u, \omega)$ is upper-semicontinuous from the strong topology of $V$ to the weak topology of $V^{\prime}$. That is, if $u_{n} \rightarrow u$ in $V$ strongly, then if $O$ is a weakly open set containing $A(u, \omega)$, it follows that $A\left(u_{n}, \omega\right) \in O$ for all $n$ large enough. Conclusion: Then, whenever $\omega \rightarrow u(\omega)$ is measurable into $V$ there is a measurable selection for $\omega \rightarrow A(u(\omega), \omega)$ into $V^{\prime}$.

Proof: Let $\omega \rightarrow u(\omega)$ be measurable into $V$, and let $u_{n}(\omega) \rightarrow u(\omega)$ in $V$ where $u_{n}$ is a simple function

$$
u_{n}(\omega)=\sum_{k=1}^{m_{n}} c_{k}^{n} \mathscr{X}_{E_{k}^{n}}(\omega), \text { the } E_{k}^{n} \text { disjoint, } \Omega=\cup_{k} E_{k}^{n}
$$

each $c_{k}^{n}$ being in $V$. We can assume that $\left\|u_{n}(\omega)\right\| \leq 2\|u(\omega)\|$ for all $\omega$. Then, by assumption, there is a measurable selection for $\omega \rightarrow A\left(c_{k}^{n}, \omega\right)$ denoted as $\omega \rightarrow y_{k}^{n}(\omega)$. Thus, $\omega \rightarrow y_{k}^{n}(\omega)$ is measurable into $V^{\prime}$ and $y_{k}^{n}(\omega) \in A\left(c_{k}^{n}, \omega\right)$ for all $\omega \in \Omega$. Consider now,

$$
y^{n}(\omega)=\sum_{k=1}^{m_{n}} y_{k}^{n}(\omega) \mathscr{X}_{E_{k}^{n}}(\omega)
$$

It is measurable and for $\omega \in E_{k}^{n}$ it equals $y_{k}^{n}(\omega) \in A\left(c_{k}^{n}, \omega\right)=A\left(u_{n}(\omega), \omega\right)$. Thus, $y^{n}$ is a measurable selection of $\omega \rightarrow A\left(u_{n}(\omega), \omega\right)$. By the assumption $A(\cdot, \omega)$ is bounded, for each $\omega$ these $y^{n}(\omega)$ lie in a bounded subset of $V^{\prime}$. The bound might depend on $\omega$ of course. It follows now from Lemma 48.2.2 that there is a subsequence $\left\{y^{n(\omega)}\right\}$ that converges weakly to $y(\omega)$, where $\omega \rightarrow y(\omega)$ is measurable. But,

$$
y^{n(\omega)}(\omega) \in A\left(u_{n(\omega)}(\omega), \omega\right)
$$

is a convex closed set for which $u \rightarrow A(u, \omega)$ is upper-semicontinuous and $u_{n(\omega)} \rightarrow u$, hence, $y(\omega) \in A(u(\omega), \omega)$. This is the claimed measurable selection.

The next lemma is about the projection map onto a set valued map whose values are closed convex sets.

Lemma 48.3.2 Let $\omega \rightarrow \mathscr{K}(\omega)$ be measurable into $\mathbb{R}^{n}$ where $\mathscr{K}(\omega)$ is closed and convex. Then $\omega \rightarrow P_{\mathscr{K}(\omega)} u(\omega)$ is also measurable into $\mathbb{R}^{n}$ if $\omega \rightarrow u(\omega)$ is measurable. Here $P_{\mathscr{K}(\omega)}$ is the projection map giving the closest point.

Proof: It follows from standard results on measurable multi-functions [70] also in Theorem 48.1.2 above that there is a countable collection $\left\{w_{n}(\omega)\right\}, \omega \rightarrow \underline{w_{n}(\omega) \text { being }}$ measurable and $w_{n}(\omega) \in \mathscr{K}(\omega)$ for each $\omega$ such that for each $\omega, \mathscr{K}(\omega)=\overline{\cup_{n} w_{n}(\omega)}$. Let

$$
d_{n}(\omega) \equiv \min \left\{\left\|u(\omega)-w_{k}(\omega)\right\|, k \leq n\right\}
$$

Let $u_{1}(\omega) \equiv w_{1}(\omega)$. Let

$$
u_{2}(\omega)=w_{1}(\omega)
$$

on the set

$$
\left\{\omega:\left\|u(\omega)-w_{1}(\omega)\right\|<\left\{\left\|u(\omega)-w_{2}(\omega)\right\|\right\}\right\}
$$

and

$$
u_{2}(\omega) \equiv w_{2}(\omega) \text { off the above set. }
$$

Thus $\left\|u_{2}(\omega)-u(\omega)\right\|=d_{2}$. Let

$$
\begin{aligned}
& u_{3}(\omega)=w_{1}(\omega) \text { on }\left\{\begin{array}{c}
\omega:\left\|u(\omega)-w_{1}(\omega)\right\| \\
<\left\|u(\omega)-w_{j}(\omega)\right\|, j=2,3
\end{array}\right\} \equiv S_{1} \\
& u_{3}(\omega)=w_{2}(\omega) \text { on } S_{1} \cap\left\{\begin{array}{c}
\omega:\left\|u(\omega)-w_{1}(\omega)\right\| \\
<\left\|u(\omega)-w_{j}(\omega)\right\|, j=3
\end{array}\right\} \\
& u_{3}(\omega)=w_{3}(\omega) \text { on the remainder of } \Omega
\end{aligned}
$$

Thus $\left\|u_{3}(\omega)-u(\omega)\right\|=d_{3}$. Continue this way, obtaining $u_{n}(\omega)$ such that

$$
\left\|u_{n}(\omega)-u(\omega)\right\|=d_{n}(\omega)
$$

and $u_{n}(\omega) \in \mathscr{K}(\omega)$ with $u_{n}$ measurable. Thus, in effect one picks the closest of all the $w_{k}(\omega)$ for $k \leq n$ as the value of $u_{n}(\omega)$ and $u_{n}$ is measurable and by density in $\mathscr{K}(\omega)$ of $\left\{w_{n}(\omega)\right\}$ for each $\omega,\left\{u_{n}(\omega)\right\}$ must be a minimizing sequence for

$$
\lambda(\omega) \equiv \inf \{\|u(\omega)-z\|: z \in \mathscr{K}(\omega)\}
$$

Then it follows that $u_{n}(\omega) \rightarrow P_{\mathscr{K}(\omega)} u(\omega)$ weakly in $\mathbb{R}^{n}$. Here is why: Suppose it fails to converge to $P_{\mathscr{K}(\omega)} u(\omega)$. Since it is minimizing, it is a bounded sequence. Thus there would be a subsequence, still denoted as $u_{n}(\omega)$ which converges to some $q(\omega) \neq P_{\mathscr{K}(\omega)} u(\omega)$. Then

$$
\lambda(\omega)=\lim _{n \rightarrow \infty}\left\|u(\omega)-u_{n}(\omega)\right\| \geq\|u(\omega)-q(\omega)\|
$$

because convex and lower semicontinuous is weakly lower semicontinuous. But this implies $q(\omega)=P_{\mathscr{K}(\omega)}(u(\omega))$ because the projection map is well defined thanks to strict convexity of the norm used. This is a contradiction. Hence $P_{\mathscr{K}(\omega)} u(\omega)=\lim _{n \rightarrow \infty} u_{n}(\omega)$ and so is a measurable function. It follows that $\omega \rightarrow P_{\mathscr{K}(\omega)}(u(\omega), \omega)$ is measurable into $\mathbb{R}^{n}$.

One way to prove the following Theorem in simpler cases is to use a measurable version of the Kakutani fixed point theorem. It is done this way in [99] without the dependence on $\omega$. See also [35] for a measurable version of this fixed point theorem. However, one can also prove it by a generalization of the proof Browder gave for a single valued case and this is summarized here. We want to include the case where $A$ is a sum of two set valued operators and this involves careful consideration of the details. Such a situation occurs when one considers operators which are a sum, one dependent on the boundary of a region, and the other from a partial differential inclusion. Also, we will need to consider finite dimensional subspaces which depend on $\omega$ which further complicates the considerations.

Theorem 48.3.3 Assumtions: Let $B(\cdot, \omega): V \rightarrow \mathscr{P}\left(V^{\prime}\right), C(\cdot, \omega): V \rightarrow \mathscr{P}\left(V^{\prime}\right)$ for $V$ a separable Banach space. Suppose that $\omega \rightarrow B(x, \omega), \omega \rightarrow C(x, \omega)$ each has a measurable selection and $x \rightarrow B(x, \omega), x \rightarrow C(x, \omega)$ each is upper semicontinuous from strong to weak topologies. Also let $E(\omega)$ be an $n$ dimensional subspace of $V$ which has a basis $\left\{b_{1}(\omega), \cdots, b_{n}(\omega)\right\}$ each of which is a measurable function into $V$, and that $K(\omega) \subseteq E(\omega)$ where $K(\omega)$ is a measurable multifunction which has convex closed bounded values. Also let $y(\omega)$ be given, a measurable function into $V^{\prime}$. Conclusion: There exist measurable functions $w_{B}(\omega), w_{C}(\omega)$ and $x(\omega)$ with $w_{B}(\omega) \in B(x(\omega), \omega), w_{C}(\omega) \in C(x(\omega), \omega)$, and $x(\omega) \in K(\omega)$ such that for all $z \in K(\omega)$,

$$
\left\langle y(\omega)-\left(w_{B}(\omega)+w_{C}(\omega)\right), z-x(\omega)\right\rangle \leq 0
$$

Proof: The argument will refer to the following commutative diagram.

$$
\begin{array}{rll}
E(\omega)^{\prime} & \xrightarrow{\theta(\omega)^{*}} & \mathbb{R}^{n} \\
i(\omega)^{*} A(\cdot, \omega) \uparrow & & \uparrow \theta(\omega)^{*} i(\omega)^{*} A(\theta(\omega) \cdot, \omega) \\
E(\omega) & \stackrel{\theta(\omega)}{\leftarrow} & \mathbb{R}^{n}
\end{array}
$$

where $A(\cdot, \omega)$ will be either $B(\cdot, \omega)$ or $C(\cdot, \omega)$. Here $\theta(\omega) \mathbf{e}_{i} \equiv b_{i}(\omega)$ and extended linearly. Then it is clear that $\theta(\omega)$ maps measurable functions to measurable functions.

What of $\theta(\omega)^{-1}$ ? Is $\omega \rightarrow \theta(\omega)^{-1} h(\omega)$ measurable into $\mathbb{R}^{n}$ whenever $h$ is measurable into $V$ ? Let $h(\omega)$ have values in $E(\omega)$ and be measurable into $V$. Thus

$$
h(\omega)=\sum_{i} a_{i}(\omega) b_{i}(\omega)
$$

The question reduces to whether the $a_{i}$ are measurable. To see that these are measurable, consider first $\|h(\omega)\|<M$ for all $\omega$. Let $S_{r} \equiv\left\{\omega: \inf _{|\mathbf{a}|>r}\left\|\sum_{i} a_{i} b_{i}(\omega)\right\|>M\right\}$. Thus this is a measurable set. Also every $\omega$ is in some $S_{r}$ because if not, you could get a sequence $\left|\mathbf{a}^{r}\right| \rightarrow$ $\infty$ and yet $\left\|\sum_{i} a_{i}^{r} b_{i}(\omega)\right\| \leq M$. But then, dividing by $\left|\mathbf{a}^{r}\right|$ and taking a suitable subsequence, one can obtain $\sum_{i} a_{i} b_{i}(\omega)=0$ for some $|\mathbf{a}|=1$. Also the $S_{r}$ are increasing in $r$. Now for $\omega \in S_{r}$, define $\Phi(\mathbf{a}, \omega)=-\left\|\sum_{i} a_{i} b_{i}(\omega)-h(\omega)\right\|$ where we will let $|\mathbf{a}| \leq r+1$. Since $\left\{b_{i}(\omega)\right\}$ is a basis, there exists $\mathbf{a}(\omega)$ such that $\Phi(\mathbf{a}(\omega), \omega)=0$. This a must satisfy $|\mathbf{a}| \leq r+1$ because if not, then you would have $\left\|\sum_{i} a_{i} b_{i}(\omega)\right\| \geq M$ since $\omega \in S_{r}$. But $\left\|\sum_{i} a_{i} b_{i}(\omega)\right\|=\|h(\omega)\|<M$. Thus the maximum of $\mathbf{a} \rightarrow \Phi(\mathbf{a}, \omega)$ occurs on the compact set $|\mathbf{a}| \leq r+1$ and is 0 . By Kuratowski's theorem, we have $\omega \rightarrow \mathbf{a}(\omega)$ is measurable where $h(\omega)=\sum_{i} a_{i}(\omega) b_{i}(\omega)$ on $S_{r}$. Thus, since every $\omega$ is in some $S_{r}$, we must have $\omega \rightarrow a_{i}(\omega)$ is measurable in case $\|h(\omega)\| \leq M$ for all $\omega$. In the general case, let $\mathbf{a}^{m}(\omega)$ be the measurable function which goes with $h_{m}(\omega)$ where $h_{m}(\omega)$ is given by a truncation of $h$ so that $\left\|h_{m}(\omega)\right\| \leq m$. For each $\omega, h_{m}(\omega)$ is eventually smaller than $m$, so $h(\omega)=h_{m}(\omega)$. Thus if $a_{i}^{m}(\omega)$ go with $h_{m}(\omega)$, these are constant for all $m$ large enough. Thus letting $a_{i}(\omega) \equiv \lim _{m \rightarrow \infty} a_{i}^{m}(\omega), a_{i}$ is measurable and

$$
h(\omega)=\lim _{m \rightarrow \infty} h_{m}(\omega)=\lim _{m \rightarrow \infty} \sum_{i} a_{i}^{m}(\omega) b_{i}(\omega)=\sum_{i} a_{i}(\omega) b_{i}(\omega)
$$

and so the $a_{i}$ are indeed measurable. Thus the $\theta(\omega)^{-1} h(\omega)=\sum_{i} a_{i}(\omega) \mathbf{e}_{i}$ which shows that $\theta(\omega)^{-1}$ does map measurable functions to measurable functions. In particular,

$$
\theta(\omega)^{-1} K(\omega)
$$

is indeed a closed, bounded, convex, and measurable multifunction which can be seen by considering a sequence $\left\{k_{i}(\omega)\right\}_{i=1}^{\infty}$ of measurable functions dense in $K(\omega)$.

Define for $A=B$ or $C$,

$$
\begin{equation*}
\hat{A}(\cdot, \omega)=\theta(\omega)^{*} i(\omega)^{*} A(\theta(\omega) \cdot, \omega), \mathbf{y}(\omega)=\theta(\omega)^{*} i(\omega)^{*} y(\omega) \tag{48.3.9}
\end{equation*}
$$

We claim that $\omega \rightarrow \hat{A}(\mathbf{x}, \omega)$ has a measurable selection and for fixed $\omega$ this is upper semicontinuous in $\mathbf{x}$. The second condition for fixed $\omega$ is obvious. Consider the first. It was shown above that $\theta(\omega) \mathbf{x}$ is measurable into $V$. Thus, by Theorem 48.3.1, it follows that $\omega \rightarrow A(\theta(\omega) \mathbf{x}, \omega)$ has a measurable selection into $V^{\prime}$. Therefore, it suffices to show that if $z(\omega)$ is measurable into $V^{\prime}$ then $\theta(\omega)^{*} i(\omega)^{*} z(\omega)$ is measurable into $\mathbb{R}^{n}$. Let $\mathbf{w} \in \mathbb{R}^{n}$.

Then

$$
\begin{aligned}
\left(\theta(\omega)^{*} i(\omega)^{*} z(\omega), \mathbf{w}\right)_{\mathbb{R}^{n}} & \equiv\left\langle i(\omega)^{*} z(\omega), \theta(\omega) \mathbf{w}\right\rangle \\
& =\left\langle i(\omega)^{*} z(\omega), \sum_{i} w_{i} b_{i}(\omega)\right\rangle \\
& =\left\langle z(\omega), \sum_{i} w_{i} b_{i}(\omega)\right\rangle_{V^{\prime}, V}
\end{aligned}
$$

which is measurable. By the Pettis theorem, $\omega \rightarrow \theta(\omega)^{*} i(\omega)^{*} z(\omega)$ is measurable. Thus $\hat{A}(\cdot, \omega)$ has the properties claimed.

Now tile $\mathbb{R}^{n}$ with $n$ simplices, each having diameter less than $\varepsilon<1$, the set of simplices being locally finite. Define for $A=B$ or $C$ the single valued function $\hat{A}_{\varepsilon}$ on all of $\mathbb{R}^{n}$ by the following rule. If

$$
\mathbf{x} \in\left[\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right]
$$

so $\mathbf{x}=\sum_{i=0}^{n} t_{i} \mathbf{x}_{i}, t_{i} \geq 0, \sum_{i} t_{i}=1$, then let $\hat{A}_{\varepsilon}\left(\mathbf{x}_{k}, \omega\right)$ be a measurable selection from $\hat{A}\left(\mathbf{x}_{k}, \omega\right)$ for each $\mathbf{x}_{k}$ a vertex of the simplex. However, we chose $\hat{A}_{\varepsilon}\left(\mathbf{x}_{k}, \omega\right)$ in the obvious way. It is $\theta(\omega)^{*} i(\omega)^{*} w_{k}^{\varepsilon B}(\omega)$ where $w_{k}^{\varepsilon B}(\omega)$ is a measurable selection of $B\left(\theta(\omega) \mathbf{x}_{k}, \omega\right)$, measurable into $V^{\prime}$ when $A=B$ and $\theta(\omega)^{*} i(\omega)^{*} w_{k}^{\varepsilon C}(\omega)$ where $w_{k}^{\varepsilon C}(\omega)$ is a measurable selection of $C\left(\theta(\omega) \mathbf{x}_{k}, \omega\right)$, measurable into $V^{\prime}$ when $A=C$. Then

$$
\begin{equation*}
\hat{B}_{\varepsilon}\left(\mathbf{x}_{k}, \omega\right)=\theta(\omega)^{*} i(\omega)^{*} w_{k}^{\varepsilon B}(\omega), \omega \rightarrow w_{k}^{\varepsilon B}(\omega) \text { measurable } \tag{48.3.10}
\end{equation*}
$$

with a similar definition holding for $\hat{C}_{\varepsilon}$.
Define single valued maps as follows. For $\mathbf{x}=\sum_{i=0}^{n} t_{i} \mathbf{x}_{i}, \sum_{i} t_{i}=1, t_{i} \geq 0,\left[\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right]$ in the tiling,

$$
\begin{gather*}
\hat{B}_{\varepsilon}(\mathbf{x}, \omega) \equiv \sum_{k=0}^{n} t_{k}\left(\hat{B}_{\varepsilon}\left(\mathbf{x}_{k}, \omega\right)\right), \hat{C}_{\varepsilon}(\mathbf{x}, \omega) \equiv \sum_{k=0}^{n} t_{k}\left(\hat{C}_{\varepsilon}\left(\mathbf{x}_{k}, \omega\right)\right) \\
\hat{A}_{\varepsilon}(\mathbf{x}, \omega) \equiv \hat{B}_{\varepsilon}(\mathbf{x}, \omega)+\hat{C}_{\varepsilon}(\mathbf{x}, \omega) \tag{48.3.11}
\end{gather*}
$$

Thus $\hat{A}_{\varepsilon}(\cdot, \omega)$ is a continuous map defined on $\mathbb{R}^{n}$ thanks to the local finiteness of the tiling, and $\omega \rightarrow \hat{A}_{\mathcal{E}}(\mathbf{x}, \omega)$ is measurable.

Let $P_{\theta(\omega)^{-1} K(\omega)}$ denote the projection onto the closed convex set $\theta(\omega)^{-1} K(\omega)$. This is a continuous mapping by Hilbert space considerations. Therefore,

$$
\mathbf{x} \rightarrow P_{\theta(\omega)^{-1} K(\omega)}\left(\mathbf{y}(\omega)-\hat{A}_{\mathcal{E}}(\mathbf{x}, \omega)+\mathbf{x}\right)
$$

is continuous and by Lemma 48.3.2, $\omega \rightarrow P_{\theta(\omega)^{-1} K(\omega)}\left(\mathbf{y}(\omega)-\hat{A}_{\varepsilon}(\mathbf{x}, \omega)+\mathbf{x}\right)$ is measurable, and for each $\omega$, this function of $\mathbf{x}$ maps into $\theta(\omega)^{-1} K(\omega)$. Therefore by Corollary 48.2.14, there exists a fixed point $\mathbf{x}_{\varepsilon}(\omega) \in \theta(\omega)^{-1} K(\omega)$ such that $\omega \rightarrow \mathbf{x}_{\varepsilon}(\omega)$ is measurable and

$$
P_{\theta(\omega)^{-1} K(\omega)}\left(\mathbf{y}(\omega)-\left(\hat{B}_{\varepsilon}\left(\mathbf{x}_{\varepsilon}(\omega), \omega\right)+\hat{C}_{\varepsilon}\left(\mathbf{x}_{\varepsilon}(\omega), \omega\right)\right)+\mathbf{x}_{\varepsilon}(\omega)\right)=\mathbf{x}_{\varepsilon}(\omega)
$$

This requires

$$
\begin{equation*}
\left(\mathbf{y}(\omega)-\left(\hat{B}_{\varepsilon}\left(\mathbf{x}_{\varepsilon}(\omega), \omega\right)+\hat{C}_{\varepsilon}\left(\mathbf{x}_{\varepsilon}(\omega), \omega\right)\right), \mathbf{z}-\mathbf{x}_{\varepsilon}(\omega)\right)_{\mathbb{R}^{n}} \leq 0 \tag{48.3.12}
\end{equation*}
$$

for all $\mathbf{z} \in \theta(\omega)^{-1} K(\omega)$. Note that this implies $\omega \rightarrow \hat{B}_{\varepsilon}\left(\mathbf{x}_{\varepsilon}(\omega), \omega\right), \hat{C}_{\varepsilon}\left(\mathbf{x}_{\varepsilon}(\omega), \omega\right)$ are measurable because of the continuity in first argument and measurability of $\omega \rightarrow \mathbf{x}_{\varepsilon}(\omega)$

We have

$$
\begin{equation*}
\mathbf{x}_{\varepsilon}(\omega)=\sum_{k=0}^{n} t_{k}^{\varepsilon}(\omega) \mathbf{x}_{k}^{\varepsilon}(\omega) \tag{48.3.13}
\end{equation*}
$$

where the $\mathbf{x}_{k}^{\varepsilon}(\omega)$ are vertices of the tiling corresponding to $\varepsilon$.
Claim: The vertices $\mathbf{x}_{k}^{\varepsilon}(\omega)$ and coordinates $t_{k}^{\varepsilon}(\omega)$ can be considered measurable.
Proof of claim: Let the simplices in the tiling be $\left\{\sigma_{k}\right\}_{k=1}^{\infty}$ and let the vertices of simplices in the tiling be $\left\{\mathbf{z}_{j}\right\}_{j=1}^{\infty}$. Say the vertices of $\sigma_{k}$ are $\left\{\mathbf{x}_{0}^{k}, \cdots, \mathbf{x}_{n}^{k}\right\}$ listed in the order of the given enumeraton of vertices of simplices in the tiling. Let $F_{k}$ and $E_{k}$ be defined as follows.

$$
F_{k} \equiv \mathbf{x}_{\varepsilon}^{-1}\left(\sigma_{k}\right), E_{1} \equiv F_{1}, \cdots, E_{k} \equiv F_{k} \backslash \cup_{i=1}^{k-1} F_{i}
$$

Then each $\omega$ is in exactly one of these measurable sets $E_{k}$ which partition $\Omega$. For $\omega \in$ $E_{k}, \mathbf{x}_{\varepsilon}(\omega) \in \sigma_{k}(\omega)$. Thus $\sigma_{k}(\omega)$ is the first simplex which contains $\mathbf{x}_{\varepsilon}(\omega)$ and the ordered vertices of this simplex are constant on the measurable set $E_{k}$. These vertices are determined this way on a measurable set $E_{k}$ and so they must be measurable $\mathbb{R}^{n}$ valued functions. Then $\omega \rightarrow t_{k}^{\varepsilon}(\omega)$ is also measurable because there is a continuous mapping to these scalars from $\mathbf{x}_{\varepsilon}(\omega)$ which was obtained measurable. This shows the claim.

Recall 48.3.10. Let $\mathbf{W}^{\varepsilon}(\omega)$ be defined as follows.

$$
\mathbf{W}^{\varepsilon}(\omega) \equiv\binom{t_{0}^{\varepsilon}(\omega), \cdots, t_{n}^{\varepsilon}(\omega), \mathbf{x}_{0}^{\varepsilon}(\omega), \cdots, \mathbf{x}_{n}^{\varepsilon}(\omega),}{\mathbf{x}_{\varepsilon}(\omega), w_{0}^{\varepsilon \varepsilon}(\omega), \cdots, w_{n}^{\varepsilon B}(\omega), w_{0}^{\varepsilon C}(\omega), \cdots, w_{n}^{\varepsilon C}(\omega)}
$$

This is in $\mathbb{R}^{2(n+1)} \times\left(V^{\prime}\right)^{2(n+1)}$. Then by Theorem 48.2.2, since $\mathbf{W}^{\varepsilon}(\omega)$ is bounded in a reflexive separable Banach space, there is a subsequence $\varepsilon(\omega) \rightarrow 0$ such that $\mathbf{W}^{\varepsilon(\omega)}(\omega) \rightarrow$ $\mathbf{W}(\omega)$ weakly and given by

$$
\mathbf{W}(\omega) \equiv\binom{t_{0}(\omega), \cdots, t_{n}(\omega), \mathbf{x}_{0}(\omega), \cdots, \mathbf{x}_{n}(\omega),}{\mathbf{x}(\omega), w_{0}^{B}(\omega), \cdots, w_{n}^{B}(\omega), w_{0}^{C}(\omega), \cdots, w_{n}^{C}(\omega)}
$$

where each of these components is measurable into the appropriate space. Of course, in the finite dimensional components, the convergence is strong because strong and weak convergence is the same in finite dimensions. Since the diameter of the simplex containing the fixed point $\mathbf{x}_{\varepsilon(\omega)}(\omega)$ converges to 0 , it follows that

$$
\lim _{\varepsilon(\omega) \rightarrow 0} \mathbf{x}_{k}^{\varepsilon(\omega)}(\omega)=\mathbf{x}(\omega)
$$

By upper semicontinuity, for $A=B, C$, it follows that $\hat{A}\left(\mathbf{x}_{k}^{\varepsilon(\omega)}(\omega), \omega\right) \subseteq \hat{A}(\mathbf{x}(\omega), \omega)+$ $B(\mathbf{0}, r)$ for all $\varepsilon(\omega)$ small enough. Since, by the construction,

$$
\hat{B}_{\varepsilon(\omega)}\left(\mathbf{x}_{k}^{\varepsilon(\omega)}(\omega), \omega\right)=\theta(\omega)^{*} i(\omega)^{*} w_{k}^{\varepsilon(\omega) B}(\omega) \in \hat{B}\left(\mathbf{x}_{k}^{\varepsilon(\omega)}(\omega), \omega\right)
$$

a similar statement for $\hat{C}$, it follows that $\hat{B}_{\varepsilon(\omega)}\left(\mathbf{x}_{k}^{\varepsilon(\omega)}(\omega), \omega\right)=\theta(\omega)^{*} i(\omega)^{*} w_{k}^{\varepsilon(\omega) B}(\omega)$ is within $r$ of the closed convex bounded set $\hat{B}(\mathbf{x}(\omega), \omega)$ whenever $\varepsilon(\omega)$ is small enough, similar for $\hat{C}$. Thus

$$
\theta(\omega)^{*} i(\omega)^{*} w_{k}^{B}(\omega) \in \hat{B}(\mathbf{x}(\omega), \omega)
$$

similar for $\hat{C}$. Since this last set is convex, it follows that

$$
\theta(\omega)^{*} i(\omega)^{*} \sum_{k} t_{k}(\omega) w_{k}^{B}(\omega) \in \hat{B}(\mathbf{x}(\omega), \omega)
$$

similar for $\hat{C}$.
Now recall 48.3 .9 and the inequality 48.3 .12 which imply that for $\mathbf{z} \in \theta(\omega)^{-1} K(\omega)$,

$$
\begin{align*}
& \left(\mathbf{y}(\omega)-\theta(\omega)^{*} i(\omega)^{*}\left(\sum_{k=0}^{n} t_{k}(\omega) w_{k}^{B}(\omega)+\sum_{k=0}^{n} t_{k}(\omega) w_{k}^{C}(\omega)\right), \mathbf{z}-\mathbf{x}(\omega)\right)  \tag{48.3.14}\\
& =\lim _{\varepsilon(\omega) \rightarrow 0}\left(\mathbf{y}(\omega)-\binom{\sum_{k=0}^{n} t_{k}^{\varepsilon(\omega)}(\omega) \theta(\omega)^{*} i(\omega)^{*} w_{k}^{\varepsilon(\omega) B}(\omega)}{+\sum_{k=0}^{n} t_{k}^{\varepsilon(\omega)}(\omega) \theta(\omega)^{*} i(\omega)^{*} w_{k}^{\varepsilon(\omega) C}(\omega)}, \mathbf{z}-\mathbf{x}_{\varepsilon(\omega)}(\omega)\right) \\
& =\lim _{\varepsilon(\omega) \rightarrow 0}\left(\mathbf{y}(\omega)-\binom{\sum_{k=0}^{n} t_{k}^{\varepsilon(\omega)}(\omega) \hat{B}_{\varepsilon(\omega)}\left(\mathbf{x}_{k}^{\varepsilon(\omega)}, \omega\right)}{+\sum_{k=0}^{n} t_{k}^{\varepsilon(\omega)}(\omega) \hat{C}_{\varepsilon(\omega)}\left(\mathbf{x}_{k}^{\varepsilon(\omega)}, \omega\right)}, \mathbf{z}-\mathbf{x}_{\varepsilon(\omega)}(\omega)\right)
\end{align*}
$$

Recall 48.3 .11 and 48.3 .13 which imply from the above conventions that the sum in the above equals $\hat{B}_{\varepsilon(\omega)}\left(\mathbf{x}_{\varepsilon(\omega)}, \omega\right)+\hat{C}_{\varepsilon(\omega)}\left(\mathbf{x}_{\varepsilon(\omega)}, \omega\right)$. Thus the above equals

$$
\lim _{\varepsilon(\omega) \rightarrow 0}\left(\mathbf{y}(\omega)-\left(\hat{B}_{\varepsilon(\omega)}\left(\mathbf{x}_{\varepsilon(\omega)}, \omega\right)+\hat{C}_{\varepsilon(\omega)}\left(\mathbf{x}_{\varepsilon(\omega)}, \omega\right)\right), \mathbf{z}-\mathbf{x}_{\varepsilon(\omega)}(\omega)\right) \leq 0
$$

Now $w_{k}^{\varepsilon(\omega) B}(\omega) \in B\left(\theta(\omega) \mathbf{x}_{k}^{\varepsilon(\omega)}(\omega), \omega\right)$ and the weak uppersemicontinuity must then imply that $w_{k}^{B}(\omega) \in B(\theta(\omega) \mathbf{x}(\omega), \omega)$, a similar statement holding for $C$. By convexity,

$$
w_{B}(\omega) \equiv \sum_{k=0}^{n} t_{k}(\omega) w_{k}^{B}(\omega) \in B(\theta(\omega) \mathbf{x}(\omega), \omega)
$$

similar for $C$. Then from 48.3.14,

$$
\begin{aligned}
& \left(\mathbf{y}(\omega)-\theta(\omega)^{*} i(\omega)^{*}\left(\sum_{k=0}^{n} t_{k}(\omega) w_{k}^{B}(\omega)+\sum_{k=0}^{n} t_{k}(\omega) w_{k}^{C}(\omega)\right), \mathbf{z}-\mathbf{x}(\omega)\right) \\
= & \left(\theta(\omega)^{*} i(\omega)^{*}\left(y(\omega)-\left(w_{B}(\omega)+w_{C}(\omega)\right)\right), \mathbf{z}-\mathbf{x}(\omega)\right) \leq 0
\end{aligned}
$$

It follows that if $x(\omega) \equiv \theta(\omega) \mathbf{x}(\omega)$,

$$
\left\langle y(\omega)-\left(w_{B}(\omega)+w_{C}(\omega)\right), \theta(\omega) \mathbf{z}-x(\omega)\right\rangle \leq 0
$$

each of $x(\omega)$,

$$
w_{B}(\omega), w_{C}(\omega)
$$

are measurable and $w_{B}(\omega) \in B(x(\omega), \omega), w_{C}(\omega) \in C(x(\omega), \omega)$. Since $\theta(\omega) \mathbf{z}$ is a generic element of $K(\omega)$, this proves the theorem.

Obviously one could have any finite sum of operators having the same properties as $B, C$ above and one could get a similar result.

### 48.4 A Measurable Kakutani Theorem

Recall the Kakutani theorem, Theorem 25.4.4.
Theorem 48.4.1 Let $K$ be a compact convex subset of $\mathbb{R}^{n}$ and let $A: K \rightarrow \mathscr{P}(K)$ such that $A \mathbf{x}$ is a closed convex subset of $K$ and $A$ is upper semicontinuous. Then there exists $\mathbf{x}$ such that $\mathbf{x} \in A \mathbf{x}$. This is the "fixed point".

Here is a measurable version of this theorem. It is just like the proof of the above Browder lemma.

Theorem 48.4.2 Let $K(\omega)$ be compact, convex, and $\omega \rightarrow K(\omega)$ a measurable multifunction. Let $A(\cdot, \omega): K(\omega) \rightarrow K(\omega)$ be upper semicontinuous, and let $\omega \rightarrow A(\mathbf{x}, \omega)$ have a measurable selection for each $\mathbf{x} \in \mathbb{R}^{n}$. Then there exists $\mathbf{x}(\omega) \in K(\omega) \cap A(K(\omega), \omega)$ such that $\omega \rightarrow \mathbf{x}(\omega)$ is measurable.

Proof: Tile $\mathbb{R}^{n}$ with $n$ simplices such that the collection is locally finite and each simplex has diameter less than $\varepsilon<1$. This collection of simplices is determined by a countable collection of vertices so there exists a one to one and onto map from $\mathbb{N}$ to the collection of vertices. By assumption, for each vertex $\mathbf{x}$, there exists $A_{\varepsilon}(\mathbf{x}, \omega) \in A\left(P_{K(\omega)} \mathbf{x}, \omega\right)$. By Lemma 48.3.2, $\omega \rightarrow P_{K(\omega)} \mathbf{x}$ is measurable and by Theorem 48.3.1, there is a measurable selection for $\omega \rightarrow A\left(P_{K(\omega)} \mathbf{x}, \omega\right)$ which is denoted as $A_{\varepsilon}(\mathbf{x}, \omega)$. By local finiteness, this function is continuous in $\mathbf{x}$ on the set of vertices. Define $A_{\varepsilon}$ on all of $\mathbb{R}^{n}$ by the following rule. If

$$
\mathbf{x} \in\left[\mathbf{x}_{0}, \cdots, \mathbf{x}_{n}\right]
$$

so $\mathbf{x}=\sum_{i=0}^{n} t_{i} \mathbf{x}_{i}$, then

$$
A_{\varepsilon}(\mathbf{x}, \omega) \equiv \sum_{k=0}^{n} t_{k} A_{\varepsilon}\left(\mathbf{x}_{k}, \omega\right)
$$

By local finiteness, this function satisfies $\omega \rightarrow A_{\mathcal{E}}(\mathbf{x}, \omega)$ is measurable and also $\mathbf{x} \rightarrow A_{\varepsilon}(\mathbf{x}, \varepsilon)$ is continuous. It also maps $\mathbb{R}^{n}$ to $K(\omega)$. By Corollary 48.2 .14 there is a measurable fixed point $\mathbf{x}_{\varepsilon}(\omega)$ satisfying $\mathbf{x}_{\varepsilon}(\omega) \in K(\omega)$ and $A_{\varepsilon}\left(\mathbf{x}_{\varepsilon}(\omega), \omega\right)=\mathbf{x}_{\varepsilon}(\omega)$.

Suppose $\mathbf{x}_{\varepsilon}(\omega) \in\left[\mathbf{x}_{0}^{\varepsilon}(\omega), \cdots, \mathbf{x}_{n}^{\varepsilon}(\omega)\right]$ so $\mathbf{x}_{\varepsilon}(\omega)=\sum_{k=0}^{n} t_{k}^{\varepsilon}(\omega) \mathbf{x}_{k}^{\varepsilon}(\omega)$.
claim: The vertices $\mathbf{x}_{k}^{\varepsilon}(\omega)$ can be considered measurable also as is $t_{k}^{\varepsilon}(\omega)$.
Proof of claim: Let the simplices in the tiling be $\left\{\sigma_{k}\right\}_{k=1}^{\infty}$ and let the vertices of simplices in the tiling be $\left\{\mathbf{z}_{j}\right\}_{j=1}^{\infty}$. Let

$$
F_{k}:=\mathbf{x}_{\varepsilon}^{-1}\left(\sigma_{k}\right), E_{1}:=F_{1}, \cdots, E_{k}:=F_{k} \backslash \cup_{i=1}^{k} F_{i}
$$

Then $\omega$ is in exactly one of these measurable sets $E_{k}$. These measurable sets partition $\Omega$. Let $\sigma_{k}(\omega)$ be the unique simplex for $\omega \in E_{k}$. Thus $\mathbf{x}_{\varepsilon}(\omega) \in \sigma_{k}(\omega)$ on the measurable set $E_{k}$. Its vertices, are $\mathbf{z}_{i_{0}}(\omega), \mathbf{z}_{i_{1}}(\omega), \cdots, \mathbf{z}_{i_{n}}(\omega)$. These are $\mathbf{x}_{0}^{\varepsilon}(\omega), \cdots, \mathbf{x}_{n}^{\varepsilon}(\omega)$ in order. They are determined in this way on a measurable set so they are measurable $\mathbb{R}^{n}$ valued functions. Then $\omega \rightarrow t_{k}^{\varepsilon}(\omega)$ is also measurable because there is a continuous mapping to these scalars from $\mathbf{x}_{\varepsilon}(\omega)$.

Then since $\mathbf{x}_{\varepsilon}(\omega)$ is contained in $K(\omega)$, a compact set, and the diameter of each simplex is less than 1 , it follows that $A_{\varepsilon}\left(\mathbf{x}_{k}^{\varepsilon}(\omega), \omega\right)$ is contained in

$$
A(\overline{K(\omega)+B(\mathbf{0}, 1)}, \omega)
$$

which is a compact set. Let $\mathbf{W}^{\varepsilon}(\omega) \in \mathbb{R}^{2 n+2 n^{2}}$ be defined as follows.

$$
\mathbf{W}^{\varepsilon}(\omega):=\binom{t_{1}^{\varepsilon}(\omega), \cdots, t_{n}^{\varepsilon}(\omega), \mathbf{x}_{0}^{\varepsilon}(\omega), \cdots, \mathbf{x}_{n}^{\varepsilon}(\omega), \mathbf{x}_{\varepsilon}(\omega),}{A_{\varepsilon}\left(\mathbf{x}_{1}^{\varepsilon}(\omega), \omega\right) \cdots A_{\varepsilon_{m}}\left(\mathbf{x}_{n}^{\varepsilon_{m}}(\omega), \omega\right)}
$$

Thus $\mathbf{W}^{\varepsilon}$ has values in a compact subset of $\mathbb{R}^{2 n+2 n^{2}}$ and is measurable. By Lemma 48.2.2 there exists a subsequence $\varepsilon(\omega) \rightarrow 0$ and a measurable function $\omega \rightarrow \mathbf{W}(\omega)$ such that

$$
\mathbf{W}^{\varepsilon(\omega)}(\omega) \rightarrow \mathbf{W}(\omega)=\binom{t_{1}(\omega), \cdots, t_{n}(\omega), \mathbf{x}_{0}(\omega), \cdots,}{\mathbf{x}_{n}(\omega), \mathbf{x}(\omega), \mathbf{y}_{1}(\omega), \cdots, \mathbf{y}_{n}(\omega)}
$$

as $\varepsilon(\omega) \rightarrow 0$. Recall also that

$$
A_{\varepsilon}\left(\mathbf{x}_{k}^{\varepsilon}(\omega), \omega\right) \subseteq A\left(P_{K(\omega)} \mathbf{x}_{k}^{\varepsilon}, \omega\right)
$$

Now

$$
\left|P_{K(\omega)} \mathbf{x}_{k}^{\varepsilon}(\omega)-\mathbf{x}_{\varepsilon}(\omega)\right|=\left|P_{K(\omega)} \mathbf{x}_{k}^{\varepsilon}(\omega)-P_{K(\omega)} \mathbf{x}_{\varepsilon}(\omega)\right| \leq\left|\mathbf{x}_{k}^{\varepsilon}-\mathbf{x}_{\varepsilon}\right|<\varepsilon
$$

Both $\mathbf{x}_{k}^{\varepsilon(\omega)}(\omega)$ and $\mathbf{x}_{\varepsilon(\omega)}(\omega)$ converge to $\mathbf{x}(\omega)$ and so the above shows that also,

$$
P_{K(\omega)} \mathbf{x}_{k}^{\varepsilon}(\omega) \rightarrow \mathbf{x}(\omega)
$$

Therefore,

$$
A_{\varepsilon(\omega)}\left(\mathbf{x}_{k}^{\varepsilon(\omega)}(\omega), \omega\right) \subseteq A(\mathbf{x}(\omega), \omega)+B(0, r)
$$

whenever $\varepsilon(\omega)$ is small enough. Since $A(\mathbf{x}(\omega), \omega)$ is closed, this implies

$$
\mathbf{y}_{k}(\omega) \in A(\mathbf{x}(\omega), \omega)
$$

Since $A(\mathbf{x}(\omega), \omega)$ is convex,

$$
\sum_{k=1}^{n} t_{k}(\omega) \mathbf{y}_{k}(\omega) \in A(\mathbf{x}(\omega), \omega)
$$

Also, from the construction,

$$
\mathbf{x}_{\varepsilon}(\omega)=A_{\varepsilon}\left(\mathbf{x}_{\varepsilon}(\omega), \omega\right) \equiv \sum_{k=0}^{n} t_{k}^{\varepsilon}(\omega) A_{\varepsilon}\left(\mathbf{x}_{k}^{\varepsilon}(\omega), \omega\right)
$$

so passing to the limit as $\varepsilon(\omega) \rightarrow 0$, we get

$$
\mathbf{x}(\omega)=\sum_{k=0}^{n} t_{k}(\omega) \mathbf{y}_{k}(\omega) \in A(\mathbf{x}(\omega), \omega)
$$

and this is the measurable fixed point.

### 48.5 Some Variational Inequalities

In the following, $V$ will be a reflexive separable Banach space. Following [99], here is a definition of a pseudomonotone operator. Actually, we will consider a slight generalization of the usual definition in 25.4.17 which involves an assumption that there exists a subsequence such that the liminf condition holds rather than use the original sequence.

Definition 48.5.1 Let $V$ be a reflexive Banach space. Then $A: V \rightarrow \mathscr{P}\left(V^{\prime}\right)$ is pseudomonotone if the following conditions hold.
Au is closed, nonempty, convex.

If $F$ is a finite dimensional subspace of $V$, then if $u \in F$ and $W \supseteq A u$ for $W$ a weakly open set in $V^{\prime}$, then there exists $\delta>0$ such that

$$
\begin{equation*}
v \in B(u, \delta) \cap F \text { implies } A v \subseteq W \tag{48.5.16}
\end{equation*}
$$

If $u_{k} \rightharpoonup u$ and if $u_{k}^{*} \in A u_{k}$ is such that

$$
\limsup _{k \rightarrow \infty}\left\langle u_{k}^{*}, u_{k}-u\right\rangle \leq 0,
$$

Then there exists a subsequence still denoted with $k$ such that for all $v \in V$, there exists $u^{*}(v) \in A u$ such that

$$
\begin{equation*}
\lim _{\inf _{k \rightarrow \infty}}\left\langle u_{k}^{*}, u_{k}-v\right\rangle \geq\left\langle u^{*}(v),(u-v)\right\rangle \tag{48.5.17}
\end{equation*}
$$

We say $A$ is coercive if

$$
\begin{equation*}
\lim _{\|v\| \rightarrow \infty} \inf \left\{\frac{\left\langle z^{*}, v\right\rangle}{\|v\|}: z^{*} \in A v\right\}=\infty . \tag{48.5.18}
\end{equation*}
$$

If one assumes $A$ is bounded, then the weak upper semicontinuity condition 48.5.16 can be proved from the other conditions. It has been known for a long time that these operators are useful in the study of variational inequalities. In this section, we give a short example to show how one can obtain measurable solutions to variational inequalities from the measurable Browder lemma given above. This is the following theorem which gives a measurable version of old results of Brezis dating from the late 1960s. This will involve the following assumptions.

## 1. Measurability condition

For each $u \in V$, there is a measurable selection $z(\omega)$ such that

$$
z(\omega) \in A(u, \omega)
$$

## 2. Values of $A$

$A(\cdot, \omega): V \rightarrow \mathscr{P}\left(V^{\prime}\right)$ has bounded, closed, nonempty, convex values. $A(\cdot, \omega)$ maps bounded sets to bounded sets.

## 3. Limit conditions

$$
\text { If } u_{n} \rightharpoonup u \text { and } \lim \sup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle \leq 0, z_{n} \in A\left(u_{n}, \omega\right)
$$

then for given $v$, there exists $z(v) \in A(u, \omega)$ such that

$$
\lim _{\inf _{k \rightarrow \infty}}\left\langle z_{n}, u_{n}-v\right\rangle \geq\langle z(v), u-v\rangle
$$

Thus, for fixed $\omega, A(\cdot, \omega)$ is a set valued bounded pseudomonotone operator. Recall that the sum of two of these is also a set valued bounded pseudomonotone operator.

By Theorem 48.3.1 if $\omega \rightarrow u(\omega)$ is measurable, then $A(u(\omega), \omega)$ has a measurable selection. Also, the limit condition implies that $A(\cdot, \omega)$ is upper semicontinuous from the strong to the weak topology. The overall approach to the following theorem is well known. The new ingredients are Lemma 48.2.2 and Theorem 48.3.3 which are what allows us to obtain measurable solutions. First is a standard result on the sum of two pseudomonotone bounded operators. See Theorem 25.5.1 on Page 855.

Theorem 48.5.2 Say $A, B$ are set valued bounded pseudomonotone operators. Then their sum is also a set valued bounded pseudomonotone operator. Also, if $u_{n} \rightarrow u$ weakly, $z_{n} \rightarrow z$ weakly, $z_{n} \in A\left(u_{n}\right)$, and $w_{n} \rightarrow w$ weakly with $w_{n} \in B\left(u_{n}\right)$, then if

$$
\lim \sup _{n \rightarrow \infty}\left\langle z_{n}+w_{n}, u_{n}-u\right\rangle \leq 0
$$

it follows that

$$
\lim _{n \rightarrow \infty} \inf _{n \rightarrow w_{n}}\left\langle u_{n}-v\right\rangle \geq\langle z(v)+w(v), u-v\rangle, z(v) \in A(u), w(v) \in B(u)
$$

and $z \in A(u), w \in B(u)$.
Theorem 48.5.3 Let $V$ be a reflexive separable Banach space. Let $\omega \rightarrow K(\omega)$ be a measurable multifunction, $K(\omega)$ convex, closed, and bounded. Also for $A=B, C$ let $A(\cdot, \cdot)$ satisfy 1-3. Let $\omega \rightarrow f(\omega)$ be measurable with values in $V^{\prime}$. Then there exists measurable $\omega \rightarrow u(\omega) \in K(\omega)$ and $\omega \rightarrow w_{B}(\omega), \omega \rightarrow w_{C}(\omega)$ with $w_{B}(\omega) \in B(u(\omega), \omega), w_{C}(\omega) \in$ $C(u(\omega), \omega)$ such that

$$
\left\langle f(\omega)-\left(w_{B}(\omega)+w_{C}(\omega)\right), z-u(\omega)\right\rangle \leq 0
$$

for all $z \in K(\omega)$. If it is only known that $K(\omega)$ is closed and convex, the same conclusion can be obtained if it is also known that for some $z(\omega) \in K(\omega), B(\cdot, \omega)+C(\cdot, \omega)$ is coercive meaning

$$
\lim _{\|v\| \rightarrow \infty} \inf \left\{\frac{\left\langle z^{*}, v-z\right\rangle}{\|v\|}: z^{*} \in B(v, \omega)+C(v, \omega)\right\}=\infty .
$$

Proof: Let $V_{n}=V_{n}(\omega)$ denote an increasing sequence of finite dimensional subspaces whose union is dense in $V$. Let $V_{n}(\omega)$ contain the first $n$ vectors of $\left\{d_{k}(\omega)\right\}_{k=1}^{\infty}$ where the closure of this sequence equals $K(\omega)$ for each $\omega$, each function being measurable. Let

$$
V_{n}(\omega)=\operatorname{span}\left(v_{1}, \cdots, v_{n}, d_{1}(\omega), \cdots, d_{n}(\omega)\right)
$$

where $\left\{v_{k}\right\}_{k=1}^{\infty}$ is dense in $V$. Also let

$$
\gamma_{n}(\omega)=\max \left\{\left\|d_{1}(\omega)\right\|, \cdots,\left\|d_{n}(\omega)\right\|,\left\|v_{1}\right\|, \cdots,\left\|v_{n}\right\|\right\}
$$

and $B_{n}(\omega)$ defined as $\overline{B\left(0, \gamma_{n}(\omega)\right)}$. Then let

$$
K_{n}(\omega)=V_{n} \cap K(\omega) \cap B_{n}(\omega) \neq \emptyset \text { so } \cup_{n} K_{n}(\omega) \text { dense in } K(\omega)
$$

Now each $K_{n}(\omega)$ is a set valued compact convex subset of $V_{n}(\omega)$ which is a measurable multifunction. It is a measurable multifunction because the linear combinations of the measurable functions $\left\{v_{1}, \cdots, v_{n}, d_{1}(\omega), \cdots, d_{n}(\omega)\right\}$ having a subset of the rational numbers as coefficients is a dense subset of $K_{n}(\omega)$. Then by Theorem 48.3.3, there exist measurable functions

$$
u_{n}(\omega) \in K_{n}(\omega), w_{n}^{B}(\omega) \in B\left(u_{n}(\omega), \omega\right), w_{n}^{C}(\omega) \in C\left(u_{n}(\omega), \omega\right)
$$

such that

$$
\begin{equation*}
\left\langle f(\omega)-\left(w_{n}^{B}(\omega)+w_{n}^{C}(\omega)\right), z-u_{n}(\omega)\right\rangle \leq 0 \tag{*}
\end{equation*}
$$

for all $z \in K_{n}(\omega)$.
Thus for all $w \in K_{n}(\omega)$,

$$
\left\langle w_{n}^{B}(\omega)+w_{n}^{C}(\omega), u_{n}(\omega)-w\right\rangle \leq\left\langle f(\omega), u_{n}(\omega)-w\right\rangle
$$

These $u_{n}(\omega)$ are bounded because $K(\omega)$ is a bounded set or in the other case, one can pick the special $z(\omega)$ in the definition for coercivity to obtain that these $u_{n}(\omega)$ are bounded. Thus, since $A(\cdot, \omega)$ is assumed to be bounded for $A=B, C$, each of $u_{n}(\omega), w_{n}^{B}(\omega), w_{n}^{C}(\omega)$ are bounded for each $\omega$.

By Lemma 48.2.2, there is a subsequence $n(\omega)$ such that

$$
\left(u_{n(\omega)}(\omega), w_{n(\omega)}^{B}(\omega), w_{n(\omega)}^{C}(\omega)\right)
$$

converges weakly to $\left(u(\omega), w_{B}(\omega), w_{C}(\omega)\right)$ in $V \times V^{\prime} \times V^{\prime}$ and

$$
\omega \rightarrow\left(u(\omega), w_{B}(\omega), w_{C}(\omega)\right)
$$

is measurable into $V \times V^{\prime} \times V^{\prime}$. It is now only a matter of verifying the desired variational inequality for each $\omega$.

By convexity, $u(\omega) \in K(\omega)$. Now for fixed $\omega$, let $\hat{u}_{n(\omega)} \rightarrow u(\omega)$ strongly in $V$ where $\hat{u}_{n(\omega)} \in K_{n}(\omega)$. Then

$$
\begin{gathered}
\quad \lim \sup _{n(\omega) \rightarrow \infty}\left\langle w_{n(\omega)}^{B}(\omega)+w_{n(\omega)}^{C}(\omega), u_{n(\omega)}(\omega)-u(\omega)\right\rangle \\
=\lim \sup _{n(\omega) \rightarrow \infty}\left\langle w_{n(\omega)}^{B}(\omega)+w_{n(\omega)}^{C}(\omega), u_{n(\omega)}(\omega)-\hat{u}_{n(\omega)}\right\rangle \\
\leq \lim \sup _{n(\omega) \rightarrow \infty}\left\langle f(\omega), u_{n(\omega)}(\omega)-\hat{u}_{n(\omega)}\right\rangle \leq 0 .
\end{gathered}
$$

By Theorem 48.5.2,

$$
w_{B}(\omega) \in B(u(\omega), \omega), w_{C}(\omega) \in C(u(\omega), \omega)
$$

Also, there is a subsequence, still denoted with $n(\omega)$ such that

$$
\begin{aligned}
& \lim \inf _{n(\omega) \rightarrow \infty}\left\langle w_{n(\omega)}^{B}(\omega)+w_{n(\omega)}^{C}(\omega), u_{n(\omega)}(\omega)-u(\omega)\right\rangle \\
\geq & \langle w(u(\omega)), u(\omega)-u(\omega)\rangle=0
\end{aligned}
$$

for some $w(u(\omega)) \in B(u(\omega), \omega)+C(u(\omega), \omega)$ because the sum of pseudomonotone operators is pseudomonotone. Thus for this subsequence, since

$$
\begin{gathered}
\quad \lim \sup _{n(\omega) \rightarrow \infty}\left\langle w_{n(\omega)}^{B}(\omega)+w_{n(\omega)}^{C}(\omega), u_{n(\omega)}(\omega)-u(\omega)\right\rangle \\
\leq \quad 0 \leq \lim \inf _{n(\omega) \rightarrow \infty}\left\langle w_{n(\omega)}^{B}(\omega)+w_{n(\omega)}^{C}(\omega), u_{n(\omega)}(\omega)-u(\omega)\right\rangle,
\end{gathered}
$$

it follows that

$$
\lim _{n(\omega) \rightarrow \infty}\left\langle w_{n(\omega)}^{B}(\omega)+w_{n(\omega)}^{C}(\omega), u_{n(\omega)}(\omega)-u(\omega)\right\rangle=0
$$

We will consider this subsequence or a further subsequence. Then since the limsup condition holds for this subsequence, there exists for any $v \in V$,

$$
w_{B}(v) \in B(u(\omega), \omega), w_{C}(v) \in C(u(\omega), \omega)
$$

such that

$$
\begin{aligned}
& \left\langle w_{B}(\omega)+w_{C}(\omega), u(\omega)-v\right\rangle \\
\geq & \lim _{\inf _{n(\omega) \rightarrow \infty}}\binom{\left\langle w_{n(\omega)}^{B}(\omega)+w_{n(\omega)}^{C}(\omega), u_{n(\omega)}(\omega)-u(\omega)\right\rangle}{+\left\langle w_{n(\omega)}^{B}(\omega)+w_{n(\omega)}^{C}(\omega), u(\omega)-v\right\rangle} \\
= & \lim \inf _{n(\omega) \rightarrow \infty}\left\langle w_{n(\omega)}^{B}(\omega)+w_{n(\omega)}^{C}(\omega), u_{n(\omega)}(\omega)-v\right\rangle \\
\geq & \left\langle w_{B}(v)+w_{C}(v), u(\omega)-v\right\rangle
\end{aligned}
$$

Finally, let $v \in K(\omega)$. Then it follows that there exists a sequence $\left\{\hat{v}_{n}\right\}$ such that $\hat{v}_{n} \in$ $K_{n}(\omega)$ which converges strongly to $v$. Thus

$$
\left\langle w_{n}^{B}(\omega)+w_{n}^{C}(\omega), u_{n}(\omega)-\hat{v}_{n}\right\rangle \leq\left\langle f(\omega), u_{n}(\omega)-\hat{v}_{n}\right\rangle
$$

Then

$$
\begin{gathered}
\left\langle w_{B}(\omega)+w_{C}(\omega), u(\omega)-v\right\rangle= \\
\lim \sup _{n(\omega) \rightarrow \infty}\left(\begin{array}{c}
\left\langle{ }^{(\omega)} \begin{array}{c}
\left.w_{n(\omega)}^{B}(\omega)+w_{n(\omega)}^{C}(\omega), u_{n(\omega)}(\omega)-u(\omega)\right\rangle \\
+\left\langle w_{n(\omega)}^{B}(\omega)+w_{n(\omega)}^{C}(\omega), u(\omega)-v\right\rangle
\end{array}\right)
\end{array}\right) .\left\{\begin{array}{c}
\end{array}\right)
\end{gathered}
$$

$$
\begin{aligned}
& =\lim \sup _{n(\omega) \rightarrow \infty}\left\langle w_{n(\omega)}^{B}(\omega)+w_{n(\omega)}^{C}(\omega), u_{n(\omega)}(\omega)-v\right\rangle \\
& =\lim \sup _{n(\omega) \rightarrow \infty}\left\langle w_{n(\omega)}^{B}(\omega)+w_{n(\omega)}^{C}(\omega), u_{n(\omega)}(\omega)-\hat{v}_{n(\omega)}\right\rangle \\
& \leq \lim \sup _{n(\omega) \rightarrow \infty}\left\langle f(\omega), u_{n(\omega)}(\omega)-\hat{v}_{n(\omega)}\right\rangle=\langle f(\omega), u(\omega)-v\rangle
\end{aligned}
$$

Note that from standard results, in the case of coercivity and $V=K(\omega)$, the above shows that for $f$ measurable there is a measurable $u$ such that $f(\omega) \in B(u(\omega), \omega)+$ $C(u(\omega), \omega)$. Specifically, there are measurable functions

$$
w_{B}(\omega) \in B(u(\omega), \omega) \text { and } w_{C}(\omega) \in C(u(\omega), \omega)
$$

such that $f(\omega)=w_{C}(\omega)+w_{B}(\omega)$.
Example 48.5.4 We can let $\Omega=[0, T]$ and let the measurable sets be the Lebesgue measurable sets, $t \rightarrow f(t)$ measurable into $V^{\prime}$. Then for $A(\cdot, \cdot)$ satisfying $1-3$ the above theorem gives the solution $u, w(t), u(t) \in K(t)$ to variational inequalities of the form

$$
\langle f(t)-w(t), z-u(t)\rangle \leq 0, w(t) \in A(u(t), t)
$$

for all $z \in K(t)$ where $K(t)$ is a closed bounded convex subset of $V$ for $t \rightarrow K(t)$ a measurable multifunction. Here both $u$ and $w$ are measurable. If $u \rightarrow A(u, t)$ is coercive, this allows for $K(t)$ only closed and convex. If $A$ is the sum of $B, C$ and $u \rightarrow B(u, t)$ and $u \rightarrow C(u, t)$, these each satisfying the conditions $1-3, w(t)=w_{B}(t)+w_{C}(t)$ where both of these summands are measurable and $w_{B}(t) \in B(u(t), t), w_{C}(t) \in C(u(t), t)$. If suitable estimates hold, and $f \in L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)$ then one can conclude that $w \in L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)$ and $u \in L^{p}([0, T] ; V)$. This paper has resolved the only difficult issue which is existence of a measurable solution with no monotonicity or uniqueness assumptions on the problem for fixed $t$.

Example 48.5.5 We can let $\Omega$ be of the form $[0, T] \times \Omega$ where $(\Omega, \mathscr{F})$ is a measurable space. In this case, we could let the $\sigma$ algebra be $\mathscr{B} \times \mathscr{F}$ the product measurable sets and obtain product measurable solutions to the same variational inequalities.

Example 48.5.6 In the case of a filtration $\left\{\mathscr{F}_{t}\right\}$, one could let the $\sigma$ algebra consist of the progressively measurable sets and obtain the same conclusions. Thus the variational inequality would be of the form

$$
\langle f(t, \omega)-w(t, \omega), z-u(t, \omega)\rangle \leq 0, w(t, \omega) \in A(u(t, \omega), t, \omega), z \in K(t, \omega)
$$

This result is quite interesting because it is describing a situation where there is no uniqueness or monotonicity and all that is required are conditions of measurability on $f$. Also, one only needs to check the limit conditions on $u \rightarrow A(u, \omega)$ for fixed $\omega$ so all Sobolev embedding theorems are available. Nor is it necessary to assume that $\omega \rightarrow A(u, \omega)$ is a measurable multifunction as is often done. It suffices to check that it has a measurable selection. This is a strictly more general condition.

### 48.6 An Example

Let $\sigma(r, t)$ be a continuous function of $r$ which satisfies

$$
\begin{aligned}
r & \rightarrow \sigma(r, t) \text { is continuous, } t \rightarrow \sigma(r, t) \text { is measurable } \\
0 & <\delta(t) \leq \sigma(r, t) \leq 1 / \delta(t)
\end{aligned}
$$

There is no uniform lower bound needed for $\delta(t)$. Then we will let $V$ be a closed subspace of $H^{1}(\Omega)$ where $\Omega$ is a bounded open set with Lipschitz boundary.

$$
V \equiv\left\{u \in H^{1}(\omega): \gamma u=0 \text { on } \Sigma_{0}\right\}
$$

where $\alpha\left(\Sigma_{0}\right)>0$ for $\alpha$ the surface measure, $\Sigma_{0}$ a closed subset of $\partial \Omega$ and $\gamma$ is the trace map. Thus an equivalent norm for $V$ is

$$
\|u\|_{V}^{2}=\int_{\Omega}|\nabla u|^{2} d x
$$

Also let $H=L^{2}(\Omega)$ and let $H=H^{\prime}$ so that $V \subseteq H=H^{\prime} \subseteq V^{\prime}$. Then let $A(\cdot, t): V \rightarrow V^{\prime}$ be defined by

$$
\langle A(u, t), v\rangle \equiv \int_{\Omega} \sigma(u, t) \nabla u \cdot \nabla v
$$

Is this a bounded pseudomonotone map? It is clearly bounded thanks to the bounds on $\sigma$. Suppose then that $u_{n} \rightarrow u$ weakly in $V$ and

$$
\lim \sup _{n \rightarrow \infty}\left\langle A\left(u_{n}, t\right), u_{n}-u\right\rangle \leq 0
$$

Does the liminf condition hold? If not, then there exists a subsequence and $v \in V$ such that

$$
\lim _{n \rightarrow \infty}\left\langle A\left(u_{n}, t\right), u_{n}-v\right\rangle<\langle A(u, t), u-v\rangle
$$

By compactness, there is a further subsequence still denoted with $n$ such that $u_{n} \rightarrow u$ strongly in $L^{2}(\Omega)$ and pointwise. Consider

$$
\int_{\Omega} \sigma\left(u_{n}, t\right) \nabla u_{n} \cdot\left(\nabla u_{n}-\nabla v\right)
$$

Now by the dominated convergence theorem,

$$
\int_{\Omega}\left|\sigma\left(u_{n}, t\right)-\sigma(u, t)\right|^{2} \rightarrow 0
$$

and so in fact $\sigma\left(u_{n}, t\right) \nabla u_{n} \rightarrow \sigma(u, t) \nabla u$ weakly in $H^{3}$. Then

$$
\begin{aligned}
\int_{\Omega} \sigma\left(u_{n}, t\right) \nabla u_{n} \cdot\left(\nabla u_{n}-\nabla v\right)= & \int_{\Omega} \sigma\left(u_{n}, t\right) \nabla u_{n} \cdot\left(\nabla u_{n}-\nabla u\right) \\
& +\int_{\Omega} \sigma\left(u_{n}, t\right) \nabla u_{n} \cdot(\nabla u-\nabla v)
\end{aligned}
$$

$$
\geq \int_{\Omega} \sigma\left(u_{n}, t\right) \nabla u \cdot\left(\nabla u_{n}-\nabla u\right)+\int_{\Omega} \sigma\left(u_{n}, t\right) \nabla u_{n} \cdot(\nabla u-\nabla v)
$$

The second term in the above converges to $\int_{\Omega} \sigma(u, t) \nabla u \cdot(\nabla u-\nabla v)$.
Consider the first term after $\geq$. It equals

$$
\begin{equation*}
\int_{\Omega}\left(\sigma\left(u_{n}, t\right)-\sigma(u, t)\right) \nabla u \cdot\left(\nabla u_{n}-\nabla u\right)+\int_{\Omega} \sigma(u, t) \nabla u \cdot\left(\nabla u_{n}-\nabla u\right) \tag{*}
\end{equation*}
$$

The second of these terms converges to 0 because of weak convergence of $u_{n}$ to $u$. As to the first, if the measure of $E$ is small enough, then

$$
\left(\int_{E}|\nabla u|^{2}\right)^{1 / 2}<\delta
$$

By Egoroff's theorem, there is a set $E$ having measure this small such that off this set, $\sigma\left(u_{n}(x), t\right)-\sigma(u(x), t) \rightarrow 0$ uniformly for $x \notin E$. Thus an application of Holder's inequality shows that $\left|\int_{E^{C}}\left(\sigma\left(u_{n}, t\right)-\sigma(u, t)\right) \nabla u \cdot\left(\nabla u_{n}-\nabla u\right)\right| \leq \delta$ whenever $n$ is sufficiently large thanks to the weak convergence of $u_{n}$ to $u$ which implies that $\nabla u_{n}-\nabla u$ is bounded in $L^{2}(\Omega)^{3}$. As to the integral over $E$, the fact that $\sigma$ is bounded for fixed $t$ implies the existence of a constant $C$ independent of $n$ such that

$$
\left|\int_{\Omega}\left(\sigma\left(u_{n}, t\right)-\sigma(u, t)\right) \nabla u \mathscr{X}_{E} \cdot\left(\nabla u_{n}-\nabla u\right)\right| \leq C\left(\int_{E}|\nabla u|^{2}\right)^{1 / 2}<C \delta
$$

Thus the first term in * has absolute value no larger than $(C+1) \delta$ provided $n$ is sufficiently large. Since $\delta$ is arbitrary, the limit of this term is 0 . Thus,

$$
\lim _{n \rightarrow \infty} \inf _{n \rightarrow} \sigma\left(u_{n}, t\right) \nabla u_{n} \cdot\left(\nabla u_{n}-\nabla v\right) \geq \int_{\Omega} \sigma(u, t) \nabla u \cdot(\nabla u-\nabla v)
$$

This is a contradiction. Thus the liminf condition must hold.
Next consider another operator. Let $\Sigma_{1}$ be $\partial \Omega \backslash \Sigma_{0}$ and has positive surface measure. Let $r \rightarrow a(r, t)$ be lower semicontinuous and $r \rightarrow b(r, t)$ be upper semicontinuous. Let $0<\delta(t) \leq a(r, t) \leq b(r, t) \leq \frac{1}{\delta(t)}$. Also let both of these functions be measurable in $t$. Now $\gamma: V \rightarrow L^{2}\left(\Sigma_{1}\right)$ and so $\gamma^{*}: L^{2}\left(\Sigma_{1}\right) \rightarrow V^{\prime}$ defined in the usual way. Then $z \in B(u, t)$ will mean $z=\gamma^{*} w$ for some $w \in L^{2}\left(\Sigma_{1}\right)$ with

$$
w(\mathbf{x}) \in[a(\gamma u(\mathbf{x}), t), b(\gamma u(\mathbf{x}), t)]
$$

for a.e. $\mathbf{x}$ such that

$$
\langle z, v\rangle=\int_{\Sigma_{1}} w(\mathbf{x}) \gamma v(\mathbf{x})
$$

Using Sobolev embedding theorems, if $u_{n} \rightarrow u$ weakly in $V$, then from the Sobolev embedding theorem $u_{n} \rightarrow u$ strongly in a suitable Sobolev space of fractional order such that the embedding of $V$ into this space is compact and the trace map is still continuous. Thus there is a subsequence such that $\gamma u_{n}(\mathbf{x}) \rightarrow \gamma u(\mathbf{x})$ pointwise a.e. and $w_{n} \rightarrow w$ in $L^{2}\left(\Sigma_{1}\right)$.

Then by the semicontinuity properties of $a, b$ we obtain from routine considerations that $w(\mathbf{x}) \in[a(\gamma u(\mathbf{x}), t), b(\gamma u(\mathbf{x}), t)]$ a.e. To see how you can do this, let

$$
E=\left\{\mathbf{x}: w(\mathbf{x}) \geq b(\gamma u(\mathbf{x}), t)+\frac{1}{k}\right\} .
$$

Then

$$
\begin{aligned}
\int_{\Sigma_{1}} \mathscr{X}_{E}(\mathbf{x})\left(-b\left(\gamma u_{n}(\mathbf{x}), t\right)\right) & \leq \int_{\Sigma_{1}} \mathscr{X}_{E}(\mathbf{x})\left(-w_{n}(\mathbf{x})\right) \rightarrow \int_{\Sigma_{1}} \mathscr{X}_{E}(\mathbf{x})(-w(\mathbf{x})) \\
& \leq \int_{\Sigma_{1}} \mathscr{X}_{E}(\mathbf{x})\left(-b(\gamma u(\mathbf{x}), t)-\frac{1}{k}\right)
\end{aligned}
$$

By lower semicontinuity of $-b(\cdot, t)$ and the boundedness assumption, we can use Fatou's lemma to take liminf of both sides and conclude that

$$
\int_{\Sigma_{1}} \mathscr{X}_{E}(\mathbf{x})(-b(\gamma u(\mathbf{x}), t)) \leq \int_{\Sigma_{1}} \mathscr{X}_{E}(\mathbf{x})\left(-b(\gamma u(\mathbf{x}), t)-\frac{1}{k}\right)
$$

an obvious contradiction unless $\alpha(E)=0$. Then taking the union of the exceptional sets for all $k$, it follows that $w(\mathbf{x}) \leq b(\gamma u(\mathbf{x}), t)$ a.e. The other side of the inequality can be shown similarly. Letting $z_{n} \in B\left(u_{n}, t\right)$ and $v \in V$, is it true that

$$
\lim \inf _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-v\right\rangle \geq\left\langle z(v), u_{n}-v\right\rangle
$$

for some $z(v) \in B(u, t)$ ? Suppose not. Then from the above, there is a subsequence such that the limit equals the liminf but which has the inequality turned around for some $v$ and all $z \in B(u, t)$. Then from what was just shown, letting $w_{n}$ go with $z_{n}$, there is a further subsequence such that $w_{n} \rightarrow w$ weakly in $L^{2}\left(\Sigma_{1}\right)$ and and $\gamma u_{n} \rightarrow \gamma u$ strongly in $L^{2}\left(\Sigma_{1}\right)$ and

$$
w(\mathbf{x}) \in[a(\gamma u(\mathbf{x}), t), b(\gamma u(\mathbf{x}), t)] \text { a.e. } \mathbf{x}
$$

Then

$$
\int_{\Sigma_{1}} w_{n}(\mathbf{x})\left(\gamma u_{n}(\mathbf{x})-\gamma v(\mathbf{x})\right) \rightarrow \int_{\Sigma_{1}} w(\mathbf{x})(\gamma u(\mathbf{x})-\gamma v(\mathbf{x}))=\langle z, u-v\rangle
$$

where $w \in B(u, t)$ and $z=\gamma^{*} w$ so the liminf condition holds. Thus this second operator is pseudomonotone.

Do these have measurable selections? This is obvious. Letting $u \in V, t \rightarrow \gamma^{*} a(\gamma u(\mathbf{x}), t)$ is measurable into $V^{\prime}$ and is in $B(u, t)$. Similarly $t \rightarrow A(u, t)$ is measurable into $V^{\prime}$. Note that on the second operator, it was really only necessary to assume that there exists $t \rightarrow c(r, t)$ measurable with $c(r, t) \in[a(r, t), b(r, t)]$ and totally eliminate the assumption that either $a$ or $b$ is measurable in $t$.

Now let $t \rightarrow f(t)$ be measurable into $V^{\prime}$. Say

$$
\langle f(t), v\rangle=\int_{\Omega} h(t) v d x+\int_{\Sigma_{1}} \beta(t) v d \alpha
$$

and let $K(t) \subseteq V$ be a closed convex subset of $V$. There is obviously a coercivity condition holding for the sum of these two operators $A(u, t)+B(u, t)$ and so there exists $u(t) \in K(t)$ such that for all $v \in K(t)$

$$
\begin{equation*}
\langle f(t)-(A(u(t), t)+z(t)), v-u(t)\rangle \leq 0 \tag{*}
\end{equation*}
$$

where $t \rightarrow u(t)$ is measurable into $V, t \rightarrow z(t)$ measurable into $V^{\prime}, t \rightarrow A(u(t), t)$ measurable into $V^{\prime}$. Is $t \rightarrow w(t)$ measurable where $z(t)=\gamma^{*} w(t)$ ? Let $\phi \in V$. Then

$$
\langle z(t), \phi\rangle_{V^{\prime}, V}=(w(t), \gamma \phi)_{L^{2}\left(\Sigma_{1}\right)}
$$

and is given to be a measurable function of $t$. However, since $\Sigma_{1}$ is open, the image of the trace is dense in $L^{2}\left(\Sigma_{1}\right)$ and so by this density and Pettis theorem, $t \rightarrow w(t)$ must be measurable into $L^{2}\left(\Sigma_{1}\right)$. Thus the variational inequality $*$ is of the form

$$
\binom{(h(t), v-u(t))_{H}+(\beta(t)-w(t), \gamma v-\gamma u(t))_{L^{2}\left(\Sigma_{1}\right)}}{-(\sigma(u(t), t) \nabla u(t), \nabla v-\nabla u(t))_{H^{3}}} \leq 0
$$

for all $v \in K(t)$. If the inequality which gives coercivity were eliminated, we would still have the above if $K(t)$ were assumed bounded.

What equation is satisfied if $K(t)=V$ ? We would have

$$
\begin{gathered}
\int_{\Omega} \sigma(u(t), t) \nabla u(t) \cdot \nabla v d x+\int_{\Sigma_{1}} w(t) v d x=\int_{\Omega}\langle f(t), v\rangle d x \\
w(t) \in[a(\gamma u(t), t), b(\gamma u(t), t)]
\end{gathered}
$$

Then proceding formally, we obtain

$$
\begin{aligned}
& \int_{\Omega} \nabla \cdot(\sigma(u(t), t) \nabla u(t) v) d x-\int_{\Omega} \nabla \cdot(\sigma(u(t), t) \nabla u(t)) v+\int_{\Sigma_{1}} w(t) v d x \\
= & \int_{\Omega} h(t) v+\int_{\Sigma_{1}} \beta(t) v
\end{aligned}
$$

Then a formal application of the divergence theorem yields the boundary conditions

$$
\begin{aligned}
\sigma(u(t), t) \nabla u(t) \cdot \mathbf{n}+w(t) & =\beta(t) \text { on } \Sigma_{1} \\
u(t) & =0 \text { on } \Sigma_{0}
\end{aligned}
$$

where $w(t) \in[a(\gamma u(t), t), b(\gamma u(t), t)]$ along with the partial differential equation

$$
-\nabla \cdot(\sigma(u(t), t) \nabla u(t))=h(t)
$$

Note that if either $a(\cdot, t)$ or $b(\cdot, t)$ were continuous, there would have been no point in considering the second operator as a set valued map. One could simply replace the closed interval $[a(r, t), b(r, t)]$ with either $a(r, t)$ or $b(r, t)$ and obtain the desired solutions. Also note that nothing is needed about the integrability of either $t \rightarrow h(t)$ or $t \rightarrow \beta(t)$. The following lemma is convenient in considering the convex sets $K(t)$.

Lemma 48.6.1 Let $F$ be those points where $u=0$. Then for a.e. $\mathbf{x} \in F, \nabla u(\mathbf{x})=\mathbf{0}$. Here $u \in W^{1, p}(\Omega)$ and we assume $\partial \Omega$ has measure zero.

Proof: It suffices to consider $u=0$ on $F$ contained in the interior of $\Omega$. First I show that $u_{, x_{n}}=0$ a.e.

$$
\frac{u\left(t_{1}, \cdots, t_{n-1}, t_{n}\right)-u\left(t_{1}, \cdots, t_{n-1}, t_{n}-h\right)}{h}=\frac{1}{h} \int_{t_{n}-h}^{t_{n}} u,_{n}\left(t_{1}, \cdots, t_{n-1}, s\right) d s
$$

Then a.e. $t_{n}$ is a Lebesgue point of $u, x_{n}$ for $F_{\left(t_{1}, \cdots, t_{n-1}\right)}$ where $F_{\left(t_{1}, \cdots, t_{n-1}\right)}$ consists of those points of $F$ where $\left(t_{1}, \cdots, t_{n-1}\right)$ is fixed. Also let $t_{n}$ be a point of density of $F_{\left(t_{1}, \cdots, t_{n-1}\right)}$. Of course $m_{1}$ a.e. points of $F_{\left(t_{1}, \cdots, t_{n-1}\right)}$ are points of density. Therefore, there exists a sequence $h_{k} \rightarrow 0+$ such that $t_{n}-h_{k} \rightarrow t_{n}$ as $k \rightarrow \infty$ and $\left(t_{n}-h_{k}\right) \in F_{\left(t_{1}, \cdots, t_{n-1}\right)}$. Otherwise there would be some open set about $t_{n}$ which excludes points of $F_{\left(t_{1}, \cdots, t_{n-1}\right)}$ which would imply that $t_{n}$ is not actually a point of density. Then using the fundamental theorem of calculus, we get for such points which are points of $F_{\left(t_{1}, \cdots, t_{n-1}\right)}$ the fact that $u, x_{n}\left(t_{1}, \cdots, t_{n-1}, t_{n}\right)=0$. Thus for a.e. $t_{n} \in F_{\left(t_{1}, \cdots, t_{n-1}\right)}, u, x_{n}\left(t_{1}, \cdots, t_{n-1}, t_{n}\right)=0$. Thus $u, x_{n}\left(t_{1}, \cdots, t_{n-1}, t_{n}\right)=0$ for a.e. $t_{n}$ in $F_{\left(t_{1}, \cdots, t_{n-1}\right)}$. Similar reasoning holds for differentiation with respect to the other variables. Thus $\nabla u=\mathbf{0}$ a.e. on $F$.

Lemma 48.6.2 Let $V$ be a closed subset of $W^{1, p}(\Omega), p>1$ and let $k \in V$. Then $\max (k, u) \in$ $V$ and if $u_{n} \rightarrow u$ in $V$, then $\max \left(u_{n}, k\right) \rightarrow \max (u, k)$ in $V$.

Proof: We consider $\psi(r)=|r|, \psi_{\varepsilon}(r)=\sqrt{\varepsilon+r^{2}}$. Then for $\phi \in C_{c}^{\infty}(\Omega)$,

$$
\begin{aligned}
\int_{\Omega} \psi(u(\mathbf{x})) \phi,{x_{k}}_{k}(\mathbf{x}) & =\lim _{\varepsilon \rightarrow 0} \int_{\Omega} \psi_{\varepsilon}(u(\mathbf{x})) \phi, x_{k}(\mathbf{x}) \\
& =-\lim _{\varepsilon \rightarrow 0} \int_{\Omega} \frac{u(\mathbf{x})}{\sqrt{\varepsilon+u^{2}(\mathbf{x})}} u, x_{k}(\mathbf{x}) \phi(\mathbf{x}) \\
& =-\int_{\Omega} \xi(u(\mathbf{x})) u, x_{k}(\mathbf{x}) \phi(\mathbf{x})
\end{aligned}
$$

where $\xi(r)=1$ if $r>0,-1$ if $r<0$ and 0 if $r=0$. Thus $\psi(u), x_{k}=\xi(u(\mathbf{x})) u, x_{k}(\mathbf{x})$ a.e. and so $\psi \circ u$ is clearly in $W^{1, p}(\Omega)$. Of course $\max (u, k)=\frac{|k-u|+(k+u)}{2}$ so this shows that $\max (u, k)$ is in $W^{1, p}(\Omega)$.

Next suppose $u_{n} \rightarrow u$ in $W^{1, p}(\Omega)$. Does

$$
\xi\left(u_{n}\right) u_{n}, x_{k} \rightarrow \xi(u) u_{, x_{k}} ?
$$

Let $G=\{\mathbf{x}: u(\mathbf{x}) \neq 0\}$. A subsequence, still denoted by $u_{n}$ converges pointwise a.e . to $u$ and $u_{n}, x_{k} \rightarrow u, x_{k}$ pointwise a.e. Therefore, off a set of measure zero, $\xi\left(u_{n}(\mathbf{x})\right)=\xi(u(\mathbf{x}))$ for all $n$ large enough on $G$. Also,

$$
\begin{gather*}
\left(\int_{\Omega}\left|\xi\left(u_{n}\right) u_{n}, x_{k}-\xi(u) u_{, x_{k}}\right|^{p}\right)^{1 / p} \leq\left(\int_{\Omega}\left|\xi\left(u_{n}\right) u_{n, x_{k}}-\xi\left(u_{n}\right) u_{, x_{k}}\right|^{p}\right)^{1 / p} \\
+\left(\int_{\Omega}\left|\xi\left(u_{n}\right)-\xi(u)\right|^{p}\left|u, x_{k}\right|^{p}\right)^{1 / p} \tag{*}
\end{gather*}
$$

That second term on the right converges to 0 . It equals

$$
\left(\int_{G^{C}}\left|\xi\left(u_{n}\right)-\xi(u)\right|^{p}\left|u, x_{k}\right|^{p}+\int_{G}\left|\xi\left(u_{n}\right)-\xi(u)\right|^{p}\left|u, x_{k}\right|^{p}\right)^{1 / p}
$$

Now on $G^{C}, u, x_{k}(\mathbf{x})=0$ a.e. and so the first term in the parentheses is 0 . The second converges to 0 by the dominated convergence theorem. Then this shows that the second term in * converges to 0 . The first obviously converges to 0 from the convergence of $u_{n}, x_{k}$ to $u, x_{k}$. Now consider whether $\psi\left(u_{n}\right), x_{k}$ converges to $\psi(u), x_{k}$. Those functions $\psi\left(u_{n}\right), x_{k}$ are bounded in $L^{p}(\Omega)$ from the above description and so if it fails to converge to $\psi(u), x_{k}$ in $L^{p}$ a subsequence converges weakly to $\zeta \neq \psi(u),,_{k}$. But then, the above argument shows that a further subsequence does converge strongly to $\psi(u), x_{k}$ contrary to $\zeta \neq \psi(u), x_{k}$.

Now, from the description of the maximum of two functions given above, we obtain that $\max \left(u_{n}, k\right) \rightarrow \max (u, k)$ in $V$ provided $u_{n} \rightarrow u$ in $V$.

Consider $k \in C([0, T] ; V)$. Let

$$
K(t) \equiv\{u \in V: u(\mathbf{x}) \geq k(t, \mathbf{x}) \text { a.e. } \mathbf{x}\}
$$

This is clearly a convex subset of $V$. Is it closed and convex? Is $t \rightarrow K(t)$ a set valued measurable function?

Claim: $K(t)$ is closed and convex.
Proof: It is obvious it is convex. Suppose $u_{n} \rightarrow u$ in $V, u_{n} \in K(t)$. Then there is a subsequence, still denoted as $u_{n}$ such that $u_{n}(\mathbf{x}) \rightarrow u(\mathbf{x})$ a.e. Hence $K(t)$ is closed.

Claim: $t \rightarrow K(t)$ is a measurable multifunction.
Proof: Consider the subset of $C([0, T] ; V)$ defined by

$$
\{u \in C([0, T] ; V): \text { for a.e. } t, u(t, \mathbf{x}) \geq k(t, \mathbf{x}) \text { a.e. } \mathbf{x}\}
$$

This is a subset of the completely separable set $C([0, T] ; V)$ and so it is also separable. Let $\left\{d_{i}\right\}_{i=1}^{\infty}$ be a dense subset of $C([0, T] ; V)$. Then let $\left\{b_{i}\right\}_{i=1}^{\infty}$ be defined by $b_{i}(t, \mathbf{x}) \equiv$ $\max \left(k(t, \mathbf{x}), d_{i}(t, \mathbf{x})\right)$. Thus the functions $\mathbf{x} \rightarrow b_{i}(t, \mathbf{x})$ are each in $K(t)$ because of the above Lemma. They are also measurable into $V$ because $k, d_{i} \in C([0, T] ; V)$. Is $\left\{b_{i}(t, \cdot)\right\}_{i=1}^{\infty}$ dense in $K(t)$ ? Suppose $u \in K(t)$. Then $t \rightarrow v(t, \mathbf{x}) \equiv u(\mathbf{x})$ is in $C([0, T] ; V)$ and so there is a subsequence denoted by $d_{i}$ which converges pointwise to $u$ in $C([0, T] ; V)$. Therefore, we can get a subsequence such that by the above lemma, $\max \left(k(t, \cdot), d_{i}(t, \cdot)\right) \rightarrow u(\cdot)$ in $V$. Thus $\left\{b_{i}(t, \cdot)\right\}_{i=1}^{\infty}$ is dense in $K(t)$ and so $t \rightarrow K(t)$ is a measurable multifunction.

As an example, you could simply take $k$ to be the restriction to $\Omega \times[0, T]$ of a smooth function.

This is an example of an obstacle problem in which the obstacle changes in $t$ and there is no uniqueness even though there exists a measurable solution to the variational inequality for each $t$.

One could also replace $\sigma(u, t)$ with a graph having a jump as in the second of the two operators and get similar results by beginning with the above solutions and then using Lemma 48.2.2, and the arguments used in the second operator to pass to a limit.

The next section is an interesting result on the pseudomonotone condition for Nemytskii operators defined in this section.

### 48.7 Limit Conditions For Nemytskii Operators

This is about the following problem. You know

$$
u \rightarrow A(u, t)
$$

is pseudomonotone. You can also define $\hat{A}: \mathscr{V} \rightarrow \mathscr{V}^{\prime}$ by

$$
\hat{A}(u)(t) \equiv A(u(t), t) \text { a.e. }
$$

Then when can you obtain a useable limit condition for $\hat{A}$ ? I think the earliest solution to this problem was given in [17]. These ideas were extended to set valued maps in [18] and to another situation in [85].

Define $\mathscr{V} \equiv \mathscr{V}_{p}$ by

$$
\mathscr{V}=L^{p}([0, T] ; V), p>1
$$

where $V$ is a separable Banach space and $H$ is a Hilbert space such that

$$
V \subseteq H=H^{\prime} \subseteq V^{\prime}
$$

with each space dense in the following one. The measure space is chosen to be

$$
([0, T], \mathscr{B}([0, T]), m)
$$

where $m$ is the Lebesgue measure and $\mathscr{B}([0, T])$ consists of all the Borel sets, although one could use the $\sigma$ algebra of Lebesgue measurable sets as well. We denote by $\mathscr{V}_{p}$ or $\mathscr{V}$ the above space. If $U$ is a Banach space, $\mathscr{U}_{r}$ will denote $L^{r}([0, T], U)$.

We will assume the following measurability condition. For each $u \in \mathscr{V}$,

$$
\begin{equation*}
t \rightarrow A(u(t), t) \text { is a measurable multifunction } \tag{48.7.19}
\end{equation*}
$$

In the case when $A(\cdot, t)$ is single-valued, bounded and pseudomonotone, this measurability condition is satisfied and so it is measurable. Thus, this definition is a generalization of what would be expected for single-valued operators. We use the following lemma.

Lemma 48.7.1 Let $U$ be a separable reflexive Banach space. Suppose there is a sequence $\left\{u_{j}(\omega)\right\}_{j=1}^{\infty}$ in $U$, where each $\omega \rightarrow u_{j}(\omega)$ is measurable and for each $\omega, \sup _{i}\left\|u_{i}(\omega)\right\|<$ $\infty$. Then, there exists $u(\omega) \in U$ such that $\omega \rightarrow u(\omega)$ is measurable, and a subsequence $n(\omega)$, that depends on $\omega$, such that the weak limit

$$
\lim _{n(\omega) \rightarrow \infty} u_{n(\omega)}(\omega)=u(\omega)
$$

holds.
Proof. Let $\left\{z_{i}\right\}_{i=1}^{\infty}$ be a countable dense subset of $U^{\prime}$. Let $\mathbf{h}: U \rightarrow \prod_{i=1}^{\infty} \mathbb{R}$ be defined by

$$
\mathbf{h}(u)=\prod_{i=1}^{\infty}\left\langle z_{i}, u\right\rangle .
$$

Let $X=\prod_{i=1}^{\infty} \mathbb{R}$ with the product topology. Then, this is a Polish space with the metric defined as $d(\mathbf{x}, \mathbf{y})=\sum_{i=1}^{\infty} \frac{\left|x_{i}-y_{i}\right|}{1+\left|x_{i}-y_{i}\right|} 2^{-i}$. By compactness, for a fixed $\omega$, the $\mathbf{h}\left(u_{n}(\omega)\right)$ are contained in a compact subset of $X$. Next, define

$$
\Gamma_{n}(\omega)=\overline{\cup_{k \geq n} \mathbf{h}\left(u_{k}(\omega)\right)}
$$

which is a nonempty compact subset of $X$.
Next, we claim that $\omega \rightarrow \Gamma_{n}(\omega)$ is a measurable multifunction.
The proof of the claim is as follows. It is necessary to show that $\Gamma_{n}^{-}(O)$ defined as $\left\{\omega: \Gamma_{n}(\omega) \cap O \neq \emptyset\right\}$ is measurable whenever $O$ is open. It suffices to verify this for $O$ a basic open set in the topology of $X$. Thus let $O=\prod_{i=1}^{\infty} O_{i}$ where each $O_{i}$ is a proper open subset of $\mathbb{R}$ only for $i \in\left\{j_{1}, \cdots, j_{m}\right\}$. Then,

$$
\Gamma_{n}^{-}(O)=\cup_{k \geq n} \cap_{r=1}^{m}\left\{\omega:\left\langle z_{j_{r}}, u_{k}(\omega)\right\rangle \in O_{j_{r}}\right\}
$$

which is a measurable set since $u_{k}$ is measurable.
Then, it follows that $\omega \rightarrow \Gamma_{n}(\omega)$ is strongly measurable because it has compact values in $X$, thanks to Tychonoff's theorem. Thus $\Gamma_{n}^{-}(H)=\left\{\omega: H \cap \Gamma_{n}(\omega) \neq \emptyset\right\}$ is measurable whenever $H$ is a closed set. Now, let $\Gamma(\omega)$ be defined as $\cap_{n} \Gamma_{n}(\omega)$ and then for $H$ closed,

$$
\Gamma^{-}(H)=\cap_{n} \Gamma_{n}^{-}(H)
$$

and each set in the intersection is measurable, so this shows that $\omega \rightarrow \Gamma(\omega)$ is also measurable. Therefore, it has a measurable selection $\mathbf{g}(\omega)$. It follows from the definition of $\Gamma(\omega)$ that there exists a subsequence $n(\omega)$ such that

$$
\mathbf{g}(\omega)=\lim _{n(\omega) \rightarrow \infty} \mathbf{h}\left(u_{n(\omega)}(\omega)\right) \quad \text { in } X
$$

In terms of components, we have

$$
g_{i}(\omega)=\lim _{n(\omega) \rightarrow \infty}\left\langle z_{i}, u_{n(\omega)}(\omega)\right\rangle
$$

Furthermore, there is a further subsequence, still denoted with $n(\omega)$, such that $u_{n(\omega)}(\omega) \rightarrow$ $u(\omega)$ weakly. This means that for each $i$,

$$
g_{i}(\omega)=\lim _{n(\omega) \rightarrow \infty}\left\langle z_{i}, u_{n(\omega)}(\omega)\right\rangle=\left\langle z_{i}, u(\omega)\right\rangle
$$

Thus, for each $z_{i}$ in a dense set, $\omega \rightarrow\left\langle z_{i}, u(\omega)\right\rangle$ is measurable. Since the $z_{i}$ are dense, this implies $\omega \rightarrow\langle z, u(\omega)\rangle$ is measurable for every $z \in U^{\prime}$ and so by the Pettis theorem, $\omega \rightarrow u(\omega)$ is measurable.

Also is a definition.
Definition 48.7.2 Let $A(\cdot, t): V \rightarrow \mathscr{P}\left(V^{\prime}\right)$. Then, the Nemytskii operator associated with A,

$$
\hat{A}: L^{p}([0, T] ; V) \rightarrow \mathscr{P}\left(L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)\right)
$$

is given by

$$
z \in \hat{A}(u) \text { if and only if } z \in L^{p^{\prime}}\left([0, T] ; V^{\prime}\right) \text { and } z(t) \in A(u(t), t) \text { a.e. } t .
$$

## Growth and coercivity

The next three conditions on the operator $A$ are similar to the conditions proposed by Bian and Webb, [18] See also Berkovitz and Mustonen [17] which seems to be the paper where these ideas originated. These specific and reasonable conditions, together with a fourth one we add below, allow us to prove an appropriate limit condition that is based on the assumption that $u \rightarrow A(u, t)$ is a set-valued, bounded and pseudomonotone map from $V$ to $\mathscr{P}\left(V^{\prime}\right)$ and $t \rightarrow A(u, t)$ has a measurable selection.

Our aim is to provide reasonable conditions under which an assumption of pseudomonotonicity on $u \rightarrow A(u, t)$ transfers to a useable limit condition for the operator $\hat{A}$ defined on $\mathscr{V}=L^{p}([0, T] ; V)$.

It is obvious that $\hat{A} u$ is convex because this is true of $A(u, t)$. It is also closed. To see this, suppose $z_{n} \in \hat{A}(u)$. Then $z_{n}(t) \in A(u(t), t)$ for a.e.t. Taking the union of the exceptional sets, we can assume this inclusion holds off a single set of measure zero for all $n$. If you have $z_{n} \rightarrow w$ strongly in $\mathscr{V}^{\prime}$, then a subsequence converges pointwise a.e. Therefore, by upper semicontinuity of the pointwise operator $u \rightarrow A(u, t)$, it follows that $w(t) \in A(u(t), t)$ for a.e. $t$. Thus $\hat{A} u$ is convex and strongly closed.

We assume the following conditions on $A$.

1. $A(\cdot, t): V \rightarrow \mathscr{P}\left(V^{\prime}\right)$ is pseudomonotone and bounded: $A(u, t)$ is a closed convex set for each $t, u \rightarrow A(u, t)$ is bounded, and if

$$
\lim \sup _{n \rightarrow \infty}\left\langle A\left(u_{n}, t\right), u_{n}-u\right\rangle \leq 0
$$

then for any $v \in V$,

$$
\lim _{n \rightarrow \infty}\left\langle A\left(u_{n}, t\right), u_{n}-v\right\rangle \geq\langle z(v), u-v\rangle \text { some } z(v) \in A(u, t)
$$

2. $A(\cdot, t)$ satisfies the estimates: There exists $b_{1} \geq 0$ and $b_{2} \geq 0$, such that

$$
\begin{equation*}
\|z\|_{V^{\prime}} \leq b_{1}\|u\|_{V}^{p-1}+b_{2}(t) \tag{48.7.20}
\end{equation*}
$$

for all $z \in A(u, t), b_{2}(\cdot) \in L^{p^{\prime}}([0, T])$.
3. There exist a positive constant $b_{3}$ and a nonnegative function $b_{4}$ that is $\mathscr{B}([0, T])$ measurable and also $b_{4}(\cdot) \in L^{1}([0, T])$, such that

$$
\begin{equation*}
\inf _{z \in A(u, t)}\langle z, u\rangle \geq b_{3}\|u\|_{V}^{p}-b_{4}(t)-\lambda|u|_{H}^{2} . \tag{48.7.21}
\end{equation*}
$$

4. The operators $t \rightarrow A(u(t), t)$ are measurable in the sense that

$$
t \rightarrow A(u(t), t)
$$

is a measurable multifunction with respect to $\mathscr{F}$ where $\mathscr{F}$ will be the $\sigma$ algebra of Lebesgue measurable sets whenever $t \rightarrow u(t)$ is in $\mathscr{V}_{p}$.
5. For $u \in \mathscr{V}_{p}$, we define $\hat{A}(u) \in \mathscr{P}\left(\mathscr{V}_{p}^{\prime}\right)$ as follows: $z \in \hat{A}(u)$ means that $z(t) \in$ $A(u(t), t)$ a.e. $t$. Thus this is the Nemytskii operator for $A(\cdot, t)$.

In the following theorem and in arguments which take place below, $U$ will be a Hilbert space dense in $V$ with the inclusion map compact. Such a Hilbert space always exists and is important in probability theory where $(i, U, V)$ is an abstract Wiener space. However, in most applications from partial differential equations, it suffices to take $U$ as a suitable Sobolev space.

Theorem 48.7.3 Suppose conditions 1 - 5 hold. Also, suppose

$$
V \subseteq H=H^{\prime} \subseteq V^{\prime}, \text { where } V \text { is dense in } H
$$

Then, the operator $\hat{A}$ satisfies the following.
Hypotheses:

$$
u_{n} \rightarrow u \text { weakly in } \mathscr{V}, \lim \sup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \leq 0
$$

for $z_{n} \in \hat{A} u_{n}$, and there exists a set of zero measure $\Sigma$ such that for $t \notin \Sigma$, every subsequence of $\left\{u_{n}\right\}$ has a further subsequence, possibly depending on $t \notin \Sigma$ such that

$$
u_{n}(t) \rightarrow u(t) \text { weakly in } U^{\prime}
$$

where $U$ is a Banach space dense in $V$. In 3, if $\lambda>0$, then assume also that

$$
\begin{equation*}
\sup _{n} \sup _{t \in[0, T]}\left|u_{n}(t)\right|_{H}<\infty . \tag{48.7.22}
\end{equation*}
$$

Conclusion: If the above conditions hold, then for each $v \in \mathscr{V}$, there exists $z(v)$ with

$$
\lim _{n \rightarrow \infty} \inf _{n \rightarrow}\left\langle z_{n}, u_{n}-v\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \geq\langle z(v), u-v\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}
$$

where $z(v) \in \hat{A}(u)$. Furthermore, $\hat{A} u$ is a nonempty, closed and convex set in $\mathscr{V}^{\prime}$.
Proof: It was argued above that $\hat{A}(u)$ is closed and convex.
Enlarge the set of measure zero $\Sigma$, if needed, so that for each $n$,

$$
z_{n}(t) \in A\left(u_{n}(t), t\right)
$$

for each $t \notin \Sigma$.
Next, we claim that if $t \notin \Sigma$, then

$$
\lim _{n \rightarrow \infty}\left\langle z_{n}(t), u_{n}(t)-u(t)\right\rangle \geq 0
$$

Proof of the claim: Let $t \notin \Sigma$ be fixed and suppose to the contrary that

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\langle z_{n}(t), u_{n}(t)-u(t)\right\rangle<0 \tag{48.7.23}
\end{equation*}
$$

Then, there exists a subsequence $\left\{n_{k}\right\}$, which may depend on $t$, such that

$$
\begin{equation*}
\lim _{k \rightarrow \infty}\left\langle z_{n_{k}}(t), u_{n_{k}}(t)-u(t)\right\rangle=\lim \inf _{n \rightarrow \infty}\left\langle z_{n}(t), u_{n}(t)-u(t)\right\rangle<0 \tag{48.7.24}
\end{equation*}
$$

Now, condition 3 implies that for all $k$ large enough,

$$
\begin{gathered}
b_{3}\left\|u_{n_{k}}(t)\right\|_{V}^{p}-b_{4}(t)-\lambda\left|u_{n_{k}}(t)\right|_{H}^{2}<\left\|z_{n_{k}}(t)\right\|_{V^{\prime}}\|u(t)\|_{V} \\
\leq\left(b_{1}\left\|u_{n_{k}}(t)\right\|_{V}^{p-1}+b_{2}(t)\right)\|u(t)\|_{V}
\end{gathered}
$$

therefore, $\left\|u_{n_{k}}(t)\right\|_{V}$ and consequently $\left\|z_{n_{k}}(t)\right\|_{V^{\prime}}$ are bounded. This follows from 48.7.22 in case $\lambda>0$. Note that $\left\|z_{n_{k}}(t)\right\|_{V^{\prime}}$ is bounded independently of $n_{k}$ because of the assumption that $A(\cdot, t)$ is bounded and we just showed that $\left\|u_{n_{k}}(t)\right\|_{V}$ is bounded.

Taking a further subsequence if necessary, let $u_{n_{k}}(t) \rightarrow u(t)$ weakly in $U^{\prime}$ and $u_{n_{k}}(t) \rightarrow$ $\xi$ weakly in $V$. Thus, by density considerations, $\xi=u(t)$. Now, 48.7.24 and the limit conditions for pseudomonotone operators imply that the liminf condition holds.There exists $z_{\infty} \in A(u(t), t)$ such that

$$
\begin{aligned}
\liminf _{k \rightarrow \infty}\left\langle z_{n_{k}}(t), u_{n_{k}}(t)-u(t)\right\rangle & \geq\left\langle z_{\infty}, u(t)-u(t)\right\rangle=0 \\
& >\lim _{k \rightarrow \infty}\left\langle z_{n_{k}}(t), u_{n_{k}}(t)-u(t)\right\rangle
\end{aligned}
$$

which is a contradiction. This completes the proof of the claim.
We continue with the proof of the theorem. It follows from this claim that for every $t \notin \Sigma$,

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\langle z_{n}(t), u_{n}(t)-u(t)\right\rangle \geq 0 \tag{48.7.25}
\end{equation*}
$$

Also, it is assumed that

$$
\lim _{n \rightarrow \infty} \sup _{n \rightarrow}\left\langle z_{n}, u_{n}-u\right\rangle_{\mathscr{V}} \leq 0
$$

Then from the estimates,

$$
\int_{0}^{T}\left(b_{3}\left\|u_{n}(t)\right\|_{V}^{p}-b_{4}(t)-\lambda\left|u_{n}(t)\right|_{H}^{2}\right) d t \leq \int_{0}^{T}\|u(t)\|_{V}\left(\left\|u_{n}\right\|^{p-1} b_{1}+b_{2}\right) d t
$$

so it is routine to get $\left\|u_{n}\right\|_{\mathscr{V}}$ is bounded. This follows from the assumptions, in particular 48.7.22.

Now, the coercivity condition 3 shows that if $y \in L^{p}([0, T] ; V)$, then

$$
\begin{aligned}
\left\langle z_{n}(t), u_{n}(t)-y(t)\right\rangle \geq & b_{3}\left\|u_{n}(t)\right\|_{V}^{p}-b_{4}(t)-\lambda\left|u_{n}(t)\right|_{H}^{2} \\
& -\left(b_{1}\left\|u_{n}(t)\right\|^{p-1}+b_{2}(t)\right)\|y(t)\|_{V}
\end{aligned}
$$

Using $p-1=\frac{p}{p^{\prime}}$, where $\frac{1}{p}+\frac{1}{p^{\prime}}=1$, the right-hand side of this inequality equals

$$
b_{3}\left\|u_{n}(t)\right\|_{V}^{p}-b_{4}(t)-b_{1}\left\|u_{n}(t)\right\|^{p / p^{\prime}}\|y(t)\|_{V}-b_{2}(t)\|y(t)\|_{V}-\lambda\left|u_{n}(t)\right|_{H}^{2}
$$

the last term being bounded independent of $t, n$ by assumption. Thus there exists $c \in$ $L^{1}(0, T)$ and a positive constant $C$ such that

$$
\begin{equation*}
\left\langle z_{n}(t), u_{n}(t)-y(t)\right\rangle \geq-c(t)-C\|y(t)\|_{V}^{p} . \tag{48.7.26}
\end{equation*}
$$

Letting $y=u$, we use Fatou's lemma to write

$$
\begin{gathered}
{\lim \inf _{n \rightarrow \infty} \int_{0}^{T}\left(\left\langle z_{n}(t), u_{n}(t)-u(t)\right\rangle+c(t)+C\|u(t)\|_{V}^{p}\right) d t \geq}_{\int_{0}^{T} \lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty}\left\langle z_{n}(t), u_{n}(t)-u(t)\right\rangle+\left(c(t)+C\|u(t)\|_{V}^{p}\right) d t} \begin{array}{c}
\geq \int_{0}^{T}\left(c(t)+C\|u(t)\|_{V}^{p}\right) d t
\end{array} .
\end{gathered}
$$

Here, we added the term $c(t)+C\|u(t)\|_{V}^{p}$ to make the integrand nonnegative in order to apply Fatou's lemma. Thus,

$$
\lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty} \int_{0}^{T}\left\langle z_{n}(t), u_{n}(t)-u(t)\right\rangle d t \geq 0
$$

Consequently, using the claim in the last inequality,

$$
\begin{aligned}
0 & \geq \lim _{n \rightarrow \infty} \sup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \\
& \geq \lim _{n \rightarrow \infty} \inf _{0}^{T}\left\langle z_{n}(t), u_{n}(t)-u(t)\right\rangle d t \\
& =\lim _{n \rightarrow \infty} \inf _{n}\left\langle z_{n}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \\
& \geq \int_{0}^{T} \lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty}\left\langle z_{n}(t), u_{n}(t)-u(t)\right\rangle d t \geq 0
\end{aligned}
$$

hence, we find that

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}=0 \tag{48.7.27}
\end{equation*}
$$

We need to show that if $y$ is given in $\mathscr{V}$ then

$$
\left.\lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty}, u_{n}-y\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \geq\langle z(y), u-y\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}, \quad z(y) \in \hat{A} u
$$

Suppose to the contrary that there exists $y$ such that

$$
\begin{equation*}
\eta=\lim \inf _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-y\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}<\langle z, u-y\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \tag{48.7.28}
\end{equation*}
$$

for all $z \in \hat{A} u$. Take a subsequence, denoted still with subscript $n$ such that

$$
\eta=\lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-y\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}
$$

Thus

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-y\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}<\langle z, u-y\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \tag{48.7.29}
\end{equation*}
$$

We will obtain a contradiction to this. In what follows we continue to use the subsequence just described which satisfies the above inequality.

The estimate 48.7.26 implies,

$$
\begin{equation*}
0 \leq\left\langle z_{n}(t), u_{n}(t)-u(t)\right\rangle^{-} \leq c(t)+C\|u(t)\|_{V}^{p}, \tag{48.7.30}
\end{equation*}
$$

where $c$ is a function in $L^{1}(0, T)$. Thanks to (48.7.25),

$$
\lim _{n \rightarrow \infty}\left\langle z_{n}(t), u_{n}(t)-u(t)\right\rangle \geq 0
$$

and, therefore, the following pointwise limit exists,

$$
\lim _{n \rightarrow \infty}\left\langle z_{n}(t), u_{n}(t)-u(t)\right\rangle^{-}=0
$$

and so we may apply the dominated convergence theorem using (48.7.30) and conclude

$$
\lim _{n \rightarrow \infty} \int_{0}^{T}\left\langle z_{n}(t), u_{n}(t)-u(t)\right\rangle^{-} d t=\int_{0}^{T} \lim _{n \rightarrow \infty}\left\langle z_{n}(t), u_{n}(t)-u(t)\right\rangle^{-} d t=0
$$

Now, it follows from (48.7.27) and the above equation, that

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \int_{0}^{T}\left\langle z_{n}(t), u_{n}(t)-u(t)\right\rangle^{+} d t \\
= & \lim _{n \rightarrow \infty} \int_{0}^{T}\left\langle z_{n}(t), u_{n}(t)-u(t)\right\rangle+\left\langle z_{n}(t), u_{n}(t)-u(t)\right\rangle^{-} d t \\
= & \lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}=0 .
\end{aligned}
$$

Therefore, both $\int_{0}^{T}\left\langle z_{n}(t), u_{n}(t)-u(t)\right\rangle^{+} d t$ and $\int_{0}^{T}\left\langle z_{n}(t), u_{n}(t)-u(t)\right\rangle^{-} d t$ converge to 0, thus,

$$
\begin{align*}
\lim _{n \rightarrow \infty} \int_{0}^{T}\left|\left\langle z_{n}(t), u_{n}(t)-u(t)\right\rangle\right| d t & =0  \tag{48.7.31}\\
\lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} & =0
\end{align*}
$$

From the above, it follows that there exists a further subsequence $\left\{n_{k}\right\}$ not depending on $t$ such that

$$
\begin{equation*}
\left|\left\langle z_{n_{k}}(t), u_{n_{k}}(t)-u(t)\right\rangle\right| \rightarrow 0 \quad \text { a.e. } t . \tag{48.7.32}
\end{equation*}
$$

Therefore, by the pseudomonotone limit condition for $A$ there exists $w_{t} \in A(u(t), t)$ such that for a.e. $t$,

$$
\begin{aligned}
\alpha(t) & \equiv \lim \inf _{k \rightarrow \infty}\left\langle z_{n_{k}}(t), u_{n_{k}}(t)-y(t)\right\rangle \\
& =\lim \inf _{k \rightarrow \infty}\left\langle z_{n_{k}}(t), u(t)-y(t)\right\rangle \geq\left\langle w_{t}, u(t)-y(t)\right\rangle .
\end{aligned}
$$

Then on the exceptional set, let $\alpha(t) \equiv \infty$, and consider the set

$$
F(t) \equiv\{w \in A(u(t), t):\langle w, u(t)-y(t)\rangle \leq \alpha(t)\}
$$

which then satisfies $F(t) \neq \emptyset$. Now $F(t)$ is closed and convex in $V^{\prime}$.
Claim: $t \rightarrow F(t)$ has a measurable selection off a set of measure zero.
Proof of claim: Letting $B(0, C(t))$ contain $A(u(t), t)$, we can assume $t \rightarrow C(t)$ is measurable by using the estimates and the measurability of $u$. For $p \in \mathbb{N}$, let $S_{p} \equiv\{t: C(t)<p\}$. If it is shown that $F$ has a measurable selection on $S_{p}$, then it follows that it has a measurable selection. Thus in what follows, assume that $t \in S_{p}$.

Define

$$
G(t) \equiv\left\{w:\langle w, u(t)-y(t)\rangle<\alpha(t)+\frac{1}{n}, t \notin \Sigma\right\} \cap B(0, p)
$$

Thus, it was shown above that this $G(t) \neq \emptyset$. For $U$ open,

$$
\begin{equation*}
G^{-}(U) \equiv\left\{t \in S_{p}: \text { for some } w \in U \cap B(0, p),\langle w, u(t)-y(t)\rangle<\alpha(t)+\frac{1}{n}\right\} \tag{*}
\end{equation*}
$$

Let $\left\{w_{j}\right\}$ be a dense subset of $U \cap B(0, p)$. This is possible because $V^{\prime}$ is separable. The expression in $*$ equals

$$
\cup_{k=1}^{\infty}\left\{t \in S_{p}:\left\langle w_{k}, u(t)-y(t)\right\rangle<\alpha(t)+\frac{1}{n}\right\}
$$

which is measurable. Thus $G$ is a measurable multifunction.
Since $t \rightarrow G(t)$ is measurable, there is a sequence $\left\{w_{n}(t)\right\}$ of measurable functions such that $\overline{\cup_{n=1}^{\infty} w_{n}(t)}$ equals

$$
\overline{G(t)}=\left\{w:\langle w, u(t)-y(t)\rangle \leq \alpha(t)+\frac{1}{n}, t \notin \Sigma\right\} \cap \overline{B(0, p)}
$$

As shown above, there exists $w_{t}$ in $A(u(t), t)$ as well as $G(t)$. Thus there is a sequence of $w_{r}(t)$ converging to $w_{t}$. Since $t \rightarrow A(u(t), t)$ is a measurable multifunction, it has a countable subset of measurable functions $\left\{z_{m}(t)\right\}$ which is dense in $A(u(t), t)$. Let

$$
U_{k}(t) \equiv \cup_{m} B\left(z_{m}(t), \frac{1}{k}\right) \subseteq A(u(t), t)+B\left(0, \frac{2}{k}\right)
$$

Now define $A_{1 k}=\left\{t: w_{1}(t) \in U_{k}(t)\right\}$. Then let $A_{2 k}=\left\{t \notin A_{1 k}: w_{2}(t) \in U_{k}(t)\right\}$ and $A_{3 k}=\left\{t \notin \cup_{i=1}^{2} A_{i k}: w_{3}(t) \in U_{k}(t)\right\}$ and so forth. Any $t \in S_{p}$ must be contained in one of these $A_{r k}$ for some $r$ since if not so, there would not be a sequence $w_{r}(t)$ converging to $w_{t} \in A(u(t), t)$. These $A_{r p}$ partition $S_{p}$ and each is measurable since the $\left\{z_{k}(t)\right\}$ are measurable. Let

$$
\hat{w}_{k}(t) \equiv \sum_{r=1}^{\infty} \mathscr{X}_{A_{r k}}(t) w_{r}(t)
$$

Thus $\hat{w}_{k}(t)$ is in $U_{k}(t)$ for all $t \in S_{p}$ and equals exactly one of the $w_{m}(t) \in \overline{G(t)}$.
Also, by construction, the $\hat{w}_{k}(\cdot)$ are bounded in $L^{\infty}\left(S_{p} ; V^{\prime}\right)$. Therefore, there is a subsequence of these, still called $\hat{w}_{k}$ which converges weakly to a function $w$ in $L^{2}\left(S_{p} ; V^{\prime}\right)$. Thus $w$ is a weak limit point of $c o\left(\cup_{j=k}^{\infty} \hat{w}_{j}\right)$ for each $k$. Therefore, in the open ball
$B\left(w, \frac{1}{k}\right) \subseteq L^{2}\left(S_{p} ; V^{\prime}\right)$ with respect to the strong topology, there is a convex combination $\sum_{j=k}^{\infty} c_{j k} \hat{w}_{j}$ (the $c_{j k}$ add to 1 and only finitely many are nonzero). Since $\overline{G(t)}$ is convex and closed, this convex combination is in $\overline{G(t)}$. Off a set of measure zero, we can assume this convergence of $\sum_{j=k}^{\infty} c_{j k} \hat{w}_{j}$ as $k \rightarrow \infty$ happens pointwise for a suitable subsequence. However,

$$
\sum_{j=k}^{\infty} c_{j k} \hat{w}_{j}(t) \in U_{k}(t) \subseteq A(u(t), t)+B\left(0, \frac{2}{k}\right)
$$

Thus $w(t) \in A(u(t), t)$ a.e. $t$ because $A(u(t), t)$ is a closed set. Since $w$ is the pointwise limit of measurable functions off a set of measure zero, it can be assumed measurable and for a.e. $t, w(t) \in A(u(t), t) \cap \overline{G(t)}$. Now denote this measurable function $w_{n}$. Then

$$
w_{n}(t) \in A(u(t), t),\left\langle w_{n}(t), u(t)-y(t)\right\rangle \leq \alpha(t)+\frac{1}{n} \text { a.e. } t
$$

These $w_{n}(t)$ are bounded for each $t$ off a set of measure zero and so by Lemma 48.7.1, there is a measurable function $t \rightarrow z(t)$ and a subsequence $w_{n(t)}(t) \rightarrow z(t)$ weakly as $n(t) \rightarrow \infty$. Now $A(u(t), t)$ is closed and convex, hence weakly closed as well. Thus $z(t) \in A(u(t), t)$ and

$$
\begin{equation*}
\langle z(t), u(t)-y(t)\rangle \leq \alpha(t)=\lim \inf _{k \rightarrow \infty}\left\langle z_{n_{k}}(t), u_{n_{k}}(t)-y(t)\right\rangle \tag{**}
\end{equation*}
$$

Therefore, $t \rightarrow F(t)$ has a measurable selection on $S_{p}$ excluding a set of measure zero, namely $z(t)$ which will be called $z_{p}(t)$ in what follows.

It follows that $F(t)$ has a measurable selection on $[0, T]$ other than a set of measure zero. To see this, enlarge $\Sigma$ to include the exceptional sets of measure zero in the above argument for each $p$. Then partition $[0, T] \backslash \Sigma$ as follows. For $p=1,2, \cdots$, consider $S_{p} \backslash$ $S_{p-1}, p=1,2, \cdots$ for $S_{0}$ defined as $\emptyset$. Then letting $z_{p}$ be the selection for $t \in S_{p}$, let $z(t)=$ $\sum_{p=1}^{\infty} z_{p}(t) \mathscr{X}_{S_{p} \backslash S_{p-1}}(t)$. The estimates imply $z \in \mathscr{V}^{\prime}$ and so $z \in \hat{A}(u)$.

From the estimates, there exists $h \in L^{1}(0, T)$ such that $\langle z(t), u(t)-y(t)\rangle \geq-|h(t)|$ Thus, from the above inequality,

$$
\begin{aligned}
& \|h\|_{L^{1}}+\langle z, u-y\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \leq \int_{0}^{T} \lim \inf _{k \rightarrow \infty}\left\langle z_{n_{k}}(t), u_{n_{k}}(t)-y(t)\right\rangle+|h(t)| d t \\
& \leq \lim \inf _{k \rightarrow \infty}\left\langle z_{n_{k}}, u_{n_{k}}-y\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}+\|h\|_{L^{1}}=\lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-y\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}+\|h\|_{L^{1}}
\end{aligned}
$$

which contradicts 48.7.29.
This all works for progressively measurable operators. These are discussed more later in the material on probability. A filtration is $\left\{\mathscr{F}_{t}\right\}, t \in[0, T]$ where each $\mathscr{F}_{t}$ is a $\sigma$ algebra of sets in $\Omega$ usually a probability space and these are increasing in $t$. Then the progressively measurable sets $\mathscr{P}$ are $S \subseteq \Omega$ such that

$$
S \cap[0, t] \times \Omega \text { is } \mathscr{B}([0, t]) \times \mathscr{F}_{t} \text { measurable }
$$

You can verify that this is indeed a $\sigma$ algebra of sets in $[0, T] \times \Omega$. Now here is a generalization of the above which will work for this situation.

In what follows, $\mathscr{P}$ will be the $\sigma$ algebra of progressively measurable sets. Thus there is a filtration $\left\{\mathscr{F}_{t}\right\}$ and a set $S$ is $\mathscr{P}$ measurable means that

$$
(t, \omega) \rightarrow \mathscr{X}_{[0, t] \times \Omega}(t, \omega) \mathscr{X}_{S}(t, \omega) \text { is a } \mathscr{B}([0, t]) \times \mathscr{F}_{t}
$$

meaurable function.
We assume the following conditions on $A$.

1. $A(\cdot, t, \omega): V \rightarrow \mathscr{P}\left(V^{\prime}\right)$ is pseudomonotone and bounded: $A(u, t, \omega)$ is a closed convex set for each $(t, \omega), u \rightarrow A(u, t, \omega)$ is bounded, and if $u_{n} \rightarrow u$ weakly and

$$
\lim \sup _{n \rightarrow \infty}\left\langle A\left(u_{n}, t, \omega\right), u_{n}-u\right\rangle \leq 0
$$

then for any $v \in V$,

$$
\lim _{n \rightarrow \infty}\left\langle A\left(u_{n}, t, \omega\right), u_{n}-v\right\rangle \geq\langle z(v), u-v\rangle \text { some } z(v) \in A(u, t, \omega)
$$

2. $A(\cdot, t, \omega)$ satisfies the estimates: There exists $b_{1} \geq 0$ and $b_{2} \geq 0$, such that

$$
\begin{equation*}
\|z\|_{V^{\prime}} \leq b_{1}\|u\|_{V}^{p-1}+b_{2}(t, \omega) \tag{48.7.33}
\end{equation*}
$$

for all $z \in A(u, t, \omega), b_{2}(\cdot, \cdot) \in L^{p^{\prime}}([0, T] \times \Omega)$.
3. There exist a positive constant $b_{3}$ and a nonnegative function $b_{4}$ that is $\mathscr{B}([0, T]) \times$ $\mathscr{F}_{T}$ measurable and also $b_{4}(\cdot, \cdot) \in L^{1}([0, T] \times \Omega)$, such that for some $\lambda \geq 0$,

$$
\begin{equation*}
\inf _{z \in A(u, t, \omega)}\langle z, u\rangle \geq b_{3}\|u\|_{V}^{p}-b_{4}(t, \omega)-\lambda|u|_{H}^{2} \tag{48.7.34}
\end{equation*}
$$

4. The mapping $(t, \omega) \rightarrow A(u(t, \omega), t, \omega)$ is measurable in the sense that

$$
(t, \omega) \rightarrow A(u(t, \omega), t, \omega)
$$

is a measurable multifunction with respect to $\mathscr{P}$ whenever $(t, \omega) \rightarrow u(t, \omega)$ is in $\mathscr{V} \equiv \mathscr{V}_{p} \equiv L^{p}([0, T] \times \Omega ; V, \mathscr{P})$.
5. For $u \in \mathscr{V}_{p}$, we define $\hat{A}(u) \in \mathscr{P}\left(\mathscr{V}_{p}^{\prime}\right)$ as follows: $z \in \hat{A}(u)$ means that $z(t, \omega) \in$ $A(u(t, \omega), t, \omega)$ a.e. $(t, \omega)$. Thus this is the Nemytskii operator for $A(\cdot, t, \omega)$.

In the following theorem and in arguments which take place below, $U$ will be a Hilbert space dense in $V$ with the inclusion map compact. Such a Hilbert space always exists and is important in probability theory where $(i, U, V)$ is an abstract Wiener space. However, in most applications from partial differential equations, it suffices to take $U$ as a suitable Sobolev space. Also for $S$ a set in $[0, T] \times \Omega, S_{\omega}$ will denote $\{t:(t, \omega) \in S\}$.

Theorem 48.7.4 Suppose conditions 1 - 5 hold. Also, suppose $U$ is a separable Hilbert space dense in $V$, a reflexive separable Banach space with the inclusion map compact and $V$ is dense in a Hilbert space $H$. Thus

$$
U \subseteq V \subseteq H=H^{\prime} \subseteq V^{\prime} \subseteq U^{\prime}
$$

Then, the operator $\hat{A}$ in the definition 5 satisfies the following.
Hypotheses:

$$
u_{n} \rightarrow u \text { weakly in } \mathscr{V}, \lim \sup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \leq 0
$$

for $z_{n} \in \hat{A} u_{n}$. For each $\omega$, off a set of P measure zero $N$, every subsequence of $u_{n}(t, \omega)$ has a further subsequence, possibly depending on $t, \omega$ such that

$$
u_{n}(t, \omega) \rightarrow u(t, \omega) \text { weakly in } U^{\prime}
$$

Assume also that

$$
\begin{equation*}
\sup _{\omega \in \Omega \backslash N} \sup _{n} \sup _{t \in[0, T]} \lambda\left|u_{n}(t, \omega)\right|_{H}<\infty . \tag{48.7.35}
\end{equation*}
$$

Conclusion: If the above conditions hold, then there exists $z(v)$ with

$$
\lim _{n \rightarrow \infty} \inf _{n}\left\langle z_{n}, u_{n}-v\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \geq\langle z(v), u-v\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}
$$

where $z(v) \in \hat{A}(u)$.
Proof: It was argued above that $\hat{A}(u)$ is closed and convex. Let $\Sigma$ have measure zero and for each $(t, \omega) \notin \Sigma, z_{n}(t, \omega) \in A\left(u_{n}(t, \omega), t, \omega\right)$ for each $n$. Now $\Sigma_{\omega}$ has measure zero for a.e. $\omega$ since otherwise $\Sigma$ would not have measure zero. These are the $\omega$ of interest in the following argument, and we can simply include the exceptional $\omega$ in the set of measure zero $N$ which is being ignored since it has measure zero.

First we claim that if $t \notin \Sigma_{\omega}$, then

$$
\lim _{n \rightarrow \infty} \inf _{n \rightarrow}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle \geq 0
$$

Proof of the claim: Let $t \notin \Sigma_{\omega}$ be fixed and suppose to the contrary that

$$
\begin{equation*}
\lim \inf _{n \rightarrow \infty}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle<0 \tag{48.7.36}
\end{equation*}
$$

Then, there exists a subsequence $\left\{n_{k}\right\}$, which may depend on $t, \omega$, such that

$$
\begin{align*}
& \lim _{k \rightarrow \infty}\left\langle z_{n_{k}}(t, \omega), u_{n_{k}}(t, \omega)-u(t, \omega)\right\rangle  \tag{48.7.37}\\
= & \lim \inf _{n \rightarrow \infty}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle<0 . \tag{48.7.38}
\end{align*}
$$

Now, condition 3 implies that for all $k$ large enough,

$$
b_{3}\left\|\left.u_{n_{k}}(t, \omega)\right|_{V} ^{p}-b_{4}(t, \omega)-\lambda\left|u_{n_{k}}(t, \omega)\right|_{H}^{2}<\right\| z_{n_{k}}(t, \omega)\left\|_{V^{\prime}}\right\| u(t, \omega) \|_{V}
$$

$$
\leq\left(b_{1}\left\|u_{n_{k}}(t, \omega)\right\|_{V}^{p-1}+b_{2}(t, \omega)\right)\|u(t, \omega)\|_{V}
$$

therefore, $\left\|u_{n_{k}}(t, \omega)\right\|_{V}$ and consequently $\left\|z_{n_{k}}(t, \omega)\right\|_{V^{\prime}}$ are bounded. This follows from 48.7.35. Note that $\left|\mid z_{n_{k}}(t, \omega) \|_{V^{\prime}}\right.$ is bounded independently of $n_{k}$ because of the assumption that $A(\cdot, t, \omega)$ is bounded and we just showed that $\left\|u_{n_{k}}(t, \omega)\right\|_{V}$ is bounded.

Taking a further subsequence if necessary, let $u_{n_{k}}(t, \omega) \rightarrow u(t, \omega)$ weakly in $U^{\prime}$ and $u_{n_{k}}(t, \omega) \rightarrow \xi$ weakly in $V$. Thus, by density considerations, $\xi=u(t, \omega)$. Now, 48.7.37 and the limit conditions for pseudomonotone operators imply that the liminf condition holds.There exists $z_{\infty} \in A(u(t, \omega), t, \omega)$ such that

$$
\begin{aligned}
\lim _{k \rightarrow \infty}\left\langle z_{n_{k}}(t, \omega), u_{n_{k}}(t, \omega)-u(t, \omega)\right\rangle & \geq\left\langle z_{\infty}, u(t, \omega)-u(t, \omega)\right\rangle=0 \\
& >\lim _{k \rightarrow \infty}\left\langle z_{n_{k}}(t, \omega), u_{n_{k}}(t, \omega)-u(t, \omega)\right\rangle
\end{aligned}
$$

which is a contradiction. This completes the proof of the claim.
We continue with the proof of the theorem. It follows from this claim that for given $\omega$, every $t \notin \Sigma_{\omega}$,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \inf _{n \rightarrow}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle \geq 0 \tag{48.7.39}
\end{equation*}
$$

Also, it is assumed that

$$
\limsup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle_{\mathscr{V}} \leq 0
$$

Then from the estimates,

$$
\begin{aligned}
& \int_{\Omega} \int_{0}^{T}\left(b_{3}\left\|u_{n}(t, \omega)\right\|_{V}^{p}-b_{4}(t, \omega)-\lambda\left|u_{n}(t, \omega)\right|_{H}^{2}\right) d t d P \\
\leq & \int_{\Omega} \int_{0}^{T}\|u(t, \omega)\|_{V}\left(\left\|u_{n}(t, \omega)\right\|^{p-1} b_{1}+b_{2}\right) d t d P
\end{aligned}
$$

so it is routine to get $\left\|u_{n}\right\|_{\mathscr{V}}$ is bounded. This follows from the assumptions, in particular 48.7.35.

Now, the coercivity condition 3 shows that if $y \in \mathscr{V}$, then

$$
\begin{aligned}
\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-y(t, \omega)\right\rangle \geq & b_{3}\left\|u_{n}(t, \omega)\right\|_{V}^{p}-b_{4}(t, \omega)-\lambda\left|u_{n}(t, \omega)\right|_{H}^{2} \\
& -\left(b_{1}\left\|u_{n}(t, \omega)\right\|^{p-1}+b_{2}(t, \omega)\right)\|y(t, \omega)\|_{V}
\end{aligned}
$$

Using $p-1=\frac{p}{p^{\prime}}$, where $\frac{1}{p}+\frac{1}{p^{\prime}}=1$, the right-hand side of this inequality equals

$$
\begin{aligned}
& b_{3}\left\|u_{n}(t, \omega)\right\|_{V}^{p}-b_{4}(t, \omega)-b_{1}\left\|u_{n}(t, \omega)\right\|^{p / p^{\prime}}\|y(t, \omega)\|_{V} \\
& -b_{2}(t, \omega)\|y(t, \omega)\|_{V}-\lambda\left|u_{n}(t, \omega)\right|_{H}^{2}
\end{aligned}
$$

the last term being bounded independent of $t, n$ by assumption. Thus there exists $c(\cdot, \cdot) \in$ $L^{1}([0, T] \times \Omega)$ and a positive constant $C$ such that

$$
\begin{equation*}
\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-y(t, \omega)\right\rangle \geq-c(t, \omega)-C\|y(t, \omega)\|_{V}^{p} \tag{48.7.40}
\end{equation*}
$$

Letting $y=u$, we use Fatou's lemma to write

$$
\begin{gathered}
{\lim \inf _{n \rightarrow \infty} \int_{\Omega} \int_{0}^{T}\left(\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle+c(t, \omega)+C\|u(t, \omega)\|_{V}^{p}\right) d t d P \geq}_{\int_{\Omega} \int_{0}^{T} \lim \inf _{n \rightarrow \infty}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle+\left(c(t, \omega)+C\|u(t, \omega)\|_{V}^{p}\right) d t d P} \begin{array}{c}
\geq \int_{\Omega} \int_{0}^{T}\left(c(t, \omega)+C\|u(t, \omega)\|_{V}^{p}\right) d t d P
\end{array} .
\end{gathered}
$$

Here, we added the term $c(t, \omega)+C\|u(t, \omega)\|_{V}^{p}$ to make the integrand nonnegative in order to apply Fatou's lemma. Thus,

$$
\lim \inf _{n \rightarrow \infty} \int_{\Omega} \int_{0}^{T}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle d t d P \geq 0
$$

Consequently, using the claim in the last inequality,

$$
\begin{aligned}
0 & \geq \lim \sup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \\
& \geq \lim _{n \rightarrow \infty} \int_{\Omega} \int_{0}^{T}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle d t d P \\
& =\lim \inf _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \\
& \geq \int_{\Omega} \int_{0}^{T} \lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle d t d P \geq 0
\end{aligned}
$$

hence, we find that

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}=0 \tag{48.7.41}
\end{equation*}
$$

We need to show that if $y$ is given in $\mathscr{V}$ then

$$
\lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-y\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \geq\langle z(y), u-y\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}, \quad z(y) \in \hat{A} u
$$

Suppose to the contrary that there exists $y$ such that

$$
\begin{equation*}
\eta=\lim \inf _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-y\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}<\langle z, u-y\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \tag{48.7.42}
\end{equation*}
$$

for all $z \in \hat{A} u$. Take a subsequence, denoted still with subscript $n$ such that

$$
\eta=\lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-y\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}
$$

Note that this subsequence does not depend on $(t, \omega)$. Thus

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-y\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}<\langle z, u-y\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \tag{48.7.43}
\end{equation*}
$$

We will obtain a contradiction to this. In what follows, we continue to use the subsequence just described which satisfies the above inequality.

The estimate 48.7.40 implies,

$$
\begin{equation*}
0 \leq\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle^{-} \leq c(t, \omega)+C\|u(t, \omega)\|_{V}^{p}, \tag{48.7.44}
\end{equation*}
$$

where $c$ is a function in $L^{1}(0, T)$. Thanks to (48.7.39),

$$
\lim _{n \rightarrow \infty}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle \geq 0, \text { a.e. }
$$

and, therefore, the following pointwise limit exists,

$$
\lim _{n \rightarrow \infty}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle^{-}=0, \text { a.e. }
$$

and so we may apply the dominated convergence theorem using (48.7.44) and conclude

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \int_{\Omega} \int_{0}^{T}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle^{-} d t d P \\
= & \int_{\Omega} \int_{0}^{T} \lim _{n \rightarrow \infty}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle^{-} d t d P=0
\end{aligned}
$$

Now, it follows from (48.7.41) and the above equation, that

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \int_{\Omega} \int_{0}^{T}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle^{+} d t d P \\
= & \lim _{n \rightarrow \infty} \int_{\Omega} \int_{0}^{T}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle \\
& +\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle^{-} d t d P \\
= & \lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}=0 .
\end{aligned}
$$

Therefore, both

$$
\int_{\Omega} \int_{0}^{T}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle^{+} d t d P
$$

and

$$
\int_{\Omega} \int_{0}^{T}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle^{-} d t d P
$$

converge to 0 , thus,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \int_{\Omega} \int_{0}^{T}\left|\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle\right| d t d P=0 \tag{48.7.45}
\end{equation*}
$$

From the above, it follows that there exists a further subsequence $\left\{n_{k}\right\}$ not depending on $t, \omega$ such that

$$
\begin{equation*}
\left|\left\langle z_{n_{k}}(t, \omega), u_{n_{k}}(t, \omega)-u(t, \omega)\right\rangle\right| \rightarrow 0 \quad \text { a.e. }(t, \omega) . \tag{48.7.46}
\end{equation*}
$$

Therefore, by the pseudomonotone limit condition for $A$ there exists

$$
w_{t, \omega} \in A(u(t, \omega), t, \omega)
$$

such that for a.e. $(t, \omega)$

$$
\begin{aligned}
\alpha(t, \omega) & \equiv \lim \inf _{k \rightarrow \infty}\left\langle z_{n_{k}}(t, \omega), u_{n_{k}}(t, \omega)-y(t, \omega)\right\rangle \\
& =\lim \inf _{k \rightarrow \infty}\left\langle z_{n_{k}}(t, \omega), u(t, \omega)-y(t, \omega)\right\rangle \geq\left\langle w_{t, \omega}, u(t, \omega)-y(t, \omega)\right\rangle
\end{aligned}
$$

Then on the exceptional set, let $\alpha(t, \omega) \equiv \infty$, and consider the set

$$
F(t, \omega) \equiv\{w \in A(u(t, \omega), t, \omega):\langle w, u(t, \omega)-y(t, \omega)\rangle \leq \alpha(t, \omega)\}
$$

which then satisfies $F(t, \omega) \neq \emptyset$. Now $F(t, \omega)$ is closed and convex in $V^{\prime}$.
Claim : $(t, x) \rightarrow F(t, \omega)$ has a measurable selection off a set of measure zero.
Proof of claim: Letting $B(0, C(t, \omega))$ contain $A(u(t, \omega), t, \omega)$, we can assume

$$
(t, \omega) \rightarrow C(t, \omega)
$$

is $\mathscr{P}$ measurable by using the estimates and the measurability of $u$. For $\gamma \in \mathbb{N}$, let $S_{\gamma} \equiv$ $\{(t, \omega): C(t, \omega)<\gamma\}$. If it is shown that $F$ has a measurable selection on $S_{\gamma}$, then it follows that it has a measurable selection. Thus in what follows, assume that $(t, \omega) \in S_{\gamma}$.

Define

$$
G(t, \omega) \equiv\left\{w:\langle w, u(t, \omega)-y(t, \omega)\rangle<\alpha(t, \omega)+\frac{1}{n},(t, \omega) \notin \Sigma\right\} \cap B(0, \gamma)
$$

Thus, it was shown above that this $G(t, \omega) \neq \emptyset$ at least for large enough $\gamma$. For $U$ open,

$$
G^{-}(U) \equiv\left\{\begin{array}{c}
(t, \omega) \in S_{\gamma}: \text { for some } w \in U \cap B(0, \gamma)  \tag{*}\\
\langle w, u(t, \omega)-y(t, \omega)\rangle<\alpha(t, \omega)+\frac{1}{n}
\end{array}\right\}
$$

Let $\left\{w_{j}\right\}$ be a dense subset of $U \cap B(0, \gamma)$. This is possible because $V^{\prime}$ is separable. The expression in $*$ equals

$$
\cup_{k=1}^{\infty}\left\{(t, \omega) \in S_{\gamma}:\left\langle w_{k}, u(t, \omega)-y(t, \omega)\right\rangle<\alpha(t, \omega)+\frac{1}{n}\right\}
$$

which is measurable. Thus $G$ is a measurable multifunction.
Since $(t, \omega) \rightarrow \underline{G(t, \omega)}$ is measurable, there is a sequence $\left\{w_{n}(t, \omega)\right\}$ of measurable functions such that $\cup_{n=1}^{\infty} w_{n}(t, \omega)$ equals

$$
\overline{G(t, \omega)}=\left\{w:\langle w, u(t, \omega)-y(t, \omega)\rangle \leq \alpha(t, \omega)+\frac{1}{n}, t \notin \Sigma\right\} \cap \overline{B(0, \gamma)}
$$

As shown above, there exists $w_{t, \omega}$ in $A(u(t, \omega), t, \omega)$ as well as $G(t, \omega)$. Thus there is a sequence of $w_{r}(t, \omega)$ converging to $w_{t, \omega}$. Of course $r$ will need to depend on $t, \omega$. Since $(t, \omega) \rightarrow A(u(t, \omega), t, \omega)$ is a measurable multifunction, it has a countable subset of $\mathscr{P}$ measurable functions $\left\{z_{k}(t, \omega)\right\}$ which is dense in $A(u(t, \omega), t, \omega)$. Let

$$
U_{k}(t, \omega) \equiv \cup_{m} B\left(z_{m}(t, \omega), \frac{1}{k}\right) \subseteq A(u(t, \omega), t, \omega)+B\left(0, \frac{2}{k}\right)
$$

Now define $A_{1 k}=\left\{(t, \omega): w_{1}(t, \omega) \in U_{k}(t, \omega)\right\}$. Then let

$$
A_{2 k}=\left\{(t, \omega) \notin A_{1 k}: w_{2}(t, \omega) \in U_{k}(t, \omega)\right\}
$$

and

$$
A_{3 k}=\left\{(t, \omega) \notin \cup_{i=1}^{2} A_{i k}: w_{3}(t, \omega) \in U_{k}(t, \omega)\right\}
$$

and so forth. Any $(t, \omega) \in S_{\gamma}$ must be contained in one of these $A_{r k}$ for some $r$ since if not so, there would not be a sequence $w_{r}(t, \omega)$ converging to $w_{t, \omega} \in A(u(t, \omega), t, \omega)$. These $A_{r \gamma}$ partition $S_{\gamma}$ and each is measurable since the $\left\{z_{k}(t, \omega)\right\}$ are measurable. Let

$$
\hat{w}_{k}(t, \omega) \equiv \sum_{r=1}^{\infty} \mathscr{X}_{A_{r k}}(t, \omega) w_{r}(t, \omega)
$$

Thus $\hat{w}_{k}(t, \omega)$ is in $U_{k}(t, \omega)$ for all $(t, \omega) \in S_{\gamma}$ and equals exactly one of the $w_{m}(t, \omega) \in$ $\overline{G(t, \omega)}$.

Also, by construction, the $\hat{w}_{k}(\cdot, \cdot)$ are bounded in $L^{\infty}\left(S_{\gamma} ; V^{\prime}\right)$. Therefore, there is a subsequence of these, still called $\hat{w}_{k}$ which converges weakly to a function $w$ in $L^{2}\left(S_{\gamma} ; V^{\prime}\right)$. Thus $w$ is a weak limit point of $\operatorname{co}\left(\cup_{j=k}^{\infty} \hat{w}_{j}\right)$ for each $k$. Therefore, in the open ball $B\left(w, \frac{1}{k}\right) \subseteq L^{2}\left(S_{\gamma} ; V^{\prime}\right)$ with respect to the strong topology, there is a convex combination $\sum_{j=k}^{\infty} c_{j k} \hat{w}_{j}$ (the $c_{j k}$ add to 1 and only finitely many are nonzero). Since $\overline{G(t, \omega)}$ is convex and closed, this convex combination is in $\overline{G(t, \omega)}$. Off a set of $\mathscr{P}$ measure zero, we can assume this convergence of $\sum_{j=k}^{\infty} c_{j k} \hat{w}_{j}$ as $k \rightarrow \infty$ happens pointwise a.e. for a suitable subsequence. However,

$$
\sum_{j=k}^{\infty} c_{j k} \hat{w}_{j}(t, \omega) \in U_{k}(t, \omega) \subseteq A(u(t, \omega), t, \omega)+B\left(0, \frac{2}{k}\right)
$$

Thus $w(t, \omega) \in A(u(t, \omega), t, \omega)$ a.e. $(t, \omega)$ because $A(u(t, \omega), t, \omega)$ is a closed set. Since $w$ is the pointwise limit of measurable functions off a set of measure zero, it can be assumed measurable and for a.e. $(t, \omega), w(t, \omega) \in A(u(t, \omega), t, \omega) \cap \overline{G(t, \omega)}$. Now denote this measurable function $w_{n}$. Then

$$
w_{n}(t, \omega) \in A(u(t, \omega), t, \omega),\left\langle w_{n}(t, \omega), u(t, \omega)-y(t, \omega)\right\rangle \leq \alpha(t, \omega)+\frac{1}{n} \text { a.e. }(t, \omega)
$$

These $w_{n}(t, \omega)$ are bounded for each $(t, \omega)$ off a set of measure zero and so by Lemma 48.7.1, there is a $\mathscr{P}$ measurable function $(t, \omega) \rightarrow z(t, \omega)$ and a subsequence $w_{n(t, \omega)}(t, \omega) \rightarrow$ $z(t, \omega)$ weakly as $n(t, \omega) \rightarrow \infty$. Now $A(u(t, \omega), t, \omega)$ is closed and convex, and $w_{n(t, \omega)}(t, \omega)$ is in $A(u(t, \omega), t, \omega)$, and so $z(t, \omega) \in A(u(t, \omega), t, \omega)$ and

$$
\begin{equation*}
\langle z(t, \omega), u(t, \omega)-y(t, \omega)\rangle \leq \alpha(t, \omega)=\liminf _{k \rightarrow \infty}\left\langle z_{n_{k}}(t, \omega), u_{n_{k}}(t, \omega)-y(t, \omega)\right\rangle \tag{**}
\end{equation*}
$$

Therefore, $t \rightarrow F(t, \omega)$ has a measurable selection on $S_{\gamma}$ excluding a set of measure zero, namely $z(t, \omega)$ which will be called $z_{\gamma}(t, \omega)$ in what follows.

Then $F(t, \omega)$ has a measurable selection on $[0, T] \times \Omega$ other than a set of measure zero. To see this, enlarge $\Sigma$ to include the exceptional sets of measure zero in the above
argument for each $\gamma$. Then partition $[0, T] \times \Omega \backslash \Sigma$ as follows. For $\gamma=1,2, \cdots$, consider $S_{\gamma} \backslash S_{\gamma-1}, \gamma=1,2, \cdots$ for $S_{0}$ defined as $\emptyset$. Then letting $z_{\gamma}$ be the selection for $(t, \omega) \in S_{\gamma}$, let $z(t, \omega)=\sum_{\gamma=1}^{\infty} z_{\gamma}(t, \omega) \mathscr{X}_{S_{\gamma} \backslash S_{\gamma-1}}(t, \omega)$. The estimates imply $z \in \mathscr{V}^{\prime}$ and so $z \in \hat{A}(u)$.

From the estimates, there exists $h \in L^{1}([0, T] \times \Omega)$ such that

$$
\langle z(t, \omega), u(t, \omega)-y(t, \omega)\rangle \geq-|h(t, \omega)|
$$

Thus, from the above inequality,

$$
\begin{gathered}
\|h\|_{L^{1}}+\langle z, u-y\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \\
\leq \int_{\Omega} \int_{0}^{T} \lim \inf _{k \rightarrow \infty}\left\langle z_{n_{k}}(t, \omega), u_{n_{k}}(t, \omega)-y(t, \omega)\right\rangle+|h(t, \omega)| d t d P \\
\leq \liminf _{k \rightarrow \infty}\left\langle z_{n_{k}}, u_{n_{k}}-y\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}+\|h\|_{L^{1}} \\
=\lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-y\right\rangle_{\mathscr{V}^{1}, \mathscr{V}}+\|h\|_{L^{1}}
\end{gathered}
$$

which contradicts 48.7.42.
The difficulty with this, is that it is hard to get the hypotheses holding. If you have $u_{n} \rightarrow u$ weakly in $\mathscr{V}$, then how do you get $u_{n}(t, \omega) \rightarrow u(t, \omega)$ weakly in $U^{\prime}$, for a subsequence, this for each $\omega$ not in a set of measure zero? The weak convergence does not seem to give pointwise weak convergence of the sort you need. More precisely, the pointwise convergence you get, might not be the right thing because in $u_{n}$ it will be $u_{n(\omega)}$. This is the problem with this theorem. It seems correct but not very useful.

## Part V

## Complex Analysis

## Chapter 49

## The Complex Numbers

The reader is presumed familiar with the algebraic properties of complex numbers, including the operation of conjugation. Here a short review of the distance in $\mathbb{C}$ is presented.

The length of a complex number, referred to as the modulus of $z$ and denoted by $|z|$ is given by

$$
|z| \equiv\left(x^{2}+y^{2}\right)^{1 / 2}=(z \bar{z})^{1 / 2},
$$

Then $\mathbb{C}$ is a metric space with the distance between two complex numbers, $z$ and $w$ defined as

$$
d(z, w) \equiv|z-w|
$$

This metric on $\mathbb{C}$ is the same as the usual metric of $\mathbb{R}^{2}$. A sequence, $z_{n} \rightarrow z$ if and only if $x_{n} \rightarrow x$ in $\mathbb{R}$ and $y_{n} \rightarrow y$ in $\mathbb{R}$ where $z=x+i y$ and $z_{n}=x_{n}+i y_{n}$. For example if $z_{n}=\frac{n}{n+1}+i \frac{1}{n}$, then $z_{n} \rightarrow 1+0 i=1$.

Definition 49.0.1 A sequence of complex numbers, $\left\{z_{n}\right\}$ is a Cauchy sequence if for every $\varepsilon>0$ there exists $N$ such that $n, m>N$ implies $\left|z_{n}-z_{m}\right|<\varepsilon$.

This is the usual definition of Cauchy sequence. There are no new ideas here.
Proposition 49.0.2 The complex numbers with the norm just mentioned forms a complete normed linear space.

Proof: Let $\left\{z_{n}\right\}$ be a Cauchy sequence of complex numbers with $z_{n}=x_{n}+i y_{n}$. Then $\left\{x_{n}\right\}$ and $\left\{y_{n}\right\}$ are Cauchy sequences of real numbers and so they converge to real numbers, $x$ and $y$ respectively. Thus $z_{n}=x_{n}+i y_{n} \rightarrow x+i y . \mathbb{C}$ is a linear space with the field of scalars equal to $\mathbb{C}$. It only remains to verify that $|\mid$ satisfies the axioms of a norm which are:

$$
\begin{gathered}
|z+w| \leq|z|+|w| \\
|z| \geq 0 \text { for all } z \\
|z|=0 \text { if and only if } z=0 \\
|\alpha z|=|\alpha||z|
\end{gathered}
$$

The only one of these axioms of a norm which is not completely obvious is the first one, the triangle inequality. Let $z=x+i y$ and $w=u+i v$

$$
\begin{aligned}
|z+w|^{2} & =(z+w)(\bar{z}+\bar{w})=|z|^{2}+|w|^{2}+2 \operatorname{Re}(z \bar{w}) \\
& \leq|z|^{2}+|w|^{2}+2|(z \bar{w})|=(|z|+|w|)^{2}
\end{aligned}
$$

and this verifies the triangle inequality.
Definition 49.0.3 An infinite sum of complex numbers is defined as the limit of the sequence of partial sums. Thus,

$$
\sum_{k=1}^{\infty} a_{k} \equiv \lim _{n \rightarrow \infty} \sum_{k=1}^{n} a_{k}
$$

Just as in the case of sums of real numbers, an infinite sum converges if and only if the sequence of partial sums is a Cauchy sequence.

From now on, when $f$ is a function of a complex variable, it will be assumed that $f$ has values in $X$, a complex Banach space. Usually in complex analysis courses, $f$ has values in $\mathbb{C}$ but there are many important theorems which don't require this so I will leave it fairly general for a while. Later the functions will have values in $\mathbb{C}$. If you are only interested in this case, think $\mathbb{C}$ whenever you see $X$.

Definition 49.0.4 A sequence of functions of a complex variable, $\left\{f_{n}\right\}$ converges uniformly to a function, $g$ for $z \in S$ iffor every $\varepsilon>0$ there exists $N_{\varepsilon}$ such that if $n>N_{\varepsilon}$, then

$$
\left\|f_{n}(z)-g(z)\right\|<\varepsilon
$$

for all $z \in S$. The infinite sum $\sum_{k=1}^{\infty} f_{n}$ converges uniformly on $S$ if the partial sums converge uniformly on $S$. Here $\|\cdot\|$ refers to the norm in $X$, the Banach space in which $f$ has its values.

The following proposition is also a routine application of the above definition. Neither the definition nor this proposition say anything new.

Proposition 49.0.5 A sequence of functions, $\left\{f_{n}\right\}$ defined on a set $S$, converges uniformly to some function, $g$ if and only iffor all $\varepsilon>0$ there exists $N_{\varepsilon}$ such that whenever $m, n>N_{\varepsilon}$,

$$
\left\|f_{n}-f_{m}\right\|_{\infty}<\varepsilon
$$

Here $\|f\|_{\infty} \equiv \sup \{\|f(z)\|: z \in S\}$.
Just as in the case of functions of a real variable, one of the important theorems is the Weierstrass M test. Again, there is nothing new here. It is just a review of earlier material.

Theorem 49.0.6 Let $\left\{f_{n}\right\}$ be a sequence of complex valued functions defined on $S \subseteq \mathbb{C}$. Suppose there exists $M_{n}$ such that $\left\|f_{n}\right\|_{\infty}<M_{n}$ and $\sum M_{n}$ converges. Then $\sum f_{n}$ converges uniformly on $S$.

Proof: Let $z \in S$. Then letting $m<n$

$$
\left\|\sum_{k=1}^{n} f_{k}(z)-\sum_{k=1}^{m} f_{k}(z)\right\| \leq \sum_{k=m+1}^{n}\left\|f_{k}(z)\right\| \leq \sum_{k=m+1}^{\infty} M_{k}<\varepsilon
$$

whenever $m$ is large enough. Therefore, the sequence of partial sums is uniformly Cauchy on $S$ and therefore, converges uniformly to $\sum_{k=1}^{\infty} f_{k}(z)$ on $S$.

### 49.1 The Extended Complex Plane

The set of complex numbers has already been considered along with the topology of $\mathbb{C}$ which is nothing but the topology of $\mathbb{R}^{2}$. Thus, for $z_{n}=x_{n}+i y_{n}, z_{n} \rightarrow z \equiv x+i y$ if and only if $x_{n} \rightarrow x$ and $y_{n} \rightarrow y$. The norm in $\mathbb{C}$ is given by

$$
|x+i y| \equiv((x+i y)(x-i y))^{1 / 2}=\left(x^{2}+y^{2}\right)^{1 / 2}
$$

which is just the usual norm in $\mathbb{R}^{2}$ identifying $(x, y)$ with $x+i y$. Therefore, $\mathbb{C}$ is a complete metric space topologically like $\mathbb{R}^{2}$ and so the Heine Borel theorem that compact sets are those which are closed and bounded is valid. Thus, as far as topology is concerned, there is nothing new about $\mathbb{C}$.

The extended complex plane, denoted by $\widehat{\mathbb{C}}$, consists of the complex plane, $\mathbb{C}$ along with another point not in $\mathbb{C}$ known as $\infty$. For example, $\infty$ could be any point in $\mathbb{R}^{3}$ with nonzero third component. A sequence of complex numbers, $z_{n}$, converges to $\infty$ if, whenever $K$ is a compact set in $\mathbb{C}$, there exists a number, $N$ such that for all $n>N, z_{n} \notin K$. Since compact sets in $\mathbb{C}$ are closed and bounded, this is equivalent to saying that for all $R>0$, there exists $N$ such that if $n>N$, then $z_{n} \notin B(0, R)$ which is the same as saying $\lim _{n \rightarrow \infty}\left|z_{n}\right|=\infty$ where this last symbol has the same meaning as it does in calculus.

A geometric way of understanding this in terms of more familiar objects involves a concept known as the Riemann sphere.

Consider the unit sphere, $S^{2}$ given by $(z-1)^{2}+y^{2}+x^{2}=1$. Define a map from the complex plane to the surface of this sphere as follows. Extend a line from the point, $p$ in the complex plane to the point $(0,0,2)$ on the top of this sphere and let $\theta(p)$ denote the point of this sphere which the line intersects. Define $\theta(\infty) \equiv(0,0,2)$.


Then $\theta^{-1}$ is sometimes called sterographic projection. The mapping $\theta$ is clearly continuous because it takes converging sequences, to converging sequences. Furthermore, it is clear that $\theta^{-1}$ is also continuous. In terms of the extended complex plane, $\widehat{\mathbb{C}}$, a sequence, $z_{n}$ converges to $\infty$ if and only if $\theta z_{n}$ converges to $(0,0,2)$ and a sequence, $z_{n}$ converges to $z \in \mathbb{C}$ if and only if $\theta\left(z_{n}\right) \rightarrow \theta(z)$.

In fact this makes it easy to define a metric on $\widehat{\mathbb{C}}$.

Definition 49.1.1 Let $z, w \in \widehat{\mathbb{C}}$ including possibly $w=\infty$. Then let $d(x, w) \equiv|\theta(z)-\theta(w)|$ where this last distance is the usual distance measured in $\mathbb{R}^{3}$.

Theorem 49.1.2 $(\widehat{\mathbb{C}}, d)$ is a compact, hence complete metric space.

Proof: Suppose $\left\{z_{n}\right\}$ is a sequence in $\widehat{\mathbb{C}}$. This means $\left\{\theta\left(z_{n}\right)\right\}$ is a sequence in $S^{2}$ which is compact. Therefore, there exists a subsequence, $\left\{\theta z_{n_{k}}\right\}$ and a point, $z \in S^{2}$ such that $\theta z_{n_{k}} \rightarrow \theta z$ in $S^{2}$ which implies immediately that $d\left(z_{n_{k}}, z\right) \rightarrow 0$. A compact metric space must be complete.

### 49.2 Exercises

1. Prove the root test for series of complex numbers. If $a_{k} \in \mathbb{C}$ and $r \equiv \lim \sup _{n \rightarrow \infty}\left|a_{n}\right|^{1 / n}$ then

$$
\sum_{k=0}^{\infty} a_{k}\left\{\begin{array}{l}
\text { converges absolutely if } r<1 \\
\text { diverges if } r>1 \\
\text { test fails if } r=1
\end{array}\right.
$$

2. Does $\lim _{n \rightarrow \infty} n\left(\frac{2+i}{3}\right)^{n}$ exist? Tell why and find the limit if it does exist.
3. Let $A_{0}=0$ and let $A_{n} \equiv \sum_{k=1}^{n} a_{k}$ if $n>0$. Prove the partial summation formula,

$$
\sum_{k=p}^{q} a_{k} b_{k}=A_{q} b_{q}-A_{p-1} b_{p}+\sum_{k=p}^{q-1} A_{k}\left(b_{k}-b_{k+1}\right)
$$

Now using this formula, suppose $\left\{b_{n}\right\}$ is a sequence of real numbers which converges to 0 and is decreasing. Determine those values of $\omega$ such that $|\omega|=1$ and $\sum_{k=1}^{\infty} b_{k} \omega^{k}$ converges.
4. Let $f: U \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be given by $f(x+i y)=u(x, y)+i v(x, y)$. Show $f$ is continuous on $U$ if and only if $u: U \rightarrow \mathbb{R}$ and $v: U \rightarrow \mathbb{R}$ are both continuous.

## Chapter 50

## Riemann Stieltjes Integrals

In the theory of functions of a complex variable, the most important results are those involving contour integration. I will base this on the notion of Riemann Stieltjes integrals as in [32], [95], and [65]. The Riemann Stieltjes integral is a generalization of the usual Riemann integral and requires the concept of a function of bounded variation.

Definition 50.0.1 Let $\gamma:[a, b] \rightarrow \mathbb{C}$ be a function. Then $\gamma$ is of bounded variation if

$$
\sup \left\{\sum_{i=1}^{n}\left|\gamma\left(t_{i}\right)-\gamma\left(t_{i-1}\right)\right|: a=t_{0}<\cdots<t_{n}=b\right\} \equiv V(\gamma,[a, b])<\infty
$$

where the sums are taken over all possible lists, $\left\{a=t_{0}<\cdots<t_{n}=b\right\}$. The set of points $\gamma([a, b])$ will also be denoted by $\gamma^{*}$.

The idea is that it makes sense to talk of the length of the curve $\gamma([a, b])$, defined as $V(\gamma,[a, b])$. For this reason, in the case that $\gamma$ is continuous, such an image of a bounded variation function is called a rectifiable curve.

Definition 50.0.2 Let $\gamma:[a, b] \rightarrow \mathbb{C}$ be of bounded variation and let $f: \gamma^{*} \rightarrow X$. Letting $\mathscr{P} \equiv\left\{t_{0}, \cdots, t_{n}\right\}$ where $a=t_{0}<t_{1}<\cdots<t_{n}=b$, define

$$
\|\mathscr{P}\| \equiv \max \left\{\left|t_{j}-t_{j-1}\right|: j=1, \cdots, n\right\}
$$

and the Riemann Steiltjes sum by

$$
S(\mathscr{P}) \equiv \sum_{j=1}^{n} f\left(\gamma\left(\tau_{j}\right)\right)\left(\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right)
$$

where $\tau_{j} \in\left[t_{j-1}, t_{j}\right]$. (Note this notation is a little sloppy because it does not identify the specific point, $\tau_{j}$ used. It is understood that this point is arbitrary.) Define $\int_{\gamma} f d \gamma$ as the unique number which satisfies the following condition. For all $\varepsilon>0$ there exists a $\delta>0$ such that if $\|\mathscr{P}\| \leq \delta$, then

$$
\left|\int_{\gamma} f d \gamma-S(\mathscr{P})\right|<\varepsilon
$$

Sometimes this is written as

$$
\int_{\gamma} f d \gamma \equiv \lim _{\|\mathscr{P}\| \rightarrow 0} S(\mathscr{P})
$$

The set of points in the curve, $\gamma([a, b])$ will be denoted sometimes by $\gamma^{*}$.
Then $\gamma^{*}$ is a set of points in $\mathbb{C}$ and as $t$ moves from $a$ to $b, \gamma(t)$ moves from $\gamma(a)$ to $\gamma(b)$. Thus $\gamma^{*}$ has a first point and a last point. If $\phi:[c, d] \rightarrow[a, b]$ is a continuous nondecreasing function, then $\gamma \circ \phi:[c, d] \rightarrow \mathbb{C}$ is also of bounded variation and yields the same set of points in $\mathbb{C}$ with the same first and last points.

Theorem 50.0.3 Let $\phi$ and $\gamma$ be as just described. Then assuming that

$$
\int_{\gamma} f d \gamma
$$

exists, so does

$$
\int_{\gamma \circ \phi} f d(\gamma \circ \phi)
$$

and

$$
\begin{equation*}
\int_{\gamma} f d \gamma=\int_{\gamma \circ \phi} f d(\gamma \circ \phi) \tag{50.0.1}
\end{equation*}
$$

Proof: There exists $\delta>0$ such that if $\mathscr{P}$ is a partition of $[a, b]$ such that $\|\mathscr{P}\|<\delta$, then

$$
\left|\int_{\gamma} f d \gamma-S(\mathscr{P})\right|<\varepsilon
$$

By continuity of $\phi$, there exists $\sigma>0$ such that if $\mathscr{Q}$ is a partition of $[c, d]$ with $\|\mathscr{Q}\|<$ $\sigma, \mathscr{Q}=\left\{s_{0}, \cdots, s_{n}\right\}$, then $\left|\phi\left(s_{j}\right)-\phi\left(s_{j-1}\right)\right|<\delta$. Thus letting $\mathscr{P}$ denote the points in $[a, b]$ given by $\phi\left(s_{j}\right)$ for $s_{j} \in \mathscr{Q}$, it follows that $\|\mathscr{P}\|<\delta$ and so

$$
\left|\int_{\gamma} f d \gamma-\sum_{j=1}^{n} f\left(\gamma\left(\phi\left(\tau_{j}\right)\right)\right)\left(\gamma\left(\phi\left(s_{j}\right)\right)-\gamma\left(\phi\left(s_{j-1}\right)\right)\right)\right|<\varepsilon
$$

where $\tau_{j} \in\left[s_{j-1}, s_{j}\right]$. Therefore, from the definition 50.0.1 holds and

$$
\int_{\gamma \circ \phi} f d(\gamma \circ \phi)
$$

exists.
This theorem shows that $\int_{\gamma} f d \gamma$ is independent of the particular $\gamma$ used in its computation to the extent that if $\phi$ is any nondecreasing continuous function from another interval, $[c, d]$, mapping to $[a, b]$, then the same value is obtained by replacing $\gamma$ with $\gamma \circ \phi$.

The fundamental result in this subject is the following theorem. We have in mind functions which have values in $\mathbb{C}$ but there is no change if the functions have values in any complete normed vector space.

Theorem 50.0.4 Let $f: \gamma^{*} \rightarrow X$ be continuous and let $\gamma:[a, b] \rightarrow \mathbb{C}$ be continuous and of bounded variation. Then $\int_{\gamma} f d \gamma$ exists. Also letting $\delta_{m}>0$ be such that $|t-s|<\delta_{m}$ implies $\|f(\gamma(t))-f(\gamma(s))\|<\frac{1}{m}$,

$$
\left|\int_{\gamma} f d \gamma-S(\mathscr{P})\right| \leq \frac{2 V(\gamma,[a, b])}{m}
$$

whenever $\|\mathscr{P}\|<\delta_{m}$.

Proof: The function, $f \circ \gamma$, is uniformly continuous because it is defined on a compact set. Therefore, there exists a decreasing sequence of positive numbers, $\left\{\boldsymbol{\delta}_{m}\right\}$ such that if $|s-t|<\delta_{m}$, then

$$
|f(\gamma(t))-f(\gamma(s))|<\frac{1}{m}
$$

Let

$$
F_{m} \equiv \overline{\left\{S(\mathscr{P}):\|\mathscr{P}\|<\delta_{m}\right\}} .
$$

Thus $F_{m}$ is a closed set. (The symbol, $S(\mathscr{P})$ in the above definition, means to include all sums corresponding to $\mathscr{P}$ for any choice of $\tau_{j}$.) It is shown that

$$
\begin{equation*}
\operatorname{diam}\left(F_{m}\right) \leq \frac{2 V(\gamma,[a, b])}{m} \tag{50.0.2}
\end{equation*}
$$

and then it will follow there exists a unique point, $I \in \cap_{m=1}^{\infty} F_{m}$. This is because $X$ is complete. It will then follow $I=\int_{\gamma} f(t) d \gamma(t)$. To verify 50.0 .2 , it suffices to verify that whenever $\mathscr{P}$ and $\mathscr{Q}$ are partitions satisfying $\|\mathscr{P}\|<\delta_{m}$ and $\|\mathscr{Q}\|<\delta_{m}$,

$$
\begin{equation*}
|S(\mathscr{P})-S(\mathscr{Q})| \leq \frac{2}{m} V(\gamma,[a, b]) \tag{50.0.3}
\end{equation*}
$$

Suppose $\|\mathscr{P}\|<\delta_{m}$ and $\mathscr{Q} \supseteq \mathscr{P}$. Then also $\|\mathscr{Q}\|<\delta_{m}$. To begin with, suppose that $\mathscr{P} \equiv\left\{t_{0}, \cdots, t_{p}, \cdots, t_{n}\right\}$ and $\mathscr{Q} \equiv\left\{t_{0}, \cdots, t_{p-1}, t^{*}, t_{p}, \cdots, t_{n}\right\}$. Thus $\mathscr{Q}$ contains only one more point than $\mathscr{P}$. Letting $S(\mathscr{Q})$ and $S(\mathscr{P})$ be Riemann Steiltjes sums,

$$
\begin{gathered}
S(\mathscr{Q}) \equiv \sum_{j=1}^{p-1} f\left(\gamma\left(\sigma_{j}\right)\right)\left(\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right)+f\left(\gamma\left(\sigma_{*}\right)\right)\left(\gamma\left(t^{*}\right)-\gamma\left(t_{p-1}\right)\right) \\
+f\left(\gamma\left(\sigma^{*}\right)\right)\left(\gamma\left(t_{p}\right)-\gamma\left(t^{*}\right)\right)+\sum_{j=p+1}^{n} f\left(\gamma\left(\sigma_{j}\right)\right)\left(\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right) \\
S(\mathscr{P}) \equiv \sum_{j=1}^{p-1} f\left(\gamma\left(\tau_{j}\right)\right)\left(\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right)+ \\
\overbrace{f\left(\gamma\left(\tau_{p}\right)\right)\left(\gamma\left(t^{*}\right)-\gamma\left(t_{p-1}\right)\right)+f\left(\gamma\left(\tau_{p}\right)\right)\left(\gamma\left(t_{p}\right)-\gamma\left(t^{*}\right)\right)}^{\left.=f\left(\tau_{p}\right)\right)\left(\gamma\left(t_{p}\right)-\gamma\left(t_{p-1}\right)\right)} \\
\quad+\sum_{j=p+1}^{n} f\left(\gamma\left(\tau_{j}\right)\right)\left(\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right) .
\end{gathered}
$$

Therefore,

$$
\begin{gather*}
|S(\mathscr{P})-S(\mathscr{Q})| \leq \sum_{j=1}^{p-1} \frac{1}{m}\left|\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right|+\frac{1}{m}\left|\gamma\left(t^{*}\right)-\gamma\left(t_{p-1}\right)\right|+ \\
\frac{1}{m}\left|\gamma\left(t_{p}\right)-\gamma\left(t^{*}\right)\right|+\sum_{j=p+1}^{n} \frac{1}{m}\left|\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right| \leq \frac{1}{m} V(\gamma,[a, b]) \tag{50.0.4}
\end{gather*}
$$

Clearly the extreme inequalities would be valid in 50.0 .4 if $\mathscr{Q}$ had more than one extra point. You simply do the above trick more than one time. Let $S(\mathscr{P})$ and $S(\mathscr{Q})$ be Riemann Steiltjes sums for which $\|\mathscr{P}\|$ and $\|\mathscr{Q}\|$ are less than $\delta_{m}$ and let $\mathscr{R} \equiv \mathscr{P} \cup \mathscr{Q}$. Then from what was just observed,

$$
|S(\mathscr{P})-S(\mathscr{Q})| \leq|S(\mathscr{P})-S(\mathscr{R})|+|S(\mathscr{R})-S(\mathscr{Q})| \leq \frac{2}{m} V(\gamma,[a, b])
$$

and this shows 50.0.3 which proves 50.0.2. Therefore, there exists a unique complex number, $I \in \cap_{m=1}^{\infty} F_{m}$ which satisfies the definition of $\int_{\gamma} f d \gamma$. This proves the theorem.

The following theorem follows easily from the above definitions and theorem.
Theorem 50.0.5 Let $f \in C\left(\gamma^{*}\right)$ and let $\gamma:[a, b] \rightarrow \mathbb{C}$ be of bounded variation and continuous. Let

$$
\begin{equation*}
M \geq \max \{\|f \circ \gamma(t)\|: t \in[a, b]\} \tag{50.0.5}
\end{equation*}
$$

Then

$$
\begin{equation*}
\left\|\int_{\gamma} f d \gamma\right\| \leq M V(\gamma,[a, b]) \tag{50.0.6}
\end{equation*}
$$

Also if $\left\{f_{n}\right\}$ is a sequence of functions of $C\left(\gamma^{*}\right)$ which is converging uniformly to the function, $f$ on $\gamma^{*}$, then

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \int_{\gamma} f_{n} d \gamma=\int_{\gamma} f d \gamma \tag{50.0.7}
\end{equation*}
$$

Proof: Let 50.0.5 hold. From the proof of the above theorem, when $\|\mathscr{P}\|<\delta_{m}$,

$$
\left\|\int_{\gamma} f d \gamma-S(\mathscr{P})\right\| \leq \frac{2}{m} V(\gamma,[a, b])
$$

and so

$$
\begin{aligned}
& \left\|\int_{\gamma} f d \gamma\right\| \leq\|S(\mathscr{P})\|+\frac{2}{m} V(\gamma,[a, b]) \\
\leq & \sum_{j=1}^{n} M\left|\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right|+\frac{2}{m} V(\gamma,[a, b]) \\
\leq & M V(\gamma,[a, b])+\frac{2}{m} V(\gamma,[a, b])
\end{aligned}
$$

This proves 50.0 .6 since $m$ is arbitrary. To verify 50.0 .7 use the above inequality to write

$$
\begin{aligned}
\left\|\int_{\gamma} f d \gamma-\int_{\gamma} f_{n} d \gamma\right\|=\left\|\int_{\gamma}\left(f-f_{n}\right) d \gamma(t)\right\| \\
\leq \max \left\{\left\|f \circ \gamma(t)-f_{n} \circ \gamma(t)\right\|: t \in[a, b]\right\} V(\gamma,[a, b]) .
\end{aligned}
$$

Since the convergence is assumed to be uniform, this proves 50.0.7.
It turns out to be much easier to evaluate such integrals in the case where $\gamma$ is also $C^{1}([a, b])$. The following theorem about approximation will be very useful but first here is an easy lemma.

Lemma 50.0.6 Let $\gamma:[a, b] \rightarrow \mathbb{C}$ be in $C^{1}([a, b])$. Then $V(\gamma,[a, b])<\infty$ so $\gamma$ is of bounded variation.

Proof: This follows from the following

$$
\begin{aligned}
\sum_{j=1}^{n}\left|\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right| & =\sum_{j=1}^{n}\left|\int_{t_{j-1}}^{t_{j}} \gamma^{\prime}(s) d s\right| \\
& \leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_{j}}\left|\gamma^{\prime}(s)\right| d s \\
& \leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_{j}}\left\|\gamma^{\prime}\right\|_{\infty} d s \\
& =\left\|\gamma^{\prime}\right\|_{\infty}(b-a)
\end{aligned}
$$

Therefore it follows $V(\gamma,[a, b]) \leq\left\|\gamma^{\prime}\right\|_{\infty}(b-a)$. Here $\|\gamma\|_{\infty}=\max \{|\gamma(t)|: t \in[a, b]\}$.
Theorem 50.0.7 Let $\gamma:[a, b] \rightarrow \mathbb{C}$ be continuous and of bounded variation. Let $\Omega$ be an open set containing $\gamma^{*}$ and let $f: \Omega \times K \rightarrow X$ be continuous for $K$ a compact set in $\mathbb{C}$, and let $\varepsilon>0$ be given. Then there exists $\eta:[a, b] \rightarrow \mathbb{C}$ such that $\eta(a)=\gamma(a), \gamma(b)=\eta(b)$, $\eta \in C^{1}([a, b])$, and

$$
\begin{gather*}
\|\gamma-\eta\|<\varepsilon  \tag{50.0.8}\\
\left|\int_{\gamma} f(\cdot, z) d \gamma-\int_{\eta} f(\cdot, z) d \eta\right|<\varepsilon  \tag{50.0.9}\\
V(\eta,[a, b]) \leq V(\gamma,[a, b]) \tag{50.0.10}
\end{gather*}
$$

where $\|\gamma-\eta\| \equiv \max \{|\gamma(t)-\eta(t)|: t \in[a, b]\}$.
Proof: Extend $\gamma$ to be defined on all $\mathbb{R}$ according to $\gamma(t)=\gamma(a)$ if $t<a$ and $\gamma(t)=\gamma(b)$ if $t>b$. Now define

$$
\gamma_{h}(t) \equiv \frac{1}{2 h} \int_{-2 h+t+\frac{2 h}{(b-a)}(t-a)}^{t+\frac{2 h}{(b-a)}(t-a)} \gamma(s) d s
$$

where the integral is defined in the obvious way. That is,

$$
\int_{a}^{b} \alpha(t)+i \beta(t) d t \equiv \int_{a}^{b} \alpha(t) d t+i \int_{a}^{b} \beta(t) d t
$$

Therefore,

$$
\begin{aligned}
& \gamma_{h}(b)=\frac{1}{2 h} \int_{b}^{b+2 h} \gamma(s) d s=\gamma(b) \\
& \gamma_{h}(a)=\frac{1}{2 h} \int_{a-2 h}^{a} \gamma(s) d s=\gamma(a)
\end{aligned}
$$

Also, because of continuity of $\gamma$ and the fundamental theorem of calculus,

$$
\gamma_{h}^{\prime}(t)=\frac{1}{2 h}\left\{\gamma\left(t+\frac{2 h}{b-a}(t-a)\right)\left(1+\frac{2 h}{b-a}\right)-\right.
$$

$$
\left.\gamma\left(-2 h+t+\frac{2 h}{b-a}(t-a)\right)\left(1+\frac{2 h}{b-a}\right)\right\}
$$

and so $\gamma_{h} \in C^{1}([a, b])$. The following lemma is significant.
Lemma 50.0.8 $V\left(\gamma_{h},[a, b]\right) \leq V(\gamma,[a, b])$.
Proof: Let $a=t_{0}<t_{1}<\cdots<t_{n}=b$. Then using the definition of $\gamma_{h}$ and changing the variables to make all integrals over $[0,2 h]$,

$$
\begin{gathered}
\sum_{j=1}^{n}\left|\gamma_{h}\left(t_{j}\right)-\gamma_{h}\left(t_{j-1}\right)\right|= \\
\sum_{j=1}^{n} \left\lvert\, \frac{1}{2 h} \int_{0}^{2 h}\left[\gamma\left(s-2 h+t_{j}+\frac{2 h}{b-a}\left(t_{j}-a\right)\right)-\right.\right. \\
\left.\gamma\left(s-2 h+t_{j-1}+\frac{2 h}{b-a}\left(t_{j-1}-a\right)\right)\right] \mid \\
\leq \frac{1}{2 h} \int_{0}^{2 h} \sum_{j=1}^{n} \left\lvert\, \gamma\left(s-2 h+t_{j}+\frac{2 h}{b-a}\left(t_{j}-a\right)\right)-\right. \\
\left.\gamma\left(s-2 h+t_{j-1}+\frac{2 h}{b-a}\left(t_{j-1}-a\right)\right) \right\rvert\, d s
\end{gathered}
$$

For a given $s \in[0,2 h]$, the points, $s-2 h+t_{j}+\frac{2 h}{b-a}\left(t_{j}-a\right)$ for $j=1, \cdots, n$ form an increasing list of points in the interval $[a-2 h, b+2 h]$ and so the integrand is bounded above by $V(\gamma,[a-2 h, b+2 h])=V(\gamma,[a, b])$. It follows

$$
\sum_{j=1}^{n}\left|\gamma_{h}\left(t_{j}\right)-\gamma_{h}\left(t_{j-1}\right)\right| \leq V(\gamma,[a, b])
$$

which proves the lemma.
With this lemma the proof of the theorem can be completed without too much trouble. Let $H$ be an open set containing $\gamma^{*}$ such that $\bar{H}$ is a compact subset of $\Omega$. Let $0<\varepsilon<$ $\operatorname{dist}\left(\gamma^{*}, H^{C}\right)$. Then there exists $\delta_{1}$ such that if $h<\delta_{1}$, then for all $t$,

$$
\begin{align*}
\left|\gamma(t)-\gamma_{h}(t)\right| & \leq \frac{1}{2 h} \int_{-2 h+t+\frac{2 h}{(b-a)}(t-a)}^{t+\frac{2 h}{(b-a)}(t-a)}|\gamma(s)-\gamma(t)| d s \\
& <\frac{1}{2 h} \int_{-2 h+t+\frac{2 h}{(b-a)}(t-a)}^{t+\frac{2 h}{(b-a)}(t-a)} \varepsilon d s=\varepsilon \tag{50.0.11}
\end{align*}
$$

due to the uniform continuity of $\gamma$. This proves 50.0.8.
From 50.0.2 and the above lemma, there exists $\boldsymbol{\delta}_{2}$ such that if $\|\mathscr{P}\|<\boldsymbol{\delta}_{2}$, then for all $z \in K$,

$$
\left\|\int_{\gamma} f(\cdot, z) d \gamma(t)-S(\mathscr{P})\right\|<\frac{\varepsilon}{3},\left\|\int_{\gamma_{h}} f(\cdot, z) d \gamma_{h}(t)-S_{h}(\mathscr{P})\right\|<\frac{\varepsilon}{3}
$$

for all $h$. Here $S(\mathscr{P})$ is a Riemann Steiltjes sum of the form

$$
\sum_{i=1}^{n} f\left(\gamma\left(\tau_{i}\right), z\right)\left(\gamma\left(t_{i}\right)-\gamma\left(t_{i-1}\right)\right)
$$

and $S_{h}(\mathscr{P})$ is a similar Riemann Steiltjes sum taken with respect to $\gamma_{h}$ instead of $\gamma$. Because of 50.0.11 $\gamma_{h}(t)$ has values in $H \subseteq \Omega$. Therefore, fix the partition, $\mathscr{P}$, and choose $h$ small enough that in addition to this, the following inequality is valid for all $z \in K$.

$$
\left|S(\mathscr{P})-S_{h}(\mathscr{P})\right|<\frac{\varepsilon}{3}
$$

This is possible because of 50.0 .11 and the uniform continuity of $f$ on $\bar{H} \times K$. It follows

$$
\begin{gathered}
\left\|\int_{\gamma} f(\cdot, z) d \gamma(t)-\int_{\gamma_{h}} f(\cdot, z) d \gamma_{h}(t)\right\| \leq \\
\left\|\int_{\gamma} f(\cdot, z) d \gamma(t)-S(\mathscr{P})\right\|+\left\|S(\mathscr{P})-S_{h}(\mathscr{P})\right\| \\
\quad+\left\|S_{h}(\mathscr{P})-\int_{\gamma_{h}} f(\cdot, z) d \gamma_{h}(t)\right\|<\varepsilon
\end{gathered}
$$

Formula 50.0.10 follows from the lemma. This proves the theorem.
Of course the same result is obtained without the explicit dependence of $f$ on $z$.
This is a very useful theorem because if $\gamma$ is $C^{1}([a, b])$, it is easy to calculate $\int_{\gamma} f d \gamma$ and the above theorem allows a reduction to the case where $\gamma$ is $C^{1}$. The next theorem shows how easy it is to compute these integrals in the case where $\gamma$ is $C^{1}$. First note that if $f$ is continuous and $\gamma \in C^{1}([a, b])$, then by Lemma 50.0.6 and the fundamental existence theorem, Theorem 50.0.4, $\int_{\gamma} f d \gamma$ exists.

Theorem 50.0.9 If $f: \gamma^{*} \rightarrow X$ is continuous and $\gamma:[a, b] \rightarrow \mathbb{C}$ is in $C^{1}([a, b])$, then

$$
\begin{equation*}
\int_{\gamma} f d \gamma=\int_{a}^{b} f(\gamma(t)) \gamma^{\prime}(t) d t \tag{50.0.12}
\end{equation*}
$$

Proof: Let $\mathscr{P}$ be a partition of $[a, b], \mathscr{P}=\left\{t_{0}, \cdots, t_{n}\right\}$ and $\|\mathscr{P}\|$ is small enough that whenever $|t-s|<\|\mathscr{P}\|$,

$$
\begin{equation*}
|f(\gamma(t))-f(\gamma(s))|<\varepsilon \tag{50.0.13}
\end{equation*}
$$

and

$$
\left\|\int_{\gamma} f d \gamma-\sum_{j=1}^{n} f\left(\gamma\left(\tau_{j}\right)\right)\left(\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right)\right\|<\varepsilon
$$

Now

$$
\sum_{j=1}^{n} f\left(\gamma\left(\tau_{j}\right)\right)\left(\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right)=\int_{a}^{b} \sum_{j=1}^{n} f\left(\gamma\left(\tau_{j}\right)\right) \mathscr{X}_{\left[t_{j-1}, t_{j}\right]}(s) \gamma^{\prime}(s) d s
$$

where here

$$
\mathscr{X}_{[a, b]}(s) \equiv\left\{\begin{array}{l}
1 \text { if } s \in[p, q] \\
0 \text { if } s \notin[p, q]
\end{array} .\right.
$$

Also,

$$
\int_{a}^{b} f(\gamma(s)) \gamma^{\prime}(s) d s=\int_{a}^{b} \sum_{j=1}^{n} f(\gamma(s)) \mathscr{X}_{\left[t_{j-1}, t_{j}\right]}(s) \gamma^{\prime}(s) d s
$$

and thanks to 50.0.13,

$$
\left.\begin{aligned}
& \|\overbrace{\int_{a}^{b} \sum_{j=1}^{n} f\left(\gamma\left(\tau_{j}\right)\right) \mathscr{X}_{\left[t_{j-1}, t_{j}\right]}(s) \gamma(s) d s}^{=\sum_{j=1}^{n} f\left(\gamma\left(\tau_{j}\right)\right)\left(\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right)}-\overbrace{\int_{a}^{b} \sum_{j=1}^{n} f(\gamma(s)) \mathscr{X}_{\left[t_{j-1}, t_{j}\right]}(s) \gamma^{\prime}(s) d s}^{=\int_{a}^{b} f(\gamma(s)) \gamma^{\prime}(s) d s}\|
\end{aligned} \right\rvert\,
$$

It follows that

$$
\begin{aligned}
& \left\|\int_{\gamma} f d \gamma-\int_{a}^{b} f(\gamma(s)) \gamma^{\prime}(s) d s\right\| \leq\left\|\int_{\gamma} f d \gamma-\sum_{j=1}^{n} f\left(\gamma\left(\tau_{j}\right)\right)\left(\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right)\right\| \\
+ & \left\|\sum_{j=1}^{n} f\left(\gamma\left(\tau_{j}\right)\right)\left(\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right)-\int_{a}^{b} f(\gamma(s)) \gamma^{\prime}(s) d s\right\| \leq \varepsilon\left\|\gamma^{\prime}\right\|_{\infty}(b-a)+\varepsilon
\end{aligned}
$$

Since $\varepsilon$ is arbitrary, this verifies 50.0.12.
Definition 50.0.10 Let $\Omega$ be an open subset of $\mathbb{C}$ and let $\gamma:[a, b] \rightarrow \Omega$ be a continuous function with bounded variation $f: \Omega \rightarrow X$ be a continuous function. Then the following notation is more customary.

$$
\int_{\gamma} f(z) d z \equiv \int_{\gamma} f d \gamma
$$

The expression, $\int_{\gamma} f(z) d z$, is called a contour integral and $\gamma$ is referred to as the contour. A function $f: \Omega \rightarrow X$ for $\Omega$ an open set in $\mathbb{C}$ has a primitive if there exists a function, $F$, the primitive, such that $F^{\prime}(z)=f(z)$. Thus $F$ is just an antiderivative. Also if $\gamma_{k}:\left[a_{k}, b_{k}\right] \rightarrow \mathbb{C}$ is continuous and of bounded variation, for $k=1, \cdots, m$ and $\gamma_{k}\left(b_{k}\right)=\gamma_{k+1}\left(a_{k}\right)$, define

$$
\begin{equation*}
\int_{\sum_{k=1}^{m} \gamma_{k}} f(z) d z \equiv \sum_{k=1}^{m} \int_{\gamma_{k}} f(z) d z \tag{50.0.14}
\end{equation*}
$$

In addition to this, for $\gamma:[a, b] \rightarrow \mathbb{C}$,
define $-\gamma:[a, b] \rightarrow \mathbb{C}$ by $-\gamma(t) \equiv \gamma(b+a-t)$. Thus $\gamma$ simply traces out the points of $\gamma^{*}$ in the opposite order.

The following lemma is useful and follows quickly from Theorem 50.0.3.
Lemma 50.0.11 In the above definition, there exists a continuous bounded variation function, $\gamma$ defined on some closed interval, $[c, d]$, such that $\gamma([c, d])=\cup_{k=1}^{m} \gamma_{k}\left(\left[a_{k}, b_{k}\right]\right)$ and $\gamma(c)=\gamma_{1}\left(a_{1}\right)$ while $\gamma(d)=\gamma_{m}\left(b_{m}\right)$. Furthermore,

$$
\int_{\gamma} f(z) d z=\sum_{k=1}^{m} \int_{\gamma_{k}} f(z) d z
$$

If $\gamma:[a, b] \rightarrow \mathbb{C}$ is of bounded variation and continuous, then

$$
\int_{\gamma} f(z) d z=-\int_{-\gamma} f(z) d z
$$

Re stating Theorem 50.0.7 with the new notation in the above definition,
Theorem 50.0.12 Let $K$ be a compact set in $\mathbb{C}$ and let $f: \Omega \times K \rightarrow X$ be continuous for $\Omega$ an open set in $\mathbb{C}$. Also let $\gamma:[a, b] \rightarrow \Omega$ be continuous with bounded variation. Then if $r>0$ is given, there exists $\eta:[a, b] \rightarrow \Omega$ such that $\eta(a)=\gamma(a), \eta(b)=\gamma(b), \eta$ is $C^{1}([a, b])$, and

$$
\left|\int_{\gamma} f(z, w) d z-\int_{\eta} f(z, w) d z\right|<r,\|\eta-\gamma\|<r
$$

It will be very important to consider which functions have primitives. It turns out, it is not enough for $f$ to be continuous in order to possess a primitive. This is in stark contrast to the situation for functions of a real variable in which the fundamental theorem of calculus will deliver a primitive for any continuous function. The reason for the interest in such functions is the following theorem and its corollary.

Theorem 50.0.13 Let $\gamma:[a, b] \rightarrow \mathbb{C}$ be continuous and of bounded variation. Also suppose $F^{\prime}(z)=f(z)$ for all $z \in \Omega$, an open set containing $\gamma^{*}$ and $f$ is continuous on $\Omega$. Then

$$
\int_{\gamma} f(z) d z=F(\gamma(b))-F(\gamma(a)) .
$$

Proof: By Theorem 50.0.12 there exists $\eta \in C^{1}([a, b])$ such that $\gamma(a)=\eta(a)$, and $\gamma(b)=\eta(b)$ such that

$$
\left\|\int_{\gamma} f(z) d z-\int_{\eta} f(z) d z\right\|<\varepsilon
$$

Then since $\eta$ is in $C^{1}([a, b])$,

$$
\begin{aligned}
\int_{\eta} f(z) d z & =\int_{a}^{b} f(\eta(t)) \eta^{\prime}(t) d t=\int_{a}^{b} \frac{d F(\eta(t))}{d t} d t \\
& =F(\eta(b))-F(\eta(a))=F(\gamma(b))-F(\gamma(a))
\end{aligned}
$$

Therefore,

$$
\left\|(F(\gamma(b))-F(\gamma(a)))-\int_{\gamma} f(z) d z\right\|<\varepsilon
$$

and since $\varepsilon>0$ is arbitrary, this proves the theorem.

Corollary 50.0.14 If $\gamma:[a, b] \rightarrow \mathbb{C}$ is continuous, has bounded variation, is a closed curve, $\gamma(a)=\gamma(b)$, and $\gamma^{*} \subseteq \Omega$ where $\Omega$ is an open set on which $F^{\prime}(z)=f(z)$, then

$$
\int_{\gamma} f(z) d z=0
$$

Another important result is a Fubini theorem for these contour integrals.
Theorem 50.0.15 Let $\gamma_{i}$ be continuous and bounded variation. Let $f$ be continuous on $\gamma_{1}^{*} \times \gamma_{2}^{*}$ having values in $X$ a complex complete normed linear space. Then

$$
\int_{\gamma_{1}} \int_{\gamma_{2}} f(z, w) d w d z=\int_{\gamma_{2}} \int_{\gamma_{1}} f(z, w) d z d w
$$

Proof: This follows quickly from the above lemma and the definition of the contour integral. Say $\gamma_{i}$ is defined on $\left[a_{i}, b_{i}\right]$. Let a partition of $\left[a_{1}, b_{1}\right]$ be denoted by $\left\{t_{0}, t_{1}, \cdots, t_{n}\right\}=$ $P_{1}$ and a partition of $\left[a_{2}, b_{2}\right]$ be denoted by $\left\{s_{0}, s_{1}, \cdots, s_{m}\right\}=P_{2}$.

$$
\begin{gathered}
\int_{\gamma_{1}} \int_{\gamma_{2}} f(z, w) d w d z=\sum_{i=1}^{n} \int_{\gamma_{1}\left(\left[t_{i-1}, t_{i}\right]\right)} \int_{\gamma_{2}} f(z, w) d w d z \\
=\sum_{i=1}^{n} \sum_{j=1}^{m} \int_{\gamma_{1}\left(\left[t_{i-1}, t_{i}\right]\right)} \int_{\gamma_{2}\left(\left[s_{j-1}, s_{j}\right]\right)} f(z, w) d w d z
\end{gathered}
$$

To save room, denote $\gamma_{1}\left(\left[t_{i-1}, t_{i}\right]\right)$ by $\gamma_{1 i}$ and $\gamma_{2}\left(\left[s_{j-1}, s_{j}\right]\right)$ by $\gamma_{2 j}$ Then if $\left\|P_{i}\right\|, i=1,2$ is small enough,

$$
\begin{gather*}
\left\|\int_{\gamma_{1 i}} \int_{\gamma_{2 j}} f(z, w) d w d z-\int_{\gamma_{1 i}} \int_{\gamma_{2 j}} f\left(\gamma_{1}\left(t_{i}\right), \gamma_{2}\left(s_{j}\right)\right) d w d z\right\| \\
=\left\|\int_{\gamma_{1 i}} \int_{\gamma_{2 j}}\left(f(z, w)-f\left(\gamma_{1}\left(t_{i}\right), \gamma_{2}\left(s_{j}\right)\right)\right) d w d z\right\| \leq \\
\max \left(\left\|\int_{\gamma_{2 j}}\left(f(z, w)-f\left(\gamma_{1}\left(t_{i}\right), \gamma_{2}\left(s_{j}\right)\right)\right) d w\right\|\right) V\left(\gamma_{1},\left[t_{i-1}, t_{i}\right]\right) \\
\leq \varepsilon V\left(\gamma_{2},\left[s_{j-1}, s_{j}\right]\right) V\left(\gamma_{1},\left[t_{i-1}, t_{i}\right]\right) \tag{50.0.15}
\end{gather*}
$$

Also from this theorem,

$$
\begin{gather*}
\left\|\int_{\gamma_{2 j}} \int_{\gamma_{1 i}} f(z, w) d z d w-\int_{\gamma_{2 j}} \int_{\gamma_{1 i}} f\left(\gamma_{1}\left(t_{i}\right), \gamma_{2}\left(s_{j}\right)\right) d z d w\right\| \\
\leq \max \left(\left\|\int_{\gamma_{1 i}}\left(f(z, w)-f\left(\gamma_{1}\left(t_{i}\right), \gamma_{2}\left(s_{j}\right)\right)\right) d z\right\|\right) V\left(\gamma_{2},\left[s_{j-1}, s_{j}\right]\right) \\
\leq \varepsilon V\left(\gamma_{2},\left[s_{j-1}, s_{j}\right]\right) V\left(\gamma_{1},\left[t_{i-1}, t_{i}\right]\right) \tag{50.0.16}
\end{gather*}
$$

Now approximating with sums and using the definition, $\int_{\gamma_{1 i}} d z=\gamma_{1}\left(t_{j}\right)-\gamma_{1}\left(t_{j-1}\right)$ and so

$$
\begin{aligned}
& \int_{\gamma_{2 j}} \int_{\gamma_{1 i}} f\left(\gamma_{1}\left(t_{i}\right), \gamma_{2}\left(s_{j}\right)\right) d z d w=f\left(\gamma_{1}\left(t_{i}\right), \gamma_{2}\left(s_{j}\right)\right) \int_{\gamma_{2 j}} \int_{\gamma_{1 i}} d z d w \\
= & f\left(\gamma_{1}\left(t_{i}\right), \gamma_{2}\left(s_{j}\right)\right) \int_{\gamma_{1 i}} \int_{\gamma_{2 j}} d w d z=\int_{\gamma_{1 i}} \int_{\gamma_{2 j}} f\left(\gamma_{1}\left(t_{i}\right), \gamma_{2}\left(s_{j}\right)\right) d w d z(50.0 .17)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \left\|\int_{\gamma_{1}} \int_{\gamma_{2}} f(z, w) d w d z-\int_{\gamma_{2}} \int_{\gamma_{1}} f(z, w) d z d w\right\| \leq \\
& \left\|\begin{array}{c}
\sum_{i=1}^{n} \sum_{j=1}^{m} \int_{\gamma_{1 i}} \int_{\gamma_{2 j}} f(z, w) d w d z \\
-\sum_{i=1}^{n} \sum_{j=1}^{m} \int_{\gamma_{1 i}} \int_{\gamma_{2 j}} f\left(\gamma_{1}\left(t_{i}\right), \gamma_{2}\left(s_{j}\right)\right) d w d z
\end{array}\right\| \\
& +\| \sum_{i=1}^{n} \sum_{j=1}^{m} \int_{\gamma_{1 i}} \int_{\gamma_{2 j}} f\left(\gamma_{1}\left(t_{i}\right), \gamma_{2}\left(s_{j}\right)\right) d w d z \\
& -\sum_{i=1}^{n} \sum_{j=1}^{m} \int_{\gamma_{2 j}} \int_{\gamma_{1 i}} f\left(\gamma_{1}\left(t_{i}\right), \gamma_{2}\left(s_{j}\right)\right) d z d w
\end{aligned} \| .
$$

From 50.0.17 the middle term is 0 . Thus, from the estimates 50.0.16 and 50.0.15,

$$
\begin{aligned}
& \left\|\int_{\gamma_{1}} \int_{\gamma_{2}} f(z, w) d w d z-\int_{\gamma_{2}} \int_{\gamma_{1}} f(z, w) d z d w\right\| \\
\leq & 2 \varepsilon V\left(\gamma_{2},\left[a_{2}, b_{2}\right]\right) V\left(\gamma_{1},\left[a_{1}, b_{1}\right]\right)
\end{aligned}
$$

Since $\varepsilon$ is arbitrary, the two integrals are equal.

### 50.1 Exercises

1. Let $\gamma:[a, b] \rightarrow \mathbb{R}$ be increasing. Show $V(\gamma,[a, b])=\gamma(b)-\gamma(a)$.
2. Suppose $\gamma:[a, b] \rightarrow \mathbb{C}$ satisfies a Lipschitz condition, $|\gamma(t)-\gamma(s)| \leq K|s-t|$. Show $\gamma$ is of bounded variation and that $V(\gamma,[a, b]) \leq K|b-a|$.
3. $\gamma:\left[c_{0}, c_{m}\right] \rightarrow \mathbb{C}$ is piecewise smooth if there exist numbers, $c_{k}, k=1, \cdots, m$ such that $c_{0}<c_{1}<\cdots<c_{m-1}<c_{m}$ such that $\gamma$ is continuous and $\gamma:\left[c_{k}, c_{k+1}\right] \rightarrow \mathbb{C}$ is $C^{1}$. Show that such piecewise smooth functions are of bounded variation and give an estimate for $V\left(\gamma,\left[c_{0}, c_{m}\right]\right)$.
4. Let $\gamma:[0,2 \pi] \rightarrow \mathbb{C}$ be given by $\gamma(t)=r(\cos m t+i \sin m t)$ for $m$ an integer. Find $\int_{\gamma} \frac{d z}{z}$.
5. Show that if $\gamma:[a, b] \rightarrow \mathbb{C}$ then there exists an increasing function $h:[0,1] \rightarrow[a, b]$ such that $\gamma \circ h([0,1])=\gamma^{*}$.
6. Let $\gamma:[a, b] \rightarrow \mathbb{C}$ be an arbitrary continuous curve having bounded variation and let $f, g$ have continuous derivatives on some open set containing $\gamma^{*}$. Prove the usual integration by parts formula.

$$
\int_{\gamma} f g^{\prime} d z=f(\gamma(b)) g(\gamma(b))-f(\gamma(a)) g(\gamma(a))-\int_{\gamma} f^{\prime} g d z
$$

7. Let $f(z) \equiv|z|^{-(1 / 2)} e^{-i \frac{\theta}{2}}$ where $z=|z| e^{i \theta}$. This function is called the principle branch of $z^{-(1 / 2)}$. Find $\int_{\gamma} f(z) d z$ where $\gamma$ is the semicircle in the upper half plane which goes from $(1,0)$ to $(-1,0)$ in the counter clockwise direction. Next do the integral in which $\gamma$ goes in the clockwise direction along the semicircle in the lower half plane.
8. Prove an open set, $U$ is connected if and only if for every two points in $U$, there exists a $C^{1}$ curve having values in $U$ which joins them.
9. Let $\mathscr{P}, \mathscr{Q}$ be two partitions of $[a, b]$ with $\mathscr{P} \subseteq \mathscr{Q}$. Each of these partitions can be used to form an approximation to $V(\gamma,[a, b])$ as described above. Recall the total variation was the supremum of sums of a certain form determined by a partition. How is the sum associated with $\mathscr{P}$ related to the sum associated with $\mathscr{Q}$ ? Explain.
10. Consider the curve,

$$
\gamma(t)=\left\{\begin{array}{l}
t+i t^{2} \sin \left(\frac{1}{t}\right) \text { if } t \in(0,1] \\
0 \text { if } t=0
\end{array}\right.
$$

Is $\gamma$ a continuous curve having bounded variation? What if the $t^{2}$ is replaced with $t$ ? Is the resulting curve continuous? Is it a bounded variation curve?
11. Suppose $\gamma:[a, b] \rightarrow \mathbb{R}$ is given by $\gamma(t)=t$. What is $\int_{\gamma} f(t) d \gamma$ ? Explain.

## Chapter 51

## Fundamentals Of Complex Analysis

### 51.1 Analytic Functions

Definition 51.1.1 Let $\Omega$ be an open set in $\mathbb{C}$ and let $f: \Omega \rightarrow X$. Then $f$ is analytic on $\Omega$ if for every $z \in \Omega$,

$$
\lim _{h \rightarrow 0} \frac{f(z+h)-f(z)}{h} \equiv f^{\prime}(z)
$$

exists and is a continuous function of $z \in \Omega$. Here $h \in \mathbb{C}$.
Note that if $f$ is analytic, it must be the case that $f$ is continuous. It is more common to not include the requirement that $f^{\prime}$ is continuous but it is shown later that the continuity of $f^{\prime}$ follows.

What are some examples of analytic functions? In the case where $X=\mathbb{C}$, the simplest example is any polynomial. Thus

$$
p(z) \equiv \sum_{k=0}^{n} a_{k} z^{k}
$$

is an analytic function and

$$
p^{\prime}(z)=\sum_{k=1}^{n} a_{k} k z^{k-1}
$$

More generally, power series are analytic. This will be shown soon but first here is an important definition and a convergence theorem called the root test.

Definition 51.1.2 Let $\left\{a_{k}\right\}$ be a sequence in $X$. Then $\sum_{k=1}^{\infty} a_{k} \equiv \lim _{n \rightarrow \infty} \sum_{k=1}^{n} a_{k}$ whenever this limit exists. When the limit exists, the series is said to converge.

Theorem 51.1.3 Consider $\sum_{k=1}^{\infty} a_{k}$ and let $\rho \equiv \limsup _{k \rightarrow \infty}\left\|a_{k}\right\|^{1 / k}$. Then if $\rho<1$, the series converges absolutely and if $\rho>1$ the series diverges spectacularly in the sense that $\lim _{k \rightarrow \infty} a_{k} \neq 0$. If $\rho=1$ the test fails. Also $\sum_{k=1}^{\infty} a_{k}(z-a)^{k}$ converges on some disk $B(a, R)$. It converges absolutely if $|z-a|<R$ and uniformly on $B\left(a, r_{1}\right)$ whenever $r_{1}<R$. The function $f(z)=\sum_{k=1}^{\infty} a_{k}(z-a)^{k}$ is continuous on $B(a, R)$.

Proof: Suppose $\rho<1$. Then there exists $r \in(\rho, 1)$. Therefore, $\left\|a_{k}\right\| \leq r^{k}$ for all $k$ large enough and so by a comparison test, $\sum_{k}\left\|a_{k}\right\|$ converges because the partial sums are bounded above. Therefore, the partial sums of the original series form a Cauchy sequence in $X$ and so they also converge due to completeness of $X$.

Now suppose $\rho>1$. Then letting $\rho>r>1$, it follows $\left\|a_{k}\right\|^{1 / k} \geq r$ infinitely often. Thus $\left\|a_{k}\right\| \geq r^{k}$ infinitely often. Thus there exists a subsequence for which $\left\|a_{n_{k}}\right\|$ converges to $\infty$. Therefore, the series cannot converge.

Now consider $\sum_{k=1}^{\infty} a_{k}(z-a)^{k}$. This series converges absolutely if

$$
\limsup _{k \rightarrow \infty}\left\|a_{k}\right\|^{1 / k}|z-a|<1
$$

which is the same as saying $|z-a|<1 / \rho$ where $\rho \equiv \limsup _{k \rightarrow \infty}\left\|a_{k}\right\|^{1 / k}$. Let $R=1 / \rho$.
Now suppose $r_{1}<R$. Consider $|z-a| \leq r_{1}$. Then for such $z$,

$$
\left\|a_{k}\right\||z-a|^{k} \leq\left\|a_{k}\right\| r_{1}^{k}
$$

and

$$
\lim \sup _{k \rightarrow \infty}\left(\left\|a_{k}\right\| r_{1}^{k}\right)^{1 / k}=\lim \sup _{k \rightarrow \infty}\left\|a_{k}\right\|^{1 / k} r_{1}=\frac{r_{1}}{R}<1
$$

so $\sum_{k}\left\|a_{k}\right\| r_{1}^{k}$ converges. By the Weierstrass $M$ test, $\sum_{k=1}^{\infty} a_{k}(z-a)^{k}$ converges uniformly for $|z-a| \leq r_{1}$. Therefore, $f$ is continuous on $B(a, R)$ as claimed because it is the uniform limit of continuous functions, the partial sums of the infinite series.

What if $\rho=0$ ? In this case,

$$
\lim \sup _{k \rightarrow \infty}\left\|a_{k}\right\|^{1 / k}|z-a|=0 \cdot|z-a|=0
$$

and so $R=\infty$ and the series, $\sum\left\|a_{k}\right\||z-a|^{k}$ converges everywhere.
What if $\rho=\infty$ ? Then in this case, the series converges only at $z=a$ because if $z \neq a$,

$$
\limsup _{k \rightarrow \infty}\left\|a_{k}\right\|^{1 / k}|z-a|=\infty
$$

Theorem 51.1.4 Let $f(z) \equiv \sum_{k=1}^{\infty} a_{k}(z-a)^{k}$ be given in Theorem 51.1.3 where $R>0$. Then $f$ is analytic on $B(a, R)$. So are all its derivatives.

Proof: Consider $g(z)=\sum_{k=2}^{\infty} a_{k} k(z-a)^{k-1}$ on $B(a, R)$ where $R=\rho^{-1}$ as above. Let
$r_{1}<r<R$. Then letting $|z-a|<r_{1}$ and $h<r-r_{1}$,

$$
\begin{aligned}
& \| \frac{f(z+h)-f(z)}{h}-g(z)| | \\
\leq & \sum_{k=2}^{\infty}\left\|a_{k}\right\|\left|\frac{(z+h-a)^{k}-(z-a)^{k}}{h}-k(z-a)^{k-1}\right| \\
\leq & \sum_{k=2}^{\infty}\left\|a_{k}\right\|\left|\frac{1}{h}\left(\sum_{i=0}^{k}\binom{k}{i}(z-a)^{k-i} h^{i}-(z-a)^{k}\right)-k(z-a)^{k-1}\right| \\
= & \sum_{k=2}^{\infty}\left\|a_{k}\right\|\left|\frac{1}{h}\left(\sum_{i=1}^{k}\binom{k}{i}(z-a)^{k-i} h^{i}\right)-k(z-a)^{k-1}\right| \\
\leq & \sum_{k=2}^{\infty} \| a_{k}| |\left|\left(\sum_{i=2}^{k}\binom{k}{i}(z-a)^{k-i} h^{i-1}\right)\right| \\
\leq & |h| \sum_{k=2}^{\infty} \| a_{k}| |\left(\sum_{i=0}^{k-2}\binom{k}{i+2}|z-a|^{k-2-i}|h|^{i}\right) \\
= & |h| \sum_{k=2}^{\infty} \| a_{k}| |\left(\sum_{i=0}^{k-2}\binom{k-2}{i} \frac{k(k-1)}{(i+2)(i+1)}|z-a|^{k-2-i}|h|^{i}\right) \\
\leq & |h| \sum_{k=2}^{\infty} \| a_{k}| | \frac{k(k-1)}{2}\left(\sum_{i=0}^{k-2}\binom{k-2}{i}|z-a|^{k-2-i}|h|^{i}\right) \\
= & |h| \sum_{k=2}^{\infty}\left\|a_{k}\right\| \frac{k(k-1)}{2}(|z-a|+|h|)^{k-2}<|h| \sum_{k=2}^{\infty}\left\|a_{k}\right\| \frac{k(k-1)}{2} r^{k-2} .
\end{aligned}
$$

Then

$$
\limsup _{k \rightarrow \infty}\left(\left\|a_{k}\right\| \frac{k(k-1)}{2} r^{k-2}\right)^{1 / k}=\rho r<1
$$

and so

$$
\left\|\frac{f(z+h)-f(z)}{h}-g(z)\right\| \leq C|h| .
$$

therefore, $g(z)=f^{\prime}(z)$. Now by Theorem 51.1.3 it also follows that $f^{\prime}$ is continuous. Since $r_{1}<R$ was arbitrary, this shows that $f^{\prime}(z)$ is given by the differentiated series above for $|z-a|<R$. Now a repeat of the argument shows all the derivatives of $f$ exist and are continuous on $B(a, R)$.

### 51.1.1 Cauchy Riemann Equations

Next consider the very important Cauchy Riemann equations which give conditions under which complex valued functions of a complex variable are analytic.
Theorem 51.1.5 Let $\Omega$ be an open subset of $\mathbb{C}$ and let $f: \Omega \rightarrow \mathbb{C}$ be a function, such that for $z=x+i y \in \Omega$,

$$
f(z)=u(x, y)+i v(x, y) .
$$

Then $f$ is analytic if and only if $u, v$ are $C^{1}(\Omega)$ and

$$
\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}
$$

## Furthermore,

$$
f^{\prime}(z)=\frac{\partial u}{\partial x}(x, y)+i \frac{\partial v}{\partial x}(x, y)
$$

Proof: Suppose $f$ is analytic first. Then letting $t \in \mathbb{R}$,

$$
\begin{gathered}
f^{\prime}(z)=\lim _{t \rightarrow 0} \frac{f(z+t)-f(z)}{t}= \\
\lim _{t \rightarrow 0}\left(\frac{u(x+t, y)+i v(x+t, y)}{t}-\frac{u(x, y)+i v(x, y)}{t}\right) \\
=\frac{\partial u(x, y)}{\partial x}+i \frac{\partial v(x, y)}{\partial x}
\end{gathered}
$$

But also

$$
\begin{gathered}
f^{\prime}(z)=\lim _{t \rightarrow 0} \frac{f(z+i t)-f(z)}{i t}= \\
\lim _{t \rightarrow 0}\left(\frac{u(x, y+t)+i v(x, y+t)}{i t}-\frac{u(x, y)+i v(x, y)}{i t}\right) \\
\frac{1}{i}\left(\frac{\partial u(x, y)}{\partial y}+i \frac{\partial v(x, y)}{\partial y}\right) \\
=\frac{\partial v(x, y)}{\partial y}-i \frac{\partial u(x, y)}{\partial y}
\end{gathered}
$$

This verifies the Cauchy Riemann equations. We are assuming that $z \rightarrow f^{\prime}(z)$ is continuous. Therefore, the partial derivatives of $u$ and $v$ are also continuous. To see this, note that from the formulas for $f^{\prime}(z)$ given above, and letting $z_{1}=x_{1}+i y_{1}$

$$
\left|\frac{\partial v(x, y)}{\partial y}-\frac{\partial v\left(x_{1}, y_{1}\right)}{\partial y}\right| \leq\left|f^{\prime}(z)-f^{\prime}\left(z_{1}\right)\right|
$$

showing that $(x, y) \rightarrow \frac{\partial v(x, y)}{\partial y}$ is continuous since $\left(x_{1}, y_{1}\right) \rightarrow(x, y)$ if and only if $z_{1} \rightarrow z$. The other cases are similar.

Now suppose the Cauchy Riemann equations hold and the functions, $u$ and $v$ are $C^{1}(\Omega)$. Then letting $h=h_{1}+i h_{2}$,

$$
\begin{gathered}
f(z+h)-f(z)=u\left(x+h_{1}, y+h_{2}\right) \\
+i v\left(x+h_{1}, y+h_{2}\right)-(u(x, y)+i v(x, y))
\end{gathered}
$$

We know $u$ and $v$ are both differentiable and so

$$
f(z+h)-f(z)=\frac{\partial u}{\partial x}(x, y) h_{1}+\frac{\partial u}{\partial y}(x, y) h_{2}+
$$

$$
i\left(\frac{\partial v}{\partial x}(x, y) h_{1}+\frac{\partial v}{\partial y}(x, y) h_{2}\right)+o(h) .
$$

Dividing by $h$ and using the Cauchy Riemann equations,

$$
\begin{gathered}
\frac{f(z+h)-f(z)}{h}=\frac{\frac{\partial u}{\partial x}(x, y) h_{1}+i \frac{\partial v}{\partial y}(x, y) h_{2}}{h}+ \\
\frac{i \frac{\partial v}{\partial x}(x, y) h_{1}+\frac{\partial u}{\partial y}(x, y) h_{2}}{h}+\frac{o(h)}{h} \\
=\frac{\partial u}{\partial x}(x, y) \frac{h_{1}+i h_{2}}{h}+i \frac{\partial v}{\partial x}(x, y) \frac{h_{1}+i h_{2}}{h}+\frac{o(h)}{h}
\end{gathered}
$$

Taking the limit as $h \rightarrow 0$,

$$
f^{\prime}(z)=\frac{\partial u}{\partial x}(x, y)+i \frac{\partial v}{\partial x}(x, y)
$$

It follows from this formula and the assumption that $u, v$ are $C^{1}(\Omega)$ that $f^{\prime}$ is continuous.
It is routine to verify that all the usual rules of derivatives hold for analytic functions. In particular, the product rule, the chain rule, and quotient rule.

### 51.1.2 An Important Example

An important example of an analytic function is $e^{z} \equiv \exp (z) \equiv e^{x}(\cos y+i \sin y)$ where $z=x+i y$. You can verify that this function satisfies the Cauchy Riemann equations and that all the partial derivatives are continuous. Also from the above discussion, $\left(e^{z}\right)^{\prime}=$ $e^{x} \cos (y)+i e^{x} \sin y=e^{z}$. Later I will show that $e^{z}$ is given by the usual power series. An important property of this function is that it can be used to parameterize the circle centered at $z_{0}$ having radius $r$.

Lemma 51.1.6 Let $\gamma$ denote the closed curve which is a circle of radius $r$ centered at $z_{0}$. Then a parameterization this curve is $\gamma(t)=z_{0}+r e^{i t}$ where $t \in[0,2 \pi]$.

Proof: $\left|\gamma(t)-z_{0}\right|^{2}=\left|r e^{i t} r e^{-i t}\right|=r^{2}$. Also, you can see from the definition of the sine and cosine that the point described in this way moves counter clockwise over this circle.

### 51.2 Exercises

1. Verify all the usual rules of differentiation including the product and chain rules.
2. Suppose $f$ and $f^{\prime}: U \rightarrow \mathbb{C}$ are analytic and $f(z)=u(x, y)+i v(x, y)$. Verify $u_{x x}+$ $u_{y y}=0$ and $v_{x x}+v_{y y}=0$. This partial differential equation satisfied by the real and imaginary parts of an analytic function is called Laplace's equation. We say these functions satisfying Laplace's equation are harmonic functions. If $u$ is a harmonic function defined on $B(0, r)$ show that $v(x, y) \equiv \int_{0}^{y} u_{x}(x, t) d t-\int_{0}^{x} u_{y}(t, 0) d t$ is such that $u+i v$ is analytic.
3. Let $f: U \rightarrow \mathbb{C}$ be analytic and $f(z)=u(x, y)+i v(x, y)$. Show $u, v$ and $u v$ are all harmonic although it can happen that $u^{2}$ is not. Recall that a function, $w$ is harmonic if $w_{x x}+w_{y y}=0$.
4. Define a function $f(z) \equiv \bar{z} \equiv x-i y$ where $z=x+i y$. Is $f$ analytic?
5. If $f(z)=u(x, y)+i v(x, y)$ and $f$ is analytic, verify that

$$
\operatorname{det}\left(\begin{array}{ll}
u_{x} & u_{y} \\
v_{x} & v_{y}
\end{array}\right)=\left|f^{\prime}(z)\right|^{2}
$$

6. Show that if $u(x, y)+i v(x, y)=f(z)$ is analytic, then $\nabla u \cdot \nabla v=0$. Recall

$$
\nabla u(x, y)=\left\langle u_{x}(x, y), u_{y}(x, y)\right\rangle .
$$

7. Show that every polynomial is analytic.
8. If $\gamma(t)=x(t)+i y(t)$ is a $C^{1}$ curve having values in $U$, an open set of $\mathbb{C}$, and if $f: U \rightarrow \mathbb{C}$ is analytic, we can consider $f \circ \gamma$, another $C^{1}$ curve having values in $\mathbb{C}$. Also, $\gamma^{\prime}(t)$ and $(f \circ \gamma)^{\prime}(t)$ are complex numbers so these can be considered as vectors in $\mathbb{R}^{2}$ as follows. The complex number, $x+i y$ corresponds to the vector, $\langle x, y\rangle$. Suppose that $\gamma$ and $\eta$ are two such $C^{1}$ curves having values in $U$ and that $\gamma\left(t_{0}\right)=\eta\left(s_{0}\right)=z$ and suppose that $f: U \rightarrow \mathbb{C}$ is analytic. Show that the angle between $(f \circ \gamma)^{\prime}\left(t_{0}\right)$ and $(f \circ \eta)^{\prime}\left(s_{0}\right)$ is the same as the angle between $\gamma^{\prime}\left(t_{0}\right)$ and $\eta^{\prime}\left(s_{0}\right)$ assuming that $f^{\prime}(z) \neq 0$. Thus analytic mappings preserve angles at points where the derivative is nonzero. Such mappings are called isogonal. . Hint: To make this easy to show, first observe that $\langle x, y\rangle \cdot\langle a, b\rangle=\frac{1}{2}(z \bar{w}+\bar{z} w)$ where $z=x+i y$ and $w=a+i b$.
9. Analytic functions are even better than what is described in Problem 8. In addition to preserving angles, they also preserve orientation. To verify this show that if $z=x+i y$ and $w=a+i b$ are two complex numbers, then $\langle x, y, 0\rangle$ and $\langle a, b, 0\rangle$ are two vectors in $\mathbb{R}^{3}$. Recall that the cross product, $\langle x, y, 0\rangle \times\langle a, b, 0\rangle$, yields a vector normal to the two given vectors such that the triple, $\langle x, y, 0\rangle,\langle a, b, 0\rangle$, and $\langle x, y, 0\rangle \times\langle a, b, 0\rangle$ satisfies the right hand rule and has magnitude equal to the product of the sine of the included angle times the product of the two norms of the vectors. In this case, the cross product either points in the direction of the positive $z$ axis or in the direction of the negative $z$ axis. Thus, either the vectors $\langle x, y, 0\rangle,\langle a, b, 0\rangle, \mathbf{k}$ form a right handed system or the vectors $\langle a, b, 0\rangle,\langle x, y, 0\rangle, \mathbf{k}$ form a right handed system. These are the two possible orientations. Show that in the situation of Problem 8 the orientation of $\gamma^{\prime}\left(t_{0}\right), \eta^{\prime}\left(s_{0}\right), \mathbf{k}$ is the same as the orientation of the vectors $(f \circ \gamma)^{\prime}\left(t_{0}\right),(f \circ \eta)^{\prime}\left(s_{0}\right), \mathbf{k}$. Such mappings are called conformal. If $f$ is analytic and $f^{\prime}(z) \neq 0$, then we know from this problem and the above that $f$ is a conformal map. Hint: You can do this by verifying that $(f \circ \gamma)^{\prime}\left(t_{0}\right) \times(f \circ \eta)^{\prime}\left(s_{0}\right)=$ $\left|f^{\prime}\left(\gamma\left(t_{0}\right)\right)\right|^{2} \gamma^{\prime}\left(t_{0}\right) \times \eta^{\prime}\left(s_{0}\right)$. To make the verification easier, you might first establish the following simple formula for the cross product where here $x+i y=z$ and $a+i b=w$.

$$
(x, y, 0) \times(a, b, 0)=\operatorname{Re}(z i \bar{w}) \mathbf{k}
$$

10. Write the Cauchy Riemann equations in terms of polar coordinates. Recall the polar coordinates are given by

$$
x=r \cos \theta, y=r \sin \theta
$$

This means, letting $u(x, y)=u(r, \theta), v(x, y)=v(r, \theta)$, write the Cauchy Riemann equations in terms of $r$ and $\theta$. You should eventually show the Cauchy Riemann equations are equivalent to

$$
\frac{\partial u}{\partial r}=\frac{1}{r} \frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r}=-\frac{1}{r} \frac{\partial u}{\partial \theta}
$$

11. Show that a real valued analytic function must be constant.

### 51.3 Cauchy's Formula For A Disk

The Cauchy integral formula is the most important theorem in complex analysis. It will be established for a disk in this chapter and later will be generalized to much more general situations but the version given here will suffice to prove many interesting theorems needed in the later development of the theory. The following are some advanced calculus results.

Lemma 51.3.1 Let $f:[a, b] \rightarrow \mathbb{C}$. Then $f^{\prime}(t)$ exists if and only if $\operatorname{Re} f^{\prime}(t)$ and $\operatorname{Im} f^{\prime}(t)$ exist. Furthermore,

$$
f^{\prime}(t)=\operatorname{Re} f^{\prime}(t)+i \operatorname{Im} f^{\prime}(t)
$$

Proof: The if part of the equivalence is obvious.
Now suppose $f^{\prime}(t)$ exists. Let both $t$ and $t+h$ be contained in $[a, b]$

$$
\left|\frac{\operatorname{Re} f(t+h)-\operatorname{Re} f(t)}{h}-\operatorname{Re}\left(f^{\prime}(t)\right)\right| \leq\left|\frac{f(t+h)-f(t)}{h}-f^{\prime}(t)\right|
$$

and this converges to zero as $h \rightarrow 0$. Therefore, $\operatorname{Re} f^{\prime}(t)=\operatorname{Re}\left(f^{\prime}(t)\right) . \operatorname{Similarly}, \operatorname{Im} f^{\prime}(t)=$ $\operatorname{Im}\left(f^{\prime}(t)\right)$.

Lemma 51.3.2 If $g:[a, b] \rightarrow \mathbb{C}$ and $g$ is continuous on $[a, b]$ and differentiable on $(a, b)$ with $g^{\prime}(t)=0$, then $g(t)$ is a constant.

Proof: From the above lemma, you can apply the mean value theorem to the real and imaginary parts of $g$.

Applying the above lemma to the components yields the following lemma.
Lemma 51.3.3 If $g:[a, b] \rightarrow \mathbb{C}^{n}=X$ and $g$ is continuous on $[a, b]$ and differentiable on $(a, b)$ with $g^{\prime}(t)=0$, then $g(t)$ is a constant.

If you want to have $X$ be a complex Banach space, the result is still true.
Lemma 51.3.4 If $g:[a, b] \rightarrow X$ and $g$ is continuous on $[a, b]$ and differentiable on $(a, b)$ with $g^{\prime}(t)=0$, then $g(t)$ is a constant.

Proof: Let $\Lambda \in X^{\prime}$. Then $\Lambda g:[a, b] \rightarrow \mathbb{C}$. Therefore, from Lemma 51.3.2, for each $\Lambda \in X^{\prime}, \Lambda g(s)=\Lambda g(t)$ and since $X^{\prime}$ separates the points, it follows $g(s)=g(t)$ so $g$ is constant.

Lemma 51.3.5 Let $\phi:[a, b] \times[c, d] \rightarrow \mathbb{R}$ be continuous and let

$$
\begin{equation*}
g(t) \equiv \int_{a}^{b} \phi(s, t) d s \tag{51.3.1}
\end{equation*}
$$

Then $g$ is continuous. If $\frac{\partial \phi}{\partial t}$ exists and is continuous on $[a, b] \times[c, d]$, then

$$
\begin{equation*}
g^{\prime}(t)=\int_{a}^{b} \frac{\partial \phi(s, t)}{\partial t} d s \tag{51.3.2}
\end{equation*}
$$

Proof: The first claim follows from the uniform continuity of $\phi$ on $[a, b] \times[c, d]$, which uniform continuity results from the set being compact. To establish 51.3.2, let $t$ and $t+h$ be contained in $[c, d]$ and form, using the mean value theorem,

$$
\begin{aligned}
\frac{g(t+h)-g(t)}{h} & =\frac{1}{h} \int_{a}^{b}[\phi(s, t+h)-\phi(s, t)] d s \\
& =\frac{1}{h} \int_{a}^{b} \frac{\partial \phi(s, t+\theta h)}{\partial t} h d s \\
& =\int_{a}^{b} \frac{\partial \phi(s, t+\theta h)}{\partial t} d s
\end{aligned}
$$

where $\theta$ may depend on $s$ but is some number between 0 and 1 . Then by the uniform continuity of $\frac{\partial \phi}{\partial t}$, it follows that 51.3.2 holds.

Corollary 51.3.6 Let $\phi:[a, b] \times[c, d] \rightarrow \mathbb{C}$ be continuous and let

$$
\begin{equation*}
g(t) \equiv \int_{a}^{b} \phi(s, t) d s \tag{51.3.3}
\end{equation*}
$$

Then $g$ is continuous. If $\frac{\partial \phi}{\partial t}$ exists and is continuous on $[a, b] \times[c, d]$, then

$$
\begin{equation*}
g^{\prime}(t)=\int_{a}^{b} \frac{\partial \phi(s, t)}{\partial t} d s \tag{51.3.4}
\end{equation*}
$$

Proof: Apply Lemma 51.3.5 to the real and imaginary parts of $\phi$.
Applying the above corollary to the components, you can also have the same result for $\phi$ having values in $\mathbb{C}^{n}$.

Corollary 51.3.7 Let $\phi:[a, b] \times[c, d] \rightarrow \mathbb{C}^{n}$ be continuous and let

$$
\begin{equation*}
g(t) \equiv \int_{a}^{b} \phi(s, t) d s \tag{51.3.5}
\end{equation*}
$$

Then $g$ is continuous. If $\frac{\partial \phi}{\partial t}$ exists and is continuous on $[a, b] \times[c, d]$, then

$$
\begin{equation*}
g^{\prime}(t)=\int_{a}^{b} \frac{\partial \phi(s, t)}{\partial t} d s \tag{51.3.6}
\end{equation*}
$$

If you want to consider $\phi$ having values in $X$, a complex Banach space a similar result holds.

Corollary 51.3.8 Let $\phi:[a, b] \times[c, d] \rightarrow X$ be continuous and let

$$
\begin{equation*}
g(t) \equiv \int_{a}^{b} \phi(s, t) d s \tag{51.3.7}
\end{equation*}
$$

Then $g$ is continuous. If $\frac{\partial \phi}{\partial t}$ exists and is continuous on $[a, b] \times[c, d]$, then

$$
\begin{equation*}
g^{\prime}(t)=\int_{a}^{b} \frac{\partial \phi(s, t)}{\partial t} d s \tag{51.3.8}
\end{equation*}
$$

Proof: Let $\Lambda \in X^{\prime}$. Then $\Lambda \phi:[a, b] \times[c, d] \rightarrow \mathbb{C}$ is continuous and $\frac{\partial \Lambda \phi}{\partial t}$ exists and is continuous on $[a, b] \times[c, d]$. Therefore, from 51.3.8,

$$
\Lambda\left(g^{\prime}(t)\right)=(\Lambda g)^{\prime}(t)=\int_{a}^{b} \frac{\partial \Lambda \phi(s, t)}{\partial t} d s=\Lambda \int_{a}^{b} \frac{\partial \phi(s, t)}{\partial t} d s
$$

and since $X^{\prime}$ separates the points, it follows 51.3.8 holds.
You can give a different proof of this.
Theorem 51.3.9 Let $\phi:[a, b] \times[c, d] \rightarrow X$ be continuous and suppose $\phi_{t}$ is continuous. Then

$$
\left(\int_{a}^{b} \phi(s, t) d s\right)_{, t}=\int_{a}^{b} \frac{\partial \phi}{\partial t}(s, t) d s
$$

Here $X$ is a complex Banach space.
Proof: Consider the following set $P$ which is where the ordered pair $(t, h)$ will be.


This is so that both $t$ and $t+h$ are in $[a, b]$. Then for such an ordered pair, consider

$$
\Delta(s, t, h) \equiv\left\{\begin{array}{l}
\frac{\phi(s, t+h)-\phi(s, t)}{h} \text { if } h \neq 0 \\
\phi_{t}(s, t) \text { if } h=0
\end{array}\right.
$$

Claim: $\Delta$ is continuous on the compact set $[a, b] \times P$.
Proof of claim: It is obvious unless $h=0$. Therefore, consider the point $(s, t, 0)$.

$$
\left\|\Delta\left(s^{\prime}, t^{\prime}, h\right)-\Delta(s, t, 0)\right\|=\left\|\frac{\phi\left(s^{\prime}, t^{\prime}+h\right)-\phi\left(s^{\prime}, t^{\prime}\right)}{h}-\phi_{t}(s, t)\right\|
$$

$$
=\left\|\frac{1}{h} \int_{t^{\prime}}^{t^{\prime}+h} \phi_{t}\left(s^{\prime}, r\right) d r-\phi_{t}(s, t)\right\| \leq \frac{1}{h} \int_{t^{\prime}}^{t^{\prime}+h}\left\|\phi_{t}\left(s^{\prime}, r\right)-\phi_{t}(s, t)\right\| d r<\varepsilon
$$

provided $\left|\left(s^{\prime}, t^{\prime}, h\right)-(s, t, 0)\right|$ is small enough, this by continuity of $\phi_{t}$. Therefore, $\Delta(s, t, h)$ is uniformly continuous.

$$
\begin{aligned}
\left\|\frac{1}{h}\left(\int_{a}^{b} \phi(s, t+h) d s-\int_{a}^{b} \phi(s, t) d s\right)-\int_{a}^{b} \phi_{t}(s, t) d s\right\| \\
\leq \int_{a}^{b}\left\|\frac{\phi(s, t+h)-\phi(s, t)}{h}-\phi_{t}(s, t)\right\| d s=\int_{a}^{b}\left\|\Delta(s, t, h)-\phi_{t}(s, t)\right\| d s
\end{aligned}
$$

Then by uniform continuity, if $h$ is small enough, the integrand on the right is smaller than $\varepsilon$.

The following is Cauchy's integral formula for a disk.
Theorem 51.3.10 Let $f: \Omega \rightarrow X$ be analytic on the open set, $\Omega$ and let

$$
\overline{B\left(z_{0}, r\right)} \subseteq \Omega
$$

Let $\gamma(t) \equiv z_{0}+r e^{i t}$ for $t \in[0,2 \pi]$. Then if $z \in B\left(z_{0}, r\right)$,

$$
\begin{equation*}
f(z)=\frac{1}{2 \pi i} \int_{\gamma} \frac{f(w)}{w-z} d w \tag{51.3.9}
\end{equation*}
$$

Proof: Consider for $\alpha \in[0,1]$,

$$
g(\alpha) \equiv \int_{0}^{2 \pi} \frac{f\left(z+\alpha\left(z_{0}+r e^{i t}-z\right)\right)}{r e^{i t}+z_{0}-z} r i e^{i t} d t
$$

If $\alpha$ equals one, this reduces to the integral in 51.3.9. The idea is to show $g$ is a constant and that $g(0)=f(z) 2 \pi i$. First consider the claim about $g(0)$.

$$
\begin{aligned}
g(0) & =\left(\int_{0}^{2 \pi} \frac{r e^{i t}}{r e^{i t}+z_{0}-z} d t\right) i f(z) \\
& =i f(z)\left(\int_{0}^{2 \pi} \frac{1}{1-\frac{z-z_{0}}{r e^{i t}}} d t\right) \\
& =i f(z) \int_{0}^{2 \pi} \sum_{n=0}^{\infty} r^{-n} e^{-i n t}\left(z-z_{0}\right)^{n} d t
\end{aligned}
$$

because $\left|\frac{z-z_{0}}{r e^{i t}}\right|<1$. Since this sum converges uniformly you can interchange the sum and the integral to obtain

$$
\begin{aligned}
g(0) & =\text { if }(z) \sum_{n=0}^{\infty} r^{-n}\left(z-z_{0}\right)^{n} \int_{0}^{2 \pi} e^{-i n t} d t \\
& =2 \pi i f(z)
\end{aligned}
$$

because $\int_{0}^{2 \pi} e^{-i n t} d t=0$ if $n>0$.
Next consider the claim that $g$ is constant. By Corollary 51.3.7, for $\alpha \in(0,1)$,

$$
\begin{aligned}
g^{\prime}(\alpha) & =\int_{0}^{2 \pi} \frac{f^{\prime}\left(z+\alpha\left(z_{0}+r e^{i t}-z\right)\right)\left(r e^{i t}+z_{0}-z\right)}{r e^{i t}+z_{0}-z} r i e^{i t} d t \\
& =\int_{0}^{2 \pi} f^{\prime}\left(z+\alpha\left(z_{0}+r e^{i t}-z\right)\right) r i e^{i t} d t \\
& =\int_{0}^{2 \pi} \frac{d}{d t}\left(f\left(z+\alpha\left(z_{0}+r e^{i t}-z\right)\right) \frac{1}{\alpha}\right) d t \\
& =f\left(z+\alpha\left(z_{0}+r e^{i 2 \pi}-z\right)\right) \frac{1}{\alpha}-f\left(z+\alpha\left(z_{0}+r e^{0}-z\right)\right) \frac{1}{\alpha}=0
\end{aligned}
$$

Now $g$ is continuous on $[0,1]$ and $g^{\prime}(t)=0$ on $(0,1)$ so by Lemma 51.3.3, $g$ equals a constant. This constant can only be $g(0)=2 \pi i f(z)$. Thus,

$$
g(1)=\int_{\gamma} \frac{f(w)}{w-z} d w=g(0)=2 \pi i f(z)
$$

This proves the theorem.
This is a very significant theorem. A few applications are given next.

Theorem 51.3.11 Let $f: \Omega \rightarrow X$ be analytic where $\Omega$ is an open set in $\mathbb{C}$. Then $f$ has infinitely many derivatives on $\Omega$. Furthermore, for all $z \in B\left(z_{0}, r\right)$,

$$
\begin{equation*}
f^{(n)}(z)=\frac{n!}{2 \pi i} \int_{\gamma} \frac{f(w)}{(w-z)^{n+1}} d w \tag{51.3.10}
\end{equation*}
$$

where $\gamma(t) \equiv z_{0}+r e^{i t}, t \in[0,2 \pi]$ for $r$ small enough that $B\left(z_{0}, r\right) \subseteq \Omega$.
Proof: Let $z \in B\left(z_{0}, r\right) \subseteq \Omega$ and let $\overline{B\left(z_{0}, r\right)} \subseteq \Omega$. Then, letting $\gamma(t) \equiv z_{0}+r e^{i t}, t \in$ $[0,2 \pi]$, and $h$ small enough,

$$
f(z)=\frac{1}{2 \pi i} \int_{\gamma} \frac{f(w)}{w-z} d w, f(z+h)=\frac{1}{2 \pi i} \int_{\gamma} \frac{f(w)}{w-z-h} d w
$$

Now

$$
\frac{1}{w-z-h}-\frac{1}{w-z}=\frac{h}{(-w+z+h)(-w+z)}
$$

and so

$$
\begin{aligned}
\frac{f(z+h)-f(z)}{h} & =\frac{1}{2 \pi h i} \int_{\gamma} \frac{h f(w)}{(-w+z+h)(-w+z)} d w \\
& =\frac{1}{2 \pi i} \int_{\gamma} \frac{f(w)}{(-w+z+h)(-w+z)} d w
\end{aligned}
$$

Now for all $h$ sufficiently small, there exists a constant $C$ independent of such $h$ such that

$$
\begin{aligned}
& \left|\frac{1}{(-w+z+h)(-w+z)}-\frac{1}{(-w+z)(-w+z)}\right| \\
= & \left|\frac{h}{(w-z-h)(w-z)^{2}}\right| \leq C|h|
\end{aligned}
$$

and so, the integrand converges uniformly as $h \rightarrow 0$ to

$$
=\frac{f(w)}{(w-z)^{2}}
$$

Therefore, the limit as $h \rightarrow 0$ may be taken inside the integral to obtain

$$
f^{\prime}(z)=\frac{1}{2 \pi i} \int_{\gamma} \frac{f(w)}{(w-z)^{2}} d w
$$

Continuing in this way, yields 51.3.10.
This is a very remarkable result. It shows the existence of one continuous derivative implies the existence of all derivatives, in contrast to the theory of functions of a real variable. Actually, more than what is stated in the theorem was shown. The above proof establishes the following corollary.

Corollary 51.3.12 Suppose $f$ is continuous on $\partial B\left(z_{0}, r\right)$ and suppose that for all $z \in$ $B\left(z_{0}, r\right)$,

$$
f(z)=\frac{1}{2 \pi i} \int_{\gamma} \frac{f(w)}{w-z} d w
$$

where $\gamma(t) \equiv z_{0}+r e^{i t}, t \in[0,2 \pi]$. Then $f$ is analytic on $B\left(z_{0}, r\right)$ and in fact has infinitely many derivatives on $B\left(z_{0}, r\right)$.

Another application is the following lemma.
Lemma 51.3.13 Let $\gamma(t)=z_{0}+r e^{i t}$, for $t \in[0,2 \pi]$, suppose $f_{n} \rightarrow f$ uniformly on $\overline{B\left(z_{0}, r\right)}$, and suppose

$$
\begin{equation*}
f_{n}(z)=\frac{1}{2 \pi i} \int_{\gamma} \frac{f_{n}(w)}{w-z} d w \tag{51.3.11}
\end{equation*}
$$

for $z \in B\left(z_{0}, r\right)$. Then

$$
\begin{equation*}
f(z)=\frac{1}{2 \pi i} \int_{\gamma} \frac{f(w)}{w-z} d w \tag{51.3.12}
\end{equation*}
$$

implying that $f$ is analytic on $B\left(z_{0}, r\right)$.
Proof: From 51.3.11 and the uniform convergence of $f_{n}$ to $f$ on $\gamma([0,2 \pi])$, the integrals in 51.3.11 converge to

$$
\frac{1}{2 \pi i} \int_{\gamma} \frac{f(w)}{w-z} d w
$$

Therefore, the formula 51.3.12 follows.
Uniform convergence on a closed disk of the analytic functions implies the target function is also analytic. This is amazing. Think of the Weierstrass approximation theorem for polynomials. You can obtain a continuous nowhere differentiable function as the uniform limit of polynomials.

The conclusions of the following proposition have all been obtained earlier in Theorem 51.1.4 but they can be obtained more easily if you use the above theorem and lemmas.

Proposition 51.3.14 Let $\left\{a_{n}\right\}$ denote a sequence in $X$. Then there exists $R \in[0, \infty]$ such that

$$
\sum_{k=0}^{\infty} a_{k}\left(z-z_{0}\right)^{k}
$$

converges absolutely if $\left|z-z_{0}\right|<R$, diverges if $\left|z-z_{0}\right|>R$ and converges uniformly on $B\left(z_{0}, r\right)$ for all $r<R$. Furthermore, if $R>0$, the function,

$$
f(z) \equiv \sum_{k=0}^{\infty} a_{k}\left(z-z_{0}\right)^{k}
$$

is analytic on $B\left(z_{0}, R\right)$.
Proof: The assertions about absolute convergence are routine from the root test if

$$
R \equiv\left(\limsup _{n \rightarrow \infty}\left|a_{n}\right|^{1 / n}\right)^{-1}
$$

with $R=\infty$ if the quantity in parenthesis equals zero. The root test can be used to verify absolute convergence which then implies convergence by completeness of $X$.

The assertion about uniform convergence follows from the Weierstrass $M$ test and $M_{n} \equiv$ $\left|a_{n}\right| r^{n}$. ( $\sum_{n=0}^{\infty}\left|a_{n}\right| r^{n}<\infty$ by the root test). It only remains to verify the assertion about $f(z)$ being analytic in the case where $R>0$.

Let $0<r<R$ and define $f_{n}(z) \equiv \sum_{k=0}^{n} a_{k}\left(z-z_{0}\right)^{k}$. Then $f_{n}$ is a polynomial and so it is analytic. Thus, by the Cauchy integral formula above,

$$
f_{n}(z)=\frac{1}{2 \pi i} \int_{\gamma} \frac{f_{n}(w)}{w-z} d w
$$

where $\gamma(t)=z_{0}+r e^{i t}$, for $t \in[0,2 \pi]$. By Lemma 51.3.13 and the first part of this proposition involving uniform convergence,

$$
f(z)=\frac{1}{2 \pi i} \int_{\gamma} \frac{f(w)}{w-z} d w
$$

Therefore, $f$ is analytic on $B\left(z_{0}, r\right)$ by Corollary 51.3.12. Since $r<R$ is arbitrary, this shows $f$ is analytic on $B\left(z_{0}, R\right)$.

This proposition shows that all functions having values in $X$ which are given as power series are analytic on their circle of convergence, the set of complex numbers, $z$, such that $\left|z-z_{0}\right|<R$. In fact, every analytic function can be realized as a power series.

Theorem 51.3.15 If $f: \Omega \rightarrow X$ is analytic and if $B\left(z_{0}, r\right) \subseteq \Omega$, then

$$
\begin{equation*}
f(z)=\sum_{n=0}^{\infty} a_{n}\left(z-z_{0}\right)^{n} \tag{51.3.13}
\end{equation*}
$$

for all $\left|z-z_{0}\right|<r$. Furthermore,

$$
\begin{equation*}
a_{n}=\frac{f^{(n)}\left(z_{0}\right)}{n!} \tag{51.3.14}
\end{equation*}
$$

Proof: Consider $\left|z-z_{0}\right|<r$ and let $\gamma(t)=z_{0}+r e^{i t}, t \in[0,2 \pi]$. Then for $w \in \gamma([0,2 \pi])$,

$$
\left|\frac{z-z_{0}}{w-z_{0}}\right|<1
$$

and so, by the Cauchy integral formula,

$$
\begin{aligned}
f(z) & =\frac{1}{2 \pi i} \int_{\gamma} \frac{f(w)}{w-z} d w \\
& =\frac{1}{2 \pi i} \int_{\gamma} \frac{f(w)}{\left(w-z_{0}\right)\left(1-\frac{z-z_{0}}{w-z_{0}}\right)} d w \\
& =\frac{1}{2 \pi i} \int_{\gamma} \frac{f(w)}{\left(w-z_{0}\right)} \sum_{n=0}^{\infty}\left(\frac{z-z_{0}}{w-z_{0}}\right)^{n} d w .
\end{aligned}
$$

Since the series converges uniformly, you can interchange the integral and the sum to obtain

$$
\begin{aligned}
f(z) & =\sum_{n=0}^{\infty}\left(\frac{1}{2 \pi i} \int_{\gamma} \frac{f(w)}{\left(w-z_{0}\right)^{n+1}}\right)\left(z-z_{0}\right)^{n} \\
& \equiv \sum_{n=0}^{\infty} a_{n}\left(z-z_{0}\right)^{n}
\end{aligned}
$$

By Theorem 51.3.11, 51.3.14 holds.
Note that this also implies that if a function is analytic on an open set, then all of its derivatives are also analytic. This follows from Theorem 51.1.4 which says that a function given by a power series has all derivatives on the disk of convergence.

### 51.4 Exercises

1. Show that if $\left|e_{k}\right| \leq \varepsilon$, then $\left|\sum_{k=m}^{\infty} e_{k}\left(r^{k}-r^{k+1}\right)\right|<\varepsilon$ if $0 \leq r<1$. Hint: Let $|\theta|=1$ and verify that

$$
\theta \sum_{k=m}^{\infty} e_{k}\left(r^{k}-r^{k+1}\right)=\left|\sum_{k=m}^{\infty} e_{k}\left(r^{k}-r^{k+1}\right)\right|=\sum_{k=m}^{\infty} \operatorname{Re}\left(\theta e_{k}\right)\left(r^{k}-r^{k+1}\right)
$$

where $-\varepsilon<\operatorname{Re}\left(\theta e_{k}\right)<\varepsilon$.
2. Abel's theorem says that if $\sum_{n=0}^{\infty} a_{n}(z-a)^{n}$ has radius of convergence equal to 1 and if $A=\sum_{n=0}^{\infty} a_{n}$, then $\lim _{r \rightarrow 1-} \sum_{n=0}^{\infty} a_{n} r^{n}=A$. Hint: Show

$$
\sum_{k=0}^{\infty} a_{k} r^{k}=\sum_{k=0}^{\infty} A_{k}\left(r^{k}-r^{k+1}\right)
$$

where $A_{k}$ denotes the $k^{\text {th }}$ partial sum of $\sum a_{j}$. Thus

$$
\sum_{k=0}^{\infty} a_{k} r^{k}=\sum_{k=m+1}^{\infty} A_{k}\left(r^{k}-r^{k+1}\right)+\sum_{k=0}^{m} A_{k}\left(r^{k}-r^{k+1}\right)
$$

where $\left|A_{k}-A\right|<\varepsilon$ for all $k \geq m$. In the first sum, write $A_{k}=A+e_{k}$ and use Problem 1. Use this theorem to verify that $\arctan (1)=\sum_{k=0}^{\infty}(-1)^{k} \frac{1}{2 k+1}$.
3. Find the integrals using the Cauchy integral formula.
(a) $\int_{\gamma} \frac{\sin z}{z-i} d z$ where $\gamma(t)=2 e^{i t}: t \in[0,2 \pi]$.
(b) $\int_{\gamma} \frac{1}{z-a} d z$ where $\gamma(t)=a+r e^{i t}: t \in[0,2 \pi]$
(c) $\int_{\gamma} \frac{\cos z}{z^{2}} d z$ where $\gamma(t)=e^{i t}: t \in[0,2 \pi]$
(d) $\int_{\gamma} \frac{\log (z)}{z^{n}} d z$ where $\gamma(t)=1+\frac{1}{2} e^{i t}: t \in[0,2 \pi]$ and $n=0,1,2$. In this problem, $\log (z) \equiv \ln |z|+i \arg (z)$ where $\arg (z) \in(-\pi, \pi)$ and $z=|z| e^{i \arg (z)}$. Thus $e^{\log (z)}=z$ and $\log (z)^{\prime}=\frac{1}{z}$.
4. Let $\gamma(t)=4 e^{i t}: t \in[0,2 \pi]$ and find $\int_{\gamma} \frac{z^{2}+4}{z\left(z^{2}+1\right)} d z$.
5. Suppose $f(z)=\sum_{n=0}^{\infty} a_{n} z^{n}$ for all $|z|<R$. Show that then

$$
\frac{1}{2 \pi} \int_{0}^{2 \pi}\left|f\left(r e^{i \theta}\right)\right|^{2} d \theta=\sum_{n=0}^{\infty}\left|a_{n}\right|^{2} r^{2 n}
$$

for all $r \in[0, R)$. Hint: Let

$$
f_{n}(z) \equiv \sum_{k=0}^{n} a_{k} z^{k}
$$

show

$$
\frac{1}{2 \pi} \int_{0}^{2 \pi}\left|f_{n}\left(r e^{i \theta}\right)\right|^{2} d \theta=\sum_{k=0}^{n}\left|a_{k}\right|^{2} r^{2 k}
$$

and then take limits as $n \rightarrow \infty$ using uniform convergence.
6. The Cauchy integral formula, marvelous as it is, can actually be improved upon. The Cauchy integral formula involves representing $f$ by the values of $f$ on the boundary of the disk, $B(a, r)$. It is possible to represent $f$ by using only the values of $\operatorname{Re} f$ on the boundary. This leads to the Schwarz formula. Supply the details in the following outline.

Suppose $f$ is analytic on $|z|<R$ and

$$
\begin{equation*}
f(z)=\sum_{n=0}^{\infty} a_{n} z^{n} \tag{51.4.15}
\end{equation*}
$$

with the series converging uniformly on $|z|=R$. Then letting $|w|=R$,

$$
2 u(w)=f(w)+\overline{f(w)}
$$

and so

$$
\begin{equation*}
2 u(w)=\sum_{k=0}^{\infty} a_{k} w^{k}+\sum_{k=0}^{\infty} \overline{a_{k}}(\bar{w})^{k} \tag{51.4.16}
\end{equation*}
$$

Now letting $\gamma(t)=R e^{i t}, t \in[0,2 \pi]$

$$
\begin{aligned}
\int_{\gamma} \frac{2 u(w)}{w} d w & =\left(a_{0}+\overline{a_{0}}\right) \int_{\gamma} \frac{1}{w} d w \\
& =2 \pi i\left(a_{0}+\overline{a_{0}}\right)
\end{aligned}
$$

Thus, multiplying 51.4.16 by $w^{-1}$,

$$
\frac{1}{\pi i} \int_{\gamma} \frac{u(w)}{w} d w=a_{0}+\overline{a_{0}}
$$

Now multiply 51.4.16 by $w^{-(n+1)}$ and integrate again to obtain

$$
a_{n}=\frac{1}{\pi i} \int_{\gamma} \frac{u(w)}{w^{n+1}} d w
$$

Using these formulas for $a_{n}$ in 51.4.15, we can interchange the sum and the integral (Why can we do this?) to write the following for $|z|<R$.

$$
\begin{aligned}
f(z) & =\frac{1}{\pi i} \int_{\gamma} \frac{1}{z} \sum_{k=0}^{\infty}\left(\frac{z}{w}\right)^{k+1} u(w) d w-\overline{a_{0}} \\
& =\frac{1}{\pi i} \int_{\gamma} \frac{u(w)}{w-z} d w-\overline{a_{0}}
\end{aligned}
$$

which is the Schwarz formula. Now $\operatorname{Re} a_{0}=\frac{1}{2 \pi i} \int_{\gamma} \frac{u(w)}{w} d w$ and $\overline{a_{0}}=\operatorname{Re} a_{0}-i \operatorname{Im} a_{0}$. Therefore, we can also write the Schwarz formula as

$$
\begin{equation*}
f(z)=\frac{1}{2 \pi i} \int_{\gamma} \frac{u(w)(w+z)}{(w-z) w} d w+i \operatorname{Im} a_{0} \tag{51.4.17}
\end{equation*}
$$

7. Take the real parts of the second form of the Schwarz formula to derive the Poisson formula for a disk,

$$
\begin{equation*}
u\left(r e^{i \alpha}\right)=\frac{1}{2 \pi} \int_{0}^{2 \pi} \frac{u\left(R e^{i \theta}\right)\left(R^{2}-r^{2}\right)}{R^{2}+r^{2}-2 R r \cos (\theta-\alpha)} d \theta \tag{51.4.18}
\end{equation*}
$$

8. Suppose that $u(w)$ is a given real continuous function defined on $\partial B(0, R)$ and define $f(z)$ for $|z|<R$ by 51.4.17. Show that $f$, so defined is analytic. Explain why $u$ given in 51.4.18 is harmonic. Show that

$$
\lim _{r \rightarrow R_{-}} u\left(r e^{i \alpha}\right)=u\left(R e^{i \alpha}\right)
$$

Thus $u$ is a harmonic function which approaches a given function on the boundary and is therefore, a solution to the Dirichlet problem.
9. Suppose $f(z)=\sum_{k=0}^{\infty} a_{k}\left(z-z_{0}\right)^{k}$ for all $\left|z-z_{0}\right|<R$. Show that

$$
f^{\prime}(z)=\sum_{k=0}^{\infty} a_{k} k\left(z-z_{0}\right)^{k-1}
$$

for all $\left|z-z_{0}\right|<R$. Hint: Let $f_{n}(z)$ be a partial sum of $f$. Show that $f_{n}^{\prime}$ converges uniformly to some function, $g$ on $\left|z-z_{0}\right| \leq r$ for any $r<R$. Now use the Cauchy integral formula for a function and its derivative to identify $g$ with $f^{\prime}$.
10. Use Problem 9 to find the exact value of $\sum_{k=0}^{\infty} k^{2}\left(\frac{1}{3}\right)^{k}$.
11. Prove the binomial formula,

$$
(1+z)^{\alpha}=\sum_{n=0}^{\infty}\binom{\alpha}{n} z^{n}
$$

where

$$
\binom{\alpha}{n} \equiv \frac{\alpha \cdots(\alpha-n+1)}{n!}
$$

Can this be used to give a proof of the binomial formula,

$$
(a+b)^{n}=\sum_{k=0}^{n}\binom{n}{k} a^{n-k} b^{k} ?
$$

Explain.
12. Suppose $f$ is analytic on $B\left(z_{0}, r\right)$ and continuous on $\overline{B\left(z_{0}, r\right)}$ and $|f(z)| \leq M$ on $\overline{B\left(z_{0}, r\right)}$. Show that then $\left|f^{(n)}(a)\right| \leq \frac{M n!}{r^{n}}$.

### 51.5 Zeros Of An Analytic Function

In this section we give a very surprising property of analytic functions which is in stark contrast to what takes place for functions of a real variable.

Definition 51.5.1 A region is a connected open set.
It turns out the zeros of an analytic function which is not constant on some region cannot have a limit point. This is also a good time to define the order of a zero.

Definition 51.5.2 Suppose $f$ is an analytic function defined near a point, $\alpha$ where $f(\alpha)=$ 0 . Thus $\alpha$ is a zero of the function, $f$. The zero is of order $m$ if $f(z)=(z-\alpha)^{m} g(z)$ where $g$ is an analytic function which is not equal to zero at $\alpha$.

Theorem 51.5.3 Let $\Omega$ be a connected open set (region) and let $f: \Omega \rightarrow X$ be analytic. Then the following are equivalent.

1. $f(z)=0$ for all $z \in \Omega$
2. There exists $z_{0} \in \Omega$ such that $f^{(n)}\left(z_{0}\right)=0$ for all $n$.
3. There exists $z_{0} \in \Omega$ which is a limit point of the set,

$$
Z \equiv\{z \in \Omega: f(z)=0\}
$$

Proof: It is clear the first condition implies the second two. Suppose the third holds. Then for $z$ near $z_{0}$

$$
f(z)=\sum_{n=k}^{\infty} \frac{f^{(n)}\left(z_{0}\right)}{n!}\left(z-z_{0}\right)^{n}
$$

where $k \geq 1$ since $z_{0}$ is a zero of $f$. Suppose $k<\infty$. Then,

$$
f(z)=\left(z-z_{0}\right)^{k} g(z)
$$

where $g\left(z_{0}\right) \neq 0$. Letting $z_{n} \rightarrow z_{0}$ where $z_{n} \in Z, z_{n} \neq z_{0}$, it follows

$$
0=\left(z_{n}-z_{0}\right)^{k} g\left(z_{n}\right)
$$

which implies $g\left(z_{n}\right)=0$. Then by continuity of $g$, we see that $g\left(z_{0}\right)=0$ also, contrary to the choice of $k$. Therefore, $k$ cannot be less than $\infty$ and so $z_{0}$ is a point satisfying the second condition.

Now suppose the second condition and let

$$
S \equiv\left\{z \in \Omega: f^{(n)}(z)=0 \text { for all } n\right\}
$$

It is clear that $S$ is a closed set which by assumption is nonempty. However, this set is also open. To see this, let $z \in S$. Then for all $w$ close enough to $z$,

$$
f(w)=\sum_{k=0}^{\infty} \frac{f^{(k)}(z)}{k!}(w-z)^{k}=0
$$

Thus $f$ is identically equal to zero near $z \in S$. Therefore, all points near $z$ are contained in $S$ also, showing that $S$ is an open set. Now $\Omega=S \cup(\Omega \backslash S)$, the union of two disjoint open sets, $S$ being nonempty. It follows the other open set, $\Omega \backslash S$, must be empty because $\Omega$ is connected. Therefore, the first condition is verified. This proves the theorem. (See the following diagram.)


Note how radically different this is from the theory of functions of a real variable. Consider, for example the function

$$
f(x) \equiv\left\{\begin{array}{l}
x^{2} \sin \left(\frac{1}{x}\right) \text { if } x \neq 0 \\
0 \text { if } x=0
\end{array}\right.
$$

which has a derivative for all $x \in \mathbb{R}$ and for which 0 is a limit point of the set, $Z$, even though $f$ is not identically equal to zero.

Here is a very important application called Euler's formula. Recall that

$$
\begin{equation*}
e^{z} \equiv e^{x}(\cos (y)+i \sin (y)) \tag{51.5.19}
\end{equation*}
$$

Is it also true that $e^{z}=\sum_{k=0}^{\infty} \frac{z^{k}}{k!}$ ?
Theorem 51.5.4 (Euler's Formula) Let $z=x+i y$. Then

$$
e^{z}=\sum_{k=0}^{\infty} \frac{z^{k}}{k!}
$$

Proof: It was already observed that $e^{z}$ given by 51.5 .19 is analytic. So is $\exp (z) \equiv$ $\sum_{k=0}^{\infty} \frac{z^{k}}{k!}$. In fact the power series converges for all $z \in \mathbb{C}$. Furthermore the two functions, $e^{z}$ and $\exp (z)$ agree on the real line which is a set which contains a limit point. Therefore, they agree for all values of $z \in \mathbb{C}$.

This formula shows the famous two identities,

$$
e^{i \pi}=-1 \text { and } e^{2 \pi i}=1
$$

### 51.6 Liouville's Theorem

The following theorem pertains to functions which are analytic on all of $\mathbb{C}$, "entire" functions.

Definition 51.6.1 A function, $f: \mathbb{C} \rightarrow \mathbb{C}$ or more generally, $f: \mathbb{C} \rightarrow X$ is entire means it is analytic on $\mathbb{C}$.

Theorem 51.6.2 (Liouville's theorem) If $f$ is a bounded entire function having values in $X$, then $f$ is a constant.

Proof: Since $f$ is entire, pick any $z \in \mathbb{C}$ and write

$$
f^{\prime}(z)=\frac{1}{2 \pi i} \int_{\gamma_{R}} \frac{f(w)}{(w-z)^{2}} d w
$$

where $\gamma_{R}(t)=z+R e^{i t}$ for $t \in[0,2 \pi]$. Therefore,

$$
\left\|f^{\prime}(z)\right\| \leq C \frac{1}{R}
$$

where $C$ is some constant depending on the assumed bound on $f$. Since $R$ is arbitrary, let $R \rightarrow \infty$ to obtain $f^{\prime}(z)=0$ for any $z \in \mathbb{C}$. It follows from this that $f$ is constant for if $z_{j} j=1,2$ are two complex numbers, let $h(t)=f\left(z_{1}+t\left(z_{2}-z_{1}\right)\right)$ for $t \in[0,1]$. Then $h^{\prime}(t)=f^{\prime}\left(z_{1}+t\left(z_{2}-z_{1}\right)\right)\left(z_{2}-z_{1}\right)=0$. By Lemmas 51.3.2-51.3.4 $h$ is a constant on $[0,1]$ which implies $f\left(z_{1}\right)=f\left(z_{2}\right)$.

With Liouville's theorem it becomes possible to give an easy proof of the fundamental theorem of algebra. It is ironic that all the best proofs of this theorem in algebra come from the subjects of analysis or topology. Out of all the proofs that have been given of this very important theorem, the following one based on Liouville's theorem is the easiest.

Theorem 51.6.3 (Fundamental theorem of Algebra) Let

$$
p(z)=z^{n}+a_{n-1} z^{n-1}+\cdots+a_{1} z+a_{0}
$$

be a polynomial where $n \geq 1$ and each coefficient is a complex number. Then there exists $z_{0} \in \mathbb{C}$ such that $p\left(z_{0}\right)=0$.

Proof: Suppose not. Then $p(z)^{-1}$ is an entire function. Also

$$
|p(z)| \geq|z|^{n}-\left(\left|a_{n-1}\right||z|^{n-1}+\cdots+\left|a_{1}\right||z|+\left|a_{0}\right|\right)
$$

and so $\lim _{|z| \rightarrow \infty}|p(z)|=\infty$ which implies $\lim _{|z| \rightarrow \infty}\left|p(z)^{-1}\right|=0$. It follows that, since $p(z)^{-1}$ is bounded for $z$ in any bounded set, we must have that $p(z)^{-1}$ is a bounded entire function. But then it must be constant. However since $p(z)^{-1} \rightarrow 0$ as $|z| \rightarrow \infty$, this constant can only be 0 . However, $\frac{1}{p(z)}$ is never equal to zero. This proves the theorem.

### 51.7 The General Cauchy Integral Formula

### 51.7.1 The Cauchy Goursat Theorem

This section gives a fundamental theorem which is essential to the development which follows and is closely related to the question of when a function has a primitive. First of all, if you have two points in $\mathbb{C}, z_{1}$ and $z_{2}$, you can consider $\gamma(t) \equiv z_{1}+t\left(z_{2}-z_{1}\right)$ for $t \in[0,1]$ to obtain a continuous bounded variation curve from $z_{1}$ to $z_{2}$. More generally, if $z_{1}, \cdots, z_{m}$ are points in $\mathbb{C}$ you can obtain a continuous bounded variation curve from $z_{1}$ to $z_{m}$ which consists of first going from $z_{1}$ to $z_{2}$ and then from $z_{2}$ to $z_{3}$ and so on, till in the end one goes from $z_{m-1}$ to $z_{m}$. We denote this piecewise linear curve as $\gamma\left(z_{1}, \cdots, z_{m}\right)$. Now let $T$ be a triangle with vertices $z_{1}, z_{2}$ and $z_{3}$ encountered in the counter clockwise direction as shown.


Denote by $\int_{\partial T} f(z) d z$, the expression, $\int_{\gamma\left(z_{1}, z_{2}, z_{3}, z_{1}\right)} f(z) d z$. Consider the following picture.


By Lemma 50.0.11

$$
\begin{equation*}
\int_{\partial T} f(z) d z=\sum_{k=1}^{4} \int_{\partial T_{k}^{1}} f(z) d z \tag{51.7.20}
\end{equation*}
$$

On the "inside lines" the integrals cancel as claimed in Lemma 50.0.11 because there are two integrals going in opposite directions for each of these inside lines.

Theorem 51.7.1 (Cauchy Goursat) Let $f: \Omega \rightarrow X$ have the property that $f^{\prime}(z)$ exists for all $z \in \Omega$ and let $T$ be a triangle contained in $\Omega$. Then

$$
\int_{\partial T} f(w) d w=0
$$

Proof: Suppose not. Then

$$
\left\|\int_{\partial T} f(w) d w\right\|=\alpha \neq 0
$$

From 51.7.20 it follows

$$
\alpha \leq \sum_{k=1}^{4}\left\|\int_{\partial T_{k}^{1}} f(w) d w\right\|
$$

and so for at least one of these $T_{k}^{1}$, denoted from now on as $T_{1}$,

$$
\left\|\int_{\partial T_{1}} f(w) d w\right\| \geq \frac{\alpha}{4}
$$

Now let $T_{1}$ play the same role as $T$, subdivide as in the above picture, and obtain $T_{2}$ such that

$$
\left\|\int_{\partial T_{2}} f(w) d w\right\| \geq \frac{\alpha}{4^{2}}
$$

Continue in this way, obtaining a sequence of triangles,

$$
T_{k} \supseteq T_{k+1}, \operatorname{diam}\left(T_{k}\right) \leq \operatorname{diam}(T) 2^{-k}
$$

and

$$
\left\|\int_{\partial T_{k}} f(w) d w\right\| \geq \frac{\alpha}{4^{k}}
$$

Then let $z \in \cap_{k=1}^{\infty} T_{k}$ and note that by assumption, $f^{\prime}(z)$ exists. Therefore, for all $k$ large enough,

$$
\int_{\partial T_{k}} f(w) d w=\int_{\partial T_{k}} f(z)+f^{\prime}(z)(w-z)+g(w) d w
$$

where $\|g(w)\|<\varepsilon|w-z|$. Now observe that $w \rightarrow f(z)+f^{\prime}(z)(w-z)$ has a primitive, namely,

$$
F(w)=f(z) w+f^{\prime}(z)(w-z)^{2} / 2 .
$$

Therefore, by Corollary 50.0.14.

$$
\int_{\partial T_{k}} f(w) d w=\int_{\partial T_{k}} g(w) d w .
$$

From the definition, of the integral,

$$
\begin{aligned}
\frac{\alpha}{4^{k}} & \leq\left\|\int_{\partial T_{k}} g(w) d w\right\| \leq \varepsilon \operatorname{diam}\left(T_{k}\right)\left(\text { length of } \partial T_{k}\right) \\
& \leq \varepsilon 2^{-k}(\text { length of } T) \operatorname{diam}(T) 2^{-k}
\end{aligned}
$$

and so

$$
\alpha \leq \varepsilon(\text { length of } T) \operatorname{diam}(T)
$$

Since $\varepsilon$ is arbitrary, this shows $\alpha=0$, a contradiction. Thus $\int_{\partial T} f(w) d w=0$ as claimed. This fundamental result yields the following important theorem.

Theorem 51.7.2 (Morera ${ }^{1}$ ) Let $\Omega$ be an open set and let $f^{\prime}(z)$ exist for all $z \in \Omega$. Let $D \equiv \overline{B\left(z_{0}, r\right)} \subseteq \Omega$. Then there exists $\varepsilon>0$ such that $f$ has a primitive on $B\left(z_{0}, r+\varepsilon\right)$.

Proof: Choose $\varepsilon>0$ small enough that $B\left(z_{0}, r+\varepsilon\right) \subseteq \Omega$. Then for $w \in B\left(z_{0}, r+\varepsilon\right)$, define

$$
F(w) \equiv \int_{\gamma\left(z_{0}, w\right)} f(u) d u
$$

Then by the Cauchy Goursat theorem, and $w \in B\left(z_{0}, r+\varepsilon\right)$, it follows that for $|h|$ small enough,

$$
\begin{aligned}
& \frac{F(w+h)-F(w)}{h}=\frac{1}{h} \int_{\gamma(w, w+h)} f(u) d u \\
& =\frac{1}{h} \int_{0}^{1} f(w+t h) h d t=\int_{0}^{1} f(w+t h) d t
\end{aligned}
$$

which converges to $f(w)$ due to the continuity of $f$ at $w$. This proves the theorem.
The following is a slight generalization of the above theorem which is also referred to as Morera's theorem.

[^32]Corollary 51.7.3 Let $\Omega$ be an open set and suppose that whenever

$$
\gamma\left(z_{1}, z_{2}, z_{3}, z_{1}\right)
$$

is a closed curve bounding a triangle $T$, which is contained in $\Omega$, and $f$ is a continuous function defined on $\Omega$, it follows that

$$
\int_{\gamma\left(z_{1}, z_{2}, z_{3}, z_{1}\right)} f(z) d z=0
$$

then $f$ is analytic on $\Omega$.
Proof: As in the proof of Morera's theorem, let $\overline{B\left(z_{0}, r\right)} \subseteq \Omega$ and use the given condition to construct a primitive, $F$ for $f$ on $B\left(z_{0}, r\right)$. Then $F$ is analytic and so by Theorem 51.3.11, it follows that $F$ and hence $f$ have infinitely many derivatives, implying that $f$ is analytic on $B\left(z_{0}, r\right)$. Since $z_{0}$ is arbitrary, this shows $f$ is analytic on $\Omega$.

### 51.7.2 A Redundant Assumption

Earlier in the definition of analytic, it was assumed the derivative is continuous. This assumption is redundant.

Theorem 51.7.4 Let $\Omega$ be an open set in $\mathbb{C}$ and suppose $f: \Omega \rightarrow X$ has the property that $f^{\prime}(z)$ exists for each $z \in \Omega$. Then $f$ is analytic on $\Omega$.

Proof: Let $z_{0} \in \Omega$ and let $B\left(z_{0}, r\right) \subseteq \Omega$. By Morera's theorem $f$ has a primitive, $F$ on $B\left(z_{0}, r\right)$. It follows that $F$ is analytic because it has a derivative, $f$, and this derivative is continuous. Therefore, by Theorem 51.3.11 $F$ has infinitely many derivatives on $B\left(z_{0}, r\right)$ implying that $f$ also has infinitely many derivatives on $B\left(z_{0}, r\right)$. Thus $f$ is analytic as claimed.

It follows a function is analytic on an open set, $\Omega$ if and only if $f^{\prime}(z)$ exists for $z \in \Omega$. This is because it was just shown the derivative, if it exists, is automatically continuous.

The same proof used to prove Theorem 51.7.2 implies the following corollary.
Corollary 51.7.5 Let $\Omega$ be a convex open set and suppose that $f^{\prime}(z)$ exists for all $z \in \Omega$. Then $f$ has a primitive on $\Omega$.

Note that this implies that if $\Omega$ is a convex open set on which $f^{\prime}(z)$ exists and if $\gamma$ : $[a, b] \rightarrow \Omega$ is a closed, continuous curve having bounded variation, then letting $F$ be a primitive of $f$ Theorem 50.0.13 implies

$$
\int_{\gamma} f(z) d z=F(\gamma(b))-F(\gamma(a))=0
$$

Notice how different this is from the situation of a function of a real variable! It is possible for a function of a real variable to have a derivative everywhere and yet the derivative can be discontinuous. A simple example is the following.

$$
f(x) \equiv\left\{\begin{array}{l}
x^{2} \sin \left(\frac{1}{x}\right) \text { if } x \neq 0 \\
0 \text { if } x=0
\end{array} .\right.
$$

Then $f^{\prime}(x)$ exists for all $x \in \mathbb{R}$. Indeed, if $x \neq 0$, the derivative equals $2 x \sin \frac{1}{x}-\cos \frac{1}{x}$ which has no limit as $x \rightarrow 0$. However, from the definition of the derivative of a function of one variable, $f^{\prime}(0)=0$.

### 51.7.3 Classification Of Isolated Singularities

First some notation.
Definition 51.7.6 Let $B^{\prime}(a, r) \equiv\{z \in \mathbb{C}$ such that $0<|z-a|<r\}$. Thus this is the usual ball without the center. A function is said to have an isolated singularity at the point $a \in \mathbb{C}$ if $f$ is analytic on $B^{\prime}(a, r)$ for some $r>0$.

It turns out isolated singularities can be neatly classified into three types, removable singularities, poles, and essential singularities. The next theorem deals with the case of a removable singularity.
Definition 51.7.7 An isolated singularity of $f$ is said to be removable if there exists an analytic function, $g$ analytic at a and near a such that $f=g$ at all points near $a$.

Theorem 51.7.8 Let $f: B^{\prime}(a, r) \rightarrow X$ be analytic. Thus $f$ has an isolated singularity at $a$. Suppose also that

$$
\lim _{z \rightarrow a} f(z)(z-a)=0
$$

Then there exists a unique analytic function, $g: B(a, r) \rightarrow X$ such that $g=f$ on $B^{\prime}(a, r)$. Thus the singularity at a is removable.

Proof: Let $h(z) \equiv(z-a)^{2} f(z), h(a) \equiv 0$. Then $h$ is analytic on $B(a, r)$ because it is easy to see that $h^{\prime}(a)=0$. It follows $h$ is given by a power series,

$$
h(z)=\sum_{k=2}^{\infty} a_{k}(z-a)^{k}
$$

where $a_{0}=a_{1}=0$ because of the observation above that $h^{\prime}(a)=h(a)=0$. It follows that for $|z-a|>0$

$$
f(z)=\sum_{k=2}^{\infty} a_{k}(z-a)^{k-2} \equiv g(z)
$$

This proves the theorem.
What of the other case where the singularity is not removable? This situation is dealt with by the amazing Casorati Weierstrass theorem.

Theorem 51.7.9 (Casorati Weierstrass) Let a be an isolated singularity and suppose for some $r>0, f\left(B^{\prime}(a, r)\right)$ is not dense in $\mathbb{C}$. Then either a is a removable singularity or there exist finitely many $b_{1}, \cdots, b_{M}$ for some finite number, $M$ such that for $z$ near $a$,

$$
\begin{equation*}
f(z)=g(z)+\sum_{k=1}^{M} \frac{b_{k}}{(z-a)^{k}} \tag{51.7.21}
\end{equation*}
$$

where $g(z)$ is analytic near $a$.

Proof: Suppose $B\left(z_{0}, \boldsymbol{\delta}\right)$ has no points of $f\left(B^{\prime}(a, r)\right)$. If $f\left(B^{\prime}(a, r)\right)$ is not dense, then such a ball exists. Then for $z \in B^{\prime}(a, r),\left|f(z)-z_{0}\right| \geq \delta>0$. It follows from Theorem 51.7.8 that $\frac{1}{f(z)-z_{0}}$ has a removable singularity at $a$. Hence, there exists $h$ an analytic function such that for $z$ near $a$,

$$
\begin{equation*}
h(z)=\frac{1}{f(z)-z_{0}} . \tag{51.7.22}
\end{equation*}
$$

There are two cases. First suppose $h(a)=0$. Then $\sum_{k=1}^{\infty} a_{k}(z-a)^{k}=\frac{1}{f(z)-z_{0}}$ for $z$ near $a$. If all the $a_{k}=0$, this would be a contradiction because then the left side would equal zero for $z$ near $a$ but the right side could not equal zero. Therefore, there is a first $m$ such that $a_{m} \neq 0$. Hence there exists an analytic function, $k(z)$ which is not equal to zero in some ball, $B(a, \varepsilon)$ such that

$$
k(z)(z-a)^{m}=\frac{1}{f(z)-z_{0}}
$$

Hence, taking both sides to the -1 power,

$$
f(z)-z_{0}=\frac{1}{(z-a)^{m}} \sum_{k=0}^{\infty} b_{k}(z-a)^{k}
$$

and so 51.7.21 holds.
The other case is that $h(a) \neq 0$. In this case, raise both sides of 51.7.22 to the -1 power and obtain

$$
f(z)-z_{0}=h(z)^{-1}
$$

a function analytic near $a$. Therefore, the singularity is removable. This proves the theorem.
This theorem is the basis for the following definition which classifies isolated singularities.

Definition 51.7.10 Let a be an isolated singularity of a complex valued function, $f$. When 51.7.21 holds for $z$ near $a$, then $a$ is called a pole. The order of the pole in 51.7 .21 is $M$. If for every $r>0, f\left(B^{\prime}(a, r)\right)$ is dense in $\mathbb{C}$ then a is called an essential singularity.

In terms of the above definition, isolated singularities are either removable, a pole, or essential. There are no other possibilities.

Theorem 51.7.11 Suppose $f: \Omega \rightarrow \mathbb{C}$ has an isolated singularity at $a \in \Omega$. Then $a$ is $a$ pole if and only if

$$
\lim _{z \rightarrow a} d(f(z), \infty)=0
$$

in $\widehat{\mathbb{C}}$.
Proof: Suppose first $f$ has a pole at $a$. Then by definition, $f(z)=g(z)+\sum_{k=1}^{M} \frac{b_{k}}{(z-a)^{k}}$ for $z$ near $a$ where $g$ is analytic. Then

$$
\begin{aligned}
|f(z)| & \geq \frac{\left|b_{M}\right|}{|z-a|^{M}}-|g(z)|-\sum_{k=1}^{M-1} \frac{\left|b_{k}\right|}{|z-a|^{k}} \\
& =\frac{1}{|z-a|^{M}}\left(\left|b_{M}\right|-\left(|g(z)||z-a|^{M}+\sum_{k=1}^{M-1}\left|b_{k}\right||z-a|^{M-k}\right)\right)
\end{aligned}
$$

Now $\lim _{z \rightarrow a}\left(|g(z)||z-a|^{M}+\sum_{k=1}^{M-1}\left|b_{k}\right||z-a|^{M-k}\right)=0$ and so the above inequality proves $\lim _{z \rightarrow a}|f(z)|=\infty$. Referring to the diagram on Page 1597, you see this is the same as saying

$$
\lim _{z \rightarrow a}|\theta f(z)-(0,0,2)|=\lim _{z \rightarrow a}|\theta f(z)-\theta(\infty)|=\lim _{z \rightarrow a} d(f(z), \infty)=0
$$

Conversely, suppose $\lim _{z \rightarrow a} d(f(z), \infty)=0$. Then from the diagram on Page 1597, it follows $\lim _{z \rightarrow a}|f(z)|=\infty$ and in particular, $a$ cannot be either removable or an essential singularity by the Casorati Weierstrass theorem, Theorem 51.7.9. The only case remaining is that $a$ is a pole. This proves the theorem.

Definition 51.7.12 Let $f: \Omega \rightarrow \mathbb{C}$ where $\Omega$ is an open subset of $\mathbb{C}$. Then $f$ is called meromorphic if all singularities are isolated and are either poles or removable and this set of singularities has no limit point. It is convenient to regard meromorphic functions as having values in $\widehat{\mathbb{C}}$ where if $a$ is a pole, $f(a) \equiv \infty$. From now on, this will be assumed when $a$ meromorphic function is being considered.

The usefulness of the above convention about $f(a) \equiv \infty$ at a pole is made clear in the following theorem.

Theorem 51.7.13 Let $\Omega$ be an open subset of $\mathbb{C}$ and let $f: \Omega \rightarrow \widehat{\mathbb{C}}$ be meromorphic. Then $f$ is continuous with respect to the metric, $d$ on $\widehat{\mathbb{C}}$.

Proof: Let $z_{n} \rightarrow z$ where $z \in \Omega$. Then if $z$ is a pole, it follows from Theorem 51.7.11 that

$$
d\left(f\left(z_{n}\right), \infty\right) \equiv d\left(f\left(z_{n}\right), f(z)\right) \rightarrow 0
$$

If $z$ is not a pole, then $f\left(z_{n}\right) \rightarrow f(z)$ in $\mathbb{C}$ which implies

$$
\left|\theta\left(f\left(z_{n}\right)\right)-\theta(f(z))\right|=d\left(f\left(z_{n}\right), f(z)\right) \rightarrow 0
$$

Recall that $\theta$ is continuous on $\mathbb{C}$.

### 51.7.4 The Cauchy Integral Formula

This section presents the general version of the Cauchy integral formula valid for arbitrary closed rectifiable curves. The key idea in this development is the notion of the winding number. This is the number also called the index, defined in the following theorem. This winding number, along with the earlier results, especially Liouville's theorem, yields an extremely general Cauchy integral formula.

Definition 51.7.14 Let $\gamma:[a, b] \rightarrow \mathbb{C}$ and suppose $z \notin \gamma^{*}$. The winding number, $n(\gamma, z)$ is defined by

$$
n(\gamma, z) \equiv \frac{1}{2 \pi i} \int_{\gamma} \frac{d w}{w-z}
$$

The main interest is in the case where $\gamma$ is closed curve. However, the same notation will be used for any such curve.

Theorem 51.7.15 Let $\gamma:[a, b] \rightarrow \mathbb{C}$ be continuous and have bounded variation with $\gamma(a)=$ $\gamma(b)$. Also suppose that $z \notin \gamma^{*}$. Define

$$
\begin{equation*}
n(\gamma, z) \equiv \frac{1}{2 \pi i} \int_{\gamma} \frac{d w}{w-z} \tag{51.7.23}
\end{equation*}
$$

Then $n(\gamma, \cdot)$ is continuous and integer valued. Furthermore, there exists a sequence, $\eta_{k}$ : $[a, b] \rightarrow \mathbb{C}$ such that $\eta_{k}$ is $C^{1}([a, b])$,

$$
\left\|\eta_{k}-\gamma\right\|<\frac{1}{k}, \eta_{k}(a)=\eta_{k}(b)=\gamma(a)=\gamma(b)
$$

and $n\left(\eta_{k}, z\right)=n(\gamma, z)$ for all $k$ large enough. Also $n(\gamma, \cdot)$ is constant on every connected component of $\mathbb{C} \backslash \gamma^{*}$ and equals zero on the unbounded component of $\mathbb{C} \backslash \gamma^{*}$.

Proof: First consider the assertion about continuity.

$$
\begin{aligned}
\left|n(\gamma, z)-n\left(\gamma, z_{1}\right)\right| & \leq C\left|\int_{\gamma}\left(\frac{1}{w-z}-\frac{1}{w-z_{1}}\right) d w\right| \\
& \leq \widetilde{C}(\text { Length of } \gamma)\left|z_{1}-z\right|
\end{aligned}
$$

whenever $z_{1}$ is close enough to $z$. This proves the continuity assertion. Note this did not depend on $\gamma$ being closed.

Next it is shown that for a closed curve the winding number equals an integer. To do so, use Theorem 50.0.12 to obtain $\eta_{k}$, a function in $C^{1}([a, b])$ such that $z \notin \eta_{k}([a, b])$ for all $k$ large enough, $\eta_{k}(x)=\gamma(x)$ for $x=a, b$, and

$$
\left|\frac{1}{2 \pi i} \int_{\gamma} \frac{d w}{w-z}-\frac{1}{2 \pi i} \int_{\eta_{k}} \frac{d w}{w-z}\right|<\frac{1}{k},\left\|\eta_{k}-\gamma\right\|<\frac{1}{k}
$$

It is shown that each of $\frac{1}{2 \pi i} \int_{\eta_{k}} \frac{d w}{w-z}$ is an integer. To simplify the notation, write $\eta$ instead of $\eta_{k}$.

$$
\int_{\eta} \frac{d w}{w-z}=\int_{a}^{b} \frac{\eta^{\prime}(s) d s}{\eta(s)-z}
$$

Define

$$
\begin{equation*}
g(t) \equiv \int_{a}^{t} \frac{\eta^{\prime}(s) d s}{\eta(s)-z} \tag{51.7.24}
\end{equation*}
$$

Then

$$
\begin{aligned}
\left(e^{-g(t)}(\eta(t)-z)\right)^{\prime} & =e^{-g(t)} \eta^{\prime}(t)-e^{-g(t)} g^{\prime}(t)(\eta(t)-z) \\
& =e^{-g(t)} \eta^{\prime}(t)-e^{-g(t)} \eta^{\prime}(t)=0
\end{aligned}
$$

It follows that $e^{-g(t)}(\eta(t)-z)$ equals a constant. In particular, using the fact that $\eta(a)=$ $\eta(b)$,

$$
e^{-g(b)}(\eta(b)-z)=e^{-g(a)}(\eta(a)-z)=(\eta(a)-z)=(\eta(b)-z)
$$

and so $e^{-g(b)}=1$. This happens if and only if $-g(b)=2 m \pi i$ for some integer $m$. Therefore, 51.7.24 implies

$$
2 m \pi i=\int_{a}^{b} \frac{\eta^{\prime}(s) d s}{\eta(s)-z}=\int_{\eta} \frac{d w}{w-z}
$$

Therefore, $\frac{1}{2 \pi i} \int_{\eta_{k}} \frac{d w}{w-z}$ is a sequence of integers converging to $\frac{1}{2 \pi i} \int_{\gamma} \frac{d w}{w-z} \equiv n(\gamma, z)$ and so $n(\gamma, z)$ must also be an integer and $n\left(\eta_{k}, z\right)=n(\gamma, z)$ for all $k$ large enough.

Since $n(\gamma, \cdot)$ is continuous and integer valued, it follows from Corollary 7.13.11 on Page 172 that it must be constant on every connected component of $\mathbb{C} \backslash \gamma^{*}$. It is clear that $n(\gamma, z)$ equals zero on the unbounded component because from the formula,

$$
\lim _{z \rightarrow \infty}|n(\gamma, z)| \leq \lim _{z \rightarrow \infty} V(\gamma,[a, b])\left(\frac{1}{|z|-c}\right)
$$

where $c \geq \max \left\{|w|: w \in \gamma^{*}\right\}$. This proves the theorem.
Corollary 51.7.16 Suppose $\gamma:[a, b] \rightarrow \mathbb{C}$ is a continuous bounded variation curve and $n(\gamma, z)$ is an integer where $z \notin \gamma^{*}$. Then $\gamma(a)=\gamma(b)$. Also $z \rightarrow n(\gamma, z)$ for $z \notin \gamma^{*}$ is continuous.

Proof: Letting $\eta$ be a $C^{1}$ curve for which $\eta(a)=\gamma(a)$ and $\eta(b)=\gamma(b)$ and which is close enough to $\gamma$ that $n(\eta, z)=n(\gamma, z)$, the argument is similar to the above. Let

$$
\begin{equation*}
g(t) \equiv \int_{a}^{t} \frac{\eta^{\prime}(s) d s}{\eta(s)-z} \tag{51.7.25}
\end{equation*}
$$

Then

$$
\begin{aligned}
\left(e^{-g(t)}(\eta(t)-z)\right)^{\prime} & =e^{-g(t)} \eta^{\prime}(t)-e^{-g(t)} g^{\prime}(t)(\eta(t)-z) \\
& =e^{-g(t)} \eta^{\prime}(t)-e^{-g(t)} \eta^{\prime}(t)=0
\end{aligned}
$$

Hence

$$
\begin{equation*}
e^{-g(t)}(\eta(t)-z)=c \neq 0 \tag{51.7.26}
\end{equation*}
$$

By assumption

$$
g(b)=\int_{\eta} \frac{1}{w-z} d w=2 \pi i m
$$

for some integer, $m$. Therefore, from 51.7.26

$$
1=e^{2 \pi m i}=\frac{\eta(b)-z}{c}
$$

Thus $c=\eta(b)-z$ and letting $t=a$ in 51.7.26,

$$
1=\frac{\eta(a)-z}{\eta(b)-z}
$$

which shows $\eta(a)=\eta(b)$. This proves the corollary since the assertion about continuity was already observed.

It is a good idea to consider a simple case to get an idea of what the winding number is measuring. To do so, consider $\gamma:[a, b] \rightarrow \mathbb{C}$ such that $\gamma$ is continuous, closed and bounded variation. Suppose also that $\gamma$ is one to one on $(a, b)$. Such a curve is called a simple closed curve. It can be shown that such a simple closed curve divides the plane into exactly two components, an "inside" bounded component and an "outside" unbounded component. This is called the Jordan Curve theorem. This is a difficult theorem which requires some very hard topology such as homology theory or degree theory. It won't be used here beyond making reference to it. For now, it suffices to simply assume that $\gamma$ is such that this result holds. This will usually be obvious anyway. Also suppose that it is possible to change the parameter to be in $[0,2 \pi]$, in such a way that $\gamma(t)+\lambda\left(z+r e^{i t}-\gamma(t)\right)-z \neq 0$ for all $t \in[0,2 \pi]$ and $\lambda \in[0,1]$. (As $t$ goes from 0 to $2 \pi$ the point $\gamma(t)$ traces the curve $\gamma([0,2 \pi])$ in the counter clockwise direction.) Suppose $z \in D$, the inside of the simple closed curve and consider the curve $\delta(t)=z+r e^{i t}$ for $t \in[0,2 \pi]$ where $r$ is chosen small enough that $\overline{B(z, r)} \subseteq D$. Then it happens that $n(\delta, z)=n(\gamma, z)$.

Proposition 51.7.17 Under the above conditions,

$$
n(\delta, z)=n(\gamma, z)
$$

and $n(\delta, z)=1$.
Proof: By changing the parameter, assume that $[a, b]=[0,2 \pi]$. From Theorem 51.7.15 it suffices to assume also that $\gamma$ is $C^{1}$. Define $h_{\lambda}(t) \equiv \gamma(t)+\lambda\left(z+r e^{i t}-\gamma(t)\right)$ for $\lambda \in$ $[0,1]$. (This function is called a homotopy of the curves $\gamma$ and $\delta$.) Note that for each $\lambda \in$ $[0,1], t \rightarrow h_{\lambda}(t)$ is a closed $C^{1}$ curve. Also,

$$
\frac{1}{2 \pi i} \int_{h_{\lambda}} \frac{1}{w-z} d w=\frac{1}{2 \pi i} \int_{0}^{2 \pi} \frac{\gamma^{\prime}(t)+\lambda\left(r i e^{i t}-\gamma^{\prime}(t)\right)}{\gamma(t)+\lambda\left(z+r e^{i t}-\gamma(t)\right)-z} d t
$$

This number is an integer and it is routine to verify that it is a continuous function of $\lambda$. When $\lambda=0$ it equals $n(\gamma, z)$ and when $\lambda=1$ it equals $n(\boldsymbol{\delta}, z)$. Therefore, $n(\boldsymbol{\delta}, z)=n(\gamma, z)$. It only remains to compute $n(\boldsymbol{\delta}, z)$.

$$
n(\delta, z)=\frac{1}{2 \pi i} \int_{0}^{2 \pi} \frac{r i e^{i t}}{r e^{i t}} d t=1
$$

This proves the proposition.
Now if $\gamma$ was not one to one but caused the point, $\gamma(t)$ to travel around $\gamma^{*}$ twice, you could modify the above argument to have the parameter interval, $[0,4 \pi]$ and still find $n(\delta, z)=n(\gamma, z)$ only this time, $n(\delta, z)=2$. Thus the winding number is just what its name suggests. It measures the number of times the curve winds around the point. One might ask why bother with the winding number if this is all it does. The reason is that the notion of counting the number of times a curve winds around a point is rather vague. The winding number is precise. It is also the natural thing to consider in the general Cauchy integral formula presented below. Consider a situation typified by the following picture in which $\Omega$ is the open set between the dotted curves and $\gamma_{j}$ are closed rectifiable curves in $\Omega$.


The following theorem is the general Cauchy integral formula.
Definition 51.7.18 Let $\left\{\gamma_{k}\right\}_{k=1}^{n}$ be continuous oriented curves having bounded variation. Then this is called a cycle if whenever, $z \notin \cup_{k=1}^{n} \gamma_{k}^{*}, \sum_{k=1}^{n} n\left(\gamma_{k}, z\right)$ is an integer.

By Theorem 51.7.15 if each $\gamma_{k}$ is a closed curve, then $\left\{\gamma_{k}\right\}_{k=1}^{n}$ is a cycle.
Theorem 51.7.19 Let $\Omega$ be an open subset of the plane and let $f: \Omega \rightarrow X$ be analytic. If $\gamma_{k}:\left[a_{k}, b_{k}\right] \rightarrow \Omega, k=1, \cdots, m$ are continuous curves having bounded variation such that for all $z \notin \cup_{k=1}^{m} \gamma_{k}\left(\left[a_{k}, b_{k}\right]\right)$

$$
\sum_{k=1}^{m} n\left(\gamma_{k}, z\right) \text { equals an integer }
$$

and for all $z \notin \Omega$,

$$
\sum_{k=1}^{m} n\left(\gamma_{k}, z\right)=0
$$

Then for all $z \in \Omega \backslash \cup_{k=1}^{m} \gamma_{k}\left(\left[a_{k}, b_{k}\right]\right)$,

$$
f(z) \sum_{k=1}^{m} n\left(\gamma_{k}, z\right)=\sum_{k=1}^{m} \frac{1}{2 \pi i} \int_{\gamma_{k}} \frac{f(w)}{w-z} d w .
$$

Proof: Let $\phi$ be defined on $\Omega \times \Omega$ by

$$
\phi(z, w) \equiv\left\{\begin{array}{l}
\frac{f(w)-f(z)}{w-z} \text { if } w \neq z \\
f^{\prime}(z) \text { if } w=z
\end{array}\right.
$$

Then $\phi$ is analytic as a function of both $z$ and $w$ and is continuous in $\Omega \times \Omega$. Here is why: It is clear that $\frac{d}{d w} \phi(z, \cdot)(w)$ exists if $w \neq z$. It remains to consider whether $\frac{d}{d z} \phi(\cdot, z)(z)$ exists. One needs to consider

$$
\frac{\phi(z+h, z)-\phi(z, z)}{h}=\frac{\frac{f(z+h)-f(z)}{h}-f^{\prime}(z)}{h}
$$

We can write $f(z+h)$ as a power series in $h$ whenever $h$ is suitably small.

$$
\frac{\frac{f(z+h)-f(z)}{h}-f^{\prime}(z)}{h}=
$$

$$
\begin{aligned}
& =\frac{1}{h}\left(\frac{1}{h}\left(f^{\prime}(z) h+\frac{1}{2!} f^{\prime \prime}(z) h^{2}+\frac{1}{3!} f^{\prime \prime \prime}(z) h^{3}+\cdots\right)-f^{\prime}(z)\right) \\
& =\frac{1}{h}\left(\left(f^{\prime}(z)+\frac{1}{2!} f^{\prime \prime}(z) h+\frac{1}{3!} f^{\prime \prime \prime}(z) h^{2}+\cdots\right)-f^{\prime}(z)\right) \\
& =\frac{1}{2!} f^{\prime \prime}(z)+\frac{1}{3!} f^{\prime \prime \prime}(z) h+\text { higher order terms }
\end{aligned}
$$

Thus the limit of the difference quotient exists and is $\frac{1}{2!} f^{\prime \prime}(z)$.
Define

$$
h(z) \equiv \frac{1}{2 \pi i} \sum_{k=1}^{m} \int_{\gamma_{k}} \phi(z, w) d w
$$

Is $h$ analytic on $\Omega$ ? To show this is the case, verify

$$
\int_{\partial T} h(z) d z=0
$$

for every triangle, $T$, contained in $\Omega$ and apply Corollary 51.7.3. This is an application of the Fubini theorem of Theorem 50.0.15. By Theorem 50.0.15,

$$
\int_{\partial T} \int_{\gamma_{k}} \phi(z, w) d w d z=\int_{\gamma_{k}} \overbrace{\int_{\partial T} \phi(z, w) d z}^{=0} d w=0
$$

because $\phi$ is analytic. By Corollary 51.7.3, $h$ is analytic on $\Omega$ as claimed.
Now let $H$ denote the set,

$$
\begin{aligned}
H & \equiv\left\{z \in \mathbb{C} \backslash \cup_{k=1}^{m} \gamma_{k}^{*}: \sum_{k=1}^{m} n\left(\gamma_{k}, z\right)=0\right\} \\
& =\left\{z \in \mathbb{C} \backslash \cup_{k=1}^{m} \gamma_{k}^{*}: \sum_{k=1}^{m} n\left(\gamma_{k}, z\right) \in(-1 / 2,1 / 2)\right\}
\end{aligned}
$$

the second equality holding because it is given that the sum of these is integer valued. Thus $H$ is an open set because $z \rightarrow \sum_{k=1}^{m} n\left(\gamma_{k}, z\right)$ is continuous. This is obvious from the formula for $n\left(\gamma_{k}, z\right)$. Also, $\Omega \cup H=\mathbb{C}$ because by assumption, $\Omega^{C} \subseteq H$. Extend $h(z)$ to all of $\mathbb{C}$ as follows:

$$
g(z) \equiv\left\{\begin{array}{l}
h(z) \equiv \frac{1}{2 \pi i} \sum_{k=1}^{m} \int_{\gamma_{k}} \phi(z, w) d w \text { if } z \in \Omega  \tag{51.7.27}\\
\frac{1}{2 \pi i} \sum_{k=1}^{m} \int_{\gamma_{k}} \frac{f(w)}{w-z} d w \text { if } z \in H
\end{array}\right.
$$

Why is $g(z)$ well defined? Then on $\Omega \cap H, z \notin \cup_{k=1}^{m} \gamma_{k}^{*}$ and so

$$
\begin{aligned}
g(z) & =\frac{1}{2 \pi i} \sum_{k=1}^{m} \int_{\gamma_{k}} \phi(z, w) d w=\frac{1}{2 \pi i} \sum_{k=1}^{m} \int_{\gamma_{k}} \frac{f(w)-f(z)}{w-z} d w \\
& =\frac{1}{2 \pi i} \sum_{k=1}^{m} \int_{\gamma_{k}} \frac{f(w)}{w-z} d w-\frac{1}{2 \pi i} \sum_{k=1}^{m} \int_{\gamma_{k}} \frac{f(z)}{w-z} d w \\
& =\frac{1}{2 \pi i} \sum_{k=1}^{m} \int_{\gamma_{k}} \frac{f(w)}{w-z} d w
\end{aligned}
$$

because $z \in H$. This shows $g(z)$ is well defined. Also, $g$ is analytic on $\Omega$ because it equals $h$ there. It is routine to verify that $g$ is analytic on $H$ also because of the second line of 51.7.27. (See discussion at the end if this is not clear. )

Therefore, $g$ is an entire function, meaning that it is analytic on all of $\mathbb{C}$.
Now note that $\sum_{k=1}^{m} n\left(\gamma_{k}, z\right)=0$ for all $z$ contained in the unbounded component of $\mathbb{C} \backslash \cup_{k=1}^{m} \gamma_{k}^{*}$ which component contains $B(0, r)^{C}$ for $r$ large enough. It follows that for $|z|>r$, it must be the case that $z \in H$ and so for such $z$, the bottom description of $g(z)$ found in 51.7.27 is valid. Therefore, it follows

$$
\lim _{|z| \rightarrow \infty}\|g(z)\|=0
$$

and so $g$ is bounded and analytic on all of $\mathbb{C}$. By Liouville's theorem, $g$ is a constant. Hence, from the above equation, the constant can only equal zero.

For $z \in \Omega \backslash \cup_{k=1}^{m} \gamma_{k}^{*}$, since it was just shown that $h(z)=g(z)=0$ on $\Omega$

$$
\begin{aligned}
0=h(z)= & \frac{1}{2 \pi i} \sum_{k=1}^{m} \int_{\gamma_{k}} \phi(z, w) d w=\frac{1}{2 \pi i} \sum_{k=1}^{m} \int_{\gamma_{k}} \frac{f(w)-f(z)}{w-z} d w= \\
& \frac{1}{2 \pi i} \sum_{k=1}^{m} \int_{\gamma_{k}} \frac{f(w)}{w-z} d w-f(z) \sum_{k=1}^{m} n\left(\gamma_{k}, z\right) .
\end{aligned}
$$

In case it is not obvious why $g$ is analytic on $H$, use the formula. It reduces to showing that

$$
z \rightarrow \int_{\gamma_{k}} \frac{f(w)}{w-z} d w
$$

is analytic. However, taking a difference quotient and simplifying a little, one obtains

$$
\frac{\int_{\gamma_{k}} \frac{f(w)}{w-(z+h)} d w-\int_{\gamma_{k}} \frac{f(w)}{w-z} d w}{h}=\int_{\gamma_{k}} \frac{f(w)}{(w-z)(w-(z+h))} d w
$$

considering only small $h$, the denominator is bounded below by some $\delta>0$ and also $f(w)$ is bounded on the compact set $\gamma_{k}^{*},|f(w)| \leq M$. Then for such small $h$,

$$
\begin{aligned}
& \left|\frac{f(w)}{(w-z)(w-(z+h))}-\frac{f(w)}{(w-z)^{2}}\right| \\
= & \left|\frac{1}{w-z}\left(\frac{1}{(w-(z+h))}-\frac{1}{(w-z)}\right) f(w)\right| \\
\leq & \left|\frac{1}{w-z}\right| \frac{1}{\delta} h M
\end{aligned}
$$

it follows that one obtains uniform convergence as $h \rightarrow 0$ of the integrand to $\frac{f(w)}{(w-z)^{2}}$ for any sequence $h \rightarrow 0$ and so the integral converges to

$$
\int_{\gamma_{k}} \frac{f(w)}{(w-z)^{2}} d w
$$

Corollary 51.7.20 Let $\Omega$ be an open set and let $\gamma_{k}:\left[a_{k}, b_{k}\right] \rightarrow \Omega, k=1, \cdots, m$, be closed, continuous and of bounded variation. Suppose also that $\sum_{k=1}^{m} n\left(\gamma_{k}, z\right)=0$ for all $z \notin \Omega$. Then if $f: \Omega \rightarrow \mathbb{C}$ is analytic, $\sum_{k=1}^{m} \int_{\gamma_{k}} f(w) d w=0$.

Proof: This follows from Theorem 51.7.19 as follows. Let

$$
g(w)=f(w)(w-z)
$$

where $z \in \Omega \backslash \cup_{k=1}^{m} \gamma_{k}\left(\left[a_{k}, b_{k}\right]\right)$. Then by this theorem,

$$
\begin{gathered}
0=0 \sum_{k=1}^{m} n\left(\gamma_{k}, z\right)=g(z) \sum_{k=1}^{m} n\left(\gamma_{k}, z\right)= \\
\sum_{k=1}^{m} \frac{1}{2 \pi i} \int_{\gamma_{k}} \frac{g(w)}{w-z} d w=\frac{1}{2 \pi i} \sum_{k=1}^{m} \int_{\gamma_{k}} f(w) d w .
\end{gathered}
$$

Another simple corollary to the above theorem is Cauchy's theorem for a simply connected region.

Definition 51.7.21 An open set, $\Omega \subseteq \mathbb{C}$ is a region if it is open and connected. A region, $\Omega$ is simply connected if $\widehat{\mathbb{C}} \backslash \Omega$ is connected where $\widehat{\mathbb{C}}$ is the extended complex plane. In the future, the term simply connected open set will be an open set which is connected and $\widehat{\mathbb{C}}$ $\backslash \Omega$ is connected.

Corollary 51.7.22 Let $\gamma:[a, b] \rightarrow \Omega$ be a continuous closed curve of bounded variation where $\Omega$ is a simply connected region in $\mathbb{C}$ and let $f: \Omega \rightarrow X$ be analytic. Then

$$
\int_{\gamma} f(w) d w=0
$$

Proof: Let $D$ denote the unbounded component of $\widehat{\mathbb{C}} \backslash \gamma^{*}$. Thus $\infty \in \widehat{\mathbb{C}} \backslash \gamma^{*}$. Then the connected set, $\widehat{\mathbb{C}} \backslash \Omega$ is contained in $D$ since every point of $\widehat{\mathbb{C}} \backslash \Omega$ must be in some component of $\widehat{\mathbb{C}} \backslash \gamma^{*}$ and $\infty$ is contained in both $\widehat{\mathbb{C}} \backslash \Omega$ and $D$. Thus $D$ must be the component that contains $\widehat{\mathbb{C}} \backslash \Omega$. It follows that $n(\gamma, \cdot)$ must be constant on $\widehat{\mathbb{C}} \backslash \Omega$, its value being its value on $D$. However, for $z \in D, n(\gamma, z)=\frac{1}{2 \pi i} \int_{\gamma} \frac{1}{w-z} d w$ and so $\lim _{|z| \rightarrow \infty} n(\gamma, z)=0$ showing $n(\gamma, z)=0$ on $D$. Therefore this verifies the hypothesis of Theorem 51.7.19. Let $z \in \Omega \cap D$ and define $g(w) \equiv f(w)(w-z)$. Thus $g$ is analytic on $\Omega$ and by Theorem 51.7.19,

$$
0=n(z, \gamma) g(z)=\frac{1}{2 \pi i} \int_{\gamma} \frac{g(w)}{w-z} d w=\frac{1}{2 \pi i} \int_{\gamma} f(w) d w
$$

This proves the corollary.
The following is a very significant result which will be used later.
Corollary 51.7.23 Suppose $\Omega$ is a simply connected open set and $f: \Omega \rightarrow X$ is analytic. Then $f$ has a primitive, $F$, on $\Omega$. Recall this means there exists $F$ such that $F^{\prime}(z)=f(z)$ for all $z \in \Omega$.

Proof: Pick a point, $z_{0} \in \Omega$ and let $V$ denote those points, $z$ of $\Omega$ for which there exists a curve, $\gamma:[a, b] \rightarrow \Omega$ such that $\gamma$ is continuous, of bounded variation, $\gamma(a)=z_{0}$, and $\gamma(b)=z$. Then it is easy to verify that $V$ is both open and closed in $\Omega$ and therefore, $V=\Omega$ because $\Omega$ is connected. Denote by $\gamma_{z_{0}, z}$ such a curve from $z_{0}$ to $z$ and define $F(z) \equiv$ $\int_{\gamma_{z_{0}, z}} f(w) d w$. Then $F$ is well defined because if $\gamma_{j}, j=1,2$ are two such curves, it follows from Corollary 51.7.22 that $\int_{\gamma_{1}} f(w) d w+\int_{-\gamma_{2}} f(w) d w=0$, implying that $\int_{\gamma_{1}} f(w) d w=$ $\int_{\gamma_{2}} f(w) d w$.Now this function, $F$ is a primitive because, thanks to Corollary 51.7.22

$$
(F(z+h)-F(z)) h^{-1}=\frac{1}{h} \int_{\gamma_{z, z+h}} f(w) d w=\frac{1}{h} \int_{0}^{1} f(z+t h) h d t
$$

and so, taking the limit as $h \rightarrow 0, F^{\prime}(z)=f(z)$.

### 51.7.5 An Example Of A Cycle

The next theorem deals with the existence of a cycle with nice properties. Basically, you go around the compact subset of an open set with suitable contours while staying in the open set. The method involves the following simple concept.

Definition 51.7.24 A tiling of $\mathbb{R}^{2}=\mathbb{C}$ is the union of infinitely many equally spaced vertical and horizontal lines. You can think of the small squares which result as tiles. To tile the plane or $\mathbb{R}^{2}=\mathbb{C}$ means to consider such a union of horizontal and vertical lines. It is like graph paper. See the picture below for a representation of part of a tiling of $\mathbb{C}$.


Theorem 51.7.25 Let $K$ be a compact subset of an open set, $\Omega$. Then there exist continuous, closed, bounded variation oriented curves $\left\{\Gamma_{j}\right\}_{j=1}^{m}$ for which $\Gamma_{j}^{*} \cap K=\emptyset$ for each $j$, $\Gamma_{j}^{*} \subseteq \Omega$, and for all $p \in K, \sum_{k=1}^{m} n\left(\Gamma_{k}, p\right)=1$ while for all $z \notin \Omega, \sum_{k=1}^{m} n\left(\Gamma_{k}, z\right)=0$.

Proof: Let $\delta=\operatorname{dist}\left(K, \Omega^{C}\right)$. Since $K$ is compact, $\delta>0$. Now tile the plane with squares, each of which has diameter less than $\delta / 2$.


Let $S$ denote the set of all the closed squares in this tiling which have nonempty intersection with $K$.Thus, all the squares of $S$ are contained in $\Omega$. First suppose $p$ is a point of $K$ which is in the interior of one of these squares in the tiling. Denote by $\partial S_{k}$ the boundary of $S_{k}$ one of the squares in $S$, oriented in the counter clockwise direction and $S_{m}$ denote the square of $S$ which contains the point, $p$ in its interior. Let the edges of the square, $S_{j}$ be $\left\{\gamma_{k}^{j}\right\}_{k=1}^{4}$. Thus a short computation shows $n\left(\partial S_{m}, p\right)=1$ but $n\left(\partial S_{j}, p\right)=0$ for all $j \neq m$. The reason for this is that for $z$ in $S_{j}$, the values $\left\{z-p: z \in S_{j}\right\}$ lie in an open square, $Q$ which is located at a positive distance from 0 . Then $\widehat{\mathbb{C}} \backslash Q$ is connected and $1 /(z-p)$ is analytic on $Q$. It follows from Corollary 51.7.23 that this function has a primitive on $Q$ and so

$$
\int_{\partial S_{j}} \frac{1}{z-p} d z=0
$$

Similarly, if $z \notin \Omega, n\left(\partial S_{j}, z\right)=0$. On the other hand, a direct computation will verify that $n\left(p, \partial S_{m}\right)=1$. Thus $1=\sum_{j, k} n\left(p, \gamma_{k}^{j}\right)=\sum_{S_{j} \in S} n\left(p, \partial S_{j}\right)$ and if $z \notin \Omega, 0=\sum_{j, k} n\left(z, \gamma_{k}^{j}\right)=$ $\sum_{S_{j} \in S} n\left(z, \partial S_{j}\right)$.

If $\gamma_{k}^{j *}$ coincides with $\gamma_{l}^{l *}$, then the contour integrals taken over this edge are taken in opposite directions and so the edge the two squares have in common can be deleted without changing $\sum_{j, k} n\left(z, \gamma_{k}^{j}\right)$ for any $z$ not on any of the lines in the tiling. For example, see the picture,


From the construction, if any of the $\gamma_{k}^{j *}$ contains a point of $K$ then this point is on one
of the four edges of $S_{j}$ and at this point, there is at least one edge of some $S_{l}$ which also contains this point. As just discussed, this shared edge can be deleted without changing $\sum_{k, j} n\left(z, \gamma_{k}^{j}\right)$. Delete the edges of the $S_{k}$ which intersect $K$ but not the endpoints of these edges. That is, delete the open edges. When this is done, delete all isolated points. Let the resulting oriented curves be denoted by $\left\{\gamma_{k}\right\}_{k=1}^{m}$. Note that you might have $\gamma_{k}^{*}=\gamma_{l}^{*}$. The construction is illustrated in the following picture.


Then as explained above, $\sum_{k=1}^{m} n\left(p, \gamma_{k}\right)=1$. It remains to prove the claim about the closed curves.

Each orientation on an edge corresponds to a direction of motion over that edge. Call such a motion over the edge a route. Initially, every vertex, (corner of a square in $S$ ) has the property there are the same number of routes to and from that vertex. When an open edge whose closure contains a point of $K$ is deleted, every vertex either remains unchanged as to the number of routes to and from that vertex or it loses both a route away and a route to. Thus the property of having the same number of routes to and from each vertex is preserved by deleting these open edges. The isolated points which result lose all routes to and from. It follows that upon removing the isolated points you can begin at any of the remaining vertices and follow the routes leading out from this and successive vertices according to orientation and eventually return to that end. Otherwise, there would be a vertex which would have only one route leading to it which does not happen. Now if you have used all the routes out of this vertex, pick another vertex and do the same process. Otherwise, pick an unused route out of the vertex and follow it to return. Continue this way till all routes are used exactly once, resulting in closed oriented curves, $\Gamma_{k}$. Then $\sum_{k} n\left(\Gamma_{k}, p\right)=\sum_{j} n\left(\gamma_{j}, p\right)=1$.

In case $p \in K$ is on some line of the tiling, it is not on any of the $\Gamma_{k}$ because $\Gamma_{k}^{*} \cap K=\emptyset$ and so the continuity of $z \rightarrow n\left(\Gamma_{k}, z\right)$ yields the desired result in this case also. This proves the lemma.

### 51.8 Exercises

1. If $U$ is simply connected, $f$ is analytic on $U$ and $f$ has no zeros in $U$, show there exists an analytic function, $F$, defined on $U$ such that $e^{F}=f$.
2. Let $f$ be defined and analytic near the point $a \in \mathbb{C}$. Show that then

$$
f(z)=\sum_{k=0}^{\infty} b_{k}(z-a)^{k}
$$

whenever $|z-a|<R$ where $R$ is the distance between $a$ and the nearest point where $f$ fails to have a derivative. The number $R$, is called the radius of convergence and the power series is said to be expanded about $a$.
3. Find the radius of convergence of the function $\frac{1}{1+z^{2}}$ expanded about $a=2$. Note there is nothing wrong with the function, $\frac{1}{1+x^{2}}$ when considered as a function of a real variable, $x$ for any value of $x$. However, if you insist on using power series, you find there is a limitation on the values of $x$ for which the power series converges due to the presence in the complex plane of a point, $i$, where the function fails to have a derivative.
4. Suppose $f$ is analytic on all of $\mathbb{C}$ and satisfies $|f(z)|<A+B|z|^{1 / 2}$. Show $f$ is constant.
5. What if you defined an open set, $U$ to be simply connected if $\mathbb{C} \backslash U$ is connected. Would it amount to the same thing? Hint: Consider the outside of $B(0,1)$.
6. Let $\gamma(t)=e^{i t}: t \in[0,2 \pi]$. Find $\int_{\gamma} \frac{1}{z^{n}} d z$ for $n=1,2, \cdots$.
7. Show $i \int_{0}^{2 \pi}(2 \cos \theta)^{2 n} d \theta=\int_{\gamma}\left(z+\frac{1}{z}\right)^{2 n}\left(\frac{1}{z}\right) d z$ where $\gamma(t)=e^{i t}: t \in[0,2 \pi]$. Then evaluate this integral using the binomial theorem and the previous problem.
8. Suppose that for some constants $a, b \neq 0, a, b \in \mathbb{R}, f(z+i b)=f(z)$ for all $z \in \mathbb{C}$ and $f(z+a)=f(z)$ for all $z \in \mathbb{C}$. If $f$ is analytic, show that $f$ must be constant. Can you generalize this? Hint: This uses Liouville's theorem.
9. Suppose $f(z)=u(x, y)+i v(x, y)$ is analytic for $z \in U$, an open set. Let $g(z)=$ $u^{*}(x, y)+i v^{*}(x, y)$ where

$$
\binom{u^{*}}{v^{*}}=Q\binom{u}{v}
$$

where $Q$ is a unitary matrix. That is $Q Q^{*}=Q^{*} Q=I$. When will $g$ be analytic?
10. Suppose $f$ is analytic on an open set, $U$, except for $\gamma^{*} \subset U$ where $\gamma$ is a one to one continuous function having bounded variation, but it is known that $f$ is continuous on $\gamma^{*}$. Show that in fact $f$ is analytic on $\gamma^{*}$ also. Hint: Pick a point on $\gamma^{*}$, say $\gamma\left(t_{0}\right)$ and suppose for now that $t_{0} \in(a, b)$. Pick $r>0$ such that $B=B\left(\gamma\left(t_{0}\right), r\right) \subseteq U$. Then show there exists $t_{1}<t_{0}$ and $t_{2}>t_{0}$ such that $\gamma\left(\left[t_{1}, t_{2}\right]\right) \subseteq \bar{B}$ and $\gamma\left(t_{i}\right) \notin B$. Thus $\gamma\left(\left[t_{1}, t_{2}\right]\right)$ is a path across $B$ going through the center of $B$ which divides $B$ into two open sets, $B_{1}$,
and $B_{2}$ along with $\gamma^{*}$. Let the boundary of $B_{k}$ consist of $\gamma\left(\left[t_{1}, t_{2}\right]\right)$ and a circular arc, $C_{k}$. Now letting $z \in B_{k}$, the line integral of $\frac{f(w)}{w-z}$ over $\gamma^{*}$ in two different directions cancels. Therefore, if $z \in B_{k}$, you can argue that $f(z)=\frac{1}{2 \pi i} \int_{C} \frac{f(w)}{w-z} d w$. By continuity, this continues to hold for $z \in \gamma\left(\left(t_{1}, t_{2}\right)\right)$. Therefore, $f$ must be analytic on $\gamma\left(\left(t_{1}, t_{1}\right)\right)$ also. This shows that $f$ must be analytic on $\gamma((a, b))$. To get the endpoints, simply extend $\gamma$ to have the same properties but defined on $[a-\varepsilon, b+\varepsilon]$ and repeat the above argument or else do this at the beginning and note that you get $[a, b] \subseteq(a-\varepsilon, b+\varepsilon)$.
11. Let $U$ be an open set contained in the upper half plane and suppose that there are finitely many line segments on the $x$ axis which are contained in the boundary of $U$. Now suppose that $f$ is defined, real, and continuous on these line segments and is defined and analytic on $U$. Now let $\widetilde{U}$ denote the reflection of $U$ across the $x$ axis. Show that it is possible to extend $f$ to a function, $g$ defined on all of

$$
W \equiv \widetilde{U} \cup U \cup\{\text { the line segments mentioned earlier }\}
$$

such that $g$ is analytic in $W$. Hint: For $z \in \widetilde{U}$, the reflection of $U$ across the $x$ axis, let $g(z) \equiv \overline{f(\bar{z})}$. Show that $g$ is analytic on $\widetilde{U} \cup U$ and continuous on the line segments. Then use Problem 10 or Morera's theorem to argue that $g$ is analytic on the line segments also. The result of this problem is know as the Schwarz reflection principle.
12. Show that rotations and translations of analytic functions yield analytic functions and use this observation to generalize the Schwarz reflection principle to situations in which the line segments are part of a line which is not the $x$ axis. Thus, give a version which involves reflection about an arbitrary line.

## Chapter 52

## The Open Mapping Theorem

### 52.1 A Local Representation

The open mapping theorem, is an even more surprising result than the theorem about the zeros of an analytic function. The following proof of this important theorem uses an interesting local representation of the analytic function.

Theorem 52.1.1 (Open mapping theorem) Let $\Omega$ be a region in $\mathbb{C}$ and suppose $f: \Omega \rightarrow \mathbb{C}$ is analytic. Then $f(\Omega)$ is either a point or a region. In the case where $f(\Omega)$ is a region, it follows that for each $z_{0} \in \Omega$, there exists an open set, $V$ containing $z_{0}$ and $m \in \mathbb{N}$ such that for all $z \in V$,

$$
\begin{equation*}
f(z)=f\left(z_{0}\right)+\phi(z)^{m} \tag{52.1.1}
\end{equation*}
$$

where $\phi: V \rightarrow B(0, \delta)$ is one to one, analytic and onto, $\phi\left(z_{0}\right)=0, \phi^{\prime}(z) \neq 0$ on $V$ and $\phi^{-1}$ analytic on $B(0, \delta)$. If $f$ is one to one then $m=1$ for each $z_{0}$ and $f^{-1}: f(\Omega) \rightarrow \Omega$ is analytic.

Proof: Suppose $f(\Omega)$ is not a point. Then if $z_{0} \in \Omega$ it follows there exists $r>0$ such that $f(z) \neq f\left(z_{0}\right)$ for all $z \in B\left(z_{0}, r\right) \backslash\left\{z_{0}\right\}$. Otherwise, $z_{0}$ would be a limit point of the set,

$$
\left\{z \in \Omega: f(z)-f\left(z_{0}\right)=0\right\}
$$

which would imply from Theorem 51.5 .3 that $f(z)=f\left(z_{0}\right)$ for all $z \in \Omega$. Therefore, making $r$ smaller if necessary and using the power series of $f$,

$$
f(z)=f\left(z_{0}\right)+\left(z-z_{0}\right)^{m} g(z)\left(\stackrel{?}{=}\left(\left(z-z_{0}\right) g(z)^{1 / m}\right)^{m}\right)
$$

for all $z \in B\left(z_{0}, r\right)$, where $g(z) \neq 0$ on $B\left(z_{0}, r\right)$. As implied in the above formula, one wonders if you can take the $m^{\text {th }}$ root of $g(z)$.
$\frac{g^{\prime}}{g}$ is an analytic function on $B\left(z_{0}, r\right)$ and so by Corollary 51.7.5 it has a primitive on $B\left(z_{0}, r\right), h$. Therefore by the product rule and the chain rule, $\left(g e^{-h}\right)^{\prime}=0$ and so there exists a constant, $C=e^{a+i b}$ such that on $B\left(z_{0}, r\right)$,

$$
g e^{-h}=e^{a+i b}
$$

Therefore,

$$
g(z)=e^{h(z)+a+i b}
$$

and so, modifying $h$ by adding in the constant, $a+i b, g(z)=e^{h(z)}$ where $h^{\prime}(z)=\frac{g^{\prime}(z)}{g(z)}$ on $B\left(z_{0}, r\right)$. Letting

$$
\phi(z)=\left(z-z_{0}\right) e^{\frac{h(z)}{m}}
$$

implies formula 52.1.1 is valid on $B\left(z_{0}, r\right)$. Now

$$
\phi^{\prime}\left(z_{0}\right)=e^{\frac{h\left(z_{0}\right)}{m}} \neq 0 .
$$

Shrinking $r$ if necessary you can assume $\phi^{\prime}(z) \neq 0$ on $B\left(z_{0}, r\right)$. Is there an open set $V$ contained in $B\left(z_{0}, r\right)$ such that $\phi$ maps $V$ onto $B(0, \delta)$ for some $\delta>0$ ?

Let $\phi(z)=u(x, y)+i v(x, y)$ where $z=x+i y$. Consider the mapping

$$
\binom{x}{y} \rightarrow\binom{u(x, y)}{v(x, y)}
$$

where $u, v$ are $C^{1}$ because $\phi$ is given to be analytic. The Jacobian of this map at $(x, y) \in$ $B\left(z_{0}, r\right)$ is

$$
\begin{gathered}
\left|\begin{array}{cc}
u_{x}(x, y) & u_{y}(x, y) \\
v_{x}(x, y) & v_{y}(x, y)
\end{array}\right|=\left|\begin{array}{cc}
u_{x}(x, y) & -v_{x}(x, y) \\
v_{x}(x, y) & u_{x}(x, y)
\end{array}\right| \\
=u_{x}(x, y)^{2}+v_{x}(x, y)^{2}=\left|\phi^{\prime}(z)\right|^{2} \neq 0
\end{gathered}
$$

This follows from a use of the Cauchy Riemann equations. Also

$$
\binom{u\left(x_{0}, y_{0}\right)}{v\left(x_{0}, y_{0}\right)}=\binom{0}{0}
$$

Therefore, by the inverse function theorem there exists an open set, $V$, containing $z_{0}$ and $\delta>0$ such that $(u, v)^{T}$ maps $V$ one to one onto $B(0, \delta)$. Thus $\phi$ is one to one onto $B(0, \delta)$ as claimed. Applying the same argument to other points, $z$ of $V$ and using the fact that $\phi^{\prime}(z) \neq 0$ at these points, it follows $\phi$ maps open sets to open sets. In other words, $\phi^{-1}$ is continuous.

It also follows that $\phi^{m}$ maps $V$ onto $B\left(0, \delta^{m}\right)$. Therefore, the formula 52.1.1 implies that $f$ maps the open set, $V$, containing $z_{0}$ to an open set. This shows $f(\Omega)$ is an open set because $z_{0}$ was arbitrary. It is connected because $f$ is continuous and $\Omega$ is connected. Thus $f(\Omega)$ is a region. It remains to verify that $\phi^{-1}$ is analytic on $B(0, \delta)$. Since $\phi^{-1}$ is continuous,

$$
\lim _{\phi\left(z_{1}\right) \rightarrow \phi(z)} \frac{\phi^{-1}\left(\phi\left(z_{1}\right)\right)-\phi^{-1}(\phi(z))}{\phi\left(z_{1}\right)-\phi(z)}=\lim _{z_{1} \rightarrow z} \frac{z_{1}-z}{\phi\left(z_{1}\right)-\phi(z)}=\frac{1}{\phi^{\prime}(z)}
$$

Therefore, $\phi^{-1}$ is analytic as claimed.
It only remains to verify the assertion about the case where $f$ is one to one. If $m>1$, then $e^{\frac{2 \pi i}{m}} \neq 1$ and so for $z_{1} \in V$,

$$
\begin{equation*}
e^{\frac{2 \pi i}{m}} \phi\left(z_{1}\right) \neq \phi\left(z_{1}\right) \tag{52.1.2}
\end{equation*}
$$

But $e^{\frac{2 \pi i}{m}} \phi\left(z_{1}\right) \in B(0, \delta)$ and so there exists $z_{2} \neq z_{1}$ (since $\phi$ is one to one) such that $\phi\left(z_{2}\right)=$ $e^{\frac{2 \pi i}{m}} \phi\left(z_{1}\right)$. But then

$$
\phi\left(z_{2}\right)^{m}=\left(e^{\frac{2 \pi i}{m}} \phi\left(z_{1}\right)\right)^{m}=\phi\left(z_{1}\right)^{m}
$$

implying $f\left(z_{2}\right)=f\left(z_{1}\right)$ contradicting the assumption that $f$ is one to one. Thus $m=1$ and $f^{\prime}(z)=\phi^{\prime}(z) \neq 0$ on $V$. Since $f$ maps open sets to open sets, it follows that $f^{-1}$ is
continuous and so

$$
\begin{aligned}
\left(f^{-1}\right)^{\prime}(f(z)) & =\lim _{f\left(z_{1}\right) \rightarrow f(z)} \frac{f^{-1}\left(f\left(z_{1}\right)\right)-f^{-1}(f(z))}{f\left(z_{1}\right)-f(z)} \\
& =\lim _{z_{1} \rightarrow z} \frac{z_{1}-z}{f\left(z_{1}\right)-f(z)}=\frac{1}{f^{\prime}(z)}
\end{aligned}
$$

This proves the theorem.
One does not have to look very far to find that this sort of thing does not hold for functions mapping $\mathbb{R}$ to $\mathbb{R}$. Take for example, the function $f(x)=x^{2}$. Then $f(\mathbb{R})$ is neither a point nor a region. In fact $f(\mathbb{R})$ fails to be open.

Corollary 52.1.2 Suppose in the situation of Theorem 52.1.1 $m>1$ for the local representation of $f$ given in this theorem. Then there exists $\delta>0$ such that if $w \in B\left(f\left(z_{0}\right), \delta\right)=$ $f(V)$ for $V$ an open set containing $z_{0}$, then $f^{-1}(w)$ consists of $m$ distinct points in $V$. ( $f$ is $m$ to one on $V$ )

Proof: Let $w \in B\left(f\left(z_{0}\right), \boldsymbol{\delta}\right)$. Then $w=f(\widehat{z})$ where $\widehat{z} \in V$. Thus $f(\widehat{z})=f\left(z_{0}\right)+\phi(\widehat{z})^{m}$. Consider the $m$ distinct numbers, $\left\{e^{\frac{2 k \pi i}{m}} \phi(\widehat{z})\right\}_{k=1}^{m}$. Then each of these numbers is in $B(0, \delta)$ and so since $\phi$ maps $V$ one to one onto $B(0, \delta)$, there are $m$ distinct numbers in $V,\left\{z_{k}\right\}_{k=1}^{m}$ such that $\phi\left(z_{k}\right)=e^{\frac{2 k \pi i}{m}} \phi(\widehat{z})$. Then

$$
\begin{aligned}
f\left(z_{k}\right) & =f\left(z_{0}\right)+\phi\left(z_{k}\right)^{m}=f\left(z_{0}\right)+\left(e^{\frac{2 k \pi i}{m}} \phi(\widehat{z})\right)^{m} \\
& =f\left(z_{0}\right)+e^{2 k \pi i} \phi(\widehat{z})^{m}=f\left(z_{0}\right)+\phi(\widehat{z})^{m}=f(\widehat{z})=w
\end{aligned}
$$

This proves the corollary.

### 52.2 Branches Of The Logarithm

The argument used in to prove the next theorem was used in the proof of the open mapping theorem. It is a very important result and deserves to be stated as a theorem.

Theorem 52.2.1 Let $\Omega$ be a simply connected region and suppose $f: \Omega \rightarrow \mathbb{C}$ is analytic and nonzero on $\Omega$. Then there exists an analytic function, $g$ such that $e^{g(z)}=f(z)$ for all $z \in \Omega$.

Proof: The function, $f^{\prime} / f$ is analytic on $\Omega$ and so by Corollary 51.7.23 there is a primitive for $f^{\prime} / f$, denoted as $g_{1}$. Then

$$
\left(e^{-g_{1}} f\right)^{\prime}=-\frac{f^{\prime}}{f} e^{-g_{1}} f+e^{-g_{1}} f^{\prime}=0
$$

and so since $\Omega$ is connected, it follows $e^{-g_{1}} f$ equals a constant, $e^{a+i b}$. Therefore, $f(z)=$ $e^{g_{1}(z)+a+i b}$. Define $g(z) \equiv g_{1}(z)+a+i b$.

The function, $g$ in the above theorem is called a branch of the logarithm of $f$ and is written as $\log (f(z))$.

Definition 52.2.2 Let $\rho$ be a ray starting at 0 . Thus $\rho$ is a straight line of infinite length extending in one direction with its initial point at 0 .

A special case of the above theorem is the following.

Theorem 52.2.3 Let $\rho$ be a ray starting at 0 . Then there exists an analytic function, $L(z)$ defined on $\mathbb{C} \backslash \rho$ such that

$$
e^{L(z)}=z .
$$

This function, L is called a branch of the logarithm. This branch of the logarithm satisfies the usual formula for logarithms, $L(z w)=L(z)+L(w)$ provided $z w \notin \rho$.

Proof: $\mathbb{C} \backslash \rho$ is a simply connected region because its complement with respect to $\widehat{\mathbb{C}}$ is connected. Furthermore, the function, $f(z)=z$ is not equal to zero on $\mathbb{C} \backslash \rho$. Therefore, by Theorem 52.2.1 there exists an analytic function $L(z)$ such that $e^{L(z)}=f(z)=z$. Now consider the problem of finding a description of $L(z)$. Each $z \in \mathbb{C} \backslash \rho$ can be written in a unique way in the form

$$
z=|z| e^{i \arg _{\theta}(z)}
$$

where $\arg _{\theta}(z)$ is the angle in $(\theta, \theta+2 \pi)$ associated with $z$. (You could of course have considered this to be the angle in $(\theta-2 \pi, \theta)$ associated with $z$ or in infinitely many other open intervals of length $2 \pi$. The description of the $\log$ is not unique.) Then letting $L(z)=$ $a+i b$

$$
z=|z| e^{i \arg _{\theta}(z)}=e^{L(z)}=e^{a} e^{i b}
$$

and so you can let $L(z)=\ln |z|+i \arg _{\theta}(z)$.
Does $L(z)$ satisfy the usual properties of the logarithm? That is, for $z, w \in \mathbb{C} \backslash \rho$, is $L(z w)=L(z)+L(w)$ ? This follows from the usual rules of exponents. You know $e^{z+w}=$ $e^{z} e^{w}$. (You can verify this directly or you can reduce to the case where $z, w$ are real. If $z$ is a fixed real number, then the equation holds for all real $w$. Therefore, it must also hold for all complex $w$ because the real line contains a limit point. Now for this fixed $w$, the equation holds for all $z$ real. Therefore, by similar reasoning, it holds for all complex z.)

Now suppose $z, w \in \mathbb{C} \backslash \rho$ and $z w \notin \rho$. Then

$$
e^{L(z w)}=z w, e^{L(z)+L(w)}=e^{L(z)} e^{L(w)}=z w
$$

and so $L(z w)=L(z)+L(w)$ as claimed. This proves the theorem.
In the case where the ray is the negative real axis, it is called the principal branch of the $\operatorname{logarithm}$. Thus $\arg (z)$ is a number between $-\pi$ and $\pi$.

Definition 52.2.4 Let $\log$ denote the branch of the logarithm which corresponds to the ray for $\theta=\pi$. That is, the ray is the negative real axis. Sometimes this is called the principal branch of the logarithm.

### 52.3 Maximum Modulus Theorem

Here is another very significant theorem known as the maximum modulus theorem which follows immediately from the open mapping theorem.

Theorem 52.3.1 (maximum modulus theorem) Let $\Omega$ be a bounded region and let $f: \Omega \rightarrow$ $\mathbb{C}$ be analytic and $f: \bar{\Omega} \rightarrow \mathbb{C}$ continuous. Then if $z \in \Omega$,

$$
\begin{equation*}
|f(z)| \leq \max \{|f(w)|: w \in \partial \Omega\} \tag{52.3.3}
\end{equation*}
$$

If equality is achieved for any $z \in \Omega$, then $f$ is a constant.
Proof: Suppose $f$ is not a constant. Then $f(\Omega)$ is a region and so if $z \in \Omega$, there exists $r>0$ such that $B(f(z), r) \subseteq f(\Omega)$. It follows there exists $z_{1} \in \Omega$ with $\left|f\left(z_{1}\right)\right|>|f(z)|$. Hence $\max \{|f(w)|: w \in \bar{\Omega}\}$ is not achieved at any interior point of $\Omega$. Therefore, the point at which the maximum is achieved must lie on the boundary of $\Omega$ and so

$$
\max \{|f(w)|: w \in \partial \Omega\}=\max \{|f(w)|: w \in \bar{\Omega}\}>|f(z)|
$$

for all $z \in \Omega$ or else $f$ is a constant. This proves the theorem.
You can remove the assumption that $\Omega$ is bounded and give a slightly different version.
Theorem 52.3.2 Let $f: \Omega \rightarrow \mathbb{C}$ be analytic on a region, $\Omega$ and suppose $\overline{B(a, r)} \subseteq \Omega$. Then

$$
|f(a)| \leq \max \left\{\left|f\left(a+r e^{i \theta}\right)\right|: \theta \in[0,2 \pi]\right\}
$$

Equality occurs for some $r>0$ and $a \in \Omega$ if and only if $f$ is constant in $\Omega$ hence equality occurs for all such $a, r$.

Proof: The claimed inequality holds by Theorem 52.3.1. Suppose equality in the above is achieved for some $\overline{B(a, r)} \subseteq \Omega$. Then by Theorem 52.3.1 $f$ is equal to a constant, $w$ on $B(a, r)$. Therefore, the function, $f(\cdot)-w$ has a zero set which has a limit point in $\Omega$ and so by Theorem 51.5.3 $f(z)=w$ for all $z \in \Omega$.

Conversely, if $f$ is constant, then the equality in the above inequality is achieved for all $\overline{B(a, r)} \subseteq \Omega$.

Next is yet another version of the maximum modulus principle which is in Conway [32]. Let $\Omega$ be an open set.

Definition 52.3.3 Define $\partial_{\infty} \Omega$ to equal $\partial \Omega$ in the case where $\Omega$ is bounded and $\partial \Omega \cup\{\infty\}$ in the case where $\Omega$ is not bounded.

Definition 52.3.4 Let $f$ be a complex valued function defined on a set $S \subseteq \mathbb{C}$ and let a be a limit point of $S$.

$$
\limsup |f(z)| \equiv \lim _{z \rightarrow 0}\left\{\sup |f(w)|: w \in B^{\prime}(a, r) \cap S\right\}
$$

The limit exists because $\left\{\sup |f(w)|: w \in B^{\prime}(a, r) \cap S\right\}$ is decreasing in $r$. In case $a=\infty$,

$$
\lim \sup _{z \rightarrow \infty}|f(z)| \equiv \lim _{r \rightarrow \infty}\{\sup |f(w)|:|w|>r, w \in S\}
$$

Note that if $\limsup _{z \rightarrow a}|f(z)| \leq M$ and $\delta>0$, then there exists $r>0$ such that if $z \in$ $B^{\prime}(a, r) \cap S$, then $|f(z)|<M+\delta$. If $a=\infty$, there exists $r>0$ such that if $|z|>r$ and $z \in S$, then $|f(z)|<M+\delta$.

Theorem 52.3.5 Let $\Omega$ be an open set in $\mathbb{C}$ and let $f: \Omega \rightarrow \mathbb{C}$ be analytic. Suppose also that for every $a \in \partial_{\infty} \Omega$,

$$
\limsup _{z \rightarrow a}|f(z)| \leq M<\infty
$$

Then in fact $|f(z)| \leq M$ for all $z \in \Omega$.
Proof: Let $\delta>0$ and let $H \equiv\{z \in \Omega:|f(z)|>M+\delta\}$. Suppose $H \neq \emptyset$. Then $H$ is an open subset of $\Omega$. I claim that $H$ is actually bounded. If $\Omega$ is bounded, there is nothing to show so assume $\Omega$ is unbounded. Then the condition involving the limsup implies there exists $r>0$ such that if $|z|>r$ and $z \in \Omega$, then $|f(z)| \leq M+\delta / 2$. It follows $H$ is contained in $\overline{B(0, r)}$ and so it is bounded. Now consider the components of $\Omega$. One of these components contains points from $H$. Let this component be denoted as $V$ and let $H_{V} \equiv H \cap V$. Thus $H_{V}$ is a bounded open subset of $V$. Let $U$ be a component of $H_{V}$. First suppose $\bar{U} \subseteq V$. In this case, it follows that on $\partial U,|f(z)|=M+\delta$ and so by Theorem 52.3.1 $|f(z)| \leq M+\delta$ for all $z \in U$ contradicting the definition of $H$. Next suppose $\partial U$ contains a point of $\partial V, a$. Then in this case, $a$ violates the condition on limsup. Either way you get a contradiction. Hence $H=\emptyset$ as claimed. Since $\delta>0$ is arbitrary, this shows $|f(z)| \leq M$.

### 52.4 Extensions Of Maximum Modulus Theorem

### 52.4.1 Phragmên Lindelöf Theorem

This theorem is an extension of Theorem 52.3.5. It uses a growth condition near the extended boundary to conclude that $f$ is bounded. I will present the version found in Conway [32]. It seems to be more of a method than an actual theorem. There are several versions of it.

Theorem 52.4.1 Let $\Omega$ be a simply connected region in $\mathbb{C}$ and suppose $f$ is analytic on $\Omega$. Also suppose there exists a function, $\phi$ which is nonzero and uniformly bounded on $\Omega$. Let $M$ be a positive number. Now suppose $\partial_{\infty} \Omega=A \cup B$ such that for every $a \in A$, $\limsup _{z \rightarrow a}|f(z)| \leq M$ and for every $b \in B$, and $\eta>0, \limsup _{z \rightarrow b}|f(z)||\phi(z)|^{\eta} \leq M$. Then $|f(z)| \leq M$ for all $z \in \Omega$.

Proof: By Theorem 52.2.1 there exists $\log (\phi(z))$ analytic on $\Omega$. Now define $g(z) \equiv$ $\exp (\eta \log (\phi(z)))$ so that $g(z)=\phi(z)^{\eta}$. Now also

$$
|g(z)|=|\exp (\eta \log (\phi(z)))|=|\exp (\eta \ln |\phi(z)|)|=|\phi(z)|^{\eta}
$$

Let $m \geq|\phi(z)|$ for all $z \in \Omega$. Define $F(z) \equiv f(z) g(z) m^{-\eta}$. Thus $F$ is analytic and for $b \in B$,

$$
\limsup \sup _{z \rightarrow b}|F(z)|=\lim \sup _{z \rightarrow b}|f(z)||\phi(z)|^{\eta} m^{-\eta} \leq M m^{-\eta}
$$

while for $a \in A$,

$$
\lim \sup _{z \rightarrow a}|F(z)| \leq M
$$

Therefore, for $\alpha \in \partial_{\infty} \Omega, \limsup _{z \rightarrow \alpha}|F(z)| \leq \max \left(M, M \eta^{-\eta}\right)$ and so by Theorem 52.3.5, $|f(z)| \leq\left(\frac{m^{\eta}}{|\phi(z)|^{\eta}}\right) \max \left(M, M \eta^{-\eta}\right)$. Now let $\eta \rightarrow 0$ to obtain $|f(z)| \leq M$.

In applications, it is often the case that $B=\{\infty\}$.
Now here is an interesting case of this theorem. It involves a particular form for $\Omega$, in this case $\Omega=\left\{z \in \mathbb{C}:|\arg (z)|<\frac{\pi}{2 a}\right\}$ where $a \geq \frac{1}{2}$.


Then $\partial \Omega$ equals the two slanted lines. Also on $\Omega$ you can define a $\operatorname{logarithm,\operatorname {log}(z)=}$ $\ln |z|+i \arg (z)$ where $\arg (z)$ is the angle associated with $z$ between $-\pi$ and $\pi$. Therefore, if $c$ is a real number you can define $z^{c}$ for such $z$ in the usual way:

$$
\begin{aligned}
z^{c} & \equiv \exp (c \log (z))=\exp (c[\ln |z|+i \arg (z)]) \\
& =|z|^{c} \exp (i c \arg (z))=|z|^{c}(\cos (c \arg (z))+i \sin (c \arg (z)))
\end{aligned}
$$

If $|c|<a$, then $|c \arg (z)|<\frac{\pi}{2}$ and so $\cos (c \arg (z))>0$. Therefore, for such $c$,

$$
\begin{aligned}
\left|\exp \left(-\left(z^{c}\right)\right)\right| & =\left|\exp \left(-|z|^{c}(\cos (c \arg (z))+i \sin (c \arg (z)))\right)\right| \\
& =\left|\exp \left(-|z|^{c}(\cos (c \arg (z)))\right)\right|
\end{aligned}
$$

which is bounded since $\cos (c \arg (z))>0$.
Corollary 52.4.2 Let $\Omega=\left\{z \in \mathbb{C}:|\arg (z)|<\frac{\pi}{2 a}\right\}$ where $a \geq \frac{1}{2}$ and suppose $f$ is analytic on $\Omega$ and satisfies $\limsup _{z \rightarrow a}|f(z)| \leq M$ on $\partial \Omega$ and suppose there are positive constants, $P, b$ where $b<a$ and

$$
|f(z)| \leq P \exp \left(|z|^{b}\right)
$$

for all $|z|$ large enough. Then $|f(z)| \leq M$ for all $z \in \Omega$.
Proof: Let $b<c<a$ and let $\phi(z) \equiv \exp \left(-\left(z^{c}\right)\right)$. Then as discussed above, $\phi(z) \neq 0$ on $\Omega$ and $|\phi(z)|$ is bounded on $\Omega$. Now

$$
\begin{gathered}
|\phi(z)|^{\eta}=\left|\exp \left(-|z|^{c} \eta(\cos (c \arg (z)))\right)\right| \\
\lim \sup _{z \rightarrow \infty}|f(z)||\phi(z)|^{\eta} \leq \lim \sup _{z \rightarrow \infty} \frac{P \exp \left(|z|^{b}\right)}{\left|\exp \left(|z|^{c} \eta(\cos (c \arg (z)))\right)\right|}=0 \leq M
\end{gathered}
$$

and so by Theorem 52.4.1 $|f(z)| \leq M$.
The following is another interesting case. This case is presented in Rudin [113]

Corollary 52.4.3 Let $\Omega$ be the open set consisting of $\{z \in \mathbb{C}: a<\operatorname{Re} z<b\}$ and suppose $f$ is analytic on $\Omega$, continuous on $\bar{\Omega}$, and bounded on $\Omega$. Suppose also that $f(z) \leq 1$ on the two lines $\operatorname{Re} z=a$ and $\operatorname{Re} z=b$. Then $|f(z)| \leq 1$ for all $z \in \Omega$.

Proof: This time let $\phi(z)=\frac{1}{1+z-a}$. Thus $|\phi(z)| \leq 1$ because $\operatorname{Re}(z-a)>0$ and $\phi(z) \neq 0$ for all $z \in \Omega$. Also, $\limsup _{z \rightarrow \infty}|\phi(z)|^{\eta}=0$ for every $\eta>0$. Therefore, if $a$ is a point of the sides of $\Omega, \limsup _{z \rightarrow a}|f(z)| \leq 1$ while $\limsup _{z \rightarrow \infty}|f(z)||\phi(z)|^{\eta}=0 \leq 1$ and so by Theorem 52.4.1, $|f(z)| \leq 1$ on $\Omega$.

This corollary yields an interesting conclusion.

Corollary 52.4.4 Let $\Omega$ be the open set consisting of $\{z \in \mathbb{C}: a<\operatorname{Re} z<b\}$ and suppose $f$ is analytic on $\Omega$, continuous on $\bar{\Omega}$, and bounded on $\Omega$. Define

$$
M(x) \equiv \sup \{|f(z)|: \operatorname{Re} z=x\}
$$

Then for $x \in(a, b)$.

$$
M(x) \leq M(a)^{\frac{b-x}{b-a}} M(b)^{\frac{x-a}{b-a}}
$$

Proof: Let $\varepsilon>0$ and define

$$
g(z) \equiv(M(a)+\varepsilon)^{\frac{b-z}{b-a}}(M(b)+\varepsilon)^{\frac{z-a}{b-a}}
$$

where for $M>0$ and $z \in \mathbb{C}, M^{z} \equiv \exp (z \ln (M))$. Thus $g \neq 0$ and so $f / g$ is analytic on $\Omega$ and continuous on $\bar{\Omega}$. Also on the left side,

$$
\left|\frac{f(a+i y)}{g(a+i y)}\right|=\left|\frac{f(a+i y)}{(M(a)+\varepsilon)^{\frac{b-a-i y}{b-a}}}\right|=\left|\frac{f(a+i y)}{(M(a)+\varepsilon)^{\frac{b-a}{b-a}}}\right| \leq 1
$$

while on the right side a similar computation shows $\left|\frac{f}{g}\right| \leq 1$ also. Therefore, by Corollary 52.4.3 $|f / g| \leq 1$ on $\Omega$. Therefore, letting $x+i y=z$,

$$
|f(z)| \leq\left|(M(a)+\varepsilon)^{\frac{b-z}{b-a}}(M(b)+\varepsilon)^{\frac{z-a}{b-a}}\right|=\left|(M(a)+\varepsilon)^{\frac{b-x}{b-a}}(M(b)+\varepsilon)^{\frac{x-a}{b-a}}\right|
$$

and so

$$
M(x) \leq(M(a)+\varepsilon)^{\frac{b-x}{b-a}}(M(b)+\varepsilon)^{\frac{x-a}{b-a}}
$$

Since $\varepsilon>0$ is arbitrary, it yields the conclusion of the corollary.
Another way of saying this is that $x \rightarrow \ln (M(x))$ is a convex function.
This corollary has an interesting application known as the Hadamard three circles theorem.

### 52.4.2 Hadamard Three Circles Theorem

Let $0<R_{1}<R_{2}$ and suppose $f$ is analytic on $\left\{z \in \mathbb{C}: R_{1}<|z|<R_{2}\right\}$. Then letting $R_{1}<$ $a<b<R_{2}$, note that $g(z) \equiv \exp (z)$ satisfies

$$
g:\{z \in \mathbb{C}: \ln a<\operatorname{Re} z<b\} \rightarrow\{z \in \mathbb{C}: a<|z|<b\}
$$

is onto and that in fact, $g$ maps the line $\ln r+i y$ onto the circle $r e^{i \theta}$. Now let $M(x)$ be defined as above and $m$ be defined by $m(r) \equiv \max _{\theta}\left|f\left(r e^{i \theta}\right)\right|$.Then for $a<r<b$, Corollary 52.4.4 implies

$$
\begin{aligned}
m(r) & =\sup _{y}\left|f\left(e^{\ln r+i y}\right)\right|=M(\ln r) \leq M(\ln a)^{\frac{\ln b-\ln r}{\ln b-\ln a}} M(\ln b)^{\frac{\ln r-\ln a}{\ln b-\ln a}} \\
& =m(a)^{\ln (b / r) / \ln (b / a)} m(b)^{\ln (r / a) / \ln (b / a)}
\end{aligned}
$$

and so $m(r)^{\ln (b / a)} \leq m(a)^{\ln (b / r)} m(b)^{\ln (r / a)}$. Taking logarithms, this yields

$$
\ln \left(\frac{b}{a}\right) \ln (m(r)) \leq \ln \left(\frac{b}{r}\right) \ln (m(a))+\ln \left(\frac{r}{a}\right) \ln (m(b))
$$

which says the same as $r \rightarrow \ln (m(r))$ is a convex function of $\ln r$.
The next example, also in Rudin [113] is very dramatic. An unbelievably weak assumption is made on the growth of the function and still you get a uniform bound in the conclusion.

Corollary 52.4.5 Let $\Omega=\left\{z \in \mathbb{C}:|\operatorname{Im}(z)|<\frac{\pi}{2}\right\}$. Suppose $f$ is analytic on $\Omega$, continuous on $\bar{\Omega}$, and there exist constants, $\alpha<1$ and $A<\infty$ such that

$$
|f(z)| \leq \exp (A \exp (\alpha|x|)) \text { for } z=x+i y
$$

and

$$
\left|f\left(x \pm i \frac{\pi}{2}\right)\right| \leq 1
$$

for all $x \in \mathbb{R}$. Then $|f(z)| \leq 1$ on $\Omega$.
Proof: This time let $\phi(z)=[\exp (A \exp (\beta z)) \exp (A \exp (-\beta z))]^{-1}$ where $\alpha<\beta<1$. Then $\phi(z) \neq 0$ on $\Omega$ and for $\eta>0$

$$
|\phi(z)|^{\eta}=\frac{1}{|\exp (\eta A \exp (\beta z)) \exp (\eta A \exp (-\beta z))|}
$$

Now

$$
\begin{aligned}
& \exp (\eta A \exp (\beta z)) \exp (\eta A \exp (-\beta z)) \\
= & \exp (\eta A(\exp (\beta z)+\exp (-\beta z))) \\
= & \exp \left[\eta A\left(\cos (\beta y)\left(e^{\beta x}+e^{-\beta x}\right)+i \sin (\beta y)\left(e^{\beta x}-e^{-\beta x}\right)\right)\right]
\end{aligned}
$$

and so

$$
|\phi(z)|^{\eta}=\frac{1}{\exp \left[\eta A\left(\cos (\beta y)\left(e^{\beta x}+e^{-\beta x}\right)\right)\right]}
$$

Now $\cos \beta y>0$ because $\beta<1$ and $|y|<\frac{\pi}{2}$. Therefore, $\limsup _{z \rightarrow \infty}|f(z)||\phi(z)|^{\eta} \leq 0 \leq 1$ and so by Theorem 52.4.1, $|f(z)| \leq 1$.

### 52.4.3 Schwarz's Lemma

This interesting lemma comes from the maximum modulus theorem. It will be used later as part of the proof of the Riemann mapping theorem.

Lemma 52.4.6 Suppose $F: B(0,1) \rightarrow B(0,1), F$ is analytic, and $F(0)=0$. Then for all $z \in B(0,1)$,

$$
\begin{equation*}
|F(z)| \leq|z| \tag{52.4.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|F^{\prime}(0)\right| \leq 1 \tag{52.4.5}
\end{equation*}
$$

If equality holds in 52.4 .5 then there exists $\lambda \in \mathbb{C}$ with $|\lambda|=1$ and

$$
\begin{equation*}
F(z)=\lambda z \tag{52.4.6}
\end{equation*}
$$

Proof: First note that by assumption, $F(z) / z$ has a removable singularity at 0 if its value at 0 is defined to be $F^{\prime}(0)$. By the maximum modulus theorem, if $|z|<r<1$,

$$
\left|\frac{F(z)}{z}\right| \leq \max _{t \in[0,2 \pi]} \frac{\left|F\left(r e^{i t}\right)\right|}{r} \leq \frac{1}{r}
$$

Then letting $r \rightarrow 1$,

$$
\left|\frac{F(z)}{z}\right| \leq 1
$$

this shows 52.4 .4 and it also verifies 52.4 .5 on taking the limit as $z \rightarrow 0$. If equality holds in 52.4.5, then $|F(z) / z|$ achieves a maximum at an interior point so $F(z) / z$ equals a constant, $\lambda$ by the maximum modulus theorem. Since $F(z)=\lambda z$, it follows $F^{\prime}(0)=\lambda$ and so $|\lambda|=1$.

Rudin [113] gives a memorable description of what this lemma says. It says that if an analytic function maps the unit ball to itself, keeping 0 fixed, then it must do one of two things, either be a rotation or move all points closer to 0 . (This second part follows in case $\left|F^{\prime}(0)\right|<1$ because in this case, you must have $|F(z)| \neq|z|$ and so by 52.4.4, $\left.|F(z)|<|z|\right)$

### 52.4.4 One To One Analytic Maps On The Unit Ball

The transformation in the next lemma is of fundamental importance.
Lemma 52.4.7 Let $\alpha \in B(0,1)$ and define

$$
\phi_{\alpha}(z) \equiv \frac{z-\alpha}{1-\bar{\alpha} z}
$$

Then $\phi_{\alpha}: B(0,1) \rightarrow B(0,1), \phi_{\alpha}: \partial B(0,1) \rightarrow \partial B(0,1)$, and is one to one and onto. Also $\phi_{-\alpha}=\phi_{\alpha}^{-1}$. Also

$$
\phi_{\alpha}^{\prime}(0)=1-|\alpha|^{2}, \phi^{\prime}(\alpha)=\frac{1}{1-|\alpha|^{2}}
$$

Proof: First of all, for $|z|<1 /|\alpha|$,

$$
\phi_{\alpha} \circ \phi_{-\alpha}(z) \equiv \frac{\left(\frac{z+\alpha}{1+\bar{\alpha} z}\right)-\alpha}{1-\bar{\alpha}\left(\frac{z+\alpha}{1+\bar{\alpha} z}\right)}=z
$$

after a few computations. If I show that $\phi_{\alpha}$ maps $B(0,1)$ to $B(0,1)$ for all $|\alpha|<1$, this will have shown that $\phi_{\alpha}$ is one to one and onto $B(0,1)$.

Consider $\left|\phi_{\alpha}\left(e^{i \theta}\right)\right|$. This yields

$$
\left|\frac{e^{i \theta}-\alpha}{1-\bar{\alpha} e^{i \theta}}\right|=\left|\frac{1-\alpha e^{-i \theta}}{1-\bar{\alpha} e^{i \theta}}\right|=1
$$

where the first equality is obtained by multiplying by $\left|e^{-i \theta}\right|=1$. Therefore, $\phi_{\alpha}$ maps $\partial B(0,1)$ one to one and onto $\partial B(0,1)$. Now notice that $\phi_{\alpha}$ is analytic on $B(0,1)$ because the only singularity, a pole is at $z=1 / \bar{\alpha}$. By the maximum modulus theorem, it follows

$$
\left|\phi_{\alpha}(z)\right|<1
$$

whenever $|z|<1$. The same is true of $\phi_{-\alpha}$.
It only remains to verify the assertions about the derivatives. Long division gives $\phi_{\alpha}(z)=(-\bar{\alpha})^{-1}+\left(\frac{-\alpha+(\bar{\alpha})^{-1}}{1-\bar{\alpha} z}\right)$ and so

$$
\begin{aligned}
\phi_{\alpha}^{\prime}(z) & =(-1)(1-\bar{\alpha} z)^{-2}\left(-\alpha+(\bar{\alpha})^{-1}\right)(-\bar{\alpha}) \\
& =\bar{\alpha}(1-\bar{\alpha} z)^{-2}\left(-\alpha+(\bar{\alpha})^{-1}\right) \\
& =(1-\bar{\alpha} z)^{-2}\left(-|\alpha|^{2}+1\right)
\end{aligned}
$$

Hence the two formulas follow. This proves the lemma.
One reason these mappings are so important is the following theorem.
Theorem 52.4.8 Suppose $f$ is an analytic function defined on $B(0,1)$ and $f$ maps $B(0,1)$ one to one and onto $B(0,1)$. Then there exists $\theta$ such that

$$
f(z)=e^{i \theta} \phi_{\alpha}(z)
$$

for some $\alpha \in B(0,1)$.
Proof: Let $f(\alpha)=0$. Then $h(z) \equiv f \circ \phi_{-\alpha}(z)$ maps $B(0,1)$ one to one and onto $B(0,1)$ and has the property that $h(0)=0$. Therefore, by the Schwarz lemma,

$$
|h(z)| \leq|z|
$$

but it is also the case that $h^{-1}(0)=0$ and $h^{-1}$ maps $B(0,1)$ to $B(0,1)$. Therefore, the same inequality holds for $h^{-1}$. Therefore,

$$
|z|=\left|h^{-1}(h(z))\right| \leq|h(z)|
$$

and so $|h(z)|=|z|$. By the Schwarz lemma again, $h(z) \equiv f\left(\phi_{-\alpha}(z)\right)=e^{i \theta} z$. Letting $z=$ $\phi_{\alpha}$, you get $f(z)=e^{i \theta} \phi_{\alpha}(z)$.

### 52.5 Exercises

1. Consider the function, $g(z)=\frac{z-i}{z+i}$. Show this is analytic on the upper half plane, $P+$ and maps the upper half plane one to one and onto $B(0,1)$. Hint: First show $g$ maps the real axis to $\partial B(0,1)$. This is really easy because you end up looking at a complex number divided by its conjugate. Thus $|g(z)|=1$ for $z$ on $\partial(P+)$. Now show that $\limsup _{z \rightarrow \infty}|g(z)|=1$. Then apply a version of the maximum modulus theorem. You might note that $g(z)=1+\frac{-2 i}{z+i}$. This will show $|g(z)| \leq 1$. Next pick $w \in B(0,1)$ and solve $g(z)=w$. You just have to show there exists a unique solution and its imaginary part is positive.
2. Does there exist an entire function $f$ which maps $\mathbb{C}$ onto the upper half plane?
3. Letting $g$ be the function of Problem 1 show that $\left(g^{-1}\right)^{\prime}(0)=2$. Also note that $g^{-1}(0)=i$. Now suppose $f$ is an analytic function defined on the upper half plane which has the property that $|f(z)| \leq 1$ and $f(i)=\beta$ where $|\beta|<1$. Find an upper bound to $\left|f^{\prime}(i)\right|$. Also find all functions, $f$ which satisfy the condition, $f(i)=$ $\beta,|f(z)| \leq 1$, and achieve this maximum value. Hint: You could consider the function, $h(z) \equiv \phi_{\beta} \circ f \circ g^{-1}(z)$ and check the conditions for the Schwarz lemma for this function, $h$.
4. This and the next two problems follow a presentation of an interesting topic in Rudin [113]. Let $\phi_{\alpha}$ be given in Lemma 52.4.7. Suppose $f$ is an analytic function defined on $B(0,1)$ which satisfies $|f(z)| \leq 1$. Suppose also there are $\alpha, \beta \in B(0,1)$ and it is required $f(\alpha)=\beta$. If $f$ is such a function, show that $\left|f^{\prime}(\alpha)\right| \leq \frac{1-|\beta|^{2}}{1-|\alpha|^{2}}$. Hint: To show this consider $g=\phi_{\beta} \circ f \circ \phi_{-\alpha}$. Show $g(0)=0$ and $|g(z)| \leq 1$ on $B(0,1)$. Now use Lemma 52.4.6.
5. In Problem 4 show there exists a function, $f$ analytic on $B(0,1)$ such that $f(\alpha)=$ $\beta,|f(z)| \leq 0$, and $\left|f^{\prime}(\alpha)\right|=\frac{1-|\beta|^{2}}{1-|\alpha|^{2}}$. Hint: You do this by choosing $g$ in the above problem such that equality holds in Lemma 52.4.6. Thus you need $g(z)=\lambda z$ where $|\lambda|=1$ and solve $g=\phi_{\beta} \circ f \circ \phi_{-\alpha}$ for $f$.
6. Suppose that $f: B(0,1) \rightarrow B(0,1)$ and that $f$ is analytic, one to one, and onto with $f(\alpha)=0$. Show there exists $\lambda,|\lambda|=1$ such that $f(z)=\lambda \phi_{\alpha}(z)$. This gives a different way to look at Theorem 52.4.8. Hint: Let $g=f^{-1}$. Then $g^{\prime}(0) f^{\prime}(\alpha)=1$. However, $f(\alpha)=0$ and $g(0)=\alpha$. From Problem 4 with $\beta=0$, you can conclude an inequality for $\left|f^{\prime}(\alpha)\right|$ and another one for $\left|g^{\prime}(0)\right|$. Then use the fact that the product of these two equals 1 which comes from the chain rule to conclude that equality must take place. Now use Problem 5 to obtain the form of $f$.
7. In Corollary 52.4 .5 show that it is essential that $\alpha<1$. That is, show there exists an example where the conclusion is not satisfied with a slightly weaker growth condition. Hint: Consider $\exp (\exp (z))$.
8. Suppose $\left\{f_{n}\right\}$ is a sequence of functions which are analytic on $\Omega$, a bounded region such that each $f_{n}$ is also continuous on $\bar{\Omega}$. Suppose that $\left\{f_{n}\right\}$ converges uniformly on
$\partial \Omega$. Show that then $\left\{f_{n}\right\}$ converges uniformly on $\bar{\Omega}$ and that the function to which the sequence converges is analytic on $\Omega$ and continuous on $\bar{\Omega}$.
9. Suppose $\Omega$ is a bounded region and there exists a point $z_{0} \in \Omega$ such that $\left|f\left(z_{0}\right)\right|=$ $\min \{|f(z)|: z \in \bar{\Omega}\}$. Can you conclude $f$ must equal a constant?
10. Suppose $f$ is continuous on $\overline{B(a, r)}$ and analytic on $B(a, r)$ and that $f$ is not constant. Suppose also $|f(z)|=C \neq 0$ for all $|z-a|=r$. Show that there exists $\alpha \in B(a, r)$ such that $f(\alpha)=0$. Hint: If not, consider $f / C$ and $C / f$. Both would be analytic on $B(a, r)$ and are equal to 1 on the boundary.
11. Suppose $f$ is analytic on $B(0,1)$ but for every $a \in \partial B(0,1), \lim _{z \rightarrow a}|f(z)|=\infty$. Show there exists a sequence, $\left\{z_{n}\right\} \subseteq B(0,1)$ such that $\lim _{n \rightarrow \infty}\left|z_{n}\right|=1$ and $f\left(z_{n}\right)=0$.

### 52.6 Counting Zeros

The above proof of the open mapping theorem relies on the very important inverse function theorem from real analysis. There are other approaches to this important theorem which do not rely on the big theorems from real analysis and are more oriented toward the use of the Cauchy integral formula and specialized techniques from complex analysis. One of these approaches is given next which involves the notion of "counting zeros". The next theorem is the one about counting zeros. It will also be used later in the proof of the Riemann mapping theorem.

Theorem 52.6.1 Let $\Omega$ be an open set in $\mathbb{C}$ and let $\gamma:[a, b] \rightarrow \Omega$ be closed, continuous, bounded variation, and $n(\gamma, z)=0$ for all $z \notin \Omega$. Suppose also that $f$ is analytic on $\Omega$ having zeros $a_{1}, \cdots, a_{m}$ where the zeros are repeated according to multiplicity, and suppose that none of these zeros are on $\gamma^{*}$. Then

$$
\frac{1}{2 \pi i} \int_{\gamma} \frac{f^{\prime}(z)}{f(z)} d z=\sum_{k=1}^{m} n\left(\gamma, a_{k}\right) .
$$

Proof: Let $f(z)=\prod_{j=1}^{m}\left(z-a_{j}\right) g(z)$ where $g(z) \neq 0$ on $\Omega$. Note that some of the $a_{j}$ could be repeated. Hence

$$
\frac{f^{\prime}(z)}{f(z)}=\sum_{j=1}^{m} \frac{1}{z-a_{j}}+\frac{g^{\prime}(z)}{g(z)}
$$

and so

$$
\frac{1}{2 \pi i} \int_{\gamma} \frac{f^{\prime}(z)}{f(z)} d z=\sum_{j=1}^{m} n\left(\gamma, a_{j}\right)+\frac{1}{2 \pi i} \int_{\gamma} \frac{g^{\prime}(z)}{g(z)} d z .
$$

But the function, $z \rightarrow \frac{g^{\prime}(z)}{g(z)}$ is analytic and so by Corollary 51.7.20, the last integral in the above expression equals 0 . Therefore, this proves the theorem.

The following picture is descriptive of the situation described in the next theorem.


Theorem 52.6.2 Let $\Omega$ be a region, let $\gamma:[a, b] \rightarrow \Omega$ be closed continuous, and bounded variation such that $n(\gamma, z)=0$ for all $z \notin \Omega$. Also suppose $f: \Omega \rightarrow \mathbb{C}$ is analytic and that $\alpha \notin$ $f\left(\gamma^{*}\right)$. Then $f \circ \gamma:[a, b] \rightarrow \mathbb{C}$ is continuous, closed, and bounded variation. Also suppose $\left\{a_{1}, \cdots, a_{m}\right\}=f^{-1}(\alpha)$ where these points are counted according to their multiplicities as zeros of the function $f-\alpha$ Then

$$
n(f \circ \gamma, \alpha)=\sum_{k=1}^{m} n\left(\gamma, a_{k}\right)
$$

Proof: It is clear that $f \circ \gamma$ is continuous. It only remains to verify that it is of bounded variation. Suppose first that $\gamma^{*} \subseteq B \subseteq \bar{B} \subseteq \Omega$ where $B$ is a ball. Then

$$
\begin{gathered}
|f(\gamma(t))-f(\gamma(s))|= \\
\leq\left|\int_{0}^{1} f^{\prime}(\gamma(s)+\lambda(\gamma(t)-\gamma(s)))(\gamma(t)-\gamma(s)) d \lambda\right| \\
\leq C|\gamma(t)-\gamma(s)|
\end{gathered}
$$

where $C \geq \max \left\{\left|f^{\prime}(z)\right|: z \in \bar{B}\right\}$. Hence, in this case,

$$
V(f \circ \gamma,[a, b]) \leq C V(\gamma,[a, b]) .
$$

Now let $\varepsilon$ denote the distance between $\gamma^{*}$ and $\mathbb{C} \backslash \Omega$. Since $\gamma^{*}$ is compact, $\varepsilon>0$. By uniform continuity there exists $\delta=\frac{b-a}{p}$ for $p$ a positive integer such that if $|s-t|<\delta$, then $|\gamma(s)-\gamma(t)|<\frac{\varepsilon}{2}$. Then

$$
\gamma([t, t+\delta]) \subseteq \overline{B\left(\gamma(t), \frac{\varepsilon}{2}\right)} \subseteq \Omega
$$

Let $C \geq \max \left\{\left|f^{\prime}(z)\right|: z \in \cup_{j=1}^{p} \overline{B\left(\gamma\left(t_{j}\right), \frac{\varepsilon}{2}\right)}\right\}$ where $t_{j} \equiv \frac{j}{p}(b-a)+a$. Then from what was just shown,

$$
\begin{aligned}
V(f \circ \gamma,[a, b]) & \leq \sum_{j=0}^{p-1} V\left(f \circ \gamma,\left[t_{j}, t_{j+1}\right]\right) \\
& \leq C \sum_{j=0}^{p-1} V\left(\gamma,\left[t_{j}, t_{j+1}\right]\right)<\infty
\end{aligned}
$$

showing that $f \circ \gamma$ is bounded variation as claimed. Now from Theorem 51.7.15 there exists $\eta \in C^{1}([a, b])$ such that

$$
\eta(a)=\gamma(a)=\gamma(b)=\eta(b), \eta([a, b]) \subseteq \Omega
$$

and

$$
\begin{equation*}
n\left(\eta, a_{k}\right)=n\left(\gamma, a_{k}\right), n(f \circ \gamma, \alpha)=n(f \circ \eta, \alpha) \tag{52.6.7}
\end{equation*}
$$

for $k=1, \cdots, m$. Then

$$
\begin{aligned}
& n(f \circ \gamma, \alpha)=n(f \circ \eta, \alpha) \\
= & \frac{1}{2 \pi i} \int_{f \circ \eta} \frac{d w}{w-\alpha} \\
= & \frac{1}{2 \pi i} \int_{a}^{b} \frac{f^{\prime}(\eta(t))}{f(\eta(t))-\alpha} \eta^{\prime}(t) d t \\
= & \frac{1}{2 \pi i} \int_{\eta} \frac{f^{\prime}(z)}{f(z)-\alpha} d z \\
= & \sum_{k=1}^{m} n\left(\eta, a_{k}\right)
\end{aligned}
$$

By Theorem 52.6.1. By 52.6.7, this equals $\sum_{k=1}^{m} n\left(\gamma, a_{k}\right)$ which proves the theorem.
The next theorem is incredible and is very interesting for its own sake. The following picture is descriptive of the situation of this theorem.


Theorem 52.6.3 Let $f: B(a, R) \rightarrow \mathbb{C}$ be analytic and let

$$
f(z)-\alpha=(z-a)^{m} g(z), \infty>m \geq 1
$$

where $g(z) \neq 0$ in $B(a, R) .(f(z)-\alpha$ has a zero of order $m$ at $z=a$.) Then there exist $\varepsilon, \delta>0$ with the property that for each $z$ satisfying $0<|z-\alpha|<\delta$, there exist points,

$$
\left\{a_{1}, \cdots, a_{m}\right\} \subseteq B(a, \varepsilon),
$$

such that

$$
f^{-1}(z) \cap B(a, \varepsilon)=\left\{a_{1}, \cdots, a_{m}\right\}
$$

and each $a_{k}$ is a zero of order 1 for the function $f(\cdot)-z$.
Proof: By Theorem 51.5.3 $f$ is not constant on $B(a, R)$ because it has a zero of order $m$. Therefore, using this theorem again, there exists $\varepsilon>0$ such that $\overline{B(a, 2 \varepsilon)} \subseteq B(a, R)$ and there are no solutions to the equation $f(z)-\alpha=0$ for $z \in \overline{B(a, 2 \varepsilon)}$ except $a$. Also assume $\varepsilon$ is small enough that for $0<|z-a| \leq 2 \varepsilon, f^{\prime}(z) \neq 0$. This can be done since otherwise, $a$
would be a limit point of a sequence of points, $z_{n}$, having $f^{\prime}\left(z_{n}\right)=0$ which would imply, by Theorem 51.5.3 that $f^{\prime}=0$ on $B(a, R)$, contradicting the assumption that $f-\alpha$ has a zero of order $m$ and is therefore not constant. Thus the situation is described by the following picture.


Now pick $\gamma(t)=a+\varepsilon e^{i t}, t \in[0,2 \pi]$. Then $\alpha \notin f\left(\gamma^{*}\right)$ so there exists $\delta>0$ with

$$
\begin{equation*}
B(\alpha, \delta) \cap f\left(\gamma^{*}\right)=\emptyset \tag{52.6.8}
\end{equation*}
$$

Therefore, $B(\alpha, \delta)$ is contained on one component of $\mathbb{C} \backslash f(\gamma([0,2 \pi]))$. Therefore,

$$
n(f \circ \gamma, \alpha)=n(f \circ \gamma, z)
$$

for all $z \in B(\alpha, \delta)$. Now consider $f$ restricted to $B(a, 2 \varepsilon)$. For $z \in B(\alpha, \delta), f^{-1}(z)$ must consist of a finite set of points because $f^{\prime}(w) \neq 0$ for all $w$ in $\overline{B(a, 2 \varepsilon)} \backslash\{a\}$ implying that the zeros of $f(\cdot)-z$ in $\overline{B(a, 2 \varepsilon)}$ have no limit point. Since $\overline{B(a, 2 \varepsilon)}$ is compact, this means there are only finitely many. By Theorem 52.6.2,

$$
\begin{equation*}
n(f \circ \gamma, z)=\sum_{k=1}^{p} n\left(\gamma, a_{k}\right) \tag{52.6.9}
\end{equation*}
$$

where $\left\{a_{1}, \cdots, a_{p}\right\}=f^{-1}(z)$. Each point, $a_{k}$ of $f^{-1}(z)$ is either inside the circle traced out by $\gamma$, yielding $n\left(\gamma, a_{k}\right)=1$, or it is outside this circle yielding $n\left(\gamma, a_{k}\right)=0$ because of 52.6.8. It follows the sum in 52.6 .9 reduces to the number of points of $f^{-1}(z)$ which are contained in $B(a, \varepsilon)$. Thus, letting those points in $f^{-1}(z)$ which are contained in $B(a, \varepsilon)$ be denoted by $\left\{a_{1}, \cdots, a_{r}\right\}$

$$
n(f \circ \gamma, \alpha)=n(f \circ \gamma, z)=r
$$

Also, by Theorem 52.6.1, $m=n(f \circ \gamma, \alpha)$ because $a$ is a zero of $f-\alpha$ of order $m$. Therefore, for $z \in B(\alpha, \delta)$

$$
m=n(f \circ \gamma, \alpha)=n(f \circ \gamma, z)=r
$$

It is required to show $r=m$, the order of the zero of $f-\alpha$. Therefore, $r=m$. Each of these $a_{k}$ is a zero of order 1 of the function $f(\cdot)-z$ because $f^{\prime}\left(a_{k}\right) \neq 0$. This proves the theorem.

This is a very fascinating result partly because it implies that for values of $f$ near a value, $\alpha$, at which $f(\cdot)-\alpha$ has a zero of order $m$ for $m>1$, the inverse image of these values includes at least $m$ points, not just one. Thus the topological properties of the inverse image changes radically. This theorem also shows that $f(B(a, \varepsilon)) \supseteq B(\alpha, \delta)$.

Theorem 52.6.4 (open mapping theorem) Let $\Omega$ be a region and $f: \Omega \rightarrow \mathbb{C}$ be analytic. Then $f(\Omega)$ is either a point or a region. If $f$ is one to one, then $f^{-1}: f(\Omega) \rightarrow \Omega$ is analytic.

Proof: If $f$ is not constant, then for every $\alpha \in f(\Omega)$, it follows from Theorem 51.5.3 that $f(\cdot)-\alpha$ has a zero of order $m<\infty$ and so from Theorem 52.6.3, for each $a \in \Omega$ there exist $\varepsilon, \delta>0$ such that $f(B(a, \varepsilon)) \supseteq B(\alpha, \delta)$ which clearly implies that $f$ maps open sets to open sets. Therefore, $f(\Omega)$ is open, connected because $f$ is continuous. If $f$ is one to one, Theorem 52.6.3 implies that for every $\alpha \in f(\Omega)$ the zero of $f(\cdot)-\alpha$ is of order 1. Otherwise, that theorem implies that for $z$ near $\alpha$, there are $m$ points which $f$ maps to $z$ contradicting the assumption that $f$ is one to one. Therefore, $f^{\prime}(z) \neq 0$ and since $f^{-1}$ is continuous, due to $f$ being an open map, it follows

$$
\begin{aligned}
\left(f^{-1}\right)^{\prime}(f(z)) & =\lim _{f\left(z_{1}\right) \rightarrow f(z)} \frac{f^{-1}\left(f\left(z_{1}\right)\right)-f^{-1}(f(z))}{f\left(z_{1}\right)-f(z)} \\
& =\lim _{z_{1} \rightarrow z} \frac{z_{1}-z}{f\left(z_{1}\right)-f(z)}=\frac{1}{f^{\prime}(z)}
\end{aligned}
$$

This proves the theorem.

### 52.7 An Application To Linear Algebra

Gerschgorin's theorem gives a convenient way to estimate eigenvalues of a matrix from easy to obtain information. For $A$ an $n \times n$ matrix, denote by $\sigma(A)$ the collection of all eigenvalues of $A$.

Theorem 52.7.1 Let A be an $n \times n$ matrix. Consider the $n$ Gerschgorin discs defined as

$$
D_{i} \equiv\left\{\lambda \in \mathbb{C}:\left|\lambda-a_{i i}\right| \leq \sum_{j \neq i}\left|a_{i j}\right|\right\} .
$$

Then every eigenvalue is contained in some Gerschgorin disc.
This theorem says to add up the absolute values of the entries of the $i^{t h}$ row which are off the main diagonal and form the disc centered at $a_{i i}$ having this radius. The union of these discs contains $\sigma(A)$.

Proof: Suppose $A \mathbf{x}=\lambda \mathbf{x}$ where $\mathbf{x} \neq \mathbf{0}$. Then for $A=\left(a_{i j}\right)$

$$
\sum_{j \neq i} a_{i j} x_{j}=\left(\lambda-a_{i i}\right) x_{i} .
$$

Therefore, if we pick $k$ such that $\left|x_{k}\right| \geq\left|x_{j}\right|$ for all $x_{j}$, it follows that $\left|x_{k}\right| \neq 0$ since $|\mathbf{x}| \neq 0$ and

$$
\left|x_{k}\right| \sum_{j \neq k}\left|a_{k j}\right| \geq \sum_{j \neq k}\left|a_{k j}\right|\left|x_{j}\right| \geq\left|\lambda-a_{k k}\right|\left|x_{k}\right| .
$$

Now dividing by $\left|x_{k}\right|$ we see that $\lambda$ is contained in the $k^{t h}$ Gerschgorin disc.

More can be said using the theory about counting zeros. To begin with the distance between two $n \times n$ matrices, $A=\left(a_{i j}\right)$ and $B=\left(b_{i j}\right)$ as follows.

$$
\|A-B\|^{2} \equiv \sum_{i j}\left|a_{i j}-b_{i j}\right|^{2}
$$

Thus two matrices are close if and only if their corresponding entries are close.
Let $A$ be an $n \times n$ matrix. Recall the eigenvalues of $A$ are given by the zeros of the polynomial, $p_{A}(z)=\operatorname{det}(z I-A)$ where $I$ is the $n \times n$ identity. Then small changes in $A$ will produce small changes in $p_{A}(z)$ and $p_{A}^{\prime}(z)$. Let $\gamma_{k}$ denote a very small closed circle which winds around $z_{k}$, one of the eigenvalues of $A$, in the counter clockwise direction so that $n\left(\gamma_{k}, z_{k}\right)=1$. This circle is to enclose only $z_{k}$ and is to have no other eigenvalue on it. Then apply Theorem 52.6.1. According to this theorem

$$
\frac{1}{2 \pi i} \int_{\gamma} \frac{p_{A}^{\prime}(z)}{p_{A}(z)} d z
$$

is always an integer equal to the multiplicity of $z_{k}$ as a root of $p_{A}(t)$. Therefore, small changes in $A$ result in no change to the above contour integral because it must be an integer and small changes in $A$ result in small changes in the integral. Therefore whenever every entry of the matrix $B$ is close enough to the corresponding entry of the matrix $A$, the two matrices have the same number of zeros inside $\gamma_{k}$ under the usual convention that zeros are to be counted according to multiplicity. By making the radius of the small circle equal to $\varepsilon$ where $\varepsilon$ is less than the minimum distance between any two distinct eigenvalues of $A$, this shows that if $B$ is close enough to $A$, every eigenvalue of $B$ is closer than $\varepsilon$ to some eigenvalue of $A$. The next theorem is about continuous dependence of eigenvalues.

Theorem 52.7.2 If $\lambda$ is an eigenvalue of $A$, then if $\|B-A\|$ is small enough, some eigenvalue of $B$ will be within $\varepsilon$ of $\lambda$.

Consider the situation that $A(t)$ is an $n \times n$ matrix and that $t \rightarrow A(t)$ is continuous for $t \in[0,1]$.

Lemma 52.7.3 Let $\lambda(t) \in \sigma(A(t))$ for $t<1$ and let $\Sigma_{t}=\cup_{s \geq t} \sigma(A(s))$. Also let $K_{t}$ be the connected component of $\lambda(t)$ in $\Sigma_{t}$. Then there exists $\eta>0$ such that $K_{t} \cap \sigma(A(s)) \neq \emptyset$ for all $s \in[t, t+\eta]$.

Proof: Denote by $D(\lambda(t), \delta)$ the disc centered at $\lambda(t)$ having radius $\delta>0$, with other occurrences of this notation being defined similarly. Thus

$$
D(\lambda(t), \delta) \equiv\{z \in \mathbb{C}:|\lambda(t)-z| \leq \delta\}
$$

Suppose $\delta>0$ is small enough that $\lambda(t)$ is the only element of $\sigma(A(t))$ contained in $D(\lambda(t), \delta)$ and that $p_{A(t)}$ has no zeroes on the boundary of this disc. Then by continuity, and the above discussion and theorem, there exists $\eta>0, t+\eta<1$, such that for $s \in[t, t+\eta], p_{A(s)}$ also has no zeroes on the boundary of this disc and that $A(s)$ has the
same number of eigenvalues, counted according to multiplicity, in the disc as $A(t)$. Thus $\sigma(A(s)) \cap D(\lambda(t), \delta) \neq \emptyset$ for all $s \in[t, t+\eta]$. Now let

$$
H=\bigcup_{s \in[t, t+\eta]} \sigma(A(s)) \cap D(\lambda(t), \delta) .
$$

I will show $H$ is connected. Suppose not. Then $H=P \cup Q$ where $P, Q$ are separated and $\lambda(t) \in P$. Let

$$
s_{0} \equiv \inf \{s: \lambda(s) \in Q \text { for some } \lambda(s) \in \sigma(A(s))\}
$$

There exists $\lambda\left(s_{0}\right) \in \sigma\left(A\left(s_{0}\right)\right) \cap D(\lambda(t), \delta)$. If $\lambda\left(s_{0}\right) \notin Q$, then from the above discussion there are

$$
\lambda(s) \in \sigma(A(s)) \cap Q
$$

for $s>s_{0}$ arbitrarily close to $\lambda\left(s_{0}\right)$. Therefore, $\lambda\left(s_{0}\right) \in Q$ which shows that $s_{0}>t$ because $\lambda(t)$ is the only element of $\sigma(A(t))$ in $D(\lambda(t), \delta)$ and $\lambda(t) \in P$. Now let $s_{n} \uparrow s_{0}$. Then $\lambda\left(s_{n}\right) \in P$ for any

$$
\lambda\left(s_{n}\right) \in \sigma\left(A\left(s_{n}\right)\right) \cap D(\lambda(t), \delta)
$$

and from the above discussion, for some choice of $s_{n} \rightarrow s_{0}, \lambda\left(s_{n}\right) \rightarrow \lambda\left(s_{0}\right)$ which contradicts $P$ and $Q$ separated and nonempty. Since $P$ is nonempty, this shows $Q=\emptyset$. Therefore, $H$ is connected as claimed. But $K_{t} \supseteq H$ and so $K_{t} \cap \sigma(A(s)) \neq \emptyset$ for all $s \in[t, t+\eta]$. This proves the lemma.

The following is the necessary theorem.
Theorem 52.7.4 Suppose $A(t)$ is an $n \times n$ matrix and that $t \rightarrow A(t)$ is continuous for $t \in[0,1]$. Let $\lambda(0) \in \sigma(A(0))$ and define $\Sigma \equiv \cup_{t \in[0,1]} \sigma(A(t))$. Let $K_{\lambda(0)}=K_{0}$ denote the connected component of $\lambda(0)$ in $\Sigma$. Then $K_{0} \cap \sigma(A(t)) \neq \emptyset$ for all $t \in[0,1]$.

Proof: Let $S \equiv\left\{t \in[0,1]: K_{0} \cap \sigma(A(s)) \neq \emptyset\right.$ for all $\left.s \in[0, t]\right\}$. Then $0 \in S$. Let $t_{0}=$ $\sup (S)$. Say $\sigma\left(A\left(t_{0}\right)\right)=\lambda_{1}\left(t_{0}\right), \cdots, \lambda_{r}\left(t_{0}\right)$. I claim at least one of these is a limit point of $K_{0}$ and consequently must be in $K_{0}$ which will show that $S$ has a last point. Why is this claim true? Let $s_{n} \uparrow t_{0}$ so $s_{n} \in S$. Now let the discs, $D\left(\lambda_{i}\left(t_{0}\right), \delta\right), i=1, \cdots, r$ be disjoint with $p_{A\left(t_{0}\right)}$ having no zeroes on $\gamma_{i}$ the boundary of $D\left(\lambda_{i}\left(t_{0}\right), \delta\right)$. Then for $n$ large enough it follows from Theorem 52.6 .1 and the discussion following it that $\sigma\left(A\left(s_{n}\right)\right)$ is contained in $\cup_{i=1}^{r} D\left(\lambda_{i}\left(t_{0}\right), \delta\right)$. Therefore, $K_{0} \cap\left(\sigma\left(A\left(t_{0}\right)\right)+D(0, \delta)\right) \neq \emptyset$ for all $\delta$ small enough. This requires at least one of the $\lambda_{i}\left(t_{0}\right)$ to be in $\overline{K_{0}}$. Therefore, $t_{0} \in S$ and $S$ has a last point.

Now by Lemma 52.7.3, if $t_{0}<1$, then $K_{0} \cup K_{t}$ would be a strictly larger connected set containing $\lambda(0)$. (The reason this would be strictly larger is that $K_{0} \cap \sigma(A(s))=\emptyset$ for some $s \in(t, t+\eta)$ while $K_{t} \cap \sigma(A(s)) \neq \emptyset$ for all $s \in[t, t+\eta]$.) Therefore, $t_{0}=1$ and this proves the theorem.

The following is an interesting corollary of the Gerschgorin theorem.
Corollary 52.7.5 Suppose one of the Gerschgorin discs, $D_{i}$ is disjoint from the union of the others. Then $D_{i}$ contains an eigenvalue of A. Also, if there are $n$ disjoint Gerschgorin discs, then each one contains an eigenvalue of $A$.

Proof: Denote by $A(t)$ the matrix $\left(a_{i j}^{t}\right)$ where if $i \neq j, a_{i j}^{t}=t a_{i j}$ and $a_{i i}^{t}=a_{i i}$. Thus to get $A(t)$ we multiply all non diagonal terms by $t$. Let $t \in[0,1]$. Then

$$
A(0)=\operatorname{diag}\left(a_{11}, \cdots, a_{n n}\right)
$$

and $A(1)=A$. Furthermore, the map, $t \rightarrow A(t)$ is continuous. Denote by $D_{j}^{t}$ the Gerschgorin disc obtained from the $j^{t h}$ row for the matrix, $A(t)$. Then it is clear that $D_{j}^{t} \subseteq D_{j}$ the $j^{t h}$ Gerschgorin disc for $A$. Then $a_{i i}$ is the eigenvalue for $A(0)$ which is contained in the disc, consisting of the single point $a_{i i}$ which is contained in $D_{i}$. Letting $K$ be the connected component in $\Sigma$ for $\Sigma$ defined in Theorem 52.7 .4 which is determined by $a_{i i}$, it follows by Gerschgorin's theorem that $K \cap \sigma(A(t)) \subseteq \cup_{j=1}^{n} D_{j}^{t} \subseteq \cup_{j=1}^{n} D_{j}=D_{i} \cup\left(\cup_{j \neq i} D_{j}\right)$ and also, since $K$ is connected, there are no points of $K$ in both $D_{i}$ and $\left(\cup_{j \neq i} D_{j}\right)$. Since at least one point of $K$ is in $D_{i},\left(a_{i i}\right)$ it follows all of $K$ must be contained in $D_{i}$. Now by Theorem 52.7.4 this shows there are points of $K \cap \sigma(A)$ in $D_{i}$. The last assertion follows immediately.

Actually, this can be improved slightly. It involves the following lemma.
Lemma 52.7.6 In the situation of Theorem 52.7.4 suppose $\lambda(0)=K_{0} \cap \sigma(A(0))$ and that $\lambda(0)$ is a simple root of the characteristic equation of $A(0)$. Then for all $t \in[0,1]$,

$$
\sigma(A(t)) \cap K_{0}=\lambda(t)
$$

where $\lambda(t)$ is a simple root of the characteristic equation of $A(t)$.
Proof: Let $S \equiv$

$$
\left\{t \in[0,1]: K_{0} \cap \sigma(A(s))=\lambda(s), \text { a simple eigenvalue for all } s \in[0, t]\right\}
$$

Then $0 \in S$ so it is nonempty. Let $t_{0}=\sup (S)$ and suppose $\lambda_{1} \neq \lambda_{2}$ are two elements of $\sigma\left(A\left(t_{0}\right)\right) \cap K_{0}$. Then choosing $\eta>0$ small enough, and letting $D_{i}$ be disjoint discs containing $\lambda_{i}$ respectively, similar arguments to those of Lemma 52.7.3 imply

$$
H_{i} \equiv \cup_{s \in\left[t_{0}-\eta, t_{0}\right]} \sigma(A(s)) \cap D_{i}
$$

is a connected and nonempty set for $i=1,2$ which would require that $H_{i} \subseteq K_{0}$. But then there would be two different eigenvalues of $A(s)$ contained in $K_{0}$, contrary to the definition of $t_{0}$. Therefore, there is at most one eigenvalue, $\lambda\left(t_{0}\right) \in K_{0} \cap \sigma\left(A\left(t_{0}\right)\right)$. The possibility that it could be a repeated root of the characteristic equation must be ruled out. Suppose then that $\lambda\left(t_{0}\right)$ is a repeated root of the characteristic equation. As before, choose a small disc, $D$ centered at $\lambda\left(t_{0}\right)$ and $\eta$ small enough that

$$
H \equiv \cup_{s \in\left[t_{0}-\eta, t_{0}\right]} \sigma(A(s)) \cap D
$$

is a nonempty connected set containing either multiple eigenvalues of $A(s)$ or else a single repeated root to the characteristic equation of $A(s)$. But since $H$ is connected and contains $\lambda\left(t_{0}\right)$ it must be contained in $K_{0}$ which contradicts the condition for $s \in S$ for all these $s \in\left[t_{0}-\eta, t_{0}\right]$. Therefore, $t_{0} \in S$ as hoped. If $t_{0}<1$, there exists a small disc centered at $\lambda\left(t_{0}\right)$ and $\eta>0$ such that for all $s \in\left[t_{0}, t_{0}+\eta\right], A(s)$ has only simple eigenvalues in
$D$ and the only eigenvalues of $A(s)$ which could be in $K_{0}$ are in $D$. (This last assertion follows from noting that $\lambda\left(t_{0}\right)$ is the only eigenvalue of $A\left(t_{0}\right)$ in $K_{0}$ and so the others are at a positive distance from $K_{0}$. For $s$ close enough to $t_{0}$, the eigenvalues of $A(s)$ are either close to these eigenvalues of $A\left(t_{0}\right)$ at a positive distance from $K_{0}$ or they are close to the eigenvalue, $\lambda\left(t_{0}\right)$ in which case it can be assumed they are in $D$.) But this shows that $t_{0}$ is not really an upper bound to $S$. Therefore, $t_{0}=1$ and the lemma is proved.

With this lemma, the conclusion of the above corollary can be improved.
Corollary 52.7.7 Suppose one of the Gerschgorin discs, $D_{i}$ is disjoint from the union of the others. Then $D_{i}$ contains exactly one eigenvalue of $A$ and this eigenvalue is a simple root to the characteristic polynomial of $A$.

Proof: In the proof of Corollary 52.7.5, first note that $a_{i i}$ is a simple root of $A(0)$ since otherwise the $i^{t h}$ Gerschgorin disc would not be disjoint from the others. Also, $K$, the connected component determined by $a_{i i}$ must be contained in $D_{i}$ because it is connected and by Gerschgorin's theorem above, $K \cap \sigma(A(t))$ must be contained in the union of the Gerschgorin discs. Since all the other eigenvalues of $A(0)$, the $a_{j j}$, are outside $D_{i}$, it follows that $K \cap \sigma(A(0))=a_{i i}$. Therefore, by Lemma 52.7.6, $K \cap \sigma(A(1))=K \cap \sigma(A)$ consists of a single simple eigenvalue. This proves the corollary.

Example 52.7.8 Consider the matrix,

$$
\left(\begin{array}{lll}
5 & 1 & 0 \\
1 & 1 & 1 \\
0 & 1 & 0
\end{array}\right)
$$

The Gerschgorin discs are $D(5,1), D(1,2)$, and $D(0,1)$. Then $D(5,1)$ is disjoint from the other discs. Therefore, there should be an eigenvalue in $D(5,1)$. The actual eigenvalues are not easy to find. They are the roots of the characteristic equation, $t^{3}-6 t^{2}+3 t+5=0$. The numerical values of these are $-.66966,1.4231$, and 5.24655 , verifying the predictions of Gerschgorin's theorem.

### 52.8 Exercises

1. Use Theorem 52.6.1 to give an alternate proof of the fundamental theorem of algebra. Hint: Take a contour of the form $\gamma_{r}=r e^{i t}$ where $t \in[0,2 \pi]$. Consider $\int_{\gamma_{r}} \frac{p^{\prime}(z)}{p(z)} d z$ and consider the limit as $r \rightarrow \infty$.
2. Let $M$ be an $n \times n$ matrix. Recall that the eigenvalues of $M$ are given by the zeros of the polynomial, $p_{M}(z)=\operatorname{det}(M-z I)$ where $I$ is the $n \times n$ identity. Formulate a theorem which describes how the eigenvalues depend on small changes in $M$. Hint: You could define a norm on the space of $n \times n$ matrices as $\|M\| \equiv \operatorname{tr}\left(M M^{*}\right)^{1 / 2}$ where $M^{*}$ is the conjugate transpose of $M$. Thus

$$
\|M\|=\left(\sum_{j, k}\left|M_{j k}\right|^{2}\right)^{1 / 2}
$$

Argue that small changes will produce small changes in $p_{M}(z)$. Then apply Theorem 52.6.1 using $\gamma_{k}$ a very small circle surrounding $z_{k}$, the $k^{t h}$ eigenvalue.
3. Suppose that two analytic functions defined on a region are equal on some set, $S$ which contains a limit point. (Recall $p$ is a limit point of $S$ if every open set which contains $p$, also contains infinitely many points of $S$. ) Show the two functions coincide. We defined $e^{z} \equiv e^{x}(\cos y+i \sin y)$ earlier and we showed that $e^{z}$, defined this way was analytic on $\mathbb{C}$. Is there any other way to define $e^{z}$ on all of $\mathbb{C}$ such that the function coincides with $e^{x}$ on the real axis?
4. You know various identities for real valued functions. For example $\cosh ^{2} x-\sinh ^{2} x=$ 1. If you define $\cosh z \equiv \frac{e^{z}+e^{-z}}{2}$ and $\sinh z \equiv \frac{e^{z}-e^{-z}}{2}$, does it follow that

$$
\cosh ^{2} z-\sinh ^{2} z=1
$$

for all $z \in \mathbb{C}$ ? What about

$$
\sin (z+w)=\sin z \cos w+\cos z \sin w ?
$$

Can you verify these sorts of identities just from your knowledge about what happens for real arguments?
5. Was it necessary that $U$ be a region in Theorem 51.5.3? Would the same conclusion hold if $U$ were only assumed to be an open set? Why? What about the open mapping theorem? Would it hold if $U$ were not a region?
6. Let $f: U \rightarrow \mathbb{C}$ be analytic and one to one. Show that $f^{\prime}(z) \neq 0$ for all $z \in U$. Does this hold for a function of a real variable?
7. We say a real valued function, $u$ is subharmonic if $u_{x x}+u_{y y} \geq 0$. Show that if $u$ is subharmonic on a bounded region, (open connected set) $U$, and continuous on $\bar{U}$ and $u \leq m$ on $\partial U$, then $u \leq m$ on $U$. Hint: If not, $u$ achieves its maximum at $\left(x_{0}, y_{0}\right) \in U$. Let $u\left(x_{0}, y_{0}\right)>m+\delta$ where $\delta>0$. Now consider $u_{\varepsilon}(x, y)=\varepsilon x^{2}+u(x, y)$ where $\varepsilon$ is small enough that $0<\varepsilon x^{2}<\delta$ for all $(x, y) \in U$. Show that $u_{\varepsilon}$ also achieves its maximum at some point of $U$ and that therefore, $u_{\varepsilon x x}+u_{\varepsilon y y} \leq 0$ at that point implying that $u_{x x}+u_{y y} \leq-\varepsilon$, a contradiction.
8. If $u$ is harmonic on some region, $U$, show that $u$ coincides locally with the real part of an analytic function and that therefore, $u$ has infinitely many derivatives on $U$. Hint: Consider the case where $0 \in U$. You can always reduce to this case by a suitable translation. Now let $B(0, r) \subseteq U$ and use the Schwarz formula to obtain an analytic function whose real part coincides with $u$ on $\partial B(0, r)$. Then use Problem 7.
9. Show the solution to the Dirichlet problem of Problem 8 on Page 1627 is unique. You need to formulate this precisely and then prove uniqueness.

## Chapter 53

## Residues

Definition 53.0.1 The residue of $f$ at an isolated singularity $\alpha$ which is a pole, written $\operatorname{res}(f, \alpha)$ is the coefficient of $(z-\alpha)^{-1}$ where

$$
f(z)=g(z)+\sum_{k=1}^{m} \frac{b_{k}}{(z-\alpha)^{k}} .
$$

Thus $\operatorname{res}(f, \alpha)=b_{1}$ in the above.
At this point, recall Corollary 51.7.20 which is stated here for convenience.
Corollary 53.0.2 Let $\Omega$ be an open set and let $\gamma_{k}:\left[a_{k}, b_{k}\right] \rightarrow \Omega, k=1, \cdots, m$, be closed, continuous and of bounded variation. Suppose also that

$$
\sum_{k=1}^{m} n\left(\gamma_{k}, z\right)=0
$$

for all $z \notin \Omega$. Then if $f: \Omega \rightarrow \mathbb{C}$ is analytic,

$$
\sum_{k=1}^{m} \int_{\gamma_{k}} f(w) d w=0
$$

The following theorem is called the residue theorem. Note the resemblance to Corollary 51.7.20.

Theorem 53.0.3 Let $\Omega$ be an open set and let $\gamma_{k}:\left[a_{k}, b_{k}\right] \rightarrow \Omega, k=1, \cdots, m$, be closed, continuous and of bounded variation. Suppose also that

$$
\sum_{k=1}^{m} n\left(\gamma_{k}, z\right)=0
$$

for all $z \notin \Omega$. Then if $f: \Omega \rightarrow \widehat{\mathbb{C}}$ is meromorphic such that no $\gamma_{k}^{*}$ contains any poles of $f$,

$$
\begin{equation*}
\frac{1}{2 \pi i} \sum_{k=1}^{m} \int_{\gamma_{k}} f(w) d w=\sum_{\alpha \in A} \operatorname{res}(f, \alpha) \sum_{k=1}^{m} n\left(\gamma_{k}, \alpha\right) \tag{53.0.1}
\end{equation*}
$$

where here $A$ denotes the set of poles of $f$ in $\Omega$. The sum on the right is a finite sum.
Proof: First note that there are at most finitely many $\alpha$ which are not in the unbounded component of $\mathbb{C} \backslash \cup_{k=1}^{m} \gamma_{k}\left(\left[a_{k}, b_{k}\right]\right)$. Thus there exists a finite set, $\left\{\alpha_{1}, \cdots, \alpha_{N}\right\} \subseteq A$ such that these are the only possibilities for which $\sum_{k=1}^{n} n\left(\gamma_{k}, \alpha\right)$ might not equal zero. Therefore, 53.0.1 reduces to

$$
\frac{1}{2 \pi i} \sum_{k=1}^{m} \int_{\gamma_{k}} f(w) d w=\sum_{j=1}^{N} \operatorname{res}\left(f, \alpha_{j}\right) \sum_{k=1}^{n} n\left(\gamma_{k}, \alpha_{j}\right)
$$

and it is this last equation which is established. Near $\alpha_{j}$,

$$
f(z)=g_{j}(z)+\sum_{r=1}^{m_{j}} \frac{b_{r}^{j}}{\left(z-\alpha_{j}\right)^{r}} \equiv g_{j}(z)+Q_{j}(z)
$$

where $g_{j}$ is analytic at and near $\alpha_{j}$. Now define

$$
G(z) \equiv f(z)-\sum_{j=1}^{N} Q_{j}(z)
$$

It follows that $G(z)$ has a removable singularity at each $\alpha_{j}$. Therefore, by Corollary 51.7.20,

$$
0=\sum_{k=1}^{m} \int_{\gamma_{k}} G(z) d z=\sum_{k=1}^{m} \int_{\gamma_{k}} f(z) d z-\sum_{j=1}^{N} \sum_{k=1}^{m} \int_{\gamma_{k}} Q_{j}(z) d z
$$

Now

$$
\begin{aligned}
\sum_{k=1}^{m} \int_{\gamma_{k}} Q_{j}(z) d z & =\sum_{k=1}^{m} \int_{\gamma_{k}}\left(\frac{b_{1}^{j}}{\left(z-\alpha_{j}\right)}+\sum_{r=2}^{m_{j}} \frac{b_{r}^{j}}{\left(z-\alpha_{j}\right)^{r}}\right) d z \\
& =\sum_{k=1}^{m} \int_{\gamma_{k}} \frac{b_{1}^{j}}{\left(z-\alpha_{j}\right)} d z \equiv \sum_{k=1}^{m} n\left(\gamma_{k}, \alpha_{j}\right) \operatorname{res}\left(f, \alpha_{j}\right)(2 \pi i)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sum_{k=1}^{m} \int_{\gamma_{k}} f(z) d z & =\sum_{j=1}^{N} \sum_{k=1}^{m} \int_{\gamma_{k}} Q_{j}(z) d z \\
& =\sum_{j=1}^{N} \sum_{k=1}^{m} n\left(\gamma_{k}, \alpha_{j}\right) \operatorname{res}\left(f, \alpha_{j}\right)(2 \pi i) \\
& =2 \pi i \sum_{j=1}^{N} \operatorname{res}\left(f, \alpha_{j}\right) \sum_{k=1}^{m} n\left(\gamma_{k}, \alpha_{j}\right) \\
& =(2 \pi i) \sum_{\alpha \in A} \operatorname{res}(f, \alpha) \sum_{k=1}^{m} n\left(\gamma_{k}, \alpha\right)
\end{aligned}
$$

which proves the theorem.
The following is an important example. This example can also be done by real variable methods and there are some who think that real variable methods are always to be preferred to complex variable methods. However, I will use the above theorem to work this example.

Example 53.0.4 Find $\lim _{R \rightarrow \infty} \int_{-R}^{R} \frac{\sin (x)}{x} d x$
Things are easier if you write it as

$$
\lim _{R \rightarrow \infty} \frac{1}{i}\left(\int_{-R}^{-R^{-1}} \frac{e^{i x}}{x} d x+\int_{R^{-1}}^{R} \frac{e^{i x}}{x} d x\right)
$$

This gives the same answer because $\cos (x) / x$ is odd. Consider the following contour in which the orientation involves counterclockwise motion exactly once around.


Denote by $\gamma_{R^{-1}}$ the little circle and $\gamma_{R}$ the big one. Then on the inside of this contour there are no singularities of $e^{i z} / z$ and it is contained in an open set with the property that the winding number with respect to this contour about any point not in the open set equals zero. By Theorem 51.7.22

$$
\begin{equation*}
\frac{1}{i}\left(\int_{-R}^{-R^{-1}} \frac{e^{i x}}{x} d x+\int_{\gamma_{R^{-1}}} \frac{e^{i z}}{z} d z+\int_{R^{-1}}^{R} \frac{e^{i x}}{x} d x+\int_{\gamma_{R}} \frac{e^{i z}}{z} d z\right)=0 \tag{53.0.2}
\end{equation*}
$$

Now

$$
\left|\int_{\gamma_{R}} \frac{e^{i z}}{z} d z\right|=\left|\int_{0}^{\pi} e^{R(i \cos \theta-\sin \theta)} i d \theta\right| \leq \int_{0}^{\pi} e^{-R \sin \theta} d \theta
$$

and this last integral converges to 0 by the dominated convergence theorem. Now consider the other circle. By the dominated convergence theorem again,

$$
\int_{\gamma_{R^{-1}}} \frac{e^{i z}}{z} d z=\int_{\pi}^{0} e^{R^{-1}(i \cos \theta-\sin \theta)} i d \theta \rightarrow-i \pi
$$

as $R \rightarrow \infty$. Then passing to the limit in 53.0.2,

$$
\begin{aligned}
& \lim _{R \rightarrow \infty} \int_{-R}^{R} \frac{\sin (x)}{x} d x \\
= & \lim _{R \rightarrow \infty} \frac{1}{i}\left(\int_{-R}^{-R^{-1}} \frac{e^{i x}}{x} d x+\int_{R^{-1}}^{R} \frac{e^{i x}}{x} d x\right) \\
= & \lim _{R \rightarrow \infty} \frac{1}{i}\left(-\int_{\gamma_{R^{-1}}} \frac{e^{i z}}{z} d z-\int_{\gamma_{R}} \frac{e^{i z}}{z} d z\right)=\frac{-1}{i}(-i \pi)=\pi .
\end{aligned}
$$

Example 53.0.5 Find $\lim _{R \rightarrow \infty} \int_{-R}^{R} e^{i x t} \frac{\sin x}{x} d x$. Note this is essentially finding the inverse Fourier transform of the function, $\sin (x) / x$.

This equals

$$
\begin{aligned}
& \lim _{R \rightarrow \infty} \int_{-R}^{R}(\cos (x t)+i \sin (x t)) \frac{\sin (x)}{x} d x \\
= & \lim _{R \rightarrow \infty} \int_{-R}^{R} \cos (x t) \frac{\sin (x)}{x} d x \\
= & \lim _{R \rightarrow \infty} \int_{-R}^{R} \cos (x t) \frac{\sin (x)}{x} d x \\
= & \lim _{R \rightarrow \infty} \frac{1}{2} \int_{-R}^{R} \frac{\sin (x(t+1))+\sin (x(1-t))}{x} d x
\end{aligned}
$$

Let $t \neq 1,-1$. Then changing variables yields

$$
\lim _{R \rightarrow \infty}\left(\frac{1}{2} \int_{-R(1+t)}^{R(1+t)} \frac{\sin (u)}{u} d u+\frac{1}{2} \int_{-R(1-t)}^{R(1-t)} \frac{\sin (u)}{u} d u\right)
$$

In case $|t|<1$ Example 53.0.4 implies this limit is $\pi$. However, if $t>1$ the limit equals 0 and this is also the case if $t<-1$. Summarizing,

$$
\lim _{R \rightarrow \infty} \int_{-R}^{R} e^{i x t} \frac{\sin x}{x} d x=\left\{\begin{array}{l}
\pi \text { if }|t|<1 \\
0 \text { if }|t|>1
\end{array}\right.
$$

### 53.1 Rouche's Theorem And The Argument Principle

### 53.1.1 Argument Principle

A simple closed curve is just one which is homeomorphic to the unit circle. The Jordan Curve theorem states that every simple closed curve in the plane divides the plane into exactly two connected components, one bounded and the other unbounded. This is a very hard theorem to prove. However, in most applications the conclusion is obvious. Nevertheless, to avoid using this big topological result and to attain some extra generality, I will state the following theorem in terms of the winding number to avoid using it. This theorem is called the argument principle. First recall that $f$ has a zero of order $m$ at $\alpha$ if $f(z)=g(z)(z-\alpha)^{m}$ where $g$ is an analytic function which is not equal to zero at $\alpha$. This is equivalent to having $f(z)=\sum_{k=m}^{\infty} a_{k}(z-\alpha)^{k}$ for $z$ near $\alpha$ where $a_{m} \neq 0$. Also recall that $f$ has a pole of order $m$ at $\alpha$ if for $z$ near $\alpha, f(z)$ is of the form

$$
\begin{equation*}
f(z)=h(z)+\sum_{k=1}^{m} \frac{b_{k}}{(z-\alpha)^{k}} \tag{53.1.3}
\end{equation*}
$$

where $b_{m} \neq 0$ and $h$ is a function analytic near $\alpha$.
Theorem 53.1.1 (argument principle) Let $f$ be meromorphic in $\Omega$. Also suppose $\gamma^{*}$ is a closed bounded variation curve containing none of the poles or zeros of $f$ with the property that for all $z \notin \Omega, n(\gamma, z)=0$ and for all $z \in \Omega, n(\gamma, z)$ either equals 0 or 1 . Now let $\left\{p_{1}, \cdots, p_{m}\right\}$ and $\left\{z_{1}, \cdots, z_{n}\right\}$ be respectively the poles and zeros for which the winding
number of $\gamma$ about these points equals 1 . Let $z_{k}$ be a zero of order $r_{k}$ and let $p_{k}$ be a pole of order $l_{k}$. Then

$$
\frac{1}{2 \pi i} \int_{\gamma} \frac{f^{\prime}(z)}{f(z)} d z=\sum_{k=1}^{n} r_{k}-\sum_{k=1}^{m} l_{k}
$$

Proof: This theorem follows from computing the residues of $f^{\prime} / f$ which has residues only at poles and zeros. I will do this now. First suppose $f$ has a pole of order $p$ at $\alpha$. Then $f$ has the form given in 53.1.3. Therefore,

$$
\begin{gathered}
\frac{f^{\prime}(z)}{f(z)}=\frac{h^{\prime}(z)-\sum_{k=1}^{p} \frac{k b_{k}}{(z-\alpha)^{k+1}}}{h(z)+\sum_{k=1}^{p} \frac{b_{k}}{(z-\alpha)^{k}}} \\
=\frac{h^{\prime}(z)(z-\alpha)^{p}-\sum_{k=1}^{p-1} k b_{k}(z-\alpha)^{-k-1+p}-\frac{p b_{p}}{(z-\alpha)}}{h(z)(z-\alpha)^{p}+\sum_{k=1}^{p-1} b_{k}(z-\alpha)^{p-k}+b_{p}} \\
=\frac{r(z)-\frac{p b_{p}}{(z-\alpha)}}{s(z)+b_{p}}
\end{gathered}
$$

where $s(\alpha)=0, \lim _{z \rightarrow \alpha}(z-\alpha) r(\alpha)=0$. It has a simple pole at $\alpha$ and so the residue is

$$
\operatorname{res}\left(\frac{f^{\prime}}{f}, \alpha\right)=\lim _{z \rightarrow \alpha}(z-\alpha) \frac{r(z)-\frac{p b_{p}}{(z-\alpha)}}{s(z)+b_{p}}=-p
$$

the order of the pole.
Next suppose $f$ has a zero of order $p$ at $\alpha$. Then

$$
\frac{f^{\prime}(z)}{f(z)}=\frac{k}{z-\alpha} \frac{\sum_{k=p}^{\infty} a_{k}(z-\alpha)^{k-1}}{\sum_{k=p}^{\infty} a_{k}(z-\alpha)^{k-1}}=\frac{k}{z-\alpha}
$$

and from this it is clear res $\left(f^{\prime} / f\right)=p$, the order of the zero. The conclusion of this theorem now follows from the residue theorem Theorem 53.0.3.

One can also generalize the theorem to the case where there are many closed curves involved. This is proved in the same way as the above.

Theorem 53.1.2 (argument principle) Let $f$ be meromorphic in $\Omega$ and let $\gamma_{k}:\left[a_{k}, b_{k}\right] \rightarrow \Omega$, $k=1, \cdots, m$, be closed, continuous and of bounded variation. Suppose also that

$$
\sum_{k=1}^{m} n\left(\gamma_{k}, z\right)=0
$$

and for all $z \notin \Omega$ and for $z \in \Omega, \sum_{k=1}^{m} n\left(\gamma_{k}, z\right)$ either equals 0 or 1 . Now let $\left\{p_{1}, \cdots, p_{m}\right\}$ and $\left\{z_{1}, \cdots, z_{n}\right\}$ be respectively the poles and zeros for which the above sum of winding numbers equals 1. Let $z_{k}$ be a zero of order $r_{k}$ and let $p_{k}$ be a pole of order $l_{k}$. Then

$$
\frac{1}{2 \pi i} \int_{\gamma} \frac{f^{\prime}(z)}{f(z)} d z=\sum_{k=1}^{n} r_{k}-\sum_{k=1}^{m} l_{k}
$$

There is also a simple extension of this important principle which I found in [65].
Theorem 53.1.3 (argument principle) Let $f$ be meromorphic in $\Omega$. Also suppose $\gamma^{*}$ is a closed bounded variation curve with the property that for all $z \notin \Omega, n(\gamma, z)=0$ and for all $z \in \Omega, n(\gamma, z)$ either equals 0 or 1 . Now let $\left\{p_{1}, \cdots, p_{m}\right\}$ and $\left\{z_{1}, \cdots, z_{n}\right\}$ be respectively the poles and zeros for which the winding number of $\gamma$ about these points equals 1 listed according to multiplicity. Thus if there is a pole of order $m$ there will be this value repeated $m$ times in the list for the poles. Also let $g(z)$ be an analytic function. Then

$$
\frac{1}{2 \pi i} \int_{\gamma} g(z) \frac{f^{\prime}(z)}{f(z)} d z=\sum_{k=1}^{n} g\left(z_{k}\right)-\sum_{k=1}^{m} g\left(p_{k}\right)
$$

Proof: This theorem follows from computing the residues of $g\left(f^{\prime} / f\right)$. It has residues at poles and zeros. I will do this now. First suppose $f$ has a pole of order $m$ at $\alpha$. Then $f$ has the form given in 53.1.3. Therefore,

$$
\begin{aligned}
& g(z) \frac{f^{\prime}(z)}{f(z)} \\
= & \frac{g(z)\left(h^{\prime}(z)-\sum_{k=1}^{m} \frac{k b_{k}}{(z-\alpha)^{k+1}}\right)}{h(z)+\sum_{k=1}^{m} \frac{b_{k}}{(z-\alpha)^{k}}} \\
= & g(z) \frac{h^{\prime}(z)(z-\alpha)^{m}-\sum_{k=1}^{m-1} k b_{k}(z-\alpha)^{-k-1+m}-\frac{m b_{m}}{(z-\alpha)}}{h(z)(z-\alpha)^{m}+\sum_{k=1}^{m-1} b_{k}(z-\alpha)^{m-k}+b_{m}}
\end{aligned}
$$

From this, it is clear res $\left(g\left(f^{\prime} / f\right), \alpha\right)=-m g(\alpha)$, where $m$ is the order of the pole. Thus $\alpha$ would have been listed $m$ times in the list of poles. Hence the residue at this point is equivalent to adding $-g(\alpha) m$ times.

Next suppose $f$ has a zero of order $m$ at $\alpha$. Then

$$
g(z) \frac{f^{\prime}(z)}{f(z)}=g(z) \frac{\sum_{k=m}^{\infty} a_{k} k(z-\alpha)^{k-1}}{\sum_{k=m}^{\infty} a_{k}(z-\alpha)^{k}}=g(z) \frac{\sum_{k=m}^{\infty} a_{k} k(z-\alpha)^{k-1-m}}{\sum_{k=m}^{\infty} a_{k}(z-\alpha)^{k-m}}
$$

and from this it is clear res $\left(g\left(f^{\prime} / f\right)\right)=g(\alpha) m$, where $m$ is the order of the zero. The conclusion of this theorem now follows from the residue theorem, Theorem 53.0.3.

The way people usually apply these theorems is to suppose $\gamma^{*}$ is a simple closed bounded variation curve, often a circle. Thus it has an inside and an outside, the outside being the unbounded component of $\mathbb{C} \backslash \gamma^{*}$. The orientation of the curve is such that you go around it once in the counterclockwise direction. Then letting $r_{k}$ and $l_{k}$ be as described, the conclusion of the theorem follows. In applications, this is likely the way it will be.

### 53.1.2 Rouche's Theorem

With the argument principle, it is possible to prove Rouche's theorem. In the argument principle, denote by $Z_{f}$ the quantity $\sum_{k=1}^{m} r_{k}$ and by $P_{f}$ the quantity $\sum_{k=1}^{n} l_{k}$. Thus $Z_{f}$ is the
number of zeros of $f$ counted according to the order of the zero with a similar definition holding for $P_{f}$. Thus the conclusion of the argument principle is.

$$
\frac{1}{2 \pi i} \int_{\gamma} \frac{f^{\prime}(z)}{f(z)} d z=Z_{f}-P_{f}
$$

Rouche's theorem allows the comparison of $Z_{h}-P_{h}$ for $h=f, g$. It is a wonderful and amazing result.

Theorem 53.1.4 (Rouche's theorem)Let $f, g$ be meromorphic in an open set $\Omega$. Also suppose $\gamma^{*}$ is a closed bounded variation curve with the property that for all $z \notin \Omega, n(\gamma, z)=0$, no zeros or poles are on $\gamma^{*}$, and for all $z \in \Omega, n(\gamma, z)$ either equals 0 or 1. Let $Z_{f}$ and $P_{f}$ denote respectively the numbers of zeros and poles of $f$, which have the property that the winding number equals 1 , counted according to order, with $Z_{g}$ and $P_{g}$ being defined similarly. Also suppose that for $z \in \gamma^{*}$

$$
\begin{equation*}
|f(z)+g(z)|<|f(z)|+|g(z)| . \tag{53.1.4}
\end{equation*}
$$

Then

$$
Z_{f}-P_{f}=Z_{g}-P_{g}
$$

Proof: From the hypotheses,

$$
\left|1+\frac{f(z)}{g(z)}\right|<1+\left|\frac{f(z)}{g(z)}\right|
$$

which shows that for all $z \in \gamma^{*}$,

$$
\frac{f(z)}{g(z)} \in \mathbb{C} \backslash[0, \infty)
$$

Letting $l$ denote a branch of the logarithm defined on $\mathbb{C} \backslash[0, \infty)$, it follows that $l\left(\frac{f(z)}{g(z)}\right)$ is a primitive for the function,

$$
\frac{(f / g)^{\prime}}{(f / g)}=\frac{f^{\prime}}{f}-\frac{g^{\prime}}{g}
$$

Therefore, by the argument principle,

$$
\begin{aligned}
0 & =\frac{1}{2 \pi i} \int_{\gamma} \frac{(f / g)^{\prime}}{(f / g)} d z=\frac{1}{2 \pi i} \int_{\gamma}\left(\frac{f^{\prime}}{f}-\frac{g^{\prime}}{g}\right) d z \\
& =Z_{f}-P_{f}-\left(Z_{g}-P_{g}\right)
\end{aligned}
$$

This proves the theorem.
Often another condition other than 53.1.4 is used.
Corollary 53.1.5 In the situation of Theorem 53.1.4 change 53.1.4 to the condition,

$$
|f(z)-g(z)|<|f(z)|
$$

for $z \in \gamma^{*}$. Then the conclusion is the same.
Proof: The new condition implies $\left|1-\frac{g}{f}(z)\right|<\left|\frac{g(z)}{f(z)}\right|$ on $\gamma^{*}$. Therefore, $\frac{g(z)}{f(z)} \notin(-\infty, 0]$ and so you can do the same argument with a branch of the logarithm.

### 53.1.3 A Different Formulation

In [115] I found this modification for Rouche's theorem concerned with the counting of zeros of analytic functions. This is a very useful form of Rouche's theorem because it makes no mention of a contour.

Theorem 53.1.6 Let $\Omega$ be a bounded open set and suppose $f, g$ are continuous on $\bar{\Omega}$ and analytic on $\Omega$. Also suppose $|f(z)|<|g(z)|$ on $\partial \Omega$. Then $g$ and $f+g$ have the same number of zeros in $\Omega$ provided each zero is counted according to multiplicity.

Proof: Let $K=\{z \in \bar{\Omega}:|f(z)| \geq|g(z)|\}$. Then letting $\lambda \in[0,1]$, if $z \notin K$, then $|f(z)|<$ $|g(z)|$ and so

$$
0<|g(z)|-|f(z)| \leq|g(z)|-\lambda|f(z)| \leq|g(z)+\lambda f(z)|
$$

which shows that all zeros of $g+\lambda f$ are contained in $K$ which must be a compact subset of $\Omega$ due to the assumption that $|f(z)|<|g(z)|$ on $\partial \Omega$. By Theorem 51.7.25 on Page 1644 there exists a cycle, $\left\{\gamma_{k}\right\}_{k=1}^{n}$ such that $\cup_{k=1}^{n} \gamma_{k}^{*} \subseteq \Omega \backslash K, \sum_{k=1}^{n} n\left(\gamma_{k}, z\right)=1$ for every $z \in K$ and $\sum_{k=1}^{n} n\left(\gamma_{k}, z\right)=0$ for all $z \notin \Omega$. Then as above, it follows from the residue theorem or more directly, Theorem 53.1.2,

$$
\sum_{k=1}^{n} \frac{1}{2 \pi i} \int_{\gamma_{k}} \frac{\lambda f^{\prime}(z)+g^{\prime}(z)}{\lambda f(z)+g(z)} d z=\sum_{j=1}^{p} m_{j}
$$

where $m_{j}$ is the order of the $j^{\text {th }}$ zero of $\lambda f+g$ in $K$, hence in $\Omega$. However,

$$
\lambda \rightarrow \sum_{k=1}^{n} \frac{1}{2 \pi i} \int_{\gamma_{k}} \frac{\lambda f^{\prime}(z)+g^{\prime}(z)}{\lambda f(z)+g(z)} d z
$$

is integer valued and continuous so it gives the same value when $\lambda=0$ as when $\lambda=1$. When $\lambda=0$ this gives the number of zeros of $g$ in $\Omega$ and when $\lambda=1$ it is the number of zeros of $f+g$. This proves the theorem.

Here is another formulation of this theorem.
Corollary 53.1.7 Let $\Omega$ be a bounded open set and suppose $f, g$ are continuous on $\bar{\Omega}$ and analytic on $\Omega$. Also suppose $|f(z)-g(z)|<|g(z)|$ on $\partial \Omega$. Then $f$ and $g$ have the same number of zeros in $\Omega$ provided each zero is counted according to multiplicity.

Proof: You let $f-g$ play the role of $f$ in Theorem 53.1.6. Thus $f-g+g=f$ and $g$ have the same number of zeros. Alternatively, you can give a proof of this directly as follows.

Let $K=\{z \in \Omega:|f(z)-g(z)| \geq|g(z)|\}$. Then if $g(z)+\lambda(f(z)-g(z))=0$ it follows

$$
\begin{aligned}
0 & =|g(z)+\lambda(f(z)-g(z))| \geq|g(z)|-\lambda|f(z)-g(z)| \\
& \geq|g(z)|-|f(z)-g(z)|
\end{aligned}
$$

and so $z \in K$. Thus all zeros of $g(z)+\lambda(f(z)-g(z))$ are contained in $K$. By Theorem 51.7.25 on Page 1644 there exists a cycle, $\left\{\gamma_{k}\right\}_{k=1}^{n}$ such that $\cup_{k=1}^{n} \gamma_{k}^{*} \subseteq \Omega \backslash K, \sum_{k=1}^{n} n\left(\gamma_{k}, z\right)=$ 1 for every $z \in K$ and $\sum_{k=1}^{n} n\left(\gamma_{k}, z\right)=0$ for all $z \notin \Omega$. Then by Theorem 53.1.2,

$$
\sum_{k=1}^{n} \frac{1}{2 \pi i} \int_{\gamma_{k}} \frac{\lambda\left(f^{\prime}(z)-g^{\prime}(z)\right)+g^{\prime}(z)}{\lambda(f(z)-g(z))+g(z)} d z=\sum_{j=1}^{p} m_{j}
$$

where $m_{j}$ is the order of the $j^{\text {th }}$ zero of $\lambda(f-g)+g$ in $K$, hence in $\Omega$. The left side is continuous as a function of $\lambda$ and so the number of zeros of $g$ corresponding to $\lambda=0$ equals the number of zeros of $f$ corresponding to $\lambda=1$. This proves the corollary.

### 53.2 Singularities And The Laurent Series

### 53.2.1 What Is An Annulus?

In general, when you consider singularities, isolated or not, the fundamental tool is the Laurent series. This series is important for many other reasons also. In particular, it is fundamental to the spectral theory of various operators in functional analysis and is one way to obtain relationships between algebraic and analytical conditions essential in various convergence theorems. A Laurent series lives on an annulus. In all this $f$ has values in $X$ where $X$ is a complex Banach space. If you like, let $X=\mathbb{C}$.

Definition 53.2.1 Define ann $\left(a, R_{1}, R_{2}\right) \equiv\left\{z: R_{1}<|z-a|<R_{2}\right\}$.
Thus ann $(a, 0, R)$ would denote the punctured ball, $B(a, R) \backslash\{0\}$ and when $R_{1}>0$, the annulus looks like the following.


The annulus is the stuff between the two circles.
Here is an important lemma which is concerned with the situation described in the following picture.


Lemma 53.2.2 Let $\gamma_{r}(t) \equiv a+r e^{i t}$ for $t \in[0,2 \pi]$ and let $|z-a|<r$. Then $n\left(\gamma_{r}, z\right)=1$. If $|z-a|>r$, then $n\left(\gamma_{r}, z\right)=0$.

Proof: For the first claim, consider for $t \in[0,1]$,

$$
f(t) \equiv n\left(\gamma_{r}, a+t(z-a)\right)
$$

Then from properties of the winding number derived earlier, $f(t) \in \mathbb{Z}, f$ is continuous, and $f(0)=1$. Therefore, $f(t)=1$ for all $t \in[0,1]$. This proves the first claim because $f(1)=n\left(\gamma_{r}, z\right)$.

For the second claim,

$$
\begin{aligned}
n\left(\gamma_{r}, z\right) & =\frac{1}{2 \pi i} \int_{\gamma_{r}} \frac{1}{w-z} d w \\
& =\frac{1}{2 \pi i} \int_{\gamma_{r}} \frac{1}{w-a-(z-a)} d w \\
& =\frac{1}{2 \pi i} \frac{-1}{z-a} \int_{\gamma_{r}} \frac{1}{1-\left(\frac{w-a}{z-a}\right)} d w \\
& =\frac{-1}{2 \pi i(z-a)} \int_{\gamma_{r}} \sum_{k=0}^{\infty}\left(\frac{w-a}{z-a}\right)^{k} d w
\end{aligned}
$$

The series converges uniformly for $w \in \gamma_{r}$ because

$$
\left|\frac{w-a}{z-a}\right|=\frac{r}{r+c}
$$

for some $c>0$ due to the assumption that $|z-a|>r$. Therefore, the sum and the integral can be interchanged to give

$$
n\left(\gamma_{r}, z\right)=\frac{-1}{2 \pi i(z-a)} \sum_{k=0}^{\infty} \int_{\gamma_{r}}\left(\frac{w-a}{z-a}\right)^{k} d w=0
$$

because $w \rightarrow\left(\frac{w-a}{z-a}\right)^{k}$ has an antiderivative. This proves the lemma.
Now consider the following picture which pertains to the next lemma.


Lemma 53.2.3 Let $g$ be analytic on $\operatorname{ann}\left(a, R_{1}, R_{2}\right)$. Then if $\gamma_{r}(t) \equiv a+r e^{i t}$ for $t \in[0,2 \pi]$ and $r \in\left(R_{1}, R_{2}\right)$, then $\int_{\gamma_{r}} g(z) d z$ is independent of $r$.

Proof: Let $R_{1}<r_{1}<r_{2}<R_{2}$ and denote by $-\gamma_{r}(t)$ the curve, $-\gamma_{r}(t) \equiv a+r e^{i(2 \pi-t)}$ for $t \in[0,2 \pi]$. Then if $z \in \overline{B\left(a, R_{1}\right)}$, Lemma 53.2.2 implies both $n\left(\gamma_{r_{2}}, z\right)$ and $n\left(\gamma_{r_{1}}, z\right)=1$ and so

$$
n\left(-\gamma_{r_{1}}, z\right)+n\left(\gamma_{r_{2}}, z\right)=-1+1=0 .
$$

Also if $z \notin B\left(a, R_{2}\right)$, then Lemma 53.2.2 implies $n\left(\gamma_{r_{j}}, z\right)=0$ for $j=1,2$. Therefore, whenever $z \notin \operatorname{ann}\left(a, R_{1}, R_{2}\right)$, the sum of the winding numbers equals zero. Therefore, by Theorem 51.7.19 applied to the function, $f(w)=g(z)(w-z)$ and $z \in$ ann $\left(a, R_{1}, R_{2}\right) \backslash$ $\cup_{j=1}^{2} \gamma_{r_{j}}([0,2 \pi])$,

$$
\begin{gathered}
f(z)\left(n\left(\gamma_{r_{2}}, z\right)+n\left(-\gamma_{r_{1}}, z\right)\right)=0\left(n\left(\gamma_{r_{2}}, z\right)+n\left(-\gamma_{r_{1}}, z\right)\right)= \\
\frac{1}{2 \pi i} \int_{\gamma_{r_{2}}} \frac{g(w)(w-z)}{w-z} d w-\frac{1}{2 \pi i} \int_{\gamma_{r_{1}}} \frac{g(w)(w-z)}{w-z} d w \\
=\frac{1}{2 \pi i} \int_{\gamma_{r_{2}}} g(w) d w-\frac{1}{2 \pi i} \int_{\gamma_{r_{1}}} g(w) d w
\end{gathered}
$$

which proves the desired result.

### 53.2.2 The Laurent Series

The Laurent series is like a power series except it allows for negative exponents. First here is a definition of what is meant by the convergence of such a series.

Definition 53.2.4 $\sum_{n=-\infty}^{\infty} a_{n}(z-a)^{n}$ converges if both the series,

$$
\sum_{n=0}^{\infty} a_{n}(z-a)^{n} \text { and } \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n}
$$

converge. When this is the case, the symbol, $\sum_{n=-\infty}^{\infty} a_{n}(z-a)^{n}$ is defined as

$$
\sum_{n=0}^{\infty} a_{n}(z-a)^{n}+\sum_{n=1}^{\infty} a_{-n}(z-a)^{-n}
$$

Lemma 53.2.5 Suppose

$$
f(z)=\sum_{n=-\infty}^{\infty} a_{n}(z-a)^{n}
$$

for all $|z-a| \in\left(R_{1}, R_{2}\right)$. Then both $\sum_{n=0}^{\infty} a_{n}(z-a)^{n}$ and $\sum_{n=1}^{\infty} a_{-n}(z-a)^{-n}$ converge absolutely and uniformly on $\left\{z: r_{1} \leq|z-a| \leq r_{2}\right\}$ for any $r_{1}<r_{2}$ satisfying $R_{1}<r_{1}<r_{2}<R_{2}$.

Proof: Let $R_{1}<|w-a|=r_{1}-\delta<r_{1}$. Then $\sum_{n=1}^{\infty} a_{-n}(w-a)^{-n}$ converges and so

$$
\lim _{n \rightarrow \infty}\left|a_{-n}\right||w-a|^{-n}=\lim _{n \rightarrow \infty}\left|a_{-n}\right|\left(r_{1}-\delta\right)^{-n}=0
$$

which implies that for all $n$ sufficiently large,

$$
\left|a_{-n}\right|\left(r_{1}-\delta\right)^{-n}<1
$$

Therefore,

$$
\sum_{n=1}^{\infty}\left|a_{-n}\right||z-a|^{-n}=\sum_{n=1}^{\infty}\left|a_{-n}\right|\left(r_{1}-\delta\right)^{-n}\left(r_{1}-\delta\right)^{n}|z-a|^{-n}
$$

Now for $|z-a| \geq r_{1}$,

$$
|z-a|^{-n} \leq \frac{1}{r_{1}^{n}}
$$

and so for all sufficiently large $n$

$$
\left|a_{-n}\right||z-a|^{-n} \leq \frac{\left(r_{1}-\delta\right)^{n}}{r_{1}^{n}}
$$

Therefore, by the Weierstrass $M$ test, the series, $\sum_{n=1}^{\infty} a_{-n}(z-a)^{-n}$ converges absolutely and uniformly on the set

$$
\left\{z \in \mathbb{C}:|z-a| \geq r_{1}\right\}
$$

Similar reasoning shows the series, $\sum_{n=0}^{\infty} a_{n}(z-a)^{n}$ converges uniformly on the set

$$
\left\{z \in \mathbb{C}:|z-a| \leq r_{2}\right\}
$$

This proves the Lemma.
Theorem 53.2.6 Let $f$ be analytic on ann $\left(a, R_{1}, R_{2}\right)$. Then there exist numbers, $a_{n} \in \mathbb{C}$ such that for all $z \in$ ann $\left(a, R_{1}, R_{2}\right)$,

$$
\begin{equation*}
f(z)=\sum_{n=-\infty}^{\infty} a_{n}(z-a)^{n} \tag{53.2.5}
\end{equation*}
$$

where the series converges absolutely and uniformly on $\overline{\operatorname{ann}\left(a, r_{1}, r_{2}\right)}$ whenever $R_{1}<r_{1}<$ $r_{2}<R_{2}$. Also

$$
\begin{equation*}
a_{n}=\frac{1}{2 \pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} d w \tag{53.2.6}
\end{equation*}
$$

where $\gamma(t)=a+r e^{i t}, t \in[0,2 \pi]$ for any $r \in\left(R_{1}, R_{2}\right)$. Furthermore the series is unique in the sense that if 53.2.5 holds for $z \in$ ann $\left(a, R_{1}, R_{2}\right)$, then $a_{n}$ is given in 53.2.6.

Proof: Let $R_{1}<r_{1}<r_{2}<R_{2}$ and define $\gamma_{1}(t) \equiv a+\left(r_{1}-\varepsilon\right) e^{i t}$ and $\gamma_{2}(t) \equiv a+$ $\left(r_{2}+\varepsilon\right) e^{i t}$ for $t \in[0,2 \pi]$ and $\varepsilon$ chosen small enough that $R_{1}<r_{1}-\varepsilon<r_{2}+\varepsilon<R_{2}$.


Then using Lemma 53.2.2, if $z \notin$ ann $\left(a, R_{1}, R_{2}\right)$ then

$$
n\left(-\gamma_{1}, z\right)+n\left(\gamma_{2}, z\right)=0
$$

and if $z \in \operatorname{ann}\left(a, r_{1}, r_{2}\right)$,

$$
n\left(-\gamma_{1}, z\right)+n\left(\gamma_{2}, z\right)=1
$$

Therefore, by Theorem 51.7.19, for $z \in$ ann $\left(a, r_{1}, r_{2}\right)$

$$
\begin{gather*}
f(z)=\frac{1}{2 \pi i}\left[\int_{-\gamma_{1}} \frac{f(w)}{w-z} d w+\int_{\gamma_{2}} \frac{f(w)}{w-z} d w\right] \\
=\frac{1}{2 \pi i}\left[\int_{\gamma_{1}} \frac{f(w)}{(z-a)\left[1-\frac{w-a}{z-a}\right]} d w+\int_{\gamma_{2}} \frac{f(w)}{(w-a)\left[1-\frac{z-a}{w-a}\right]} d w\right] \\
=\frac{1}{2 \pi i} \int_{\gamma_{2}} \frac{f(w)}{w-a} \sum_{n=0}^{\infty}\left(\frac{z-a}{w-a}\right)^{n} d w+ \\
\frac{1}{2 \pi i} \int_{\gamma_{1}} \frac{f(w)}{(z-a)} \sum_{n=0}^{\infty}\left(\frac{w-a}{z-a}\right)^{n} d w . \tag{53.2.7}
\end{gather*}
$$

From the formula 53.2.7, it follows that for $z \in \overline{\operatorname{ann}\left(a, r_{1}, r_{2}\right)}$, the terms in the first sum are bounded by an expression of the form $C\left(\frac{r_{2}}{r_{2}+\varepsilon}\right)^{n}$ while those in the second are bounded by one of the form $C\left(\frac{r_{1}-\varepsilon}{r_{1}}\right)^{n}$ and so by the Weierstrass M test, the convergence is uniform and so the integrals and the sums in the above formula may be interchanged and after renaming the variable of summation, this yields

$$
\begin{align*}
f(z)= & \sum_{n=0}^{\infty}\left(\frac{1}{2 \pi i} \int_{\gamma_{2}} \frac{f(w)}{(w-a)^{n+1}} d w\right)(z-a)^{n}+ \\
& \sum_{n=-\infty}^{-1}\left(\frac{1}{2 \pi i} \int_{\gamma_{1}} \frac{f(w)}{(w-a)^{n+1}}\right)(z-a)^{n} . \tag{53.2.8}
\end{align*}
$$

Therefore, by Lemma 53.2.3, for any $r \in\left(R_{1}, R_{2}\right)$,

$$
\begin{align*}
f(z)= & \sum_{n=0}^{\infty}\left(\frac{1}{2 \pi i} \int_{\gamma_{r}} \frac{f(w)}{(w-a)^{n+1}} d w\right)(z-a)^{n}+ \\
& \sum_{n=-\infty}^{-1}\left(\frac{1}{2 \pi i} \int_{\gamma_{r}} \frac{f(w)}{(w-a)^{n+1}}\right)(z-a)^{n} \tag{53.2.9}
\end{align*}
$$

and so

$$
f(z)=\sum_{n=-\infty}^{\infty}\left(\frac{1}{2 \pi i} \int_{\gamma_{r}} \frac{f(w)}{(w-a)^{n+1}} d w\right)(z-a)^{n}
$$

where $r \in\left(R_{1}, R_{2}\right)$ is arbitrary. This proves the existence part of the theorem. It remains to characterize $a_{n}$.

If $f(z)=\sum_{n=-\infty}^{\infty} a_{n}(z-a)^{n}$ on ann $\left(a, R_{1}, R_{2}\right)$ let

$$
\begin{equation*}
f_{n}(z) \equiv \sum_{k=-n}^{n} a_{k}(z-a)^{k} \tag{53.2.10}
\end{equation*}
$$

This function is analytic in ann $\left(a, R_{1}, R_{2}\right)$ and so from the above argument,

$$
\begin{equation*}
f_{n}(z)=\sum_{k=-\infty}^{\infty}\left(\frac{1}{2 \pi i} \int_{\gamma_{r}} \frac{f_{n}(w)}{(w-a)^{k+1}} d w\right)(z-a)^{k} \tag{53.2.11}
\end{equation*}
$$

Also if $k>n$ or if $k<-n$,

$$
\left(\frac{1}{2 \pi i} \int_{\gamma_{r}} \frac{f_{n}(w)}{(w-a)^{k+1}} d w\right)=0
$$

and so

$$
f_{n}(z)=\sum_{k=-n}^{n}\left(\frac{1}{2 \pi i} \int_{\gamma_{r}} \frac{f_{n}(w)}{(w-a)^{k+1}} d w\right)(z-a)^{k}
$$

which implies from 53.2.10 that for each $k \in[-n, n]$,

$$
\frac{1}{2 \pi i} \int_{\gamma_{r}} \frac{f_{n}(w)}{(w-a)^{k+1}} d w=a_{k}
$$

However, from the uniform convergence of the series,

$$
\sum_{n=0}^{\infty} a_{n}(w-a)^{n}
$$

and

$$
\sum_{n=1}^{\infty} a_{-n}(w-a)^{-n}
$$

ensured by Lemma 53.2 .5 which allows the interchange of sums and integrals, if $k \in$
$[-n, n]$,

$$
\begin{aligned}
& \frac{1}{2 \pi i} \int_{\gamma_{r}} \frac{f(w)}{(w-a)^{k+1}} d w \\
= & \frac{1}{2 \pi i} \int_{\gamma_{r}} \frac{\sum_{m=0}^{\infty} a_{m}(w-a)^{m}+\sum_{m=1}^{\infty} a_{-m}(w-a)^{-m}}{(w-a)^{k+1}} d w \\
= & \sum_{m=0}^{\infty} a_{m} \frac{1}{2 \pi i} \int_{\gamma_{r}}(w-a)^{m-(k+1)} d w \\
& +\sum_{m=1}^{\infty} a_{-m} \int_{\gamma_{r}}(w-a)^{-m-(k+1)} d w \\
= & \sum_{m=0}^{n} a_{m} \frac{1}{2 \pi i} \int_{\gamma_{r}}(w-a)^{m-(k+1)} d w \\
& +\sum_{m=1}^{n} a_{-m} \int_{\gamma_{r}}(w-a)^{-m-(k+1)} d w \\
= & \frac{1}{2 \pi i} \int_{\gamma_{r}} \frac{f_{n}(w)}{(w-a)^{k+1}} d w
\end{aligned}
$$

because if $l>n$ or $l<-n$,

$$
\int_{\gamma_{r}} \frac{a_{l}(w-a)^{l}}{(w-a)^{k+1}} d w=0
$$

for all $k \in[-n, n]$. Therefore,

$$
a_{k}=\frac{1}{2 \pi i} \int_{\gamma_{r}} \frac{f(w)}{(w-a)^{k+1}} d w
$$

and so this establishes uniqueness. This proves the theorem.

### 53.2.3 Contour Integrals And Evaluation Of Integrals

Here are some examples of hard integrals which can be evaluated by using residues. This will be done by integrating over various closed curves having bounded variation.

Example 53.2.7 The first example we consider is the following integral.

$$
\int_{-\infty}^{\infty} \frac{1}{1+x^{4}} d x
$$

One could imagine evaluating this integral by the method of partial fractions and it should work out by that method. However, we will consider the evaluation of this integral by the method of residues instead. To do so, consider the following picture.


Let $\gamma_{r}(t)=r e^{i t}, t \in[0, \pi]$ and let $\sigma_{r}(t)=t: t \in[-r, r]$. Thus $\gamma_{r}$ parameterizes the top curve and $\sigma_{r}$ parameterizes the straight line from $-r$ to $r$ along the $x$ axis. Denoting by $\Gamma_{r}$ the closed curve traced out by these two, we see from simple estimates that

$$
\lim _{r \rightarrow \infty} \int_{\gamma_{r}} \frac{1}{1+z^{4}} d z=0
$$

This follows from the following estimate.

$$
\left|\int_{\gamma_{r}} \frac{1}{1+z^{4}} d z\right| \leq \frac{1}{r^{4}-1} \pi r
$$

Therefore,

$$
\int_{-\infty}^{\infty} \frac{1}{1+x^{4}} d x=\lim _{r \rightarrow \infty} \int_{\Gamma_{r}} \frac{1}{1+z^{4}} d z
$$

We compute $\int_{\Gamma_{r}} \frac{1}{1+z^{4}} d z$ using the method of residues. The only residues of the integrand are located at points, $z$ where $1+z^{4}=0$. These points are

$$
\begin{aligned}
& z=-\frac{1}{2} \sqrt{2}-\frac{1}{2} i \sqrt{2}, z=\frac{1}{2} \sqrt{2}-\frac{1}{2} i \sqrt{2} \\
& z=\frac{1}{2} \sqrt{2}+\frac{1}{2} i \sqrt{2}, z=-\frac{1}{2} \sqrt{2}+\frac{1}{2} i \sqrt{2}
\end{aligned}
$$

and it is only the last two which are found in the inside of $\Gamma_{r}$. Therefore, we need to calculate the residues at these points. Clearly this function has a pole of order one at each of these points and so we may calculate the residue at $\alpha$ in this list by evaluating

$$
\lim _{z \rightarrow \alpha}(z-\alpha) \frac{1}{1+z^{4}}
$$

Thus

$$
\begin{aligned}
& \operatorname{Res}\left(f, \frac{1}{2} \sqrt{2}+\frac{1}{2} i \sqrt{2}\right) \\
= & \lim _{z \rightarrow \frac{1}{2} \sqrt{2}+\frac{1}{2} i \sqrt{2}}\left(z-\left(\frac{1}{2} \sqrt{2}+\frac{1}{2} i \sqrt{2}\right)\right) \frac{1}{1+z^{4}} \\
= & -\frac{1}{8} \sqrt{2}-\frac{1}{8} i \sqrt{2}
\end{aligned}
$$

Similarly we may find the other residue in the same way

$$
\begin{aligned}
& \operatorname{Res}\left(f,-\frac{1}{2} \sqrt{2}+\frac{1}{2} i \sqrt{2}\right) \\
= & \lim _{z \rightarrow-\frac{1}{2} \sqrt{2}+\frac{1}{2} i \sqrt{2}}\left(z-\left(-\frac{1}{2} \sqrt{2}+\frac{1}{2} i \sqrt{2}\right)\right) \frac{1}{1+z^{4}} \\
= & -\frac{1}{8} i \sqrt{2}+\frac{1}{8} \sqrt{2} .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int_{\Gamma_{r}} \frac{1}{1+z^{4}} d z & =2 \pi i\left(-\frac{1}{8} i \sqrt{2}+\frac{1}{8} \sqrt{2}+\left(-\frac{1}{8} \sqrt{2}-\frac{1}{8} i \sqrt{2}\right)\right) \\
& =\frac{1}{2} \pi \sqrt{2}
\end{aligned}
$$

Thus, taking the limit we obtain $\frac{1}{2} \pi \sqrt{2}=\int_{-\infty}^{\infty} \frac{1}{1+x^{4}} d x$.
Obviously many different variations of this are possible. The main idea being that the integral over the semicircle converges to zero as $r \rightarrow \infty$.

Sometimes we don't blow up the curves and take limits. Sometimes the problem of interest reduces directly to a complex integral over a closed curve. Here is an example of this.

Example 53.2.8 The integral is

$$
\int_{0}^{\pi} \frac{\cos \theta}{2+\cos \theta} d \theta
$$

This integrand is even and so it equals

$$
\frac{1}{2} \int_{-\pi}^{\pi} \frac{\cos \theta}{2+\cos \theta} d \theta
$$

For $z$ on the unit circle, $z=e^{i \theta}, \bar{z}=\frac{1}{z}$ and therefore, $\cos \theta=\frac{1}{2}\left(z+\frac{1}{z}\right)$. Thus $d z=i e^{i \theta} d \theta$ and so $d \theta=\frac{d z}{i z}$. Note this is proceeding formally to get a complex integral which reduces to the one of interest. It follows that a complex integral which reduces to the one desired is

$$
\frac{1}{2 i} \int_{\gamma} \frac{\frac{1}{2}\left(z+\frac{1}{z}\right)}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)} \frac{d z}{z}=\frac{1}{2 i} \int_{\gamma} \frac{z^{2}+1}{z\left(4 z+z^{2}+1\right)} d z
$$

where $\gamma$ is the unit circle. Now the integrand has poles of order 1 at those points where $z\left(4 z+z^{2}+1\right)=0$. These points are

$$
0,-2+\sqrt{3},-2-\sqrt{3}
$$

Only the first two are inside the unit circle. It is also clear the function has simple poles at these points. Therefore,

$$
\operatorname{Res}(f, 0)=\lim _{z \rightarrow 0} z\left(\frac{z^{2}+1}{z\left(4 z+z^{2}+1\right)}\right)=1
$$

$$
\begin{gathered}
\operatorname{Res}(f,-2+\sqrt{3})= \\
\lim _{z \rightarrow-2+\sqrt{3}}(z-(-2+\sqrt{3})) \frac{z^{2}+1}{z\left(4 z+z^{2}+1\right)}=-\frac{2}{3} \sqrt{3}
\end{gathered}
$$

It follows

$$
\begin{aligned}
\int_{0}^{\pi} \frac{\cos \theta}{2+\cos \theta} d \theta & =\frac{1}{2 i} \int_{\gamma} \frac{z^{2}+1}{z\left(4 z+z^{2}+1\right)} d z \\
& =\frac{1}{2 i} 2 \pi i\left(1-\frac{2}{3} \sqrt{3}\right) \\
& =\pi\left(1-\frac{2}{3} \sqrt{3}\right)
\end{aligned}
$$

Other rational functions of the trig functions will work out by this method also.
Sometimes you have to be clever about which version of an analytic function that reduces to a real function you should use. The following is such an example.

Example 53.2.9 The integral here is

$$
\int_{0}^{\infty} \frac{\ln x}{1+x^{4}} d x
$$

The same curve used in the integral involving $\frac{\sin x}{x}$ earlier will create problems with the $\log$ since the usual version of the log is not defined on the negative real axis. This does not need to be of concern however. Simply use another branch of the logarithm. Leave out the ray from 0 along the negative $y$ axis and use Theorem 52.2 .3 to define $L(z)$ on this set. Thus $L(z)=\ln |z|+i \arg _{1}(z)$ where $\arg _{1}(z)$ will be the angle, $\theta$, between $-\frac{\pi}{2}$ and $\frac{3 \pi}{2}$ such that $z=|z| e^{i \theta}$. Now the only singularities contained in this curve are

$$
\frac{1}{2} \sqrt{2}+\frac{1}{2} i \sqrt{2},-\frac{1}{2} \sqrt{2}+\frac{1}{2} i \sqrt{2}
$$

and the integrand, $f$ has simple poles at these points. Thus using the same procedure as in the other examples,

$$
\begin{gathered}
\operatorname{Res}\left(f, \frac{1}{2} \sqrt{2}+\frac{1}{2} i \sqrt{2}\right)= \\
\frac{1}{32} \sqrt{2} \pi-\frac{1}{32} i \sqrt{2} \pi
\end{gathered}
$$

and

$$
\begin{gathered}
\operatorname{Res}\left(f, \frac{-1}{2} \sqrt{2}+\frac{1}{2} i \sqrt{2}\right)= \\
\frac{3}{32} \sqrt{2} \pi+\frac{3}{32} i \sqrt{2} \pi
\end{gathered}
$$

Consider the integral along the small semicircle of radius $r$. This reduces to

$$
\int_{\pi}^{0} \frac{\ln |r|+i t}{1+\left(r e^{i t}\right)^{4}}\left(r i e^{i t}\right) d t
$$

which clearly converges to zero as $r \rightarrow 0$ because $r \ln r \rightarrow 0$. Therefore, taking the limit as $r \rightarrow 0$,

$$
\begin{gathered}
\int_{\text {large semicircle }} \frac{L(z)}{1+z^{4}} d z+\lim _{r \rightarrow 0+} \int_{-R}^{-r} \frac{\ln (-t)+i \pi}{1+t^{4}} d t+ \\
\lim _{r \rightarrow 0+} \int_{r}^{R} \frac{\ln t}{1+t^{4}} d t=2 \pi i\left(\frac{3}{32} \sqrt{2} \pi+\frac{3}{32} i \sqrt{2} \pi+\frac{1}{32} \sqrt{2} \pi-\frac{1}{32} i \sqrt{2} \pi\right) .
\end{gathered}
$$

Observing that $\int_{\text {large semicircle }} \frac{L(z)}{1+z^{4}} d z \rightarrow 0$ as $R \rightarrow \infty$,

$$
e(R)+2 \lim _{r \rightarrow 0+} \int_{r}^{R} \frac{\ln t}{1+t^{4}} d t+i \pi \int_{-\infty}^{0} \frac{1}{1+t^{4}} d t=\left(-\frac{1}{8}+\frac{1}{4} i\right) \pi^{2} \sqrt{2}
$$

where $e(R) \rightarrow 0$ as $R \rightarrow \infty$. From an earlier example this becomes

$$
e(R)+2 \lim _{r \rightarrow 0+} \int_{r}^{R} \frac{\ln t}{1+t^{4}} d t+i \pi\left(\frac{\sqrt{2}}{4} \pi\right)=\left(-\frac{1}{8}+\frac{1}{4} i\right) \pi^{2} \sqrt{2}
$$

Now letting $r \rightarrow 0+$ and $R \rightarrow \infty$,

$$
\begin{aligned}
2 \int_{0}^{\infty} \frac{\ln t}{1+t^{4}} d t & =\left(-\frac{1}{8}+\frac{1}{4} i\right) \pi^{2} \sqrt{2}-i \pi\left(\frac{\sqrt{2}}{4} \pi\right) \\
& =-\frac{1}{8} \sqrt{2} \pi^{2}
\end{aligned}
$$

and so

$$
\int_{0}^{\infty} \frac{\ln t}{1+t^{4}} d t=-\frac{1}{16} \sqrt{2} \pi^{2}
$$

which is probably not the first thing you would thing of. You might try to imagine how this could be obtained using elementary techniques.

The next example illustrates the use of what is referred to as a branch cut. It includes many examples.

Example 53.2.10 Mellin transformations are of the form

$$
\int_{0}^{\infty} f(x) x^{\alpha} \frac{d x}{x}
$$

Sometimes it is possible to evaluate such a transform in terms of the constant, $\alpha$.
Assume $f$ is an analytic function except at isolated singularities, none of which are on $(0, \infty)$. Also assume that $f$ has the growth conditions,

$$
|f(z)| \leq \frac{C}{|z|^{b}}, b>\alpha
$$

for all large $|z|$ and assume that

$$
|f(z)| \leq \frac{C^{\prime}}{|z|^{b_{1}}}, b_{1}<\alpha
$$

for all $|z|$ sufficiently small. It turns out there exists an explicit formula for this Mellin transformation under these conditions. Consider the following contour.


In this contour the small semicircle in the center has radius $\varepsilon$ which will converge to 0 . Denote by $\gamma_{R}$ the large circular path which starts at the upper edge of the slot and continues to the lower edge. Denote by $\gamma_{\varepsilon}$ the small semicircular contour and denote by $\gamma_{\varepsilon R+}$ the straight part of the contour from 0 to $R$ which provides the top edge of the slot. Finally denote by $\gamma_{\varepsilon R-}$ the straight part of the contour from $R$ to 0 which provides the bottom edge of the slot. The interesting aspect of this problem is the definition of $f(z) z^{\alpha-1}$. Let

$$
z^{\alpha-1} \equiv e^{(\ln |z|+i \arg (z))(\alpha-1)}=e^{(\alpha-1) \log (z)}
$$

where $\arg (z)$ is the angle of $z$ in $(0,2 \pi)$. Thus you use a branch of the logarithm which is defined on $\mathbb{C} \backslash(0, \infty)$. Then it is routine to verify from the assumed estimates that

$$
\lim _{R \rightarrow \infty} \int_{\gamma_{R}} f(z) z^{\alpha-1} d z=0
$$

and

$$
\lim _{\varepsilon \rightarrow 0+} \int_{\gamma_{\varepsilon}} f(z) z^{\alpha-1} d z=0
$$

Also, it is routine to verify

$$
\lim _{\varepsilon \rightarrow 0+} \int_{\gamma_{\varepsilon R+}} f(z) z^{\alpha-1} d z=\int_{0}^{R} f(x) x^{\alpha-1} d x
$$

and

$$
\lim _{\varepsilon \rightarrow 0+} \int_{\gamma_{\varepsilon R-}} f(z) z^{\alpha-1} d z=-e^{i 2 \pi(\alpha-1)} \int_{0}^{R} f(x) x^{\alpha-1} d x
$$

Therefore, letting $\Sigma_{R}$ denote the sum of the residues of $f(z) z^{\alpha-1}$ which are contained in the disk of radius $R$ except for the possible residue at 0 ,

$$
e(R)+\left(1-e^{i 2 \pi(\alpha-1)}\right) \int_{0}^{R} f(x) x^{\alpha-1} d x=2 \pi i \Sigma_{R}
$$

where $e(R) \rightarrow 0$ as $R \rightarrow \infty$. Now letting $R \rightarrow \infty$,

$$
\lim _{R \rightarrow \infty} \int_{0}^{R} f(x) x^{\alpha-1} d x=\frac{2 \pi i}{1-e^{i 2 \pi(\alpha-1)}} \Sigma=\frac{\pi e^{-\pi i \alpha}}{\sin (\pi \alpha)} \Sigma
$$

where $\Sigma$ denotes the sum of all the residues of $f(z) z^{\alpha-1}$ except for the residue at 0 .
The next example is similar to the one on the Mellin transform. In fact it is a Mellin transform but is worked out independently of the above to emphasize a slightly more informal technique related to the contour.

Example 53.2.11 $\int_{0}^{\infty} \frac{x^{p-1}}{1+x} d x, p \in(0,1)$.
Since the exponent of $x$ in the numerator is larger than -1 . The integral does converge. However, the techniques of real analysis don't tell us what it converges to. The contour to be used is as follows: From $(\varepsilon, 0)$ to $(r, 0)$ along the $x$ axis and then from $(r, 0)$ to $(r, 0)$ counter clockwise along the circle of radius $r$, then from $(r, 0)$ to $(\varepsilon, 0)$ along the $x$ axis and from $(\varepsilon, 0)$ to $(\varepsilon, 0)$, clockwise along the circle of radius $\varepsilon$. You should draw a picture of this contour. The interesting thing about this is that $z^{p-1}$ cannot be defined all the way around 0 . Therefore, use a branch of $z^{p-1}$ corresponding to the branch of the logarithm obtained by deleting the positive $x$ axis. Thus

$$
z^{p-1}=e^{(\ln |z|+i A(z))(p-1)}
$$

where $z=|z| e^{i A(z)}$ and $A(z) \in(0,2 \pi)$. Along the integral which goes in the positive direction on the $x$ axis, let $A(z)=0$ while on the one which goes in the negative direction, take $A(z)=2 \pi$. This is the appropriate choice obtained by replacing the line from $(\varepsilon, 0)$ to $(r, 0)$ with two lines having a small gap joined by a circle of radius $\varepsilon$ and then taking a limit as the gap closes. You should verify that the two integrals taken along the circles of radius $\varepsilon$ and $r$ converge to 0 as $\varepsilon \rightarrow 0$ and as $r \rightarrow \infty$. Therefore, taking the limit,

$$
\int_{0}^{\infty} \frac{x^{p-1}}{1+x} d x+\int_{\infty}^{0} \frac{x^{p-1}}{1+x}\left(e^{2 \pi i(p-1)}\right) d x=2 \pi i \operatorname{Res}(f,-1)
$$

Calculating the residue of the integrand at -1 , and simplifying the above expression,

$$
\left(1-e^{2 \pi i(p-1)}\right) \int_{0}^{\infty} \frac{x^{p-1}}{1+x} d x=2 \pi i e^{(p-1) i \pi}
$$

Upon simplification

$$
\int_{0}^{\infty} \frac{x^{p-1}}{1+x} d x=\frac{\pi}{\sin p \pi}
$$

Example 53.2.12 The Fresnel integrals are

$$
\int_{0}^{\infty} \cos \left(x^{2}\right) d x, \int_{0}^{\infty} \sin \left(x^{2}\right) d x
$$

To evaluate these integrals consider $f(z)=e^{i z^{2}}$ on the curve which goes from the origin to the point $r$ on the $x$ axis and from this point to the point $r\left(\frac{1+i}{\sqrt{2}}\right)$ along a circle of radius $r$, and from there back to the origin as illustrated in the following picture.


Thus the curve to integrate over is shaped like a slice of pie. Denote by $\gamma_{r}$ the curved part. Since $f$ is analytic,

$$
\begin{aligned}
0 & =\int_{\gamma_{r}} e^{i z^{2}} d z+\int_{0}^{r} e^{i x^{2}} d x-\int_{0}^{r} e^{i\left(t\left(\frac{1+i}{\sqrt{2}}\right)\right)^{2}}\left(\frac{1+i}{\sqrt{2}}\right) d t \\
& =\int_{\gamma_{r}} e^{i z^{2}} d z+\int_{0}^{r} e^{i x^{2}} d x-\int_{0}^{r} e^{-t^{2}}\left(\frac{1+i}{\sqrt{2}}\right) d t \\
& =\int_{\gamma_{r}} e^{i z^{2}} d z+\int_{0}^{r} e^{i x^{2}} d x-\frac{\sqrt{\pi}}{2}\left(\frac{1+i}{\sqrt{2}}\right)+e(r)
\end{aligned}
$$

where $e(r) \rightarrow 0$ as $r \rightarrow \infty$. Here we used the fact that $\int_{0}^{\infty} e^{-t^{2}} d t=\frac{\sqrt{\pi}}{2}$. Now consider the first of these integrals.

$$
\begin{aligned}
& \left|\int_{\gamma_{r}} e^{i z^{2}} d z\right|
\end{aligned}=\left\lvert\, \int_{0}^{\frac{\pi}{4}} e^{i\left(r e^{i t}\right)^{2} r i e^{i t} d t \mid} \begin{aligned}
& \leq r \int_{0}^{\frac{\pi}{4}} e^{-r^{2} \sin 2 t} d t \\
&=\frac{r}{2} \int_{0}^{1} \frac{e^{-r^{2} u}}{\sqrt{1-u^{2}}} d u \\
& \leq \frac{r}{2} \int_{0}^{r^{-(3 / 2)}} \frac{1}{\sqrt{1-u^{2}}} d u+\frac{r}{2}\left(\int_{0}^{1} \frac{1}{\sqrt{1-u^{2}}}\right) e^{-\left(r^{1 / 2}\right)}
\end{aligned}\right.
$$

which converges to zero as $r \rightarrow \infty$. Therefore, taking the limit as $r \rightarrow \infty$,

$$
\frac{\sqrt{\pi}}{2}\left(\frac{1+i}{\sqrt{2}}\right)=\int_{0}^{\infty} e^{i x^{2}} d x
$$

and so

$$
\int_{0}^{\infty} \sin x^{2} d x=\frac{\sqrt{\pi}}{2 \sqrt{2}}=\int_{0}^{\infty} \cos x^{2} d x
$$

The following example is one of the most interesting. By an auspicious choice of the contour it is possible to obtain a very interesting formula for $\cot \pi z$ known as the MittagLeffler expansion of $\cot \pi z$.

Example 53.2.13 Let $\gamma_{N}$ be the contour which goes from $-N-\frac{1}{2}-N i$ horizontally to $N+\frac{1}{2}-N i$ and from there, vertically to $N+\frac{1}{2}+N i$ and then horizontally to $-N-\frac{1}{2}+N i$ and finally vertically to $-N-\frac{1}{2}-N i$. Thus the contour is a large rectangle and the direction of integration is in the counter clockwise direction. Consider the following integral.

$$
I_{N} \equiv \int_{\gamma_{N}} \frac{\pi \cos \pi z}{\sin \pi z\left(\alpha^{2}-z^{2}\right)} d z
$$

where $\alpha \in \mathbb{R}$ is not an integer. This will be used to verify the formula of Mittag Leffler,

$$
\begin{equation*}
\frac{1}{\alpha^{2}}+\sum_{n=1}^{\infty} \frac{2}{\alpha^{2}-n^{2}}=\frac{\pi \cot \pi \alpha}{\alpha} \tag{53.2.12}
\end{equation*}
$$

You should verify that $\cot \pi z$ is bounded on this contour and that therefore, $I_{N} \rightarrow 0$ as $N \rightarrow \infty$. Now you compute the residues of the integrand at $\pm \alpha$ and at $n$ where $|n|<$ $N+\frac{1}{2}$ for $n$ an integer. These are the only singularities of the integrand in this contour and therefore, you can evaluate $I_{N}$ by using these. It is left as an exercise to calculate these residues and find that the residue at $\pm \alpha$ is

$$
\frac{-\pi \cos \pi \alpha}{2 \alpha \sin \pi \alpha}
$$

while the residue at $n$ is

$$
\frac{1}{\alpha^{2}-n^{2}}
$$

Therefore,

$$
0=\lim _{N \rightarrow \infty} I_{N}=\lim _{N \rightarrow \infty} 2 \pi i\left[\sum_{n=-N}^{N} \frac{1}{\alpha^{2}-n^{2}}-\frac{\pi \cot \pi \alpha}{\alpha}\right]
$$

which establishes the following formula of Mittag Leffler.

$$
\lim _{N \rightarrow \infty} \sum_{n=-N}^{N} \frac{1}{\alpha^{2}-n^{2}}=\frac{\pi \cot \pi \alpha}{\alpha}
$$

Writing this in a slightly nicer form, yields 53.2.12.

### 53.3 Exercises

1. Example 53.2.7 found the integral of a rational function of a certain sort. The technique used in this example typically works for rational functions of the form $\frac{f(x)}{g(x)}$ where $\operatorname{deg}(g(x)) \geq \operatorname{deg} f(x)+2$ provided the rational function has no poles on the real axis. State and prove a theorem based on these observations.
2. Fill in the missing details of Example 53.2.13 about $I_{N} \rightarrow 0$. Note how important it was that the contour was chosen just right for this to happen. Also verify the claims about the residues.
3. Suppose $f$ has a pole of order $m$ at $z=a$. Define $g(z)$ by

$$
g(z)=(z-a)^{m} f(z) .
$$

Show

$$
\operatorname{Res}(f, a)=\frac{1}{(m-1)!} g^{(m-1)}(a)
$$

Hint: Use the Laurent series.
4. Give a proof of Theorem 53.1.1. Hint: Let $p$ be a pole. Show that near $p$, a pole of order $m$,

$$
\frac{f^{\prime}(z)}{f(z)}=\frac{-m+\sum_{k=1}^{\infty} b_{k}(z-p)^{k}}{(z-p)+\sum_{k=2}^{\infty} c_{k}(z-p)^{k}}
$$

Show that $\operatorname{Res}(f, p)=-m$. Carry out a similar procedure for the zeros.
5. Use Rouche's theorem to prove the fundamental theorem of algebra which says that if $p(z)=z^{n}+a_{n-1} z^{n-1} \cdots+a_{1} z+a_{0}$, then $p$ has $n$ zeros in $\mathbb{C}$. Hint: Let $q(z)=-z^{n}$ and let $\gamma$ be a large circle, $\gamma(t)=r e^{i t}$ for $r$ sufficiently large.
6. Consider the two polynomials $z^{5}+3 z^{2}-1$ and $z^{5}+3 z^{2}$. Show that on $|z|=1$, the conditions for Rouche's theorem hold. Now use Rouche's theorem to verify that $z^{5}+3 z^{2}-1$ must have two zeros in $|z|<1$.
7. Consider the polynomial, $z^{11}+7 z^{5}+3 z^{2}-17$. Use Rouche's theorem to find a bound on the zeros of this polynomial. In other words, find $r$ such that if $z$ is a zero of the polynomial, $|z|<r$. Try to make $r$ fairly small if possible.
8. Verify that $\int_{0}^{\infty} e^{-t^{2}} d t=\frac{\sqrt{\pi}}{2}$. Hint: Use polar coordinates.
9. Use the contour described in Example 53.2.7 to compute the exact values of the following improper integrals.
(a) $\int_{-\infty}^{\infty} \frac{x}{\left(x^{2}+4 x+13\right)^{2}} d x$
(b) $\int_{0}^{\infty} \frac{x^{2}}{\left(x^{2}+a^{2}\right)^{2}} d x$
(c) $\int_{-\infty}^{\infty} \frac{d x}{\left(x^{2}+a^{2}\right)\left(x^{2}+b^{2}\right)}, a, b>0$
10. Evaluate the following improper integrals.
(a) $\int_{0}^{\infty} \frac{\cos a x}{\left(x^{2}+b^{2}\right)^{2}} d x$
(b) $\int_{0}^{\infty} \frac{x \sin x}{\left(x^{2}+a^{2}\right)^{2}} d x$
11. Find the Cauchy principle value of the integral

$$
\int_{-\infty}^{\infty} \frac{\sin x}{\left(x^{2}+1\right)(x-1)} d x
$$

defined as

$$
\lim _{\varepsilon \rightarrow 0+}\left(\int_{-\infty}^{1-\varepsilon} \frac{\sin x}{\left(x^{2}+1\right)(x-1)} d x+\int_{1+\varepsilon}^{\infty} \frac{\sin x}{\left(x^{2}+1\right)(x-1)} d x\right)
$$

12. Find a formula for the integral $\int_{-\infty}^{\infty} \frac{d x}{\left(1+x^{2}\right)^{n+1}}$ where $n$ is a nonnegative integer.
13. Find $\int_{-\infty}^{\infty} \frac{\sin ^{2} x}{x^{2}} d x$.
14. If $m<n$ for $m$ and $n$ integers, show

$$
\int_{0}^{\infty} \frac{x^{2 m}}{1+x^{2 n}} d x=\frac{\pi}{2 n} \frac{1}{\sin \left(\frac{2 m+1}{2 n} \pi\right)}
$$

15. Find $\int_{-\infty}^{\infty} \frac{1}{\left(1+x^{4}\right)^{2}} d x$.
16. Find $\int_{0}^{\infty} \frac{\ln (x)}{1+x^{2}} d x=0$.
17. Suppose $f$ has an isolated singularity at $\alpha$. Show the singularity is essential if and only if the principal part of the Laurent series of $f$ has infinitely many terms. That is, show $f(z)=\sum_{k=0}^{\infty} a_{k}(z-\alpha)^{k}+\sum_{k=1}^{\infty} \frac{b_{k}}{(z-\alpha)^{k}}$ where infinitely many of the $b_{k}$ are nonzero.
18. Suppose $\Omega$ is a bounded open set and $f_{n}$ is analytic on $\Omega$ and continuous on $\bar{\Omega}$. Suppose also that $f_{n} \rightarrow f$ uniformly on $\bar{\Omega}$ and that $f \neq 0$ on $\partial \Omega$. Show that for all $n$ large enough, $f_{n}$ and $f$ have the same number of zeros on $\Omega$ provided the zeros are counted according to multiplicity.

## Chapter 54

## Functional Analysis Applications

### 54.1 The Spectral Radius

As a very important application of the theory of Laurent series, I will give a short description of the spectral radius. This is a fundamental result which must be understood in order to prove convergence of various important numerical methods such as the Gauss Seidel or Jacobi methods.

Definition 54.1.1 Let $X$ be a complex Banach space and let $A \in \mathscr{L}(X, X)$. Then

$$
r(A) \equiv\left\{\lambda \in \mathbb{C}:(\lambda I-A)^{-1} \in \mathscr{L}(X, X)\right\}
$$

This is called the resolvent set. The spectrum of $A$, denoted by $\sigma(A)$ is defined as all the complex numbers which are not in the resolvent set. Thus

$$
\sigma(A) \equiv \mathbb{C} \backslash r(A)
$$

Lemma 54.1.2 $\lambda \in r(A)$ if and only if $\lambda I-A$ is one to one and onto $X$. Also if $|\lambda|>||A||$, then $\lambda \in \sigma(A)$. If the Neumann series,

$$
\frac{1}{\lambda} \sum_{k=0}^{\infty}\left(\frac{A}{\lambda}\right)^{k}
$$

converges, then

$$
\frac{1}{\lambda} \sum_{k=0}^{\infty}\left(\frac{A}{\lambda}\right)^{k}=(\lambda I-A)^{-1}
$$

Proof: Note that to be in $r(A), \lambda I-A$ must be one to one and map $X$ onto $X$ since otherwise, $(\lambda I-A)^{-1} \notin \mathscr{L}(X, X)$.

By the open mapping theorem, if these two algebraic conditions hold, then $(\lambda I-A)^{-1}$ is continuous and so this proves the first part of the lemma. Now suppose $|\lambda|>||A||$. Consider the Neumann series

$$
\frac{1}{\lambda} \sum_{k=0}^{\infty}\left(\frac{A}{\lambda}\right)^{k}
$$

By the root test, Theorem 51.1.3 on Page 1611 this series converges to an element of $\mathscr{L}(X, X)$ denoted here by $B$. Now suppose the series converges. Letting $B_{n} \equiv \frac{1}{\lambda} \sum_{k=0}^{n}\left(\frac{A}{\lambda}\right)^{k}$,

$$
\begin{aligned}
(\lambda I-A) B_{n} & =B_{n}(\lambda I-A)=\sum_{k=0}^{n}\left(\frac{A}{\lambda}\right)^{k}-\sum_{k=0}^{n}\left(\frac{A}{\lambda}\right)^{k+1} \\
& =I-\left(\frac{A}{\lambda}\right)^{n+1} \rightarrow I
\end{aligned}
$$

as $n \rightarrow \infty$ because the convergence of the series requires the $n^{\text {th }}$ term to converge to 0 . Therefore,

$$
(\lambda I-A) B=B(\lambda I-A)=I
$$

which shows $\lambda I-A$ is both one to one and onto and the Neumann series converges to $(\lambda I-A)^{-1}$. This proves the lemma.

This lemma also shows that $\sigma(A)$ is bounded. In fact, $\sigma(A)$ is closed.
Lemma 54.1.3 $r(A)$ is open. In fact, if $\lambda \in r(A)$ and $|\mu-\lambda|<\left\|(\lambda I-A)^{-1}\right\|^{-1}$, then $\mu \in r(A)$.

Proof: First note

$$
\begin{align*}
(\mu I-A) & =\left(I-(\lambda-\mu)(\lambda I-A)^{-1}\right)(\lambda I-A)  \tag{54.1.1}\\
& =(\lambda I-A)\left(I-(\lambda-\mu)(\lambda I-A)^{-1}\right) \tag{54.1.2}
\end{align*}
$$

Also from the assumption about $|\lambda-\mu|$,

$$
\left\|(\lambda-\mu)(\lambda I-A)^{-1}\right\| \leq|\lambda-\mu|\left\|(\lambda I-A)^{-1}\right\|<1
$$

and so by the root test,

$$
\sum_{k=0}^{\infty}\left((\lambda-\mu)(\lambda I-A)^{-1}\right)^{k}
$$

converges to an element of $\mathscr{L}(X, X)$. As in Lemma 54.1.2,

$$
\sum_{k=0}^{\infty}\left((\lambda-\mu)(\lambda I-A)^{-1}\right)^{k}=\left(I-(\lambda-\mu)(\lambda I-A)^{-1}\right)^{-1}
$$

Therefore, from 54.1.1,

$$
(\mu I-A)^{-1}=(\lambda I-A)^{-1}\left(I-(\lambda-\mu)(\lambda I-A)^{-1}\right)^{-1}
$$

This proves the lemma.
Corollary 54.1.4 $\sigma(A)$ is a compact set.
Proof: Lemma 54.1.2 shows $\sigma(A)$ is bounded and Lemma 54.1.3 shows it is closed.
Definition 54.1.5 The spectral radius, denoted by $\rho(A)$ is defined by

$$
\rho(A) \equiv \max \{|\lambda|: \lambda \in \sigma(A)\}
$$

Since $\sigma(A)$ is compact, this maximum exists. Note from Lemma 54.1.2, $\rho(A) \leq\|A\|$.
There is a simple formula for the spectral radius.

Lemma 54.1.6 If $|\lambda|>\rho(A)$, then the Neumann series,

$$
\frac{1}{\lambda} \sum_{k=0}^{\infty}\left(\frac{A}{\lambda}\right)^{k}
$$

converges.
Proof: This follows directly from Theorem 53.2.6 on Page 1682 and the observation above that $\frac{1}{\lambda} \sum_{k=0}^{\infty}\left(\frac{A}{\lambda}\right)^{k}=(\lambda I-A)^{-1}$ for all $|\lambda|>||A||$. Thus the analytic function, $\lambda \rightarrow$ $(\lambda I-A)^{-1}$ has a Laurent expansion on $|\lambda|>\rho(A)$ by Theorem 53.2.6 and it must coincide with $\frac{1}{\lambda} \sum_{k=0}^{\infty}\left(\frac{A}{\lambda}\right)^{k}$ on $|\lambda|>\|A\|$ so the Laurent expansion of $\lambda \rightarrow(\lambda I-A)^{-1}$ must equal $\frac{1}{\lambda} \sum_{k=0}^{\infty}\left(\frac{A}{\lambda}\right)^{k}$ on $|\lambda|>\rho(A)$. This proves the lemma.

The theorem on the spectral radius follows. It is due to Gelfand.
Theorem 54.1.7 $\rho(A)=\lim _{n \rightarrow \infty}\left\|A^{n}\right\|^{1 / n}$.
Proof: If

$$
|\lambda|<\lim \sup _{n \rightarrow \infty}\left\|A^{n}\right\|^{1 / n}
$$

then by the root test, the Neumann series does not converge and so by Lemma 54.1.6 $|\lambda| \leq \rho(A)$. Thus

$$
\rho(A) \geq \lim \sup _{n \rightarrow \infty}\left\|A^{n}\right\|^{1 / n}
$$

Now let $p$ be a positive integer. Then $\lambda \in \sigma(A)$ implies $\lambda^{p} \in \sigma\left(A^{p}\right)$ because

$$
\begin{aligned}
\lambda^{p} I-A^{p} & =(\lambda I-A)\left(\lambda^{p-1}+\lambda^{p-2} A+\cdots+A^{p-1}\right) \\
& =\left(\lambda^{p-1}+\lambda^{p-2} A+\cdots+A^{p-1}\right)(\lambda I-A)
\end{aligned}
$$

It follows from Lemma 54.1.2 applied to $A^{p}$ that for $\lambda \in \sigma(A),\left|\lambda^{p}\right| \leq\left|\left|A^{p}\right|\right|$ and so $|\lambda| \leq$ $\left\|A^{p}\right\|^{1 / p}$. Therefore, $\rho(A) \leq\left\|A^{p}\right\|^{1 / p}$ and since $p$ is arbitrary,

$$
\lim \inf _{p \rightarrow \infty}\left\|A^{p}\right\|^{1 / p} \geq \rho(A) \geq \lim \sup _{n \rightarrow \infty}\left\|A^{n}\right\|^{1 / n}
$$

This proves the theorem.

### 54.2 Analytic Semigroups

### 54.3 Sectorial Operators and Analytic Semigroups

In solving ordinary differential equations, the main result involves the fundamental matrix $\Phi(t)$ where $\Phi^{\prime}(t)=A \Phi(t), \Phi(0)=I$, or $\Phi^{\prime}(t)+A \Phi(t)=0, \Phi(0)=I$ and the variation of constants formula. Recall that $\Phi(t+s)=\Phi(t) \Phi(s)$. This idea generalizes to the situation where $A$ is a closed densely defined operator defined on $D(A) \subseteq X$, a Banach space under some conditions which are sufficiently general to include what was done above with $A$ an
$n \times n$ matrix as a special case. The identity $\Phi(t) \Phi(s)=\Phi(t+s)$ holds for any $t, s \in \mathbb{R}$ and so is called a group of transformations. However, in the more general case, the identity only holds for $t, s \geq 0$ which is why it is called a semigroup. In this more general setting, I will call it $S(t)$. I am mostly following the presentation in Henry [63] in this short introduction. In what follows $H$ will be a Banach space unless specified to be a Hilbert space. This new material differs in letting $A$ be only a closed densely defined operator. It might not be a bounded operator.

These semigroups are useful in considering various partial differential equations which can be considered just like they were ordinary differential equations in the form $u^{\prime}+A u=$ $f(u)$. The semigroups discussed here, when applied to actual examples, have the property of allowing one to begin with a very un-smooth initial condition, something in $H$, and making $S(t) x$ in $D(A)$ for all $t>0$. When applied to partial differential equations, this typically has the effect of making a solution $t \rightarrow S(t) x$ smoother for positive $t$ than the initial condition.

One can show that $\lambda \rightarrow(\lambda I-A)^{-1}$ is analytic on its so called resolvent set. This follows from two things, the resolvant identity

$$
(\lambda I-A)^{-1}(\mu I-A)^{-1}=(\mu-\lambda)^{-1}\left((\lambda I-A)^{-1}-(\mu I-A)^{-1}\right)
$$

which follows from an observation that $(\mu I-A),(\lambda I-A)$ are onto so the identity holds if and only if

$$
(\lambda I-A)^{-1}(\mu I-A)^{-1}(\mu I-A)=(\mu-\lambda)^{-1}\left((\lambda I-A)^{-1}-(\mu I-A)^{-1}\right)(\mu I-A)
$$

if and only if

$$
\begin{aligned}
(\lambda I-A)^{-1} & =(\mu-\lambda)^{-1}\left((\lambda I-A)^{-1}(\mu I-A)-I\right) \\
& =(\mu-\lambda)^{-1}\left((\lambda I-A)^{-1}((\mu-\lambda) I+(\lambda I-A))-I\right)
\end{aligned}
$$

if and only if

$$
\begin{aligned}
(\mu-\lambda)(\lambda I-A)^{-1} & =(\lambda I-A)^{-1}((\mu-\lambda) I+(\lambda I-A))-I \\
& =(\mu-\lambda)(\lambda I-A)^{-1}+I-I
\end{aligned}
$$

and an assumption that $\sup _{\lambda}\left\|(\lambda I-A)^{-1} x\right\|<\infty$ for all $\lambda$ near $\mu$ which by the Uniform boundedness theorem implies $\left\|(\lambda I-A)^{-1}\right\|$ is bounded for $\lambda$ near $\mu$.

Thus I will always assume this resolvent $\lambda \rightarrow(\lambda I-A)^{-1}$ is analytic for $\lambda$ on its resolvent set, where this function is analytic. As to the resolvent set, the following describes it in this case of sectorial operators.

Definition 54.3.1 Let $\phi<\pi / 2$ and for $a \in \mathbb{R}$, let $S_{a \phi}$ denote the sector in the complex plane

$$
\{z \in \mathbb{C} \backslash\{a\}:|\arg (z-a)| \leq \pi-\phi\}
$$

This sector is as shown below.


A closed, densely defined linear operator $A$ is called sectorial if for some sector as described above, it follows that for all $\lambda \in S_{a \phi}$,

$$
\begin{equation*}
(\lambda I-A)^{-1} \in \mathscr{L}(H, H) \tag{54.3.3}
\end{equation*}
$$

and for some $M$

$$
\begin{equation*}
\left\|(\lambda I-A)^{-1}\right\| \leq \frac{M}{|\lambda-a|} \tag{54.3.4}
\end{equation*}
$$

The following perturbation theorem is very useful for sectorial operators. I won't use it here, but in applications of this theory, it is useful. First note that for $\lambda \in S_{a \phi}$,

$$
\begin{equation*}
A(\lambda I-A)^{-1}=-I+\lambda(\lambda I-A)^{-1} \tag{54.3.5}
\end{equation*}
$$

Also, if $x \in D(A)$,

$$
\begin{equation*}
(\lambda-A)^{-1} A x=-x+\lambda(\lambda I-A)^{-1} x \tag{54.3.6}
\end{equation*}
$$

This follows from algebra and noting that $\lambda I-A$ maps $D(A)$ onto $H$ because $(\lambda I-A)^{-1} \in$ $\mathscr{L}(H, H)$. Thus the above is true if and only if $A=\left(-I+\lambda(\lambda I-A)^{-1}\right)(\lambda I-A)$ which is obviously true. 54.3.6 is similar. Thus from 54.3.5,

$$
\begin{equation*}
\left\|A(\lambda I-A)^{-1}\right\| \leq 1+|\lambda|\left\|(\lambda I-A)^{-1}\right\| \leq 1+|\lambda| \frac{M}{|\lambda-a|} \leq C \tag{54.3.7}
\end{equation*}
$$

for some constant $C$ whenever $|\lambda|$ is large enough and in $S_{a \phi}$.
Proposition 54.3.2 Suppose $A$ is a sectorial operator as defined above so it is a densely defined closed operator on $D(A) \subseteq H$ which satisfies

$$
\begin{equation*}
\left\|A(\lambda I-A)^{-1}\right\| \leq C \tag{54.3.8}
\end{equation*}
$$

whenever $|\lambda|, \lambda \in S_{a \phi}$, is sufficiently large and suppose $B$ is a densely defined closed operator such that $D(B) \supseteq D(A)$ and for all $x \in D(A)$,

$$
\begin{equation*}
\|B x\| \leq \varepsilon\|A x\|+K\|x\| \tag{54.3.9}
\end{equation*}
$$

where $\varepsilon C<1$. Then $A+B$ is also sectorial.

Proof: I need to consider $(\lambda I-(A+B))^{-1}$. This equals

$$
\begin{equation*}
\left(\left(I-B(\lambda I-A)^{-1}\right)(\lambda I-A)\right)^{-1} \tag{54.3.10}
\end{equation*}
$$

The issue is whether this makes any sense for all $\lambda \in S_{b \phi}$ for some $b \in \mathbb{R}$. Let $b>a$ be very large so that if $\lambda \in S_{b \phi}$, then 54.3.8 holds. Then from 54.3.9, it follows that for $\|x\| \leq 1$,

$$
\begin{aligned}
\left\|B(\lambda I-A)^{-1} x\right\| & \leq \varepsilon\left\|A(\lambda I-A)^{-1} x\right\|+K\left\|(\lambda I-A)^{-1} x\right\| \\
& \leq \varepsilon C+K /|\lambda-a|
\end{aligned}
$$

and so if $b$ is made sufficiently large and $\lambda \in S_{b \phi}$, then for all $\|x\| \leq 1$,

$$
\left\|B(\lambda I-A)^{-1} x\right\| \leq \varepsilon C+K /|\lambda-a|<r<1
$$

Therefore, for such $b$,

$$
\left(I-B(\lambda I-A)^{-1}\right)^{-1}=\sum_{k=0}^{\infty}\left(B(\lambda I-A)^{-1}\right)^{k}
$$

exists and so for such $b$, the expression in 54.3.10 makes sense and equals

$$
(\lambda I-A)^{-1}\left(I-B(\lambda I-A)^{-1}\right)^{-1}
$$

and furthermore,

$$
\left\|(\lambda I-A)^{-1}\left(I-B(\lambda I-A)^{-1}\right)^{-1}\right\| \leq \frac{M}{|\lambda-a|} \frac{1}{1-r} \leq \frac{M^{\prime}}{|\lambda-b|}
$$

by adjusting the constants because

$$
\frac{M}{|\lambda-a|} \frac{|\lambda-b|}{1-r}
$$

is bounded for $\lambda \in S_{b \phi}$.
In finite dimensions, this kind of thing just shown always holds. There you have $D(A)$ is the whole space typically and $B$ will satisfy such an inequality in 54.3.9. The following example shows that all the bounded operators are sectorial.

Example 54.3.3 If $A \in \mathscr{L}(H, H)$, then $A$ is sectorial.
The spectrum $\sigma(A)$ is bounded by $\|A\|$ and so there is clearly a sector of the above form contained in the resolvent set of $A$. As to the estimate 54.3.4, let $a$ be larger than $2\|A\|$ and let $S_{a \phi}$ be contained in the resolvent set. Then for $\lambda \in S_{a \phi},|\lambda|>2\|A\|$ and so

$$
\left\|(\lambda I-A)^{-1}\right\|=|\lambda|^{-1}\left\|\left(I-\frac{A}{\lambda}\right)^{-1}\right\| \leq|\lambda|^{-1}\left\|\sum_{k=0}^{\infty}\left(\frac{A}{\lambda}\right)^{k}\right\| \leq|\lambda|^{-1} 2
$$

Now for $\lambda \in S_{a \phi},\left|\frac{\lambda-a}{\lambda}\right| \leq M$ for some constant $M$ and so

$$
\left\|(\lambda I-A)^{-1}\right\| \leq \frac{2 M}{|\lambda-a|}
$$

Definition 54.3.4 Let $\varepsilon>0$ and for a sectorial operator as defined above, let the contour $\gamma_{\varepsilon, \phi}$ be as shown next where the orientation is also as shown by the arrow, a being the center of the small circle.


The little circle has radius $\varepsilon$ in the above contour but $\varepsilon$ is not necessarily small.
Definition 54.3.5 For $t \in S_{0(\phi+\pi / 2)}^{0}$ the open sector shown in the following picture,

define

$$
\begin{equation*}
S(t) \equiv \frac{1}{2 \pi i} \int_{\gamma_{\varepsilon, \phi}} e^{\lambda t}(\lambda I-A)^{-1} d \lambda \tag{54.3.11}
\end{equation*}
$$

where $\varepsilon$ is some positive number. Since the integrand is analytic, two different values for $\varepsilon$ give the same result in 54.3.11. The following picture shows $S_{0(\phi+\pi / 2)}^{0}$ and $S_{0 \phi}$. Note how the dotted line is at right angles to the solid line.


Also define $S(0) \equiv I$.
I need to move $A$ in and out of an integral.
Lemma 54.3.6 Let $f(\lambda), A f(y)$ be bounded and continuous on $\gamma_{\varepsilon, \phi}^{*}$ and have values in $D(A)$. Then $A \int_{\gamma_{\varepsilon, \phi}} e^{\lambda t} f(\lambda) d \lambda=\int_{\gamma_{\varepsilon, \phi}} e^{\lambda t} A f(\lambda) d \lambda$ provided $t \in S_{0(\phi+\pi / 2)}^{0}$. Also, for large $R$, and $\Gamma_{R}$ the circle $a+R e^{i \theta}$ with $\theta \in[\pi-\phi, \pi+\phi], \lim _{R \rightarrow \infty} \int_{\Gamma_{R}} e^{\lambda t} f(\lambda) d \lambda=0$.

Proof: On one of the straight lines making up the contour, we have $\lambda=a+y w$ where $|w|=1, y \geq \varepsilon$. Then $e^{\lambda t}=e^{a t+y w t}=e^{a t} e^{y \mid t e^{i(\arg w+\arg t)}}=e^{a t} e^{y(\cos (\arg w+\arg t)+i \sin (\arg w+\arg t))}$. Now for $t \in S_{0(\phi+\pi / 2)}^{0}$,

$$
\arg w+\arg t>\frac{\pi}{2}
$$

and so $\cos (\arg w+\arg t)<0$. Therefore, $\left|e^{\lambda t}\right| \leq\left|e^{a t}\right| e^{-y|t| \delta_{t}}$ where $\delta_{t}>0$. Thus this part of the contour integral is of the form $\int_{\varepsilon}^{\infty} e^{a t+y t w} f(a+y w) w d y$ and

$$
\left\|\int_{R}^{\infty} e^{a t+y t w} f(a+y w) w d y\right\| \leq C\left|e^{a t}\right| \int_{R}^{\infty} e^{-y|t| \delta_{t}} d y<\eta
$$

if $R$ is large enough. Now consider $A \int_{\varepsilon}^{R} \overbrace{e^{a t+y t w} f(a+y w) w}^{g(y)} d y$. There is a sequence of Riemann sums converging to the integral, $\left\{S\left(g, P_{n}\right)\right\}$ as $\left\|P_{n}\right\| \rightarrow 0$ for $P_{n}$ a partition. Each of these sums is in the $D(A)$. Then

$$
S\left(g, P_{n}\right) \rightarrow \int_{\varepsilon}^{R} g(y) d y, A S\left(g, P_{n}\right)=S\left(A g, P_{n}\right) \rightarrow \int_{\varepsilon}^{R} A g(y) d y
$$

Since $A$ is a closed operator, $\int_{\varepsilon}^{R} g(y) d y \in D(A)$ and $A\left(\int_{\varepsilon}^{R} g(y) d y\right)=\int_{\varepsilon}^{R} A g(y) d y$. Now $\int_{\varepsilon}^{R} g(y) d y \in D(A)$ and $\lim _{R \rightarrow \infty} \int_{\varepsilon}^{R} g(y) d y=\int_{\varepsilon}^{\infty} g(y) d y$ while

$$
\lim _{R \rightarrow \infty} A \int_{\varepsilon}^{R} g(y) d y=\lim _{R \rightarrow \infty} \int_{\varepsilon}^{R} A g(y) d y=\int_{\varepsilon}^{\infty} A g(y) d y
$$

Since $A$ is closed, $\int_{\varepsilon}^{\infty} A g(y) d y=A \int_{\varepsilon}^{\infty} g(y) d y$ and $\int_{\varepsilon}^{\infty} g(y) d y \in D(A)$. The other straight line is similar. As to the circular part, it is easier because it is not an improper integral. The argument for taking $A$ on the inside is similar, approximating with Riemann sums and then passing to a limit.

It remains to consider the other claim. On the circle, $\lambda=a+\operatorname{Re}^{i \theta}$ so $d \lambda=\operatorname{Ri}^{i \theta} d \theta$ and

$$
\int_{\Gamma_{R}} e^{\lambda t} f(\lambda) d \lambda=\int_{\pi-\phi}^{\pi+\phi} e^{a t+|t| R(\cos (\theta+\arg t)+i \sin (\theta+\arg t))} f(\lambda) R i e^{i \theta} d \theta
$$

Now for $t \in S_{0(\phi+\pi / 2)}^{0}$ and $\theta$ as indicated, $\theta+\arg t>\frac{\pi}{2}$ and $\theta+\arg t<\frac{3 \pi}{2}$ and so the magnitude of the above integral is no more than an expresson of the form

$$
\left|e^{a t}\right| C \int_{\pi-\phi}^{\pi+\phi} e^{-|t| R \delta} R d \theta
$$

which clearly converges to 0 as $R \rightarrow \infty$.
Because of this lemma, I will move $A$ into and out of the integrals which occur in what follows. Also, it is possible to approximate contour integrals over $\gamma_{\varepsilon, \phi}$ with closed contours and use the Cauchy integral formula.

Next is consideration of the above definition along with estimates.

Lemma 54.3.7 The above definition is well defined for $t \in S_{0(\phi+\pi / 2)}^{0}$. Also there is a constant $M_{r}$ such that

$$
\|S(t)\| \leq M_{r} e^{a t}
$$

for every $t \in S_{0(\phi+\pi / 2)}^{0}$ such that $|\arg t| \leq r<\left(\frac{\pi}{2}-\phi\right)$. If $S_{r}$ is the sector just described, $t$ such that $|\arg t| \leq r<\left(\frac{\pi}{2}-\phi\right)$, then for any $x \in H$,

$$
\begin{equation*}
\lim _{t \rightarrow 0, t \in S_{r}} S(t) x=x \tag{54.3.12}
\end{equation*}
$$

Also, for $|\arg t| \leq r<\left(\frac{\pi}{2}-\phi\right)$

$$
\begin{equation*}
\|A S(t)\| \leq M_{r}\left|e^{a t}\right| \frac{1}{|t|}+N_{r}\left|e^{a t}\right||a| \tag{54.3.13}
\end{equation*}
$$

Proof: In the definition of $S(t)$

$$
S(t) \equiv \frac{1}{2 \pi i} \int_{\gamma_{\varepsilon, \phi}} e^{\lambda t}(\lambda I-A)^{-1} d \lambda
$$

Since $S(t)$ does not depend on $\varepsilon$, we can take $\varepsilon=1 /|t|$. Then the circular part of the contour is $\lambda=a+\frac{1}{|t|} e^{i \theta}$. Then $e^{\lambda t}=e^{\left(a+\frac{1}{|t|} e^{i \theta}\right)\left(|t|\left(e^{i \arg t}\right)\right)}=e^{a t} e^{e^{i(\theta+\arg (t))}}$. Then on the circle which is part of $\gamma_{\varepsilon, \phi}$ the contour integral equals

$$
\frac{1}{2 \pi} \int_{\phi-\pi}^{\pi-\phi} e^{a t} e^{i(\theta+\arg (t))}\left(\left(a+\frac{1}{|t|} e^{i \theta}\right) I-A\right)^{-1} \frac{1}{|t|} e^{i \theta} d \theta
$$

Now

$$
\left|e^{e^{i(\theta+\arg (t))}}\right|=\left|e^{\cos (\theta+\arg t)+i \sin (\theta+\arg t)}\right| \leq e
$$

and by assumption, the norm of the integrand is no larger than $\frac{e e^{a t} M}{1 /|t|} \frac{1}{|t|}$ and so the norm of this integral is dominated by

$$
\frac{e e^{a t} M}{2 \pi} \int_{\phi-\pi}^{\pi-\phi} d \theta=\frac{e e^{a t} M}{2 \pi}(2 \pi-2 \phi) \leq e^{a t} M
$$

where $M$ is independent of $t$.
Now consider the part of the contour used to define $S(t)$ which is the top line segment. $\lambda=y w+a$ where $\arg (w)=\pi-\phi, y>1 /|t|$. This part of the contour integral equals

$$
\frac{1}{2 \pi i} \int_{1 /|t|}^{\infty} e^{(y w+a) t}((y w+a) I-A)^{-1} w d y
$$

Then from the resolvent estimate 54.3.4, the norm of this is dominated by

$$
e^{a t} \frac{1}{2 \pi i} \int_{1 /|t|}^{\infty} e^{y e^{i \arg w}\left(|t| e^{i \arg t}\right)} \frac{M}{y} d y=e^{a t} \frac{1}{2 \pi i} \int_{1 /|t|}^{\infty} e^{|t| y e^{i(\arg w+\arg t)}} \frac{M}{y} d y
$$

By assumption $|\arg (t)| \leq r<\left(\frac{\pi}{2}-\phi\right)$ and so

$$
\begin{gathered}
\arg (w)+\arg (t) \geq(\pi-\phi)-r=\frac{\pi}{2}+\left(\frac{\pi}{2}-\phi\right)-r \equiv \frac{\pi}{2}+\delta(r), \delta(r)>0 \\
e^{i(\arg w+\arg t)}=\cos (\arg w+\arg t)+i \sin (\arg w+\arg t), \text { so } \cos (\arg w+\arg t)<0
\end{gathered}
$$

It follows the integral dominated by an expression of the form

$$
\begin{aligned}
& e^{a t} \frac{1}{2 \pi} \int_{1 /|t|}^{\infty} \exp (-c(r)|t| y) \frac{M}{y} d y=e^{a t} \frac{1}{2 \pi} \int_{1}^{\infty} \exp (-c(r) x) \frac{M|t|}{x} \frac{1}{|t|} d x \\
= & e^{a t} \frac{1}{2 \pi} \int_{1}^{\infty} \exp (-c(r) x) \frac{M}{x} d x
\end{aligned}
$$

where $c(r)<0$ independent of $|\arg (t)| \leq r$. A similar estimate holds for the integral on the bottom segment. Thus for $|\arg (t)| \leq r,\|S(t)\|$ is bounded by $M e^{a t}$ for some constant $M$. In particular, $\|S(t)\| e^{-a t}$ is bounded for $t \in[0, \infty)$.

Now let $x \in D(A)$. From 54.3.6,

$$
\begin{equation*}
\frac{e^{\lambda t}}{\lambda}(\lambda-A)^{-1} A x+\frac{e^{\lambda t}}{\lambda} x=e^{\lambda t}(\lambda I-A)^{-1} x \tag{54.3.14}
\end{equation*}
$$

On the circular part of the contour, $\lambda=a+\frac{1}{|t|} e^{i \theta}$. Consider the first term on the left in the above equation. The contour integral is of the form

$$
\int_{\phi-\pi}^{\pi-\phi} e^{a t} e^{e^{i(\theta+\arg (t))}} \frac{1}{a+\frac{1}{|t|} e^{i \theta}}\left(\left(a+\frac{1}{|t|} e^{i \theta}\right) I-A\right)^{-1} A x \frac{i}{|t|} e^{i \theta} d \theta
$$

which is dominated by

$$
\begin{aligned}
e\left|e^{a t}\right| \int_{\phi-\pi}^{\pi-\phi} \frac{1}{\left|a+\frac{1}{|t|} e^{i \theta}\right|} \frac{M}{\left|\frac{1}{|t|} e^{i \theta}\right|}\|A x\| \frac{1}{|t|} & \leq e^{a t} \hat{M}\|A x\| \int_{\phi-\pi}^{\pi-\phi} \frac{|t|}{|a| t \mid+e^{i \theta \mid}} d \theta \\
& \leq e^{a t} \hat{M}\|A x\| \int_{\phi-\pi}^{\pi-\phi} \frac{|t|}{1-|a||t|} d \theta
\end{aligned}
$$

which converges to 0 as $t \rightarrow 0$. On the other part of the contour, $\lambda=y w+a$ where $\arg (w)=$ $\pi-\phi, y>1 /|t|$.

$$
\frac{e^{a t}}{2 \pi i} \int_{1 /|t|}^{\infty} e^{y w t} \frac{1}{y w+a}((y w+a) I-A)^{-1} w d y
$$

As above, $\arg (w)+\arg (t)>\frac{\pi}{2}+\delta(r), \delta(r)>0$ for $|\arg t| \leq r<\left(\frac{\pi}{2}-\phi\right)$. Thus, as above, this integral is dominated by

$$
\frac{e^{a t}}{2 \pi} \int_{1 /|t|}^{\infty} e^{-y|t| c(r)} \frac{1}{|y w+a|} \frac{M}{|y|} d y=\frac{e^{a t}}{2 \pi} \int_{1}^{\infty} e^{-u c(r)} \frac{|t|}{|u w+a| t| |} \frac{M}{|u|} d u
$$

Which converges to 0 as $t \rightarrow 0$ in the sector $|\arg t| \leq r<\left(\frac{\pi}{2}-\phi\right)$. Also note that for $t$ close to 0 , the contour contains 0 on the left. Similarly the integral over the other straight line converges to 0 as $t \rightarrow 0$ in that sector.

$$
S(t) x=\varepsilon(t)+\frac{1}{2 \pi i} \int_{\gamma_{1 /|t|, \phi}} \frac{e^{\lambda t}}{\lambda} x d \lambda, \lim _{t \rightarrow 0+} \varepsilon(t)=0
$$

Now approximate $\gamma_{1 /|t|, \phi}$ with a closed contour having a large circular arc of radius $R$ such that the resulting bounded contour $\Gamma_{R}$ has 0 on its inside and

$$
\left|\frac{1}{2 \pi i} \int_{\gamma_{1| | t \mid, \phi}} \frac{e^{\lambda t}}{\lambda} x d \lambda-\frac{1}{2 \pi i} \int_{\Gamma_{R}} \frac{e^{\lambda t}}{\lambda} x d \lambda\right|<\eta(R)
$$

By the Cauchy integral formula, this shows that

$$
S(t) x=\varepsilon(t)+\eta(R)+x, \quad \lim _{t \rightarrow 0+} \varepsilon(t)=0=\lim _{R \rightarrow \infty} \eta(R)
$$

So let $R \rightarrow \infty$ and obtain $S(t) x=\varepsilon(t)+x$ and now let $t \rightarrow 0$ to obtain $S(t) x \rightarrow x$. By the first part, $\|S(t)\|$ is bounded for small $t$ in that sector so it follows that for any $x \in H$,

$$
\begin{aligned}
\|S(t) x-x\| & \leq\|S(t) x-S(t) y\|+\|S(t) y-y\|+\|y-x\| \\
& \leq C\|x-y\|+\|S(t) y-y\|
\end{aligned}
$$

Choosing $\|x-y\|$ small enough for $y \in D(A)$, the above is no more than $\varepsilon / 2+\|S(t) y-y\|$ and the second term converges to 0 from what was just shown. Hence, for all $x \in H$,

$$
\lim _{t \rightarrow 0} S(t) x=x
$$

where $t$ is in the sector $|\arg t| \leq r<\left(\frac{\pi}{2}-\phi\right)$.
Now for $|\arg t| \leq r<\left(\frac{\pi}{2}-\phi\right), A S(t)=\frac{1}{2 \pi i} \int_{\gamma_{\varepsilon, \phi}} e^{\lambda t} A(\lambda I-A)^{-1} d \lambda$. From 54.3.5 this is

$$
\frac{1}{2 \pi i} \int_{\gamma_{\varepsilon, \phi}} e^{\lambda t}\left(-I+\lambda(\lambda I-A)^{-1}\right) d \lambda
$$

As above, let $\varepsilon=1 /|t|$. On the circle, $\lambda=a+\frac{1}{|t|} e^{i \theta}$ and as above, this is

$$
\int_{\phi-\pi}^{\pi-\phi} e^{a t} e^{e^{i(\theta+\arg (t))}}\left(-I+\left(a+\frac{1}{|t|} e^{i \theta}\right)\left(\left(a+\frac{1}{|t|} e^{i \theta}\right) I-A\right)^{-1}\right) \frac{i}{|t|} e^{i \theta} d \theta
$$

As before, because of the choice of $t$, the above is dominated by

$$
\begin{aligned}
e\left|e^{a t}\right| \int_{\phi-\pi}^{\pi-\phi}\left(1+M \frac{\left|a+\frac{1}{|t|} e^{i \theta}\right|}{1 /|t|}\right) \frac{1}{|t|} d \theta & =e\left|e^{a t}\right| M \int_{\phi-\pi}^{\pi-\phi}\left(1+|a| t\left|+e^{i \theta}\right|\right) \frac{1}{|t|} d \theta \\
& =e\left|e^{a t}\right| M \int_{\phi-\pi}^{\pi-\phi}\left(\frac{1}{|t|}+\left|a+e^{i \theta} \frac{1}{|t|}\right|\right) d \theta \\
& \leq e\left|e^{a t} M\right| 2 \pi \frac{2}{|t|}+M e\left|e^{a t}\right||a| 2 \pi(54.3 .15)
\end{aligned}
$$

Now consider one of the straight lines. On either of these $\lambda=a+w y$ where $|w|=1$ and $y \geq 1 /|t|$. Then the contour integral is

$$
\frac{e^{a t}}{2 \pi i} \int_{1 /|t|}^{\infty} e^{y w t}\left(-I+(a+w y)((a+w y) I-A)^{-1}\right) w d y
$$

As earlier, the norm of this is dominated by $\frac{\left|e^{a t}\right|}{2 \pi} \int_{1 /|t|}^{\infty} e^{-y|t| c(r)}\left(1+M \frac{|a+w y|}{|w y|}\right) d y=$

$$
\begin{gathered}
=\frac{\left|e^{a t}\right|}{2 \pi} \int_{1}^{\infty} e^{-x c(r)}\left(1+M \frac{|a+w(x /|t|)|}{|w(x /|t|)|}\right) \frac{1}{|t|} d x \\
=\frac{\left|e^{a t}\right|}{2 \pi} \int_{1}^{\infty} e^{-x c(r)}\left(1+M \frac{|a| t|+w x|}{|x|}\right) \frac{1}{|t|} d x \leq \frac{\left|e^{a t}\right|}{2 \pi}\left(M_{r} \frac{1}{|t|}\right)+N_{r}|a| \frac{\left|e^{a t}\right|}{2 \pi}
\end{gathered}
$$

Combining this with 54.3 .15 and adjusting constants,

$$
\|A S(t)\| \leq M_{r}\left|e^{a t}\right| \frac{1}{|t|}+N_{r}\left|e^{a t}\right||a|
$$

Also note that if the contour is shifted to the right slightly, the integral over the shifted contour, $\gamma_{\varepsilon, \phi}^{\prime}$ coincides with the integral over $\gamma_{\varepsilon, \phi}$ thanks to the Cauchy integral formula and Lemma 54.3.6 which allows the approximation of the above integrals with one on a closed contour. The following is the main result.

Theorem 54.3.8 Let A be a sectorial operator as defined in Definition 54.3.1 for the sector $S_{a, \phi}$. Then there exists a semigroup $S(t)$ for $t \in|\arg z| \leq r<\left(\frac{\pi}{2}-\phi\right)$ which satisfies the following conditions.

1. Then $S(t)$ given above in 54.3 .11 is analytic for $t \in S_{0,(\phi+\pi / 2)}^{0}$.
2. For any $x \in H$ and $t \in S_{0,(\phi+\pi / 2)}^{0}$, then for $n$ a positive integer, $S^{(n)}(t) x=A^{n} S(t) x$
3. $S$ is a semigroup on the open sector, $S_{0,(\phi+\pi / 2)}^{0}$. That is, for all $t, s \in S_{0(\phi+\pi / 2)}^{0}$,

$$
S(t+s)=S(t) S(s)
$$

4. $\lim _{t \rightarrow 0, t \in S_{r}} S(t) x=x$ for all $x \in H$ where $|\arg t| \leq r<\left(\frac{\pi}{2}-\phi\right)$
5. For some constants $M, N$, ift is positive and real, $\|S(t)\| \leq M e^{a t},\|A S(t)\| \leq M e^{a t} \frac{1}{|t|}+$ $N\left|e^{a t}\right||a|$

Proof: Consider the first claim. This follows right away from the formula: $S(t) \equiv$ $\frac{1}{2 \pi i} \int_{\gamma_{\varepsilon, \phi}} e^{\lambda t}(\lambda I-A)^{-1} d \lambda$. One can differentiate under the integral sign using the dominated convergence theorem to obtain

$$
\begin{gathered}
S^{\prime}(t) \equiv \frac{1}{2 \pi i} \int_{\gamma_{\varepsilon, \phi}} \lambda e^{\lambda t}(\lambda I-A)^{-1} d \lambda=\frac{1}{2 \pi i} \int_{\gamma_{\varepsilon, \phi}} e^{\lambda t}\left(I+A(\lambda I-A)^{-1}\right) d \lambda \\
=\frac{1}{2 \pi i} \int_{\gamma_{\varepsilon, \phi}} e^{\lambda t} A(\lambda I-A)^{-1} d \lambda
\end{gathered}
$$

because of Lemma 54.3.6 the Cauchy integral theorem, and approximating $\gamma_{\varepsilon, \phi}$ with closed contours.


Now from Lemma 54.3.6 one can take $A$ out of the integral and

$$
S^{\prime}(t)=A\left(\frac{1}{2 \pi i} \int_{\gamma_{\varepsilon, \phi}} e^{\lambda t}(\lambda I-A)^{-1} d \lambda\right)=A S(t)
$$

To get the bigher derivatives, note $S(t)$ has infinitely many derivatives due to $t$ being a complex variable. Therefore,

$$
S^{\prime \prime}(t)=\lim _{h \rightarrow 0} \frac{S^{\prime}(t+h)-S^{\prime}(t)}{h}=\lim _{h \rightarrow 0} A \frac{S(t+h)-S(t)}{h}
$$

and $\frac{S(t+h)-S(t)}{h} \rightarrow A S(t)$ and so since $A$ is closed, $A S(t) \in D(A)$ and the above becomes $A^{2} S(t)$. Continuing this way yields the claims 1.) and 2.). Note this also implies $S(t) x \in$ $D(A)$ for each $t \in S_{0(\phi+\pi / 2)}^{0}$ which says more than $S(t) x \in H$. In practice this has the effect of regularizing the solution to an initial value problem.

Next consider the semigroup property. Let $s, t \in S_{0,(\phi+\pi / 2)}^{0}$. As described above let $\gamma_{\varepsilon, \phi}^{\prime}$ denote the contour shifted slightly to the right. Then

$$
\begin{equation*}
S(t) S(s)=\left(\frac{1}{2 \pi i}\right)^{2} \int_{\gamma_{\varepsilon, \phi}} \int_{\gamma_{\varepsilon, \phi}^{\prime}} e^{\lambda t}(\lambda I-A)^{-1} e^{\mu s}(\mu I-A)^{-1} d \mu d \lambda \tag{54.3.16}
\end{equation*}
$$

Using the resolvent identity,

$$
(\lambda I-A)^{-1}(\mu I-A)^{-1}=(\mu-\lambda)^{-1}\left((\lambda I-A)^{-1}-(\mu I-A)^{-1}\right),
$$

then substituting this resolvent identity in 54.3.16, it equals

$$
\begin{gathered}
\left(\frac{1}{2 \pi i}\right)^{2} \int_{\gamma_{\varepsilon, \phi}} \int_{\gamma_{\varepsilon, \phi}^{\prime}} e^{\mu s} e^{\lambda t}\left((\mu-\lambda)^{-1}\left((\lambda I-A)^{-1}-(\mu I-A)^{-1}\right)\right) d \mu d \lambda \\
=-\left(\frac{1}{2 \pi i}\right)^{2} \int_{\gamma_{\varepsilon, \phi}} e^{\lambda t} \int_{\gamma_{\varepsilon, \phi}^{\prime}} e^{\mu s}(\mu-\lambda)^{-1}(\mu I-A)^{-1} d \mu d \lambda \\
+\left(\frac{1}{2 \pi i}\right)^{2} \int_{\gamma_{\varepsilon, \phi}} \int_{\gamma_{\varepsilon, \phi}^{\prime}} e^{\mu s} e^{\lambda t}(\mu-\lambda)^{-1}(\lambda I-A)^{-1} d \mu d \lambda
\end{gathered}
$$

The order of integration can be interchanged because of the absolute convergence and Fubini's theorem. Then this reduces to

$$
\begin{aligned}
= & -\left(\frac{1}{2 \pi i}\right)^{2} \int_{\gamma_{\varepsilon, \phi}^{\prime}}(\mu I-A)^{-1} e^{\mu s} \int_{\gamma_{\varepsilon, \phi}} e^{\lambda t}(\mu-\lambda)^{-1} d \lambda d \mu \\
& +\left(\frac{1}{2 \pi i}\right)^{2} \int_{\gamma_{\varepsilon, \phi}}(\lambda I-A)^{-1} e^{\lambda t} \int_{\gamma_{\varepsilon, \phi}^{\prime}} e^{\mu s}(\mu-\lambda)^{-1} d \mu d \lambda
\end{aligned}
$$

Now the following diagram might help in drawing some interesting conclusions.


The first iterated integral equals 0 . This can be seen from the above picture. By Lemma 54.3.6, the inner integral taken over $\gamma_{\varepsilon, \phi}$ is essentially equal to the integral over the closed contour in the above picture provided the radius of the part of the large circle in the above closed contour is large enough. This closed contour integral equals 0 by the Cauchy integral theorem. The second iterated integral equals

$$
\frac{1}{2 \pi i} \int_{\gamma_{\varepsilon, \phi}}(\lambda I-A)^{-1} e^{\lambda t} e^{\lambda s} d \lambda=S(t+s)
$$

from the Cauchy integral formula. This verifies the semigroup identity.
4.) is done in Lemma 54.3 .7 which also includes 5.) when you let $t$ be positive and real.

### 54.3.1 The Numerical Range

In Hilbert space, there is a useful easy to check criterion which implies an operator is sectorial.

Definition 54.3.9 Let A be a closed densely defined operator $A: D(A) \rightarrow H$ for $H$ a Hilbert space. The numerical range is the following set.

$$
\{(A u, u): u \in D(A)\}
$$

Also recall the resolvent set, $r(A)$ consists of those $\lambda \in \mathbb{C}$ such that $(\lambda I-A)^{-1} \in \mathscr{L}(H, H)$. Thus, to be in this set $\lambda I-A$ is one to one and onto with continuous inverse.

Proposition 54.3.10 Suppose the numerical range of A, a closed densely defined operator $A: D(A) \rightarrow H$ for $H$ a Hilbert space is contained in the set

$$
\{z \in \mathbb{C}:|\arg (z)| \geq \pi-\phi\}
$$

where $0<\phi<\pi / 2$ and suppose $A^{-1} \in \mathscr{L}(H, H),(0 \in r(A))$. Then $A$ is sectorial with the sector $S_{0, \phi^{\prime}}$ where $\pi / 2>\phi^{\prime}>\phi$. Here $\arg (z)$ is the angle which is between $-\pi$ and $\pi$.

Proof: Here is a picture of the situation along with details used to motivate the proof.


In the picture the angle which is a little larger than $\phi$ is $\phi^{\prime}$. Let $\lambda$ be as shown with $|\arg \lambda| \leq \pi-\phi^{\prime}$. Then from the picture and trigonometry, if $u \in D(A)$,

$$
|\lambda| \sin \left(\phi^{\prime}-\phi\right)<\left|\lambda-\left(A \frac{u}{|u|}, \frac{u}{|u|}\right)\right|
$$

and so $|u||\lambda| \sin \left(\phi^{\prime}-\phi\right)<\left|\left(\lambda u-A u, \frac{u}{|u|}\right)\right| \leq\|(\lambda I-A) u\|$. Hence for all $\lambda$ such that $|\arg \lambda| \leq \pi-\phi^{\prime}$ and $u \in D(A)$,

$$
|u|<\left(\frac{1}{\sin \left(\phi^{\prime}-\phi\right)}\right) \frac{1}{|\lambda|}|(\lambda I-A) u| \equiv \frac{M}{|\lambda|}|(\lambda I-A) u|
$$

Thus $(\lambda I-A)$ is one to one on $S_{0, \phi^{\prime}}$ and if $\lambda \in r(A)$, then

$$
\left\|(\lambda I-A)^{-1}\right\|<\frac{M}{|\lambda|} .
$$

By assumption $0 \in r(A)$. Now if $|\mu|$ is small, $(\mu I-A)^{-1}$ must exist because it equals $\left(\left(\mu A^{-1}-I\right) A\right)^{-1}$ and for $|\mu|<\left\|A^{-1}\right\|,\left(\mu A^{-1}-I\right)^{-1} \in \mathscr{L}(H, H)$ since the infinite series

$$
\sum_{k=0}^{\infty}(-1)^{k}\left(\mu A^{-1}\right)^{k}
$$

converges and must equal to $\left(\mu A^{-1}-I\right)^{-1}$. Therefore, there exists $\mu \in S_{0, \phi^{\prime}}$ such that $\mu \neq 0$ and $\mu \in r(A)$. Also if $\mu \neq 0$ and $\mu \in S_{0, \phi^{\prime}}$, then if $|\lambda-\mu|<\frac{|\mu|}{M},(\lambda I-A)^{-1}$ must exist because

$$
(\lambda I-A)^{-1}=\left[\left((\lambda-\mu)(\mu I-A)^{-1}-I\right)(\mu I-A)\right]^{-1}
$$

where $\left((\lambda-\mu)(\mu I-A)^{-1}-I\right)^{-1}$ exists because

$$
\left\|(\lambda-\mu)(\mu I-A)^{-1}\right\|=|\lambda-\mu|\left\|(\mu I-A)^{-1}\right\|<\frac{|\mu|}{M} \cdot \frac{M}{|\mu|}=1
$$

It follows that if $S \equiv\left\{\lambda \in S_{0, \phi^{\prime}}: \lambda \in r(A)\right\}$, then $S$ is open in $S_{0, \phi}$. However, $S$ is also closed because if $\lambda=\lim _{n \rightarrow \infty} \lambda_{n}$ where $\lambda_{n} \in S$, then if $\lambda=0$, it is given $\lambda \in S$. If $\lambda \neq 0$, then for large enough $n,\left|\lambda-\lambda_{n}\right|<\frac{\left|\lambda_{n}\right|}{M}$ and so $\lambda \in S$. Since $S_{0, \phi^{\prime}}$ is connected, it follows $S=S_{0, \phi^{\prime}}$.

Corollary 54.3.11 If for some $a \in \mathbb{R}$, the numerical values of $-a I+A$ are in the set $\{\lambda:|\lambda| \geq \pi-\phi\}$ where $0<\phi<\pi / 2$, and $a \in r(A)$ then $A$ is sectorial.

Proof: By assumption, $0 \in r(-a I+A)$ and also from Proposition 54.3.10, for $\mu \in S_{0, \phi^{\prime}}$ where $\pi / 2>\phi^{\prime}>\phi$,

$$
((-a I+A)-\mu I)^{-1} \in \mathscr{L}(H, H),\left\|((-a I+A)-\mu I)^{-1}\right\| \leq \frac{M}{|\mu|}
$$

Therefore, for $\mu \in S_{0, \phi^{\prime}}, \mu+a \in r(A)$. Therefore, if $\lambda \in S_{a, \phi^{\prime}}, \lambda-a \in S_{0, \phi^{\prime}}$

$$
\left\|(A-\lambda I)^{-1}\right\|=\left\|(A-a I-(\lambda-a) I)^{-1}\right\| \leq \frac{M}{|\lambda-a|}
$$

### 54.3.2 An Interesting Example

In this section related to this example, for $V$ a Banach space, $V^{\prime}$ will denote the space of continuous conjugate linear functions defined on $V$. Usually the symbol has meant the space of continuous linear functions but here they will be conjugate linear. That is $f \in V^{\prime}$ means

$$
f(a x+b y)=\bar{a} f(x)+\bar{b} f(y)
$$

and $f$ is continuous.
Let $\Omega$ be a bounded open set in $\mathbb{R}^{n}$ and define

$$
V_{0} \equiv\left\{u \in C^{\infty}(\bar{\Omega}): u=0 \text { on } \Gamma\right\}
$$

where $\Gamma$ is some measurable subset of the boundary of $\Omega$ and $C^{\infty}(\bar{\Omega})$ denotes the restrictions of functions in $C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$ to $\Omega$. By Corollary 15.5.11 $V_{0}$ is dense in $L^{2}(\Omega)$. Now define the following for $u, v \in V_{0}$.

$$
A_{0} u(v) \equiv-a \int_{\Omega} u \bar{v} d x-\int_{\Omega} a(x) \nabla u \cdot \overline{\nabla v} d x
$$

where $a>0$ and $a(x) \geq 0$ is a $C^{1}(\bar{\Omega})$ function. Also define the following inner product on $V_{0}$.

$$
(u, v)_{1} \equiv \int_{\Omega}(a u \bar{v}+a(x) \nabla u \cdot \overline{\nabla v}) d x
$$

Let $\|\cdot\|_{1}$ denote the corresponding norm.
Of course $V_{0}$ is not a Banach space because it fails to be complete. $u \in V$ will mean that $u \in L^{2}(\Omega)$ and there exists a sequence $\left\{u_{n}\right\} \subseteq V_{0}$ such that

$$
\lim _{m, n \rightarrow \infty}\left\|u_{n}-u_{m}\right\|_{1}=0
$$

and

$$
\lim _{n \rightarrow \infty}\left|u_{n}-u\right|_{L^{2}(\Omega)}=0
$$

For $u \in V$, define $\nabla u$ to be that element of $L^{2}\left(\Omega ; \mathbb{C}^{n}, a(x) d m_{n}\right)$, the space of vector valued $L^{2}$ functions taken with respect to the measure $a(x) d m_{n}$ which satisfies

$$
\left|\nabla u-\nabla u_{n}\right|_{L^{2}\left(\Omega ; \mathbb{C}^{n}, a(x) d m_{n}\right)} \rightarrow 0 .
$$

Denote this space by $W$ for simplicity of notation.
Observation 54.3.12 $V$ is a Hilbert space with inner product given by

$$
(u, v)_{1} \equiv \int_{\Omega}(a u \bar{v}+a(x) \nabla u \cdot \overline{\nabla v}) d x
$$

Everything is obvious except completeness. Suppose then that $\left\{u_{n}\right\}$ is a Cauchy sequence in $V$. Then there exists a unique $u \in L^{2}(\Omega)$ such that $\left|u_{n}-u\right|_{L^{2}(\Omega)} \rightarrow 0$. Now let

$$
\left|w_{n}-u_{n}\right|_{L^{2}(\Omega)}+\left|\nabla w_{n}-\nabla u_{n}\right|_{W}<1 / 2^{n}
$$

It follows $\left\{\nabla w_{n}\right\}$ is also a Cauchy sequence in $W$ while $\left\{w_{n}\right\}$ is a Cauchy sequence in $L^{2}(\Omega)$ converging to $u$. Thus the thing to which $\nabla w_{n}$ converges in $W$ is the definition of $\nabla u$ and $u \in V$. Thus

$$
\begin{aligned}
\left\|u_{n}-u\right\|_{1} & \leq\left\|u_{n}-w_{n}\right\|_{1}+\left\|w_{n}-u\right\|_{1} \\
& <\frac{1}{2^{n}}+\left\|w_{n}-u\right\|_{1}
\end{aligned}
$$

and the last term converges to 0 . Hence $V$ is complete as claimed.
Then it is clear $V$ is a Hilbert space. The next observation is a simple one involving the Riesz map.

Definition 54.3.13 Let $V$ be a Hilbert space and let $V^{\prime}$ be the space of continuous conjugate linear functions defined on $V$. Then define $R: V \rightarrow V^{\prime}$ by

$$
R x(y) \equiv(x, y)
$$

This is called the Riesz map.
Lemma 54.3.14 The Riesz map is one to one and onto and linear.
Proof: It is obvious it is one to one and linear. The only challenge is to show it is onto. Let $z^{*} \in V^{\prime}$. If $z^{*}(V)=\{0\}$, then letting $z=0$, it follows $R z=z^{*}$. If $z^{*}(V) \neq 0$, then

$$
\operatorname{ker}\left(z^{*}\right) \equiv\left\{x \in V: z^{*}(x)=0\right\}
$$

is a closed subspace. It is closed because $z^{*}$ is continuous and it is just $z^{*-1}(0)$. Since $\operatorname{ker}\left(z^{*}\right)$ is not everything in $V$ there exists

$$
w \in \operatorname{ker}\left(z^{*}\right)^{\perp} \equiv\left\{x:(x, y)=0 \text { for all } y \in \operatorname{ker}\left(z^{*}\right)\right\}
$$

and $w \neq 0$. Then

$$
z^{*}\left(\overline{z^{*}(x)} w-\overline{z^{*}(w)} x\right)=z^{*}(x) z^{*}(w)-z^{*}(w) z^{*}(x)=0
$$

and so $\overline{z^{*}(x)} w-\overline{z^{*}(w)} x \in \operatorname{ker}\left(z^{*}\right)$. Therefore, for any $x \in V$,

$$
\begin{aligned}
& 0=\left(w, \overline{z^{*}(x)} w-\overline{z^{*}(w)} x\right) \\
& =z^{*}(x)(w, w)-z^{*}(w)(w, x)
\end{aligned}
$$

and so

$$
z^{*}(x)=\left(\frac{z^{*}(w)}{\|w\|^{2}}, x\right)
$$

so let $z=w /\|w\|^{2}$. Then $R z=z^{*}$ and so $R$ is onto. This proves the lemma.
Now for the $V$ described above,

$$
R u(v)=\int_{\Omega}(a u \bar{v}+a(x) \nabla u \cdot \overline{\nabla v}) d x
$$

Also, as noted above $V$ is dense in $H \equiv L^{2}(\Omega)$ and so if $H$ is identified with $H^{\prime}$, it follows

$$
V \subseteq H=H^{\prime} \subseteq V^{\prime}
$$

Let $A: D(A) \rightarrow H$ be given by

$$
D(A) \equiv\{u \in V: R u \in H\}
$$

and

$$
A \equiv-R
$$

on $D(A)$. Then the numerical range for $A$ is contained in $(-\infty,-a]$ and so $A$ is sectorial by Proposition 54.3.10 provided $A$ is closed and densely defined.

Why is $D(A)$ dense? It is because it contains $C_{c}^{\infty}(\Omega)$ which is dense in $L^{2}(\Omega)$. This follows from integration by parts which shows that for $u, v \in C_{c}^{\infty}(\Omega)$,

$$
\begin{aligned}
& -\int_{\Omega} a u \bar{v} d x-\int_{\Omega} a(x) \nabla u \cdot \overline{\nabla v} d x \\
= & -\int_{\Omega} a u \bar{v} d x+\int_{\Omega} \nabla \cdot(a(x) \nabla u) \bar{v} d x
\end{aligned}
$$

and since $C_{c}^{\infty}(\Omega)$ is dense in $H$,

$$
A u=-a u+\nabla \cdot(a(x) \nabla u) \in L^{2}(\Omega)=H
$$

Why is $A$ closed? If $u_{n} \in D(A)$ and $u_{n} \rightarrow u$ in $H$ while $A u_{n} \rightarrow \xi$ in $H$, then it follows from the definition that $R u_{n} \rightarrow-\xi$ and $\left\{u_{n}\right\}$ converges to $u$ in $V$ so for any $v \in V$,

$$
R u(v)=\lim _{n \rightarrow \infty} R u_{n}(v)=\lim _{n \rightarrow \infty}\left(R u_{n}, v\right)_{H}=(-\xi, v)_{H}
$$

which shows $R u=-\xi \in H$ and so $u \in D(A)$ and $A u=\xi$. Thus $A$ is closed. This completes the example.

Obviously you could follow identical reasoning to include many other examples of more complexity. What does it mean for $u \in D(A)$ ? It means that in a weak sense

$$
-a u+\nabla \cdot(a(x) \nabla u) \in H
$$

Since $A$ is sectorial for $S_{-a, \phi}$ for any $0<\phi<\pi / 2$, this has shown the existence of a weak solution to the partial differential equation along with appropriate boundary conditions,

$$
-a u+\nabla \cdot(a(x) \nabla u)=f, u \in V
$$

What are these appropriate boundary conditions? $u=0$ on $\Gamma$ is one. the other would be a variational boundary condition which comes from integration by parts. Letting $v \in V$, formally do the following using the divergence theorem.

$$
\begin{aligned}
& \quad(f, v)_{H}=\int_{\Omega}(-a u+\nabla \cdot(a(x) \nabla u)) v d x \\
& =\int_{\Omega}-a u v d x+\int_{\partial \Omega}(a(x) \nabla u v) \cdot n d s-\int_{\Omega} a(x) \nabla u(x) \cdot \nabla v(x) d x \\
& =(f, v)_{H}+\int_{\partial \Omega \backslash \Gamma}(a(x) \nabla u) \cdot n v d s
\end{aligned}
$$

and so the other boundary condition is

$$
a(x) \frac{\partial u}{\partial n}=0 \text { on } \partial \Omega \backslash \Gamma .
$$

To what extent this weak solution is really a classical solution depends on more technical considerations.

### 54.3.3 Fractional Powers Of Sectorial Operators

It will always be assumed in this section that $A$ is sectorial for the sector $S_{-a, \phi}$ where $a>0$. To begin with, here is a useful lemma which will be used in the presentation of these fractional powers.

Lemma 54.3.15 The following holds for $\alpha \in(0,1)$ and $\sigma<t$.

$$
\int_{\sigma}^{t}(t-s)^{\alpha-1}(s-\sigma)^{-\alpha} d s=\frac{\pi}{\sin (\pi \alpha)}
$$

In particular,

$$
\int_{0}^{1}(1-s)^{\alpha-1} s^{-\alpha} d s=\frac{\pi}{\sin (\pi \alpha)}
$$

Also for $\alpha, \beta>0$

$$
\Gamma(\alpha) \Gamma(\beta)=\left(\int_{0}^{1} x^{\alpha-1}(1-x)^{\beta-1} d x\right) \Gamma(\alpha+\beta)
$$

Proof: First change variables to get rid of the $\sigma$. Let $y=(t-\sigma)^{-1}(s-\sigma)$. Then the integral becomes

$$
\begin{aligned}
& \int_{0}^{1}(t-[(t-\sigma) y+\sigma])^{\alpha-1}(t-\sigma)^{-\alpha} y^{-\alpha}(t-\sigma) d y \\
= & \int_{0}^{1}((t-\sigma)(1-y))^{\alpha-1}(t-\sigma)^{-\alpha} y^{-\alpha}(t-\sigma) d y \\
= & \int_{0}^{1}(1-y)^{\alpha-1} y^{-\alpha} d y
\end{aligned}
$$

Next let $y=x^{2}$. The integral is

$$
2 \int_{0}^{1}\left(1-x^{2}\right)^{\alpha-1} x^{1-2 \alpha} d x
$$

Next let $x=\sin \theta$

$$
2 \int_{0}^{\frac{1}{2} \pi}(\cos (\theta))^{2 \alpha-1} \sin ^{(1-2 \alpha)}(\theta) d \theta=2 \int_{0}^{\frac{1}{2} \pi}\left(\frac{\cos (\theta)}{\sin (\theta)}\right)^{2 \alpha-1} d \theta
$$

Now change the variable again. Let $u=\cot (\theta)$. Then this yields

$$
2 \int_{0}^{\infty} \frac{u^{2 \alpha-1}}{1+u^{2}} d u
$$

This is fairly easy to evaluate using contour integrals. Consider the following contour called $\Gamma_{R}$ for large $R$. As $R \rightarrow \infty$, the integral over the little circle converges to 0 and so does the integral over the big circle. There is one singularity at $i$.


Thus

$$
\begin{aligned}
& \lim _{R \rightarrow \infty} \int_{\Gamma_{R}} \frac{e^{(\ln |z|+i \arg (z))(1-2 \alpha)}}{1+z^{2}} d z= \\
& =\quad(1+\cos (1-2 \alpha) \pi) \int_{0}^{\infty} \frac{u^{2 \alpha-1}}{1+u^{2}} d u \\
& \quad+i \sin ((1-2 \alpha) \pi) \int_{0}^{\infty} \frac{u^{2 \alpha-1}}{1+u^{2}} d u
\end{aligned}
$$

$$
=\pi\left(\cos \left(\frac{\pi}{2}(1-2 \alpha)\right)+i \sin \left(\frac{\pi}{2}(1-2 \alpha)\right)\right)
$$

Then equating the imaginary parts yields

$$
\sin ((1-2 \alpha) \pi) \int_{0}^{\infty} \frac{u^{2 \alpha-1}}{1+u^{2}} d u=\pi \sin \left(\frac{\pi}{2}(1-2 \alpha)\right)
$$

and so using the trig identities for the sum of two angles,

$$
\begin{aligned}
\int_{0}^{\infty} \frac{u^{2 \alpha-1}}{1+u^{2}} d u & =\frac{\pi\left(\sin \left(\frac{\pi}{2}(1-2 \alpha)\right)\right)}{2 \sin \left(\frac{\pi}{2}(1-2 \alpha)\right) \cos \left(\frac{\pi}{2}(1-2 \alpha)\right)} \\
& =\frac{\pi}{2 \cos \left(\frac{\pi}{2}(1-2 \alpha)\right)}=\frac{\pi}{2 \sin (\pi \alpha)}
\end{aligned}
$$

It remains to verify the last identity.

$$
\begin{aligned}
\Gamma(\alpha) \Gamma(\beta) & \equiv \int_{0}^{\infty} \int_{0}^{\infty} t^{\alpha-1} e^{-t} s^{\beta-1} e^{-s} d s d t \\
& =\int_{0}^{\infty} \int_{t}^{\infty} t^{\alpha-1} e^{-u}(u-t)^{\beta-1} d u d t \\
& =\int_{0}^{\infty} e^{-u} \int_{0}^{u} t^{\alpha-1}(u-t)^{\beta-1} d t d u \\
& =\int_{0}^{1} x^{\alpha-1}(1-x)^{\beta-1} d x \int_{0}^{\infty} e^{-u} u^{\alpha+\beta-1} d u \\
& =\left(\int_{0}^{1} x^{\alpha-1}(1-x)^{\beta-1} d x\right) \Gamma(\alpha+\beta)
\end{aligned}
$$

This proves the lemma.
If it is not stated otherwise, in all that follows $\alpha>0$.
Definition 54.3.16 Let A be a sectorial operator corresponding to the sector $S_{-a \phi}$ where $-a<0$. Then define for $\alpha>0$,

$$
(-A)^{-\alpha} \equiv \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} t^{\alpha-1} S(t) d t
$$

where $S(t)$ is the analytic semigroup generated by $A$ as in Corollary 54.3.8. Note that from the estimate, $\|S(t)\| \leq M e^{-a t}$ of this corollary, the integral is well defined and is in $\mathscr{L}(H, H)$.

Theorem 54.3.17 For $(-A)^{-\alpha}$ as defined in Definition 54.3.16

$$
\begin{equation*}
(-A)^{-\alpha}(-A)^{-\beta}=(-A)^{-(\alpha+\beta)} \tag{54.3.17}
\end{equation*}
$$

Also

$$
\begin{equation*}
(-A)^{-1}(-A)=I,(-A)(-A)^{-1}=I \tag{54.3.18}
\end{equation*}
$$

and $(-A)^{-\alpha}$ is one to one if $\alpha \geq 0$, defining $A^{0} \equiv I$.

If $\alpha<\beta$, then

$$
\begin{equation*}
(-A)^{-\beta}(H) \subseteq(-A)^{-\alpha}(H) \tag{54.3.19}
\end{equation*}
$$

If $\alpha \in(0,1)$, then

$$
\begin{equation*}
(-A)^{-\alpha}=\frac{\sin (\pi \alpha)}{\pi} \int_{0}^{\infty} \lambda^{-\alpha}(\lambda I-A)^{-1} d \lambda \tag{54.3.20}
\end{equation*}
$$

Proof: Consider 54.3.17.

$$
(-A)^{-\alpha}(-A)^{-\beta} \equiv \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_{0}^{\infty} \int_{0}^{\infty} t^{\alpha-1} s^{\beta-1} S(t+s) d s d t
$$

Changing variables and using Fubini's theorem which is justified because of the abolute convergence of the iterated integrals, which follows from Corollary 54.3.8, this becomes

$$
\begin{aligned}
& \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_{0}^{\infty} \int_{t}^{\infty} t^{\alpha-1}(u-t)^{\beta-1} S(u) d u d t \\
= & \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_{0}^{\infty} \int_{0}^{u} t^{\alpha-1}(u-t)^{\beta-1} S(u) d t d u \\
= & \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_{0}^{\infty} S(u) \int_{0}^{1}(u x)^{\alpha-1}(u-u x)^{\beta-1} u d x d u \\
= & \frac{1}{\Gamma(\alpha) \Gamma(\beta)}\left(\int_{0}^{1} x^{\alpha-1}(1-x)^{\beta-1} d x\right) \int_{0}^{\infty} S(u) u^{\alpha+\beta-1} d u \\
= & \frac{1}{\Gamma(\alpha) \Gamma(\beta)}\left(\int_{0}^{1} x^{\alpha-1}(1-x)^{\beta-1} d x\right) \Gamma(\alpha+\beta)(-A)^{-(\alpha+\beta)} \\
= & (-A)^{-(\alpha+\beta)}
\end{aligned}
$$

This proves the first part of the theorem.
Consider 54.3.18. Since $A$ is a closed operator, and approximating the integral with an appropriate sequence of Riemann sums, $(-A)$ can be taken inside the integral and so

$$
\begin{aligned}
(-A) & \frac{1}{\Gamma(1)} \int_{0}^{\infty} t^{1-1} S(t) d t
\end{aligned}=\int_{0}^{\infty}(-A) S(t) d t t .
$$

Next let $x \in D(-A)$. Then

$$
\begin{aligned}
& \frac{1}{\Gamma(1)} \int_{0}^{\infty} t^{1-1} S(t) d t(-A) x=-\int_{0}^{\infty} S(t) A x d t \\
& =-\int_{0}^{\infty} A S(t) x d t=\int_{0}^{\infty}-\frac{d}{d t}(S(t)) d t=I x
\end{aligned}
$$

This shows that the integral in which $\alpha=1$ deserves to be called $A^{-1}$ so the definition is not bad notation. Also, by assumption, $A^{-1}$ is one to one. Thus

$$
(-A)^{-1}(-A)^{-1} x=0
$$

implies

$$
(-A)^{-1} x=0
$$

hence $x=0$ so that $(-A)^{-2}$ is also one to one. Similarly, $(-A)^{-m}$ is one to one for all positive integers $m$.

From what was just shown, if $(-A)^{-\alpha} x=0$ for $\alpha \in(0,1)$, then

$$
(-A)^{-1} x=(-A)^{-(1-\alpha)}(-A)^{-\alpha} x=0
$$

and so $x=0$. This shows $(-A)^{-\alpha}$ is one to one for all $\alpha \in[0,1]$ if is defined as $(-A)^{0} \equiv I$.
What about $\alpha>1$ ? For such $\alpha$, it is of the form $m+\beta$ where $\beta \in[0,1)$ and $m$ is a positive integer. Therefore, if

$$
(-A)^{-(m+\beta)} x=0
$$

then

$$
(-A)^{-\beta}\left((-A)^{-m}\right) x=0
$$

and so from what was just shown,

$$
\left((-A)^{-m}\right) x=0
$$

and now this implies $x=0$ so that $(-A)^{-\alpha}$ is one to one for all $\alpha \geq 0$.
Consider 54.3.19. It was shown above that

$$
(-A)^{-\alpha}(-A)^{-\beta}=(-A)^{-(\alpha+\beta)}
$$

Let $x=(-A)^{-(\alpha+\beta)} y$. Then

$$
x=(-A)^{-\alpha}(-A)^{-\beta} y \subseteq(-A)^{-\alpha}(-A)^{-\beta}(H) \subseteq(-A)^{-\alpha}(H)
$$

This proves 54.3.19. If $\alpha<\beta,(-A)^{-\beta}(H) \subseteq(-A)^{-\alpha}(H)$.
Now consider the problem of writing $(-A)^{-\alpha}$ for $\alpha \in(0,1)$ in terms of $A$, not mentioning $S(t)$. By Proposition 19.14.5,

$$
(\lambda I-A)^{-1} x=\int_{0}^{\infty} e^{-\lambda t} S(t) x d t
$$

Then

$$
\begin{aligned}
\int_{0}^{\infty} \lambda^{-\alpha}(\lambda I-A)^{-1} d \lambda & =\int_{0}^{\infty} \lambda^{-\alpha} \int_{0}^{\infty} e^{-\lambda t} S(t) d t d \lambda \\
& =\int_{0}^{\infty} S(t) \int_{0}^{\infty} \lambda^{-\alpha} e^{-\lambda t} d \lambda d t \\
& =\int_{0}^{\infty} S(t) \int_{0}^{\infty} \lambda^{\beta-1} e^{-\lambda t} d \lambda d t
\end{aligned}
$$

where $\beta \equiv 1-\alpha$. Then using Lemma 54.3.15, this equals

$$
\int_{0}^{\infty} S(t) \int_{0}^{\infty} \mu^{\beta-1} t^{1-\beta} e^{-\mu} t^{-1} d \mu d t=\int_{0}^{\infty} t^{-\beta} S(t) \int_{0}^{\infty} \mu^{\beta-1} e^{-\mu} d \mu d t
$$

$$
\begin{aligned}
& =\Gamma(1-\alpha) \int_{0}^{\infty} t^{\alpha-1} S(t) d t=\Gamma(\alpha) \Gamma(1-\alpha)(-A)^{-\alpha} \\
& =\left(\int_{0}^{1} x^{\alpha-1}(1-x)^{-\alpha} d x\right)(-A)^{-\alpha}=\frac{\pi}{\sin (\pi \alpha)}(-A)^{-\alpha}
\end{aligned}
$$

and so this gives the formula

$$
(-A)^{-\alpha}=\frac{\sin (\pi \alpha)}{\pi} \int_{0}^{\infty} \lambda^{-\alpha}(\lambda I-A)^{-1} d \lambda
$$

This proves 54.3.20.
Definition 54.3.18 For $\alpha \geq 0$, define $(-A)^{\alpha}$ on $D\left((-A)^{\alpha}\right) \equiv(-A)^{-\alpha}(H)$ by

$$
(-A)^{\alpha} \equiv\left((-A)^{-\alpha}\right)^{-1}
$$

Note that if $\alpha, \beta>0$, then if $x \in D\left((-A)^{\alpha+\beta}\right)$,

$$
\begin{gather*}
(-A)^{\alpha+\beta} x=\left((-A)^{-(\alpha+\beta)}\right)^{-1} x= \\
\left((-A)^{-\alpha}(-A)^{-\beta}\right)^{-1} x=(-A)^{\beta}(-A)^{\alpha} x . \tag{54.3.21}
\end{gather*}
$$

Next let $\beta>\alpha>0$ and let $x \in D\left((-A)^{\beta}\right)$. Then from what was just shown,

$$
(-A)^{\alpha}(-A)^{\beta-\alpha} x=(-A)^{\beta} x
$$

and so

$$
(-A)^{\beta-\alpha} x=(-A)^{-\alpha}(-A)^{\beta} x
$$

If $x \in D\left((-A)^{\beta}\right)$, does it follow that $(-A)^{-\alpha} x \in D\left((-A)^{\beta}\right)$ ? Note $x=(-A)^{-\beta} y$ and so

$$
(-A)^{-\alpha} x=(-A)^{-\alpha}(-A)^{-\beta} y=(-A)^{-(\alpha+\beta)} y \in D\left((-A)^{\alpha+\beta}\right) .
$$

Therefore, from 54.3.21,

$$
(-A)^{\beta-\alpha} x=(-A)^{\beta-\alpha}(-A)^{\alpha}\left((-A)^{-\alpha} x\right)=(-A)^{\beta}(-A)^{-\alpha} x
$$

Theorem 54.3.19 The definition of $(-A)^{\alpha}$ is well defined and $(-A)^{\alpha}$ is densely defined and closed. Also for any $\alpha>0$,

$$
\begin{equation*}
\left\|(-A)^{\alpha} S(t)\right\| \leq \frac{C_{\alpha}}{\delta} \frac{1}{t^{\alpha}} e^{-\delta t} \tag{54.3.22}
\end{equation*}
$$

where $-\delta>-a$. Furthermore, $C_{\alpha}$ is bounded as $\alpha \rightarrow 0+$ and is bounded on compact intervals of $(0, \infty)$. Also for $\alpha \in(0,1)$ and $x \in D\left((-A)^{\alpha}\right)$,

$$
\begin{equation*}
\|(S(t)-I) x\| \leq \frac{C_{1-\alpha}}{\alpha \delta} t^{\alpha}\left\|(-A)^{\alpha} x\right\| \tag{54.3.23}
\end{equation*}
$$

There exists a constant $C$ independent of $\alpha \in[0,1)$ such that for $x \in D(A)$ and $\varepsilon>0$,

$$
\begin{equation*}
\left\|(-A)^{\alpha} x\right\| \leq \varepsilon\|(-A) x\|+C \varepsilon^{-\alpha /(1-\alpha)}\|x\| \tag{54.3.24}
\end{equation*}
$$

There exists a constant $C^{\prime}$ independent of $\alpha \in[0,1]$ such that for $x \in D(A)$,

$$
\begin{equation*}
\left\|(-A)^{\alpha} x\right\| \leq C^{\prime}\|(-A) x\|^{\alpha}\|x\|^{1-\alpha} \tag{54.3.25}
\end{equation*}
$$

The formula 54.3.25 is called an interpolation inequality.
Proof: It is obvious $(-A)^{\alpha}$ is densely defined because its domain is at least as large as $D(A)$ which was assumed to be dense. It is a closed operator because if $x_{n} \in D\left((-A)^{\alpha}\right)$ and

$$
x_{n} \rightarrow x,(-A)^{\alpha} x_{n} \rightarrow y
$$

then

$$
(-A)^{-\alpha} x_{n} \rightarrow(-A)^{-\alpha} x, x_{n}=(-A)^{-\alpha}(-A)^{\alpha} x_{n} \rightarrow(-A)^{-\alpha} y
$$

and so

$$
(-A)^{-\alpha} y=x
$$

showing $x \in D\left((-A)^{\alpha}\right)$ and $y=(-A)^{-\alpha} x$. Thus $(-A)^{\alpha}$ is closed and densely defined.
Let $-\delta>-a$ where the sector for $A$ was $S_{-a, \phi}, a>0$. Then recall from Corollary 54.3.8 there is a constant, $N$ such that

$$
\|(-A) S(t)\| \leq \frac{N}{t} e^{-\delta t}
$$

What about $\left\|(-A)^{\alpha} S(t)\right\|$ ? First note that for $\alpha \in[0,1)$ this at least makes sense because $S(t)$ maps into $D(A)$. For any $\alpha>0$,

$$
S(t)(-A)^{-\alpha}=(-A)^{-\alpha} S(t)
$$

follows from the definiton of $(-A)^{-\alpha}$. Therefore,

$$
\begin{equation*}
(-A)^{\alpha} S(t)(-A)^{-\alpha}=S(t) \tag{54.3.26}
\end{equation*}
$$

Note this implies that on $D\left((-A)^{\alpha}\right)$,

$$
(-A)^{\alpha} S(t)=S(t)(-A)^{\alpha}
$$

Also

$$
(-A)^{-1} S(t)=S(t)(-A)^{-1}=S(t)(-A)^{-\alpha}(-A)^{-(1-\alpha)}
$$

and so

$$
S(t)=(-A) S(t)(-A)^{-\alpha}(-A)^{-(1-\alpha)}
$$

From 54.3.26 it follows

$$
\begin{align*}
(-A)^{\alpha} S(t) & =(-A)(-A)^{\alpha} S(t)(-A)^{-\alpha}(-A)^{-(1-\alpha)} \\
& =(-A) S(t)(-A)^{-(1-\alpha)} \tag{54.3.27}
\end{align*}
$$

Then with this formula,

$$
\begin{aligned}
& \left\|(-A)^{\alpha} S(t)\right\|=\left\|(-A) S(t)(-A)^{-(1-\alpha)}\right\| \\
& =\left\|\frac{1}{\Gamma(1-\alpha)} \int_{0}^{\infty} s^{1-\alpha}(-A) S(t+s) d s\right\| \\
& \leq \frac{N}{\Gamma(1-\alpha)} \int_{0}^{\infty} \frac{s^{1-\alpha}}{(t+s)} e^{-\delta(s+t)} d s \\
& =\frac{N}{\Gamma(1-\alpha)} \int_{t}^{\infty} \frac{(u-t)^{1-\alpha}}{u} e^{-\delta u} d s \\
& \leq \frac{N}{\Gamma(1-\alpha)} \int_{t}^{\infty}\left(1-\frac{t}{u}\right)^{1-\alpha} \frac{1}{u^{\alpha}} e^{-\delta u} d s \\
& \leq \frac{N}{\Gamma(1-\alpha)} \frac{1}{t^{\alpha}} \int_{t}^{\infty} e^{-\delta u} d s=\frac{N}{\Gamma(1-\alpha) \delta} \frac{1}{t^{\alpha}} e^{-\delta t} \\
& \equiv \frac{C_{\alpha}}{\delta} \frac{1}{t^{\alpha}} e^{-\delta t} .
\end{aligned}
$$

this establishes the formula when $\alpha \in[0,1)$. Next suppose $\alpha=m$, a positive integer.

$$
\begin{aligned}
\left\|A^{m} S(t)\right\| & =\left\|A^{m} S\left(\frac{t}{m}\right)^{m}\right\| \\
& =\left\|\left(A S\left(\frac{t}{m}\right)\right)^{m}\right\| \leq \frac{N}{t^{m}} m^{m}
\end{aligned}
$$

This is why the above inequality holds.
If $\alpha, \beta>0$,

$$
\begin{aligned}
\left\|A^{\alpha+\beta} S(t)\right\| & =\left\|A^{\alpha+\beta} S\left(\frac{t}{2}\right) S\left(\frac{t}{2}\right)\right\| \\
& =\left\|A^{\alpha} S\left(\frac{t}{2}\right) A^{\beta} S\left(\frac{t}{2}\right)\right\| \\
& \leq \frac{C_{\alpha}}{t^{\alpha}} \frac{C_{\beta}}{t^{\beta}} e^{-2 \delta t}=\frac{C}{t^{\alpha+\beta}} e^{-\delta t}
\end{aligned}
$$

Suppose now that $\alpha>0$. Then

$$
\alpha=m+\beta
$$

where $\beta \in[0,1)$. Then from what was just shown,

$$
\left\|A^{m+\beta} S(t)\right\| \leq \frac{C}{t^{m+\beta}} e^{-\delta t}
$$

Next consider 54.3.23. First note that whenever $\alpha>0$,

$$
(-A)^{-\alpha} S(s)=S(s)(-A)^{-\alpha}
$$

and so on $D\left((-A)^{\alpha}\right)$,

$$
S(s)=(-A)^{\alpha} S(s)(-A)^{-\alpha}, S(s)(-A)^{\alpha}=(-A)^{\alpha} S(s)
$$

Now for $x \in D\left((-A)^{\alpha}\right)$,

$$
\begin{aligned}
\|(S(t)-I) x\| & =\left\|-\int_{0}^{t}(-A) S(s) x d s\right\| \\
& =\left\|-\int_{0}^{t}(-A)^{1-\alpha}(-A)^{\alpha} S(s) x d s\right\| \\
& =\left\|-\int_{0}^{t}(-A)^{1-\alpha} S(s)(-A)^{\alpha} x d s\right\| \\
& \leq \int_{0}^{t}\left\|(-A)^{1-\alpha} S(s)\right\| d s\left\|(-A)^{\alpha} x\right\| \\
\leq \int_{0}^{t} & \frac{C_{1-\alpha}}{\delta} \frac{1}{s^{1-\alpha}} e^{-\delta s} d s\left\|(-A)^{\alpha} x\right\| \\
& \leq \frac{C_{1-\alpha}}{\delta} \frac{1}{\alpha} t^{\alpha}\left\|(-A)^{\alpha} x\right\|
\end{aligned}
$$

and this shows 54.3.23.
Next consider 54.3.24. Let $x \in H$ and $\beta \in(0,1)$. Then

$$
\begin{gathered}
\left\|(-A)^{-\beta} x\right\|=\frac{1}{\Gamma(\beta)}\left\|\int_{0}^{\infty} t^{\beta-1} S(t) x d t\right\| \\
=\frac{1}{\Gamma(\beta)}\left\|\int_{0}^{\eta} t^{\beta-1} S(t) x d t+\int_{\eta}^{\infty} t^{\beta-1} S(t) x d t\right\| \\
\leq \frac{1}{\Gamma(\beta)} \int_{0}^{\eta} t^{\beta-1}\|S(t) x\| d t+\frac{1}{\Gamma(\beta)}\left\|\int_{\eta}^{\infty} t^{\beta-1} S(t) x d t\right\| \\
\leq \frac{C}{\Gamma(\beta)} \frac{\eta^{\beta}}{\beta}\|x\|+\frac{1}{\Gamma(\beta)}\left\|\int_{\eta}^{\infty} t^{\beta-1} S(t) x d t\right\| \\
\leq \frac{C}{\Gamma(\beta)} \frac{\eta^{\beta}}{\beta}\|x\|+ \\
\quad \frac{1}{\Gamma(\beta)}\left\|\eta^{\beta-1} S(\eta) A^{-1} x+(1-\beta) \int_{\eta}^{\infty} t^{\beta-2} S(t) A^{-1} x d t\right\| \\
\leq \frac{1}{\Gamma(\beta)}\left(\frac{C \eta^{\beta}}{\beta}\|x\|+\eta^{\beta-1}\left\|A^{-1} x\right\|+(1-\beta)\left\|A^{-1} x\right\| \int_{\eta}^{\infty} t^{\beta-2} d t\right) \\
=\frac{1}{\Gamma(\beta)}\left(\frac{C \eta^{\beta}}{\beta}\|x\|+2 \eta^{\beta-1}\left\|A^{-1} x\right\|\right) .
\end{gathered}
$$

Now let $\delta=C \eta^{\beta}$ so $\eta=C^{-1 / \beta} \delta^{1 / \beta}$ and $\eta^{\beta-1}=C^{\frac{1-\beta}{\beta}} \delta^{(\beta-1) / \beta}$. Thus for all $x \in H$,

$$
\left\|(-A)^{-\beta} x\right\| \leq \frac{1}{\Gamma(\beta)}\left(\frac{\delta}{\beta}\|x\|+2 C^{\frac{1-\beta}{\beta}} \delta^{(\beta-1) / \beta}\left\|A^{-1} x\right\|\right)
$$

Let $\varepsilon=\frac{\delta}{\beta \Gamma(\beta)}=\frac{\delta}{\Gamma(1+\beta)}$. Then the above is of the form

$$
\begin{aligned}
\left\|(-A)^{-\beta} x\right\| & \leq \varepsilon\|x\|+2 \frac{C^{\frac{1-\beta}{\beta}}}{\Gamma(\beta)}(\varepsilon \Gamma(1+\beta))^{(\beta-1) / \beta}\left\|A^{-1} x\right\| \\
& \leq \varepsilon\|x\|+2 C^{\frac{1-\beta}{\beta}}(\varepsilon \Gamma(1+\beta))^{(\beta-1) / \beta}\left\|A^{-1} x\right\|
\end{aligned}
$$

because $\Gamma$ is decreasing on $(0,1)$. I need to verify that for $\beta \in(0,1)$,

$$
\Gamma(1+\beta)^{(\beta-1) / \beta}
$$

is bounded. It is continuous on $(0,1]$ and so if I can show $\lim _{\beta \rightarrow 0+} \Gamma(1+\beta)^{(\beta-1) / \beta}$ exists, then it will follow the function is bounded. It suffices to show

$$
\lim _{\beta \rightarrow 0+} \frac{\beta-1}{\beta} \ln \Gamma(1+\beta)=-\lim _{\beta \rightarrow 0+} \frac{\ln \Gamma(1+\beta)}{\beta}
$$

exists. Consider this. By L'Hospital's rule and dominated convergence theorem, this is

$$
\begin{aligned}
\lim _{\beta \rightarrow 0+} \frac{\int_{0}^{\infty} \ln (t) t^{\beta} e^{-t} d t}{\Gamma(1+\beta)} & =\lim _{\beta \rightarrow 0+} \int_{0}^{\infty} \ln (t) t^{\beta} e^{-t} d t \\
& =\lim _{\beta \rightarrow 0+} \int_{0}^{\infty} \ln (t) e^{-t} d t
\end{aligned}
$$

Thus the function is bounded independent of $\beta \in(0,1)$. This shows there is a constant $C$ which is independent of $\beta \in(0,1)$ such that for any $x \in H$,

$$
\begin{equation*}
\left\|(-A)^{-\beta} x\right\| \leq \varepsilon\|x\|+C \varepsilon^{(\beta-1) / \beta}\left\|A^{-1} x\right\| \tag{54.3.28}
\end{equation*}
$$

Now let $y \in D(A)=D((-A))$ and let $x=(-A) y$. Then the above becomes

$$
\left\|(-A)^{-\beta}(-A) y\right\| \leq \varepsilon\|(-A) y\|+C \varepsilon^{(\beta-1) / \beta}\|y\|
$$

I claim that

$$
(-A)^{-\beta}(-A) y=(-A)^{1-\beta} y
$$

The reason for this is as follows.

$$
(-A)^{\beta}(-A)^{1-\beta} y=(-A) y
$$

and so the desired result follows from multiplying on the left by $(-A)^{-\beta}$. Hence

$$
\left\|(-A)^{1-\beta} y\right\| \leq \varepsilon\|(-A) y\|+C \varepsilon^{(\beta-1) / \beta}\|y\|
$$

Now let $1-\beta=\alpha$ and obtain

$$
\left\|(-A)^{\alpha} y\right\| \leq \varepsilon\|(-A) y\|+C \varepsilon^{-\alpha /(1-\alpha)}\|y\|
$$

This proves 54.3.24.
Finally choose $\varepsilon$ to minimize the right side of the above expression. Thus let

$$
\varepsilon=\left(\frac{\alpha\|y\| C}{\|(-A) y\|(1-\alpha)}\right)^{1-\alpha}
$$

Then the above expression becomes

$$
\begin{aligned}
&\left\|(-A)^{\alpha} y\right\| \leq\|(-A) y\|\left(\frac{\alpha\|y\| C}{\|(-A) y\|(1-\alpha)}\right)^{1-\alpha} \\
&+C\left(\left(\frac{\alpha\|y\| C}{\|(-A) y\|(1-\alpha)}\right)^{1-\alpha}\right)^{-\alpha /(1-\alpha)}\|y\| \\
&=\|(-A) y\|^{\alpha}\|y\|^{1-\alpha}\left(\frac{\alpha C}{(1-\alpha)}\right)^{1-\alpha} \\
& \quad+\|(-A) y\|^{\alpha}\|y\|^{1-\alpha}\left(\frac{\alpha C}{(1-\alpha)}\right)^{-\alpha} \\
&=\left(\left(\frac{\alpha C}{(1-\alpha)}\right)^{1-\alpha}+\left(\frac{\alpha C}{(1-\alpha)}\right)^{-\alpha}\right)\|(-A) y\|^{\alpha}\|y\|^{1-\alpha} \\
& \leq C^{\prime}\|(-A) y\|^{\alpha}\|y\|^{1-\alpha}
\end{aligned}
$$

where $C^{\prime}$ does not depend on $\alpha \in(0,1)$. To see such a constant exists, note

$$
\lim _{\alpha \rightarrow 1}\left(\frac{\alpha C}{(1-\alpha)}\right)^{1-\alpha}=1
$$

and

$$
\lim _{\alpha \rightarrow 1}\left(\frac{\alpha C}{(1-\alpha)}\right)^{-\alpha}=0
$$

while

$$
\lim _{\alpha \rightarrow 0}\left(\frac{\alpha C}{(1-\alpha)}\right)^{1-\alpha}=0, \lim _{\alpha \rightarrow 0}\left(\frac{\alpha C}{(1-\alpha)}\right)^{-\alpha}=1
$$

Of course $C^{\prime}$ depends on $C$ but as shown above, this did not depend on $\alpha \in(0,1)$. This proves 54.3.25.

The following corollary follows from the proof of the above theorem.
Corollary 54.3.20 Let $\alpha \in(0,1)$. Then for all $\varepsilon>0$, there exists a constant $C(\alpha, \varepsilon)$ such that

$$
\left\|(-A)^{-\alpha} x\right\| \leq \varepsilon\|x\|+C(\varepsilon, \alpha)\left\|(-A)^{-1} x\right\|
$$

Also if $A^{-1}$ is compact, then so is $(-A)^{-\alpha}$ for all $\alpha \in(0,1)$.

Proof: The first part is done in the above theorem. Let $S$ be a bounded set and let $\eta>0$. Then let $\varepsilon>0$ be small enough that for all $x \in S, \varepsilon\|x\|<\eta / 4$. Let $\left\{(-A)^{-1} x_{n}\right\}$ be a $\eta /(2+2 C(\varepsilon, \alpha))$ net for $(-A)^{-1}(S)$. Then if $(-A)^{-\alpha} x \in(-A)^{-\alpha} S$, there exists $x_{n}$ such that

$$
\left\|(-A)^{-1} x_{n}-(-A)^{-1} x\right\|<\frac{\eta}{2+2 C(\varepsilon, \alpha)}
$$

Then

$$
\begin{aligned}
\left\|(-A)^{-\alpha} x_{n}-(-A)^{-\alpha} x\right\| & \leq \varepsilon\left\|x_{n}-x\right\|+C(\varepsilon, \alpha)\left\|(-A)^{-1} x_{n}-(-A)^{-1} x\right\| \\
& <\frac{\eta}{2}+\frac{\eta}{2}=\eta
\end{aligned}
$$

showing $(-A)^{-\alpha}(S)$ has a $\eta$ net. Thus $(-A)^{-\alpha}$ is compact. This proves the corollary.
The next proposition gives a general interpolation inequality.
Proposition 54.3.21 Let $0<\alpha<\beta$ and let

$$
\gamma=\theta \beta+(1-\theta) \alpha, \theta \in(0,1)
$$

Then there exists a constant, $C$ such that for all $x \in D\left((-A)^{\beta}\right)$,

$$
\left\|(-A)^{\gamma} x\right\| \leq C\left\|(-A)^{\beta} x\right\|^{\theta}\left\|(-A)^{\alpha} x\right\|^{1-\theta}
$$

Proof: This is an exercise in using 54.3.22. Letting $x \in D\left((-A)^{\beta}\right)$,

$$
(-A)^{\gamma} x=(-A)^{\theta}(-A)^{-\theta}(-A)^{\gamma} x
$$

Therefore, letting $C$ denote a generic constant, it follows since $(-A)^{\theta}$ is closed,

$$
\begin{gathered}
\Gamma(\theta)\left\|(-A)^{\gamma} x\right\|=\left\|\int_{0}^{\infty} t^{\theta-1}(-A)^{\theta} S(t)(-A)^{\gamma} x d t\right\| \\
\leq \int_{0}^{\eta} t^{\theta-1}\left\|(-A)^{\theta}(-A)^{\gamma-\beta} S(t)(-A)^{\beta} x\right\| d t \\
+\int_{\eta}^{\infty} t^{\theta-1}\left\|(-A)^{\theta}(-A)^{\gamma-\alpha} S(t)(-A)^{\alpha} x\right\| d t \\
\leq C \int_{0}^{\eta} t^{\theta-1} t^{-\theta} t^{\beta-\gamma} d t\left\|(-A)^{\beta} x\right\|+C \int_{\eta}^{\infty} t^{\theta-1} t^{-\theta} t^{\alpha-\gamma} d t\left\|(-A)^{\alpha} x\right\| \\
=C\left(\frac{\eta^{\beta-\gamma}}{\beta-\gamma}\left\|(-A)^{\beta} x\right\|+\frac{\eta^{-(\gamma-\alpha)}}{\gamma-\alpha}\left\|(-A)^{\alpha} x\right\|\right)
\end{gathered}
$$

and now writing in what $\gamma$ is in terms of $\theta$ yields

$$
\Gamma(\theta)\left\|(-A)^{\gamma} x\right\| \leq C\left(\frac{1}{\beta-\alpha}\right)\left(\frac{\left(\eta^{\beta-\alpha}\right)^{1-\theta}}{(1-\theta)}\left\|(-A)^{\beta} x\right\|+\frac{\left(\eta^{\beta-\alpha}\right)^{-\theta}}{\theta}\left\|(-A)^{\alpha} x\right\|\right)
$$

Letting $\lambda=\eta^{\beta-\alpha}$, it follows

$$
\Gamma(\theta)\left\|(-A)^{\gamma} x\right\| \leq C\left(\frac{1}{\beta-\alpha}\right)\left(\frac{\lambda^{1-\theta}}{(1-\theta)}\left\|(-A)^{\beta} x\right\|+\frac{\lambda^{-\theta}}{\theta}\left\|(-A)^{\alpha} x\right\|\right)
$$

then let

$$
\lambda=\frac{\left\|(-A)^{\alpha} x\right\|}{\left\|(-A)^{\beta} x\right\|}
$$

which is obtained from minimizing the expression on the right in the above. then placing this in the inequality yields

$$
\begin{gathered}
\Gamma(\theta)\left\|(-A)^{\gamma} x\right\| \\
\leq C\left(\frac{1}{\beta-\alpha}\right)\left(\frac{\left(\frac{\left\|(-A)^{\alpha} x\right\|}{\left\|(-A)^{\beta} x\right\|}\right)^{1-\theta}}{(1-\theta)}\left\|(-A)^{\beta} x\right\|\right. \\
\left.+\frac{\left(\frac{\left\|(-A)^{\alpha} x\right\|}{\left\|(-A)^{\beta} x\right\|}\right)^{-\theta}}{\theta}\left\|(-A)^{\alpha} x\right\|\right) \\
=C\left(\frac{1}{\beta-\alpha}\right)\left(\frac{1}{(1-\theta)}+\frac{1}{\theta}\right)\left\|(-A)^{\alpha} x\right\|^{1-\theta}\left\|(-A)^{\beta} x\right\|^{\theta}
\end{gathered}
$$

and this proves the proposition.
Note that the constant is not bounded as $\theta \rightarrow 1$.
Here is another interesting result about compactness.

Proposition 54.3.22 Let $A$ be sectorial for $S_{-a, \phi}$ where $-a<0$. Then the following are equivalent.

1. $(-A)^{-\alpha}$ is compact for all $\alpha>0$.
2. $S(t)$ is compact for each $t>0$.

Proof: First suppose $(-A)^{-\alpha}$ is compact for all $\alpha>0$. Then

$$
\begin{aligned}
& \Gamma(\alpha)(-A)^{-\alpha}=\int_{0}^{t} s^{\alpha-1} S(s) d s+\int_{t}^{\infty} s^{\alpha-1} S(s) d s \\
&= \frac{t^{\alpha}}{\alpha} S(t)-\int_{0}^{t} \frac{s^{\alpha}}{\alpha} A S(s) d s+\left.s^{\alpha-1} S(s) A^{-1}\right|_{t} ^{\infty} \\
& \quad-(\alpha-1) \int_{t}^{\infty} s^{\alpha-2} S(s) A^{-1} d s
\end{aligned}
$$

Now

$$
\left\|\frac{s^{\alpha}}{\alpha} A S(s)\right\| \leq C \frac{s^{\alpha-1}}{\alpha}
$$

and so the second integral satisfies

$$
\begin{aligned}
& \left\|\int_{0}^{t} \frac{s^{\alpha}}{\alpha} A S(s) d s\right\| \leq C \frac{t^{\alpha}}{\alpha^{2}} \\
\Gamma(\alpha)(-A)^{-\alpha}= & O\left(\frac{t^{\alpha}}{\alpha^{2}}\right)+\frac{t^{\alpha}}{\alpha} S(t) \\
& -t^{\alpha-1} A^{-1} S(t)-(\alpha-1) \int_{t}^{\infty} s^{\alpha-2} S(s) d s A^{-1}
\end{aligned}
$$

It follows that for $t>0$, and $\varepsilon>0$ given,

$$
\begin{aligned}
S(t)= & \left(\frac{t^{\alpha}}{\alpha}-t^{\alpha-1}\right)^{-1}\left(\Gamma(\alpha)(-A)^{-\alpha}\right. \\
& \left.+(\alpha-1) \int_{t}^{\infty} s^{\alpha-2} S(s) d s A^{-1}+O\left(\frac{t^{\alpha}}{\alpha^{2}}\right)\right) \\
= & \left(\frac{t^{\alpha}}{\alpha}-t^{\alpha-1}\right)^{-1}\left(\Gamma(\alpha)(-A)^{-\alpha}\right. \\
& \left.+(\alpha-1) \int_{t}^{\infty} s^{\alpha-2} S(s) d s A^{-1}\right)+O\left(\frac{1}{\alpha}\right) \\
= & N_{\alpha}+O\left(\frac{1}{\alpha}\right)
\end{aligned}
$$

where $N_{\alpha}$ is a compact operator. Now let $B$ be a bounded set in $H,\|x\| \leq M$ for all $x \in B$ and let $\eta>0$ be given. Then choose $\alpha$ large enough that $\left\|O\left(\frac{1}{\alpha}\right)\right\|<\frac{\eta}{4+4 M}$. Then there exists a $\eta / 2$ net, $\left\{N_{\alpha} x_{n}\right\}_{n=1}^{N}$ for $N_{\alpha}(B)$. Then consider $\left\{S(t) x_{n}\right\}_{n=1}^{N}$. For $x \in B$, there exists $x_{n}$ such that $\left\|N_{\alpha} x_{n}-N_{\alpha} x\right\|<\eta / 2$. Then

$$
\begin{aligned}
\left\|S(t) x-S(t) x_{n}\right\| \leq & \left\|S(t) x-N_{\alpha} x\right\| \\
& +\left\|N_{\alpha} x-N_{\alpha} x_{n}\right\|+\left\|N_{\alpha} x_{n}-S(t) x_{n}\right\|
\end{aligned}
$$

$$
\leq \frac{\eta}{4+4 M} M+\frac{\eta}{2}+\frac{\eta}{4+4 M} M<\eta
$$

Thus $S(t)(B)$ has an $\eta$ net for every $\eta>0$ and so $S(t)$ is compact.
Next suppose $S(t)$ is compact for all $t>0$. Then

$$
(-A)^{-\alpha}=\frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} t^{\alpha-1} S(t) d t
$$

and the integral is a limit in norm of Riemann sums of the form

$$
\sum_{k=1}^{m} t_{k}^{\alpha-1} S\left(t_{k}\right) \Delta t_{k}
$$

and each of these operators is compact. Since $(-A)^{-\alpha}$ is the limit in norm of compact operators, it must also be compact. This proves the proposition.

Here are some observations which are listed in the book by Henry [63]. Like the above proposition, these are exercises in this book.

Observation 54.3.23 For each $x \in H, t \rightarrow t A S(t)$ is continuous and $\lim _{t \rightarrow 0+} t A S(t) x=0$.
The reason for this is that if $x \in D(A)$, then

$$
t A S(t) x=|t S(t) A x| \rightarrow 0
$$

as $t \rightarrow 0$. Now suppose $y \in H$ is arbitrary. Then letting $x \in D(A)$,

$$
\begin{aligned}
|t A S(t) y| & \leq|t A S(t)(y-x)|+|t A S(t) x| \\
& \leq \varepsilon+|t A S(t) x|
\end{aligned}
$$

provided $x$ is close enough to $y$. The last term converges to 0 and so

$$
\lim _{\sup _{t \rightarrow 0+}}|t A S(t) y| \leq \varepsilon
$$

where $\varepsilon>0$ is arbitrary. Thus

$$
\lim _{t \rightarrow 0+}|t A S(t) y|=0
$$

Why is $t \rightarrow t A S(t) x$ continuous on $[0, T]$ ? This is true if $x \in D(A)$ because $t \rightarrow t S(t) A x$ is continuous. If $y \in H$ is arbitrary, let $x_{n}$ converge to $y$ in $H$ where $x_{n} \in D(A)$. Then

$$
\left|t A S(t) y-t A S(t) x_{n}\right| \leq C\left|y-x_{n}\right|
$$

and so the convergence is uniform. Thus $t \rightarrow t A S(t) y$ is continuous because it is the uniform limit of a sequence of continuous functions.

Observation 54.3.24 If $x \in H$ and $A$ is sectorial for $S_{-a, \phi},-a<0$, then for any $\alpha \in[0,1]$,

$$
\lim _{t \rightarrow 0+} t^{\alpha}\left\|(-A)^{\alpha} S(t) x\right\|=0
$$

This follows as above because you can verify this is true for $x \in D(A)$ and then use the fact shown above that

$$
t^{\alpha}\left\|(-A)^{\alpha} S(t)\right\| \leq C
$$

to extend it to $x$ arbitrary.

### 54.3.4 A Scale Of Banach Spaces

Next I will present an important and interesting theorem which can be used to prove equivalence of certain norms.

Theorem 54.3.25 Let $A, B$ be sectorial for $S_{-a, \phi}$ where $-a<0$ and suppose $D(A)=$ $D(B)$. Also suppose

$$
(A-B)(-A)^{-\alpha},(A-B)(-B)^{-\alpha}
$$

are both bounded on $D(A)$ for some $\alpha \in(0,1)$. Then for all $\beta \in[0,1]$,

$$
(-A)^{\beta}(-B)^{-\beta},(-B)^{\beta}(-A)^{-\beta}
$$

are both bounded on $D(A)=D(B)$. Also $D\left((-A)^{\beta}\right)=D\left((-B)^{\beta}\right)$.
Proof: First of all it is a good idea to verify $(A-B)(-A)^{-\alpha},(A-B)(-B)^{-\alpha}$ make sense on $D(A)$. If $x \in D(A)$, then why is $(-A)^{-\alpha} x \in D(A)$ ? Here is why. Since $x \in D(A)$,

$$
x=(-A)^{-1} y
$$

for some $y \in H$. Then

$$
(-A)^{-\alpha} x=(-A)^{-\alpha}(-A)^{-1} y=(-A)^{-1}(-A)^{-\alpha} y \in D(A)
$$

The case of $(A-B)(-B)^{-\alpha}$ is similar.
Next for $\beta \in(0,1)$ and $\lambda>0$, use 54.3.25 to write

$$
\begin{align*}
& \left\|(-A)^{\beta}(\lambda I-A)^{-1} x\right\| \\
\leq & C\left\|(-A)(\lambda I-A)^{-1} x\right\|^{\beta}\left\|(\lambda I-A)^{-1} x\right\|^{1-\beta} \\
\leq & C\left\|(-A)(\lambda I-A)^{-1}\right\|^{\beta}\left\|(\lambda I-A)^{-1}\right\|^{1-\beta}\|x\| \\
\leq & C\left\|I-\lambda(\lambda I-A)^{-1}\right\|^{\beta} \frac{M}{(\lambda+\delta)^{1-\beta}}\|x\| \\
\leq & C\left(1+\frac{\lambda}{(\lambda+\delta)}\right)^{\beta} \frac{M}{(\lambda+\delta)^{1-\beta}}\|x\| \equiv \frac{C}{(\lambda+\delta)^{1-\beta}}\|x\| \tag{54.3.29}
\end{align*}
$$

where $-a<-\delta<0$ where $C$ denotes a generic constant. Similarly, for all $\beta \in(0,1)$,

$$
\begin{equation*}
\left\|(-B)^{\beta}(\lambda I-B)^{-1} x\right\| \leq \frac{C}{(\lambda+\delta)^{1-\beta}}\|x\| \tag{54.3.30}
\end{equation*}
$$

Now from Theorem 54.3.17 and letting $\beta \in(0,1)$,

$$
(-B)^{-\beta}-(-A)^{-\beta}=\frac{\sin (\pi \beta)}{\pi} \int_{0}^{\infty} \lambda^{-\beta}\left((\lambda I-B)^{-1}-(\lambda I-A)^{-1}\right) d \lambda
$$

$$
\begin{equation*}
=\frac{\sin (\pi \beta)}{\pi} \int_{0}^{\infty} \lambda^{-\beta}(\lambda I-B)^{-1}(A-B)(\lambda I-A)^{-1} d \lambda . \tag{54.3.31}
\end{equation*}
$$

Therefore, letting $x \in D(A)$ and letting $C$ denote a generic constant which can be changed from line to line and using 54.3.29 and 54.3.30,

$$
\begin{aligned}
& \left\|x-(-B)^{\beta}(-A)^{-\beta} x\right\| \\
\leq & C \int_{0}^{\infty} \frac{1}{\lambda^{\beta}}\left\|(-B)^{\beta}(\lambda I-B)^{-1}(A-B)(\lambda I-A)^{-1} x\right\| d \lambda
\end{aligned}
$$

The reason $(-B)^{\beta}$ goes inside the integral is that it is a closed operator. Then the above

$$
\begin{aligned}
& \leq C \int_{0}^{\infty} \frac{1}{\lambda^{\beta}(\lambda+\delta)^{1-\beta}}\left\|(A-B)(-A)^{-\alpha}(-A)^{\alpha}(\lambda I-A)^{-1} x\right\| d \lambda \\
& \leq C \int_{0}^{\infty} \frac{1}{\lambda^{\beta}(\lambda+\delta)^{1-\beta}}\left\|(-A)^{\alpha}(\lambda I-A)^{-1} x\right\| d \lambda \\
& \leq C \int_{0}^{\infty} \frac{1}{\lambda^{\beta}(\lambda+\delta)^{1-\beta}} \frac{1}{(\lambda+\delta)^{1-\alpha}} d \lambda\|x\|=C\|x\| .
\end{aligned}
$$

It follows $(-B)^{\beta}(-A)^{-\beta}$ is bounded on $D(A)$.
Next reverse $A$ and $B$ in 54.3.31. This yields

$$
(-A)^{-\beta}-(-B)^{-\beta}=\frac{\sin (\pi \beta)}{\pi} \int_{0}^{\infty} \lambda^{-\beta}(\lambda I-A)^{-1}(B-A)(\lambda I-B)^{-1} d \lambda
$$

Letting $x \in D(A)$,

$$
\begin{align*}
& \left\|x-(-A)^{\beta}(-B)^{-\beta} x\right\| \\
\leq & C \int_{0}^{\infty} \lambda^{-\beta}\left\|(-A)^{\beta}(\lambda I-A)^{-1}(B-A)(\lambda I-B)^{-1} x\right\| d \lambda \\
\leq & C \int_{0}^{\infty} \frac{1}{\lambda^{\beta}(\lambda+\delta)^{1-\beta}}\left\|(B-A)(-B)^{-\alpha}(-B)^{\alpha}(\lambda I-B)^{-1} x\right\| d \lambda(54.3 .32) \\
\leq & C \int_{0}^{\infty} \frac{1}{\lambda^{\beta}(\lambda+\delta)^{1-\beta}(\lambda+\delta)^{1-\alpha}} d \lambda\|x\|=C\|x\| \tag{54.3.33}
\end{align*}
$$

This shows $(-A)^{\beta}(-B)^{-\beta}$ is bounded on $D(A)=D(B)$. Note the assertion these are bounded refers to the norm on $H$.

It remains to verify $D\left((-A)^{\beta}\right)=D\left((-B)^{\beta}\right)$. Since $D(A)$ is dense in $H$ there exists a unique $L(A, B) \in \mathscr{L}(H, H)$ such that $L(A, B)=(-A)^{\beta}(-B)^{-\beta}$ on $D(A)$. Let $L(B, A)$ be defined similarly as a continuous linear map which equals $(-B)^{\beta}(-A)^{-\beta}$ on $D(A)$. Then

$$
\begin{aligned}
& (-A)^{-\beta} L(A, B)=(-B)^{-\beta} \\
& (-B)^{-\beta} L(B, A)=(-A)^{-\beta}
\end{aligned}
$$

The first of these equations shows $D\left((-B)^{\beta}\right) \subseteq D\left((-A)^{\beta}\right)$ and the second turns the inclusion around. Thus they are equal as claimed.

Next consider the case where $\beta=1$. In this case

$$
(A-B) B^{-\alpha}
$$

is bounded on $D(A)$ and so

$$
(A-B) B^{-\alpha} B^{-1+\alpha}
$$

is also bounded on $D(A)$. But this equals

$$
(A-B) B^{-1}
$$

Thus $A B^{-1}$ is bounded on $D(A)$. Similarly you can show

$$
(B-A) A^{-1}
$$

is bounded which implies $B A^{-1}$ is bounded on $D(A)$. This proves the theorem.
Definition 54.3.26 Let $A$ be sectorial for the sector $S_{a, \phi}$. Let $b>a$ so that $A-b I$ is sectorial for $S_{-\delta, \phi}$ where $\delta=b-a$. Then for each $\alpha \in[0,1]$, define a norm on $D\left((b I-A)^{\alpha}\right) \equiv$ $H_{\alpha}$ by

$$
\|x\|_{\alpha} \equiv\left\|(b I-A)^{\alpha} x\right\|
$$

The $\left\{H_{\alpha}\right\}_{\alpha \in[0,1]}$ is called a scale of Banach spaces.
Proposition 54.3.27 The $H_{\alpha}$ above are Banach spaces and they decrease in $\alpha$. Furthermore, if $b_{i}>$ a for $i=1,2$ then the two norms associated with the $b_{i}$ are equivalent.

Proof: That the $H_{\alpha}$ are decreasing was shown above in Theorem 54.3.17. They are Banach spaces because $(b I-A)^{\alpha}$ is a closed mapping which is also one to one.

It only remains to verify the claim about the equivalence of the norms. Let $b_{2}>b_{1}>a$. Then if $\alpha \in(0,1)$,

$$
\begin{aligned}
& \left(\left(b_{1} I-A\right)-\left(b_{2} I-A\right)\right)\left(b_{2} I-A\right)^{-\alpha} \\
= & \left(b_{1}-b_{2}\right)\left(b_{2} I-A\right)^{-\alpha} \in \mathscr{L}(H, H)
\end{aligned}
$$

and so by Theorem 54.3.25, for each $\beta \in[0,1]$,

$$
D\left(\left(b_{1} I-A\right)^{\beta}\right)=D\left(\left(b_{2} I-A\right)^{\beta}\right)
$$

so the spaces, $H_{\beta}$ are the same for either choice of $b>a$. Also from this theorem,

$$
\left(b_{1} I-A\right)^{\beta}\left(b_{2} I-A\right)^{-\beta},\left(b_{2} I-A\right)^{\beta}\left(b_{1} I-A\right)^{-\beta}
$$

are both bounded on $D(A)$. Therefore, for $x \in H_{\beta}$

$$
\begin{aligned}
\left\|\left(b_{1} I-A\right)^{\beta} x\right\| & =\left\|\left(b_{1} I-A\right)^{\beta}\left(b_{2} I-A\right)^{-\beta}\left(b_{2} I-A\right)^{\beta} x\right\| \\
& \leq C\left\|\left(b_{2} I-A\right)^{\beta} x\right\|
\end{aligned}
$$

Similarly using the boundedness of $\left(b_{2} I-A\right)^{\beta}\left(b_{1} I-A\right)^{-\beta}$, it follows

$$
\left\|\left(b_{2} I-A\right)^{\beta} x\right\| \leq C^{\prime}\left\|\left(b_{1} I-A\right)^{\beta} x\right\|
$$

Thus showing the two norms are equivalent. This proves the proposition.

## Chapter 55

## Complex Mappings

### 55.1 Conformal Maps

If $\gamma(t)=x(t)+i y(t)$ is a $C^{1}$ curve having values in $U$, an open set of $\mathbb{C}$, and if $f: U \rightarrow \mathbb{C}$ is analytic, consider $f \circ \gamma$, another $C^{1}$ curve having values in $\mathbb{C}$. Also, $\gamma^{\prime}(t)$ and $(f \circ \gamma)^{\prime}(t)$ are complex numbers so these can be considered as vectors in $\mathbb{R}^{2}$ as follows. The complex number, $x+i y$ corresponds to the vector, $(x, y)$. Suppose that $\gamma$ and $\eta$ are two such $C^{1}$ curves having values in $U$ and that $\gamma\left(t_{0}\right)=\eta\left(s_{0}\right)=z$ and suppose that $f: U \rightarrow \mathbb{C}$ is analytic. What can be said about the angle between $(f \circ \gamma)^{\prime}\left(t_{0}\right)$ and $(f \circ \eta)^{\prime}\left(s_{0}\right)$ ? It turns out this angle is the same as the angle between $\gamma^{\prime}\left(t_{0}\right)$ and $\eta^{\prime}\left(s_{0}\right)$ assuming that $f^{\prime}(z) \neq 0$. To see this, note $(x, y) \cdot(a, b)=\frac{1}{2}(z \bar{w}+\bar{z} w)$ where $z=x+i y$ and $w=a+i b$. Therefore, letting $\theta$ be the cosine between the two vectors, $(f \circ \gamma)^{\prime}\left(t_{0}\right)$ and $(f \circ \eta)^{\prime}\left(s_{0}\right)$, it follows from calculus that

$$
\begin{aligned}
& \cos \theta \\
= & \frac{(f \circ \gamma)^{\prime}\left(t_{0}\right) \cdot(f \circ \eta)^{\prime}\left(s_{0}\right)}{\left|(f \circ \eta)^{\prime}\left(s_{0}\right)\right|\left|(f \circ \gamma)^{\prime}\left(t_{0}\right)\right|} \\
= & \frac{1}{2} \frac{f^{\prime}\left(\gamma\left(t_{0}\right)\right) \gamma^{\prime}\left(t_{0}\right) \overline{f^{\prime}\left(\eta\left(s_{0}\right)\right) \eta^{\prime}\left(s_{0}\right)}+\overline{f^{\prime}\left(\gamma\left(t_{0}\right)\right) \gamma^{\prime}\left(t_{0}\right)} f^{\prime}\left(\eta\left(s_{0}\right)\right) \eta^{\prime}\left(s_{0}\right)}{\left|f^{\prime}\left(\gamma\left(t_{0}\right)\right)\right|\left|f^{\prime}\left(\eta\left(s_{0}\right)\right)\right|} \\
= & \frac{1}{2} \frac{f^{\prime}(z) \overline{f^{\prime}(z)} \gamma^{\prime}\left(t_{0}\right) \overline{\eta^{\prime}\left(s_{0}\right)}+\overline{f^{\prime}(z)} f^{\prime}(z) \overline{\gamma^{\prime}\left(t_{0}\right)} \eta^{\prime}\left(s_{0}\right)}{\left|f^{\prime}(z)\right|\left|f^{\prime}(z)\right|} \\
= & \frac{1}{2} \frac{\gamma^{\prime}\left(t_{0}\right) \overline{\eta^{\prime}\left(s_{0}\right)}+\eta^{\prime}\left(s_{0}\right) \overline{\gamma^{\prime}\left(t_{0}\right)}}{1}
\end{aligned}
$$

which equals the angle between the vectors, $\gamma^{\prime}\left(t_{0}\right)$ and $\eta^{\prime}\left(t_{0}\right)$. Thus analytic mappings preserve angles at points where the derivative is nonzero. Such mappings are called isogonal.

Actually, they also preserve orientations. If $z=x+i y$ and $w=a+i b$ are two complex numbers, then $(x, y, 0)$ and $(a, b, 0)$ are two vectors in $\mathbb{R}^{3}$. Recall that the cross product, $(x, y, 0) \times(a, b, 0)$, yields a vector normal to the two given vectors such that the triple, $(x, y, 0),(a, b, 0)$, and $(x, y, 0) \times(a, b, 0)$ satisfies the right hand rule and has magnitude equal to the product of the sine of the included angle times the product of the two norms of the vectors. In this case, the cross product will produce a vector which is a multiple of $\mathbf{k}$, the unit vector in the direction of the $z$ axis. In fact, you can verify by computing both sides that, letting $z=x+i y$ and $w=a+i b$,

$$
(x, y, 0) \times(a, b, 0)=\operatorname{Re}(z i \bar{w}) \mathbf{k} .
$$

Therefore, in the above situation,

$$
\begin{aligned}
& (f \circ \gamma)^{\prime}\left(t_{0}\right) \times(f \circ \eta)^{\prime}\left(s_{0}\right) \\
= & \operatorname{Re}\left(f^{\prime}\left(\gamma\left(t_{0}\right)\right) \gamma\left(t_{0}\right) \overline{i f^{\prime}\left(\eta\left(s_{0}\right)\right) \eta^{\prime}\left(s_{0}\right)}\right) \mathbf{k} \\
= & \left|f^{\prime}(z)\right|^{2} \operatorname{Re}\left(\gamma^{\prime}\left(t_{0}\right) \overline{i \eta^{\prime}\left(s_{0}\right)}\right) \mathbf{k}
\end{aligned}
$$

which shows that the orientation of $\gamma^{\prime}\left(t_{0}\right), \eta^{\prime}\left(s_{0}\right)$ is the same as the orientation of

$$
(f \circ \gamma)^{\prime}\left(t_{0}\right),(f \circ \eta)^{\prime}\left(s_{0}\right)
$$

Mappings which preserve both orientation and angles are called conformal mappings and this has shown that analytic functions are conformal mappings if the derivative does not vanish.

### 55.2 Fractional Linear Transformations

### 55.2.1 Circles And Lines

These mappings map lines and circles to either lines or circles.
Definition 55.2.1 A fractional linear transformation is a function of the form

$$
\begin{equation*}
f(z)=\frac{a z+b}{c z+d} \tag{55.2.1}
\end{equation*}
$$

where $a d-b c \neq 0$.
Note that if $c=0$, this reduces to a linear transformation $(a / d) z+(b / d)$. Special cases of these are defined as follows.

$$
\begin{gathered}
\text { dilations: } z \rightarrow \delta z, \delta \neq 0, \text { inversions: } z \rightarrow \frac{1}{z} \\
\text { translations: } z \rightarrow z+\rho
\end{gathered}
$$

The next lemma is the key to understanding fractional linear transformations.
Lemma 55.2.2 The fractional linear transformation, 55.2.1 can be written as a finite composition of dilations, inversions, and translations.

Proof: Let

$$
S_{1}(z)=z+\frac{d}{c}, S_{2}(z)=\frac{1}{z}, S_{3}(z)=\frac{(b c-a d)}{c^{2}} z
$$

and

$$
S_{4}(z)=z+\frac{a}{c}
$$

in the case where $c \neq 0$. Then $f(z)$ given in 55.2.1 is of the form

$$
f(z)=S_{4} \circ S_{3} \circ S_{2} \circ S_{1}
$$

Here is why.

$$
S_{2}\left(S_{1}(z)\right)=S_{2}\left(z+\frac{d}{c}\right) \equiv \frac{1}{z+\frac{d}{c}}=\frac{c}{z c+d}
$$

Now consider

$$
S_{3}\left(\frac{c}{z c+d}\right) \equiv \frac{(b c-a d)}{c^{2}}\left(\frac{c}{z c+d}\right)=\frac{b c-a d}{c(z c+d)}
$$

Finally, consider

$$
S_{4}\left(\frac{b c-a d}{c(z c+d)}\right) \equiv \frac{b c-a d}{c(z c+d)}+\frac{a}{c}=\frac{b+a z}{z c+d}
$$

In case that $c=0, f(z)=\frac{a}{d} z+\frac{b}{d}$ which is a translation composed with a dilation. Because of the assumption that $a d-b c \neq 0$, it follows that since $c=0$, both $a$ and $d \neq 0$. This proves the lemma.

This lemma implies the following corollary.
Corollary 55.2.3 Fractional linear transformations map circles and lines to circles or lines.

Proof: It is obvious that dilations and translations map circles to circles and lines to lines. What of inversions? If inversions have this property, the above lemma implies a general fractional linear transformation has this property as well.

Note that all circles and lines may be put in the form

$$
\alpha\left(x^{2}+y^{2}\right)-2 a x-2 b y=r^{2}-\left(a^{2}+b^{2}\right)
$$

where $\alpha=1$ gives a circle centered at $(a, b)$ with radius $r$ and $\alpha=0$ gives a line. In terms of complex variables you may therefore consider all possible circles and lines in the form

$$
\begin{equation*}
\alpha z \bar{z}+\beta z+\bar{\beta} \bar{z}+\gamma=0 \tag{55.2.2}
\end{equation*}
$$

To see this let $\beta=\beta_{1}+i \beta_{2}$ where $\beta_{1} \equiv-a$ and $\beta_{2} \equiv b$. Note that even if $\alpha$ is not 0 or 1 the expression still corresponds to either a circle or a line because you can divide by $\alpha$ if $\alpha \neq 0$. Now I verify that replacing $z$ with $\frac{1}{z}$ results in an expression of the form in 55.2.2. Thus, let $w=\frac{1}{z}$ where $z$ satisfies 55.2.2. Then

$$
(\alpha+\beta \bar{w}+\bar{\beta} w+\gamma w \bar{w})=\frac{1}{z \bar{z}}(\alpha z \bar{z}+\beta z+\bar{\beta} \bar{z}+\gamma)=0
$$

and so $w$ also satisfies a relation like 55.2.2. One simply switches $\alpha$ with $\gamma$ and $\beta$ with $\bar{\beta}$. Note the situation is slightly different than with dilations and translations. In the case of an inversion, a circle becomes either a line or a circle and similarly, a line becomes either a circle or a line. This proves the corollary.

The next example is quite important.
Example 55.2.4 Consider the fractional linear transformation, $w=\frac{z-i}{z+i}$.
First consider what this mapping does to the points of the form $z=x+i 0$. Substituting into the expression for $w$,

$$
w=\frac{x-i}{x+i}=\frac{x^{2}-1-2 x i}{x^{2}+1}
$$

a point on the unit circle. Thus this transformation maps the real axis to the unit circle.
The upper half plane is composed of points of the form $x+i y$ where $y>0$. Substituting in to the transformation,

$$
w=\frac{x+i(y-1)}{x+i(y+1)}
$$

which is seen to be a point on the interior of the unit disk because $|y-1|<|y+1|$ which implies $|x+i(y+1)|>|x+i(y-1)|$. Therefore, this transformation maps the upper half plane to the interior of the unit disk.

One might wonder whether the mapping is one to one and onto. The mapping is clearly one to one because it has an inverse, $z=-i \frac{w+1}{w-1}$ for all $w$ in the interior of the unit disk. Also, a short computation verifies that $z$ so defined is in the upper half plane. Therefore, this transformation maps $\{z \in \mathbb{C}$ such that $\operatorname{Im} z>0\}$ one to one and onto the unit disk $\{z \in \mathbb{C}$ such that $|z|<1\}$.

A fancy way to do part of this is to use Theorem 52.3.5. $\lim \sup _{z \rightarrow a}\left|\frac{z-i}{z+i}\right| \leq 1$ whenever $a$ is the real axis or $\infty$. Therefore, $\left|\frac{z-i}{z+i}\right| \leq 1$. This is a little shorter.

### 55.2.2 Three Points To Three Points

There is a simple procedure for determining fractional linear transformations which map a given set of three points to another set of three points. The problem is as follows: There are three distinct points in the extended complex plane, $z_{1}, z_{2}$, and $z_{3}$ and it is desired to find a fractional linear transformation such that $z_{i} \rightarrow w_{i}$ for $i=1,2,3$ where here $w_{1}, w_{2}$, and $w_{3}$ are three distinct points in the extended complex plane. Then the procedure says that to find the desired fractional linear transformation solve the following equation for $w$.

$$
\frac{w-w_{1}}{w-w_{3}} \cdot \frac{w_{2}-w_{3}}{w_{2}-w_{1}}=\frac{z-z_{1}}{z-z_{3}} \cdot \frac{z_{2}-z_{3}}{z_{2}-z_{1}}
$$

The result will be a fractional linear transformation with the desired properties. If any of the points equals $\infty$, then the quotient containing this point should be adjusted.

Why should this procedure work? Here is a heuristic argument to indicate why you would expect this to happen rather than a rigorous proof. The reader may want to tighten the argument to give a proof. First suppose $z=z_{1}$. Then the right side equals zero and so the left side also must equal zero. However, this requires $w=w_{1}$. Next suppose $z=z_{2}$. Then the right side equals 1 . To get a 1 on the left, you need $w=w_{2}$. Finally suppose $z=z_{3}$. Then the right side involves division by 0 . To get the same bad behavior, on the left, you need $w=w_{3}$.

Example 55.2.5 Let $\operatorname{Im} \xi>0$ and consider the fractional linear transformation which takes $\xi$ to $0, \bar{\xi}$ to $\infty$ and 0 to $\xi / \bar{\xi}$, .

The equation for $w$ is

$$
\frac{w-0}{w-(\xi / \bar{\xi})}=\frac{z-\xi}{z-0} \cdot \frac{\bar{\xi}-0}{\bar{\xi}-\xi}
$$

After some computations,

$$
w=\frac{z-\xi}{z-\bar{\xi}}
$$

Note that this has the property that $\frac{x-\xi}{x-\bar{\xi}}$ is always a point on the unit circle because it is a complex number divided by its conjugate. Therefore, this fractional linear transformation
maps the real line to the unit circle. It also takes the point, $\xi$ to 0 and so it must map the upper half plane to the unit disk. You can verify the mapping is onto as well.

Example 55.2.6 Let $z_{1}=0, z_{2}=1$, and $z_{3}=2$ and let $w_{1}=0, w_{2}=i$, and $w_{3}=2 i$.

Then the equation to solve is

$$
\frac{w}{w-2 i} \cdot \frac{-i}{i}=\frac{z}{z-2} \cdot \frac{-1}{1}
$$

Solving this yields $w=i z$ which clearly works.

### 55.3 Riemann Mapping Theorem

From the open mapping theorem analytic functions map regions to other regions or else to single points. The Riemann mapping theorem states that for every simply connected region, $\Omega$ which is not equal to all of $\mathbb{C}$ there exists an analytic function, $f$ such that $f(\Omega)=B(0,1)$ and in addition to this, $f$ is one to one. The proof involves several ideas which have been developed up to now. The proof is based on the following important theorem, a case of Montel's theorem. Before, beginning, note that the Riemann mapping theorem is a classic example of a major existence theorem. In mathematics there are two sorts of questions, those related to whether something exists and those involving methods for finding it. The real questions are often related to questions of existence. There is a long and involved history for proofs of this theorem. The first proofs were based on the Dirichlet principle and turned out to be incorrect, thanks to Weierstrass who pointed out the errors. For more on the history of this theorem, see Hille [65].

The following theorem is really wonderful. It is about the existence of a subsequence having certain salubrious properties. It is this wonderful result which will give the existence of the mapping desired. The other parts of the argument are technical details to set things up and use this theorem.

### 55.3.1 Montel's Theorem

Theorem 55.3.1 Let $\Omega$ be an open set in $\mathbb{C}$ and let $\mathscr{F}$ denote a set of analytic functions mapping $\Omega$ to $B(0, M) \subseteq \mathbb{C}$. Then there exists a sequence of functions from $\mathscr{F},\left\{f_{n}\right\}_{n=1}^{\infty}$ and an analytic function, $f$ such that $f_{n}^{(k)}$ converges uniformly to $f^{(k)}$ on every compact subset of $\Omega$.

Proof: First note there exists a sequence of compact sets, $K_{n}$ such that $K_{n} \subseteq \operatorname{int} K_{n+1} \subseteq$ $\Omega$ for all $n$ where here int $K$ denotes the interior of the set $K$, the union of all open sets contained in $K$ and $\cup_{n=1}^{\infty} K_{n}=\Omega$. In fact, you can verify that $\overline{B(0, n)} \cap\left\{z \in \Omega: \operatorname{dist}\left(z, \Omega^{C}\right) \leq \frac{1}{n}\right\}$ works for $K_{n}$. Then there exist positive numbers, $\delta_{n}$ such that if $z \in K_{n}$, then $\overline{B\left(z, \delta_{n}\right)} \subseteq$ int $K_{n+1}$. Now denote by $\mathscr{F}_{n}$ the set of restrictions of functions of $\mathscr{F}$ to $K_{n}$. Then let $z \in K_{n}$
and let $\gamma(t) \equiv z+\delta_{n} e^{i t}, t \in[0,2 \pi]$. It follows that for $z_{1} \in B\left(z, \delta_{n}\right)$, and $f \in \mathscr{F}$,

$$
\begin{aligned}
\left|f(z)-f\left(z_{1}\right)\right| & =\left|\frac{1}{2 \pi i} \int_{\gamma} f(w)\left(\frac{1}{w-z}-\frac{1}{w-z_{1}}\right) d w\right| \\
& \leq \frac{1}{2 \pi}\left|\int_{\gamma} f(w) \frac{z-z_{1}}{(w-z)\left(w-z_{1}\right)} d w\right|
\end{aligned}
$$

Letting $\left|z_{1}-z\right|<\frac{\delta_{n}}{2}$,

$$
\begin{aligned}
\left|f(z)-f\left(z_{1}\right)\right| & \leq \frac{M}{2 \pi} 2 \pi \delta_{n} \frac{\left|z-z_{1}\right|}{\delta_{n}^{2} / 2} \\
& \leq 2 M \frac{\left|z-z_{1}\right|}{\delta_{n}}
\end{aligned}
$$

It follows that $\mathscr{F}_{n}$ is equicontinuous and uniformly bounded so by the Arzela Ascoli theorem there exists a sequence, $\left\{f_{n k}\right\}_{k=1}^{\infty} \subseteq \mathscr{F}$ which converges uniformly on $K_{n}$. Let $\left\{f_{1 k}\right\}_{k=1}^{\infty}$ converge uniformly on $K_{1}$. Then use the Arzela Ascoli theorem applied to this sequence to get a subsequence, denoted by $\left\{f_{2 k}\right\}_{k=1}^{\infty}$ which also converges uniformly on $K_{2}$. Continue in this way to obtain $\left\{f_{n k}\right\}_{k=1}^{\infty}$ which converges uniformly on $K_{1}, \cdots, K_{n}$. Now the sequence $\left\{f_{n n}\right\}_{n=m}^{\infty}$ is a subsequence of $\left\{f_{m k}\right\}_{k=1}^{\infty}$ and so it converges uniformly on $K_{m}$ for all $m$. Denoting $f_{n n}$ by $f_{n}$ for short, this is the sequence of functions promised by the theorem. It is clear $\left\{f_{n}\right\}_{n=1}^{\infty}$ converges uniformly on every compact subset of $\Omega$ because every such set is contained in $K_{m}$ for all $m$ large enough. Let $f(z)$ be the point to which $f_{n}(z)$ converges. Then $f$ is a continuous function defined on $\Omega$. Is $f$ is analytic? Yes it is by Lemma 51.3.13. Alternatively, you could let $T \subseteq \Omega$ be a triangle. Then

$$
\int_{\partial T} f(z) d z=\lim _{n \rightarrow \infty} \int_{\partial T} f_{n}(z) d z=0
$$

Therefore, by Morera's theorem, $f$ is analytic.
As for the uniform convergence of the derivatives of $f$, recall Theorem 51.7.25 about the existence of a cycle. Let $K$ be a compact subset of int $\left(K_{n}\right)$ and let $\left\{\gamma_{k}\right\}_{k=1}^{m}$ be closed oriented curves contained in

$$
\operatorname{int}\left(K_{n}\right) \backslash K
$$

such that $\sum_{k=1}^{m} n\left(\gamma_{k}, z\right)=1$ for every $z \in K$. Also let $\eta$ denote the distance between $\cup_{j} \gamma_{j}^{*}$ and $K$. Then for $z \in K$,

$$
\begin{aligned}
\left|f^{(k)}(z)-f_{n}^{(k)}(z)\right| & =\left|\frac{k!}{2 \pi i} \sum_{j=1}^{m} \int_{\gamma_{j}} \frac{f(w)-f_{n}(w)}{(w-z)^{k+1}} d w\right| \\
& \leq \frac{k!}{2 \pi}\left\|f_{k}-f\right\|_{K_{n}} \sum_{j=1}^{m}\left(\text { length of } \gamma_{k}\right) \frac{1}{\eta^{k+1}}
\end{aligned}
$$

where here $\left|\mid f_{k}-f \|_{K_{n}} \equiv \sup \left\{\left|f_{k}(z)-f(z)\right|: z \in K_{n}\right\}\right.$. Thus you get uniform convergence of the derivatives.

Since the family, $\mathscr{F}$ satisfies the conclusion of Theorem 55.3.1 it is known as a normal family of functions. More generally,

Definition 55.3.2 Let $\mathscr{F}$ denote a collection offunctions which are analytic on $\Omega$, a region. Then $\mathscr{F}$ is normal if every sequence contained in $\mathscr{F}$ has a subsequence which converges uniformly on compact subsets of $\Omega$.

The following result is about a certain class of fractional linear transformations. Recall Lemma 52.4.7 which is listed here for convenience.

Lemma 55.3.3 For $\alpha \in B(0,1)$, let

$$
\phi_{\alpha}(z) \equiv \frac{z-\alpha}{1-\bar{\alpha} z}
$$

Then $\phi_{\alpha}$ maps $B(0,1)$ one to one and onto $B(0,1), \phi_{\alpha}^{-1}=\phi_{-\alpha}$, and

$$
\phi_{\alpha}^{\prime}(\alpha)=\frac{1}{1-|\alpha|^{2}}
$$

The next lemma, known as Schwarz's lemma is interesting for its own sake but will also be an important part of the proof of the Riemann mapping theorem. It was stated and proved earlier but for convenience it is given again here.

Lemma 55.3.4 Suppose $F: B(0,1) \rightarrow B(0,1), F$ is analytic, and $F(0)=0$. Then for all $z \in B(0,1)$,

$$
\begin{equation*}
|F(z)| \leq|z| \tag{55.3.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|F^{\prime}(0)\right| \leq 1 \tag{55.3.4}
\end{equation*}
$$

If equality holds in 55.3 .4 then there exists $\lambda \in \mathbb{C}$ with $|\lambda|=1$ and

$$
\begin{equation*}
F(z)=\lambda z \tag{55.3.5}
\end{equation*}
$$

Proof: First note that by assumption, $F(z) / z$ has a removable singularity at 0 if its value at 0 is defined to be $F^{\prime}(0)$. By the maximum modulus theorem, if $|z|<r<1$,

$$
\left|\frac{F(z)}{z}\right| \leq \max _{t \in[0,2 \pi]} \frac{\left|F\left(r e^{i t}\right)\right|}{r} \leq \frac{1}{r}
$$

Then letting $r \rightarrow 1$,

$$
\left|\frac{F(z)}{z}\right| \leq 1
$$

this shows 55.3 .3 and it also verifies 55.3 .4 on taking the limit as $z \rightarrow 0$. If equality holds in 55.3.4, then $|F(z) / z|$ achieves a maximum at an interior point so $F(z) / z$ equals a constant, $\lambda$ by the maximum modulus theorem. Since $F(z)=\lambda z$, it follows $F^{\prime}(0)=\lambda$ and so $|\lambda|=1$. This proves the lemma.

Definition 55.3.5 A region, $\Omega$ has the square root property if whenever $f, \frac{1}{f}: \Omega \rightarrow \mathbb{C}$ are both analytic ${ }^{1}$, it follows there exists $\phi: \Omega \rightarrow \mathbb{C}$ such that $\phi$ is analytic and $f(z)=\phi^{2}(z)$.

The next theorem will turn out to be equivalent to the Riemann mapping theorem.

[^33]
### 55.3.2 Regions With Square Root Property

Theorem 55.3.6 Let $\Omega \neq \mathbb{C}$ for $\Omega$ a region and suppose $\Omega$ has the square root property. Then for $z_{0} \in \Omega$ there exists $h: \Omega \rightarrow B(0,1)$ such that $h$ is one to one, onto, analytic, and $h\left(z_{0}\right)=0$.

Proof: Define $\mathscr{F}$ to be the set of functions, $f$ such that $f: \Omega \rightarrow B(0,1)$ is one to one and analytic. The first task is to show $\mathscr{F}$ is nonempty. Then, using Montel's theorem it will be shown there is a function in $\mathscr{F}, h$, such that $\left|h^{\prime}\left(z_{0}\right)\right| \geq\left|\psi^{\prime}\left(z_{0}\right)\right|$ for all $\psi \in \mathscr{F}$. When this has been done it will be shown that $h$ is actually onto. This will prove the theorem.

Claim 1: $\mathscr{F}$ is nonempty.
Proof of Claim 1: Since $\Omega \neq \mathbb{C}$ it follows there exists $\xi \notin \Omega$. Then it follows $z-\xi$ and $\frac{1}{z-\xi}$ are both analytic on $\Omega$. Since $\Omega$ has the square root property, there exists an analytic function, $\phi: \Omega \rightarrow \mathbb{C}$ such that $\phi^{2}(z)=z-\xi$ for all $z \in \Omega, \phi(z)=\sqrt{z-\xi}$. Since $z-\xi$ is not constant, neither is $\phi$ and it follows from the open mapping theorem that $\phi(\Omega)$ is a region. Note also that $\phi$ is one to one because if $\phi\left(z_{1}\right)=\phi\left(z_{2}\right)$, then you can square both sides and conclude $z_{1}-\xi=z_{2}-\xi$ implying $z_{1}=z_{2}$.

Now pick $a \in \phi(\Omega)$. Thus $\sqrt{z_{a}-\xi}=a$. I claim there exists a positive lower bound to $|\sqrt{z-\xi}+a|$ for $z \in \Omega$. If not, there exists a sequence, $\left\{z_{n}\right\} \subseteq \Omega$ such that

$$
\sqrt{z_{n}-\xi}+a=\sqrt{z_{n}-\xi}+\sqrt{z_{a}-\xi} \equiv \varepsilon_{n} \rightarrow 0
$$

Then

$$
\begin{equation*}
\sqrt{z_{n}-\xi}=\left(\varepsilon_{n}-\sqrt{z_{a}-\xi}\right) \tag{55.3.6}
\end{equation*}
$$

and squaring both sides,

$$
z_{n}-\xi=\varepsilon_{n}^{2}+z_{a}-\xi-2 \varepsilon_{n} \sqrt{z_{a}-\xi}
$$

Consequently, $\left(z_{n}-z_{a}\right)=\varepsilon_{n}^{2}-2 \varepsilon_{n} \sqrt{z_{a}-\xi}$ which converges to 0 . Taking the limit in 55.3.6, it follows $2 \sqrt{z_{a}-\xi}=0$ and so $\xi=z_{a}$, a contradiction to $\xi \notin \Omega$. Choose $r>0$ such that for all $z \in \Omega,|\sqrt{z-\xi}+a|>r>0$. Then consider

$$
\begin{equation*}
\psi(z) \equiv \frac{r}{\sqrt{z-\xi}+a} . \tag{55.3.7}
\end{equation*}
$$

This is one to one, analytic, and maps $\Omega$ into $B(0,1)(|\sqrt{z-\xi}+a|>r)$. Thus $\mathscr{F}$ is not empty and this proves the claim.

Claim 2: Let $z_{0} \in \Omega$. There exists a finite positive real number, $\eta$, defined by

$$
\begin{equation*}
\eta \equiv \sup \left\{\left|\psi^{\prime}\left(z_{0}\right)\right|: \psi \in \mathscr{F}\right\} \tag{55.3.8}
\end{equation*}
$$

and an analytic function, $h \in \mathscr{F}$ such that $\left|h^{\prime}\left(z_{0}\right)\right|=\eta$. Furthermore, $h\left(z_{0}\right)=0$.
Proof of Claim 2: First you show $\eta<\infty$. Let $\gamma(t)=z_{0}+r e^{i t}$ for $t \in[0,2 \pi]$ and $r$ is small enough that $B\left(z_{0}, r\right) \subseteq \Omega$. Then for $\psi \in \mathscr{F}$, the Cauchy integral formula for the derivative implies

$$
\psi^{\prime}\left(z_{0}\right)=\frac{1}{2 \pi i} \int_{\gamma} \frac{\psi(w)}{\left(w-z_{0}\right)^{2}} d w
$$

and so $\left|\psi^{\prime}\left(z_{0}\right)\right| \leq(1 / 2 \pi) 2 \pi r\left(1 / r^{2}\right)=1 / r$. Therefore, $\eta<\infty$ as desired. For $\psi$ defined above in 55.3.7

$$
\psi^{\prime}\left(z_{0}\right)=\frac{-r \phi^{\prime}\left(z_{0}\right)}{\left(\phi\left(z_{0}\right)+a\right)^{2}}=\frac{-r(1 / 2)\left(\sqrt{z_{0}-\xi}\right)^{-1}}{\left(\phi\left(z_{0}\right)+a\right)^{2}} \neq 0
$$

Therefore, $\eta>0$. It remains to verify the existence of the function, $h$.
By Theorem 55.3.1, there exists a sequence, $\left\{\psi_{n}\right\}$, of functions in $\mathscr{F}$ and an analytic function, $h$, such that

$$
\begin{equation*}
\left|\psi_{n}^{\prime}\left(z_{0}\right)\right| \rightarrow \eta \tag{55.3.9}
\end{equation*}
$$

and

$$
\begin{equation*}
\psi_{n} \rightarrow h, \psi_{n}^{\prime} \rightarrow h^{\prime} \tag{55.3.10}
\end{equation*}
$$

uniformly on all compact subsets of $\Omega$. It follows

$$
\begin{equation*}
\left|h^{\prime}\left(z_{0}\right)\right|=\lim _{n \rightarrow \infty}\left|\psi_{n}^{\prime}\left(z_{0}\right)\right|=\eta \tag{55.3.11}
\end{equation*}
$$

and for all $z \in \Omega$,

$$
\begin{equation*}
|h(z)|=\lim _{n \rightarrow \infty}\left|\psi_{n}(z)\right| \leq 1 \tag{55.3.12}
\end{equation*}
$$

By 55.3.11, $h$ is not a constant. Therefore, in fact, $|h(z)|<1$ for all $z \in \Omega$ in 55.3 .12 by the open mapping theorem.

Next it must be shown that $h$ is one to one in order to conclude $h \in \mathscr{F}$. Pick $z_{1} \in \Omega$ and suppose $z_{2}$ is another point of $\Omega$. Since the zeros of $h-h\left(z_{1}\right)$ have no limit point, there exists a circular contour bounding a circle which contains $z_{2}$ but not $z_{1}$ such that $\gamma^{*}$ contains no zeros of $h-h\left(z_{1}\right)$.


Using the theorem on counting zeros, Theorem 52.6.1, and the fact that $\psi_{n}$ is one to one,

$$
\begin{aligned}
0 & =\lim _{n \rightarrow \infty} \frac{1}{2 \pi i} \int_{\gamma} \frac{\psi_{n}^{\prime}(w)}{\psi_{n}(w)-\psi_{n}\left(z_{1}\right)} d w \\
& =\frac{1}{2 \pi i} \int_{\gamma} \frac{h^{\prime}(w)}{h(w)-h\left(z_{1}\right)} d w
\end{aligned}
$$

which shows that $h-h\left(z_{1}\right)$ has no zeros in $B\left(z_{2}, r\right)$. In particular $z_{2}$ is not a zero of $h-$ $h\left(z_{1}\right)$. This shows that $h$ is one to one since $z_{2} \neq z_{1}$ was arbitrary. Therefore, $h \in \mathscr{F}$. It only remains to verify that $h\left(z_{0}\right)=0$.

If $h\left(z_{0}\right) \neq 0$,consider $\phi_{h\left(z_{0}\right)} \circ h$ where $\phi_{\alpha}$ is the fractional linear transformation defined in Lemma 55.3.3. By this lemma it follows $\phi_{h\left(z_{0}\right)} \circ h \in \mathscr{F}$. Now using the chain rule,

$$
\begin{aligned}
\left|\left(\phi_{h\left(z_{0}\right)} \circ h\right)^{\prime}\left(z_{0}\right)\right| & =\left|\phi_{h\left(z_{0}\right)}^{\prime}\left(h\left(z_{0}\right)\right)\right|\left|h^{\prime}\left(z_{0}\right)\right| \\
& =\left|\frac{1}{1-\left|h\left(z_{0}\right)\right|^{2}}\right|\left|h^{\prime}\left(z_{0}\right)\right| \\
& =\left|\frac{1}{1-\left|h\left(z_{0}\right)\right|^{2}}\right| \eta>\eta
\end{aligned}
$$

Contradicting the definition of $\eta$. This proves Claim 2.
Claim 3: The function, $h$ just obtained maps $\Omega$ onto $B(0,1)$.
Proof of Claim 3: To show $h$ is onto, use the fractional linear transformation of Lemma 55.3.3. Suppose $h$ is not onto. Then there exists $\alpha \in B(0,1) \backslash h(\Omega)$. Then $0 \neq \phi_{\alpha} \circ h(z)$ for all $z \in \Omega$ because

$$
\phi_{\alpha} \circ h(z)=\frac{h(z)-\alpha}{1-\bar{\alpha} h(z)}
$$

and it is assumed $\alpha \notin h(\Omega)$. Therefore, since $\Omega$ has the square root property, you can consider an analytic function $z \rightarrow \sqrt{\phi_{\alpha} \circ h(z)}$. This function is one to one because both $\phi_{\alpha}$ and $h$ are. Also, the values of this function are in $B(0,1)$ by Lemma 55.3.3 so it is in $\mathscr{F}$.

Now let

$$
\begin{equation*}
\psi \equiv \phi_{\sqrt{\phi_{\alpha} \circ h\left(z_{0}\right)}} \circ \sqrt{\phi_{\alpha} \circ h} \tag{55.3.13}
\end{equation*}
$$

Thus

$$
\psi\left(z_{0}\right)=\phi \sqrt{\phi_{\alpha} \circ h\left(z_{0}\right)} \circ \sqrt{\phi_{\alpha} \circ h\left(z_{0}\right)}=0
$$

and $\psi$ is a one to one mapping of $\Omega$ into $B(0,1)$ so $\psi$ is also in $\mathscr{F}$. Therefore,

$$
\begin{equation*}
\left|\psi^{\prime}\left(z_{0}\right)\right| \leq \eta,\left|\left(\sqrt{\phi_{\alpha} \circ h}\right)^{\prime}\left(z_{0}\right)\right| \leq \eta \tag{55.3.14}
\end{equation*}
$$

Define $s(w) \equiv w^{2}$. Then using Lemma 55.3.3, in particular, the description of $\phi_{\alpha}^{-1}=\phi_{-\alpha}$, you can solve 55.3.13 for $h$ to obtain

$$
\begin{align*}
h(z) & =\phi_{-\alpha} \circ S \circ \phi_{-\sqrt{\phi_{\alpha} \circ h\left(z_{0}\right)}} \circ \psi \\
& =(\overbrace{\left.\phi_{-\alpha} \circ S \circ \phi_{-\sqrt{\phi_{\alpha} \circ h\left(z_{0}\right)}} \circ \psi\right)(z)}=F \\
& =(F \circ \psi)(z) \tag{55.3.15}
\end{align*}
$$

Now

$$
F(0)=\phi_{-\alpha} \circ s \circ \phi_{-\sqrt{\phi_{\alpha} \circ h\left(z_{0}\right)}}(0)=\phi_{\alpha}^{-1}\left(\phi_{\alpha} \circ h\left(z_{0}\right)\right)=h\left(z_{0}\right)=0
$$

and $F$ maps $B(0,1)$ into $B(0,1)$. Also, $F$ is not one to one because it maps $B(0,1)$ to $B(0,1)$ and has $s$ in its definition. Thus there exists $z_{1} \in B(0,1)$ such that $\phi_{-\sqrt{\phi_{\alpha} \circ h\left(z_{0}\right)}}\left(z_{1}\right)=$ $-\frac{1}{2}$ and another point $z_{2} \in B(0,1)$ such that $\phi_{-\sqrt{\phi_{\alpha} \circ h\left(z_{0}\right)}}\left(z_{2}\right)=\frac{1}{2}$. However, thanks to $s, F\left(z_{1}\right)=F\left(z_{2}\right)$.

Since $F(0)=h\left(z_{0}\right)=0$, you can apply the Schwarz lemma to $F$. Since $F$ is not one to one, it can't be true that $F(z)=\lambda z$ for $|\lambda|=1$ and so by the Schwarz lemma it must be the case that $\left|F^{\prime}(0)\right|<1$. But this implies from 55.3 .15 and 55.3.14 that

$$
\begin{aligned}
\eta & =\left|h^{\prime}\left(z_{0}\right)\right|=\left|F^{\prime}\left(\psi\left(z_{0}\right)\right)\right|\left|\psi^{\prime}\left(z_{0}\right)\right| \\
& =\left|F^{\prime}(0)\right|\left|\psi^{\prime}\left(z_{0}\right)\right|<\left|\psi^{\prime}\left(z_{0}\right)\right| \leq \eta
\end{aligned}
$$

a contradiction. This proves the theorem.
The following lemma yields the usual form of the Riemann mapping theorem.

Lemma 55.3.7 Let $\Omega$ be a simply connected region with $\Omega \neq \mathbb{C}$. Then $\Omega$ has the square root property.

Proof: Let $f$ and $\frac{1}{f}$ both be analytic on $\Omega$. Then $\frac{f^{\prime}}{f}$ is analytic on $\Omega$ so by Corollary 51.7.23, there exists $\widetilde{F}$, analytic on $\Omega$ such that $\widetilde{F}^{\prime}=\frac{f^{\prime}}{f}$ on $\Omega$. Then $\left(f e^{-\widetilde{F}}\right)^{\prime}=0$ and so $f(z)=C e^{\widetilde{F}}=e^{a+i b} e^{\widetilde{F}}$. Now let $F=\widetilde{F}+a+i b$. Then $F$ is still a primitive of $f^{\prime} / f$ and $f(z)=e^{F(z)}$. Now let $\phi(z) \equiv e^{\frac{1}{2} F(z)}$. Then $\phi$ is the desired square root and so $\Omega$ has the square root property.

Corollary 55.3.8 (Riemann mapping theorem) Let $\Omega$ be a simply connected region with $\Omega \neq \mathbb{C}$ and let $z_{0} \in \Omega$. Then there exists a function, $f: \Omega \rightarrow B(0,1)$ such that $f$ is one to one, analytic, and onto with $f\left(z_{0}\right)=0$. Furthermore, $f^{-1}$ is also analytic.

Proof: From Theorem 55.3.6 and Lemma 55.3.7 there exists a function, $f: \Omega \rightarrow B(0,1)$ which is one to one, onto, and analytic such that $f\left(z_{0}\right)=0$. The assertion that $f^{-1}$ is analytic follows from the open mapping theorem.

### 55.4 Analytic Continuation

### 55.4.1 Regular And Singular Points

Given a function which is analytic on some set, can you extend it to an analytic function defined on a larger set? Sometimes you can do this. It was done in the proof of the Cauchy integral formula. There are also reflection theorems like those discussed in the exercises starting with Problem 10 on Page 1647. Here I will give a systematic way of extending an analytic function to a larger set. I will emphasize simply connected regions. The subject of analytic continuation is much larger than the introduction given here. A good source for much more on this is found in Alfors [3]. The approach given here is suggested by Rudin [113] and avoids many of the standard technicalities.

Definition 55.4.1 Let $f$ be analytic on $B(a, r)$ and let $\beta \in \partial B(a, r)$. Then $\beta$ is called $a$ regular point of $f$ if there exists some $\delta>0$ and a function, $g$ analytic on $B(\beta, \delta)$ such that $g=f$ on $B(\beta, \delta) \cap B(a, r)$. Those points of $\partial B(a, r)$ which are not regular are called singular.


Theorem 55.4.2 Suppose $f$ is analytic on $B(a, r)$ and the power series

$$
f(z)=\sum_{k=0}^{\infty} a_{k}(z-a)^{k}
$$

has radius of convergence $r$. Then there exists a singular point on $\partial B(a, r)$.
Proof: If not, then for every $z \in \partial B(a, r)$ there exists $\delta_{z}>0$ and $g_{z}$ analytic on $B\left(z, \delta_{z}\right)$ such that $g_{z}=f$ on $B\left(z, \delta_{z}\right) \cap B(a, r)$. Since $\partial B(a, r)$ is compact, there exist $z_{1}, \cdots, z_{n}$, points in $\partial B(a, r)$ such that $\left\{B\left(z_{k}, \delta_{z_{k}}\right)\right\}_{k=1}^{n}$ covers $\partial B(a, r)$. Now define

$$
g(z) \equiv\left\{\begin{array}{l}
f(z) \text { if } z \in B(a, r) \\
g_{z_{k}}(z) \text { if } z \in B\left(z_{k}, \delta_{z_{k}}\right)
\end{array}\right.
$$

Is this well defined? If $z \in B\left(z_{i}, \delta_{z_{i}}\right) \cap B\left(z_{j}, \delta_{z_{j}}\right)$, is $g_{z_{i}}(z)=g_{z_{j}}(z)$ ? Consider the following picture representing this situation.


You see that if $z \in B\left(z_{i}, \delta_{z_{i}}\right) \cap B\left(z_{j}, \delta_{z_{j}}\right)$ then $I \equiv B\left(z_{i}, \delta_{z_{i}}\right) \cap B\left(z_{j}, \delta_{z_{j}}\right) \cap B(a, r)$ is a nonempty open set. Both $g_{z_{i}}$ and $g_{z_{j}}$ equal $f$ on $I$. Therefore, they must be equal on $B\left(z_{i}, \delta_{z_{i}}\right) \cap B\left(z_{j}, \delta_{z_{j}}\right)$ because $I$ has a limit point. Therefore, $g$ is well defined and analytic on an open set containing $\overline{B(a, r)}$. Since $g$ agrees with $f$ on $B(a, r)$, the power series for $g$ is the same as the power series for $f$ and converges on a ball which is larger than $B(a, r)$ contrary to the assumption that the radius of convergence of the above power series equals $r$. This proves the theorem.

### 55.4.2 Continuation Along A Curve

Next I will describe what is meant by continuation along a curve. The following definition is standard and is found in Rudin [113].

Definition 55.4.3 A function element is an ordered pair, $(f, D)$ where $D$ is an open ball and $f$ is analytic on $D$. $\left(f_{0}, D_{0}\right)$ and $\left(f_{1}, D_{1}\right)$ are direct continuations of each other if $D_{1} \cap D_{0} \neq \emptyset$ and $f_{0}=f_{1}$ on $D_{1} \cap D_{0}$. In this case I will write $\left(f_{0}, D_{0}\right) \sim\left(f_{1}, D_{1}\right)$. A chain is a finite sequence, of disks, $\left\{D_{0}, \cdots, D_{n}\right\}$ such that $D_{i-1} \cap D_{i} \neq \emptyset$. If $\left(f_{0}, D_{0}\right)$ is a given function element and there exist function elements, $\left(f_{i}, D_{i}\right)$ such that $\left\{D_{0}, \cdots, D_{n}\right\}$ is a chain and $\left(f_{j-1}, D_{j-1}\right) \sim\left(f_{j}, D_{j}\right)$ then $\left(f_{n}, D_{n}\right)$ is called the analytic continuation of $\left(f_{0}, D_{0}\right)$ along the chain $\left\{D_{0}, \cdots, D_{n}\right\}$. Now suppose $\gamma$ is an oriented curve with parameter interval $[a, b]$ and there exists a chain, $\left\{D_{0}, \cdots, D_{n}\right\}$ such that $\gamma^{*} \subseteq \cup_{k=1}^{n} D_{k}, \gamma(a)$ is the center of $D_{0}, \gamma(b)$ is the center of $D_{n}$, and there is an increasing list of numbers in $[a, b], a=$ $s_{0}<s_{1} \cdots<s_{n}=b$ such that $\gamma\left(\left[s_{i}, s_{i+1}\right]\right) \subseteq D_{i}$ and $\left(f_{n}, D_{n}\right)$ is an analytic continuation of $\left(f_{0}, D_{0}\right)$ along the chain. Then $\left(f_{n}, D_{n}\right)$ is called an analytic continuation of $\left(f_{0}, D_{0}\right)$ along the curve $\gamma$. ( $\gamma$ will always be a continuous curve. Nothing more is needed.)

In the above situation it does not follow that if $D_{n} \cap D_{0} \neq \emptyset$, that $f_{n}=f_{0}$ ! However, there are some cases where this will happen. This is the monodromy theorem which follows. This is as far as I will go on the subject of analytic continuation. For more on this subject including a development of the concept of Riemann surfaces, see Alfors [3].

Lemma 55.4.4 Suppose $(f, B(0, r))$ for $r<1$ is a function element and $(f, B(0, r))$ can be analytically continued along every curve in $B(0,1)$ that starts at 0 . Then there exists an analytic function, $g$ defined on $B(0,1)$ such that $g=f$ on $B(0, r)$.

Proof: Let

$$
\begin{aligned}
R= & \sup \left\{r_{1} \geq r \text { such that there exists } g_{r_{1}}\right. \\
& \text { analytic on } \left.B\left(0, r_{1}\right) \text { which agrees with } f \text { on } B(0, r) \cdot\right\}
\end{aligned}
$$

Define $g_{R}(z) \equiv g_{r_{1}}(z)$ where $|z|<r_{1}$. This is well defined because if you use $r_{1}$ and $r_{2}$, both $g_{r_{1}}$ and $g_{r_{2}}$ agree with $f$ on $B(0, r)$, a set with a limit point and so the two functions agree at every point in both $B\left(0, r_{1}\right)$ and $B\left(0, r_{2}\right)$. Thus $g_{R}$ is analytic on $B(0, R)$. If $R<1$, then by the assumption there are no singular points on $B(0, R)$ and so Theorem 55.4.2 implies the radius of convergence of the power series for $g_{R}$ is larger than $R$ contradicting the choice of $R$. Therefore, $R=1$ and this proves the lemma. Let $g=g_{R}$.

The following theorem is the main result in this subject, the monodromy theorem.
Theorem 55.4.5 Let $\Omega$ be a simply connected proper subset of $\mathbb{C}$ and suppose $(f, B(a, r))$ is a function element with $B(a, r) \subseteq \Omega$. Suppose also that this function element can be analytically continued along every curve through $a$. Then there exists $G$ analytic on $\Omega$ such that $G$ agrees with $f$ on $B(a, r)$.

Proof: By the Riemann mapping theorem, there exists $h: \Omega \rightarrow B(0,1)$ which is analytic, one to one and onto such that $f(a)=0$. Since $h$ is an open map, there exists $\delta>0$ such that

$$
B(0, \delta) \subseteq h(B(a, r))
$$

It follows $f \circ h^{-1}$ can be analytically continued along every curve through 0 . By Lemma 55.4.4 there exists $g$ analytic on $B(0,1)$ which agrees with $f \circ h^{-1}$ on $B(0, \delta)$. Define $G(z) \equiv g(h(z))$. For $z=h^{-1}(w)$, it follows $G\left(h^{-1}(w)\right)=g(w)$. If $w \in B(0, \delta)$, then $G\left(h^{-1}(w)\right)=f \circ h^{-1}(w)$ and so $G=f$ on $h^{-1}(B(0, \delta))$, an open set contained in $B(a, r)$. Therefore, $G=f$ on $B(a, r)$ because $h^{-1}(B(0, \delta))$ has a limit point. This proves the theorem.

Actually, you sometimes want to consider the case where $\Omega=\mathbb{C}$. This requires a small modification to obtain from the above theorem.

Corollary 55.4.6 Suppose $(f, B(a, r))$ is a function element with $B(a, r) \subseteq \mathbb{C}$. Suppose also that this function element can be analytically continued along every curve through $a$. Then there exists $G$ analytic on $\mathbb{C}$ such that $G$ agrees with $f$ on $B(a, r)$.

Proof: Let $\Omega_{1} \equiv\{z \in \mathbb{C}: a+i t: t>a\}$ and $\Omega_{2} \equiv\{z \in \mathbb{C}: a-i t: t>a\}$. Here is a picture of $\Omega_{1}$.


A picture of $\Omega_{2}$ is similar except the line extends down from the boundary of $B(a, r)$.
Thus $B(a, r) \subseteq \Omega_{i}$ and $\Omega_{i}$ is simply connected and proper. By Theorem 55.4.5 there exist analytic functions, $G_{i}$ analytic on $\Omega_{i}$ such that $G_{i}=f$ on $B(a, r)$. Thus $G_{1}=G_{2}$ on $B(a, r)$, a set with a limit point. Therefore, $G_{1}=G_{2}$ on $\Omega_{1} \cap \Omega_{2}$. Now let $G(z)=G_{i}(z)$ where $z \in \Omega_{i}$. This is well defined and analytic on $\mathbb{C}$. This proves the corollary.

### 55.5 The Picard Theorems

The Picard theorem says that if $f$ is an entire function and there are two complex numbers not contained in $f(\mathbb{C})$, then $f$ is constant. This is certainly one of the most amazing things which could be imagined. However, this is only the little Picard theorem. The big Picard theorem is even more incredible. This one asserts that to be non constant the entire function must take every value of $\mathbb{C}$ but two infinitely many times! I will begin with the little Picard theorem. The method of proof I will use is the one found in Saks and Zygmund [115], Conway [32] and Hille [65]. This is not the way Picard did it in 1879. That approach is very different and is presented at the end of the material on elliptic functions. This approach is much more recent dating it appears from around 1924.

Lemma 55.5.1 Let $f$ be analytic on a region containing $\overline{B(0, r)}$ and suppose

$$
\left|f^{\prime}(0)\right|=b>0, f(0)=0
$$

and $|f(z)| \leq M$ for all $z \in \overline{B(0, r)}$. Then $f(B(0, r)) \supseteq B\left(0, \frac{r^{2} b^{2}}{6 M}\right)$.

Proof: By assumption,

$$
\begin{equation*}
f(z)=\sum_{k=0}^{\infty} a_{k} z^{k},|z| \leq r \tag{55.5.16}
\end{equation*}
$$

Then by the Cauchy integral formula for the derivative,

$$
a_{k}=\frac{1}{2 \pi i} \int_{\partial B(0, r)} \frac{f(w)}{w^{k+1}} d w
$$

where the integral is in the counter clockwise direction. Therefore,

$$
\left|a_{k}\right| \leq \frac{1}{2 \pi} \int_{0}^{2 \pi} \frac{\left|f\left(r e^{i \theta}\right)\right|}{r^{k}} d \theta \leq \frac{M}{r^{k}}
$$

In particular, $b r \leq M$. Therefore, from 55.5.16

$$
\begin{aligned}
|f(z)| & \geq b|z|-\sum_{k=2}^{\infty} \frac{M}{r^{k}}|z|^{k}=b|z|-\frac{M\left(\frac{|z|}{r}\right)^{2}}{1-\frac{|z|}{r}} \\
& =b|z|-\frac{M|z|^{2}}{r^{2}-r|z|}
\end{aligned}
$$

Suppose $|z|=\frac{r^{2} b}{4 M}<r$. Then this is no larger than

$$
\frac{1}{4} b^{2} r^{2} \frac{3 M-b r}{M(4 M-b r)} \geq \frac{1}{4} b^{2} r^{2} \frac{3 M-M}{M(4 M-M)}=\frac{r^{2} b^{2}}{6 M}
$$

Let $|w|<\frac{r^{2} b}{4 M}$. Then for $|z|=\frac{r^{2} b}{4 M}$ and the above,

$$
|w|=|(f(z)-w)-f(z)|<\frac{r^{2} b}{4 M} \leq|f(z)|
$$

and so by Rouche's theorem, $z \rightarrow f(z)-w$ and $z \rightarrow f(z)$ have the same number of zeros in $B\left(0, \frac{r^{2} b}{4 M}\right)$. But $f$ has at least one zero in this ball and so this shows there exists at least one $z \in B\left(0, \frac{r^{2} b}{4 M}\right)$ such that $f(z)-w=0$. This proves the lemma.

### 55.5.1 Two Competing Lemmas

Lemma 55.5 .1 is a really nice lemma but there is something even better, Bloch's lemma. This lemma does not depend on the bound of $f$. Like the above two lemmas it is interesting for its own sake and in addition is the key to a fairly short proof of Picard's theorem. It features the number $\frac{1}{24}$. The best constant is not currently known.

Lemma 55.5.2 Let $f$ be analytic on an open set containing $\overline{B(0, R)}$ and suppose $\left|f^{\prime}(0)\right|>$ 0 . Then there exists $a \in B(0, R)$ such that

$$
f(B(0, R)) \supseteq B\left(f(a), \frac{\left|f^{\prime}(0)\right| R}{24}\right) .
$$

Proof: Let $K(\rho) \equiv \max \left\{\left|f^{\prime}(z)\right|:|z|=\rho\right\}$. For simplicity, let $C_{\rho} \equiv\{z:|z|=\rho\}$.
Claim: $K$ is continuous from the left.
Proof of claim: Let $z_{\rho} \in C_{\rho}$ such that $\left|f^{\prime}\left(z_{\rho}\right)\right|=K(\rho)$. Then by the maximum modulus theorem, if $\lambda \in(0,1)$,

$$
\left|f^{\prime}\left(\lambda z_{\rho}\right)\right| \leq K(\lambda \rho) \leq K(\rho)=\left|f^{\prime}\left(z_{\rho}\right)\right|
$$

Letting $\lambda \rightarrow 1$ yields the claim.
Let $\rho_{0}$ be the largest such that

$$
\left(R-\rho_{0}\right) K\left(\rho_{0}\right)=R\left|f^{\prime}(0)\right|
$$

(Note $(R-0) K(0)=R\left|f^{\prime}(0)\right|$.) Thus $\rho_{0}<R$ because $(R-R) K(R)=0$. Let $|a|=\rho_{0}$ such that $\left|f^{\prime}(a)\right|=K\left(\rho_{0}\right)$. Thus

$$
\begin{equation*}
\left|f^{\prime}(a)\right|\left(R-\rho_{0}\right)=\left|f^{\prime}(0)\right| R \tag{55.5.17}
\end{equation*}
$$

Now let $r=\frac{R-\rho_{0}}{2}$. From 55.5.17,

$$
\begin{equation*}
\left|f^{\prime}(a)\right| r=\frac{1}{2}\left|f^{\prime}(0)\right| R, B(a, r) \subseteq B\left(0, \rho_{0}+r\right) \subseteq B(0, R) \tag{55.5.18}
\end{equation*}
$$



Therefore, if $z \in B(a, r)$, it follows from the maximum modulus theorem and the definition of $\rho_{0}$ that

$$
\begin{align*}
\left|f^{\prime}(z)\right| & \leq K\left(\rho_{0}+r\right)<\frac{R\left|f^{\prime}(0)\right|}{R-\rho_{0}-r}=\frac{2 R\left|f^{\prime}(0)\right|}{R-\rho_{0}} \\
& =\frac{2 R\left|f^{\prime}(0)\right|}{2 r}=\frac{R\left|f^{\prime}(0)\right|}{r} \tag{55.5.19}
\end{align*}
$$

$\frac{\text { Let }}{B(0, r)} g(z)=f(a+z)-f(a)$ where $z \in B(0, r)$. Then $\left|g^{\prime}(0)\right|=\left|f^{\prime}(a)\right|>0$ and for $z \in$ $\overline{B(0, r)}$,

$$
|g(z)| \leq\left|\int_{\gamma(a, z)} g^{\prime}(w) d w\right| \leq|z-a| \frac{R\left|f^{\prime}(0)\right|}{r}=R\left|f^{\prime}(0)\right|
$$

By Lemma 55.5.1 and 55.5.18,

$$
\begin{aligned}
g(B(0, r)) & \supseteq B\left(0, \frac{r^{2}\left|f^{\prime}(a)\right|^{2}}{6 R\left|f^{\prime}(0)\right|}\right) \\
& =B\left(0, \frac{r^{2}\left(\frac{1}{2 r}\left|f^{\prime}(0)\right| R\right)^{2}}{6 R\left|f^{\prime}(0)\right|}\right)=B\left(0, \frac{\left|f^{\prime}(0)\right| R}{24}\right)
\end{aligned}
$$

Now $g(B(0, r))=f(B(a, r))-f(a)$ and so this implies

$$
f(B(0, R)) \supseteq f(B(a, r)) \supseteq B\left(f(a), \frac{\left|f^{\prime}(0)\right| R}{24}\right) .
$$

This proves the lemma.
Here is a slightly more general version which allows the center of the open set to be arbitrary.

Lemma 55.5.3 Let $f$ be analytic on an open set containing $\overline{B\left(z_{0}, R\right)}$ and suppose $\left|f^{\prime}\left(z_{0}\right)\right|>$ 0 . Then there exists $a \in B\left(z_{0}, R\right)$ such that

$$
f\left(B\left(z_{0}, R\right)\right) \supseteq B\left(f(a), \frac{\left|f^{\prime}\left(z_{0}\right)\right| R}{24}\right)
$$

Proof: You look at $g(z) \equiv f\left(z_{0}+z\right)-f\left(z_{0}\right)$ for $z \in B(0, R)$. Then $g^{\prime}(0)=f^{\prime}\left(z_{0}\right)$ and so by Lemma 55.5.2 there exists $a_{1} \in B(0, R)$ such that

$$
g(B(0, R)) \supseteq B\left(g\left(a_{1}\right), \frac{\left|f^{\prime}\left(z_{0}\right)\right| R}{24}\right)
$$

Now $g(B(0, R))=f\left(B\left(z_{0}, R\right)\right)-f\left(z_{0}\right)$ and $g\left(a_{1}\right)=f(a)-f\left(z_{0}\right)$ for some $a \in B\left(z_{0}, R\right)$ and so

$$
\begin{aligned}
f\left(B\left(z_{0}, R\right)\right)-f\left(z_{0}\right) & \supseteq B\left(g\left(a_{1}\right), \frac{\left|f^{\prime}\left(z_{0}\right)\right| R}{24}\right) \\
& =B\left(f(a)-f\left(z_{0}\right), \frac{\left|f^{\prime}\left(z_{0}\right)\right| R}{24}\right)
\end{aligned}
$$

which implies

$$
f\left(B\left(z_{0}, R\right)\right) \supseteq B\left(f(a), \frac{\left|f^{\prime}\left(z_{0}\right)\right| R}{24}\right)
$$

as claimed. This proves the lemma.
No attempt was made to find the best number to multiply by $R\left|f^{\prime}\left(z_{0}\right)\right|$. A discussion of this is given in Conway [32]. See also [65]. Much larger numbers than $1 / 24$ are available and there is a conjecture due to Alfors about the best value. The conjecture is that $1 / 24$ can be replaced with

$$
\frac{\Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{11}{12}\right)}{(1+\sqrt{3})^{1 / 2} \Gamma\left(\frac{1}{4}\right)} \approx .47186
$$

You can see there is quite a gap between the constant for which this lemma is proved above and what is thought to be the best constant.

Bloch's lemma above gives the existence of a ball of a certain size inside the image of a ball. By contrast the next lemma leads to conditions under which the values of a function do not contain a ball of certain radius. It concerns analytic functions which do not achieve the values 0 and 1 .

Lemma 55.5.4 Let $\mathscr{F}$ denote the set of functions, $f$ defined on $\Omega$, a simply connected region which do not achieve the values 0 and 1. Then for each such function, it is possible to define a function analytic on $\Omega, H(z)$ by the formula

$$
H(z) \equiv \log \left[\sqrt{\frac{\log (f(z))}{2 \pi i}}-\sqrt{\frac{\log (f(z))}{2 \pi i}-1}\right]
$$

There exists a constant $C$ independent of $f \in \mathscr{F}$ such that $H(\Omega)$ does not contain any ball of radius $C$.

Proof: Let $f \in \mathscr{F}$. Then since $f$ does not take the value 0 , there exists $g_{1}$ a primitive of $f^{\prime} / f$. Thus

$$
\frac{d}{d z}\left(e^{-g_{1}} f\right)=0
$$

so there exists $a, b$ such that $f(z) e^{-g_{1}(z)}=e^{a+b i}$. Letting $g(z)=g_{1}(z)+a+i b$, it follows $e^{g(z)}=f(z)$. Let $\log (f(z))=g(z)$. Then for $n \in \mathbb{Z}$, the integers,

$$
\frac{\log (f(z))}{2 \pi i}, \frac{\log (f(z))}{2 \pi i}-1 \neq n
$$

because if equality held, then $f(z)=1$ which does not happen. It follows $\frac{\log (f(z))}{2 \pi i}$ and $\frac{\log (f(z))}{2 \pi i}-1$ are never equal to zero. Therefore, using the same reasoning, you can define a logarithm of these two quantities and therefore, a square root. Hence there exists a function analytic on $\Omega$,

$$
\begin{equation*}
\sqrt{\frac{\log (f(z))}{2 \pi i}}-\sqrt{\frac{\log (f(z))}{2 \pi i}-1} \tag{55.5.20}
\end{equation*}
$$

For $n$ a positive integer, this function cannot equal $\sqrt{n} \pm \sqrt{n-1}$ because if it did, then

$$
\begin{equation*}
\left(\sqrt{\frac{\log (f(z))}{2 \pi i}}-\sqrt{\frac{\log (f(z))}{2 \pi i}-1}\right)=\sqrt{n} \pm \sqrt{n-1} \tag{55.5.21}
\end{equation*}
$$

and you could take reciprocals of both sides to obtain

$$
\begin{equation*}
\left(\sqrt{\frac{\log (f(z))}{2 \pi i}}+\sqrt{\frac{\log (f(z))}{2 \pi i}-1}\right)=\sqrt{n} \mp \sqrt{n-1} \tag{55.5.22}
\end{equation*}
$$

Then adding 55.5.21 and 55.5.22

$$
2 \sqrt{\frac{\log (f(z))}{2 \pi i}}=2 \sqrt{n}
$$

which contradicts the above observation that $\frac{\log (f(z))}{2 \pi i}$ is not equal to an integer.
Also, the function of 55.5 .20 is never equal to zero. Therefore, you can define the logarithm of this function also. It follows

$$
H(z) \equiv \log \left(\sqrt{\frac{\log (f(z))}{2 \pi i}}-\sqrt{\frac{\log (f(z))}{2 \pi i}-1}\right) \neq \ln (\sqrt{n} \pm \sqrt{n-1})+2 m \pi i
$$

where $m$ is an arbitrary integer and $n$ is a positive integer. Now

$$
\lim _{n \rightarrow \infty} \ln (\sqrt{n}+\sqrt{n-1})=\infty
$$

and $\lim _{n \rightarrow \infty} \ln (\sqrt{n}-\sqrt{n-1})=-\infty$ and so $\mathbb{C}$ is covered by rectangles having vertices at points $\ln (\sqrt{n} \pm \sqrt{n-1})+2 m \pi i$ as described above. Each of these rectangles has height equal to $2 \pi$ and a short computation shows their widths are bounded. Therefore, there exists $C$ independent of $f \in \mathscr{F}$ such that $C$ is larger than the diameter of all these rectangles. Hence $H(\Omega)$ cannot contain any ball of radius larger than $C$.

### 55.5.2 The Little Picard Theorem

Now here is the little Picard theorem. It is easy to prove from the above.
Theorem 55.5.5 If $h$ is an entire function which omits two values then $h$ is a constant.
Proof: Suppose the two values omitted are $a$ and $b$ and that $h$ is not constant. Let $f(z)=(h(z)-a) /(b-a)$. Then $f$ omits the two values 0 and 1 . Let $H$ be defined in Lemma 55.5.4. Then $H(z)$ is clearly not of the form $a z+b$ because then it would have values equal to the vertices $\ln (\sqrt{n} \pm \sqrt{n-1})+2 m \pi i$ or else be constant neither of which happen if $h$ is not constant. Therefore, by Liouville's theorem, $H^{\prime}$ must be unbounded. Pick $\xi$ such that $\left|H^{\prime}(\xi)\right|>24 C$ where $C$ is such that $H(\mathbb{C})$ contains no balls of radius larger than $C$. But by Lemma 55.5.3 $H(B(\xi, 1))$ must contain a ball of radius $\frac{\left|H^{\prime}(\xi)\right|}{24}>\frac{24 C}{24}=C$, a contradiction. This proves Picard's theorem.

The following is another formulation of this theorem.
Corollary 55.5.6 If $f$ is a meromophic function defined on $\mathbb{C}$ which omits three distinct values, $a, b, c$, then $f$ is a constant.

Proof: Let $\phi(z) \equiv \frac{z-a}{z-c} \frac{b-c}{b-a}$. Then $\phi(c)=\infty, \phi(a)=0$, and $\phi(b)=1$. Now consider the function, $h=\phi \circ f$. Then $h$ misses the three points $\infty, 0$, and 1 . Since $h$ is meromorphic and does not have $\infty$ in its values, it must actually be analytic. Thus $h$ is an entire function which misses the two values 0 and 1 . Therefore, $h$ is constant by Theorem 55.5.5.

### 55.5.3 Schottky's Theorem

Lemma 55.5.7 Let $f$ be analytic on an open set containing $\overline{B(0, R)}$ and suppose that $f$ does not take on either of the two values 0 or 1 . Also suppose $|f(0)| \leq \beta$. Then letting $\theta \in(0,1)$, it follows

$$
|f(z)| \leq M(\beta, \theta)
$$

for all $z \in B(0, \theta R)$, where $M(\beta, \theta)$ is a function of only the two variables $\beta$, $\theta$. (In particular, there is no dependence on $R$.)

Proof: Consider the function, $H(z)$ used in Lemma 55.5.4 given by

$$
\begin{equation*}
H(z) \equiv \log \left(\sqrt{\frac{\log (f(z))}{2 \pi i}}-\sqrt{\frac{\log (f(z))}{2 \pi i}-1}\right) \tag{55.5.23}
\end{equation*}
$$

You notice there are two explicit uses of logarithms. Consider first the logarithm inside the radicals. Choose this logarithm such that

$$
\begin{equation*}
\log (f(0))=\ln |f(0)|+i \arg (f(0)), \arg (f(0)) \in(-\pi, \pi] . \tag{55.5.24}
\end{equation*}
$$

You can do this because

$$
e^{\log (f(0))}=f(0)=e^{\ln |f(0)|} e^{i \alpha}=e^{\ln |f(0)|+i \alpha}
$$

and by replacing $\alpha$ with $\alpha+2 m \pi$ for a suitable integer, $m$ it follows the above equation still holds. Therefore, you can assume 55.5.24. Similar reasoning applies to the logarithm on the outside of the parenthesis. It can be assumed $H(0)$ equals

$$
\begin{equation*}
\ln \left|\sqrt{\frac{\log (f(0))}{2 \pi i}}-\sqrt{\frac{\log (f(0))}{2 \pi i}-1}\right|+i \arg \left(\sqrt{\frac{\log (f(0))}{2 \pi i}}-\sqrt{\frac{\log (f(0))}{2 \pi i}-1}\right) \tag{55.5.25}
\end{equation*}
$$

where the imaginary part is no larger than $\pi$ in absolute value.
Now if $\xi \in B(0, R)$ is a point where $H^{\prime}(\xi) \neq 0$, then by Lemma 55.5.2

$$
H(B(\xi, R-|\xi|)) \supseteq B\left(H(a), \frac{\left|H^{\prime}(\xi)\right|(R-|\xi|)}{24}\right)
$$

where $a$ is some point in $B(\xi, R-|\xi|)$. But by Lemma 55.5.4 $H(B(\xi, R-|\xi|))$ contains no balls of radius $C$ where $C$ depended only on the maximum diameters of those rectangles having vertices $\ln (\sqrt{n} \pm \sqrt{n-1})+2 m \pi i$ for $n$ a positive integer and $m$ an integer. Therefore,

$$
\frac{\left|H^{\prime}(\xi)\right|(R-|\xi|)}{24}<C
$$

and consequently

$$
\left|H^{\prime}(\xi)\right|<\frac{24 C}{R-|\xi|}
$$

Even if $H^{\prime}(\xi)=0$, this inequality still holds. Therefore, if $z \in B(0, R)$ and $\gamma(0, z)$ is the straight segment from 0 to $z$,

$$
\begin{aligned}
|H(z)-H(0)| & =\left|\int_{\gamma(0, z)} H^{\prime}(w) d w\right|=\left|\int_{0}^{1} H^{\prime}(t z) z d t\right| \\
& \leq \int_{0}^{1}\left|H^{\prime}(t z) z\right| d t \leq \int_{0}^{1} \frac{24 C}{R-t|z|}|z| d t \\
& =24 C \ln \left(\frac{R}{R-|z|}\right)
\end{aligned}
$$

Therefore, for $z \in \partial B(0, \theta R)$,

$$
\begin{equation*}
|H(z)| \leq|H(0)|+24 C \ln \left(\frac{1}{1-\theta}\right) \tag{55.5.26}
\end{equation*}
$$

By the maximum modulus theorem, the above inequality holds for all $|z|<\theta R$ also.
Next I will use 55.5.23 to get an inequality for $|f(z)|$ in terms of $|H(z)|$. From 55.5.23,

$$
H(z)=\log \left(\sqrt{\frac{\log (f(z))}{2 \pi i}}-\sqrt{\frac{\log (f(z))}{2 \pi i}-1}\right)
$$

and so

$$
\begin{aligned}
2 H(z) & =\log \left(\sqrt{\frac{\log (f(z))}{2 \pi i}}-\sqrt{\frac{\log (f(z))}{2 \pi i}-1}\right)^{2} \\
-2 H(z) & =\log \left(\sqrt{\frac{\log (f(z))}{2 \pi i}}-\sqrt{\frac{\log (f(z))}{2 \pi i}-1}\right)^{-2} \\
& =\log \left(\sqrt{\frac{\log (f(z))}{2 \pi i}}+\sqrt{\frac{\log (f(z))}{2 \pi i}-1}\right)^{2}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \left(\sqrt{\frac{\log (f(z))}{2 \pi i}}+\sqrt{\frac{\log (f(z))}{2 \pi i}-1}\right)^{2} \\
& +\left(\sqrt{\frac{\log (f(z))}{2 \pi i}}-\sqrt{\frac{\log (f(z))}{2 \pi i}-1}\right)^{2} \\
= & \exp (2 H(z))+\exp (-2 H(z))
\end{aligned}
$$

and

$$
\left(\frac{\log (f(z))}{\pi i}-1\right)=\frac{1}{2}(\exp (2 H(z))+\exp (-2 H(z)))
$$

Thus

$$
\log (f(z))=\pi i+\frac{\pi i}{2}(\exp (2 H(z))+\exp (-2 H(z)))
$$

which shows

$$
\begin{aligned}
|f(z)| & =\left|\exp \left[\frac{\pi i}{2}(\exp (2 H(z))+\exp (-2 H(z)))\right]\right| \\
& \leq \exp \left|\frac{\pi i}{2}(\exp (2 H(z))+\exp (-2 H(z)))\right| \\
& \leq \exp \left|\frac{\pi}{2}(|\exp (2 H(z))|+|\exp (-2 H(z))|)\right| \\
& \leq \exp \left|\frac{\pi}{2}(\exp (2|H(z)|)+\exp (|-2 H(z)|))\right| \\
& =\exp (\pi \exp 2|H(z)|)
\end{aligned}
$$

Now from 55.5.26 this is dominated by

$$
\begin{align*}
& \exp \left(\pi \exp 2\left(|H(0)|+24 C \ln \left(\frac{1}{1-\theta}\right)\right)\right) \\
= & \exp \left(\pi \exp (2|H(0)|) \exp \left(48 C \ln \left(\frac{1}{1-\theta}\right)\right)\right) \tag{55.5.27}
\end{align*}
$$

Consider $\exp (2|H(0)|)$. I want to obtain an inequality for this which involves $\beta$. This is where I will use the convention about the logarithms discussed above. From 55.5.25,

$$
\begin{align*}
& 2|H(0)|=2\left|\log \left(\sqrt{\frac{\log (f(0))}{2 \pi i}}-\sqrt{\frac{\log (f(0))}{2 \pi i}-1}\right)\right| \\
\leq & 2\left(\left(\ln \left|\sqrt{\frac{\log (f(0))}{2 \pi i}}-\sqrt{\frac{\log (f(0))}{2 \pi i}-1}\right|\right)^{2}+\pi^{2}\right)^{1 / 2} \\
\leq & 2\left(\left|\ln \left(\left|\sqrt{\frac{\log (f(0))}{2 \pi i}}\right|+\left|\sqrt{\frac{\log (f(0))}{2 \pi i}-1}\right|\right)\right|^{2}+\pi^{2}\right)^{1 / 2} \\
\leq & 2\left|\ln \left(\left|\sqrt{\frac{\log (f(0))}{2 \pi i}}\right|+\left|\sqrt{\frac{\log (f(0))}{2 \pi i}-1}\right|\right)\right|+2 \pi \\
\leq & \ln \left(2\left(\left|\frac{\log (f(0))}{2 \pi i}\right|+\left|\frac{\log (f(0))}{2 \pi i}-1\right|\right)\right)+2 \pi \\
= & \ln \left(\left(\left|\frac{\log (f(0))}{\pi i}\right|+\left|\frac{\log (f(0))}{\pi i}-2\right|\right)\right)+2 \pi \tag{55.5.28}
\end{align*}
$$

Consider $\left|\frac{\log (f(0))}{\pi i}\right|$

$$
\frac{\log (f(0))}{\pi i}=-\frac{\ln |f(0)|}{\pi} i+\frac{\arg (f(0))}{\pi}
$$

and so

$$
\begin{aligned}
\left|\frac{\log (f(0))}{\pi i}\right| & =\left(\left|\frac{\ln |f(0)|}{\pi}\right|^{2}+\left(\frac{\arg (f(0))}{\pi}\right)^{2}\right)^{1 / 2} \\
& \leq\left(\left|\frac{\ln \beta}{\pi}\right|^{2}+\left(\frac{\pi}{\pi}\right)^{2}\right)^{1 / 2} \\
& =\left(\left|\frac{\ln \beta}{\pi}\right|^{2}+1\right)^{1 / 2}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\left|\frac{\log (f(0))}{\pi i}-2\right| & \leq\left(\left|\frac{\ln \beta}{\pi}\right|^{2}+(2+1)^{2}\right)^{1 / 2} \\
& =\left(\left|\frac{\ln \beta}{\pi}\right|^{2}+9\right)^{1 / 2}
\end{aligned}
$$

It follows from 55.5.28 that

$$
2|H(0)| \leq \ln \left(2\left(\left|\frac{\ln \beta}{\pi}\right|^{2}+9\right)^{1 / 2}\right)+2 \pi
$$

Hence from 55.5.27

$$
\begin{gathered}
|f(z)| \leq \\
\exp \left(\pi \exp \left(\ln \left(2\left(\left|\frac{\ln \beta}{\pi}\right|^{2}+9\right)^{1 / 2}\right)+2 \pi\right) \exp \left(48 C \ln \left(\frac{1}{1-\theta}\right)\right)\right)
\end{gathered}
$$

and so, letting $M(\beta, \theta)$ be given by the above expression on the right, the lemma is proved.
The following theorem will be referred to as Schottky's theorem. It looks just like the above lemma except it is only assumed that $f$ is analytic on $B(0, R)$ rather than on an open set containing $\overline{B(0, R)}$. Also, the case of an arbitrary center is included along with arbitrary points which are not attained as values of the function.
Theorem 55.5.8 Let $f$ be analytic on $B\left(z_{0}, R\right)$ and suppose that $f$ does not take on either of the two distinct values $a$ or $b$. Also suppose $\left|f\left(z_{0}\right)\right| \leq \beta$. Then letting $\theta \in(0,1)$, it follows

$$
|f(z)| \leq M(a, b, \beta, \theta)
$$

for all $z \in B\left(z_{0}, \theta R\right)$, where $M(a, b, \beta, \theta)$ is a function of only the variables $\beta, \theta, a, b$. (In particular, there is no dependence on $R$.)

Proof: First you can reduce to the case where the two values are 0 and 1 by considering

$$
h(z) \equiv \frac{f(z)-a}{b-a}
$$

If there exists an estimate of the desired sort for $h$, then there exists such an estimate for $f$. Of course here the function, $M$ would depend on $a$ and $b$. Therefore, there is no loss of generality in assuming the points which are missed are 0 and 1 .

Apply Lemma 55.5.7 to $B\left(0, R_{1}\right)$ for the function, $g(z) \equiv f\left(z_{0}+z\right)$ and $R_{1}<R$. Then if $\beta \geq\left|f\left(z_{0}\right)\right|=|g(0)|$, it follows $|g(z)|=\left|f\left(z_{0}+z\right)\right| \leq M(\beta, \theta)$ for every $z \in B\left(0, \theta R_{1}\right)$. Now let $\theta \in(0,1)$ and choose $R_{1}<R$ large enough that $\theta R=\theta_{1} R_{1}$ where $\theta_{1} \in(0,1)$. Then if $\left|z-z_{0}\right|<\theta R$, it follows

$$
|f(z)| \leq M\left(\beta, \theta_{1}\right)
$$

Now let $R_{1} \rightarrow R$ so $\theta_{1} \rightarrow \theta$.

### 55.5.4 A Brief Review

First recall the definition of the metric on $\widehat{\mathbb{C}}$. For convenience it is listed here again. Consider the unit sphere, $S^{2}$ given by $(z-1)^{2}+y^{2}+x^{2}=1$. Define a map from the complex plane to the surface of this sphere as follows. Extend a line from the point, $p$ in the complex plane to the point $(0,0,2)$ on the top of this sphere and let $\theta(p)$ denote the point of this sphere which the line intersects. Define $\theta(\infty) \equiv(0,0,2)$.


Then $\theta^{-1}$ is sometimes called sterographic projection. The mapping $\theta$ is clearly continuous because it takes converging sequences, to converging sequences. Furthermore, it is clear that $\theta^{-1}$ is also continuous. In terms of the extended complex plane, $\widehat{\mathbb{C}}$, a sequence, $z_{n}$ converges to $\infty$ if and only if $\theta z_{n}$ converges to $(0,0,2)$ and a sequence, $z_{n}$ converges to $z \in \mathbb{C}$ if and only if $\theta\left(z_{n}\right) \rightarrow \theta(z)$.

In fact this makes it easy to define a metric on $\widehat{\mathbb{C}}$.
Definition 55.5.9 Let $z, w \in \widehat{\mathbb{C}}$. Then let $d(x, y) \equiv|\theta(z)-\theta(w)|$ where this last distance is the usual distance measured in $\mathbb{R}^{3}$.

Theorem 55.5.10 $(\widehat{\mathbb{C}}, d)$ is a compact, hence complete metric space.
Proof: Suppose $\left\{z_{n}\right\}$ is a sequence in $\widehat{\mathbb{C}}$. This means $\left\{\theta\left(z_{n}\right)\right\}$ is a sequence in $S^{2}$ which is compact. Therefore, there exists a subsequence, $\left\{\theta z_{n_{k}}\right\}$ and a point, $z \in S^{2}$ such that $\theta z_{n_{k}} \rightarrow \theta z$ in $S^{2}$ which implies immediately that $d\left(z_{n_{k}}, z\right) \rightarrow 0$. A compact metric space must be complete.

Also recall the interesting fact that meromorphic functions are continuous with values in $\widehat{\mathbb{C}}$ which is reviewed here for convenience. It came from the theory of classification of isolated singularities.

Theorem 55.5.11 Let $\Omega$ be an open subset of $\mathbb{C}$ and let $f: \Omega \rightarrow \widehat{\mathbb{C}}$ be meromorphic. Then $f$ is continuous with respect to the metric, $d$ on $\widehat{\mathbb{C}}$.

Proof: Let $z_{n} \rightarrow z$ where $z \in \Omega$. Then if $z$ is a pole, it follows from Theorem 51.7.11 that

$$
d\left(f\left(z_{n}\right), \infty\right) \equiv d\left(f\left(z_{n}\right), f(z)\right) \rightarrow 0
$$

If $z$ is not a pole, then $f\left(z_{n}\right) \rightarrow f(z)$ in $\mathbb{C}$ which implies

$$
\left|\theta\left(f\left(z_{n}\right)\right)-\theta(f(z))\right|=d\left(f\left(z_{n}\right), f(z)\right) \rightarrow 0
$$

Recall that $\theta$ is continuous on $\mathbb{C}$.
The fundamental result behind all the theory about to be presented is the Ascoli Arzela theorem also listed here for convenience.

Definition 55.5.12 Let $(X, d)$ be a complete metric space. Then it is said to be locally compact if $\overline{B(x, r)}$ is compact for each $r>0$.

Thus if you have a locally compact metric space, then if $\left\{a_{n}\right\}$ is a bounded sequence, it must have a convergent subsequence.

Let $K$ be a compact subset of $\mathbb{R}^{n}$ and consider the continuous functions which have values in a locally compact metric space, $(X, d)$ where $d$ denotes the metric on $X$. Denote this space as $C(K, X)$.

Definition 55.5.13 For $f, g \in C(K, X)$, where $K$ is a compact subset of $\mathbb{R}^{n}$ and $X$ is a locally compact complete metric space define

$$
\rho_{K}(f, g) \equiv \sup \{d(f(\mathbf{x}), g(\mathbf{x})): \mathbf{x} \in K\}
$$

The Ascoli Arzela theorem, Theorem 7.8.4 is a major result which tells which subsets of $C(K, X)$ are sequentially compact.

Definition 55.5.14 Let $A \subseteq C(K, X)$ for $K$ a compact subset of $\mathbb{R}^{n}$. Then $A$ is said to be uniformly equicontinuous iffor every $\varepsilon>0$ there exists a $\delta>0$ such that whenever $\mathbf{x}, \mathbf{y} \in K$ with $|\mathbf{x}-\mathbf{y}|<\delta$ and $f \in A$,

$$
d(f(\mathbf{x}), f(\mathbf{y}))<\varepsilon
$$

The set, $A$ is said to be uniformly bounded if for some $M<\infty$, and $a \in X$,

$$
f(\mathbf{x}) \in B(a, M)
$$

for all $f \in A$ and $\mathbf{x} \in K$.
The Ascoli Arzela theorem follows.
Theorem 55.5.15 Suppose $K$ is a nonempty compact subset of $\mathbb{R}^{n}$ and $A \subseteq C(K, X)$, is uniformly bounded and uniformly equicontinuous where $X$ is a locally compact complete metric space. Then if $\left\{f_{k}\right\} \subseteq A$, there exists a function, $f \in C(K, X)$ and a subsequence, $f_{k_{l}}$ such that

$$
\lim _{l \rightarrow \infty} \rho_{K}\left(f_{k_{l}}, f\right)=0
$$

In the cases of interest here, $X=\widehat{\mathbb{C}}$ with the metric defined above.

### 55.5.5 Montel's Theorem

The following lemma is another version of Montel's theorem. It is this which will make possible a proof of the big Picard theorem.

Lemma 55.5.16 Let $\Omega$ be a region and let $\mathscr{F}$ be a set of functions analytic on $\Omega$ none of which achieve the two distinct values, $a$ and $b$. If $\left\{f_{n}\right\} \subseteq \mathscr{F}$ then one of the following hold: Either there exists a function, $f$ analytic on $\Omega$ and a subsequence, $\left\{f_{n_{k}}\right\}$ such that for any compact subset, $K$ of $\Omega$,

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \mid\left\|f_{n_{k}}-f\right\|_{K, \infty}=0 \tag{55.5.29}
\end{equation*}
$$

or there exists a subsequence $\left\{f_{n_{k}}\right\}$ such that for all compact subsets $K$,

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \rho_{K}\left(f_{n_{k}}, \infty\right)=0 \tag{55.5.30}
\end{equation*}
$$

Proof: Let $B\left(z_{0}, 2 R\right) \subseteq \Omega$. There are two cases to consider. The first case is that there exists a subsequence, $n_{k}$ such that $\left\{f_{n_{k}}\left(z_{0}\right)\right\}$ is bounded. The second case is that $\lim _{n \rightarrow \infty}\left|f_{n_{k}}\left(z_{0}\right)\right|=\infty$.

Consider the first case. By Theorem 55.5.8 $\left\{f_{n_{k}}(z)\right\}$ is uniformly bounded on $\overline{B\left(z_{0}, R\right)}$ because by this theorem, and letting $\theta=1 / 2$ applied to $B\left(z_{0}, 2 R\right)$, it follows $\left|f_{n_{k}}(z)\right| \leq$ $M\left(a, b, \frac{1}{2}, \beta\right)$ where $\beta$ is an upper bound to the numbers, $\left|f_{n_{k}}\left(z_{0}\right)\right|$. The Cauchy integral formula implies the existence of a uniform bound on the $\left\{f_{n_{k}}^{\prime}\right\}$ which implies the functions are equicontinuous and uniformly bounded. Therefore, by the Ascoli Arzela theorem there exists a further subsequence which converges uniformly on $\overline{B\left(z_{0}, R\right)}$ to a function, $f$ analytic on $B\left(z_{0}, R\right)$. Thus denoting this subsequence by $\left\{f_{n_{k}}\right\}$ to save on notation,

$$
\begin{equation*}
\lim _{k \rightarrow \infty}| | f_{n_{k}}-\left.f\right|_{\overline{B\left(z_{0}, R\right), \infty}}=0 \tag{55.5.31}
\end{equation*}
$$

Consider the second case. In this case, it follows $\left\{1 / f_{n}\left(z_{0}\right)\right\}$ is bounded on $\overline{B\left(z_{0}, R\right)}$ and so by the same argument just given $\left\{1 / f_{n}(z)\right\}$ is uniformly bounded on $\overline{B\left(z_{0}, R\right)}$. Therefore, a subsequence converges uniformly on $\overline{B\left(z_{0}, R\right)}$. But $\left\{1 / f_{n}(z)\right\}$ converges to 0 and so this requires that $\left\{1 / f_{n}(z)\right\}$ must converge uniformly to 0 . Therefore,

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \rho_{\overline{B\left(z_{0}, R\right)}}\left(f_{n_{k}}, \infty\right)=0 \tag{55.5.32}
\end{equation*}
$$

Now let $\left\{D_{k}\right\}$ denote a countable set of closed balls, $D_{k}=\overline{B\left(z_{k}, R_{k}\right)}$ such that

$$
B\left(z_{k}, 2 R_{k}\right) \subseteq \Omega
$$

and $\cup_{k=1}^{\infty} \operatorname{int}\left(D_{k}\right)=\Omega$. Using a Cantor diagonal process, there exists a subsequence, $\left\{f_{n_{k}}\right\}$ of $\left\{f_{n}\right\}$ such that for each $D_{j}$, one of the above two alternatives holds. That is, either

$$
\begin{equation*}
\lim _{k \rightarrow \infty}\left\|f_{n_{k}}-g_{j}\right\|_{D_{j}, \infty}=0 \tag{55.5.33}
\end{equation*}
$$

or,

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \rho_{D_{j}}\left(f_{n_{k}}, \infty\right) \tag{55.5.34}
\end{equation*}
$$

Let $A=\left\{\cup \operatorname{int}\left(D_{j}\right): 55.5 .33\right.$ holds $\}, B=\left\{\cup \operatorname{int}\left(D_{j}\right): 55.5 .34\right.$ holds $\}$. Note that the balls whose union is $A$ cannot intersect any of the balls whose union is $B$. Therefore, one of $A$ or $B$ must be empty since otherwise, $\Omega$ would not be connected.

If $K$ is any compact subset of $\Omega$, it follows $K$ must be a subset of some finite collection of the $D_{j}$. Therefore, one of the alternatives in the lemma must hold. That the limit function, $f$ must be analytic follows easily in the same way as the proof in Theorem 55.3.1 on Page 1739. You could also use Morera's theorem. This proves the lemma.

### 55.5.6 The Great Big Picard Theorem

The next theorem is the main result which the above lemmas lead to. It is the Big Picard theorem, also called the Great Picard theorem.Recall $B^{\prime}(a, r)$ is the deleted ball consisting of all the points of the ball except the center.

Theorem 55.5.17 Suppose $f$ has an isolated essential singularity at 0 . Then for every $R>0$, and $\beta \in \mathbb{C}, f^{-1}(\beta) \cap B^{\prime}(0, R)$ is an infinite set except for one possible exceptional $\beta$.

Proof: Suppose this is not true. Then there exists $R_{1}>0$ and two points, $\alpha$ and $\beta$ such that $f^{-1}(\beta) \cap B^{\prime}\left(0, R_{1}\right)$ and $f^{-1}(\alpha) \cap B^{\prime}\left(0, R_{1}\right)$ are both finite sets. Then shrinking $R_{1}$ and calling the result $R$, there exists $B(0, R)$ such that

$$
f^{-1}(\beta) \cap B^{\prime}(0, R)=\emptyset, f^{-1}(\alpha) \cap B^{\prime}(0, R)=\emptyset
$$

Now let $A_{0}$ denote the annulus $\left\{z \in \mathbb{C}: \frac{R}{2^{2}}<|z|<\frac{3 R}{2^{2}}\right\}$ and let $A_{n}$ denote the annulus $\left\{z \in \mathbb{C}: \frac{R}{2^{2+n}}<|z|<\frac{3 R}{2^{2+n}}\right\}$. The reason for the 3 is to insure that $A_{n} \cap A_{n+1} \neq \emptyset$. This follows from the observation that $3 R / 2^{2+1+n}>R / 2^{2+n}$. Now define a set of functions on $A_{0}$ as follows:

$$
f_{n}(z) \equiv f\left(\frac{z}{2^{n}}\right)
$$

By the choice of $R$, this set of functions missed the two points $\alpha$ and $\beta$. Therefore, by Lemma 55.5.16 there exists a subsequence such that one of the two options presented there holds.

First suppose $\lim _{k \rightarrow \infty}\left\|f_{n_{k}}-f\right\|_{K, \infty}=0$ for all $K$ a compact subset of $A_{0}$ and $f$ is analytic on $A_{0}$. In particular, this happens for $\gamma_{0}$ the circular contour having radius $R / 2$. Thus $f_{n_{k}}$ must be bounded on this contour. But this says the same thing as $f\left(z / 2^{n_{k}}\right)$ is bounded for $|z|=R / 2$, this holding for each $k=1,2, \cdots$. Thus there exists a constant, $M$ such that on each of a shrinking sequence of concentric circles whose radii converge to $0,|f(z)| \leq M$. By the maximum modulus theorem, $|f(z)| \leq M$ at every point between successive circles in this sequence. Therefore, $|f(z)| \leq M$ in $B^{\prime}(0, R)$ contradicting the Weierstrass Casorati theorem.

The other option which might hold from Lemma 55.5.16 is that $\lim _{k \rightarrow \infty} \rho_{K}\left(f_{n_{k}}, \infty\right)=0$ for all $K$ compact subset of $A_{0}$. Since $f$ has an essential singularity at 0 the zeros of $f$ in $B(0, R)$ are isolated. Therefore, for all $k$ large enough, $f_{n_{k}}$ has no zeros for $|z|<3 R / 2^{2}$. This is because the values of $f_{n_{k}}$ are the values of $f$ on $A_{n_{k}}$, a small anulus which avoids all the zeros of $f$ whenever $k$ is large enough. Only consider $k$ this large. Then use the above argument on the analytic functions $1 / f_{n_{k}}$. By the assumption that $\lim _{k \rightarrow \infty} \rho_{K}\left(f_{n_{k}}, \infty\right)=0$, it follows $\lim _{k \rightarrow \infty}\left\|1 / f_{n_{k}}-0\right\|_{K, \infty}=0$ and so as above, there exists a shrinking sequence of concentric circles whose radii converge to 0 and a constant, $M$ such that for $z$ on any of these circles, $|1 / f(z)| \leq M$. This implies that on some deleted ball, $B^{\prime}(0, r)$ where $r \leq R,|f(z)| \geq 1 / M$ which again violates the Weierstrass Casorati theorem. This proves the theorem.

As a simple corollary, here is what this remarkable theorem says about entire functions.

Corollary 55.5.18 Suppose $f$ is entire and nonconstant and not a polynomial. Then $f$ assumes every complex value infinitely many times with the possible exception of one.

Proof: Since $f$ is entire, $f(z)=\sum_{n=0}^{\infty} a_{n} z^{n}$. Define for $z \neq 0$,

$$
g(z) \equiv f\left(\frac{1}{z}\right)=\sum_{n=0}^{\infty} a_{n}\left(\frac{1}{z}\right)^{n}
$$

Thus 0 is an isolated essential singular point of $g$. By the big Picard theorem, Theorem 55.5.17 it follows $g$ takes every complex number but possibly one an infinite number of times. This proves the corollary.

Note the difference between this and the little Picard theorem which says that an entire function which is not constant must achieve every value but two.

### 55.6 Exercises

1. Prove that in Theorem 55.3.1 it suffices to assume $\mathscr{F}$ is uniformly bounded on each compact subset of $\Omega$.
2. Find conditions on $a, b, c, d$ such that the fractional linear transformation, $\frac{a z+b}{c z+d}$ maps the upper half plane onto the upper half plane.
3. Let $D$ be a simply connected region which is a proper subset of $\mathbb{C}$. Does there exist an entire function, $f$ which maps $\mathbb{C}$ onto $D$ ? Why or why not?
4. Verify the conclusion of Theorem 55.3.1 involving the higher order derivatives.
5. What if $\Omega=\mathbb{C}$ ? Does there exist an analytic function, $f$ mapping $\Omega$ one to one and onto $B(0,1)$ ? Explain why or why not. Was $\Omega \neq \mathbb{C}$ used in the proof of the Riemann mapping theorem?
6. Verify that $\left|\phi_{\alpha}(z)\right|=1$ if $|z|=1$. Apply the maximum modulus theorem to conclude that $\left|\phi_{\alpha}(z)\right| \leq 1$ for all $|z|<1$.
7. Suppose that $|f(z)| \leq 1$ for $|z|=1$ and $f(\alpha)=0$ for $|\alpha|<1$. Show that $|f(z)| \leq$ $\left|\phi_{\alpha}(z)\right|$ for all $z \in B(0,1)$. Hint: Consider $\frac{f(z)(1-\bar{\alpha} z)}{z-\alpha}$ which has a removable singularity at $\alpha$. Show the modulus of this function is bounded by 1 on $|z|=1$. Then apply the maximum modulus theorem.
8. Let $U$ and $V$ be open subsets of $\mathbb{C}$ and suppose $u: U \rightarrow \mathbb{R}$ is harmonic while $h$ is an analytic map which takes $V$ one to one onto $U$. Show that $u \circ h$ is harmonic on $V$.
9. Show that for a harmonic function, $u$ defined on $B(0, R)$, there exists an analytic function, $h=u+i v$ where

$$
v(x, y) \equiv \int_{0}^{y} u_{x}(x, t) d t-\int_{0}^{x} u_{y}(t, 0) d t
$$

10. Suppose $\Omega$ is a simply connected region and $u$ is a real valued function defined on $\Omega$ such that $u$ is harmonic. Show there exists an analytic function, $f$ such that $u=\operatorname{Re} f$. Show this is not true if $\Omega$ is not a simply connected region. Hint: You might use the Riemann mapping theorem and Problems 8 and 9. For the second part it might be good to try something like $u(x, y)=\ln \left(x^{2}+y^{2}\right)$ on the annulus $1<|z|<2$.
11. Show that $w=\frac{1+z}{1-z}$ maps $\{z \in \mathbb{C}: \operatorname{Im} z>0$ and $|z|<1\}$ to the first quadrant,

$$
\{z=x+i y: x, y>0\} .
$$

12. Let $f(z)=\frac{a z+b}{c z+d}$ and let $g(z)=\frac{a_{1} z+b_{1}}{c_{1} z+d_{1}}$. Show that $f \circ g(z)$ equals the quotient of two expressions, the numerator being the top entry in the vector

$$
\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)\left(\begin{array}{ll}
a_{1} & b_{1} \\
c_{1} & d_{1}
\end{array}\right)\binom{z}{1}
$$

and the denominator being the bottom entry. Show that if you define

$$
\phi\left(\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)\right) \equiv \frac{a z+b}{c z+d}
$$

then $\phi(A B)=\phi(A) \circ \phi(B)$. Find an easy way to find the inverse of $f(z)=\frac{a z+b}{c z+d}$ and give a condition on the $a, b, c, d$ which insures this function has an inverse.
13. The modular group ${ }^{2}$ is the set of fractional linear transformations, $\frac{a z+b}{c z+d}$ such that $a, b, c, d$ are integers and $a d-b c=1$. Using Problem 12 or brute force show this modular group is really a group with the group operation being composition. Also show the inverse of $\frac{a z+b}{c z+d}$ is $\frac{d z-b}{-c z+a}$.
14. Let $\Omega$ be a region and suppose $f$ is analytic on $\Omega$ and that the functions $f_{n}$ are also analytic on $\Omega$ and converge to $f$ uniformly on compact subsets of $\Omega$. Suppose $f$ is one to one. Can it be concluded that for an arbitrary compact set, $K \subseteq \Omega$ that $f_{n}$ is one to one for all $n$ large enough?
15. The Vitali theorem says that if $\Omega$ is a region and $\left\{f_{n}\right\}$ is a uniformly bounded sequence of functions which converges pointwise on a set, $S \subseteq \Omega$ which has a limit point in $\Omega$, then in fact, $\left\{f_{n}\right\}$ must converge uniformly on compact subsets of $\Omega$ to an analytic function. Prove this theorem. Hint: If the sequence fails to converge, show you can get two different subsequences converging uniformly on compact sets to different functions. Then argue these two functions coincide on $S$.
16. Does there exist a function analytic on $B(0,1)$ which maps $B(0,1)$ onto $B^{\prime}(0,1)$, the open unit ball in which 0 has been deleted?

[^34]
## Chapter 56

## Approximation By Rational Functions

### 56.1 Runge's Theorem

Consider the function, $\frac{1}{z}=f(z)$ for $z$ defined on $\Omega \equiv B(0,1) \backslash\{0\}=B^{\prime}(0,1)$. Clearly $f$ is analytic on $\Omega$. Suppose you could approximate $f$ uniformly by polynomials on ann $\left(0, \frac{1}{2}, \frac{3}{4}\right)$, a compact subset of $\Omega$. Then, there would exist a suitable polynomial $p(z)$, such that $\left|\frac{1}{2 \pi i} \int_{\gamma} f(z)-p(z) d z\right|<\frac{1}{10}$ where here $\gamma$ is a circle of radius $\frac{2}{3}$. However, this is impossible because $\frac{1}{2 \pi i} \int_{\gamma} f(z) d z=1$ while $\frac{1}{2 \pi i} \int_{\gamma} p(z) d z=0$. This shows you can't expect to be able to uniformly approximate analytic functions on compact sets using polynomials. This is just horrible! In real variables, you can approximate any continuous function on a compact set with a polynomial. However, that is just the way it is. It turns out that the ability to approximate an analytic function on $\Omega$ with polynomials is dependent on $\Omega$ being simply connected.

All these theorems work for $f$ having values in a complex Banach space. However, I will present them in the context of functions which have values in $\mathbb{C}$. The changes necessary to obtain the extra generality are very minor.

Definition 56.1.1 Approximation will be taken with respect to the following norm.

$$
\|f-g\|_{K, \infty} \equiv \sup \{\|f(z)-g(z)\|: z \in K\}
$$

### 56.1.1 Approximation With Rational Functions

It turns out you can approximate analytic functions by rational functions, quotients of polynomials. The resulting theorem is one of the most profound theorems in complex analysis. The basic idea is simple. The Riemann sums for the Cauchy integral formula are rational functions. The idea used to implement this observation is that if you have a compact subset, $K$ of an open set, $\Omega$ there exists a cycle composed of closed oriented curves $\left\{\gamma_{j}\right\}_{j=1}^{n}$ which are contained in $\Omega \backslash K$ such that for every $z \in K, \sum_{k=1}^{n} n\left(\gamma_{k}, z\right)=1$. One more ingredient is needed and this is a theorem which lets you keep the approximation but move the poles.

To begin with, consider the part about the cycle of closed oriented curves. Recall Theorem 51.7.25 which is stated for convenience.

Theorem 56.1.2 Let $K$ be a compact subset of an open set, $\Omega$. Then there exist continuous, closed, bounded variation oriented curves $\left\{\gamma_{j}\right\}_{j=1}^{m}$ for which $\gamma_{j}^{*} \cap K=\emptyset$ for each $j, \gamma_{j}^{*} \subseteq$ $\Omega$, and for all $p \in K$,

$$
\sum_{k=1}^{m} n\left(p, \gamma_{k}\right)=1
$$

and

$$
\sum_{k=1}^{m} n\left(z, \gamma_{k}\right)=0
$$

for all $z \notin \Omega$.

This theorem implies the following.
Theorem 56.1.3 Let $K \subseteq \Omega$ where $K$ is compact and $\Omega$ is open. Then there exist oriented closed curves, $\gamma_{k}$ such that $\gamma_{k}^{*} \cap K=\emptyset$ but $\gamma_{k}^{*} \subseteq \Omega$, such that for all $z \in K$,

$$
\begin{equation*}
f(z)=\frac{1}{2 \pi i} \sum_{k=1}^{p} \int_{\gamma_{k}} \frac{f(w)}{w-z} d w \tag{56.1.1}
\end{equation*}
$$

Proof: This follows from Theorem 51.7.25 and the Cauchy integral formula. As shown in the proof, you can assume the $\gamma_{k}$ are linear mappings but this is not important.

Next I will show how the Cauchy integral formula leads to approximation by rational functions, quotients of polynomials.

Lemma 56.1.4 Let $K$ be a compact subset of an open set, $\Omega$ and let $f$ be analytic on $\Omega$. Then there exists a rational function, $Q$ whose poles are not in $K$ such that

$$
\|Q-f\|_{K, \infty}<\varepsilon
$$

Proof: By Theorem 56.1.3 there are oriented curves, $\gamma_{k}$ described there such that for all $z \in K$,

$$
\begin{equation*}
f(z)=\frac{1}{2 \pi i} \sum_{k=1}^{p} \int_{\gamma_{k}} \frac{f(w)}{w-z} d w \tag{56.1.2}
\end{equation*}
$$

Defining $g(w, z) \equiv \frac{f(w)}{w-z}$ for $(w, z) \in \cup_{k=1}^{p} \gamma_{k}^{*} \times K$, it follows since the distance between $K$ and $\cup_{k} \gamma_{k}^{*}$ is positive that $g$ is uniformly continuous and so there exists a $\delta>0$ such that if $\|\mathscr{P}\|<\delta$, then for all $z \in K$,

$$
\left|f(z)-\frac{1}{2 \pi i} \sum_{k=1}^{p} \sum_{j=1}^{n} \frac{f\left(\gamma_{k}\left(\tau_{j}\right)\right)\left(\gamma_{k}\left(t_{i}\right)-\gamma_{k}\left(t_{i-1}\right)\right)}{\gamma_{k}\left(\tau_{j}\right)-z}\right|<\frac{\varepsilon}{2}
$$

The complicated expression is obtained by replacing each integral in 56.1 .2 with a Riemann sum. Simplifying the appearance of this, it follows there exists a rational function of the form

$$
R(z)=\sum_{k=1}^{M} \frac{A_{k}}{w_{k}-z}
$$

where the $w_{k}$ are elements of components of $\mathbb{C} \backslash K$ and $A_{k}$ are complex numbers or in the case where $f$ has values in $X$, these would be elements of $X$ such that

$$
\|R-f\|_{K, \infty}<\frac{\varepsilon}{2}
$$

This proves the lemma.

### 56.1.2 Moving The Poles And Keeping The Approximation

Lemma 56.1.4 is a nice lemma but needs refining. In this lemma, the Riemann sum handed you the poles. It is much better if you can pick the poles. The following theorem from advanced calculus, called Merten's theorem, will be used

### 56.1.3 Merten's Theorem.

Theorem 56.1.5 Suppose $\sum_{i=r}^{\infty} a_{i}$ and $\sum_{j=r}^{\infty} b_{j}$ both converge absolutely ${ }^{1}$. Then

$$
\left(\sum_{i=r}^{\infty} a_{i}\right)\left(\sum_{j=r}^{\infty} b_{j}\right)=\sum_{n=r}^{\infty} c_{n}
$$

where

$$
c_{n}=\sum_{k=r}^{n} a_{k} b_{n-k+r}
$$

Proof: Let $p_{n k}=1$ if $r \leq k \leq n$ and $p_{n k}=0$ if $k>n$. Then

$$
c_{n}=\sum_{k=r}^{\infty} p_{n k} a_{k} b_{n-k+r}
$$

Also,

$$
\begin{aligned}
\sum_{k=r}^{\infty} \sum_{n=r}^{\infty} p_{n k}\left|a_{k}\right|\left|b_{n-k+r}\right| & =\sum_{k=r}^{\infty}\left|a_{k}\right| \sum_{n=r}^{\infty} p_{n k}\left|b_{n-k+r}\right| \\
& =\sum_{k=r}^{\infty}\left|a_{k}\right| \sum_{n=k}^{\infty}\left|b_{n-k+r}\right| \\
& =\sum_{k=r}^{\infty}\left|a_{k}\right| \sum_{n=k}^{\infty}\left|b_{n-(k-r)}\right| \\
& =\sum_{k=r}^{\infty}\left|a_{k}\right| \sum_{m=r}^{\infty}\left|b_{m}\right|<\infty
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sum_{n=r}^{\infty} c_{n} & =\sum_{n=r}^{\infty} \sum_{k=r}^{n} a_{k} b_{n-k+r}=\sum_{n=r}^{\infty} \sum_{k=r}^{\infty} p_{n k} a_{k} b_{n-k+r} \\
& =\sum_{k=r}^{\infty} a_{k} \sum_{n=r}^{\infty} p_{n k} b_{n-k+r}=\sum_{k=r}^{\infty} a_{k} \sum_{n=k}^{\infty} b_{n-k+r} \\
& =\sum_{k=r}^{\infty} a_{k} \sum_{m=r}^{\infty} b_{m}
\end{aligned}
$$

and this proves the theorem.
It follows that $\sum_{n=r}^{\infty} c_{n}$ converges absolutely. Also, you can see by induction that you can multiply any number of absolutely convergent series together and obtain a series which is absolutely convergent. Next, here are some similar results related to Merten's theorem.

[^35]Lemma 56.1.6 Let $\sum_{n=0}^{\infty} a_{n}(z)$ and $\sum_{n=0}^{\infty} b_{n}(z)$ be two convergent series for $z \in K$ which satisfy the conditions of the Weierstrass $M$ test. Thus there exist positive constants, $A_{n}$ and $B_{n}$ such that $\left|a_{n}(z)\right| \leq A_{n},\left|b_{n}(z)\right| \leq B_{n}$ for all $z \in K$ and $\sum_{n=0}^{\infty} A_{n}<\infty, \sum_{n=0}^{\infty} B_{n}<\infty$. Then defining the Cauchy product,

$$
c_{n}(z) \equiv \sum_{k-0}^{n} a_{n-k}(z) b_{k}(z)
$$

it follows $\sum_{n=0}^{\infty} c_{n}(z)$ also converges absolutely and uniformly on $K$ because $c_{n}(z)$ satisfies the conditions of the Weierstrass $M$ test. Therefore,

$$
\begin{equation*}
\sum_{n=0}^{\infty} c_{n}(z)=\left(\sum_{k=0}^{\infty} a_{k}(z)\right)\left(\sum_{n=0}^{\infty} b_{n}(z)\right) \tag{56.1.3}
\end{equation*}
$$

Proof:

$$
\left|c_{n}(z)\right| \leq \sum_{k=0}^{n}\left|a_{n-k}(z)\right|\left|b_{k}(z)\right| \leq \sum_{k=0}^{n} A_{n-k} B_{k} .
$$

Also,

$$
\begin{aligned}
\sum_{n=0}^{\infty} \sum_{k=0}^{n} A_{n-k} B_{k} & =\sum_{k=0}^{\infty} \sum_{n=k}^{\infty} A_{n-k} B_{k} \\
& =\sum_{k=0}^{\infty} B_{k} \sum_{n=0}^{\infty} A_{n}<\infty
\end{aligned}
$$

The claim of 56.1.3 follows from Merten's theorem. This proves the lemma.
Corollary 56.1.7 Let $P$ be a polynomial and let $\sum_{n=0}^{\infty} a_{n}(z)$ converge uniformly and absolutely on $K$ such that the $a_{n}$ satisfy the conditions of the Weierstrass $M$ test. Then there exists a series for $P\left(\sum_{n=0}^{\infty} a_{n}(z)\right), \sum_{n=0}^{\infty} c_{n}(z)$, which also converges absolutely and uniformly for $z \in K$ because $c_{n}(z)$ also satisfies the conditions of the Weierstrass $M$ test.

The following picture is descriptive of the following lemma. This lemma says that if you have a rational function with one pole off a compact set, then you can approximate on the compact set with another rational function which has a different pole.


Lemma 56.1.8 Let $R$ be a rational function which has a pole only at $a \in V$, a component of $\mathbb{C} \backslash K$ where $K$ is a compact set. Suppose $b \in V$. Then for $\varepsilon>0$ given, there exists $a$ rational function, $Q$, having a pole only at $b$ such that

$$
\begin{equation*}
\|R-Q\|_{K, \infty}<\varepsilon \tag{56.1.4}
\end{equation*}
$$

If it happens that $V$ is unbounded, then there exists a polynomial, $P$ such that

$$
\begin{equation*}
\|R-P\|_{K, \infty}<\varepsilon \tag{56.1.5}
\end{equation*}
$$

Proof: Say that $b \in V$ satisfies $\mathscr{P}$ if for all $\varepsilon>0$ there exists a rational function, $Q_{b}$, having a pole only at $b$ such that

$$
\left\|R-Q_{b}\right\|_{K, \infty}<\varepsilon
$$

Now define a set,

$$
S \equiv\{b \in V: b \text { satisfies } \mathscr{P}\}
$$

Observe that $S \neq \emptyset$ because $a \in S$.
I claim $S$ is open. Suppose $b_{1} \in S$. Then there exists a $\delta>0$ such that

$$
\begin{equation*}
\left|\frac{b_{1}-b}{z-b}\right|<\frac{1}{2} \tag{56.1.6}
\end{equation*}
$$

for all $z \in K$ whenever $b \in B\left(b_{1}, \boldsymbol{\delta}\right)$. In fact, it suffices to take $\left|b-b_{1}\right|<\operatorname{dist}\left(b_{1}, K\right) / 4$ because then

$$
\begin{aligned}
\left|\frac{b_{1}-b}{z-b}\right| & <\left|\frac{\operatorname{dist}\left(b_{1}, K\right) / 4}{z-b}\right| \leq \frac{\operatorname{dist}\left(b_{1}, K\right) / 4}{\left|z-b_{1}\right|-\left|b_{1}-b\right|} \\
& \leq \frac{\operatorname{dist}\left(b_{1}, K\right) / 4}{\operatorname{dist}\left(b_{1}, K\right)-\operatorname{dist}\left(b_{1}, K\right) / 4} \leq \frac{1}{3}<\frac{1}{2}
\end{aligned}
$$

Since $b_{1}$ satisfies $\mathscr{P}$, there exists a rational function $Q_{b_{1}}$ with the desired properties. It is shown next that you can approximate $Q_{b_{1}}$ with $Q_{b}$ thus yielding an approximation to $R$ by the use of the triangle inequality,

$$
\left\|R-Q_{b_{1}}\right\|_{K, \infty}+\left\|Q_{b_{1}}-Q_{b}\right\|_{K, \infty} \geq\left\|R-Q_{b}\right\|_{K, \infty}
$$

Since $Q_{b_{1}}$ has poles only at $b_{1}$, it follows it is a sum of functions of the form $\frac{\alpha_{n}}{\left(z-b_{1}\right)^{n}}$. Therefore, it suffices to consider the terms of $Q_{b_{1}}$ or that $Q_{b_{1}}$ is of the special form

$$
Q_{b_{1}}(z)=\frac{1}{\left(z-b_{1}\right)^{n}}
$$

However,

$$
\frac{1}{\left(z-b_{1}\right)^{n}}=\frac{1}{(z-b)^{n}\left(1-\frac{b_{1}-b}{z-b}\right)^{n}}
$$

Now from the choice of $b_{1}$, the series

$$
\sum_{k=0}^{\infty}\left(\frac{b_{1}-b}{z-b}\right)^{k}=\frac{1}{\left(1-\frac{b_{1}-b}{z-b}\right)}
$$

converges absolutely independent of the choice of $z \in K$ because

$$
\left|\left(\frac{b_{1}-b}{z-b}\right)^{k}\right|<\frac{1}{2^{k}}
$$

By Corollary 56.1.7 the same is true of the series for $\frac{1}{\left(1-\frac{b_{1}-b}{z-b}\right)^{n}}$. Thus a suitable partial sum can be made uniformly on $K$ as close as desired to $\frac{1}{\left(z-b_{1}\right)^{n}}$. This shows that $b$ satisfies $\mathscr{P}$ whenever $b$ is close enough to $b_{1}$ verifying that $S$ is open.

Next it is shown $S$ is closed in $V$. Let $b_{n} \in S$ and suppose $b_{n} \rightarrow b \in V$. Then since $b_{n} \in S$, there exists a rational function, $Q_{b_{n}}$ such that

$$
\left\|Q_{b_{n}}-R\right\|_{K, \infty}<\frac{\varepsilon}{2}
$$

Then for all $n$ large enough,

$$
\frac{1}{2} \operatorname{dist}(b, K) \geq\left|b_{n}-b\right|
$$

and so for all $n$ large enough,

$$
\left|\frac{b-b_{n}}{z-b_{n}}\right|<\frac{1}{2}
$$

for all $z \in K$. Pick such a $b_{n}$. As before, it suffices to assume $Q_{b_{n}}$, is of the form $\frac{1}{\left(z-b_{n}\right)^{n}}$. Then

$$
Q_{b_{n}}(z)=\frac{1}{\left(z-b_{n}\right)^{n}}=\frac{1}{(z-b)^{n}\left(1-\frac{b_{n}-b}{z-b}\right)^{n}}
$$

and because of the estimate, there exists $M$ such that for all $z \in K$

$$
\begin{equation*}
\left|\frac{1}{\left(1-\frac{b_{n}-b}{z-b}\right)^{n}}-\sum_{k=0}^{M} a_{k}\left(\frac{b_{n}-b}{z-b}\right)^{k}\right|<\frac{\varepsilon(\operatorname{dist}(b, K))^{n}}{2} \tag{56.1.7}
\end{equation*}
$$

Therefore, for all $z \in K$

$$
\begin{aligned}
\left|Q_{b_{n}}(z)-\frac{1}{(z-b)^{n}} \sum_{k=0}^{M} a_{k}\left(\frac{b_{n}-b}{z-b}\right)^{k}\right| & = \\
\left|\frac{1}{(z-b)^{n}\left(1-\frac{b_{n}-b}{z-b}\right)^{n}}-\frac{1}{(z-b)^{n}} \sum_{k=0}^{M} a_{k}\left(\frac{b_{n}-b}{z-b}\right)^{k}\right| & \leq \\
\frac{\varepsilon(\operatorname{dist}(b, K))^{n}}{2} \frac{1}{\operatorname{dist}(b, K)^{n}} & =\frac{\varepsilon}{2}
\end{aligned}
$$

and so, letting $Q_{b}(z)=\frac{1}{(z-b)^{n}} \sum_{k=0}^{M} a_{k}\left(\frac{b_{n}-b}{z-b}\right)^{k}$,

$$
\begin{aligned}
\left\|R-Q_{b}\right\|_{K, \infty} & \leq\left\|R-Q_{b_{n}}\right\|_{K, \infty}+\left\|Q_{b_{n}}-Q_{b}\right\|_{K, \infty} \\
& <\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon
\end{aligned}
$$

showing that $b \in S$. Since $S$ is both open and closed in $V$ it follows that, since $S \neq \emptyset, S=V$. Otherwise $V$ would fail to be connected.

It remains to consider the case where $V$ is unbounded. Pick $b \in V$ large enough that

$$
\begin{equation*}
\left|\frac{z}{b}\right|<\frac{1}{2} \tag{56.1.8}
\end{equation*}
$$

for all $z \in K$. From what was just shown, there exists a rational function, $Q_{b}$ having a pole only at $b$ such that $\left\|Q_{b}-R\right\|_{K, \infty}<\frac{\varepsilon}{2}$. It suffices to assume that $Q_{b}$ is of the form

$$
\begin{aligned}
Q_{b}(z) & =\frac{p(z)}{(z-b)^{n}}=p(z)(-1)^{n} \frac{1}{b^{n}} \frac{1}{\left(1-\frac{z}{b}\right)^{n}} \\
& =p(z)(-1)^{n} \frac{1}{b^{n}}\left(\sum_{k=0}^{\infty}\left(\frac{z}{b}\right)^{k}\right)^{n}
\end{aligned}
$$

Then by an application of Corollary 56.1.7 there exists a partial sum of the power series for $Q_{b}$ which is uniformly close to $Q_{b}$ on $K$. Therefore, you can approximate $Q_{b}$ and therefore also $R$ uniformly on $K$ by a polynomial consisting of a partial sum of the above infinite sum. This proves the theorem.

If $f$ is a polynomial, then $f$ has a pole at $\infty$. This will be discussed more later.

### 56.1.4 Runge's Theorem

Now what follows is the first form of Runge's theorem.
Theorem 56.1.9 Let $K$ be a compact subset of an open set, $\Omega$ and let $\left\{b_{j}\right\}$ be a set which consists of one point from each component of $\widehat{\mathbb{C}} \backslash K$. Let $f$ be analytic on $\Omega$. Then for each $\varepsilon>0$, there exists a rational function, $Q$ whose poles are all contained in the set, $\left\{b_{j}\right\}$ such that

$$
\begin{equation*}
\|Q-f\|_{K, \infty}<\varepsilon \tag{56.1.9}
\end{equation*}
$$

If $\widehat{\mathbb{C}} \backslash K$ has only one component, then $Q$ may be taken to be a polynomial.
Proof: By Lemma 56.1.4 there exists a rational function of the form

$$
R(z)=\sum_{k=1}^{M} \frac{A_{k}}{w_{k}-z}
$$

where the $w_{k}$ are elements of components of $\mathbb{C} \backslash K$ and $A_{k}$ are complex numbers such that

$$
\|R-f\|_{K, \infty}<\frac{\varepsilon}{2}
$$

Consider the rational function, $R_{k}(z) \equiv \frac{A_{k}}{w_{k}-z}$ where $w_{k} \in V_{j}$, one of the components of $\mathbb{C} \backslash K$, the given point of $V_{j}$ being $b_{j}$. By Lemma 56.1 .8 , there exists a function, $Q_{k}$ which is either a rational function having its only pole at $b_{j}$ or a polynomial, depending on whether $V_{j}$ is bounded such that

$$
\left\|R_{k}-Q_{k}\right\|_{K, \infty}<\frac{\varepsilon}{2 M}
$$

Letting $Q(z) \equiv \sum_{k=1}^{M} Q_{k}(z)$,

$$
\|R-Q\|_{K, \infty}<\frac{\varepsilon}{2}
$$

It follows

$$
\|f-Q\|_{K, \infty} \leq\|f-R\|_{K, \infty}+\|R-Q\|_{K, \infty}<\varepsilon .
$$

In the case of only one component of $\mathbb{C} \backslash K$, this component is the unbounded component and so you can take $Q$ to be a polynomial. This proves the theorem.

The next version of Runge's theorem concerns the case where the given points are contained in $\widehat{\mathbb{C}} \backslash \Omega$ for $\Omega$ an open set rather than a compact set. Note that here there could be uncountably many components of $\widehat{\mathbb{C}} \backslash \Omega$ because the components are no longer open sets. An easy example of this phenomenon in one dimension is where $\Omega=[0,1] \backslash P$ for $P$ the Cantor set. Then you can show that $\mathbb{R} \backslash \Omega$ has uncountably many components. Nevertheless, Runge's theorem will follow from Theorem 56.1.9 with the aid of the following interesting lemma.

Lemma 56.1.10 Let $\Omega$ be an open set in $\mathbb{C}$. Then there exists a sequence of compact sets, $\left\{K_{n}\right\}$ such that

$$
\begin{equation*}
\Omega=\cup_{k=1}^{\infty} K_{n}, \cdots, K_{n} \subseteq \operatorname{int} K_{n+1} \cdots \tag{56.1.10}
\end{equation*}
$$

and for any $K \subseteq \Omega$,

$$
\begin{equation*}
K \subseteq K_{n} \tag{56.1.11}
\end{equation*}
$$

for all $n$ sufficiently large, and every component of $\widehat{\mathbb{C}} \backslash K_{n}$ contains a component of $\widehat{\mathbb{C}} \backslash \Omega$.
Proof: Let

$$
V_{n} \equiv\{z:|z|>n\} \cup \bigcup_{z \notin \Omega} B\left(z, \frac{1}{n}\right)
$$

Thus $\{z:|z|>n\}$ contains the point, $\infty$. Now let

$$
K_{n} \equiv \widehat{\mathbb{C}} \backslash V_{n}=\mathbb{C} \backslash V_{n} \subseteq \Omega
$$

You should verify that 56.1 .10 and 56.1.11 hold. It remains to show that every component of $\widehat{\mathbb{C}} \backslash K_{n}$ contains a component of $\widehat{\mathbb{C}} \backslash \Omega$. Let $D$ be a component of $\widehat{\mathbb{C}} \backslash K_{n} \equiv V_{n}$.

If $\infty \notin D$, then $D$ contains no point of $\{z:|z|>n\}$ because this set is connected and $D$ is a component. (If it did contain a point of this set, it would have to contain the whole set.) Therefore, $D \subseteq \bigcup_{z \notin \Omega} B\left(z, \frac{1}{n}\right)$ and so $D$ contains some point of $B\left(z, \frac{1}{n}\right)$ for some $z \notin$ $\Omega$. Therefore, since this ball is connected, it follows $D$ must contain the whole ball and consequently $D$ contains some point of $\Omega^{C}$. (The point $z$ at the center of the ball will do.)

Since $D$ contains $z \notin \Omega$, it must contain the component, $H_{z}$, determined by this point. The reason for this is that

$$
H_{z} \subseteq \widehat{\mathbb{C}} \backslash \Omega \subseteq \widehat{\mathbb{C}} \backslash K_{n}
$$

and $H_{z}$ is connected. Therefore, $H_{z}$ can only have points in one component of $\widehat{\mathbb{C}} \backslash K_{n}$. Since it has a point in $D$, it must therefore, be totally contained in $D$. This verifies the desired condition in the case where $\infty \notin D$.

Now suppose that $\infty \in D . \infty \notin \Omega$ because $\Omega$ is given to be a set in $\mathbb{C}$. Letting $H_{\infty}$ denote the component of $\widehat{\mathbb{C}} \backslash \Omega$ determined by $\infty$, it follows both $D$ and $H_{\infty}$ contain $\infty$. Therefore, the connected set, $H_{\infty}$ cannot have any points in another component of $\widehat{\mathbb{C}} \backslash K_{n}$ and it is a set which is contained in $\widehat{\mathbb{C}} \backslash K_{n}$ so it must be contained in $D$. This proves the lemma.

The following picture is a very simple example of the sort of thing considered by Runge's theorem. The picture is of a region which has a couple of holes.

However, there could be many more holes than two. In fact, there could be infinitely many. Nor does it follow that the components of the complement of $\Omega$ need to have any interior points. Therefore, the picture is certainly not representative.

Theorem 56.1.11 (Runge) Let $\Omega$ be an open set, and let A be a set which has one point in each component of $\widehat{\mathbb{C}} \backslash \Omega$ and let $f$ be analytic on $\Omega$. Then there exists a sequence of rational functions, $\left\{R_{n}\right\}$ having poles only in $A$ such that $R_{n}$ converges uniformly to $f$ on compact subsets of $\Omega$.

Proof: Let $K_{n}$ be the compact sets of Lemma 56.1 .10 where each component of $\widehat{\mathbb{C}} \backslash K_{n}$ contains a component of $\widehat{\mathbb{C}} \backslash \Omega$. It follows each component of $\widehat{\mathbb{C}} \backslash K_{n}$ contains a point of $A$. Therefore, by Theorem 56.1.9 there exists $R_{n}$ a rational function with poles only in $A$ such that

$$
\left\|R_{n}-f\right\|_{K_{n}, \infty}<\frac{1}{n}
$$

It follows, since a given compact set, $K$ is a subset of $K_{n}$ for all $n$ large enough, that $R_{n} \rightarrow f$ uniformly on $K$. This proves the theorem.

Corollary 56.1.12 Let $\Omega$ be simply connected and $f$ analytic on $\Omega$. Then there exists $a$ sequence of polynomials, $\left\{p_{n}\right\}$ such that $p_{n} \rightarrow f$ uniformly on compact sets of $\Omega$.

Proof: By definition of what is meant by simply connected, $\widehat{\mathbb{C}} \backslash \Omega$ is connected and so there are no bounded components of $\widehat{\mathbb{C}} \backslash \Omega$. Therefore, in the proof of Theorem 56.1.11 when you use Theorem 56.1.9, you can always have $R_{n}$ be a polynomial by Lemma 56.1.8.

### 56.2 The Mittag-Leffler Theorem

### 56.2.1 A Proof From Runge's Theorem

This theorem is fairly easy to prove once you have Theorem 56.1.9. Given a set of complex numbers, does there exist a meromorphic function having its poles equal to this set of numbers? The Mittag-Leffler theorem provides a very satisfactory answer to this question. Actually, it says somewhat more. You can specify, not just the location of the pole but also the kind of singularity the meromorphic function is to have at that pole.

Theorem 56.2.1 Let $P \equiv\left\{z_{k}\right\}_{k=1}^{\infty}$ be a set of points in an open subset of $\mathbb{C}, \Omega$. Suppose also that $P \subseteq \Omega \subseteq \mathbb{C}$. For each $z_{k}$, denote by $S_{k}(z)$ a function of the form

$$
S_{k}(z)=\sum_{j=1}^{m_{k}} \frac{a_{j}^{k}}{\left(z-z_{k}\right)^{j}}
$$

Then there exists a meromorphic function, $Q$ defined on $\Omega$ such that the poles of $Q$ are the points, $\left\{z_{k}\right\}_{k=1}^{\infty}$ and the singular part of the Laurent expansion of $Q$ at $z_{k}$ equals $S_{k}(z)$. In other words, for $z$ near $z_{k}, Q(z)=g_{k}(z)+S_{k}(z)$ for some function, $g_{k}$ analytic near $z_{k}$.

Proof: Let $\left\{K_{n}\right\}$ denote the sequence of compact sets described in Lemma 56.1.10. Thus $\cup_{n=1}^{\infty} K_{n}=\Omega, K_{n} \subseteq \operatorname{int}\left(K_{n+1}\right) \subseteq K_{n+1} \cdots$, and the components of $\widehat{\mathbb{C}} \backslash K_{n}$ contain the components of $\widehat{\mathbb{C}} \backslash \Omega$. Renumbering if necessary, you can assume each $K_{n} \neq \emptyset$. Also let $K_{0}=\emptyset$. Let $P_{m} \equiv P \cap\left(K_{m} \backslash K_{m-1}\right)$ and consider the rational function, $R_{m}$ defined by

$$
R_{m}(z) \equiv \sum_{z_{k} \in K_{m} \backslash K_{m-1}} S_{k}(z)
$$

Since each $K_{m}$ is compact, it follows $P_{m}$ is finite and so the above really is a rational function. Now for $m>1$, this rational function is analytic on some open set containing $K_{m-1}$. There exists a set of points, $A$ one point in each component of $\widehat{\mathbb{C}} \backslash \Omega$. Consider $\widehat{\mathbb{C}} \backslash K_{m-1}$. Each of its components contains a component of $\widehat{\mathbb{C}} \backslash \Omega$ and so for each of these components of $\widehat{\mathbb{C}} \backslash K_{m-1}$, there exists a point of $A$ which is contained in it. Denote the resulting set of points by $A^{\prime}$. By Theorem 56.1.9 there exists a rational function, $Q_{m}$ whose poles are all contained in the set, $A^{\prime} \subseteq \Omega^{C}$ such that

$$
\left\|R_{m}-Q_{m}\right\|_{K_{m-1, \infty}}<\frac{1}{2^{m}}
$$

The meromorphic function is

$$
Q(z) \equiv R_{1}(z)+\sum_{k=2}^{\infty}\left(R_{k}(z)-Q_{k}(z)\right)
$$

It remains to verify this function works. First consider $K_{1}$. Then on $K_{1}$, the above sum converges uniformly. Furthermore, the terms of the sum are analytic in some open set containing $K_{1}$. Therefore, the infinite sum is analytic on this open set and so for $z \in K_{1}$

The function, $f$ is the sum of a rational function, $R_{1}$, having poles at $P_{1}$ with the specified singular terms and an analytic function. Therefore, $Q$ works on $K_{1}$. Now consider $K_{m}$ for $m>1$. Then

$$
Q(z)=R_{1}(z)+\sum_{k=2}^{m+1}\left(R_{k}(z)-Q_{k}(z)\right)+\sum_{k=m+2}^{\infty}\left(R_{k}(z)-Q_{k}(z)\right) .
$$

As before, the infinite sum converges uniformly on $K_{m+1}$ and hence on some open set, $O$ containing $K_{m}$. Therefore, this infinite sum equals a function which is analytic on $O$. Also,

$$
R_{1}(z)+\sum_{k=2}^{m+1}\left(R_{k}(z)-Q_{k}(z)\right)
$$

is a rational function having poles at $\cup_{k=1}^{m} P_{k}$ with the specified singularities because the poles of each $Q_{k}$ are not in $\Omega$. It follows this function is meromorphic because it is analytic except for the points in $P$. It also has the property of retaining the specified singular behavior.

### 56.2.2 A Direct Proof Without Runge's Theorem

There is a direct proof of this important theorem which is not dependent on Runge's theorem in the case where $\Omega=\mathbb{C}$. I think it is arguably easier to understand and the MittagLeffler theorem is very important so I will give this proof here.

Theorem 56.2.2 Let $P \equiv\left\{z_{k}\right\}_{k=1}^{\infty}$ be a set of points in $\mathbb{C}$ which satisfies $\lim _{n \rightarrow \infty}\left|z_{n}\right|=\infty$. For each $z_{k}$, denote by $S_{k}(z)$ a polynomial in $\frac{1}{z-z_{k}}$ which is of the form

$$
S_{k}(z)=\sum_{j=1}^{m_{k}} \frac{a_{j}^{k}}{\left(z-z_{k}\right)^{j}}
$$

Then there exists a meromorphic function, $Q$ defined on $\mathbb{C}$ such that the poles of $Q$ are the points, $\left\{z_{k}\right\}_{k=1}^{\infty}$ and the singular part of the Laurent expansion of $Q$ at $z_{k}$ equals $S_{k}(z)$. In other words, for $z$ near $z_{k}$,

$$
Q(z)=g_{k}(z)+S_{k}(z)
$$

for some function, $g_{k}$ analytic in some open set containing $z_{k}$.
Proof: First consider the case where none of the $z_{k}=0$. Letting

$$
K_{k} \equiv\left\{z:|z| \leq\left|z_{k}\right| / 2\right\}
$$

there exists a power series for $\frac{1}{z-z_{k}}$ which converges uniformly and absolutely on this set. Here is why:

$$
\frac{1}{z-z_{k}}=\left(\frac{-1}{1-\frac{z}{z_{k}}}\right) \frac{1}{z_{k}}=\frac{-1}{z_{k}} \sum_{l=0}^{\infty}\left(\frac{z}{z_{k}}\right)^{l}
$$

and the Weierstrass $M$ test can be applied because

$$
\left|\frac{z}{z_{k}}\right|<\frac{1}{2}
$$

on this set. Therefore, by Corollary 56.1.7, $S_{k}(z)$, being a polynomial in $\frac{1}{z-z_{k}}$, has a power series which converges uniformly to $S_{k}(z)$ on $K_{k}$. Therefore, there exists a polynomial, $P_{k}(z)$ such that

$$
\left\|P_{k}-S_{k}\right\|_{\overline{B\left(0,\left|z_{k}\right| / 2\right), \infty}}<\frac{1}{2^{k}}
$$

Let

$$
\begin{equation*}
Q(z) \equiv \sum_{k=1}^{\infty}\left(S_{k}(z)-P_{k}(z)\right) \tag{56.2.12}
\end{equation*}
$$

Consider $z \in K_{m}$ and let $N$ be large enough that if $k>N$, then $\left|z_{k}\right|>2|z|$

$$
Q(z)=\sum_{k=1}^{N}\left(S_{k}(z)-P_{k}(z)\right)+\sum_{k=N+1}^{\infty}\left(S_{k}(z)-P_{k}(z)\right) .
$$

On $K_{m}$, the second sum converges uniformly to a function analytic on int $\left(K_{m}\right)$ (interior of $K_{m}$ ) while the first is a rational function having poles at $z_{1}, \cdots, z_{N}$. Since any compact set is contained in $K_{m}$ for large enough $m$, this shows $Q(z)$ is meromorphic as claimed and has poles with the given singularities.

Now consider the case where the poles are at $\left\{z_{k}\right\}_{k=0}^{\infty}$ with $z_{0}=0$. Everything is similar in this case. Let

$$
Q(z) \equiv S_{0}(z)+\sum_{k=1}^{\infty}\left(S_{k}(z)-P_{k}(z)\right)
$$

The series converges uniformly on every compact set because of the assumption that

$$
\lim _{n \rightarrow \infty}\left|z_{n}\right|=\infty
$$

which implies that any compact set is contained in $K_{k}$ for $k$ large enough. Choose $N$ such that $z \in \operatorname{int}\left(K_{N}\right)$ and $z_{n} \notin K_{N}$ for all $n \geq N+1$. Then

$$
Q(z)=S_{0}(z)+\sum_{k=1}^{N}\left(S_{k}(z)-P_{k}(z)\right)+\sum_{k=N+1}^{\infty}\left(S_{k}(z)-P_{k}(z)\right)
$$

The last sum is analytic on $\operatorname{int}\left(K_{N}\right)$ because each function in the sum is analytic due to the fact that none of its poles are in $K_{N}$. Also, $S_{0}(z)+\sum_{k=1}^{N}\left(S_{k}(z)-P_{k}(z)\right)$ is a finite sum of rational functions so it is a rational function and $P_{k}$ is a polynomial so $z_{m}$ is a pole of this function with the correct singularity whenever $z_{m} \in \operatorname{int}\left(K_{N}\right)$.

### 56.2.3 Functions Meromorphic On $\widehat{\mathbb{C}}$

Sometimes it is useful to think of isolated singular points at $\infty$.

Definition 56.2.3 Suppose $f$ is analytic on $\{z \in \mathbb{C}:|z|>r\}$. Then $f$ is said to have a removable singularity at $\infty$ if the function, $g(z) \equiv f\left(\frac{1}{z}\right)$ has a removable singularity at 0 . $f$ is said to have a pole at $\infty$ if the function, $g(z)=f\left(\frac{1}{z}\right)$ has a pole at 0 . Then $f$ is said to be meromorphic on $\widehat{\mathbb{C}}$ if all its singularities are isolated and either poles or removable.

So what is $f$ like for these cases? First suppose $f$ has a removable singularity at $\infty$ $\left(f\left(\frac{1}{z}\right)=g(z)\right.$ has a removable singularity at 0 ). Then $z g(z)$ converges to 0 as $z \rightarrow 0$. It follows $g(z)$ must be analytic near 0 and so can be given as a power series. Thus $f(z)$ is of the form $f(z)=g\left(\frac{1}{z}\right)=\sum_{n=0}^{\infty} a_{n}\left(\frac{1}{z}\right)^{n}$. Next suppose $f$ has a pole at $\infty$. This means $g(z)$ has a pole at 0 so $g(z)$ is of the form $g(z)=\sum_{k=1}^{m} \frac{b_{k}}{z^{k}}+h(z)$ where $h(z)$ is analytic near 0 . Thus in the case of a pole at $\infty, f(z)$ is of the form $f(z)=g\left(\frac{1}{z}\right)=\sum_{k=1}^{m} b_{k} z^{k}+\sum_{n=0}^{\infty} a_{n}\left(\frac{1}{z}\right)^{n}$.

It turns out that the functions which are meromorphic on $\widehat{\mathbb{C}}$ are all rational functions. To see this, suppose $f$ is meromorphic on $\widehat{\mathbb{C}}$ and note that there exists $r>0$ such that $f(z)$ is analytic for $|z|>r$. This is required if $\infty$ is to be isolated. Therefore, there are only finitely many poles of $f$ for $|z| \leq r,\left\{a_{1}, \cdots, a_{m}\right\}$, because by assumption, these poles are isolated and this is a compact set. Let the singular part of $f$ at $a_{k}$ be denoted by $S_{k}(z)$. Then $f(z)-\sum_{k=1}^{m} S_{k}(z)$ is analytic on all of $\mathbb{C}$. Therefore, it is bounded on $|z| \leq r$. In one case, $f$ has a removable singularity at $\infty$. In this case, $f$ is bounded as $z \rightarrow \infty$ and $\sum_{k} S_{k}$ also converges to 0 as $z \rightarrow \infty$. Therefore, by Liouville's theorem, $f(z)-\sum_{k=1}^{m} S_{k}(z)$ equals a constant and so $f-\sum_{k} S_{k}$ is a constant. Thus $f$ is a rational function. In the other case that $f$ has a pole at $\infty, f(z)-\sum_{k=1}^{m} S_{k}(z)-\sum_{k=1}^{m} b_{k} z^{k}=\sum_{n=0}^{\infty} a_{n}\left(\frac{1}{z}\right)^{n}-\sum_{k=1}^{m} S_{k}(z)$. Now $f(z)-\sum_{k=1}^{m} S_{k}(z)-\sum_{k=1}^{m} b_{k} z^{k}$ is analytic on $\mathbb{C}$ and so is bounded on $|z| \leq r$. But now $\sum_{n=0}^{\infty} a_{n}\left(\frac{1}{z}\right)^{n}-\sum_{k=1}^{m} S_{k}(z)$ converges to 0 as $z \rightarrow \infty$ and so by Liouville's theorem, $f(z)-$ $\sum_{k=1}^{m} S_{k}(z)-\sum_{k=1}^{m} b_{k} z^{k}$ must equal a constant and again, $f(z)$ equals a rational function.

### 56.2.4 Great And Glorious Theorem, Simply Connected Regions

Here is given a laundry list of properties which are equivalent to an open set being simply connected. Recall Definition 51.7.21 on Page 1643 which said that an open set, $\Omega$ is simply connected means $\widehat{\mathbb{C}} \backslash \Omega$ is connected. Recall also that this is not the same thing at all as saying $\mathbb{C} \backslash \Omega$ is connected. Consider the outside of a disk for example. I will continue to use this definition for simply connected because it is the most convenient one for complex analysis. However, there are many other equivalent conditions. First here is an interesting lemma which is interesting for its own sake. Recall $n(p, \gamma)$ means the winding number of $\gamma$ about $p$. Now recall Theorem 51.7.25 implies the following lemma in which $B^{C}$ is playing the role of $\Omega$ in Theorem 51.7.25.

Lemma 56.2.4 Let $K$ be a compact subset of $B^{C}$, the complement of a closed set. Then there exist continuous, closed, bounded variation oriented curves $\left\{\Gamma_{j}\right\}_{j=1}^{m}$ for which $\Gamma_{j}^{*} \cap$ $K=\emptyset$ for each $j, \Gamma_{j}^{*} \subseteq \Omega$, and for all $p \in K$,

$$
\sum_{k=1}^{m} n\left(\Gamma_{k}, p\right)=1
$$

while for all $z \in B$

$$
\sum_{k=1}^{m} n\left(\Gamma_{k}, z\right)=0
$$

Definition 56.2.5 Let $\gamma$ be a closed curve in an open set, $\Omega, \gamma:[a, b] \rightarrow \Omega$. Then $\gamma$ is said to be homotopic to a point, $p$ in $\Omega$ if there exists a continuous function, $H:[0,1] \times[a, b] \rightarrow \Omega$ such that $H(0, t)=p, H(\alpha, a)=H(\alpha, b)$, and $H(1, t)=\gamma(t)$. This function, $H$ is called a homotopy.

Lemma 56.2.6 Suppose $\gamma$ is a closed continuous bounded variation curve in an open set, $\Omega$ which is homotopic to a point. Then if $a \notin \Omega$, it follows $n(a, \gamma)=0$.

Proof: Let $H$ be the homotopy described above. The problem with this is that it is not known that $H(\alpha, \cdot)$ is of bounded variation. There is no reason it should be. Therefore, it might not make sense to take the integral which defines the winding number. There are various ways around this. Extend $H$ as follows. $H(\alpha, t)=H(\alpha, a)$ for $t<a, H(\alpha, t)=$ $H(\alpha, b)$ for $t>b$. Let $\varepsilon>0$.

$$
H_{\varepsilon}(\alpha, t) \equiv \frac{1}{2 \varepsilon} \int_{-2 \varepsilon+t+\frac{2 \varepsilon}{(b-a)}(t-a)}^{t+\frac{2 \varepsilon}{(b-a)}(t-a)} H(\alpha, s) d s, H_{\varepsilon}(0, t)=p
$$

Thus $H_{\mathcal{E}}(\alpha, \cdot)$ is a closed curve which has bounded variation and when $\alpha=1$, this converges to $\gamma$ uniformly on $[a, b]$. Therefore, for $\varepsilon$ small enough, $n\left(a, H_{\varepsilon}(1, \cdot)\right)=n(a, \gamma)$ because they are both integers and as $\varepsilon \rightarrow 0, n\left(a, H_{\varepsilon}(1, \cdot)\right) \rightarrow n(a, \gamma)$. Also, $H_{\varepsilon}(\alpha, t) \rightarrow$ $H(\alpha, t)$ uniformly on $[0,1] \times[a, b]$ because of uniform continuity of $H$. Therefore, for small enough $\varepsilon$, you can also assume $H_{\mathcal{E}}(\alpha, t) \in \Omega$ for all $\alpha, t$. Now $\alpha \rightarrow n\left(a, H_{\mathcal{E}}(\alpha, \cdot)\right)$ is continuous. Hence it must be constant because the winding number is integer valued. But

$$
\lim _{\alpha \rightarrow 0} \frac{1}{2 \pi i} \int_{H_{\varepsilon}(\alpha, \cdot)} \frac{1}{z-a} d z=0
$$

because the length of $H_{\varepsilon}(\alpha, \cdot)$ converges to 0 and the integrand is bounded because $a \notin \Omega$. Therefore, the constant can only equal 0 . This proves the lemma.

Now it is time for the great and glorious theorem on simply connected regions. The following equivalence of properties is taken from Rudin [113]. There is a slightly different list in Conway [32] and a shorter list in Ash [7].

Theorem 56.2.7 The following are equivalent for an open set, $\Omega \neq \mathbb{C}$.

1. $\Omega$ is homeomorphic to the unit disk, $B(0,1)$.
2. Every closed curve contained in $\Omega$ is homotopic to a point in $\Omega$.
3. If $z \notin \Omega$, and if $\gamma$ is a closed bounded variation continuous curve in $\Omega$, then $n(\gamma, z)=$ 0 .
4. $\Omega$ is simply connected, ( $\widehat{\mathbb{C}} \backslash \Omega$ is connected and $\Omega$ is connected.)
5. Every function analytic on $\Omega$ can be uniformly approximated by polynomials on compact subsets.
6. For every $f$ analytic on $\Omega$ and every closed continuous bounded variation curve, $\gamma$,

$$
\int_{\gamma} f(z) d z=0
$$

7. Every function analytic on $\Omega$ has a primitive on $\Omega$.
8. If $f, 1 / f$ are both analytic on $\Omega$, then there exists an analytic, $g$ on $\Omega$ such that $f=\exp (g)$.
9. If $f, 1 / f$ are both analytic on $\Omega$, then there exists $\phi$ analytic on $\Omega$ such that $f=\phi^{2}$.

Proof: $1 \Rightarrow 2$. Assume 1 and let $\gamma$ be a closed curve in $\Omega$. Let $h$ be the homeomorphism, $h: B(0,1) \rightarrow \Omega$. Let $H(\alpha, t)=h\left(\alpha\left(h^{-1} \gamma(t)\right)\right)$. This works.
$2 \Rightarrow 3$ This is Lemma 56.2.6.
$3 \Rightarrow 4$. Suppose 3 but 4 fails to hold. Then if $\widehat{\mathbb{C}} \backslash \Omega$ is not connected, there exist disjoint nonempty sets, $A$ and $B$ such that $\bar{A} \cap B=A \cap \bar{B}=\emptyset$. It follows each of these sets must be closed because neither can have a limit point in $\Omega$ nor in the other. Also, one and only one of them contains $\infty$. Let this set be $B$. Thus $A$ is a closed set which must also be bounded. Otherwise, there would exist a sequence of points in $A,\left\{a_{n}\right\}$ such that $\lim _{n \rightarrow \infty} a_{n}=\infty$ which would contradict the requirement that no limit points of $A$ can be in $B$. Therefore, $A$ is a compact set contained in the open set, $B^{C} \equiv\{z \in \mathbb{C}: z \notin B\}$. Pick $p \in A$. By Lemma 56.2.4 there exist continuous bounded variation closed curves $\left\{\Gamma_{k}\right\}_{k=1}^{m}$ which are contained in $B^{C}$, do not intersect $A$ and such that

$$
1=\sum_{k=1}^{m} n\left(p, \Gamma_{k}\right)
$$

However, if these curves do not intersect $A$ and they also do not intersect $B$ then they must be all contained in $\Omega$. Since $p \notin \Omega$, it follows by 3 that for each $k, n\left(p, \Gamma_{k}\right)=0$, a contradiction.
$4 \Rightarrow 5$ This is Corollary 56.1.12 on Page 1773.
$5 \Rightarrow 6$ Every polynomial has a primitive and so the integral over any closed bounded variation curve of a polynomial equals 0 . Let $f$ be analytic on $\Omega$. Then let $\left\{f_{n}\right\}$ be a sequence of polynomials converging uniformly to $f$ on $\gamma^{*}$. Then

$$
0=\lim _{n \rightarrow \infty} \int_{\gamma} f_{n}(z) d z=\int_{\gamma} f(z) d z
$$

$6 \Rightarrow 7$ Pick $z_{0} \in \Omega$. Letting $\gamma\left(z_{0}, z\right)$ be a bounded variation continuous curve joining $z_{0}$ to $z$ in $\Omega$, you define a primitive for $f$ as follows.

$$
F(z)=\int_{\gamma\left(z_{0}, z\right)} f(w) d w
$$

This is well defined by 6 and is easily seen to be a primitive. You just write the difference quotient and take a limit using 6 .

$$
\begin{aligned}
\lim _{w \rightarrow 0} \frac{F(z+w)-F(z)}{w} & =\lim _{w \rightarrow 0} \frac{1}{w}\left(\int_{\gamma\left(z_{0}, z+w\right)} f(u) d u-\int_{\gamma\left(z_{0}, z\right)} f(u) d u\right) \\
& =\lim _{w \rightarrow 0} \frac{1}{w} \int_{\gamma(z, z+w)} f(u) d u \\
& =\lim _{w \rightarrow 0} \frac{1}{w} \int_{0}^{1} f(z+t w) w d t=f(z) .
\end{aligned}
$$

$7 \Rightarrow 8$ Suppose then that $f, 1 / f$ are both analytic. Then $f^{\prime} / f$ is analytic and so it has a primitive by 7 . Let this primitive be $g_{1}$. Then

$$
\begin{aligned}
\left(e^{-g_{1}} f\right)^{\prime} & =e^{-g_{1}}\left(-g_{1}^{\prime}\right) f+e^{-g_{1}} f^{\prime} \\
& =-e^{-g_{1}}\left(\frac{f^{\prime}}{f}\right) f+e^{-g_{1}} f^{\prime}=0 .
\end{aligned}
$$

Therefore, since $\Omega$ is connected, it follows $e^{-g_{1}} f$ must equal a constant. (Why?) Let the constant be $e^{a+i b i}$. Then $f(z)=e^{g_{1}(z)} e^{a+i b}$. Therefore, you let $g(z)=g_{1}(z)+a+i b$.
$8 \Rightarrow 9$ Suppose then that $f, 1 / f$ are both analytic on $\Omega$. Then by $8 f(z)=e^{g(z)}$. Let $\phi(z) \equiv e^{g(z) / 2}$.
$9 \Rightarrow 1$ There are two cases. First suppose $\Omega=\mathbb{C}$. This satisfies condition 9 because if $f, 1 / f$ are both analytic, then the same argument involved in $8 \Rightarrow 9$ gives the existence of a square root. A homeomorphism is $h(z) \equiv \frac{z}{\sqrt{1+|z|^{2}}}$. It obviously maps onto $B(0,1)$ and is continuous. To see it is $1-1$ consider the case of $z_{1}$ and $z_{2}$ having different arguments. Then $h\left(z_{1}\right) \neq h\left(z_{2}\right)$. If $z_{2}=t z_{1}$ for a positive $t \neq 1$, then it is also clear $h\left(z_{1}\right) \neq h\left(z_{2}\right)$. To show $h^{-1}$ is continuous, note that if you have an open set in $\mathbb{C}$ and a point in this open set, you can get a small open set containing this point by allowing the modulus and the argument to lie in some open interval. Reasoning this way, you can verify $h$ maps open sets to open sets. In the case where $\Omega \neq \mathbb{C}$, there exists a one to one analytic map which maps $\Omega$ onto $B(0,1)$ by the Riemann mapping theorem. This proves the theorem.

### 56.3 Exercises

1. Let $a \in \mathbb{C}$. Show there exists a sequence of polynomials, $\left\{p_{n}\right\}$ such that $p_{n}(a)=1$ but $p_{n}(z) \rightarrow 0$ for all $z \neq a$.
2. Let $l$ be a line in $\mathbb{C}$. Show there exists a sequence of polynomials $\left\{p_{n}\right\}$ such that $p_{n}(z) \rightarrow 1$ on one side of this line and $p_{n}(z) \rightarrow-1$ on the other side of the line. Hint: The complement of this line is simply connected.
3. Suppose $\Omega$ is a simply connected region, $f$ is analytic on $\Omega, f \neq 0$ on $\Omega$, and $n \in \mathbb{N}$. Show that there exists an analytic function, $g$ such that $g(z)^{n}=f(z)$ for all $z \in \Omega$. That is, you can take the $n^{\text {th }}$ root of $f(z)$. If $\Omega$ is a region which contains 0 , is it possible to find $g(z)$ such that $g$ is analytic on $\Omega$ and $g(z)^{2}=z$ ?
4. Suppose $\Omega$ is a region (connected open set) and $f$ is an analytic function defined on $\Omega$ such that $f(z) \neq 0$ for any $z \in \Omega$. Suppose also that for every positive integer, $n$ there exists an analytic function, $g_{n}$ defined on $\Omega$ such that $g_{n}^{n}(z)=f(z)$. Show that then it is possible to define an analytic function, $L$ on $f(\Omega)$ such that $e^{L(f(z))}=f(z)$ for all $z \in \Omega$.
5. You know that $\phi(z) \equiv \frac{z-i}{z+i}$ maps the upper half plane onto the unit ball. Its inverse, $\psi(z)=i \frac{1+z}{1-z}$ maps the unit ball onto the upper half plane. Also for $z$ in the upper half plane, you can define a square root as follows. If $z=|z| e^{i \theta}$ where $\theta \in(0, \pi)$, let $z^{1 / 2} \equiv|z|^{1 / 2} e^{i \theta / 2}$ so the square root maps the upper half plane to the first quadrant. Now consider

$$
\begin{equation*}
z \rightarrow \exp \left(-i \log \left[i\left(\frac{1+z}{1-z}\right)\right]^{1 / 2}\right) \tag{56.3.13}
\end{equation*}
$$

Show this is an analytic function which maps the unit ball onto an annulus. Is it possible to find a one to one analytic map which does this?

## Chapter 57

## Infinite Products

The Mittag-Leffler theorem gives existence of a meromorphic function which has specified singular part at various poles. It would be interesting to do something similar to zeros of an analytic function. That is, given the order of the zero at various points, does there exist an analytic function which has these points as zeros with the specified orders? You know that if you have the zeros of the polynomial, you can factor it. Can you do something similar with analytic functions which are just limits of polynomials? These questions involve the concept of an infinite product.

Definition 57.0.1 $\prod_{n=1}^{\infty}\left(1+u_{n}\right) \equiv \lim _{n \rightarrow \infty} \prod_{k=1}^{n}\left(1+u_{k}\right)$ whenever this limit exists. If $u_{n}=u_{n}(z)$ for $z \in H$, we say the infinite product converges uniformly on $H$ if the partial products, $\prod_{k=1}^{n}\left(1+u_{k}(z)\right)$ converge uniformly on $H$.

The main theorem is the following.
Theorem 57.0.2 Let $H \subseteq \mathbb{C}$ and suppose that $\sum_{n=1}^{\infty}\left|u_{n}(z)\right|$ converges uniformly on $H$ where $u_{n}(z)$ bounded on $H$. Then

$$
P(z) \equiv \prod_{n=1}^{\infty}\left(1+u_{n}(z)\right)
$$

converges uniformly on $H$. If $\left(n_{1}, n_{2}, \cdots\right)$ is any permutation of $(1,2, \cdots)$, then for all $z \in H$,

$$
P(z)=\prod_{k=1}^{\infty}\left(1+u_{n_{k}}(z)\right)
$$

and $P$ has a zero at $z_{0}$ if and only if $u_{n}\left(z_{0}\right)=-1$ for some $n$.
Proof: First a simple estimate:

$$
\begin{aligned}
& \prod_{k=m}^{n}\left(1+\left|u_{k}(z)\right|\right) \\
= & \exp \left(\ln \left(\prod_{k=m}^{n}\left(1+\left|u_{k}(z)\right|\right)\right)\right)=\exp \left(\sum_{k=m}^{n} \ln \left(1+\left|u_{k}(z)\right|\right)\right) \\
\leq & \exp \left(\sum_{k=m}^{\infty}\left|u_{k}(z)\right|\right)<e
\end{aligned}
$$

for all $z \in H$ provided $m$ is large enough. Since $\sum_{k=1}^{\infty}\left|u_{k}(z)\right|$ converges uniformly on $H$, $\left|u_{k}(z)\right|<\frac{1}{2}$ for all $z \in H$ provided $k$ is large enough. Thus you can take $\log \left(1+u_{k}(z)\right)$. Pick $N_{0}$ such that for $n>m \geq N_{0}$,

$$
\begin{equation*}
\left|u_{m}(z)\right|<\frac{1}{2}, \prod_{k=m}^{n}\left(1+\left|u_{k}(z)\right|\right)<e \tag{57.0.1}
\end{equation*}
$$

Now having picked $N_{0}$, the assumption the $u_{n}$ are bounded on $H$ implies there exists a constant, $C$, independent of $z \in H$ such that for all $z \in H$,

$$
\begin{equation*}
\prod_{k=1}^{N_{0}}\left(1+\left|u_{k}(z)\right|\right)<C \tag{57.0.2}
\end{equation*}
$$

Let $N_{0}<M<N$. Then

$$
\begin{aligned}
& \left|\prod_{k=1}^{N}\left(1+u_{k}(z)\right)-\prod_{k=1}^{M}\left(1+u_{k}(z)\right)\right| \\
\leq & \prod_{k=1}^{N_{0}}\left(1+\left|u_{k}(z)\right|\right)\left|\prod_{k=N_{0}+1}^{N}\left(1+u_{k}(z)\right)-\prod_{k=N_{0}+1}^{M}\left(1+u_{k}(z)\right)\right| \\
\leq & C\left|\prod_{k=N_{0}+1}^{N}\left(1+u_{k}(z)\right)-\prod_{k=N_{0}+1}^{M}\left(1+u_{k}(z)\right)\right| \\
\leq & C\left(\prod_{k=N_{0}+1}^{M}\left(1+\left|u_{k}(z)\right|\right)\right)\left|\prod_{k=M+1}^{N}\left(1+u_{k}(z)\right)-1\right| \\
\leq & C e\left|\prod_{k=M+1}^{N}\left(1+\left|u_{k}(z)\right|\right)-1\right|
\end{aligned}
$$

Since $1 \leq \prod_{k=M+1}^{N}\left(1+\left|u_{k}(z)\right|\right) \leq e$, it follows the term on the far right is dominated by

$$
\begin{aligned}
& C e^{2}\left|\ln \left(\prod_{k=M+1}^{N}\left(1+\left|u_{k}(z)\right|\right)\right)-\ln 1\right| \\
\leq & C e^{2} \sum_{k=M+1}^{N} \ln \left(1+\left|u_{k}(z)\right|\right) \\
\leq & C e^{2} \sum_{k=M+1}^{N}\left|u_{k}(z)\right|<\varepsilon
\end{aligned}
$$

uniformly in $z \in H$ provided $M$ is large enough. This follows from the simple observation that if $1<x<e$, then $x-1 \leq e(\ln x-\ln 1)$. Therefore, $\left\{\prod_{k=1}^{m}\left(1+u_{k}(z)\right)\right\}_{m=1}^{\infty}$ is uniformly Cauchy on $H$ and therefore, converges uniformly on $H$. Let $P(z)$ denote the function it converges to.

What about the permutations? Let $\left\{n_{1}, n_{2}, \cdots\right\}$ be a permutation of the indices. Let $\varepsilon>0$ be given and let $N_{0}$ be such that if $n>N_{0}$,

$$
\left|\prod_{k=1}^{n}\left(1+u_{k}(z)\right)-P(z)\right|<\varepsilon
$$

for all $z \in H$. Let $\{1,2, \cdots, n\} \subseteq\left\{n_{1}, n_{2}, \cdots, n_{p(n)}\right\}$ where $p(n)$ is an increasing sequence.

Then from 57.0.1 and 57.0.2,

$$
\begin{aligned}
& \left|P(z)-\prod_{k=1}^{p(n)}\left(1+u_{n_{k}}(z)\right)\right| \\
\leq & \left|P(z)-\prod_{k=1}^{n}\left(1+u_{k}(z)\right)\right|+\left|\prod_{k=1}^{n}\left(1+u_{k}(z)\right)-\prod_{k=1}^{p(n)}\left(1+u_{n_{k}}(z)\right)\right| \\
\leq & \varepsilon+\left|\prod_{k=1}^{n}\left(1+u_{k}(z)\right)-\prod_{k=1}^{p(n)}\left(1+u_{n_{k}}(z)\right)\right| \\
\leq & \varepsilon+\left|\prod_{k=1}^{n}\left(1+\left|u_{k}(z)\right|\right)\right|\left|1-\prod_{n_{k}>n}\left(1+u_{n_{k}}(z)\right)\right| \\
\leq & \varepsilon+\left|\prod_{k=1}^{N_{0}}\left(1+\left|u_{k}(z)\right|\right)\right|\left|\prod_{k=N_{0}+1}^{n}\left(1+\left|u_{k}(z)\right|\right)\right|\left|1-\prod_{n_{k}>n}\left(1+u_{n_{k}}(z)\right)\right| \\
\leq & \varepsilon+C e\left|\prod_{n_{k}>n}\left(1+\left|u_{n_{k}}(z)\right|\right)-1\right| \leq \varepsilon+C e\left|\prod_{k=n+1}^{M(p(n))}\left(1+\left|u_{n_{k}}(z)\right|\right)-1\right|
\end{aligned}
$$

where $M(p(n))$ is the largest index in the permuted list, $\left\{n_{1}, n_{2}, \cdots, n_{p(n)}\right\}$. then from 57.0.1, this last term is dominated by

$$
\begin{aligned}
& \varepsilon+C e^{2}\left|\ln \left(\prod_{k=n+1}^{M(p(n))}\left(1+\left|u_{n_{k}}(z)\right|\right)\right)-\ln 1\right| \\
\leq & \varepsilon+C e^{2} \sum_{k=n+1}^{\infty} \ln \left(1+\left|u_{n_{k}}\right|\right) \leq \varepsilon+C e^{2} \sum_{k=n+1}^{\infty}\left|u_{n_{k}}\right|<2 \varepsilon
\end{aligned}
$$

for all $n$ large enough uniformly in $z \in H$. Therefore, $\left|P(z)-\prod_{k=1}^{p(n)}\left(1+u_{n_{k}}(z)\right)\right|<2 \varepsilon$ whenever $n$ is large enough. This proves the part about the permutation.

It remains to verify the assertion about the points, $z_{0}$, where $P\left(z_{0}\right)=0$. Obviously, if $u_{n}\left(z_{0}\right)=-1$, then $P\left(z_{0}\right)=0$. Suppose then that $P\left(z_{0}\right)=0$ and $M>N_{0}$. Then

$$
\left|\prod_{k=1}^{M}\left(1+u_{k}\left(z_{0}\right)\right)\right|=
$$

$$
\begin{aligned}
&\left|\prod_{k=1}^{M}\left(1+u_{k}\left(z_{0}\right)\right)-\prod_{k=1}^{\infty}\left(1+u_{k}\left(z_{0}\right)\right)\right| \\
& \leq\left|\prod_{k=1}^{M}\left(1+u_{k}\left(z_{0}\right)\right)\right|\left|1-\prod_{k=M+1}^{\infty}\left(1+u_{k}\left(z_{0}\right)\right)\right| \\
& \leq\left|\prod_{k=1}^{M}\left(1+u_{k}\left(z_{0}\right)\right)\right|\left|\prod_{k=M+1}^{\infty}\left(1+\left|u_{k}\left(z_{0}\right)\right|\right)-1\right| \\
& \leq e\left|\prod_{k=1}^{M}\left(1+u_{k}\left(z_{0}\right)\right)\right|\left|\ln \prod_{k=M+1}^{\infty}\left(1+\left|u_{k}\left(z_{0}\right)\right|\right)-\ln 1\right| \\
& \leq e\left(\sum_{k=M+1}^{\infty} \ln \left(1+\left|u_{k}(z)\right|\right)\right)\left|\prod_{k=1}^{M}\left(1+u_{k}\left(z_{0}\right)\right)\right| \\
& \leq e \sum_{k=M+1}^{\infty}\left|u_{k}(z)\right|\left|\prod_{k=1}^{M}\left(1+u_{k}\left(z_{0}\right)\right)\right| \\
& \leq \frac{1}{2}\left|\prod_{k=1}^{M}\left(1+u_{k}\left(z_{0}\right)\right)\right|
\end{aligned}
$$

whenever $M$ is large enough. Therefore, for such $M$,

$$
\prod_{k=1}^{M}\left(1+u_{k}\left(z_{0}\right)\right)=0
$$

and so $u_{k}\left(z_{0}\right)=-1$ for some $k \leq M$. This proves the theorem.

### 57.1 Analytic Function With Prescribed Zeros

Suppose you are given complex numbers, $\left\{z_{n}\right\}$ and you want to find an analytic function, $f$ such that these numbers are the zeros of $f$. How can you do it? The problem is easy if there are only finitely many of these zeros, $\left\{z_{1}, z_{2}, \cdots, z_{m}\right\}$. You just write $\left(z-z_{1}\right)\left(z-z_{2}\right) \cdots\left(z-z_{m}\right)$. Now if none of the $z_{k}=0$ you could also write it as

$$
\prod_{k=1}^{m}\left(1-\frac{z}{z_{k}}\right)
$$

and this might have a better chance of success in the case of infinitely many prescribed zeros. However, you would need to verify something like $\sum_{n=1}^{\infty}\left|\frac{z}{z_{n}}\right|<\infty$ which might not be so. The way around this is to adjust the product, making it $\prod_{k=1}^{\infty}\left(1-\frac{z}{z_{k}}\right) e^{g_{k}(z)}$ where $g_{k}(z)$ is some analytic function. Recall also that for $|x|<1, \ln \left((1-x)^{-1}\right)=\sum_{n=1}^{\infty} \frac{x^{n}}{n}$. If you had $x / x_{n}$ small and real, then $1=\left(1-x / x_{n}\right) \exp \left(\ln \left(\left(1-x / x_{n}\right)^{-1}\right)\right)$ and $\prod_{k=1}^{\infty} 1$ of course converges but loses all the information about zeros. However, this is why it is not
too unreasonable to consider factors of the form

$$
\left(1-\frac{z}{z_{k}}\right) e^{\sum_{k=1}^{p_{k}}\left(\frac{z}{z_{k}}\right)^{k} \frac{1}{k}}
$$

where $p_{k}$ is suitably chosen.
First here are some estimates.
Lemma 57.1.1 For $z \in \mathbb{C}$,

$$
\begin{equation*}
\left|e^{z}-1\right| \leq|z| e^{|z|} \tag{57.1.3}
\end{equation*}
$$

and if $|z| \leq 1 / 2$,

$$
\begin{equation*}
\left|\sum_{k=m}^{\infty} \frac{z^{k}}{k}\right| \leq \frac{1}{m} \frac{|z|^{m}}{1-|z|} \leq \frac{2}{m}|z|^{m} \leq \frac{1}{m} \frac{1}{2^{m-1}} \tag{57.1.4}
\end{equation*}
$$

Proof: Consider 57.1.3.

$$
\left|e^{z}-1\right|=\left|\sum_{k=1}^{\infty} \frac{z^{k}}{k!}\right| \leq \sum_{k=1}^{\infty} \frac{|z|^{k}}{k!}=e^{|z|}-1 \leq|z| e^{|z|}
$$

the last inequality holding by the mean value theorem. Now consider 57.1.4.

$$
\begin{aligned}
\left|\sum_{k=m}^{\infty} \frac{z^{k}}{k}\right| & \leq \sum_{k=m}^{\infty} \frac{|z|^{k}}{k} \leq \frac{1}{m} \sum_{k=m}^{\infty}|z|^{k} \\
& =\frac{1}{m} \frac{|z|^{m}}{1-|z|} \leq \frac{2}{m}|z|^{m} \leq \frac{1}{m} \frac{1}{2^{m-1}}
\end{aligned}
$$

This proves the lemma.
The functions, $E_{p}$ in the next definition are called the elementary factors.
Definition 57.1.2 Let $E_{0}(z) \equiv 1-z$ and for $p \geq 1$,

$$
E_{p}(z) \equiv(1-z) \exp \left(z+\frac{z^{2}}{2}+\cdots+\frac{z^{p}}{p}\right)
$$

In terms of this new symbol, here is another estimate. A sharper inequality is available in Rudin [113] but it is more difficult to obtain.

Corollary 57.1.3 For $E_{p}$ defined above and $|z| \leq 1 / 2$,

$$
\left|E_{p}(z)-1\right| \leq 3|z|^{p+1}
$$

Proof: From elementary calculus, $\ln (1-x)=-\sum_{n=1}^{\infty} \frac{x^{n}}{n}$ for all $|x|<1$. Therefore, for $|z|<1$,

$$
\log (1-z)=-\sum_{n=1}^{\infty} \frac{z^{n}}{n}, \log \left((1-z)^{-1}\right)=\sum_{n=1}^{\infty} \frac{z^{n}}{n}
$$

because the function $\log (1-z)$ and the analytic function, $-\sum_{n=1}^{\infty} \frac{z^{n}}{n}$ both are equal to $\ln (1-x)$ on the real line segment $(-1,1)$, a set which has a limit point. Therefore, using Lemma 57.1.1,

$$
\begin{aligned}
& \left|E_{p}(z)-1\right| \\
= & \left|(1-z) \exp \left(z+\frac{z^{2}}{2}+\cdots+\frac{z^{p}}{p}\right)-1\right| \\
= & \left|(1-z) \exp \left(\log \left((1-z)^{-1}\right)-\sum_{n=p+1}^{\infty} \frac{z^{n}}{n}\right)-1\right| \\
= & \left|\exp \left(-\sum_{n=p+1}^{\infty} \frac{z^{n}}{n}\right)-1\right| \\
\leq & \left|-\sum_{n=p+1}^{\infty} \frac{z^{n}}{n}\right| e^{\left|-\sum_{n=p+1}^{\infty} \frac{z^{n}}{n}\right|} \\
\leq & \frac{1}{p+1} \cdot 2 \cdot e^{1 /(p+1)}|z|^{p+1} \cdot \leq 3|z|^{p+1}
\end{aligned}
$$

This proves the corollary.
With this estimate, it is easy to prove the Weierstrass product formula.
Theorem 57.1.4 Let $\left\{z_{n}\right\}$ be a sequence of nonzero complex numbers which have no limit point in $\mathbb{C}$ and let $\left\{p_{n}\right\}$ be a sequence of nonnegative integers such that

$$
\begin{equation*}
\sum_{n=1}^{\infty}\left(\frac{R}{\left|z_{n}\right|}\right)^{p_{n}+1}<\infty \tag{57.1.5}
\end{equation*}
$$

for all $R \in \mathbb{R}$. Then

$$
P(z) \equiv \prod_{n=1}^{\infty} E_{p_{n}}\left(\frac{z}{z_{n}}\right)
$$

is analytic on $\mathbb{C}$ and has a zero at each point, $z_{n}$ and at no others. If $w$ occurs $m$ times in $\left\{z_{n}\right\}$, then $P$ has a zero of order $m$ at $w$.

Proof: Since $\left\{z_{n}\right\}$ has no limit point, it follows $\lim _{n \rightarrow \infty}\left|z_{n}\right|=\infty$. Therefore, if $p_{n}=$ $n-1$ the condition, 57.1.5 holds for this choice of $p_{n}$. Now by Theorem 57.0.2, the infinite product in this theorem will converge uniformly on $|z| \leq R$ if the same is true of the sum,

$$
\begin{equation*}
\sum_{n=1}^{\infty}\left|E_{p_{n}}\left(\frac{z}{z_{n}}\right)-1\right| . \tag{57.1.6}
\end{equation*}
$$

But by Corollary 57.1.3 the $n^{t h}$ term of this sum satisfies

$$
\left|E_{p_{n}}\left(\frac{z}{z_{n}}\right)-1\right| \leq 3\left|\frac{z}{z_{n}}\right|^{p_{n}+1}
$$

Since $\left|z_{n}\right| \rightarrow \infty$, there exists $N$ such that for $n>N,\left|z_{n}\right|>2 R$. Therefore, for $|z|<R$ and letting $0<a=\min \left\{\left|z_{n}\right|: n \leq N\right\}$,

$$
\begin{aligned}
& \sum_{n=1}^{\infty}\left|E_{p_{n}}\left(\frac{z}{z_{n}}\right)-1\right| \leq 3 \sum_{n=1}^{N}\left|\frac{R}{a}\right|^{p_{n}+1} \\
& +3 \sum_{n=N}^{\infty}\left(\frac{R}{2 R}\right)^{p_{n}+1}<\infty
\end{aligned}
$$

By the Weierstrass $M$ test, the series in 57.1 .6 converges uniformly for $|z|<R$ and so the same is true of the infinite product. It follows from Lemma 51.3.13 on Page 1622 that $P(z)$ is analytic on $|z|<R$ because it is a uniform limit of analytic functions.

Also by Theorem 57.0.2 the zeros of the analytic $P(z)$ are exactly the points, $\left\{z_{n}\right\}$, listed according to multiplicity. That is, if $z_{n}$ is a zero of order $m$, then if it is listed $m$ times in the formula for $P(z)$, then it is a zero of order $m$ for $P$. This proves the theorem.

The following corollary is an easy consequence and includes the case where there is a zero at 0 .

Corollary 57.1.5 Let $\left\{z_{n}\right\}$ be a sequence of nonzero complex numbers which have no limit point and let $\left\{p_{n}\right\}$ be a sequence of nonnegative integers such that

$$
\begin{equation*}
\sum_{n=1}^{\infty}\left(\frac{r}{\left|z_{n}\right|}\right)^{1+p_{n}}<\infty \tag{57.1.7}
\end{equation*}
$$

for all $r \in \mathbb{R}$. Then

$$
P(z) \equiv z^{m} \prod_{n=1}^{\infty} E_{p_{n}}\left(\frac{z}{z_{n}}\right)
$$

is analytic $\Omega$ and has a zero at each point, $z_{n}$ and at no others along with a zero of order $m$ at 0 . If $w$ occurs $m$ times in $\left\{z_{n}\right\}$, then $P$ has a zero of order $m$ at $w$.

The above theory can be generalized to include the case of an arbitrary open set. First, here is a lemma.

Lemma 57.1.6 Let $\Omega$ be an open set. Also let $\left\{z_{n}\right\}$ be a sequence of points in $\Omega$ which is bounded and which has no point repeated more than finitely many times such that $\left\{z_{n}\right\}$ has no limit point in $\Omega$. Then there exist $\left\{w_{n}\right\} \subseteq \partial \Omega$ such that $\lim _{n \rightarrow \infty}\left|z_{n}-w_{n}\right|=0$.

Proof: Since $\partial \Omega$ is closed, there exists $w_{n} \in \partial \Omega$ such that $\operatorname{dist}\left(z_{n}, \partial \Omega\right)=\left|z_{n}-w_{n}\right|$. Now if there is a subsequence, $\left\{z_{n_{k}}\right\}$ such that $\left|z_{n_{k}}-w_{n_{k}}\right| \geq \varepsilon$ for all $k$, then $\left\{z_{n_{k}}\right\}$ must possess a limit point because it is a bounded infinite set of points. However, this limit point can only be in $\Omega$ because $\left\{z_{n_{k}}\right\}$ is bounded away from $\partial \Omega$. This is a contradiction. Therefore, $\lim _{n \rightarrow \infty}\left|z_{n}-w_{n}\right|=0$. This proves the lemma.

Corollary 57.1.7 Let $\left\{z_{n}\right\}$ be a sequence of complex numbers contained in $\Omega$, an open subset of $\mathbb{C}$ which has no limit point in $\Omega$. Suppose each $z_{n}$ is repeated no more than finitely many times. Then there exists a function $f$ which is analytic on $\Omega$ whose zeros are exactly $\left\{z_{n}\right\}$. If $w \in\left\{z_{n}\right\}$ and $w$ is listed $m$ times, then $w$ is a zero of order $m$ of $f$.

Proof: There is nothing to prove if $\left\{z_{n}\right\}$ is finite. You just let $f(z)=\prod_{j=1}^{m}\left(z-z_{j}\right)$ where $\left\{z_{n}\right\}=\left\{z_{1}, \cdots, z_{m}\right\}$.

Pick $w \in \Omega \backslash\left\{z_{n}\right\}_{n=1}^{\infty}$ and let $h(z) \equiv \frac{1}{z-w}$. Since $w$ is not a limit point of $\left\{z_{n}\right\}$, there exists $r>0$ such that $B(w, r)$ contains no points of $\left\{z_{n}\right\}$. Let $\Omega_{1} \equiv \Omega \backslash\{w\}$. Now $h$ is not constant and so $h\left(\Omega_{1}\right)$ is an open set by the open mapping theorem. In fact, $h$ maps each component of $\Omega$ to a region. $\left|z_{n}-w\right|>r$ for all $z_{n}$ and so $\left|h\left(z_{n}\right)\right|<r^{-1}$. Thus the sequence, $\left\{h\left(z_{n}\right)\right\}$ is a bounded sequence in the open set $h\left(\Omega_{1}\right)$. It has no limit point in $h\left(\Omega_{1}\right)$ because this is true of $\left\{z_{n}\right\}$ and $\Omega_{1}$. By Lemma 57.1.6 there exist $w_{n} \in \partial\left(h\left(\Omega_{1}\right)\right)$ such that $\lim _{n \rightarrow \infty}\left|w_{n}-h\left(z_{n}\right)\right|=0$. Consider for $z \in \Omega_{1}$

$$
\begin{equation*}
f(z) \equiv \prod_{n=1}^{\infty} E_{n}\left(\frac{h\left(z_{n}\right)-w_{n}}{h(z)-w_{n}}\right) . \tag{57.1.8}
\end{equation*}
$$

Letting $K$ be a compact subset of $\Omega_{1}, h(K)$ is a compact subset of $h\left(\Omega_{1}\right)$ and so if $z \in K$, then $\left|h(z)-w_{n}\right|$ is bounded below by a positive constant. Therefore, there exists $N$ large enough that for all $z \in K$ and $n \geq N$,

$$
\left|\frac{h\left(z_{n}\right)-w_{n}}{h(z)-w_{n}}\right|<\frac{1}{2}
$$

and so by Corollary 57.1.3, for all $z \in K$ and $n \geq N$,

$$
\begin{equation*}
\left|E_{n}\left(\frac{h\left(z_{n}\right)-w_{n}}{h(z)-w_{n}}\right)-1\right| \leq 3\left(\frac{1}{2}\right)^{n} \tag{57.1.9}
\end{equation*}
$$

Therefore,

$$
\sum_{n=1}^{\infty}\left|E_{n}\left(\frac{h\left(z_{n}\right)-w_{n}}{h(z)-w_{n}}\right)-1\right|
$$

converges uniformly for $z \in K$. This implies $\prod_{n=1}^{\infty} E_{n}\left(\frac{h\left(z_{n}\right)-w_{n}}{h(z)-w_{n}}\right)$ also converges uniformly for $z \in K$ by Theorem 57.0.2. Since $K$ is arbitrary, this shows $f$ defined in 57.1.8 is analytic on $\Omega_{1}$.

Also if $z_{n}$ is listed $m$ times so it is a zero of multiplicity $m$ and $w_{n}$ is the point from $\partial\left(h\left(\Omega_{1}\right)\right)$ closest to $h\left(z_{n}\right)$, then there are $m$ factors in 57.1.8 which are of the form

$$
\begin{align*}
E_{n}\left(\frac{h\left(z_{n}\right)-w_{n}}{h(z)-w_{n}}\right) & =\left(1-\frac{h\left(z_{n}\right)-w_{n}}{h(z)-w_{n}}\right) e^{g_{n}(z)} \\
& =\left(\frac{h(z)-h\left(z_{n}\right)}{h(z)-w_{n}}\right) e^{g_{n}(z)} \\
& =\frac{z_{n}-z}{(z-w)\left(z_{n}-w\right)}\left(\frac{1}{h(z)-w_{n}}\right) e^{g_{n}(z)} \\
& =\left(z-z_{n}\right) G_{n}(z) \tag{57.1.10}
\end{align*}
$$

where $G_{n}$ is an analytic function which is not zero at and near $z_{n}$. Therefore, $f$ has a zero of order $m$ at $z_{n}$. This proves the theorem except for the point, $w$ which has been left out
of $\Omega_{1}$. It is necessary to show $f$ is analytic at this point also and right now, $f$ is not even defined at $w$.

The $\left\{w_{n}\right\}$ are bounded because $\left\{h\left(z_{n}\right)\right\}$ is bounded and $\lim _{n \rightarrow \infty}\left|w_{n}-h\left(z_{n}\right)\right|=0$ which implies $\left|w_{n}-h\left(z_{n}\right)\right| \leq C$ for some constant, $C$. Therefore, there exists $\delta>0$ such that if $z \in B^{\prime}(w, \boldsymbol{\delta})$, then for all $n$,

$$
\left|\frac{h\left(z_{n}\right)-w}{\left(\frac{1}{z-w}\right)-w_{n}}\right|=\left|\frac{h\left(z_{n}\right)-w_{n}}{h(z)-w_{n}}\right|<\frac{1}{2}
$$

Thus 57.1.9 holds for all $z \in B^{\prime}(w, \boldsymbol{\delta})$ and $n$ so by Theorem 57.0.2, the infinite product in 57.1.8 converges uniformly on $B^{\prime}(w, \boldsymbol{\delta})$. This implies $f$ is bounded in $B^{\prime}(w, \boldsymbol{\delta})$ and so $w$ is a removable singularity and $f$ can be extended to $w$ such that the result is analytic. It only remains to verify $f(w) \neq 0$. After all, this would not do because it would be another zero other than those in the given list. By 57.1.10, a partial product is of the form

$$
\begin{equation*}
\prod_{n=1}^{N}\left(\frac{h(z)-h\left(z_{n}\right)}{h(z)-w_{n}}\right) e^{g_{n}(z)} \tag{57.1.11}
\end{equation*}
$$

where

$$
g_{n}(z) \equiv\left(\frac{h\left(z_{n}\right)-w_{n}}{h(z)-w_{n}}+\frac{1}{2}\left(\frac{h\left(z_{n}\right)-w_{n}}{h(z)-w_{n}}\right)^{2}+\cdots+\frac{1}{n}\left(\frac{h\left(z_{n}\right)-w_{n}}{h(z)-w_{n}}\right)^{n}\right)
$$

Each of the quotients in the definition of $g_{n}(z)$ converges to 0 as $z \rightarrow w$ and so the partial product of 57.1.11 converges to 1 as $z \rightarrow w$ because $\left(\frac{h(z)-h\left(z_{n}\right)}{h(z)-w_{n}}\right) \rightarrow 1$ as $z \rightarrow w$.

If $f(w)=0$, then if $z$ is close enough to $w$, it follows $|f(z)|<\frac{1}{2}$. Also, by the uniform convergence on $B^{\prime}(w, \boldsymbol{\delta})$, it follows that for some $N$, the partial product up to $N$ must also be less than $1 / 2$ in absolute value for all $z$ close enough to $w$ and as noted above, this does not occur because such partial products converge to 1 as $z \rightarrow w$. Hence $f(w) \neq 0$. This proves the corollary.

Recall the definition of a meromorphic function on Page 1636. It was a function which is analytic everywhere except at a countable set of isolated points at which the function has a pole. It is clear that the quotient of two analytic functions yields a meromorphic function but is this the only way it can happen?

Theorem 57.1.8 Suppose $Q$ is a meromorphic function on an open set, $\Omega$. Then there exist analytic functions on $\Omega, f(z)$ and $g(z)$ such that $Q(z)=f(z) / g(z)$ for all $z$ not in the set of poles of $Q$.

Proof: Let $Q$ have a pole of order $m(z)$ at $z$. Then by Corollary 57.1.7 there exists an analytic function, $g$ which has a zero of order $m(z)$ at every $z \in \Omega$. It follows $g Q$ has a removable singularity at the poles of $Q$. Therefore, there is an analytic function, $f$ such that $f(z)=g(z) Q(z)$. This proves the theorem.

Corollary 57.1.9 Suppose $\Omega$ is a region and $Q$ is a meromorphic function defined on $\Omega$ such that the set, $\{z \in \Omega: Q(z)=c\}$ has a limit point in $\Omega$. Then $Q(z)=c$ for all $z \in \Omega$.

Proof: From Theorem 57.1.8 there are analytic functions, $f, g$ such that $Q=\frac{f}{g}$. Therefore, the zero set of the function, $f(z)-c g(z)$ has a limit point in $\Omega$ and so $f(z)-c g(z)=0$ for all $z \in \Omega$. This proves the corollary.

### 57.2 Factoring A Given Analytic Function

The next theorem is the Weierstrass factorization theorem which can be used to factor a given analytic function $f$. If $f$ has a zero of order $m$ when $z=0$, then you could factor out a $z^{m}$ and from there consider the factorization of what remains when you have factored out the $z^{m}$. Therefore, the following is the main thing of interest.

Theorem 57.2.1 Let $f$ be analytic on $\mathbb{C}, f(0) \neq 0$, and let the zeros of $f$, be $\left\{z_{k}\right\}$, listed according to order. (Thus if $z$ is a zero of order $m$, it will be listed $m$ times in the list, $\left\{z_{k}\right\}$.) Choosing nonnegative integers, $p_{n}$ such that for all $r>0$,

$$
\sum_{n=1}^{\infty}\left(\frac{r}{\left|z_{n}\right|}\right)^{p_{n}+1}<\infty
$$

There exists an entire function, $g$ such that

$$
\begin{equation*}
f(z)=e^{g(z)} \prod_{n=1}^{\infty} E_{p_{n}}\left(\frac{z}{z_{n}}\right) \tag{57.2.12}
\end{equation*}
$$

Note that $e^{g(z)} \neq 0$ for any $z$ and this is the interesting thing about this function.
Proof: $\left\{z_{n}\right\}$ cannot have a limit point because if there were a limit point of this sequence, it would follow from Theorem 51.5.3 that $f(z)=0$ for all $z$, contradicting the hypothesis that $f(0) \neq 0$. Hence $\lim _{n \rightarrow \infty}\left|z_{n}\right|=\infty$ and so

$$
\sum_{n=1}^{\infty}\left(\frac{r}{\left|z_{n}\right|}\right)^{1+n-1}=\sum_{n=1}^{\infty}\left(\frac{r}{\left|z_{n}\right|}\right)^{n}<\infty
$$

by the root test. Therefore, by Theorem 57.1.4

$$
P(z)=\prod_{n=1}^{\infty} E_{p_{n}}\left(\frac{z}{z_{n}}\right)
$$

a function analytic on $\mathbb{C}$ by picking $p_{n}=n-1$ or perhaps some other choice. ( $p_{n}=n-1$ works but there might be another choice that would work.) Then $f / P$ has only removable singularities in $\mathbb{C}$ and no zeros thanks to Theorem 57.1.4. Thus, letting $h(z)=f(z) / P(z)$, Corollary 51.7.23 implies that $h^{\prime} / h$ has a primitive, $\widetilde{g}$. Then

$$
\left(h e^{-\widetilde{g}}\right)^{\prime}=0
$$

and so

$$
h(z)=e^{a+i b} e^{\widetilde{g}(z)}
$$

for some constants, $a, b$. Therefore, letting $g(z)=\widetilde{g}(z)+a+i b, h(z)=e^{g(z)}$ and thus 57.2.12 holds. This proves the theorem.

Corollary 57.2.2 Let $f$ be analytic on $\mathbb{C}$, $f$ has a zero of order $m$ at 0 , and let the other zeros of $f$ be $\left\{z_{k}\right\}$, listed according to order. (Thus if $z$ is a zero of order $l$, it will be listed $l$ times in the list, $\left\{z_{k}\right\}$.) Also let

$$
\begin{equation*}
\sum_{n=1}^{\infty}\left(\frac{r}{\left|z_{n}\right|}\right)^{1+p_{n}}<\infty \tag{57.2.13}
\end{equation*}
$$

for any choice of $r>0$. Then there exists an entire function, $g$ such that

$$
\begin{equation*}
f(z)=z^{m} e^{g(z)} \prod_{n=1}^{\infty} E_{p_{n}}\left(\frac{z}{z_{n}}\right) \tag{57.2.14}
\end{equation*}
$$

Proof: Since $f$ has a zero of order $m$ at 0 , it follows from Theorem 51.5.3 that $\left\{z_{k}\right\}$ cannot have a limit point in $\mathbb{C}$ and so you can apply Theorem 57.2.1 to the function, $f(z) / z^{m}$ which has a removable singularity at 0 . This proves the corollary.

### 57.2.1 Factoring Some Special Analytic Functions

Factoring a polynomial is in general a hard task. It is true it is easy to prove the factors exist but finding them is another matter. Corollary 57.2 .2 gives the existence of factors of a certain form but it does not tell how to find them. This should not be surprising. You can't expect things to get easier when you go from polynomials to analytic functions. Nevertheless, it is possible to factor some popular analytic functions. These factorizations are based on the following Mitag-Leffler expansions. By an auspicious choice of the contour and the method of residues it is possible to obtain a very interesting formula for $\cot \pi z$.

Example 57.2.3 Let $\gamma_{N}$ be the contour which goes from $-N-\frac{1}{2}-N i$ horizontally to $N+$ $\frac{1}{2}-N i$ and from there, vertically to $N+\frac{1}{2}+N i$ and then horizontally to $-N-\frac{1}{2}+N i$ and finally vertically to $-N-\frac{1}{2}-N i$. Thus the contour is a large rectangle and the direction of integration is in the counter clockwise direction. Consider the integral

$$
I_{N} \equiv \int_{\gamma_{N}} \frac{\pi \cos \pi z}{\sin \pi z\left(\alpha^{2}-z^{2}\right)} d z
$$

where $\alpha \in \mathbb{R}$ is not an integer. This will be used to verify the formula of Mittag-Leffler,

$$
\begin{equation*}
\frac{1}{\alpha}+\sum_{n=1}^{\infty} \frac{2 \alpha}{\alpha^{2}-n^{2}}=\pi \cot \pi \alpha \tag{57.2.15}
\end{equation*}
$$

First you show that $\cot \pi z$ is bounded on this contour. This is easy using the formula for $\cot (z)=\frac{e^{i z}+e^{-i z}}{e^{i z}-e^{-i z}}$. Therefore, $I_{N} \rightarrow 0$ as $N \rightarrow \infty$ because the integrand is of order $1 / N^{2}$ while the diameter of $\gamma_{N}$ is of order $N$. Next you compute the residues of the integrand at $\pm \alpha$ and at $n$ where $|n|<N+\frac{1}{2}$ for $n$ an integer. These are the only singularities of the integrand in this contour and therefore, using the residue theorem, you can evaluate $I_{N}$ by using these. You can calculate these residues and find that the residue at $\pm \alpha$ is
$\frac{-\pi \cos \pi \alpha}{2 \alpha \sin \pi \alpha}$
while the residue at $n$ is

$$
\frac{1}{\alpha^{2}-n^{2}}
$$

Therefore

$$
0=\lim _{N \rightarrow \infty} I_{N}=\lim _{N \rightarrow \infty} 2 \pi i\left[\sum_{n=-N}^{N} \frac{1}{\alpha^{2}-n^{2}}-\frac{\pi \cot \pi \alpha}{\alpha}\right]
$$

which establishes the following formula of Mittag Leffler.

$$
\lim _{N \rightarrow \infty} \sum_{n=-N}^{N} \frac{1}{\alpha^{2}-n^{2}}=\frac{\pi \cot \pi \alpha}{\alpha}
$$

Writing this in a slightly nicer form, you obtain 57.2.15.
This is a very interesting formula. This will be used to factor $\sin (\pi z)$. The zeros of this function are at the integers. Therefore, considering 57.2.13 you can pick $p_{n}=1$ in the Weierstrass factorization formula. Therefore, by Corollary 57.2.2 there exists an analytic function $g(z)$ such that

$$
\begin{equation*}
\sin (\pi z)=z e^{g(z)} \prod_{n=1}^{\infty}\left(1-\frac{z}{z_{n}}\right) e^{z / z_{n}} \tag{57.2.16}
\end{equation*}
$$

where the $z_{n}$ are the nonzero integers. Remember you can permute the factors in these products. Therefore, this can be written more conveniently as

$$
\sin (\pi z)=z e^{g(z)} \prod_{n=1}^{\infty}\left(1-\left(\frac{z}{n}\right)^{2}\right)
$$

and it is necessary to find $g(z)$. Differentiating both sides of 57.2.16

$$
\begin{aligned}
\pi \cos (\pi z)= & e^{g(z)} \prod_{n=1}^{\infty}\left(1-\left(\frac{z}{n}\right)^{2}\right)+z g^{\prime}(z) e^{g(z)} \prod_{n=1}^{\infty}\left(1-\left(\frac{z}{n}\right)^{2}\right) \\
& +z e^{g(z)} \sum_{n=1}^{\infty}-\left(\frac{2 z}{n^{2}}\right) \prod_{k \neq n}\left(1-\left(\frac{z}{k}\right)^{2}\right)
\end{aligned}
$$

Now divide both sides by $\sin (\pi z)$ to obtain

$$
\begin{aligned}
\pi \cot (\pi z) & =\frac{1}{z}+g^{\prime}(z)-\sum_{n=1}^{\infty} \frac{2 z / n^{2}}{\left(1-z^{2} / n^{2}\right)} \\
& =\frac{1}{z}+g^{\prime}(z)+\sum_{n=1}^{\infty} \frac{2 z}{z^{2}-n^{2}}
\end{aligned}
$$

By 57.2.15, this yields $g^{\prime}(z)=0$ for $z$ not an integer and so $g(z)=c$, a constant. So far this yields

$$
\sin (\pi z)=z e^{c} \prod_{n=1}^{\infty}\left(1-\left(\frac{z}{n}\right)^{2}\right)
$$

and it only remains to find $c$. Divide both sides by $\pi z$ and take a limit as $z \rightarrow 0$. Using the power series of $\sin (\pi z)$, this yields

$$
1=\frac{e^{c}}{\pi}
$$

and so $c=\ln \pi$. Therefore,

$$
\begin{equation*}
\sin (\pi z)=z \pi \prod_{n=1}^{\infty}\left(1-\left(\frac{z}{n}\right)^{2}\right) . \tag{57.2.17}
\end{equation*}
$$

Example 57.2.4 Find an interesting formula for $\tan (\pi z)$.
This is easy to obtain from the formula for $\cot (\pi z)$.

$$
\cot \left(\pi\left(z+\frac{1}{2}\right)\right)=-\tan \pi z
$$

for $z$ real and therefore, this formula holds for $z$ complex also. Therefore, for $z+\frac{1}{2}$ not an integer

$$
\pi \cot \left(\pi\left(z+\frac{1}{2}\right)\right)=\frac{2}{2 z+1}+\sum_{n=1}^{\infty} \frac{2 z+1}{\left(\frac{2 z+1}{2}\right)^{2}-n^{2}}
$$

### 57.3 The Existence Of An Analytic Function With Given Values

The Weierstrass product formula, Theorem 57.1.4, along with the Mittag-Leffler theorem, Theorem 56.2.1 can be used to obtain an analytic function which has given values on a countable set of points, having no limit point. This is clearly an amazing result and indicates how potent these theorems are. In fact, you can show that it isn't just the values of the function which may be specified at the points in this countable set of points but the derivatives up to any finite order.

Theorem 57.3.1 Let $P \equiv\left\{z_{k}\right\}_{k=1}^{\infty}$ be a set of points in $\mathbb{C}$, which has no limit point. For each $z_{k}$, consider

$$
\begin{equation*}
\sum_{j=0}^{m_{k}} a_{j}^{k}\left(z-z_{k}\right)^{j} \tag{57.3.18}
\end{equation*}
$$

Then there exists an analytic function defined on $\mathbb{C}$ such that the Taylor series of $f$ at $z_{k}$ has the first $m_{k}$ terms given by 57.3.18. ${ }^{1}$

Proof: By the Weierstrass product theorem, Theorem 57.1.4, there exists an analytic function, $f$ defined on all of $\Omega$ such that $f$ has a zero of order $m_{k}+1$ at $z_{k}$. Consider this $z_{k}$ Thus for $z$ near $z_{k}$,

$$
f(z)=\sum_{j=m_{k}+1}^{\infty} c_{j}\left(z-z_{k}\right)^{j}
$$

[^36]where $c_{m_{k}+1} \neq 0$. You choose $b_{1}, b_{2}, \cdots, b_{m_{k}+1}$ such that
$$
f(z)\left(\sum_{l=1}^{m_{k}+1} \frac{b_{l}}{\left(z-z_{k}\right)^{k}}\right)=\sum_{j=0}^{m_{k}} a_{j}^{k}\left(z-z_{k}\right)^{j}+\sum_{k=m_{k}+1}^{\infty} c_{j}^{k}\left(z-z_{k}\right)^{j} .
$$

Thus you need

$$
\sum_{l=1}^{m_{k}+1} \sum_{j=m_{k}+1}^{\infty} c_{j} b_{l}\left(z-z_{k}\right)^{j-l}=\sum_{r=0}^{m_{k}} a_{r}^{k}\left(z-z_{k}\right)^{r}+\text { Higher order terms }
$$

It follows you need to solve the following system of equations for $b_{1}, \cdots, b_{m_{k}+1}$.

$$
\begin{gathered}
c_{m_{k}+1} b_{m_{k}+1}=a_{0}^{k} \\
c_{m_{k}+2} b_{m_{k}+1}+c_{m_{k}+1} b_{m_{k}}=a_{1}^{k} \\
c_{m_{k}+3} b_{m_{k}+1}+c_{m_{k}+2} b_{m_{k}}+c_{m_{k}+1} b_{m_{k}-1}=a_{2}^{k} \\
\vdots \\
c_{m_{k}+m_{k}+1} b_{m_{k}+1}+c_{m_{k}+m_{k}} b_{m_{k}}+\cdots+c_{m_{k}+1} b_{1}=a_{m_{k}}^{k}
\end{gathered}
$$

Since $c_{m_{k}+1} \neq 0$, it follows there exists a unique solution to the above system. You first solve for $b_{m_{k}+1}$ in the top. Then, having found it, you go to the next and use $c_{m_{k}+1} \neq 0$ again to find $b_{m_{k}}$ and continue in this manner. Let $S_{k}(z)$ be determined in this manner for each $z_{k}$. By the Mittag-Leffler theorem, there exists a Meromorphic function, $g$ such that $g$ has exactly the singularities, $S_{k}(z)$. Therefore, $f(z) g(z)$ has removable singularities at each $z_{k}$ and for $z$ near $z_{k}$, the first $m_{k}$ terms of $f g$ are as prescribed. This proves the theorem.

Corollary 57.3.2 Let $P \equiv\left\{z_{k}\right\}_{k=1}^{\infty}$ be a set of points in $\Omega$, an open set such that $P$ has no limit points in $\Omega$. For each $z_{k}$, consider

$$
\begin{equation*}
\sum_{j=0}^{m_{k}} a_{j}^{k}\left(z-z_{k}\right)^{j} \tag{57.3.19}
\end{equation*}
$$

Then there exists an analytic function defined on $\Omega$ such that the Taylor series of $f$ at $z_{k}$ has the first $m_{k}$ terms given by 57.3.19.

Proof: The proof is identical to the above except you use the versions of the MittagLeffler theorem and Weierstrass product which pertain to open sets.

Definition 57.3.3 Denote by $H(\Omega)$ the analytic functions defined on $\Omega$, an open subset of $\mathbb{C}$. Then $H(\Omega)$ is a commutative ring ${ }^{2}$ with the usual operations of addition and multiplication. A set, $I \subseteq H(\Omega)$ is called a finitely generated ideal of the ring if $I$ is of the form

$$
\left\{\sum_{k=1}^{n} g_{k} f_{k}: f_{k} \in H(\Omega) \text { for } k=1,2, \cdots, n\right\}
$$

[^37]where $g_{1}, \cdots, g_{n}$ are given functions in $H(\Omega)$. This ideal is also denoted as $\left[g_{1}, \cdots, g_{n}\right]$ and is called the ideal generated by the functions, $\left\{g_{1}, \cdots, g_{n}\right\}$. Since there are finitely many of these functions it is called a finitely generated ideal. A principal ideal is one which is generated by a single function. An example of such a thing is $[1]=H(\Omega)$.

Then there is the following interesting theorem.
Theorem 57.3.4 Every finitely generated ideal in $H(\Omega)$ for $\Omega$ a connected open set (region) is a principal ideal.

Proof: Let $I=\left[g_{1}, \cdots, g_{n}\right]$ be a finitely generated ideal as described above. Then if any of the functions has no zeros, this ideal would consist of $H(\Omega)$ because then $g_{i}^{-1} \in H(\Omega)$ and so $1 \in I$. It follows all the functions have zeros. If any of the functions has a zero of infinite order, then the function equals zero on $\Omega$ because $\Omega$ is connected and can be deleted from the list. Similarly, if the zeros of any of these functions have a limit point in $\Omega$, then the function equals zero and can be deleted from the list. Thus, without loss of generality, all zeros are of finite order and there are no limit points of the zeros in $\Omega$. Let $m\left(g_{i}, z\right)$ denote the order of the zero of $g_{i}$ at $z$. If $g_{i}$ has no zero at $z$, then $m\left(g_{i}, z\right)=0$.

I claim that if no point of $\Omega$ is a zero of all the $g_{i}$, then the conclusion of the theorem is true and in fact $\left[g_{1}, \cdots, g_{n}\right]=[1]=H(\Omega)$. The claim is obvious if $n=1$ because this assumption that no point is a zero of all the functions implies $g \neq 0$ and so $g^{-1}$ is analytic. Hence $1 \in\left[g_{1}\right]$. Suppose it is true for $n-1$ and consider $\left[g_{1}, \cdots, g_{n}\right]$ where no point of $\Omega$ is a zero of all the $g_{i}$. Even though this may be true of $\left\{g_{1}, \cdots, g_{n}\right\}$, it may not be true of $\left\{g_{1}, \cdots, g_{n-1}\right\}$. By Corollary 57.1.7 there exists $\phi$, a function analytic on $\Omega$ such that $m(\phi, z)=\min \left\{m\left(g_{i}, z\right), i=1,2, \cdots, n-1\right\}$. Thus the functions $\left\{g_{1} / \phi, \cdots, g_{n-1} / \phi\right\}$.are all analytic. Could they all equal zero at some point, $z$ ? If so, pick $i$ where $m(\phi, z)=$ $m\left(g_{i}, z\right)$. Thus $g_{i} / \phi$ is not equal to zero at $z$ after all and so these functions are analytic there is no point of $\Omega$ which is a zero of all of them. By induction, $\left[g_{1} / \phi, \cdots, g_{n-1} / \phi\right]=H(\Omega)$. (Also there are no new zeros obtained in this way.)

Now this means there exist functions $f_{i} \in H(\Omega)$ such that

$$
\sum_{i=1}^{n} f_{i}\left(\frac{g_{i}}{\phi}\right)=1
$$

and so $\phi=\sum_{i=1}^{n} f_{i} g_{i}$. Therefore, $[\phi] \subseteq\left[g_{1}, \cdots, g_{n-1}\right]$. On the other hand, if $\sum_{k=1}^{n-1} h_{k} g_{k} \in$ [ $g_{1}, \cdots, g_{n-1}$ ] you could define $h \equiv \sum_{k=1}^{n-1} h_{k}\left(g_{k} / \phi\right)$, an analytic function with the property that $h \phi=\sum_{k=1}^{n-1} h_{k} g_{k}$ which shows $[\phi]=\left[g_{1}, \cdots, g_{n-1}\right]$. Therefore,

$$
\left[g_{1}, \cdots, g_{n}\right]=\left[\phi, g_{n}\right]
$$

Now $\phi$ has no zeros in common with $g_{n}$ because the zeros of $\phi$ are contained in the set of zeros for $g_{1}, \cdots, g_{n-1}$. Now consider a zero, $\alpha$ of $\phi$. It is not a zero of $g_{n}$ and so near $\alpha$, these functions have the form

$$
\phi(z)=\sum_{k=m}^{\infty} a_{k}(z-\alpha)^{k}, g_{n}(z)=\sum_{k=0}^{\infty} b_{k}(z-\alpha)^{k}, b_{0} \neq 0
$$

I want to determine coefficients for an analytic function, $h$ such that

$$
\begin{equation*}
m\left(1-h g_{n}, \alpha\right) \geq m(\phi, \alpha) \tag{57.3.20}
\end{equation*}
$$

Let

$$
h(z)=\sum_{k=0}^{\infty} c_{k}(z-\alpha)^{k}
$$

and the $c_{k}$ must be determined. Using Merten's theorem, the power series for $1-h g_{n}$ is of the form

$$
1-b_{0} c_{0}-\sum_{j=1}^{\infty}\left(\sum_{r=0}^{j} b_{j-r} c_{r}\right)(z-\alpha)^{j}
$$

First determine $c_{0}$ such that $1-c_{0} b_{0}=0$. This is no problem because $b_{0} \neq 0$. Next you need to get the coefficients of $(z-\alpha)$ to equal zero. This requires

$$
b_{1} c_{0}+b_{0} c_{1}=0
$$

Again, there is no problem because $b_{0} \neq 0$. In fact, $c_{1}=\left(-b_{1} c_{0} / b_{0}\right)$. Next consider the second order terms if $m \geq 2$.

$$
b_{2} c_{0}+b_{1} c_{1}+b_{0} c_{2}=0
$$

Again there is no problem in solving, this time for $c_{2}$ because $b_{0} \neq 0$. Continuing this way, you see that in every step, the $c_{k}$ which needs to be solved for is multiplied by $b_{0} \neq$ 0 . Therefore, by Corollary 57.1.7 there exists an analytic function, $h$ satisfying 57.3.20. Therefore, $\left(1-h g_{n}\right) / \phi$ has a removable singularity at every zero of $\phi$ and so may be considered an analytic function. Therefore,

$$
1=\frac{1-h g_{n}}{\phi} \phi+h g_{n} \in\left[\phi, g_{n}\right]=\left[g_{1} \cdots g_{n}\right]
$$

which shows $\left[g_{1} \cdots g_{n}\right]=H(\Omega)=[1]$. It follows the claim is established.
Now suppose $\left\{g_{1} \cdots g_{n}\right\}$ are just elements of $H(\Omega)$. As explained above, it can be assumed they all have zeros of finite order and the zeros have no limit point in $\Omega$ since if these occur, you can delete the function from the list. By Corollary 57.1.7 there exists $\phi \in H(\Omega)$ such that $m(\phi, z) \leq \min \left\{m\left(g_{i}, z\right): i=1, \cdots, n\right\}$. Then $g_{k} / \phi$ has a removable singularity at each zero of $g_{k}$ and so can be regarded as an analytic function. Also, as before, there is no point which is a zero of each $g_{k} / \phi$ and so by the first part of this argument, $\left[g_{1} / \phi \cdots g_{n} / \phi\right]=H(\Omega)$. As in the first part of the argument, this implies $\left[g_{1} \cdots g_{n}\right]=[\phi]$ which proves the theorem. $\left[g_{1} \cdots g_{n}\right]$ is a principal ideal as claimed.

The following corollary follows from the above theorem. You don't need to assume $\Omega$ is connected.

Corollary 57.3.5 Every finitely generated ideal in $H(\Omega)$ for $\Omega$ an open set is a principal ideal.

Proof: Let $\left[g_{1}, \cdots, g_{n}\right]$ be a finitely generated ideal in $H(\Omega)$. Let $\left\{U_{k}\right\}$ be the components of $\Omega$. Then applying the above to each component, there exists $h_{k} \in H\left(U_{k}\right)$ such that restricting each $g_{i}$ to $U_{k},\left[g_{1}, \cdots, g_{n}\right]=\left[h_{k}\right]$. Then let $h(z)=h_{k}(z)$ for $z \in U_{k}$. This is an analytic function which works.

### 57.4 Jensen's Formula

This interesting formula relates the zeros of an analytic function to an integral. The proof given here follows Alfors, [3]. First, here is a technical lemma.

## Lemma 57.4.1

$$
\int_{-\pi}^{\pi} \ln \left|1-e^{i \theta}\right| d \theta=0
$$

Proof: First note that the only problem with the integrand occurs when $\theta=0$. However, this is an integrable singularity so the integral will end up making sense. Letting $z=e^{i \theta}$, you could get the above integral as a limit as $\varepsilon \rightarrow 0$ of the following contour integral where $\gamma_{\varepsilon}$ is the contour shown in the following picture with the radius of the big circle equal to 1 and the radius of the little circle equal to $\varepsilon$..

$$
\int_{\gamma_{\varepsilon}} \frac{\ln |1-z|}{i z} d z
$$



On the indicated contour, $1-z$ lies in the half plane $\operatorname{Re} z>0$ and so $\log (1-z)=$ $\ln |1-z|+i \arg (1-z)$. The above integral equals

$$
\int_{\gamma_{\varepsilon}} \frac{\log (1-z)}{i z} d z-\int_{\gamma_{\varepsilon}} \frac{\arg (1-z)}{z} d z
$$

The first of these integrals equals zero because the integrand has a removable singularity at 0 . The second equals

$$
\begin{aligned}
& i \int_{-\pi}^{-\eta_{\varepsilon}} \arg \left(1-e^{i \theta}\right) d \theta+i \int_{\eta_{\varepsilon}}^{\pi} \arg \left(1-e^{i \theta}\right) d \theta \\
& +\varepsilon i \int_{-\frac{\pi}{2}-\lambda_{\varepsilon}}^{-\pi} \theta d \theta+\varepsilon i \int_{\pi}^{\frac{\pi}{2}-\lambda_{\varepsilon}} \theta d \theta
\end{aligned}
$$

where $\eta_{\varepsilon}, \lambda_{\varepsilon} \rightarrow 0$ as $\varepsilon \rightarrow 0$. The last two terms converge to 0 as $\varepsilon \rightarrow 0$ while the first two add to zero. To see this, change the variable in the first integral and then recall that when you multiply complex numbers you add the arguments. Thus you end up integrating $\arg$ (real valued function) which equals zero.

In this material on Jensen's equation, $\varepsilon$ will denote a small positive number. Its value is not important as long as it is positive. Therefore, it may change from place to place.

Now suppose $f$ is analytic on $B(0, r+\varepsilon)$, and $f$ has no zeros on $\overline{B(0, r)}$. Then you can define a branch of the logarithm which makes sense for complex numbers near $f(z)$. Thus $z \rightarrow \log (f(z))$ is analytic on $B(0, r+\varepsilon)$. Therefore, its real part, $u(x, y) \equiv \ln |f(x+i y)|$ must be harmonic. Consider the following lemma.

Lemma 57.4.2 Let u be harmonic on $B(0, r+\varepsilon)$. Then

$$
u(0)=\frac{1}{2 \pi} \int_{-\pi}^{\pi} u\left(r e^{i \theta}\right) d \theta
$$

Proof: For a harmonic function, $u$ defined on $B(0, r+\varepsilon)$, there exists an analytic function, $h=u+i v$ where

$$
v(x, y) \equiv \int_{0}^{y} u_{x}(x, t) d t-\int_{0}^{x} u_{y}(t, 0) d t
$$

By the Cauchy integral theorem,

$$
h(0)=\frac{1}{2 \pi i} \int_{\gamma_{r}} \frac{h(z)}{z} d z=\frac{1}{2 \pi} \int_{-\pi}^{\pi} h\left(r e^{i \theta}\right) d \theta
$$

Therefore, considering the real part of $h$,

$$
u(0)=\frac{1}{2 \pi} \int_{-\pi}^{\pi} u\left(r e^{i \theta}\right) d \theta
$$

This proves the lemma.
Now this shows the following corollary.
Corollary 57.4.3 Suppose $f$ is analytic on $B(0, r+\varepsilon)$ and has no zeros on $\overline{B(0, r)}$. Then

$$
\begin{equation*}
\ln |f(0)|=\frac{1}{2 \pi} \int_{-\pi}^{\pi} \ln \left|f\left(r e^{i \theta}\right)\right| \tag{57.4.21}
\end{equation*}
$$

What if $f$ has some zeros on $|z|=r$ but none on $B(0, r)$ ? It turns out 57.4.21 is still valid. Suppose the zeros are at $\left\{r e^{i \theta_{k}}\right\}_{k=1}^{m}$, listed according to multiplicity. Then let

$$
g(z)=\frac{f(z)}{\prod_{k=1}^{m}\left(z-r e^{i \theta_{k}}\right)}
$$

It follows $g$ is analytic on $B(0, r+\varepsilon)$ but has no zeros in $\overline{B(0, r)}$. Then 57.4.21 holds for $g$ in place of $f$. Thus

$$
\begin{aligned}
& \ln |f(0)|-\sum_{k=1}^{m} \ln |r| \\
= & \frac{1}{2 \pi} \int_{-\pi}^{\pi} \ln \left|f\left(r e^{i \theta}\right)\right| d \theta-\frac{1}{2 \pi} \int_{-\pi}^{\pi} \sum_{k=1}^{m} \ln \left|r e^{i \theta}-r e^{i \theta_{k}}\right| d \theta \\
= & \frac{1}{2 \pi} \int_{-\pi}^{\pi} \ln \left|f\left(r e^{i \theta}\right)\right| d \theta-\frac{1}{2 \pi} \int_{-\pi}^{\pi} \sum_{k=1}^{m} \ln \left|e^{i \theta}-e^{i \theta_{k}}\right| d \theta-\sum_{k=1}^{m} \ln |r| \\
= & \frac{1}{2 \pi} \int_{-\pi}^{\pi} \ln \left|f\left(r e^{i \theta}\right)\right| d \theta-\frac{1}{2 \pi} \int_{-\pi}^{\pi} \sum_{k=1}^{m} \ln \left|e^{i \theta}-1\right| d \theta-\sum_{k=1}^{m} \ln |r|
\end{aligned}
$$

Therefore, 57.4.21 will continue to hold exactly when $\frac{1}{2 \pi} \int_{-\pi}^{\pi} \sum_{k=1}^{m} \ln \left|e^{i \theta}-1\right| d \theta=0$. But this is the content of Lemma 57.4.1. This proves the following lemma.

Lemma 57.4.4 Suppose $f$ is analytic on $B(0, r+\varepsilon)$ and has no zeros on $B(0, r)$. Then

$$
\begin{equation*}
\ln |f(0)|=\frac{1}{2 \pi} \int_{-\pi}^{\pi} \ln \left|f\left(r e^{i \theta}\right)\right| \tag{57.4.22}
\end{equation*}
$$

With this preparation, it is now not too hard to prove Jensen's formula. Suppose there are $n$ zeros of $f$ in $B(0, r),\left\{a_{k}\right\}_{k=1}^{n}$, listed according to multiplicity, none equal to zero. Let

$$
F(z) \equiv f(z) \prod_{i=1}^{n} \frac{r^{2}-\overline{a_{i}} z}{r\left(z-a_{i}\right)}
$$

Then $F$ is analytic on $B(0, r+\varepsilon)$ and has no zeros in $B(0, r)$. The reason for this is that $f(z) / \prod_{i=1}^{n} r\left(z-a_{i}\right)$ has no zeros there and $r^{2}-\overline{a_{i}} z$ cannot equal zero if $|z|<r$ because if this expression equals zero, then

$$
|z|=\frac{r^{2}}{\left|a_{i}\right|}>r
$$

The other interesting thing about $F(z)$ is that when $z=r e^{i \theta}$,

$$
\begin{aligned}
F\left(r e^{i \theta}\right) & =f\left(r e^{i \theta}\right) \prod_{i=1}^{n} \frac{r^{2}-\overline{a_{i}} r e^{i \theta}}{r\left(r e^{i \theta}-a_{i}\right)} \\
& =f\left(r e^{i \theta}\right) \prod_{i=1}^{n} \frac{r-\overline{a_{i}} e^{i \theta}}{\left(r e^{i \theta}-a_{i}\right)}=f\left(r e^{i \theta}\right) e^{i \theta} \prod_{i=1}^{n} \frac{r e^{-i \theta}-\overline{a_{i}}}{r e^{i \theta}-a_{i}}
\end{aligned}
$$

so $\left|F\left(r e^{i \theta}\right)\right|=\left|f\left(r e^{i \theta}\right)\right|$.
Theorem 57.4.5 Let $f$ be analytic on $B(0, r+\varepsilon)$ and suppose $f(0) \neq 0$. If the zeros of $f$ in $B(0, r)$ are $\left\{a_{k}\right\}_{k=1}^{n}$, listed according to multiplicity, then

$$
\ln |f(0)|=-\sum_{i=1}^{n} \ln \left(\frac{r}{\left|a_{i}\right|}\right)+\frac{1}{2 \pi} \int_{0}^{2 \pi} \ln \left|f\left(r e^{i \theta}\right)\right| d \theta
$$

Proof: From the above discussion and Lemma 57.4.4,

$$
\ln |F(0)|=\frac{1}{2 \pi} \int_{-\pi}^{\pi} \ln \left|f\left(r e^{i \theta}\right)\right| d \theta
$$

But $F(0)=f(0) \prod_{i=1}^{n} \frac{r}{a_{i}}$ and so $\ln |F(0)|=\ln |f(0)|+\sum_{i=1}^{n} \ln \left|\frac{r}{a_{i}}\right|$. Therefore,

$$
\ln |f(0)|=-\sum_{i=1}^{n} \ln \left|\frac{r}{a_{i}}\right|+\frac{1}{2 \pi} \int_{0}^{2 \pi} \ln \left|f\left(r e^{i \theta}\right)\right| d \theta
$$

as claimed.
Written in terms of exponentials this is

$$
|f(0)| \prod_{k=1}^{n}\left|\frac{r}{a_{k}}\right|=\exp \left(\frac{1}{2 \pi} \int_{0}^{2 \pi} \ln \left|f\left(r e^{i \theta}\right)\right| d \theta\right)
$$

### 57.5 Blaschke Products

The Blaschke ${ }^{3}$ product is a way to produce a function which is bounded and analytic on $B(0,1)$ which also has given zeros in $B(0,1)$. The interesting thing here is that there may be infinitely many of these zeros. Thus, unlike the above case of Jensen's inequality, the function is not analytic on $B(0,1)$. Recall for purposes of comparison, Liouville's theorem which says bounded entire functions are constant. The Blaschke product gives examples of bounded functions on $B(0,1)$ which are definitely not constant.

Theorem 57.5.1 Let $\left\{\alpha_{n}\right\}$ be a sequence of nonzero points in $B(0,1)$ with the property that

$$
\sum_{n=1}^{\infty}\left(1-\left|\alpha_{n}\right|\right)<\infty
$$

Then for $k \geq 0$, an integer

$$
B(z) \equiv z^{k} \prod_{k=1}^{\infty} \frac{\alpha_{n}-z}{1-\overline{\alpha_{n}} z} \frac{\left|\alpha_{n}\right|}{\alpha_{n}}
$$

is a bounded function which is analytic on $B(0,1)$ which has zeros only at 0 if $k>0$ and at the $\alpha_{n}$.

Proof: From Theorem 57.0.2 the above product will converge uniformly on $B(0, r)$ for $r<1$ to an analytic function if

$$
\sum_{k=1}^{\infty}\left|\frac{\alpha_{n}-z}{1-\overline{\alpha_{n}} z} \frac{\left|\alpha_{n}\right|}{\alpha_{n}}-1\right|
$$

converges uniformly on $B(0, r)$. But for $|z|<r$,

[^38]and so the assumption on the sum gives uniform convergence of the product on $B(0, r)$ to an analytic function. Since $r<1$ is arbitrary, this shows $B(z)$ is analytic on $B(0,1)$ and has the specified zeros because the only place the factors equal zero are at the $\alpha_{n}$ or 0 .

Now consider the factors in the product. The claim is that they are all no larger in absolute value than 1 . This is very easy to see from the maximum modulus theorem. Let $|\alpha|<1$ and $\phi(z)=\frac{\alpha-z}{1-\bar{\alpha} z}$. Then $\phi$ is analytic near $B(0,1)$ because its only pole is $1 / \bar{\alpha}$. Consider $z=e^{i \theta}$. Then

$$
\left|\phi\left(e^{i \theta}\right)\right|=\left|\frac{\alpha-e^{i \theta}}{1-\bar{\alpha} e^{i \theta}}\right|=\left|\frac{1-\alpha e^{-i \theta}}{1-\bar{\alpha} e^{i \theta}}\right|=1 .
$$

Thus the modulus of $\phi(z)$ equals 1 on $\partial B(0,1)$. Therefore, by the maximum modulus theorem, $|\phi(z)|<1$ if $|z|<1$. This proves the claim that the terms in the product are no larger than 1 and shows the function determined by the Blaschke product is bounded. This proves the theorem.

Note in the conditions for this theorem the one for the sum, $\sum_{n=1}^{\infty}\left(1-\left|\alpha_{n}\right|\right)<\infty$. The Blaschke product gives an analytic function, whose absolute value is bounded by 1 and which has the $\alpha_{n}$ as zeros. What if you had a bounded function, analytic on $B(0,1)$ which had zeros at $\left\{\alpha_{k}\right\}$ ? Could you conclude the condition on the sum? The answer is yes. In fact, you can get by with less than the assumption that $f$ is bounded but this will not be presented here. See Rudin [113]. This theorem is an exciting use of Jensen's equation.

Theorem 57.5.2 Suppose $f$ is an analytic function on $B(0,1), f(0) \neq 0$, and $|f(z)| \leq$ $M$ for all $z \in B(0,1)$. Suppose also that the zeros of $f$ are $\left\{\alpha_{k}\right\}_{k=1}^{\infty}$, listed according to multiplicity. Then $\sum_{k=1}^{\infty}\left(1-\left|\alpha_{k}\right|\right)<\infty$.

Proof: If there are only finitely many zeros, there is nothing to prove so assume there are infinitely many. Also let the zeros be listed such that $\left|\alpha_{n}\right| \leq\left|\alpha_{n+1}\right| \cdots$ Let $n(r)$ denote the number of zeros in $B(0, r)$. By Jensen's formula,

$$
\ln |f(0)|+\sum_{i=1}^{n(r)} \ln r-\ln \left|\alpha_{i}\right|=\frac{1}{2 \pi} \int_{0}^{2 \pi} \ln \left|f\left(r e^{i \theta}\right)\right| d \theta \leq \ln (M)
$$

Therefore, by the mean value theorem,

$$
\sum_{i=1}^{n(r)} \frac{1}{r}\left(r-\left|\alpha_{i}\right|\right) \leq \sum_{i=1}^{n(r)} \ln r-\ln \left|\alpha_{i}\right| \leq \ln (M)-\ln |f(0)|
$$

As $r \rightarrow 1-, n(r) \rightarrow \infty$, and so an application of Fatous lemma yields

$$
\sum_{i=1}^{\infty}\left(1-\left|\alpha_{i}\right|\right) \leq \lim \inf _{r \rightarrow 1-} \sum_{i=1}^{n(r)} \frac{1}{r}\left(r-\left|\alpha_{i}\right|\right) \leq \ln (M)-\ln |f(0)|
$$

This proves the theorem.
You don't need the assumption that $f(0) \neq 0$.

Corollary 57.5.3 Suppose $f$ is an analytic function on $B(0,1)$ and $|f(z)| \leq M$ for all $z \in B(0,1)$. Suppose also that the nonzero zeros ${ }^{4}$ of $f$ are $\left\{\alpha_{k}\right\}_{k=1}^{\infty}$, listed according to multiplicity. Then $\sum_{k=1}^{\infty}\left(1-\left|\alpha_{k}\right|\right)<\infty$.

Proof: Suppose $f$ has a zero of order $m$ at 0 . Then consider the analytic function, $g(z) \equiv f(z) / z^{m}$ which has the same zeros except for 0 . The argument goes the same way except here you use $g$ instead of $f$ and only consider $r>r_{0}>0$. Thus from Jensen's equation,

$$
\begin{aligned}
& \ln |g(0)|+\sum_{i=1}^{n(r)} \ln r-\ln \left|\alpha_{i}\right| \\
= & \frac{1}{2 \pi} \int_{0}^{2 \pi} \ln \left|g\left(r e^{i \theta}\right)\right| d \theta \\
= & \frac{1}{2 \pi} \int_{0}^{2 \pi} \ln \left|f\left(r e^{i \theta}\right)\right| d \theta-\frac{1}{2 \pi} \int_{0}^{2 \pi} m \ln (r) \\
\leq & M+\frac{1}{2 \pi} \int_{0}^{2 \pi} m \ln \left(r^{-1}\right) \\
\leq & M+m \ln \left(\frac{1}{r_{0}}\right)
\end{aligned}
$$

Now the rest of the argument is the same.
An interesting restatement yields the following amazing result.
Corollary 57.5.4 Suppose $f$ is analytic and bounded on $B(0,1)$ having zeros $\left\{\alpha_{n}\right\}$. Then if $\sum_{k=1}^{\infty}\left(1-\left|\alpha_{n}\right|\right)=\infty$, it follows $f$ is identically equal to zero.

### 57.5.1 The Müntz-Szasz Theorem Again

Corollary 57.5.4 makes possible an easy proof of a remarkable theorem named above which yields a wonderful generalization of the Weierstrass approximation theorem. In what follows $b>0$. The Weierstrass approximation theorem states that linear combinations of $1, t, t^{2}, t^{3}, \cdots$ (polynomials) are dense in $C([0, b])$. Let $\lambda_{1}<\lambda_{2}<\lambda_{3}<\cdots$ be an increasing list of positive real numbers. This theorem tells when linear combinations of $1, t^{\lambda_{1}}, t^{\lambda_{2}}, \cdots$ are dense in $C([0, b])$. The proof which follows is like the one given in Rudin [113]. There is a much longer one in Cheney [33] which discusses more aspects of the subject. See also Page 533 where the version given in Cheney is presented. This other approach is much more elementary and does not depend in any way on the theory of functions of a complex variable. There are those of us who automatically prefer real variable techniques. Nevertheless, this proof by Rudin is a very nice and insightful application of the preceding material. Cheney refers to the theorem as the second Müntz theorem. I guess Szasz must also have been involved.

[^39]Theorem 57.5.5 Let $\lambda_{1}<\lambda_{2}<\lambda_{3}<\cdots$ be an increasing list of positive real numbers and let $a>0$. If

$$
\begin{equation*}
\sum_{n=1}^{\infty} \frac{1}{\lambda_{n}}=\infty \tag{57.5.23}
\end{equation*}
$$

then linear combinations of $1, t^{\lambda_{1}}, t^{\lambda_{2}}, \cdots$ are dense in $C([0, b])$.
Proof: Let $X$ denote the closure of linear combinations of $\left\{1, t^{\lambda_{1}}, t^{\lambda_{2}}, \cdots\right\}$ in $C([0, b])$. If $X \neq C([0, b])$, then letting $f \in C([0, b]) \backslash X$, define $\Lambda \in C([0, b])^{\prime}$ as follows. First let $\Lambda_{0}: X+\mathbb{C} f$ be given by $\Lambda_{0}(g+\alpha f)=\alpha\|f\|_{\infty}$. Then

$$
\begin{aligned}
\sup _{\|g+\alpha f\| \leq 1}\left|\Lambda_{0}(g+\alpha f)\right| & =\sup _{\|g+\alpha f\| \leq 1}|\alpha|\|f\|_{\infty} \\
& =\sup _{\|g / \alpha+f\| \leq \frac{1}{|\alpha|}}|\alpha|\|f\|_{\infty} \\
& =\sup _{\|g+f\| \leq \frac{1}{|\alpha|}}|\alpha|\|f\|_{\infty}
\end{aligned}
$$

Now $\operatorname{dist}(f, X)>0$ because $X$ is closed. Therefore, there exists a lower bound, $\eta>0$ to $\|g+f\|$ for $g \in X$. Therefore, the above is no larger than

$$
\sup _{|\alpha| \leq \frac{1}{\eta}}|\alpha|\|f\|_{\infty}=\left(\frac{1}{\eta}\right)\|f\|_{\infty}
$$

which shows that $\left\|\Lambda_{0}\right\| \leq\left(\frac{1}{\eta}\right)\|f\|_{\infty}$. By the Hahn Banach theorem $\Lambda_{0}$ can be extended to $\Lambda \in C([0, b])^{\prime}$ which has the property that $\Lambda(X)=0$ but $\Lambda(f)=\|f\| \neq 0$. By the Weierstrass approximation theorem, Theorem 9.1.7 or one of its cases, there exists a polynomial, $p$ such that $\Lambda(p) \neq 0$. Therefore, if it can be shown that whenever $\Lambda(X)=0$, it is the case that $\Lambda(p)=0$ for all polynomials, it must be the case that $X$ is dense in $C([0, b])$.

By the Riesz representation theorem the elements of $C([0, b])^{\prime}$ are complex measures. Suppose then that for $\mu$ a complex measure it follows that for all $t^{\lambda_{k}}$,

$$
\int_{[0, b]} t^{\lambda_{k}} d \mu=0
$$

I want to show that then

$$
\int_{[0, b]} t^{k} d \mu=0
$$

for all positive integers. It suffices to modify $\mu$ is necessary to have $\mu(\{0\})=0$ since this will not change any of the above integrals. Let $\mu_{1}(E)=\mu(E \cap(0, b])$ and use $\mu_{1}$. I will continue using the symbol, $\mu$.

For $\operatorname{Re}(z)>0$, define

$$
F(z) \equiv \int_{[0, b]} t^{z} d \mu=\int_{(0, b]} t^{z} d \mu
$$

The function $t^{z}=\exp (z \ln (t))$ is analytic. I claim that $F(z)$ is also analytic for $\operatorname{Re} z>0$. Apply Morera's theorem. Let $T$ be a triangle in $\operatorname{Re} z>0$. Then

$$
\int_{\partial T} F(z) d z=\int_{\partial T} \int_{(0, b]} e^{(z \ln (t))} \xi d|\mu| d z
$$

Now $\int_{\partial T}$ can be split into three integrals over intervals of $\mathbb{R}$ and so this integral is essentially a Lebesgue integral taken with respect to Lebesgue measure. Furthermore, $e^{(z \ln (t))}$ is a continuous function of the two variables and $\xi$ is a function of only the one variable, $t$. Thus the integrand is product measurable. The iterated integral is also absolutely integrable because $\left|e^{(z \ln (t))}\right| \leq e^{x \ln t} \leq e^{x \ln b}$ where $x+i y=z$ and $x$ is given to be positive. Thus the integrand is actually bounded. Therefore, you can apply Fubini's theorem and write

$$
\begin{aligned}
\int_{\partial T} F(z) d z & =\int_{\partial T} \int_{(0, b]} e^{(z \ln (t))} \xi d|\mu| d z \\
& =\int_{(0, b]} \xi \int_{\partial T} e^{(z \ln (t))} d z d|\mu|=0
\end{aligned}
$$

By Morera's theorem, $F$ is analytic on $\operatorname{Re} z>0$ which is given to have zeros at the $\lambda_{k}$.
Now let $\phi(z)=\frac{1+z}{1-z}$. Then $\phi$ maps $B(0,1)$ one to one onto $\operatorname{Re} z>0$. To see this let $0<r<1$.

$$
\phi\left(r e^{i \theta}\right)=\frac{1+r e^{i \theta}}{1-r e^{i \theta}}=\frac{1-r^{2}+i 2 r \sin \theta}{1+r^{2}-2 r \cos \theta}
$$

and so $\operatorname{Re} \phi\left(r e^{i \theta}\right)>0$. Now the inverse of $\phi$ is $\phi^{-1}(z)=\frac{z-1}{z+1}$. For $\operatorname{Re} z>0$,

$$
\left|\phi^{-1}(z)\right|^{2}=\frac{z-1}{z+1} \cdot \frac{\bar{z}-1}{\bar{z}+1}=\frac{|z|^{2}-2 \operatorname{Re} z+1}{|z|^{2}+2 \operatorname{Re} z+1}<1
$$

Consider $F \circ \phi$, an analytic function defined on $B(0,1)$. This function is given to have zeros at $z_{n}$ where $\phi\left(z_{n}\right)=\frac{1+z_{n}}{1-z_{n}}=\lambda_{n}$. This reduces to $z_{n}=\frac{-1+\lambda_{n}}{1+\lambda_{n}}$. Now

$$
1-\left|z_{n}\right| \geq \frac{c}{1+\lambda_{n}}
$$

for a positive constant, $c$. It is given that $\sum \frac{1}{\lambda_{n}}=\infty$. so it follows $\sum\left(1-\left|z_{n}\right|\right)=\infty$ also. Therefore, by Corollary 57.5.4, $F \circ \phi=0$. It follows $F=0$ also. In particular, $F(k)$ for $k$ a positive integer equals zero. This has shown that if $\Lambda \in C([0, b])^{\prime}$ and $\Lambda$ sends 1 and all the $t^{\lambda_{n}}$ to 0 , then $\Lambda$ sends 1 and all $t^{k}$ for $k$ a positive integer to zero. As explained above, $X$ is dense in $C((0, b])$.

The converse of this theorem is also true and is proved in Rudin [113].

### 57.6 Exercises

1. Suppose $f$ is an entire function with $f(0)=1$. Let

$$
M(r)=\max \{|f(z)|:|z|=r\}
$$

Use Jensen's equation to establish the following inequality.

$$
M(2 r) \geq 2^{n(r)}
$$

where $n(r)$ is the number of zeros of $f$ in $\overline{B(0, r)}$.
2. The version of the Blaschke product presented above is that found in most complex variable texts. However, there is another one in [89]. Instead of $\frac{\alpha_{n}-z}{1-\overline{\alpha_{n} z}} \frac{\left|\alpha_{n}\right|}{\alpha_{n}}$ you use

$$
\frac{\alpha_{n}-z}{\frac{1}{\overline{\alpha_{n}}}-z}
$$

Prove a version of Theorem 57.5.1 using this modification.
3. The Weierstrass approximation theorem holds for polynomials of $n$ variables on any compact subset of $\mathbb{R}^{n}$. Give a multidimensional version of the Müntz-Szasz theorem which will generalize the Weierstrass approximation theorem for $n$ dimensions. You might just pick a compact subset of $\mathbb{R}^{n}$ in which all components are positive. You have to do something like this because otherwise, $t^{\lambda}$ might not be defined.
4. Show $\cos (\pi z)=\prod_{k=1}^{\infty}\left(1-\frac{4 z^{2}}{(2 k-1)^{2}}\right)$.
5. Recall $\sin (\pi z)=z \pi \prod_{n=1}^{\infty}\left(1-\left(\frac{z}{n}\right)^{2}\right)$. Use this to derive Wallis product,

$$
\frac{\pi}{2}=\prod_{k=1}^{\infty} \frac{4 k^{2}}{(2 k-1)(2 k+1)}
$$

6. The order of an entire function, $f$ is defined as

$$
\inf \left\{a \geq 0:|f(z)| \leq e^{|z|^{a}} \text { for all large enough }|z|\right\}
$$

If no such $a$ exists, the function is said to be of infinite order. Show the order of an entire function is also equal to

$$
\lim \sup _{r \rightarrow \infty} \frac{\ln (\ln (M(r)))}{\ln (r)}
$$

where $M(r) \equiv \max \{|f(z)|:|z|=r\}$.
7. Suppose $\Omega$ is a simply connected region and let $f$ be meromorphic on $\Omega$. Suppose also that the set, $S \equiv\{z \in \Omega: f(z)=c\}$ has a limit point in $\Omega$. Can you conclude $f(z)=c$ for all $z \in \Omega$ ?
8. This and the next collection of problems are dealing with the gamma function. Show that

$$
\left|\left(1+\frac{z}{n}\right) e^{\frac{-z}{n}}-1\right| \leq \frac{C(z)}{n^{2}}
$$

and therefore,

$$
\sum_{n=1}^{\infty}\left|\left(1+\frac{z}{n}\right) e^{\frac{-z}{n}}-1\right|<\infty
$$

with the convergence uniform on compact sets.
9. $\uparrow$ Show $\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right) e^{\frac{-z}{n}}$ converges to an analytic function on $\mathbb{C}$ which has zeros only at the negative integers and that therefore,

$$
\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1} e^{\frac{z}{n}}
$$

is a meromorphic function (Analytic except for poles) having simple poles at the negative integers.
10. $\uparrow$ Show there exists $\gamma$ such that if

$$
\Gamma(z) \equiv \frac{e^{-\gamma_{z}}}{z} \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1} e^{\frac{z}{n}}
$$

then $\Gamma(1)=1$. Thus $\Gamma$ is a meromorphic function having simple poles at the negative integers. Hint: $\prod_{n=1}^{\infty}(1+n) e^{-1 / n}=c=e^{\gamma}$.
11. $\uparrow$ Now show that

$$
\gamma=\lim _{n \rightarrow \infty}\left[\sum_{k=1}^{n} \frac{1}{k}-\ln n\right]
$$

12. $\uparrow$ Justify the following argument leading to Gauss’s formula

$$
\begin{aligned}
& \Gamma(z)=\lim _{n \rightarrow \infty}\left(\prod_{k=1}^{n}\left(\frac{k}{k+z}\right) e^{\frac{z}{k}}\right) \frac{e^{-\gamma z}}{z} \\
= & \lim _{n \rightarrow \infty}\left(\frac{n!}{(1+z)(2+z) \cdots(n+z)} e^{z\left(\sum_{k=1}^{n} \frac{1}{k}\right)}\right) \frac{e^{-\gamma z}}{z} \\
= & \lim _{n \rightarrow \infty} \frac{n!}{(1+z)(2+z) \cdots(n+z)} e^{z\left(\sum_{k=1}^{n} \frac{1}{k}\right)} e^{-z\left[\sum_{k=1}^{n} \frac{1}{k}-\ln n\right]} \\
= & \lim _{n \rightarrow \infty} \frac{n!n^{z}}{(1+z)(2+z) \cdots(n+z)} .
\end{aligned}
$$

13. $\uparrow$ Verify from the Gauss formula above that $\Gamma(z+1)=\Gamma(z) z$ and that for $n$ a nonnegative integer, $\Gamma(n+1)=n!$.
14. $\uparrow$ The usual definition of the gamma function for positive $x$ is

$$
\Gamma_{1}(x) \equiv \int_{0}^{\infty} e^{-t} t^{x-1} d t
$$

Show $\left(1-\frac{t}{n}\right)^{n} \leq e^{-t}$ for $t \in[0, n]$. Then show

$$
\int_{0}^{n}\left(1-\frac{t}{n}\right)^{n} t^{x-1} d t=\frac{n!n^{x}}{x(x+1) \cdots(x+n)}
$$

Use the first part to conclude that

$$
\Gamma_{1}(x)=\lim _{n \rightarrow \infty} \frac{n!n^{x}}{x(x+1) \cdots(x+n)}=\Gamma(x)
$$

Hint: To show $\left(1-\frac{t}{n}\right)^{n} \leq e^{-t}$ for $t \in[0, n]$, verify this is equivalent to showing $(1-u)^{n} \leq e^{-n u}$ for $u \in[0,1]$.
15. $\uparrow$ Show $\Gamma(z)=\int_{0}^{\infty} e^{-t} t^{z-1} d t$. whenever $\operatorname{Re} z>0$. Hint: You have already shown that this is true for positive real numbers. Verify this formula for $\operatorname{Re} z>0$ yields an analytic function.
16. $\uparrow$ Show $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$. Then find $\Gamma\left(\frac{5}{2}\right)$.
17. Show that $\int_{-\infty}^{\infty} e^{\frac{-s^{2}}{2}} d s=\sqrt{2 \pi}$. Hint: Denote this integral by $I$ and observe that $I^{2}=\int_{\mathbb{R}^{2}} e^{-\left(x^{2}+y^{2}\right) / 2} d x d y$. Then change variables to polar coordinates, $x=r \cos (\theta)$, $y=r \sin \theta$.
18. $\uparrow$ Now that you know what the gamma function is, consider in the formula for $\Gamma(\alpha+1)$ the following change of variables. $t=\alpha+\alpha^{1 / 2} s$. Then in terms of the new variable, $s$, the formula for $\Gamma(\alpha+1)$ is

$$
\begin{aligned}
& e^{-\alpha} \alpha^{\alpha+\frac{1}{2}} \int_{-\sqrt{\alpha}}^{\infty} e^{-\sqrt{\alpha} s}\left(1+\frac{s}{\sqrt{\alpha}}\right)^{\alpha} d s \\
& =e^{-\alpha} \alpha^{\alpha+\frac{1}{2}} \int_{-\sqrt{\alpha}}^{\infty} e^{\alpha\left[\ln \left(1+\frac{s}{\sqrt{\alpha}}\right)-\frac{s}{\sqrt{\alpha}}\right]} d s
\end{aligned}
$$

Show the integrand converges to $e^{-\frac{s^{2}}{2}}$. Show that then

$$
\lim _{\alpha \rightarrow \infty} \frac{\Gamma(\alpha+1)}{e^{-\alpha} \alpha^{\alpha+(1 / 2)}}=\int_{-\infty}^{\infty} e^{\frac{-s^{2}}{2}} d s=\sqrt{2 \pi}
$$

Hint: You will need to obtain a dominating function for the integral so that you can use the dominated convergence theorem. You might try considering $s \in(-\sqrt{\alpha}, \sqrt{\alpha})$ first and consider something like $e^{1-\left(s^{2} / 4\right)}$ on this interval. Then look for another function for $s>\sqrt{\alpha}$. This formula is known as Stirling's formula.
19. This and the next several problems develop the zeta function and give a relation between the zeta and the gamma function. Define for $0<r<2 \pi$

$$
\begin{align*}
I_{r}(z) \equiv & \int_{0}^{2 \pi} \frac{e^{(z-1)(\ln r+i \theta)}}{e^{r e^{i \theta}}-1} i r e^{i \theta} d \theta+\int_{r}^{\infty} \frac{e^{(z-1)(\ln t+2 \pi i)}}{e^{t}-1} d t  \tag{57.6.24}\\
& +\int_{\infty}^{r} \frac{e^{(z-1) \ln t}}{e^{t}-1} d t
\end{align*}
$$

Show that $I_{r}$ is an entire function. The reason $0<r<2 \pi$ is that this prevents $e^{r e^{i \theta}}-1$ from equaling zero. The above is just a precise description of the contour integral, $\int_{\gamma} \frac{w^{z^{z-1}}}{e^{w}-1} d w$ where $\gamma$ is the contour shown below.

in which on the integrals along the real line, the argument is different in going from $r$ to $\infty$ than it is in going from $\infty$ to $r$. Now I have not defined such contour integrals over contours which have infinite length and so have chosen to simply write out explicitly what is involved. You have to work with these integrals given above anyway but the contour integral just mentioned is the motivation for them. Hint: You may want to use convergence theorems from real analysis if it makes this more convenient but you might not have to.
20. $\uparrow$ In the context of Problem 19 define for small $\delta>0$

$$
I_{r \delta}(z) \equiv \int_{\gamma_{r, \delta}} \frac{w^{z-1}}{e^{w}-1} d w
$$

where $\gamma_{r \delta}$ is shown below.


Show that $\lim _{\delta \rightarrow 0} I_{r \delta}(z)=I_{r}(z)$. Hint: Use the dominated convergence theorem if it makes this go easier. This is not a hard problem if you use these theorems but you can probably do it without them with more work.
21. $\uparrow$ In the context of Problem 20 show that for $r_{1}<r, I_{r \delta}(z)-I_{r_{1} \delta}(z)$ is a contour integral,

$$
\int_{\gamma_{r, r_{1}, \delta}} \frac{w^{z-1}}{e^{w}-1} d w
$$

where the oriented contour is shown below.


In this contour integral, $w^{z-1}$ denotes $e^{(z-1) \log (w)}$ where $\log (w)=\ln |w|+i \arg (w)$ for $\arg (w) \in(0,2 \pi)$. Explain why this integral equals zero. From Problem 20 it follows that $I_{r}=I_{r_{1}}$. Therefore, you can define an entire function, $I(z) \equiv I_{r}(z)$ for all $r$ positive but sufficiently small. Hint: Remember the Cauchy integral formula for analytic functions defined on simply connected regions. You could argue there is a simply connected region containing $\gamma_{r, r_{1}, \delta}$.
22. $\uparrow$ In case $\operatorname{Re} z>1$, you can get an interesting formula for $I(z)$ by taking the limit as $r \rightarrow 0$. Recall that

$$
\begin{align*}
I_{r}(z) \equiv & \int_{0}^{2 \pi} \frac{e^{(z-1)(\ln r+i \theta)}}{e^{r e^{i \theta}}-1} i r e^{i \theta} d \theta+\int_{r}^{\infty} \frac{e^{(z-1)(\ln t+2 \pi i)}}{e^{t}-1} d t  \tag{57.6.25}\\
& +\int_{\infty}^{r} \frac{e^{(z-1) \ln t}}{e^{t}-1} d t
\end{align*}
$$

and now it is desired to take a limit in the case where $\operatorname{Re} z>1$. Show the first integral above converges to 0 as $r \rightarrow 0$. Next argue the sum of the two last integrals converges to

$$
\left(e^{(z-1) 2 \pi i}-1\right) \int_{0}^{\infty} \frac{e^{(z-1) \ln (t)}}{e^{t}-1} d t
$$

Thus

$$
\begin{equation*}
I(z)=\left(e^{z 2 \pi i}-1\right) \int_{0}^{\infty} \frac{e^{(z-1) \ln (t)}}{e^{t}-1} d t \tag{57.6.26}
\end{equation*}
$$

when $\operatorname{Re} z>1$.
23. $\uparrow$ So what does all this have to do with the zeta function and the gamma function? The zeta function is defined for $\operatorname{Re} z>1$ by

$$
\sum_{n=1}^{\infty} \frac{1}{n^{z}} \equiv \zeta(z)
$$

By Problem 15, whenever $\operatorname{Re} z>0$,

$$
\Gamma(z)=\int_{0}^{\infty} e^{-t} t^{z-1} d t
$$

Change the variable and conclude

$$
\Gamma(z) \frac{1}{n^{z}}=\int_{0}^{\infty} e^{-n s} s^{z-1} d s
$$

Therefore, for $\operatorname{Re} z>1$,

$$
\zeta(z) \Gamma(z)=\sum_{n=1}^{\infty} \int_{0}^{\infty} e^{-n s} s^{z-1} d s
$$

Now show that you can interchange the order of the sum and the integral. This is possibly most easily done by using Fubini's theorem. Show that $\sum_{n=1}^{\infty} \int_{0}^{\infty}\left|e^{-n s} s^{z-1}\right| d s<$ $\infty$ and then use Fubini's theorem. I think you could do it other ways though. It is possible to do it without any reference to Lebesgue integration. Thus

$$
\begin{aligned}
\zeta(z) \Gamma(z) & =\int_{0}^{\infty} s^{z-1} \sum_{n=1}^{\infty} e^{-n s} d s \\
& =\int_{0}^{\infty} \frac{s^{z-1} e^{-s}}{1-e^{-s}} d s=\int_{0}^{\infty} \frac{s^{z-1}}{e^{s}-1} d s
\end{aligned}
$$

By 57.6.26,

$$
\begin{aligned}
I(z) & =\left(e^{z 2 \pi i}-1\right) \int_{0}^{\infty} \frac{e^{(z-1) \ln (t)}}{e^{t}-1} d t \\
& =\left(e^{z 2 \pi i}-1\right) \zeta(z) \Gamma(z) \\
& =\left(e^{2 \pi i z}-1\right) \zeta(z) \Gamma(z)
\end{aligned}
$$

whenever $\operatorname{Re} z>1$.
24. $\uparrow$ Now show there exists an entire function, $h(z)$ such that

$$
\zeta(z)=\frac{1}{z-1}+h(z)
$$

for $\operatorname{Re} z>1$. Conclude $\zeta(z)$ extends to a meromorphic function defined on all of $\mathbb{C}$ which has a simple pole at $z=1$, namely, the right side of the above formula. Hint: Use Problem 10 to observe that $\Gamma(z)$ is never equal to zero but has simple poles at every nonnegative integer. Then for $\operatorname{Re} z>1$,

$$
\zeta(z) \equiv \frac{I(z)}{\left(e^{2 \pi i z}-1\right) \Gamma(z)}
$$

By 57.6.26 $\zeta$ has no poles for $\operatorname{Re} z>1$. The right side of the above equation is defined for all $z$. There are no poles except possibly when $z$ is a nonnegative integer. However, these points are not poles either because of Problem 10 which states that $\Gamma$ has simple poles at these points thus cancelling the simple zeros of $\left(e^{2 \pi i z}-1\right)$. The
only remaining possibility for a pole for $\zeta$ is at $z=1$. Show it has a simple pole at this point. You can use the formula for $I(z)$

$$
\begin{align*}
I(z) \equiv & \int_{0}^{2 \pi} \frac{e^{(z-1)(\ln r+i \theta)}}{e^{r e^{i \theta}}-1} i r e^{i \theta} d \theta+\int_{r}^{\infty} \frac{e^{(z-1)(\ln t+2 \pi i)}}{e^{t}-1} d t  \tag{57.6.27}\\
& +\int_{\infty}^{r} \frac{e^{(z-1) \ln t}}{e^{t}-1} d t
\end{align*}
$$

Thus $I(1)$ is given by

$$
I(1) \equiv \int_{0}^{2 \pi} \frac{1}{e^{r e^{i \theta}}-1} i r e^{i \theta} d \theta+\int_{r}^{\infty} \frac{1}{e^{t}-1} d t+\int_{\infty}^{r} \frac{1}{e^{t}-1} d t
$$

$=\int_{\gamma_{r}} \frac{d w}{e^{w}-1}$ where $\gamma_{r}$ is the circle of radius $r$. This contour integral equals $2 \pi i$ by the residue theorem. Therefore,

$$
\frac{I(z)}{\left(e^{2 \pi i z}-1\right) \Gamma(z)}=\frac{1}{z-1}+h(z)
$$

where $h(z)$ is an entire function. People worry a lot about where the zeros of $\zeta$ are located. In particular, the zeros for $\operatorname{Re} z \in(0,1)$ are of special interest. The Riemann hypothesis says they are all on the line $\operatorname{Re} z=1 / 2$. This is a good problem for you to do next.
25. There is an important relation between prime numbers and the zeta function due to Euler. Let $\left\{p_{n}\right\}_{n=1}^{\infty}$ be the prime numbers. Then for $\operatorname{Re} z>1$,

$$
\prod_{n=1}^{\infty} \frac{1}{1-p_{n}^{-z}}=\zeta(z)
$$

To see this, consider a partial product.

$$
\prod_{n=1}^{N} \frac{1}{1-p_{n}^{-z}}=\prod_{n=1}^{N} \sum_{j_{n}=1}^{\infty}\left(\frac{1}{p_{n}^{z}}\right)^{j_{n}}
$$

Let $S_{N}$ denote all positive integers which use only $p_{1}, \cdots, p_{N}$ in their prime factorization. Then the above equals $\sum_{n \in S_{N}} \frac{1}{n^{2}}$. Letting $N \rightarrow \infty$ and using the fact that $\operatorname{Re} z>1$ so that the order in which you sum is not important (See Theorem 58.0.1 or recall advanced calculus. ) you obtain the desired equation. Show $\sum_{n=1}^{\infty} \frac{1}{p_{n}}=\infty$.

## Chapter 58

## Elliptic Functions

This chapter is to give a short introduction to elliptic functions. There is much more available. There are books written on elliptic functions. What I am presenting here follows Alfors [3] although the material is found in many books on complex analysis. Hille, [65] has a much more extensive treatment than what I will attempt here. There are also many references and historical notes available in the book by Hille. Another good source for more having much the same emphasis as what is presented here is in the book by Saks and Zygmund [115]. This is a very interesting subject because it has considerable overlap with algebra.

Before beginning, recall that an absolutely convergent series can be summed in any order and you always get the same answer. The easy way to see this is to think of the series as a Lebesgue integral with respect to counting measure and apply convergence theorems as needed. The following theorem provides the necessary results.

Theorem 58.0.1 Suppose $\sum_{n=1}^{\infty}\left|a_{n}\right|<\infty$ and let $\theta, \phi: \mathbb{N} \rightarrow \mathbb{N}$ be one to one and onto mappings. Then $\sum_{n=1}^{\infty} a_{\phi(n)}$ and $\sum_{n=1}^{\infty} a_{\theta(n)}$ both converge and the two sums are equal.

Proof: By the monotone convergence theorem,

$$
\sum_{n=1}^{\infty}\left|a_{n}\right|=\lim _{n \rightarrow \infty} \sum_{k=1}^{n}\left|a_{\phi(k)}\right|=\lim _{n \rightarrow \infty} \sum_{k=1}^{n}\left|a_{\theta(k)}\right|
$$

but these last two equal $\sum_{k=1}^{\infty}\left|a_{\phi(k)}\right|$ and $\sum_{k=1}^{\infty}\left|a_{\theta(k)}\right|$ respectively. Therefore, $\sum_{k=1}^{\infty} a_{\theta(k)}$ and $\sum_{k=1}^{\infty} a_{\phi(k)}$ exist ( $n \rightarrow a_{\theta(n)}$ is in $L^{1}$ with respect to counting measure.) It remains to show the two are equal. There exists $M$ such that if $n>M$ then

$$
\begin{gathered}
\sum_{k=n+1}^{\infty}\left|a_{\theta(k)}\right|<\varepsilon, \quad \sum_{k=n+1}^{\infty}\left|a_{\phi(k)}\right|<\varepsilon \\
\left|\sum_{k=1}^{\infty} a_{\phi(k)}-\sum_{k=1}^{n} a_{\phi(k)}\right|<\varepsilon,\left|\sum_{k=1}^{\infty} a_{\theta(k)}-\sum_{k=1}^{n} a_{\theta(k)}\right|<\varepsilon
\end{gathered}
$$

Pick such an $n$ denoted by $n_{1}$. Then pick $n_{2}>n_{1}>M$ such that

$$
\left\{\theta(1), \cdots, \theta\left(n_{1}\right)\right\} \subseteq\left\{\phi(1), \cdots, \phi\left(n_{2}\right)\right\}
$$

Then

$$
\sum_{k=1}^{n_{2}} a_{\phi(k)}=\sum_{k=1}^{n_{1}} a_{\theta(k)}+\sum_{\phi(k) \notin\left\{\theta(1), \cdots, \theta\left(n_{1}\right)\right\}} a_{\phi(k)}
$$

Therefore,

$$
\left|\sum_{k=1}^{n_{2}} a_{\phi(k)}-\sum_{k=1}^{n_{1}} a_{\theta(k)}\right|=\left|\sum_{\phi(k) \notin\left\{\theta(1), \cdots, \theta\left(n_{1}\right)\right\}, k \leq n_{2}} a_{\phi(k)}\right|
$$

Now all of these $\phi(k)$ in the last sum are contained in $\left\{\theta\left(n_{1}+1\right), \cdots\right\}$ and so the last sum above is dominated by

$$
\leq \sum_{k=n_{1}+1}^{\infty}\left|a_{\theta(k)}\right|<\varepsilon
$$

Therefore,

$$
\begin{aligned}
& \left|\sum_{k=1}^{\infty} a_{\phi(k)}-\sum_{k=1}^{\infty} a_{\theta(k)}\right| \leq \\
& +\left|\sum_{k=1}^{\infty} a_{\phi(k)}-\sum_{k=1}^{n_{2}} a_{\phi(k)}\right| \\
+ & \left|\sum_{k=1}^{n_{1}} a_{\theta(k)}-\sum_{k=1}^{n_{1}} a_{\theta(k)}\right| \\
\sum_{k=1}^{\infty} a_{\theta(k)} \mid< & \varepsilon+\varepsilon+\varepsilon=3 \varepsilon
\end{aligned}
$$

and since $\varepsilon$ is arbitrary, it follows $\sum_{k=1}^{\infty} a_{\phi(k)}=\sum_{k=1}^{\infty} a_{\theta(k)}$ as claimed. This proves the theorem.

### 58.1 Periodic Functions

Definition 58.1.1 A function defined on $\mathbb{C}$ is said to be periodic if there exists $w$ such that $f(z+w)=f(z)$ for all $z \in \mathbb{C}$. Denote by $M$ the set of all periods. Thus if $w_{1}, w_{2} \in M$ and $a, b \in \mathbb{Z}$, then $a w_{1}+b w_{2} \in M$. For this reason $M$ is called the module of periods. ${ }^{1}$ In all which follows it is assumed $f$ is meromorphic.

Theorem 58.1.2 Let $f$ be a meromorphic function and let $M$ be the module of periods. Then if $M$ has a limit point, then $f$ equals a constant. If this does not happen then either there exists $w_{1} \in M$ such that $\mathbb{Z} w_{1}=M$ or there exist $w_{1}, w_{2} \in M$ such that $M=$ $\left\{a w_{1}+b w_{2}: a, b \in \mathbb{Z}\right\}$ and $w_{1} / w_{2}$ is not real. Also if $\tau=w_{2} / w_{1}$,

$$
|\tau| \geq 1, \frac{-1}{2} \leq \operatorname{Re} \tau \leq \frac{1}{2}
$$

Proof: Suppose $f$ is meromorphic and $M$ has a limit point, $w_{0}$. By Theorem 57.1.8 on Page 1791 there exist analytic functions, $p, q$ such that $f(z)=\frac{p(z)}{q(z)}$. Now pick $z_{0}$ such that $z_{0}$ is not a pole of $f$. Then letting $w_{n} \rightarrow w_{0}$ where $\left\{w_{n}\right\} \subseteq M, f\left(z_{0}+w_{n}\right)=f\left(z_{0}\right)$. Therefore, $p\left(z_{0}+w_{n}\right)=f\left(z_{0}\right) q\left(z_{0}+w_{n}\right)$ and so the analytic function, $p(z)-f\left(z_{0}\right) q(z)$ has a zero set which has a limit point. Therefore, this function is identically equal to zero because of Theorem 51.5.3 on Page 1628. Thus $f$ equals a constant as claimed.

This has shown that if $f$ is not constant, then $M$ is discrete. Therefore, there exists $w_{1} \in M$ such that $\left|w_{1}\right|=\min \{|w|: w \in M\}$. Suppose first that every element of $M$ is a real multiple of $w_{1}$. Thus, if $w \in M$, it follows there exists a real number, $x$ such that $w=x w_{1}$. Then there exist positive integers, $k, k+1$ such that $k \leq x<k+1$. If $x>k$, then $w-k w_{1}=(x-k) w_{1}$ is a period having smaller absolute value than $\left|w_{1}\right|$ which would be a contradiction. Hence, $x=k$ and so $M=\mathbb{Z} w_{1}$.

[^40]Now suppose there exists $w_{2} \in M$ which is not a real multiple of $w_{1}$. You can let $w_{2}$ be the element of $M$ having this property which has smallest absolute value. Now let $w \in M$. Since $w_{1}$ and $w_{2}$ point in different directions, it follows $w=x w_{1}+y w_{2}$ for some real numbers, $x, y$. Let $|m-x| \leq \frac{1}{2}$ and $|n-y| \leq \frac{1}{2}$ where $m, n$ are integers. Therefore,

$$
w=m w_{1}+n w_{2}+(x-m) w_{1}+(y-n) w_{2}
$$

and so

$$
\begin{equation*}
w-m w_{1}-n w_{2}=(x-m) w_{1}+(y-n) w_{2} \tag{58.1.1}
\end{equation*}
$$

Now since $w_{2} / w_{1} \notin \mathbb{R}$,

$$
\begin{aligned}
\left|(x-m) w_{1}+(y-n) w_{2}\right| & <\left|(x-m) w_{1}\right|+\left|(y-n) w_{2}\right| \\
& =\frac{1}{2}\left|w_{1}\right|+\frac{1}{2}\left|w_{2}\right| .
\end{aligned}
$$

Therefore, from 58.1.1,

$$
\begin{aligned}
\left|w-m w_{1}-n w_{2}\right| & =\left|(x-m) w_{1}+(y-n) w_{2}\right| \\
& <\frac{1}{2}\left|w_{1}\right|+\frac{1}{2}\left|w_{2}\right| \leq\left|w_{2}\right|
\end{aligned}
$$

and so the period, $w-m w_{1}-n w_{2}$ cannot be a non real multiple of $w_{1}$ because $w_{2}$ is the one which has smallest absolute value and this period has smaller absolute value than $w_{2}$. Therefore, the ratio $w-m w_{1}-n w_{2} / w_{1}$ must be a real number, $x$. Thus

$$
w-m w_{1}-n w_{2}=x w_{1}
$$

Since $w_{1}$ has minimal absolute value of all periods, it follows $|x| \geq 1$. Let $k \leq x<k+1$ for some integer, $k$. If $x>k$, then

$$
w-m w_{1}-n w_{2}-k w_{1}=(x-k) w_{1}
$$

which would contradict the choice of $w_{1}$ as being the period having minimal absolute value because the expression on the left in the above is a period and it equals something which has absolute value less than $\left|w_{1}\right|$. Therefore, $x=k$ and $w$ is an integer linear combination of $w_{1}$ and $w_{2}$. It only remains to verify the claim about $\tau$.

From the construction, $\left|w_{1}\right| \leq\left|w_{2}\right|$ and $\left|w_{2}\right| \leq\left|w_{1}-w_{2}\right|,\left|w_{2}\right| \leq\left|w_{1}+w_{2}\right|$. Therefore,

$$
|\tau| \geq 1,|\tau| \leq|1-\tau|,|\tau| \leq|1+\tau|
$$

The last two of these inequalities imply $-1 / 2 \leq \operatorname{Re} \tau \leq 1 / 2$.
This proves the theorem.
Definition 58.1.3 For $f$ a meromorphic function which has the last of the above alternatives holding in which $M=\left\{a w_{1}+b w_{2}: a, b \in \mathbb{Z}\right\}$, the function, $f$ is called elliptic. This is also called doubly periodic.

Theorem 58.1.4 Suppose $f$ is an elliptic function which has no poles. Then $f$ is constant.

Proof: Since $f$ has no poles it is analytic. Now consider the parallelograms determined by the vertices, $m w_{1}+n w_{2}$ for $m, n \in \mathbb{Z}$. By periodicity of $f$ it must be bounded because its values are identical on each of these parallelograms. Therefore, it equals a constant by Liouville's theorem.

Definition 58.1.5 Define $P_{a}$ to be the parallelogram determined by the points

$$
\begin{aligned}
& a+m w_{1}+n w_{2}, a+(m+1) w_{1}+n w_{2}, a+m w_{1}+(n+1) w_{2}, \\
& a+(m+1) w_{1}+(n+1) w_{2}
\end{aligned}
$$

Such $P_{a}$ will be referred to as a period parallelogram. The sum of the orders of the poles in a period parallelogram which contains no poles or zeros of $f$ on its boundary is called the order of the function. This is well defined because of the periodic property of $f$.

Theorem 58.1.6 The sum of the residues of any elliptic function, $f$ equals zero on every $P_{a}$ if a is chosen so that there are no poles on $\partial P_{a}$.

Proof: Choose $a$ such that there are no poles of $f$ on the boundary of $P_{a}$. By periodicity,

$$
\int_{\partial P_{a}} f(z) d z=0
$$

because the integrals over opposite sides of the parallelogram cancel out because the values of $f$ are the same on these sides and the orientations are opposite. It follows from the residue theorem that the sum of the residues in $P_{a}$ equals 0 .

Theorem 58.1.7 Let $P_{a}$ be a period parallelogram for a nonconstant elliptic function, $f$ which has order equal to $m$. Then $f$ assumes every value in $f\left(P_{a}\right)$ exactly $m$ times.

Proof: Let $c \in f\left(P_{a}\right)$ and consider $P_{a^{\prime}}$ such that $f^{-1}(c) \cap P_{a^{\prime}}=f^{-1}(c) \cap P_{a}$ and $P_{a^{\prime}}$ contains the same poles and zeros of $f-c$ as $P_{a}$ but $P_{a^{\prime}}$ has no zeros of $f(z)-c$ or poles of $f$ on its boundary. Thus $f^{\prime}(z) /(f(z)-c)$ is also an elliptic function and so Theorem 58.1.6 applies. Consider

$$
\frac{1}{2 \pi i} \int_{\partial P_{a^{\prime}}} \frac{f^{\prime}(z)}{f(z)-c} d z
$$

By the argument principle, this equals $N_{z}-N_{p}$ where $N_{z}$ equals the number of zeros of $f(z)-c$ and $N_{p}$ equals the number of the poles of $f(z)$. From Theorem 58.1.6 this must equal zero because it is the sum of the residues of $f^{\prime} /(f-c)$ and so $N_{z}=N_{p}$. Now $N_{p}$ equals the number of poles in $P_{a}$ counted according to multiplicity.

There is an even better theorem than this one.

Theorem 58.1.8 Let $f$ be a non constant elliptic function with poles $p_{1}, \cdots, p_{m}$ and zeros, $z_{1}, \cdots, z_{m}$ in $P_{\alpha}$, listed according to multiplicity where $\partial P_{\alpha}$ contains no poles or zeros of $f$. Then $\sum_{k=1}^{m} z_{k}-\sum_{k=1}^{m} p_{k} \in M$, the module of periods.

Proof: You can assume $\partial P_{a}$ contains no poles or zeros of $f$ because if it did, then you could consider a slightly shifted period parallelogram, $P_{a^{\prime}}$ which contains no new zeros and poles but which has all the old ones but no poles or zeros on its boundary. By Theorem 53.1.3 on Page 1676

$$
\begin{equation*}
\frac{1}{2 \pi i} \int_{\partial P_{a}} z \frac{f^{\prime}(z)}{f(z)} d z=\sum_{k=1}^{m} z_{k}-\sum_{k=1}^{m} p_{k} . \tag{58.1.2}
\end{equation*}
$$

Denoting by $\gamma(z, w)$ the straight oriented line segment from $z$ to $w$,

$$
\begin{gathered}
\int_{\partial P_{a}} z \frac{f^{\prime}(z)}{f(z)} d z \\
=\int_{\gamma\left(a, a+w_{1}\right)} z \frac{f^{\prime}(z)}{f(z)} d z+\int_{\gamma\left(a+w_{1}+w_{2}, a+w_{2}\right)} z \frac{f^{\prime}(z)}{f(z)} d z \\
+\int_{\gamma\left(a+w_{1}, a+w_{2}+w_{1}\right)} z \frac{f^{\prime}(z)}{f(z)} d z+\int_{\gamma\left(a+w_{2}, a\right)} z \frac{f^{\prime}(z)}{f(z)} d z \\
=\int_{\gamma\left(a, a+w_{1}\right)}\left(z-\left(z+w_{2}\right)\right) \frac{f^{\prime}(z)}{f(z)} d z \\
\quad+\int_{\gamma\left(a, a+w_{2}\right)}\left(z-\left(z+w_{1}\right)\right) \frac{f^{\prime}(z)}{f(z)} d z
\end{gathered}
$$

Now near these line segments $\frac{f^{\prime}(z)}{f(z)}$ is analytic and so there exists a primitive, $g_{w_{i}}(z)$ on $\gamma\left(a, a+w_{i}\right)$ by Corollary 51.7.5 on Page 1633 which satisfies $e^{g_{w_{i}}(z)}=f(z)$. Therefore,

$$
=-w_{2}\left(g_{w_{1}}\left(a+w_{1}\right)-g_{w_{1}}(a)\right)-w_{1}\left(g_{w_{2}}\left(a+w_{2}\right)-g_{w_{2}}(a)\right) .
$$

Now by periodicity of $f$ it follows $f\left(a+w_{1}\right)=f(a)=f\left(a+w_{2}\right)$. Hence

$$
g_{w_{i}}\left(a+w_{1}\right)-g_{w_{i}}(a)=2 m \pi i
$$

for some integer, $m$ because

$$
e^{g_{w_{i}}\left(a+w_{i}\right)}-e^{g_{w_{i}}(a)}=f\left(a+w_{i}\right)-f(a)=0 .
$$

Therefore, from 58.1.2, there exist integers, $k, l$ such that

$$
\begin{aligned}
& \frac{1}{2 \pi i} \int_{\partial P_{a}} z \frac{f^{\prime}(z)}{f(z)} d z \\
= & \frac{1}{2 \pi i}\left[-w_{2}\left(g_{w_{1}}\left(a+w_{1}\right)-g_{w_{1}}(a)\right)-w_{1}\left(g_{w_{2}}\left(a+w_{2}\right)-g_{w_{2}}(a)\right)\right] \\
= & \frac{1}{2 \pi i}\left[-w_{2}(2 k \pi i)-w_{1}(2 l \pi i)\right] \\
= & -w_{2} k-w_{1} l \in M
\end{aligned}
$$

From 58.1.2 it follows

$$
\sum_{k=1}^{m} z_{k}-\sum_{k=1}^{m} p_{k} \in M
$$

This proves the theorem.
Hille says this relation is due to Liouville. There is also a simple corollary which follows from the above theorem applied to the elliptic function, $f(z)-c$.

Corollary 58.1.9 Let $f$ be a non constant elliptic function and suppose the function, $f(z)-$ $c$ has poles $p_{1}, \cdots, p_{m}$ and zeros, $z_{1}, \cdots, z_{m}$ on $P_{\alpha}$, listed according to multiplicity where $\partial P_{\alpha}$ contains no poles or zeros of $f(z)-c$. Then $\sum_{k=1}^{m} z_{k}-\sum_{k=1}^{m} p_{k} \in M$, the module of periods.

### 58.1.1 The Unimodular Transformations

Definition 58.1.10 Suppose $f$ is a nonconstant elliptic function and the module of periods is of the form $\left\{a w_{1}+b w_{2}\right\}$ where $a, b$ are integers and $w_{1} / w_{2}$ is not real. Then by analogy with linear algebra, $\left\{w_{1}, w_{2}\right\}$ is referred to as a basis. The unimodular transformations will refer to matrices of the form

$$
\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)
$$

where all entries are integers and

$$
a d-b c= \pm 1
$$

These linear transformations are also called the modular group.
The following is an interesting lemma which ties matrices with the fractional linear transformations.

Lemma 58.1.11 Define

$$
\phi\left(\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)\right) \equiv \frac{a z+b}{c z+d}
$$

Then

$$
\begin{equation*}
\phi(A B)=\phi(A) \circ \phi(B), \tag{58.1.3}
\end{equation*}
$$

$\phi(A)(z)=z$ if and only if

$$
A=c I
$$

where $I$ is the identity matrix and $c \neq 0$. Also if $f(z)=\frac{a z+b}{c z+d}$, then $f^{-1}(z)$ exists if and only if $a d-c b \neq 0$. Furthermore it is easy to find $f^{-1}$.

Proof: The equation in 58.1.3 is just a simple computation. Now suppose $\phi(A)(z)=z$. Then letting $A=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$, this requires

$$
a z+b=z(c z+d)
$$

and so $a z+b=c z^{2}+d z$. Since this is to hold for all $z$ it follows $c=0=b$ and $a=d$. The other direction is obvious.

Consider the claim about the existence of an inverse. Let $a d-c b \neq 0$ for $f(z)=\frac{a z+b}{c z+d}$. Then

$$
f(z)=\phi\left(\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)\right)
$$

It follows $\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)^{-1}$ exists and equals $\frac{1}{a d-b c}\left(\begin{array}{cc}d & -b \\ -c & a\end{array}\right)$. Therefore,

$$
\begin{aligned}
z & =\phi(I)(z)=\phi\left(\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)\left(\frac{1}{a d-b c}\left(\begin{array}{cc}
d & -b \\
-c & a
\end{array}\right)\right)\right)(z) \\
& =\phi\left(\left(\begin{array}{cc}
a & b \\
c & d
\end{array}\right)\right) \circ \phi\left(\left(\frac{1}{a d-b c}\left(\begin{array}{cc}
d & -b \\
-c & a
\end{array}\right)\right)\right)(z) \\
& =f \circ f^{-1}(z)
\end{aligned}
$$

which shows $f^{-1}$ exists and it is easy to find.
Next suppose $f^{-1}$ exists. I need to verify the condition $a d-c b \neq 0$. If $f^{-1}$ exists, then from the process used to find it, you see that it must be a fractional linear transformation. Letting $A=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$ so $\phi(A)=f$, it follows there exists a matrix $B$ such that

$$
\phi(B A)(z)=\phi(B) \circ \phi(A)(z)=z .
$$

However, it was shown that this implies $B A$ is a nonzero multiple of $I$ which requires that $A^{-1}$ must exist. Hence the condition must hold.

Theorem 58.1.12 If $f$ is a nonconstant elliptic function with a basis $\left\{w_{1}, w_{2}\right\}$ for the module of periods, then $\left\{w_{1}^{\prime}, w_{2}^{\prime}\right\}$ is another basis, if and only if there exists a unimodular transformation, $\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)=A$ such that

$$
\binom{w_{1}^{\prime}}{w_{2}^{\prime}}=\left(\begin{array}{ll}
a & b  \tag{58.1.4}\\
c & d
\end{array}\right)\binom{w_{1}}{w_{2}}
$$

Proof: Since $\left\{w_{1}, w_{2}\right\}$ is a basis, there exist integers, $a, b, c, d$ such that 58.1.4 holds. It remains to show the transformation determined by the matrix is unimodular. Taking conjugates,

$$
\binom{\overline{w_{1}^{\prime}}}{\overline{w_{2}^{\prime}}}=\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)\binom{\overline{w_{1}}}{\overline{w_{2}}} .
$$

Therefore,

$$
\left(\begin{array}{ll}
w_{1}^{\prime} & \overline{w_{1}^{\prime}} \\
w_{2}^{\prime} & \overline{w_{2}^{\prime}}
\end{array}\right)=\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)\left(\begin{array}{ll}
w_{1} & \overline{w_{1}} \\
w_{2} & \overline{w_{2}}
\end{array}\right)
$$

Now since $\left\{w_{1}^{\prime}, w_{2}^{\prime}\right\}$ is also given to be a basis, there exits another matrix having all integer entries, $\left(\begin{array}{ll}e & f \\ g & h\end{array}\right)$ such that

$$
\binom{\overline{w_{1}}}{\overline{w_{2}}}=\left(\begin{array}{ll}
e & f \\
g & h
\end{array}\right)\binom{\overline{w_{1}^{\prime}}}{\overline{w_{2}^{\prime}}}
$$

and

$$
\binom{w_{1}}{w_{2}}=\left(\begin{array}{ll}
e & f \\
g & h
\end{array}\right)\binom{w_{1}^{\prime}}{w_{2}^{\prime}}
$$

Therefore,

$$
\left(\begin{array}{ll}
w_{1}^{\prime} & \overline{w_{1}^{\prime}} \\
w_{2}^{\prime} & \overline{w_{2}^{\prime}}
\end{array}\right)=\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)\left(\begin{array}{ll}
e & f \\
g & h
\end{array}\right)\left(\begin{array}{ll}
w_{1}^{\prime} & \overline{w_{1}^{\prime}} \\
w_{2}^{\prime} & \overline{w_{2}^{\prime}}
\end{array}\right) .
$$

However, since $w_{1}^{\prime} / w_{2}^{\prime}$ is not real, it is routine to verify that

$$
\operatorname{det}\left(\begin{array}{ll}
w_{1}^{\prime} & \overline{w_{1}^{\prime}} \\
w_{2}^{\prime} & \overline{w_{2}^{\prime}}
\end{array}\right) \neq 0
$$

Therefore,

$$
\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right)=\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)\left(\begin{array}{ll}
e & f \\
g & h
\end{array}\right)
$$

and so $\operatorname{det}\left(\begin{array}{ll}a & b \\ c & d\end{array}\right) \operatorname{det}\left(\begin{array}{ll}e & f \\ g & h\end{array}\right)=1$. But the two matrices have all integer entries and so both determinants must equal either 1 or -1 .

Next suppose

$$
\binom{w_{1}^{\prime}}{w_{2}^{\prime}}=\left(\begin{array}{ll}
a & b  \tag{58.1.5}\\
c & d
\end{array}\right)\binom{w_{1}}{w_{2}}
$$

where $\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$ is unimodular. I need to verify that $\left\{w_{1}^{\prime}, w_{2}^{\prime}\right\}$ is a basis. If $w \in M$, there exist integers, $m, n$ such that

$$
w=m w_{1}+n w_{2}=\left(\begin{array}{ll}
m & n
\end{array}\right)\binom{w_{1}}{w_{2}}
$$

From 58.1.5

$$
\pm\left(\begin{array}{cc}
d & -b \\
-c & a
\end{array}\right)\binom{w_{1}^{\prime}}{w_{2}^{\prime}}=\binom{w_{1}}{w_{2}}
$$

and so

$$
w= \pm\left(\begin{array}{cc}
m & n
\end{array}\right)\left(\begin{array}{cc}
d & -b \\
-c & a
\end{array}\right)\binom{w_{1}^{\prime}}{w_{2}^{\prime}}
$$

which is an integer linear combination of $\left\{w_{1}^{\prime}, w_{2}^{\prime}\right\}$. It only remains to verify that $w_{1}^{\prime} / w_{2}^{\prime}$ is not real.

Claim: Let $w_{1}$ and $w_{2}$ be nonzero complex numbers. Then $w_{2} / w_{1}$ is not real if and only if

$$
w_{1} \overline{w_{2}}-\overline{w_{1}} w_{2}=\operatorname{det}\left(\begin{array}{ll}
w_{1} & \overline{w_{1}} \\
w_{2} & \overline{w_{2}}
\end{array}\right) \neq 0
$$

Proof of the claim: Let $\lambda=w_{2} / w_{1}$. Then

$$
w_{1} \overline{w_{2}}-\overline{w_{1}} w_{2}=\bar{\lambda} w_{1} \overline{w_{1}}-\overline{w_{1}} \lambda w_{1}=(\bar{\lambda}-\lambda)\left|w_{1}\right|^{2}
$$

Thus the ratio is not real if and only if $(\bar{\lambda}-\lambda) \neq 0$ if and only if $w_{1} \overline{w_{2}}-\overline{w_{1}} w_{2} \neq 0$.
Now to verify $w_{2}^{\prime} / w_{1}^{\prime}$ is not real,

$$
\begin{aligned}
& \operatorname{det}\left(\begin{array}{ll}
w_{1}^{\prime} & \overline{w_{1}^{\prime}} \\
w_{2}^{\prime} & \overline{w_{2}^{\prime}}
\end{array}\right) \\
= & \operatorname{det}\left(\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)\left(\begin{array}{ll}
w_{1} & \overline{w_{1}} \\
w_{2} & \overline{w_{2}}
\end{array}\right)\right) \\
= & \pm \operatorname{det}\left(\begin{array}{ll}
w_{1} & \overline{w_{1}} \\
w_{2} & \overline{w_{2}}
\end{array}\right) \neq 0
\end{aligned}
$$

This proves the theorem.

### 58.1.2 The Search For An Elliptic Function

By Theorem 58.1.4 and 58.1.6 if you want to find a nonconstant elliptic function it must fail to be analytic and also have either no terms in its Laurent expansion which are of the form $b_{1}(z-a)^{-1}$ or else these terms must cancel out. It is simplest to look for a function which simply does not have them. Weierstrass looked for a function of the form

$$
\begin{equation*}
\wp(z) \equiv \frac{1}{z^{2}}+\sum_{w \neq 0}\left(\frac{1}{(z-w)^{2}}-\frac{1}{w^{2}}\right) \tag{58.1.6}
\end{equation*}
$$

where $w$ consists of all numbers of the form $a w_{1}+b w_{2}$ for $a, b$ integers. Sometimes people write this as $\wp\left(z, w_{1}, w_{2}\right)$ to emphasize its dependence on the periods, $w_{1}$ and $w_{2}$ but I won't do so. It is understood there exist these periods, which are given. This is a reasonable thing to try. Suppose you formally differentiate the right side. Never mind whether this is justified for now. This yields

$$
\wp^{\prime}(z)=\frac{-2}{z^{3}}-\sum_{w \neq 0} \frac{-2}{(z-w)^{3}}=\sum_{w} \frac{-2}{(z-w)^{3}}
$$

which is clearly periodic having both periods $w_{1}$ and $w_{2}$. Therefore, $\wp\left(z+w_{1}\right)-\wp(z)$ and $\wp\left(z+w_{2}\right)-\wp(z)$ are both constants, $c_{1}$ and $c_{2}$ respectively. The reason for this is that since $\wp^{\prime}$ is periodic with periods $w_{1}$ and $w_{2}$, it follows $\not \wp^{\prime}\left(z+w_{i}\right)-\not{ }^{\prime}(z)=0$ as long as $z$ is not a period. From 58.1.6 you can see right away that

$$
\wp(z)=\wp(-z)
$$

Indeed

$$
\begin{aligned}
\wp(-z) & =\frac{1}{z^{2}}+\sum_{w \neq 0}\left(\frac{1}{(-z-w)^{2}}-\frac{1}{w^{2}}\right) \\
& =\frac{1}{z^{2}}+\sum_{w \neq 0}\left(\frac{1}{(-z+w)^{2}}-\frac{1}{w^{2}}\right)=\wp(z) .
\end{aligned}
$$

and so

$$
\begin{aligned}
c_{1} & =\wp\left(-\frac{w_{1}}{2}+w_{1}\right)-\wp\left(-\frac{w_{1}}{2}\right) \\
& =\wp\left(\frac{w_{1}}{2}\right)-\wp\left(-\frac{w_{1}}{2}\right)=0
\end{aligned}
$$

which shows the constant for $\wp\left(z+w_{1}\right)-\wp(z)$ must equal zero. Similarly the constant for $\wp\left(z+w_{2}\right)-\wp(z)$ also equals zero. Thus $\wp$ is periodic having the two periods $w_{1}, w_{2}$.

Of course to justify this, you need to consider whether the series of 58.1.6 converges. Consider the terms of the series.

$$
\left|\frac{1}{(z-w)^{2}}-\frac{1}{w^{2}}\right|=|z|\left|\frac{2 w-z}{(z-w)^{2} w^{2}}\right|
$$

If $|w|>2|z|$, this can be estimated more. For such $w$,

$$
\begin{aligned}
& \left|\frac{1}{(z-w)^{2}}-\frac{1}{w^{2}}\right| \\
= & |z|\left|\frac{2 w-z}{(z-w)^{2} w^{2}}\right| \leq|z| \frac{(5 / 2)|w|}{|w|^{2}(|w|-|z|)^{2}} \\
\leq & |z| \frac{(5 / 2)|w|}{|w|^{2}((1 / 2)|w|)^{2}}=|z| \frac{10}{|w|^{3}} .
\end{aligned}
$$

It follows the series in 58.1 .6 converges uniformly and absolutely on every compact set, $K$ provided $\sum_{w \neq 0} \frac{1}{|w|^{3}}$ converges. This question is considered next.

Claim: There exists a positive number, $k$ such that for all pairs of integers, $m, n$, not both equal to zero,

$$
\frac{\left|m w_{1}+n w_{2}\right|}{|m|+|n|} \geq k>0
$$

Proof of claim: If not, there exists $m_{k}$ and $n_{k}$ such that

$$
\lim _{k \rightarrow \infty} \frac{m_{k}}{\left|m_{k}\right|+\left|n_{k}\right|} w_{1}+\frac{n_{k}}{\left|m_{k}\right|+\left|n_{k}\right|} w_{2}=0
$$

However, $\left(\frac{m_{k}}{\left|m_{k}\right|+\left|n_{k}\right|}, \frac{n_{k}}{\left|m_{k}\right|+\left|n_{k}\right|}\right)$ is a bounded sequence in $\mathbb{R}^{2}$ and so, taking a subsequence, still denoted by $k$, you can have

$$
\left(\frac{m_{k}}{\left|m_{k}\right|+\left|n_{k}\right|}, \frac{n_{k}}{\left|m_{k}\right|+\left|n_{k}\right|}\right) \rightarrow(x, y) \in \mathbb{R}^{2}
$$

and so there are real numbers, $x, y$ such that $x w_{1}+y w_{2}=0$ contrary to the assumption that $w_{2} / w_{1}$ is not equal to a real number. This proves the claim.

Now from the claim,

$$
\begin{aligned}
& \sum_{w \neq 0} \frac{1}{|w|^{3}} \\
= & \sum_{(m, n) \neq(0,0)} \frac{1}{\left|m w_{1}+n w_{2}\right|^{3}} \leq \sum_{(m, n) \neq(0,0)} \frac{1}{k^{3}(|m|+|n|)^{3}} \\
= & \frac{1}{k^{3}} \sum_{j=1}^{\infty} \sum_{|m|+|n|=j} \frac{1}{(|m|+|n|)^{3}}=\frac{1}{k^{3}} \sum_{j=1}^{\infty} \frac{4 j}{j^{3}}<\infty .
\end{aligned}
$$

Now consider the series in 58.1.6. Letting $z \in B(0, R)$,

$$
\begin{aligned}
\wp(z) \equiv & \frac{1}{z^{2}}+\sum_{w \neq 0,|w| \leq R}\left(\frac{1}{(z-w)^{2}}-\frac{1}{w^{2}}\right) \\
& +\sum_{w \neq 0,|w|>R}\left(\frac{1}{(z-w)^{2}}-\frac{1}{w^{2}}\right)
\end{aligned}
$$

and the last series converges uniformly on $B(0, R)$ to an analytic function. Thus $\wp$ is a meromorphic function and also the argument given above involving differentiation of the series termwise is valid. Thus $\wp$ is an elliptic function as claimed. This is called the Weierstrass $\wp$ function. This has proved the following theorem.

Theorem 58.1.13 The function $\wp$ defined above is an example of an elliptic function. On any compact set, $\wp ~ e q u a l s ~ a ~ r a t i o n a l ~ f u n c t i o n ~ a d d e d ~ t o ~ a ~ s e r i e s ~ w h i c h ~ i s ~ u n i f o r m l y ~ a n d ~$ absolutely convergent on the compact set.

### 58.1.3 The Differential Equation Satisfied By $\wp$

For $z$ not a pole,

$$
\wp^{\prime}(z)=\frac{-2}{z^{3}}-\sum_{w \neq 0} \frac{2}{(z-w)^{3}}
$$

Also since there are no poles of order 1 you can obtain a primitive for $\wp,-\zeta .{ }^{2}$ To do so, recall

$$
\wp(z) \equiv \frac{1}{z^{2}}+\sum_{w \neq 0}\left(\frac{1}{(z-w)^{2}}-\frac{1}{w^{2}}\right)
$$

where for $|z|<R$ this is the sum of a rational function with a uniformly convergent series. Therefore, you can take the integral along any path, $\gamma(0, z)$ from 0 to $z$ which misses the poles of $\wp$. By the uniform convergence of the above integral, you can interchange the sum with the integral and obtain

$$
\begin{equation*}
\zeta(z)=\frac{1}{z}+\sum_{w \neq 0} \frac{1}{z-w}+\frac{z}{w^{2}}+\frac{1}{w} \tag{58.1.7}
\end{equation*}
$$

[^41]This function is odd. Here is why.

$$
\zeta(-z)=\frac{1}{-z}+\sum_{w \neq 0} \frac{1}{-z-w}-\frac{z}{w^{2}}+\frac{1}{w}
$$

while

$$
\begin{aligned}
-\zeta(z) & =\frac{1}{-z}+\sum_{w \neq 0} \frac{-1}{z-w}-\frac{z}{w^{2}}-\frac{1}{w} \\
& =\frac{1}{-z}+\sum_{w \neq 0} \frac{-1}{z+w}-\frac{z}{w^{2}}+\frac{1}{w}
\end{aligned}
$$

Now consider 58.1.7. It will be used to find the Laurent expansion about the origin for $\zeta$ which will then be differentiated to obtain the Laurent expansion for $\wp$ at the origin. Since $w \neq 0$ and the interest is for $z$ near 0 so $|z|<|w|$,

$$
\begin{aligned}
\frac{1}{z-w}+\frac{z}{w^{2}}+\frac{1}{w} & =\frac{z}{w^{2}}+\frac{1}{w}-\frac{1}{w} \frac{1}{1-\frac{z}{w}} \\
& =\frac{z}{w^{2}}+\frac{1}{w}-\frac{1}{w} \sum_{k=0}^{\infty}\left(\frac{z}{w}\right)^{k} \\
& =-\frac{1}{w} \sum_{k=2}^{\infty}\left(\frac{z}{w}\right)^{k}
\end{aligned}
$$

From 58.1.7

$$
\begin{aligned}
\zeta(z) & =\frac{1}{z}+\sum_{w \neq 0}\left(-\sum_{k=2}^{\infty} \frac{z^{k}}{w^{k+1}}\right) \\
& =\frac{1}{z}-\sum_{k=2}^{\infty} \sum_{w \neq 0} \frac{z^{k}}{w^{k+1}}=\frac{1}{z}-\sum_{k=2}^{\infty} \sum_{w \neq 0} \frac{z^{2 k-1}}{w^{2 k}}
\end{aligned}
$$

because the sum over odd powers must be zero because for each $w \neq 0$, there exists $-w \neq 0$ such that the two terms $\frac{z^{2 k}}{w^{2 k+1}}$ and $\frac{z^{2 k}}{(-w)^{2 k+1}}$ cancel each other. Hence

$$
\zeta(z)=\frac{1}{z}-\sum_{k=2}^{\infty} G_{k} z^{2 k-1}
$$

where $G_{k}=\sum_{w \neq 0} \frac{1}{w^{2 k}}$. Now with this,

$$
\begin{aligned}
-\zeta^{\prime}(z) & =\wp(z)=\frac{1}{z^{2}}+\sum_{k=2}^{\infty} G_{k}(2 k-1) z^{2 k-2} \\
& =\frac{1}{z^{2}}+3 G_{2} z^{2}+5 G_{3} z^{4}+\cdots
\end{aligned}
$$

Therefore,

$$
\wp^{\prime}(z)=\frac{-2}{z^{3}}+6 G_{2} z+20 G_{3} z^{3}+\cdots
$$

$$
\begin{aligned}
\wp(z)^{2} & =\frac{4}{z^{6}}-\frac{24 G_{2}}{z^{2}}-80 G_{3}+\cdots \\
4 \wp(z)^{3} & =4\left(\frac{1}{z^{2}}+3 G_{2} z^{2}+5 G_{3} z^{4} \cdots\right)^{3} \\
& =\frac{4}{z^{6}}+\frac{36}{z^{2}} G_{2}+60 G_{3}+\cdots
\end{aligned}
$$

and finally

$$
60 G_{2} \wp(z)=\frac{60 G_{2}}{z^{2}}+0+\cdots
$$

where in the above, the positive powers of $z$ are not listed explicitly. Therefore,

$$
\wp^{\prime}(z)^{2}-4 \wp(z)^{3}+60 G_{2} \wp(z)+140 G_{3}=\sum_{n=1}^{\infty} a_{n} z^{n}
$$

In deriving the equation it was assumed $|z|<|w|$ for all $w=a w_{1}+b w_{2}$ where $a, b$ are integers not both zero. The left side of the above equation is periodic with respect to $w_{1}$ and $w_{2}$ where $w_{2} / w_{1}$ is not a real number. The only possible poles of the left side are at $0, w_{1}, w_{2}$, and $w_{1}+w_{2}$, the vertices of the parallelogram determined by $w_{1}$ and $w_{2}$. This follows from the original formula for $\wp(z)$. However, the above equation shows the left side has no pole at 0 . Since the left side is periodic with periods $w_{1}$ and $w_{2}$, it follows it has no pole at the other vertices of this parallelogram either. Therefore, the left side is periodic and has no poles. Consequently, it equals a constant by Theorem 58.1.4. But the right side of the above equation shows this constant is 0 because this side equals zero when $z=0$. Therefore, $\wp$ satisfies the differential equation,

$$
\wp^{\prime}(z)^{2}-4 \wp(z)^{3}+60 G_{2} \wp(z)+140 G_{3}=0
$$

It is traditional to define $60 G_{2} \equiv g_{2}$ and $140 G_{3} \equiv g_{3}$. Then in terms of these new quantities the differential equation is

$$
\wp^{\prime}(z)^{2}=4 \wp(z)^{3}-g_{2} \wp(z)-g_{3}
$$

Suppose $e_{1}, e_{2}$ and $e_{3}$ are zeros of the polynomial $4 w^{3}-g_{2} w-g_{3}=0$. Then the above equation can be written in the form

$$
\begin{equation*}
\wp(z)^{2}=4\left(\wp(z)-e_{1}\right)\left(\wp(z)-e_{2}\right)\left(\wp(z)-e_{3}\right) \tag{58.1.8}
\end{equation*}
$$

### 58.1.4 A Modular Function

The next task is to find the $e_{i}$ in 58.1.8. First recall that $\wp$ is an even function. That is $\wp(-z)=\wp(z)$. This follows from 58.1.6 which is listed here for convenience.

$$
\begin{equation*}
\wp(z) \equiv \frac{1}{z^{2}}+\sum_{w \neq 0}\left(\frac{1}{(z-w)^{2}}-\frac{1}{w^{2}}\right) \tag{58.1.9}
\end{equation*}
$$

Thus

$$
\begin{aligned}
\wp(-z) & =\frac{1}{z^{2}}+\sum_{w \neq 0}\left(\frac{1}{(-z-w)^{2}}-\frac{1}{w^{2}}\right) \\
& =\frac{1}{z^{2}}+\sum_{w \neq 0}\left(\frac{1}{(-z+w)^{2}}-\frac{1}{w^{2}}\right)=\wp(z)
\end{aligned}
$$

Therefore, $\wp\left(w_{1}-z\right)=\wp\left(z-w_{1}\right)=\wp(z)$ and so $-\wp^{\prime}\left(w_{1}-z\right)=\wp^{\prime}(z)$. Letting $z=$ $w_{1} / 2$, it follows $\wp^{\prime}\left(w_{1} / 2\right)=0$. Similarly, $\not \wp^{\prime}\left(w_{2} / 2\right)=0$ and $\wp^{\prime}\left(\left(w_{1}+w_{2}\right) / 2\right)=0$. Therefore, from 58.1.8

$$
0=4\left(\wp\left(w_{1} / 2\right)-e_{1}\right)\left(\wp\left(w_{1} / 2\right)-e_{2}\right)\left(\wp\left(w_{1} / 2\right)-e_{3}\right) .
$$

It follows one of the $e_{i}$ must equal $\wp\left(w_{1} / 2\right)$. Similarly, one of the $e_{i}$ must equal $\wp\left(w_{2} / 2\right)$ and one must equal $\wp\left(\left(w_{1}+w_{2}\right) / 2\right)$.

Lemma 58.1.14 The numbers, $\wp\left(w_{1} / 2\right), \wp\left(w_{2} / 2\right)$, and $\wp\left(\left(w_{1}+w_{2}\right) / 2\right)$ are distinct.
Proof: Choose $P_{a}$, a period parallelogram which contains the pole 0 , and the points $w_{1} / 2, w_{2} / 2$, and $\left(w_{1}+w_{2}\right) / 2$ but no other pole of $\wp(z)$. Also $\partial P_{a}^{*}$ does not contain any zeros of the elliptic function, $z \rightarrow \wp(z)-\wp\left(w_{1} / 2\right)$. This can be done by shifting $P_{0}$ slightly because the poles are only at the points $a w_{1}+b w_{2}$ for $a, b$ integers and the zeros of $\wp(z)-$ $\wp\left(w_{1} / 2\right)$ are discrete.


If $\wp\left(w_{2} / 2\right)=\wp\left(w_{1} / 2\right)$, then $\wp(z)-\wp\left(w_{1} / 2\right)$ has two zeros, $w_{2} / 2$ and $w_{1} / 2$ and since the pole at 0 is of order 2, this is the order of $\wp(z)-\wp\left(w_{1} / 2\right)$ on $P_{a}$ hence by Theorem 58.1.7 on Page 1818 these are the only zeros of this function on $P_{a}$. It follows by Corollary 58.1.9 on Page 1820 which says the sum of the zeros minus the sum of the poles is in $M$, $\frac{w_{1}}{2}+\frac{w_{2}}{2} \in M$. Thus there exist integers, $a, b$ such that

$$
\frac{w_{1}+w_{2}}{2}=a w_{1}+b w_{2}
$$

which implies $(2 a-1) w_{1}+(2 b-1) w_{2}=0$ contradicting $w_{2} / w_{1}$ not being real. Similar reasoning applies to the other pairs of points in $\left\{w_{1} / 2, w_{2} / 2,\left(w_{1}+w_{2}\right) / 2\right\}$. For example,
consider $\left(w_{1}+w_{2}\right) / 2$ and choose $P_{a}$ such that its boundary contains no zeros of the elliptic function, $z \rightarrow \wp(z)-\wp\left(\left(w_{1}+w_{2}\right) / 2\right)$ and $P_{a}$ contains no poles of $\wp$ on its interior other than 0 . Then if $\wp\left(w_{2} / 2\right)=\wp\left(\left(w_{1}+w_{2}\right) / 2\right)$, it follows from Theorem 58.1.7 on Page $1818 w_{2} / 2$ and $\left(w_{1}+w_{2}\right) / 2$ are the only two zeros of $\wp(z)-\wp\left(\left(w_{1}+w_{2}\right) / 2\right)$ on $P_{a}$ and by Corollary 58.1.9 on Page 1820

$$
\frac{w_{1}+w_{1}+w_{2}}{2}=a w_{1}+b w_{2} \in M
$$

for some integers $a, b$ which leads to the same contradiction as before about $w_{1} / w_{2}$ not being real. The other cases are similar. This proves the lemma.

Lemma 58.1.14 proves the $e_{i}$ are distinct. Number the $e_{i}$ such that

$$
e_{1}=\wp\left(w_{1} / 2\right), e_{2}=\wp\left(w_{2} / 2\right)
$$

and

$$
e_{3}=\wp\left(\left(w_{1}+w_{2}\right) / 2\right)
$$

To summarize, it has been shown that for complex numbers, $w_{1}$ and $w_{2}$ with $w_{2} / w_{1}$ not real, an elliptic function, $\wp$ has been defined. Denote this function as $\wp(z)=\wp\left(z, w_{1}, w_{2}\right)$. This in turn determined numbers, $e_{i}$ as described above. Thus these numbers depend on $w_{1}$ and $w_{2}$ and as described above,

$$
\begin{aligned}
& e_{1}\left(w_{1}, w_{2}\right)=\wp\left(\frac{w_{1}}{2}, w_{1}, w_{2}\right), e_{2}\left(w_{1}, w_{2}\right)=\wp\left(\frac{w_{2}}{2}, w_{1}, w_{2}\right) \\
& e_{3}\left(w_{1}, w_{2}\right)=\wp\left(\frac{w_{1}+w_{2}}{2}, w_{1}, w_{2}\right) .
\end{aligned}
$$

Therefore, using the formula for $\wp, 58.1 .9$,

$$
\wp(z) \equiv \frac{1}{z^{2}}+\sum_{w \neq 0}\left(\frac{1}{(z-w)^{2}}-\frac{1}{w^{2}}\right)
$$

you see that if the two periods $w_{1}$ and $w_{2}$ are replaced with $t w_{1}$ and $t w_{2}$ respectively, then

$$
e_{i}\left(t w_{1}, t w_{2}\right)=t^{-2} e_{i}\left(w_{1}, w_{2}\right)
$$

Let $\tau$ denote the complex number which equals the ratio, $w_{2} / w_{1}$ which was assumed in all this to not be real. Then

$$
e_{i}\left(w_{1}, w_{2}\right)=w_{1}^{-2} e_{i}(1, \tau)
$$

Now define the function, $\lambda(\tau)$

$$
\begin{equation*}
\lambda(\tau) \equiv \frac{e_{3}(1, \tau)-e_{2}(1, \tau)}{e_{1}(1, \tau)-e_{2}(1, \tau)}\left(=\frac{e_{3}\left(w_{1}, w_{2}\right)-e_{2}\left(w_{1}, w_{2}\right)}{e_{1}\left(w_{1}, w_{2}\right)-e_{2}\left(w_{1}, w_{2}\right)}\right) . \tag{58.1.10}
\end{equation*}
$$

This function is meromorphic for $\operatorname{Im} \tau>0$ or for $\operatorname{Im} \tau<0$. However, since the denominator is never equal to zero the function must actually be analytic on both the upper half plane and the lower half plane. It never is equal to 0 because $e_{3} \neq e_{2}$ and it never equals 1 because $e_{3} \neq e_{1}$. This is stated as an observation.

Observation 58.1.15 The function, $\lambda(\tau)$ is analytic for $\tau$ in the upper half plane and never assumes the values 0 and 1 .

This is a very interesting function. Consider what happens when

$$
\binom{w_{1}^{\prime}}{w_{2}^{\prime}}=\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)\binom{w_{1}}{w_{2}}
$$

and the matrix is unimodular. By Theorem 58.1.12 on Page $1821\left\{w_{1}^{\prime}, w_{2}^{\prime}\right\}$ is just another basis for the same module of periods. Therefore, $\wp\left(z, w_{1}, w_{2}\right)=\wp\left(z, w_{1}^{\prime}, w_{2}^{\prime}\right)$ because both are defined as sums over the same values of $w$, just in different order which does not matter because of the absolute convergence of the sums on compact subsets of $\mathbb{C}$. Since $\wp$ is unchanged, it follows $\wp^{\prime}(z)$ is also unchanged and so the numbers, $e_{i}$ are also the same. However, they might be permuted in which case the function $\lambda(\tau)$ defined above would change. What would it take for $\lambda(\tau)$ to not change? In other words, for which unimodular transformations will $\lambda$ be left unchanged? This happens if and only if no permuting takes place for the $e_{i}$. This occurs if $\wp\left(\frac{w_{1}}{2}\right)=\wp\left(\frac{w_{1}^{\prime}}{2}\right)$ and $\wp\left(\frac{w_{2}}{2}\right)=\wp\left(\frac{w_{2}^{\prime}}{2}\right)$. If

$$
\frac{w_{1}^{\prime}}{2}-\frac{w_{1}}{2} \in M, \frac{w_{2}^{\prime}}{2}-\frac{w_{2}}{2} \in M
$$

then $\wp\left(\frac{w_{1}}{2}\right)=\wp\left(\frac{w_{1}^{\prime}}{2}\right)$ and so $e_{1}$ will be unchanged and similarly for $e_{2}$ and $e_{3}$. This occurs exactly when

$$
\frac{1}{2}\left((a-1) w_{1}+b w_{2}\right) \in M, \frac{1}{2}\left(c w_{1}+(d-1) w_{2}\right) \in M
$$

This happens if $a$ and $d$ are odd and if $b$ and $c$ are even. Of course the stylish way to say this is

$$
\begin{equation*}
a \equiv 1 \bmod 2, d \equiv 1 \bmod 2, b \equiv 0 \bmod 2, c \equiv 0 \bmod 2 \tag{58.1.11}
\end{equation*}
$$

This has shown that for unimodular transformations satisfying 58.1.11 $\lambda$ is unchanged. Letting $\tau$ be defined as above,

$$
\tau^{\prime}=\frac{w_{2}^{\prime}}{w_{1}^{\prime}} \equiv \frac{c w_{1}+d w_{2}}{a w_{1}+b w_{2}}=\frac{c+d \tau}{a+b \tau}
$$

Thus for unimodular transformations, $\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$ satisfying 58.1.11, or more succinctly,

$$
\left(\begin{array}{ll}
a & b  \tag{58.1.12}\\
c & d
\end{array}\right) \sim\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right) \bmod 2
$$

it follows that

$$
\begin{equation*}
\lambda\left(\frac{c+d \tau}{a+b \tau}\right)=\lambda(\tau) \tag{58.1.13}
\end{equation*}
$$

Furthermore, this is the only way this can happen.

Lemma 58.1.16 $\lambda(\tau)=\lambda\left(\tau^{\prime}\right)$ if and only if

$$
\tau^{\prime}=\frac{a \tau+b}{c \tau+d}
$$

where 58.1.12 holds.
Proof: It only remains to verify that if $\wp\left(w_{1}^{\prime} / 2\right)=\wp\left(w_{1} / 2\right)$ then it is necessary that

$$
\frac{w_{1}^{\prime}}{2}-\frac{w_{1}}{2} \in M
$$

with a similar requirement for $w_{2}$ and $w_{2}^{\prime}$. If $\frac{w_{1}^{\prime}}{2}-\frac{w_{1}}{2} \notin M$, then there exist integers, $m, n$ such that

$$
\frac{-w_{1}^{\prime}}{2}+m w_{1}+n w_{2}
$$

is in the interior of $P_{0}$, the period parallelogram whose vertices are $0, w_{1}, w_{1}+w_{2}$, and $w_{2}$. Therefore, it is possible to choose small $a$ such that $P_{a}$ contains the pole, $0, \frac{w_{1}}{2}$, and $\frac{-w_{1}^{\prime}}{2}+m w_{1}+n w_{2}$ but no other poles of $\wp$ and in addition, $\partial P_{a}^{*}$ contains no zeros of $z \rightarrow$ $\wp(z)-\wp\left(\frac{w_{1}}{2}\right)$. Then the order of this elliptic function is 2 . By assumption, and the fact that $\wp$ is even,

$$
\wp\left(\frac{-w_{1}^{\prime}}{2}+m w_{1}+n w_{2}\right)=\wp\left(\frac{-w_{1}^{\prime}}{2}\right)=\wp\left(\frac{w_{1}^{\prime}}{2}\right)=\wp\left(\frac{w_{1}}{2}\right) .
$$

It follows both $\frac{-w_{1}^{\prime}}{2}+m w_{1}+n w_{2}$ and $\frac{w_{1}}{2}$ are zeros of $\wp(z)-\wp\left(\frac{w_{1}}{2}\right)$ and so by Theorem 58.1.7 on Page 1818 these are the only two zeros of this function in $P_{a}$. Therefore, from Corollary 58.1.9 on Page 1820

$$
\frac{w_{1}}{2}-\frac{w_{1}^{\prime}}{2}+m w_{1}+n w_{2} \in M
$$

which shows $\frac{w_{1}}{2}-\frac{w_{1}^{\prime}}{2} \in M$. This completes the proof of the lemma.
Note the condition in the lemma is equivalent to the condition 58.1 .13 because you can relabel the coefficients. The message of either version is that the coefficient of $\tau$ in the numerator and denominator is odd while the constant in the numerator and denominator is even.

$$
\begin{array}{r}
\text { Next, }\left(\begin{array}{ll}
1 & 0 \\
2 & 1
\end{array}\right) \sim\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right) \bmod 2 \text { and therefore, } \\
 \tag{58.1.14}\\
\lambda\left(\frac{2+\tau}{1}\right)=\lambda(\tau+2)=\lambda(\tau) .
\end{array}
$$

Thus $\lambda$ is periodic of period 2.
Thus $\lambda$ leaves invariant a certain subgroup of the unimodular group. According to the next definition, $\lambda$ is an example of something called a modular function.

Definition 58.1.17 When an analytic or meromorphic function is invariant under a group of linear transformations, it is called an automorphic function. A function which is automorphic with respect to a subgroup of the modular group is called a modular function or an elliptic modular function.

Now consider what happens for some other unimodular matrices which are not congruent to the identity mod 2 . This will yield other functional equations for $\lambda$ in addition to the fact that $\lambda$ is periodic of period 2 . As before, these functional equations come about because $\wp$ is unchanged when you change the basis for $M$, the module of periods. In particular, consider the unimodular matrices

$$
\left(\begin{array}{ll}
1 & 0  \tag{58.1.15}\\
1 & 1
\end{array}\right),\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)
$$

Consider the first of these. Thus

$$
\binom{w_{1}^{\prime}}{w_{2}^{\prime}}=\binom{w_{1}}{w_{1}+w_{2}}
$$

Hence $\tau^{\prime}=w_{2}^{\prime} / w_{1}^{\prime}=\left(w_{1}+w_{2}\right) / w_{1}=1+\tau$. Then from the definition of $\lambda$,

$$
\left.\begin{array}{rl}
\lambda\left(\tau^{\prime}\right) & =\lambda(1+\tau) \\
& =\frac{\wp\left(\frac{w_{1}^{\prime}+w_{2}^{\prime}}{2}\right)-\wp\left(\frac{w_{2}^{\prime}}{2}\right)}{\wp\left(\frac{w_{1}^{\prime}}{2}\right)-\wp\left(\frac{w_{2}^{\prime}}{2}\right)} \\
& =\frac{\wp\left(\frac{w_{1}+w_{2}+w_{1}}{2}\right)-\wp\left(\frac{w_{1}+w_{2}}{2}\right)}{\wp\left(\frac{w_{1}}{2}\right)-\wp\left(\frac{w_{1}+w_{2}}{2}\right)} \\
& =\frac{\wp\left(\frac{w_{2}}{2}+w_{1}\right)-\wp\left(\frac{w_{1}+w_{2}}{2}\right)}{\wp\left(\frac{w_{1}}{2}\right)-\wp\left(\frac{w_{1}+w_{2}}{2}\right)} \\
& =\frac{\wp\left(\frac{w_{2}}{2}\right)-\wp\left(\frac{w_{1}+w_{2}}{2}\right)}{\wp\left(\frac{w_{1}}{2}\right)-\wp\left(\frac{w_{1}+w_{2}}{2}\right)} \\
& =-\frac{\wp\left(\frac{w_{1}+w_{2}}{2}\right)-\wp\left(\frac{w_{2}}{2}\right)}{\wp\left(\frac{w_{1}}{2}\right)-\wp\left(\frac{w_{1}+w_{2}}{2}\right)} \\
& =-\frac{\wp\left(\frac{w_{1}+w_{2}}{2}\right)-\wp\left(\frac{w_{2}}{2}\right)}{\wp\left(\frac{w_{1}}{2}\right)-\wp\left(\frac{w_{2}}{2}\right)+\wp\left(\frac{w_{2}}{2}\right)-\wp\left(\frac{w_{1}+w_{2}}{2}\right)} \\
& =-\frac{\wp\left(\frac{\wp\left(\frac{w_{1}+w_{2}}{2}\right)-\wp\left(\frac{w_{2}}{2}\right)}{\wp\left(\frac{w_{1}}{2}\right)-\wp\left(\frac{w_{2}}{2}\right)}\right)}{\wp\left(\frac{w_{2}}{2}\right)-\wp\left(\frac{w_{1}+w_{2}}{2}\right)} \\
\wp\left(\frac{w_{1}}{2}\right)-\wp\left(\frac{w_{2}}{2}\right)
\end{array}\right)
$$

Summarizing the important feature of the above,

$$
\begin{equation*}
\lambda(1+\tau)=\frac{\lambda(\tau)}{\lambda(\tau)-1} \tag{58.1.17}
\end{equation*}
$$

Next consider the other unimodular matrix in 58.1.15. In this case $w_{1}^{\prime}=w_{2}$ and $w_{2}^{\prime}=w_{1}$.

Therefore, $\tau^{\prime}=w_{2}^{\prime} / w_{1}^{\prime}=w_{1} / w_{2}=1 / \tau$. Then

$$
\begin{align*}
\lambda\left(\tau^{\prime}\right) & =\lambda(1 / \tau) \\
& =\frac{\wp\left(\frac{w_{1}^{\prime}+w_{2}^{\prime}}{2}\right)-\wp\left(\frac{w_{2}^{\prime}}{2}\right)}{\wp\left(\frac{w_{1}^{\prime}}{2}\right)-\wp\left(\frac{w_{2}^{\prime}}{2}\right)} \\
& =\frac{\wp\left(\frac{w_{1}+w_{2}}{2}\right)-\wp\left(\frac{w_{1}}{2}\right)}{\wp\left(\frac{w_{2}}{2}\right)-\wp\left(\frac{w_{1}}{2}\right)} \\
& =\frac{e_{3}-e_{1}}{e_{2}-e_{1}}=-\frac{e_{3}-e_{2}+e_{2}-e_{1}}{e_{1}-e_{2}} \\
& =-(\lambda(\tau)-1)=-\lambda(\tau)+1 . \tag{58.1.18}
\end{align*}
$$

You could try other unimodular matrices and attempt to find other functional equations if you like but this much will suffice here.

### 58.1.5 A Formula For $\lambda$

Recall the formula of Mittag-Leffler for $\cot (\pi \alpha)$ given in 57.2.15. For convenience, here it is.

$$
\frac{1}{\alpha}+\sum_{n=1}^{\infty} \frac{2 \alpha}{\alpha^{2}-n^{2}}=\pi \cot \pi \alpha
$$

As explained in the derivation of this formula it can also be written as

$$
\sum_{n=-\infty}^{\infty} \frac{\alpha}{\alpha^{2}-n^{2}}=\pi \cot \pi \alpha
$$

Differentiating both sides yields

$$
\begin{align*}
\pi^{2} \csc ^{2}(\pi \alpha) & =\sum_{n=-\infty}^{\infty} \frac{\alpha^{2}+n^{2}}{\left(\alpha^{2}-n^{2}\right)^{2}} \\
& =\sum_{n=-\infty}^{\infty} \frac{(\alpha+n)^{2}-2 \alpha n}{(\alpha+n)^{2}(\alpha-n)^{2}} \\
& =\sum_{n=-\infty}^{\infty} \frac{(\alpha+n)^{2}}{(\alpha+n)^{2}(\alpha-n)^{2}}-\overbrace{\sum_{n=-\infty}^{\infty} \frac{2 \alpha n}{\left(\alpha^{2}-n^{2}\right)^{2}}}^{=0} \\
& =\sum_{n=-\infty}^{\infty} \frac{1}{(\alpha-n)^{2}} \tag{58.1.19}
\end{align*}
$$

Now this formula can be used to obtain a formula for $\lambda(\tau)$. As pointed out above, $\lambda$ depends only on the ratio $w_{2} / w_{1}$ and so it suffices to take $w_{1}=1$ and $w_{2}=\tau$. Thus

$$
\begin{equation*}
\lambda(\tau)=\frac{\wp\left(\frac{1+\tau}{2}\right)-\wp\left(\frac{\tau}{2}\right)}{\wp\left(\frac{1}{2}\right)-\wp\left(\frac{\tau}{2}\right)} \tag{58.1.20}
\end{equation*}
$$

From the original formula for $\wp$,

$$
\begin{align*}
& \wp\left(\frac{1+\tau}{2}\right)-\wp\left(\frac{\tau}{2}\right) \\
= & \frac{1}{\left(\frac{1+\tau}{2}\right)^{2}}-\frac{1}{\left(\frac{\tau}{2}\right)^{2}}+\sum_{(k, m) \neq(0,0)} \frac{1}{\left(k-\frac{1}{2}+\left(m-\frac{1}{2}\right) \tau\right)^{2}}-\frac{1}{\left(k+\left(m-\frac{1}{2}\right) \tau\right)^{2}} \\
= & \sum_{(k, m) \in \mathbb{Z}^{2}} \frac{1}{\left(k-\frac{1}{2}+\left(m-\frac{1}{2}\right) \tau\right)^{2}}-\frac{1}{\left(k+\left(m-\frac{1}{2}\right) \tau\right)^{2}} \\
= & \sum_{(k, m) \in \mathbb{Z}^{2}} \frac{1}{\left(k-\frac{1}{2}+\left(m-\frac{1}{2}\right) \tau\right)^{2}}-\frac{1}{\left(k+\left(m-\frac{1}{2}\right) \tau\right)^{2}} \\
= & \sum_{(k, m) \in \mathbb{Z}^{2}} \frac{1}{\left(k-\frac{1}{2}+\left(-m-\frac{1}{2}\right) \tau\right)^{2}}-\frac{1}{\left(k+\left(-m-\frac{1}{2}\right) \tau\right)^{2}} \\
= & \sum_{(k, m) \in \mathbb{Z}^{2}} \frac{1}{\left(\frac{1}{2}+\left(m+\frac{1}{2}\right) \tau-k\right)^{2}}-\frac{1}{\left(\left(m+\frac{1}{2}\right) \tau-k\right)^{2}} . \tag{58.1.21}
\end{align*}
$$

Similarly,

$$
\begin{align*}
& \wp\left(\frac{1}{2}\right)-\wp\left(\frac{\tau}{2}\right) \\
= & \frac{1}{\left(\frac{1}{2}\right)^{2}}-\frac{1}{\left(\frac{\tau}{2}\right)^{2}}+\sum_{(k, m) \neq(0,0)} \frac{1}{\left(k-\frac{1}{2}+m \tau\right)^{2}}-\frac{1}{\left(k+\left(m-\frac{1}{2}\right) \tau\right)^{2}} \\
= & \sum_{(k, m) \in \mathbb{Z}^{2}} \frac{1}{\left(k-\frac{1}{2}+m \tau\right)^{2}}-\frac{1}{\left(k+\left(m-\frac{1}{2}\right) \tau\right)^{2}} \\
= & \sum_{(k, m) \in \mathbb{Z}^{2}} \frac{1}{\left(k-\frac{1}{2}-m \tau\right)^{2}}-\frac{1}{\left(k+\left(-m-\frac{1}{2}\right) \tau\right)^{2}} \\
= & \sum_{(k, m) \in \mathbb{Z}^{2}} \frac{1}{\left(\frac{1}{2}+m \tau-k\right)^{2}}-\frac{1}{\left(\left(m+\frac{1}{2}\right) \tau-k\right)^{2}} . \tag{58.1.22}
\end{align*}
$$

Now use 58.1.19 to sum these over $k$. This yields,

$$
\begin{aligned}
& \wp\left(\frac{1+\tau}{2}\right)-\wp\left(\frac{\tau}{2}\right) \\
= & \sum_{m} \frac{\pi^{2}}{\sin ^{2}\left(\pi\left(\frac{1}{2}+\left(m+\frac{1}{2}\right) \tau\right)\right)}-\frac{\pi^{2}}{\sin ^{2}\left(\pi\left(m+\frac{1}{2}\right) \tau\right)} \\
= & \sum_{m} \frac{\pi^{2}}{\cos ^{2}\left(\pi\left(m+\frac{1}{2}\right) \tau\right)}-\frac{\pi^{2}}{\sin ^{2}\left(\pi\left(m+\frac{1}{2}\right) \tau\right)}
\end{aligned}
$$

and

$$
\begin{aligned}
\wp\left(\frac{1}{2}\right)-\wp\left(\frac{\tau}{2}\right) & =\sum_{m} \frac{\pi^{2}}{\sin ^{2}\left(\pi\left(\frac{1}{2}+m \tau\right)\right)}-\frac{\pi^{2}}{\sin ^{2}\left(\pi\left(m+\frac{1}{2}\right) \tau\right)} \\
& =\sum_{m} \frac{\pi^{2}}{\cos ^{2}(\pi m \tau)}-\frac{\pi^{2}}{\sin ^{2}\left(\pi\left(m+\frac{1}{2}\right) \tau\right)}
\end{aligned}
$$

The following interesting formula for $\lambda$ results.

$$
\begin{equation*}
\lambda(\tau)=\frac{\sum_{m} \frac{1}{\cos ^{2}\left(\pi\left(m+\frac{1}{2}\right) \tau\right)}-\frac{1}{\sin ^{2}\left(\pi\left(m+\frac{1}{2}\right) \tau\right)}}{\sum_{m} \frac{1}{\cos ^{2}(\pi m \tau)}-\frac{1}{\sin ^{2}\left(\pi\left(m+\frac{1}{2}\right) \tau\right)}} \tag{58.1.23}
\end{equation*}
$$

From this it is obvious $\lambda(-\tau)=\lambda(\tau)$. Therefore, from 58.1.18,

$$
\begin{equation*}
-\lambda(\tau)+1=\lambda\left(\frac{1}{\tau}\right)=\lambda\left(\frac{-1}{\tau}\right) \tag{58.1.24}
\end{equation*}
$$

(It is good to recall that $\lambda$ has been defined for $\tau \notin \mathbb{R}$.)

### 58.1.6 Mapping Properties Of $\boldsymbol{\lambda}$

The two functional equations, 58.1.24 and 58.1.17 along with some other properties presented above are of fundamental importance. For convenience, they are summarized here in the following lemma.

Lemma 58.1.18 The following functional equations hold for $\lambda$.

$$
\begin{gather*}
\lambda(1+\tau)=\frac{\lambda(\tau)}{\lambda(\tau)-1}, 1=\lambda(\tau)+\lambda\left(\frac{-1}{\tau}\right)  \tag{58.1.25}\\
\lambda(\tau+2)=\lambda(\tau) \tag{58.1.26}
\end{gather*}
$$

$\lambda(z)=\lambda(w)$ if and only if there exists a unimodular matrix,

$$
\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right) \sim\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right) \bmod 2
$$

such that

$$
\begin{equation*}
w=\frac{a z+b}{c z+d} \tag{58.1.27}
\end{equation*}
$$

Consider the following picture.


In this picture, $l_{1}$ is the $y$ axis and $l_{2}$ is the line, $x=1$ while $C$ is the top half of the circle centered at $\left(\frac{1}{2}, 0\right)$ which has radius $1 / 2$. Note the above formula implies $\lambda$ has real values on $l_{1}$ which are between 0 and 1 . This is because 58.1 .23 implies

$$
\begin{align*}
\lambda(i b) & =\frac{\sum_{m} \frac{1}{\cos ^{2}\left(\pi\left(m+\frac{1}{2}\right) i b\right)}-\frac{1}{\sin ^{2}\left(\pi\left(m+\frac{1}{2}\right) i b\right)}}{\sum_{m} \frac{1}{\cos ^{2}(\pi m i b)}-\frac{1}{\sin ^{2}\left(\pi\left(m+\frac{1}{2}\right) i b\right)}} \\
& =\frac{\sum_{m} \frac{1}{\cosh ^{2}\left(\pi\left(m+\frac{1}{2}\right) b\right)}+\frac{1}{\sinh ^{2}\left(\pi\left(m+\frac{1}{2}\right) b\right)}}{\sum_{m} \frac{1}{\cosh ^{2}(\pi m b)}+\frac{1}{\sinh ^{2}\left(\pi\left(m+\frac{1}{2}\right) b\right)}} \in(0,1) \tag{58.1.28}
\end{align*}
$$

This follows from the observation that

$$
\cos (i x)=\cosh (x), \sin (i x)=i \sinh (x)
$$

Thus it is clear from 58.1.28 that $\lim _{b \rightarrow 0+} \lambda(i b)=1$.
Next I need to consider the behavior of $\lambda(\tau)$ as $\operatorname{Im}(\tau) \rightarrow \infty$. From 58.1.23 listed here for convenience,

$$
\begin{equation*}
\lambda(\tau)=\frac{\sum_{m} \frac{1}{\cos ^{2}\left(\pi\left(m+\frac{1}{2}\right) \tau\right)}-\frac{1}{\sin ^{2}\left(\pi\left(m+\frac{1}{2}\right) \tau\right)}}{\sum_{m} \frac{1}{\cos ^{2}(\pi m \tau)}-\frac{1}{\sin ^{2}\left(\pi\left(m+\frac{1}{2}\right) \tau\right)}} \tag{58.1.29}
\end{equation*}
$$

it follows

$$
\begin{align*}
\lambda(\tau) & =\frac{\frac{1}{\cos ^{2}\left(\pi\left(-\frac{1}{2}\right) \tau\right)}-\frac{1}{\sin ^{2}\left(\pi\left(-\frac{1}{2}\right) \tau\right)}+\frac{1}{\cos ^{2}\left(\pi \frac{1}{2} \tau\right)}-\frac{1}{\sin ^{2}\left(\pi \frac{1}{2} \tau\right)}+A(\tau)}{1+B(\tau)} \\
& =\frac{\frac{2}{\cos ^{2}\left(\pi\left(\frac{1}{2}\right) \tau\right)}-\frac{2}{\sin ^{2}\left(\pi\left(\frac{1}{2}\right) \tau\right)}+A(\tau)}{1+B(\tau)} \tag{58.1.30}
\end{align*}
$$

Where $A(\tau), B(\tau) \rightarrow 0$ as $\operatorname{Im}(\tau) \rightarrow \infty$. I took out the $m=0$ term involving $1 / \cos ^{2}(\pi m \tau)$ in the denominator and the $m=-1$ and $m=0$ terms in the numerator of 58.1.29. In fact, $e^{-i \pi(a+i b)} A(a+i b), e^{-i \pi(a+i b)} B(a+i b)$ converge to zero uniformly in $a$ as $b \rightarrow \infty$.

Lemma 58.1.19 For $A, B$ defined in 58.1.30, $e^{-i \pi(a+i b)} C(a+i b) \rightarrow 0$ uniformly in a for $C=A, B$.

Proof: From 58.1.23,

$$
e^{-i \pi \tau} A(\tau)=\sum_{\substack{m \neq 0 \\ m \neq-1}} \frac{e^{-i \pi \tau}}{\cos ^{2}\left(\pi\left(m+\frac{1}{2}\right) \tau\right)}-\frac{e^{-i \pi \tau}}{\sin ^{2}\left(\pi\left(m+\frac{1}{2}\right) \tau\right)}
$$

Now let $\tau=a+i b$. Then letting $\alpha_{m}=\pi\left(m+\frac{1}{2}\right)$,

$$
\begin{aligned}
\cos \left(\alpha_{m} a+i \alpha_{m} b\right) & =\cos \left(\alpha_{m} a\right) \cosh \left(\alpha_{m} b\right)-i \sinh \left(\alpha_{m} b\right) \sin \left(\alpha_{m} a\right) \\
\sin \left(\alpha_{m} a+i \alpha_{m} b\right) & =\sin \left(\alpha_{m} a\right) \cosh \left(\alpha_{m} b\right)+i \cos \left(\alpha_{m} a\right) \sinh \left(\alpha_{m} b\right)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left|\cos ^{2}\left(\alpha_{m} a+i \alpha_{m} b\right)\right| & =\cos ^{2}\left(\alpha_{m} a\right) \cosh ^{2}\left(\alpha_{m} b\right)+\sinh ^{2}\left(\alpha_{m} b\right) \sin ^{2}\left(\alpha_{m} a\right) \\
& \geq \sinh ^{2}\left(\alpha_{m} b\right)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\left|\sin ^{2}\left(\alpha_{m} a+i \alpha_{m} b\right)\right| & =\sin ^{2}\left(\alpha_{m} a\right) \cosh ^{2}\left(\alpha_{m} b\right)+\cos ^{2}\left(\alpha_{m} a\right) \sinh ^{2}\left(\alpha_{m} b\right) \\
& \geq \sinh ^{2}\left(\alpha_{m} b\right)
\end{aligned}
$$

It follows that for $\tau=a+i b$ and $b$ large

$$
\begin{aligned}
& \left|e^{-i \pi \tau} A(\tau)\right| \\
\leq & \sum_{\substack{m \neq 0 \\
m \neq-1}} \frac{2 e^{\pi b}}{\sinh ^{2}\left(\pi\left(m+\frac{1}{2}\right) b\right)} \\
\leq & \sum_{m=1}^{\infty} \frac{2 e^{\pi b}}{\sinh ^{2}\left(\pi\left(m+\frac{1}{2}\right) b\right)}+\sum_{m=-\infty}^{-2} \frac{2 e^{\pi b}}{\sinh ^{2}\left(\pi\left(m+\frac{1}{2}\right) b\right)} \\
= & 2 \sum_{m=1}^{\infty} \frac{2 e^{\pi b}}{\sinh ^{2}\left(\pi\left(m+\frac{1}{2}\right) b\right)}=4 \sum_{m=1}^{\infty} \frac{e^{\pi b}}{\sinh ^{2}\left(\pi\left(m+\frac{1}{2}\right) b\right)}
\end{aligned}
$$

Now a short computation shows

$$
\frac{\frac{e^{\pi b}}{\sinh ^{2}\left(\pi\left(m+1+\frac{1}{2}\right) b\right)}}{\frac{e^{\pi b}}{\sinh ^{2}\left(\pi\left(m+\frac{1}{2}\right) b\right)}}=\frac{\sinh ^{2}\left(\pi\left(m+\frac{1}{2}\right) b\right)}{\sinh ^{2}\left(\pi\left(m+\frac{3}{2}\right) b\right)} \leq \frac{1}{e^{3 \pi b}}
$$

Therefore, for $\tau=a+i b$,

$$
\begin{aligned}
\left|e^{-i \pi \tau} A(\tau)\right| & \leq 4 \frac{e^{\pi b}}{\sinh \left(\frac{3 \pi b}{2}\right)} \sum_{m=1}^{\infty}\left(\frac{1}{e^{3 \pi b}}\right)^{m} \\
& \leq 4 \frac{e^{\pi b}}{\sinh \left(\frac{3 \pi b}{2}\right)} \frac{1 / e^{3 \pi b}}{1-\left(1 / e^{3 \pi b}\right)}
\end{aligned}
$$

which converges to zero as $b \rightarrow \infty$. Similar reasoning will establish the claim about $B(\tau)$. This proves the lemma.

Lemma 58.1.20 $\lim _{b \rightarrow \infty} \lambda(a+i b) e^{-i \pi(a+i b)}=16$ uniformly in $a \in \mathbb{R}$.
Proof: From 58.1.30 and Lemma 58.1.19, this lemma will be proved if it is shown

$$
\lim _{b \rightarrow \infty}\left(\frac{2}{\cos ^{2}\left(\pi\left(\frac{1}{2}\right)(a+i b)\right)}-\frac{2}{\sin ^{2}\left(\pi\left(\frac{1}{2}\right)(a+i b)\right)}\right) e^{-i \pi(a+i b)}=16
$$

uniformly in $a \in \mathbb{R}$. Let $\tau=a+i b$ to simplify the notation. Then the above expression equals

$$
\begin{aligned}
& \left(\frac{8}{\left(e^{i \frac{\pi}{2} \tau}+e^{-i \frac{\pi}{2} \tau}\right)^{2}}+\frac{8}{\left(e^{i \frac{\pi}{2} \tau}-e^{-i \frac{\pi}{2} \tau}\right)^{2}}\right) e^{-i \pi \tau} \\
= & \left(\frac{8 e^{i \pi \tau}}{\left(e^{i \pi \tau}+1\right)^{2}}+\frac{8 e^{i \pi \tau}}{\left(e^{i \pi \tau}-1\right)^{2}}\right) e^{-i \pi \tau} \\
= & \frac{8}{\left(e^{i \pi \tau}+1\right)^{2}}+\frac{8}{\left(e^{i \pi \tau}-1\right)^{2}} \\
= & 16 \frac{1+e^{2 \pi i \tau}}{\left(1-e^{2 \pi i \tau}\right)^{2}}
\end{aligned}
$$

Now

$$
\begin{aligned}
\left|\frac{1+e^{2 \pi i \tau}}{\left(1-e^{2 \pi i \tau}\right)^{2}}-1\right| & =\left|\frac{1+e^{2 \pi i \tau}}{\left(1-e^{2 \pi i \tau}\right)^{2}}-\frac{\left(1-e^{2 \pi i \tau}\right)^{2}}{\left(1-e^{2 \pi i \tau}\right)^{2}}\right| \\
& \leq \frac{\left|3 e^{2 \pi i \tau}-e^{4 \pi i \tau}\right|}{\left(1-e^{-2 \pi b}\right)^{2}} \leq \frac{3 e^{-2 \pi b}+e^{-4 \pi b}}{\left(1-e^{-2 \pi b}\right)^{2}}
\end{aligned}
$$

and this estimate proves the lemma.
Corollary 58.1.21 $\lim _{b \rightarrow \infty} \lambda(a+i b)=0$ uniformly in $a \in \mathbb{R}$. Also $\lambda(i b)$ for $b>0$ is real and is between 0 and $1, \lambda$ is real on the line, $l_{2}$ and on the curve, $C$ and $\lim _{b \rightarrow 0+} \lambda(1+i b)=$ $-\infty$.

Proof: From Lemma 58.1.20,

$$
\left|\lambda(a+i b) e^{-i \pi(a+i b)}-16\right|<1
$$

for all $a$ provided $b$ is large enough. Therefore, for such $b$,

$$
|\lambda(a+i b)| \leq 17 e^{-\pi b} .
$$

58.1.28 proves the assertion about $\lambda(-b i)$ real.

By the first part, $\lim _{b \rightarrow \infty}|\lambda(i b)|=0$. Now from 58.1.24

$$
\begin{equation*}
\lim _{b \rightarrow 0+} \lambda(i b)=\lim _{b \rightarrow 0+}\left(1-\lambda\left(\frac{-1}{i b}\right)\right)=\lim _{b \rightarrow 0+}\left(1-\lambda\left(\frac{i}{b}\right)\right)=1 \tag{58.1.31}
\end{equation*}
$$

by Corollary 58.1.21.
Next consider the behavior of $\lambda$ on line $l_{2}$ in the above picture. From 58.1.17 and 58.1.28,

$$
\lambda(1+i b)=\frac{\lambda(i b)}{\lambda(i b)-1}<0
$$

and so as $b \rightarrow 0+$ in the above, $\lambda(1+i b) \rightarrow-\infty$.
It is left as an exercise to show that the map $\tau \rightarrow 1-\frac{1}{\tau}$ maps $l_{2}$ onto the curve, $C$. Therefore, by 58.1.25, for $\tau \in l_{2}$,

$$
\begin{align*}
\lambda\left(1-\frac{1}{\tau}\right) & =\frac{\lambda\left(\frac{-1}{\tau}\right)}{\lambda\left(\frac{-1}{\tau}\right)-1}  \tag{58.1.32}\\
& =\frac{1-\lambda(\tau)}{(1-\lambda(\tau))-1}=\frac{\lambda(\tau)-1}{\lambda(\tau)} \in \mathbb{R} \tag{58.1.33}
\end{align*}
$$

It follows $\lambda$ is real on the boundary of $\Omega$ in the above picture. This proves the corollary.
Now, following Alfors [3], cut off $\Omega$ by considering the horizontal line segment, $z=$ $a+i b_{0}$ where $b_{0}$ is very large and positive and $a \in[0,1]$. Also cut $\Omega$ off by the images of this horizontal line, under the transformations $z=\frac{1}{\tau}$ and $z=1-\frac{1}{\tau}$. These are arcs of circles because the two transformations are fractional linear transformations. It is left as an exercise for you to verify these arcs are situated as shown in the following picture. The important thing to notice is that for $b_{0}$ large the points of these circles are close to the origin and $(1,0)$ respectively. The following picture is a summary of what has been obtained so far on the mapping by $\lambda$.


In the picture, the descriptions are of $\lambda$ acting on points of the indicated boundary of $\Omega$. Consider the oriented contour which results from $\lambda(z)$ as $z$ moves first up $l_{2}$ as indicated, then along the line $z=a+i b$ and then down $l_{1}$ and then along $C_{1}$ to $C$ and along $C$ till $C_{2}$ and then along $C_{2}$ to $l_{2}$. As indicated in the picture, this involves going from a large negative real number to a small negative real number and then over a smooth curve which stays small to a real positive number and from there to a real number near $1 . \lambda(z)$ stays
fairly near 1 on $C_{1}$ provided $b_{0}$ is large so that the circle, $C_{1}$ has very small radius. Then along $C, \lambda(z)$ is real until it hits $C_{2}$. What about the behavior of $\lambda$ on $C_{2}$ ? For $z \in C_{2}$, it follows from the definition of $C_{2}$ that $z=1-\frac{1}{\tau}$ where $\tau$ is on the line, $a+i b_{0}$. Therefore, by Lemma 58.1.20, 58.1.17, and 58.1.24

$$
\begin{aligned}
\lambda(z) & =\lambda\left(1-\frac{1}{\tau}\right)=\frac{\lambda\left(\frac{-1}{\tau}\right)}{\lambda\left(\frac{-1}{\tau}\right)-1}=\frac{\lambda\left(\frac{1}{\tau}\right)}{\lambda\left(\frac{1}{\tau}\right)-1} \\
& =\frac{1-\lambda(\tau)}{(1-\lambda(\tau))-1}=\frac{\lambda(\tau)-1}{\lambda(\tau)}=1-\frac{1}{\lambda(\tau)}
\end{aligned}
$$

which is approximately equal to

$$
1-\frac{1}{16 e^{i \pi\left(a+i b_{0}\right)}}=1-\frac{e^{\pi b_{0}} e^{-i a \pi}}{16}
$$

These points are essentially on a large half circle in the upper half plane which has radius approximately $\frac{e^{\pi b_{0}}}{16}$.

Now let $w \in \mathbb{C}$ with $\operatorname{Im}(w) \neq 0$. Then for $b_{0}$ large enough, the motion over the boundary of the truncated region indicated in the above picture results in $\lambda$ tracing out a large simple closed curve oriented in the counter clockwise direction which includes $w$ on its interior if $\operatorname{Im}(w)>0$ but which excludes $w$ if $\operatorname{Im}(w)<0$.

Theorem 58.1.22 Let $\Omega$ be the domain described above. Then $\lambda$ maps $\Omega$ one to one and onto the upper half plane of $\mathbb{C},\{z \in \mathbb{C}$ such that $\operatorname{Im}(z)>0\}$. Also, the line $\lambda\left(l_{1}\right)=$ $(0,1), \lambda\left(l_{2}\right)=(-\infty, 0)$, and $\lambda(C)=(1, \infty)$.

Proof: Let $\operatorname{Im}(w)>0$ and denote by $\gamma$ the oriented contour described above and illustrated in the above picture. Then the winding number of $\lambda \circ \gamma$ about $w$ equals 1. Thus

$$
\frac{1}{2 \pi i} \int_{\lambda \circ \gamma} \frac{1}{z-w} d z=1
$$

But, splitting the contour integrals into $l_{2}$, the top line, $l_{1}, C_{1}, C$, and $C_{2}$ and changing variables on each of these, yields

$$
1=\frac{1}{2 \pi i} \int_{\gamma} \frac{\lambda^{\prime}(z)}{\lambda(z)-w} d z
$$

and by the theorem on counting zeros, Theorem 52.6.1 on Page 1661 , the function, $z \rightarrow$ $\lambda(z)-w$ has exactly one zero inside the truncated $\Omega$. However, this shows this function has exactly one zero inside $\Omega$ because $b_{0}$ was arbitrary as long as it is sufficiently large. Since $w$ was an arbitrary element of the upper half plane, this verifies the first assertion of the theorem. The remaining claims follow from the above description of $\lambda$, in particular the estimate for $\lambda$ on $C_{2}$. This proves the theorem.

Note also that the argument in the above proof shows that if $\operatorname{Im}(w)<0$, then $w$ is not in $\lambda(\Omega)$. However, if you consider the reflection of $\Omega$ about the $y$ axis, then it will follow that $\lambda$ maps this set one to one onto the lower half plane. The argument will make significant use of Theorem 52.6.3 on Page 1663 which is stated here for convenience.

Theorem 58.1.23 Let $f: B(a, R) \rightarrow \mathbb{C}$ be analytic and let

$$
f(z)-\alpha=(z-a)^{m} g(z), \infty>m \geq 1
$$

where $g(z) \neq 0$ in $B(a, R) .(f(z)-\alpha$ has a zero of order $m$ at $z=a$.) Then there exist $\varepsilon, \delta>0$ with the property that for each z satisfying $0<|z-\alpha|<\delta$, there exist points,

$$
\left\{a_{1}, \cdots, a_{m}\right\} \subseteq B(a, \varepsilon)
$$

such that

$$
f^{-1}(z) \cap B(a, \varepsilon)=\left\{a_{1}, \cdots, a_{m}\right\}
$$

and each $a_{k}$ is a zero of order 1 for the function $f(\cdot)-z$.
Corollary 58.1.24 Let $\Omega$ be the region above. Consider the set of points, $Q=\bar{\Omega} \cup \Omega^{\prime} \backslash$ $\{0,1\}$ described by the following picture.


Then $\lambda(Q)=\mathbb{C} \backslash\{0,1\}$. Also $\lambda^{\prime}(z) \neq 0$ for every $z$ in $\cup_{k=-\infty}^{\infty}(Q+2 k) \equiv H$.
Proof: By Theorem 58.1.22, this will be proved if it can be shown that $\lambda\left(\Omega^{\prime}\right)=$ $\{z \in \mathbb{C}: \operatorname{Im}(z)<0\}$. Consider $\lambda_{1}$ defined on $\Omega^{\prime}$ by

$$
\lambda_{1}(x+i y) \equiv \overline{\lambda(-x+i y)}
$$

Claim: $\lambda_{1}$ is analytic.
Proof of the claim: You just verify the Cauchy Riemann equations. Letting $\lambda(x+i y)=$ $u(x, y)+i v(x, y)$,

$$
\begin{aligned}
\lambda_{1}(x+i y) & =u(-x, y)-i v(-x, y) \\
& \equiv u_{1}(x, y)+i v(x, y)
\end{aligned}
$$

Then $u_{1 x}(x, y)=-u_{x}(-x, y)$ and $v_{1 y}(x, y)=-v_{y}(-x, y)=-u_{x}(-x, y)$ since $\lambda$ is analytic. Thus $u_{1 x}=v_{1 y}$. Next, $u_{1 y}(x, y)=u_{y}(-x, y)$ and $v_{1 x}(x, y)=v_{x}(-x, y)=-u_{y}(-x, y)$ and so $u_{1 y}=-v_{x}$.

Now recall that on $l_{1}, \lambda$ takes real values. Therefore, $\lambda_{1}=\lambda$ on $l_{1}$, a set with a limit point. It follows $\lambda=\lambda_{1}$ on $\Omega^{\prime} \cup \Omega$. By Theorem 58.1.22 $\lambda$ maps $\Omega$ one to one onto the upper half plane. Therefore, from the definition of $\lambda_{1}=\lambda$, it follows $\lambda$ maps $\Omega^{\prime}$ one to one onto the lower half plane as claimed. This has shown that $\lambda$ is one to one on $\Omega \cup \Omega^{\prime}$. This also verifies from Theorem 52.6.3 on Page 1663 that $\lambda^{\prime} \neq 0$ on $\Omega \cup \Omega^{\prime}$.

Now consider the lines $l_{2}$ and $C$. If $\lambda^{\prime}(z)=0$ for $z \in l_{2}$, a contradiction can be obtained. Pick such a point. If $\lambda^{\prime}(z)=0$, then $z$ is a zero of order $m \geq 2$ of the function, $\lambda-\lambda(z)$. Then by Theorem 52.6 .3 there exist $\delta, \varepsilon>0$ such that if $w \in B(\lambda(z), \delta)$, then $\lambda^{-1}(w) \cap$ $B(z, \varepsilon)$ contains at least $m$ points.


In particular, for $z_{1} \in \Omega \cap B(z, \varepsilon)$ sufficiently close to $z, \lambda\left(z_{1}\right) \in B(\lambda(z), \delta)$ and so the function $\lambda-\lambda\left(z_{1}\right)$ has at least two distinct zeros. These zeros must be in $B(z, \varepsilon) \cap \Omega$ because $\lambda\left(z_{1}\right)$ has positive imaginary part and the points on $l_{2}$ are mapped by $\lambda$ to a real number while the points of $B(z, \varepsilon) \backslash \bar{\Omega}$ are mapped by $\lambda$ to the lower half plane thanks to the relation, $\lambda(z+2)=\lambda(z)$. This contradicts $\lambda$ one to one on $\Omega$. Therefore, $\lambda^{\prime} \neq 0$ on $l_{2}$. Consider $C$. Points on $C$ are of the form $1-\frac{1}{\tau}$ where $\tau \in l_{2}$. Therefore, using 58.1.33,

$$
\lambda\left(1-\frac{1}{\tau}\right)=\frac{\lambda(\tau)-1}{\lambda(\tau)}
$$

Taking the derivative of both sides,

$$
\lambda^{\prime}\left(1-\frac{1}{\tau}\right)\left(\frac{1}{\tau^{2}}\right)=\frac{\lambda^{\prime}(\tau)}{\lambda(\tau)^{2}} \neq 0
$$

Since $\lambda$ is periodic of period 2 it follows $\lambda^{\prime}(z) \neq 0$ for all $z \in \cup_{k=-\infty}^{\infty}(Q+2 k)$.
Lemma 58.1.25 If $\operatorname{Im}(\tau)>0$ then there exists a unimodular $\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$ such that

$$
\frac{c+d \tau}{a+b \tau}
$$

is contained in the interior of $Q$. In fact, $\left|\frac{c+d \tau}{a+b \tau}\right| \geq 1$ and

$$
-1 / 2 \leq \operatorname{Re}\left(\frac{c+d \tau}{a+b \tau}\right) \leq 1 / 2
$$

Proof: Letting a basis for the module of periods of $\wp$ be $\{1, \tau\}$, it follows from Theorem 58.1.2 on Page 1816 that there exists a basis for the same module of periods, $\left\{w_{1}^{\prime}, w_{2}^{\prime}\right\}$ with the property that for $\tau^{\prime}=w_{2}^{\prime} / w_{1}^{\prime}$

$$
\left|\tau^{\prime}\right| \geq 1, \frac{-1}{2} \leq \operatorname{Re} \tau^{\prime} \leq \frac{1}{2}
$$

Since this is a basis for the same module of periods, there exists a unimodular matrix, $\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$ such that

$$
\binom{w_{1}^{\prime}}{w_{2}^{\prime}}=\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)\binom{1}{\tau} .
$$

Hence,

$$
\tau^{\prime}=\frac{w_{2}^{\prime}}{w_{1}^{\prime}}=\frac{c+d \tau}{a+b \tau}
$$

Thus $\tau^{\prime}$ is in the interior of $H$. In fact, it is on the interior of $\Omega^{\prime} \cup \Omega \equiv Q$.


### 58.1.7 A Short Review And Summary

With this lemma, it is easy to extend Corollary 58.1.24. First, a simple observation and review is a good idea. Recall that when you change the basis for the module of periods, the Weierstrass $\wp$ function does not change and so the set of $e_{i}$ used in defining $\lambda$ also do not change. Letting the new basis be $\left\{w_{1}^{\prime}, w_{2}^{\prime}\right\}$, it was shown that

$$
\binom{w_{1}^{\prime}}{w_{2}^{\prime}}=\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)\binom{w_{1}}{w_{2}}
$$

for some unimodular transformation, $\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$. Letting $\tau=w_{2} / w_{1}$ and $\tau^{\prime}=w_{2}^{\prime} / w_{1}^{\prime}$

$$
\tau^{\prime}=\frac{c+d \tau}{a+b \tau} \equiv \phi(\tau)
$$

Now as discussed earlier

$$
\begin{aligned}
\lambda\left(\tau^{\prime}\right) & =\lambda(\phi(\tau)) \equiv \frac{\wp\left(\frac{w_{1}^{\prime}+w_{2}^{\prime}}{2}\right)-\wp\left(\frac{w_{2}^{\prime}}{2}\right)}{\wp\left(\frac{w_{1}^{\prime}}{2}\right)-\wp\left(\frac{w_{2}^{\prime}}{2}\right)} \\
& =\frac{\wp\left(\frac{1+\tau^{\prime}}{2}\right)-\wp\left(\frac{\tau^{\prime}}{2}\right)}{\wp\left(\frac{1}{2}\right)-\wp\left(\frac{\tau^{\prime}}{2}\right)}
\end{aligned}
$$

These numbers in the above fraction must be the same as $\wp\left(\frac{1+\tau}{2}\right), \wp\left(\frac{\tau}{2}\right)$, and $\wp\left(\frac{1}{2}\right)$ but they might occur differently. This is because $\wp$ does not change and these numbers are the zeros of a polynomial having coefficients involving only numbers and $\wp(z)$. It could happen for example that $\wp\left(\frac{1+\tau^{\prime}}{2}\right)=\wp\left(\frac{\tau}{2}\right)$ in which case this would change the value of $\lambda$. In effect, you can keep track of all possibilities by simply permuting the $e_{i}$ in the formula for $\lambda(\tau)$ given by $\frac{e_{3}-e_{2}}{e_{1}-e_{2}}$. Thus consider the following permutation table.

| 1 | 2 | 3 |
| :--- | :--- | :--- |
| 2 | 3 | 1 |
| 3 | 1 | 2 |
| 2 | 1 | 3 |
| 1 | 3 | 2 |
| 3 | 2 | 1 |.

Corresponding to this list of 6 permutations, all possible formulas for $\lambda(\phi(\tau))$ can be obtained as follows. Letting $\tau^{\prime}=\phi(\tau)$ where $\phi$ is a unimodular matrix corresponding to a change of basis,

$$
\begin{gather*}
\lambda\left(\tau^{\prime}\right)=\frac{e_{3}-e_{2}}{e_{1}-e_{2}}=\lambda(\tau)  \tag{58.1.34}\\
\begin{aligned}
& \lambda\left(\tau^{\prime}\right)=\frac{e_{1}-e_{3}}{e_{2}-e_{3}}= \frac{e_{3}-e_{2}+e_{2}-e_{1}}{e_{3}-e_{2}}=1-\frac{1}{\lambda(\tau)}=\frac{\lambda(\tau)-1}{\lambda(\tau)} \\
& \lambda\left(\tau^{\prime}\right)= \frac{e_{2}-e_{1}}{e_{3}-e_{1}}=-\left[\frac{e_{3}-e_{2}-\left(e_{1}-e_{2}\right)}{e_{1}-e_{2}}\right]^{-1} \\
&=-[\lambda(\tau)-1]^{-1}=\frac{1}{1-\lambda(\tau)} \\
& \lambda\left(\tau^{\prime}\right)=\frac{e_{3}-e_{1}}{e_{2}-e_{1}}=-\left[\frac{e_{3}-e_{2}-\left(e_{1}-e_{2}\right)}{e_{1}-e_{2}}\right] \\
&=\frac{-[\lambda(\tau)-1]=1-\lambda(\tau)}{1\left(\tau^{\prime}\right)=\frac{e_{2}-e_{3}}{e_{1}-e_{3}}=\frac{e_{3}-e_{2}}{e_{3}-e_{2}-\left(e_{1}-e_{2}\right)}=\frac{1}{1-\frac{1}{\lambda(\tau)}}=\frac{\lambda(\tau)}{\lambda(\tau)-1}} \\
& \lambda\left(\tau^{\prime}\right)=\frac{e_{1}-e_{3}}{e_{3}-e_{2}}=\frac{1}{\lambda(\tau)}
\end{aligned} \tag{58.1.35}
\end{gather*}
$$

Corollary 58.1.26 $\lambda^{\prime}(\tau) \neq 0$ for all $\tau$ in the upper half plane, denoted by $P_{+}$.
Proof: Let $\tau \in P_{+}$. By Lemma 58.1.25 there exists $\phi$ a unimodular transformation and $\tau^{\prime}$ in the interior of $Q$ such that $\tau^{\prime}=\phi(\tau)$. Now from the definition of $\lambda$ in terms of the $e_{i}$, there is at worst a permutation of the $e_{i}$ and so it might be the case that $\lambda(\phi(\tau)) \neq \lambda(\tau)$ but it is the case that $\lambda(\phi(\tau))=\xi(\lambda(\tau))$ where $\xi^{\prime}(z) \neq 0$. Here $\xi$ is one of the functions determined by 58.1.34-58.1.39. (Since $\lambda(\tau) \notin\{0,1\}, \xi^{\prime}(\lambda(z)) \neq 0$. This follows from the above possibilities for $\xi$ listed above in $58.1 .34-58.1 .39$.) All the possibilities are $\xi(z)=$

$$
z, \frac{z-1}{z}, \frac{1}{1-z}, 1-z, \frac{z}{z-1}, \frac{1}{z}
$$

and these are the same as the possibilities for $\xi^{-1}$. Therefore,

$$
\lambda^{\prime}(\phi(\tau)) \phi^{\prime}(\tau)=\xi^{\prime}(\lambda(\tau)) \lambda^{\prime}(\tau)
$$

and so $\lambda^{\prime}(\tau) \neq 0$ as claimed.
Now I will present a lemma which is of major significance. It depends on the remarkable mapping properties of the modular function and the monodromy theorem from analytic continuation. A review of the monodromy theorem will be listed here for convenience. First recall the definition of the concept of function elements and analytic continuation.

Definition 58.1.27 A function element is an ordered pair, $(f, D)$ where $D$ is an open ball and $f$ is analytic on $D .\left(f_{0}, D_{0}\right)$ and $\left(f_{1}, D_{1}\right)$ are direct continuations of each other if $D_{1} \cap D_{0} \neq \emptyset$ and $f_{0}=f_{1}$ on $D_{1} \cap D_{0}$. In this case I will write $\left(f_{0}, D_{0}\right) \sim\left(f_{1}, D_{1}\right)$. A chain is a finite sequence, of disks, $\left\{D_{0}, \cdots, D_{n}\right\}$ such that $D_{i-1} \cap D_{i} \neq \emptyset$. If $\left(f_{0}, D_{0}\right)$ is a given function element and there exist function elements, $\left(f_{i}, D_{i}\right)$ such that $\left\{D_{0}, \cdots, D_{n}\right\}$ is a chain and $\left(f_{j-1}, D_{j-1}\right) \sim\left(f_{j}, D_{j}\right)$ then $\left(f_{n}, D_{n}\right)$ is called the analytic continuation of $\left(f_{0}, D_{0}\right)$ along the chain $\left\{D_{0}, \cdots, D_{n}\right\}$. Now suppose $\gamma$ is an oriented curve with parameter interval $[a, b]$ and there exists a chain, $\left\{D_{0}, \cdots, D_{n}\right\}$ such that $\gamma^{*} \subseteq \cup_{k=1}^{n} D_{k}, \gamma(a)$ is the center of $D_{0}, \gamma(b)$ is the center of $D_{n}$, and there is an increasing list of numbers in $[a, b], a=$ $s_{0}<s_{1} \cdots<s_{n}=b$ such that $\gamma\left(\left[s_{i}, s_{i+1}\right]\right) \subseteq D_{i}$ and $\left(f_{n}, D_{n}\right)$ is an analytic continuation of $\left(f_{0}, D_{0}\right)$ along the chain. Then $\left(f_{n}, D_{n}\right)$ is called an analytic continuation of $\left(f_{0}, D_{0}\right)$ along the curve $\gamma$. ( $\gamma$ will always be a continuous curve. Nothing more is needed. )

Then the main theorem is the monodromy theorem listed next, Theorem 55.4.5 and its corollary on Page 1747.

Theorem 58.1.28 Let $\Omega$ be a simply connected subset of $\mathbb{C}$ and suppose $(f, B(a, r))$ is a function element with $B(a, r) \subseteq \Omega$. Suppose also that this function element can be analytically continued along every curve through $a$. Then there exists $G$ analytic on $\Omega$ such that $G$ agrees with $f$ on $B(a, r)$.

Here is the lemma.
Lemma 58.1.29 Let $\lambda$ be the modular function defined on $P_{+}$the upper half plane. Let $V$ be a simply connected region in $\mathbb{C}$ and let $f: V \rightarrow \mathbb{C} \backslash\{0,1\}$ be analytic and nonconstant. Then there exists an analytic function, $g: V \rightarrow P_{+}$such that $\lambda \circ g=f$.

Proof: Let $a \in V$ and choose $r_{0}$ small enough that $f\left(B\left(a, r_{0}\right)\right)$ contains neither 0 nor 1 . You need only let $B\left(a, r_{0}\right) \subseteq V$. Now there exists a unique point in $Q, \tau_{0}$ such that $\lambda\left(\tau_{0}\right)=$ $f(a)$. By Corollary 58.1.24, $\lambda^{\prime}\left(\tau_{0}\right) \neq 0$ and so by the open mapping theorem, Theorem 52.6.3 on Page 1663, There exists $B\left(\tau_{0}, R_{0}\right) \subseteq P_{+}$such that $\lambda$ is one to one on $B\left(\tau_{0}, R_{0}\right)$ and has a continuous inverse. Then picking $r_{0}$ still smaller, it can be assumed $f\left(B\left(a, r_{0}\right)\right) \subseteq$ $\lambda\left(B\left(\tau_{0}, R_{0}\right)\right)$. Thus there exists a local inverse for $\lambda, \lambda_{0}^{-1}$ defined on $f\left(B\left(a, r_{0}\right)\right)$ having values in $B\left(\tau_{0}, R_{0}\right) \cap \lambda^{-1}\left(f\left(B\left(a, r_{0}\right)\right)\right)$. Then defining $g_{0} \equiv \lambda_{0}^{-1} \circ f,\left(g_{0}, B\left(a, r_{0}\right)\right)$ is a function element. I need to show this can be continued along every curve starting at $a$ in such a way that each function in each function element has values in $P_{+}$.

Let $\gamma:[\alpha, \beta] \rightarrow V$ be a continuous curve starting at $a,(\gamma(\alpha)=a)$ and suppose that if $t<T$ there exists a nonnegative integer $m$ and a function element $\left(g_{m}, B\left(\gamma(t), r_{m}\right)\right)$ which is an analytic continuation of $\left(g_{0}, B\left(a, r_{0}\right)\right)$ along $\gamma$ where $g_{m}(\gamma(t)) \in P_{+}$and each function in every function element for $j \leq m$ has values in $P_{+}$. Thus for some small $T>0$ this has been achieved.

Then consider $f(\gamma(T)) \in \mathbb{C} \backslash\{0,1\}$. As in the first part of the argument, there exists a unique $\tau_{T} \in Q$ such that $\lambda\left(\tau_{T}\right)=f(\gamma(T))$ and for $r$ small enough there is an analytic local inverse, $\lambda_{T}^{-1}$ between $f(B(\gamma(T), r))$ and $\lambda^{-1}(f(B(\gamma(T), r))) \cap B\left(\tau_{T}, R_{T}\right) \subseteq P_{+}$for some $R_{T}>0$. By the assumption that the analytic continuation can be carried out for $t<T$, there exists $\left\{t_{0}, \cdots, t_{m}=t\right\}$ and function elements $\left(g_{j}, B\left(\gamma\left(t_{j}\right), r_{j}\right)\right), j=0, \cdots, m$ as just described with $g_{j}\left(\gamma\left(t_{j}\right)\right) \in P_{+}, \lambda \circ g_{j}=f$ on $B\left(\gamma\left(t_{j}\right), r_{j}\right)$ such that for $t \in\left[t_{m}, T\right], \gamma(t) \in$ $B(\gamma(T), r)$. Let

$$
I=B\left(\gamma\left(t_{m}\right), r_{m}\right) \cap B(\gamma(T), r)
$$

Then since $\lambda_{T}^{-1}$ is a local inverse, it follows for all $z \in I$

$$
\lambda\left(g_{m}(z)\right)=f(z)=\lambda\left(\lambda_{T}^{-1} \circ f(z)\right)
$$

Pick $z_{0} \in I$. Then by Lemma 58.1.18 on Page 1836 there exists a unimodular mapping of the form

$$
\phi(z)=\frac{a z+b}{c z+d}
$$

where

$$
\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right) \sim\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right) \bmod 2
$$

such that

$$
g_{m}\left(z_{0}\right)=\phi\left(\lambda_{T}^{-1} \circ f\left(z_{0}\right)\right) .
$$

Since both $g_{m}\left(z_{0}\right)$ and $\phi\left(\lambda_{T}^{-1} \circ f\left(z_{0}\right)\right)$ are in the upper half plane, it follows $a d-c b=1$ and $\phi$ maps the upper half plane to the upper half plane. Note the pole of $\phi$ is real and all the sets being considered are contained in the upper half plane so $\phi$ is analytic where it needs to be.

Claim: For all $z \in I$,

$$
\begin{equation*}
g_{m}(z)=\phi \circ \lambda_{T}^{-1} \circ f(z) \tag{58.1.40}
\end{equation*}
$$

Proof: For $z=z_{0}$ the equation holds. Let

$$
A=\left\{z \in I: g_{m}(z)=\phi\left(\lambda_{T}^{-1} \circ f(z)\right)\right\} .
$$

Thus $z_{0} \in I$. If $z \in I$ and if $w$ is close enough to $z$, then $w \in I$ also and so both sides of 58.1.40 with $w$ in place of $z$ are in $\lambda_{m}^{-1}(f(I))$. But by construction, $\lambda$ is one to one on this set and since $\lambda$ is invariant with respect to $\phi$,

$$
\lambda\left(g_{m}(w)\right)=\lambda\left(\lambda_{T}^{-1} \circ f(w)\right)=\lambda\left(\phi \circ \lambda_{T}^{-1} \circ f(w)\right)
$$

and consequently, $w \in A$. This shows $A$ is open. But $A$ is also closed in $I$ because the functions are continuous. Therefore, $A=I$ and so 58.1.40 is obtained.

Letting $f(z) \in f(B(\gamma(T)), r)$,

$$
\lambda\left(\phi\left(\lambda_{T}^{-1}(f(z))\right)\right)=\lambda\left(\lambda_{T}^{-1}(f(z))\right)=f(z)
$$

and so $\phi \circ \lambda_{T}^{-1}$ is a local inverse for $\lambda$ on $\left.f(B(\gamma)), r\right)$. Let the new function element be $(\overbrace{\phi \circ \lambda_{T}^{-1} \circ f}^{g_{m+1}}, B(\gamma(T), r))$. This has shown the initial function element can be continued along every curve through $a$.

By the monodromy theorem, there exists $g$ analytic on $V$ such that $g$ has values in $P_{+}$ and $g=g_{0}$ on $B\left(a, r_{0}\right)$. By the construction, it also follows $\lambda \circ g=f$. This last claim is easy to see because $\lambda \circ g=f$ on $B\left(a, r_{0}\right)$, a set with a limit point so the equation holds for all $z \in V$. This proves the lemma.

### 58.2 The Picard Theorem Again

Having done all this work on the modular function which is important for its own sake, there is an easy proof of the Picard theorem. In fact, this is the way Picard did it in 1879. I will state it slightly differently since it is no trouble to do so, [65].

Theorem 58.2.1 Let $f$ be meromorphic on $\mathbb{C}$ and suppose $f$ misses three distinct points, $a, b, c$. Then $f$ is a constant function.

Proof: Let $\phi(z) \equiv \frac{z-a}{z-c} \frac{b-c}{b-a}$. Then $\phi(c)=\infty, \phi(a)=0$, and $\phi(b)=1$. Now consider the function, $h=\phi \circ f$. Then $h$ misses the three points $\infty, 0$, and 1 . Since $h$ is meromorphic and does not have $\infty$ in its values, it must actually be analytic. Thus $h$ is an entire function which misses the two values 0 and 1 . If $h$ is not constant, then by Lemma 58.1.29 there exists a function, $g$ analytic on $\mathbb{C}$ which has values in the upper half plane, $P_{+}$such that $\lambda \circ g=h$. However, $g$ must be a constant because there exists $\psi$ an analytic map on the upper half plane which maps the upper half plane to $B(0,1)$. You can use the Riemann mapping theorem or more simply, $\psi(z)=\frac{z-i}{z+i}$. Thus $\psi \circ g$ equals a constant by Liouville's theorem. Hence $g$ is a constant and so $h$ must also be a constant because $\lambda(g(z))=h(z)$. This proves $f$ is a constant also. This proves the theorem.

### 58.3 Exercises

1. Show the set of modular transformations is a group. Also show those modular transformations which are congruent mod 2 to the identity as described above is a subgroup.
2. Suppose $f$ is an elliptic function with period module $M$. If $\left\{w_{1}, w_{2}\right\}$ and $\left\{w_{1}^{\prime}, w_{2}^{\prime}\right\}$ are two bases, show that the resulting period parallelograms resulting from the two bases have the same area.
3. Given a module of periods with basis $\left\{w_{1}, w_{2}\right\}$ and letting a typical element of this module be denoted by $w$ as described above, consider the product

$$
\sigma(z) \equiv z \prod_{w \neq 0}\left(1-\frac{z}{w}\right) e^{(z / w)+\frac{1}{2}(z / w)^{2}} .
$$

Show this product converges uniformly on compact sets, is an entire function, and satisfies

$$
\sigma^{\prime}(z) / \sigma(z)=\zeta(z)
$$

where $\zeta(z)$ was defined above as a primitive of $\wp(z)$ and is given by

$$
\zeta(z)=\frac{1}{z}+\sum_{w \neq 0} \frac{1}{z-w}+\frac{z}{w^{2}}+\frac{1}{w}
$$

4. Show $\zeta\left(z+w_{i}\right)=\zeta(z)+\eta_{i}$ where $\eta_{i}$ is a constant.
5. Let $P_{a}$ be the parallelogram shown in the following picture.


Show that $\frac{1}{2 \pi i} \int_{\partial P_{a}} \zeta(z) d z=1$ where the contour is taken once around the parallelogram in the counter clockwise direction. Next evaluate this contour integral directly to obtain Legendre's relation,

$$
\eta_{1} w_{2}-\eta_{2} w_{1}=2 \pi i
$$

6. For $\sigma$ defined in Problem 3, 4 explain the following steps. For $j=1,2$

$$
\frac{\sigma^{\prime}\left(z+w_{j}\right)}{\sigma\left(z+w_{j}\right)}=\zeta\left(z+w_{j}\right)=\zeta(z)+\eta_{j}=\frac{\sigma^{\prime}(z)}{\sigma(z)}+\eta_{j}
$$

Therefore, there exists a constant, $C_{j}$ such that

$$
\sigma\left(z+w_{j}\right)=C_{j} \sigma(z) e^{\eta_{j} z}
$$

Next show $\sigma$ is an odd function, $(\sigma(-z)=-\sigma(z))$ and then let $z=-w_{j} / 2$ to find $C_{j}=-e^{\frac{\eta_{j} \omega_{j}}{2}}$ and so

$$
\begin{equation*}
\sigma\left(z+w_{j}\right)=-\sigma(z) e^{\eta_{j}\left(z+\frac{w_{j}}{2}\right)} \tag{58.3.41}
\end{equation*}
$$

7. Show any even elliptic function, $f$ with periods $w_{1}$ and $w_{2}$ for which 0 is neither a pole nor a zero can be expressed in the form

$$
f(0) \prod_{k=1}^{n} \frac{\wp(z)-\wp\left(a_{k}\right)}{\wp(z)-\wp\left(b_{k}\right)}
$$

where $C$ is some constant. Here $\wp$ is the Weierstrass function which comes from the two periods, $w_{1}$ and $w_{2}$. Hint: You might consider the above function in terms of the poles and zeros on a period parallelogram and recall that an entire function which is elliptic is a constant.
8. Suppose $f$ is any elliptic function with $\left\{w_{1}, w_{2}\right\}$ a basis for the module of periods. Using Theorem 58.1.8 and 58.3.41 show that there exists constants $a_{1}, \cdots, a_{n}$ and $b_{1}, \cdots, b_{n}$ such that for some constant $C$,

$$
f(z)=C \prod_{k=1}^{n} \frac{\sigma\left(z-a_{k}\right)}{\sigma\left(z-b_{k}\right)}
$$

Hint: You might try something like this: By Theorem 58.1.8, it follows that if $\left\{\alpha_{k}\right\}$ are the zeros and $\left\{b_{k}\right\}$ the poles in an appropriate period parallelogram, $\sum \alpha_{k}-\sum b_{k}$ equals a period. Replace $\alpha_{k}$ with $a_{k}$ such that $\sum a_{k}-\sum b_{k}=0$. Then use 58.3.41 to show that the given formula for $f$ is bi periodic. Anyway, you try to arrange things such that the given formula has the same poles as $f$. Remember an entire elliptic function equals a constant.
9. Show that the map $\tau \rightarrow 1-\frac{1}{\tau}$ maps $l_{2}$ onto the curve, $C$ in the above picture on the mapping properties of $\lambda$.
10. Modify the proof of Theorem 58.1.22 to show that $\lambda(\Omega) \cap\{z \in \mathbb{C}: \operatorname{Im}(z)<0\}=\emptyset$.

## Part VI

## Topics In Probability

## Chapter 59

## Basic Probability

Caution: This material on probability and stochastic processes may be half baked in places. I have not yet rewritten it several times. This is not to say that nothing else is half baked. However, the probability is higher here.

### 59.1 Random Variables And Independence

Recall Lemma 14.2.3 on Page 388 which is stated here for convenience.
Lemma 59.1.1 Let $M$ be a metric space with the closed balls compact and suppose $\lambda$ is a measure defined on the Borel sets of $M$ which is finite on compact sets. Then there exists a unique Radon measure, $\bar{\lambda}$ which equals $\lambda$ on the Borel sets. In particular $\lambda$ must be both inner and outer regular on all Borel sets.

Also important is the following fundamental result which is called the Borel Cantelli lemma.

Lemma 59.1.2 Let $(\Omega, \mathscr{F}, \lambda)$ be a measure space and let $\left\{A_{i}\right\}$ be a sequence of measurable sets satisfying

$$
\sum_{i=1}^{\infty} \lambda\left(A_{i}\right)<\infty .
$$

Then letting $S$ denote the set of $\omega \in \Omega$ which are in infinitely many $A_{i}$, it follows $S$ is a measurable set and $\lambda(S)=0$.

Proof: $S=\cap_{k=1}^{\infty} \cup_{m=k}^{\infty} A_{m}$. Therefore, $S$ is measurable and also

$$
\lambda(S) \leq \lambda\left(\cup_{m=k}^{\infty} A_{m}\right) \leq \sum_{m=k}^{\infty} \lambda\left(A_{k}\right)
$$

and this converges to 0 as $k \rightarrow \infty$ because of the convergence of the series.
Here is another nice observation.
Proposition 59.1.3 Suppose $E_{i}$ is a separable Banach space. Then if $B_{i}$ is a Borel set of $E_{i}$, it follows $\prod_{i=1}^{n} B_{i}$ is a Borel set in $\prod_{i=1}^{n} E_{i}$.

Proof: An easy way to do this is to consider the projection maps.

$$
\pi_{i} \mathbf{x} \equiv x_{i}
$$

Then these projection maps are continuous. Hence for $U$ an open set,

$$
\pi_{i}^{-1}(U) \equiv \prod_{j=1}^{n} A_{j}, A_{j}=E_{j} \text { if } j \neq i \text { and } A_{i}=U
$$

Thus $\pi_{i}^{-1}$ (open) equals an open set. Let

$$
\mathscr{S} \equiv\left\{V \subseteq \mathbb{R}: \pi_{i}^{-1}(V) \text { is Borel }\right\}
$$

Then $\mathscr{S}$ contains all the open sets and is clearly a $\sigma$ algebra. Therefore, $\mathscr{S}$ contains the Borel sets. Let $B_{i}$ be a Borel set in $E_{i}$. Then

$$
\prod_{i=1}^{n} B_{i}=\cap_{i=1}^{n} \pi_{i}^{-1}\left(B_{i}\right)
$$

a finite intersection of Borel sets.
Definition 59.1.4 A probability space is a measure space, $(\Omega, \mathscr{F}, P)$ where $P$ is a measure satisfying $P(\Omega)=1$. A random vector (variable) is a measurable function, $\mathbf{X}: \Omega \rightarrow Z$ where $Z$ is some topological space. It is often the case that $Z$ will equal $\mathbb{R}^{p}$. Assume $Z$ is a separable Banach space. Define the following $\sigma$ algebra.

$$
\sigma(\mathbf{X}) \equiv\left\{\mathbf{X}^{-1}(E): E \text { is Borel in } Z\right\}
$$

Thus $\sigma(\mathbf{X}) \subseteq \mathscr{F}$. For $E$ a Borel set in $Z$ define

$$
\lambda_{\mathbf{X}}(E) \equiv P\left(\mathbf{X}^{-1}(E)\right)
$$

This is called the distribution of the random variable, $\mathbf{X}$. If

$$
\int_{\Omega}|\mathbf{X}(\omega)| d P<\infty
$$

then define

$$
E(\mathbf{X}) \equiv \int_{\Omega} \mathbf{X} d P
$$

where the integral is defined as the Bochner integral.
Recall the following fundamental result which was proved earlier but which I will give a short proof of now.

Proposition 59.1.5 Let $(\Omega, \mathscr{S}, \mu)$ be a measure space and let $\mathbf{X}: \Omega \rightarrow Z$ where $Z$ is a separable Banach space. Then $\mathbf{X}$ is strongly measurable if and only if $\mathbf{X}^{-1}(U) \in \mathscr{S}$ for all $U$ open in $Z$.

Proof: To begin with, let $D(a, r)$ be the closure of the open ball $B(a, r)$. By Lemma 21.1.6, there exists $\left\{f_{i}\right\} \subseteq B^{\prime}$, the unit ball in $Z^{\prime}$ such that

$$
\|z\|_{Z}=\sup _{i}\left\{\left|f_{i}(z)\right|\right\}
$$

Then

$$
\begin{gathered}
D(a, r)=\{z:\|a-z\| \leq r\}=\cap_{i}\left\{z:\left|f_{i}(z)-f_{i}(a)\right| \leq r\right\} \\
=\cap_{i} f_{i}^{-1}\left(\overline{B\left(f_{i}(a), r\right)}\right)
\end{gathered}
$$

It follows that

$$
\begin{aligned}
\mathbf{X}^{-1}(D(a, r)) & =\cap_{i} \mathbf{X}^{-1}\left(f_{i}^{-1}\left(\overline{B\left(f_{i}(a), r\right)}\right)\right) \\
& =\cap_{i}\left(f_{i} \circ \mathbf{X}\right)^{-1}\left(\overline{B\left(f_{i}(a), r\right)}\right)
\end{aligned}
$$

If $\mathbf{X}$ is strongly measurable, then it is weakly measurable and so each $f_{i} \circ \mathbf{X}$ is a real (complex) valued measurable function. Hence the expression on the right in the above is measurable. Now if $U$ is any open set in $Z$, then it is the countable union of such closed disks $U=\cup_{i} D_{i}$. Therefore, $\mathbf{X}^{-1}(U)=\cap_{i} \mathbf{X}^{-1}\left(D_{i}\right) \in \mathscr{S}$. It follows that strongly measurable implies inverse images of open sets are in $\mathscr{S}$.

Conversely, suppose $\mathbf{X}^{-1}(U) \in \mathscr{S}$ for every open $U$. Then for $f \in Z^{\prime}, f \circ \mathbf{X}$ is real valued and measurable. Therefore, $\mathbf{X}$ is weakly measurable. By the Pettis theorem, it follows that $f \circ \mathbf{X}$ is strongly measurable.

Proposition 59.1.6 If $\mathbf{X}: \Omega \rightarrow Z$ is measurable, then $\sigma(\mathbf{X})$ equals the smallest $\sigma$ algebra such that $\mathbf{X}$ is measurable with respect to it. Also if $X_{i}$ are random variables having values in separable Banach spaces $Z_{i}$, then $\sigma(\mathbf{X})=\sigma\left(X_{1}, \cdots, X_{n}\right)$ where $\mathbf{X}$ is the vector mapping $\Omega$ to $\prod_{i=1}^{n} Z_{i}$ and $\sigma\left(X_{1}, \cdots, X_{n}\right)$ is the smallest $\sigma$ algebra such that each $X_{i}$ is measurable with respect to it.

Proof: Let $\mathscr{G}$ denote the smallest $\sigma$ algebra such that $\mathbf{X}$ is measurable with respect to this $\sigma$ algebra. By definition $\mathbf{X}^{-1}$ (open) $\in \mathscr{G}$. Furthermore, the set of all $E$ such that $\mathbf{X}^{-1}(E) \in \mathscr{G}$ is a $\sigma$ algebra. Hence it includes all the Borel sets. Hence $\mathbf{X}^{-1}$ (Borel) $\in \mathscr{G}$ and so $\mathscr{G} \supseteq \sigma(\mathbf{X})$. However, $\sigma(\mathbf{X})$ defined above is a $\sigma$ algebra such that $\mathbf{X}$ is measurable with respect to $\sigma(\mathbf{X})$. Therefore, $\mathscr{G}=\sigma(\mathbf{X})$.

Letting $B_{i}$ be a Borel set in $Z_{i}, \prod_{i=1}^{n} B_{i}$ is a Borel set by Proposition 59.1.3 and so

$$
\mathbf{X}^{-1}\left(\prod_{i=1}^{n} B_{i}\right)=\cap_{i=1}^{n} X_{i}^{-1}\left(B_{i}\right) \in \sigma\left(X_{1}, \cdots, X_{n}\right)
$$

If $\mathscr{G}$ denotes the Borel sets $F \subseteq \prod_{i=1}^{n} Z_{i}$ such that $\mathbf{X}^{-1}(F) \in \sigma\left(X_{1}, \cdots, X_{n}\right)$, then $\mathscr{G}$ is clearly a $\sigma$ algebra which contains the open sets. Hence $\mathscr{G}=\mathscr{B}$ the Borel sets of $\prod_{i=1}^{n} Z_{i}$. This shows that $\sigma(\mathbf{X}) \subseteq \sigma\left(X_{1}, \cdots, X_{n}\right)$. Next we observe that $\sigma(\mathbf{X})$ is a $\sigma$ algebra with the property that each $X_{i}$ is measurable with respect to $\sigma(\mathbf{X})$. This follows from $X_{i}^{-1}\left(B_{i}\right)=$ $\mathbf{X}^{-1}\left(\prod_{j=1}^{n} A_{j}\right) \in \sigma(\mathbf{X})$, where each $A_{j}=Z_{j}$ except for $A_{i}=B_{i}$. Since $\sigma\left(X_{1}, \cdots, X_{n}\right)$ is defined as the smallest such $\sigma$ algebra, it follows that $\sigma(\mathbf{X}) \supseteq \sigma\left(X_{1}, \cdots, X_{n}\right)$.

For random variables having values in a separable Banach space or even more generally for a separable metric space, much can be said about regularity of $\lambda_{\mathbf{x}}$.

Definition 59.1.7 A measure, $\mu$ defined on $\mathscr{B}(E)$ will be called inner regular if for all $F \in \mathscr{B}(E)$,

$$
\mu(F)=\sup \{\mu(K): K \subseteq F \text { and } K \text { is closed }\}
$$

A measure, $\mu$ defined on $\mathscr{B}(E)$ will be called outer regular if for all $F \in \mathscr{B}(E)$,

$$
\mu(F)=\inf \{\mu(V): V \supseteq F \text { and } V \text { is open }\}
$$

When a measure is both inner and outer regular, it is called regular.
For probability measures, the above definition of regularity tends to come free. Note it is a little weaker than the usual definition of regularity because $K$ is only assumed to be closed, not compact.

Lemma 59.1.8 Let $\mu$ be a finite measure defined on $\mathscr{B}(E)$ where $E$ is a metric space. Then $\mu$ is regular.

Proof: First note every open set is the countable union of closed sets and every closed set is the countable intersection of open sets. Here is why. Let $V$ be an open set and let

$$
K_{k} \equiv\left\{x \in V: \operatorname{dist}\left(x, V^{C}\right) \geq 1 / k\right\}
$$

Then clearly the union of the $K_{k}$ equals $V$. Next, for $K$ closed let

$$
V_{k} \equiv\{x \in X: \operatorname{dist}(x, K)<1 / k\} .
$$

Clearly the intersection of the $V_{k}$ equals $K$. Therefore, letting $V$ denote an open set and $K$ a closed set,

$$
\begin{aligned}
& \mu(V)=\sup \{\mu(K): K \subseteq V \text { and } K \text { is closed }\} \\
& \mu(K)=\inf \{\mu(V): V \supseteq K \text { and } V \text { is open }\}
\end{aligned}
$$

Also since $V$ is open and $K$ is closed,

$$
\begin{aligned}
& \mu(V)=\inf \{\mu(U): U \supseteq V \text { and } U \text { is open }\} \\
& \mu(K)=\sup \{\mu(L): L \subseteq K \text { and } L \text { is closed }\}
\end{aligned}
$$

In words, $\mu$ is regular on open and closed sets. Let

$$
\mathscr{F} \equiv\{F \in \mathscr{B}(X) \text { such that } \mu \text { is regular on } F\}
$$

Then $\mathscr{F}$ contains the open sets and the closed sets.
Suppose $F \in \mathscr{F}$. Then there exists $V \supseteq F$ with $\mu(V \backslash F)<\varepsilon$. It follows $V^{C} \subseteq F^{C}$ and

$$
\mu\left(F^{C} \backslash V^{C}\right)=\mu(V \backslash F)<\varepsilon
$$

Thus $\mu$ is inner regular on $F^{C}$. Since $F \in \mathscr{F}$, there exists $K \subseteq F$ where $K$ is closed and $\mu(F \backslash K)<\varepsilon$. Then also $K^{C} \supseteq F^{C}$ and

$$
\mu\left(K^{C} \backslash F^{C}\right)=\mu(F \backslash K)<\varepsilon
$$

Thus if $F \in \mathscr{F}$ so is $F^{C}$.
Suppose now that $\left\{F_{i}\right\} \subseteq \mathscr{F}$, the $F_{i}$ being disjoint. Is $\cup F_{i} \in \mathscr{F}$ ? There exists $K_{i} \subseteq F_{i}$ such that $\mu\left(K_{i}\right)+\varepsilon / 2^{i}>\mu\left(F_{i}\right)$. Then

$$
\begin{aligned}
\mu\left(\cup_{i=1}^{\infty} F_{i}\right) & =\sum_{i=1}^{\infty} \mu\left(F_{i}\right) \leq \varepsilon+\sum_{i=1}^{\infty} \mu\left(K_{i}\right) \\
& <2 \varepsilon+\sum_{i=1}^{N} \mu\left(K_{i}\right)=2 \varepsilon+\mu\left(\cup_{i=1}^{N} K_{i}\right)
\end{aligned}
$$

provided $N$ is large enough. Thus it follows $\mu$ is inner regular on $\cup_{i=1}^{\infty} F_{i}$. Why is it outer regular? Let $V_{i} \supseteq F_{i}$ such that $\mu\left(F_{i}\right)+\varepsilon / 2^{i}>\mu\left(V_{i}\right)$ and

$$
\mu\left(\cup_{i=1}^{\infty} F_{i}\right)=\sum_{i=1}^{\infty} \mu\left(F_{i}\right)>-\varepsilon+\sum_{i=1}^{\infty} \mu\left(V_{i}\right) \geq-\varepsilon+\mu\left(\cup_{i=1}^{\infty} V_{i}\right)
$$

which shows $\mu$ is outer regular on $\cup_{i=1}^{\infty} F_{i}$. It follows $\mathscr{F}$ contains the $\pi$ system consisting of open sets and is closed with respect to countable disjoint unions and complements, and so by the Lemma on $\pi$ systems, Lemma 12.12.3, $\mathscr{F}$ contains $\sigma(\tau)$ where $\tau$ is the set of open sets. Hence $\mathscr{F}$ contains the Borel sets and is itself a subset of the Borel sets by definition. Therefore, $\mathscr{F}=\mathscr{B}(X)$.

One can say more if the metric space is complete and separable. In fact in this case the above definition of inner regularity can be shown to imply the usual one.

Lemma 59.1.9 Let $\mu$ be a finite measure on a $\sigma$ algebra containing $\mathscr{B}(X)$, the Borel sets of $X$, a separable complete metric space. Then if $C$ is a closed set,

$$
\mu(C)=\sup \{\mu(K): K \subseteq C \text { and } K \text { is compact. }\}
$$

It follows that for a finite measure on $\mathscr{B}(X)$ where $X$ is a Polish space, $\mu$ is inner regular in the sense that for all $F \in \mathscr{B}(X)$,

$$
\mu(F)=\sup \{\mu(K): K \subseteq F \text { and } K \text { is compact }\}
$$

Proof: Let $\left\{a_{k}\right\}$ be a countable dense subset of $C$. Thus $\cup_{k=1}^{\infty} B\left(a_{k}, \frac{1}{n}\right) \supseteq C$. Therefore, there exists $m_{n}$ such that

$$
\mu\left(C \backslash \cup_{k=1}^{m_{n}} B \overline{\left(a_{k}, \frac{1}{n}\right)}\right) \equiv \mu\left(C \backslash C_{n}\right)<\frac{\varepsilon}{2^{n}}
$$

Now let $K=C \cap\left(\cap_{n=1}^{\infty} C_{n}\right)$. Then $K$ is a subset of $C_{n}$ for each $n$ and so for each $\varepsilon>0$ there exists an $\varepsilon$ net for $K$ since $C_{n}$ has a $1 / n$ net, namely $a_{1}, \cdots, a_{m_{n}}$. Since $K$ is closed, it is complete and so it is also compact since it is complete and totally bounded, Theorem 7.6.5. Now

$$
\mu(C \backslash K) \leq \mu\left(\cup_{n=1}^{\infty}\left(C \backslash C_{n}\right)\right)<\sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n}}=\varepsilon
$$

Thus $\mu(C)$ can be approximated by $\mu(K)$ for $K$ a compact subset of $C$. The last claim follows from Lemma 59.1.8.

Definition 59.1.10 A measurable function $\mathbf{X}:(\Omega, \mathscr{F}, \mu) \rightarrow Z$ a topological space is called a random variable when $\mu(\Omega)=1$. For such a random variable, one can define a distribution measure $\lambda_{\mathbf{x}}$ on the Borel sets of $Z$ as follows.

$$
\lambda_{\mathbf{x}}(G) \equiv \mu\left(\mathbf{X}^{-1}(G)\right)
$$

This is a well defined measure on the Borel sets of $Z$ because it makes sense for every $G$ open and $\mathscr{G} \equiv\left\{G \subseteq Z: \mathbf{X}^{-1}(G) \in \mathscr{F}\right\}$ is a $\sigma$ algebra which contains the open sets, hence the Borel sets. Such a measurable function is also called a random vector.

Corollary 59.1.11 Let $\mathbf{X}$ be a random variable (random vector) with values in a complete metric space, $Z$. Then $\lambda_{\mathbf{x}}$ is an inner and outer regular measure defined on $\mathscr{B}(Z)$.

Proposition 59.1.12 For $\mathbf{X}$ a random vector defined above, $\mathbf{X}$ having values in a complete separable metric space $Z$, then $\lambda_{\mathbf{x}}$ is inner and outer regular and Borel.

$$
(\Omega, P) \xrightarrow{\mathbf{x}}\left(Z, \lambda_{\mathbf{x}}\right) \xrightarrow{h} E
$$

If $h$ is Borel measurable and $h \in L^{1}\left(Z, \lambda_{\mathbf{x}} ; E\right)$ for $E$ a Banach space, then

$$
\begin{equation*}
\int_{\Omega} h(\mathbf{X}(\omega)) d P=\int_{Z} h(\mathbf{x}) d \lambda \mathbf{x} \tag{59.1.1}
\end{equation*}
$$

In the case where $Z=E$, a separable Banach space, if $\mathbf{X}$ is measurable then $\mathbf{X} \in$ $L^{1}(\Omega ; E)$ if and only if the identity map on $E$ is in $L^{1}(E ; \lambda \mathbf{x})$ and

$$
\begin{equation*}
\int_{\Omega} \mathbf{X}(\omega) d P=\int_{E} \mathbf{x} d \lambda \mathbf{X}(\mathbf{x}) \tag{59.1.2}
\end{equation*}
$$

Proof: The regularity claims are established above. It remains to verify 59.1.1.
Since $h \in L^{1}(Z, E)$, it follows there exists a sequence of simple functions $\left\{h_{n}\right\}$ such that

$$
h_{n}(\mathbf{x}) \rightarrow h(\mathbf{x}), \int_{Z}\left\|h_{m}-h_{n}\right\| d \lambda \mathbf{x} \rightarrow 0 \text { as } m, n \rightarrow \infty
$$

The first convergence above implies

$$
\begin{equation*}
h_{n} \circ \mathbf{X} \rightarrow h \circ \mathbf{X} \text { pointwise on } \Omega \tag{59.1.3}
\end{equation*}
$$

Then letting $h_{n}(\mathbf{x})=\sum_{k=1}^{m} \mathbf{x}_{k} \mathscr{X}_{E_{k}}(\mathbf{x})$, where the $E_{k}$ are disjoint and Borel, it follows easily that $h_{n} \circ \mathbf{X}$ is also a simple function of the form $h_{n} \circ \mathbf{X}(\omega)=\sum_{k=1}^{m} \mathbf{x}_{k} \mathscr{X}_{\mathbf{X}^{-1}\left(E_{k}\right)}(\omega)$ and by assumption $\mathbf{X}^{-1}\left(E_{k}\right) \in \mathscr{F}$. From the definition of the integral, it is easily seen

$$
\int h_{n} \circ \mathbf{X} d P=\int h_{n} d \lambda \mathbf{x}, \int\left\|h_{n}\right\| \circ \mathbf{X} d P=\int\left\|h_{n}\right\| d \lambda \mathbf{x}
$$

Also, $h_{n} \circ \mathbf{X}-h_{m} \circ \mathbf{X}$ is a simple function and so

$$
\begin{equation*}
\int\left\|h_{n} \circ \mathbf{X}-h_{m} \circ \mathbf{X}\right\| d P=\int\left\|h_{n}-h_{m}\right\| d \lambda_{\mathbf{x}} \tag{59.1.4}
\end{equation*}
$$

It follows from the definition of the Bochner integral and 59.1.3, and 59.1.4 that $h \circ \mathbf{X}$ is in $L^{1}(\Omega ; E)$ and

$$
\int h \circ \mathbf{X} d P=\lim _{n \rightarrow \infty} \int h_{n} \circ \mathbf{X} d P=\lim _{n \rightarrow \infty} \int h_{n} d \lambda_{\mathbf{x}}=\int h d \lambda_{\mathbf{x}}
$$

Finally consider the case that $E=Z$ and suppose $\mathbf{X} \in L^{1}(\Omega ; E)$. Then letting $h$ be the identity map on $E$, it follows $h$ is obviously separably valued and $h^{-1}(U) \in \mathscr{B}(E)$ for all $U$ open and so $h$ is measurable. Why is it in $L^{1}(E ; E)$ ?

$$
\begin{aligned}
\int_{E}\|h(\mathbf{x})\| d \lambda_{\mathbf{x}} & =\int_{0}^{\infty} \lambda_{\mathbf{x}}([\|h\|>t]) d t \equiv \int_{0}^{\infty} P(\mathbf{X} \in[\|\mathbf{x}\|>t]) d t \\
& \equiv \int_{0}^{\infty} P([\|\mathbf{X}\|>t]) d t=\int_{\Omega}\|\mathbf{X}\| d P<\infty
\end{aligned}
$$

Thus the identity map on $E$ is in $L^{1}\left(E ; \lambda_{\mathbf{X}}\right)$. Next suppose the identity map $h$ is in absolutely integrable, in $L^{1}\left(E ; \lambda_{\mathbf{x}}\right)$. Then $\mathbf{X}(\omega)=h \circ \mathbf{X}(\omega)$ and so from the first part, $\mathbf{X} \in L^{1}(\Omega ; E)$ and from 59.1.1, 59.1.2 follows.

### 59.2 Kolmogorov Extension Theorem For Polish Spaces

Let $M_{t}$ be a complete separable metric space. This is called a Polish space. I will denote a totally ordered index set, (Like $\mathbb{R})$ and the interest will be in building a measure on the product space, $\prod_{t \in I} M_{t}$. By the well ordering principle, you can always put an order on any index set so this order is no restriction, but we do not insist on a well order and in fact, index sets of great interest are $\mathbb{R}$ or $[0, \infty)$. Also for $X$ a topological space, $\mathscr{B}(X)$ will denote the Borel sets.

Notation 59.2.1 The symbol $J$ will denote a finite subset of $I, J=\left(t_{1}, \cdots, t_{n}\right)$, the $t_{i}$ taken in order. $\mathbf{E}_{J}$ will denote a set which has a set $E_{t}$ of $\mathscr{B}\left(M_{t}\right)$ in the $t^{t h}$ position for $t \in J$ and for $t \notin J$, the set in the $t^{\text {th }}$ position will be $M_{t} . \mathbf{K}_{J}$ will denote a set which has a compact set in the $t^{\text {th }}$ position for $t \in J$ and for $t \notin J$, the set in the $t^{\text {th }}$ position will be $M_{t}$. Also denote by $\mathscr{R}_{J}$ the sets $\mathbf{E}_{J}$ and $\mathscr{R}$ the union of the $\mathscr{R}_{J}$. Let $\mathscr{E}_{J}$ denote finite disjoint unions of sets of $\mathscr{R}_{J}$ and let $\mathscr{E}$ denote finite disjoint unions of sets of $\mathscr{R}$. Thus if $\mathbf{F}$ is a set of $\mathscr{E}$, there exists $J$ such that $\mathbf{F}$ is a finite disjoint union of sets of $\mathscr{R}_{J}$. For $\mathbf{F} \in \Omega$, denote by $\pi_{J}(\mathbf{F})$ the set $\prod_{t \in J} F_{t}$ where $\mathbf{F}=\prod_{t \in I} F_{t}$.

Lemma 59.2.2 The sets, $\mathscr{E}$, $\mathscr{E}_{J}$ defined above form an algebra of sets of $\prod_{t \in I} M_{t}$.
Proof: First consider $\mathscr{R}_{J}$. If $\mathbf{A}, \mathbf{B} \in \mathscr{R}_{J}$, then $\mathbf{A} \cap \mathbf{B} \in \mathscr{R}_{J}$ also. Is $\mathbf{A} \backslash \mathbf{B}$ a finite disjoint union of sets of $\mathscr{R}_{J}$ ? It suffices to verify that $\pi_{J}(\mathbf{A} \backslash \mathbf{B})$ is a finite disjoint union of $\pi_{J}\left(\mathscr{R}_{J}\right)$. Let $|J|$ denote the number of indices in $J$. If $|J|=1$, then it is obvious that $\pi_{J}(\mathbf{A} \backslash \mathbf{B})$ is a finite disjoint union of sets of $\pi_{J}\left(\mathscr{R}_{J}\right)$. In fact, letting $J=(t)$ and the $t^{t h}$ entry of $\mathbf{A}$ is $A$ and the $t^{\text {th }}$ entry of $\mathbf{B}$ is $B$, then the $t^{t h}$ entry of $\mathbf{A} \backslash \mathbf{B}$ is $A \backslash B$, a Borel set of $M_{t}$, a finite disjoint union of Borel sets of $M_{t}$.

Suppose then that for $\mathbf{A}, \mathbf{B}$ sets of $\mathscr{R}_{J}, \pi_{J}(\mathbf{A} \backslash \mathbf{B})$ is a finite disjoint union of sets of $\pi_{J}\left(\mathscr{R}_{J}\right)$ for $|J| \leq n$, and consider $J=\left(t_{1}, \cdots, t_{n}, t_{n+1}\right)$. Let the $t_{i}^{t h}$ entry of $\mathbf{A}$ and $\mathbf{B}$ be respectively $A_{i}$ and $B_{i}$. It follows that $\pi_{J}(\mathbf{A} \backslash \mathbf{B})$ has the following in the entries for $J$

$$
\left(A_{1} \times A_{2} \times \cdots \times A_{n} \times A_{n+1}\right) \backslash\left(B_{1} \times B_{2} \times \cdots \times B_{n} \times B_{n+1}\right)
$$

Letting $A$ represent $A_{1} \times A_{2} \times \cdots \times A_{n}$ and $B$ represent $B_{1} \times B_{2} \times \cdots \times B_{n}$, this is of the form

$$
A \times\left(A_{n+1} \backslash B_{n+1}\right) \cup(A \backslash B) \times\left(A_{n+1} \cap B_{n+1}\right)
$$

By induction, $(A \backslash B)$ is the finite disjoint union of sets of $\mathscr{R}_{\left(t_{1}, \cdots, t_{n}\right)}$. Therefore, the above is the finite disjoint union of sets of $\mathscr{R}_{J}$. It follows that $\mathscr{E}_{J}$ is an algebra.

Now suppose $\mathbf{A}, \mathbf{B} \in \mathscr{R}$. Then for some finite set $J$, both are in $\mathscr{R}_{J}$. Then from what was just shown,

$$
\mathbf{A} \backslash \mathbf{B} \in \mathscr{E}_{J} \subseteq \mathscr{E}, \mathbf{A} \cap \mathbf{B} \in \mathscr{R} .
$$

By Lemma 12.10.2 on Page 318 this shows $\mathscr{E}$ is an algebra.

With this preparation, here is the Kolmogorov extension theorem. In the statement and proof of the theorem, $F_{i}, G_{i}$, and $E_{i}$ will denote Borel sets. Any list of indices from $I$ will always be assumed to be taken in order. Thus, if $J \subseteq I$ and $J=\left(t_{1}, \cdots, t_{n}\right)$, it will always be assumed $t_{1}<t_{2}<\cdots<t_{n}$.

Theorem 59.2.3 For each finite set

$$
J=\left(t_{1}, \cdots, t_{n}\right) \subseteq I
$$

suppose there exists a Borel probability measure, $v_{J}=v_{t_{1} \cdots t_{n}}$ defined on the Borel sets of $\prod_{t \in J} M_{t}$ such that the following consistency condition holds. If

$$
\left(t_{1}, \cdots, t_{n}\right) \subseteq\left(s_{1}, \cdots, s_{p}\right)
$$

then

$$
\begin{equation*}
v_{t_{1} \cdots t_{n}}\left(F_{t_{1}} \times \cdots \times F_{t_{n}}\right)=v_{s_{1} \cdots s_{p}}\left(G_{s_{1}} \times \cdots \times G_{s_{p}}\right) \tag{59.2.5}
\end{equation*}
$$

where if $s_{i}=t_{j}$, then $G_{s_{i}}=F_{t_{j}}$ and if $s_{i}$ is not equal to any of the indices, $t_{k}$, then $G_{s_{i}}=M_{s_{i}}$. Then for $\mathscr{E}$ defined in Definition 59.2.1, there exists a probability measure, $P$ and $a \sigma$ algebra $\mathscr{F}=\sigma(\mathscr{E})$ such that

$$
\left(\prod_{t \in I} M_{t}, P, \mathscr{F}\right)
$$

is a probability space. Also there exist measurable functions, $X_{s}: \prod_{t \in I} M_{t} \rightarrow M_{s}$ defined as

$$
X_{s} \mathbf{x} \equiv x_{s}
$$

for each $s \in I$ such that for each $\left(t_{1} \cdots t_{n}\right) \subseteq I$,

$$
\begin{gather*}
v_{t_{1} \cdots t_{n}}\left(F_{t_{1}} \times \cdots \times F_{t_{n}}\right)=P\left(\left[X_{t_{1}} \in F_{t_{1}}\right] \cap \cdots \cap\left[X_{t_{n}} \in F_{t_{n}}\right]\right) \\
\quad=P\left(\left(X_{t_{1}}, \cdots, X_{t_{n}}\right) \in \prod_{j=1}^{n} F_{t_{j}}\right)=P\left(\prod_{t \in I} F_{t}\right) \tag{59.2.6}
\end{gather*}
$$

where $F_{t}=M_{t}$ for every $t \notin\left\{t_{1} \cdots t_{n}\right\}$ and $F_{t_{i}}$ is a Borel set. Also if $f$ is a nonnegative function of finitely many variables, $x_{t_{1}}, \cdots, x_{t_{n}}$, measurable with respect to $\mathscr{B}\left(\prod_{j=1}^{n} M_{t_{j}}\right)$, then $f$ is also measurable with respect to $\mathscr{F}$ and

$$
\begin{align*}
& \int_{M_{t_{1} \times \cdots \times M_{t_{n}}}} f\left(x_{t_{1}}, \cdots, x_{t_{n}}\right) d v_{t_{1} \cdots t_{n}} \\
= & \int_{\prod_{t \in I} M_{t}} f\left(x_{t_{1}}, \cdots, x_{t_{n}}\right) d P \tag{59.2.7}
\end{align*}
$$

Proof: Let $\mathscr{E}$ be the algebra of sets defined in Definition 14.4.1. I want to define a measure on $\mathscr{E}$. For $\mathbf{F} \in \mathscr{E}$, there exists $J$ such that $\mathbf{F}$ is the finite disjoint unions of sets of $\mathscr{R}_{J}$. Define

$$
P_{0}(\mathbf{F}) \equiv v_{J}\left(\pi_{J}(\mathbf{F})\right)
$$

Then $P_{0}$ is well defined because of the consistency condition on the measures $v_{J} . P_{0}$ is clearly finitely additive because the $v_{J}$ are measures and one can pick $J$ as large as desired to include all $t$ where there may be something other than $M_{t}$. Also, from the definition,

$$
P_{0}(\Omega) \equiv P_{0}\left(\prod_{t \in I} M_{t}\right)=v_{t_{1}}\left(M_{t_{1}}\right)=1
$$

Next I will show $P_{0}$ is a finite measure on $\mathscr{E}$. After this it is only a matter of using the Caratheodory extension theorem to get the existence of the desired probability measure $P$.

Claim: Suppose $\mathbf{E}^{n}$ is in $\mathscr{E}$ and suppose $\mathbf{E}^{n} \downarrow \emptyset$. Then $P_{0}\left(\mathbf{E}^{n}\right) \downarrow 0$.
Proof of the claim: If not, there exists a sequence such that although $\mathbf{E}^{n} \downarrow \emptyset, P_{0}\left(\mathbf{E}^{n}\right) \downarrow$ $\varepsilon>0$. Let $\mathbf{E}^{n} \in \mathscr{E}_{J_{n}}$. Thus it is a finite disjoint union of sets of $\mathscr{R}_{J_{n}}$. By regularity of the measures $v_{J}$, which follows from Lemmas 59.1.8 and 59.1.9, there exists a compact set $\mathbf{K}_{J_{n}} \subseteq \mathbf{E}^{n}$ such that

$$
v_{J_{n}}\left(\pi_{J_{n}}\left(\mathbf{K}_{J_{n}}\right)\right)+\frac{\varepsilon}{2^{n+2}}>v_{J_{n}}\left(\pi_{J_{n}}\left(\mathbf{E}^{n}\right)\right)
$$

Thus

$$
\begin{aligned}
P_{0}\left(\mathbf{K}_{J_{n}}\right)+\frac{\varepsilon}{2^{n+2}} & \equiv v_{J_{n}}\left(\pi_{J_{n}}\left(\mathbf{K}_{J_{n}}\right)\right)+\frac{\varepsilon}{2^{n+2}} \\
& >v_{J_{n}}\left(\pi_{J_{n}}\left(\mathbf{E}^{n}\right)\right) \equiv P_{0}\left(\mathbf{E}^{n}\right)
\end{aligned}
$$

The interesting thing about these $\mathbf{K}_{J_{n}}$ is: they have the finite intersection property. Here is why.

$$
\begin{aligned}
\varepsilon & \leq P_{0}\left(\cap_{k=1}^{m} \mathbf{K}_{J_{k}}\right)+P_{0}\left(\mathbf{E}^{m} \backslash \cap_{k=1}^{m} \mathbf{K}_{J_{k}}\right) \\
& \leq P_{0}\left(\cap_{k=1}^{m} \mathbf{K}_{J_{k}}\right)+P_{0}\left(\cup_{k=1}^{m} \mathbf{E}^{k} \backslash \mathbf{K}_{J_{k}}\right) \\
& <P_{0}\left(\cap_{k=1}^{m} \mathbf{K}_{J_{k}}\right)+\sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k+2}}<P_{0}\left(\cap_{k=1}^{m} \mathbf{K}_{J_{k}}\right)+\varepsilon / 2
\end{aligned}
$$

and so $P_{0}\left(\cap_{k=1}^{m} \mathbf{K}_{J_{k}}\right)>\varepsilon / 2$. In considering all the $\mathbf{E}^{n}$, there are countably many entries in the product space which have something other than $M_{t}$ in them. Say these are $\left\{t_{1}, t_{2}, \cdots\right\}$. Let $p_{t_{i}}$ be a point which is in the intersection of the $t_{i}$ components of the sets $\mathbf{K}_{J_{n}}$. The compact sets in the $t_{i}$ position must have the finite intersection property also because if not, the sets $\mathbf{K}_{J_{n}}$ can't have it. Thus there is such a point. As to the other positions, use the axiom of choice to pick something in each of these. Thus the intersection of these $\mathbf{K}_{J_{n}}$ contains a point which is contrary to $\mathbf{E}^{n} \downarrow \emptyset$ because these sets are contained in the $\mathbf{E}^{n}$.

With the claim, it follows $P_{0}$ is a measure on $\mathscr{E}$. Here is why: If $\mathbf{E}=\cup_{k=1}^{\infty} \mathbf{E}^{k}$ where $\mathbf{E}, \mathbf{E}^{k} \in \mathscr{E}$, then $\left(\mathbf{E} \backslash \cup_{k=1}^{n} \mathbf{E}_{k}\right) \downarrow \emptyset$ and so

$$
P_{0}\left(\cup_{k=1}^{n} \mathbf{E}_{k}\right) \rightarrow P_{0}(\mathbf{E}) .
$$

Hence if the $\mathbf{E}_{k}$ are disjoint, $P_{0}\left(\cup_{k=1}^{n} \mathbf{E}_{k}\right)=\sum_{k=1}^{n} P_{0}\left(\mathbf{E}_{k}\right) \rightarrow P_{0}(\mathbf{E})$. Thus for disjoint $\mathbf{E}_{k}$ having $\cup_{k} \mathbf{E}_{k}=\mathbf{E} \in \mathscr{E}$,

$$
P_{0}\left(\cup_{k=1}^{\infty} \mathbf{E}_{k}\right)=\sum_{k=1}^{\infty} P_{0}\left(\mathbf{E}_{k}\right)
$$

Now to conclude the proof, apply the Caratheodory extension theorem to obtain $P$ a probability measure which extends $P_{0}$ to a $\sigma$ algebra which contains $\sigma(\mathscr{E})$ the sigma algebra generated by $\mathscr{E}$ with $P=P_{0}$ on $\mathscr{E}$. Thus for $\mathbf{E}_{J} \in \mathscr{E}, P\left(\mathbf{E}_{J}\right)=P_{0}\left(\mathbf{E}_{J}\right)=v_{J}\left(P_{J} \mathbf{E}_{j}\right)$.

Next, let $\left(\prod_{t \in I} M_{t}, \mathscr{F}, P\right)$ be the probability space and for $\mathbf{x} \in \prod_{t \in I} M_{t}$ let $X_{t}(\mathbf{x})=x_{t}$, the $t^{t h}$ entry of $\mathbf{x}$. It follows $X_{t}$ is measurable (also continuous) because if $U$ is open in $M_{t}$, then $X_{t}^{-1}(U)$ has a $U$ in the $t^{t h}$ slot and $M_{s}$ everywhere else for $s \neq t$. Thus inverse images of open sets are measurable. Also, letting $J$ be a finite subset of $I$ and for $J=\left(t_{1}, \cdots, t_{n}\right)$, and $F_{t_{1}}, \cdots, F_{t_{n}}$ Borel sets in $M_{t_{1}} \cdots M_{t_{n}}$ respectively, it follows $\mathbf{F}_{J}$, where $\mathbf{F}_{J}$ has $F_{t_{i}}$ in the $t_{i}^{t h}$ entry, is in $\mathscr{E}$ and therefore,

$$
\begin{gathered}
P\left(\left[X_{t_{1}} \in F_{t_{1}}\right] \cap\left[X_{t_{2}} \in F_{t_{2}}\right] \cap \cdots \cap\left[X_{t_{n}} \in F_{t_{n}}\right]\right)= \\
P\left(\left[\left(X_{t_{1}}, X_{t_{2}}, \cdots, X_{t_{n}}\right) \in F_{t_{1}} \times \cdots \times F_{t_{n}}\right]\right)=P\left(\mathbf{F}_{J}\right)=P_{0}\left(\mathbf{F}_{J}\right) \\
=v_{t_{1} \cdots t_{n}}\left(F_{t_{1}} \times \cdots \times F_{t_{n}}\right)
\end{gathered}
$$

Finally consider the claim about the integrals. Suppose $f\left(x_{t_{1}}, \cdots, x_{t_{n}}\right)=\mathscr{X}_{F}$ where $F$ is a Borel set of $\prod_{t \in J} M_{t}$ where $J=\left(t_{1}, \cdots, t_{n}\right)$. To begin with suppose

$$
\begin{equation*}
F=F_{t_{1}} \times \cdots \times F_{t_{n}} \tag{59.2.8}
\end{equation*}
$$

where each $F_{t_{j}}$ is in $\mathscr{B}\left(M_{t_{j}}\right)$. Then

$$
\begin{align*}
\int_{M_{t_{1}} \times \cdots \times M_{t_{n}}} & \mathscr{X}_{F}\left(x_{t_{1}}, \cdots, x_{t_{n}}\right) d v_{t_{1} \cdots t_{n}}=v_{t_{1} \cdots t_{n}}\left(F_{t_{1}} \times \cdots \times F_{t_{n}}\right) \\
& =P\left(\prod_{t \in I} F_{t}\right)=\int_{\Omega} \mathscr{X}_{\Pi_{t \in I} F_{t}}(\mathbf{x}) d P \\
& =\int_{\Omega} \mathscr{X}_{F}\left(x_{t_{1}}, \cdots, x_{t_{n}}\right) d P \tag{59.2.9}
\end{align*}
$$

where $F_{t}=M_{t}$ if $t \notin J$. Let $\mathscr{K}$ denote sets, $F$ of the sort in 59.2.8. It is clearly a $\pi$ system. Now let $\mathscr{G}$ denote those sets $F$ in $\mathscr{B}\left(\prod_{t \in J} M_{t}\right)$ such that 59.2 .9 holds. Thus $\mathscr{G} \supseteq \mathscr{K}$. It is clear that $\mathscr{G}$ is closed with respect to countable disjoint unions and complements. Hence $\mathscr{G} \supseteq \sigma(\mathscr{K})$ but $\sigma(\mathscr{K})=\mathscr{B}\left(\prod_{t \in J} M_{t}\right)$ because every open set in $\prod_{t \in J} M_{t}$ is the countable union of rectangles like 59.2.8 in which each $F_{t_{i}}$ is open. Therefore, 59.2.9 holds for every $F \in \mathscr{B}\left(\prod_{t \in J} M_{t}\right)$.

Passing to simple functions and then using the monotone convergence theorem yields the final claim of the theorem.

### 59.3 Independence

The concept of independence is probably the main idea which separates probability from analysis and causes some of us to struggle to understand what is going on.

Definition 59.3.1 Let $(\Omega, \mathscr{F}, P)$ be a probability space. The sets in $\mathscr{F}$ are called events. $A$ set of events, $\left\{A_{i}\right\}_{i \in I}$ is called independent if whenever $\left\{A_{i_{k}}\right\}_{k=1}^{m}$ is a finite subset

$$
P\left(\cap_{k=1}^{m} A_{i_{k}}\right)=\prod_{k=1}^{m} P\left(A_{i_{k}}\right)
$$

Each of these events defines a rather simple $\sigma$ algebra, $\left(A_{i}, A_{i}^{C}, \emptyset, \Omega\right)$ denoted by $\mathscr{F}_{i}$. Now the following lemma is interesting because it motivates a more general notion of independent $\sigma$ algebras.

Lemma 59.3.2 Suppose $B_{i} \in \mathscr{F}_{i}$ for $i \in I$. Then for any $m \in \mathbb{N}$

$$
P\left(\cap_{k=1}^{m} B_{i_{k}}\right)=\prod_{k=1}^{m} P\left(B_{i_{k}}\right)
$$

Proof: The proof is by induction on the number $l$ of the $B_{i_{k}}$ which are not equal to $A_{i_{k}}$. First suppose $l=0$. Then the above assertion is true by assumption. Suppose it is so for some $l$ and there are $l+1$ sets not equal to $A_{i_{k}}$. If any equals $\emptyset$ there is nothing to show. Both sides equal 0 . If any equals $\Omega$, there is also nothing to show. You can ignore that set in both sides and then you have by induction the two sides are equal because you have no more than $l$ sets different than $A_{i_{k}}$. The only remaining case is where some $B_{i_{k}}=A_{i_{k}}^{C}$. Say $B_{i_{m+1}}=A_{i_{m+1}}^{C}$ for simplicity.

$$
\begin{aligned}
& P\left(\cap_{k=1}^{m+1} B_{i_{k}}\right)=P\left(A_{i_{m+1}}^{C} \cap \cap_{k=1}^{m} B_{i_{k}}\right) \\
& =P\left(\cap_{k=1}^{m} B_{i_{k}}\right)-P\left(A_{i_{m+1}} \cap \cap_{k=1}^{m} B_{i_{k}}\right)
\end{aligned}
$$

Then by induction,

$$
\begin{aligned}
& =\prod_{k=1}^{m} P\left(B_{i_{k}}\right)-P\left(A_{i_{m+1}}\right) \prod_{k=1}^{m} P\left(B_{i_{k}}\right)=\prod_{k=1}^{m} P\left(B_{i_{k}}\right)\left(1-P\left(A_{i_{m+1}}\right)\right) \\
& =P\left(A_{i_{m+1}}^{C}\right) \prod_{k=1}^{m} P\left(B_{i_{k}}\right)=\prod_{k=1}^{m+1} P\left(B_{i_{k}}\right)
\end{aligned}
$$

thus proving it for $l+1$.
This motivates a more general notion of independence in terms of $\sigma$ algebras.
Definition 59.3.3 If $\left\{\mathscr{F}_{i}\right\}_{i \in I}$ is any set of $\sigma$ algebras contained in $\mathscr{F}$, they are said to be independent if whenever $A_{i_{k}} \in \mathscr{F}_{i_{k}}$ for $k=1,2, \cdots, m$, then

$$
P\left(\cap_{k=1}^{m} A_{i_{k}}\right)=\prod_{k=1}^{m} P\left(A_{i_{k}}\right) .
$$

A set of random variables $\left\{\mathbf{X}_{i}\right\}_{i \in I}$ is independent if the $\sigma$ algebras $\left\{\sigma\left(\mathbf{X}_{i}\right)\right\}_{i \in I}$ are independent $\sigma$ algebras. Here $\sigma(\mathbf{X})$ denotes the smallest $\sigma$ algebra such that $\mathbf{X}$ is measurable. Thus $\sigma(\mathbf{X})=\left\{\mathbf{X}^{-1}(U): U\right.$ is a Borel set $\}$. More generally, $\sigma\left(\mathbf{X}_{i}: i \in I\right)$ is the smallest $\sigma$ algebra such that each $\mathbf{X}_{i}$ is measurable.

Note that by Lemma 59.3.2 you can consider independent events in terms of independent $\sigma$ algebras. That is, a set of independent events can always be considered as events taken from a set of independent $\sigma$ algebras. This is a more general notion because here the $\sigma$ algebras might have infinitely many sets in them.

Lemma 59.3.4 Suppose the set of random variables, $\left\{\mathbf{X}_{i}\right\}_{i \in I}$ is independent. Also suppose $I_{1} \subseteq I$ and $j \notin I_{1}$. Then the $\sigma$ algebras $\sigma\left(\mathbf{X}_{i}: i \in I_{1}\right), \sigma\left(\mathbf{X}_{j}\right)$ are independent $\sigma$ algebras.

Proof: Let $B \in \sigma\left(\mathbf{X}_{j}\right)$. I want to show that for any $A \in \sigma\left(\mathbf{X}_{i}: i \in I_{1}\right)$, it follows that $P(A \cap B)=P(A) P(B)$. Let $\mathscr{K}$ consist of finite intersections of sets of the form $\mathbf{X}_{k}^{-1}\left(B_{k}\right)$ where $B_{k}$ is a Borel set and $k \in I_{1}$. Thus $\mathscr{K}$ is a $\pi$ system and $\sigma(\mathscr{K})=\sigma\left(\mathbf{X}_{i}: i \in I_{1}\right)$. Now if you have one of these sets of the form $A=\cap_{k=1}^{m} \mathbf{X}_{k}^{-1}\left(B_{k}\right)$ where without loss of generality, it can be assumed the $k$ are distinct since $\mathbf{X}_{k}^{-1}\left(B_{k}\right) \cap \mathbf{X}_{k}^{-1}\left(B_{k}^{\prime}\right)=\mathbf{X}_{k}^{-1}\left(B_{k} \cap B_{k}^{\prime}\right)$, then

$$
\begin{aligned}
P(A \cap B) & =P\left(\cap_{k=1}^{m} \mathbf{X}_{k}^{-1}\left(B_{k}\right) \cap B\right)=P(B) \prod_{k=1}^{m} P\left(\mathbf{X}_{k}^{-1}\left(B_{k}\right)\right) \\
& =P(B) P\left(\cap_{k=1}^{m} \mathbf{X}_{k}^{-1}\left(B_{k}\right)\right) .
\end{aligned}
$$

Thus $\mathscr{K}$ is contained in

$$
\mathscr{G} \equiv\left\{A \in \sigma\left(\mathbf{X}_{i}: i \in I_{1}\right): P(A \cap B)=P(A) P(B)\right\}
$$

Now $\mathscr{G}$ is closed with respect to complements and countable disjoint unions. Here is why: If each $A_{i} \in \mathscr{G}$ and the $A_{i}$ are disjoint,

$$
\begin{aligned}
P\left(\left(\cup_{i=1}^{\infty} A_{i}\right) \cap B\right) & =P\left(\cup_{i=1}^{\infty}\left(A_{i} \cap B\right)\right) \\
& =\sum_{i} P\left(A_{i} \cap B\right)=\sum_{i} P\left(A_{i}\right) P(B) \\
& =P(B) \sum_{i} P\left(A_{i}\right)=P(B) P\left(\cup_{i=1}^{\infty} A_{i}\right)
\end{aligned}
$$

If $A \in \mathscr{G}$,

$$
P\left(A^{C} \cap B\right)+P(A \cap B)=P(B)
$$

and so

$$
\begin{aligned}
P\left(A^{C} \cap B\right) & =P(B)-P(A \cap B) \\
& =P(B)-P(A) P(B) \\
& =P(B)(1-P(A))=P(B) P\left(A^{C}\right) .
\end{aligned}
$$

Therefore, from the lemma on $\pi$ systems, Lemma 12.12 .3 on Page 329 , it follows $\mathscr{G} \supseteq$ $\sigma(\mathscr{K})=\sigma\left(\mathbf{X}_{i}: i \in I_{1}\right)$.

Lemma 59.3.5 If $\left\{\mathbf{X}_{k}\right\}_{k=1}^{r}$ are independent random variables having values in $Z$ a separable metric space, and if $g_{k}$ is a Borel measurable function, then $\left\{g_{k}\left(\mathbf{X}_{k}\right)\right\}_{k=1}^{r}$ is also
independent. Furthermore, if the random variables have values in $\mathbb{R}$, and they are all bounded, then

$$
E\left(\prod_{i=1}^{r} X_{i}\right)=\prod_{i=1}^{r} E\left(X_{i}\right)
$$

More generally, the above formula holds if it is only known that each $X_{i} \in L^{1}(\Omega ; \mathbb{R})$ and

$$
\prod_{i=1}^{r} X_{i} \in L^{1}(\Omega ; \mathbb{R})
$$

Proof: First consider the claim about $\left\{g_{k}\left(\mathbf{X}_{k}\right)\right\}_{k=1}^{r}$. Letting $O$ be an open set in $Z$,

$$
\left(g_{k} \circ \mathbf{X}_{k}\right)^{-1}(O)=\mathbf{X}_{k}^{-1}\left(g_{k}^{-1}(O)\right)=\mathbf{X}_{k}^{-1}(\text { Borel set }) \in \sigma\left(\mathbf{X}_{k}\right) .
$$

It follows $\left(g_{k} \circ \mathbf{X}_{k}\right)^{-1}(E)$ is in $\sigma\left(\mathbf{X}_{k}\right)$ whenever $E$ is Borel because the sets whose inverse images are measurable includes the Borel sets. Thus $\sigma\left(g_{k} \circ \mathbf{X}_{k}\right) \subseteq \sigma\left(\mathbf{X}_{k}\right)$ and this proves the first part of the lemma.

Let $X_{1}=\sum_{i=1}^{m} c_{i} \mathscr{X}_{E_{i}}, X_{2}=\sum_{j=1}^{m} d_{j} \mathscr{X}_{F_{j}}$ where $P\left(E_{i} F_{j}\right)=P\left(E_{i}\right) P\left(F_{j}\right)$. Then

$$
\int X_{1} X_{2} d P=\sum_{i, j} d_{j} c_{i} P\left(E_{i}\right) P\left(F_{j}\right)=\left(\int X_{1} d P\right)\left(\int X_{2} d P\right)
$$

In general for $X_{1}, X_{2}$ independent, there exist sequences of bounded simple functions

$$
\left\{s_{n}\right\},\left\{t_{n}\right\}
$$

measurable with respect to $\sigma\left(X_{1}\right)$ and $\sigma\left(X_{2}\right)$ respectively such that $s_{n} \rightarrow X_{1}$ pointwise and $t_{n} \rightarrow X_{2}$ pointwise. Then from the above and the dominated convergence theorem,

$$
\begin{aligned}
\int X_{1} X_{2} d P & =\lim _{n \rightarrow \infty} \int s_{n} t_{n} d P=\lim _{n \rightarrow \infty}\left(\int s_{n} d P\right)\left(\int t_{n} d P\right) \\
& =\left(\int X_{1} d P\right)\left(\int X_{2} d P\right)
\end{aligned}
$$

Next suppose there are $m$ of these independent bounded random variables. Then $\prod_{i=2}^{m} X_{i} \in$ $\sigma\left(X_{2}, \cdots, X_{m}\right)$ and by Lemma 59.3.4 the two random variables $X_{1}$ and $\prod_{i=2}^{m} X_{i}$ are independent. Hence from the above and induction,

$$
\int \prod_{i=1}^{m} X_{i} d P=\int X_{1} \prod_{i=2}^{m} X_{i} d P=\int X_{1} d P \int \prod_{i=2}^{m} X_{i} d P=\prod_{i=1}^{m} \int X_{i} d P
$$

Now consider the last claim. Replace each $X_{i}$ with $X_{i}^{n}$ where this is just a truncation of the form

$$
X_{i}^{n} \equiv\left\{\begin{array}{c}
X_{i} \text { if }\left|X_{i}\right| \leq n \\
n \text { if } X_{i}>n \\
-n \text { if } X_{i}<n
\end{array}\right.
$$

Then by the first part

$$
E\left(\prod_{i=1}^{r} X_{i}^{n}\right)=\prod_{i=1}^{r} E\left(X_{i}^{n}\right)
$$

Now $\left|\prod_{i=1}^{r} X_{i}^{n}\right| \leq\left|\prod_{i=1}^{r} X_{i}\right| \in L^{1}$ and so by the dominated convergence theorem, you can pass to the limit in both sides to get the desired result.

Maybe this would be a good place to put a really interesting result known as the Doob Dynkin lemma. This amazing result is illustrated with the following diagram in which $\mathbf{X}=\left(X_{1}, \cdots, X_{m}\right)$. By Proposition 59.1.6 $\sigma(\mathbf{X})=\sigma\left(X_{1}, \cdots, X_{n}\right)$.


You start with $X$ and can write it as the composition $g \circ \mathbf{X}$ provided $X$ is $\sigma(\mathbf{X})$ measurable.
Lemma 59.3.6 Let $(\Omega, \mathscr{F})$ be a measure space and let $X_{i}: \Omega \rightarrow E_{i}$ where $E_{i}$ is a separable Banach space. Suppose also that $X: \Omega \rightarrow F$ where $F$ is a separable Banach space. Then $X$ is $\sigma\left(X_{1}, \cdots, X_{m}\right)$ measurable if and only if there exists a Borel measurable function $g: \prod_{i=1}^{m} E_{i} \rightarrow F$ such that $X=g\left(X_{1}, \cdots, X_{m}\right)$.

Proof: First suppose $X(\omega)=f \mathscr{X}_{W}(\omega)$ where $f \in F$ and $W \in \sigma\left(X_{1}, \cdots, X_{m}\right)$. Then by Proposition 59.1.6, $W$ is of the form $\left(X_{1}, \cdots, X_{m}\right)^{-1}(B) \equiv \mathbf{X}^{-1}(B)$ where $B$ is Borel in $\prod_{i=1}^{m} E_{i}$. Therefore,

$$
X(\omega)=f \mathscr{X}_{\mathbf{X}^{-1}(B)}(\omega)=f \mathscr{X}_{B}(\mathbf{X}(\omega))
$$

Now suppose $X$ is measurable with respect to $\sigma\left(X_{1}, \cdots, X_{m}\right)$. Then there exist simple functions

$$
X_{n}(\omega)=\sum_{k=1}^{m_{n}} f_{k} \mathscr{X}_{B_{k}}(\mathbf{X}(\omega)) \equiv g_{n}(\mathbf{X}(\omega))
$$

where the $B_{k}$ are Borel sets in $\prod_{i=1}^{m} E_{i}$, such that $X_{n}(\omega) \rightarrow X(\omega)$, each $g_{n}$ being Borel. Thus $g_{n}$ converges on $\mathbf{X}(\Omega)$. Furthermore, the set on which $g_{n}$ does converge is a Borel set equal to

$$
\cap_{n=1}^{\infty} \cup_{m=1}^{\infty} \cap_{p, q \geq m}\left[\left\|g_{p}-g_{q}\right\|<\frac{1}{n}\right]
$$

which contains $\mathbf{X}(\Omega)$. Therefore, modifying $g_{n}$ by multiplying it by the indicator function of this Borel set containing $\mathbf{X}(\Omega)$, we can conclude that $g_{n}$ converges to a Borel function $g$ and, passing to a limit in the above,

$$
X(\omega)=g(\mathbf{X}(\omega))
$$

Conversely, suppose $X(\omega)=g(\mathbf{X}(\omega))$. Why is $X \sigma(\mathbf{X})$ measurable?

$$
X^{-1}(\text { open })=\mathbf{X}^{-1}\left(g^{-1}(\text { open })\right)=\mathbf{X}^{-1}(\text { Borel }) \in \sigma(\mathbf{X})
$$

### 59.4 Banach Space Valued Random Variables

Recall that for $X$ a random variable, $\sigma(X)$ is the smallest $\sigma$ algebra containing all the sets of the form $X^{-1}(F)$ where $F$ is Borel. Since such sets, $X^{-1}(F)$ for $F$ Borel form a $\sigma$ algebra it follows $\sigma(X)=\left\{X^{-1}(F): F\right.$ is Borel $\}$.

Next consider the case where you have a set of $\sigma$ algebras. The following lemma is helpful when you try to verify such a set of $\sigma$ algebras is independent. It says you only need to check things on $\pi$ systems contained in the $\sigma$ algebras. This is really nice because it is much easier to consider the smaller $\pi$ systems than the whole $\sigma$ algebra.

Lemma 59.4.1 Suppose $\left\{\mathscr{F}_{i}\right\}_{i \in I}$ is a set of $\sigma$ algebras contained in $\mathscr{F}$ where $\mathscr{F}$ is a $\sigma$ algebra of sets of $\Omega$. Suppose that $\mathscr{K}_{i} \subseteq \mathscr{F}_{i}$ is a $\pi$ system and $\mathscr{F}_{i}=\sigma\left(\mathscr{K}_{i}\right)$. Suppose also that whenever $J$ is a finite subset of I and $A_{j} \in \mathscr{K}_{j}$ for $j \in J$, it follows

$$
P\left(\cap_{j \in J} A_{j}\right)=\prod_{j \in J} P\left(A_{j}\right) .
$$

Then $\left\{\mathscr{F}_{i}\right\}_{i \in I}$ is independent.
Proof: I need to verify that under the given conditions, if $\left\{j_{1}, j_{2}, \cdots, j_{n}\right\} \subseteq I$ and $A_{j_{k}} \subseteq$ $\mathscr{F}_{j_{k}}$, then

$$
P\left(\cap_{k=1}^{n} A_{j_{k}}\right)=\prod_{k=1}^{n} P\left(A_{j_{k}}\right) .
$$

By hypothesis, this is true if each $A_{j_{k}} \in \mathscr{K}_{j_{k}}$. Suppose it is true whenever there are at most $r-1 \geq 0$ of the $A_{j_{k}}$ which are not in $\mathscr{K}_{j_{k}}$. Consider

$$
\cap_{k=1}^{n} A_{j_{k}}
$$

where there are $r$ sets which are not in the corresponding $\mathscr{K}_{j_{k}}$. Without loss of generality, say there are at most $r-1$ sets in the first $n-1$ which are not in the corresponding $\mathscr{K}_{j_{k}}$.

Pick $\left(A_{j_{1}} \cdots, A_{j_{n-1}}\right)$ let

$$
\mathscr{G}_{\left(A_{j_{1}} \cdots A_{j_{n-1}}\right)} \equiv\left\{B \in \mathscr{F}_{j_{n}}: P\left(\cap_{k=1}^{n-1} A_{j_{k}} \cap B\right)=\prod_{k=1}^{n-1} P\left(A_{j_{k}}\right) P(B)\right\}
$$

I am going to show $\mathscr{G}_{\left(A_{j_{1}} \cdots A_{j_{n-1}}\right)}$ is closed with respect to complements and countable disjoint unions and then apply the Lemma on $\pi$ systems. By the induction hypothesis,

$$
\begin{aligned}
& \mathscr{K}_{j_{n}} \subseteq \mathscr{G}_{\left(A_{j_{1}} \cdots A_{j_{n-1}}\right)} . \text { If } B \in \mathscr{G}\left(A_{j_{1}} \cdots A_{j_{n-1}}\right) \\
& \prod_{k=1}^{n-1} P\left(A_{j_{k}}\right)=P\left(\cap_{k=1}^{n-1} A_{j_{k}}\right) \\
&=P\left(\left(\cap_{k=1}^{n-1} A_{j_{k}} \cap B^{C}\right) \cup\left(\cap_{k=1}^{n-1} A_{j_{k}} \cap B\right)\right) \\
&=P\left(\cap_{k=1}^{n-1} A_{j_{k}} \cap B^{C}\right)+P\left(\cap_{k=1}^{n-1} A_{j_{k}} \cap B\right) \\
&=P\left(\cap_{k=1}^{n-1} A_{j_{k}} \cap B^{C}\right)+\prod_{k=1}^{n-1} P\left(A_{j_{k}}\right) P(B)
\end{aligned}
$$

and so

$$
\begin{aligned}
P\left(\cap_{k=1}^{n-1} A_{j_{k}} \cap B^{C}\right) & =\prod_{k=1}^{n-1} P\left(A_{j_{k}}\right)(1-P(B)) \\
& =\prod_{k=1}^{n-1} P\left(A_{j_{k}}\right) P\left(B^{C}\right)
\end{aligned}
$$

showing if $B \in \mathscr{G}_{\left(A_{j_{1}} \cdots, A_{j_{n-1}}\right)}$, then so is $B^{C}$. It is clear that $\mathscr{G}_{\left(A_{j_{1}} \cdots, A_{j_{n-1}}\right)}$ is closed with respect to disjoint unions also. Here is why. If $\left\{B_{j}\right\}_{j=1}^{\infty}$ are disjoint sets in $\mathscr{G}_{\left(A_{j_{1}} \cdots A_{j_{n-1}}\right)}$,

$$
\begin{aligned}
P\left(\cup_{i=1}^{\infty} B_{i} \cap \cap_{k=1}^{n-1} A_{j_{k}}\right) & =\sum_{i=1}^{\infty} P\left(B_{i} \cap \cap_{k=1}^{n-1} A_{j_{k}}\right) \\
& =\sum_{i=1}^{\infty} P\left(B_{i}\right) \prod_{k=1}^{n-1} P\left(A_{j_{k}}\right) \\
& =\prod_{k=1}^{n-1} P\left(A_{j_{k}}\right) \sum_{i=1}^{\infty} P\left(B_{i}\right) \\
& =\prod_{k=1}^{n-1} P\left(A_{j_{k}}\right) P\left(\cup_{i=1}^{\infty} B_{i}\right)
\end{aligned}
$$

Therefore, by the $\pi$ system lemma, Lemma $12.12 .3^{\mathscr{G}}{\left(A_{j_{1} \cdots A_{j_{n-1}}}\right)}=\mathscr{F}_{j_{n}}$. This proves the induction step in going from $r-1$ to $r$.

What is a useful $\pi$ system for $\mathscr{B}(E)$, the Borel sets of $E$ where $E$ is a Banach space?
Recall the fundamental lemma used to prove the Pettis theorem. It was proved on Page 645.

Lemma 59.4.2 Let E be a separable real Banach space. Sets of the form

$$
\left\{x \in E: x_{i}^{*}(x) \leq \alpha_{i}, i=1,2, \cdots, m\right\}
$$

where $x_{i}^{*} \in D^{\prime}$, a dense subspace of the unit ball of $E^{\prime}$ and $\alpha_{i} \in[-\infty, \infty)$ are a $\pi$ system, and denoting this $\pi$ system by $\mathscr{K}$, it follows $\sigma(\mathscr{K})=\mathscr{B}(E)$. The sets of $\mathscr{K}$ are examples of cylindrical sets. The $D^{\prime}$ is that set for the proof of the Pettis theorem.

Proof: The sets described are obviously a $\pi$ system. I want to show $\sigma(\mathscr{K})$ contains the closed balls because then $\sigma(\mathscr{K})$ contains the open balls and hence the open sets and the result will follow. Let $D^{\prime}$ be described in Lemma 21.1.6. As pointed out earlier it can
be any dense subset of $B^{\prime}$. Then

$$
\begin{aligned}
& \{x \in E:\|x-a\| \leq r\} \\
= & \left\{x \in E: \sup _{f \in D^{\prime}}|f(x-a)| \leq r\right\} \\
= & \left\{x \in E: \sup _{f \in D^{\prime}}|f(x)-f(a)| \leq r\right\} \\
= & \cap_{f \in D^{\prime}}\{x \in E: f(a)-r \leq f(x) \leq f(a)+r\} \\
= & \cap_{f \in D^{\prime}}\{x \in E: f(x) \leq f(a)+r \text { and }(-f)(x) \leq r-f(a)\}
\end{aligned}
$$

which equals a countable intersection of sets of the given $\pi$ system. Therefore, every closed ball is contained in $\sigma(\mathscr{K})$. It follows easily that every open ball is also contained in $\sigma(\mathscr{K})$ because

$$
B(a, r)=\cup_{n=1}^{\infty} \overline{B\left(a, r-\frac{1}{n}\right)} .
$$

Since the Banach space is separable, it is completely separable and so every open set is the countable union of balls. This shows the open sets are in $\sigma(\mathscr{K})$ and so $\sigma(\mathscr{K}) \supseteq \mathscr{B}(E)$. However, all the sets in the $\pi$ system are closed hence Borel because they are inverse images of closed sets. Therefore, $\sigma(\mathscr{K}) \subseteq \mathscr{B}(E)$ and so $\sigma(\mathscr{K})=\mathscr{B}(E)$.

As mentioned above, we can replace $D^{\prime}$ in the above with $M$, any dense subset of $E^{\prime}$.
Observation 59.4.3 Denote by $C_{\alpha, n}$ the set $\left\{\beta \in \mathbb{R}^{n}: \beta_{i} \leq \alpha_{i}\right\}$. Also denote by $\mathbf{g}_{n}$ an element of $M^{n}$ with the understanding that $\mathbf{g}_{n}: E \rightarrow \mathbb{R}^{n}$ according to the rule

$$
\mathbf{g}_{n}(x) \equiv\left(g_{1}(x), \cdots, g_{n}(x)\right)
$$

Then the sets in the above lemma can be written as $\mathbf{g}_{n}^{-1}\left(C_{\alpha, n}\right)$. In other words, sets of the form $\mathbf{g}_{n}^{-1}\left(C_{\alpha, n}\right)$ form a $\pi$ system for $\mathscr{B}(E)$.

Next suppose you have some random variables having values in a separable Banach space, $E,\left\{X_{i}\right\}_{i \in I}$. How can you tell if they are independent? To show they are independent, you need to verify that

$$
P\left(\cap_{k=1}^{n} X_{i_{k}}^{-1}\left(F_{i_{k}}\right)\right)=\prod_{k=1}^{n} P\left(X_{i_{k}}^{-1}\left(F_{i_{k}}\right)\right)
$$

whenever the $F_{i_{k}}$ are Borel sets in $E$. It is desirable to find a way to do this easily.
Lemma 59.4.4 Let $\mathscr{K}$ be a $\pi$ system of sets of $E$, a separable real Banach space and let $(\Omega, \mathscr{F}, P)$ be a probability space and $X: \Omega \rightarrow E$ be a random variable. Then

$$
X^{-1}(\sigma(\mathscr{K}))=\sigma\left(X^{-1}(\mathscr{K})\right)
$$

Proof: First note that $X^{-1}(\sigma(\mathscr{K}))$ is a $\sigma$ algebra which contains $X^{-1}(\mathscr{K})$ and so it contains $\sigma\left(X^{-1}(\mathscr{K})\right)$. Thus

$$
X^{-1}(\sigma(\mathscr{K})) \supseteq \sigma\left(X^{-1}(\mathscr{K})\right)
$$

Now let

$$
\mathscr{G} \equiv\left\{A \in \sigma(\mathscr{K}): X^{-1}(A) \in \sigma\left(X^{-1}(\mathscr{K})\right)\right\}
$$

Then $\mathscr{G} \supseteq \mathscr{K}$. If $A \in \mathscr{G}$, then

$$
X^{-1}(A) \in \sigma\left(X^{-1}(\mathscr{K})\right)
$$

and so

$$
X^{-1}(A)^{C}=X^{-1}\left(A^{C}\right) \in \sigma\left(X^{-1}(\mathscr{K})\right)
$$

because $\sigma\left(X^{-1}(\mathscr{K})\right)$ is a $\sigma$ algebra. Hence $A^{C} \in \mathscr{G}$. Finally suppose $\left\{A_{i}\right\}$ is a sequence of disjoint sets of $\mathscr{G}$. Then

$$
X^{-1}\left(\cup_{i=1}^{\infty} A_{i}\right)=\cup_{i=1}^{\infty} X^{-1}\left(A_{i}\right) \in \sigma\left(X^{-1}(\mathscr{K})\right)
$$

again because $\sigma\left(X^{-1}(\mathscr{K})\right)$ is a $\sigma$ algebra. It follows from Lemma 12.12.3 on Page 329 that $\mathscr{G} \supseteq \sigma(\mathscr{K})$ and this shows that whenever

$$
A \in \sigma(\mathscr{K}), X^{-1}(A) \in \sigma\left(X^{-1}(\mathscr{K})\right)
$$

Thus $X^{-1}(\sigma(\mathscr{K})) \subseteq \sigma\left(X^{-1}(\mathscr{K})\right)$.
With this lemma, here is the desired result about independent random variables. Essentially, you can reduce to the case of random vectors having values in $\mathbb{R}^{n}$.

### 59.5 Reduction To Finite Dimensions

Let $E$ be a Banach space and let $\mathbf{g} \in\left(E^{\prime}\right)^{n}$. Then for $x \in E, \mathbf{g} \circ x$ is the vector in $\mathbb{F}^{n}$ which equals $\left(g_{1}(x), g_{2}(x), \cdots, g_{n}(x)\right)$.

Theorem 59.5.1 Let $X_{i}$ be a random variable having values in E a real separable Banach space. The random variables $\left\{X_{i}\right\}_{i \in I}$ are independent if whenever

$$
\left\{i_{1}, \cdots, i_{n}\right\} \subseteq I
$$

$m_{i_{1}}, \cdots, m_{i_{n}}$ are positive integers, and $\mathbf{g}_{m_{i_{1}}}, \cdots, \mathbf{g}_{m_{i_{n}}}$ are respectively in

$$
(M)^{m_{i_{1}}}, \cdots,(M)^{m_{i_{n}}}
$$

for $M$ a dense subspace of $E^{\prime},\left\{\mathbf{g}_{m_{i_{j}}} \circ X_{i_{j}}\right\}_{j=1}^{n}$ are independent random vectors having values in $\mathbb{R}^{m_{i_{1}}}, \cdots, \mathbb{R}^{m_{i n}}$ respectively.

Proof: It is necessary to show that the events $X_{i_{j}}^{-1}\left(B_{i_{j}}\right)$ are independent events whenever $B_{i_{j}}$ are Borel sets. By Lemma 59.4.1 and the above Lemma 59.4.2, it suffices to verify that the events

$$
X_{i_{j}}^{-1}\left(\mathbf{g}_{m_{i j}}^{-1}\left(C_{\tilde{\alpha}, m_{i j}}\right)\right)=\left(\mathbf{g}_{m_{i_{j}}} \circ X_{i_{j}}\right)^{-1}\left(C_{\tilde{\alpha}, m_{i j}}\right)
$$

are independent where $C_{\tilde{\alpha}, m_{i j}}$ are the cones described in Lemma 59.4.2. Thus

$$
\begin{aligned}
\tilde{\alpha} & =\left(\alpha_{k_{1}}, \cdots, \alpha_{k_{m}}\right) \\
C_{\tilde{\alpha}, m_{i_{j}}} & =\prod_{i=1}^{m_{i_{j}}}\left(-\infty, \alpha_{k_{i}}\right]
\end{aligned}
$$

But this condition is implied when the finite dimensional valued random vectors $\mathbf{g}_{m_{i j}} \circ X_{i_{j}}$ are independent.

The above assertion also goes the other way as you may want to show.

### 59.6 0, 1 Laws

I am following [120] for the proof of many of the following theorems. Recall the set of $\omega$ which are in infinitely many of the sets $\left\{A_{n}\right\}$ is

$$
\cap_{n=1}^{\infty} \cup_{m=n}^{\infty} A_{m} .
$$

This is because $\omega$ is in the above set if and only if for every $n$ there exists $m \geq n$ such that it is in $A_{m}$.

Theorem 59.6.1 Suppose $A_{n} \in \mathscr{F}_{n}$ where the $\sigma$ algebras $\left\{\mathscr{F}_{n}\right\}_{n=1}^{\infty}$ are independent. Suppose also that

$$
\sum_{k=1}^{\infty} P\left(A_{k}\right)=\infty
$$

Then

$$
P\left(\cap_{n=1}^{\infty} \cup_{m=n}^{\infty} A_{m}\right)=1
$$

Proof: It suffices to verify that

$$
P\left(\cup_{n=1}^{\infty} \cap_{m=n}^{\infty} A_{m}^{C}\right)=0
$$

which can be accomplished by showing

$$
P\left(\cap_{m=n}^{\infty} A_{m}^{C}\right)=0
$$

for each $n$. The sets $\left\{A_{k}^{C}\right\}$ satisfy $A_{k}^{C} \in \mathscr{F}_{k}$. Therefore, noting that $e^{-x} \geq 1-x$,

$$
\begin{aligned}
P\left(\cap_{m=n}^{\infty} A_{m}^{C}\right) & =\lim _{N \rightarrow \infty} P\left(\cap_{m=n}^{N} A_{m}^{C}\right)=\lim _{N \rightarrow \infty} \prod_{m=n}^{N} P\left(A_{m}^{C}\right) \\
& =\lim _{N \rightarrow \infty} \prod_{m=n}^{N}\left(1-P\left(A_{m}\right)\right) \leq \lim _{N \rightarrow \infty} \prod_{m=n}^{N} e^{-P\left(A_{m}\right)} \\
& =\lim _{N \rightarrow \infty} \exp \left(-\sum_{m=n}^{N} P\left(A_{m}\right)\right)=0
\end{aligned}
$$

The Kolmogorov zero one law follows next. It has to do with something called a tail event.

Definition 59.6.2 Let $\left\{\mathscr{F}_{n}\right\}$ be a sequence of $\sigma$ algebras. Then $\mathscr{T}_{n} \equiv \sigma\left(\cup_{k=n}^{\infty} \mathscr{F}_{k}\right)$ where this means the smallest $\sigma$ algebra which contains each $\mathscr{F}_{k}$ for $k \geq n$. Then a tail event is a set which is in the $\sigma$ algebra, $\mathscr{T} \equiv \cap_{n=1}^{\infty} \mathscr{T}_{n}$.

As usual, $(\Omega, \mathscr{F}, P)$ is the underlying probability space such that all $\sigma$ algebras are contained in $\mathscr{F}$.

Lemma 59.6.3 Suppose $\left\{\mathscr{F}_{n}\right\}_{n=1}^{\infty}$ are independent $\sigma$ algebras and suppose $A$ is a tail event and $A_{k_{i}} \in \mathscr{F}_{k_{i}}, i=1, \cdots, m$ are given sets. Then

$$
P\left(A_{k_{1}} \cap \cdots \cap A_{k_{m}} \cap A\right)=P\left(A_{k_{1}} \cap \cdots \cap A_{k_{m}}\right) P(A)
$$

Proof: Let $\mathscr{K}$ be the $\pi$ system consisting of finite intersections of the form

$$
B_{m_{1}} \cap B_{m_{2}} \cap \cdots \cap B_{m_{j}}
$$

where $m_{i} \in \mathscr{F}_{k_{i}}$ for $k_{i}>\max \left\{k_{1}, \cdots, k_{m}\right\} \equiv N$. Thus $\sigma(\mathscr{K})=\sigma\left(\cup_{i=N+1}^{\infty} \mathscr{F}_{i}\right)$. Now let

$$
\mathscr{G} \equiv\left\{B \in \sigma(\mathscr{K}): P\left(A_{k_{1}} \cap \cdots \cap A_{k_{m}} \cap B\right)=P\left(A_{k_{1}} \cap \cdots \cap A_{k_{m}}\right) P(B)\right\}
$$

Then clearly $\mathscr{K} \subseteq \mathscr{G}$. It is also true that $\mathscr{G}$ is closed with respect to complements and countable disjoint unions. By the lemma on $\pi$ systems, $\mathscr{G}=\sigma(\mathscr{K})=\sigma\left(\cup_{i=N+1}^{\infty} \mathscr{F}_{i}\right)$. Since $A$ is in $\sigma\left(\cup_{i=N+1}^{\infty} \mathscr{F}_{i}\right)$ due to the assumption that it is a tail event, it follows that

$$
P\left(A_{k_{1}} \cap \cdots \cap A_{k_{m}} \cap A\right)=P\left(A_{k_{1}} \cap \cdots \cap A_{k_{m}}\right) P(A)
$$

Theorem 59.6.4 Suppose the $\sigma$ algebras, $\left\{\mathscr{F}_{n}\right\}_{n=1}^{\infty}$ are independent and suppose $A$ is a tail event. Then $P(A)$ either equals 0 or 1 .

Proof: Let $A \in \mathscr{T}$. I want to show that $P(A)=P(A)^{2}$. Let $\mathscr{K}$ denote sets of the form $A_{k_{1}} \cap \cdots \cap A_{k_{m}}$ for some $m, A_{k_{j}} \in \mathscr{F}_{k_{j}}$ where each $k_{j}>n$. Thus $\mathscr{K}$ is a $\pi$ system and

$$
\sigma(\mathscr{K})=\sigma\left(\cup_{k=n+1}^{\infty} \mathscr{F}_{k}\right) \equiv \mathscr{T}_{n+1}
$$

Let

$$
\mathscr{G} \equiv\left\{B \in \mathscr{T}_{n+1} \equiv \sigma\left(\cup_{k=n+1}^{\infty} \mathscr{F}_{k}\right): P(A \cap B)=P(A) P(B)\right\}
$$

Thus $\mathscr{K} \subseteq \mathscr{G}$ because

$$
P\left(A_{k_{1}} \cap \cdots \cap A_{k_{m}} \cap A\right)=P\left(A_{k_{1}} \cap \cdots \cap A_{k_{m}}\right) P(A)
$$

by Lemma 59.6.3. However, $\mathscr{G}$ is closed with respect to countable disjoint unions and complements. Here is why. If $B \in \mathscr{G}$,

$$
P\left(A \cap B^{C}\right)+P(A \cap B)=P(A)
$$

and so

$$
P\left(A \cap B^{C}\right)=P(A)-P(A \cap B)=P(A)(1-P(B))=P(A) P\left(B^{C}\right)
$$

and so $B^{C} \in \mathscr{G}$. If $\left\{B_{i}\right\}_{i=1}^{\infty}$ are disjoint sets in $\mathscr{G}$,

$$
\begin{aligned}
P\left(A \cap \cup_{k=1}^{\infty} B_{k}\right) & =\sum_{k=1}^{\infty} P\left(A \cap B_{k}\right)=P(A) \sum_{k=1}^{\infty} P\left(B_{k}\right) \\
& =P(A) P\left(\cup_{k=1}^{\infty} B_{k}\right)
\end{aligned}
$$

and so $\cup_{k=1}^{\infty} B_{k} \in \mathscr{G}$. Therefore by the Lemma on $\pi$ systems Lemma 12.12.3 on Page 329, it follows $\mathscr{G}=\sigma(\mathscr{K})=\sigma\left(\cup_{k=n+1}^{\infty} \mathscr{F}_{k}\right)$.

Thus for any $B \in \sigma\left(\cup_{k=n+1}^{\infty} \mathscr{F}_{k}\right)=\mathscr{T}_{n+1}, P(A \cap B)=P(A) P(B)$. However, $A$ is in all of these $\mathscr{T}_{n+1}$ and so $P(A \cap A)=P(A)=P(A)^{2}$ so $P(A)$ equals either 0 or 1 .

What sorts of things are tail events of independent $\sigma$ algebras?
Theorem 59.6.5 Let $\left\{\mathbf{X}_{k}\right\}$ be a sequence of independent random variables having values in $Z$ a Banach space. Then

$$
A \equiv\left\{\omega:\left\{\mathbf{X}_{k}(\omega)\right\} \text { converges }\right\}
$$

is a tail event of the independent $\sigma$ algebras $\left\{\sigma\left(\mathbf{X}_{k}\right)\right\}$. So is

$$
B \equiv\left\{\omega:\left\{\sum_{k=1}^{\infty} \mathbf{X}_{k}(\omega)\right\} \text { converges }\right\}
$$

Proof: Since $Z$ is complete, $A$ is the same as the set where $\left\{\mathbf{X}_{k}(\omega)\right\}$ is a Cauchy sequence. This set is

$$
\cap_{n=1}^{\infty} \cap_{p=1}^{\infty} \cup_{m=p}^{\infty} \cap_{l, k \geq m}\left\{\omega:\left\|\mathbf{X}_{k}(\omega)-\mathbf{X}_{l}(\omega)\right\|<1 / n\right\}
$$

Note that

$$
\cup_{m=p}^{\infty} \cap_{l, k \geq m}\left\{\omega:\left\|\mathbf{X}_{k}(\omega)-\mathbf{X}_{l}(\omega)\right\|<1 / n\right\} \in \sigma\left(\cup_{j=p}^{\infty} \sigma\left(\mathbf{X}_{j}\right)\right)
$$

for every $p$ is the set where ultimately any pair of $\mathbf{X}_{k}, \mathbf{X}_{l}$ are closer together than $1 / n$,

$$
\cap_{p=1}^{\infty} \cup_{m=p}^{\infty} \cap_{l, k \geq m}\left\{\omega:\left\|\mathbf{X}_{k}(\omega)-\mathbf{X}_{l}(\omega)\right\|<1 / n\right\}
$$

is a tail event. The set where $\left\{\mathbf{X}_{k}(\omega)\right\}$ is a Cauchy sequence is the intersection of all these and is therefore, also a tail event.

Now consider $B$. This set is the same as the set where the partial sums are Cauchy sequences. Let $\mathbf{S}_{n} \equiv \sum_{k=1}^{n} \mathbf{X}_{k}$. The set where the sum converges is then

$$
\cap_{n=1}^{\infty} \cap_{p=2}^{\infty} \cup_{m=p}^{\infty} \cap_{l, k \geq m}\left\{\omega:\left\|\mathbf{S}_{k}(\omega)-\mathbf{S}_{l}(\omega)\right\|<1 / n\right\}
$$

Say $k<l$ and consider for $m \geq p$

$$
\left\{\omega:\left\|\mathbf{S}_{k}(\omega)-\mathbf{S}_{l}(\omega)\right\|<1 / n, k \geq m\right\}
$$

This is the same as

$$
\left\{\omega:\left\|\sum_{j=k-1}^{l} \mathbf{X}_{j}(\omega)\right\|<1 / n, k \geq m\right\} \in \sigma\left(\cup_{j=p-1}^{\infty} \sigma\left(\mathbf{X}_{j}\right)\right)
$$

Thus

$$
\cup_{m=p}^{\infty} \cap_{l, k \geq m}\left\{\omega:\left\|\mathbf{S}_{k}(\omega)-\mathbf{S}_{l}(\omega)\right\|<1 / n\right\} \in \sigma\left(\cup_{j=p-1}^{\infty} \sigma\left(\mathbf{X}_{j}\right)\right)
$$

and so the intersection for all $p$ of these is a tail event. Then the intersection over all $n$ of these tail events is a tail event.

From this it can be concluded that if you have a sequence of independent random variables, $\left\{\mathbf{X}_{k}\right\}$ the set where it converges is either of probability 1 or probability 0 . A similar conclusion holds for the set where the infinite sum of these random variables converges. This is stated in the next corollary. This incredible assertion is the next corollary.

Corollary 59.6.6 Let $\left\{\mathbf{X}_{k}\right\}$ be a sequence of random variables having values in a Banach space. Then

$$
\lim _{n \rightarrow \infty} \mathbf{X}_{n}(\omega)
$$

either exists for a.e. $\omega$ or the convergence fails to take place for a.e. $\omega$. Also if

$$
A \equiv\left\{\omega: \sum_{k=1}^{\infty} \mathbf{X}_{k}(\omega) \text { converges }\right\}
$$

then $P(A)=0$ or 1 .

### 59.7 Kolmogorov's Inequality, Strong Law of Large Numbers

Kolmogorov's inequality is a very interesting inequality which depends on independence of a set of random vectors. The random vectors have values in $\mathbb{R}^{n}$ or more generally some real separable Hilbert space.

Lemma 59.7.1 If $\mathbf{Y}, \mathbf{X}$ are independent random variables having values in a real separable Hilbert space, $H$ with $E\left(|\mathbf{X}|^{2}\right), E\left(|\mathbf{Y}|^{2}\right)<\infty$, then

$$
\int_{\Omega}(\mathbf{X}, \mathbf{Y}) d P=\left(\int_{\Omega} \mathbf{X} d P, \int_{\Omega} \mathbf{Y} d P\right)
$$

Proof: Let $\left\{\mathbf{e}_{k}\right\}$ be a complete orthonormal basis. Thus

$$
\int_{\Omega}(\mathbf{X}, \mathbf{Y}) d P=\int_{\Omega} \sum_{k=1}^{\infty}\left(\mathbf{X}, \mathbf{e}_{k}\right)\left(\mathbf{Y}, \mathbf{e}_{k}\right) d P
$$

Now

$$
\int_{\Omega} \sum_{k=1}^{\infty}\left|\left(\mathbf{X}, \mathbf{e}_{k}\right)\left(\mathbf{Y}, \mathbf{e}_{k}\right)\right| d P \leq \int_{\Omega}\left(\sum_{k}\left|\left(\mathbf{X}, \mathbf{e}_{k}\right)\right|^{2}\right)^{1 / 2}\left(\sum_{k}\left|\left(\mathbf{Y}, \mathbf{e}_{k}\right)\right|^{2}\right)^{1 / 2} d P
$$

$$
=\int_{\Omega}|\mathbf{X}||\mathbf{Y}| d P \leq\left(\int_{\Omega}|\mathbf{X}|^{2} d P\right)^{1 / 2}\left(\int_{\Omega}|\mathbf{Y}|^{2} d P\right)^{1 / 2}<\infty
$$

and so by Fubini's theorem,

$$
\begin{gathered}
\int_{\Omega}(\mathbf{X}, \mathbf{Y}) d P=\int_{\Omega} \sum_{k=1}^{\infty}\left(\mathbf{X}, \mathbf{e}_{k}\right)\left(\mathbf{Y}, \mathbf{e}_{k}\right) d P=\sum_{k=1}^{\infty} \int_{\Omega}\left(\mathbf{X}, \mathbf{e}_{k}\right)\left(\mathbf{Y}, \mathbf{e}_{k}\right) d P \\
=\sum_{k=1}^{\infty} \int_{\Omega}\left(\mathbf{X}, \mathbf{e}_{k}\right) d P \int_{\Omega}\left(\mathbf{Y}, \mathbf{e}_{k}\right) d P=\sum_{k=1}^{\infty}\left(\int_{\Omega} \mathbf{X} d P, \mathbf{e}_{k}\right)\left(\int_{\Omega} \mathbf{Y} d P, \mathbf{e}_{k}\right) d P \\
=\left(\int_{\Omega} \mathbf{X} d P, \int_{\Omega} \mathbf{Y} d P\right)
\end{gathered}
$$

Now here is Kolmogorov's inequality.
Theorem 59.7.2 Suppose $\left\{\mathbf{X}_{k}\right\}_{k=1}^{n}$ are independent with $E\left(\left|\mathbf{X}_{k}\right|\right)<\infty, E\left(\mathbf{X}_{k}\right)=\mathbf{0}$. Then for any $\varepsilon>0$,

$$
P\left(\left[\max _{1 \leq k \leq n}\left|\sum_{j=1}^{k} \mathbf{X}_{j}\right| \geq \varepsilon\right]\right) \leq \frac{1}{\varepsilon^{2}} \sum_{j=1}^{n} E\left(\left|\mathbf{X}_{k}\right|^{2}\right)
$$

Proof: Let

$$
A=\left[\max _{1 \leq k \leq n}\left|\sum_{j=1}^{k} \mathbf{X}_{j}\right| \geq \varepsilon\right]
$$

Now let $A_{1} \equiv\left[\left|\mathbf{X}_{1}\right| \geq \boldsymbol{\varepsilon}\right]$ and if $A_{1}, \cdots, A_{m}$ have been chosen,

$$
A_{m+1} \equiv\left[\left|\sum_{j=1}^{m+1} \mathbf{X}_{j}\right| \geq \varepsilon\right] \cap \bigcap_{r=1}^{m}\left[\left|\sum_{j=1}^{r} \mathbf{X}_{j}\right|<\varepsilon\right]
$$

Thus the $A_{k}$ partition $A$ and $\omega \in A_{k}$ means

$$
\left|\sum_{j=1}^{k} \mathbf{X}_{j}\right| \geq \varepsilon
$$

but this did not happen for $\left|\sum_{j=1}^{r} \mathbf{X}_{j}\right|$ for any $r<k$. Note also that $A_{k} \in \sigma\left(\mathbf{X}_{1}, \cdots, \mathbf{X}_{k}\right)$. Then from algebra,

$$
\begin{array}{r}
\left|\sum_{j=1}^{n} \mathbf{X}_{j}\right|^{2}=\left(\sum_{i=1}^{k} \mathbf{X}_{i}+\sum_{j=k+1}^{n} \mathbf{X}_{j}, \sum_{i=1}^{k} \mathbf{X}_{i}+\sum_{j=k+1}^{n} \mathbf{X}_{j}\right) \\
=\left|\sum_{j=1}^{k} \mathbf{X}_{j}\right|^{2}+\sum_{i \leq k, j>k}\left(\mathbf{X}_{i}, \mathbf{X}_{j}\right)+\sum_{i \leq k, j>k}\left(\mathbf{X}_{j}, \mathbf{X}_{i}\right)+\sum_{i>k, j>k}\left(\mathbf{X}_{j}, \mathbf{X}_{i}\right)
\end{array}
$$

Written more succinctly,

$$
\left|\sum_{j=1}^{n} \mathbf{X}_{j}\right|^{2}=\left|\sum_{j=1}^{k} \mathbf{X}_{j}\right|^{2}+\sum_{j>k \text { or } i>k}\left(\mathbf{X}_{i}, \mathbf{X}_{j}\right)
$$

Now multiply both sides by $\mathscr{X}_{A_{k}}$ and integrate. Suppose $i \leq k$ for one of the terms in the second sum. Then by Lemma 59.3.4 and $A_{k} \in \sigma\left(\mathbf{X}_{1}, \cdots, \mathbf{X}_{k}\right)$, the two random vectors $\mathscr{X}_{A_{k}} \mathbf{X}_{i}, \mathbf{X}_{j}$ are independent,

$$
\int_{\Omega} \mathscr{X}_{A_{k}}\left(\mathbf{X}_{i}, \mathbf{X}_{j}\right) d P=\left(\int_{\Omega} \mathscr{X}_{A_{k}} \mathbf{X}_{i} d P, \int_{\Omega} \mathbf{X}_{j} d P\right)=0
$$

the last equality holding because by assumption $E\left(\mathbf{X}_{j}\right)=\mathbf{0}$. Therefore, it can be assumed both $i, j$ are larger than $k$ and

$$
\begin{align*}
\int_{\Omega} \mathscr{X}_{A_{k}}\left|\sum_{j=1}^{n} \mathbf{X}_{j}\right|^{2} d P= & \int_{\Omega} \mathscr{X}_{A_{k}}\left|\sum_{j=1}^{k} \mathbf{X}_{j}\right|^{2} d P \\
& +\sum_{j>k, i>k} \int_{\Omega} \mathscr{X}_{A_{k}}\left(\mathbf{X}_{i}, \mathbf{X}_{j}\right) d P \tag{59.7.10}
\end{align*}
$$

The last term on the right is interesting. Suppose $i>j$. The integral inside the sum is of the form

$$
\begin{equation*}
\int_{\Omega}\left(\mathbf{X}_{i}, \mathscr{X}_{A_{k}} \mathbf{X}_{j}\right) d P \tag{59.7.11}
\end{equation*}
$$

The second factor in the inner product is in

$$
\sigma\left(\mathbf{X}_{1}, \cdots, \mathbf{X}_{k}, \mathbf{X}_{j}\right)
$$

and $\mathbf{X}_{i}$ is not included in the list of random vectors. Thus by Lemma 59.3.4, the two random vectors $\mathbf{X}_{i}, \mathscr{X}_{A_{k}} \mathbf{X}_{j}$ are independent and so 59.7.11 reduces to

$$
\left(\int_{\Omega} \mathbf{X}_{i} d P, \int_{\Omega} \mathscr{X}_{A_{k}} \mathbf{X}_{j} d P\right)=\left(\mathbf{0}, \int_{\Omega} \mathscr{X}_{A_{k}} \mathbf{X}_{j} d P\right)=0
$$

A similar result holds if $j>i$. Thus the mixed terms in the last term of 59.7.10 are all equal to 0 . Hence 59.7.10 reduces to

$$
\begin{aligned}
\int_{\Omega} \mathscr{X}_{A_{k}}\left|\sum_{j=1}^{n} \mathbf{X}_{j}\right|^{2} d P= & \int_{\Omega} \mathscr{X}_{A_{k}}\left|\sum_{j=1}^{k} \mathbf{X}_{j}\right|^{2} d P \\
& +\sum_{i>k} \int_{\Omega} \mathscr{X}_{A_{k}}\left|\mathbf{X}_{i}\right|^{2} d P
\end{aligned}
$$

and so

$$
\int_{\Omega} \mathscr{X}_{A_{k}}\left|\sum_{j=1}^{n} \mathbf{X}_{j}\right|^{2} d P \geq \int_{\Omega} \mathscr{X}_{A_{k}}\left|\sum_{j=1}^{k} \mathbf{X}_{j}\right|^{2} d P \geq \varepsilon^{2} P\left(A_{k}\right)
$$

Now, summing these yields

$$
\begin{gathered}
\varepsilon^{2} P(A) \leq \int_{\Omega} \mathscr{X}_{A}\left|\sum_{j=1}^{n} \mathbf{X}_{j}\right|^{2} d P \leq \int_{\Omega}\left|\sum_{j=1}^{n} \mathbf{X}_{j}\right|^{2} d P \\
=\sum_{i, j} \int_{\Omega}\left(\mathbf{X}_{i}, \mathbf{X}_{j}\right) d P
\end{gathered}
$$

By independence of the random vectors the mixed terms of the above sum equal zero and so it reduces to

$$
\sum_{i=1}^{n} \int_{\Omega}\left|\mathbf{X}_{i}\right|^{2} d P
$$

This theorem implies the following amazing result.
Theorem 59.7.3 Let $\left\{\mathbf{X}_{k}\right\}_{k=1}^{\infty}$ be independent random vectors having values in a separable real Hilbert space and suppose $E\left(\left|\mathbf{X}_{k}\right|\right)<\infty$ for each $k$ and $E\left(\mathbf{X}_{k}\right)=\mathbf{0}$. Suppose also that

$$
\sum_{j=1}^{\infty} E\left(\left|\mathbf{X}_{j}\right|^{2}\right)<\infty
$$

Then

$$
\sum_{j=1}^{\infty} \mathbf{X}_{j}
$$

converges a.e.
Proof: Let $\varepsilon>0$ be given. By Kolmogorov's inequality, Theorem 59.7.2, it follows that for $p \leq m<n$

$$
\begin{aligned}
P\left(\left[\max _{m \leq k \leq n}\left|\sum_{j=m}^{k} \mathbf{X}_{j}\right| \geq \varepsilon\right]\right) & \leq \frac{1}{\varepsilon^{2}} \sum_{j=p}^{n} E\left(\left|\mathbf{X}_{j}\right|^{2}\right) \\
& \leq \frac{1}{\varepsilon^{2}} \sum_{j=p}^{\infty} E\left(\left|\mathbf{X}_{j}\right|^{2}\right)
\end{aligned}
$$

Therefore, letting $n \rightarrow \infty$ it follows that for all $m, n$ such that $p \leq m \leq n$

$$
P\left(\left[\max _{p \leq m \leq n}\left|\sum_{j=m}^{n} \mathbf{X}_{j}\right| \geq \varepsilon\right]\right) \leq \frac{1}{\varepsilon^{2}} \sum_{j=p}^{\infty} E\left(\left|\mathbf{X}_{j}\right|^{2}\right)
$$

It follows from the assumption

$$
\sum_{j=1}^{\infty} E\left(\left|\mathbf{X}_{j}\right|^{2}\right)<\infty
$$

there exists a sequence, $\left\{p_{n}\right\}$ such that if $m \geq p_{n}$

$$
P\left(\left[\max _{k \geq m \geq p_{n}}\left|\sum_{j=m}^{k} \mathbf{X}_{j}\right| \geq 2^{-n}\right]\right) \leq 2^{-n}
$$

By the Borel Cantelli lemma, Lemma 59.1.2, there is a set of measure $0, N$ such that for $\omega \notin N, \omega$ is in only finitely many of the sets,

$$
\left[\max _{k \geq m \geq p_{n}}\left|\sum_{j=m}^{k} \mathbf{X}_{j}\right| \geq 2^{-n}\right]
$$

and so for $\omega \notin N$, it follows that for large enough $n$,

$$
\left[\max _{k \geq m \geq p_{n}}\left|\sum_{j=m}^{k} \mathbf{X}_{j}(\omega)\right|<2^{-n}\right]
$$

However, this says the partial sums $\left\{\sum_{j=1}^{k} \mathbf{X}_{j}(\omega)\right\}_{k=1}^{\infty}$ are a Cauchy sequence. Therefore, they converge.

With this amazing result, there is a simple proof of the strong law of large numbers. In the following lemma, $s_{k}$ and $a_{j}$ could have values in any normed linear space.

Lemma 59.7.4 Suppose $s_{k} \rightarrow s$. Then

$$
\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{n} s_{k}=s
$$

Also if

$$
\sum_{j=1}^{\infty} \frac{a_{j}}{j}
$$

converges, then

$$
\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{n} a_{j}=0
$$

Proof: Consider the first part. Since $s_{k} \rightarrow s$, it follows there is some constant, $C$ such that $\left|s_{k}\right|<C$ for all $k$ and $|s|<C$ also. Choose $K$ so large that if $k \geq K$, then for $n>K$,

$$
\begin{gathered}
\left|s-s_{k}\right|<\varepsilon / 2 \\
\left|s-\frac{1}{n} \sum_{k=1}^{n} s_{k}\right| \leq \frac{1}{n} \sum_{k=1}^{n}\left|s_{k}-s\right| \\
=\frac{1}{n} \sum_{k=1}^{K}\left|s_{k}-s\right|+\frac{1}{n} \sum_{k=K}^{n}\left|s_{k}-s\right| \\
\leq \frac{2 C K}{n}+\frac{\varepsilon}{2} \frac{n-K}{n}<\frac{2 C K}{n}+\frac{\varepsilon}{2}
\end{gathered}
$$

Therefore, whenever $n$ is large enough,

$$
\left|s-\frac{1}{n} \sum_{k=1}^{n} s_{k}\right|<\varepsilon .
$$

Now consider the second claim. Let

$$
s_{k}=\sum_{j=1}^{k} \frac{a_{j}}{j}
$$

and $s=\lim _{k \rightarrow \infty} s_{k}$ Then by the first part,

$$
\begin{aligned}
s & =\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{n} s_{k}=\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} \frac{a_{j}}{j} \\
& =\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{n} \frac{a_{j}}{j} \sum_{k=j}^{n} 1=\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{n} \frac{a_{j}}{j}(n-j) \\
& =\lim _{n \rightarrow \infty}\left(\sum_{j=1}^{n} \frac{a_{j}}{j}-\frac{1}{n} \sum_{j=1}^{n} a_{j}\right)=s-\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{n} a_{j}
\end{aligned}
$$

Now here is the strong law of large numbers.
Theorem 59.7.5 Suppose $\left\{\mathbf{X}_{k}\right\}$ are independent random variables and $E\left(\left|\mathbf{X}_{k}\right|\right)<\infty$ for each $k$ and $E\left(\mathbf{X}_{k}\right)=\mathbf{m}_{k}$. Suppose also

$$
\begin{equation*}
\sum_{j=1}^{\infty} \frac{1}{j^{2}} E\left(\left|\mathbf{X}_{j}-\mathbf{m}_{j}\right|^{2}\right)<\infty \tag{59.7.12}
\end{equation*}
$$

Then

$$
\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{n}\left(\mathbf{X}_{j}-\mathbf{m}_{j}\right)=\mathbf{0}
$$

Proof: Consider the sum

$$
\sum_{j=1}^{\infty} \frac{\mathbf{X}_{j}-\mathbf{m}_{j}}{j}
$$

This sum converges a.e. because of 59.7.12 and Theorem 59.7.3 applied to the random vectors $\left\{\frac{\mathbf{x}_{j}-\mathbf{m}_{j}}{j}\right\}$. Therefore, from Lemma 59.7.4 it follows that for a.e. $\omega$,

$$
\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{n}\left(\mathbf{X}_{j}(\omega)-\mathbf{m}_{j}\right)=\mathbf{0}
$$

The next corollary is often called the strong law of large numbers. It follows immediately from the above theorem.

Corollary 59.7.6 Suppose $\left\{\mathbf{X}_{j}\right\}_{j=1}^{\infty}$ are independent having mean $\mathbf{m}$ and variance equal to

$$
\sigma^{2} \equiv \int_{\Omega}\left|\mathbf{X}_{j}-\mathbf{m}\right|^{2} d P<\infty
$$

Then for a.e. $\omega \in \Omega$

$$
\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{n} \mathbf{X}_{j}(\omega)=\mathbf{m}
$$

### 59.8 The Characteristic Function

One of the most important tools in probability is the characteristic function. To begin with, assume the random variables have values in $\mathbb{R}^{p}$.

Definition 59.8.1 Let $\mathbf{X}$ be a random variable as above. The characteristic function is

$$
\phi_{\mathbf{X}}(\mathbf{t}) \equiv E\left(e^{i \mathbf{t} \cdot \mathbf{X}}\right) \equiv \int_{\Omega} e^{i \mathbf{t} \cdot \mathbf{X}(\omega)} d P=\int_{\mathbb{R} p} e^{i \mathbf{t} \cdot \mathbf{x}} d \lambda \mathbf{X}
$$

the last equation holding by Proposition 59.1.12.
Recall the following fundamental lemma and definition, Lemma 32.3.4 on Page 1101.
Definition 59.8.2 For $T \in \mathscr{G}^{*}$, define $F T, F^{-1} T \in \mathscr{G}^{*}$ by

$$
F T(\phi) \equiv T(F \phi), F^{-1} T(\phi) \equiv T\left(F^{-1} \phi\right)
$$

Lemma 59.8.3 $F$ and $F^{-1}$ are both one to one, onto, and are inverses of each other.
The main result on characteristic functions is the following.
Theorem 59.8.4 Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors with values in $\mathbb{R}^{p}$ and suppose $E\left(e^{i \cdot \mathbf{X}}\right)$ $=E\left(e^{i \cdot \mathbf{t} \cdot \mathbf{Y}}\right)$ for all $\mathbf{t} \in \mathbb{R}^{p}$. Then $\lambda_{\mathbf{x}}=\lambda_{\mathbf{Y}}$.

Proof: For $\psi \in \mathscr{G}$, let $\lambda_{\mathbf{X}}(\psi) \equiv \int_{\mathbb{R}^{p}} \psi d \lambda_{\mathbf{X}}$ and $\lambda_{\mathbf{Y}}(\psi) \equiv \int_{\mathbb{R}^{p}} \psi d \lambda_{\mathbf{Y}}$. Thus both $\lambda_{\mathbf{X}}$ and $\lambda_{\mathbf{Y}}$ are in $\mathscr{G}^{*}$. Then letting $\psi \in \mathscr{G}$ and using Fubini's theorem,

$$
\begin{aligned}
\int_{\mathbb{R}^{p}} \int_{\mathbb{R}^{p}} e^{i \cdot \mathbf{y}} \psi(\mathbf{t}) d t d \lambda_{\mathbf{Y}} & =\int_{\mathbb{R}^{p}} \int_{\mathbb{R}^{p}} e^{i \cdot \mathbf{y}} d \lambda_{\mathbf{Y}} \psi(\mathbf{t}) d t \\
& =\int_{\mathbb{R}^{p}} E\left(e^{i \cdot \mathbf{Y}}\right) \psi(\mathbf{t}) d t \\
& =\int_{\mathbb{R}^{p}} E\left(e^{i \mathbf{t} \cdot \mathbf{X}}\right) \psi(\mathbf{t}) d t \\
& =\int_{\mathbb{R}^{p}} \int_{\mathbb{R}^{p}} e^{i \mathbf{t} \cdot \mathbf{x}} d \lambda_{\mathbf{X}} \psi(\mathbf{t}) d t \\
& =\int_{\mathbb{R}^{p}} \int_{\mathbb{R}^{p}} e^{i \mathbf{t} \cdot \mathbf{x}} \psi(\mathbf{t}) d t d \lambda_{\mathbf{X}}
\end{aligned}
$$

Thus $\lambda_{\mathbf{Y}}\left(F^{-1} \psi\right)=\lambda_{\mathbf{X}}\left(F^{-1} \psi\right)$. Since $\psi \in \mathscr{G}$ is arbitrary and $F^{-1}$ is onto, this implies $\lambda_{\mathbf{X}}$ $=\lambda_{\mathbf{Y}}$ in $\mathscr{G}^{*}$. But $\mathscr{G}$ is dense in $C_{0}\left(\mathbb{R}^{p}\right)$ from the Stone Weierstrass theorem and so $\lambda_{\mathbf{X}}=\lambda_{\mathbf{Y}}$ as measures. Recall from real analysis the dual space of $C_{0}\left(\mathbb{R}^{n}\right)$ is the space of complex measures.

Alternatively, the above shows that since $F^{-1}$ is onto, for all $\psi \in \mathscr{G}$,

$$
\int_{\mathbb{R}^{p}} \psi d \lambda_{\mathbf{Y}}=\int_{\mathbb{R}^{p}} \psi d \lambda_{\mathbf{x}}
$$

and then, by a use of the Stone Weierstrass theorem, the above will hold for all $\psi \in C_{c}\left(\mathbb{R}^{n}\right)$ and now, by the Riesz representation theorem for positive linear functionals, the two measures are equal.

You can also give a version of this theorem in which reference is made only to the probability distribution measures.

Definition 59.8.5 For $\mu$ a probability measure on the Borel sets of $\mathbb{R}^{n}$,

$$
\phi_{\mu}(\mathbf{t}) \equiv \int_{\mathbb{R}^{n}} e^{i \mathbf{t} \cdot \mathbf{x}} d \mu
$$

Theorem 59.8.6 Let $\mu$ and $v$ be probability measures on the Borel sets of $\mathbb{R}^{p}$ and suppose $\phi_{\mu}(\mathbf{t})=\phi_{v}(\mathbf{t})$. Then $\mu=v$.

Proof: The proof is identical to the above. Just replace $\lambda_{\mathbf{X}}$ with $\mu$ and $\lambda_{\mathbf{Y}}$ with $v$.

### 59.9 Conditional Probability

Here I will consider the concept of conditional probability depending on the theory of differentiation of general Radon measures. This leads to a different way of thinking about independence.

If $\mathbf{X}, \mathbf{Y}$ are two random vectors defined on a probability space having values in $\mathbb{R}^{p_{1}}$ and $\mathbb{R}^{p_{2}}$ respectively, and if $E$ is a Borel set in the appropriate space, then $(\mathbf{X}, \mathbf{Y})$ is a random vector with values in $\mathbb{R}^{p_{1}} \times \mathbb{R}^{p_{2}}$ and $\lambda_{(\mathbf{X}, \mathbf{Y})}\left(E \times \mathbb{R}^{p_{2}}\right)=\lambda_{\mathbf{X}}(E), \lambda_{(\mathbf{X}, \mathbf{Y})}\left(\mathbb{R}^{p_{1}} \times E\right)=\lambda_{\mathbf{Y}}(E)$. Thus, by Theorem 31.2.3 on Page 1085, there exist probability measures, denoted here by $\lambda_{\mathbf{X} \mid \mathbf{y}}$ and $\lambda_{\mathbf{Y} \mid \mathbf{x}}$, such that whenever $E$ is a Borel set in $\mathbb{R}^{p_{1}} \times \mathbb{R}^{p_{2}}$,

$$
\int_{\mathbb{R}^{p_{1} \times \mathbb{R}^{p_{2}}}} \mathscr{X}_{E} d \lambda_{(\mathbf{X}, \mathbf{Y})}=\int_{\mathbb{R}^{p_{1}}} \int_{\mathbb{R}^{p_{2}}} \mathscr{X}_{E} d \lambda_{\mathbf{Y} \mid \mathbf{x}} d \lambda_{\mathbf{x}}
$$

and

$$
\int_{\mathbb{R}^{p_{1} \times \mathbb{R}^{p_{2}}}} \mathscr{X}_{E} d \lambda_{(\mathbf{X}, \mathbf{Y})}=\int_{\mathbb{R}^{p_{2}}} \int_{\mathbb{R}^{p_{1}}} \mathscr{X}_{E} d \lambda_{\mathbf{x} \mid \mathbf{y}} d \lambda_{\mathbf{Y}}
$$

Definition 59.9.1 Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors defined on a probability space. The conditional probability measure of $\mathbf{Y}$ given $\mathbf{X}$ is the measure $\lambda_{\mathbf{Y} \mid \mathbf{x}}$ in the above. Similarly the conditional probability measure of $\mathbf{X}$ given $\mathbf{Y}$ is the measure $\lambda_{\mathbf{x | y}}^{\mathbf{y}}$.

More generally, one can use the theory of slicing measures to consider any finite list of random vectors, $\left\{\mathbf{X}_{i}\right\}$, defined on a probability space with $\mathbf{X}_{i} \in \mathbb{R}^{p_{i}}$, and write the following for $E$ a Borel set in $\prod_{i=1}^{n} \mathbb{R}^{p_{i}}$.

$$
\begin{gather*}
\int_{\mathbb{R}^{p_{1 \times \cdots}} \cdots \times \mathbb{R}^{p_{n}}} \mathscr{X}_{E} d \lambda_{\left(\mathbf{X}_{1}, \cdots, \mathbf{x}_{n}\right)}=\int_{\mathbb{R}^{p_{1}} \times \cdots \times \mathbb{R}^{p_{n-1}}} \int_{\mathbb{R}^{p_{n}}} \mathscr{X}_{E} d \lambda_{\mathbf{x}_{n} \mid\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{n-1}\right)} d \lambda_{\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{n-1}\right)} \\
=\int_{\mathbb{R}^{p_{1}} \times \cdots \times \mathbb{R}^{p_{n-2}}} \int_{\mathbb{R}^{p_{n-1}}} \int_{\mathbb{R}^{p_{n}}} \mathscr{X}_{E} d \lambda_{\mathbf{x}_{n} \mid\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{n-1}\right)} d \lambda_{\mathbf{x}_{n-1} \mid\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{n-2}\right)} d \lambda_{\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{n-2}\right)} \\
\vdots  \tag{59.9.13}\\
\int_{\mathbb{R}^{p_{1}}} \cdots \int_{\mathbb{R}^{p_{n}}} \mathscr{X}_{E} d \lambda_{\mathbf{x}_{n} \mid\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{n-1}\right)} d \lambda_{\mathbf{x}_{n-1} \mid\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{n-2}\right)} \cdots d \lambda_{\mathbf{x}_{2} \mid \mathbf{x}_{1}} d \lambda_{\mathbf{x}_{1}} .
\end{gather*}
$$

Obviously, this could have been done in any order in the iterated integrals by simply modifying the "given" variables, those occurring after the symbol |, to be those which have been integrated in an outer level of the iterated integral. For simplicity, write

$$
\lambda_{\mathbf{x}_{n} \mid\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{n-1}\right)}=\lambda_{\mathbf{x}_{n} \mid \mathbf{x}_{1}, \cdots, \mathbf{x}_{n-1}}
$$

Definition 59.9.2 Let $\left\{\mathbf{X}_{1}, \cdots, \mathbf{X}_{n}\right\}$ be random vectors defined on a probability space having values in $\mathbb{R}^{p_{1}}, \cdots, \mathbb{R}^{p_{n}}$ respectively. The random vectors are independent if for every $E$ a Borel set in $\mathbb{R}^{p_{1}} \times \cdots \times \mathbb{R}^{p_{n}}$,

$$
\begin{gather*}
\int_{\mathbb{R}^{p_{1 \times \cdots \times \mathbb{R}^{p_{n}}}}} \mathscr{X}_{E} d \lambda_{\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{n}\right)} \\
=\int_{\mathbb{R}^{p_{1}}} \cdots \int_{\mathbb{R}^{p_{n}}} \mathscr{X}_{E} d \lambda \mathbf{x}_{n} d \lambda \mathbf{x}_{n-1} \cdots d \lambda_{\mathbf{x}_{2}} d \lambda \mathbf{x}_{1} \tag{59.9.14}
\end{gather*}
$$

and the iterated integration may be taken in any order. If $\mathscr{A}$ is any set of random vectors defined on a probability space, $\mathscr{A}$ is independent if any finite set of random vectors from $\mathscr{A}$ is independent.

Thus, the random vectors are independent exactly when the dependence on the givens in 59.9.13 can be dropped.

Does this amount to the same thing as discussed earlier? Suppose you have three random variables $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$. Let $A=\mathbf{X}^{-1}(E), B=\mathbf{Y}^{-1}(F), C=\mathbf{Z}^{-1}(G)$ where $E, F, G$ are Borel sets. Thus these inverse images are typical sets in $\sigma(\mathbf{X}), \sigma(\mathbf{Y}), \sigma(\mathbf{Z})$ respectively. First suppose that the random variables are independent in the earlier sense. Then

$$
\begin{gathered}
P(A \cap B \cap C)=P(A) P(B) P(C) \\
=\int_{\mathbb{R}^{p_{1}}} \mathscr{X}_{E}(\mathbf{x}) d \lambda_{\mathbf{x}} \int_{\mathbb{R}^{p_{2}}} \mathscr{X}_{F}(\mathbf{y}) d \lambda_{\mathbf{Y}} \int_{\mathbb{R}^{p_{3}}} \mathscr{X}_{G}(\mathbf{z}) d \lambda_{\mathbf{Z}} \\
=\int_{\mathbb{R}^{p_{1}}} \int_{\mathbb{R}^{p_{2}}} \int_{\mathbb{R}^{p_{3}}} \mathscr{X}_{E}(\mathbf{x}) \mathscr{X}_{F}(\mathbf{y}) \mathscr{X}_{G}(\mathbf{z}) d \lambda_{\mathbf{Z}} d \lambda_{\mathbf{Y}} d \lambda_{\mathbf{x}}
\end{gathered}
$$

Also

$$
\begin{aligned}
P & (A \cap B \cap C)=\int_{\mathbb{R}^{p_{1}} \times \mathbb{R}^{p_{2}} \times \mathbb{R}^{p_{3}}} \mathscr{X}_{E}(\mathbf{x}) \mathscr{X}_{F}(\mathbf{y}) \mathscr{X}_{G}(\mathbf{z}) d \lambda_{(\mathbf{X}, \mathbf{Y}, \mathbf{Z})} \\
& =\int_{\mathbb{R}^{p_{1}}} \int_{\mathbb{R}^{p_{2}}} \int_{\mathbb{R}^{p_{3}}} \mathscr{X}_{E}(\mathbf{x}) \mathscr{X}_{F}(\mathbf{y}) \mathscr{X}_{G}(\mathbf{z}) d \lambda_{\mathbf{Z} \mid \mathbf{x y}} d \lambda_{\mathbf{Y} \mid \mathbf{x}} d \lambda_{\mathbf{x}}
\end{aligned}
$$

Thus

$$
\begin{aligned}
& \int_{\mathbb{R}^{p_{1}}} \int_{\mathbb{R}^{p_{2}}} \int_{\mathbb{R}^{p_{3}}} \mathscr{X}_{E}(\mathbf{x}) \mathscr{X}_{F}(\mathbf{y}) \mathscr{X}_{G}(\mathbf{z}) d \lambda_{\mathbf{Z}} d \lambda_{\mathbf{Y}} d \lambda_{\mathbf{x}} \\
= & \int_{\mathbb{R}^{p_{1}}} \int_{\mathbb{R}^{p_{2}}} \int_{\mathbb{R}^{p_{3}}} \mathscr{X}_{E}(\mathbf{x}) \mathscr{X}_{F}(\mathbf{y}) \mathscr{X}_{G}(\mathbf{z}) d \lambda_{\mathbf{Z} \mid \mathbf{x y}} d \lambda_{\mathbf{Y} \mid \mathbf{x}} d \lambda_{\mathbf{x}}
\end{aligned}
$$

Now letting $G=\mathbb{R}^{p_{3}}$, it follows that

$$
\begin{aligned}
& \int_{\mathbb{R}^{p_{1}}} \int_{\mathbb{R}^{p_{2}}} \mathscr{X}_{E}(\mathbf{x}) \mathscr{X}_{F}(\mathbf{y}) d \lambda_{\mathbf{Y}} d \lambda_{\mathbf{x}} \\
= & \int_{\mathbb{R}^{p_{1}}} \int_{\mathbb{R}^{p_{2}}} \mathscr{X}_{E}(\mathbf{x}) \mathscr{X}_{F}(\mathbf{y}) d \lambda_{\mathbf{Y} \mid \mathbf{x}} d \lambda_{\mathbf{x}}
\end{aligned}
$$

By uniqueness of the slicing measures or an application of the Besikovitch differentiation theorem, it follows that for $\lambda_{\mathbf{x}}$ a.e. $\mathbf{x}$,

$$
\lambda_{\mathbf{Y}}=\lambda_{\mathbf{Y} \mid \mathbf{X}}
$$

Thus, using this in the above,

$$
\begin{aligned}
& \int_{\mathbb{R}^{p_{1}}} \int_{\mathbb{R}^{p_{2}}} \int_{\mathbb{R}^{p_{3}}} \mathscr{X}_{E}(\mathbf{x}) \mathscr{X}_{F}(\mathbf{y}) \mathscr{X}_{G}(\mathbf{z}) d \lambda_{\mathbf{Z}} d \lambda_{\mathbf{Y}} d \lambda_{\mathbf{x}} \\
= & \int_{\mathbb{R}^{p_{1}}} \int_{\mathbb{R}^{p_{2}}} \int_{\mathbb{R}^{p_{3}}} \mathscr{X}_{E}(\mathbf{x}) \mathscr{X}_{F}(\mathbf{y}) \mathscr{X}_{G}(\mathbf{z}) d \lambda_{\mathbf{Z} \mid \mathbf{x}} d \lambda_{\mathbf{Y}} d \lambda_{\mathbf{x}}
\end{aligned}
$$

and also it reduces to

$$
\begin{aligned}
& \int_{\mathbb{R}^{p_{1} \times \mathbb{R}^{p_{2}}}} \int_{\mathbb{R}^{p_{3}}} \mathscr{X}_{E}(\mathbf{x}) \mathscr{X}_{F}(\mathbf{y}) \mathscr{X}_{G}(\mathbf{z}) d \lambda_{\mathbf{Z}} d \lambda_{(\mathbf{X}, \mathbf{Y})} \\
= & \int_{\mathbb{R}^{p_{1} \times \mathbb{R}^{p_{2}}}} \int_{\mathbb{R}^{p_{3}}} \mathscr{X}_{E}(\mathbf{x}) \mathscr{X}_{F}(\mathbf{y}) \mathscr{X}_{G}(\mathbf{z}) d \lambda_{\mathbf{Z} \mid \mathbf{x y}} d \lambda_{(\mathbf{X}, \mathbf{Y})}
\end{aligned}
$$

Now by uniqueness of the slicing measures again, for $\lambda_{(\mathbf{X}, \mathbf{Y})}$ a.e. $(\mathbf{x}, \mathbf{y})$, it follows that

$$
\lambda_{\mathbf{Z}}=\lambda_{\mathbf{Z} \mid \mathbf{x y}}
$$

Similar conclusions hold for $\lambda_{\mathbf{X}}, \lambda_{\mathbf{Y}}$. In each case, off a set of measure zero the distribution measures equal the slicing measures.

Conversely, if the distribution measures equal the slicing measures off sets of measure zero as described above, then it is obvious that the random variables are independent. The same reasoning applies for any number of random variables.

Thus this gives a different and more analytical way to think of independence of finitely many random variables. Clearly, the argument given above will apply to any finite set of random variables.

Proposition 59.9.3 Equations 59.9.14 and 59.9.13 hold with $\mathscr{X}_{E}$ replaced by any nonnegative Borel measurable function and for any bounded continuous function or for any function in $L^{1}$.

Proof: The two equations hold for simple functions in place of $\mathscr{X}_{E}$ and so an application of the monotone convergence theorem applied to an increasing sequence of simple functions converging pointwise to a given nonnegative Borel measurable function yields the conclusion of the proposition in the case of the nonnegative Borel function. For a bounded continuous function or one in $L^{1}$, one can apply the result just established to the positive and negative parts of the real and imaginary parts of the function.

Lemma 59.9.4 Let $\mathbf{X}_{1}, \cdots, \mathbf{X}_{n}$ be random vectors with values in $\mathbb{R}^{p_{1}}, \cdots, \mathbb{R}^{p_{n}}$ respectively and let $\mathbf{g}: \mathbb{R}^{p_{1}} \times \cdots \times \mathbb{R}^{p_{n}} \rightarrow \mathbb{R}^{k}$ be Borel measurable. Then $\mathbf{g}\left(\mathbf{X}_{1}, \cdots, \mathbf{X}_{n}\right)$ is a random vector with values in $\mathbb{R}^{k}$ and if $h: \mathbb{R}^{k} \rightarrow[0, \infty)$, then

$$
\begin{gather*}
\int_{\mathbb{R}^{k}} h(\mathbf{y}) d \lambda_{\mathbf{g}\left(\mathbf{X}_{1}, \cdots, \mathbf{x}_{n}\right)}(y)= \\
\int_{\mathbb{R}^{p_{1}} \times \cdots \times \mathbb{R}^{p_{n}}} h\left(\mathbf{g}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{n}\right)\right) d \lambda_{\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{n}\right) .} \tag{59.9.15}
\end{gather*}
$$

If $\mathbf{X}_{i}$ is a random vector with values in $\mathbb{R}^{p_{i}}, i=1,2, \cdots$ and if $\mathbf{g}_{i}: \mathbb{R}^{p_{i}} \rightarrow \mathbb{R}^{k_{i}}$, where $\mathbf{g}_{i}$ is Borel measurable, then the random vectors $\mathbf{g}_{i}\left(\mathbf{X}_{i}\right)$ are also independent whenever the $\mathbf{X}_{i}$ are independent.

Proof: First let $E$ be a Borel set in $\mathbb{R}^{k}$. From the definition,

$$
\begin{aligned}
\lambda_{\mathbf{g}\left(\mathbf{X}_{1}, \cdots, \mathbf{X}_{n}\right)}(E) & =P\left(\mathbf{g}\left(\mathbf{X}_{1}, \cdots, \mathbf{X}_{n}\right) \in E\right) \\
= & P\left(\left(\mathbf{X}_{1}, \cdots, \mathbf{X}_{n}\right) \in \mathbf{g}^{-1}(E)\right)=\lambda_{\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{n}\right)}\left(\mathbf{g}^{-1}(E)\right) \\
\int_{\mathbb{R}^{k}} \mathscr{X}_{E} d \lambda_{\mathbf{g}\left(\mathbf{X}_{1}, \cdots, \mathbf{x}_{n}\right)} & =\int_{\mathbb{R}^{p_{1 \times}} \cdots \times \mathbb{R}^{p_{n}}} \mathscr{X}_{\mathbf{g}^{-1}(E)} d \lambda_{\left(\mathbf{X}_{1}, \cdots, \mathbf{x}_{n}\right)} \\
& =\int_{\mathbb{R}^{p_{1 \times \cdots} \times \mathbb{R}^{p_{n}}}} \mathscr{X}_{E}\left(\mathbf{g}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{n}\right)\right) d \lambda_{\left(\mathbf{X}_{1}, \cdots, \mathbf{X}_{n}\right)}
\end{aligned}
$$

This proves 59.9 .15 in the case when $h$ is $\mathscr{X}_{E}$. To prove it in the general case, approximate the nonnegative Borel measurable function with simple functions for which the formula is true, and use the monotone convergence theorem.

It remains to prove the last assertion that functions of independent random vectors are also independent random vectors. Let $E$ be a Borel set in $\mathbb{R}^{k_{1}} \times \cdots \times \mathbb{R}^{k_{n}}$. Then for

$$
\begin{gathered}
\pi_{i}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{n}\right) \equiv \mathbf{x}_{i}, \\
\int_{\mathbb{R}^{k_{1} \times \cdots \times \mathbb{R}^{k_{n}}}} \mathscr{X}_{E} d \lambda_{\left(\mathbf{g}_{1}\left(\mathbf{x}_{1}\right), \cdots, \mathbf{g}_{n}\left(\mathbf{x}_{n}\right)\right)} \\
\equiv \int_{\mathbb{R}^{p_{1}} \times \cdots \times \mathbb{R}^{p_{n}}} \mathscr{X}_{E} \circ\left(\mathbf{g}_{1} \circ \pi_{1}, \cdots, \mathbf{g}_{n} \circ \pi_{n}\right) d \lambda_{\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{n}\right)} \\
=\int_{\mathbb{R}^{p_{1}}} \cdots \int_{\mathbb{R}^{p_{n}}} \mathscr{X}_{E} \circ\left(\mathbf{g}_{1} \circ \pi_{1}, \cdots, \mathbf{g}_{n} \circ \pi_{n}\right) d \lambda \mathbf{x}_{n} \cdots d \lambda_{\mathbf{x}_{1}} \\
=\int_{\mathbb{R}^{k_{1}}} \cdots \int_{\mathbb{R}^{k_{n}}} \mathscr{X}_{E} d \lambda_{\mathbf{g}_{n}\left(\mathbf{x}_{n}\right)} \cdots d \lambda_{\mathbf{g}_{1}\left(\mathbf{x}_{1}\right)}
\end{gathered}
$$

and this proves the last assertion.
Proposition 59.9.5 Let $v_{1}, \cdots, v_{n}$ be Radon probability measures defined on $\mathbb{R}^{p}$. Then there exists a probability space and independent random vectors $\left\{\mathbf{X}_{1}, \cdots, \mathbf{X}_{n}\right\}$ defined on this probability space such that $\lambda_{\mathbf{x}_{i}}=v_{i}$.

Proof: Let $(\Omega, \mathscr{S}, P) \equiv\left(\left(\mathbb{R}^{p}\right)^{n}, \mathscr{S}_{1} \times \cdots \times \mathscr{S}_{n}, v_{1} \times \cdots \times v_{n}\right)$ where this is just the product $\sigma$ algebra and product measure which satisfies the following for measurable rectangles.

$$
\left(v_{1} \times \cdots \times v_{n}\right)\left(\prod_{i=1}^{n} E_{i}\right)=\prod_{i=1}^{n} v_{i}\left(E_{i}\right)
$$

Now let $\mathbf{X}_{i}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{i}, \cdots, \mathbf{x}_{n}\right)=\mathbf{x}_{i}$. Then from the definition, if $E$ is a Borel set in $\mathbb{R}^{p}$,

$$
\begin{gathered}
\lambda_{\mathbf{x}_{i}}(E) \equiv P\left\{\mathbf{X}_{i} \in E\right\} \\
=\left(v_{1} \times \cdots \times v_{n}\right)\left(\mathbb{R}^{p} \times \cdots \times E \times \cdots \times \mathbb{R}^{p}\right)=v_{i}(E) .
\end{gathered}
$$

Let $\mathscr{M}$ consist of all Borel sets of $\left(\mathbb{R}^{p}\right)^{n}$ such that

$$
\int_{\mathbb{R}^{p}} \cdots \int_{\mathbb{R}^{p}} \mathscr{X}_{E}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{n}\right) d \lambda_{\mathbf{x}_{1}} \cdots d \lambda_{\mathbf{x}_{n}}=\int_{\left(\mathbb{R}^{p}\right)^{n}} \mathscr{X}_{E} d \lambda_{\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{n}\right)} .
$$

From what was just shown and the definition of $\left(v_{1} \times \cdots \times v_{n}\right)$ that $\mathscr{M}$ contains all sets of the form $\prod_{i=1}^{n} E_{i}$ where each $E_{i} \in$ Borel sets of $\mathbb{R}^{p}$. Therefore, $\mathscr{M}$ contains the algebra of all finite disjoint unions of such sets. It is also clear that $\mathscr{M}$ is a monotone class and so by the theorem on monotone classes, $\mathscr{M}$ equals the Borel sets. You could also note that $\mathscr{M}$ is closed with respect to complements and countable disjoint unions and apply Lemma 12.12.3. Therefore, the given random vectors are independent and this proves the proposition.

The following Lemma was proved earlier in a different way.
Lemma 59.9.6 If $\left\{X_{i}\right\}_{i=1}^{n}$ are independent random variables having values in $\mathbb{R}$,

$$
E\left(\prod_{i=1}^{n} X_{i}\right)=\prod_{i=1}^{n} E\left(X_{i}\right)
$$

Proof: By Lemma 59.9.4 and denoting by $P$ the product, $\prod_{i=1}^{n} X_{i}$,

$$
\begin{aligned}
E\left(\prod_{i=1}^{n} X_{i}\right) & =\int_{\mathbb{R}} z d \lambda_{P}(z)=\int_{\mathbb{R}^{n}} \prod_{i=1}^{n} x_{i} d \lambda_{\left(X_{1}, \cdots, X_{n}\right)} \\
& =\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{i=1}^{n} x_{i} d \lambda_{X_{1}} \cdots d \lambda_{X_{n}}=\prod_{i=1}^{n} E\left(X_{i}\right)
\end{aligned}
$$

### 59.10 Conditional Expectation

Definition 59.10.1 Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors having values in $\mathbb{F}^{p_{1}}$ and $\mathbb{F}^{p_{2}}$ respectively. Then if

$$
\int|\mathbf{x}| d \lambda_{\mathbf{x} \mid \mathbf{y}}(x)<\infty
$$

we define

$$
E(\mathbf{X} \mid \mathbf{y}) \equiv \int \mathbf{x} d \lambda \mathbf{X} \mid \mathbf{y}(x)
$$

Proposition 59.10.2 Suppose $\int_{\mathbb{F}^{p_{1}} \times \mathbb{F}^{p_{2}}}|\mathbf{x}| d \lambda_{(\mathbf{X}, \mathbf{Y})}(x)<\infty$. Then $E(\mathbf{X} \mid \mathbf{y})$ exists for $\lambda_{\mathbf{Y}}$ a.e. $\mathbf{y}$ and

$$
\int_{\mathbb{F}^{p_{2}}} E(\mathbf{X} \mid \mathbf{y}) d \lambda_{\mathbf{Y}}=\int_{\mathbb{F}^{p_{1}}} \mathbf{x} d \lambda_{\mathbf{X}}(x)=E(\mathbf{X})
$$

Proof: $\infty>\int_{\mathbb{F}^{p_{1}} \times \mathbb{F}^{p_{2}}}|\mathbf{x}| d \lambda_{(\mathbf{X}, \mathbf{Y})}=\int_{\mathbb{F}^{p_{2}}} \int_{\mathbb{F}^{p_{1}}}|\mathbf{x}| d \lambda_{\mathbf{X} \mid \mathbf{y}}(x) d \lambda_{\mathbf{Y}}(y)$ and so

$$
\int_{\mathbb{F}^{p_{1}}}|\mathbf{x}| d \lambda \mathbf{x} \mid \mathbf{y}(x)<\infty
$$

, $\lambda_{\mathbf{Y}}$ a.e. Now

$$
\int_{\mathbb{F}^{p_{2}}} E(\mathbf{X} \mid \mathbf{y}) d \lambda_{\mathbf{Y}}
$$

$$
\begin{aligned}
& =\int_{\mathbb{F}^{p_{2}}} \int_{\mathbb{F}^{p_{1}}} \mathbf{x} d \lambda_{\mathbf{X} \mid \mathbf{y}}(x) d \lambda_{\mathbf{Y}}(y)=\int_{\mathbb{F}^{p_{1 \times ~}} \mathbb{F}^{p_{2}}} \mathbf{x} d \lambda_{(\mathbf{X}, \mathbf{Y})} \\
& =\int_{\mathbb{F}^{p_{1}}} \int_{\mathbb{F}^{p_{2}}} \mathbf{x} d \lambda_{\mathbf{Y} \mid \mathbf{x}}(y) d \lambda_{\mathbf{X}}(x)=\int_{\mathbb{F}^{p_{2}}} \mathbf{x} d \lambda_{\mathbf{x}}(x)=E(\mathbf{X}) .
\end{aligned}
$$

Definition 59.10.3 Let $\left\{X_{n}\right\}$ be any sequence, finite or infinite, of random variables with values in $\mathbb{R}$ which are defined on some probability space, $(\Omega, \mathscr{S}, P)$. We say $\left\{X_{n}\right\}$ is a Martingale if

$$
E\left(X_{n} \mid x_{n-1}, \cdots, x_{1}\right)=x_{n-1}
$$

and we say $\left\{X_{n}\right\}$ is a submartingale if

$$
E\left(X_{n} \mid x_{n-1}, \cdots, x_{1}\right) \geq x_{n-1}
$$

Next we define what is meant by an upcrossing.
Definition 59.10.4 Let $\left\{x_{i}\right\}_{i=1}^{I}$ be any sequence of real numbers, $I \leq \infty$. Define an increasing sequence of integers $\left\{m_{k}\right\}$ as follows. $m_{1}$ is the first integer $\geq 1$ such that $x_{m_{1}} \leq a, m_{2}$ is the first integer larger than $m_{1}$ such that $x_{m_{2}} \geq b, m_{3}$ is the first integer larger than $m_{2}$ such that $x_{m_{3}} \leq a$, etc. Then each sequence, $\left\{x_{m_{2 k-1}}, \cdots, x_{m_{2 k}}\right\}$, is called an upcrossing of $[a, b]$.

Proposition 59.10.5 Let $\left\{X_{i}\right\}_{i=1}^{n}$ be a finite sequence of real random variables defined on $\Omega$ where $(\Omega, \mathscr{S}, P)$ is a probability space. Let $U_{[a, b]}(\omega)$ denote the number of upcrossings of $X_{i}(\omega)$ of the interval $[a, b]$. Then $U_{[a, b]}$ is a random variable.

Proof: Let $X_{0}(\omega) \equiv a+1$, let $Y_{0}(\omega) \equiv 0$, and let $Y_{k}(\omega)$ remain 0 for $k=0, \cdots, l$ until $X_{l}(\omega) \leq a$. When this happens (if ever), $Y_{l+1}(\omega) \equiv 1$. Then let $Y_{i}(\omega)$ remain 1 for $i=$ $l+1, \cdots, r$ until $X_{r}(\omega) \geq b$ when $Y_{r+1}(\omega) \equiv 0$. Let $Y_{k}(\omega)$ remain 0 for $k \geq r+1$ until $X_{k}(\omega) \leq a$ when $Y_{k}(\omega) \equiv 1$ and continue in this way. Thus the upcrossings of $X_{i}(\omega)$ are identified as unbroken strings of ones with a zero at each end, with the possible exception of the last string of ones which may be missing the zero at the upper end and may or may not be an upcrossing.

Note also that $Y_{0}$ is measurable because it is identically equal to 0 and that if $Y_{k}$ is measurable, then $Y_{k+1}$ is measurable because the only change in going from $k$ to $k+1$ is a change from 0 to 1 or from 1 to 0 on a measurable set determined by $X_{k}$. Now let

$$
Z_{k}(\omega)=\left\{\begin{array}{l}
1 \text { if } Y_{k}(\omega)=1 \text { and } Y_{k+1}(\omega)=0 \\
0 \text { otherwise }
\end{array}\right.
$$

if $k<n$ and

$$
Z_{n}(\omega)=\left\{\begin{array}{l}
1 \text { if } Y_{n}(\omega)=1 \text { and } X_{n}(\omega) \geq b \\
0 \text { otherwise }
\end{array}\right.
$$

Thus $Z_{k}(\omega)=1$ exactly when an upcrossing has been completed and each $Z_{i}$ is a random variable.

$$
U_{[a, b]}(\omega)=\sum_{k=1}^{n} Z_{k}(\omega)
$$

so $U_{[a, b]}$ is a random variable as claimed.
The following corollary collects some key observations found in the above construction.

Corollary 59.10.6 $U_{[a, b]}(\omega) \leq$ the number of unbroken strings of ones in the sequence, $\left\{Y_{k}(\omega)\right\}$ there being at most one unbroken string of ones which produces no upcrossing. Also

$$
\begin{equation*}
Y_{i}(\omega)=\psi_{i}\left(\left\{X_{j}(\omega)\right\}_{j=1}^{i-1}\right) \tag{59.10.16}
\end{equation*}
$$

where $\psi_{i}$ is some function of the past values of $X_{j}(\omega)$.
Lemma 59.10.7 (upcrossing lemma) Let $\left\{X_{i}\right\}_{i=1}^{n}$ be a submartingale and suppose

$$
E\left(\left|X_{n}\right|\right)<\infty .
$$

Then

$$
E\left(U_{[a, b]}\right) \leq \frac{E\left(\left|X_{n}\right|\right)+|a|}{b-a}
$$

Proof: Let $\phi(x) \equiv a+(x-a)^{+}$. Thus $\phi$ is a convex and increasing function.

$$
\begin{gathered}
\phi\left(X_{k+r}\right)-\phi\left(X_{k}\right)=\sum_{i=k+1}^{k+r} \phi\left(X_{i}\right)-\phi\left(X_{i-1}\right) \\
=\sum_{i=k+1}^{k+r}\left(\phi\left(X_{i}\right)-\phi\left(X_{i-1}\right)\right) Y_{i}+\sum_{i=k+1}^{k+r}\left(\phi\left(X_{i}\right)-\phi\left(X_{i-1}\right)\right)\left(1-Y_{i}\right) .
\end{gathered}
$$

The upcrossings of $\phi\left(X_{i}\right)$ are exactly the same as the upcrossings of $X_{i}$ and from Formula 59.10.16,

$$
\begin{gathered}
E\left(\sum_{i=k+1}^{k+r}\left(\phi\left(X_{i}\right)-\phi\left(X_{i-1}\right)\right)\left(1-Y_{i}\right)\right) \\
=\sum_{i=k+1}^{k+r} \int_{\mathbb{R}^{i}}\left(\phi\left(x_{i}\right)-\phi\left(x_{i-1}\right)\right)\left(1-\psi_{i}\left(\left\{x_{j}\right\}_{j=1}^{i-1}\right)\right) d \lambda_{\left(X_{1}, \cdots, X_{i}\right)} \\
=\sum_{i=k+1}^{k+r} \int_{\mathbb{R}^{i-1}} \int_{\mathbb{R}}\left(\phi\left(x_{i}\right)-\phi\left(x_{i-1}\right)\right) \\
\left(1-\psi_{i}\left(\left\{x_{j}\right\}_{j=1}^{i-1}\right)\right) d \lambda_{X_{i} \mid x_{1} \cdots x_{i-1}} d \lambda_{\left(X_{1}, \cdots, X_{i-1}\right)} \\
=\sum_{i=k+1}^{k+r} \int_{\mathbb{R}^{i-1}}\left(1-\psi_{i}\left(\left\{x_{j}\right\}_{j=1}^{i-1}\right)\right) . \\
\int_{\mathbb{R}}\left(\phi\left(x_{i}\right)-\phi\left(x_{i-1}\right)\right) d \lambda_{X_{i} \mid x_{1} \cdots x_{i-1}} d \lambda_{\left(X_{1}, \cdots, X_{i-1}\right)}
\end{gathered}
$$

By Jensen's inequality, Problem 10 of Chapter 15,

$$
\begin{gathered}
\geq \sum_{i=k+1}^{k+r} \int_{\mathbb{R}^{i-1}}\left(1-\psi_{i}\left(\left\{x_{j}\right\}_{j=1}^{i-1}\right)\right) \\
{\left[\phi\left(E\left(X_{i} \mid x_{1}, \cdots, x_{i-1}\right)\right)-\phi\left(x_{i-1}\right)\right] d \lambda_{\left(X_{1}, \cdots, X_{i-1}\right)}}
\end{gathered}
$$

$$
\geq \sum_{i=k+1}^{k+r} \int_{\mathbb{R}^{i-1}}\left(1-\psi_{i}\left(\left\{x_{j}\right\}_{j=1}^{i-1}\right)\right)\left[\phi\left(x_{i-1}\right)-\phi\left(x_{i-1}\right)\right] d \lambda_{\left(X_{1}, \cdots, X_{i-1}\right)}=0
$$

because of the assumption that our sequence of random variables is a submartingale and the observation that $\phi$ is both convex and increasing.

Now let the unbroken strings of ones for $\left\{Y_{i}(\omega)\right\}$ be

$$
\begin{equation*}
\left\{k_{1}, \cdots, k_{1}+r_{1}\right\},\left\{k_{2}, \cdots, k_{2}+r_{2}\right\}, \cdots,\left\{k_{m}, \cdots, k_{m}+r_{m}\right\} \tag{59.10.17}
\end{equation*}
$$

where $m=V(\omega) \equiv$ the number of unbroken strings of ones in the sequence $\left\{Y_{i}(\omega)\right\}$. By Corollary 59.10.6 $V(\omega) \geq U_{[a, b]}(\omega)$.

$$
\begin{gathered}
\phi\left(X_{n}(\omega)\right)-\phi\left(X_{1}(\omega)\right) \\
=\sum_{k=1}^{n}\left(\phi\left(X_{k}(\omega)\right)-\phi\left(X_{k-1}(\omega)\right)\right) Y_{k}(\omega) \\
+\sum_{k=1}^{n}\left(\phi\left(X_{k}(\omega)\right)-\phi\left(X_{k-1}(\omega)\right)\right)\left(1-Y_{k}(\omega)\right)
\end{gathered}
$$

Summing the first sum over the unbroken strings of ones (the terms in which $Y_{i}(\omega)=0$ contribute nothing), implies

$$
\begin{gather*}
\phi\left(X_{n}(\omega)\right)-\phi\left(X_{1}(\omega)\right) \\
\geq U_{[a, b]}(\omega)(b-a)+0+ \\
\sum_{k=1}^{n}\left(\phi\left(X_{k}(\omega)\right)-\phi\left(X_{k-1}(\omega)\right)\right)\left(1-Y_{k}(\omega)\right) \tag{59.10.18}
\end{gather*}
$$

where the zero on the right side results from a string of ones which does not produce an upcrossing. It is here that we use $\phi(x) \geq a$. Such a string begins with $\phi\left(X_{k}(\omega)\right)=a$ and results in an expression of the form $\phi\left(X_{k+m}(\omega)\right)-\phi\left(X_{k}(\omega)\right) \geq 0$ since $\phi\left(X_{k+m}(\omega)\right) \geq a$. If we had not replaced $X_{k}$ with $\phi\left(X_{k}\right)$, it would have been possible for $\phi\left(X_{k+m}(\omega)\right)$ to be less than $a$ and the zero in the above could have been a negative number.

Therefore from Formula 59.10.18,

$$
\begin{aligned}
(b-a) E\left(U_{[a, b]}\right) & \leq E\left(\phi\left(X_{n}\right)-\phi\left(X_{1}\right)\right) \leq E\left(\phi\left(X_{n}\right)-a\right) \\
& =E\left(\left(X_{n}-a\right)^{+}\right) \leq|a|+E\left(\left|X_{n}\right|\right)
\end{aligned}
$$

and this proves the lemma.
Theorem 59.10.8 (submartingale convergence theorem) Let $\left\{X_{i}\right\}_{i=1}^{\infty}$ be a submartingale with $K \equiv \sup \left\{E\left(\left|X_{n}\right|\right): n \geq 1\right\}<\infty$. Then there exists a random variable, $X_{\infty}$, such that $E\left(\left|X_{\infty}\right|\right) \leq K$ and $\lim _{n \rightarrow \infty} X_{n}(\omega)=X_{\infty}(\omega)$ a.e.

Proof: Let $a, b \in \mathbb{Q}$ and let $a<b$. Let $U_{[a, b]}^{n}(\omega)$ be the number of upcrossings of $\left\{X_{i}(\omega)\right\}_{i=1}^{n}$. Then let

$$
U_{[a, b]}(\omega) \equiv \lim _{n \rightarrow \infty} U_{[a, b]}^{n}(\omega)=\text { number of upcrossings of }\left\{X_{i}\right\}
$$

By the upcrossing lemma,

$$
E\left(U_{[a, b]}^{n}\right) \leq \frac{E\left(\left|X_{n}\right|\right)+|a|}{b-a} \leq \frac{K+|a|}{b-a}
$$

and so by the monotone convergence theorem,

$$
E\left(U_{[a, b]}\right) \leq \frac{K+|a|}{b-a}<\infty
$$

which shows $U_{[a, b]}(\omega)$ is finite a.e., for all $\omega \notin S_{[a, b]}$ where $P\left(S_{[a, b]}\right)=0$. Define

$$
S \equiv \cup\left\{S_{[a, b]}: a, b \in \mathbb{Q}, a<b\right\}
$$

Then $P(S)=0$ and if $\omega \notin S,\left\{X_{k}\right\}_{k=1}^{\infty}$ has only finitely many upcrossings of every interval having rational endpoints. Thus, for $\omega \notin S, \limsup _{k \rightarrow \infty} X_{k}(\omega)=\liminf _{k \rightarrow \infty} X_{k}(\omega)=$ $\lim _{k \rightarrow \infty} X_{k}(\omega) \equiv X_{\infty}(\omega)$. Letting $X_{\infty}(\omega)=0$ for $\omega \in S$, Fatou's lemma implies

$$
\int_{\Omega}\left|X_{\infty}\right| d P=\int_{\Omega} \lim \inf _{n \rightarrow \infty}\left|X_{n}\right| d P \leq \lim \inf _{n \rightarrow \infty} \int_{\Omega}\left|X_{n}\right| d P \leq K
$$

and so this proves the theorem.

### 59.11 Characteristic Functions, Independence

There is a way to tell if random vectors are independent by using their characteristic functions.

Proposition 59.11.1 If $\mathbf{X}_{i}$ is a random vector having values in $\mathbb{R}^{p_{i}}$, then the random vectors are independent if and only if

$$
E\left(e^{i P}\right)=\prod_{j=1}^{n} E\left(e^{i \mathbf{t}_{j} \cdot \mathbf{x}_{j}}\right)
$$

where $P \equiv \sum_{j=1}^{n} \mathbf{t}_{j} \cdot \mathbf{X}_{j}$ for $\mathbf{t}_{j} \in \mathbb{R}^{p_{j}}$.
The proof of this proposition will depend on the following lemma.
Lemma 59.11.2 Let $\mathbf{Y}$ be a random vector with values in $\mathbb{R}^{p}$ and let $f$ be bounded and measurable with respect to the Radon measure $\lambda_{\mathbf{Y}}$, and satisfy

$$
\int f(\mathbf{y}) e^{i t \cdot \mathbf{y}} d \lambda_{\mathbf{Y}}=0
$$

for all $\mathbf{t} \in \mathbb{R}^{p}$. Then $f(\mathbf{y})=0$ for $\lambda_{\mathbf{Y}}$ a.e. $\mathbf{y}$.
Proof: You could write the following for $\phi \in \mathscr{G}$

$$
\int \phi(\mathbf{t}) \int f(\mathbf{y}) e^{i t \cdot \mathbf{y}} d \lambda_{\mathbf{Y}} d t=0=\int f(\mathbf{y})\left(\int \phi(\mathbf{t}) e^{i \mathbf{t} \cdot \mathbf{y}} d t\right) d \lambda_{\mathbf{Y}}
$$

and now recall that the inverse Fourier transform maps $\mathscr{G}$ onto $\mathscr{G}$. Hence

$$
\int f(\mathbf{y}) \psi(\mathbf{y}) d \lambda_{\mathbf{Y}}=0
$$

for all $\psi \in \mathscr{G}$. Thus this is also so for every $\psi \in C_{0}^{\infty}\left(\mathbb{R}^{p}\right)$ by an obvious application of the Stone Weierstrass theorem. Let $\left\{\phi_{k}\right\}$ be a sequence of functions in $C_{c}^{\infty}\left(\mathbb{R}^{p}\right)$ which converges to

$$
\operatorname{sgn}(f) \equiv\left\{\begin{array}{l}
\bar{f} /|f| \text { if } f \neq 0 \\
0 \text { if } f=0
\end{array}\right.
$$

pointwise and in $L^{1}\left(\mathbb{R}^{p}, \lambda_{\mathbf{Y}}\right)$, each $\left|\phi_{k}\right| \leq 2$. Then for any $\psi \in C_{0}^{\infty}\left(\mathbb{R}^{p}\right)$,

$$
0=\int f(\mathbf{y}) \phi_{n}(\mathbf{y}) \psi(\mathbf{y}) d \lambda_{\mathbf{Y}} \rightarrow \int|f(\mathbf{y})| \psi(\mathbf{y}) d \lambda_{\mathbf{Y}}
$$

Also, the above holds for any $\psi \in C_{c}\left(\mathbb{R}^{p}\right)$ as can be seen by taking such a $\psi$ and convolving with a mollifier. By the Riesz representation theorem, $f(\mathbf{y})=0 \lambda_{\mathbf{Y}}$ a.e. (The measure $\mu(E) \equiv \int_{E}|f(\mathbf{y})| d \lambda_{\mathbf{Y}}$ equals 0 .)

Proof of the proposition: If the $\mathbf{X}_{j}$ are independent, the formula follows from Lemma 59.9.6 and Lemma 59.9.4.

Now suppose the formula holds. Thus

$$
\begin{gather*}
\prod_{j=1}^{n} E\left(e^{i \mathbf{t}_{j} \cdot \mathbf{x}_{j}}\right)= \\
\int_{\mathbb{R}^{p_{n}}} \cdots \int_{\mathbb{R}^{p_{2}}} \int_{\mathbb{R}^{p_{1}}} e^{i \mathbf{t}_{1} \cdot \mathbf{x}_{1}} e^{i \mathbf{t}_{2} \cdot \mathbf{x}_{2}} \cdots e^{i \mathbf{t}_{n} \cdot \mathbf{x}_{n}} d \lambda_{\mathbf{x}_{1}} d \lambda_{\mathbf{x}_{2}} \cdots d \lambda_{\mathbf{x}_{n}}=E\left(e^{i P}\right) \\
=\int_{\mathbb{R}^{p_{n}}} \cdots \int_{\mathbb{R}^{p_{2}}} \int_{\mathbb{R}^{p_{1}}} e^{i \mathbf{t}_{1} \cdot \mathbf{x}_{1}} e^{i \mathbf{t}_{2} \cdot \mathbf{x}_{2}} \cdots e^{i \mathbf{t}_{n} \cdot \mathbf{x}_{n}} d \lambda \lambda_{\mathbf{x}_{1} \mid \mathbf{x}_{2} \cdots \mathbf{x}_{n}} d \lambda_{\mathbf{x}_{2} \mid \mathbf{x}_{3} \cdots \mathbf{x}_{n}} \cdots d \lambda_{\mathbf{x}_{n}} \tag{59.11.19}
\end{gather*}
$$

Then from the above Lemma 59.11.2, the following equals 0 for $\lambda_{\mathbf{x}_{n}}$ a.e. $\mathbf{x}_{n}$.

$$
\begin{gathered}
\int_{\mathbb{R}^{p_{n-1}}} \cdots \int_{\mathbb{R}^{p_{2}}} \int_{\mathbb{R}^{p_{1}}} e^{i \mathbf{t}_{1} \cdot \mathbf{x}_{1}} e^{i \mathbf{t}_{2} \cdot \mathbf{x}_{2}} \cdots e^{i \mathbf{t}_{n-1} \cdot \mathbf{x}_{n-1}} d \lambda \mathbf{x}_{1} d \lambda \mathbf{x}_{2} \cdots d \lambda \mathbf{x}_{n-1}- \\
\int_{\mathbb{R}^{p_{n-1}}} \cdots \int_{\mathbb{R}^{p_{2}}} \int_{\mathbb{R}^{p_{1}}} e^{i \mathbf{t}_{1} \cdot \mathbf{x}_{1}} e^{i \mathbf{t}_{2} \cdot \mathbf{x}_{2}} \cdots e^{i \mathbf{t}_{n-1} \cdot \mathbf{x}_{n-1}} d \lambda \mathbf{x}_{1} \mid \mathbf{x}_{2} \ldots \mathbf{x}_{n} d \lambda \\
\mathbf{x}_{2} \mid \mathbf{x}_{3} \cdots \mathbf{x}_{n}
\end{gathered} \cdots d \lambda \lambda_{\mathbf{x}_{n-1} \mid \mathbf{x}_{n}} .
$$

Let $\mathbf{t}_{i}=\mathbf{0}$ for $i=1,2, \cdots, n-2$. Then this implies

$$
\int_{\mathbb{R}^{p_{n-1}}} e^{i \mathbf{t}_{n-1} \cdot \mathbf{x}_{n-1}} d \lambda \mathbf{x}_{n-1}=\int_{\mathbb{R}^{p_{n-1}}} e^{i \mathbf{t}_{n-1} \cdot \mathbf{x}_{n-1}} d \lambda_{\mathbf{x}_{n-1} \mid \mathbf{x}_{n}}
$$

By the fact that the characteristic function determines the distribution measure, Theorem 59.8.4, it follows that for these $\mathbf{x}_{n}$ off a set of $\lambda_{\mathbf{x}_{n}}$ measure zero, $\lambda_{\mathbf{x}_{n-1}}=\lambda \mathbf{x}_{n-1} \mid \mathbf{x}_{n}$. Returning to 59.11.19, one can replace $\lambda_{\mathbf{x}_{n-1} \mid \mathbf{x}_{n}}$ with $\lambda_{\mathbf{x}_{n-1}}$ to obtain

$$
\int_{\mathbb{R}^{p_{n}}} \cdots \int_{\mathbb{R}^{p_{2}}} \int_{\mathbb{R}^{p_{1}}} e^{i \mathbf{t}_{1} \cdot \mathbf{x}_{1}} e^{i \mathbf{t}_{2} \cdot \mathbf{x}_{2}} \cdots e^{i \mathbf{t}_{n} \cdot \mathbf{x}_{n}} d \lambda \mathbf{x}_{1} d \lambda \mathbf{x}_{2} \cdots d \lambda \mathbf{x}_{n-1} d \lambda \mathbf{x}_{n}
$$

$$
=\int_{\mathbb{R} p_{n}} \cdots \int_{\mathbb{R}^{p_{2}}} \int_{\mathbb{R}^{p_{1}}} e^{i i_{1} \cdot \mathbf{x}_{1}} e^{i \mathbf{t}_{2} \cdot \mathbf{x}_{2}} \cdots e^{i t_{n} \cdot \mathbf{x}_{n}} d \lambda_{\mathbf{x}_{1} \mid \mathbf{x}_{2} \ldots \mathbf{x}_{n}} d \lambda_{\mathbf{x}_{2} \mid \mathbf{x}_{3} \cdots \mathbf{x}_{n}} \cdots d \lambda_{\mathbf{x}_{n-1}} d \lambda_{\mathbf{x}_{n}}
$$

Next let $\mathbf{t}_{n}=0$ and applying the above Lemma 59.11.2 again, this implies that for $\lambda_{\mathbf{x}_{n-1}}$ a.e. $\mathbf{x}_{n-1}$, the following equals 0 .

$$
\begin{gathered}
\int_{\mathbb{R}^{p_{n-2}}} \cdots \int_{\mathbb{R}^{p_{2}}} \int_{\mathbb{R}^{p_{1}}} e^{i \mathbf{t}_{1} \cdot \mathbf{x}_{1}} e^{i \mathbf{t}_{2} \cdot \mathbf{x}_{2}} \cdots e^{i \mathbf{t}_{n-2} \cdot \mathbf{x}_{n-2}} d \lambda \mathbf{x}_{1} d \lambda \lambda_{\mathbf{x}_{2}} \cdots d \lambda \mathbf{x}_{n-2}- \\
\int_{\mathbb{R}^{p_{n-2}}} \cdots \int_{\mathbb{R}^{p_{2}}} \int_{\mathbb{R}^{p_{1}}} e^{i \mathbf{t}_{1} \cdot \mathbf{x}_{1}} e^{i \mathbf{t}_{2} \cdot \mathbf{x}_{2}} \cdots e^{i \mathbf{t}_{n-2} \cdot \mathbf{x}_{n-2}} d \lambda \lambda_{\mathbf{x}_{1} \mid \mathbf{x}_{2} \ldots \mathbf{x}_{n}} d \lambda \mathbf{x}_{2}\left|\mathbf{x}_{3} \cdots \mathbf{x}_{n} \cdots d \lambda \mathbf{x}_{n-2}\right| \mathbf{x}_{n} \mathbf{x}_{n-1}
\end{gathered}
$$

Let $\mathbf{t}_{i}=\mathbf{0}$ for $i=1,2, \cdots, n-3$. Then you obtain

$$
\int_{\mathbb{R}^{p_{n-2}}} e^{i \mathbf{t}_{n-2} \cdot \mathbf{x}_{n-2}} d \lambda \lambda_{\mathbf{x}_{n-2}}=\int_{\mathbb{R}^{p_{n-2}}} e^{i \mathbf{t}_{n-2} \cdot \mathbf{x}_{n-2}} d \lambda \mathbf{x}_{n-2} \mid \mathbf{x}_{n} \mathbf{x}_{n-1}
$$

and so $\lambda_{\mathbf{x}_{n-2}}=\lambda_{\mathbf{x}_{n-2} \mid \mathbf{x}_{n} \mathbf{x}_{n-1}}$ for $\mathbf{x}_{n-1}$ off a set of $\lambda_{\mathbf{x}_{n-1}}$ measure zero. Continuing this way, it follows that

$$
\lambda_{\mathbf{x}_{n-k}}=\lambda_{\mathbf{x}_{n-k} \mid \mathbf{x}_{n} \mathbf{x}_{n-1} \cdots \mathbf{x}_{n-k+1}}
$$

for $\mathbf{x}_{n-k+1}$ off a set of $\lambda_{\mathbf{x}_{n-k+1}}$ measure zero. Thus if $E$ is Borel in $\mathbb{R}^{p_{n-1}} \times \cdots \times \mathbb{R}^{p_{1}}$,

$$
\begin{gathered}
\int_{\mathbb{R}^{p_{n} \times \cdots \times \mathbb{R}^{p_{1}}}} \mathscr{X}_{E} d \lambda_{\left(\mathbf{x}_{1} \cdots \mathbf{x}_{n}\right)}= \\
\int_{\mathbb{R}^{p_{n}}} \cdots \int_{\mathbb{R}^{p_{2}}} \int_{\mathbb{R}^{p_{1}}} \mathscr{X}_{E} d \lambda_{\mathbf{x}_{1} \mid \mathbf{x}_{2} \cdots \mathbf{x}_{n}} d \lambda_{\mathbf{x}_{2} \mid \mathbf{x}_{3} \cdots \mathbf{x}_{n}} \cdots d \lambda_{\mathbf{x}_{n-1} \mid \mathbf{x}_{n}} d \lambda_{\mathbf{x}_{n}} \\
\int_{\mathbb{R}^{p_{n}}} \cdots \int_{\mathbb{R}^{p_{2}}} \int_{\mathbb{R}^{p_{1}}} \mathscr{X}_{E} d \lambda_{\mathbf{x}_{1} \mid \mathbf{x}_{2} \cdots \mathbf{x}_{n}} d \lambda_{\mathbf{x}_{2} \mid \mathbf{x}_{3} \cdots \mathbf{x}_{n}} \cdots d \lambda_{\mathbf{x}_{n-1}} d \lambda_{\mathbf{x}_{n}} \\
\vdots \\
=\int_{\mathbb{R}^{p_{n}}} \cdots \int_{\mathbb{R}^{p_{2}}} \int_{\mathbb{R}^{p_{1}}} \mathscr{X}_{E} d \lambda_{\mathbf{x}_{1}} d \lambda_{\mathbf{x}_{2}} \cdots d \lambda_{\mathbf{x}_{n}}
\end{gathered}
$$

One could achieve this iterated integral in any order by similar arguments to the above. By Definition 59.9.2 and the discussion which follows, this implies that the random variables $\mathbf{X}_{i}$ are independent.

Here is another proof of the Doob Dynkin lemma based on differentiation theory.
Lemma 59.11.3 Suppose $\mathbf{X}, \mathbf{Y}_{1}, \mathbf{Y}_{2}, \cdots, \mathbf{Y}_{k}$ are random vectors $\mathbf{X}$ having values in $\mathbb{R}^{n}$ and $\mathbf{Y}_{j}$ having values in $\mathbb{R}^{p_{j}}$ and

$$
\mathbf{X}, \mathbf{Y}_{j} \in L^{1}(\Omega)
$$

Suppose $\mathbf{X}$ is $\sigma\left(\mathbf{Y}_{1}, \cdots, \mathbf{Y}_{k}\right)$ measurable. Thus

$$
\left\{\mathbf{X}^{-1}(E): E \text { Borel }\right\} \subseteq\left\{\left(\mathbf{Y}_{1}, \cdots, \mathbf{Y}_{k}\right)^{-1}(F): F \text { is Borel in } \prod_{j=1}^{k} \mathbb{R}^{p_{j}}\right\}
$$

Then there exists a Borel function, $\mathbf{g}: \prod_{j=1}^{k} \mathbb{R}^{p_{j}} \rightarrow \mathbb{R}^{n}$ such that

$$
\mathbf{X}=\mathbf{g}\left(\mathbf{Y}_{1}, \mathbf{Y}_{2}, \cdots, \mathbf{Y}_{k}\right)
$$

Proof: For the sake of brevity, denote by $\mathbf{Y}$ the vector $\left(\mathbf{Y}_{1}, \cdots, \mathbf{Y}_{k}\right)$ and by $\mathbf{y}$ the vector $\left(\mathbf{y}_{1}, \cdots, \mathbf{y}_{k}\right)$ and let $\prod_{j=1}^{k} \mathbb{R}^{p_{j}} \equiv \mathbb{R}^{P}$. For $E$ a Borel set of $\mathbb{R}^{n}$,

$$
\begin{align*}
\int_{\mathbf{Y}^{-1}(E)} \mathbf{X} d P & =\int_{\mathbb{R}^{n} \times \mathbb{R}^{P}} \mathscr{X}_{\mathbb{R}^{n} \times E}(\mathbf{x}, \mathbf{y}) \mathbf{x} d \lambda_{(\mathbf{X}, \mathbf{Y})} \\
& =\int_{E} \int_{\mathbb{R}^{n}} \mathbf{x} d \lambda_{\mathbf{X} \mid \mathbf{y}} d \lambda_{\mathbf{Y}} \tag{59.11.20}
\end{align*}
$$

Consider the function

$$
\mathbf{y} \rightarrow \int_{\mathbb{R}^{n}} \mathbf{x} d \lambda_{\mathbf{x} \mid \mathbf{y}}
$$

Since $d \lambda_{\mathbf{Y}}$ is a Radon measure having inner and outer regularity, it follows the above function is equal to a Borel function for $\lambda_{\mathbf{Y}}$ a.e. $\mathbf{y}$. This function will be denoted by $\mathbf{g}$. Then from 59.11.20

$$
\begin{aligned}
\int_{\mathbf{Y}^{-1}(E)} \mathbf{X} d P & =\int_{E} \mathbf{g}(\mathbf{y}) d \lambda_{\mathbf{Y}}=\int_{\mathbb{R}^{P}} \mathscr{X}_{E}(\mathbf{y}) \mathbf{g}(\mathbf{y}) d \lambda_{\mathbf{Y}} \\
& =\int_{\Omega} \mathscr{X}_{E}(\mathbf{Y}(\omega)) \mathbf{g}(\mathbf{Y}(\omega)) d P \\
& =\int_{\mathbf{Y}^{-1}(E)} \mathbf{g}(\mathbf{Y}(\omega)) d P
\end{aligned}
$$

and since $\mathbf{Y}^{-1}(E)$ is an arbitrary element of $\sigma(\mathbf{Y})$, this shows that since $\mathbf{X}$ is $\sigma(\mathbf{Y})$ measurable,

$$
\mathbf{X}=\mathbf{g}(\mathbf{Y}) P \text { a.e. }
$$

What about the case where $\mathbf{X}$ is not necessarily measurable in $\sigma\left(\mathbf{Y}_{1}, \cdots, \mathbf{Y}_{k}\right)$ ?
Lemma 59.11.4 There exists a unique function $\mathbf{Z}(\omega)$ which satisfies

$$
\int_{F} \mathbf{X}(\omega) d P=\int_{F} \mathbf{Z}(\omega) d P
$$

for all $F \in \sigma\left(\mathbf{Y}_{1}, \cdots, \mathbf{Y}_{k}\right)$ such that $\mathbf{Z}$ is $\sigma\left(\mathbf{Y}_{1}, \cdots, \mathbf{Y}_{k}\right)$ measurable. It is denoted by

$$
E\left(\mathbf{X} \mid \sigma\left(\mathbf{Y}_{1}, \cdots, \mathbf{Y}_{k}\right)\right)
$$

Proof: It is like the above. Letting $E$ be a Borel set in $\mathbb{R}^{p}$,

$$
\begin{aligned}
\int_{\mathbf{Y}^{-1}(E)} \mathbf{X} d P & =\int_{\mathbb{R}^{n} \times \mathbb{R}^{P}} \mathscr{X}_{\mathbb{R}^{n} \times E}(\mathbf{x}, \mathbf{y}) \mathbf{x} d \lambda_{(\mathbf{X}, \mathbf{Y})} \\
& =\int_{E} \int_{\mathbb{R}^{n}} \mathbf{x} d \lambda_{\mathbf{X} \mid \mathbf{y}} d \lambda_{\mathbf{Y}}
\end{aligned}
$$

Now let $\mathbf{g}(\mathbf{y}) \equiv E\left(\mathbf{X} \mid \mathbf{y}_{1}, \cdots, \mathbf{y}_{k}\right)$ be a Borel representative of

$$
\int_{\mathbb{R}^{n}} \mathbf{x} d \lambda_{\mathbf{x} \mid \mathbf{y}}
$$

It follows $\omega \rightarrow \mathbf{g}(\mathbf{Y}(\omega))=E\left(\mathbf{X} \mid \mathbf{Y}_{1}(\omega), \cdots, \mathbf{Y}_{k}(\omega)\right)$ is $\sigma\left(\mathbf{Y}_{1}, \cdots, \mathbf{Y}_{k}\right)$ measurable because by definition $\omega \rightarrow \mathbf{Y}(\omega)$ is $\sigma\left(\mathbf{Y}_{1}, \cdots, \mathbf{Y}_{k}\right)$ measurable and a Borel measurable function composed with a measurable one is still measurable. It follows that for all $E$ Borel in $\mathbb{R}^{p}$,

$$
\begin{aligned}
& \int_{\mathbf{Y}^{-1}(E)} \mathbf{X} d P=\int_{E} E\left(\mathbf{X} \mid \mathbf{y}_{1}, \cdots, \mathbf{y}_{k}\right) d \lambda \mathbf{Y} \\
& =\int_{\mathbf{Y}^{-1}(E)} E\left(\mathbf{X} \mid \mathbf{Y}_{1}(\omega), \cdots, \mathbf{Y}_{k}(\omega)\right) d P
\end{aligned}
$$

and so $\mathbf{Z}(\omega)=E\left(\mathbf{X} \mid \mathbf{Y}_{1}(\omega), \cdots, \mathbf{Y}_{k}(\omega)\right)$ works because a generic set of $\sigma\left(\mathbf{Y}_{1}, \cdots, \mathbf{Y}_{k}\right)$ is $\mathbf{Y}^{-1}(E)$ for $E$ a Borel set in $\mathbb{R}^{p}$. If both $\mathbf{Z}, \mathbf{Z}_{1}$ work, then for all $F \in \sigma\left(\mathbf{Y}_{1}, \cdots, \mathbf{Y}_{k}\right)$,

$$
\int_{F}\left(\mathbf{Z}-\mathbf{Z}_{1}\right) d P=0
$$

Since $F$ is arbitrary, some routine computations show $\mathbf{Z}=\mathbf{Z}_{1}$ a.e.
Observation 59.11.5 Note that a.e.

$$
E\left(\mathbf{X} \mid \mathbf{Y}_{1}(\omega), \cdots, \mathbf{Y}_{k}(\omega)\right)=E\left(\mathbf{X} \mid \sigma\left(\mathbf{Y}_{1}, \cdots, \mathbf{Y}_{k}\right)\right)
$$

where the one on the left is the expected value of $\mathbf{X}$ given values of $\mathbf{Y}_{j}(\omega)$. This one corresponds to the sort of thing we say in words. The one on the right is an abstract concept which is usually obtained using the Radon Nikodym theorem and its description is given in the lemma. This lemma shows that its meaning is really to take the expected value of $\mathbf{X}$ given values for the $\mathbf{Y}_{k}$.

### 59.12 Characteristic Functions For Measures

Recall the characteristic function for a random variable having values in $\mathbb{R}^{n}$. I will give a review of this to begin with. Then the concept will be generalized to random variables (vectors) which have values in a real separable Banach space.

Definition 59.12.1 Let $\mathbf{X}$ be a random variable. The characteristic function is

$$
\phi_{\mathbf{X}}(\mathbf{t}) \equiv E\left(e^{i \mathbf{t} \cdot \mathbf{X}}\right) \equiv \int_{\Omega} e^{i \mathbf{t} \cdot \mathbf{X}(\omega)} d P=\int_{\mathbb{R}^{p}} e^{i \mathbf{t} \cdot \mathbf{x}} d \lambda_{\mathbf{X}}
$$

the last equation holding by Proposition 59.1.12 on Page 1858.
Recall the following fundamental lemma and definition, Lemma 32.3.4 on Page 1101.
Definition 59.12.2 For $T \in \mathscr{G}^{*}$, define $F T, F^{-1} T \in \mathscr{G}^{*}$ by

$$
F T(\phi) \equiv T(F \phi), F^{-1} T(\phi) \equiv T\left(F^{-1} \phi\right)
$$

Lemma 59.12.3 $F$ and $F^{-1}$ are both one to one, onto, and are inverses of each other.
The main result on characteristic functions is the following in Theorem 59.8.4 on Page 1880 which is stated here for convenience.

Theorem 59.12.4 Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors with values in $\mathbb{R}^{p}$ and suppose $E\left(e^{i t \cdot \mathbf{X}}\right)$ $=E\left(e^{i \mathbf{t} \cdot \mathbf{Y}}\right)$ for all $\mathbf{t} \in \mathbb{R}^{p}$. Then $\lambda_{\mathbf{X}}=\lambda_{\mathbf{Y}}$.

I want to do something similar for random variables which have values in a separable real Banach space, $E$ instead of $\mathbb{R}^{p}$.

Corollary 59.12.5 Let $\mathscr{K}$ be a $\pi$ system of subsets of $\Omega$ and suppose two probability measures, $\mu$ and $v$ defined on $\sigma(\mathscr{K})$ are equal on $\mathscr{K}$. Then $\mu=v$.

Proof: This follows from the Lemma 12.12.3 on Page 329. Let

$$
\mathscr{G} \equiv\{E \in \sigma(\mathscr{K}): \mu(E)=v(E)\}
$$

Then $\mathscr{K} \subseteq \mathscr{G}$, since $\mu$ and $v$ are both probability measures, it follows that if $E \in \mathscr{G}$, then so is $E^{C}$. Since these are measures, if $\left\{A_{i}\right\}$ is a sequence of disjoint sets from $\mathscr{G}$ then

$$
\mu\left(\cup_{i=1}^{\infty} A_{i}\right)=\sum_{i} \mu\left(A_{i}\right)=\sum_{i} v\left(A_{i}\right)=v\left(\cup_{i=1}^{\infty} A\right)
$$

and so from Lemma 12.12.3, $\mathscr{G}=\sigma(\mathscr{K})$.
Next recall the following fundamental lemma used to prove Pettis' theorem. It is proved on Page 645 but is stated here for convenience.

Lemma 59.12.6 If $E$ is a separable Banach space with $B^{\prime}$ the closed unit ball in $E^{\prime}$, then there exists a sequence $\left\{f_{n}\right\}_{n=1}^{\infty} \equiv D^{\prime} \subseteq B^{\prime}$ with the property that for every $x \in E$,

$$
\|x\|=\sup _{f \in D^{\prime}}|f(x)|
$$

Definition 59.12.7 Let E be a separable real Banach space. A cylindrical set is one which is of the form

$$
\left\{x \in E: x_{i}^{*}(x) \in \Gamma_{i}, i=1,2, \cdots, m\right\}
$$

where here $x_{i}^{*} \in E^{\prime}$ and $\Gamma_{i}$ is a Borel set in $\mathbb{R}$.
It is obvious that $\emptyset$ is a cylindrical set and that the intersection of two cylindrical sets is another cylindrical set. Thus the cylindrical sets form a $\pi$ system. What is the smallest $\sigma$ algebra containing the cylindrical sets? It is the Borel sets of $E$. This is a special case of Lemma 59.4.2. Recall why this was. Letting $\left\{f_{n}\right\}_{n=1}^{\infty}=D^{\prime}$ be the sequence of Lemma 59.12.6 it follows that

$$
\begin{aligned}
& \{x \in E:| | x-a \| \leq \delta\} \\
= & \left\{x \in E: \sup _{f \in D^{\prime}}|f(x-a)| \leq \delta\right\} \\
= & \left\{x \in E: \sup _{f \in D^{\prime}}|f(x)-f(a)| \leq \delta\right\} \\
= & \cap_{n=1}^{\infty}\left\{x \in E: f_{n}(x) \in \overline{B\left(f_{n}(a), \delta\right)}\right\}
\end{aligned}
$$

which yields a countable intersection of cylindrical sets. It follows the smallest $\sigma$ algebra containing the cylindrical sets contains the closed balls and hence the open balls and consequently the open sets and so it contains the Borel sets. However, each cylindrical set is a Borel set and so in fact this $\sigma$ algebra equals $\mathscr{B}(E)$.

From Corollary 59.12.5 it follows that two probability measures which are equal on the cylindrical sets are equal on the Borel sets $\mathscr{B}(E)$.

Definition 59.12.8 Let $\mu$ be a probability measure on $\mathscr{B}(E)$ where $E$ is a real separable Banach space. Then for $x^{*} \in E^{\prime}$,

$$
\phi_{\mu}\left(x^{*}\right) \equiv \int_{E} e^{i x^{*}(x)} d \mu(x)
$$

$\phi_{\mu}$ is called the characteristic function for the measure $\mu$.
Note this is a little different than earlier when the symbol $\phi_{X}(\mathbf{t})$ was used and $X$ was a random variable. Here the focus is more on the measure than a random variable, $X$ such that $\mathscr{L}(X)=\mu$. It might appear this is a more general concept but in fact this is not the case. You could just consider the separable Banach space or Polish space with the Borel $\sigma$ algebra as your probabililty space and then consider the identity map as a random variable having the given measure as a distribution measure. Of course a major result is the one which says that the characteristic function determines the measures.

Theorem 59.12.9 Let $\mu$ and $v$ be two probability measures on $\mathscr{B}(E)$ where $E$ is a separable real Banach space. Suppose

$$
\phi_{\mu}\left(x^{*}\right)=\phi_{v}\left(x^{*}\right)
$$

for all $x^{*} \in E^{\prime}$. Then $\mu=v$.
Proof: It suffices to verify that $\mu(A)=v(A)$ for all $A \in \mathscr{K}$ where $\mathscr{K}$ is the set of cylindrical sets. Fix $\mathbf{g}_{n} \in\left(E^{\prime}\right)^{n}$. Thus the two measures are equal if for all such $\mathbf{g}_{n}, n \in \mathbb{N}$,

$$
\mu\left(\mathbf{g}_{n}^{-1}(B)\right)=v\left(\mathbf{g}_{n}^{-1}(B)\right)
$$

for $B$ a Borel set in $\mathbb{R}^{n}$. Of course, for such a choice of $\mathbf{g}_{n} \in\left(E^{\prime}\right)^{n}$, there are measures defined on the Borel sets of $\mathbb{R}^{n} \mu_{n}$ and $v_{n}$ which are given by

$$
\mu_{n}(B) \equiv \mu\left(\mathbf{g}_{n}^{-1}(B)\right), v_{n}(B) \equiv v\left(\mathbf{g}_{n}^{-1}(B)\right)
$$

and so it suffices to verify that these two measures are equal. So what are their characteristic functions? Note that $\mathbf{g}_{n}$ is a random variable taking $E$ to $\mathbb{R}^{n}$ and $\mu_{n}, v_{n}$ are just the probability distribution measures of this random variable. Therefore,

$$
\phi_{\mu_{n}}(\mathbf{t}) \equiv \int_{\mathbb{R}^{n}} e^{i \mathbf{t} \cdot \mathbf{s}} d \mu_{n}=\int_{E} e^{i \mathbf{t} \cdot \mathbf{g}_{n}(x)} d \mu
$$

Similarly,

$$
\phi_{v_{n}}(\mathbf{t}) \equiv \int_{\mathbb{R}^{n}} e^{i \mathbf{t} \cdot \mathbf{s}} d v_{n}=\int_{E} e^{i \mathbf{t} \cdot \mathbf{g}_{n}(x)} d v
$$

Now $\mathbf{t} \cdot \mathbf{g}_{n} \in E^{\prime}$ and so by assumption, the two ends of the above are equal. Hence $\phi_{\mu_{n}}(\mathbf{t})=$ $\phi_{v_{n}}(\mathbf{t})$ and so by Theorem 59.8.6, $\mu_{n}=v_{n}$ which, as shown above, implies $\mu=v$.

### 59.13 Characteristic Functions In Banach Space

I will consider the relation between the characteristic function and independence of random variables having values in a Banach space. Recall an earlier proposition which relates independence of random vectors with characteristic functions. It is proved starting on Page 1889.

Proposition 59.13.1 Let $\left\{\mathbf{X}_{k}\right\}_{k=1}^{n}$ be random vectors such that $\mathbf{X}_{k}$ has values in $\mathbb{R}^{p_{k}}$. Then the random vectors are independent if and only if

$$
E\left(e^{i P}\right)=\prod_{j=1}^{n} E\left(e^{i \boldsymbol{i t}_{j} \cdot \mathbf{x}_{j}}\right)
$$

where $P \equiv \sum_{j=1}^{n} \mathbf{t}_{j} \cdot \mathbf{X}_{j}$ for $\mathbf{t}_{j} \in \mathbb{R}^{p_{j}}$.
It turns out there is a generalization of the above proposition to the case where the random variables have values in a real separable Banach space. Before proving this recall an earlier theorem which had to do with reducing to the case where the random variables had values in $\mathbb{R}^{n}$, Theorem 59.5.1. It is restated here for convenience.

Theorem 59.13.2 The random variables $\left\{X_{i}\right\}_{i \in I}$ are independent if whenever

$$
\left\{i_{1}, \cdots, i_{n}\right\} \subseteq I
$$

$m_{i_{1}}, \cdots, m_{i_{n}}$ are positive integers, and $\mathbf{g}_{m_{i_{1}}}, \cdots, \mathbf{g}_{m_{i_{n}}}$ are in

$$
\left(E^{\prime}\right)^{m_{i_{1}}}, \cdots,\left(E^{\prime}\right)^{m_{i_{n}}}
$$

respectively, $\left\{\mathbf{g}_{m_{i_{j}}} \circ X_{i_{j}}\right\}_{j=1}^{n}$ are independent random vectors having values in

$$
\mathbb{R}^{m_{i_{1}}}, \cdots, \mathbb{R}^{m_{i_{n}}}
$$

respectively.
Now here is the theorem about independence and the characteristic functions.
Theorem 59.13.3 Let $\left\{X_{k}\right\}_{k=1}^{n}$ be random variables such that $X_{k}$ has values in $E_{k}$, a real separable Banach space. Then the random variables are independent if and only if

$$
E\left(e^{i P}\right)=\prod_{j=1}^{n} E\left(e^{i t_{j}^{*}\left(X_{j}\right)}\right)
$$

where $P \equiv \sum_{j=1}^{n} t_{j}^{*}\left(X_{j}\right)$ for $t_{j}^{*} \in E_{j}^{\prime}$.
Proof: If the random variables are independent, then so are the random variables, $t_{j}^{*}\left(X_{j}\right)$ and so the equation follows.

The interesting case is when the equation holds.

It suffices to consider only the case where each $E_{k}=E$. This is because you can consider each $X_{j}$ to have values in $\prod_{k=1}^{n} E_{k}$ by letting $X_{j}$ take its values in the $j^{\text {th }}$ component of the product and 0 in the other components. Can you draw the conclusion the random variables are independent? By Theorem 59.5.1, it suffices to show the random variables $\left\{\mathbf{g}_{m_{k}} \circ X_{k}\right\}_{k=1}^{n}$ are independent where $\mathbf{g}_{m_{k}}=\left(x_{1}^{*}, \cdots, x_{m_{k}}^{*}\right) \in\left(E^{\prime}\right)^{m_{k}}$. This happens if whenever $\mathbf{t}_{m_{k}} \in \mathbb{R}^{m_{k}}$ and

$$
P=\sum_{k=1}^{n} \mathbf{t}_{m_{k}} \cdot\left(\mathbf{g}_{m_{k}} \circ X_{k}\right)
$$

it follows

$$
\begin{equation*}
E\left(e^{i P}\right)=\prod_{k=1}^{n} E\left(e^{i \mathbf{t}_{m_{k}} \cdot\left(\mathbf{g}_{m_{k}} \circ X_{k}\right)}\right) \tag{59.13.21}
\end{equation*}
$$

However, the expression on the right in 59.13.21 equals

$$
\prod_{k=1}^{n} E\left(e^{i\left(\mathbf{t}_{m_{k}} \cdot \mathbf{g}_{m_{k}}\right) \circ X_{k}}\right)
$$

and $\mathbf{t}_{m_{k}} \cdot \mathbf{g}_{m_{k}} \equiv \sum_{j=1}^{m_{k}} t_{j} x_{j}^{*} \in E^{\prime}$. Also the expression on the left equals $E\left(e^{i \sum_{k=1}^{n} \mathbf{t}_{m_{k}} \cdot \mathbf{g}_{m_{k}} \circ X_{k}}\right)$ Therefore, by assumption, 59.13.21 holds.

There is an obvious corollary which is useful.
Corollary 59.13.4 Let $\left\{X_{k}\right\}_{k=1}^{n}$ be random variables such that $X_{k}$ has values in $E_{k}$, a real separable Banach space. Then the random variables are independent if and only if

$$
E\left(e^{i P}\right)=\prod_{j=1}^{n} E\left(e^{i i_{j}^{*}\left(X_{j}\right)}\right)
$$

where $P \equiv \sum_{j=1}^{n} t_{j}^{*}\left(X_{j}\right)$ for $t_{j}^{*} \in M_{j}$ where $M_{j}$ is a dense subset of $E_{j}^{\prime}$.
Proof: The easy direction follows from Theorem 59.13.3. Suppose then the above equation holds for all $t_{j}^{*} \in M_{j}$. Then let $t_{j}^{*} \in E^{\prime}$ and let $\left\{t_{n j}^{*}\right\}$ be a sequence in $M_{j}$ such that

$$
\lim _{n \rightarrow \infty} t_{n j}^{*}=t_{j}^{*} \text { in } E^{\prime}
$$

Then define

$$
P \equiv \sum_{j=1}^{n} t_{j}^{*} X_{j}, P_{n} \equiv \sum_{j=1}^{n} t_{n j}^{*} X_{j}
$$

It follows

$$
\begin{aligned}
E\left(e^{i P}\right) & =\lim _{n \rightarrow \infty} E\left(e^{i P_{n}}\right) \\
& =\lim _{n \rightarrow \infty} \prod_{j=1}^{n} E\left(e^{i t_{n j}^{*}\left(X_{j}\right)}\right) \\
& =\prod_{j=1}^{n} E\left(e^{i t_{j}^{*}\left(X_{j}\right)}\right) \square
\end{aligned}
$$

### 59.14 Convolution And Sums

Lemma 59.1.9 on Page 1857 makes possible a definition of convolution of two probability measures defined on $\mathscr{B}(E)$ where $E$ is a separable Banach space as well as some other interesting theorems which held earlier in the context of locally compact spaces. I will first show a little theorem about density of continuous functions in $L^{p}(E)$ and then define the convolution of two finite measures. First here is a simple technical lemma.

Lemma 59.14.1 Suppose $K$ is a compact subset of $U$ an open set in $E$ a metric space. Then there exists $\delta>0$ such that

$$
\operatorname{dist}(x, K)+\operatorname{dist}\left(x, U^{C}\right) \geq \delta \text { for all } x \in E
$$

Proof: For each $x \in K$, there exists a ball, $B\left(x, \delta_{x}\right)$ such that $B\left(x, 3 \delta_{x}\right) \subseteq U$. Finitely many of these balls cover $K$ because $K$ is compact, say $\left\{B\left(x_{i}, \delta_{x_{i}}\right)\right\}_{i=1}^{m}$. Let

$$
0<\delta<\min \left(\delta_{x_{i}}: i=1,2, \cdots, m\right)
$$

Now pick any $x \in K$. Then $x \in B\left(x_{i}, \delta_{x_{i}}\right)$ for some $x_{i}$ and so $B(x, \delta) \subseteq B\left(x_{i}, 2 \delta_{x_{i}}\right) \subseteq$ $U$. Therefore, for any $x \in K$, $\operatorname{dist}\left(x, U^{C}\right) \geq \delta$. If $x \in B\left(x_{i}, 2 \delta_{x_{i}}\right)$ for some $x_{i}$, it follows dist $\left(x, U^{C}\right) \geq \delta$ because then $B(x, \delta) \subseteq B\left(x_{i}, 3 \delta_{x_{i}}\right) \subseteq U$. If $x \notin B\left(x_{i}, 2 \delta_{x_{i}}\right)$ for any of the $x_{i}$, then $x \notin B(y, \delta)$ for any $y \in K$ because all these sets are contained in some $B\left(x_{i}, 2 \delta_{x_{i}}\right)$. Consequently dist $(x, K) \geq \delta$. This proves the lemma.

From this lemma, there is an easy corollary.
Corollary 59.14.2 Suppose $K$ is a compact subset of $U$, an open set in $E$ a metric space. Then there exists a uniformly continuous function $f$ defined on all of $E$, having values in $[0,1]$ such that $f(x)=0$ if $x \notin U$ and $f(x)=1$ if $x \in K$.

Proof: Consider

$$
f(x) \equiv \frac{\operatorname{dist}\left(x, U^{C}\right)}{\operatorname{dist}\left(x, U^{C}\right)+\operatorname{dist}(x, K)}
$$

Then some algebra yields

$$
\begin{gathered}
\left|f(x)-f\left(x^{\prime}\right)\right| \leq \\
\frac{1}{\delta}\left(\left|\operatorname{dist}\left(x, U^{C}\right)-\operatorname{dist}\left(x^{\prime}, U^{C}\right)\right|+\left|\operatorname{dist}(x, K)-\operatorname{dist}\left(x^{\prime}, K\right)\right|\right)
\end{gathered}
$$

where $\delta$ is the constant of Lemma 59.14.1. Now it is a general fact that

$$
\left|\operatorname{dist}(x, S)-\operatorname{dist}\left(x^{\prime}, S\right)\right| \leq d\left(x, x^{\prime}\right)
$$

Therefore,

$$
\left|f(x)-f\left(x^{\prime}\right)\right| \leq \frac{2}{\delta} d\left(x, x^{\prime}\right)
$$

and this proves the corollary.
Now suppose $\mu$ is a finite measure defined on the Borel sets of a separable Banach space, $E$. It was shown above that $\mu$ is inner and outer regular. Lemma 59.1.9 on Page 1857 shows that $\mu$ is inner regular in the usual sense with respect to compact sets. This makes possible the following theorem.

Theorem 59.14.3 Let $\mu$ be a finite measure on $\mathscr{B}(E)$ where $E$ is a separable Banach space and let $f \in L^{p}(E ; \mu)$. Then for any $\varepsilon>0$, there exists a uniformly continuous, bounded $g$ defined on $E$ such that

$$
\|f-g\|_{L^{p}(E)}<\varepsilon .
$$

Proof: As usual in such situations, it suffices to consider only $f \geq 0$. Then by Theorem 11.3.9 on Page 241 and an application of the monotone convergence theorem, there exists a simple measurable function,

$$
s(x) \equiv \sum_{k=1}^{m} c_{k} \mathscr{X}_{A_{k}}(x)
$$

such that $\|f-s\|_{L^{p}(E)}<\varepsilon / 2$. Now by regularity of $\mu$ there exist compact sets, $K_{k}$ and open sets, $V_{k}$ such that $2 \sum_{k=1}^{m}\left|c_{k}\right| \mu\left(V_{k} \backslash K\right)^{1 / p}<\varepsilon / 2$ and by Corollary 59.14.2 there exist uniformly continuous functions $g_{k}$ having values in $[0,1]$ such that $g_{k}=1$ on $K_{k}$ and 0 on $V_{k}^{C}$. Then consider

$$
g(x)=\sum_{k=1}^{m} c_{k} g_{k}(x) .
$$

This function is bounded and uniformly continuous. Furthermore,

$$
\begin{aligned}
\|s-g\|_{L^{p}(E)} & \leq\left(\int_{E}\left|\sum_{k=1}^{m} c_{k} \mathscr{X}_{A_{k}}(x)-\sum_{k=1}^{m} c_{k} g_{k}(x)\right|^{p} d \mu\right)^{1 / p} \\
& \leq\left(\int_{E}\left(\sum_{k=1}^{m}\left|c_{k}\right|\left|\mathscr{X}_{A_{k}}(x)-g_{k}(x)\right|\right)^{p}\right)^{1 / p} \\
& \leq \sum_{k=1}^{m}\left|c_{k}\right|\left(\int_{E}\left|\mathscr{X}_{A_{k}}(x)-g_{k}(x)\right|^{p} d \mu\right)^{1 / p} \\
& \leq \sum_{k=1}^{m}\left|c_{k}\right|\left(\int_{V_{k} \backslash K_{k}} 2^{p} d \mu\right)^{1 / p} \\
& =2 \sum_{k=1}^{m}\left|c_{k}\right| \mu\left(V_{k} \backslash K\right)^{1 / p}<\varepsilon / 2
\end{aligned}
$$

Therefore,

$$
\|f-g\|_{L^{p}} \leq\|f-s\|_{L^{p}}+\|s-g\|_{L^{p}}<\varepsilon / 2+\varepsilon / 2 .
$$

This proves the theorem.
Lemma 59.14.4 Let $A \in \mathscr{B}(E)$ where $\mu$ is a finite measure on $\mathscr{B}(E)$ for $E$ a separable Banach space. Also let $x_{i} \in E$ for $i=1,2, \cdots, m$. Then for $\mathbf{x} \in E^{m}$,

$$
\mathbf{x} \rightarrow \mu\left(A+\sum_{i=1}^{m} x_{i}\right), \mathbf{x} \rightarrow \mu\left(A-\sum_{i=1}^{m} x_{i}\right)
$$

are Borel measurable functions. Furthermore, the above functions are

$$
\mathscr{B}(E) \times \cdots \times \mathscr{B}(E)
$$

measurable where the above denotes the product measurable sets as described in Theorem 12.12.6 on Page 332.

Proof: First consider the case where $A=U$, an open set. Let

$$
\begin{equation*}
\mathbf{y} \in\left\{\mathbf{x} \in E^{m}: \mu\left(U+\sum_{i=1}^{m} x_{i}\right)>\alpha\right\} \tag{59.14.22}
\end{equation*}
$$

Then from Lemma 59.1.9 on Page 1857 there exists a compact set, $K \subseteq U+\sum_{i=1}^{m} y_{i}$ such that $\mu(K)>\alpha$. Then if $\mathbf{y}^{\prime}$ is close enough to $\mathbf{y}$, it follows $K \subseteq U+\sum_{i=1}^{m} y_{i}^{\prime}$ also. Therefore, for all $\mathbf{y}^{\prime}$ close enough to $\mathbf{y}$,

$$
\mu\left(U+\sum_{i=1}^{m} y_{i}^{\prime}\right) \geq \mu(K)>\alpha
$$

In other words the set described in 59.14 .22 is an open set and so $\mathbf{y} \rightarrow \mu\left(U+\sum_{i=1}^{m} y_{i}\right)$ is Borel measurable whenever $U$ is an open set in $E$.

Define a $\pi$ system, $\mathscr{K}$ to consist of all open sets in $E$. Then define $\mathscr{G}$ as

$$
\left\{A \in \sigma(\mathscr{K})=\mathscr{B}(E): \mathbf{y} \rightarrow \mu\left(A+\sum_{i=1}^{m} y_{i}\right) \text { is Borel measurable }\right\}
$$

I just showed $\mathscr{G} \supseteq \mathscr{K}$. Now suppose $A \in \mathscr{G}$. Then

$$
\mu\left(A^{C}+\sum_{i=1}^{m} y_{i}\right)=\mu(E)-\mu\left(A+\sum_{i=1}^{m} y_{i}\right)
$$

and so $A^{C} \in \mathscr{G}$ whenever $A \in \mathscr{G}$. Next suppose $\left\{A_{i}\right\}$ is a sequence of disjoint sets of $\mathscr{G}$. Then

$$
\begin{aligned}
\mu\left(\left(\cup_{i=1}^{\infty} A_{i}\right)+\sum_{j=1}^{m} y_{j}\right) & =\mu\left(\cup_{i=1}^{\infty}\left(A_{i}+\sum_{j=1}^{m} y_{j}\right)\right) \\
& =\sum_{i=1}^{\infty} \mu\left(A_{i}+\sum_{j=1}^{m} y_{j}\right)
\end{aligned}
$$

and so $\cup_{i=1}^{\infty} A_{i} \in \mathscr{G}$ because the above is the sum of Borel measurable functions. By the lemma on $\pi$ systems, Lemma 12.12.3 on Page 329, it follows $\mathscr{G}=\sigma(\mathscr{K})=\mathscr{B}(E)$. Similarly, $\mathbf{x} \rightarrow \mu\left(A-\sum_{j=1}^{m} x_{j}\right)$ is also Borel measurable whenever $A \in \mathscr{B}(E)$. Finally note that

$$
\mathscr{B}(E) \times \cdots \times \mathscr{B}(E)
$$

contains the open sets of $E^{m}$ because the separability of $E$ implies the existence of a countable basis for the topology of $E^{m}$ consisting of sets of the form $\prod_{i=1}^{m} U_{i}$ where the $U_{i}$ come from a countable basis for $E$. Since every open set is the countable union of sets like the above, each being a measurable box, the open sets are contained in

$$
\mathscr{B}(E) \times \cdots \times \mathscr{B}(E)
$$

which implies $\mathscr{B}\left(E^{m}\right) \subseteq \mathscr{B}(E) \times \cdots \times \mathscr{B}(E)$ also. This proves the lemma.
With this lemma, it is possible to define the convolution of two finite measures.
Definition 59.14.5 Let $\mu$ and $v$ be two finite measures on $\mathscr{B}(E)$, for $E$ a separable Banach space. Then define a new measure, $\mu * v$ on $\mathscr{B}(E)$ as follows

$$
\mu * v(A) \equiv \int_{E} v(A-x) d \mu(x)
$$

This is well defined because of Lemma 59.14.4 which says that $x \rightarrow v(A-x)$ is Borel measurable.

Here is an interesting theorem about convolutions. However, first here is a little lemma. The following picture is descriptive of the set described in the following lemma.


Lemma 59.14.6 For A a Borel set in E, a separable Banach space, define

$$
S_{A} \equiv\{(x, y) \in E \times E: x+y \in A\}
$$

Then $S_{A} \in \mathscr{B}(E) \times \mathscr{B}(E)$, the $\sigma$ algebra of product measurable sets, the smallest $\sigma$ algebra which contains all the sets of the form $A \times B$ where $A$ and $B$ are Borel.

Proof: Let $\mathscr{K}$ denote the open sets in $E$. Then $\mathscr{K}$ is a $\pi$ system. Let

$$
\mathscr{G} \equiv\left\{A \in \sigma(\mathscr{K})=\mathscr{B}(E): S_{A} \in \mathscr{B}(E) \times \mathscr{B}(E)\right\}
$$

Then $\mathscr{K} \subseteq \mathscr{G}$ because if $U \in \mathscr{K}$ then $S_{U}$ is an open set in $E \times E$ and all open sets are in $\mathscr{B}(E) \times \mathscr{B}(E)$ because a countable basis for the topology of $E \times E$ are sets of the form $B \times C$ where $B$ and $C$ come from a countable basis for $E$. Therefore, $\mathscr{K} \subseteq \mathscr{G}$. Now let
$A \in \mathscr{G}$. For $(x, y) \in E \times E$, either $x+y \in A$ or $x+y \notin A$. Hence $E \times E=S_{A} \cup S_{A C}$ which shows that if $A \in \mathscr{G}$ then so is $A^{C}$. Finally if $\left\{A_{i}\right\}$ is a sequence of disjoint sets of $\mathscr{G}$

$$
S_{\cup_{i=1}^{\infty} A_{i}}=\cup_{i=1}^{\infty} S_{A_{i}}
$$

and this shows that $\mathscr{G}$ is also closed with respect to countable unions of disjoint sets. Therefore, by the lemma on $\pi$ systems, Lemma 12.12 .3 on Page 329 it follows $\mathscr{G}=\sigma(\mathscr{K})=$ $\mathscr{B}(E)$. This proves the lemma.

Theorem 59.14.7 Let $\mu, v$, and $\lambda$ be finite measures on $\mathscr{B}(E)$ for $E$ a separable Banach space. Then

$$
\begin{align*}
\mu * v & =v * \mu  \tag{59.14.23}\\
(\mu * v) * \lambda & =\mu *(v * \lambda) \tag{59.14.24}
\end{align*}
$$

If $\mu$ is the distribution for an $E$ valued random variable, $X$ and if $v$ is the distribution for an $E$ valued random variable, $Y$, and $X$ and $Y$ are independent, then $\mu * v$ is the distribution for the random variable, $X+Y$. Also the characteristic function of a convolution equals the product of the characteristic functions.

Proof: First consider 59.14.23. Letting $A \in \mathscr{B}(E)$, the following computation holds from Fubini's theorem and Lemma 59.14.6

$$
\begin{aligned}
\mu * v(A) & \equiv \int_{E} v(A-x) d \mu(x)=\int_{E} \int_{E} \mathscr{X}_{S_{A}}(x, y) d v(y) d \mu(x) \\
& =\int_{E} \int_{E} \mathscr{X}_{S_{A}}(x, y) d \mu(x) d v(y)=v * \mu(A) .
\end{aligned}
$$

Next consider 59.14.24. Using 59.14.23 whenever convenient,

$$
\begin{aligned}
(\mu * v) * \lambda(A) & \equiv \int_{E}(\mu * v)(A-x) d \lambda(x) \\
& =\int_{E} \int_{E} v(A-x-y) d \mu(y) d \lambda(x)
\end{aligned}
$$

while

$$
\begin{aligned}
\mu *(v * \lambda)(A) & \equiv \int_{E}(v * \lambda)(A-y) d \mu(y) \\
& =\int_{E} \int_{E} v(A-y-x) d \lambda(x) d \mu(y) \\
& =\int_{E} \int_{E} v(A-y-x) d \mu(y) d \lambda(x)
\end{aligned}
$$

The necessary product measurability comes from Lemma 59.14.4.
Recall

$$
(\mu * v)(A) \equiv \int_{E} v(A-x) d \mu(x)
$$

Therefore, if $s$ is a simple function, $s(x)=\sum_{k=1}^{n} c_{k} \mathscr{X}_{A_{k}}(x)$,

$$
\begin{aligned}
\int_{E} s d(\mu * v) & =\sum_{k=1}^{n} c_{k} \int_{E} v\left(A_{k}-x\right) d \mu(x) \\
& =\int_{E} \sum_{k=1}^{n} c_{k} v\left(A_{k}-x\right) d \mu(x) \\
& =\int_{E} \sum_{k=1}^{n} c_{k} \mathscr{X}_{A_{k}-x}(y) d v(y) d \mu(x) \\
& =\int_{E} \int_{E} s(x+y) d v(y) d \mu(x)
\end{aligned}
$$

Approximating with simple functions it follows that whenever $f$ is bounded and measurable or nonnegative and measurable,

$$
\begin{equation*}
\int_{E} f d(\mu * v)=\int_{E} \int_{E} f(x+y) d v(y) d \mu(x) \tag{59.14.25}
\end{equation*}
$$

Therefore, letting $Z=X+Y$, and $\lambda$ the distribution of $Z$, it follows from independence of $X$ and $Y$ that for $t^{*} \in E^{\prime}$,

$$
\phi_{\lambda}\left(t^{*}\right) \equiv E\left(e^{i t^{*}(Z)}\right)=E\left(e^{i t^{*}(X+Y)}\right)=E\left(e^{i t^{*}(X)}\right) E\left(e^{i t^{*}(Y)}\right)
$$

But also, it follows from 59.14.25

$$
\begin{aligned}
\phi_{(\mu * v)}\left(t^{*}\right) & =\int_{E} e^{i t^{*}(z)} d(\mu * v)(z) \\
& =\int_{E} \int_{E} e^{i t^{*}(x+y)} d v(y) d \mu(x) \\
& =\int_{E} \int_{E} e^{i t^{*}(x)} e^{i t^{*}(y)} d v(y) d \mu(x) \\
& =\left(\int_{E} e^{i t^{*}(y)} d v(y)\right)\left(\int_{E} e^{i t^{*}(x)} d \mu(x)\right) \\
& =E\left(e^{i t^{*}(X)}\right) E\left(e^{i t^{*}(Y)}\right)
\end{aligned}
$$

Since $\phi_{\lambda}\left(t^{*}\right)=\phi_{(\mu * v)}\left(t^{*}\right)$, it follows $\lambda=\mu * v$.
Note the last part of this argument shows the characteristic function of a convolution equals the product of the characteristic functions. This proves the theorem.

### 59.15 The Convergence Of Sums

It turns out that when random variables have symmetric distributions, some remarkable things can be said about infinite sums of these random variables. Conditions are given here that enable one to conclude the convergence of the sequence of partial sums from the convergence of some subsequence of partial sums.

The following lemma is like an earlier result but is proved here for convenience.

Definition 59.15.1 Let $\mathbf{X}$ be a random variable. $\mathscr{L}(\mathbf{X})=\mu$ means $\lambda_{\mathbf{X}}=\mu$. This is called the law of $\mathbf{X}$. It is the same as saying the distribution measure of $\mathbf{X}$ is $\mu$.

Lemma 59.15.2 Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $\mathbf{X}: \Omega \rightarrow E$ be a random variable, where $E$ is a real separable Banach space. Also let $\mathscr{L}(\mathbf{X})=\mu$, a probability measure defined on $\mathscr{B}(E)$, the Borel sets of $E$. Suppose $h: E \rightarrow \mathbb{R}$ is in $L^{1}(E ; \mu)$ or is nonnegative and Borel measurable. Then

$$
\int_{\Omega}(h \circ \mathbf{X}) d P=\int_{E} h(\mathbf{x}) d \mu .
$$

Proof: First suppose $A$ is a Borel set in $E$. Then

$$
\begin{aligned}
\int_{E} \mathscr{X}_{A}(\mathbf{x}) d \mu & \equiv \mu(A) \equiv P([\mathbf{X} \in A]) \\
\int_{\Omega}\left(\mathscr{X}_{A} \circ \mathbf{X}\right) d P & =\int_{\Omega} \mathscr{X}_{\mathbf{X}^{-1}(A)}(\omega) d P \equiv P\left(\mathbf{X}^{-1}(A)\right) \equiv P([\mathbf{X} \in A])
\end{aligned}
$$

Thus for nonnegative simple Borel measurable functions $s$, it follows

$$
\int_{\Omega}(s \circ \mathbf{X}) d P=\int_{E} s(\mathbf{x}) d \mu
$$

Now approximating with an increasing sequence of nonnegative simple functions and using the monotone convergence theorem, the desired formula holds for nonnegative Borel measurable functions $h$.

If $h$ is Borel measurable and in $L^{1}(E ; \mu)$, then you can consider the formula for the positive and negative parts and get the result in this case also. This proves the lemma.

Here is a simple definition and lemma about random variables whose distribution is symmetric.

Definition 59.15.3 Let $X$ be a random variable defined on a probability space, $(\Omega, \mathscr{F}, P)$ having values in a Banach space, $E$. Then it has a symmetric distribution if whenever $A$ is a Borel set,

$$
P([X \in A])=P([X \in-A])
$$

In terms of the distribution,

$$
\lambda_{X}=\lambda_{-X}
$$

It is good to observe that if $X, Y$ are independent random variables defined on a probability space, $(\Omega, \mathscr{F}, P)$ such that each has symmetric distribution, then $X+Y$ also has symmetric distribution. Here is why. Let $A$ be a Borel set in $E$. Then by Theorem 59.14.7 on Page 1902,

$$
\begin{aligned}
\lambda_{X+Y}(A) & =\int_{E} \lambda_{X}(A-z) d \lambda_{Y}(z) \\
& =\int_{E} \lambda_{-X}(A-z) d \lambda_{-Y}(z) \\
& =\lambda_{-(X+Y)}(A)=\lambda_{X+Y}(-A)
\end{aligned}
$$

By induction, it follows that if you have $n$ independent random variables each having symmetric distribution, then their sum has symmetric distribution.

Here is a simple lemma about random variables having symmetric distributions. It will depend on Lemma 59.15.2 on Page 1904.

Lemma 59.15.4 Let $\mathbf{X} \equiv\left(X_{1}, \cdots, X_{n}\right)$ and $Y$ be random variables defined on a probability space, $(\Omega, \mathscr{F}, P)$ such that $X_{i}, i=1,2, \cdots, n$ and $Y$ have values in $E$ a separable Banach space. Thus $\mathbf{X}$ has values in $E^{n}$. Suppose also that $\left\{X_{1}, \cdots, X_{n}, Y\right\}$ are independent and that $Y$ has symmetric distribution. Then if $A \in \mathscr{B}\left(E^{n}\right)$, it follows

$$
\begin{aligned}
& P\left([\mathbf{X} \in A] \cap\left[\left\|\sum_{i=1}^{n} X_{i}+Y\right\|<r\right]\right) \\
= & P\left([\mathbf{X} \in A] \cap\left[\left\|\sum_{i=1}^{n} X_{i}-Y\right\|<r\right]\right)
\end{aligned}
$$

You can also change the inequalities in the obvious way, $<$ to $\leq,>$ or $\geq$.
Proof: Denote by $\lambda_{\mathbf{X}}$ and $\lambda_{Y}$ the distribution measures for $\mathbf{X}$ and $Y$ respectively. Since the random variables are independent, the distribution for the random variable, $(\mathbf{X}, Y)$ mapping into $E^{n+1}$ is $\lambda_{\mathbf{x}} \times \lambda_{Y}$ where this denotes product measure. Since the Banach space is separable, the Borel sets are contained in the product measurable sets. Then by symmetry of the distribution of $Y$

$$
\begin{aligned}
& P\left([\mathbf{X} \in A] \cap\left[\left\|\sum_{i=1}^{n} X_{i}+Y\right\|<r\right]\right) \\
= & \int_{E^{n} \times E} \mathscr{X}_{A}(\mathbf{x}) \mathscr{X}_{B(0, r)}\left(\sum_{i=1}^{n} x_{i}+y\right) d\left(\lambda \mathbf{x} \times \lambda_{Y}\right)(\mathbf{x}, y) \\
= & \int_{E} \int_{E^{n}} \mathscr{X}_{A}(\mathbf{x}) \mathscr{X}_{B(0, r)}\left(\sum_{i=1}^{n} x_{i}+y\right) d \lambda_{\mathbf{x}} d \lambda_{Y} \\
= & \int_{E} \int_{E^{n}} \mathscr{X}_{A}(\mathbf{x}) \mathscr{X}_{B(0, r)}\left(\sum_{i=1}^{n} x_{i}+y\right) d \lambda_{\mathbf{x}} d \lambda_{-Y} \\
= & \int_{E^{n} \times E} \mathscr{X}_{A}(\mathbf{x}) \mathscr{X}_{B(0, r)}\left(\sum_{i=1}^{n} x_{i}+y\right) d\left(\lambda \mathbf{x} \times \lambda_{-Y}\right)(\mathbf{x}, y) \\
= & P\left([\mathbf{X} \in A] \cap\left[\left\|\sum_{i=1}^{n} X_{i}+(-Y)\right\|<r\right]\right)
\end{aligned}
$$

This proves the lemma. Other cases are similar.
Now here is a really interesting lemma.

Lemma 59.15.5 Let $E$ be a real separable Banach space. Assume $\xi_{1}, \cdots, \xi_{N}$ are independent random variables having values in $E$, a separable Banach space which have symmetric distributions. Also let $S_{k}=\sum_{i=1}^{k} \xi_{i}$. Then for any $r>0$,

$$
P\left(\left[\sup _{k \leq N}\left\|S_{k}\right\|>r\right]\right) \leq 2 P\left(\left[\left\|S_{N}\right\|>r\right]\right)
$$

Proof: First of all,

$$
\begin{gather*}
P\left(\left[\sup _{k \leq N}\left\|S_{k}\right\|>r\right]\right) \\
=P\left(\left[\sup _{k \leq N}\left\|S_{k}\right\|>r \text { and }\left\|S_{N}\right\|>r\right]\right) \\
+P\left(\left[\sup _{k \leq N-1}\left\|S_{k}\right\|>r \text { and }\left\|S_{N}\right\| \leq r\right]\right) \\
\leq P\left(\left[\left\|S_{N}\right\|>r\right]\right)+P\left(\left[\sup _{k \leq N-1}\left\|S_{k}\right\|>r \text { and }\left\|S_{N}\right\| \leq r\right]\right) \tag{59.15.26}
\end{gather*}
$$

I need to estimate the second of these terms. Let

$$
A_{1} \equiv\left[\left\|S_{1}\right\|>r\right], \cdots, A_{k} \equiv\left[\left\|S_{k}\right\|>r,\left\|S_{j}\right\| \leq r \text { for } j<k\right]
$$

Thus $A_{k}$ consists of those $\omega$ where $\left\|S_{k}(\omega)\right\|>r$ for the first time at $k$. Thus

$$
\left[\sup _{k \leq N-1}\left\|S_{k}\right\|>r \text { and }\left\|S_{N}\right\| \leq r\right]=\cup_{j=1}^{N-1} A_{j} \cap\left[\left\|S_{N}\right\| \leq r\right]
$$

and the sets in the above union are disjoint. Consider $A_{j} \cap\left[\left\|S_{N}\right\| \leq r\right]$. For $\omega$ in this set,

$$
\left\|S_{j}(\omega)\right\|>r,\left\|S_{i}(\omega)\right\| \leq r \text { if } i<j
$$

Since $\left\|S_{N}(\omega)\right\| \leq r$ in this set, it follows

$$
\left\|S_{N}(\omega)\right\|=\left\|S_{j}(\omega)+\sum_{i=j+1}^{N} \xi_{i}(\omega)\right\| \leq r
$$

Thus

$$
\begin{gather*}
P\left(A_{j} \cap\left[\left\|S_{N}\right\| \leq r\right]\right)  \tag{59.15.27}\\
=P\left(\cap_{i=1}^{j-1}\left[\left\|S_{i}\right\| \leq r\right] \cap\left[\left\|S_{j}\right\|>r\right] \cap\left[\left\|S_{j}+\sum_{i=j+1}^{N} \xi_{i}\right\| \leq r\right]\right) \tag{59.15.28}
\end{gather*}
$$

Now $\cap_{i=1}^{j-1}\left[\left\|S_{i}\right\| \leq r\right] \cap\left[\left\|S_{j}\right\|>r\right]$ is of the form

$$
\left[\left(\xi_{1}, \cdots, \xi_{j}\right) \in A\right]
$$

for some Borel set, $A$. Then letting $Y=\sum_{i=j+1}^{N} \xi_{i}$ in Lemma 59.15.4 and $X_{i}=\xi_{i}, 59.15 .28$ equals

$$
\begin{aligned}
& P\left(\cap_{i=1}^{j-1}\left[\left\|S_{i}\right\| \leq r\right] \cap\left[\left\|S_{j}\right\|>r\right] \cap\left[\left\|S_{j}-\sum_{i=j+1}^{N} \xi_{i}\right\| \leq r\right]\right) \\
= & P\left(\cap_{i=1}^{j-1}\left[\left\|S_{i}\right\| \leq r\right] \cap\left[\left\|S_{j}\right\|>r\right] \cap\left[\left\|S_{j}-\left(S_{N}-S_{j}\right)\right\| \leq r\right]\right) \\
= & P\left(\cap_{i=1}^{j-1}\left[\left\|S_{i}\right\| \leq r\right] \cap\left[\left\|S_{j}\right\|>r\right] \cap\left[\left\|2 S_{j}-S_{N}\right\| \leq r\right]\right)
\end{aligned}
$$

Now since $\left\|S_{j}(\omega)\right\|>r$,

$$
\begin{aligned}
{\left[\left\|2 S_{j}-S_{N}\right\| \leq r\right] } & \subseteq\left[2\left\|S_{j}\right\|-\left\|S_{N}\right\| \leq r\right] \\
& \subseteq\left[2 r-\left\|S_{N}\right\|<r\right] \\
& =\left[\left\|S_{N}\right\|>r\right]
\end{aligned}
$$

and so, referring to 59.15.27, this has shown

$$
\begin{aligned}
& \quad P\left(A_{j} \cap\left[\left\|S_{N}\right\| \leq r\right]\right) \\
& =P\left(\cap_{i=1}^{j-1}\left[| | S_{i} \| \leq r\right] \cap\left[\left\|S_{j}\right\|>r\right] \cap\left[\left\|2 S_{j}-S_{N}\right\| \leq r\right]\right) \\
& \leq P\left(\cap_{i=1}^{j-1}\left[\left\|S_{i}\right\| \leq r\right] \cap\left[\left\|S_{j}\right\|>r\right] \cap\left[\left\|S_{N}\right\|>r\right]\right) \\
& =P\left(A_{j} \cap\left[\left\|S_{N}\right\|>r\right]\right) .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& P\left(\left[\sup _{k \leq N-1}\left\|S_{k}\right\|>r \text { and }\left\|S_{N}\right\| \leq r\right]\right)=\sum_{i=1}^{N-1} P\left(A_{j} \cap\left[\left\|S_{N}\right\| \leq r\right]\right) \\
& \leq \sum_{i=1}^{N-1} P\left(A_{j} \cap\left[\left\|S_{N}\right\|>r\right]\right) \leq P\left(\left[\left\|S_{N}\right\|>r\right]\right)
\end{aligned}
$$

and using 59.15.26, this proves the lemma.
This interesting lemma will now be used to prove the following which concludes a sequence of partial sums converges given a subsequence of the sequence of partial sums converges.

Lemma 59.15.6 Let $\left\{\zeta_{k}\right\}$ be a sequence of independent random variables having values in a separable real Banach space, $E$ whose distributions are symmetric. Letting $S_{k} \equiv \sum_{i=1}^{k} \zeta_{i}$, suppose $\left\{S_{n_{k}}\right\}$ converges a.e. Also suppose that for every $m>n_{k}$,

$$
\begin{equation*}
P\left(\left[\left|\mid S_{m}-S_{n_{k}} \|_{E}>2^{-k}\right]\right)<2^{-k}\right. \tag{59.15.29}
\end{equation*}
$$

Then in fact,

$$
\begin{equation*}
S_{k}(\omega) \rightarrow S(\omega) \text { a.e. } \omega \tag{59.15.30}
\end{equation*}
$$

and off a set of measure zero, the convergence of $S_{k}$ to $S$ is uniform.
Proof: Let $n_{k} \leq l \leq m$. Then by Lemma 59.15.5

$$
P\left(\left[\sup _{n_{k}<l \leq m}\left\|S_{l}-S_{n_{k}}\right\|>2^{-k}\right]\right) \leq 2 P\left(\left[\| S_{m}-S_{n_{k}}| |>2^{-k}\right]\right)
$$

In using this lemma, you could renumber the $\zeta_{i}$ so that the sum

$$
\sum_{j=n_{k}+1}^{l} \zeta_{j}
$$

corresponds to

$$
\sum_{j=1}^{l-n_{k}} \xi_{j}
$$

where $\xi_{j}=\zeta_{j+n_{k}}$.
Then using 59.15.29,

$$
P\left(\left[\sup _{n_{k}<l \leq m}\left\|S_{l}-S_{n_{k}}\right\|>2^{-k}\right]\right) \leq 2 P\left(\left[\left\|S_{m}-S_{n_{k}}\right\|>2^{-k}\right]\right)<2^{-(k-1)}
$$

If $S_{l}(\omega)$ fails to converge then $\omega$ must be in infinitely many of the sets,

$$
\left[\sup _{n_{k}<l} \|\left|S_{l}-S_{n_{k}}\right| \mid>2^{-k}\right]
$$

each of which has measure no more than $2^{-(k-1)}$. Thus $\omega$ must be in a set of measure zero. This proves the lemma.

### 59.16 The Multivariate Normal Distribution

Definition 59.16.1 A random vector, $\mathbf{X}$, with values in $\mathbb{R}^{p}$ has a multivariate normal distribution written as $\mathbf{X} \sim N_{p}(\mathbf{m}, \Sigma)$ if for all Borel $E \subseteq \mathbb{R}^{p}$,

$$
\lambda_{\mathbf{x}}(E)=\int_{\mathbb{R}^{p}} \mathscr{X}_{E}(\mathbf{x}) \frac{1}{(2 \pi)^{p / 2} \operatorname{det}(\Sigma)^{1 / 2}} e^{\frac{-1}{2}(\mathbf{x}-\mathbf{m})^{*} \Sigma^{-1}(\mathbf{x}-\mathbf{m})} d x
$$

for $\mu$ a given vector and $\Sigma$ a given positive definite symmetric matrix.
Theorem 59.16.2 For $\mathbf{X} \sim N_{p}(\mathbf{m}, \Sigma), \mathbf{m}=E(\mathbf{X})$ and

$$
\Sigma=E\left((\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^{*}\right) .
$$

Proof: Let $R$ be an orthogonal transformation such that

$$
R \Sigma R^{*}=D=\operatorname{diag}\left(\sigma_{1}^{2}, \cdots, \sigma_{p}^{2}\right)
$$

Changing the variable by $\mathbf{x}-\mathbf{m}=R^{*} \mathbf{y}$,

$$
\begin{aligned}
E(\mathbf{X}) & \equiv \int_{\mathbb{R}^{p}} \mathbf{x} e^{\frac{-1}{2}(\mathbf{x}-\mathbf{m})^{*} \Sigma^{-1}(\mathbf{x}-\mathbf{m})} d x\left(\frac{1}{(2 \pi)^{p / 2} \operatorname{det}(\Sigma)^{1 / 2}}\right) \\
& =\int_{\mathbb{R}^{p}}\left(R^{*} \mathbf{y}+\mathbf{m}\right) e^{-\frac{1}{2} \mathbf{y}^{*} D^{-1} \mathbf{y}} d y\left(\frac{1}{(2 \pi)^{p / 2} \prod_{i=1}^{p} \sigma_{i}}\right) \\
& =\mathbf{m} \int_{\mathbb{R}^{p}} e^{-\frac{1}{2} \mathbf{y}^{*} D^{-1} \mathbf{y}} d y\left(\frac{1}{(2 \pi)^{p / 2} \prod_{i=1}^{p} \sigma_{i}}\right)=\mathbf{m}
\end{aligned}
$$

by Fubini's theorem and the easy to establish formula

$$
\frac{1}{\sqrt{2 \pi} \sigma} \int_{\mathbb{R}} e^{-\frac{y^{2}}{2 \sigma^{2}}} d y=1
$$

Next let $M \equiv E\left((\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^{*}\right)$. Thus, changing the variable as above by $\mathbf{x}-\mathbf{m}=$ $R^{*} \mathbf{y}$

$$
\begin{aligned}
M & =\int_{\mathbb{R}^{p}}(\mathbf{x}-\mathbf{m})(\mathbf{x}-\mathbf{m})^{*} e^{\frac{-1}{2}(\mathbf{x}-\mathbf{m})^{*} \Sigma^{-1}(\mathbf{x}-\mathbf{m})} d x\left(\frac{1}{(2 \pi)^{p / 2} \operatorname{det}(\Sigma)^{1 / 2}}\right) \\
& =R^{*} \int_{\mathbb{R}^{p}} \mathbf{y y}^{*} e^{-\frac{1}{2} \mathbf{y}^{*} D^{-1} \mathbf{y}} d y\left(\frac{1}{(2 \pi)^{p / 2} \prod_{i=1}^{p} \sigma_{i}}\right) R
\end{aligned}
$$

Therefore,

$$
\left(R M R^{*}\right)_{i j}=\int_{\mathbb{R}^{p}} y_{i} y_{j} e^{-\frac{1}{2} \mathbf{y}^{*} D^{-1} \mathbf{y}} d y\left(\frac{1}{(2 \pi)^{p / 2} \prod_{i=1}^{p} \sigma_{i}}\right)=0
$$

so; $R M R^{*}$ is a diagonal matrix.

$$
\left(R M R^{*}\right)_{i i}=\int_{\mathbb{R}^{p}} y_{i}^{2} e^{-\frac{1}{2} \mathbf{y}^{*} D^{-1} \mathbf{y}} d y\left(\frac{1}{(2 \pi)^{p / 2} \prod_{i=1}^{p} \sigma_{i}}\right)
$$

Using Fubini's theorem and the easy to establish equations,

$$
\frac{1}{\sqrt{2 \pi} \sigma} \int_{\mathbb{R}} e^{-\frac{y^{2}}{2 \sigma^{2}}} d y=1, \frac{1}{\sqrt{2 \pi} \sigma} \int_{\mathbb{R}} y^{2} e^{-\frac{y^{2}}{2 \sigma^{2}}} d y=\sigma^{2}
$$

it follows $\left(R M R^{*}\right)_{i i}=\sigma_{i}^{2}$. Hence $R M R^{*}=D$ and so $M=R^{*} D R=\Sigma$.
Theorem 59.16.3 Suppose $\mathbf{X}_{1} \sim N_{p}\left(\mathbf{m}_{1}, \Sigma_{1}\right), \mathbf{X}_{2} \sim N_{p}\left(\mathbf{m}_{2}, \Sigma_{2}\right)$ and the two random vectors are independent. Then

$$
\begin{equation*}
\mathbf{X}_{1}+\mathbf{X}_{2} \sim N_{p}\left(\mathbf{m}_{1}+\mathbf{m}_{2}, \Sigma_{1}+\Sigma_{2}\right) \tag{59.16.31}
\end{equation*}
$$

Also, if $\mathbf{X} \sim N_{p}(\mathbf{m}, \Sigma)$ then $-\mathbf{X} \sim N_{p}(-\mathbf{m}, \Sigma)$. Furthermore, if $\mathbf{X} \sim N_{p}(\mathbf{m}, \Sigma)$ then

$$
\begin{equation*}
E\left(e^{i \mathbf{t} \cdot \mathbf{X}}\right)=e^{i \mathbf{t} \cdot \mathbf{m}} e^{-\frac{1}{2} \mathbf{t}^{*} \Sigma \mathbf{t}} \tag{59.16.32}
\end{equation*}
$$

Also if $a$ is a constant and $\mathbf{X} \sim N_{p}(\mathbf{m}, \Sigma)$ then $a \mathbf{X} \sim N_{p}\left(a \mathbf{m}, a^{2} \Sigma\right)$.
Proof: Consider $E\left(e^{i t \cdot \mathbf{X}}\right)$ for $\mathbf{X} \sim N_{p}(\mathbf{m}, \Sigma)$.

$$
E\left(e^{i \mathbf{t} \cdot \mathbf{X}}\right) \equiv \frac{1}{(2 \pi)^{p / 2}(\operatorname{det} \Sigma)^{1 / 2}} \int_{\mathbb{R}^{p}} e^{i \mathbf{t} \cdot \mathbf{x}} e^{-\frac{1}{2}(\mathbf{x}-\mathbf{m})^{*} \Sigma^{-1}(\mathbf{x}-\mathbf{m})} d x
$$

Let $R$ be an orthogonal transformation such that

$$
R \Sigma R^{*}=D=\operatorname{diag}\left(\sigma_{1}^{2}, \cdots, \sigma_{p}^{2}\right)
$$

Then let $R(\mathbf{x}-\mathbf{m})=\mathbf{y}$. Then

$$
E\left(e^{i \mathbf{t} \cdot \mathbf{X}}\right)=\frac{1}{(2 \pi)^{p / 2} \prod_{i=1}^{p} \sigma_{i}} \int_{\mathbb{R} p} e^{i \mathbf{t} \cdot\left(R^{*} \mathbf{y}+\mathbf{m}\right)} e^{-\frac{1}{2} \mathbf{y}^{*} D^{-1} \mathbf{y}} d y
$$

Therefore

$$
E\left(e^{i \mathbf{t} \cdot \mathbf{X}}\right)=\frac{1}{(2 \pi)^{p / 2} \prod_{i=1}^{p} \sigma_{i}} \int_{\mathbb{R}^{p}} e^{i \mathbf{s} \cdot(\mathbf{y}+R \mathbf{m})} e^{-\frac{1}{2} \mathbf{y}^{*} D^{-1} \mathbf{y}} d y
$$

where $\mathbf{s}=R \mathbf{t}$. This equals

$$
\begin{gathered}
e^{i \mathbf{t} \cdot \mathbf{m}} \prod_{i=1}^{p}\left(\int_{\mathbb{R}} e^{i s_{i} y_{i}} e^{-\frac{1}{2 \sigma_{i}^{2}} y_{i}^{2}} d y_{i}\right) \frac{1}{\sqrt{2 \pi} \sigma_{i}} \\
=e^{i \mathbf{t} \cdot \mathbf{m}} \prod_{i=1}^{p}\left(\int_{\mathbb{R}} e^{i s_{i} \sigma_{i} u} e^{-\frac{1}{2} u^{2}} d u\right) \frac{1}{\sqrt{2 \pi}} \\
=e^{i \mathbf{t} \cdot \mathbf{m}} \prod_{i=1}^{p} e^{-\frac{1}{2} s_{i}^{2} \sigma_{i}^{2}} \frac{1}{\sqrt{2 \pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}\left(u-i s_{i} \sigma_{i}\right)^{2}} d u \\
=e^{i \mathbf{t} \cdot \mathbf{m}} e^{-\frac{1}{2} \sum_{i=1}^{p} s_{i}^{2} \sigma_{i}^{2}}=e^{i \mathbf{t} \cdot \mathbf{m}} e^{-\frac{1}{2} \mathbf{t}^{*} \Sigma \mathbf{t}}
\end{gathered}
$$

This proves 59.16.32.
Since $\mathbf{X}_{1}$ and $\mathbf{X}_{2}$ are independent, $e^{i t \cdot \mathbf{X}_{1}}$ and $e^{i \mathbf{t} \cdot \mathbf{X}_{2}}$ are also independent. Hence

$$
E\left(e^{i \boldsymbol{t} \cdot \mathbf{X}_{1}+\mathbf{X}_{2}}\right)=E\left(e^{i \mathbf{t} \cdot \mathbf{X}_{1}}\right) E\left(e^{i \boldsymbol{t} \cdot \mathbf{X}_{2}}\right)
$$

Thus,

$$
\begin{aligned}
E\left(e^{i \mathbf{t} \cdot \mathbf{X}_{1}+\mathbf{X}_{2}}\right) & =E\left(e^{i \mathbf{t} \cdot \mathbf{X}_{1}}\right) E\left(e^{i \mathbf{t} \cdot \mathbf{X}_{2}}\right) \\
& =e^{i \mathbf{t} \cdot \mathbf{m}_{1}} e^{-\frac{1}{2} \mathbf{t}^{*} \Sigma_{1} \mathbf{t}} e^{i \mathbf{t} \cdot \mathbf{m}_{2}} e^{-\frac{1}{2} \mathbf{t}^{*} \Sigma_{2} \mathbf{t}} \\
& =e^{i \mathbf{t} \cdot\left(\mathbf{m}_{1}+\mathbf{m}_{2}\right)} e^{-\frac{1}{2} \mathbf{t}^{*}\left(\Sigma_{1}+\Sigma_{2}\right) \mathbf{t}}
\end{aligned}
$$

which is the characteristic function of a random vector distributed as

$$
N_{p}\left(\mathbf{m}_{1}+\mathbf{m}_{2}, \Sigma_{1}+\Sigma_{2}\right)
$$

Now it follows that $\mathbf{X}_{1}+\mathbf{X}_{2} \sim N_{p}\left(\mathbf{m}_{1}+\mathbf{m}_{2}, \Sigma_{1}+\Sigma_{2}\right)$ by Theorem 59.8.4. This proves 59.16.31.

The assertion about $-\mathbf{X}$ is also easy to see because

$$
\begin{aligned}
E\left(e^{i \mathbf{t} \cdot(-\mathbf{X})}\right) & =E\left(e^{i(-\mathbf{t}) \cdot \mathbf{x}}\right) \\
& =\frac{1}{(2 \pi)^{p / 2}(\operatorname{det} \Sigma)^{1 / 2}} \int_{\mathbb{R}^{p}} e^{i(-\mathbf{t}) \cdot \mathbf{x}} e^{-\frac{1}{2}(\mathbf{x}-\mathbf{m})^{*} \Sigma^{-1}(\mathbf{x}-\mathbf{m})} d x \\
& =\frac{1}{(2 \pi)^{p / 2}(\operatorname{det} \Sigma)^{1 / 2}} \int_{\mathbb{R}^{p}} e^{i \mathbf{t} \cdot \mathbf{x}} e^{-\frac{1}{2}(\mathbf{x}+\mathbf{m})^{*} \Sigma^{-1}(\mathbf{x}+\mathbf{m})} d x
\end{aligned}
$$

which is the characteristic function of a random variable which is $N(-\mathbf{m}, \Sigma)$. Theorem 59.8.4 again implies $-\mathbf{X} \sim N(-\mathbf{m}, \Sigma)$. Finally consider the last claim. You apply what is known about $\mathbf{X}$ with $\mathbf{t}$ replaced with $a \mathbf{t}$ and then massage things. This gives the characteristic function for $a \mathbf{X}$ is given by

$$
E(\exp (i \mathbf{t} \cdot a \mathbf{X}))=\exp (i \mathbf{t} \cdot a \mathbf{m}) \exp \left(-\frac{1}{2} \mathbf{t}^{*} \Sigma a^{2} \mathbf{t}\right)
$$

which is the characteristic function of a normal random vector having mean $a \mathbf{m}$ and covariance $a^{2} \Sigma$. This proves the theorem.

Following [103] a random vector has a generalized normal distribution if its characteristic function is given as

$$
\begin{equation*}
e^{i \mathbf{t} \cdot \mathbf{m}} e^{-\frac{1}{2} \mathbf{t}^{*} \Sigma \mathbf{t}} \tag{59.16.33}
\end{equation*}
$$

where $\Sigma$ is symmetric and has nonnegative eigenvalues. For a random real valued variable, $\mathbf{m}$ is scalar and so is $\Sigma$ so the characteristic function of such a generalized normally distributed random variable is

$$
\begin{equation*}
e^{i t \mu} e^{-\frac{1}{2} t^{2} \sigma^{2}} \tag{59.16.34}
\end{equation*}
$$

These generalized normal distributions do not require $\Sigma$ to be invertible, only that the eigenvalues be nonnegative. In one dimension this would correspond the characteristic function of a dirac measure having point mass 1 at $\mu$. In higher dimensions, it could be a mixture of such things with more familiar things. I won't try very hard to distinguish between generalized normal distributions and normal distributions in which the covariance matrix has all positive eigenvalues.

Here are some other interesting results about normal distributions found in [103]. The next theorem has to do with the question whether a random vector is normally distributed in the above generalized sense.

Theorem 59.16.4 Let $\mathbf{X}=\left(X_{1}, \cdots, X_{p}\right)$ where each $X_{i}$ is a real valued random variable. Then $\mathbf{X}$ is normally distributed in the above generalized sense if and only if every linear combination, $\sum_{j=1}^{p} a_{i} X_{i}$ is normally distributed. In this case the mean of $\mathbf{X}$ is

$$
\mathbf{m}=\left(E\left(X_{1}\right), \cdots, E\left(X_{p}\right)\right)
$$

and the covariance matrix for $\mathbf{X}$ is

$$
\Sigma_{j k}=E\left(\left(X_{j}-m_{j}\right)\left(X_{k}-m_{k}\right)^{*}\right)
$$

Proof: Suppose first $\mathbf{X}$ is normally distributed. Then its characteristic function is of the form

$$
\phi_{\mathbf{X}}(\mathbf{t})=E\left(e^{i \mathbf{t} \cdot \mathbf{X}}\right)=e^{i \mathbf{t} \cdot \mathbf{m}} e^{-\frac{1}{2} \mathbf{t}^{*} \Sigma \mathbf{t}}
$$

Then letting $\mathbf{a}=\left(a_{1}, \cdots, a_{p}\right)$

$$
E\left(e^{i t \sum_{j=1}^{p} a_{i} X_{i}}\right)=E\left(e^{i t \mathbf{a} \cdot \mathbf{X}}\right)=e^{i t \mathbf{a} \cdot \mathbf{m}} e^{-\frac{1}{2} \mathbf{a}^{*} \Sigma \mathbf{a} t^{2}}
$$

which is the characteristic function of a normally distributed random variable with mean $\mathbf{a} \cdot \mathbf{m}$ and variance $\sigma^{2}=\mathbf{a}^{*} \Sigma \mathbf{a}$. This proves half of the theorem. If $\mathbf{X}$ is normally distributed, then every linear combination is normally distributed.

Next suppose $\sum_{j=1}^{p} a_{j} X_{j}=\mathbf{a} \cdot \mathbf{X}$ is normally distributed with mean $\mu$ and variance $\sigma^{2}$ so that its characteristic function is given in 59.16.34. I will now relate $\mu$ and $\sigma^{2}$ to various quantities involving the $X_{j}$. Letting $m_{j}=E\left(X_{j}\right), \mathbf{m}=\left(m_{1}, \cdots, m_{p}\right)^{*}$

$$
\begin{aligned}
\mu & =\sum_{j=1}^{p} a_{j} E\left(X_{j}\right)=\sum_{j=1}^{p} a_{j} m_{j}, \sigma^{2}=E\left(\left(\sum_{j=1}^{p} a_{j} X_{j}-\sum_{j=1}^{p} a_{j} m_{j}\right)^{2}\right) \\
& =E\left(\left(\sum_{j=1}^{p} a_{j}\left(X_{j}-m_{j}\right)\right)^{2}\right)=\sum_{j, k} a_{j} a_{k} E\left(\left(X_{j}-m_{j}\right)\left(X_{k}-m_{k}\right)\right)
\end{aligned}
$$

It follows the mean of the normally distributed random variable, $\mathbf{a} \cdot \mathbf{X}$ is

$$
\mu=\sum_{j} a_{j} m_{j}=\mathbf{a} \cdot \mathbf{m}
$$

and its variance is

$$
\sigma^{2}=\mathbf{a}^{*} E\left((\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^{*}\right) \mathbf{a}
$$

Therefore,

$$
\begin{gathered}
E\left(e^{i t \mathbf{t} \cdot \mathbf{X}}\right)=e^{i t \mu} e^{-\frac{1}{2} t^{2} \sigma^{2}} \\
=e^{i t \mathbf{a} \cdot \mathbf{m}} e^{-\frac{1}{2} t^{2} \mathbf{a}^{*} E\left((\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^{*}\right) \mathbf{a}}
\end{gathered}
$$

Then letting $\mathbf{s}=t \mathbf{a}$ this shows

$$
\begin{aligned}
E\left(e^{i s \cdot \mathbf{X}}\right) & =e^{i \mathbf{s} \cdot \mathbf{m}} e^{-\frac{1}{2} \mathbf{s}^{*} E\left((\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^{*}\right) \mathbf{s}} \\
& =e^{i \mathbf{s} \cdot \mathbf{m}} e^{-\frac{1}{2} \mathbf{s}^{*} \Sigma \mathbf{s}}
\end{aligned}
$$

which is the characteristic function of a normally distributed random variable with $\mathbf{m}$ given above and $\Sigma$ given by

$$
\Sigma_{j k}=E\left(\left(X_{j}-m_{j}\right)\left(X_{k}-m_{k}\right)\right)
$$

By assumption, a is completely arbitrary and so it follows that $\mathbf{s}$ is also. Hence, $\mathbf{X}$ is normally distributed as claimed.

Corollary 59.16.5 Let $\mathbf{X}=\left(X_{1}, \cdots, X_{p}\right), \mathbf{Y}=\left(Y_{1}, \cdots, Y_{p}\right)$ where each $X_{i}, Y_{i}$ is a real valued random variable. Suppose also that for every $\mathbf{a} \in \mathbb{R}^{p}, \mathbf{a} \cdot \mathbf{X}$ and $\mathbf{a} \cdot \mathbf{Y}$ are both normally distributed with the same mean and variance. Then $\mathbf{X}$ and $\mathbf{Y}$ are both multivariate normal random vectors with the same mean and variance.

Proof: In the Proof of Theorem 59.16.4 the proof implies that the characteristic functions of $\mathbf{a} \cdot \mathbf{X}$ and $\mathbf{a} \cdot \mathbf{Y}$ are both of the form

$$
e^{i t m} e^{-\frac{1}{2} \sigma^{2} t^{2}}
$$

Then as in the proof of that theorem, it must be the case that

$$
m=\sum_{j=1}^{p} a_{j} m_{j}
$$

where $E\left(X_{i}\right)=m_{i}=E\left(Y_{i}\right)$ and

$$
\begin{aligned}
\sigma^{2} & =\mathbf{a}^{*} E\left((\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^{*}\right) \mathbf{a} \\
& =\mathbf{a}^{*} E\left((\mathbf{Y}-\mathbf{m})(\mathbf{Y}-\mathbf{m})^{*}\right) \mathbf{a}
\end{aligned}
$$

and this last equation must hold for every $\mathbf{a}$. Therefore,

$$
E\left((\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^{*}\right)=E\left((\mathbf{Y}-\mathbf{m})(\mathbf{Y}-\mathbf{m})^{*}\right) \equiv \Sigma
$$

and so the characteristic function of both $\mathbf{X}$ and $\mathbf{Y}$ is $e^{i s \cdot m} e^{-\frac{1}{2} s^{*} \Sigma s}$ as in the proof of Theorem 59.16.4.

Theorem 59.16.6 Suppose $\mathbf{X}=\left(X_{1}, \cdots, X_{p}\right)$ is normally distributed with mean $\mathbf{m}$ and covariance $\Sigma$. Then if $X_{1}$ is uncorrelated with any of the $X_{i}$, meaning

$$
E\left(\left(X_{1}-m_{1}\right)\left(X_{j}-m_{j}\right)\right)=0 \text { for } j>1
$$

then $X_{1}$ and $\left(X_{2}, \cdots, X_{p}\right)$ are both normally distributed and the two random vectors are independent. Here $m_{j} \equiv E\left(X_{j}\right)$. More generally, if the covariance matrix is a diagonal matrix, the random variables, $\left\{X_{1}, \cdots, X_{p}\right\}$ are independent.

Proof: From Theorem 59.16.2

$$
\Sigma=E\left((\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^{*}\right) .
$$

Then by assumption,

$$
\Sigma=\left(\begin{array}{cc}
\sigma_{1}^{2} & \mathbf{0}  \tag{59.16.35}\\
\mathbf{0} & \Sigma_{p-1}
\end{array}\right)
$$

I need to verify that if $E \in \sigma\left(X_{1}\right)$ and $F \in \sigma\left(X_{2}, \cdots, X_{p}\right)$, then

$$
P(E \cap F)=P(E) P(F) .
$$

Let $E=X_{1}^{-1}(A)$ and

$$
F=\left(X_{2}, \cdots, X_{p}\right)^{-1}(B)
$$

where $A$ and $B$ are Borel sets in $\mathbb{R}$ and $\mathbb{R}^{p-1}$ respectively. Thus I need to verify that

$$
\begin{array}{r}
P\left(\left[\left(X_{1},\left(X_{2}, \cdots, X_{p}\right)\right) \in(A, B)\right]\right)= \\
\mu_{\left(X_{1},\left(X_{2}, \cdots, X_{p}\right)\right)}(A \times B)=\mu_{X_{1}}(A) \mu_{\left(X_{2}, \cdots, X_{p}\right)}(B) . \tag{59.16.36}
\end{array}
$$

Using 59.16.35, Fubini's theorem, and definitions,

$$
\begin{gathered}
\mu_{\left(X_{1},\left(X_{2}, \cdots, X_{p}\right)\right)}(A \times B)= \\
\int_{\mathbb{R}^{p}} \mathscr{X}_{A \times B}(\mathbf{x}) \frac{1}{(2 \pi)^{p / 2} \operatorname{det}(\Sigma)^{1 / 2}} e^{\frac{-1}{2}(\mathbf{x}-\mathbf{m})^{*} \Sigma^{-1}(\mathbf{x}-\mathbf{m})} d x \\
=\int_{\mathbb{R}} \mathscr{X}_{A}\left(x_{1}\right) \int_{\mathbb{R}^{p-1}} \mathscr{X}_{B}\left(X_{2}, \cdots, X_{p}\right) . \\
\frac{1}{(2 \pi)^{(p-1) / 2} \sqrt{2 \pi}\left(\sigma_{1}^{2}\right)^{1 / 2} \operatorname{det}\left(\Sigma_{p-1}\right)^{1 / 2}} e^{\frac{-\left(x_{1}-m_{1}\right)^{2}}{2 \sigma_{1}^{2}}} . \\
e^{\frac{-1}{2}\left(\mathbf{x}^{\prime}-\mathbf{m}^{\prime}\right)^{*} \Sigma_{p-1}^{-1}\left(\mathbf{x}^{\prime}-\mathbf{m}^{\prime}\right)} d x^{\prime} d x_{1}
\end{gathered}
$$

where $\mathbf{x}^{\prime}=\left(x_{2}, \cdots, x_{p}\right)$ and $\mathbf{m}^{\prime}=\left(m_{2}, \cdots, m_{p}\right)$. Now this equals

$$
\begin{align*}
& \int_{\mathbb{R}} \mathscr{X}_{A}\left(x_{1}\right) \frac{1}{\sqrt{2 \pi \sigma_{1}^{2}}} e^{\frac{-\left(x_{1}-m_{1}\right)^{2}}{2 \sigma_{1}^{2}}} \int_{B} \frac{1}{(2 \pi)^{(p-1) / 2} \operatorname{det}\left(\Sigma_{p-1}\right)^{1 / 2}}  \tag{59.16.37}\\
& e^{\frac{-1}{2}\left(\mathbf{x}^{\prime}-\mathbf{m}^{\prime}\right)^{*} \Sigma_{p-1}^{-1}\left(\mathbf{x}^{\prime}-\mathbf{m}^{\prime}\right)} d x^{\prime} d x \tag{59.16.38}
\end{align*}
$$

In case $B=\mathbb{R}^{p-1}$, the inside integral equals 1 and

$$
\begin{aligned}
\mu_{X_{1}}(A) & =\mu_{\left(X_{1},\left(X_{2}, \cdots, X_{p}\right)\right)}\left(A \times \mathbb{R}^{p-1}\right) \\
& =\int_{\mathbb{R}} \mathscr{X}_{A}\left(x_{1}\right) \frac{1}{\sqrt{2 \pi \sigma_{1}^{2}}} e^{\frac{-\left(x_{1}-m_{1}\right)^{2}}{2 \sigma_{1}^{2}}} d x_{1}
\end{aligned}
$$

which shows $X_{1}$ is normally distributed as claimed. Similarly, letting $A=\mathbb{R}$,

$$
\begin{aligned}
& \mu_{\left(X_{2}, \cdots, X_{p}\right)}(B) \\
= & \mu_{\left(X_{1},\left(X_{2}, \cdots, X_{p}\right)\right)}(\mathbb{R} \times B) \\
= & \int_{B} \frac{1}{(2 \pi)^{(p-1) / 2} \operatorname{det}\left(\Sigma_{p-1}\right)^{1 / 2}} e^{\frac{-1}{2}\left(\mathbf{x}^{\prime}-\mathbf{m}^{\prime}\right)^{*} \Sigma_{p-1}^{-1}\left(\mathbf{x}^{\prime}-\mathbf{m}^{\prime}\right)} d x^{\prime}
\end{aligned}
$$

and $\left(X_{2}, \cdots, X_{p}\right)$ is also normally distributed with mean $\mathbf{m}^{\prime}$ and covariance $\Sigma_{p-1}$. Now from 59.16.37, 59.16 .36 follows. In case the covariance matrix is diagonal, the above reasoning extends in an obvious way to prove the random variables, $\left\{X_{1}, \cdots, X_{p}\right\}$ are independent.

However, another way to prove this is to use Proposition 59.11.1 on Page 1889 and consider the characteristic function. Let $E\left(X_{j}\right)=m_{j}$ and

$$
P=\sum_{j=1}^{p} t_{j} X_{j}
$$

Then since $\mathbf{X}$ is normally distributed and the covariance is a diagonal,

$$
\begin{align*}
D & \equiv\left(\begin{array}{ccc}
\sigma_{1}^{2} & & 0 \\
& \ddots & \\
0 & & \sigma_{p}^{2}
\end{array}\right) \\
E\left(e^{i P}\right) & =E\left(e^{i \mathbf{t} \cdot \mathbf{X}}\right)=e^{i \mathbf{t} \cdot \mathbf{m}} e^{-\frac{1}{2} \mathbf{t}^{*} \Sigma \mathbf{t}} \\
& =\exp \left(\sum_{j=1}^{p} i t_{j} m_{j}-\frac{1}{2} t_{j}^{2} \sigma_{j}^{2}\right)  \tag{59.16.39}\\
& =\prod_{j=1}^{p} \exp \left(i t_{j} m_{j}-\frac{1}{2} t_{j}^{2} \sigma_{j}^{2}\right)
\end{align*}
$$

Also,

$$
\begin{aligned}
E\left(e^{i t_{j} X_{j}}\right) & =E\left(\exp \left(i t_{j} X_{j}+\sum_{k \neq j} i 0 X_{k}\right)\right) \\
& =\exp \left(i t_{j} m_{j}-\frac{1}{2} t_{j}^{2} \sigma_{j}^{2}\right)
\end{aligned}
$$

With 59.16.39, this shows

$$
E\left(e^{i P}\right)=\prod_{j=1}^{p} E\left(e^{i t_{j} X_{j}}\right)
$$

which shows by Proposition 59.11.1 that the random variables,

$$
\left\{X_{1}, \cdots, X_{p}\right\}
$$

are independent.

### 59.17 Use Of Characteristic Functions To Find Moments

Let $X$ be a random variable with characteristic function

$$
\phi_{X}(t) \equiv E(\exp (i t X))
$$

Then this can be used to find moments of the random variable assuming they exist. The $k^{t h}$ moment is defined as

$$
E\left(X^{k}\right)
$$

This can be done by using the dominated convergence theorem to differentiate the characteristic function with respect to $t$ and then plugging in $t=0$. For example,

$$
\phi_{X}^{\prime}(t)=E(i X \exp (i t X))
$$

and now plugging in $t=0$ you get $i E(X)$. Doing another differentiation you obtain

$$
\phi_{X}^{\prime \prime}(t)=E\left(-X^{2} \exp (i t X)\right)
$$

and plugging in $t=0$ you get $-E\left(X^{2}\right)$ and so forth.
An important case is where $X$ is normally distributed with mean 0 and variance $\sigma^{2}$. In this case, as shown above, the characteristic function is

$$
e^{-\frac{1}{2} t^{2} \sigma^{2}}
$$

Also all moments exist when $X$ is normally distributed. So what are these moments?

$$
D_{t}\left(e^{-\frac{1}{2} t^{2} \sigma^{2}}\right)=-t \sigma^{2} e^{-\frac{1}{2} t^{2} \sigma^{2}}
$$

and plugging in $t=0$ you find the mean equals 0 as expected.

$$
D_{t}\left(-t \sigma^{2} e^{-\frac{1}{2} t^{2} \sigma^{2}}\right)=-\sigma^{2} e^{-\frac{1}{2} t^{2} \sigma^{2}}+t^{2} \sigma^{4} e^{-\frac{1}{2} t^{2} \sigma^{2}}
$$

and plugging in $t=0$ you find the second moment is $\sigma^{2}$. Then do it again.

$$
D_{t}\left(-\sigma^{2} e^{-\frac{1}{2} t^{2} \sigma^{2}}+t^{2} \sigma^{4} e^{-\frac{1}{2} t^{2} \sigma^{2}}\right)=3 \sigma^{4} t e^{-\frac{1}{2} t^{2} \sigma^{2}}-t^{3} \sigma^{6} e^{-\frac{1}{2} t^{2} \sigma^{2}}
$$

Then $E\left(X^{3}\right)=0$.

$$
\begin{aligned}
& D_{t}\left(3 \sigma^{4} t e^{-\frac{1}{2} t^{2} \sigma^{2}}-t^{3} \sigma^{6} e^{-\frac{1}{2} t^{2} \sigma^{2}}\right) \\
= & 3 \sigma^{4} e^{-\frac{1}{2} t^{2} \sigma^{2}}-6 \sigma^{6} t^{2} e^{-\frac{1}{2} t^{2} \sigma^{2}}+t^{4} \sigma^{8} e^{-\frac{1}{2} t^{2} \sigma^{2}}
\end{aligned}
$$

and so $E\left(X^{4}\right)=3 \sigma^{4}$. By now you can see the pattern. If you continue this way, you find the odd moments are all 0 and

$$
\begin{equation*}
E\left(X^{2 m}\right)=C_{m}\left(\sigma^{2}\right)^{m} \tag{59.17.40}
\end{equation*}
$$

This is an important observation.

### 59.18 The Central Limit Theorem

The central limit theorem is one of the most marvelous theorems in mathematics. It can be proved through the use of characteristic functions. Recall for $\mathbf{x} \in \mathbb{R}^{p}$,

$$
\|\mathbf{x}\|_{\infty} \equiv \max \left\{\left|x_{j}\right|, j=1, \cdots, p\right\} .
$$

Also recall the definition of the distribution function for a random vector, $\mathbf{X}$.

$$
F_{\mathbf{X}}(\mathbf{x}) \equiv P\left(X_{j} \leq x_{j}, j=1, \cdots, p\right)
$$

Definition 59.18.1 Let $\left\{\mathbf{X}_{n}\right\}$ be random vectors with values in $\mathbb{R}^{p}$. Then $\left\{\lambda_{\mathbf{x}_{n}}\right\}_{n=1}^{\infty}$ is called "tight" if for all $\varepsilon>0$ there exists a compact set, $K_{\mathcal{\varepsilon}}$ such that

$$
\lambda_{\mathbf{x}_{n}}\left(\left[\mathbf{x} \notin K_{\mathcal{\varepsilon}}\right]\right)<\boldsymbol{\varepsilon}
$$

for all $\lambda_{\mathbf{x}_{n}}$. Similarly, if $\left\{\mu_{n}\right\}$ is a sequence of probability measures defined on the Borel sets of $\mathbb{R}^{p}$, then this sequence is "tight" if for each $\varepsilon>0$ there exists a compact set, $K_{\mathcal{E}}$ such that

$$
\mu_{n}\left(\left[\mathbf{x} \notin K_{\varepsilon}\right]\right)<\varepsilon
$$

for all $\mu_{n}$.
Lemma 59.18.2 If $\left\{\mathbf{X}_{n}\right\}$ is a sequence of random vectors with values in $\mathbb{R}^{p}$ such that

$$
\lim _{n \rightarrow \infty} \phi_{\mathbf{X}_{n}}(\mathbf{t})=\psi(\mathbf{t})
$$

for all $\mathbf{t}$, where $\psi(\mathbf{0})=1$ and $\psi$ is continuous at $\mathbf{0}$, then $\left\{\lambda_{\mathbf{x}_{n}}\right\}_{n=1}^{\infty}$ is tight.
Proof: Let $\mathbf{e}_{j}$ be the $j^{t h}$ standard unit basis vector.

$$
\begin{aligned}
& \left|\frac{1}{u} \int_{-u}^{u}\left(1-\phi_{\mathbf{X}_{n}}\left(t \mathbf{e}_{j}\right)\right) d t\right| \\
= & \left|\frac{1}{u} \int_{-u}^{u}\left(1-\int_{\mathbb{R}^{p}} e^{i t x_{j}} d \lambda \mathbf{x}_{n}\right) d t\right| \\
= & \left|\frac{1}{u} \int_{-u}^{u}\left(\int_{\mathbb{R}^{p}}\left(1-e^{i t x_{j}}\right) d \lambda \mathbf{x}_{n}\right) d t\right| \\
= & \left|\int_{\mathbb{R}^{p}} \frac{1}{u} \int_{-u}^{u}\left(1-e^{i t x_{j}}\right) d t d \lambda_{\mathbf{x}_{n}}(x)\right| \\
= & \left|2 \int_{\mathbb{R}^{p}}\left(1-\frac{\sin \left(u x_{j}\right)}{u x_{j}}\right) d \lambda \mathbf{x}_{n}(x)\right| \\
\geq & 2 \int_{\left[\left|x_{j}\right| \geq \frac{2}{u}\right]}\left(1-\frac{1}{\left|u x_{j}\right|}\right) d \lambda \mathbf{x}_{n}(x)
\end{aligned}
$$

$$
\begin{gathered}
\geq 2 \int_{\left[\left|x_{j}\right| \geq \frac{2}{u}\right]}\left(1-\frac{1}{|u|(2 / u)}\right) d \lambda \mathbf{x}_{n}(x) \\
\quad=\int_{\left[\left|x_{j}\right| \geq \frac{2}{u}\right]} 1 d \lambda_{\mathbf{x}_{n}}(x) \\
\quad=\lambda_{\mathbf{x}_{n}}\left(\left[\mathbf{x}:\left|x_{j}\right| \geq \frac{2}{u}\right]\right) .
\end{gathered}
$$

If $\varepsilon>0$ is given, there exists $r>0$ such that if $u \leq r$,

$$
\frac{1}{u} \int_{-u}^{u}\left(1-\psi\left(t \mathbf{e}_{j}\right)\right) d t<\varepsilon / p
$$

for all $j=1, \cdots, p$ and so, by the dominated convergence theorem, the same is true with $\phi_{\mathbf{X}_{n}}$ in place of $\psi$ provided $n$ is large enough, say $n \geq N(u)$. Thus, if $u \leq r$, and $n \geq N(u)$,

$$
\lambda \mathbf{x}_{n}\left(\left[\mathbf{x}:\left|x_{j}\right| \geq \frac{2}{u}\right]\right)<\varepsilon / p
$$

for all $j \in\{1, \cdots, p\}$. It follows that for $u \leq r$ and $n \geq N(u)$,

$$
\lambda_{\mathbf{x}_{n}}\left(\left[\mathbf{x}:\|\mathbf{x}\|_{\infty} \geq \frac{2}{u}\right]\right)<\varepsilon
$$

because

$$
\left[\mathbf{x}:\|\mathbf{x}\|_{\infty} \geq \frac{2}{u}\right] \subseteq \cup_{j=1}^{p}\left[\mathbf{x}:\left|x_{j}\right| \geq \frac{2}{u}\right]
$$

This proves the lemma because there are only finitely many measures, $\lambda_{\mathbf{x}_{n}}$ for $n<N(u)$ and the compact set can be enlarged finitely many times to obtain a single compact set, $K_{\varepsilon}$ such that for all $n, \lambda_{\mathbf{x}_{n}}\left(\left[\mathbf{x} \notin K_{\varepsilon}\right]\right)<\varepsilon$. This proves the lemma.
Lemma 59.18.3 If $\phi_{\mathbf{X}_{n}}(\mathbf{t}) \rightarrow \phi_{\mathbf{X}}(\mathbf{t})$ for all $\mathbf{t}$, then whenever $\psi \in \mathfrak{S}$,

$$
\lambda_{\mathbf{x}_{n}}(\psi) \equiv \int_{\mathbb{R}^{p}} \psi(\mathbf{y}) d \lambda_{\mathbf{x}_{n}}(y) \rightarrow \int_{\mathbb{R}^{p}} \psi(\mathbf{y}) d \lambda_{\mathbf{X}}(y) \equiv \lambda_{\mathbf{X}}(\psi)
$$

as $n \rightarrow \infty$.
Proof: Recall that if $\mathbf{X}$ is any random vector, its characteristic function is given by

$$
\phi_{\mathbf{X}}(\mathbf{y}) \equiv \int_{\mathbb{R}^{p}} e^{i \mathbf{y} \cdot \mathbf{x}} d \lambda_{\mathbf{X}}(x)
$$

Also remember the inverse Fourier transform. Letting $\psi \in \mathfrak{S}$, the Schwartz class,

$$
\begin{aligned}
F^{-1}\left(\lambda_{\mathbf{x}}\right)(\psi) & \equiv \lambda_{\mathbf{x}}\left(F^{-1} \psi\right) \equiv \int_{\mathbb{R}^{p}} F^{-1} \psi d \lambda_{\mathbf{x}} \\
& =\frac{1}{(2 \pi)^{p / 2}} \int_{\mathbb{R}^{p}} \int_{\mathbb{R}^{p}} e^{i \mathbf{y} \cdot \mathbf{x}} \psi(\mathbf{x}) d x d \lambda_{\mathbf{x}}(y) \\
& =\frac{1}{(2 \pi)^{p / 2}} \int_{\mathbb{R}^{p}} \psi(\mathbf{x}) \int_{\mathbb{R}^{p}} e^{i \mathbf{y} \cdot \mathbf{x}} d \lambda_{\mathbf{x}}(y) d x \\
& =\frac{1}{(2 \pi)^{p / 2}} \int_{\mathbb{R}^{p}} \psi(\mathbf{x}) \phi_{\mathbf{X}}(\mathbf{x}) d x
\end{aligned}
$$

and so, considered as elements of $\mathfrak{S}^{*}$,

$$
F^{-1}\left(\lambda_{\mathbf{x}}\right)=\phi_{\mathbf{X}}(\cdot)(2 \pi)^{-(p / 2)} \in L^{\infty}
$$

By the dominated convergence theorem

$$
\begin{aligned}
(2 \pi)^{p / 2} F^{-1}\left(\lambda_{\mathbf{x}_{n}}\right)(\psi) & \equiv \int_{\mathbb{R}^{p}} \phi_{\mathbf{X}_{n}}(\mathbf{t}) \psi(\mathbf{t}) d t \\
& \rightarrow \int_{\mathbb{R}^{p}} \phi_{\mathbf{X}}(\mathbf{t}) \psi(\mathbf{t}) d t \\
& =(2 \pi)^{p / 2} F^{-1}(\lambda \mathbf{x})(\psi)
\end{aligned}
$$

whenever $\psi \in \mathfrak{S}$. Thus

$$
\begin{aligned}
\lambda_{\mathbf{x}_{n}}(\psi) & =F F^{-1} \lambda_{\mathbf{x}_{n}}(\psi) \equiv F^{-1} \lambda_{\mathbf{x}_{n}}(F \psi) \rightarrow F^{-1} \lambda_{\mathbf{X}}(F \psi) \\
& \equiv F^{-1} F \lambda_{\mathbf{x}}(\psi)=\lambda_{\mathbf{X}}(\psi)
\end{aligned}
$$

This proves the lemma.
Lemma 59.18.4 If $\phi_{\mathbf{X}_{n}}(\mathbf{t}) \rightarrow \phi_{\mathbf{X}}(\mathbf{t})$, then if $\psi$ is any bounded uniformly continuous function,

$$
\lim _{n \rightarrow \infty} \int_{\mathbb{R}^{p}} \psi d \lambda \mathbf{x}_{n}=\int_{\mathbb{R}^{p}} \psi d \lambda \mathbf{x}
$$

Proof: Let $\varepsilon>0$ be given, let $\psi$ be a bounded function in $C^{\infty}\left(\mathbb{R}^{p}\right)$. Now let $\eta \in C_{c}^{\infty}\left(Q_{r}\right)$ where $Q_{r} \equiv[-r, r]^{p}$ satisfy the additional requirement that $\eta=1$ on $Q_{r / 2}$ and $\eta(\mathbf{x}) \in[0,1]$ for all $\mathbf{x}$. By Lemma 59.18.2 the set, $\left\{\lambda_{\mathbf{x}_{n}}\right\}_{n=1}^{\infty}$, is tight and so if $\varepsilon>0$ is given, there exists $r$ sufficiently large such that for all $n$,

$$
\int_{\left[\mathbf{x} \notin Q_{r / 2}\right]}|1-\eta||\psi| d \lambda \mathbf{x}_{n}<\frac{\varepsilon}{3}
$$

and

$$
\int_{\left[\mathbf{x} \notin \varrho_{r / 2}\right]}|1-\eta||\psi| d \lambda \mathbf{x}<\frac{\varepsilon}{3} .
$$

Thus,

$$
\begin{gathered}
\left|\int_{\mathbb{R}^{p}} \psi d \lambda_{\mathbf{x}_{n}}-\int_{\mathbb{R}^{p}} \psi d \lambda\right| \leq\left|\int_{\mathbb{R}^{p}} \psi d \lambda_{\mathbf{x}_{n}}-\int_{\mathbb{R}^{p}} \psi \eta d \lambda \mathbf{x}_{n}\right|+ \\
\left|\int_{\mathbb{R}^{p}} \psi \eta d \lambda_{\mathbf{x}_{n}}-\int_{\mathbb{R}^{p}} \psi \eta d \lambda_{\mathbf{x}}\right|+\left|\int_{\mathbb{R}^{p}} \psi \eta d \lambda_{\mathbf{x}}-\int_{\mathbb{R}^{p}} \psi d \lambda \mathbf{x}\right| \\
\leq \frac{2 \varepsilon}{3}+\left|\int_{\mathbb{R}^{p}} \psi \eta d \lambda_{\mathbf{x}_{n}}-\int_{\mathbb{R}^{p}} \psi \eta d \lambda \mathbf{x}\right|<\varepsilon
\end{gathered}
$$

whenever $n$ is large enough by Lemma 59.18 .3 because $\psi \eta \in \mathfrak{S}$. This establishes the conclusion of the lemma in the case where $\psi$ is also infinitely differentiable. To consider the general case, let $\psi$ only be uniformly continuous and let $\psi_{k}=\psi * \phi_{k}$ where $\phi_{k}$ is a mollifier whose support is in $(-(1 / k),(1 / k))^{p}$. Then $\psi_{k}$ converges uniformly to $\psi$ and so the desired conclusion follows for $\psi$ after a routine estimate. This proves the lemma.

Definition 59.18.5 Let $\mu$ be a Radon measure on $\mathbb{R}^{p}$. A Borel set, $A$, is a $\mu$ continuity set if $\mu(\partial A)=0$ where $\partial A \equiv \bar{A} \backslash \operatorname{int}(A)$ and int denotes the interior.

The main result is the following continuity theorem. More can be said about the equivalence of various criteria [19].

Theorem 59.18.6 If $\phi_{\mathbf{X}_{n}}(\mathbf{t}) \rightarrow \phi_{\mathbf{X}}(\mathbf{t})$ then $\lambda_{\mathbf{X}_{n}}(A) \rightarrow \lambda_{\mathbf{X}}(A)$ whenever $A$ is a $\lambda_{\mathbf{X}}$ continuity set.

Proof: First suppose $K$ is a closed set and let

$$
\psi_{k}(\mathbf{x}) \equiv(1-k \operatorname{dist}(\mathbf{x}, K))^{+}
$$

Thus, since $K$ is closed $\lim _{k \rightarrow \infty} \psi_{k}(\mathbf{x})=\mathscr{X}_{K}(\mathbf{x})$. Choose $k$ large enough that

$$
\int_{\mathbb{R}^{p}} \psi_{k} d \lambda_{\mathbf{x}} \leq \lambda_{\mathbf{x}}(K)+\varepsilon
$$

Then by Lemma 59.18.4, applied to the bounded uniformly continuous function $\psi_{k}$,

$$
\lim \sup _{n \rightarrow \infty} \lambda_{\mathbf{x}_{n}}(K) \leq \lim \sup _{n \rightarrow \infty} \int \psi_{k} d \lambda_{\mathbf{x}_{n}}=\int \psi_{k} d \lambda_{\mathbf{x}} \leq \lambda_{\mathbf{x}}(K)+\varepsilon
$$

Since $\varepsilon$ is arbitrary, this shows

$$
\lim \sup _{n \rightarrow \infty} \lambda_{\mathbf{x}_{n}}(K) \leq \lambda_{\mathbf{x}}(K)
$$

for all $K$ closed.
Next suppose $V$ is open and let

$$
\psi_{k}(\mathbf{x})=1-\left(1-k \operatorname{dist}\left(\mathbf{x}, \mathbf{V}^{C}\right)\right)^{+}
$$

Thus $\psi_{k}(\mathbf{x}) \in[0,1], \psi_{k}=1$ if $\operatorname{dist}\left(\mathbf{x}, V^{C}\right) \geq 1 / k$, and $\psi_{k}=0$ on $V^{C}$. Since $V$ is open, it follows

$$
\lim _{k \rightarrow \infty} \psi_{k}(\mathbf{x})=\mathscr{X}_{V}(\mathbf{x})
$$

Choose $k$ large enough that

$$
\int \psi_{k} d \lambda_{\mathbf{x}} \geq \lambda_{\mathbf{x}}(V)-\varepsilon
$$

Then by Lemma 59.18.4,

$$
\begin{gathered}
{\lim \inf _{n \rightarrow \infty}}^{\lambda_{\mathbf{x}_{n}}(V) \geq \lim _{\inf _{n \rightarrow \infty}} \int \psi_{k}(\mathbf{x}) d \lambda_{\mathbf{x}_{n}}=} \\
=\int \psi_{k}(\mathbf{x}) d \lambda_{\mathbf{x}} \geq \lambda_{\mathbf{x}}(V)-\varepsilon
\end{gathered}
$$

and since $\varepsilon$ is arbitrary,

$$
\lim _{n \rightarrow \infty} \inf _{\mathbf{x}_{n}}(V) \geq \lambda_{\mathbf{x}}(V)
$$

Now let $\lambda_{\mathbf{x}}(\partial A)=0$ for $A$ a Borel set.

$$
\begin{aligned}
\lambda_{\mathbf{X}}(\operatorname{int}(A)) & \leq \lim \inf _{n \rightarrow \infty} \lambda_{\mathbf{x}_{n}}(\operatorname{int}(A)) \leq \lim \inf _{n \rightarrow \infty} \lambda_{\mathbf{x}_{n}}(A) \leq \\
\lim \sup _{n \rightarrow \infty} \lambda_{\mathbf{x}_{n}}(A) & \leq \lim \sup _{n \rightarrow \infty} \lambda_{\mathbf{x}_{n}}(\bar{A}) \leq \lambda_{\mathbf{x}}(\bar{A})
\end{aligned}
$$

But $\lambda_{\mathbf{X}}(\operatorname{int}(A))=\lambda_{\mathbf{X}}(\bar{A})$ by assumption and so $\lim _{n \rightarrow \infty} \lambda_{\mathbf{x}_{n}}(A)=\lambda_{\mathbf{X}}(A)$ as claimed. This proves the theorem.

As an application of this theorem the following is a version of the central limit theorem in the situation in which the limit distribution is multivariate normal. It concerns a sequence of random vectors, $\left\{\mathbf{X}_{k}\right\}_{k=1}^{\infty}$, which are identically distributed, have finite mean $\mathbf{m}$, and satisfy

$$
\begin{equation*}
E\left(\left|\mathbf{X}_{k}\right|^{2}\right)<\infty \tag{59.18.41}
\end{equation*}
$$

Definition 59.18.7 For $\mathbf{X}$ a random vector with values in $\mathbb{R}^{p}$, let

$$
F_{\mathbf{X}}(\mathbf{x}) \equiv P\left(\left\{X_{j} \leq x_{j} \text { for each } j=1,2, \ldots, p\right\}\right)
$$

Theorem 59.18.8 Let $\left\{\mathbf{X}_{k}\right\}_{k=1}^{\infty}$ be random vectors satisfying 59.18.41, which are independent and identically distributed with mean $\mathbf{m}$ and positive definite covariance $\mathbf{\square} \equiv$ $E\left((\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^{*}\right)$. Let

$$
\begin{equation*}
\mathbf{Z}_{n} \equiv \sum_{j=1}^{n} \frac{\mathbf{X}_{j}-\mathbf{m}}{\sqrt{n}} \tag{59.18.42}
\end{equation*}
$$

Then for $\mathbf{Z} \sim N_{p}(\mathbf{0}, \llbracket)$,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} F_{\mathbf{Z}_{n}}(\mathbf{x})=F_{\mathbf{Z}}(\mathbf{x}) \tag{59.18.43}
\end{equation*}
$$

for all $\mathbf{x}$.
Proof: The characteristic function of $\mathbf{Z}_{n}$ is given by

$$
\phi_{\mathbf{Z}_{n}}(\mathbf{t})=E\left(e^{i \mathbf{t} \cdot \sum_{j=1}^{n} \frac{\mathbf{x}_{j}-\mathbf{m}}{\sqrt{n}}}\right)=\prod_{j=1}^{n} E\left(e^{i \mathbf{t} \cdot\left(\frac{\mathbf{x}_{j}-\mathbf{m}}{\sqrt{n}}\right)}\right) .
$$

By Taylor's theorem applied to real and imaginary parts of $e^{i x}$, it follows

$$
e^{i x}=1+i x-f(x) \frac{x^{2}}{2}
$$

where $|f(x)|<2$ and

$$
\lim _{x \rightarrow 0} f(x)=1
$$

Denoting $\mathbf{X}_{j}$ as $\mathbf{X}$, this implies

$$
e^{i \mathbf{t} \cdot\left(\frac{\mathbf{X}-\mathbf{m}}{\sqrt{n}}\right)}=1+i \mathbf{i} \cdot \frac{\mathbf{X}-\mathbf{m}}{\sqrt{n}}-f\left(\mathbf{t} \cdot\left(\frac{\mathbf{X}-\mathbf{m}}{\sqrt{n}}\right)\right) \frac{(\mathbf{t} \cdot(\mathbf{X}-\mathbf{m}))^{2}}{2 n}
$$

Thus

$$
\begin{aligned}
& e^{i \mathbf{t} \cdot\left(\frac{\mathbf{x}-\mathbf{m}}{\sqrt{n}}\right)}=1+i \mathbf{t} \cdot \frac{\mathbf{X}-\mathbf{m}}{\sqrt{n}}-\frac{(\mathbf{t} \cdot(\mathbf{X}-\mathbf{m}))^{2}}{2 n} \\
& +\left(1-f\left(\mathbf{t} \cdot\left(\frac{\mathbf{X}-\mathbf{m}}{\sqrt{n}}\right)\right)\right) \frac{(\mathbf{t} \cdot(\mathbf{X}-\mathbf{m}))^{2}}{2 n}
\end{aligned}
$$

Thus

$$
\begin{gather*}
\phi_{\mathbf{Z}_{n}}(\mathbf{t})=\prod_{j=1}^{n}\left[1-E\left(\frac{(\mathbf{t} \cdot(\mathbf{X}-\mathbf{m}))^{2}}{2 n}\right)\right. \\
\left.+E\left(\left(1-f\left(\mathbf{t} \cdot\left(\frac{\mathbf{X}-\mathbf{m}}{\sqrt{n}}\right)\right)\right) \frac{(\mathbf{t} \cdot(\mathbf{X}-\mathbf{m}))^{2}}{2 n}\right)\right] \\
=\prod_{j=1}^{n}\left[1-\frac{1}{2 n} \mathbf{t}^{*} \Sigma \mathbf{t}+\right. \\
\left.\frac{1}{2 n} E\left(\left(1-f\left(\mathbf{t} \cdot\left(\frac{\mathbf{X}-\mathbf{m}}{\sqrt{n}}\right)\right)\right)(\mathbf{t} \cdot(\mathbf{X}-\mathbf{m}))^{2}\right)\right] . \tag{59.18.44}
\end{gather*}
$$

(Note $(\mathbf{t} \cdot(\mathbf{X}-\mathbf{m}))^{2}=\mathbf{t}^{*}(\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^{*} \mathbf{t}$.) Now here is a simple inequality for complex numbers whose moduli are no larger than one. I will give a proof of this at the end. It follows easily by induction.

$$
\begin{equation*}
\left|z_{1} \cdots z_{n}-w_{1} \cdots w_{n}\right| \leq \sum_{k=1}^{n}\left|z_{k}-w_{k}\right| \tag{59.18.45}
\end{equation*}
$$

Also for each $\mathbf{t}$, and all $n$ large enough,

$$
\left|\frac{1}{2 n} E\left(\left(1-f\left(\mathbf{t} \cdot\left(\frac{\mathbf{X}-\mathbf{m}}{\sqrt{n}}\right)\right)\right)(\mathbf{t} \cdot(\mathbf{X}-\mathbf{m}))^{2}\right)\right|<1
$$

Applying 59.18.45 to 59.18.44,

$$
\begin{aligned}
\phi_{\mathbf{Z}_{n}}(\mathbf{t}) & =\left(\prod_{j=1}^{n}\left(1-\frac{1}{2 n} \mathbf{t}^{*} \Sigma \mathbf{t}\right)\right)+e_{n} \\
& =\left(1-\frac{1}{2 n} \mathbf{t}^{*} \Sigma \mathbf{t}\right)^{n}+e_{n}
\end{aligned}
$$

where

$$
\begin{aligned}
\left|e_{n}\right| & \leq \sum_{j=1}^{n}\left|\frac{1}{2 n} E\left(\left(1-f\left(\mathbf{t} \cdot\left(\frac{\mathbf{X}-\mathbf{m}}{\sqrt{n}}\right)\right)\right)(\mathbf{t} \cdot(\mathbf{X}-\mathbf{m}))^{2}\right)\right| \\
& =\frac{1}{2}\left|E\left(\left(1-f\left(\mathbf{t} \cdot\left(\frac{\mathbf{X}-\mathbf{m}}{\sqrt{n}}\right)\right)\right)(\mathbf{t} \cdot(\mathbf{X}-\mathbf{m}))^{2}\right)\right|
\end{aligned}
$$

which converges to 0 as $n \rightarrow \infty$ by the Dominated Convergence theorem. Therefore,

$$
\lim _{n \rightarrow \infty}\left|\phi_{\mathbf{Z}_{n}}(\mathbf{t})-\left(1-\frac{\mathbf{t}^{*} \Sigma \mathbf{t}}{2 n}\right)^{n}\right|=0
$$

and so

$$
\lim _{n \rightarrow \infty} \phi_{\mathbf{Z}_{n}}(\mathbf{t})=e^{-\frac{1}{2} \mathbf{t}^{*} \Sigma \mathbf{t}}=\phi_{\mathbf{Z}}(\mathbf{t})
$$

where $\mathbf{Z} \sim N_{p}(\mathbf{0}, \boldsymbol{\square})$. Therefore, $F_{\mathbf{Z}_{n}}(\mathbf{x}) \rightarrow F_{\mathbf{Z}}(\mathbf{x})$ for all $\mathbf{x}$ because $R_{\mathbf{x}} \equiv \prod_{k=1}^{p}\left(-\infty, x_{k}\right]$ is a set of $\lambda_{\mathbf{Z}}$ continuity due to the assumption that $\lambda_{\mathbf{Z}} \ll m_{p}$ which is implied by $\mathbf{Z} \sim N_{p}(\mathbf{0}, \boldsymbol{\square})$. This proves the theorem.

Here is the proof of the little inequality used above. The inequality is obviously true if $n=1$. Assume it is true for $n$. Then since all the numbers have absolute value no larger than one,

$$
\begin{aligned}
& \left|\prod_{i=1}^{n+1} z_{i}-\prod_{i=1}^{n+1} w_{i}\right| \leq\left|\prod_{i=1}^{n+1} z_{i}-z_{n+1} \prod_{i=1}^{n} w_{i}\right| \\
& \quad+\left|z_{n+1} \prod_{i=1}^{n} w_{i}-\prod_{i=1}^{n+1} w_{i}\right| \\
& \leq\left|\prod_{i=1}^{n} z_{i}-\prod_{i=1}^{n} w_{i}\right|+\left|z_{n+1}-w_{n+1}\right| \\
& \leq \sum_{k=1}^{n+1}\left|z_{k}-w_{k}\right|
\end{aligned}
$$

by induction.
Suppose $\mathbf{X}$ is a random vector with covariance $\Sigma$ and mean $\mathbf{m}$, and suppose also that $\Sigma^{-1}$ exists. Consider $\Sigma^{-(1 / 2)}(\mathbf{X}-\mathbf{m}) \equiv \mathbf{Y}$. Then $E(\mathbf{Y})=0$ and

$$
\begin{aligned}
E\left(\mathbf{Y} \mathbf{Y}^{*}\right) & =E\left(\Sigma^{-(1 / 2)}(\mathbf{X}-\mathbf{m})\left(\mathbf{X}^{*}-\mathbf{m}\right) \Sigma^{-(1 / 2)}\right) \\
& =\Sigma^{-(1 / 2)} E\left((\mathbf{X}-\mathbf{m})\left(\mathbf{X}^{*}-\mathbf{m}\right)\right) \Sigma^{-(1 / 2)}=I
\end{aligned}
$$

Thus $\mathbf{Y}$ has zero mean and covariance $I$. This implies the following corollary to Theorem 59.18.8.

Corollary 59.18.9 Let independent identically distributed random variables,

$$
\left\{\mathbf{X}_{j}\right\}_{j=1}^{\infty}
$$

have mean $\mathbf{m}$ and positive definite covariance $\llbracket$ where $\mathbf{■}^{-1}$ exists. Then if

$$
\mathbf{Z}_{n} \equiv \sum_{j=1}^{n} \mathbf{m}^{-(1 / 2)} \frac{\left(\mathbf{X}_{j}-\mathbf{m}\right)}{\sqrt{n}}
$$

it follows that for $\mathbf{Z} \sim N_{p}(\mathbf{0}, I)$,

$$
F_{\mathbf{Z}_{n}}(\mathbf{x}) \rightarrow F_{\mathbf{Z}}(\mathbf{x})
$$

for all $\mathbf{x}$.

### 59.19 Characteristic Functions, Prokhorov Theorem

Recall one can define the characteristic function of a probability measure. In a sense it is more natural.

Definition 59.19.1 Let $\mu$ be a probability measure defined on the Borel sets of $\mathbb{R}^{p}$. Then

$$
\phi_{\mu}(\mathbf{t}) \equiv \int_{\mathbb{R}^{p}} e^{i t \cdot \mathbf{x}} d \mu
$$

Also $\left\{\mu_{n}\right\}_{n=1}^{\infty}$ is called "tight" if for all $\varepsilon>0$ there exists a compact set, $K_{\varepsilon}$ such that

$$
\mu_{n}\left(\left[\mathbf{x} \notin K_{\varepsilon}\right]\right)<\varepsilon
$$

for all $\mu_{n}$.
Then there is a version of Lemma 59.18.2 whose proof is identical to the proof of that lemma.

Lemma 59.19.2 If $\left\{\mu_{n}\right\}$ is a sequence of Borel probability measures defined on the Borel sets of $\mathbb{R}^{p}$ such that

$$
\lim _{n \rightarrow \infty} \phi_{\mu_{n}}(\mathbf{t})=\psi(\mathbf{t})
$$

for all $\mathbf{t}$, where $\psi(\mathbf{0})=1$ and $\psi$ is continuous at $\mathbf{0}$, then $\left\{\mu_{n}\right\}_{n=1}^{\infty}$ is tight.
Proof: Let $\mathbf{e}_{j}$ be the $j^{t h}$ standard unit basis vector. Letting $\mathbf{t}=t \mathbf{e}_{j}$ in the definition,

$$
\begin{align*}
& \left|\frac{1}{u} \int_{-u}^{u}\left(1-\phi_{\mu_{n}}\left(t \mathbf{e}_{j}\right)\right) d t\right|  \tag{59.19.46}\\
= & \left|\frac{1}{u} \int_{-u}^{u}\left(1-\int_{\mathbb{R}^{p}} e^{i t x_{j}} d \mu_{n}(x)\right) d t\right| \\
= & \left|\frac{1}{u} \int_{-u}^{u}\left(\int_{\mathbb{R}^{p}}\left(1-e^{i t x_{j}}\right) d \mu_{n}(x)\right) d t\right| \\
= & \left|\int_{\mathbb{R}^{p}} \frac{1}{u} \int_{-u}^{u}\left(1-e^{i t x_{j}}\right) d t d \mu_{n}(x)\right| \\
= & \left|2 \int_{\mathbb{R}^{p}}\left(1-\frac{\sin \left(u x_{j}\right)}{u x_{j}}\right) d \mu_{n}(x)\right| \\
\geq & 2 \int_{\left[\left|x_{j}\right| \geq \frac{2}{u}\right]}\left(1-\frac{1}{\left|u x_{j}\right|}\right) d \mu_{n}(x) \\
\geq & 2 \int_{\left[\left|x_{j}\right| \geq \frac{2}{u}\right]}\left(1-\frac{1}{|u|(2 / u)}\right) d \mu_{n}(x)
\end{align*}
$$

$$
\begin{aligned}
& =\int_{\left[\left|x_{j}\right| \geq \frac{2}{u}\right]} 1 d \mu_{n}(x) \\
& =\mu_{n}\left(\left[\mathbf{x}:\left|x_{j}\right| \geq \frac{2}{u}\right]\right)
\end{aligned}
$$

If $\varepsilon>0$ is given, there exists $r>0$ such that if $u \leq r$,

$$
\frac{1}{u} \int_{-u}^{u}\left(1-\psi\left(t \mathbf{e}_{j}\right)\right) d t<\varepsilon / p
$$

for all $j=1, \cdots, p$ and so, by the dominated convergence theorem, the same is true with $\phi_{\mu_{n}}$ in place of $\psi$ provided $n$ is large enough, say $n \geq N(u)$. Thus, from 59.19.46, if $u \leq r$, and $n \geq N(u)$,

$$
\mu_{n}\left(\left[\mathbf{x}:\left|x_{j}\right| \geq \frac{2}{u}\right]\right)<\varepsilon / p
$$

for all $j \in\{1, \cdots, p\}$. It follows that for $u \leq r$ and $n \geq N(u)$,

$$
\mu_{n}\left(\left[\mathbf{x}:\|\mathbf{x}\|_{\infty} \geq \frac{2}{u}\right]\right)<\varepsilon .
$$

because

$$
\left[\mathbf{x}:\|\mathbf{x}\|_{\infty} \geq \frac{2}{u}\right] \subseteq \cup_{j=1}^{p}\left[\mathbf{x}:\left|x_{j}\right| \geq \frac{2}{u}\right]
$$

This proves the lemma because there are only finitely many measures, $\mu_{n}$ for $n<N(u)$ and the compact set can be enlarged finitely many times to obtain a single compact set, $K_{\mathcal{\varepsilon}}$ such that for all $n, \mu_{n}\left(\left[\mathbf{x} \notin K_{\varepsilon}\right]\right)<\varepsilon$.

As before, there are simple modifications of Lemmas 59.18.3 and 59.18.4. The first of these is as follows.

Lemma 59.19.3 If $\phi_{\mu_{n}}(\mathbf{t}) \rightarrow \phi_{\mu}(\mathbf{t})$ for all $\mathbf{t}$, then whenever $\psi \in \mathfrak{S}$, the Schwartz class,

$$
\mu_{n}(\psi) \equiv \int_{\mathbb{R}^{p}} \psi(\mathbf{y}) d \mu_{n}(y) \rightarrow \int_{\mathbb{R}^{p}} \psi(\mathbf{y}) d \mu(y) \equiv \mu(\psi)
$$

as $n \rightarrow \infty$.
Proof: By definition,

$$
\phi_{\mu}(\mathbf{y}) \equiv \int_{\mathbb{R}^{p}} e^{i \mathbf{y} \cdot \mathbf{x}} d \mu(x)
$$

Also remember the inverse Fourier transform. Letting $\psi \in \mathfrak{S}$, the Schwartz class,

$$
\begin{aligned}
F^{-1}(\mu)(\psi) & \equiv \mu\left(F^{-1} \psi\right) \equiv \int_{\mathbb{R}^{p}} F^{-1} \psi d \mu \\
& =\frac{1}{(2 \pi)^{p / 2}} \int_{\mathbb{R}^{p}} \int_{\mathbb{R}^{p}} e^{i \mathbf{y} \cdot \mathbf{x}} \psi(\mathbf{x}) d x d \mu(y) \\
& =\frac{1}{(2 \pi)^{p / 2}} \int_{\mathbb{R}^{p}} \psi(\mathbf{x}) \int_{\mathbb{R}^{p}} e^{i \mathbf{y} \cdot \mathbf{x}} d \mu(y) d x \\
& =\frac{1}{(2 \pi)^{p / 2}} \int_{\mathbb{R}^{p}} \psi(\mathbf{x}) \phi_{\mu}(\mathbf{x}) d x
\end{aligned}
$$

and so, considered as elements of $\mathfrak{S}^{*}$,

$$
F^{-1}(\mu)=\phi_{\mu}(\cdot)(2 \pi)^{-(p / 2)} \in L^{\infty}
$$

By the dominated convergence theorem

$$
\begin{aligned}
(2 \pi)^{p / 2} F^{-1}\left(\mu_{n}\right)(\psi) & \equiv \int_{\mathbb{R}^{p}} \phi_{\mu_{n}}(\mathbf{t}) \psi(\mathbf{t}) d t \\
& \rightarrow \int_{\mathbb{R}^{p}} \phi_{\mu}(\mathbf{t}) \psi(\mathbf{t}) d t \\
& =(2 \pi)^{p / 2} F^{-1}(\mu)(\psi)
\end{aligned}
$$

whenever $\psi \in \mathfrak{S}$. Thus

$$
\begin{aligned}
\mu_{n}(\psi) & =F F^{-1} \mu_{n}(\psi) \equiv F^{-1} \mu_{n}(F \psi) \rightarrow F^{-1} \mu(F \psi) \\
& \equiv F^{-1} F \mu(\psi)=\mu(\psi)
\end{aligned}
$$

The version of Lemma 59.18.4 is the following.
Lemma 59.19.4 If $\phi_{\mu_{n}}(\mathbf{t}) \rightarrow \phi_{\mu}(\mathbf{t})$ where $\left\{\mu_{n}\right\}$ and $\mu$ are probability measures defined on the Borel sets of $\mathbb{R}^{p}$, then if $\psi$ is any bounded uniformly continuous function,

$$
\lim _{n \rightarrow \infty} \int_{\mathbb{R}^{p}} \psi d \mu_{n}=\int_{\mathbb{R}^{p}} \psi d \mu
$$

Proof: Let $\varepsilon>0$ be given, let $\psi$ be a bounded function in $C^{\infty}\left(\mathbb{R}^{p}\right)$. Now let $\eta \in$ $C_{c}^{\infty}\left(Q_{r}\right)$ where $Q_{r} \equiv[-r, r]^{p}$ satisfy the additional requirement that $\eta=1$ on $Q_{r / 2}$ and $\eta(\mathbf{x}) \in[0,1]$ for all $\mathbf{x}$. By Lemma 59.19.2 the set, $\left\{\mu_{n}\right\}_{n=1}^{\infty}$, is tight and so if $\varepsilon>0$ is given, there exists $r$ sufficiently large such that for all $n$,

$$
\int_{\left[\mathbf{x} \notin Q_{r / 2}\right]}|1-\eta||\psi| d \mu_{n}<\frac{\varepsilon}{3}
$$

and

$$
\int_{\left[\mathbf{x} \notin Q_{r / 2}\right]}|1-\eta||\psi| d \mu<\frac{\varepsilon}{3}
$$

Thus,

$$
\begin{gathered}
\left|\int_{\mathbb{R}^{p}} \psi d \mu_{n}-\int_{\mathbb{R}^{p}} \psi d \mu\right| \leq\left|\int_{\mathbb{R}^{p}} \psi d \mu_{n}-\int_{\mathbb{R}^{p}} \psi \eta d \mu_{n}\right|+ \\
\left|\int_{\mathbb{R}^{p}} \psi \eta d \mu_{n}-\int_{\mathbb{R}^{p}} \psi \eta d \mu\right|+\left|\int_{\mathbb{R}^{p}} \psi \eta d \mu-\int_{\mathbb{R}^{p}} \psi d \mu\right| \\
\leq \frac{2 \varepsilon}{3}+\left|\int_{\mathbb{R}^{p}} \psi \eta d \mu_{n}-\int_{\mathbb{R}^{p}} \psi \eta d \mu\right|<\varepsilon
\end{gathered}
$$

whenever $n$ is large enough by Lemma 59.19 .3 because $\psi \eta \in \mathfrak{S}$. This establishes the conclusion of the lemma in the case where $\psi$ is also infinitely differentiable. To consider the general case, let $\psi$ only be uniformly continuous and let $\psi_{k}=\psi * \phi_{k}$ where $\phi_{k}$ is a
mollifier whose support is in $(-(1 / k),(1 / k))^{p}$. Then $\psi_{k}$ converges uniformly to $\psi$ and so the desired conclusion follows for $\psi$ after a routine estimate.

The next theorem is really important. It gives the existence of a measure based on an assumption that a set of measures is tight. The next theorem is Prokhorov's theorem about a tight set of measures. Recall that $\Lambda$ is tight means that for every $\varepsilon>0$ there exists $K$ compact such that $\mu\left(K^{C}\right)<\varepsilon$ for all $\mu \in \Lambda$.

Theorem 59.19.5 Let $\Lambda=\left\{\mu_{n}\right\}_{n=1}^{\infty}$ be a sequence of probability measures defined on the Borel sets of $\mathbb{R}^{p}$. If $\Lambda$ is tight then there exists a probability measure, $\lambda$ and a subsequence of $\left\{\mu_{n}\right\}_{n=1}^{\infty}$, still denoted by $\left\{\mu_{n}\right\}_{n=1}^{\infty}$ such that whenever $\phi$ is a continuous bounded complex valued function defined on $E$,

$$
\lim _{n \rightarrow \infty} \int \phi d \mu_{n}=\int \phi d \lambda
$$

Proof: By tightness, there exists an increasing sequence of compact sets, $\left\{K_{n}\right\}$ such that

$$
\mu\left(K_{n}\right)>1-\frac{1}{n}
$$

for all $\mu \in \Lambda$. Now letting $\mu \in \Lambda$ and $\phi \in C\left(K_{n}\right)$ such that $\|\phi\|_{\infty} \leq 1$, it follows

$$
\left|\int_{K_{n}} \phi d \mu\right| \leq \mu\left(K_{n}\right) \leq 1
$$

and so the restrictions of the measures of $\Lambda$ to $K_{n}$ are contained in the unit ball of $C\left(K_{n}\right)^{\prime}$. Recall from the Riesz representation theorem, the dual space of $C\left(K_{n}\right)$ is a space of complex Borel measures. Theorem 17.5 .5 on Page 462 implies the unit ball of $C\left(K_{n}\right)^{\prime}$ is weak * sequentially compact. This follows from the observation that $C\left(K_{n}\right)$ is separable which follows easily from the Weierstrass approximation theorem. Thus the unit ball in $C\left(K_{n}\right)^{\prime}$ is actually metrizable by Theorem 17.5.5 on Page 462. Therefore, there exists a subsequence of $\Lambda,\left\{\mu_{1 k}\right\}$ such that their restrictions to $K_{1}$ converge weak $*$ to a measure, $\lambda_{1} \in C\left(K_{1}\right)^{\prime}$. That is, for every $\phi \in C\left(K_{1}\right)$,

$$
\lim _{k \rightarrow \infty} \int_{K_{1}} \phi d \mu_{1 k}=\int_{K_{1}} \phi d \lambda_{1}
$$

By the same reasoning, there exists a further subsequence $\left\{\mu_{2 k}\right\}$ such that the restrictions of these measures to $K_{2}$ converge weak $*$ to a measure $\lambda_{2} \in C\left(K_{2}\right)^{\prime}$ etc. Continuing this way,

$$
\begin{gathered}
\mu_{11}, \mu_{12}, \mu_{13}, \cdots \rightarrow \text { Weak } * \text { in } C\left(K_{1}\right)^{\prime} \\
\mu_{21}, \mu_{22}, \mu_{23}, \cdots \rightarrow \text { Weak } * \text { in } C\left(K_{2}\right)^{\prime} \\
\mu_{31}, \mu_{32}, \mu_{33}, \cdots \rightarrow \text { Weak } * \text { in } C\left(K_{3}\right)^{\prime} \\
\vdots
\end{gathered}
$$

Here the $j^{t h}$ sequence is a subsequence of the $(j-1)^{t h}$. Let $\lambda_{n}$ denote the measure in $C\left(K_{n}\right)^{\prime}$ to which the sequence $\left\{\mu_{n k}\right\}_{k=1}^{\infty}$ converges weak $*$. Let $\left\{\mu_{n}\right\} \equiv\left\{\mu_{n n}\right\}$, the diagonal sequence. Thus this sequence is ultimately a subsequence of every one of the above sequences and so $\mu_{n}$ converges weak $*$ in $C\left(K_{m}\right)^{\prime}$ to $\lambda_{m}$ for each $m$.

Claim: For $p>n$, the restriction of $\lambda_{p}$ to the Borel sets of $K_{n}$ equals $\lambda_{n}$.
Proof of claim: Let $H$ be a compact subset of $K_{n}$. Then there are sets, $V_{l}$ open in $K_{n}$ which are decreasing and whose intersection equals $H$. This follows because this is a metric space. Then let $H \prec \phi_{l} \prec V_{l}$. It follows

$$
\begin{aligned}
\lambda_{n}\left(V_{l}\right) & \geq \int_{K_{n}} \phi_{l} d \lambda_{n}=\lim _{k \rightarrow \infty} \int_{K_{n}} \phi_{l} d \mu_{k} \\
& =\lim _{k \rightarrow \infty} \int_{K_{p}} \phi_{l} d \mu_{k}=\int_{K_{p}} \phi_{l} d \lambda_{p} \geq \lambda_{p}(H) .
\end{aligned}
$$

Now considering the ends of this inequality, let $l \rightarrow \infty$ and pass to the limit to conclude

$$
\lambda_{n}(H) \geq \lambda_{p}(H)
$$

Similarly,

$$
\begin{aligned}
\lambda_{n}(H) & \leq \int_{K_{n}} \phi_{l} d \lambda_{n}=\lim _{k \rightarrow \infty} \int_{K_{n}} \phi_{l} d \mu_{k} \\
& =\lim _{k \rightarrow \infty} \int_{K_{p}} \phi_{l} d \mu_{k}=\int_{K_{p}} \phi_{l} d \lambda_{p} \leq \lambda_{p}\left(V_{l}\right) .
\end{aligned}
$$

Then passing to the limit as $l \rightarrow \infty$, it follows

$$
\lambda_{n}(H) \leq \lambda_{p}(H)
$$

Thus the restriction of $\lambda_{p},\left.\lambda_{p}\right|_{K_{n}}$ to the compact sets of $K_{n}$ equals $\lambda_{n}$. Then by inner regularity it follows the two measures, $\left.\lambda_{p}\right|_{K_{n}}$, and $\lambda_{n}$ are equal on all Borel sets of $K_{n}$. Recall that for finite measures on the Borel sets of separable metric spaces, regularity is obtained for free.

It is fairly routine to exploit regularity of the measures to verify that $\lambda_{m}(F) \geq 0$ for all $F$ a Borel subset of $K_{m}$. (Whenever $\phi \geq 0, \int_{K_{m}} \phi d \lambda_{m} \geq 0$ because $\int_{K_{m}} \phi d \mu_{k} \geq 0$. Now you can approximate $\mathscr{X}_{F}$ with a suitable nonnegative $\phi$ using regularity of the measure.) Also, letting $\phi \equiv 1$,

$$
\begin{equation*}
1 \geq \lambda_{m}\left(K_{m}\right) \geq 1-\frac{1}{m} \tag{59.19.47}
\end{equation*}
$$

Define for $F$ a Borel set,

$$
\lambda(F) \equiv \lim _{n \rightarrow \infty} \lambda_{n}\left(F \cap K_{n}\right)
$$

The limit exists because the sequence on the right is increasing due to the above observation that $\lambda_{n}=\lambda_{m}$ on the Borel subsets of $K_{m}$ whenever $n>m$. Thus for $n>m$

$$
\lambda_{n}\left(F \cap K_{n}\right) \geq \lambda_{n}\left(F \cap K_{m}\right)=\lambda_{m}\left(F \cap K_{m}\right)
$$

Now let $\left\{F_{k}\right\}$ be a sequence of disjoint Borel sets. Then

$$
\begin{aligned}
\lambda\left(\cup_{k=1}^{\infty} F_{k}\right) & \equiv \lim _{n \rightarrow \infty} \lambda_{n}\left(\cup_{k=1}^{\infty} F_{k} \cap K_{n}\right)=\lim _{n \rightarrow \infty} \lambda_{n}\left(\cup_{k=1}^{\infty}\left(F_{k} \cap K_{n}\right)\right) \\
& =\lim _{n \rightarrow \infty} \sum_{k=1}^{\infty} \lambda_{n}\left(F_{k} \cap K_{n}\right)=\sum_{k=1}^{\infty} \lambda\left(F_{k}\right)
\end{aligned}
$$

the last equation holding by the monotone convergence theorem.
It remains to verify

$$
\lim _{k \rightarrow \infty} \int \phi d \mu_{k}=\int \phi d \lambda
$$

for every $\phi$ bounded and continuous. This is where tightness is used again. Suppose $\|\phi\|_{\infty}<M$. Then as noted above,

$$
\lambda_{n}\left(K_{n}\right)=\lambda\left(K_{n}\right)
$$

because for $p>n, \lambda_{p}\left(K_{n}\right)=\lambda_{n}\left(K_{n}\right)$ and so letting $p \rightarrow \infty$, the above is obtained. Also, from 59.19.47,

$$
\begin{aligned}
\lambda\left(K_{n}^{C}\right) & =\lim _{p \rightarrow \infty} \lambda_{p}\left(K_{n}^{C} \cap K_{p}\right) \\
& \leq \lim _{p \rightarrow \infty}\left(\lambda_{p}\left(K_{p}\right)-\lambda_{p}\left(K_{n}\right)\right) \\
& \leq \lim _{p \rightarrow \infty}\left(\lambda_{p}\left(K_{p}\right)-\lambda_{n}\left(K_{n}\right)\right) \\
& \leq \lim \sup _{p \rightarrow \infty}\left(1-\left(1-\frac{1}{n}\right)\right)=\frac{1}{n}
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\mid \int \phi d \mu_{k} & -\int \phi d \lambda\left|\leq\left|\int_{K_{n}^{C}} \phi d \mu_{k}+\int_{K_{n}} \phi d \mu_{k}-\left(\int_{K_{n}} \phi d \lambda+\int_{K_{n}^{C}} \phi d \lambda\right)\right|\right. \\
& \leq\left|\int_{K_{n}} \phi d \mu_{k}-\int_{K_{n}} \phi d \lambda_{n}\right|+\left|\int_{K_{n}^{C}} \phi d \mu_{k}-\int_{K_{n}^{C}} \phi d \lambda\right| \\
& \leq\left|\int_{K_{n}} \phi d \mu_{k}-\int_{K_{n}} \phi d \lambda_{n}\right|+\left|\int_{K_{n}^{C}} \phi d \mu_{k}\right|+\left|\int_{K_{n}^{C}} \phi d \lambda\right| \\
& \leq\left|\int_{K_{n}} \phi d \mu_{k}-\int_{K_{n}} \phi d \lambda_{n}\right|+\frac{M}{n}+\frac{M}{n}
\end{aligned}
$$

First let $n$ be so large that $2 M / n<\varepsilon / 2$ and then pick $k$ large enough that the above expression is less than $\varepsilon$.

Definition 59.19.6 Let $\mu,\left\{\mu_{n}\right\}$ be probability measures defined on the Borel sets of $\mathbb{R}^{p}$ and let the sequence of probability measures, $\left\{\mu_{n}\right\}$ satisfy

$$
\lim _{n \rightarrow \infty} \int \phi d \mu_{n}=\int \phi d \mu
$$

for every $\phi$ a bounded continuous function. Then $\mu_{n}$ is said to converge weakly to $\mu$.
With the above, it is possible to prove the following amazing theorem of Levy.

Theorem 59.19.7 Suppose $\left\{\mu_{n}\right\}$ is a sequence of probability measures defined on the Borel sets of $\mathbb{R}^{p}$ and let $\left\{\phi_{\mu_{n}}\right\}$ denote the corresponding sequence of characteristic functions. If there exists $\psi$ which is continuous at $\mathbf{0}, \psi(\mathbf{0})=1$, and for all $\mathbf{t}$,

$$
\phi_{\mu_{n}}(\mathbf{t}) \rightarrow \psi(\mathbf{t})
$$

then there exists a probability measure, $\lambda$ defined on the Borel sets of $\mathbb{R}^{p}$ and

$$
\phi_{\lambda}(\mathbf{t})=\psi(\mathbf{t}) .
$$

That is, $\psi$ is a characteristic function of a probability measure. Also, $\left\{\mu_{n}\right\}$ converges weakly to $\lambda$.

Proof: By Lemma 59.19.2 $\left\{\mu_{n}\right\}$ is tight. Therefore, there exists a subsequence $\left\{\mu_{n_{k}}\right\}$ converging weakly to a probability measure, $\lambda$. In particular,

$$
\begin{aligned}
\phi_{\lambda}(\mathbf{t}) & =\int e^{i \mathbf{t} \cdot \mathbf{x}} d \lambda(x)=\lim _{n \rightarrow \infty} \int e^{i \mathbf{t} \cdot \mathbf{x}} d \mu_{n_{k}}(x) \\
& =\lim _{n \rightarrow \infty} \phi_{\mu_{n_{k}}}(\mathbf{t})=\psi(\mathbf{t})
\end{aligned}
$$

The last claim follows from this and Lemma 59.19.4.
Note how it was only necessary to assume $\psi(\mathbf{0})=1$ and $\psi$ is continuous at $\mathbf{0}$ in order to conclude that $\psi$ is a characteristic function. Thus you find that $|\psi(\mathbf{t})| \leq 1$ for free. This helps to see why Prokhorov's and Levy's theorems are so amazing.

### 59.20 Generalized Multivariate Normal

In this section is a further explanation of generalized multivariable normal random variables. Recall that these have characteristic function equal to $e^{i \cdot \mathbf{m}} e^{-\frac{1}{2} \mathbf{t}^{*} \Sigma \mathbf{t}}$ where $\Sigma \geq 0, \Sigma=$ $\Sigma^{*}$. The new detail is the case that $\operatorname{det}(\Sigma)=0$.

Definition 59.20.1 A random vector, $\mathbf{X}$, with values in $\mathbb{R}^{p}$ has a multivariate normal distribution written as

$$
\mathbf{X} \sim N_{p}(\mathbf{m}, \Sigma)
$$

if for all Borel $E \subseteq \mathbb{R}^{p}$, the distribution measure is given by

$$
\lambda_{\mathbf{x}}(E)=\int_{\mathbb{R}^{p}} \mathscr{X}_{E}(\mathbf{x}) \frac{1}{(2 \pi)^{p / 2} \operatorname{det}(\Sigma)^{1 / 2}} e^{\frac{-1}{2}(\mathbf{x}-\mathbf{m})^{*} \Sigma^{-1}(\mathbf{x}-\mathbf{m})} d x
$$

for $\mathbf{m}$ a given vector and $\Sigma$ a given positive definite symmetric matrix. Recall also that the characteristic function of this random variable is

$$
\begin{equation*}
E\left(e^{i \mathbf{t} \cdot \mathbf{X}}\right)=e^{i \mathbf{t} \cdot \mathbf{m}} e^{-\frac{1}{2} \mathbf{t}^{*} \Sigma \mathbf{t}} \tag{59.20.48}
\end{equation*}
$$

So what if $\operatorname{det}(\Sigma)=0$ ? Is there a probability measure having characteristic equation

$$
e^{i \mathbf{t} \cdot \mathbf{m}} e^{-\frac{1}{2} \mathbf{t}^{*} \Sigma \mathbf{t}} ?
$$

Let $\Sigma_{n} \rightarrow \Sigma$ in the Frobenius norm, $\operatorname{det}\left(\Sigma_{n}\right)>0$. That is the $i j^{t h}$ components converge. Let $\mathbf{X}_{n}$ be the random variable which is associated with $\mathbf{m}$ and $\Sigma_{n}$. Thus for $\phi \in C_{0}\left(\mathbb{R}^{p}\right)$,

$$
\left|\lambda \mathbf{x}_{n}(\phi)\right| \equiv\left|\int_{\mathbb{R}^{p}} \phi(\mathbf{x}) \frac{1}{(2 \pi)^{p / 2} \operatorname{det}\left(\Sigma_{n}\right)^{1 / 2}} e^{\frac{-1}{2}(\mathbf{x}-\mathbf{m})^{*} \Sigma_{n}^{-1}(\mathbf{x}-\mathbf{m})} d x\right| \leq\|\phi\|_{C_{0}\left(\mathbb{R}^{p}\right)}
$$

Thus these $\lambda_{\mathbf{X}_{n}}$ are bounded in the weak $*$ topology of $C_{0}\left(\mathbb{R}^{p}\right)^{\prime}$ which is the space of signed measures. By the separability of $C_{0}\left(\mathbb{R}^{p}\right)$ and the Banach Alaoglu theorem and the Riesz representation theorem for $C_{0}\left(\mathbb{R}^{p}\right)^{\prime}$, there is a subsequence still denoted as $\lambda_{\mathbf{x}_{n}}$ which converges weak $*$ to a finite measure $\mu$. Is $\mu$ a probability measure? Is the characteristic function of this measure $e^{i \mathbf{t} \cdot \mathbf{m}} e^{-\frac{1}{2} \mathbf{t}^{*} \Sigma \mathbf{t}}$.

Note that $E\left(e^{i \mathbf{t} \cdot \mathbf{X}_{n}}\right)=e^{i \mathbf{t} \cdot \mathbf{m}} e^{-\frac{1}{2} \mathbf{t}^{*} \dot{\Sigma}_{n} \mathbf{t}} \rightarrow e^{i \mathbf{t} \cdot \mathbf{m}} e^{-\frac{1}{2} \mathbf{t}^{*} \Sigma \mathbf{t}}$ and this last function of $\mathbf{t}$ is continuous at $\mathbf{0}$. Therefore, by Lemma 59.18.2, these measures $\lambda_{\mathbf{x}_{n}}$ are also tight. Let $\varepsilon>0$ be given. Then there is a compact set $K_{\mathcal{\varepsilon}}$ such that $\lambda \mathbf{x}_{n}\left(\mathbf{x} \notin K_{\varepsilon}\right)<\varepsilon$. Now let $\phi=1$ on $K_{\mathcal{\varepsilon}}$ and $\phi \in C_{c}\left(\mathbb{R}^{p}\right), \phi \geq 0, \phi(\mathbf{x}) \in[0,1]$. Then

$$
(1-\varepsilon) \leq \int_{\mathbb{R}^{p}} \phi d \lambda_{\mathbf{x}_{n}} \rightarrow \int_{\mathbb{R}^{p}} \phi d \mu \leq \mu\left(\mathbb{R}^{p}\right)
$$

and so, since $\varepsilon$ is arbitrary, this shows that $\mu\left(\mathbb{R}^{p}\right) \geq 1$. However, $\mu\left(\mathbb{R}^{p}\right) \leq 1$ because

$$
\mu\left(\mathbb{R}^{n}\right) \leq \int_{\mathbb{R}^{p}} \psi d \mu+\varepsilon \leq \int_{\mathbb{R}^{p}} \psi d \lambda \mathbf{x}_{n}+2 \varepsilon \leq 1+2 \varepsilon
$$

for suitable $\psi \in C_{c}\left(\mathbb{R}^{p}\right)$ having values in $[0,1]$ and $n$. Thus $\mu$ is indeed a probability measure.

Now what of its characteristic function?

$$
\begin{equation*}
e^{i \mathbf{t} \cdot \mathbf{m}} e^{-\frac{1}{2} \mathbf{t}^{*} \Sigma \mathbf{t}}=\lim _{n \rightarrow \infty} e^{i \mathbf{t} \cdot \mathbf{m}} e^{-\frac{1}{2} \mathbf{t}^{*} \Sigma_{n} \mathbf{t}}=\lim _{n \rightarrow \infty} \int_{\mathbb{R}^{p}} e^{i \mathbf{t} \cdot \mathbf{x}} d \lambda \mathbf{x}_{n}(x) \tag{59.20.49}
\end{equation*}
$$

Is this equal to

$$
\int_{\mathbb{R}^{p}} e^{i \mathbf{t} \cdot \mathbf{x}} d \mu(x) ?
$$

Using tightness again,

$$
\begin{gathered}
\left|\int_{\mathbb{R}^{p}} e^{i \mathbf{t} \cdot \mathbf{x}} d \mu(x)-\int_{\mathbb{R}^{p}} e^{i \mathbf{t} \cdot \mathbf{x}} d \lambda \mathbf{x}_{n}(x)\right| \leq\left|\int_{\mathbb{R}^{p}} e^{i \mathbf{t} \cdot \mathbf{x}} d \mu(x)-\int_{\mathbb{R}^{p}} \psi e^{i \cdot \mathbf{x}} d \mu(x)\right| \\
+\left|\int_{\mathbb{R}^{p}} \psi e^{i \boldsymbol{t} \cdot \mathbf{x}} d \mu(x)-\int_{\mathbb{R}^{p}} \psi e^{i \mathbf{t} \cdot \mathbf{x}} d \lambda \mathbf{x}_{n}(x)\right|+\left|\int_{\mathbb{R}^{p}} \psi e^{i \mathbf{t} \cdot \mathbf{x}} d \lambda \mathbf{x}_{n}(x)-\int_{\mathbb{R}^{p}} e^{i \mathbf{t} \cdot \mathbf{x}} d \lambda \mathbf{x}_{n}(x)\right| \\
\leq \varepsilon+\left|\int_{\mathbb{R}^{p}} \psi e^{i \mathbf{t} \cdot \mathbf{x}} d \mu(x)-\int_{\mathbb{R}^{p}} \psi e^{i \mathbf{t} \cdot \mathbf{x}} d \lambda \mathbf{x}_{n}(x)\right|+\varepsilon
\end{gathered}
$$

for a suitable choice of $\psi \in C_{c}\left(\mathbb{R}^{p}\right)$ having values in $[0,1]$. The middle term is less than $\varepsilon$ if $n$ large enough thanks to the weak $*$ convergence of $\lambda \mathbf{x}_{n}$ to $\mu$. Hence the last limit in 59.20.49 equals $\int_{\mathbb{R}^{p}} e^{i \mathbf{t} \mathbf{x}} d \mu(x)$ as hoped. Letting $\mathbf{X}$ be a random variable having $\mu$ as its distribution measure, (You could take $\Omega=\mathbb{R}^{p}$ and the measurable sets the Borel sets.) what about $E\left((\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^{*}\right)$ ? Is it equal to $\Sigma$ ? What about the question whether $\mathbf{X} \in L^{q}\left(\Omega ; \mathbb{R}^{p}\right)$ for all $q>1$ ? This is clearly true for the case where $\Sigma^{-1}$ exists, but what of the case where $\operatorname{det}(\Sigma)=0$ ?

For simplicity, say $\mathbf{m}=\mathbf{0}$.

$$
\begin{gathered}
\int_{\Omega}|\mathbf{X}|^{q} d P=\int_{0}^{\infty} P\left(|\mathbf{X}|^{q}>\lambda\right) d \lambda=\int_{0}^{\infty} \mu\left(|\mathbf{x}|^{q}>\lambda\right) d \lambda \\
\leq \int_{0}^{\infty} \mu\left(|\mathbf{x}|^{q}>\lambda\right) d \lambda \leq \int_{0}^{\infty} \int_{\mathbb{R}^{p}}\left(1-\psi_{\lambda}\right) d \mu d \lambda
\end{gathered}
$$

where $\psi_{\lambda}=1$ on $B\left(\mathbf{0}, \frac{1}{2} \lambda^{1 / q}\right)$ is nonnegative, and is in $C_{c}\left(B\left(\mathbf{0}, \lambda^{1 / q}\right)\right)$. Now from the above, $\mu\left(\mathbb{R}^{p}\right)=\lambda_{\mathbf{x}_{n}}\left(\mathbb{R}^{p}\right)=1$ and so the inside integral satisfies

$$
\begin{equation*}
\int_{\mathbb{R}^{p}}\left(1-\psi_{\lambda}\right) d \mu=\lim _{n \rightarrow \infty} \int_{\mathbb{R}^{p}}\left(1-\psi_{\lambda}\right) d \lambda \mathbf{x}_{n} \tag{59.20.50}
\end{equation*}
$$

because

$$
\int_{\mathbb{R}^{p}} d \mu=\int_{\mathbb{R}^{p}} d \lambda_{\mathbf{x}_{n}}=1
$$

and as to the other terms, the weak $*$ convergence gives

$$
\int_{\mathbb{R}^{p}} \psi_{\lambda} d \mu=\lim _{n \rightarrow \infty} \int_{\mathbb{R}^{p}} \psi_{\lambda} d \lambda \mathbf{x}_{n}
$$

Each of these integrals in 59.20 .50 is no larger than 1. Hence from Fatou's lemma,

$$
\int_{\Omega}|\mathbf{X}|^{q} d P \leq \int_{0}^{\infty} \int_{\mathbb{R}^{p}}\left(1-\psi_{\lambda}\right) d \mu d \lambda \leq \lim \inf _{n \rightarrow \infty} \int_{0}^{\infty} \int_{\mathbb{R}^{p}}\left(1-\psi_{\lambda}\right) d \lambda \mathbf{x}_{n} d \lambda
$$

Is this on the right finite? It is dominated by

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \int_{0}^{\infty} \lambda_{\mathbf{x}_{n}}\left(|\mathbf{x}|^{q}>\frac{1}{2^{q}} \lambda\right) d \lambda & =\lim _{n \rightarrow \infty} 2^{q} \int_{0}^{\infty} \lambda_{\mathbf{x}_{n}}\left(|\mathbf{x}|^{q}>\delta\right) d \delta \\
& =\liminf _{n \rightarrow \infty} 2^{q} E\left(\left|\mathbf{X}_{n}\right|^{q}\right)
\end{aligned}
$$

So is a subsequence of $\left\{E\left(\left|\mathbf{X}_{n}\right|^{q}\right)\right\}$ bounded? It equals

$$
\int_{\mathbb{R}^{p}}|\mathbf{x}|^{q} \frac{1}{(2 \pi)^{p / 2} \operatorname{det}\left(\Sigma_{n}\right)^{1 / 2}} e^{\frac{-1}{2}(\mathbf{x}-\mathbf{m})^{*} \Sigma_{n}^{-1}(\mathbf{x}-\mathbf{m})} d x
$$

and for $q$ an even integer, this moment can be computed using the characteristic function.

$$
e^{-\frac{1}{2} \mathbf{t}^{*} \Sigma_{n} \mathbf{t}}=\int_{\mathbb{R}^{p}} e^{i \cdot \mathbf{x}} d \lambda \mathbf{x}_{n}
$$

Also, it suffices to consider $E\left(X_{k}^{q}\right)$. Differentiate both sides. Using the repeated index summation convention,

$$
e^{-\frac{1}{2} \mathbf{t}^{*} \Sigma \mathbf{t}}\left(-\Sigma_{n k j} t_{j}\right)=\int_{\mathbb{R}^{p}} i x_{k} e^{i \mathbf{t} \cdot \mathbf{x}} d \mu
$$

Now differentiate again.

$$
e^{-\frac{1}{2} \mathbf{t}^{*} \Sigma \mathbf{t}}\left(-\Sigma_{n k j} t_{j}\right)\left(-\Sigma_{n k j} t_{j}\right)+\left(-\Sigma_{n k k}\right)=-\int_{\mathbb{R}^{p}} x_{k}^{2} e^{i \mathbf{t} \cdot \mathbf{x}} d \lambda \mathbf{x}_{n}
$$

Next let $\mathbf{t}=\mathbf{0}$ to conclude that $E\left(X_{n k}^{2}\right)=\Sigma_{n k k}$. Of course you can continue differentiating as long as desired and obtain $E\left(X_{n k}^{2 m}\right)$ is equal to some polynomial formula involving $\Sigma_{n k k}$ and these are given to converge to $\Sigma_{k k}$. Therefore, for any $q>1,\left\{E\left(\left|\mathbf{X}_{n}\right|^{q}\right)\right\}$ is bounded and so from the above,

$$
\int_{\Omega}|\mathbf{X}|^{q} d P \leq \lim \inf _{n \rightarrow \infty} 2^{q} E\left(\left|\mathbf{X}_{n}\right|^{q}\right)<\infty
$$

So yes, $\mathbf{X}$ is indeed in $L^{q}\left(\Omega, \mathbb{R}^{p}\right)$ for every $q$. What about the covariance?
From the definition of the characteristic function,

$$
e^{-\frac{1}{2} *^{*} \Sigma \mathbf{t}}=\int_{\mathbb{R}^{p}} e^{i \mathbf{t} \cdot \mathbf{x}} d \mu
$$

and so taking the derivative with respect to $t_{k}$ of both sides,

$$
e^{-\frac{1}{2} \mathbf{t}^{*} \Sigma \mathbf{t}}\left(-\Sigma_{k j} t_{j}\right)=\int_{\mathbb{R}^{p}} i x_{k} e^{i \mathbf{t} \cdot \mathbf{x}} d \mu
$$

Now differentiate with respect to $t_{l}$ on both sides.

$$
\begin{aligned}
& e^{-\frac{1}{2} \mathbf{t}^{*} \Sigma \mathbf{t}}\left(-\Sigma_{l i} t_{i}\right)\left(-\Sigma_{k j} t_{j}\right)+e^{-\frac{1}{2} *^{*} \Sigma \mathbf{t}}\left(-\Sigma_{k l}\right) \\
= & \int_{\mathbb{R}^{p}} i x_{k}\left(i x_{l}\right) e^{i \mathbf{t} \cdot \mathbf{x}} d \mu=-\int_{\mathbb{R}^{p}} x_{k} x_{l} e^{i \mathbf{t} \cdot \mathbf{x}} d \mu
\end{aligned}
$$

Now let $\mathbf{t}=\mathbf{0}$ to obtain

$$
\Sigma_{k l}=\int_{\mathbb{R}^{p}} x_{k} x_{l} e^{i t \cdot \mathbf{x}} d \mu=E\left(X_{k} X_{l}\right)
$$

If $\mathbf{m} \neq \mathbf{0}$, the same kind of argument holds with a little more details. This proves the following theorem.

Theorem 59.20.2 Let $\Sigma$ be nonnegative and self adjoint $p \times p$ matrix. Then there exists $a$ random variable $\mathbf{X}$ whose distribution measure $\lambda_{\mathbf{X}}$ has characteristic function

$$
e^{i \mathbf{t} \cdot \mathbf{m}} e^{-\frac{1}{2} \mathbf{t}^{*} \Sigma \mathbf{t}}
$$

Also

$$
E\left((\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^{*}\right)=\Sigma
$$

that is

$$
E\left((X-m)_{i}(X-m)_{j}\right)=\Sigma_{i j}
$$

This is generalized normally distributed random variable.

There is an interesting corollary to this theorem.
Corollary 59.20.3 Let $H$ be a real Hilbert space. There exist random variables $W$ (h) for $h \in H$ such that each is normally distributed with mean 0 and for every $h, g,(W(h), W(g))$ is normally distributed and

$$
E(W(h) W(g))=(h, g)_{H}
$$

Furthermore, if $\left\{e_{i}\right\}$ is an orthogonal set of vectors of $H$, then $\left\{W\left(e_{i}\right)\right\}$ are independent random variables. Also for any finite set $\left\{f_{1}, f_{2}, \cdots, f_{n}\right\}$,

$$
\left(W\left(f_{1}\right), W\left(f_{2}\right), \cdots, W\left(f_{n}\right)\right)
$$

is normally distributed.
Proof: Let $\mu_{h_{1} \cdots h_{m}}$ be a multivariate normal distribution with covariance $\Sigma_{i j}=\left(h_{i}, h_{j}\right)$ and mean 0 . Thus the characteristic function of this measure is

$$
e^{-\frac{1}{2} t^{*} \Sigma \mathbf{t}}
$$

Now suppose $\mu_{k_{1} \cdots k_{n}}$ is another such measure where for simplicity,

$$
\left\{h_{1} \cdots h_{m}, k_{m+1} \cdots k_{n}\right\}=\left\{k_{1} \cdots k_{n}\right\}
$$

Let $v$ be a measure on $\mathscr{B}\left(\mathbb{R}^{m}\right)$ which is given by

$$
v(E) \equiv \mu_{k_{1} \cdots k_{n}}\left(E \times \mathbb{R}^{n-m}\right)
$$

Then does it follow that $v=\mu_{h_{1} \cdots h_{m}}$ ? If so, then the Kolmogorov consistency condition will hold for these measures $\mu_{h_{1} \ldots h_{m}}$. To determine whether this is so, take the characteristic function of $v$. Let $\Sigma_{1}$ be the $n \times n$ matrix which comes from the $\left\{k_{1} \cdots k_{n}\right\}$ and let $\Sigma_{2}$ be the one which comes from the $\left\{h_{1} \cdots h_{m}\right\}$.

$$
\begin{aligned}
\int_{\mathbb{R}^{m}} e^{i \mathbf{t} \cdot \mathbf{x}} d v(x) & \equiv \int_{\mathbb{R}^{n-m}} \int_{\mathbb{R}^{m}} e^{i(\mathbf{t}, \mathbf{0}) \cdot(\mathbf{x}, \mathbf{y})} d \mu_{k_{1} \cdots k_{n}}(x, y) \\
& =e^{-\frac{1}{2}\left(\mathbf{t}^{*}, \mathbf{0}^{*}\right) \Sigma_{1}(\mathbf{t}, \mathbf{0})}=e^{-\frac{1}{2} \mathbf{t}^{*} \Sigma_{2} \mathbf{t}}
\end{aligned}
$$

which is the characteristic function for $\mu_{h_{1} \cdots h_{m}}$. Therefore, these two measures are the same and the Kolmogorov consistency condition holds. It follows that there exists a measure $\mu$ defined on the Borel sets of $\prod_{h \in H} \mathbb{R}$ which extends all of these measures. This argument also shows that if a random vector $\mathbf{X}$ has characteristic function $e^{-\frac{1}{2} t^{*} \Sigma \mathbf{t}}$, then if $X_{k}$ is one of its components, then the characteristic function of $X_{k}$ is $e^{-\frac{1}{2} t^{2}\left|h_{k}\right|^{2}}$ so this scalar valued random variable has mean zero and variance $\left|h_{k}\right|^{2}$. Then if $\omega \in \prod_{h \in H} \mathbb{R}$

$$
W(h)(\omega) \equiv \pi_{h}(\omega)
$$

where $\pi_{h}$ denotes the projection onto position $h$ in this product. Also define

$$
\left(W\left(f_{1}\right), W\left(f_{2}\right), \cdots, W\left(f_{n}\right)\right) \equiv \pi_{f_{1} \cdots f_{n}}(\omega)
$$

Then this is a random variable whose covariance matrix is just $\Sigma_{i j}=\left(f_{i}, f_{j}\right)_{H}$ and whose characteristic equation is $e^{-\frac{1}{2} t^{*} \Sigma t}$ so this verifies that

$$
\left(W\left(f_{1}\right), W\left(f_{2}\right), \cdots, W\left(f_{n}\right)\right)
$$

is normally distributed with covariance $\Sigma$. If you have two of them, $W(g), W(h)$, then $E(W(h) W(g))=(h, g)_{H}$. This follows from what was just shown that $(W(f), W(g))$ is normally distributed and so the covariance will be

$$
\left(\begin{array}{cc}
|f|^{2} & (f, g) \\
(f, g) & |g|^{2}
\end{array}\right)=\left(\begin{array}{cc}
E\left(W(f)^{2}\right) & E(W(f) W(g)) \\
E(W(f) W(g)) & E\left(W(g)^{2}\right)
\end{array}\right)
$$

Finally consider the claim about independence. Any finite subset of $\left\{W\left(e_{i}\right)\right\}$ is generalized normal with the covariance matrix being a diagonal. Therefore, writing in terms of the distribution measures, this diagonal matrix allows the iterated integrals to be split apart and it follows that

$$
E\left(\exp \left(i \sum_{k=1}^{m} t_{k} W\left(e_{k}\right)\right)\right)=\prod_{k=1}^{m} \exp \left(i t_{k} W\left(e_{k}\right)\right)
$$

and so this follows from Proposition 59.11.1. Note that in this case, the covariance matrix will not have zero determinant.

### 59.21 Positive Definite Functions, Bochner's Theorem

First here is a nice little lemma about matrices.
Lemma 59.21.1 Suppose $M$ is an $n \times n$ matrix. Suppose also that

$$
\alpha^{*} M \alpha=0
$$

for all $\alpha \in \mathbb{C}^{n}$. Then $M=0$.
Proof: Suppose $\lambda$ is an eigenvalue for $M$ and let $\alpha$ be an associated eigenvector.

$$
0=\alpha^{*} M \alpha=\alpha^{*} \lambda \alpha=\lambda \alpha^{*} \alpha=\lambda|\alpha|^{2}
$$

and so all the eigenvalues of $M$ equal zero. By Schur's theorem there is a unitary matrix $U$ such that

$$
M=U\left(\begin{array}{ccc}
0 & & *_{1}  \tag{59.21.51}\\
& \ddots & \\
0 & & 0
\end{array}\right) U^{*}
$$

where the matrix in the middle has zeros down the main diagonal and zeros below the main diagonal. Thus

$$
M^{*}=U\left(\begin{array}{ccc}
0 & & 0 \\
& \ddots & \\
*_{2} & & 0
\end{array}\right) U^{*}
$$

where $M^{*}$ has zeros down the main diagonal and zeros above the main diagonal. Also taking the adjoint of the given equation for $M$, it follows that for all $\alpha$,

$$
\alpha^{*} M^{*} \alpha=0
$$

Therefore, $M+M^{*}$ is Hermitian and has the property that

$$
\alpha^{*}\left(M+M^{*}\right) \alpha=0
$$

Thus $M+M^{*}=0$ because it is unitarily similar to a diagonal matrix and the above equation can only hold for all $\alpha$ if $M+M^{*}$ has all zero eigenvalues which implies the diagonal matrix has zeros down the main diagonal. Therefore, from the formulas for $M, M^{*}$,

$$
0=U\left(\left(\begin{array}{ccc}
0 & & 0 \\
& \ddots & \\
*_{2} & & 0
\end{array}\right)+\left(\begin{array}{ccc}
0 & & *_{1} \\
& \ddots & \\
0 & & 0
\end{array}\right)\right) U^{*}
$$

and so the sum of the two matrices in the middle must also equal 0 . Hence the entries of the matrix in the middle in 59.21 .51 are all equal to zero. Thus $M=0$ as claimed.

Definition 59.21.2 A Borel measurable function, $f: \mathbb{R}^{n} \rightarrow \mathbb{C}$ is called positive definite if whenever $\left\{\mathbf{t}_{\mathbf{k}}\right\}_{k=1}^{p} \subseteq \mathbb{R}^{n}, \alpha \in \mathbb{C}^{p}$

$$
\begin{equation*}
\sum_{k, j} f\left(\mathbf{t}_{j}-\mathbf{t}_{k}\right) \alpha_{j} \overline{\alpha_{k}} \geq 0 \tag{59.21.52}
\end{equation*}
$$

The first thing to notice about a positive definite function is the following which implies these functions are automatically bounded.

Lemma 59.21.3 If $f$ is positive definite then whenever $\left\{\mathbf{t}_{k}\right\}_{k=1}^{p}$ are $p$ points in $\mathbb{R}^{n}$,

$$
\left|f\left(\mathbf{t}_{j}-\mathbf{t}_{k}\right)\right| \leq f(\mathbf{0})
$$

In particular, for all $\mathbf{t},|f(\mathbf{t})| \leq f(\mathbf{0})$.
Proof: Let $F$ be the $p \times p$ matrix such that

$$
F_{k j}=f\left(\mathbf{t}_{j}-\mathbf{t}_{k}\right)
$$

Then 59.21 .52 is of the form

$$
\begin{equation*}
\alpha^{*} F \alpha=(F \alpha, \alpha) \geq 0 \tag{59.21.53}
\end{equation*}
$$

where this is the inner product in $\mathbb{C}^{p}$. Letting $[\alpha, \beta] \equiv(F \alpha, \beta) \equiv \beta^{*} F \alpha$, it is obvious that $[\alpha, \beta]$ satisfies

$$
[\alpha, \alpha] \geq 0,[a \alpha+b \beta, \gamma]=a[\alpha, \gamma]+b[\beta, \gamma]
$$

I claim it also satisfies

$$
[\alpha, \beta]=\overline{[\beta, \alpha]}
$$

To verify this last claim, note that since $\alpha^{*} F \alpha$ is real,

$$
\alpha^{*} F^{*} \alpha=\alpha^{*} F \alpha \geq \mathbf{0}
$$

and so for all $\alpha \in \mathbb{C}^{p}$,

$$
\alpha^{*}\left(F^{*}-F\right) \alpha=0
$$

which from Lemma 59.21.1 implies $F^{*}=F$. Hence $F$ is self adjoint and it follows

$$
[\alpha, \beta] \equiv \beta^{*} F \alpha=\beta^{*} F^{*} \alpha=\alpha^{T} F^{* T} \bar{\beta}=\overline{\alpha^{*} F \beta}=\overline{[\beta, \alpha]} .
$$

Therefore, the Cauchy Schwarz inequality holds for $[\cdot, \cdot]$ and it follows

$$
|[\alpha, \beta]|=|(F \alpha, \beta)| \leq(F \alpha, \alpha)^{1 / 2}(F \beta, \beta)^{1 / 2}
$$

Letting $\alpha=\mathbf{e}_{k}$ and $\beta=\mathbf{e}_{j}$, it follows $F_{s s} \geq 0$ for all $s$ and

$$
\left|F_{k j}\right| \leq F_{k k}^{1 / 2} F_{j j}^{1 / 2}
$$

which says nothing more than

$$
\left|f\left(\mathbf{t}_{j}-\mathbf{t}_{k}\right)\right| \leq f(0)^{1 / 2} f(0)^{1 / 2}=f(0)
$$

This proves the lemma.
With this information, here is another useful lemma involving positive definite functions. It is interesting because it looks like the formula which defines what it means for the function to be positive definite.

Lemma 59.21.4 Let $f$ be a positive definite function as defined above and let $\mu$ be a finite Borel measure. Then

$$
\begin{equation*}
\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} f(\mathbf{x}-\mathbf{y}) d \mu(x) d \mu(y) \geq 0 \tag{59.21.54}
\end{equation*}
$$

If $\mu$ also has the property that it is symmetric, $\mu(F)=\mu(-F)$ for all $F$ Borel, then

$$
\begin{equation*}
\int_{\mathbb{R}^{n}} f(\mathbf{x}) d(\mu * \mu)(x) \geq 0 \tag{59.21.55}
\end{equation*}
$$

Proof: By definition if $\left\{t_{j}\right\}_{j=1}^{p} \subseteq \mathbb{R}^{n}$, and letting $\alpha=(1, \cdots, 1)^{T} \in \mathbb{R}^{n}$,

$$
\sum_{j, k} f\left(\mathbf{t}_{j}-\mathbf{t}_{k}\right) \geq 0
$$

Therefore, integrating over each of the variables,

$$
0 \leq \sum_{j=1}^{p} \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} f\left(\mathbf{t}_{j}-\mathbf{t}_{j}\right) d \mu\left(t_{j}\right) d \mu\left(t_{j}\right)+\sum_{j \neq k} \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} f\left(\mathbf{t}_{j}-\mathbf{t}_{k}\right) d \mu\left(t_{j}\right) d \mu\left(t_{k}\right)
$$

and so

$$
0 \leq f(0) \mu\left(\mathbb{R}^{n}\right)^{2} p+p(p-1) \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} f(\mathbf{x}-\mathbf{y}) d \mu(x) d \mu(y)
$$

Dividing both sides by $p(p-1)$ and letting $p \rightarrow \infty$, it follows

$$
0 \leq \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} f(\mathbf{x}-\mathbf{y}) d \mu(x) d \mu(y)
$$

which shows 59.21.54.
To verify 59.21.55, use 59.14.25.

$$
\int_{\mathbb{R}^{n}} f d(\mu * \mu)=\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} f(\mathbf{x}+\mathbf{y}) d \mu(x) d \mu(y)
$$

and since $\mu$ is symmetric, this equals

$$
\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} f(\mathbf{x}-\mathbf{y}) d \mu(x) d \mu(y) \geq 0
$$

by the first part of the lemma. This proves the lemma.
Lemma 59.21.5 Let $\mu_{t}$ be the measure defined on $\mathscr{B}\left(\mathbb{R}^{n}\right)$ by

$$
\mu_{t}(F) \equiv \int_{F} \frac{1}{(\sqrt{2 \pi t})^{n}} e^{-\frac{1}{2 t}|\mathbf{x}|^{2}} d x
$$

for $t>0$. Then $\mu_{t} * \mu_{t}=\mu_{2 t}$ and each $\mu_{t}$ is a probability measure.
Proof: By Theorem 59.14.7,

$$
\phi_{\mu_{t} * \mu_{t}}(\mathbf{s})=\phi_{\mu_{t}}(\mathbf{s}) \phi_{\mu_{t}}(\mathbf{s})=\left(e^{-\frac{1}{2} t|\mathbf{s}|^{2}}\right)^{2}=e^{-\frac{1}{2}(2 t)|\mathbf{s}|^{2}}=\phi_{\mu_{2 t}}(\mathbf{s})
$$

Each $\mu_{t}$ is a probability measure because it is the distribution of a normally distributed random variable of mean $\mathbf{0}$ and covariance $t I$.

Now let $\mu$ be a probability measure on $\mathscr{B}\left(\mathbb{R}^{n}\right)$.

$$
\phi_{\mu}(\mathbf{t}) \equiv \int e^{i \cdot \mathbf{t} \cdot \mathbf{y}} d \mu(y)
$$

and so by the dominated convergence theorem, $\phi_{\mu}$ is continuous and also $\phi_{\mu}(\mathbf{0})=1$. I claim $\phi_{\mu}$ is also positive definite. Let $\alpha \in \mathbb{C}^{p}$ and $\left\{\mathbf{t}_{k}\right\}_{k=1}^{p}$ a sequence of points of $\mathbb{R}^{n}$. Then

$$
\begin{gathered}
\sum_{k, j} \phi_{\mu}\left(\mathbf{t}_{k}-\mathbf{t}_{j}\right) \alpha_{k} \overline{\alpha_{j}}=\sum_{k, j} \int e^{i \mathbf{t}_{k} \cdot \mathbf{y}} \alpha_{k} e^{-i \mathbf{t}_{j} \cdot \mathbf{y}} \overline{\alpha_{j}} d \mu(y) \\
=\int \sum_{k, j} e^{i \boldsymbol{t}_{k} \cdot \mathbf{y}} \alpha_{k} e^{i_{j} \cdot \mathbf{y}} \alpha_{j} d \mu(y) .
\end{gathered}
$$

Now let $\beta(\mathbf{y}) \equiv\left(e^{i \mathbf{t}_{1} \cdot \mathbf{y}} \alpha_{1}, \cdots, e^{i \mathbf{t}_{p} \cdot \mathbf{y}} \alpha_{p}\right)^{T}$. Then the above equals

$$
\int(1, \cdots, 1) \beta(\mathbf{y}) \beta^{*}(\mathbf{y})\left(\begin{array}{c}
1 \\
\vdots \\
1
\end{array}\right) d \mu
$$

The integrand is of the form

$$
\overline{\left(\beta^{*}\left(\begin{array}{c}
1 \\
\vdots \\
1
\end{array}\right)\right)}\left(\beta^{*}\left(\begin{array}{c}
1 \\
\vdots \\
1
\end{array}\right)\right) \geq 0
$$

because it is just a complex number times its conjugate.
Thus every characteristic function is continuous, equals 1 at $\mathbf{0}$, and is positive definite. Bochner's theorem goes the other direction.

To begin with, suppose $\mu$ is a finite measure on $\mathscr{B}\left(\mathbb{R}^{n}\right)$. Then for $\mathfrak{S}$ the Schwartz class, $\mu$ can be considered to be in the space of linear transformations defined on $\mathfrak{S}$, $\mathfrak{S}^{*}$ as follows.

$$
\mu(f) \equiv \int f d \mu
$$

Recall $F^{-1}(\mu)$ is defined as

$$
\begin{aligned}
& F^{-1}(\mu)(f) \equiv \mu\left(F^{-1} f\right)=\int_{\mathbb{R}^{n}} F^{-1} f d \mu \\
& \quad=\frac{1}{(2 \pi)^{n / 2}} \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} e^{i \mathbf{x} \cdot \mathbf{y}} f(\mathbf{y}) d y d \mu \\
& \quad=\int_{\mathbb{R}^{n}}\left(\frac{1}{(2 \pi)^{n / 2}} \int_{\mathbb{R}^{n}} e^{i \mathbf{x} \cdot \mathbf{y}} d \mu\right) f(\mathbf{y}) d y
\end{aligned}
$$

and so $F^{-1}(\mu)$ is the bounded continuous function

$$
\mathbf{y} \rightarrow \frac{1}{(2 \pi)^{n / 2}} \int_{\mathbb{R}^{n}} e^{i \mathbf{x} \cdot \mathbf{y}} d \mu
$$

Now the following lemma has the main ideas for Bochner's theorem.
Lemma 59.21.6 Suppose $\psi(\mathbf{t})$ is positive definite, $\mathbf{t} \rightarrow \psi(\mathbf{t})$ is in $L^{1}\left(\mathbb{R}^{n}, m_{n}\right)$ where $m_{n}$ is Lebesgue measure, $\psi(\mathbf{0})=1$, and $\psi$ is continuous at $\mathbf{0}$. Then there exists a unique probability measure, $\mu$ defined on the Borel sets of $\mathbb{R}^{n}$ such that

$$
\phi_{\mu}(\mathbf{t})=\psi(\mathbf{t})
$$

Proof: If the conclusion is true, then

$$
\psi(\mathbf{t})=\int_{\mathbb{R}^{n}} e^{i \mathbf{t} \cdot \mathbf{x}} d \mu(x)=(2 \pi)^{n / 2} F^{-1}(\mu)(\mathbf{t})
$$

Recall that $\mu \in \mathfrak{S}^{*}$, the algebraic dual of $\mathfrak{S}$. Therefore, in $\mathfrak{S}^{*}$,

$$
\frac{1}{(2 \pi)^{n / 2}} F(\psi)=\mu
$$

That is, for all $f \in \mathfrak{S}$,

$$
\begin{align*}
\int_{\mathbb{R}^{n}} f(\mathbf{y}) d \mu(y) & =\frac{1}{(2 \pi)^{n / 2}} \int_{\mathbb{R}^{n}} F(\psi)(\mathbf{y}) f(\mathbf{y}) d y \\
& =\frac{1}{(2 \pi)^{n}} \int_{\mathbb{R}^{n}} f(\mathbf{y})\left(\int_{\mathbb{R}^{n}} e^{-i \mathbf{y} \cdot \mathbf{x}} \psi(\mathbf{x}) d x\right) d y \tag{59.21.56}
\end{align*}
$$

I will show

$$
f \rightarrow \frac{1}{(2 \pi)^{n}} \int_{\mathbb{R}^{n}} f(\mathbf{y})\left(\int_{\mathbb{R}^{n}} e^{-i \mathbf{y} \cdot \mathbf{x}} \psi(\mathbf{x}) d x\right) d y
$$

is a positive linear functional and then it will follow from 59.21 .56 that $\mu$ is unique. Thus it is needed to show the inside integral in 59.21.56 is nonnegative. First note that the integrand is a positive definite function of $\mathbf{x}$ for each fixed $\mathbf{y}$. This follows from

$$
\begin{aligned}
& \sum_{k, j} e^{-i \mathbf{y} \cdot\left(\mathbf{x}_{k}-\mathbf{x}_{j}\right)} \psi\left(\mathbf{x}_{k}-\mathbf{x}_{j}\right) \alpha_{k} \overline{\alpha_{j}} \\
= & \sum_{k, j} \psi\left(\mathbf{x}_{k}-\mathbf{x}_{j}\right)\left(e^{-i \mathbf{y} \cdot\left(\mathbf{x}_{k}\right)} \alpha_{k}\right) \overline{e^{-i \mathbf{y} \cdot\left(\mathbf{x}_{j}\right)} \alpha_{j}} \geq 0 .
\end{aligned}
$$

Let $t>0$ and

$$
h_{2 t}(\mathbf{x}) \equiv \frac{1}{(4 \pi t)^{1 / 2}} e^{-\frac{1}{4 t}|\mathbf{x}|^{2}}
$$

Then by dominated convergence theorem,

$$
\int_{\mathbb{R}^{n}} e^{-i \mathbf{y} \cdot \mathbf{x}} \psi(\mathbf{x}) d x=\lim _{t \rightarrow \infty} \int_{\mathbb{R}^{n}} e^{-i \mathbf{y} \cdot \mathbf{x}} \psi(\mathbf{x}) h_{2 t}(\mathbf{x}) d x
$$

Letting $d \eta_{2 t}=h_{2 t}(\mathbf{x}) d x$, it follows from Lemma 59.21.5 $\eta_{2 t}=\eta_{t} * \eta_{t}$ and since these are symmetric measures, it follows from Lemma 59.21.4 the above equals

$$
\lim _{t \rightarrow \infty} \int_{\mathbb{R}^{n}} e^{-i \mathbf{y} \cdot \mathbf{x}} \psi(\mathbf{x}) d\left(\eta_{t} * \eta_{t}\right) \geq 0
$$

Thus the above functional is a positive linear functional and so there exists a unique Radon measure, $\mu$ satisfying

$$
\begin{aligned}
\int_{\mathbb{R}^{n}} f(\mathbf{y}) d \mu(y) & =\frac{1}{(2 \pi)^{n / 2}} \int_{\mathbb{R}^{n}} F(\psi)(\mathbf{y}) f(\mathbf{y}) d y \\
& =\frac{1}{(2 \pi)^{n}} \int_{\mathbb{R}^{n}} f(\mathbf{y})\left(\int_{\mathbb{R}^{n}} e^{-i \mathbf{y} \cdot \mathbf{x}} \psi(\mathbf{x}) d x\right) d y \\
& =\frac{1}{(2 \pi)^{n / 2}} \int_{\mathbb{R}^{n}} \psi(\mathbf{x})\left(\frac{1}{(2 \pi)^{n / 2}} \int_{\mathbb{R}^{n}} f(\mathbf{y}) e^{-i \mathbf{y} \cdot \mathbf{x}} d y\right) d x
\end{aligned}
$$

for all $f \in C_{c}\left(\mathbb{R}^{n}\right)$. Thus from the dominated convergence theorem, the above holds for all $f \in \mathfrak{S}$ also. Hence for all $f \in \mathfrak{S}$ and considering $\mu$ as an element of $\mathfrak{S}^{*}$,

$$
\begin{aligned}
F^{-1} \mu(F f) & =\mu(f)=\int_{\mathbb{R}^{n}} f(\mathbf{y}) d \mu(y) \\
& =\frac{1}{(2 \pi)^{n / 2}} \int_{\mathbb{R}^{n}} \psi(\mathbf{x}) F(f)(\mathbf{x}) d x \\
& =\frac{1}{(2 \pi)^{n / 2}} F(\psi)(f) \equiv \frac{1}{(2 \pi)^{n / 2}} \psi(F f) .
\end{aligned}
$$

It follows that in $\mathfrak{S}^{*}$,

$$
\psi=(2 \pi)^{n / 2} F^{-1} \mu
$$

Thus

$$
\psi(\mathbf{t})=\int_{\mathbb{R}^{n}} e^{i t \cdot x} d \mu
$$

in $L^{1}$. Since the right side is continuous and the left is given continuous at $\mathbf{t}=\mathbf{0}$ and equal to 1 there, it follows

$$
1=\psi(\mathbf{0})=\int_{\mathbb{R}^{n}} e^{i \boldsymbol{0} \cdot \mathbf{x}} d \mu=\mu\left(\mathbb{R}^{n}\right)
$$

and so $\mu$ is a probability measure as claimed. This proves the lemma.
The following is Bochner's theorem.
Theorem 59.21.7 Let $\psi$ be positive definite, continuous at $\mathbf{0}$, and $\psi(\mathbf{0})=1$. Then there exists a unique Radon probability measure $\mu$ such that $\psi=\phi_{\mu}$.

Proof: If $\psi \in L^{1}\left(\mathbb{R}^{n}, m_{n}\right)$, then the result follows from Lemma 59.21.6. By Lemma 59.21.3 $\psi$ is bounded. Consider

$$
\psi_{t}(\mathbf{x}) \equiv \psi(\mathbf{x}) \frac{1}{(2 \pi t)^{n / 2}} e^{-\frac{1}{2 t}|\mathbf{x}|^{2}}
$$

Then $\psi_{t}(\mathbf{0})=1, \mathbf{x} \rightarrow \psi_{t}(\mathbf{x})$ is continuous at $\mathbf{0}$, and $\psi_{t} \in L^{1}\left(\mathbb{R}^{n}, m_{n}\right)$. Therefore, by Lemma 59.21.6 there exists a unique Radon probability measure $\mu_{t}$ such that

$$
\psi_{t}(\mathbf{x})=\int_{\mathbb{R}^{n}} e^{i \mathbf{x} \cdot \mathbf{y}} d \mu_{t}(y)=\phi_{\mu_{t}}(\mathbf{x})
$$

Now letting $t \rightarrow \infty$,

$$
\lim _{t \rightarrow \infty} \psi_{t}(\mathbf{x})=\lim _{t \rightarrow \infty} \phi_{\mu_{t}}(\mathbf{x})=\psi(\mathbf{x})
$$

By Levy's theorem, Theorem 59.19.7 it follows there exists $\mu$, a probability measure on $\mathscr{B}\left(\mathbb{R}^{n}\right)$ such that $\psi(\mathbf{x})=\phi_{\mu}(\mathbf{x})$. The measure is unique because the characteristic functions are uniquely determined by the measure. This proves the theorem.

## Chapter 60

## Conditional, Martingales

### 60.1 Conditional Expectation

From Observation 59.11 .5 on Page 1893, it was shown that the conditional expectation of a random variable $\mathbf{X}$ given some others really is just what the words suggest. Given $\omega \in \Omega$, it results in a value for the "other" random variables and then you essentially take the expectation of $\mathbf{X}$ given this information which yields the value of the conditional expectation of $\mathbf{X}$ given the other random variables. It was also shown in Lemma 59.11.4 that this gives the same result as finding a $\sigma\left(\mathbf{X}_{1}, \cdots, \mathbf{X}_{n}\right)$ measurable function $\mathbf{Z}$ such that for all $F \in \sigma\left(\mathbf{X}_{1}, \cdots, \mathbf{X}_{n}\right)$,

$$
\int_{F} \mathbf{X} d P=\int_{F} \mathbf{Z} d P
$$

This was done for a particular type of $\sigma$ algebra but there is no need to be this specialized. The following is the general version of conditional expectation given a $\sigma$ algebra. It makes perfect sense to ask for the conditional expectation given a $\sigma$ algebra and this is what will be done from now on.

Definition 60.1.1 Let $(\Omega, \mathscr{M}, P)$ be a probability space and let $\mathscr{S} \subseteq \mathscr{F}$ be two $\sigma$ algebras contained in $\mathscr{M}$. Let $f$ be $\mathscr{F}$ measurable and in $L^{1}(\Omega)$. Then $E(f \mid \mathscr{S})$, called the conditional expectation of $f$ with respect to $\mathscr{S}$ is defined as follows:

$$
E(f \mid \mathscr{S}) \text { is } \mathscr{S} \text { measurable }
$$

For all $E \in \mathscr{S}$,

$$
\int_{E} E(f \mid \mathscr{S}) d P=\int_{E} f d P
$$

Lemma 60.1.2 The above is well defined. Also, if $\mathscr{S} \subseteq \mathscr{F}$ then

$$
\begin{equation*}
E(X \mid \mathscr{S})=E(E(X \mid \mathscr{F}) \mid \mathscr{S}) \tag{60.1.1}
\end{equation*}
$$

If $Z$ is bounded and measurable in $\mathscr{S}$ then

$$
\begin{equation*}
Z E(X \mid \mathscr{S})=E(Z X \mid \mathscr{S}) \tag{60.1.2}
\end{equation*}
$$

Proof: Let a finite measure on $\mathscr{S}, \mu$ be given by

$$
\mu(E) \equiv \int_{E} f d P
$$

Then $\mu \ll P$ and so by the Radon Nikodym theorem, there exists a unique $\mathscr{S}$ measurable function, $E(f \mid \mathscr{S})$ such that

$$
\int_{E} f d P \equiv \mu(E)=\int_{E} E(f \mid \mathscr{S}) d P
$$

for all $E \in \mathscr{S}$.

Let $F \in \mathscr{S}$. Then

$$
\begin{aligned}
\int_{F} E(E(X \mid \mathscr{F}) \mid \mathscr{S}) d P & \equiv \int_{F} E(X \mid \mathscr{F}) d P \\
& \equiv \int_{F} X d P \equiv \int_{F} E(X \mid \mathscr{S}) d P
\end{aligned}
$$

and so, by uniqueness, $E(E(X \mid \mathscr{F}) \mid \mathscr{S})=E(X \mid \mathscr{S})$. This shows 60.1.1.
To establish 60.1.2, note that if $Z=\mathscr{X}_{F}$ where $F \in \mathscr{S}$,

$$
\int \mathscr{X}_{F} E(X \mid \mathscr{S}) d P=\int \mathscr{X}_{F} X d P=\int E\left(\mathscr{X}_{F} X \mid \mathscr{S}\right) d P
$$

which shows 60.1 .2 in the case where $Z$ is the indicator function of a set in $\mathscr{S}$. It follows this also holds for simple functions. Let $\left\{s_{n}\right\}$ be a sequence of simple functions which converges uniformly to $Z$ and let $F \in \mathscr{S}$. Then by what was just shown,

$$
\int_{F} s_{n} E(X \mid \mathscr{S}) d P=\int_{F} s_{n} X d P=\int_{F} E\left(s_{n} X \mid \mathscr{S}\right) d P
$$

Now

$$
\begin{aligned}
& \left|\int_{F} E\left(s_{n} X \mid \mathscr{S}\right) d P-\int_{F} E(Z X \mid \mathscr{S}) d P\right| \\
\leq & \int_{F}\left|s_{n} X-Z X\right| d P=\int_{F}\left|s_{n}-Z\right||X| d P
\end{aligned}
$$

which converges to 0 by the dominated convergence theorem. Then passing to the limit using the dominated convergence theorem, yields

$$
\int_{F} Z E(X \mid \mathscr{S}) d P=\int_{F} Z X d P \equiv \int_{F} E(Z X \mid \mathscr{S}) d P
$$

Since this holds for every $F \in \mathscr{S}$, this shows 60.1.2.
The next major result is a generalization of Jensen's inequality whose proof depends on the following lemma about convex functions.

Lemma 60.1.3 Let $\phi$ be a convex real valued function defined on an interval I. Then for each $x \in I$, there exists $a_{x}$ such that for all $t \in I$,

$$
\phi(t) \geq a_{x}(t-x)+\phi(x) .
$$

Also $\phi$ is continuous on I.
Proof: Let $x \in I$ and let $t>x$. Then by convexity of $\phi$,

$$
\begin{gathered}
\frac{\phi(x+\lambda(t-x))-\phi(x)}{\lambda(t-x)} \leq \frac{\phi(x)(1-\lambda)+\lambda \phi(t)-\phi(x)}{\lambda(t-x)} \\
=\frac{\phi(t)-\phi(x)}{t-x}
\end{gathered}
$$

Therefore $t \rightarrow \frac{\phi(t)-\phi(x)}{t-x}$ is increasing if $t>x$. If $t<x$

$$
\begin{gathered}
\frac{\phi(x+\lambda(t-x))-\phi(x)}{\lambda(t-x)} \geq \frac{\phi(x)(1-\lambda)+\lambda \phi(t)-\phi(x)}{\lambda(t-x)} \\
=\frac{\phi(t)-\phi(x)}{t-x}
\end{gathered}
$$

and so $t \rightarrow \frac{\phi(t)-\phi(x)}{t-x}$ is increasing for $t \neq x$. Let

$$
a_{x} \equiv \inf \left\{\frac{\phi(t)-\phi(x)}{t-x}: t>x\right\}
$$

Then if $t_{1}<x$, and $t>x$,

$$
\frac{\phi\left(t_{1}\right)-\phi(x)}{t_{1}-x} \leq a_{x} \leq \frac{\phi(t)-\phi(x)}{t-x}
$$

Thus for all $t \in I$,

$$
\begin{equation*}
\phi(t) \geq a_{x}(t-x)+\phi(x) \tag{60.1.3}
\end{equation*}
$$

The continuity of $\phi$ follows easily from this and the observation that convexity simply says that the graph of $\phi$ lies below the line segment joining two points on its graph. Thus, we have the following picture which clearly implies continuity.


Lemma 60.1.4 Let I be an open interval on $\mathbb{R}$ and let $\phi$ be a convex function defined on $I$. Then there exists a sequence $\left\{\left(a_{n}, b_{n}\right)\right\}$ such that

$$
\phi(t)=\sup \left\{a_{n} t+b_{n}, n=1, \cdots\right\}
$$

Proof: Let $a_{x}$ be as defined in the above lemma. Let

$$
\psi(x) \equiv \sup \left\{a_{r}(x-r)+\phi(r): r \in \mathbb{Q} \cap I\right\}
$$

Thus if $r_{1} \in \mathbb{Q}$,

$$
\psi\left(r_{1}\right) \equiv \sup \left\{a_{r}\left(r_{1}-r\right)+\phi(r): r \in \mathbb{Q} \cap I\right\} \geq \phi\left(r_{1}\right)
$$

Then $\psi$ is convex on $I$ so $\psi$ is continuous. Therefore, $\psi(t) \geq \phi(t)$ for all $t \in I$. By 60.1.3,

$$
\psi(t) \geq \phi(t) \geq \sup \left\{a_{r}(t-r)+\phi(r), r \in \mathbb{Q} \cap I\right\} \equiv \psi(t)
$$

Thus $\psi(t)=\phi(t)$. Let $\mathbb{Q} \cap I=\left\{r_{n}\right\}, a_{n}=a_{r_{n}}$ and $b_{n}=-a_{r_{n}} r_{n}+\phi\left(r_{n}\right)$.

Lemma 60.1.5 If $X \leq Y$, then $E(X \mid \mathscr{S}) \leq E(Y \mid \mathscr{S})$ a.e. Also

$$
X \rightarrow E(X \mid \mathscr{S})
$$

is linear.
Proof: Let $A \in \mathscr{S}$.

$$
\begin{aligned}
& \int_{A} E(X \mid \mathscr{S}) d P \equiv \int_{A} X d P \\
\leq & \int_{A} Y d P \equiv \int_{A} E(Y \mid \mathscr{S}) d P
\end{aligned}
$$

Hence $E(X \mid \mathscr{S}) \leq E(Y \mid \mathscr{S})$ a.e. as claimed. It is obvious $X \rightarrow E(X \mid \mathscr{S})$ is linear.
Theorem 60.1.6 (Jensen's inequality)Let $X(\omega) \in I$ and let $\phi: I \rightarrow \mathbb{R}$ be convex. Suppose

$$
E(|X|), E(|\phi(X)|)<\infty
$$

Then

$$
\phi(E(X \mid \mathscr{S})) \leq E(\phi(X) \mid \mathscr{S})
$$

Proof: Let $\phi(x)=\sup \left\{a_{n} x+b_{n}\right\}$. Letting $A \in \mathscr{S}$,

$$
\frac{1}{P(A)} \int_{A} E(X \mid \mathscr{S}) d P=\frac{1}{P(A)} \int_{A} X d P \in I \text { a.e. }
$$

whenever $P(A) \neq 0$. Hence $E(X \mid \mathscr{S})(\omega) \in I$ a.e. and so it makes sense to consider $\phi(E(X \mid \mathscr{S}))$. Now

$$
a_{n} E(X \mid \mathscr{S})+b_{n}=E\left(a_{n} X+b_{n} \mid \mathscr{S}\right) \leq E(\phi(X) \mid \mathscr{S})
$$

Thus

$$
\begin{gathered}
\sup \left\{a_{n} E(X \mid \mathscr{S})+b_{n}\right\} \\
=\phi(E(X \mid \mathscr{S})) \leq E(\phi(X) \mid \mathscr{S}) \text { a.e. }
\end{gathered}
$$

### 60.2 Discrete Martingales

Definition 60.2.1 Let $\mathscr{S}_{k}$ be an increasing sequence of $\sigma$ algebras which are subsets of $\mathscr{S}$ and $X_{k}$ be a sequence of real-valued random variables with $E\left(\left|X_{k}\right|\right)<\infty$ such that $X_{k}$ is $\mathscr{S}_{k}$ measurable. Then this sequence is called a martingale if

$$
E\left(X_{k+1} \mid \mathscr{S}_{k}\right)=X_{k},
$$

a submartingale if

$$
E\left(X_{k+1} \mid \mathscr{S}_{k}\right) \geq X_{k},
$$

and a supermartingale if

$$
E\left(X_{k+1} \mid \mathscr{S}_{k}\right) \leq X_{k} .
$$

Saying that $X_{k}$ is $\mathscr{S}_{k}$ measurable is referred to by saying $\left\{X_{k}\right\}$ is adapted to $\mathscr{S}_{k}$.

Note that if $\left\{X_{k}\right\}$ is a martingale, then $\left\{\left|X_{k}\right|\right\}$ is a submartingale and that if $\left\{X_{k}\right\}$ is a submartingale and $\phi$ is convex and increasing, then $\left\{\phi\left(X_{k}\right)\right\}$ is a submartingale.

An upcrossing occurs when a sequence goes from $a$ up to $b$. Thus it crosses the interval, $[a, b]$ in the up direction, hence upcrossing. More precisely,

Definition 60.2.2 Let $\left\{x_{i}\right\}_{i=1}^{I}$ be any sequence of real numbers, $I \leq \infty$. Define an increasing sequence of integers $\left\{m_{k}\right\}$ as follows. $m_{1}$ is the first integer $\geq 1$ such that $x_{m_{1}} \leq a, m_{2}$ is the first integer larger than $m_{1}$ such that $x_{m_{2}} \geq b, m_{3}$ is the first integer larger than $m_{2}$ such that $x_{m_{3}} \leq a$, etc. Then each sequence, $\left\{x_{m_{2 k-1}}, \cdots, x_{m_{2 k}}\right\}$, is called an upcrossing of $[a, b]$.

Here is a picture of an upcrossing.


Proposition 60.2.3 Let $\left\{X_{i}\right\}_{i=1}^{n}$ be a finite sequence of real random variables defined on $\Omega$ where $(\Omega, \mathscr{S}, P)$ is a probability space. Let $U_{[a, b]}(\omega)$ denote the number of upcrossings of $X_{i}(\omega)$ of the interval $[a, b]$. Then $U_{[a, b]}$ is a random variable.

Proof: Let $X_{0}(\omega) \equiv a+1$, let $Y_{0}(\omega) \equiv 0$, and let $Y_{k}(\omega)$ remain 0 for $k=0, \cdots, l$ until $X_{l}(\omega) \leq a$. When this happens (if ever), $Y_{l+1}(\omega) \equiv 1$. Then let $Y_{i}(\omega)$ remain 1 for $i=l+1, \cdots, r$ until $X_{r}(\omega) \geq b$ when $Y_{r+1}(\omega) \equiv 0$. Let $Y_{k}(\omega)$ remain 0 for $k \geq r+1$ until $X_{k}(\omega) \leq a$ when $Y_{k}(\omega) \equiv 1$ and continue in this way. Thus the upcrossings of $X_{i}(\omega)$ are identified as unbroken strings of ones for $Y_{k}$ with a zero at each end, with the possible exception of the last string of ones which may be missing the zero at the upper end and may or may not be an upcrossing.

Note also that $Y_{0}$ is measurable because it is identically equal to 0 and that if $Y_{k}$ is measurable, then $Y_{k+1}$ is measurable because the only change in going from $k$ to $k+1$ is a change from 0 to 1 or from 1 to 0 on a measurable set determined by $X_{k}$. In particular,

$$
Y_{k+1}^{-1}(1)=\left(\left[Y_{k}=1\right] \cap\left[X_{k}<b\right]\right) \cup\left(\left[Y_{k}=0\right] \cap\left[X_{k} \leq a\right]\right)
$$

This set is in $\mathscr{S}$ by induction. Of course, $Y_{k+1}^{-1}(0)$ is just the complement of this set. Thus $Y_{k+1}$ is $\mathscr{S}$ measurable since 0,1 are the only two values possible. Now let

$$
Z_{k}(\omega)=\left\{\begin{array}{l}
1 \text { if } Y_{k}(\omega)=1 \text { and } Y_{k+1}(\omega)=0 \\
0 \text { otherwise }
\end{array}\right.
$$

if $k<n$ and

$$
Z_{n}(\omega)=\left\{\begin{array}{l}
1 \text { if } Y_{n}(\omega)=1 \text { and } X_{n}(\omega) \geq b \\
0 \text { otherwise }
\end{array}\right.
$$

Thus $Z_{k}(\omega)=1$ exactly when an upcrossing has been completed and each $Z_{i}$ is a random variable.

$$
U_{[a, b]}(\omega)=\sum_{k=1}^{n} Z_{k}(\omega)
$$

so $U_{[a, b]}$ is a random variable as claimed.
The following corollary collects some key observations found in the above construction.
Corollary 60.2.4 $U_{[a, b]}(\omega) \leq$ the number of unbroken strings of ones in the sequence, $\left\{Y_{k}(\omega)\right\}$ there being at most one unbroken string of ones which produces no upcrossing. Also

$$
\begin{equation*}
Y_{i}(\omega)=\psi_{i}\left(\left\{X_{j}(\omega)\right\}_{j=1}^{i-1}\right) \tag{60.2.4}
\end{equation*}
$$

where $\psi_{i}$ is some function of the past values of $X_{j}(\omega)$.
Lemma 60.2.5 Let $\phi$ be a convex and increasing function and suppose

$$
\left\{\left(X_{n}, \mathscr{S}_{n}\right)\right\}
$$

is a submartingale. Then if $E\left(\left|\phi\left(X_{n}\right)\right|\right)<\infty$, it follows

$$
\left\{\left(\phi\left(X_{n}\right), \mathscr{S}_{n}\right)\right\}
$$

is also a submartingale.
Proof: It is given that $E\left(X_{n+1}, \mathscr{S}_{n}\right) \geq X_{n}$ and so

$$
\phi\left(X_{n}\right) \leq \phi\left(E\left(X_{n+1} \mid \mathscr{S}_{n}\right)\right) \leq E\left(\phi\left(X_{n+1}\right) \mid \mathscr{S}_{n}\right)
$$

by Jensen's inequality.
The following is called the upcrossing lemma.

### 60.2.1 Upcrossings

Lemma 60.2.6 (upcrossing lemma) Let $\left\{\left(X_{i}, \mathscr{S}_{i}\right)\right\}_{i=1}^{n}$ be a submartingale and let $U_{[a, b]}(\omega)$ be the number of upcrossings of $[a, b]$. Then

$$
E\left(U_{[a, b]}\right) \leq \frac{E\left(\left|X_{n}\right|\right)+|a|}{b-a}
$$

Proof: Let $\phi(x) \equiv a+(x-a)^{+}$so that $\phi$ is an increasing convex function always at least as large as $a$. By Lemma 60.2.5 it follows that $\left\{\left(\phi\left(X_{k}\right), \mathscr{S}_{k}\right)\right\}$ is also a submartingale.

$$
\begin{gathered}
\phi\left(X_{k+r}\right)-\phi\left(X_{k}\right)=\sum_{i=k+1}^{k+r} \phi\left(X_{i}\right)-\phi\left(X_{i-1}\right) \\
=\sum_{i=k+1}^{k+r}\left(\phi\left(X_{i}\right)-\phi\left(X_{i-1}\right)\right) Y_{i}+\sum_{i=k+1}^{k+r}\left(\phi\left(X_{i}\right)-\phi\left(X_{i-1}\right)\right)\left(1-Y_{i}\right) .
\end{gathered}
$$

Observe that $Y_{i}$ is $\mathscr{S}_{i-1}$ measurable from its construction in Proposition 60.2.3, $Y_{i}$ depending only on $X_{j}$ for $j<i$.

Now let the unbroken strings of ones for $\left\{Y_{i}(\omega)\right\}$ be

$$
\begin{equation*}
\left\{k_{1}, \cdots, k_{1}+r_{1}\right\},\left\{k_{2}, \cdots, k_{2}+r_{2}\right\}, \cdots,\left\{k_{m}, \cdots, k_{m}+r_{m}\right\} \tag{60.2.5}
\end{equation*}
$$

where $m=V(\omega) \equiv$ the number of unbroken strings of ones in the sequence $\left\{Y_{i}(\omega)\right\}$. By Corollary 60.2.4 $V(\omega) \geq U_{[a, b]}(\omega)$.

$$
\begin{gathered}
\phi\left(X_{n}(\omega)\right)-\phi\left(X_{1}(\omega)\right) \\
=\sum_{k=1}^{n}\left(\phi\left(X_{k}(\omega)\right)-\phi\left(X_{k-1}(\omega)\right)\right) Y_{k}(\omega) \\
+\sum_{k=1}^{n}\left(\phi\left(X_{k}(\omega)\right)-\phi\left(X_{k-1}(\omega)\right)\right)\left(1-Y_{k}(\omega)\right) .
\end{gathered}
$$

The first sum in the above reduces to summing over the unbroken strings of ones because the terms in which $Y_{i}(\omega)=0$ contribute nothing. Therefore,

$$
\begin{gather*}
\phi\left(X_{n}(\omega)\right)-\phi\left(X_{1}(\omega)\right) \\
\geq U_{[a, b]}(\omega)(b-a)+0+ \\
\sum_{k=1}^{n}\left(\phi\left(X_{k}(\omega)\right)-\phi\left(X_{k-1}(\omega)\right)\right)\left(1-Y_{k}(\omega)\right) \tag{60.2.6}
\end{gather*}
$$

where the zero on the right side results from a string of ones which does not produce an upcrossing. It is here that it is important that $\phi(x) \geq a$. Such a string begins with $\phi\left(X_{k}(\omega)\right)=a$ and results in an expression of the form $\phi\left(X_{k+m}(\omega)\right)-\phi\left(X_{k}(\omega)\right) \geq 0$ since $\phi\left(X_{k+m}(\omega)\right) \geq a$. If $X_{k}$ had not been replaced with $\phi\left(X_{k}\right)$, it would have been possible for $\phi\left(X_{k+m}(\omega)\right)$ to be less than $a$ and the zero in the above could have been a negative number This would have been inconvenient.

Next take the expected value of both sides in 60.2.6. This results in

$$
\begin{aligned}
E\left(\phi\left(X_{n}\right)-\phi\left(X_{1}\right)\right) \geq & (b-a) E\left(U_{[a, b]}\right) \\
& +E\left(\sum_{k=1}^{n}\left(\phi\left(X_{k}\right)-\phi\left(X_{k-1}\right)\right)\left(1-Y_{k}\right)\right) \\
\geq & (b-a) E\left(U_{[a, b]}\right)
\end{aligned}
$$

The reason for the last inequality where the term at the end was dropped is

$$
\begin{gathered}
E\left(\left(\phi\left(X_{k}\right)-\phi\left(X_{k-1}\right)\right)\left(1-Y_{k}\right)\right) \\
=E\left(E\left(\left(\phi\left(X_{k}\right)-\phi\left(X_{k-1}\right)\right)\left(1-Y_{k}\right) \mid \mathscr{F}_{k-1}\right)\right) \\
=E\left(\left(1-Y_{k}\right) E\left(\phi\left(X_{k}\right) \mid \mathscr{F}_{k-1}\right)-\left(1-Y_{k}\right) E\left(\phi\left(X_{k-1}\right) \mid \mathscr{F}_{k-1}\right)\right) \\
\geq E\left(\left(1-Y_{k}\right)\left(\phi\left(X_{k-1}\right)-\phi\left(X_{k-1}\right)\right)\right)=0 .
\end{gathered}
$$

Recall that $Y_{k}$ is $\mathscr{S}_{k-1}$ measurable and that $\left(\phi\left(X_{k}\right), \mathscr{S}_{k}\right)$ is a submartingale.
The reason for this lemma is to prove the amazing submartingale convergence theorem.

### 60.2.2 The Submartingale Convergence Theorem

Theorem 60.2.7 (submartingale convergence theorem) Let

$$
\left\{\left(X_{i}, \mathscr{S}_{i}\right)\right\}_{i=1}^{\infty}
$$

be a submartingale with $K \equiv \sup E\left(\left|X_{n}\right|\right)<\infty$. Then there exists a random variable, $X$, such that $E(|X|) \leq K$ and

$$
\lim _{n \rightarrow \infty} X_{n}(\omega)=X(\omega) \text { a.e. }
$$

Proof: Let $a, b \in \mathbb{Q}$ and let $a<b$. Let $U_{[a, b]}^{n}(\omega)$ be the number of upcrossings of $\left\{X_{i}(\omega)\right\}_{i=1}^{n}$. Then let

$$
U_{[a, b]}(\omega) \equiv \lim _{n \rightarrow \infty} U_{[a, b]}^{n}(\omega)=\text { number of upcrossings of }\left\{X_{i}\right\}
$$

By the upcrossing lemma,

$$
E\left(U_{[a, b]}^{n}\right) \leq \frac{E\left(\left|X_{n}\right|\right)+|a|}{b-a} \leq \frac{K+|a|}{b-a}
$$

and so by the monotone convergence theorem,

$$
E\left(U_{[a, b]}\right) \leq \frac{K+|a|}{b-a}<\infty
$$

which shows $U_{[a, b]}(\omega)$ is finite a.e., for all $\omega \notin S_{[a, b]}$ where $P\left(S_{[a, b]}\right)=0$. Define

$$
S \equiv \cup\left\{S_{[a, b]}: a, b \in \mathbb{Q}, a<b\right\} .
$$

Then $P(S)=0$ and if $\omega \notin S,\left\{X_{k}\right\}_{k=1}^{\infty}$ has only finitely many upcrossings of every interval having rational endpoints. For such $\omega$ it cannot be the case that

$$
\lim \sup _{k \rightarrow \infty} X_{k}(\omega)>\lim \inf _{k \rightarrow \infty} X_{k}(\omega)
$$

because then you could pick rational $a, b$ such that $[a, b]$ is between the limsup and the liminf and there would be infinitely many upcrossings of $[a, b]$. Thus, for $\omega \notin S$,

$$
\begin{aligned}
\limsup _{k \rightarrow \infty} X_{k}(\omega) & =\lim _{k \rightarrow \infty} \inf _{k}(\omega) \\
& =\lim _{k \rightarrow \infty} X_{k}(\omega) \equiv X_{\infty}(\omega) .
\end{aligned}
$$

Letting $X_{\infty}(\omega) \equiv 0$ for $\omega \in S$, Fatou's lemma implies

$$
\int_{\Omega}\left|X_{\infty}\right| d P=\int_{\Omega} \lim \inf _{n \rightarrow \infty}\left|X_{n}\right| d P \leq \lim \inf _{n \rightarrow \infty} \int_{\Omega}\left|X_{n}\right| d P \leq K
$$

### 60.2.3 Doob Submartingale Estimate

Another very interesting result about submartingales is the Doob submartingale estimate.
Theorem 60.2.8 Let $\left\{\left(X_{i}, \mathscr{S}_{i}\right)\right\}_{i=1}^{\infty}$ be a submartingale. Then for $\lambda>0$,

$$
P\left(\left[\max _{1 \leq k \leq n} X_{k} \geq \lambda\right]\right) \leq \frac{1}{\lambda} \int_{\Omega} X_{n}^{+} d P
$$

Proof: Let

$$
\begin{aligned}
A_{1} & \equiv\left[X_{1} \geq \lambda\right], A_{2} \equiv\left[X_{2} \geq \lambda\right] \backslash A_{1}, \\
\cdots, A_{k} & \equiv\left[X_{k} \geq \lambda\right] \backslash\left(\cup_{i=1}^{k-1} A_{i}\right) \cdots
\end{aligned}
$$

Thus each $A_{k}$ is $\mathscr{S}_{k}$ measurable, the $A_{k}$ are disjoint, and their union equals

$$
\left[\max _{1 \leq k \leq n} X_{k} \geq \lambda\right]
$$

Therefore from the definition of a submartingale and Jensen's inequality,

$$
\begin{aligned}
P\left(\left[\max _{1 \leq k \leq n} X_{k} \geq \lambda\right]\right) & =\sum_{k=1}^{n} P\left(A_{k}\right) \leq \frac{1}{\lambda} \sum_{k=1}^{n} \int_{A_{k}} X_{k} d P \\
& \leq \frac{1}{\lambda} \sum_{k=1}^{n} \int_{A_{k}} E\left(X_{n} \mid \mathscr{S}_{k}\right) d P \\
& \leq \frac{1}{\lambda} \sum_{k=1}^{n} \int_{A_{k}} E\left(X_{n} \mid \mathscr{S}_{k}\right)^{+} d P \\
& \leq \frac{1}{\lambda} \sum_{k=1}^{n} \int_{A_{k}} E\left(X_{n}^{+} \mid \mathscr{S}_{k}\right) d P \\
& =\frac{1}{\lambda} \sum_{k=1}^{n} \int_{A_{k}} X_{n}^{+} d P \leq \frac{1}{\lambda} \int_{\Omega} X_{n}^{+} d P .
\end{aligned}
$$

### 60.3 Optional Sampling And Stopping Times

### 60.3.1 Stopping Times And Their Properties Overview

I will give a brief overview of the main ideas about stopping times first and then repeat them in what follows. I think that these things are so important that it is good to have a short synopsis of what to expect. I think that the optional sampling theorem of Doob is amazing. That is why it gets repeated quite a bit. It is one of those theorems that you read and when you get to the end, having followed the argument, you sit back and feel amazed at what you just went through. You ask yourself if it is really true or whether you made some mistake. At least this is how it effects me.

First it is necessary to define the notion of a stopping time. If you have an increasing sequence of $\sigma$ algebras $\left\{\mathscr{F}_{n}\right\}$ and a process $\left\{X_{n}\right\}$ such that $X_{n}$ is $\mathscr{F}_{n}$ measurable, the idea
of a stopping time $T$ is that $T$ is a measurable function and for such a process $\left\{X_{n}\right\}, \omega \rightarrow$ $X_{n \wedge T(\omega)}(\omega)$ is also $\mathscr{F}_{n}$ measurable. In other words, by stopping with this stopping time, we preserve the $\mathscr{F}_{n}$ measurability. We want to have

$$
X_{n \wedge T}^{-1}(O) \in \mathscr{F}_{n}
$$

where $O$ is an open set in some metric space where $X_{n}$ has its values.

$$
X_{n \wedge T}^{-1}(O)=[T \leq n] \cap\left[\omega: X_{T(\omega)}(\omega) \in O\right] \cup[T>n] \cap\left[\omega: X_{n}(\omega) \in O\right]
$$

Now

$$
[T \leq n] \cap\left[\omega: X_{T(\omega)}(\omega) \in O\right]=\cup_{k=1}^{n}[T=k] \cap\left[X_{k} \in O\right]
$$

To have this in $\mathscr{F}_{n}$, we should have $[T=k] \in \mathscr{F}_{k}$. That is $[T \leq k] \in \mathscr{F}_{k}$. Now once this is done, $[T>n]=[T \leq n]^{C} \in \mathscr{F}_{n}$ also. This motivates the following definition and shows that the requirement that $[T \leq n] \in \mathscr{F}_{n}$ implies that $\omega \rightarrow X_{n \wedge T(\omega)}(\omega)$ is $\mathscr{F}_{n}$ measurable if this is true of $X_{n}$.

Definition 60.3.1 Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $\left\{\mathscr{F}_{n}\right\}_{n=1}^{\infty}$ be an increasing sequence of $\sigma$ algebras each contained in $\mathscr{F}$. A stopping time is a measurable function, $T$ which maps $\Omega$ to $\mathbb{N}$,

$$
T^{-1}(A) \in \mathscr{F} \text { for all } A \in \mathscr{P}(\mathbb{N})
$$

such that for all $n \in \mathbb{N}$,

$$
[T \leq n] \in \mathscr{F}_{n}
$$

Note this is equivalent to saying

$$
[T=n] \in \mathscr{F}_{n}
$$

because

$$
[T=n]=[T \leq n] \backslash[T \leq n-1]
$$

For $T$ a stopping time define $\mathscr{F}_{T}$ as follows.

$$
\mathscr{F}_{T} \equiv\left\{A \in \mathscr{F}: A \cap[T \leq n] \in \mathscr{F}_{n} \text { for all } n \in \mathbb{N}\right\}
$$

These sets in $\mathscr{F}_{T}$ are referred to as "prior" to $T$.
Of course $T$ has values $i$, a countable well ordered set of numbers, $i \leq i+1$. Then we have the following about the relation with stopping times and conditional expectations.

Lemma 60.3.2 Let $X$ be in $L^{1}(\Omega)$. Then $\mathscr{F}_{T} \cap[T=i]=\mathscr{F}_{i} \cap[T=i]$ and $E\left(X \mid \mathscr{F}_{T}\right)=$ $E\left(X \mid \mathscr{F}_{i}\right)$ a.e. on the set $[T=i]$. Also if $A \in \mathscr{F}_{T}$, then $A \cap[T=i] \in \mathscr{F}_{i} \cap \mathscr{F}_{T}$. Also $\mathscr{F}_{T} \cap$ $[T \leq i]=\mathscr{F}_{i} \cap[T \leq i]$ and $E\left(X \mid \mathscr{F}_{T}\right)=E\left(X \mid \mathscr{F}_{i}\right)$ a.e. on the set $[T \leq i]$.

## Proof: Let

1. Typical set in $\mathscr{F}_{T} \cap[T=i]$ is $A \cap[T=i]$ where $A \in \mathscr{F}_{T}$. Thus $A \cap[T=i]=B \in \mathscr{F}_{i}$ so $A \cap[T=i]=A \cap[T=i] \cap[T=i]=B \cap[T=i] \in \mathscr{F}_{i} \cap[T=i]$.
2. Typical set in $\mathscr{F}_{i} \cap[T=i]$ is $A \cap[T=i]$ where $A \in \mathscr{F}_{i}$. Then $A \cap[T=i] \cap[T=j] \in$ $\mathscr{F}_{j}$ for all $j$. If $j \neq i$, you get $\emptyset$ and if $j=i$, you get $A \cap[T=i] \in \mathscr{F}_{i}=\mathscr{F}_{j}$ so $A \cap[T=i]=B \in \mathscr{F}_{T}$ and so $A \cap[T=i] \cap[T=i]=B \cap[T=i] \in \mathscr{F}_{T} \cap[T=i]$.

Now let $A \in \mathscr{F}_{T}$. Then

$$
\int_{A \cap[T=i]} E\left(X \mid \mathscr{F}_{i}\right) d P=\int_{A \cap[T=i]} X d P \equiv \int_{A \cap[T=i]} E\left(X \mid \mathscr{F}_{T}\right) d P
$$

because the set $A \cap[T=i] \in \mathscr{F}_{i}$ and is also in $\mathscr{F}_{T}$. A typical set in $\mathscr{F}_{T} \cap[T=i]=\mathscr{F}_{i} \cap$ $[T=i]$ is of this form which was just shown above and so, since this holds for all sets in $\mathscr{F}_{T} \cap[T=i]=\mathscr{F}_{i} \cap[T=i]$, it must be the case that $E\left(X \mid \mathscr{F}_{i}\right)=E\left(X \mid \mathscr{F}_{T}\right)$ a.e. on $[T=i]$. The last claim is obvious from this. Indeed, if $A \in \mathscr{F}_{T} \cap[T \leq i]$, then it is of the form

$$
A=B \cap \cup_{k \leq i}[T=k]=\cup_{k \leq i} B \cap[T=k]
$$

and each set in the union is in $\mathscr{F}_{i} \cap[T \leq i]$. For the other direction, if $A \in \mathscr{F}_{i} \cap[T \leq i]$ then

$$
A=\cup_{k \leq i} B \cap[T=k], B \in \mathscr{F}_{i}
$$

and each set in the union is in $\mathscr{F}_{T} \cap[T \leq i]$. Now note that if $A \in \mathscr{F}_{T}$, then $A \cap[T \leq i] \in \mathscr{F}_{i}$ by definition and $A \cap[T \leq i] \cap[T \leq j] \in \mathscr{F}_{j} \subseteq \mathscr{F}_{i}$ if $j \leq i$ wile if $j>i$, this set is equal to $A \cap[T \leq i]$ which is in $\mathscr{F}_{i}$ and so the same argument above gives the result that $E\left(X \mid \mathscr{F}_{T}\right)=$ $E\left(X \mid \mathscr{F}_{i}\right)$ a.e. on the set $[T \leq i]$.

One of the big results is the optional sampling theorem. Suppose $X_{n}$ is a martingale. In particular, each $X_{n} \in L^{1}(\Omega)$ and $E\left(X_{n} \mid \mathscr{F}_{k}\right)=X_{k}$ whenever $k \leq n$. We can assume $X_{n}$ has values in some separable Banach space. Then $\left\|X_{n}\right\|$ is a submartingale because if $k \leq n$, then if $A \in \mathscr{F}_{k}$,

$$
\int_{A} E\left(\left\|X_{n}\right\| \mid \mathscr{F}_{k}\right) d P \geq \int_{A}\left\|E\left(X_{n} \mid \mathscr{F}_{k}\right)\right\| d P=\int_{A}\left\|X_{k}\right\| d P
$$

Now suppose we have two stopping times $\tau$ and $\sigma$ and $\tau$ is bounded meaning it has values in $\{1,2, \cdots, n\}$. The optional sampling theorem says the following. Here $M$ is a martingale.

$$
M(\sigma \wedge \tau)=E\left(M(\tau) \mid \mathscr{F}_{\sigma}\right)
$$

Furthermore, it all makes sense. First of all, why does it make sense? We need to verify that $M(\tau)$ is integrable.

$$
\begin{aligned}
\int\|M(\tau)\| & =\sum_{k=1}^{n} \int_{[\tau=k]}\|M(k)\|=\sum_{k=1}^{n} \int_{[\tau=k]}\left\|E\left(M(n) \mid \mathscr{F}_{k}\right)\right\| \\
& \leq \sum_{k=1}^{n} \int_{[\tau=k]} E\left(\|M(n)\| \mid \mathscr{F}_{k}\right) \leq \sum_{k=1}^{n} \int E\left(\|M(n)\| \mid \mathscr{F}_{k}\right) \\
& =\sum_{k=1}^{n} E(\|M(n)\|)<\infty
\end{aligned}
$$

Similarly, $M(\sigma \wedge \tau)$ is integrable. Now let $A \in \mathscr{F} \sigma$. Then using Lemma 60.3.2 as needed,

$$
\begin{aligned}
\int_{A} M(\sigma \wedge \tau) & =\sum_{i=1}^{n} \int_{A \cap[\tau=i]} M(\sigma \wedge i)=\sum_{i=1}^{n} \sum_{j=1}^{\infty} \int_{A \cap[\tau=i] \cap[\sigma=j]} M(j \wedge i) \\
& =\sum_{i=1}^{n} \sum_{j=1}^{\infty} \int_{A \cap[\tau=i] \cap[\sigma=j]} E\left(M(i) \mid \mathscr{F}_{j}\right)
\end{aligned}
$$

There are two cases here. If $j \leq i$ it is the martingale definition. If $j>i$ the third term has integrand equal to $M(i)$ which is $\mathscr{F}_{j}$ measurable so the formula is still valid. Then the above equals

$$
\begin{gathered}
\sum_{i=1}^{n} \sum_{j=1}^{\infty} \int_{A \cap[\tau=i] \cap[\sigma=j]} E\left(M(i) \mid \mathscr{F}_{\sigma}\right)=\sum_{j=1}^{\infty} \sum_{i=1}^{n} \int_{A \cap[\tau=i] \cap[\sigma=j]} E\left(M(i) \mid \mathscr{F}_{\sigma}\right) \\
=\sum_{j=1}^{\infty} \int_{A \cap[\sigma=j]} E\left(M(\tau) \mid \mathscr{F}_{\sigma}\right)=\int_{A} E\left(M(\tau) \mid \mathscr{F}_{\sigma}\right)
\end{gathered}
$$

Since $A$ is an arbitrary element of $\mathscr{F}_{\sigma}$, this shows the optional sampling theorem that $M(\sigma \wedge \tau)=E\left(M(\tau) \mid \mathscr{F}_{\sigma}\right)$.

Proposition 60.3.3 Let $M$ be a martingale having values in some separable Banach space. Let $\tau$ be a bounded stopping time and let $\sigma$ be another stopping time. Then everything makes sense in the following formula and

$$
M(\sigma \wedge \tau)=E\left(M(\tau) \mid \mathscr{F}_{\sigma}\right) \text { a.e. }
$$

### 60.4 Stopping Times

The following lemma is fundamental to understand.
Lemma 60.4.1 In the situation of Definition 60.3.1, if $S \leq T$ for two stopping times, $S$ and $T$, then $\mathscr{F}_{S} \subseteq \mathscr{F}_{T}$. Also $\mathscr{F}_{T}$ is a $\sigma$ algebra.

Proof: Let $A \in \mathscr{F}_{S}$. Then this means

$$
A \cap[S \leq n] \in \mathscr{F}_{n} \text { for all } n
$$

Then I claim that

$$
\begin{equation*}
A \cap[T \leq n]=\cup_{i=1}^{n}(A \cap[S \leq i]) \cap[T \leq n] \tag{60.4.7}
\end{equation*}
$$

Suppose $\omega$ is in the set on the left. Then if $T(\omega)<n$, it is clearly in the set on the right. If $T(\omega)=n$, then $\omega \in[S \leq i]$ for some $i \leq n$ and it is also in $[T \leq n]$. Thus the set on the left is contained in the set on the right. Next suppose $\omega$ is in the set on the right. Then $\omega \in[T \leq n]$ and it only remains to verify $\omega \in A$. However, $\omega \in A \cap[S \leq i]$ for some $i$ and so $\omega \in A$ also.

Now from 60.4.7 it follows $A \cap[T \leq n] \in \mathscr{F}_{n}$ because

$$
A \cap[S \leq i] \in \mathscr{F}_{i} \subseteq \mathscr{F}_{n}
$$

and $[T \leq n] \in \mathscr{F}_{n}$ because $T$ is a stopping time. Since $n$ is arbitrary, this shows $\mathscr{F}_{S} \subseteq \mathscr{F}_{T}$. It remains to verify $\mathscr{F}_{T}$ is a $\sigma$ algebra. Suppose $\left\{A_{i}\right\}$ is a sequence of sets in $\mathscr{F}_{T}$. Then I need to show that $\left(\cup_{i=1}^{\infty} A_{i}\right) \cap[T \leq j] \in \mathscr{F}_{j}$ for all $j$.

$$
\cup_{i=1}^{\infty} A_{i} \cap[T \leq j]=\cup_{i=1}^{\infty}\left(A_{i} \cap[T \leq j]\right)
$$

Now each $\left(A_{i} \cap[T \leq j]\right)$ is in $\mathscr{F}_{j}$ and so the countable union of these sets is also in $\mathscr{F}_{j}$. Next suppose $A \in \mathscr{F}_{T}$. I need to verify $A^{C} \cap[T \leq j] \in \mathscr{F}_{j}$ for all $j$. However, $[T \leq j] \in \mathscr{F}_{j}$ and $\Omega \in \mathscr{F}_{j}$ so $\Omega \in \mathscr{F}_{T}$. Thus

$$
\Omega \cap[T \leq j]=(A \cap[T \leq j]) \cup\left(A^{C} \cap[T \leq j]\right)
$$

and so

$$
\left(A^{C} \cap[T \leq j]\right)=\Omega \cap[T \leq j] \backslash(A \cap[T \leq j]) \in \mathscr{F}_{j} .
$$

This proves the lemma.
Lemma 60.4.2 Let $T$ be a stopping time and let $\left\{X_{n}\right\}$ be a sequence of random variables such that $X_{n}$ is $\mathscr{F}_{n}$ measurable. Then $X_{T}(\omega) \equiv X_{T(\omega)}(\omega)$ is also a random variable and it is measurable with respect to $\mathscr{F}_{T}$.

Proof: I assume the $X_{n}$ have values in some topological space and each is measurable because the inverse image of an open set is in $\mathscr{F}_{n}$. I need to show $X_{T}^{-1}(U) \cap[T \leq n] \in \mathscr{F}_{n}$ for all $n$ whenever $U$ is open.

$$
X_{T}^{-1}(U)=\cup_{i=1}^{\infty} X_{i}^{-1}(U) \cap[T=i]
$$

It follows $X_{T}^{-1}(U) \in \mathscr{F}$. Furthermore,

$$
\begin{aligned}
X_{T}^{-1}(U) \cap[T \leq n] & =\cup_{i=1}^{\infty} X_{i}^{-1}(U) \cap[T=i] \cap[T \leq n] \\
& =\cup_{i=1}^{n} X_{i}^{-1}(U) \cap[T=i] \cap[T \leq n] \\
& =\cup_{i=1}^{n} \overbrace{X_{i}^{-1}(U) \cap[T=i]}^{\text {in } \mathscr{F}_{i}} \in \mathscr{F}_{n}
\end{aligned}
$$

and so $X_{T}$ is $\mathscr{F}_{T}$ is measurable as claimed. This proves the lemma.
Lemma 60.4.3 Let $S \leq T$ be two stopping times such that $T$ is bounded above and let $\left\{X_{n}\right\}$ be a submartingale (martingale) adapted to the increasing sequence of $\sigma$ algebras, $\left\{\mathscr{F}_{n}\right\}$. Then

$$
E\left(X_{T} \mid \mathscr{F}_{S}\right) \geq X_{S}
$$

in the case where $\left\{X_{n}\right\}$ is a submartingale and

$$
E\left(X_{T} \mid \mathscr{F}_{S}\right)=X_{S}
$$

in the case where $\left\{X_{n}\right\}$ is a martingale.

Proof: I will prove the case where $\left\{X_{n}\right\}$ is a submartingale and note the other case will only involve replacing $\geq$ with $=$. First recall that from Lemma 60.4.1 $\mathscr{F}_{S} \subseteq \mathscr{F}_{T}$. Also let $m$ be an upper bound for $T$. Then it follows from this that

$$
E\left(\left|X_{T}\right|\right)=\sum_{i=1}^{m} \int_{[T=i]}\left|X_{i}\right| d P<\infty
$$

with a similar formula holding for $E\left(\left|X_{S}\right|\right)$. Thus it makes sense to speak of $E\left(X_{T} \mid \mathscr{F}_{S}\right)$.
I need to show that if $B \in \mathscr{F}_{S}$, so that $B \cap[S \leq n] \in \mathscr{F}_{n}$ for all $n$, then

$$
\begin{equation*}
\int_{B} X_{T} d P \equiv \int_{B} E\left(X_{T} \mid \mathscr{F}_{S}\right) d P \geq \int_{B} X_{S} d P \tag{60.4.8}
\end{equation*}
$$

It suffices to do this for $B$ of the special form

$$
B=A \cap[S=i]
$$

because if this is done, then the result follows from summing over all possible values of $S$. Note that if $B=A \cap[S=m]$, then $X_{T}=X_{S}=X_{m}$ and there is nothing to prove in 60.4.8 so it can be assumed $i \leq m-1$. Then let $B$ be of this form.

$$
\begin{aligned}
& \int_{A \cap[S=i]} X_{T} d P=\sum_{j=i}^{m} \int_{A \cap[S=i] \cap[T=j]} X_{T} d P \\
= & \sum_{j=i}^{m-1} \int_{A \cap[S=i] \cap[T=j]} X_{T} d P+\int_{A \cap[S=i] \cap[T \geq m]} X_{m} d P
\end{aligned}
$$

And so

$$
\begin{align*}
\int_{A \cap[S=i]} & X_{T} d P=\sum_{j=i}^{m-1} \int_{A \cap[S=i] \cap[T=j]} X_{T} d P+\int_{A \cap[S=i] \cap[T \geq m]} X_{m} d P  \tag{60.4.9}\\
& =\sum_{j=i}^{m-1} \int_{A \cap[S=i] \cap[T=j]} X_{T} d P+\int_{A \cap[S=i] \cap[T \leq m-1]^{C}} X_{m} d P \\
& \geq \sum_{j=i}^{m-1} \int_{A \cap[S=i] \cap[T=j]} X_{T} d P+\int_{A \cap[S=i] \cap[T \leq m-1]^{C}} X_{m-1} d P \\
& =\sum_{j=i}^{m-1} \int_{A \cap[S=i] \cap[T=j]} X_{T} d P+\int_{A \cap[S=i] \cap[T>m-1]} X_{m-1} d P
\end{align*}
$$

provided $m-1 \geq i$ because $\left\{X_{n}\right\}$ is a submartingale and

$$
A \cap[S=i] \cap[T \leq m-1]^{C} \in \mathscr{F}_{m-1}
$$

Now combine the top term of the sum with the term on the right to obtain

$$
=\sum_{j=i}^{m-2} \int_{A \cap[S=i] \cap[T=j]} X_{T} d P+\int_{A \cap[S=i] \cap[T \geq m-1]} X_{m-1} d P
$$

which is exactly the same form as 60.4 .9 except $m$ is replaced with $m-1$. Now repeat this process till you get the following inequality

$$
\int_{A \cap[S=i]} X_{T} d P \geq \sum_{j=i}^{i+1} \int_{A \cap[S=i] \cap[T=j]} X_{T} d P+\int_{A \cap[S=i] \cap[T \geq i+2]} X_{i+2} d P
$$

The right hand side equals

$$
\begin{gathered}
\sum_{j=i}^{i+1} \int_{A \cap[S=i] \cap[T=j]} X_{T} d P+\int_{A \cap[S=i] \cap[T \leq i+1]^{C}} X_{i+2} d P \\
\geq \sum_{j=i}^{i+1} \int_{A \cap[S=i] \cap[T=j]} X_{T} d P+\int_{A \cap[S=i] \cap[T \leq i+1]^{C}} X_{i+1} d P \\
=\int_{A \cap[S=i] \cap[T=i]} X_{T} d P+\int_{A \cap[S=i] \cap[T=i+1]} X_{T} d P+\int_{A \cap[S=i] \cap[T \leq i+1]^{C}} X_{i+1} d P \\
=\int_{A \cap[S=i] \cap[T=i]} X_{i} d P+\int_{A \cap[S=i] \cap[T=i+1]} X_{i+1} d P+\int_{A \cap[S=i] \cap[T>i+1]} X_{i+1} d P \\
=\int_{A \cap[S=i] \cap[T=i]} X_{i} d P+\int_{A \cap[S=i] \cap[T \geq i+1]} X_{i+1} d P \\
=\int_{A \cap[S=i] \cap[T=i]} X_{i} d P+\int_{A \cap[S=i] \cap[T \leq i]^{C}} X_{i+1} d P \\
= \\
\int_{A \cap[S=i] \cap[T=i]} X_{i} d P+\int_{A \cap[S=i] \cap[T \leq i]^{C}} X_{i} d P \\
= \\
\int_{A \cap[S=i] \cap[T=i] \cap[T \geq i]} X_{i} d P+\int_{A \cap[S=i] \cap[T>i]} X_{i} d P=\int_{A \cap[S=i]} X_{i} d P=\int_{A \cap[S=i]} X_{S} d P
\end{gathered}
$$

In the case where $\left\{X_{n}\right\}$ is a martingale, you replace every occurance of $\geq$ in the above argument with $=$. This proves the lemma.

This lemma is called the optional sampling theorem. Another version of this theorem is the case where you have an increasing sequence of stopping times, $\left\{T_{n}\right\}_{n=1}^{\infty}$. Thus if $\left\{X_{n}\right\}$ is a sequence of random variables each $\mathscr{F}_{n}$ measurable, the sequence $\left\{X_{T_{n}}\right\}$ is also a sequence of random variables such that $X_{T_{n}}$ is measurable with respect to $\mathscr{F}_{T_{n}}$ where $\mathscr{F}_{T_{n}}$ is an increasing sequence of $\sigma$ fields. In the case where $X_{n}$ is a submartingale (martingale) it is reasonable to ask whether $\left\{X_{T_{n}}\right\}$ is also a submartingale (martingale). The optional sampling theorem says this is often the case.

Theorem 60.4.4 Let $\left\{T_{n}\right\}$ be an increasing bounded sequence of stopping times and let $\left\{X_{n}\right\}$ be a submartingale (martingale) adapted to the increasing sequence of $\sigma$ algebras, $\left\{\mathscr{F}_{n}\right\}$. Then $\left\{X_{T_{n}}\right\}$ is a submartingale (martingale) adapted to the increasing sequence of $\sigma$ algebras $\left\{\mathscr{F}_{T_{n}}\right\}$.

Proof: This follows from Lemma 60.4.3
Example 60.4.5 Let $\left\{X_{n}\right\}$ be a sequence of real random variables such that $X_{n}$ is $\mathscr{F}_{n}$ measurable and let A be a Borel subset of $\mathbb{R}$. Let $T(\omega)$ denote the first time $X_{n}(\omega)$ is in $A$. Then $T$ is a stopping time. It is called the first hitting time.

To see this is a stopping time,

$$
[T \leq l]=\cup_{i=1}^{l} X_{i}^{-1}(A) \in \mathscr{F}_{l} .
$$

### 60.5 Optional Stopping Times And Martingales

### 60.5.1 Stopping Times And Their Properties

The purpose of this section is to consider a special optional sampling theorem for martingales which is superior to the one presented earlier. I have presented a different treatment of the fundamental properties of stopping times also. See Kallenberg [77] for more.

Definition 60.5.1 Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $\left\{\mathscr{F}_{n}\right\}_{n=1}^{\infty}$ be an increasing sequence of $\sigma$ algebras each contained in $\mathscr{F}$. A stopping time is a measurable function, $\tau$ which maps $\Omega$ to $\mathbb{N}$,

$$
\tau^{-1}(A) \in \mathscr{F} \text { for all } A \in \mathscr{P}(\mathbb{N})
$$

such that for all $n \in \mathbb{N}$,

$$
[\tau \leq n] \in \mathscr{F}_{n}
$$

Note this is equivalent to saying

$$
[\tau=n] \in \mathscr{F}_{n}
$$

because

$$
[\tau=n]=[\tau \leq n] \backslash[\tau \leq n-1] .
$$

For $\tau$ a stopping time define $\mathscr{F}_{\tau}$ as follows.

$$
\mathscr{F}_{\tau} \equiv\left\{A \in \mathscr{F}: A \cap[\tau \leq n] \in \mathscr{F}_{n} \text { for all } n \in \mathbb{N}\right\}
$$

These sets in $\mathscr{F}_{\tau}$ are referred to as "prior" to $\tau$.
First note that for $\tau$ a stopping time, $\mathscr{F}_{\tau}$ is a $\sigma$ algebra. This is in the next proposition.
Proposition 60.5.2 For $\tau$ a stopping time, $\mathscr{F}_{\tau}$ is a $\sigma$ algebra and if $Y(k)$ is $\mathscr{F}_{k}$ measurable for all $k$, then

$$
\omega \rightarrow Y(\tau(\omega))
$$

is $\mathscr{F}_{\tau}$ measurable.
Proof: Let $A_{n} \in \mathscr{F}_{\tau}$. I need to show $\cup_{n} A_{n} \in \mathscr{F}_{\tau}$. In other words, I need to show that

$$
\cup_{n} A_{n} \cap[\tau \leq k] \in \mathscr{F}_{k}
$$

The left side equals

$$
\cup_{n}\left(A_{n} \cap[\tau \leq k]\right)
$$

which is a countable union of sets of $\mathscr{F}_{k}$ and so $\mathscr{F}_{\tau}$ is closed with respect to countable unions. Next suppose $A \in \mathscr{F} \tau$.

$$
\left(A^{C} \cap[\tau \leq k]\right) \cup(A \cap[\tau \leq k])=\Omega \cap[\tau \leq k]
$$

and $\Omega \cap[\tau \leq k] \in \mathscr{F}_{k}$. Therefore, so is $A^{C} \cap[\tau \leq k]$. It remains to verify the last claim. It suffices to verify that $[Y(\tau) \leq a] \in \mathscr{F}_{\tau}$. Is $[Y(\tau) \leq a] \cap[\tau \leq l] \in \mathscr{F}_{l}$ ?

$$
[Y(\tau) \leq a]=\cup_{k}[\tau=k] \cap[Y(k) \leq a]
$$

Thus

$$
[Y(\tau) \leq a] \cap[\tau \leq l]=\cup_{k}[\tau=k] \cap[Y(k) \leq a] \cap[\tau \leq l]
$$

Consider a term in the union. If $l \geq k$ the term reduces to $[\tau=k] \cap[Y(k) \leq a] \in \mathscr{F}_{k}$ while if $l<k$, this term reduces to $\emptyset$, also a set of $\mathscr{F}_{k}$. Therefore, $Y(\tau)$ must be $\mathscr{F}_{\tau}$ measurable. This proves the proposition.

The following lemma contains the fundamental properties of stopping times.
Lemma 60.5.3 In the situation of Definition 60.5.1, let $\sigma, \tau$ be two stopping times. Then

1. $\tau$ is $\mathscr{F}_{\tau}$ measurable
2. $\mathscr{F}_{\sigma} \cap[\sigma \leq \tau] \subseteq \mathscr{F}_{\sigma \wedge \tau}=\mathscr{F}_{\sigma} \cap \mathscr{F}_{\tau}$
3. $\mathscr{F}_{\tau}=\mathscr{F}_{k}$ on $[\tau=k]$ for all $k$. That is if $A \in \mathscr{F}_{k}$, then $A \cap[\tau=k] \in \mathscr{F}_{\tau}$ and if $A \in \mathscr{F}_{\tau}$, then $A \cap[\tau=k] \in \mathscr{F}_{k}$. Also if $A \in \mathscr{F}_{\tau}$,

$$
\int_{A \cap[\tau=k]} E\left(Y \mid \mathscr{F}_{\tau}\right) d P=\int_{A \cap[\tau=k]} E\left(Y \mid \mathscr{F}_{k}\right) d P
$$

and

$$
E\left(Y \mid \mathscr{F}_{\tau}\right)=E\left(Y \mid \mathscr{F}_{k}\right) \text { a.e. }
$$

$$
\text { on }[\tau=k] \text {. }
$$

Proof: Consider the first claim. I need to show that $[\tau \leq a] \cap[\tau \leq k] \in \mathscr{F}_{k}$ for every $k$. However, this is easy if $a \geq k$ because the left side is then $[\tau \leq k]$ which is given to be in $\mathscr{F}_{k}$ since $\tau$ is a stopping time. If $a<k$, it is also easy because then the left side is $[\tau \leq a] \in \mathscr{F}_{[a]}$ where $[a]$ is the greatest integer less than or equal to $a$.

Next consider the second claim. Let $A \in \mathscr{F} \sigma$. I want to show first that

$$
\begin{equation*}
A \cap[\sigma \leq \tau] \in \mathscr{F}_{\tau} \tag{60.5.10}
\end{equation*}
$$

In other words, I want to show

$$
A \cap[\sigma \leq \tau] \cap[\tau \leq k] \in \mathscr{F}_{k}
$$

for all $k$. This will be done if I can show

$$
A \cap[\sigma \leq j] \cap[\tau \leq k] \in \mathscr{F}_{k}
$$

for each $j \leq k$ because

$$
\cup_{j \leq k} A \cap[\sigma \leq j] \cap[\tau \leq k]=A \cap[\sigma \leq \tau] \cap[\tau \leq k]
$$

However, since $A \in \mathscr{F}_{\sigma}$, it follows $A \cap[\sigma \leq j] \in \mathscr{F}_{j} \subseteq \mathscr{F}_{k}$ for each $j \leq k$ and $[\tau \leq k] \in \mathscr{F}_{k}$ and so this has shown what I wanted to show, $A \cap[\sigma \leq \tau] \in \mathscr{F} \tau$.

Now replace the stopping time, $\tau$ with the stopping time $\tau \wedge \sigma$ in what was just shown. Note

$$
[\tau \wedge \sigma \leq n]=[\tau \leq n] \cup[\sigma \leq n] \in \mathscr{F}_{n}
$$

so $\tau \wedge \sigma$ really is a stopping time. This yields

$$
A \cap[\sigma \leq \tau \wedge \sigma] \in \mathscr{F}_{\tau \wedge \sigma}
$$

However the left side equals $A \cap[\sigma \leq \tau]$. Thus

$$
A \cap[\sigma \leq \tau] \in \mathscr{F}_{\tau \wedge \sigma}
$$

This has shown the first part of 2.), $\mathscr{F}_{\sigma} \cap[\sigma \leq \tau] \subseteq \mathscr{F}_{\tau \wedge \sigma}$. Now 60.5 .10 implies if $A \in$ $\mathscr{F}_{\sigma \wedge \tau}$,

$$
A=A \cap \overbrace{[\sigma \wedge \tau \leq \tau]}^{\text {all of } \Omega} \in \mathscr{F}_{\tau}
$$

and so $\mathscr{F}_{\sigma \wedge \tau} \subseteq \mathscr{F}_{\tau}$. Similarly, $\mathscr{F}_{\sigma \wedge \tau} \subseteq \mathscr{F}_{\sigma}$ which shows

$$
\mathscr{F}_{\sigma \wedge \tau} \subseteq \mathscr{F}_{\tau} \cap \mathscr{F}_{\sigma}
$$

Next let $A \in \mathscr{F}_{\tau} \cap \mathscr{F}_{\sigma}$. Then is it in $\mathscr{F}_{\sigma \wedge \tau}$ ? Is $A \cap[\sigma \wedge \tau \leq k] \in \mathscr{F}_{k}$ ? Of course this is so because

$$
\begin{aligned}
& A \cap[\sigma \wedge \tau \leq k]=A \cap([\sigma \leq k] \cup[\tau \leq k]) \\
& \quad=(A \cap[\sigma \leq k]) \cup(A \cap[\tau \leq k]) \in \mathscr{F}_{k}
\end{aligned}
$$

since both $\sigma, \tau$ are stopping times. This proves part 2.$)$.
Now consider part 3.). Note that $[\tau=k]$ is in both $\mathscr{F}_{k}$ and $\mathscr{F}_{\tau}$. This is because $\tau$ is a stopping time so it is in $\mathscr{F}_{k}$. Why is it in $\mathscr{F}_{\tau}$ ? Is $[\tau=k] \cap[\tau \leq j] \in \mathscr{F}_{j}$ for all $j$ ? If $j<k$, then the intersection is $\emptyset \in \mathscr{F}_{j}$. If $j \geq k$, then the intersection reduces to $[\tau=k]$ and this is in $\mathscr{F}_{k} \subseteq \mathscr{F}_{j}$ so yes, $[\tau=k]$ is in both $\mathscr{F}_{k}$ and $\mathscr{F}_{\tau}$.

Let $A \in \mathscr{F}_{k}$. I need to show

$$
\mathscr{F}_{\tau} \cap[\tau=k]=\mathscr{F}_{k} \cap[\tau=k]
$$

where $\mathscr{G} \cap[\tau=k]$ means all sets of the form $A \cap[\tau=k]$ where $A \in \mathscr{G}$. Let $A \in \mathscr{F} \tau$. Then

$$
A \cap[\tau=k]=(A \cap[\tau \leq k]) \backslash(A \cap[\tau \leq k-1]) \in \mathscr{F}_{k}
$$

Therefore, there exists $B \in \mathscr{F}_{k}$ such that $B=A \cap[\tau=k]$ and so

$$
B \cap[\tau=k]=A \cap[\tau=k]
$$

which shows $\mathscr{F}_{\tau} \cap[\tau=k] \subseteq \mathscr{F}_{k} \cap[\tau=k]$. Now let $A \in \mathscr{F}_{k}$ so that

$$
A \cap[\tau=k] \in \mathscr{F}_{k} \cap[\tau=k]
$$

Then

$$
A \cap[\tau=k] \cap[\tau \leq j] \in \mathscr{F}_{j}
$$

because in case $j<k$, the set on the left is $\emptyset$ and if $j \geq k$ it reduces to $A \cap[\tau=k]$ and both $A$ and $[\tau=k]$ are in $\mathscr{F}_{k} \subseteq \mathscr{F}_{j}$. Therefore, the two $\sigma$ algebras of subsets of $[\tau=k]$,

$$
\mathscr{F}_{\tau} \cap[\tau=k], \mathscr{F}_{k} \cap[\tau=k]
$$

are equal. Thus for $A$ in either $\mathscr{F}_{\tau}$ or $\mathscr{F}_{k}, A \cap[\tau=k]$ is a set of both $\mathscr{F}_{\tau}$ and $\mathscr{F}_{k}$ because if $A \in \mathscr{F}_{k}$, then from the above, there exists $B \in \mathscr{F}_{\tau}$ such that

$$
A \cap[\tau=k]=B \cap[\tau=k] \in \mathscr{F}_{\tau}
$$

with similar reasoning holding if $A \in \mathscr{F}_{\tau}$. In other words, if $g$ is $\mathscr{F}_{\tau}$ or $\mathscr{F}_{k}$ measurable, then the restriction of $g$ to $[\tau=k]$ is measurable with respect to $\mathscr{F}_{\tau} \cap[\tau=k]$ and $\mathscr{F}_{k} \cap[\tau=k]$. Let $Y$ be an arbitrary random variable in $L^{1}(\Omega, \mathscr{F})$. It follows

$$
\begin{aligned}
\int_{A \cap[\tau=k]} E\left(Y \mid \mathscr{F}_{\tau}\right) d P & \equiv \int_{A \cap[\tau=k]} Y d P \\
& \equiv \int_{A \cap[\tau=k]} E\left(Y \mid \mathscr{F}_{k}\right) d P
\end{aligned}
$$

Since this holds for an arbitrary set in $\mathscr{F}_{\tau} \cap[\tau=k]=\mathscr{F}_{k} \cap[\tau=k]$, it follows

$$
E\left(Y \mid \mathscr{F}_{\tau}\right)=E\left(Y \mid \mathscr{F}_{k}\right) \text { a.e. on }[\tau=k]
$$

This proves the third claim and the Lemma.
With this lemma, here is a major theorem, the optional sampling theorem of Doob. This one is special for martingales.

Theorem 60.5.4 Let $\{M(k)\}$ be a real valued martingale with respect to the increasing sequence of $\sigma$ algebras, $\left\{\mathscr{F}_{k}\right\}$ and let $\sigma, \tau$ be two stopping times such that $\tau$ is bounded. Then $M(\tau)$ defined as

$$
\omega \rightarrow M(\tau(\omega))
$$

is integrable and

$$
M(\sigma \wedge \tau)=E\left(M(\tau) \mid \mathscr{F}_{\sigma}\right)
$$

Proof: By Proposition 62.6.3 $M(\tau)$ is $\mathscr{F}_{\tau}$ measurable.
Next note that since $\tau$ is bounded by some $l$,

$$
\int_{\Omega}\|M(\tau(\omega))\| d P \leq \sum_{i=1}^{l} \int_{[\tau=i]}\|M(i)\| d P<\infty .
$$

This proves the first assertion and makes possible the consideration of conditional expectation.

Let $l \geq \tau$ as described above. Then for $k \leq l$, by Lemma 62.6.4,

$$
\mathscr{F}_{k} \cap[\tau=k]=\mathscr{F}_{\tau} \cap[\tau=k] \equiv \mathscr{G}
$$

implying that if $g$ is either $\mathscr{F}_{k}$ measurable or $\mathscr{F}_{\tau}$ measurable, then its restriction to $[\tau=k]$ is $\mathscr{G}$ measurable and so if $A \in \mathscr{F}_{k} \cap[\tau=k]=\mathscr{F}_{\tau} \cap[\tau=k]$,

$$
\begin{aligned}
\int_{A} E\left(M(l) \mid \mathscr{F}_{\tau}\right) d P & \equiv \int_{A} M(l) d P \\
& =\int_{A} E\left(M(l) \mid \mathscr{F}_{k}\right) d P \\
& =\int_{A} M(k) d P \\
& =\int_{A} M(\tau) d P(\text { on } A, \tau=k)
\end{aligned}
$$

Therefore, since $A$ was arbitrary,

$$
E\left(M(l) \mid \mathscr{F}_{\tau}\right)=M(\tau) \text { a.e. }
$$

on $[\tau=k]$ for every $k \leq l$. It follows

$$
\begin{equation*}
E\left(M(l) \mid \mathscr{F}_{\tau}\right)=M(\tau) \text { a.e. } \tag{60.5.11}
\end{equation*}
$$

since it is true on each $[\tau=k]$ for all $k \leq l$.
Now consider $E\left(M(\tau) \mid \mathscr{F}_{\sigma}\right)$ on the set $[\sigma=i] \cap[\tau=j]$. By Lemma 62.6.4, on this set,

$$
E\left(M(\tau) \mid \mathscr{F}_{\sigma}\right)=E\left(M(\tau) \mid \mathscr{F}_{i}\right)=E\left(E\left(M(l) \mid \mathscr{F}_{\tau}\right) \mid \mathscr{F}_{i}\right)=E\left(E\left(M(l) \mid \mathscr{F}_{j}\right) \mid \mathscr{F}_{i}\right)
$$

If $j \leq i$, this reduces to

$$
E\left(M(l) \mid \mathscr{F}_{j}\right)=M(j)=M(\sigma \wedge \tau)
$$

If $j>i$, this reduces to

$$
E\left(M(l) \mid \mathscr{F}_{i}\right)=M(i)=M(\sigma \wedge \tau)
$$

and since this exhausts all possibilities for values of $\sigma$ and $\tau$, it follows

$$
E\left(M(\tau) \mid \mathscr{F}_{\sigma}\right)=M(\sigma \wedge \tau) \text { a.e. }
$$

This is a really amazing theorem. Note it says $M(\sigma \wedge \tau)=E\left(M(\tau) \mid \mathscr{F}_{\sigma}\right)$. I would have expected something involving $E(M(\tau) \mid \mathscr{F} \sigma \wedge \tau)$ on the right.

What about submartingales? Recall $\{X(k)\}_{k=1}^{\infty}$ is a submartingale if

$$
E\left(X(k+1) \mid \mathscr{F}_{k}\right) \geq X(k)
$$

where the $\mathscr{F}_{k}$ are an increasing sequence of $\sigma$ algebras in the usual way. The following is a very interesting result.

Lemma 60.5.5 Let $\{X(k)\}_{k=0}^{\infty}$ be a submartingale adapted to the increasing sequence of $\sigma$ algebras, $\left\{\mathscr{F}_{k}\right\}$. Then there exists a unique increasing process $\{A(k)\}_{k=0}^{\infty}$ such that $A(0)=0$ and $A(k+1)$ is $\mathscr{F}_{k}$ measurable for all $k$ and a martingale, $\{M(k)\}_{k=0}^{\infty}$ such that

$$
X(k)=A(k)+M(k)
$$

Furthermore, for $\tau$ a stopping time, $A(\tau)$ is $\mathscr{F}_{\tau}$ measurable.
Proof: Define $\sum_{k=0}^{-1} \neq 0$. First consider the uniqueness assertion. Suppose $A$ is a process which does what is supposed to do.

$$
\begin{aligned}
& \sum_{k=0}^{n-1} E\left(X(k+1)-X(k) \mid \mathscr{F}_{k}\right)=\sum_{k=0}^{n-1} E\left(A(k+1)-A(k) \mid \mathscr{F}_{k}\right) \\
& +\sum_{k=0}^{n-1} E\left(M(k+1)-M(k) \mid \mathscr{F}_{k}\right)
\end{aligned}
$$

Then since $\{M(k)\}$ is a martingale,

$$
\sum_{k=0}^{n-1} E\left(X(k+1)-X(k) \mid \mathscr{F}_{k}\right)=\sum_{k=0}^{n-1} A(k+1)-A(k)=A(n)
$$

This shows uniqueness and gives a formula for $A(n)$ assuming it exists. It is only a matter of verifying this does work. Define

$$
A(n) \equiv \sum_{k=0}^{n-1} E\left(X(k+1)-X(k) \mid \mathscr{F}_{k}\right), A(0)=0
$$

Then $A$ is increasing because from the definition,

$$
A(n+1)-A(n)=E\left(X(n+1)-X(n) \mid \mathscr{F}_{n}\right) \geq 0
$$

Also from the definition above, $A(n)$ is $\mathscr{F}_{n-1}$ measurable, consider

$$
\{X(k)-A(k)\} .
$$

Why is this a martingale?

$$
\begin{aligned}
& E\left(X(k+1)-A(k+1) \mid \mathscr{F}_{k}\right) \\
= & E\left(X(k+1) \mid \mathscr{F}_{k}\right)-A(k+1) \\
= & E\left(X(k+1) \mid \mathscr{F}_{k}\right)-\sum_{j=0}^{k} E\left(X(j+1)-X(j) \mid \mathscr{F}_{j}\right) \\
= & E\left(X(k+1) \mid \mathscr{F}_{k}\right)-E\left(X(k+1)-X(k) \mid \mathscr{F}_{k}\right) \\
& -\sum_{j=0}^{k-1} E\left(X(j+1)-X(j) \mid \mathscr{F}_{j}\right) \\
= & X(k)-\sum_{j=0}^{k-1} E\left(X(j+1)-X(j) \mid \mathscr{F}_{j}\right)=X(k)-A(k)
\end{aligned}
$$

Let $M(k) \equiv X(k)-A(k) . A(\tau)$ is $\mathscr{F}_{\tau}$ measurable by Proposition 62.6.3.
Note the nonnegative integers could be replaced with any finite set or ordered countable set of numbers with no change in the conclusions of this lemma or the above optional sampling theorem.

Next consider the case of a submartingale.

Theorem 60.5.6 Let $\{X(k)\}$ be a submartingale with respect to the increasing sequence of $\sigma$ algebras, $\left\{\mathscr{F}_{k}\right\}$ and let $\sigma, \tau$ be two stopping times such that $\tau$ is bounded. Then $X(\tau)$ defined as

$$
\omega \rightarrow X(\tau(\omega))
$$

is integrable and

$$
X(\sigma \wedge \tau) \leq E\left(X(\tau) \mid \mathscr{F}_{\sigma}\right)
$$

Proof: The claim about $X(\tau)$ being integrable is the same as in Theorem 62.6.5. If $\tau \leq l$,

$$
E(|X(\tau(\omega))|)=\sum_{i=1}^{l} \int_{[\tau=i]}|X(i)| d P<\infty
$$

By Lemma 60.5 .5 there is a martingale, $\{M(k)\}$ and an increasing process $\{A(k)\}$ such that $A(k+1)$ is $\mathscr{F}_{k}$ measurable such that

$$
X(k)=M(k)+A(k) .
$$

Then using Theorem 62.6 .5 on the martingale and the fact $A$ is increasing

$$
\begin{aligned}
E\left(X(\tau) \mid \mathscr{F}_{\sigma}\right) & =E\left(M(\tau)+A(\tau) \mid \mathscr{F}_{\sigma}\right)=M(\tau \wedge \sigma)+E\left(A(\tau) \mid \mathscr{F}_{\sigma}\right) \\
& \geq M(\tau \wedge \sigma)+E\left(A(\tau \wedge \sigma) \mid \mathscr{F}_{\sigma}\right) \\
& =M(\tau \wedge \sigma)+A(\tau \wedge \sigma)=X(\tau \wedge \sigma)
\end{aligned}
$$

because in the above, it follows from Lemma $60.5 .5, A(\tau \wedge \sigma)$ is $\mathscr{F}_{\tau \wedge \sigma}$ measurable and from Lemma 62.6.4,

$$
\mathscr{F}_{\tau \wedge \sigma}=\mathscr{F}_{\tau} \cap \mathscr{F}_{\sigma} \subseteq \mathscr{F}_{\sigma}
$$

and so

$$
E\left(A(\tau \wedge \sigma) \mid \mathscr{F}_{\sigma}\right)=A(\tau \wedge \sigma)
$$

### 60.6 Submartingale Convergence Theorem

### 60.6.1 Upcrossings

Let $\{X(k)\}$ be an adapted stochastic process, $k=0,1,2, \cdots, M$ adapted to the increasing $\sigma$ algebras $\mathscr{F}_{k}$. Also let $[a, b]$ be an interval. An upcrossing occurs when $X(k)<a$ and you have $X(k+l)>b$ while $X(r)<b$ for all $r \in[k, k+l-1]$. In order to understand
upcrossings, consider the following:

$$
\begin{aligned}
& \tau_{0} \equiv \min (\inf \{k: X(k) \leq a\}, M), \\
& \tau_{1} \equiv \min \left(\inf \left\{k:\left(X\left(k \vee \tau_{0}\right)-X\left(\tau_{0}\right)\right)_{+} \geq b-a\right\}, M\right), \\
& \tau_{2} \equiv \min \left(\inf \left\{k:\left(X\left(\tau_{1}\right)-X\left(k \vee \tau_{1}\right)\right)_{+} \geq b-a\right\}, M\right), \\
& \tau_{3} \equiv \min \left(\inf \left\{k:\left(X\left(k \vee \tau_{2}\right)-X\left(\tau_{2}\right)\right)_{+} \geq b-a\right\}, M\right), \\
& \tau_{4} \equiv \min \left(\inf \left\{k:\left(X\left(\tau_{3}\right)-X\left(k \vee \tau_{3}\right)\right)_{+} \geq b-a\right\}, M\right), \\
& \vdots
\end{aligned}
$$

As usual, $\inf (\emptyset) \equiv \infty$. Are the above stopping times? If $\alpha \geq 0$, and $\tau$ is a stopping time, is $k \rightarrow(X(\tau)-X(k \vee \tau))_{+}$adapted?

$$
\left[(X(\tau)-X(k \vee \tau))_{+}>\alpha\right]=\left[(X(\tau)-X(k))_{+}>\alpha\right] \cap[\tau \leq k]
$$

Now

$$
\left[(X(\tau)-X(k))_{+}>\alpha\right] \cap[\tau \leq k]=\cup_{i=0}^{k}\left[(X(i)-X(k))_{+}>\alpha\right] \cap[\tau \leq k] \in \mathscr{F}_{k}
$$

If $\alpha<0$, then $\left[\left(X\left(\tau_{1}\right)-X\left(k \vee \tau_{1}\right)\right)_{+}>\alpha\right]=\Omega$ and so $k \rightarrow(X(\tau)-X(k \vee \tau))_{+}$is adapted. Similarly $k \rightarrow(X(k \vee \tau)-X(\tau))_{+}$is adapted. Therefore, all those $\tau_{k}$ are stopping times.

Now consider the following random variable for odd $M, 2 n+1=M$

$$
U_{M}^{[a, b]} \equiv \lim _{\varepsilon \rightarrow 0} \sum_{k=0}^{n} \frac{X\left(\tau_{2 k+1}\right)-X\left(\tau_{2 k}\right)}{\varepsilon+X\left(\tau_{2 k+1}\right)-X\left(\tau_{2 k}\right)} \leq \frac{1}{b-a} \sum_{k=0}^{n} X\left(\tau_{2 k+1}\right)-X\left(\tau_{2 k}\right)
$$

Now suppose $\{X(k)\}$ is a nonnegative submartingale. Then since $E\left(X(2 \tau) \mid \mathscr{F}_{2 \tau-1}\right) \geq$ $X\left(\tau_{2 k-1}\right)$

$$
E\left(\sum_{k=1}^{n} X\left(\tau_{2 k}\right)-X\left(\tau_{2 k-1}\right)\right) \geq 0
$$

Hence

$$
\begin{gathered}
E\left(U_{M}^{[a, b]}\right) \leq \frac{1}{b-a} \sum_{k=0}^{n} E\left(X\left(\tau_{2 k+1}\right)-X\left(\tau_{2 k}\right)\right) \\
\leq \frac{1}{b-a} \sum_{k=0}^{n} E\left(X\left(\tau_{2 k+1}\right)-X\left(\tau_{2 k}\right)\right)+\frac{1}{b-a} \sum_{k=1}^{n} E\left(X\left(\tau_{2 k}\right)-X\left(\tau_{2 k-1}\right)\right) \\
=\frac{1}{b-a} \sum_{k=0}^{n} E\left(X\left(\tau_{k}\right)-X\left(\tau_{k-1}\right)\right) \leq \frac{1}{b-a} E\left(X\left(\tau_{k}\right)\right)
\end{gathered}
$$

Now by the optional sampling theorem $X(0), X\left(\tau_{k}\right), X(M)$ is a submartingale. Therefore, the above is no larger than

$$
\frac{1}{b-a} E(|X(M)|)
$$

Now note that $U_{M}^{[a, b]}$ is at least as large as the number of upcrossings of $\{X(k)\}$ for $k \leq M$. This is because every time an upcrossing occurs, it will follow that $X\left(\tau_{2 k+1}\right)-X\left(\tau_{2 k}\right)>0$
and so a one will occur in the above sum which defines $U_{M}^{[a, b]}$. However, this might be larger than the number of upcrossings. The above discussion has proved the following upcrossing lemma.

Lemma 60.6.1 Let $\{X(k)\}$ be a nonnegative submartingale. Let

$$
U_{M}^{[a, b]} \equiv \lim _{\varepsilon \rightarrow 0} \sum_{k=0}^{n} \frac{X\left(\tau_{2 k+1}\right)-X\left(\tau_{2 k}\right)}{\varepsilon+X\left(\tau_{2 k+1}\right)-X\left(\tau_{2 k}\right)}, 2 n+1=M
$$

Then

$$
E\left(U_{M}^{[a, b]}\right) \leq \frac{1}{b-a} E(X(M))
$$

Suppose that there exists a constant $C \geq E(X(M))$ for all $M$. That is, $\{X(k)\}$ is bounded in $L^{1}(\Omega)$. Then letting

$$
U^{[a, b]} \equiv \lim _{M \rightarrow \infty} U_{M}^{[a, b]}
$$

it follows that

$$
E\left(U^{[a, b]}\right) \leq C \frac{1}{b-a}
$$

The second half follows from the first part and the monotone convergence theorem.
Now with this estimate, it is easy to prove the submartingale convergence theorem.
Theorem 60.6.2 Let $\{X(k)\}$ be a submartingale which is bounded in $L^{1}(\Omega)$,

$$
\|X(k)\|_{L^{1}(\Omega)} \leq C
$$

Then there is a set of measure zero $N$ such that for $\omega \notin N, \lim _{k \rightarrow \infty} X(k)(\omega)$ exists. If $X(\omega)=\lim _{k \rightarrow \infty} X(k)(\omega)$, then $X \in L^{1}(\Omega)$.

Proof: Let $a<b$ and consider the submartingale $(X(k)-a)_{+}$. Let $U^{[0, b-a]}$ be the random variable of the above lemma which is associated with this submartingale. Thus

$$
E\left(U^{[0, b-a]}\right) \leq \frac{C}{b-a}
$$

It follows that $U^{[0, b-a]}$ is finite for a.e. $\omega$. As noted above, $U^{[0, b-a]}$ is an upper bound to the number of upcrossings of $(X(k)-a)_{+}$and each of these corresponds to an upcrossing of $[a, b]$ by $X(k)$. Thus for all $\omega \notin N_{a, b}$ where $P\left(N_{a, b}\right)=0$, it follows that

$$
U^{[0, b-a]}<\infty .
$$

If $\lim _{k \rightarrow \infty} X(k)(\omega)$ fails to exist, then there exists $a<b$ both rational such that

$$
\lim _{k \rightarrow \infty} \sup _{k \rightarrow \infty} X(k)>b>a>\lim _{\inf _{k \rightarrow \infty}} X(k)
$$

Thus $\omega \in N_{a, b}$ because there are infinitely many upcrossings of $[a, b]$. Let

$$
N=\cup\left\{N_{a, b}: a, b \in \mathbb{Q}\right\}
$$

Then for $\omega \notin N$, the limit just discussed must exist. Letting $X(\omega)=\lim _{k \rightarrow \infty} X(k)(\omega)$ for $\omega \notin N$ and letting $X(\omega)=0$ on $N$, it follows from Fatou's lemma that $X$ is in $L^{1}(\Omega)$.

### 60.6.2 Maximal Inequalities

Next I will show that stopping times and the optional sampling theorem, Lemma 60.4.3, can be used to establish maximal inequalities for submartingales very easily.

Lemma 60.6.3 Let $\{X(k)\}$ be real valued and adapted to the increasing sequence of $\sigma$ algebras $\left\{\mathscr{F}_{k}\right\}$. Let

$$
T(\omega) \equiv \inf \{k: X(k) \geq \lambda\}
$$

Then $T$ is a stopping time. Similarly,

$$
T(\omega) \equiv \inf \{k: X(k) \leq \lambda\}
$$

is a stopping time.
Proof: Is $[T \leq p] \in \mathscr{F}_{p}$ for all $p$ ?

$$
[T=p]=\overbrace{\cap_{i=1}^{p-1}[X(i)<\lambda]}^{\in \mathscr{F}_{p-1}} \cap \overbrace{[X(p) \geq \lambda]}^{\in \mathscr{F}_{p}}
$$

Therefore,

$$
[T \leq p]=\cup_{i=1}^{p}[T=i] \in \mathscr{F}_{p}
$$

Theorem 60.6.4 Let $\left\{X_{k}\right\}$ be a real valued submartingale with respect to the $\sigma$ algebras $\left\{\mathscr{F}_{k}\right\}$. Then for $\lambda>0$

$$
\begin{gather*}
\lambda P\left(\left[\max _{1 \leq k \leq n} X_{k} \geq \lambda\right]\right) \leq E\left(X_{n}^{+}\right),  \tag{60.6.12}\\
\lambda P\left(\left[\min _{1 \leq k \leq n} X_{k} \leq-\lambda\right]\right) \leq E\left(\left|X_{n}\right|+\left|X_{1}\right|\right),  \tag{60.6.13}\\
\lambda P\left(\left[\max _{1 \leq k \leq n}\left|X_{k}\right| \geq \lambda\right]\right) \leq 2 E\left(\left|X_{n}\right|+\left|X_{1}\right|\right) . \tag{60.6.14}
\end{gather*}
$$

Proof: Let $T(\omega)$ be the first time $X_{k}(\omega)$ is $\geq \lambda$ or if this does not happen for $k \leq n$, then $T(\omega) \equiv n$. Thus

$$
T(\omega) \equiv \min \left(\min \left\{k: X_{k}(\omega) \geq \lambda\right\}, n\right)
$$

Note

$$
[T>k]=\cap_{i=1}^{k}\left[X_{i}<\lambda\right] \in \mathscr{F}_{k}
$$

and so the complement, $[T \leq k]$ is also in $\mathscr{F}_{k}$ which shows $T$ is indeed a stopping time.
Then $1, T(\omega), n$ are stopping times, $1 \leq T(\omega) \leq n$. Therefore, from the optional sampling theorem, Lemma 60.4.3, $X_{1}, X_{T}, X_{n}$ is a submartingale. It follows

$$
\begin{aligned}
E\left(X_{n}\right) & \geq E\left(X_{T}\right)=\int_{\left[\max _{k} X_{k} \geq \lambda\right]} X_{T} d P+\int_{\left[\max _{k} X_{k}<\lambda\right]} X_{T} d P \\
& =\int_{\left[\max _{k} X_{k} \geq \lambda\right]} X_{T} d P+\int_{\left[\max _{k} X_{k}<\lambda\right]} X_{n} d P
\end{aligned}
$$

and so, subtracting the last term on the right from both sides,

$$
\begin{aligned}
E\left(X_{n}^{+}\right) & \geq \int_{\left[\max _{k} X_{k} \geq \lambda\right]} X_{n} d P=\int_{\left[\max _{k} X_{k} \geq \lambda\right]} X_{T} d P \\
& \geq \lambda P\left(\left[\max _{k} X_{k} \geq \lambda\right]\right)
\end{aligned}
$$

because $X_{T}(\omega) \geq \lambda$ on $\left[\max _{k} X_{k} \geq \lambda\right]$ from the definition of $T$. This establishes 60.6.12.
Next let $T(\omega)$ be the first time $X_{k}(\omega)$ is $\leq-\lambda$ or if this does not happen for $k \leq n$, then $T(\omega) \equiv n$. Then this is a stopping time by similar reasoning and $1 \leq T(\omega) \leq n$ are stopping times and so by the optional stopping theorem, $X_{1}, X_{T}, X_{n}$ is a submartingale. Therefore, on

$$
\left[\min _{k} X_{k} \leq-\lambda\right], X_{T}(\omega) \leq-\lambda
$$

and $E\left(X_{T} \mid \mathscr{F}_{1}\right) \geq X_{1}$ and so

$$
E\left(X_{1}\right) \leq E\left(E\left(X_{T} \mid \mathscr{F}_{1}\right)\right)=E\left(X_{T}\right)
$$

which implies

$$
\begin{aligned}
E\left(X_{1}\right) & \leq E\left(X_{T}\right)=\int_{\left[\min _{k} X_{k} \leq-\lambda\right]} X_{T} d P+\int_{\left[\min _{k} X_{k}>-\lambda\right]} X_{T} d P \\
& =\int_{\left[\min _{k} X_{k} \leq-\lambda\right]} X_{T} d P+\int_{\left[\min _{k} X_{k}>-\lambda\right]} X_{n} d P
\end{aligned}
$$

and so

$$
\begin{aligned}
E\left(X_{1}\right)-\int_{\left[\min _{k} X_{k}>-\lambda\right]} X_{n} d P & \leq \int_{\left[\min _{k} X_{k} \leq-\lambda\right]} X_{T} d P \\
& \leq-\lambda P\left(\left[\min _{k} X_{k} \leq-\lambda\right]\right)
\end{aligned}
$$

which implies

$$
\begin{aligned}
\lambda P\left(\left[\min _{k} X_{k} \leq-\lambda\right]\right) & \leq \int_{\left[\min _{k} X_{k}>-\lambda\right]} X_{n} d P-E\left(X_{1}\right) \\
& \leq \int_{\Omega}\left(\left|X_{n}\right|+\left|X_{1}\right|\right) d P
\end{aligned}
$$

and this proves 60.6.13.
The last estimate follows from these. Here is why.

$$
\left[\max _{1 \leq k \leq n}\left|X_{k}\right| \geq \lambda\right] \subseteq\left[\max _{1 \leq k \leq n} X_{k} \geq \lambda\right] \cup\left[\min _{1 \leq k \leq n} X_{k} \leq-\lambda\right]
$$

and so

$$
\lambda P\left(\left[\max _{1 \leq k \leq n}\left|X_{k}\right| \geq \lambda\right]\right) \leq \lambda P\left(\left[\max _{1 \leq k \leq n} X_{k} \geq \lambda\right] \cup\left[\min _{1 \leq k \leq n} X_{k} \leq-\lambda\right]\right)
$$

$$
\begin{gathered}
\leq \lambda P\left(\left[\max _{1 \leq k \leq n} X_{k} \geq \lambda\right]\right)+\lambda P\left(\left[\min _{1 \leq k \leq n} X_{k} \leq-\lambda\right]\right) \\
\leq 2 E\left(\left|X_{1}\right|+\left|X_{n}\right|\right)
\end{gathered}
$$

and this proves the last estimate.

### 60.6.3 The Upcrossing Estimate

A very interesting example of stopping times is next. It has to do with upcrossings. First here is a lemma.

Lemma 60.6.5 Let $\left\{\mathscr{F}_{k}\right\}$ be an increasing sequence of $\sigma$ algebras and let $\{X(k)\}$ be adapted to this sequence. Suppose that $X(k)$ has all values in $[a, b]$ and suppose $\sigma$ is a stopping time with the property that $X(\sigma)=a$. Let $\tau(\omega)$ be the first $k>\sigma$ such that $X(k)=b$. If no such $k$ exists, then $\tau \equiv \infty$. Then $\tau$ is a stopping time. Also, you can switch $a, b$ in the above and obtain the same conclusion that $\tau$ is a stopping time.

Proof: Let $I$ be an interval and consider $X(k \vee \sigma)$. Is $k \rightarrow X(k \vee \sigma)$ adapted? Let $I$ be an interval. Is

$$
A \equiv X(k \vee \sigma)^{-1}(I) \in \mathscr{F}_{k} ?
$$

We know that this set is in $\mathscr{F}_{k \vee \sigma}$.

$$
A=A \cap[\sigma \leq k] \cup\left(X(k \vee \sigma)^{-1}(I) \cap[\sigma>k]\right)
$$

Consider the second set in $\boldsymbol{\phi}$. There are two cases, $a \in I$ and $a \notin I$. First suppose $a \notin I$. Then if $\omega \in[\sigma>k]$, it follows that $X(k \vee \sigma)=X(\sigma)=a$. Therefore, in this case, the set on the right in $\boldsymbol{\phi}$ is empty and the empty set is in $\mathscr{F}_{k}$. Next suppose $a \in I$. Then for $\omega \in[\sigma>k]$,

$$
X(k \vee \sigma(\omega))=X(\sigma(\omega))=a \in I
$$

and so each $\omega \in[\sigma>k]$ is in the set $X(k \vee \sigma)^{-1}(I)$ and so, in this case, the set on the right equals

$$
[\sigma>k] \in \mathscr{F}_{k}
$$

Now consider the first set in $\boldsymbol{\varphi}$,

$$
A \cap[\sigma \leq k]=A \cap[\sigma \vee k \leq k] \in \mathscr{F}_{k}
$$

by the definition of what it means for the set $A$ to be in $\mathscr{F}_{k \vee \sigma}$. The argument proceeds in the same way when you switch $a, b$.

Definition 60.6.6 Let $\left\{X_{k}\right\}$ be a sequence of random variables adapted to the increasing sequence of $\sigma$ algebras, $\left\{\mathscr{F}_{k}\right\}$. Let $[a, b]$ be an interval. An upcrossing is a sequence $X_{n}(\omega), \cdots, X_{n+p}(\omega)$ such that $X_{n}(\omega) \leq a, X_{n+i}(\omega)<b$ for $i<p$, and $X_{n+p}(\omega) \geq b$.

Example 60.6.7 Let $\left\{\mathscr{F}_{n}\right\}$ be an increasing sequence of $\sigma$ algebras contained in $\mathscr{F}$ where $(\Omega, \mathscr{F}, P)$ is a probability space and let $\left\{X_{n}\right\}$ be a sequence of real valued random variables such that $X_{n}$ is $\mathscr{F}_{n}$ measurable. Also let $a<b$. Define

$$
\begin{gathered}
T_{0} \equiv \inf \{n: X(n) \leq a\} \\
T_{1} \equiv \inf \left\{n>T_{0}: X(n) \geq b\right\} \\
T_{2} \equiv \inf \left\{n>T_{1}: X(n) \leq a\right\} \\
\vdots \\
T_{2 k-1} \equiv \inf \left\{n>T_{2 k-2}: X(n) \geq b\right\} \\
T_{2 k} \equiv \inf \left\{n>T_{2 k-1}: X(n) \leq a\right\}
\end{gathered}
$$

If $X_{n}(\omega)$ is never in the desired interval for any $n>T_{j}(\omega)$, then define $T_{j+1}(\omega) \equiv \infty$. Then this is an increasing sequence of stopping times.

It happens that the above gives an increasing sequence of stopping times.
Lemma 60.6.8 The above example gives an increasing sequence of stopping times.
Proof: You could consider the modified random variables

$$
Y(k) \equiv(X(k) \vee a) \wedge b
$$

Then these new random variables stay in $[a, b]$ and if you replace $X(n)$ in the above with $Y(n)$, you get the same sequence of stopping times. Now apply Lemma 60.6.5.

Now there is an interesting application of these stopping times to the concept of upcrossings. Let $\left\{X_{n}\right\}$ be a submartingale such that $X_{n}$ is $\mathscr{F}_{n}$ measurable and let $a<b$. Assume $X_{0}(\omega) \leq a$. The function, $x \rightarrow(x-a)^{+}$is increasing and convex so $\left\{\left(X_{n}-a\right)^{+}\right\}$ is also a submartingale. Furthermore, $\left\{X_{n}\right\}$ goes from $\leq a$ to $\geq b$ if and only if $\left\{\left(X_{n}-a\right)^{+}\right\}$ goes from 0 to $\geq b-a$. That is, a subsequence of the form $Y_{n}(\omega), Y_{n+1}(\omega), \cdots, Y_{n+r}(\omega)$ for $Y$ equal to either $X$ or $(X-a)^{+}$starts out below $a(0)$ and ends up above $b(b-a)$. Such a sequence is called an upcrossing of $[a, b]$. The idea is to estimate the expected number of upcrossings for $n \leq N$. For the stopping times defined in Example 60.6.7, let $T_{k}^{\prime} \equiv \min \left(T_{k}, N\right)$. Thus $T_{k}^{\prime}$, a continuous function of the stopping time, is also a stopping time which is bounded. Moreover, $T_{k}^{\prime} \leq T_{k+1}^{\prime}$. Now pick $n$ such that $2 n>N$. Then for each $\omega \in \Omega$

$$
\left(X_{N}(\omega)-a\right)^{+}-\left(X_{0}(\omega)-a\right)^{+}
$$

must equal the sum of all successive terms of the form

$$
\left(\left(X_{T_{k+1}^{\prime}}(\omega)-a\right)^{+}-\left(X_{T_{k}^{\prime}}(\omega)-a\right)^{+}\right)
$$

for $k=1,2, \cdots, 2 n$. This is because $\left\{T_{k}^{\prime}(\omega)\right\}$ is a strictly increasing sequence which starts with 0 due to the assumption $X_{0}(\omega) \leq a$ and ends with $N<2 n$. Therefore,

$$
\left(X_{N}-a\right)^{+}-\left(X_{0}-a\right)^{+}=\sum_{k=1}^{2 n}\left(X_{T_{k}^{\prime}}-a\right)^{+}-\left(X_{T_{k-1}^{\prime}}-a\right)^{+}
$$

$$
=\overbrace{\sum_{k=0}^{n-1}\left(\left(X_{T_{2 k+1}^{\prime}}-a\right)^{+}-\left(X_{T_{2 k}^{\prime}}-a\right)^{+}\right)}^{\text {odds - evens }}+\overbrace{\sum_{k=1}^{n}\left(\left(X_{T_{2 k}^{\prime}}-a\right)^{+}-\left(X_{T_{2 k-1}^{\prime}}-a\right)^{+}\right)}^{\text {evens - odds }} .
$$

Now denote by $U_{[a, b]}^{N}$ the number of upcrossings. When $T_{k}^{\prime}$ is such that $k$ is odd, $\left(X_{T_{k}^{\prime}}-a\right)^{+}$ is above $b-a$ and when $k$ is even, it equals 0 . Therefore, in the first sum $X_{T_{2 k+1}^{\prime}}-X_{T_{2 k}^{\prime}} \geq$ $b-a$ and there are $U_{[a, b]}^{N}$ terms which are nonzero in this sum. (Note this might not be $n$ because many of the terms in the sum could be 0 due to the definition of $T_{k}^{\prime}$.) Hence

$$
\begin{gather*}
\left(X_{N}-a\right)^{+}-\left(X_{0}-a\right)^{+}=\left(X_{N}-a\right)^{+} \\
\geq(b-a) U_{[a, b]}^{N}+\sum_{k=1}^{n}\left(\left(X_{T_{2 k}^{\prime}}-a\right)^{+}-\left(X_{T_{2 k-1}^{\prime}}-a\right)^{+}\right) . \tag{60.6.15}
\end{gather*}
$$

Now $U_{[a, b]}^{N}$ is a random variable. To see this, let $Z_{k}(\omega)=1$ if $T_{2 k+1}^{\prime}>T_{2 k}^{\prime}$ and 0 otherwise. Thus $U_{[a, b]}^{N}(\omega)=\sum_{k=0}^{n-1} Z_{k}(\omega)$. Therefore, it makes sense to take the expected value of both sides of 60.6.15. By the optional sampling theorem, $\left\{\left(X_{T_{k}^{\prime}}-a\right)^{+}\right\}$is a submartingale and so

$$
\begin{gathered}
E\left(\left(X_{T_{2 k}^{\prime}}-a\right)^{+}-\left(X_{T_{2 k-1}^{\prime}}-a\right)^{+}\right) \\
=\int_{\Omega} E\left(\left(X_{T_{2 k}^{\prime}}-a\right)^{+} \mid \mathscr{F}_{T_{2 k-1}^{\prime}}\right) d P-\int_{\Omega}\left(X_{T_{2 k-1}^{\prime}}-a\right)^{+} d P \geq 0 .
\end{gathered}
$$

Therefore,

$$
\begin{equation*}
E\left(\left(X_{N}-a\right)^{+}\right) \geq(b-a) E\left(U_{[a, b]}^{N}\right) \tag{60.6.16}
\end{equation*}
$$

This proves most of the following fundamental upcrossing estimate.

Theorem 60.6.9 Let $\left\{X_{n}\right\}$ be a real valued submartingale such that $X_{n}$ is $\mathscr{F}_{n}$ measurable. Then letting $U_{[a, b]}^{N}$ denote the upcrossings of $\left\{X_{n}\right\}$ from a to $b$ for $n \leq N$,

$$
E\left(U_{[a, b]}^{N}\right) \leq \frac{1}{b-a} E\left(\left(X_{N}-a\right)^{+}\right)
$$

Proof: The estimate 60.6 .16 was based on the assumption that $X_{0}(\omega) \leq a$. If this is not so, modify $X_{0}$. Change it to $\min \left(X_{0}, a\right)$. Then the inequality holds for the modified submartingale which has at least as many upcrossings. Therefore, the inequality remains.

Note this theorem holds if the submartingale starts at the index 1 rather than 0 . Just adjust the argument.

### 60.7 The Submartingale Convergence Theorem

With this estimate it is now possible to prove the amazing submartingale convergence theorem.

Theorem 60.7.1 Let $\left\{X_{n}\right\}$ be a real valued submartingale such that

$$
E\left(\left|X_{n}\right|\right)<M
$$

for all $n$. Then there exists $X \in L^{1}(\Omega, \mathscr{F})$ such that $X_{n}(\omega)$ converges to $X(\omega)$ a.e. $\omega$ and $X \in L^{1}(\Omega)$.

Proof: Let $a<b$ be two rational numbers. From Theorem 60.6 .9 it follows that for all $N$,

$$
\begin{aligned}
\int_{\Omega} U_{[a, b]}^{N} d P & \leq \frac{1}{b-a} E\left(\left(X_{N}-a\right)^{+}\right) \\
& \leq \frac{1}{b-a}\left(E\left(\left|X_{N}\right|\right)+|a|\right) \leq \frac{M+|a|}{b-a}
\end{aligned}
$$

Therefore, letting $N \rightarrow \infty$, it follows that for a.e. $\omega$, there are only finitely many upcrossings of $[a, b]$. Denote by $S_{[a, b]}$ the exceptional set. Then letting $S \equiv \cup_{a, b \in \mathbb{Q}} S_{[a, b]}$, it follows that $P(S)=0$ and for $\omega \notin S,\left\{X_{n}(\omega)\right\}$ is a Cauchy sequence because if

$$
\lim \sup _{n \rightarrow \infty} X_{n}(\omega)>\lim \inf _{n \rightarrow \infty} X_{n}(\omega)
$$

then you can pick $\liminf _{n \rightarrow \infty} X_{n}(\omega)<a<b<\limsup \sin _{n \rightarrow \infty} X_{n}(\omega)$ with $a, b$ rational and conclude $\omega \in S_{[a, b]}$.

Let $X(\omega)=\lim _{n \rightarrow \infty} X_{n}(\omega)$ if $\omega \notin S$ and let $X(\omega)=0$ if $\omega \in S$. Then it only remains to verify $X \in L^{1}(\Omega)$. Since $X$ is the pointwise limit of measurable functions, it follows $X$ is measurable. By Fatou's lemma,

$$
\int_{\Omega}|X(\omega)| d P \leq \lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty}\left|X_{n}(\omega)\right| d P
$$

Thus $X \in L^{1}(\Omega)$. This proves the theorem.
As a simple application, here is an easy proof of a nice theorem about convergence of sums of independent random variables.

Theorem 60.7.2 Let $\left\{X_{k}\right\}$ be a sequence of independent real valued random variables such that $E\left(\left|X_{k}\right|\right)<\infty, E\left(X_{k}\right)=0$, and

$$
\sum_{k=1}^{\infty} E\left(X_{k}^{2}\right)<\infty
$$

Then $\sum_{k=1}^{\infty} X_{k}$ converges a.e.

Proof: Let $\mathscr{F}_{n} \equiv \sigma\left(X_{1}, \cdots, X_{n}\right)$. Consider $S_{n} \equiv \sum_{k=1}^{n} X_{k}$.

$$
E\left(S_{n+1} \mid \mathscr{F}_{n}\right)=S_{n}+E\left(X_{n+1} \mid \mathscr{F}_{n}\right) .
$$

Letting $A \in \mathscr{F}_{n}$ it follows from independence that

$$
\begin{aligned}
\int_{A} E\left(X_{n+1} \mid \mathscr{F}_{n}\right) d P & \equiv \int_{A} X_{n+1} d P \\
& =\int_{\Omega} \mathscr{X}_{A} X_{n+1} d P \\
& =P(A) \int_{\Omega} X_{n+1} d P=0
\end{aligned}
$$

and so $E\left(X_{n+1} \mid \mathscr{F}_{n}\right)=0$. Therefore, $\left\{S_{n}\right\}$ is a martingale. Now using independence again,

$$
E\left(\left|S_{n}\right|\right) \leq E\left(\left|S_{n}^{2}\right|\right)=\sum_{k=1}^{n} E\left(X_{k}^{2}\right) \leq \sum_{k=1}^{\infty} E\left(X_{k}^{2}\right)<\infty
$$

and so $\left\{S_{n}\right\}$ is an $L^{1}$ bounded martingale. Therefore, it converges a.e. and this proves the theorem.

Corollary 60.7.3 Let $\left\{X_{k}\right\}$ be a sequence of independent real valued random variables such that $E\left(\left|X_{k}\right|\right)<\infty, E\left(X_{k}\right)=m_{k}$, and

$$
\sum_{k=1}^{\infty} E\left(\left|X_{k}-m_{k}\right|^{2}\right)<\infty .
$$

Then $\sum_{k=1}^{\infty}\left(X_{k}-m_{k}\right)$ converges a.e.
This can be extended to the case where the random variables have values in a separable Hilbert space.

Theorem 60.7.4 Let $\left\{X_{k}\right\}$ be a sequence of independent $H$ valued random variables where $H$ is a real separable Hilbert space such that $E\left(\left|X_{k}\right|_{H}\right)<\infty, E\left(X_{k}\right)=0$, and

$$
\sum_{k=1}^{\infty} E\left(\left|X_{k}\right|_{H}^{2}\right)<\infty
$$

Then $\sum_{k=1}^{\infty} X_{k}$ converges a.e.
Proof: Let $\left\{e_{k}\right\}$ be an orthonormal basis for $H$. Then $\left\{\left(X_{n}, e_{k}\right)_{H}\right\}_{n=1}^{\infty}$ are real valued, independent, and their mean equals 0 . Also

$$
\sum_{n=1}^{\infty} E\left(\left|\left(X_{n}, e_{k}\right)_{H}^{2}\right|\right) \leq \sum_{n=1}^{\infty} E\left(\left|X_{n}\right|_{H}^{2}\right)<\infty
$$

and so from Theorem 60.7.2, the series,

$$
\sum_{n=1}^{\infty}\left(X_{n}, e_{k}\right)_{H}
$$

converges a.e. Therefore, there exists a set of measure zero such that for $\omega$ not in this set, $\sum_{n}\left(X_{n}(\omega), e_{k}\right)_{H}$ converges for each $k$. For $\omega$ not in this exceptional set, define

$$
Y_{k}(\omega) \equiv \sum_{n=1}^{\infty}\left(X_{n}(\omega), e_{k}\right)_{H}
$$

Next define

$$
\begin{equation*}
S(\omega) \equiv \sum_{k=1}^{\infty} Y_{k}(\omega) e_{k} \tag{60.7.17}
\end{equation*}
$$

Of course it is not clear this even makes sense. I need to show $\sum_{k=1}^{\infty}\left|Y_{k}(\omega)\right|^{2}<\infty$. Using the independence of the $X_{n}$

$$
\begin{aligned}
E\left(\left|Y_{k}\right|^{2}\right) & =E\left(\left(\sum_{n=1}^{\infty}\left(X_{n}, e_{k}\right)_{H}\right)^{2}\right) \\
& =E\left(\left(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty}\left(X_{n}, e_{k}\right)_{H}\left(X_{m}, e_{k}\right)_{H}\right)\right) \\
& \leq \lim _{N \rightarrow \infty} E\left(\left(\sum_{n=1}^{N} \sum_{m=1}^{N}\left(X_{n}, e_{k}\right)_{H}\left(X_{m}, e_{k}\right)_{H}\right)\right) \\
& =\liminf _{N \rightarrow \infty} E\left(\sum_{n=1}^{N}\left(X_{n}, e_{k}\right)_{H}^{2}\right) \\
& =\sum_{n=1}^{\infty} E\left(\left(X_{n}, e_{k}\right)_{H}^{2}\right)
\end{aligned}
$$

Hence from the above,

$$
E\left(\sum_{k}\left|Y_{k}\right|^{2}\right)=\sum_{k} E\left(\left|Y_{k}\right|^{2}\right) \leq \sum_{k} \sum_{n} E\left(\left(X_{n}, e_{k}\right)_{H}^{2}\right)
$$

and by the monotone convergence theorem or Fubini's theorem,

$$
\begin{align*}
& =E\left(\sum_{k} \sum_{n}\left(X_{n}, e_{k}\right)_{H}^{2}\right)=E\left(\sum_{n} \sum_{k}\left(X_{n}, e_{k}\right)_{H}^{2}\right) \\
& =E\left(\sum_{n}\left|X_{n}\right|_{H}^{2}\right)=\sum_{n} E\left(\left|X_{n}\right|_{H}^{2}\right)<\infty \tag{60.7.18}
\end{align*}
$$

Therefore, for $\omega$ off a set of measure zero, and for

$$
\begin{gathered}
Y_{k}(\omega) \equiv \sum_{n=1}^{\infty}\left(X_{n}(\omega), e_{k}\right)_{H} \\
\sum_{k}\left|Y_{k}(\omega)\right|^{2}<\infty
\end{gathered}
$$

and also for these $\omega$,

$$
\sum_{n} \sum_{k}\left(X_{n}(\omega), e_{k}\right)_{H}^{2}<\infty .
$$

It follows from the estimate 60.7 .18 that for $\omega$ not on a suitable set of measure zero, $S(\omega)$ defined by 60.7.17,

$$
S(\omega) \equiv \sum_{k=1}^{\infty} Y_{k}(\omega) e_{k}
$$

makes sense. Thus for these $\omega$

$$
\begin{aligned}
S(\omega) & =\sum_{l}\left(S(\omega), e_{l}\right) e_{l}=\sum_{l} Y_{l}(\omega) e_{l} \equiv \sum_{l} \sum_{n}\left(X_{n}(\omega), e_{l}\right)_{H} e_{l} \\
& =\sum_{n} \sum_{l}\left(X_{n}(\omega), e_{l}\right) e_{l}=\sum_{n} X_{n}(\omega)
\end{aligned}
$$

This proves the theorem.
Now with this theorem, here is a strong law of large numbers.
Theorem 60.7.5 Suppose $\left\{\mathbf{X}_{k}\right\}$ are independent random variables and $E\left(\left|\mathbf{X}_{k}\right|\right)<\infty$ for each $k$ and $E\left(\mathbf{X}_{k}\right)=\mathbf{m}_{k}$. Suppose also

$$
\begin{equation*}
\sum_{j=1}^{\infty} \frac{1}{j^{2}} E\left(\left|\mathbf{X}_{j}-\mathbf{m}_{j}\right|^{2}\right)<\infty \tag{60.7.19}
\end{equation*}
$$

Then

$$
\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{n}\left(\mathbf{X}_{j}-\mathbf{m}_{j}\right)=\mathbf{0} \text { a.e. }
$$

Proof: Consider the sum

$$
\sum_{j=1}^{\infty} \frac{\mathbf{X}_{j}-\mathbf{m}_{j}}{j}
$$

This sum converges a.e. because of 60.7.19 and Theorem 60.7.4 applied to the random vectors $\left\{\frac{\mathbf{x}_{j}-\mathbf{m}_{j}}{j}\right\}$. Therefore, from Lemma 59.7.4 it follows that for a.e. $\omega$,

$$
\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{n}\left(\mathbf{X}_{j}(\omega)-\mathbf{m}_{j}\right)=\mathbf{0}
$$

This proves the theorem.
The next corollary is often called the strong law of large numbers. It follows immediately from the above theorem.

Corollary 60.7.6 Suppose $\left\{\mathbf{X}_{j}\right\}_{j=1}^{\infty}$ are independent random vectors, $\lambda_{\mathbf{x}_{i}}=\lambda_{\mathbf{x}_{j}}$ for all $i, j$ having mean $\mathbf{m}$ and variance equal to

$$
\sigma^{2} \equiv \int_{\Omega}\left|\mathbf{X}_{j}-\mathbf{m}\right|^{2} d P<\infty
$$

Then for a.e. $\omega \in \Omega$

$$
\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{n} \mathbf{X}_{j}(\omega)=\mathbf{m}
$$

### 60.8 A Reverse Submartingale Convergence Theorem

Definition 60.8.1 Let $\left\{X_{n}\right\}_{n=0}^{\infty}$ be a sequence of real random variables such that $E\left(\left|X_{n}\right|\right)<$ $\infty$ for all $n$ and let $\left\{\mathscr{F}_{n}\right\}$ be a sequence of $\sigma$ algebras such that $\mathscr{F}_{n} \supseteq \mathscr{F}_{n+1}$ for all $n$. Then $\left\{X_{n}\right\}$ is called a reverse submartingale if for all $n$,

$$
E\left(X_{n} \mid \mathscr{F}_{n+1}\right) \geq X_{n+1}
$$

Note it is just like a submartingale only the indices are going the other way. Here is an interesting lemma. This lemma gives uniform integrability for a reverse submartingale.

Lemma 60.8.2 Suppose $E\left(\left|X_{n}\right|\right)<\infty$ for all $n, X_{n}$ is $\mathscr{F}_{n}$ measurable, $\mathscr{F}_{n+1} \subseteq \mathscr{F}_{n}$ for all $n \in \mathbb{N}$, and there exist $X_{\infty} \mathscr{F}_{\infty}$ measurable such that $\mathscr{F}_{\infty} \subseteq \mathscr{F}_{n}$ for all $n$ and $X_{0} \mathscr{F}_{0}$ measurable such that $\mathscr{F}_{0} \supseteq \mathscr{F}_{n}$ for all $n$ such that for all $n \in\{0,1, \cdots\}$,

$$
E\left(X_{n} \mid \mathscr{F}_{n+1}\right) \geq X_{n+1}, E\left(X_{n} \mid \mathscr{F}_{\infty}\right) \geq X_{\infty}
$$

where $E\left(\left|X_{\infty}\right|\right)<\infty$. Then $\left\{X_{n}: n \in \mathbb{N}\right\}$ is uniformly integrable.
Proof:

$$
E\left(X_{n+1}\right) \leq E\left(E\left(X_{n} \mid \mathscr{F}_{n+1}\right)\right)=E\left(X_{n}\right)
$$

Therefore, the sequence $\left\{E\left(X_{n}\right)\right\}$ is a decreasing sequence bounded below by $E\left(X_{\infty}\right)$ so it has a limit. I am going to show the functions are equiintegrable. Let $k$ be large enough that

$$
\begin{equation*}
\left|E\left(X_{k}\right)-\lim _{m \rightarrow \infty} E\left(X_{m}\right)\right|<\varepsilon \tag{60.8.20}
\end{equation*}
$$

and suppose $n>k$. Then if $\lambda>0$,

$$
\begin{aligned}
& \int_{\left[\left|X_{n}\right| \geq \lambda\right]}\left|X_{n}\right| d P=\int_{\left[X_{n} \geq \lambda\right]} X_{n} d P+\int_{\left[X_{n} \leq-\lambda\right]}\left(-X_{n}\right) d P \\
& =\int_{\left[X_{n} \geq \lambda\right]} X_{n} d P+\int_{\Omega}\left(-X_{n}\right) d P-\int_{\left[-X_{n}<\lambda\right]}\left(-X_{n}\right) d P \\
& =\int_{\left[X_{n} \geq \lambda\right]} X_{n} d P-\int_{\Omega} X_{n} d P+\int_{\left[-X_{n}<\lambda\right]} X_{n} d P
\end{aligned}
$$

From 60.8.20,

$$
\leq \int_{\left[X_{n} \geq \lambda\right]} X_{n} d P-\int_{\Omega} X_{k} d P+\varepsilon+\int_{\left[-X_{n}<\lambda\right]} X_{n} d P
$$

By assumption,

$$
E\left(X_{k} \mid \mathscr{F}_{n}\right) \geq X_{n}
$$

and so the above

$$
\begin{aligned}
& \leq \int_{\left[X_{n} \geq \lambda\right]} E\left(X_{k} \mid \mathscr{F}_{n}\right) d P-\int_{\Omega} X_{k} d P+\varepsilon+\int_{\left[-X_{n}<\lambda\right]} E\left(X_{k} \mid \mathscr{F}_{n}\right) d P \\
& =\int_{\left[X_{n} \geq \lambda\right]} X_{k} d P-\int_{\Omega} X_{k} d P+\varepsilon+\int_{\left[-X_{n}<\lambda\right]} X_{k} d P \\
& =\int_{\left[X_{n} \geq \lambda\right]} X_{k} d P-\int_{\Omega} X_{k} d P+\varepsilon+\int_{\left[X_{n}>-\lambda\right]} X_{k} d P \\
& =\int_{\left[X_{n} \geq \lambda\right]} X_{k} d P+\left(\int_{\Omega}\left(-X_{k}\right) d P-\int_{\left[X_{n}>-\lambda\right]}\left(-X_{k}\right) d P\right)+\varepsilon \\
& =\int_{\left[X_{n} \geq \lambda\right]} X_{k} d P+\int_{\left[X_{n} \leq-\lambda\right]}\left(-X_{k}\right) d P+\varepsilon=\int_{\left[\left|X_{n}\right| \geq \lambda\right]}\left|X_{k}\right| d P+\varepsilon
\end{aligned}
$$

Applying the maximal inequality for submartingales, Theorem 60.6.4,

$$
P\left(\left[\max \left\{\left|X_{j}\right|: j=n, \cdots, 1\right\} \geq \lambda\right]\right) \leq \frac{1}{\lambda}\left(E\left(\left|X_{0}\right|\right)+E\left(\left|X_{\infty}\right|\right)\right) \leq \frac{C}{\lambda}
$$

and taking sup for all $n$,

$$
P\left(\left[\sup \left\{\left|X_{j}\right|\right\} \geq \lambda\right]\right) \leq \frac{C}{\lambda}
$$

It follows since the single function, $X_{k}$ is equiintegrable that for all $\lambda$ large enough,

$$
\int_{\left[\left|X_{n}\right| \geq \lambda\right]}\left|X_{n}\right| d P \leq 2 \varepsilon
$$

and since $\varepsilon$ is arbitrary, this shows $\left\{X_{n}\right\}$ for $n>k$ is equiintegrable. Since there are only finitely many $X_{j}$ for $j \leq k$, this shows $\left\{X_{n}\right\}$ is equiintegrable. Hence $\left\{X_{n}\right\}$ is uniformly integrable. This proves the lemma.

Now with this lemma and the upcrossing lemma it is easy to prove an important convergence theorem.

Theorem 60.8.3 Let $\left\{X_{n}, \mathscr{F}_{n}\right\}_{n=0}^{\infty}$ be a backwards submartingale as described above and suppose $\sup _{n \geq 0} E\left(\left|X_{n}\right|\right)<\infty$. Then $\left\{X_{n}\right\}$ converges a.e. and in $L^{1}(\Omega)$ to a function, $X_{\infty}$.

Proof: By the upcrossing lemma applied to the submartingale $\left\{X_{k}\right\}_{k=0}^{N}$, the number of upcrossings (Downcrossings is probably a better term. They are upcrossings as $n$ gets smaller.) of the interval $[a, b]$ satisfies the inequality

$$
E\left(U_{[a, b]}^{N}\right) \leq \frac{1}{b-a} E\left(\left(X_{0}-a\right)^{+}\right)
$$

Letting $N \rightarrow \infty$, it follows the expected number of upcrossings, $E\left(U_{[a, b]}\right)$ is bounded. Therefore, there exists a set of measure $0 N_{a b}$ such that if $\omega \notin N_{a b}, U_{[a, b]}(\omega)<\infty$. Let $N=\cup\left\{N_{a b}: a, b \in \mathbb{Q}\right\}$. Then for $\omega \notin N$,

$$
\lim \sup _{n \rightarrow \infty} X_{n}(\omega)=\lim \inf _{n \rightarrow \infty} X_{n}(\omega)
$$

because if inequality holds, then letting

$$
\lim \inf _{n \rightarrow \infty} X_{n}(\omega)<a<b<\lim \sup _{n \rightarrow \infty} X_{n}(\omega)
$$

it would follow $U_{[a, b]}(\omega)=\infty$, contrary to $\omega \notin N_{a b}$.
Let $X_{\infty}(\omega) \equiv \lim _{n \rightarrow \infty} X_{n}(\omega)$. Then by Fatou's lemma,

$$
\int_{\Omega}\left|X_{\infty}(\omega)\right| d P \leq \lim \inf _{n \rightarrow \infty} \int_{\Omega}\left|X_{n}\right| d P<\infty
$$

and so $X_{\infty}$ is in $L^{1}(\Omega)$. By the Vitali convergence theorem and Lemma 62.7 .16 which shows $\left\{\left|X_{n}\right|\right\}$ is uniformly integrable, it follows

$$
\lim _{n \rightarrow \infty} \int_{\Omega}\left|X_{\infty}(\omega)-X_{n}(\omega)\right| d P=0
$$

This proves the theorem.

### 60.9 Strong Law Of Large Numbers

There is a version of the strong law of large numbers which does not depend on the random variables having finite variance. First are some preparatory lemmas. The approach followed here is from Ash [7].

Lemma 60.9.1 Let $\left\{X_{n}\right\}$ be a sequence of independent random variables with $E\left(\left|X_{k}\right|\right)<\infty$ for all $k$ and let $S_{n} \equiv \sum_{k=1}^{n} X_{k}$. Then for $k \leq n$,

$$
E\left(X_{k} \mid \sigma\left(S_{n}\right)\right)=E\left(X_{k} \mid \sigma\left(S_{n}, \mathbf{Y}\right)\right) \text { a.e. }
$$

where $\mathbf{Y}=\left(X_{n+1}, X_{n+2}, \cdots\right) \in \mathbb{R}^{\mathbb{N}}$. Also for $k \leq n$ as above,

$$
\sigma\left(S_{n}, \mathbf{Y}\right)=\sigma\left(S_{n}, S_{n+1}, \cdots\right)
$$

Proof: Note that $\mathbb{R}^{\mathbb{N}}$ with the usual product topology has a countable basis. Here it is. Let $\mathscr{B}_{N}$ denote sets of the form $\prod_{i=1}^{\infty} D_{i}$ where for $i \leq N, D_{i} \in \mathscr{B}$, a countable basis for $\mathbb{R}$ and for $i>N, D_{i}=\mathbb{R}$. Then $\mathscr{B}_{N}$ is countable and so is $\mathscr{D} \equiv \cup_{N=1}^{\infty} \mathscr{B}_{N}$. From the definition of the product topology, this is a countable basis for the product topology.

Let $V \in \mathscr{D}$ and $U$ be an open set of $\mathbb{R}$. Then if $A \in\left(S_{n}, \mathbf{Y}\right)^{-1}(U \times V)$, by independence of the $\left\{X_{n}\right\}$,

$$
\begin{gathered}
\int_{\left(S_{n}, \mathbf{Y}\right)^{-1}(U \times V)} E\left(X_{k} \mid \sigma\left(S_{n}, \mathbf{Y}\right)\right) d P \equiv \int_{\left(S_{n}, \mathbf{Y}\right)^{-1}(U \times V)} X_{k} d P \\
=\int_{\Omega} \mathscr{X}_{S_{n}^{-1}(U)}(\omega) \mathscr{X}_{\mathbf{Y}^{-1}(V)}(\omega) X_{k} d P=P\left(\mathbf{Y}^{-1}(V)\right) \int_{\Omega} \mathscr{X}_{S_{n}^{-1}(U)}(\omega) X_{k} d P \\
=P\left(\mathbf{Y}^{-1}(V)\right) \int_{S_{n}^{-1}(U)} E\left(X_{k} \mid \sigma\left(S_{n}\right)\right) d P .
\end{gathered}
$$

Now by independence again, $\left\{S_{n}, X_{n+1}, X_{n+2}, \cdots\right\}$ are independent and so the above equals

$$
\int_{S_{n}^{-1}(U)} \mathscr{X}_{\mathbf{Y}^{-1}(V)} E\left(X_{k} \mid \sigma\left(S_{n}\right)\right) d P=\int_{\left(S_{n}, \mathbf{Y}\right)^{-1}(U \times V)} E\left(X_{k} \mid \sigma\left(S_{n}\right)\right) d P
$$

Letting

$$
\begin{gathered}
\mathscr{S} \equiv\left\{A \in \mathscr{B}\left(\mathbb{R} \times \mathbb{R}^{\mathbb{N}}\right): \int_{\left(S_{n}, \mathbf{Y}\right)^{-1}(A)} E\left(X_{k} \mid \sigma\left(S_{n}\right)\right) d P\right. \\
\left.=\int_{\left(S_{n}, \mathbf{Y}\right)^{-1}(A)} E\left(X_{k} \mid \sigma\left(S_{n}, \mathbf{Y}\right)\right) d P\right\}
\end{gathered}
$$

the above has shown this is true for all $A$ in a countable basis. Therefore, it is true for all $A$ open in $\mathbb{R} \times \mathbb{R}^{\mathbb{N}}$. Finally, it is clear that $\mathscr{S}$ is a $\sigma$ algebra which shows the above holds for all $A$ Borel in $\mathbb{R} \times \mathbb{R}^{\mathbb{N}}$. Thus, for all $B \in \sigma\left(S_{n}, \mathbf{Y}\right)$,

$$
\int_{B} E\left(X_{k} \mid \sigma\left(S_{n}\right)\right) d P=\int_{B} E\left(X_{k} \mid \sigma\left(S_{n}, \mathbf{Y}\right)\right) d P
$$

and thus $E\left(X_{k} \mid \sigma\left(S_{n}\right)\right)=E\left(X_{k} \mid \sigma\left(S_{n}, \mathbf{Y}\right)\right)$ a.e.
It only remains to prove the last assertion. For $k>0$,

$$
X_{n+k}=S_{n+k}-S_{n+k-1}
$$

Thus

$$
\begin{aligned}
\sigma\left(S_{n}, \mathbf{Y}\right) & =\sigma\left(S_{n}, X_{n+1}, \cdots\right) \\
& =\sigma\left(S_{n},\left(S_{n+1}-S_{n}\right),\left(S_{n+2}-S_{n+1}\right), \cdots\right) \\
& \subseteq \sigma\left(S_{n}, S_{n+1}, \cdots\right)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\sigma\left(S_{n}, S_{n+1}, \cdots\right) & =\sigma\left(S_{n}, X_{n+1}+S_{n}, X_{n+2}+X_{n+1}+S_{n}, \cdots\right) \\
& \subseteq \sigma\left(S_{n}, X_{n+1}, X_{n+2}, \cdots\right)
\end{aligned}
$$

To see this, note that for an open set, and hence for a Borel set, $B$,

$$
\left(S_{n}+\sum_{k=n+1}^{m} X_{k}\right)^{-1}(B)=\left(S_{n}, X_{n+1}, \cdots, X_{m}\right)^{-1}\left(B^{\prime}\right)
$$

for some $B^{\prime} \in \mathbb{R}^{m+1}$. Thus $\left(S_{n}+\sum_{k=n+1}^{m} X_{k}\right)^{-1}(B)$ for $B$ a Borel set is contained in

$$
\sigma\left(S_{n}, X_{n+1}, X_{n+2}, \cdots\right)
$$

Similar considerations apply to the other inclusion stated earlier. This proves the lemma.
Lemma 60.9.2 Let $\left\{X_{k}\right\}$ be a sequence of independent identically distributed random variables such that $E\left(\left|X_{k}\right|\right)<\infty$. Then letting $S_{n}=\sum_{k=1}^{n} X_{k}$, it follows that for $k \leq n$

$$
E\left(X_{k} \mid \sigma\left(S_{n}, S_{n+1}, \cdots\right)\right)=E\left(X_{k} \mid \sigma\left(S_{n}\right)\right)=\frac{S_{n}}{n}
$$

Proof: It was shown in Lemma 60.9.1 the first equality holds. It remains to show the second. Letting $A=S_{n}^{-1}(B)$ where $B$ is Borel, it follows there exists $B^{\prime} \subseteq \mathbb{R}^{n}$ a Borel set such that

$$
S_{n}^{-1}(B)=\left(X_{1}, \cdots, X_{n}\right)^{-1}\left(B^{\prime}\right)
$$

Then

$$
\begin{gathered}
\int_{A} E\left(X_{k} \mid \sigma\left(S_{n}\right)\right) d P=\int_{S_{n}^{-1}(B)} X_{k} d P \\
=\int_{\left(X_{1}, \cdots, X_{n}\right)^{-1}\left(B^{\prime}\right)} X_{k} d P=\int_{\left(X_{1}, \cdots, X_{n}\right)^{-1}\left(B^{\prime}\right)} x_{k} d \lambda_{\left(X_{1}, \cdots, X_{n}\right)} \\
=\int \cdots \int \mathscr{X}_{\left(X_{1}, \cdots, X_{n}\right)^{-1}\left(B^{\prime}\right)}(\mathbf{x}) x_{k} d \lambda_{X_{1}} d \lambda_{X_{2}} \cdots d \lambda_{X_{n}} \\
=\int \cdots \int \mathscr{X}_{\left(X_{1}, \cdots, X_{n}\right)^{-1}\left(B^{\prime}\right)}(\mathbf{x}) x_{l} d \lambda_{X_{1}} d \lambda_{X_{2}} \cdots d \lambda_{X_{n}} \\
=\int_{A} E\left(X_{l} \mid \sigma\left(S_{n}\right)\right) d P
\end{gathered}
$$

and so since $A \in \sigma\left(S_{n}\right)$ is arbitrary,

$$
E\left(X_{l} \mid \sigma\left(S_{n}\right)\right)=E\left(X_{k} \mid \sigma\left(S_{n}\right)\right)
$$

for each $k, l \leq n$. Therefore,

$$
S_{n}=E\left(S_{n} \mid \sigma\left(S_{n}\right)\right)=\sum_{j=1}^{n} E\left(X_{j} \mid \sigma\left(S_{n}\right)\right)=n E\left(X_{k} \mid \sigma\left(S_{n}\right)\right) \text { a.e. }
$$

and so

$$
E\left(X_{k} \mid \sigma\left(S_{n}\right)\right)=\frac{S_{n}}{n} \text { a.e. }
$$

as claimed. This proves the lemma.
With this preparation, here is the strong law of large numbers for identically distributed random variables.

Theorem 60.9.3 Let $\left\{X_{k}\right\}$ be a sequence of independent identically distributed random variables such that $E\left(\left|X_{k}\right|\right)<\infty$ for all $k$. Letting $m=E\left(X_{k}\right)$,

$$
\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{n} X_{k}(\omega)=m \text { a.e. }
$$

and convergence also takes place in $L^{1}(\Omega)$.
Proof: Consider the reverse submartingale $\left\{E\left(X_{1} \mid \sigma\left(S_{n}, S_{n+1}, \cdots\right)\right)\right\}$. By Theorem 60.8.3, this converges a.e. and in $L^{1}(\Omega)$ to a random variable, $X_{\infty}$. However, from Lemma 60.9.2, $E\left(X_{1} \mid \sigma\left(S_{n}, S_{n+1}, \cdots\right)\right)=S_{n} / n$. Therefore, $S_{n} / n$ converges a.e. and in $L^{1}(\Omega)$ to $X_{\infty}$. I need to argue that $X_{\infty}$ is constant and also that it equals $m$. For $a \in \mathbb{R}$ let

$$
E_{a} \equiv\left[X_{\infty} \geq a\right]
$$

For $a$ small enough, $P\left(E_{a}\right) \neq 0$. Then since $E_{a}$ is a tail event for the independent random variables, $\left\{X_{k}\right\}$ it follows from the Kolmogorov zero one law, Theorem 59.6.4, that $P\left(E_{a}\right)=1$. Let $b \equiv \sup \left\{a: P\left(E_{a}\right)=1\right\}$. The sets, $E_{a}$ are decreasing as $a$ increases. Let $\left\{a_{n}\right\}$ be a strictly increasing sequence converging to $b$. Then

$$
\left[X_{\infty} \geq b\right]=\cap_{n}\left[X_{\infty} \geq a_{n}\right]
$$

and so

$$
1=P\left(E_{b}\right)=\lim _{n \rightarrow \infty} P\left(E_{a_{n}}\right)
$$

On the other hand, if $c>b$, then $P\left(E_{c}\right)<1$ and so $P\left(E_{c}\right)=0$. Hence $P([X=b])=1$. It remains to show $b=m$. This is easy because by the $L^{1}$ convergence,

$$
b=\int_{\Omega} X_{\infty} d P=\lim _{n \rightarrow \infty} \int_{\Omega} \frac{S_{n}}{n} d P=\lim _{n \rightarrow \infty} m=m
$$

This proves the theorem.

## Chapter 61

## Probability In Infinite Dimensions

### 61.1 Conditional Expectation In Banach Spaces

Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $X \in L^{1}(\Omega ; \mathbb{R})$. Also let $\mathscr{G} \subseteq \mathscr{F}$ where $\mathscr{G}$ is also a $\sigma$ algebra. Then the usual conditional expectation is defined by

$$
\int_{A} X d P=\int_{A} E(X \mid \mathscr{G}) d P
$$

where $E(X \mid \mathscr{G})$ is $\mathscr{G}$ measurable and $A \in \mathscr{G}$ is arbitrary. Recall this is an application of the Radon Nikodym theorem. Also recall $E(X \mid \mathscr{G})$ is unique up to a set of measure zero.

I want to do something like this here. Denote by $L^{1}(\Omega ; E, \mathscr{G})$ those functions in $L^{1}(\Omega ; E)$ which are measurable with respect to $\mathscr{G}$.

Theorem 61.1.1 Let $E$ be a separable Banach space and let $X \in L^{1}(\Omega ; E, \mathscr{F})$ where $X$ is measurable with respect to $\mathscr{F}$ and let $\mathscr{G}$ be a $\sigma$ algebra which is contained in $\mathscr{F}$. Then there exists a unique $Z \in L^{1}(\Omega ; E, \mathscr{G})$ such that for all $A \in \mathscr{G}$,

$$
\int_{A} X d P=\int_{A} Z d P
$$

Denoting this $Z$ as $E(X \mid \mathscr{G})$, it follows

$$
\|E(X \mid \mathscr{G})\| \leq E(\|X\| \mid \mathscr{G}) .
$$

Proof: First consider uniqueness. Suppose $Z^{\prime}$ is another in $L^{1}(\Omega ; E, \mathscr{G})$ which works. Consider a dense subset of $E\left\{a_{n}\right\}_{n=1}^{\infty}$. Then the balls $\left\{B\left(a_{n}, \frac{\left\|a_{n}\right\|}{4}\right)\right\}_{n=1}^{\infty}$ must cover $E \backslash$ $\{\boldsymbol{0}\}$. Here is why. If $y \neq 0$, pick $a_{n} \in B\left(y, \frac{\|y\|}{5}\right)$.


Then $\left\|a_{n}\right\| \geq 4\|y\| / 5$ and so $\left\|a_{n}-y\right\|<\|y\| / 5$. Thus

$$
y \in B\left(a_{n},\|y\| / 5\right) \subseteq B\left(a_{n}, \frac{\left\|a_{n}\right\|}{4}\right)
$$

Now suppose $Z$ is $\mathscr{G}$ measurable and

$$
\int_{A} Z d P=0
$$

for all $A \in \mathscr{G}$. The letting $A \equiv Z^{-1}\left(B\left(a_{n}, \frac{\left\|a_{n}\right\|}{4}\right)\right)$ it follows

$$
0=\int_{A} Z-a_{n}+a_{n} d P
$$

and so

$$
\begin{aligned}
& \left\|a_{n}\right\| P(A)=\left\|\int_{A} a_{n} d P\right\|=\left\|\int_{A}\left(a_{n}-Z\right) d P\right\| \\
& \leq \int_{Z^{-1}\left(B\left(a_{n}, \frac{\left\|a_{n}\right\|}{4}\right)\right)}\left\|a_{n}-Z\right\| d P \leq \frac{\left\|a_{n}\right\|}{4} P(A)
\end{aligned}
$$

which is a contradiction unless $P(A)=0$. Therefore, letting

$$
N \equiv \cup_{n=1}^{\infty} Z^{-1}\left(B\left(a_{n}, \frac{\left\|a_{n}\right\|}{4}\right)\right)=Z^{-1}(E \backslash\{0\})
$$

it follows $N$ has measure zero and so $Z=0$ a.e. This proves uniqueness because if $Z, Z^{\prime}$ both hold, then from the above argument, $Z-Z^{\prime}=0$ a.e.

Next I will show $Z$ exists. To do this recall Theorem 21.2.4 on Page 652 which is stated below for convenience.

Theorem 61.1.2 An E valued function, $X$, is Bochner integrable if and only if $X$ is strongly measurable and

$$
\begin{equation*}
\int_{\Omega}\|X(\omega)\| d P<\infty \tag{61.1.1}
\end{equation*}
$$

In this case there exists a sequence of simple functions $\left\{X_{n}\right\}$ satisfying

$$
\begin{equation*}
\int_{\Omega}\left\|X_{n}(\omega)-X_{m}(\omega)\right\| d P \rightarrow 0 \text { as } m, n \rightarrow \infty \tag{61.1.2}
\end{equation*}
$$

$X_{n}(\omega)$ converging pointwise to $X(\omega)$,

$$
\begin{equation*}
\left\|X_{n}(\omega)\right\| \leq 2\|X(\omega)\| \tag{61.1.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \int_{\Omega}\left\|X(\omega)-X_{n}(\omega)\right\| d P=0 \tag{61.1.4}
\end{equation*}
$$

Now let $\left\{X_{n}\right\}$ be the simple functions just defined and let

$$
X_{n}(\omega)=\sum_{k=1}^{m} x_{k} \mathscr{X}_{F_{k}}(\omega)
$$

where $F_{k} \in \mathscr{F}$, the $F_{k}$ being disjoint. Then define

$$
Z_{n} \equiv \sum_{k=1}^{m} x_{k} E\left(\mathscr{X}_{F_{k}} \mid \mathscr{G}\right)
$$

Thus, if $A \in \mathscr{G}$,

$$
\begin{align*}
\int_{A} Z_{n} d P & =\sum_{k=1}^{m} x_{k} \int_{A} E\left(\mathscr{X}_{F_{k}} \mid \mathscr{G}\right) d P \\
& =\sum_{k=1}^{m} x_{k} \int_{A} \mathscr{X}_{F_{k}} d P \\
& =\sum_{k=1}^{m} x_{k} P\left(F_{k} \cap A\right)=\int_{A} X_{n} d P \tag{61.1.5}
\end{align*}
$$

Then since $E\left(\mathscr{X}_{F_{k}} \mid \mathscr{G}\right) \geq 0$,

$$
\left\|Z_{n}\right\| \leq \sum_{k=1}^{m}\left\|x_{k}\right\| E\left(\mathscr{X}_{F_{k}} \mid \mathscr{G}\right)
$$

Thus if $A \in \mathscr{G}$,

$$
\begin{align*}
E\left(\left\|Z_{n}\right\| \mathscr{X}_{A}\right) & \leq E\left(\sum_{k=1}^{m}\left\|x_{k}\right\| \mathscr{X}_{A} E\left(\mathscr{X}_{F_{k}} \mid \mathscr{G}\right)\right)=\sum_{k=1}^{m}\left\|x_{k}\right\| \int_{A} E\left(\mathscr{X}_{F_{k}} \mid \mathscr{G}\right) d P \\
& =\sum_{k=1}^{m}\left\|x_{k}\right\| \int_{A} \mathscr{X}_{F_{k}} d P=E\left(\mathscr{X}_{A}\left\|X_{n}\right\|\right) \tag{61.1.6}
\end{align*}
$$

Note the use of $\leq$ in the first step in the above. Although the $F_{k}$ are disjoint, all that is known about $E\left(\mathscr{X}_{F_{k}} \mid \mathscr{G}\right)$ is that it is nonnegative. Similarly,

$$
E\left(\left\|Z_{n}-Z_{m}\right\|\right) \leq E\left(\left\|X_{n}-X_{m}\right\|\right)
$$

and this last term converges to 0 as $n, m \rightarrow \infty$ by the properties of the $X_{n}$. Therefore, $\left\{Z_{n}\right\}$ is a Cauchy sequence in $L^{1}(\Omega ; E ; \mathscr{G})$. It follows it converges to some $Z$ in $L^{1}(\Omega ; E, \mathscr{G})$. Then letting $A \in \mathscr{G}$, and using 61.1.5,

$$
\begin{aligned}
\int_{A} Z d P & =\int \mathscr{X}_{A} Z d P=\lim _{n \rightarrow \infty} \int \mathscr{X}_{A} Z_{n} d P=\lim _{n \rightarrow \infty} \int_{A} Z_{n} d P \\
& =\lim _{n \rightarrow \infty} \int_{A} X_{n} d P=\int_{A} X d P .
\end{aligned}
$$

Then define $Z \equiv E(X \mid \mathscr{G})$.
It remains to verify $\|E(X \mid \mathscr{G})\| \equiv\|Z\| \leq E(\|X\| \mid \mathscr{G})$. This follows because, from the above,

$$
\left\|Z_{n}\right\| \rightarrow\|Z\|,\left\|X_{n}\right\| \rightarrow\|X\| \text { in } L^{1}(\Omega)
$$

and so if $A \in \mathscr{G}$, then from 61.1.6,

$$
\frac{1}{P(A)} \int_{A}\left\|Z_{n}\right\| d P \leq \frac{1}{P(A)} \int_{A}\left\|X_{n}\right\| d P
$$

and so, passing to the limit,

$$
\frac{1}{P(A)} \int_{A}\|Z\| d P \leq \frac{1}{P(A)} \int_{A}\|X\| d P=\frac{1}{P(A)} \int_{A} E(\|X\| \mid \mathscr{G}) d P
$$

Since $A$ is arbitrary, this shows that

$$
\|E(X \mid \mathscr{G})\| \equiv\|Z\| \leq E(\|X\| \mid \mathscr{G}) .
$$

In the case where $E$ is reflexive, one could also use Corollary 21.7.6 on Page 681 to get the above result. You would define a vector measure on $\mathscr{G}$,

$$
v(F) \equiv \int_{F} X d P
$$

and then you would use the fact that reflexive separable Banach spaces have the Radon Nikodym property to obtain $Z \in L^{1}(\Omega ; E, \mathscr{G})$ such that

$$
v(F)=\int_{F} X d P=\int_{F} Z d P
$$

The function, $Z$ whose existence and uniqueness is guaranteed by Theorem 61.1.2 is called $E(X \mid \mathscr{G})$.

### 61.2 Probability Measures And Tightness

Here and in what remains, $\mathscr{B}(E)$ will denote the Borel sets of $E$ where $E$ is a topological space, usually at least a Banach space. Because of the fact that probability measures are finite, you can use a simpler definition of what it means for a measure to be regular. Recall that there were two ingredients, inner regularity which said that the measure of a set is the supremum of the measures of compact subsets and outer regularity which says that the measure of a set is the infimum of the measures of the open sets which contain the given set. Here the definition will be similar but instead of using compact sets, closed sets are substituted. Thus the following definition is a little different than the earlier one. I will show, however, that in many interesting cases, this definition of regularity is actually the same as the earlier one.

Definition 61.2.1 A measure, $\mu$ defined on $\mathscr{B}(E)$ will be called inner regular if for all $F \in \mathscr{B}(E)$,

$$
\mu(F)=\sup \{\mu(K): K \subseteq F \text { and } K \text { is closed }\}
$$

A measure, $\mu$ defined on $\mathscr{B}(E)$ will be called outer regular iffor all $F \in \mathscr{B}(E)$,

$$
\mu(F)=\inf \{\mu(V): V \supseteq F \text { and } V \text { is open }\}
$$

When a measure is both inner and outer regular, it is called regular.
For probability measures, regularity tends to come free.

Lemma 61.2.2 Let $\mu$ be a finite measure defined on $\mathscr{B}(E)$ where $E$ is a metric space. Then $\mu$ is regular.

Proof: First note every open set is the countable union of closed sets and every closed set is the countable intersection of open sets. Here is why. Let $V$ be an open set and let

$$
K_{k} \equiv\left\{x \in V: \operatorname{dist}\left(x, V^{C}\right) \geq 1 / k\right\}
$$

Then clearly the union of the $K_{k}$ equals $V$. Next, for $K$ closed let

$$
V_{k} \equiv\{x \in E: \operatorname{dist}(x, K)<1 / k\}
$$

Clearly the intersection of the $V_{k}$ equals $K$. Therefore, letting $V$ denote an open set and $K$ a closed set,

$$
\begin{aligned}
& \mu(V)=\sup \{\mu(K): K \subseteq V \text { and } K \text { is closed }\} \\
& \mu(K)=\inf \{\mu(V): V \supseteq K \text { and } V \text { is open }\}
\end{aligned}
$$

Also since $V$ is open and $K$ is closed,

$$
\begin{aligned}
& \mu(V)=\inf \{\mu(U): U \supseteq V \text { and } V \text { is open }\} \\
& \mu(K)=\sup \{\mu(L): L \subseteq K \text { and } L \text { is closed }\}
\end{aligned}
$$

In words, $\mu$ is regular on open and closed sets. Let

$$
\mathscr{F} \equiv\{F \in \mathscr{B}(E) \text { such that } \mu \text { is regular on } F\} .
$$

Then $\mathscr{F}$ contains the open sets. I want to show $\mathscr{F}$ is a $\sigma$ algebra and then it will follow $\mathscr{F}=\mathscr{B}(E)$.

First I will show $\mathscr{F}$ is closed with respect to complements. Let $F \in \mathscr{F}$. Then since $\mu$ is finite and $F$ is inner regular, there exists $K \subseteq F$ such that $\mu(F \backslash K)<\varepsilon$. But $K^{C} \backslash F^{C}=F \backslash K$ and so $\mu\left(K^{C} \backslash F^{C}\right)<\varepsilon$ showning that $F^{C}$ is outer regular. I have just approximated the measure of $F^{C}$ with the measure of $K^{C}$, an open set containing $F^{C}$. A similar argument works to show $F^{C}$ is inner regular. You start with $V \supseteq F$ such that $\mu(V \backslash F)<\varepsilon$, note $F^{C} \backslash V^{C}=V \backslash F$, and then conclude $\mu\left(F^{C} \backslash V^{C}\right)<\varepsilon$, thus approximating $F^{C}$ with the closed subset, $V^{C}$.

Next I will show $\mathscr{F}$ is closed with respect to taking countable unions. Let $\left\{F_{k}\right\}$ be a sequence of sets in $\mathscr{F}$. Then $\mu$ is inner regular on each of these so there exist $\left\{K_{k}\right\}$ such that $K_{k} \subseteq F_{k}$ and $\mu\left(F_{k} \backslash K_{k}\right)<\varepsilon / 2^{k+1}$. First choose $m$ large enough that

$$
\mu\left(\left(\cup_{k=1}^{\infty} F_{k}\right) \backslash\left(\cup_{k=1}^{m} F_{k}\right)\right)<\frac{\varepsilon}{2}
$$

Then

$$
\mu\left(\left(\cup_{k=1}^{m} F_{k}\right) \backslash\left(\cup_{k=1}^{m} K_{k}\right)\right) \leq \sum_{k=1}^{m} \frac{\varepsilon}{2^{k+1}}<\frac{\varepsilon}{2}
$$

and so

$$
\begin{aligned}
\mu\left(\left(\cup_{k=1}^{\infty} F_{k}\right) \backslash\left(\cup_{k=1}^{m} K_{k}\right)\right) \leq & \mu\left(\left(\cup_{k=1}^{\infty} F_{k}\right) \backslash\left(\cup_{k=1}^{m} F_{k}\right)\right) \\
& +\mu\left(\left(\cup_{k=1}^{m} F_{k}\right) \backslash\left(\cup_{k=1}^{m} K_{k}\right)\right) \\
< & \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon
\end{aligned}
$$

showing $\mu$ is inner regular on $\cup_{k=1}^{\infty} F_{k}$. Since $\mu$ is outer regular on $F_{k}$, there exists $V_{k}$ such that $\mu\left(V_{k} \backslash F_{k}\right)<\varepsilon / 2^{k}$. Then

$$
\begin{aligned}
\mu\left(\left(\cup_{k=1}^{\infty} V_{k}\right) \backslash\left(\cup_{k=1}^{\infty} F_{k}\right)\right) & \leq \sum_{k=1}^{\infty} \mu\left(V_{k} \backslash F_{k}\right) \\
& <\sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k}}=\varepsilon
\end{aligned}
$$

and this shows $\mu$ is outer regular on $\cup_{k=1}^{\infty} F_{k}$ and this proves the lemma.
Lemma 61.2.3 Let $\mu$ be a finite measure on $\mathscr{B}(E)$, the Borel sets of $E$, a separable complete metric space. Then if $C$ is a closed set,

$$
\mu(C)=\sup \{\mu(K): K \subseteq C \text { and } K \text { is compact. }\}
$$

Proof: Let $\left\{a_{k}\right\}$ be a countable dense subset of $C$. Thus $\cup_{k=1}^{\infty} B\left(a_{k}, \frac{1}{n}\right) \supseteq C$. Therefore, there exists $m_{n}$ such that

$$
\mu\left(C \backslash \cup_{k=1}^{m_{n}} \overline{B\left(a_{k}, \frac{1}{n}\right)}\right) \equiv \mu\left(C \backslash C_{n}\right)<\frac{\varepsilon}{2^{n}}
$$

Now let $K=C \cap\left(\cap_{n=1}^{\infty} C_{n}\right)$. Then $K$ is a subset of $C_{n}$ for each $n$ and so for each $\varepsilon>0$ there exists an $\varepsilon$ net for $K$ since $C_{n}$ has a $1 / n$ net, namely $a_{1}, \cdots, a_{m_{n}}$. Since $K$ is closed, it is complete and so it is also compact. Now

$$
\mu(C \backslash K)=\mu\left(\cup_{n=1}^{\infty}\left(C \backslash C_{n}\right)\right)<\sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n}}=\varepsilon
$$

Thus $\mu(C)$ can be approximated by $\mu(K)$ for $K$ a compact subset of $C$. This proves the lemma.

This shows that for a finite measure on the Borel sets of a separable metric space, the above definition of regular coincides with the earlier one.

### 61.3 Tight Measures

Now here is a definition of what it means for a set of measures to be tight.
Definition 61.3.1 Let $\Lambda$ be a set of probability measures defined on the Borel sets of a topological space. Then $\Lambda$ is "tight" if for all $\varepsilon>0$ there exists a compact set, $K_{\varepsilon}$ such that

$$
\mu\left(\left[x \notin K_{\varepsilon}\right]\right)<\varepsilon
$$

for all $\mu \in \Lambda$.
Lemma 61.2 .3 implies a single probability measure on the Borel sets of a separable metric space is tight. The proof of that lemma generalizes slightly to give a simple criterion for a set of measures to be tight.

Lemma 61.3.2 Let E be a separable complete metric space and let $\Lambda$ be a set of Borel probability measures. Then $\Lambda$ is tight if and only iffor every $\varepsilon>0$ and $r>0$ there exists $a$ finite collection of balls, $\left\{B\left(a_{i}, r\right)\right\}_{i=1}^{m}$ such that

$$
\mu\left(\cup_{i=1}^{m} \overline{B\left(a_{i}, r\right)}\right)>1-\varepsilon
$$

for every $\mu \in \Lambda$.
Proof: If $\Lambda$ is tight, then there exists a compact set, $K_{\varepsilon}$ such that

$$
\mu\left(K_{\varepsilon}\right)>1-\varepsilon
$$

for all $\mu \in \Lambda$. Then consider the open cover, $\left\{B(x, r): x \in K_{\varepsilon}\right\}$. Finitely many of these cover $K_{\varepsilon}$ and this yields the above condition.

Now suppose the above condition and let

$$
C_{n} \equiv \cup_{i=1}^{m_{n}} \overline{B\left(a_{i}^{n}, 1 / n\right)}
$$

satisfy $\mu\left(C_{n}\right)>1-\varepsilon / 2^{n}$ for all $\mu \in \Lambda$. Then let $K_{\varepsilon} \equiv \cap_{n=1}^{\infty} C_{n}$. This set $K_{\varepsilon}$ is a compact set because it is a closed subset of a complete metric space and is therefore complete, and it is also totally bounded by construction. For $\mu \in \Lambda$,

$$
\mu\left(K_{\varepsilon}^{C}\right)=\mu\left(\cup_{n} C_{n}^{C}\right) \leq \sum_{n} \mu\left(C_{n}^{C}\right)<\sum_{n} \frac{\varepsilon}{2^{n}}=\varepsilon
$$

Therefore, $\Lambda$ is tight.
Prokhorov's theorem is an important result which also involves tightness. In order to give a proof of this important theorem, it is necessary to consider some simple results from topology which are interesting for their own sake.

Theorem 61.3.3 Let $H$ be a compact metric space. Then there exists a compact subset of $[0,1], K$ and a continuous function, $\theta$ which maps $K$ onto $H$.

Proof: Without loss of generality, it can be assumed $H$ is an infinite set since otherwise the conclusion is trivial. You could pick finitely many points of $[0,1]$ for $K$.

Since $H$ is compact, it is totally bounded. Therefore, there exists a 1 net for $H\left\{h_{i}\right\}_{i=1}^{m_{1}}$. Letting $H_{i}^{1} \equiv \overline{B\left(h_{i}, 1\right)}$, it follows $H_{i}^{1}$ is also a compact metric space and so there exists a $1 / 2$ net for each $H_{i}^{1},\left\{h_{j}^{i}\right\}_{j=1}^{m_{i}}$. Then taking the intersection of $\bar{B}\left(h_{j}^{i}, \frac{1}{2}\right)$ with $H_{i}^{1}$ to obtain sets denoted by $H_{j}^{2}$ and continuing this way, one can obtain compact subsets of $H,\left\{H_{k}^{i}\right\}$ which satisfies: each $H_{j}^{i}$ is contained in some $H_{k}^{i-1}$, each $H_{j}^{i}$ is compact with diameter less than $i^{-1}$, each $H_{j}^{i}$ is the union of sets of the form $H_{k}^{i+1}$ which are contained in it. Denoting by $\left\{H_{j}^{i}\right\}_{j=1}^{m_{i}}$ those sets corresponding to a superscript of $i$, it can also be assumed $m_{i}<m_{i+1}$. If this is not so, simply add in another point to the $i^{-1}$ net. Now let $\left\{I_{j}^{i}\right\}_{j=1}^{m_{i}}$ be disjoint closed intervals in $[0,1]$ each of length no longer than $2^{-m_{i}}$ which have the property that
$I_{j}^{i}$ is contained in $I_{k}^{i-1}$ for some $k$. Letting $K_{i} \equiv \cup_{j=1}^{m_{i}} I_{j}^{i}$, it follows $K_{i}$ is a sequence of nested compact sets. Let $K=\cap_{i=1}^{\infty} K_{i}$. Then each $x \in K$ is the intersection of a unique sequence of these closed intervals, $\left\{I_{j_{k}}^{k}\right\}_{k=1}^{\infty}$. Define $\theta x \equiv \cap_{k=1}^{\infty} H_{j_{k}}^{k}$. Since the diameters of the $H_{j}^{i}$ converge to 0 as $i \rightarrow \infty$, this function is well defined. It is continuous because if $x_{n} \rightarrow x$, then ultimately $x_{n}$ and $x$ are both in $I_{j_{k}}^{k}$, the $k^{t h}$ closed interval in the sequence whose intersection is $x$. Hence, $d\left(\theta x_{n}, \theta x\right) \leq \operatorname{diameter}\left(H_{j_{k}}^{k}\right) \leq 1 / k$. To see the map is onto, let $h \in H$. Then from the construction, there exists a sequence $\left\{H_{j_{k}}^{k}\right\}_{k=1}^{\infty}$ of the above sets whose intersection equals $h$. Then $\theta\left(\cap_{i=1}^{\infty} I_{j_{k}}^{k}\right)=h$.

Note $\theta$ is maybe not one to one.
As an important corollary, it follows that the continuous functions defined on any compact metric space is separable.

Corollary 61.3.4 Let $H$ be a compact metric space and let $C(H)$ denote the continuous functions defined on $H$ with the usual norm,

$$
\|f\|_{\infty} \equiv \max \{|f(x)|: x \in H\}
$$

Then $C(H)$ is separable.
Proof: The proof is by contradiction. Suppose $C(H)$ is not separable. Let $\mathscr{H}_{k}$ denote a maximal collection of functions of $C(H)$ with the property that if $f, g \in \mathscr{H}_{k}$, then $\|f-g\|_{\infty} \geq 1 / k$. The existence of such a maximal collection of functions is a consequence of a simple use of the Hausdorff maximality theorem. Then $\cup_{k=1}^{\infty} \mathscr{H}_{k}$ is dense. Therefore, it cannot be countable by the assumption that $C(H)$ is not separable. It follows that for some $k, \mathscr{H}_{k}$ is uncountable. Now by Theorem 61.3.3 there exists a continuous function, $\theta$ defined on a compact subset, $K$ of $[0,1]$ which maps $K$ onto $H$. Now consider the functions defined on $K$

$$
\mathscr{G}_{k} \equiv\left\{f \circ \theta: f \in \mathscr{H}_{k}\right\}
$$

Then $\mathscr{G}_{k}$ is an uncountable set of continuous functions defined on $K$ with the property that the distance between any two of them is at least as large as $1 / k$. This contradicts separability of $C(K)$ which follows from the Weierstrass approximation theorem in which the separable countable set of functions is the restrictions of polynomials that involve only rational coefficients.

Now here is Prokhorov's theorem.

Theorem 61.3.5 Let $\Lambda=\left\{\mu_{n}\right\}_{n=1}^{\infty}$ be a sequence of probability measures defined on $\mathscr{B}(E)$ where $E$ is a separable complete metric space. If $\Lambda$ is tight then there exists a probability measure, $\lambda$ and a subsequence of $\left\{\mu_{n}\right\}_{n=1}^{\infty}$, still denoted by $\left\{\mu_{n}\right\}_{n=1}^{\infty}$ such that whenever $\phi$ is a continuous bounded complex valued function defined on $E$,

$$
\lim _{n \rightarrow \infty} \int \phi d \mu_{n}=\int \phi d \lambda
$$

Proof: By tightness, there exists an increasing sequence of compact sets, $\left\{K_{n}\right\}$ such that

$$
\mu\left(K_{n}\right)>1-\frac{1}{n}
$$

for all $\mu \in \Lambda$. Now letting $\mu \in \Lambda$ and $\phi \in C\left(K_{n}\right)$ such that $\|\phi\|_{\infty} \leq 1$, it follows

$$
\left|\int_{K_{n}} \phi d \mu\right| \leq \mu\left(K_{n}\right) \leq 1
$$

and so the restrictions of the measures of $\Lambda$ to $K_{n}$ are contained in the unit ball of $C\left(K_{n}\right)^{\prime}$. Recall from the Riesz representation theorem, the dual space of $C\left(K_{n}\right)$ is a space of complex Borel measures. Theorem 17.5.5 on Page 462 implies the unit ball of $C\left(K_{n}\right)^{\prime}$ is weak * sequentially compact. This follows from the observation that $C\left(K_{n}\right)$ is separable which is proved in Corollary 61.3.4 and leads to the fact that the unit ball in $C\left(K_{n}\right)^{\prime}$ is actually metrizable by Theorem 17.5.5 on Page 462. Therefore, there exists a subsequence of $\Lambda$, $\left\{\mu_{1 k}\right\}$ such that their restrictions to $K_{1}$ converge weak $*$ to a measure, $\lambda_{1} \in C\left(K_{1}\right)^{\prime}$. That is, for every $\phi \in C\left(K_{1}\right)$,

$$
\lim _{k \rightarrow \infty} \int_{K_{1}} \phi d \mu_{1 k}=\int_{K_{1}} \phi d \lambda_{1}
$$

By the same reasoning, there exists a further subsequence $\left\{\mu_{2 k}\right\}$ such that the restrictions of these measures to $K_{2}$ converge weak $*$ to a measure $\lambda_{2} \in C\left(K_{2}\right)^{\prime}$ etc. Continuing this way,

$$
\begin{aligned}
& \mu_{11}, \mu_{12}, \mu_{13}, \cdots \rightarrow \text { Weak } * \operatorname{in} C\left(K_{1}\right)^{\prime} \\
& \mu_{21}, \mu_{22}, \mu_{23}, \cdots \rightarrow \text { Weak } * \operatorname{in} C\left(K_{2}\right)^{\prime} \\
& \mu_{31}, \mu_{32}, \mu_{33}, \cdots \rightarrow \text { Weak } * \text { in } C\left(K_{3}\right)^{\prime}
\end{aligned}
$$

Here the $j^{t h}$ sequence is a subsequence of the $(j-1)^{t h}$. Let $\lambda_{n}$ denote the measure in $C\left(K_{n}\right)^{\prime}$ to which the sequence $\left\{\mu_{n k}\right\}_{k=1}^{\infty}$ converges weak*. Let $\left\{\mu_{n}\right\} \equiv\left\{\mu_{n n}\right\}$, the diagonal sequence. Thus this sequence is ultimately a subsequence of every one of the above sequences and so $\mu_{n}$ converges weak* in $C\left(K_{m}\right)^{\prime}$ to $\lambda_{m}$ for each $m$. Note that this is all happening on different sets so there is no contradiction with something converging to two different things.

Claim: For $p>n$, the restriction of $\lambda_{p}$ to the Borel sets of $K_{n}$ equals $\lambda_{n}$.
Proof of claim: Let $H$ be a compact subset of $K_{n}$. Then there are sets, $V_{l}$ open in $K_{n}$ which are decreasing and whose intersection equals $H$. This follows because this is a metric space. Then let $H \prec \phi_{l} \prec V_{l}$. It follows

$$
\begin{aligned}
\lambda_{n}\left(V_{l}\right) & \geq \int_{K_{n}} \phi_{l} d \lambda_{n}=\lim _{k \rightarrow \infty} \int_{K_{n}} \phi_{l} d \mu_{k} \\
& =\lim _{k \rightarrow \infty} \int_{K_{p}} \phi_{l} d \mu_{k}=\int_{K_{p}} \phi_{l} d \lambda_{p} \geq \lambda_{p}(H) .
\end{aligned}
$$

Now considering the ends of this inequality, let $l \rightarrow \infty$ and pass to the limit to conclude

$$
\lambda_{n}(H) \geq \lambda_{p}(H)
$$

Similarly,

$$
\begin{aligned}
\lambda_{n}(H) & \leq \int_{K_{n}} \phi_{l} d \lambda_{n}=\lim _{k \rightarrow \infty} \int_{K_{n}} \phi_{l} d \mu_{k} \\
& =\lim _{k \rightarrow \infty} \int_{K_{p}} \phi_{l} d \mu_{k}=\int_{K_{p}} \phi_{l} d \lambda_{p} \leq \lambda_{p}\left(V_{l}\right) .
\end{aligned}
$$

Then passing to the limit as $l \rightarrow \infty$, it follows

$$
\lambda_{n}(H) \leq \lambda_{p}(H)
$$

Thus the restriction of $\lambda_{p},\left.\lambda_{p}\right|_{K_{n}}$ to the compact sets of $K_{n}$ equals $\lambda_{n}$. Then by inner regularity it follows the two measures, $\left.\lambda_{p}\right|_{K_{n}}$, and $\lambda_{n}$ are equal on all Borel sets of $K_{n}$. Recall that for finite measures on separable metric spaces, regularity is obtained for free.

It is fairly routine to exploit regularity of the measures to verify that $\lambda_{m}(F) \geq 0$ for all $F$ a Borel subset of $K_{m}$. Note that $\phi \rightarrow \int_{K_{n}} \phi d \lambda_{n}$ is a positive linear functional and so $\lambda_{n} \geq 0$. Also, letting $\phi \equiv 1$,

$$
\begin{equation*}
1 \geq \lambda_{m}\left(K_{m}\right) \geq 1-\frac{1}{m} \tag{61.3.7}
\end{equation*}
$$

Define for $F$ a Borel set,

$$
\lambda(F) \equiv \lim _{n \rightarrow \infty} \lambda_{n}\left(F \cap K_{n}\right)
$$

The limit exists because the sequence on the right is increasing due to the above observation that $\lambda_{n}=\lambda_{m}$ on the Borel subsets of $K_{m}$ whenever $n>m$. Thus for $n>m$

$$
\lambda_{n}\left(F \cap K_{n}\right) \geq \lambda_{n}\left(F \cap K_{m}\right)=\lambda_{m}\left(F \cap K_{m}\right)
$$

Now let $\left\{F_{k}\right\}$ be a sequence of disjoint Borel sets. Then

$$
\begin{aligned}
\lambda\left(\cup_{k=1}^{\infty} F_{k}\right) & \equiv \lim _{n \rightarrow \infty} \lambda_{n}\left(\cup_{k=1}^{\infty} F_{k} \cap K_{n}\right)=\lim _{n \rightarrow \infty} \lambda_{n}\left(\cup_{k=1}^{\infty}\left(F_{k} \cap K_{n}\right)\right) \\
& =\lim _{n \rightarrow \infty} \sum_{k=1}^{\infty} \lambda_{n}\left(F_{k} \cap K_{n}\right)=\sum_{k=1}^{\infty} \lambda\left(F_{k}\right)
\end{aligned}
$$

the last equation holding by the monotone convergence theorem.
It remains to verify

$$
\lim _{k \rightarrow \infty} \int \phi d \mu_{k}=\int \phi d \lambda
$$

for every $\phi$ bounded and continuous. This is where tightness is used again. Then as noted above,

$$
\lambda_{n}\left(K_{n}\right)=\lambda\left(K_{n}\right)
$$

because for $p>n, \lambda_{p}\left(K_{n}\right)=\lambda_{n}\left(K_{n}\right)$ and so letting $p \rightarrow \infty$, the above is obtained. Also,
from 61.3.7,

$$
\begin{aligned}
\lambda\left(K_{n}^{C}\right) & =\lim _{p \rightarrow \infty} \lambda_{p}\left(K_{n}^{C} \cap K_{p}\right) \\
& \leq \lim _{p \rightarrow \infty}\left(\lambda_{p}\left(K_{p}\right)-\lambda_{p}\left(K_{n}\right)\right) \\
& \leq \lim _{p \rightarrow \infty}\left(\lambda_{p}\left(K_{p}\right)-\lambda_{n}\left(K_{n}\right)\right) \\
& \leq \lim _{p \rightarrow \infty}\left(1-\left(1-\frac{1}{n}\right)\right)=\frac{1}{n}
\end{aligned}
$$

Suppose $\|\phi\|_{\infty}<M$. Then

$$
\begin{aligned}
\mid \int \phi d \mu_{k} & -\int \phi d \lambda\left|\leq\left|\int_{K_{n}^{C}} \phi d \mu_{k}+\int_{K_{n}} \phi d \mu_{k}-\left(\int_{K_{n}} \phi d \lambda+\int_{K_{n}^{C}} \phi d \lambda\right)\right|\right. \\
& \leq\left|\int_{K_{n}} \phi d \mu_{k}-\int_{K_{n}} \phi d \lambda_{n}\right|+\left|\int_{K_{n}^{C}} \phi d \mu_{k}-\int_{K_{n}^{C}} \phi d \lambda\right| \\
& \leq\left|\int_{K_{n}} \phi d \mu_{k}-\int_{K_{n}} \phi d \lambda_{n}\right|+\left|\int_{K_{n}^{C}} \phi d \mu_{k}\right|+\left|\int_{K_{n}^{C}} \phi d \lambda\right| \\
& \leq\left|\int_{K_{n}} \phi d \mu_{k}-\int_{K_{n}} \phi d \lambda_{n}\right|+\frac{M}{n}+\frac{M}{n}
\end{aligned}
$$

First let $n$ be so large that $2 M / n<\varepsilon / 2$ and then pick $k$ large enough that the above expression is less than $\varepsilon$.

Definition 61.3.6 Let $E$ be a complete separable metric space and let $\mu$ and the sequence of probability measures, $\left\{\mu_{n}\right\}$ defined on $\mathscr{B}(E)$ satisfy

$$
\lim _{n \rightarrow \infty} \int \phi d \mu_{n}=\int \phi d \mu
$$

for every $\phi$ a bounded continuous function. Then $\mu_{n}$ is said to converge weakly to $\mu$.

### 61.4 A Major Existence And Convergence Theorem

Here is an interesting lemma about weak convergence.
Lemma 61.4.1 Let $\mu_{n}$ converge weakly to $\mu$ and let $U$ be an open set with $\mu(\partial U)=0$. Then

$$
\lim _{n \rightarrow \infty} \mu_{n}(U)=\mu(U)
$$

Proof: Let $\left\{\psi_{k}\right\}$ be a sequence of bounded continuous functions which decrease to $\mathscr{X}_{\bar{U}}$. Also let $\left\{\phi_{k}\right\}$ be a sequence of bounded continuous functions which increase to $\mathscr{X}_{U}$. For example, you could let

$$
\begin{aligned}
\psi_{k}(x) & \equiv(1-k \operatorname{dist}(x, U))^{+} \\
\phi_{k}(x) & \equiv 1-\left(1-k \operatorname{dist}\left(x, U^{C}\right)\right)^{+}
\end{aligned}
$$

Let $\varepsilon>0$ be given. Then since $\mu(\partial U)=0$, the dominated convergence theorem implies there exists $\psi=\psi_{k}$ and $\phi=\phi_{k}$ such that

$$
\varepsilon>\int \psi d \mu-\int \phi d \mu
$$

Next use the weak convergence to pick $N$ large enough that if $n \geq N$,

$$
\int \psi d \mu_{n} \leq \int \psi d \mu+\varepsilon, \int \phi d \mu_{n} \geq \int \phi d \mu-\varepsilon .
$$

Therefore, for $n$ this large,

$$
\mu(U), \mu_{n}(U) \in\left[\int \phi d \mu-\varepsilon, \int \psi d \mu+\varepsilon\right]
$$

and so

$$
\left|\mu(U)-\mu_{n}(U)\right|<3 \varepsilon
$$

since $\varepsilon$ is arbitrary, this proves the lemma.
Definition 61.4.2 Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $X: \Omega \rightarrow E$ be a random variable where here $E$ is some topological space. Then one can define a probability measure, $\lambda_{X}$ on $\mathscr{B}(E)$ as follows:

$$
\lambda_{X}(F) \equiv P([X \in F])
$$

More generally, if $\mu$ is a probability measure on $\mathscr{B}(E)$, and $X$ is a random variable defined on a probability space, $\mathscr{L}(X)=\mu$ means

$$
\mu(F) \equiv P([X \in F])
$$

The following amazing theorem is due to Skorokhod. It starts with a measure, $\mu$ on $\mathscr{B}(E)$ and produces a random variable, $X$ for which $\mathscr{L}(X)=\mu$. It also has something to say about the convergence of a sequence of such random variables.

Theorem 61.4.3 Let $E$ be a separable complete metric space and let $\left\{\mu_{n}\right\}$ be a sequence of Borel probability measures defined on $\mathscr{B}(E)$ such that $\mu_{n}$ converges weakly to $\mu$ another probability measure on $\mathscr{B}(E)$. Then there exist random variables, $X_{n}, X$ defined on the probability space, $([0,1), \mathscr{B}([0,1)), m)$ where $m$ is one dimensional Lebesgue measure such that

$$
\begin{equation*}
\mathscr{L}(X)=\mu, \mathscr{L}\left(X_{n}\right)=\mu_{n} \tag{61.4.8}
\end{equation*}
$$

each random variable, $X, X_{n}$ is continuous off a set of measure zero, and

$$
X_{n}(\omega) \rightarrow X(\omega) \text { ma.e. }
$$

Proof: Let $\left\{a_{k}\right\}$ be a countable dense subset of $E$.

## Construction of sets in $E$

First I will describe a construction. Letting $C \in \mathscr{B}(E)$ and $r>0$,

$$
\begin{aligned}
C_{1}^{r} & \equiv C \cap B\left(a_{1}, r\right), C_{2}^{r} \equiv B\left(a_{2}, r\right) \cap C \backslash C_{1}^{r}, \cdots, \\
C_{n}^{r} & \equiv B\left(a_{n}, r\right) \cap C \backslash\left(\cup_{k=1}^{n-1} C_{k}^{r}\right) .
\end{aligned}
$$

Thus the sets, $C_{k}^{r}$ for $k=1,2, \cdots$ are disjoint Borel sets whose union is all of $C$. Of course many may be empty.

$$
C_{n\left(\text { index of the }\left\{a_{k}\right\} \text { it is close to }\right)}^{r(\text { size })}
$$

Now let $C=E$, the whole metric space. Also let $\left\{r_{k}\right\}$ be a decreasing sequence of positive numbers which converges to 0 . Let

$$
A_{k} \equiv E_{k}^{r_{1}}, k=1,2, \cdots
$$

Thus $\left\{A_{k}\right\}$ is a sequence of Borel sets, $A_{k} \subseteq B\left(a_{k}, r_{1}\right)$, and the union of the $A_{k}$ equals $E$. For $\left(i_{1}, \cdots, i_{m}\right) \in \mathbb{N}^{m}$, suppose $A_{i_{1}, \cdots, i_{m}}$ has been defined. Then for $k \in \mathbb{N}$,

$$
A_{i_{1}, \cdots, i_{m} k} \equiv\left(A_{i_{1}, \cdots, i_{m}}\right)_{k}^{r_{m+1}}
$$

Thus $A_{i_{1}, \cdots, i_{m} k} \subseteq B\left(a_{k}, r_{m+1}\right)$, is a Borel set, and

$$
\begin{equation*}
\cup_{k=1}^{\infty} A_{i_{1}, \cdots, i_{m} k}=A_{i_{1}, \cdots, i_{m}} \tag{61.4.9}
\end{equation*}
$$

Also note that $A_{i_{1}, \cdots, i_{m}}$ could be empty. This is because $A_{i_{1}, \cdots, i_{m} k} \subseteq B\left(a_{k}, r_{m+1}\right)$ but $A_{i_{1}, \cdots, i_{m}} \subseteq$ $B\left(a_{i_{m}}, r_{m}\right)$ which might have empty intersection with $B\left(a_{k}, r_{m+1}\right)$. Applying 61.4.9 repeatedly,

$$
E=\cup_{i_{1}} \cdots \cup_{i_{m}} A_{i_{1}, \cdots, i_{m}}
$$

and also, the construction shows the Borel sets, $A_{i_{1}, \cdots, i_{m}}$ are disjoint. Note that to get $A_{i_{1}, \cdots, i_{m} k}$, you do to $A_{i_{1}, \cdots, i_{m}}$ what was done for $E$ but you consider smaller sized pieces.

## Construction of intervals depending on the measure

Next I will construct intervals, $I_{i_{1}, \cdots, i_{n}}^{V}$ in $[0,1)$ corresponding to these $A_{i_{1}, \cdots, i_{n}}$. In what follows, $v=\mu_{n}$ or $\mu$. These intervals will depend on the measure chosen as indicated in the notation.

$$
I_{1}^{v} \equiv\left[0, v\left(A_{1}\right)\right), \cdots, I_{j}^{v} \equiv\left[\sum_{k=1}^{j-1} v\left(A_{k}\right), \sum_{k=1}^{j} v\left(A_{k}\right)\right)
$$

for $j=1,2, \cdots$. Note these are disjoint intervals whose union is $[0,1)$. Also note

$$
m\left(I_{j}^{V}\right)=v\left(A_{j}\right)
$$

The endpoints of these intervals as well as their lengths depend on the measures of the sets $A_{k}$. Now supposing $I_{i_{1}, \cdots, i_{m}}^{v}=[\alpha, \beta)$ where $\beta-\alpha=v\left(A_{i_{1} \cdots, i_{m}}\right)$, define

$$
I_{i_{1} \cdots, i_{m}, j}^{v} \equiv\left[\alpha+\sum_{k=1}^{j-1} v\left(A_{i_{1} \cdots, i_{m}, k}\right), \alpha+\sum_{k=1}^{j} v\left(A_{i_{1} \cdots, i_{m}, k}\right)\right)
$$

Thus $m\left(I_{i_{1} \cdots, i_{m}, j}^{v}\right)=v\left(A_{i_{1} \cdots, i_{m}, j}\right)$ and

$$
v\left(A_{i_{1} \cdots, i_{m}}\right)=\sum_{k=1}^{\infty} v\left(A_{i_{1} \cdots, i_{m}, k}\right)=\sum_{k=1}^{\infty} m\left(I_{i_{1} \cdots, i_{m}, k}^{v}\right)=\beta-\alpha
$$

the intervals, $I_{i_{1} \cdots, i_{m}, j}^{v}$ being disjoint and

$$
I_{i_{1} \cdots, i_{m}}^{V}=\cup_{j=1}^{\infty} I_{i_{1} \cdots, i_{m}, j}^{V}
$$

These intervals satisfy the same inclusion properties as the sets $\left\{A_{i_{1}, \cdots, i_{m}}\right\}$. They are just on $[0,1)$ rather than on $E$. The intervals $I_{i_{1} \cdots, i_{m} k}^{V}$ correspond to the sets $A_{i_{1} \cdots, i_{m}, k}$ and in fact the Lebesgue measure of the interval is the same as $v\left(A_{i_{1} \cdots, i_{m}, k}\right)$.

## Choosing the sequence $\left\{r_{k}\right\}$ in an auspicious manner

There are at most countably many positive numbers, $r$ such that for $v=\mu_{n}$ or $\mu$,

$$
v\left(\partial B\left(a_{i}, r\right)\right)>0
$$

This is because $v$ is a finite measure. Taking the countable union of these countable sets, there are only countably many $r$ such that $v\left(\partial B\left(a_{i}, r\right)\right)>0$ for some $a_{i}$. Let the sequence avoid all these bad values of $r$. Thus for

$$
F \equiv \cup_{m=1}^{\infty} \cup_{k=1}^{\infty} \partial B\left(a_{k}, r_{m}\right)
$$

and $v=\mu$ or $\mu_{n}, v(F)=0$. Here the $r_{m}$ are all good values such that for all $k, m, \partial B\left(a_{k}, r_{m}\right)$ has $\mu$ measure zero and $\mu_{n}$ measure zero.

Claim 1: $\partial A_{i_{1}, \cdots, i_{k}} \subseteq F$. This really follows from the construction. However, the details follow.

Proof of claim: Suppose $C$ is a Borel set for which $\partial C \subseteq F$. I need to show $\partial C_{k}^{r_{i}} \in F$. First consider $k=1$. Then $C_{1}^{r_{i}} \equiv B\left(a_{1}, r_{i}\right) \cap C$. If $x \in \partial C_{1}^{r_{i}}$, then $B(x, \delta)$ contains points of $B\left(a_{1}, r_{i}\right) \cap C$ and points of $B\left(a_{1}, r_{i}\right)^{C} \cup C^{C}$ for every $\delta>0$. First suppose $x \in B\left(a_{1}, r_{i}\right)$. Then a small enough neighborhood of $x$ has no points of $B\left(a_{1}, r_{i}\right)^{C}$ and so every $B(x, \delta)$ has points of $C$ and points of $C^{C}$ so that $x \in \partial C \subseteq F$ by assumption. If $x \in \partial C_{1}^{r_{i}}$, then it can't happen that $\left\|x-a_{1}\right\|>r_{i}$ because then there would be a neighborhood of $x$ having no points of $C_{1}^{r_{i}}$. The only other case to consider is that $\left\|x-a_{i}\right\|=r_{i}$ but this says $x \in F$. Now assume $\partial C_{j}^{r_{i}} \subseteq F$ for $j \leq k-1$ and consider $\partial C_{k}^{r_{i}}$.

$$
\begin{align*}
C_{k}^{r_{i}} & \equiv B\left(a_{k}, r_{i}\right) \cap C \backslash \cup_{j=1}^{k-1} C_{j}^{r_{i}} \\
& =B\left(a_{k}, r_{i}\right) \cap C \cap\left(\cap_{j=1}^{k-1}\left(C_{j}^{r_{i}}\right)^{C}\right) \tag{61.4.10}
\end{align*}
$$

Consider $x \in \partial C_{k}^{r_{i}}$. If $x \in \operatorname{int}\left(B\left(a_{k}, r_{i}\right) \cap C\right)($ int $\equiv$ interior $)$ then a small enough ball about $x$ contains no points of $\left(B\left(a_{k}, r_{i}\right) \cap C\right)^{C}$ and so every ball about $x$ must contain points of

$$
\left(\cap_{j=1}^{k-1}\left(C_{j}^{r_{i}}\right)^{C}\right)^{C}=\cup_{j=1}^{k-1} C_{j}^{r_{i}}
$$

Since there are only finitely many sets in the union, there exists $s \leq k-1$ such that every ball about $x$ contains points of $C_{s}^{r_{i}}$ but from 61.4.10, every ball about $x$ contains points of $\left(C_{s}^{r_{i}}\right)^{C}$ which implies $x \in \partial C_{s}^{r_{i}} \subseteq F$ by induction. It is not possible that $d\left(x, a_{k}\right)>r_{i}$ and yet have $x$ in $\partial C_{k}^{r_{i}}$. This follows from the description in 61.4.10. If $d\left(x, a_{k}\right)=r_{i}$ then by definition, $x \in F$. The only other case to consider is that $x \notin \operatorname{int}\left(B\left(a_{k}, r_{i}\right) \cap C\right)$ but $x \in B\left(a_{k}, r_{i}\right)$. From 61.4.10, every ball about $x$ contains points of $C$. However, since $x \in B\left(a_{k}, r_{i}\right)$, a small enough ball is contained in $B\left(a_{k}, r_{i}\right)$. Therefore, every ball about $x$ must also contain points of $C^{C}$ since otherwise, $x \in \operatorname{int}\left(B\left(a_{k}, r_{i}\right) \cap C\right)$. Thus $x \in \partial C \subseteq F$ by assumption. Now apply what was just shown to the case where $C=E$, the whole space. In this case, $\partial E \subseteq F$ because $\partial E=\emptyset$. Then keep applying what was just shown to the $A_{i_{1}, \cdots, i_{n}}$. This proves the claim.

From the claim, $v\left(\operatorname{int}\left(A_{i_{1}, \cdots, i_{n}}\right)\right)=v\left(A_{i_{1}, \cdots, i_{n}}\right)$ whenever $v=\mu$ or $\mu_{n}$. This is because that in $A_{i_{1}, \cdots, i_{n}}$ which is not in $\operatorname{int}\left(A_{i_{1}, \cdots, i_{n}}\right)$ is in $F$ which has measure zero.

## Some functions on $[0,1)$

By the axiom of choice, there exists $x_{i_{1}, \cdots, i_{m}} \in \operatorname{int}\left(A_{i_{1}, \cdots, i_{m}}\right)$ whenever

$$
\operatorname{int}\left(A_{i_{1}, \cdots, i_{m}}\right) \neq \emptyset
$$

For $v=\mu_{n}$ or $\mu$, define the following functions. For $\omega \in I_{i_{1}, \cdots, i_{m}}^{v}$

$$
Z_{m}^{v}(\omega) \equiv x_{i_{1}, \cdots, i_{m}}
$$

This defines the functions, $Z_{m}^{\mu_{n}}$ and $Z_{m}^{\mu}$. Note these functions have the same values but on slightly different intervals. Here is an important claim.

Claim 2 (Limit on $\mu_{n}$ ): For a.e. $\omega \in[0,1), \lim _{n \rightarrow \infty} Z_{m}^{\mu_{n}}(\omega)=Z_{m}^{\mu}(\omega)$.
Proof of the claim: This follows from the weak convergence of $\mu_{n}$ to $\mu$ and Lemma 61.4.1. This lemma implies $\mu_{n}\left(\operatorname{int}\left(A_{i_{1}, \cdots, i_{m}}\right)\right) \rightarrow \mu\left(\operatorname{int}\left(A_{i_{1}, \cdots, i_{m}}\right)\right)$. Thus by the construction described above, $\mu_{n}\left(A_{i_{1}, \cdots, i_{m}}\right) \rightarrow \mu\left(A_{i_{1}}, \cdots, i_{m}\right)$ because of claim 1 and the construction of $F$ in which it is always a set of measure zero. It follows that if $\omega \in \operatorname{int}\left(I_{i_{1}, \cdots, i_{m}}^{\mu}\right)$, then for all $n$ large enough, $\omega \in \operatorname{int}\left(I_{i_{1}, \cdots, i_{m}}^{\mu_{n}}\right)$ and so $Z_{m}^{\mu_{n}}(\omega)=Z_{m}^{\mu}(\omega)$. Note this convergence is very far from being uniform.

Claim 3 (Limit on size of sets, fixed measure): For $v=\mu_{n}$ or $\mu,\left\{Z_{m}^{v}\right\}_{m=1}^{\infty}$ is uniformly Cauchy independent of $v$.

Proof of the claim: For $\omega \in I_{i_{1}, \cdots, i_{m}}^{V}$, then by the construction,

$$
\omega \in I_{i_{1}, \cdots, i_{m}, i_{m+1} \cdots, i_{n}}^{V}
$$

for some $i_{m+1} \cdots, i_{n}$. Therefore, $Z_{m}^{v}(\omega)$ and $Z_{n}^{v}(\omega)$ are both contained in $A_{i_{1}, \cdots, i_{m}}$ which is contained in $B\left(a_{i_{m}}, r_{m}\right)$. Since $\omega \in[0,1)$ was arbitrary, and $r_{m} \rightarrow 0$, it follows these functions are uniformly Cauchy as claimed.

Let $X^{\nu}(\omega)=\lim _{m \rightarrow \infty} Z_{m}^{\nu}(\omega)$. Since each $Z_{m}^{V}$ is continuous off a set of measure zero, it follows from the uniform convergence that $X^{v}$ is also continuous off a set of measure zero.

Claim 4: For a.e. $\omega$,

$$
\lim _{n \rightarrow \infty} X^{\mu_{n}}(\omega)=X^{\mu}(\omega)
$$

Proof of the claim: From Claim 3 and letting $\varepsilon>0$ be given, there exists $m$ large enough that for all $n$,

$$
\sup _{\omega} d\left(Z_{m}^{\mu_{n}}(\omega), X^{\mu_{n}}(\omega)\right)<\varepsilon / 3, \sup _{\omega} d\left(Z_{m}^{\mu}(\omega), X^{\mu}(\omega)\right)<\varepsilon / 3 .
$$

for $\omega$ off a set of measure zero. Now pick $\omega \in[0,1)$ such that $\omega$ is not equal to any of the end points of any of the intervals, $\left\{I_{i_{1}, \cdots, i_{m}}^{v}\right\}$, this countable set of endpoints, a set of Lebesgue measure zero. Then by Claim 2, there exists $N$ such that if $n \geq N$, then $d\left(Z_{m}^{\mu_{n}}(\omega), Z_{m}^{\mu}(\omega)\right)<\varepsilon / 3$. Therefore, for such $n$ and this $\omega$,

$$
\begin{aligned}
d\left(X^{\mu_{n}}(\omega), X^{\mu}(\omega)\right) \leq & d\left(X^{\mu_{n}}(\omega), Z_{m}^{\mu_{n}}(\omega)\right)+d\left(Z_{m}^{\mu_{n}}(\omega), Z_{m}^{\mu}(\omega)\right) \\
& +d\left(Z_{m}^{\mu}(\omega), X^{\mu}(\omega)\right) \\
< & \varepsilon / 3+\varepsilon / 3+\varepsilon / 3=\varepsilon
\end{aligned}
$$

This proves the claim.

$$
\text { Showing } \mathscr{L}\left(X^{v}\right)=v
$$

This has mostly proved the theorem except for the claim that $\mathscr{L}\left(X^{v}\right)=v$ for $v=$ $\mu_{n}$ and $\mu$. To do this, I will first show $m\left(\left(X^{v}\right)^{-1}\left(\partial A_{i_{1}, \cdots, i_{m}}\right)\right)=0$. By the construction, $v\left(\partial A_{i_{1}, \cdots, i_{m}}\right)=0$. Let $\varepsilon>0$ be given and let $\delta>0$ be small enough that

$$
H_{\delta} \equiv\left\{x \in E: \operatorname{dist}\left(x, \partial A_{i_{1}, \cdots, i_{m}}\right) \leq \delta\right\}
$$

is a set of measure less than $\varepsilon / 2$. Denote by $\mathscr{G}_{k}$ the sets of the form $A_{i_{1}, \cdots, i_{k}}$ where $\left(i_{1}, \cdots, i_{k}\right) \in \mathbb{N}^{k}$. Recall also that corresponding to $A_{i_{1}, \cdots, i_{k}}$ is an interval, $I_{i_{1}, \cdots, i_{k}}^{v}$ having length equal to $v\left(A_{i_{1}, \cdots, i_{k}}\right)$. Denote by $\mathscr{B}_{k}$ those sets of $\mathscr{G}_{k}$ which have nonempty intersection with $H_{\delta}$ and let the corresponding intervals be denoted by $\mathscr{I}_{k}^{v}$. If $\omega \notin \cup \mathscr{I}_{k}^{v}$, then from the construction, $Z_{p}^{v}(\omega)$ is at a distance of at least $\delta$ from $\partial A_{i_{1}, \cdots, i_{m}}$ for all $p \geq k$. (If $Z_{k}^{v}(\omega)$ were in some set of $\mathscr{B}_{k}$, this would require $\omega$ to be in the corresponding $I_{k}^{v}$ and it is assumed this does not happen. Then for any $p>k, Z_{p}^{v}(\omega)$ cannot be in any set of $\mathscr{G}_{p}$ which intersects $H_{\delta}$ either. If it did, you would need to have $\omega \notin \cup \mathscr{I}_{p}^{v}$ but all of these intervals are inside the intervals $\mathscr{I}_{k}^{V}$.) Passing to the limit as $p \rightarrow \infty$, it follows $X^{v}(\omega) \notin \partial A_{i_{1}, \cdots, i_{m}}$. Therefore,

$$
\left(X^{v}\right)^{-1}\left(\partial A_{i_{1}, \cdots, i_{m}}\right) \subseteq \cup \mathscr{I}_{k}^{v}
$$

Recall that $A_{i_{1}, \cdots, i_{k}} \subseteq B\left(a_{i_{k}}, r_{k}\right)$ and the $r_{k} \rightarrow 0$. Therefore, if $k$ is large enough,

$$
v\left(\cup \mathscr{B}_{k}\right)<\varepsilon
$$

because $\cup \mathscr{B}_{k}$ approximates $H_{\delta}$ closely (In fact, $\cap_{k=1}^{\infty}\left(\cup \mathscr{B}_{k}\right)=H_{\delta}$.). Therefore,

$$
\begin{aligned}
m\left(\left(X^{v}\right)^{-1}\left(\partial A_{i_{1}, \cdots, i_{m}}\right)\right) & \leq m\left(\cup \mathscr{I}_{k}^{v}\right) \\
& =\sum_{I_{i_{1}, \cdots, i_{k}}^{v} \in \mathscr{I}_{k}^{v}} m\left(I_{i_{1}, \cdots, i_{k}}^{v}\right) \\
& =\sum_{A_{i_{1}, \cdots, i_{k} \in \mathscr{B}_{k}} v\left(A_{i_{1}, \cdots, i_{k}}\right)}=v\left(\cup \mathscr{B}_{k}\right)<\varepsilon .
\end{aligned}
$$

Since $\varepsilon>0$ is arbitrary, this shows $m\left(\left(X^{v}\right)^{-1}\left(\partial A_{i_{1}, \cdots, i_{m}}\right)\right)=0$.
If $\omega \in I_{i_{1}, \cdots, i_{m}}^{V}$, then from the construction, $Z_{p}^{V}(\omega) \in \operatorname{int}\left(A_{i_{1}, \cdots, i_{m}}\right)$ for all $p \geq k$. Therefore, taking a limit, as $p \rightarrow \infty$,

$$
X^{V}(\omega) \in \operatorname{int}\left(A_{i_{1}, \cdots, i_{m}}\right) \cup \partial A_{i_{1}, \cdots, i_{m}}
$$

and so

$$
I_{i_{1}, \cdots, i_{m}}^{V} \subseteq\left(X^{v}\right)^{-1}\left(\operatorname{int}\left(A_{i_{1}, \cdots, i_{m}}\right) \cup \partial A_{i_{1}, \cdots, i_{m}}\right)
$$

but also, if $X^{v}(\omega) \in \operatorname{int}\left(A_{i_{1}, \cdots, i_{m}}\right)$, then $Z_{p}^{v}(\omega) \in \operatorname{int}\left(A_{i_{1}, \cdots, i_{m}}\right)$ for all $p$ large enough and so

$$
\begin{aligned}
& \left(X^{v}\right)^{-1}\left(\operatorname{int}\left(A_{i_{1}, \cdots, i_{m}}\right)\right) \\
\subseteq & I_{i_{1}, \cdots, i_{m}}^{v} \subseteq\left(X^{v}\right)^{-1}\left(\operatorname{int}\left(A_{i_{1}, \cdots, i_{m}}\right) \cup \partial A_{i_{1}, \cdots, i_{m}}\right)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& m\left(\left(X^{v}\right)^{-1}\left(\operatorname{int}\left(A_{i_{1}}, \cdots, i_{m}\right)\right)\right) \\
\leq & m\left(I_{i_{1}, \cdots, i_{m}}^{V}\right) \\
\leq & m\left(\left(X^{v}\right)^{-1}\left(\operatorname{int}\left(A_{i_{1}, \cdots, i_{m}}\right)\right)\right)+m\left(\left(X^{v}\right)^{-1}\left(\partial A_{i_{1}, \cdots, i_{m}}\right)\right) \\
= & m\left(\left(X^{v}\right)^{-1}\left(\operatorname{int}\left(A_{i_{1}, \cdots, i_{m}}\right)\right)\right)
\end{aligned}
$$

which shows

$$
\begin{equation*}
m\left(\left(X^{v}\right)^{-1}\left(\operatorname{int}\left(A_{i_{1}, \cdots, i_{m}}\right)\right)\right)=m\left(I_{i_{1}, \cdots, i_{m}}^{v}\right)=v\left(A_{i_{1}, \cdots, i_{m}}\right) \tag{61.4.11}
\end{equation*}
$$

Also

$$
\begin{aligned}
& m\left(\left(X^{v}\right)^{-1}\left(\operatorname{int}\left(A_{i_{1}, \cdots, i_{m}}\right)\right)\right) \\
\leq & m\left(\left(X^{v}\right)^{-1}\left(A_{i_{1}, \cdots, i_{m}}\right)\right) \\
\leq & m\left(\left(X^{v}\right)^{-1}\left(\operatorname{int}\left(A_{i_{1}, \cdots, i_{m}}\right) \cup \partial A_{i_{1}, \cdots, i_{m}}\right)\right) \\
= & m\left(\left(X^{v}\right)^{-1}\left(\operatorname{int}\left(A_{i_{1}, \cdots, i_{m}}\right)\right)\right)
\end{aligned}
$$

Hence from 61.4.11,

$$
\begin{gather*}
v\left(A_{i_{1}, \cdots, i_{m}}\right)=m\left(\left(X^{v}\right)^{-1}\left(\operatorname{int}\left(A_{i_{1}, \cdots, i_{m}}\right)\right)\right) \\
=m\left(\left(X^{v}\right)^{-1}\left(A_{i_{1}, \cdots, i_{m}}\right)\right) \tag{61.4.12}
\end{gather*}
$$

Now let $U$ be an open set in $E$. Then letting

$$
H_{k}=\left\{x \in U: \operatorname{dist}\left(x, U^{C}\right) \geq r_{k}\right\}
$$

it follows

$$
\cup_{k} H_{k}=U
$$

Next consider the sets of $\mathscr{G}_{k}$ which have nonempty intersection with $H_{k}, \mathscr{H}_{k}$. Then $H_{k}$ is covered by $\mathscr{H}_{k}$ and every set of $\mathscr{H}_{k}$ is contained in $U$, the sets of $\mathscr{H}_{k}$ also being disjoint. Then from 61.4.12,

$$
\begin{aligned}
m\left(\left(X^{v}\right)^{-1}\left(\cup \mathscr{H}_{k}\right)\right) & =\sum_{A \in \mathscr{H}_{k}} m\left(\left(X^{v}\right)^{-1}(A)\right) \\
& =\sum_{A \in \mathscr{H}_{k}} v(A)=v\left(\cup \mathscr{H}_{k}\right)
\end{aligned}
$$

Therefore, letting $k \rightarrow \infty$ and passing to the limit in the above,

$$
m\left(\left(X^{v}\right)^{-1}(U)\right)=v(U)
$$

Since this holds for every open set, it is routine to verify using regularity that it holds for every Borel set and so $\mathscr{L}\left(X^{v}\right)=v$ as claimed.

### 61.5 Bochner's Theorem In Infinite Dimensions

Let $X$ be a real vector space and let $X^{*}$ denote the space of real valued linear mappings defined on $X$. Then you can consider each $x \in X$ as a linear transformation defined on $X^{*}$ by the convention $x^{*} \rightarrow x^{*}(x)$. Now let $\Lambda$ be a Hamel basis. For a description of what one of these is, see Page 2726. It is just the usual notion of a basis. Thus every vector of $X$ is a finite linear combination of vectors of $\Lambda$ in a unique way.

Now consider $\mathbb{R}^{\Lambda}$ the space of all mappings from $\Lambda$ to $\mathbb{R}$. In different notation, this is of the form

$$
\mathbb{R}^{\Lambda} \equiv \prod_{y \in \Lambda} \mathbb{R}
$$

Since $\Lambda$ is a Hamel basis, there exists a one to one and onto mapping, $\theta: X^{*} \rightarrow \mathbb{R}^{\Lambda}$ defined as

$$
\boldsymbol{\theta}\left(x^{*}\right) \equiv \prod_{y \in \Lambda} x^{*}(y)
$$

Now denote by $\sigma(X)$ the smallest $\sigma$ algebra of sets of $X^{*}$ such that each $x$ is measurable with respect to this $\sigma$ algebra. Thus

$$
\left\{x^{*}: x^{*}(x) \in B\right\} \in \sigma(X)
$$

whenever $B$ is a Borel set in $\mathbb{R}$.
Let $\mathscr{E}$ denote the algebra of disjoint unions of sets of $\mathbb{R}^{\Lambda}$ of the form

$$
\prod_{y \in \Lambda} A_{y}
$$

where $A_{y}=\mathbb{R}$ except for finitely many $y$.

Lemma 61.5.1 Let $\mathscr{A}$ denote sets of the form

$$
\left\{x^{*}: \theta\left(x^{*}\right) \in U\right\}
$$

where $U \in \mathscr{E}$. Then $\mathscr{A}$ is an algebra and $\sigma(\mathscr{A})=\sigma(X)$. Also

$$
\left\{\theta^{-1}(U): U \in \sigma(\mathscr{E})\right\}=\sigma(X)
$$

Proof: Since $\mathscr{E}$ is an algebra it is clear $\mathscr{A}$ is also an algebra. Also, $\mathscr{A} \subseteq \sigma(X)$ because you could let $U$ have only one $A_{y}$ not equal to $\mathbb{R}$ and all the others equal to $\mathbb{R}$ and then

$$
\left\{x^{*}: \theta\left(x^{*}\right) \in U\right\}=\left\{x^{*}: y\left(x^{*}\right) \equiv x^{*}(y) \in A_{y}\right\} \in \sigma(X) .
$$

Therefore, $\sigma(\mathscr{A}) \subseteq \sigma(X)$. I need to verify that for an arbitrary $x$, it is measurable with respect to $\sigma(\mathscr{A})$. However, this is true because if $x$ is arbitrary, it is a linear combination of $\left\{y_{1}, \cdots, y_{n}\right\}$, some finite set of functions in $\Lambda$ and so, $x$ being a linear combination of measurable functions implies it is itself measurable.

By definition, $\theta^{-1}(U)$ is in $\mathscr{A}$ whenever $U \in \mathscr{E}$. Now let $\mathscr{G}$ denote those sets, $U$ in $\sigma(\mathscr{E})$ such that $\theta^{-1}(U) \in \sigma(\mathscr{A})$. Then $\mathscr{G}$ is a $\sigma$ algebra which contains $\mathscr{E}$ and so $\mathscr{G} \supseteq \sigma(\mathscr{E}) \supseteq \mathscr{G}$. This proves the last claim. This proves the lemma.

Definition 61.5.2 Let $\psi: X \rightarrow \mathbb{C}$. Then $\psi$ is said to be pseudo continuous if whenever $\left\{x_{1}, \cdots, x_{n}\right\}$ is a finite subset of $X$ and $\mathbf{a}=\left(a_{1}, \cdots, a_{n}\right) \in \mathbb{R}^{n}$,

$$
\mathbf{a} \rightarrow \psi\left(\sum_{k=1}^{n} a_{k} x_{k}\right)
$$

is continuous. $\psi$ is said to be positive definite if

$$
\sum_{j, k} \psi\left(x_{k}-x_{j}\right) \alpha_{k} \overline{\alpha_{j}} \geq 0
$$

$\psi$ is said to be a characteristic function if there exists a probability measure, $\mu$ defined on $\sigma(X)$ such that

$$
\psi(x)=\int_{X^{*}} e^{i x^{*}(x)} d \mu\left(x^{*}\right)
$$

Note that $x^{*} \rightarrow e^{i x^{*}(x)}$ is $\sigma(X)$ measurable.
Using Kolmogorov's extension theorem on Page 59.2.3, there exists a generalization of Bochner's theorem found in [125]. For convenience, here is Kolmogorov's theorem.

Theorem 61.5.3 (Kolmogorov extension theorem) For each finite set

$$
J=\left(t_{1}, \cdots, t_{n}\right) \subseteq I
$$

suppose there exists a Borel probability measure, $v_{J}=v_{t_{1} \cdots t_{n}}$ defined on the Borel sets of $\prod_{t \in J} M_{t}$ for $M_{t}=\mathbb{R}^{n_{t}}$ for $n_{t}$ an integer, such that the following consistency condition holds. If

$$
\left(t_{1}, \cdots, t_{n}\right) \subseteq\left(s_{1}, \cdots, s_{p}\right)
$$

then

$$
\begin{equation*}
v_{t_{1} \cdots t_{n}}\left(F_{t_{1}} \times \cdots \times F_{t_{n}}\right)=v_{s_{1} \cdots s_{p}}\left(G_{s_{1}} \times \cdots \times G_{s_{p}}\right) \tag{61.5.13}
\end{equation*}
$$

where if $s_{i}=t_{j}$, then $G_{s_{i}}=F_{t_{j}}$ and if $s_{i}$ is not equal to any of the indices, $t_{k}$, then $G_{s_{i}}=M_{s_{i}}$. Then for $\mathscr{E}$ defined as in Definition 14.4.1, adjusted so that $\pm \infty$ never appears as any endpoint of any interval, there exists a probability measure, $P$ and a $\sigma$ algebra $\mathscr{F}=\sigma(\mathscr{E})$ such that

$$
\left(\prod_{t \in I} M_{t}, P, \mathscr{F}\right)
$$

is a probability space. Also there exist measurable functions, $X_{s}: \prod_{t \in I} M_{t} \rightarrow M_{s}$ defined as

$$
X_{S} \mathbf{x} \equiv x_{s}
$$

for each $s \in I$ such that for each $\left(t_{1} \cdots t_{n}\right) \subseteq I$,

$$
\begin{gather*}
v_{t_{1} \cdots t_{n}}\left(F_{t_{1}} \times \cdots \times F_{t_{n}}\right)=P\left(\left[X_{t_{1}} \in F_{t_{1}}\right] \cap \cdots \cap\left[X_{t_{n}} \in F_{t_{n}}\right]\right) \\
\quad=P\left(\left(X_{t_{1}}, \cdots, X_{t_{n}}\right) \in \prod_{j=1}^{n} F_{t_{j}}\right)=P\left(\prod_{t \in I} F_{t}\right) \tag{61.5.14}
\end{gather*}
$$

where $F_{t}=M_{t}$ for every $t \notin\left\{t_{1} \cdots t_{n}\right\}$ and $F_{t_{i}}$ is a Borel set. Also if $f$ is a nonnegative function of finitely many variables, $x_{t_{1}}, \cdots, x_{t_{n}}$, measurable with respect to $\mathscr{B}\left(\prod_{j=1}^{n} M_{t_{j}}\right)$, then $f$ is also measurable with respect to $\mathscr{F}$ and

$$
\begin{align*}
& \int_{M_{t_{1} \times \cdots \times M_{t_{n}}}} f\left(x_{t_{1}}, \cdots, x_{t_{n}}\right) d v_{t_{1} \cdots t_{n}} \\
= & \int_{\prod_{t \in I} M_{t}} f\left(x_{t_{1}}, \cdots, x_{t_{n}}\right) d P \tag{61.5.15}
\end{align*}
$$

Theorem 61.5.4 Let $X$ be a real vector space and let $X^{*}$ be the space of linear functionals defined on $X$. Also let $\psi: X \rightarrow \mathbb{C}$. Then $\psi$ is a characteristic function if and only if $\psi(0)=1$ and $\psi$ is pseudo continuous at 0 .

Proof: Suppose first $\psi$ is a characteristic function as just described. I need to show it is positive definite and pseudo continuous. It is obvious $\psi(0)=1$ in this case. Also

$$
\psi\left(\sum_{k} a_{k} x_{k}\right)=\int_{X^{*}} \exp \left(i x^{*}\left(\sum_{k} a_{k} x_{k}\right)\right) d \mu\left(x^{*}\right)
$$

and this is obviously a continuous function of a by the dominated convergence theorem. It only remains to verify the function is positive definite. However,

$$
\sum_{k, j} \exp \left(i x^{*}\left(x_{k}-x_{j}\right)\right) \alpha_{k} \overline{\alpha_{j}}=\sum_{k, j} e^{i x^{*}\left(x_{k}\right)} \alpha_{k} \overline{e^{i x^{*}\left(x_{j}\right)} \alpha_{j}} \geq 0
$$

as in the earlier discussion of what it means to be positive definite given on Page 1938.

Next suppose the conditions hold. Define for $\mathbf{t} \in \mathbb{R}^{n}$ and $\left\{y_{1}, \cdots, y_{n}\right\} \subseteq \Lambda$

$$
\psi_{\left\{y_{1}, \cdots, y_{n}\right\}}(\mathbf{t}) \equiv \psi\left(\sum_{j=1}^{n} t_{j} y_{j}\right) .
$$

Then $\psi_{\left\{y_{1}, \cdots, y_{n}\right\}}$ is continuous at $\mathbf{0}$, equals 1 there, and is positive definite. It follows from Bochner's theorem, Theorem 59.21 .7 on Page 1941 there exists a measure $\mu_{\left\{y_{1}, \cdots, y_{n}\right\}}$ defined on the Borel sets of $\mathbb{R}^{n}$ such that

$$
\begin{equation*}
\psi\left(\sum_{j=1}^{n} t_{j} y_{j}\right)=\psi_{\left\{y_{1}, \cdots, y_{n}\right\}}(\mathbf{t})=\int_{\mathbb{R}^{n}} e^{i \mathbf{t} \cdot \mathbf{x}} d \mu_{\left\{y_{1}, \cdots, y_{n}\right\}}(x) \tag{61.5.16}
\end{equation*}
$$

Thus if $\left\{y_{1}, \cdots, y_{n}, y_{n+1}, \cdots, y_{p}\right\} \subseteq \Lambda$,

$$
\begin{aligned}
\psi\left(\sum_{j=1}^{n} t_{j} y_{j}+\sum_{j=1}^{p-n} s_{j} y_{j+n}\right) & =\psi_{\left\{y_{1}, \cdots, y_{p}\right\}}(\mathbf{t}, \mathbf{s}) \\
& =\int_{\mathbb{R}^{p-n}} e^{i \mathbf{s} \cdot \mathbf{x}} \int_{\mathbb{R}^{n}} e^{i \cdot \mathbf{x}} d \mu_{\left\{y_{1}, \cdots, y_{p}\right\}}(x)
\end{aligned}
$$

I need to verify the measures are consistent to use Kolmogorov's theorem. Specifically, I need to show

$$
\mu_{\left\{y_{1}, \cdots, y_{p}\right\}}\left(A \times \mathbb{R}^{p-n}\right)=\mu_{\left\{y_{1}, \cdots, y_{n}\right\}}(A)
$$

Letting

$$
\lambda(A)=\mu_{\left\{y_{1}, \cdots, y_{p}\right\}}\left(A \times \mathbb{R}^{p-n}\right)
$$

it follows

$$
\begin{aligned}
\int_{\mathbb{R}^{n}} e^{i \cdot \mathbf{x}} d \lambda & =\int_{\mathbb{R}^{p-n}} \int_{\mathbb{R}^{n}} e^{i \mathbf{t} \cdot \mathbf{x}} d \mu_{\left\{y_{1}, \cdots, y_{p}\right\}}(x) \\
& =\int_{\mathbb{R}^{p-n}} e^{i \mathbf{0} \cdot \mathbf{x}} \int_{\mathbb{R}^{n}} e^{i \mathbf{t} \cdot \mathbf{x}} d \mu_{\left\{y_{1}, \cdots, y_{p}\right\}}(x) \\
& =\psi\left(\sum_{j=1}^{n} t_{j} y_{j}+\sum_{j=1}^{p-n} 0 y_{j+n}\right) \\
& =\psi\left(\sum_{j=1}^{n} t_{j} y_{j}\right) \\
& =\int_{\mathbb{R}^{n}} e^{i \mathbf{t} \cdot \mathbf{x}} d \mu_{\left\{y_{1}, \cdots, y_{n}\right\}}(x)
\end{aligned}
$$

and so, by uniqueness of characteristic functions,

$$
\lambda=\mu_{\left\{y_{1}, \cdots, y_{n}\right\}}
$$

which verifies the necessary consistency condition for Kolmogorov's theorem.

It follows there exists a probability measure $\mu$ defined on $\sigma(\mathscr{E})$ and random variables, $X_{y}$ for each $y \in \Lambda$ such that whenever $\left\{y_{1}, \cdots, y_{p}\right\} \subseteq \Lambda$,

$$
\mu_{\left\{y_{1}, \cdots, y_{p}\right\}}\left(A_{y_{1}} \times \cdots \times A_{y_{n}}\right)=\mu\left(\prod_{y \in \Lambda} A_{y}\right)
$$

where $A_{y}=\mathbb{R}$ whenever $y \notin\left\{y_{1}, \cdots, y_{p}\right\}$. This defines a measure on $\sigma(\mathscr{E})$ which consists of sets of $\mathbb{R}^{\Lambda}$.

By Lemma 61.5 .1 it follows $\left\{\theta^{-1}(U): U \in \sigma(\mathscr{E})\right\}=\sigma(\mathscr{A})$ which equals $\sigma(X)$. Define $v$ on $\sigma(X)$ by

$$
v(F) \equiv \mu(\theta F)
$$

Thus $v$ is a measure because $\mu$ is and $\theta$ is one to one.
I need to check whether $v$ works. Let $x=\sum_{k=1}^{m} t_{k} y_{k}$ and let a typical element of $\mathbb{R}^{\Lambda}$ be denoted by $\mathbf{z}$. Then by Kolmogorov's theorem above,

$$
\begin{gathered}
\int_{X^{*}} \exp \left(i x^{*}\left(\sum_{k=1}^{m} t_{k} y_{k}\right)\right) d v=\int_{X^{*}} \exp \left(i\left(\sum_{k=1}^{m} t_{k} x^{*}\left(y_{k}\right)\right)\right) d v \\
=\int_{X^{*}} \exp \left(i\left(\sum_{k=1}^{m} t_{k} \pi_{y_{k}}\left(\theta x^{*}\right)\right)\right) d v=\int_{\mathbb{R}^{\Lambda}} \exp \left(i\left(\sum_{k=1}^{m} t_{k} \pi_{y_{k}} \mathbf{z}\right)\right) d \mu \\
=\int_{\mathbb{R}^{m}} \exp (i(\mathbf{t} \cdot \mathbf{x})) d \mu_{\left\{y_{1}, \cdots, y_{m}\right\}}(x)=\psi\left(\sum_{k=1}^{m} t_{k} y_{k}\right)
\end{gathered}
$$

where the last equality comes from 61.5.16. Since $\Lambda$ is a Hamel basis, it follows that for every $x \in X$,

$$
\psi(x)=\int_{X^{*}} \exp \left(i x^{*}(x)\right) d \nu
$$

This proves the theorem.

### 61.6 The Multivariate Normal Distribution

Here I give a review of the main theorems and definitions about multivariate normal random variables. Recall that for a random vector (variable), $\mathbf{X}$ having values in $\mathbb{R}^{p}, \lambda_{\mathbf{X}}$ is the law of $\mathbf{X}$ defined by

$$
P([\mathbf{X} \in E])=\lambda_{\mathbf{X}}(E)
$$

for all $E$ a Borel set in $\mathbb{R}^{p}$. In different notaion, $\mathscr{L}(\mathbf{X})=\lambda_{\mathbf{x}}$. Then the following definitions and theorems are proved and presented starting on Page 1908

Definition 61.6.1 A random vector, $\mathbf{X}$, with values in $\mathbb{R}^{p}$ has a multivariate normal distribution written as $\mathbf{X} \sim N_{p}(\mathbf{m}, \Sigma)$ if for all Borel $E \subseteq \mathbb{R}^{p}$,

$$
\lambda_{\mathbf{x}}(E)=\int_{\mathbb{R}^{p}} \mathscr{X}_{E}(\mathbf{x}) \frac{1}{(2 \pi)^{p / 2} \operatorname{det}(\Sigma)^{1 / 2}} e^{\frac{-1}{2}(\mathbf{x}-\mathbf{m})^{*} \Sigma^{-1}(\mathbf{x}-\mathbf{m})} d x
$$

for $\mu$ a given vector and $\Sigma$ a given positive definite symmetric matrix.

Theorem 61.6.2 For $\mathbf{X} \sim N_{p}(\mathbf{m}, \Sigma), \mathbf{m}=E(\mathbf{X})$ and

$$
\Sigma=E\left((\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^{*}\right) .
$$

Theorem 61.6.3 Suppose $\mathbf{X}_{1} \sim N_{p}\left(\mathbf{m}_{1}, \Sigma_{1}\right), \mathbf{X}_{2} \sim N_{p}\left(\mathbf{m}_{2}, \Sigma_{2}\right)$ and the two random vectors are independent. Then

$$
\begin{equation*}
\mathbf{X}_{1}+\mathbf{X}_{2} \sim N_{p}\left(\mathbf{m}_{1}+\mathbf{m}_{2}, \Sigma_{1}+\Sigma_{2}\right) \tag{61.6.17}
\end{equation*}
$$

Also, if $\mathbf{X} \sim N_{p}(\mathbf{m}, \Sigma)$ then $-\mathbf{X} \sim N_{p}(-\mathbf{m}, \Sigma)$. Furthermore, if $\mathbf{X} \sim N_{p}(\mathbf{m}, \Sigma)$ then

$$
\begin{equation*}
E\left(e^{i \mathbf{t} \cdot \mathbf{X}}\right)=e^{i \mathbf{t} \cdot \mathbf{m}} e^{-\frac{1}{2} \mathbf{t}^{*} \Sigma \mathbf{t}} \tag{61.6.18}
\end{equation*}
$$

Also if $a$ is a constant and $\mathbf{X} \sim N_{p}(\mathbf{m}, \Sigma)$ then $a \mathbf{X} \sim N_{p}\left(a \mathbf{m}, a^{2} \Sigma\right)$.
Following [103] a random vector has a generalized normal distribution if its characteristic function is given as

$$
\begin{equation*}
e^{i \mathbf{t} \cdot \mathbf{m}} e^{-\frac{1}{2} \mathbf{t}^{*} \Sigma \mathbf{t}} \tag{61.6.19}
\end{equation*}
$$

where $\Sigma$ is symmetric and has nonnegative eigenvalues. For a random real valued variable, $\mathbf{m}$ is scalar and so is $\Sigma$ so the characteristic function of such a generalized normally distributed random variable is

$$
\begin{equation*}
e^{i t m} e^{-\frac{1}{2} t^{2} \sigma^{2}} \tag{61.6.20}
\end{equation*}
$$

These generalized normal distributions do not require $\Sigma$ to be invertible, only that the eigenvalues be nonnegative. In one dimension this would correspond the characteristic function of a dirac measure having point mass 1 at $m$. In higher dimensions, it could be a mixture of such things with more familiar things. I will often not bother to distinguish between generalized normal and normal distributions.

Here are some other interesting results about normal distributions found in [103]. The next theorem has to do with the question whether a random vector is normally distributed in the above generalized sense. It is proved on Page 1911.

Theorem 61.6.4 Let $\mathbf{X}=\left(X_{1}, \cdots, X_{p}\right)$ where each $X_{i}$ is a real valued random variable. Then $\mathbf{X}$ is normally distributed in the above generalized sense if and only if every linear combination, $\sum_{j=1}^{p} a_{i} X_{i}$ is normally distributed. In this case the mean of $\mathbf{X}$ is

$$
\mathbf{m}=\left(E\left(X_{1}\right), \cdots, E\left(X_{p}\right)\right)
$$

and the covariance matrix for $\mathbf{X}$ is

$$
\Sigma_{j k}=E\left(\left(X_{j}-m_{j}\right)\left(X_{k}-m_{k}\right)\right)
$$

where $m_{j}=E\left(X_{j}\right)$.
Also proved there is the interesting corollary listed next.
Corollary 61.6.5 Let $\mathbf{X}=\left(X_{1}, \cdots, X_{p}\right), \mathbf{Y}=\left(Y_{1}, \cdots, Y_{p}\right)$ where each $X_{i}, Y_{i}$ is a real valued random variable. Suppose also that for every $\mathbf{a} \in \mathbb{R}^{p}, \mathbf{a} \cdot \mathbf{X}$ and $\mathbf{a} \cdot \mathbf{Y}$ are both normally distributed with the same mean and variance. Then $\mathbf{X}$ and $\mathbf{Y}$ are both multivariate normal random vectors with the same mean and variance.

Theorem 61.6.6 Suppose $\mathbf{X}=\left(X_{1}, \cdots, X_{p}\right)$ is normally distributed with mean $\mathbf{m}$ and covariance $\Sigma$. Then if $X_{1}$ is uncorrelated with any of the $X_{i}$,

$$
E\left(\left(X_{1}-m_{1}\right)\left(X_{j}-m_{j}\right)\right)=0 \text { for } j>1
$$

then $X_{1}$ and $\left(X_{2}, \cdots, X_{p}\right)$ are both normally distributed and the two random vectors are independent. Here $m_{j} \equiv E\left(X_{j}\right)$.

Next I will consider the question of existence of independent random variables having a given law.

Lemma 61.6.7 Let $\mu$ be a probability measure on $\mathscr{B}(E)$, the Borel subsets of a separable real Banach space. Then there exists a probability space $(\Omega, \mathscr{F}, P)$ and two independent random variables, $X, Y$ mapping $\Omega$ to $E$ such that $\mathscr{L}(X)=\mathscr{L}(Y)=\mu$.

Proof: First note that if $A, B$ are Borel sets of $E$ then $A \times B$ is a Borel set in $E \times E$ where the norm on $E \times E$ is given by

$$
\|(x, y)\| \equiv \max (\|x\|,\|y\|)
$$

This can be proved by letting $A$ be open and considering

$$
\mathscr{G} \equiv\{B \in \mathscr{B}(E): A \times B \in \mathscr{B}(A \times B)\}
$$

Show $\mathscr{G}$ is a $\sigma$ algebra and it contains the open sets. Therefore, this will show $A \times B$ is in $\mathscr{B}(A \times B)$ whenever $A$ is open and $B$ is Borel. Next repeat a similar argument to show that this is true whenever either set is Borel. Since $E$ is separable, it is completely separable and so is $E \times E$. Thus every open set in $E \times E$ is the union of balls from a countable set. However, these balls are of the form $B_{1} \times B_{2}$ where $B_{i}$ is a ball in $E$. Now let

$$
\mathscr{K} \equiv\{A \times B: A, B \text { are Borel }\}
$$

Then $\mathscr{K} \subseteq \mathscr{B}(E \times E)$ as was just shown and also every open set from $E \times E$ is in $\sigma(\mathscr{K})$. It follows $\sigma(\mathscr{K})$ equals the $\sigma$ algebra of product measurable sets, $\mathscr{B}(E) \times \mathscr{B}(E)$ and you can consider the product measure, $\mu \times \mu$. By Skorokhod's theorem, Theorem 61.4.3, there exists $(X, Y)$ a random variable with values in $E \times E$ and a probability space, $(\Omega, \mathscr{F}, P)$ such that $\mathscr{L}((X, Y))=\mu \times \mu$. Then for $A, B$ Borel sets in $E$

$$
P(X \in A, Y \in B)=(\mu \times \mu)(A \times B)=\mu(A) \mu(B)
$$

Also, $P(X \in A)=P(X \in A, Y \in E)=\mu(A)$ and similarly, $P(Y \in B)=\mu(B)$ showing $\mathscr{L}(X)=\mathscr{L}(Y)=\mu$ and $X, Y$ are independent.

Now here is an interesting theorem in [36].
Theorem 61.6.8 Suppose $v$ is a probability measure on the Borel sets of $\mathbb{R}$ and suppose that $\xi$ and $\zeta$ are independent random variables such that $\mathscr{L}(\xi)=\mathscr{L}(\zeta)=v$ and whenever $\alpha^{2}+\beta^{2}=1$ it follows $\mathscr{L}(\alpha \xi+\beta \zeta)=v$. Then

$$
\mathscr{L}(\xi)=N\left(0, \sigma^{2}\right)
$$

for some $\sigma \geq 0$. Also if $\mathscr{L}(\xi)=\mathscr{L}(\zeta)=N\left(0, \sigma^{2}\right)$ where $\xi, \zeta$ are independent, then if $\alpha^{2}+\beta^{2}=1$, it follows $\mathscr{L}(\alpha \xi+\beta \zeta)=N\left(0, \sigma^{2}\right)$.

Proof: Let $\xi, \zeta$ be independent random variables with $\mathscr{L}(\xi)=\mathscr{L}(\zeta)=v$ and whenever $\alpha^{2}+\beta^{2}=1$ it follows $\mathscr{L}(\alpha \xi+\beta \zeta)=v$.

By independence of $\xi$ and $\zeta$,

$$
\begin{aligned}
\phi_{v}(t) & \equiv \phi_{\alpha \xi+\alpha \zeta}(t) \\
& =E\left(e^{i t(\alpha \xi+\beta \zeta)}\right) \\
& =E\left(e^{i t \alpha \xi}\right) E\left(e^{i t \beta \zeta}\right) \\
& =\phi_{\xi}(\alpha t) \phi_{\zeta}(\beta t) \\
& \equiv \phi_{v}(\alpha t) \phi_{v}(\beta t)
\end{aligned}
$$

In simpler terms and suppressing the subscript,

$$
\begin{equation*}
\phi(t)=\phi(\cos (\theta) t) \phi(\sin (\theta) t) \tag{61.6.21}
\end{equation*}
$$

Since $v$ is a probability measure, $\phi(0)=1$. Also, letting $\theta=\pi / 4$, this yields

$$
\begin{equation*}
\phi(t)=\phi\left(\frac{\sqrt{2}}{2} t\right)^{2} \tag{61.6.22}
\end{equation*}
$$

and so if $\phi$ has real values, then $\phi(t) \geq 0$.
Next I will show $\phi$ is real. To do this, it follows from the definition of $\phi_{v}$,

$$
\phi_{v}(-t) \equiv \int_{\mathbb{R}} e^{-i t x} d v=\overline{\int_{\mathbb{R}} e^{i t x} d v}=\overline{\phi_{v}(t)}
$$

Then letting $\theta=\pi$,

$$
\phi(t)=\phi(-t) \cdot \phi(0)=\phi(-t)=\overline{\phi(t)}
$$

showing $\phi$ has real values. It is positive near 0 because $\phi(0)=1$ and $\phi$ is a continuous function of $t$ thanks to the dominated convergence theorem. However, this and $61.6 .22 \mathrm{im}-$ plies it is positive everywhere. Here is why. If not, let $t_{m}$ be the smallest positive value of $t$ where $\phi(t)=0$. Then $t_{m}>0$ by continuity. Now from 61.6.22, an immediate contradiction results. Therefore, $\phi(t)>0$ for all $t>0$. Similar reasoning yields the same conclusion for $t<0$.

Next note that $\phi(t)=\phi(-t)$ also implies $\phi$ depends only on $|t|$ because it takes the same value for $t$ as for $-t$. More simply, $\phi$ depends only on $t^{2}$. Thus one can define a new function of the form $\phi(t)=f\left(t^{2}\right)$ and 61.6.21 implies the following for $\alpha \in[0,1]$.

$$
f\left(t^{2}\right)=f\left(\alpha^{2} t^{2}\right) f\left(\left(1-\alpha^{2}\right) t^{2}\right)
$$

Taking $\ln$ of both sides, one obtains the following.

$$
\ln f\left(t^{2}\right)=\ln f\left(\alpha^{2} t^{2}\right)+\ln f\left(\left(1-\alpha^{2}\right) t^{2}\right)
$$

Now let $x, y \geq 0$. Then choose $t$ such that $t^{2}=x+y$. Then for some $\alpha \in[0,1], x=\alpha^{2} t^{2}$ and so $y=t^{2}\left(1-\alpha^{2}\right)$. Thus for $x, y \geq 0$,

$$
\ln f(x+y)=\ln f(x)+\ln f(y) .
$$

Hence $\ln f(x)=k x$ and so $\ln f\left(t^{2}\right)=k t^{2}$ and so $\phi(t)=f\left(t^{2}\right)=e^{k t^{2}}$ for all $t$. The constant, $k$ must be nonpositive because $\phi(t)$ is bounded due to its definition. Therefore, the characteristic function of $v$ is

$$
\phi_{v}(t)=e^{-\frac{1}{2} t^{2} \sigma^{2}}
$$

for some $\sigma \geq 0$. That is, $v$ is the law of a generalized normal random variable.
Note the other direction of the implication is obvious. If $\xi, \zeta \sim N(0, \sigma)$ and they are independent, then if $\alpha^{2}+\beta^{2}=1$, it follows

$$
\alpha \xi+\beta \zeta \sim N\left(0, \sigma^{2}\right)
$$

because

$$
\begin{aligned}
E\left(e^{i t(\alpha \xi+\beta \zeta)}\right) & =E\left(e^{i t \alpha \xi}\right) E\left(e^{i t \beta \zeta}\right) \\
& =e^{-\frac{1}{2}(\alpha t)^{2} \sigma^{2}} e^{-\frac{1}{2}(\beta t)^{2} \sigma^{2}} \\
& =e^{-\frac{1}{2} t^{2} \sigma^{2}}
\end{aligned}
$$

the characteristic function for a random variable which is $N(0, \sigma)$. This proves the theorem.
The next theorem is a useful gimmick for showing certain random variables are independent in the context of normal distributions.

Theorem 61.6.9 Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors having values in $\mathbb{R}^{p}$ and $\mathbb{R}^{q}$ respectively. Suppose also that $(\mathbf{X}, \mathbf{Y})$ is multivariate normally distributed and

$$
E\left((\mathbf{X}-E(\mathbf{X}))(\mathbf{Y}-E(\mathbf{Y}))^{*}\right)=\mathbf{0}
$$

Then $\mathbf{X}$ and $\mathbf{Y}$ are independent random vectors.
Proof: Let $\mathbf{Z}=(\mathbf{X}, \mathbf{Y}), m=p+q$. Then by hypothesis, the characteristic function of $\mathbf{Z}$ is of the form

$$
E\left(e^{i \mathbf{t} \cdot \mathbf{Z}}\right)=e^{i \mathbf{t} \cdot \mathbf{m}} e^{-\frac{1}{2} i \mathbf{t}^{*} \Sigma \mathbf{t}}
$$

where $\mathbf{m}=\left(\mathbf{m}_{\mathbf{X}}, \mathbf{m}_{\mathbf{Y}}\right)=E(\mathbf{Z})=E(\mathbf{X}, \mathbf{Y})$ and

$$
\begin{aligned}
\Sigma & =\left(\begin{array}{cc}
E\left((\mathbf{X}-E(\mathbf{X}))(\mathbf{X}-E(\mathbf{X}))^{*}\right) & \mathbf{0} \\
& E\left((\mathbf{Y}-E(\mathbf{Y}))(\mathbf{Y}-E(\mathbf{Y}))^{*}\right)
\end{array}\right) \\
& \equiv\left(\begin{array}{cc}
\Sigma_{\mathbf{X}} & \mathbf{0} \\
\mathbf{0} & \Sigma_{\mathbf{Y}}
\end{array}\right)
\end{aligned}
$$

Therefore, letting $\mathbf{t}=(\mathbf{u}, \mathbf{v})$ where $\mathbf{u} \in \mathbb{R}^{p}$ and $\mathbf{v} \in \mathbb{R}^{q}$

$$
\begin{align*}
E\left(e^{i \mathbf{t} \cdot \mathbf{Z}}\right) & =E\left(e^{i(\mathbf{u}, \mathbf{v}) \cdot(\mathbf{X}, \mathbf{Y})}\right)=E\left(e^{i(\mathbf{u} \cdot \mathbf{X}+\mathbf{v} \cdot \mathbf{Y})}\right) \\
& =e^{i \mathbf{u} \cdot \mathbf{m}_{\mathbf{X}}} e^{-\frac{1}{2} \mathbf{u}^{*} \Sigma_{\mathbf{X}} \mathbf{u} e^{i \mathbf{v} \cdot \mathbf{m}_{\mathbf{Y}}} e^{-\frac{1}{2} \mathbf{v}^{*} \Sigma_{\mathbf{Y}} \mathbf{v}}} \\
& =E\left(e^{i \mathbf{u} \cdot \mathbf{X}}\right) E\left(e^{i \mathbf{v} \cdot \mathbf{Y}}\right) \tag{61.6.23}
\end{align*}
$$

Where the last equality needs to be justified. When this is done it will follow from Proposition 59.11.1 on Page 1889 which is proved on Page 1889 that $\mathbf{X}$ and $\mathbf{Y}$ are independent. Thus all that remains is to verify

$$
E\left(e^{i \mathbf{u} \cdot \mathbf{X}}\right)=e^{i \mathbf{u} \cdot \mathbf{m}_{\mathbf{X}}} e^{-\frac{1}{2} \mathbf{u}^{*} \Sigma_{\mathbf{X}} \mathbf{u}}, E\left(e^{i \mathbf{v} \cdot \mathbf{Y}}\right)=e^{i \mathbf{v} \cdot \mathbf{m}_{\mathbf{Y}}} e^{-\frac{1}{2} \mathbf{v}^{*} \Sigma_{\mathbf{Y}} \mathbf{v}}
$$

However, this follows from 61.6.23. To get the first formula, let $\mathbf{v}=\mathbf{0}$. To get the second, let $\mathbf{u}=\mathbf{0}$. This proves the Theorem.

Note that to verify the conclusion of this theorem, it suffices to show

$$
E\left(X_{i}-E\left(X_{i}\right)\left(Y_{j}-E\left(Y_{j}\right)\right)\right)=0
$$

### 61.7 Gaussian Measures

### 61.7.1 Definitions And Basic Properties

First suppose $\mathbf{X}$ is a random vector having values in $\mathbb{R}^{n}$ and its distribution function is $N(\mathbf{m}, \Sigma)$ where $\mathbf{m}$ is the mean and $\Sigma$ is the covariance. Then the characteristic function of $\mathbf{X}$ or equivalently, the characteristic function of its distribution is

$$
e^{i \mathbf{t} \cdot \mathbf{m}} e^{-\frac{1}{2} \mathbf{t}^{*} \Sigma \mathbf{t}}
$$

What is the distribution of $\mathbf{a} \cdot \mathbf{X}$ where $\mathbf{a} \in \mathbb{R}^{n}$ ? In other words, if you take a linear functional and do it to $\mathbf{X}$ to get a scalar valued random variable, what is the distribution of this scalar valued random variable? Let $Y=\mathbf{a} \cdot \mathbf{X}$. Then

$$
E\left(e^{i t Y}\right)=E\left(e^{i t \mathbf{a} \cdot \mathbf{X}}\right)
$$

which from the above formula is

$$
e^{i \mathbf{a} \cdot \mathbf{m} t} e^{-\frac{1}{2} \mathbf{a}^{*} \sum \mathbf{a} t^{2}}
$$

which is the characteristic function of a random variable whose distribution is the normal distribution $N\left(\mathbf{a} \cdot \mathbf{m}, \mathbf{a}^{*} \Sigma \mathbf{a}\right)$. In other words, it is normally distributed having mean equal to $\mathbf{a} \cdot \mathbf{m}$ and variance equal to $\mathbf{a}^{*} \Sigma \mathbf{a}$. Obviously such a concept generalizes to a Banach space in place of $\mathbb{R}^{n}$ and this motivates the following definition.

Definition 61.7.1 Let $E$ be a real separable Banach space. A probability measure, $\mu$ defined on $\mathscr{B}(E)$ is called a Gaussian measure if for every $h \in E^{\prime}$, the law of h considered as a random variable defined on the probability space, $(E, \mathscr{B}(E), \mu)$ is normal. That is, for $A \subseteq \mathbb{R}$ a Borel set,

$$
\lambda_{h}(A) \equiv \mu\left(h^{-1}(A)\right)
$$

is given by

$$
\int_{A} \frac{1}{\sqrt{2 \pi} \sigma} e^{-\frac{1}{2 \sigma^{2}}(x-m)^{2}} d x
$$

for some $\sigma$ and m. A Gaussian measure is called symmetric if $m$ is always equal to 0 .

There is another definition of symmetric. First here are a few simple conventions. For $f \in E^{\prime}, x \rightarrow f(x)$ is normally distributed. In particular,

$$
\int_{E}|f(x)| d \mu<\infty
$$

and so it makes sense to define

$$
m_{\mu}(f) \equiv \int_{E} f(x) d \mu
$$

Thus $m_{\mu}(f)$ is the mean of the random variable $x \rightarrow f(x)$. It is obvious that $f \rightarrow m_{\mu}(f)$ is linear. Also define the variance $\sigma^{2}(f)$ by

$$
\sigma^{2}(f) \equiv \int_{E}\left(f(x)-m_{\mu}(f)\right)^{2} d \mu
$$

This is finite because $x \rightarrow f(x)$ is normally distributed. The following lemma gives such an equivalent condition for $\mu$ to be symmetric.

Lemma 61.7.2 Let $\mu$ be a Gaussian measure defined on $\mathscr{B}(E)$. Then $\mu(F)=\mu(-F)$ for all $F$ Borel if and only if $m_{\mu}(f)=0$ for all $f \in E^{\prime}$. Such a Gaussian measure is called symmetric.

Proof: Suppose first $m_{\mu}(f)=0$ for all $f \in E^{\prime}$. Let

$$
G \equiv f_{1}^{-1}\left(F_{1}\right) \cap f_{2}^{-1}\left(F_{2}\right) \cap \cdots \cap f_{m}^{-1}\left(F_{m}\right)
$$

where $F_{i}$ is a Borel set of $\mathbb{R}$ and each $f_{i} \in E^{\prime}$. Since every linear combination of the $f_{i}$ is in $E^{\prime}$, every such linear combination is normally distributed and so $\mathbf{f} \equiv\left(f_{1}, \cdots, f_{m}\right)$ is multivariate normal. That is, $\lambda_{\mathbf{f}}$ the distribution measure, is multivariate normal. Since each $m_{\mu}(f)=0$, it follows

$$
\begin{equation*}
\mu(G)=\lambda_{\mathbf{f}}\left(\prod_{i=1}^{m} F_{i}\right)=\lambda_{\mathbf{f}}\left(\prod_{i=1}^{m}-F_{i}\right)=\mu(-G) \tag{61.7.24}
\end{equation*}
$$

By Lemma 21.1.6 on Page 645 there exists a countable subset, $D \equiv\left\{f_{k}\right\}_{k=1}^{\infty}$ of the closed unit ball such that for every $x \in E$,

$$
\|x\|=\sup _{f \in D}|f(x)|
$$

Therefore, letting $D(a, r)$ denote the closed ball centered at $a$ having radius $r$, it follows

$$
D(a, r)=\cap_{k=1}^{\infty} f_{k}^{-1}\left(D\left(f_{k}(a), r\right)\right)
$$

Let

$$
D_{n}(a, r)=\cap_{k=1}^{n} f_{k}^{-1}\left(D\left(f_{k}(a), r\right)\right)
$$

Then by 61.7.24

$$
\mu\left(D_{n}(a, r)\right)=\mu\left(-D_{n}(a, r)\right)
$$

and letting $n \rightarrow \infty$, it follows

$$
\mu(D(a, r))=\mu(-D(a, r))
$$

Therefore the same is true with $D(a, r)$ replaced with an open ball. Now consider

$$
D\left(a, r_{1}\right) \cap D\left(b, r_{2}\right)=\cap_{k=1}^{\infty} f_{k}^{-1}\left(D\left(f_{k}(a), r_{1}\right)\right) \cap \cap_{k=1}^{\infty} f_{k}^{-1}\left(D\left(f_{k}(b), r_{2}\right)\right)
$$

The intersection of these two closed balls is the intersection of sets of the form

$$
\cap_{k=1}^{n} f_{k}^{-1}\left(D\left(f_{k}(a), r_{1}\right)\right) \cap \cap_{k=1}^{n} f_{k}^{-1}\left(D\left(f_{k}(b), r_{2}\right)\right)
$$

to which 61.7.24 applies. Therefore, by continuing this way it follows that if $G$ is any finite intersection of closed balls,

$$
\mu(G)=\mu(-G)
$$

Let $\mathscr{K}$ denote the set of finite intersections of closed balls, a $\pi$ system. Thus for $G \in \mathscr{K}$ the above holds. Now let

$$
\mathscr{G} \equiv\{F \in \sigma(\mathscr{K}): \mu(F)=\mu(-F)\}
$$

Thus $\mathscr{G}$ contains $\mathscr{K}$ and it is clearly closed with respect to complements and countable disjoint unions. By the $\pi$ system lemma, $\mathscr{G} \supseteq \sigma(\mathscr{K})$ but $\sigma(\mathscr{K})$ clearly contains the open sets since every open ball is the countable union of closed disks and every open set is the countable union of open balls. Therefore, $\mu(G)=\mu(-G)$ for all Borel $G$.

Conversely suppose $\mu(G)=\mu(-G)$ for all $G$ Borel. If for some $f \in E^{\prime}, m_{\mu}(f) \neq 0$, then

$$
\begin{aligned}
\mu\left(f^{-1}(0, \infty)\right) & \equiv \lambda_{f}(0, \infty) \neq \lambda_{f}(-\infty, 0) \\
& \equiv \mu\left(f^{-1}(-\infty, 0)\right)=\mu\left(-f^{-1}(0, \infty)\right)
\end{aligned}
$$

a contradiction. This proves the lemma.
Lemma 61.7.3 Let $\mu=\mathscr{L}(X)$ where $X$ is a random variable defined on a probability space, $(\Omega, \mathscr{F}, P)$ which has values in $E$, a Banach space. Suppose also that for all $\phi \in$ $E^{\prime}, \phi \circ X$ is normally distributed. Then $\mu$ is a Gaussian measure. Conversely, suppose $\mu$ is a Gaussian measure on $\mathscr{B}(E)$ and $X$ is a random variable having values in $E$ such that $\mathscr{L}(X)=\mu$. Then for every $h \in E^{\prime}, h \circ X$ is normally distributed.

Proof: First suppose $\mu$ is a Gaussian measure and $X$ is a random variable such that $\mathscr{L}(X)=\mu$. Then if $F$ is a Borel set in $\mathbb{R}$, and $h \in E^{\prime}$

$$
\begin{aligned}
P\left((h \circ X)^{-1}(F)\right) & =P\left(X^{-1}\left(h^{-1}(F)\right)\right) \\
& =\mu\left(h^{-1}(F)\right) \\
& =\frac{1}{\sqrt{2 \pi} \sigma} \int_{F} e^{-\frac{|x-m|^{2}}{2 \sigma^{2}}} d x
\end{aligned}
$$

for some $m$ and $\sigma^{2}$ showing that $h \circ X$ is normally distributed.
Next suppose $h \circ X$ is normally distributed whenever $h \in E^{\prime}$ and $\mathscr{L}(X)=\mu$. Then letting $F$ be a Borel set in $\mathbb{R}$, I need to verify

$$
\mu\left(h^{-1}(F)\right)=\frac{1}{\sqrt{2 \pi} \sigma} \int_{F} e^{-\frac{|x-m|^{2}}{2 \sigma^{2}}} d x .
$$

However, this is easy because

$$
\begin{aligned}
\mu\left(h^{-1}(F)\right) & =P\left(X^{-1}\left(h^{-1}(F)\right)\right) \\
& =P\left((h \circ X)^{-1}(F)\right)
\end{aligned}
$$

which is given to equal

$$
\frac{1}{\sqrt{2 \pi} \sigma} \int_{F} e^{-\frac{|x-m|^{2}}{2 \sigma^{2}}} d x
$$

for some $m$ and $\sigma^{2}$. This proves the lemma.
Here is another important observation. Suppose $X$ is as just described, a random variable having values in $E$ such that $\mathscr{L}(X)=\mu$ and suppose $h_{1}, \cdots, h_{n}$ are each in $E^{\prime}$. Then for scalars, $t_{1}, \cdots, t_{n}$,

$$
\begin{aligned}
& t_{1} h_{1} \circ X+\cdots+t_{n} h_{n} \circ X \\
= & \left(t_{1} h_{1}+\cdots+t_{n} h_{n}\right) \circ X
\end{aligned}
$$

and this last is assumed to be normally distributed because $\left(t_{1} h_{1}+\cdots+t_{n} h_{n}\right) \in E^{\prime}$. Therefore, by Theorem 61.6.4

$$
\left(h_{1} \circ X, \cdots, h_{n} \circ X\right)
$$

is distributed as a multivariate normal.
Obviously there exist examples of Gaussian measures defined on $E$, a Banach space. Here is why. Let $\xi$ be a random variable defined on a probability space, $(\Omega, \mathscr{F}, P)$ which is normally distributed with mean 0 and variance $\sigma^{2}$. Then let $X(\omega) \equiv \xi(\omega) e$ where $e \in E$. Then let $\mu \equiv \mathscr{L}(X)$. For $A$ a Borel set of $\mathbb{R}$ and $h \in E^{\prime}$,

$$
\begin{aligned}
\mu([h(x) \in A]) & \equiv P([X(\omega) \in[x: h(x) \in A]]) \\
& =P([h \circ X \in A])=P([\xi(\omega) h(e) \in A]) \\
& =\frac{1}{|h(e)| \sigma \sqrt{2 \pi}} \int_{A} e^{-\frac{1}{2|h(e)|^{2} \sigma^{2}} x^{2}} d x
\end{aligned}
$$

because $h(e) \xi$ is a random variable which has variance $|h(e)|^{2} \sigma^{2}$ and mean 0 . Thus $\mu$ is indeed a Gaussian measure. Similarly, one can consider finite sums of the form

$$
\sum_{i=1}^{n} \xi_{i}(\omega) e_{i}
$$

where the $\xi_{i}$ are independent normal random variables having mean 0 for convenience. However, this is a rather trivial case.

### 61.7.2 Fernique's Theorem

The following is an interesting lemma.
Lemma 61.7.4 Suppose $\mu$ is a symmetric Gaussian measure on the real separable Banach space, $E$. Then there exists a probability space, $(\Omega, \mathscr{F}, P)$ and independent random variables, $X$ and $Y$ mapping $\Omega$ to $E$ such that $\mathscr{L}(X)=\mathscr{L}(Y)=\mu$. Also, the two random variables,

$$
\frac{1}{\sqrt{2}}(X-Y), \frac{1}{\sqrt{2}}(X+Y)
$$

are independent and

$$
\mathscr{L}\left(\frac{1}{\sqrt{2}}(X-Y)\right)=\mathscr{L}\left(\frac{1}{\sqrt{2}}(X+Y)\right)=\mu
$$

Proof: Letting $X^{\prime} \equiv \frac{1}{\sqrt{2}}(X+Y)$ and $Y^{\prime} \equiv \frac{1}{\sqrt{2}}(X-Y)$, it follows from Theorem 59.13.2 on Page 1896, that $X^{\prime}$ and $Y^{\prime}$ are independent if whenever $h_{1}, \cdots, h_{m} \in E^{\prime}$ and $g_{1}, \cdots, g_{k} \in$ $E^{\prime}$, the two random vectors,

$$
\left(h_{1} \circ X^{\prime}, \cdots, h_{m} \circ X^{\prime}\right) \text { and }\left(g_{1} \circ Y^{\prime}, \cdots, g_{k} \circ Y^{\prime}\right)
$$

are independent. Now consider linear combinations

$$
\sum_{j=1}^{m} t_{j} h_{j} \circ X^{\prime}+\sum_{i=1}^{k} s_{i} g_{i} \circ Y^{\prime}
$$

This equals

$$
\begin{array}{r}
\frac{1}{\sqrt{2}} \sum_{j=1}^{m} t_{j} h_{j}(X)+\frac{1}{\sqrt{2}} \sum_{j=1}^{m} t_{j} h_{j}(Y) \\
+\frac{1}{\sqrt{2}} \sum_{i=1}^{k} s_{i} g_{i}(X)-\frac{1}{\sqrt{2}} \sum_{i=1}^{k} s_{i} g_{i}(Y) \\
= \\
\quad \frac{1}{\sqrt{2}}\left(\sum_{j=1}^{m} t_{j} h_{j}+\sum_{i=1}^{k} s_{i} g_{i}\right)(X) \\
\quad+\frac{1}{\sqrt{2}}\left(\sum_{j=1}^{m} t_{j} h_{j}-\sum_{i=1}^{k} s_{i} g_{i}\right)(Y)
\end{array}
$$

and this is the sum of two independent normally distributed random variables so it is also normally distributed. Therefore, by Theorem 61.6.4

$$
\left(h_{1} \circ X^{\prime}, \cdots, h_{m} \circ X^{\prime}, g_{1} \circ Y^{\prime}, \cdots, g_{k} \circ Y^{\prime}\right)
$$

is a random variable with multivariate normal distribution and by Theorem 61.6.9 the two random vectors

$$
\left(h_{1} \circ X^{\prime}, \cdots, h_{m} \circ X^{\prime}\right) \text { and }\left(g_{1} \circ Y^{\prime}, \cdots, g_{k} \circ Y^{\prime}\right)
$$

are independent if

$$
E\left(\left(h_{i} \circ X^{\prime}\right)\left(g_{j} \circ Y^{\prime}\right)\right)=0
$$

for all $i, j$. This is what I will show next.

$$
\begin{align*}
& E\left(\left(h_{i} \circ X^{\prime}\right)\left(g_{j} \circ Y^{\prime}\right)\right) \\
= & \frac{1}{4} E\left(\left(h_{i}(X)+h_{i}(Y)\right)\left(g_{j}(X)-g_{j}(Y)\right)\right) \\
= & \frac{1}{4} E\left(h_{i}(X) g_{j}(X)\right)-\frac{1}{4} E\left(h_{i}(X) g_{j}(Y)\right) \\
& +\frac{1}{4} E\left(h_{i}(Y) g_{j}(X)\right)-\frac{1}{4} E\left(h_{i}(Y) g_{j}(Y)\right) \tag{61.7.25}
\end{align*}
$$

Now from the above observation after the definition of Gaussian measure $h_{i}(X) g_{j}(X)$ and $h_{i}(Y) g_{j}(Y)$ are both in $L^{1}$ because each term in each product is normally distributed. Therefore, by Lemma 59.15.2,

$$
\begin{aligned}
E\left(h_{i}(X) g_{j}(X)\right) & =\int_{\Omega} h_{i}(Y) g_{j}(Y) d P \\
& =\int_{E} h_{i}(y) g_{j}(y) d \mu \\
& =\int_{\Omega} h_{i}(X) g_{j}(X) d P \\
& =E\left(h_{i}(Y) g_{j}(Y)\right)
\end{aligned}
$$

and so 61.7.25 reduces to

$$
\frac{1}{4}\left(E\left(h_{i}(Y) g_{j}(X)-h_{i}(X) g_{j}(Y)\right)\right)=0
$$

because $h_{i}(X)$ and $g_{j}(Y)$ are independent due to the assumption that $X$ and $Y$ are independent. Thus

$$
E\left(h_{i}(X) g_{j}(Y)\right)=E\left(h_{i}(X)\right) E\left(g_{j}(Y)\right)=0
$$

due to the assumption that $\mu$ is symmetric which implies the mean of these random variables equals 0 . The other term works out similarly. This has proved the independence of the random variables, $X^{\prime}$ and $Y^{\prime}$.

Next consider the claim they have the same law and it equals $\mu$. To do this, I will use Theorem 59.12.9 on Page 1895. Thus I need to show

$$
\begin{equation*}
E\left(e^{i h\left(X^{\prime}\right)}\right)=E\left(e^{i h\left(Y^{\prime}\right)}\right)=E\left(e^{i h(X)}\right) \tag{61.7.26}
\end{equation*}
$$

for all $h \in E^{\prime}$. Pick such an $h$. Then $h \circ X$ is normally distributed and has mean 0 . Therefore, for some $\sigma$,

$$
E\left(e^{i t h \circ X}\right)=e^{-\frac{1}{2} t^{2} \sigma^{2}}
$$

Now since $X$ and $Y$ are independent,

$$
\begin{aligned}
E\left(e^{i t h o X^{\prime}}\right) & =E\left(e^{i t h\left(\frac{1}{\sqrt{2}}\right)(X+Y)}\right) \\
& =E\left(e^{i t h\left(\frac{1}{\sqrt{2}}\right) X}\right) E\left(e^{i t h\left(\frac{1}{\sqrt{2}}\right) Y}\right)
\end{aligned}
$$

the product of two characteristic functions of two random variables, $\frac{1}{\sqrt{2}} X$ and $\frac{1}{\sqrt{2}} Y$. The variance of these two random variables which are normally distributed with zero mean is $\frac{1}{2} \sigma^{2}$ and so

$$
E\left(e^{i t h \circ X^{\prime}}\right)=e^{-\frac{1}{2}\left(\frac{1}{2} \sigma^{2}\right)} e^{-\frac{1}{2}\left(\frac{1}{2} \sigma^{2}\right)}=e^{-\frac{1}{2} \sigma^{2}}=E\left(e^{i t h \circ X}\right)
$$

Similar reasoning shows $E\left(e^{i t h \circ Y^{\prime}}\right)=E\left(e^{i t h \circ Y}\right)=E\left(e^{i t h \circ X}\right)$. Letting $t=1$, this yields 61.7.26. This proves the lemma.

With this preparation, here is an incredible theorem due to Fernique.
Theorem 61.7.5 Let $\mu$ be a symmetric Gaussian measure on $\mathscr{B}(E)$ where $E$ is a real separable Banach space. Then for $\lambda$ sufficiently small and positive,

$$
\int_{E} e^{\lambda\|x\|^{2}} d \mu<\infty
$$

More specifically, if $\lambda$ and $r$ are chosen such that

$$
\ln \left(\frac{\mu([x:||x||>r])}{\mu(\overline{B(0, r)})}\right)+25 \lambda r^{2}<-1
$$

then

$$
\int_{E} e^{\lambda\|x\|^{2}} d \mu \leq \exp \left(\lambda r^{2}\right)+\frac{e^{2}}{e^{2}-1}
$$

Proof: Let $X, Y$ be independent random variables having values in $E$ such that $\mathscr{L}(X)=$ $\mathscr{L}(Y)=\mu$. Then by Lemma 61.7.4

$$
\frac{1}{\sqrt{2}}(X-Y), \frac{1}{\sqrt{2}}(X+Y)
$$

are also independent and have the same law. Now let $0 \leq s \leq t$ and use independence of the above random variables along with the fact they have the same law as $X$ and $Y$ to obtain

$$
\begin{aligned}
& P(\|X\| \leq s,\|Y\|>t)=P(\|X\| \leq s) P(\|Y\|>t) \\
= & P\left(\left\|\frac{1}{\sqrt{2}}(X-Y)\right\| \leq s\right) P\left(\left\|\frac{1}{\sqrt{2}}(X+Y)\right\|>t\right) \\
= & P\left(\left\|\frac{1}{\sqrt{2}}(X-Y)\right\| \leq s,\left\|\frac{1}{\sqrt{2}}(X+Y)\right\|>t\right) \\
\leq & P\left(\frac{1}{\sqrt{2}}\left|\|X\|-\|Y \mid\| \leq s, \frac{1}{\sqrt{2}}(\|X\|+\|Y\|)>t\right) .\right.
\end{aligned}
$$

Now consider the following picture in which the region, $R$ represents the points, $(\|X\|,\|Y\|)$ such that

$$
\frac{1}{\sqrt{2}}|\|X\|-\|Y\|| \leq s \text { and } \frac{1}{\sqrt{2}}(\|X\|+\|Y\|)>t
$$



Therefore, continuing with the chain of inequalities above,

$$
\begin{aligned}
& P(\|X\| \leq s) P(\|Y\|>t) \\
\leq & P\left(\|X\|>\frac{t-s}{\sqrt{2}},\|Y\|>\frac{t-s}{\sqrt{2}}\right) \\
= & P\left(\|X\|>\frac{t-s}{\sqrt{2}}\right)^{2} .
\end{aligned}
$$

Since $X, Y$ have the same law, this can be written as

$$
P(\|X\|>t) \leq \frac{P\left(\|X\|>\frac{t-s}{\sqrt{2}}\right)^{2}}{P(\|X\| \leq s)}
$$

Now define a sequence as follows. $t_{0} \equiv r>0$ and $t_{n+1} \equiv r+\sqrt{2} t_{n}$. Also, in the above inequality, let $s \equiv r$ and then it follows

$$
\begin{aligned}
P\left(\|X\|>t_{n+1}\right) & \leq \frac{P\left(\|X\|>\frac{t_{n+1}-r}{\sqrt{2}}\right)^{2}}{P(\|X\| \leq r)} \\
& =\frac{P\left(\|X\|>t_{n}\right)^{2}}{P(\|X\| \leq r)}
\end{aligned}
$$

Let

$$
\alpha_{n}(r) \equiv \frac{P\left(\|X\|>t_{n}\right)}{P(\|X\| \leq r)}
$$

Then it follows

$$
\alpha_{n+1}(r) \leq \alpha_{n}(r)^{2}, \alpha_{0}(r)=\frac{P(\|X\|>r)}{P(\|X\| \leq r)}
$$

Consequently, $\alpha_{n}(r) \leq \alpha_{0}(r)^{2^{n}}$ and also

$$
\begin{align*}
P\left(\|X\|>t_{n}\right) & =\alpha_{n}(r) P(\|X\| \leq r) \\
& \leq P(\|X\| \leq r) \alpha_{0}(r)^{2^{n}} \\
& =P(\|X\| \leq r) e^{\ln \left(\alpha_{0}(r)\right) 2^{n}} \tag{61.7.27}
\end{align*}
$$

Now using the distribution function technique and letting $\lambda>0$,

$$
\begin{align*}
\int_{E} e^{\lambda\|x\|^{2}} d \mu & =\int_{0}^{\infty} \mu\left(\left[e^{\lambda\|x\|^{2}}>t\right]\right) d t \\
& =1+\int_{1}^{\infty} \mu\left(\left[e^{\lambda\|x\|^{2}}>t\right]\right) d t \\
& =1+\int_{1}^{\infty} P\left(\left[e^{\lambda\|X\|^{2}}>t\right]\right) d t \tag{61.7.28}
\end{align*}
$$

From 61.7.27,

$$
P\left(\left[\exp \left(\lambda\|X\|^{2}\right)>\exp \left(\lambda t_{n}^{2}\right)\right]\right) \leq P([\|X\| \leq r]) e^{\ln \left(\alpha_{0}(r)\right) 2^{n}}
$$

Now split the above improper integral into intervals, $\left(\exp \left(\lambda t_{n}^{2}\right), \exp \left(\lambda t_{n+1}^{2}\right)\right)$ for $n=$ $0,1, \cdots$ and note that $P\left(\left[e^{\lambda\|X\|^{2}}>t\right]\right)$ is decreasing in $t$. Then from 61.7.28,

$$
\begin{aligned}
& \int_{E} e^{\lambda\|x\|^{2}} d \mu \leq \exp \left(\lambda r^{2}\right)+\sum_{n=0}^{\infty} \int_{\exp \left(\lambda t_{n}^{2}\right)}^{\exp \left(\lambda t_{n+1}^{2}\right)} P\left(\left[e^{\lambda\|X\|^{2}}>t\right]\right) d t \\
\leq & \exp \left(\lambda r^{2}\right)+\sum_{n=0}^{\infty} P\left(\left[e^{\lambda\|X\|^{2}}>\exp \left(\lambda t_{n}^{2}\right)\right]\right)\left(\exp \left(\lambda t_{n+1}^{2}\right)-\exp \left(\lambda t_{n}^{2}\right)\right) \\
\leq & \exp \left(\lambda r^{2}\right)+\sum_{n=0}^{\infty} P([\|X\| \leq r]) e^{\ln \left(\alpha_{0}(r)\right) 2^{n}} \exp \left(\lambda t_{n+1}^{2}\right) \\
\leq & \exp \left(\lambda r^{2}\right)+\sum_{n=0}^{\infty} e^{\ln \left(\alpha_{0}(r)\right) 2^{n}} \exp \left(\lambda t_{n+1}^{2}\right)
\end{aligned}
$$

It remains to estimate $t_{n+1}$. From the description of the $t_{n}$,

$$
t_{n}=\left(\sum_{k=0}^{n}(\sqrt{2})^{k}\right) r=r \frac{(\sqrt{2})^{n+1}-1}{\sqrt{2}-1} \leq \frac{\sqrt{2}}{\sqrt{2}-1} r(\sqrt{2})^{n}
$$

and so

$$
t_{n+1} \leq 5 r(\sqrt{2})^{n}
$$

Therefore,

$$
\int_{E} e^{\lambda\|x\|^{2}} d \mu \leq \exp \left(\lambda r^{2}\right)+\sum_{n=0}^{\infty} e^{\ln \left(\alpha_{0}(r)\right) 2^{n}+\lambda 25 r^{2} 2^{n}}
$$

Now first pick $r$ large enough that $\ln \left(\alpha_{0}(r)\right)<-2$ and then let $\lambda$ be small enough that $25 \lambda r^{2}<1$ or some such scheme and you obtain $\ln \left(\alpha_{0}(r)\right)+\lambda 25 r^{2}<-1$. Then for this choice of $r$ and $\lambda$, or for any other choice which makes $\ln \left(\alpha_{0}(r)\right)+\lambda 25 r^{2}<-1$,

$$
\begin{aligned}
\int_{E} e^{\lambda\|x\|^{2}} d \mu & \leq \exp \left(\lambda r^{2}\right)+\sum_{n=0}^{\infty} e^{-2^{n}} \\
& \leq \exp \left(\lambda r^{2}\right)+\sum_{n=0}^{\infty} e^{-2 n} \\
& =\exp \left(\lambda r^{2}\right)+\frac{e^{2}}{e^{2}-1}
\end{aligned}
$$

This proves the theorem.

### 61.8 Gaussian Measures For A Separable Hilbert Space

First recall the Kolmogorov extension theorem, Theorem 59.2.3 on Page 1860 which is stated here for convenience. In this theorem, $I$ is an ordered index set, possibly infinite, even uncountable.

Theorem 61.8.1 (Kolmogorov extension theorem) For each finite set

$$
J=\left(t_{1}, \cdots, t_{n}\right) \subseteq I
$$

suppose there exists a Borel probability measure, $v_{J}=v_{t_{1} \cdots t_{n}}$ defined on the Borel sets of $\prod_{t \in J} M_{t}$ where $M_{t}=\mathbb{R}^{n_{t}}$ such that if

$$
\left(t_{1}, \cdots, t_{n}\right) \subseteq\left(s_{1}, \cdots, s_{p}\right)
$$

then

$$
\begin{equation*}
v_{t_{1} \cdots t_{n}}\left(F_{t_{1}} \times \cdots \times F_{t_{n}}\right)=v_{s_{1} \cdots s_{p}}\left(G_{s_{1}} \times \cdots \times G_{s_{p}}\right) \tag{61.8.29}
\end{equation*}
$$

where if $s_{i}=t_{j}$, then $G_{s_{i}}=F_{t_{j}}$ and if $s_{i}$ is not equal to any of the indices, $t_{k}$, then $G_{s_{i}}=M_{s_{i}}$. Then there exists a probability space, $(\Omega, P, \mathscr{F})$ and measurable functions, $X_{t}: \Omega \rightarrow M_{t}$ for each $t \in I$ such that for each $\left(t_{1} \cdots t_{n}\right) \subseteq I$,

$$
\begin{equation*}
v_{t_{1} \cdots t_{n}}\left(F_{t_{1}} \times \cdots \times F_{t_{n}}\right)=P\left(\left[X_{t_{1}} \in F_{t_{1}}\right] \cap \cdots \cap\left[X_{t_{n}} \in F_{t_{n}}\right]\right) . \tag{61.8.30}
\end{equation*}
$$

Lemma 61.8.2 There exists a sequence, $\left\{\xi_{k}\right\}_{k=1}^{\infty}$ of random variables such that

$$
\mathscr{L}\left(\xi_{k}\right)=N(0,1)
$$

and $\left\{\xi_{k}\right\}_{k=1}^{\infty}$ is independent.
Proof: Let $i_{1}<i_{2} \cdots<i_{n}$ be positive integers and define

$$
\mu_{i_{1} \cdots i_{n}}\left(F_{1} \times \cdots \times F_{n}\right) \equiv \frac{1}{(\sqrt{2 \pi})^{n}} \int_{F_{1} \times \cdots \times F_{n}} e^{-|\mathbf{x}|^{2} / 2} d x
$$

Then for the index set equal to $\mathbb{N}$ the measures satisfy the necessary consistency condition for the Kolmogorov theorem above. Therefore, there exists a probability space, $(\Omega, P, \mathscr{F})$ and measurable functions, $\xi_{k}: \Omega \rightarrow \mathbb{R}$ such that

$$
\begin{aligned}
& P\left(\left[\xi_{i_{1}} \in F_{i_{1}}\right] \cap\left[\xi_{i_{2}} \in F_{i_{2}}\right] \cdots \cap\left[\xi_{i_{n}} \in F_{i_{n}}\right]\right) \\
= & \mu_{i_{1} \cdots i_{n}}\left(F_{1} \times \cdots \times F_{n}\right) \\
= & P\left(\left[\xi_{i_{1}} \in F_{i_{1}}\right]\right) \cdots P\left(\left[\xi_{i_{n}} \in F_{i_{n}}\right]\right)
\end{aligned}
$$

which shows the random variables are independent as well as normal with mean 0 and variance 1 . This proves the Lemma.

A random variable $X$ defined on a probability space $(\Omega, \mathscr{F}, P)$ is called Gaussian if

$$
P([X \in A])=\frac{1}{\sqrt{2 \pi \sigma(v)^{2}}} \int_{A} e^{-\frac{1}{2 \sigma(v)^{2}}(x-m(v))^{2}} d x
$$

for all $A$ a Borel set in $\mathbb{R}$. Therefore, for the probability space $(X, \mathscr{B}(X), \mu)$ it is natural to say $\mu$ is a Gaussian measure if every $x^{*}$ in the dual space $X^{\prime}$ is a Gaussian random variable. That is, normally distributed.

Definition 61.8.3 Let $\mu$ be a measure defined on $\mathscr{B}(X)$, the Borel sets of $X$, a separable Banach space. It is called a Gaussian measure if each of the functions in the dual space $X^{\prime}$ is normally distributed. As a special case, when $X=U$ a separable real Hilberts space, $\mu$ is called a Gaussian measure if for each $v \in U$, the function $u \rightarrow(u, v)_{U}$ is normally distributed. That is, denoting this random variable as $v^{\prime}$, it follows for $A$ a Borel set in $\mathbb{R}$

$$
\lambda_{v^{\prime}}(A) \equiv \mu\left(\left[u: v^{\prime}(u) \in A\right]\right)=\frac{1}{\sqrt{2 \pi \sigma(v)^{2}}} \int_{A} e^{-\frac{1}{2 \sigma(v)^{2}}(x-m(v))^{2}} d x
$$

in case $\sigma(v)>0$. In case $\sigma(v)=0$

$$
\lambda_{v^{\prime}} \equiv \delta_{m(v)}
$$

In other words, the random variables $v^{\prime}$ for $v \in U$ are all normally distributed on the probability space $(U, \mathscr{B}(U), \mu)$.

Also recall the definition of the characteristic function of a measure.
Definition 61.8.4 The Borel sets in a topological space $X$ will be denoted by $\mathscr{B}(X)$. For a Borel probability measure $\mu$ defined on $\mathscr{B}(U)$ for $U$ a real separable Hilbert space, define its characteristic function as follows.

$$
\begin{equation*}
\phi_{\mu}(u) \equiv \widehat{\mu}(u) \equiv \int_{U} e^{i(u, v)} d \mu(v) \tag{61.8.31}
\end{equation*}
$$

More generally, if $\mu$ is a probability measure defined on $\mathscr{B}(X)$ where $X$ is a separable Banach space, then the characteristic function is defined as

$$
\phi_{\mu}\left(x^{*}\right)=\widehat{\mu}\left(x^{*}\right) \equiv \int_{U} e^{i x^{*}(x)} d \mu(x)
$$

One can tell whether $\mu$ is a Gaussian measure by looking at its characteristic function. In fact you can show the following theorem. One part of this theorem is that if $\mu$ is Gaussian, then $m$ and $\sigma^{2}$ have a certain form.

Theorem 61.8.5 A measure $\mu$ on $\mathscr{B}(U)$ is Gaussian if and only if there exists $m \in U$ and $Q \in \mathscr{L}(U)$ such that $Q$ is nonnegative symmetric with finite trace,

$$
\sum_{k}\left(Q e_{k}, e_{k}\right)<\infty
$$

for a complete orthonormal basis for $U$, and

$$
\begin{equation*}
\phi_{\mu}(u)=\widehat{\mu}(u)=e^{i(m, u)-\frac{1}{2}(Q u, u)} \tag{61.8.32}
\end{equation*}
$$

In this case $\mu$ is called $N(m, Q)$ where $m$ is the mean and $Q$ is called the covariance. The measure $\mu$ is uniquely determined by $m$ and $Q$. Also for all $h, g \in U$

$$
\begin{gather*}
\int(x, h)_{U} d \mu(x)=(m, h)_{U}  \tag{61.8.33}\\
\int((x, h)-(m, h))((x, g)-(m, g)) d \mu(x)=(Q h, g)  \tag{61.8.34}\\
\int\|x-m\|_{U}^{2} d \mu(x)=\operatorname{trace}(Q) \tag{61.8.35}
\end{gather*}
$$

Proof: First of all suppose 61.8 .32 holds. Why is $\mu$ Gaussian? Consider the random variable $u^{\prime}$ defined by $u^{\prime}(v) \equiv(v, u)$. Why is $\lambda_{u^{\prime}}$ a Gaussian measure on $\mathbb{R}$ ? By the definition in 61.8.31,

$$
\begin{aligned}
\int_{U} e^{i t u^{\prime}(v)} d \mu(v) & \equiv \int_{U} e^{i t(v, u)} d \mu(v)=\int_{\mathbb{R}} e^{i x} d \lambda_{u^{\prime}}(x) \\
& =\int_{U} e^{i(v, t u)} d \mu(v)=e^{i t(m, u)-\frac{1}{2} t^{2}(Q u, u)}
\end{aligned}
$$

and this is the characteristic equation for a random variable having mean $(m, u)$ and variance $(Q u, u)$. In case $(Q u, u)=0$, you get $e^{i t(m, u)}$ which is the characteristic function for a random variable having distribution $\boldsymbol{\delta}_{(m, u)}$. Thus if 61.8 .32 holds, then $u^{\prime}$ is normally distributed as desired. Thus $\mu$ is Gaussian by definition.

The next task is to suppose $\mu$ is Gaussian and show the existence of $m, Q$ which have the desired properties. This involves the following lemma.
Lemma 61.8.6 Let $U$ be a real separable Hilbert space and let $\mu$ be a probability measure defined on $\mathscr{B}(U)$. Suppose for some positive integer, $k$

$$
\int_{U}|(x, z)|^{k} d \mu(x)<\infty
$$

for all $z \in U$. Then the transformation,

$$
\begin{equation*}
\left(h_{1}, \cdots, h_{k}\right) \rightarrow \int_{U}\left(h_{1}, x\right) \cdots\left(h_{k}, x\right) d \mu(x) \tag{61.8.36}
\end{equation*}
$$

is a continuous $k$-linear form.

Proof: I need to show that for each $\mathbf{h} \in U^{k}$, the integral in 61.8.36 exists. From this it is obvious it is $k$ - linear, meaning linear in each argument. Then it is shown it is continuous.

First note

$$
\left|\left(h_{1}, x\right) \cdots\left(h_{k}, x\right)\right| \leq\left|\left(h_{1}, x\right)\right|^{k}+\cdots+\left|\left(h_{k}, x\right)\right|^{k}
$$

This follows from observing that one of $\left|\left(h_{j}, x\right)\right|$ is largest. Then the left side is smaller than $\left|\left(h_{j}, x\right)\right|^{k}$. Therefore, the above inequality is valid. This inequality shows the integral in 61.8.36 makes sense.

I need to establish an estimate of the form

$$
\int_{U}|(x, h)|^{k} d \mu(x)<C<\infty
$$

for every $h \in U$ such that $\|h\|$ is small enough.
Let

$$
U_{n} \equiv\left\{z \in U: \int_{U}|(x, z)|^{k} d \mu(x) \leq n\right\}
$$

Then by assumption $U=\cup_{n=1}^{\infty} U_{n}$ and it is also clear from Fatou's lemma that each $U_{n}$ is closed. Therefore, by the Bair category theorem, at least one of these $U_{n_{0}}$ contains an open ball, $B\left(z_{0}, r\right)$. Then letting $|y|<r$,

$$
\int_{U}\left|\left(x, z_{0}+y\right)\right|^{k} d \mu(x), \int_{U}\left|\left(x, z_{0}\right)\right|^{k} d \mu(x) \leq n_{0}
$$

and so for such $y$,

$$
\begin{aligned}
\int_{U}|(x, y)|^{k} d \mu & =\int_{U}\left|\left(x, z_{0}+y\right)-\left(x, z_{0}\right)\right|^{k} d \mu \\
& \leq \int_{U} 2^{k}\left|\left(x, z_{0}+y\right)\right|^{k}+2^{k}\left|\left(x, z_{0}\right)\right|^{k} d \mu(x) \\
& \leq 2^{k}\left(n_{0}+n_{0}\right)=2^{k+1} n_{0} .
\end{aligned}
$$

It follows that for arbitrary nonzero $y \in U$

$$
\int_{U}\left|\left(x, \frac{(r / 2) y}{\|y\|}\right)\right|^{k} d \mu \leq 2^{k+1} n_{0}
$$

and so

$$
\int_{U}|(x, y)|^{k} d \mu \leq\left(2^{k+2} / r\right) n_{0}\|y\|^{k} \equiv C\|y\|^{k}
$$

Thus by Holder's inequality,

$$
\begin{aligned}
\int_{U}\left|\left(h_{1}, x\right) \cdots\left(h_{k}, x\right)\right| d \mu(x) & \leq \prod_{j=1}^{k}\left(\int_{U}\left|\left(h_{j}, x\right)\right|^{k} d \mu(x)\right)^{1 / k} \\
& \leq C \prod_{j=1}^{k} \| h_{j}| |
\end{aligned}
$$

This proves the lemma.
Now continue with the proof of the theorem. I need to identify $m$ and $Q$. It is assumed $\mu$ is Gaussian. Recall this means $h^{\prime}$ is normally distributed for each $h \in U$. Then using

$$
\begin{gathered}
|x| \leq|x-m(h)|+|m(h)| \\
\int_{U}\left|(x, h)_{U}\right| d \mu(x)=\int_{\mathbb{R}}|x| d \lambda_{h^{\prime}}(x) \\
=\frac{1}{\sqrt{2 \pi \sigma^{2}(h)}} \int_{\mathbb{R}}|x| e^{-\frac{1}{2 \sigma^{2}}(x-m(h))^{2}} d x \\
\leq \frac{1}{\sqrt{2 \pi \sigma^{2}(h)}} \int_{\mathbb{R}}|x-m(h)| e^{-\frac{1}{2 \sigma^{2}}(x-m(h))^{2}} d x \\
+|m(h)|
\end{gathered}
$$

Then using the Cauchy Schwarz inequality, with respect to the probability measure

$$
\begin{gathered}
\frac{1}{\sqrt{2 \pi \sigma^{2}(h)}} e^{-\frac{1}{2 \sigma^{2}}(x-m(h))^{2}} d x \\
\leq \frac{1}{\sqrt{2 \pi \sigma^{2}(h)}}\left(\int_{\mathbb{R}}|x-m(h)|^{2} e^{-\frac{1}{2 \sigma^{2}}(x-m(h))^{2}} d x\right)^{1 / 2}+|m(h)|<\infty
\end{gathered}
$$

Thus by Lemma 61.8.6

$$
h \rightarrow \int_{U}(x, h) d \mu(x)
$$

is a continuous linear transformation and so by the Riesz representation theorem, there exists a unique $m \in U$ such that

$$
(h, m)_{U}=\int_{U}(h, x) d \mu(x)
$$

Also the above says $(h, m)$ is the mean of the random variable $x \rightarrow(x, h)$ so in the above,

$$
m(h)=(h, m)_{U}
$$

Next it is necessary to find $Q$. To do this let $Q$ be given by 61.8.34. Thus

$$
\begin{aligned}
(Q h, g) & \equiv \int_{U}((x, h)-(m, h))((x, g)-(m, g)) d \mu(x) \\
& =\int_{U}(x-m, h)(x-m, g) d \mu(x)
\end{aligned}
$$

It is clear $Q$ is linear and the above is a bilinear form (The integral makes sense because of the assumption that $h^{\prime}, g^{\prime}$ are normally distributed.) but is it continuous? Does $(Q h, h)=$ $\sigma^{2}(h)$ ?

First, the above equals

$$
\begin{align*}
& \int_{U}(x, h)(x-m, g) d \mu-\int_{U}(m, h)(x-m, g) d \mu(x) \\
= & \int_{U}(x, h)(x-m, g) d \mu \tag{61.8.37}
\end{align*}
$$

because from the first part,

$$
\int_{U}(x-m, g) d \mu(x)=\int_{U}(x, g) d \mu(x)-(m, g)_{U}=0
$$

Now by the first part, the term in 61.8.37 is

$$
\begin{aligned}
& \int_{U}(x, h)(x, g) d \mu(x)-(m, g) \int_{U}(x, h) d \mu(x) \\
= & \int_{U}(x, h)(x, g) d \mu(x)-(m, g)(m, h) .
\end{aligned}
$$

Thus

$$
|(Q h, g)| \leq \int_{U}|(x, h)(x, g)| d \mu(x)+\|m\|^{2}\|h\|\|g\|
$$

and since the random variables $h^{\prime}$ and $g^{\prime}$ given by $x \rightarrow(x, h)$ and $x \rightarrow(x, g)$ respectively are given to be normally distributed with variance $\sigma^{2}(h)$ and $\sigma^{2}(g)$ respectively, the above integral is finite. Also for all $h$,

$$
\int_{U}|(x, h)|^{2} d \mu(x)<\infty
$$

because the random variable $h^{\prime}$ is given to be normally distributed. Therefore from Lemma 61.8.6, there exists some constant $C$ such that

$$
|(Q h, g)| \leq C\|h\|\|g\|
$$

which shows $Q$ is continuous as desired.
Why is $\sigma^{2}(h)=(Q h, h)$ ? This follows because from the above

$$
\begin{aligned}
&(Q h, h) \equiv \int_{U}(h, x-m)^{2} d \mu(x) \\
&=\int_{U}((x, h)-(h, m))^{2} d \mu(x)=\int_{\mathbb{R}}(t-(h, m))^{2} d \lambda_{h^{\prime}}(t) \\
&=\frac{1}{\sqrt{2 \pi \sigma^{2}(h)}} \int_{\mathbb{R}}(t-(h, m))^{2} e^{-\frac{1}{2 \sigma^{2}(h)}(t-(h, m))^{2}} d t=\sigma^{2}(h)
\end{aligned}
$$

from a standard result for the normal distribution function which follows from an easy change of variables argument.

Why must $Q$ have finite trace? For $h \in U$, it follows from the above that $h^{\prime}$ is normally distributed with mean $(h, m)$ and variance $(Q h, h)$. Therefore, the characteristic function of $h^{\prime}$ is known. In fact

$$
\int_{U} e^{i t(x, h)} d \mu(x)=e^{i t(h, m)} e^{-\frac{1}{2} t^{2}(Q h, h)}
$$

Thus also

$$
\int_{U} e^{i t(x-m, h)} d \mu(x)=e^{-\frac{1}{2} t^{2}(Q h, h)}
$$

and letting $t=1$ this yields

$$
\int_{U} e^{i(x-m, h)} d \mu(x)=e^{-\frac{1}{2}(Q h, h)}
$$

From this it follows

$$
\int_{U}\left(1-e^{i(x-m, h)}\right) d \mu=1-e^{-\frac{1}{2}(Q h, h)}
$$

and since the right side is real, this implies

$$
\int_{U}(1-\cos (x-m, h)) d \mu(x)=1-e^{-\frac{1}{2}(Q h, h)}
$$

Thus

$$
\begin{aligned}
1-e^{-\frac{1}{2}(Q h, h)} \leq & \int_{[\|x-m\| \leq c]}(1-\cos (x-m, h)) d \mu(x) \\
& +2 \int_{[\|x-m\|>c]} d \mu(x)
\end{aligned}
$$

Now it is routine to show

$$
1-\cos t \leq \frac{1}{2} t^{2}
$$

and so

$$
\begin{aligned}
1-e^{-\frac{1}{2}(Q h, h)} \leq & \frac{1}{2} \int_{[\|x-m\| \leq c]}|(x-m, h)|^{2} d \mu(x) \\
& +2 \mu([\|x-m\|>c])
\end{aligned}
$$

Pick $c$ large enough that the last term is smaller than $1 / 8$. This can be done because the sets decrease to $\emptyset$ as $c \rightarrow \infty$ and $\mu$ is given to be a finite measure. Then with this choice of $c$,

$$
\begin{equation*}
\frac{7}{8}-\frac{1}{2} \int_{[| | x-m \| \leq c]}|(x-m, h)|^{2} d \mu(x) \leq e^{-\frac{1}{2}(Q h, h)} \tag{61.8.38}
\end{equation*}
$$

For each $h$ the integral in the above is finite. In fact

$$
\int_{[\|x-m\| \leq c]}|(x-m, h)|^{2} d \mu(x) \leq c^{2}\|h\|^{2}
$$

Let

$$
\left(Q_{c} h, h_{1}\right) \equiv \int_{[\|x-m\| \leq c]}(x-m, h)\left(x-m, h_{1}\right) d \mu(x)
$$

and let $A$ denote those $h \in U$ such that

$$
\left(Q_{c} h, h\right)<1
$$

Then from 61.8.38 it follows that for $h \in A$,

$$
\frac{3}{8}=\frac{7}{8}-\frac{1}{2} \leq \frac{7}{8}-\frac{1}{2}\left(Q_{c} h, h\right) \leq e^{-\frac{1}{2}(Q h, h)}
$$

Therefore, for such $h$,

$$
\frac{8}{3} \geq e^{\frac{1}{2}(Q h, h)} \geq 1+\frac{1}{2}(Q h, h)
$$

and so for $h \in A$,

$$
(Q h, h) \leq\left(\frac{8}{3}-1\right) 2=\frac{10}{3}
$$

Now let $h$ be arbitrary. Then for each $\varepsilon>0$

$$
\frac{h}{\varepsilon+\sqrt{\left(Q_{c} h, h\right)}} \in A
$$

and so

$$
\left(Q\left(\frac{h}{\varepsilon+\sqrt{\left(Q_{c} h, h\right)}}\right), \frac{h}{\varepsilon+\sqrt{\left(Q_{c} h, h\right)}}\right) \leq \frac{10}{3}
$$

which implies

$$
(Q h, h) \leq \frac{10}{3}\left(\varepsilon+\sqrt{\left(Q_{c} h, h\right)}\right)^{2}
$$

Since $\varepsilon$ is arbitrary,

$$
\begin{equation*}
(Q h, h) \leq \frac{10}{3}\left(Q_{c} h, h\right) \tag{61.8.39}
\end{equation*}
$$

However, $Q_{c}$ has finite trace. To see this, let $\left\{e_{k}\right\}$ be an orthonormal basis in $U$. Then

$$
\begin{gathered}
\sum_{k}\left(Q_{c} e_{k}, e_{k}\right)=\sum_{k} \int_{[\|x-m\| \leq c]}\left|\left(x-m, e_{k}\right)\right|^{2} d \mu(x) \\
=\int_{[\|x-m\| \leq c]} \sum_{k}\left|\left(x-m, e_{k}\right)\right|^{2} d \mu(x)=\int_{[\|x-m\| \leq c]}\|x-m\|^{2} d \mu(x) \leq c^{2}
\end{gathered}
$$

It follows from 61.8.39 $Q$ that $Q$ must also have finite trace.
That $\mu$ is uniquely determined by $m$ and $Q$ follows from Theorem 59.12.9. This proves the theorem.

Suppose you have a given $Q$ having finite trace and $m \in U$. Does there exist a Gaussian measure on $\mathscr{B}(U)$ having these as the covariance and mean respectively?

Proposition 61.8.7 Let $U$ be a real separable Hilbert space and let $m \in U$ and $Q$ be a positive, symmetric operator defined on $U$ which has finite trace. Then there exists a Gaussian measure with mean $m$ and covariance $Q$.

Proof: By Lemma 61.8.2 which comes from Kolmogorov's extension theorem, there exists a probability space $(\Omega, \mathscr{F}, P)$ and a sequence $\left\{\xi_{i}\right\}$ of independent random variables which are normally distributed with mean 0 and variance 1 . Then let

$$
X(\omega) \equiv m+\sum_{j=1}^{\infty} \sqrt{\lambda_{j}} \xi_{j}(\omega) e_{j}
$$

where the $\left\{e_{j}\right\}$ are the eigenvectors of $Q$ such that $Q e_{j}=\lambda_{j} e_{j}$. The series in the above converges in $L^{2}(\Omega ; U)$ because

$$
\left\|\sum_{j=m}^{n} \sqrt{\lambda_{j}} \xi_{j} e_{j}\right\|_{L^{2}(\Omega ; U)}^{2}=\int_{\Omega} \sum_{j=m}^{n} \lambda_{j} \xi_{j}^{2}(\omega) d P=\sum_{j=m}^{n} \lambda_{j}
$$

and so the partial sums form a Cauchy sequence in $L^{2}(\Omega ; U)$.
Now if $h \in U$, I need to show that $\omega \rightarrow(X(\omega), h)$ is normally distributed. From this it will follow that $\mathscr{L}(X)$ is Gaussian. A subsequence

$$
\left\{m+\sum_{j=1}^{n_{k}} \sqrt{\lambda_{j}} \xi_{j}(\omega) e_{j}\right\} \equiv\left\{S_{n_{k}}(\omega)\right\}
$$

of the above sequence converges pointwise a.e. to $X$.

$$
\begin{aligned}
& E(\exp (i t(X, h))) \\
= & \lim _{k \rightarrow \infty} E\left(\exp \left(i t\left(S_{n_{k}}, h\right)\right)\right) \\
= & \exp (i t(m, h)) \lim _{k \rightarrow \infty} E\left(\exp \left(i t \sum_{j=1}^{n_{k}} \sqrt{\lambda_{j}} \xi_{j}(\omega)\left(e_{j}, h\right)\right)\right)
\end{aligned}
$$

Since the $\xi_{j}$ are independent,

$$
\begin{gather*}
=\exp (i t(m, h)) \lim _{k \rightarrow \infty} \prod_{j=1}^{n_{k}} E\left(\exp \left(i t \sqrt{\lambda_{j}}\left(e_{j}, h\right)\right) \xi_{j}\right) \\
=\exp (i t(m, h)) \lim _{k \rightarrow \infty} \prod_{j=1}^{n_{k}} e^{-\frac{1}{2} t^{2} \lambda_{j}\left(e_{j}, h\right)^{2}} \\
=\exp (i t(m, h)) \lim _{k \rightarrow \infty} \exp \left(-\frac{1}{2} t^{2} \sum_{j=1}^{n_{k}} \lambda_{j}\left(e_{j}, h\right)^{2}\right) . \tag{61.8.40}
\end{gather*}
$$

Now

$$
\begin{aligned}
(Q h, h) & =\left(Q \sum_{k=1}^{\infty}\left(e_{k}, h\right) e_{k}, \sum_{j=1}^{\infty}\left(e_{j}, h\right) e_{j}\right) \\
& =\left(\sum_{k=1}^{\infty}\left(e_{k}, h\right) \lambda_{k} e_{k}, \sum_{j=1}^{\infty}\left(e_{j}, h\right) e_{j}\right) \\
& =\sum_{j=1}^{\infty} \lambda_{j}\left(e_{j}, h\right)^{2}
\end{aligned}
$$

and so, passing to the limit in 61.8 .40 yields

$$
\begin{equation*}
\exp (i t(m, h)) \exp \left(-\frac{1}{2} t^{2}(Q h, h)\right) \tag{61.8.41}
\end{equation*}
$$

which implies that $\omega \rightarrow(X(\omega), h)$ is normally distributed with mean $(m, h)$ and variance $(Q h, h)$.

Now let $\mu=\mathscr{L}(X)$. That is, for all $B \in \mathscr{B}(U)$,

$$
\mu(B) \equiv P([X \in B])
$$

In particular, $B$ could be the cylindrical set

$$
B \equiv[x:(x, h) \in A]
$$

for $A$ a Borel set in $\mathbb{R}$. Then by definition, if $h \in U$, and $A$ is a Borel set in $\mathbb{R}$,

$$
\begin{aligned}
\mu(B) & =\mu([x:(x, h) \in A]) \equiv P(\{\omega:(X(\omega), h) \in A\}) \\
& =\int_{A} \frac{1}{\sqrt{2 \pi(Q h, h)}} e^{-\frac{(t-(m, h))^{2}}{2(Q h, h)}} d t
\end{aligned}
$$

and so $x \rightarrow(x, h)$ is normally distributed. Therefore by definition, $\mu$ is a Gaussian measure.
Letting $t=1$ in 61.8.41 it follows

$$
\int_{U} e^{i(x, h)} d \mu(x)=\int_{\Omega} e^{i(X(\omega), h)} d P=\exp (i(m, h)) \exp \left(-\frac{1}{2}(Q h, h)\right)
$$

which is the characteristic function of a Gaussian measure on $U$ having covariance $Q$ and mean $m$. This proves the proposition.

### 61.9 Abstract Wiener Spaces

This material follows [21], [71] and [58]. More can be found on this subject in these references. Here $H$ will be a separable real Hilbert space.

Definition 61.9.1 Cylinder sets in $H$ are of the form

$$
\left\{x \in H:\left(\left(x, e_{1}\right), \cdots,\left(x, e_{n}\right)\right) \in F\right\}
$$

where $F \in \mathscr{B}\left(\mathbb{R}^{n}\right)$, the Borel sets of $\mathbb{R}^{n}$ and $\left\{e_{k}\right\}$ are given vectors in $H$. Denote this collection of cylinder sets as $\mathscr{C}$.

Proposition 61.9.2 The cylinder sets form an algebra of sets.
Proof: First note the complement of a cylinder set is a cylinder set.

$$
\begin{aligned}
& \left\{x \in H:\left(\left(x, e_{1}\right), \cdots,\left(x, e_{n}\right)\right) \in F\right\}^{C} \\
= & \left\{x \in H:\left(\left(x, e_{1}\right), \cdots,\left(x, e_{n}\right)\right) \in F^{C}\right\} .
\end{aligned}
$$

Now consider the intersection of two of these cylinder sets. Let the cylinder sets be

$$
\begin{aligned}
& \left\{x \in H:\left(\left(x, e_{1}\right), \cdots,\left(x, e_{n}\right)\right) \in E\right\}, \\
& \left\{x \in H:\left(\left(x, f_{1}\right), \cdots,\left(x, f_{m}\right)\right) \in F\right\}
\end{aligned}
$$

The first of these equals

$$
\left\{x \in H:\left(\left(x, e_{1}\right), \cdots,\left(x, e_{n}\right),\left(x, f_{1}\right), \cdots,\left(x, f_{m}\right)\right) \in E \times \mathbb{R}^{m}\right\}
$$

and the second equals

$$
\left\{x \in H:\left(\left(x, e_{1}\right), \cdots,\left(x, e_{n}\right),\left(x, f_{1}\right), \cdots,\left(x, f_{m}\right)\right) \in \mathbb{R}^{n} \times F\right\}
$$

Therefore, their intersection equals

$$
\begin{aligned}
& \left\{x \in H:\left(\left(x, e_{1}\right), \cdots,\left(x, e_{n}\right),\left(x, f_{1}\right), \cdots,\left(x, f_{m}\right)\right)\right. \\
& \left.\in E \times \mathbb{R}^{m} \cap \mathbb{R}^{n} \times F\right\}
\end{aligned}
$$

a cylinder set.
Now it is clear the whole of $H$ and $\emptyset$ are cylinder sets given by

$$
\{x \in H:(e, x) \in \mathbb{R}\},\{x \in H:(e, x) \in \emptyset\}
$$

respectively and so if $C_{1}, C_{2}$ are two cylinder sets,

$$
C_{1} \backslash C_{2} \equiv C_{1} \cap C_{2}^{C}
$$

which was just shown to be a cylinder set. Hence

$$
C_{1} \cup C_{2}=\left(C_{1}^{C} \cap C_{2}^{C}\right)^{C}
$$

a cylinder set. This proves the proposition.
It is good to have a more geometrical description of cylinder sets. Letting $A$ be a cylinder set as just described, let $P$ denote the orthogonal projection onto span $\left(e_{1}, \cdots, e_{n}\right)$. Also let $\alpha: P H \rightarrow \mathbb{R}^{n}$ be given by

$$
\alpha(x) \equiv\left(\left(x, e_{1}\right), \cdots,\left(x, e_{n}\right)\right)
$$

This is continuous but might not be one to one if the $e_{i}$ are not a basis for example. Then consider $\alpha^{-1}(F)$, those $x \in P H$ such that

$$
\left(\left(x, e_{1}\right), \cdots,\left(x, e_{n}\right)\right) \in F .
$$

For any $x \in H$,

$$
\left((I-P) x, e_{k}\right)=0
$$

for each $k$ and so

$$
\left(\left(x, e_{1}\right), \cdots,\left(x, e_{n}\right)\right)=\left(\left(P x, e_{1}\right), \cdots,\left(P x, e_{n}\right)\right) \in F
$$

Thus $P x \in \alpha^{-1}(F)$, which is a Borel set of $P H$ and

$$
x=P x+(I-P) x
$$

so the cylinder set is contained in

$$
\alpha^{-1}(F)+(P H)^{\perp}
$$

which is of the form

$$
(\text { Borel set of } P H)+(P H)^{\perp}
$$

On the other hand, consider a set of the form

$$
G+(P H)^{\perp}
$$

where $G$ is a Borel set in $P H$. There is a basis for $P H$ consisting of a subset of $\left\{e_{1} \cdots, e_{n}\right\}$. For simplicity, suppose it is $\left\{e_{1} \cdots, e_{k}\right\}$. Then let $\alpha_{1}: P H \rightarrow \mathbb{R}^{k}$ be given by

$$
\alpha_{1}(x) \equiv\left(\left(x, e_{1}\right), \cdots,\left(x, e_{k}\right)\right)
$$

Thus $\alpha$ is a homeomorphism of $P H$ and $\mathbb{R}^{k}$ so $\alpha_{1}(G)$ is a Borel set of $\mathbb{R}^{k}$. Now

$$
\alpha^{-1}\left(\alpha_{1}(G) \times \mathbb{R}^{n-k}\right)=G
$$

and $\alpha_{1}(G) \times \mathbb{R}^{n-k}$ is a Borel set of $\mathbb{R}^{n}$. This has proved the following important Proposition illustrated by the following picture.


Proposition 61.9.3 The cylinder sets are sets of the form

$$
B+M^{\perp}
$$

where $M$ is a finite dimensional subspace and B is a Borel subset of M. Furthermore, the collection of cylinder sets is an algebra.

Lemma 61.9.4 $\sigma(\mathscr{C})$, the smallest $\sigma$ algebra containing $\mathscr{C}$, contains the Borel sets of $H, \mathscr{B}(H)$.

Proof: It follows from the definition of these cylinder sets that if $f_{i}(x) \equiv\left(x, e_{i}\right)$, so that $f_{i} \in H^{\prime}$, then with respect to $\sigma(\mathscr{C})$, each $f_{i}$ is measurable. It follows that every linear combination of the $f_{i}$ is also measurable with respect to $\sigma(\mathscr{C})$. However, this set of linear
combinations is dense in $H^{\prime}$ and so the conclusion of the lemma follows from Lemma 59.4.2 on Page 1868. This proves the lemma.

Also note that the mapping

$$
x \rightarrow\left(\left(x, e_{1}\right), \cdots,\left(x, e_{n}\right)\right)
$$

is a $\sigma(\mathscr{C})$ measurable map. Restricting it to span $\left(e_{1}, \cdots, e_{n}\right)$, it is Borel measurable. Next is a definition of a Gaussian measure defined on $\mathscr{C}$. While this is what it is called, it is a fake measure in general because it cannot be extended to a countably additive measure on $\sigma(\mathscr{C})$. This will be shown below.

Definition 61.9.5 Let $Q \in \mathscr{L}(H, H)$ be self adjoint and satisfy

$$
(Q x, x)>0
$$

for all $x \in H, x \neq 0$. Define $v$ on the cylinder sets, $\mathscr{C}$ by the following rule. For $\left\{e_{k}\right\}_{k=1}^{n}$ an orthonormal set in $H$,

$$
\begin{aligned}
& v\left(\left\{x \in H:\left(\left(x, e_{1}\right), \cdots,\left(x, e_{n}\right)\right) \in F\right\}\right) \\
\equiv & \frac{1}{(2 \pi)^{n / 2}\left(\operatorname{det}\left(\theta^{*} Q \theta\right)\right)^{1 / 2}} \int_{F} e^{-\frac{1}{2} *^{*} \theta^{*} Q^{-1} \theta \mathbf{t}} d t .
\end{aligned}
$$

where here

$$
\theta \mathbf{t} \equiv \sum_{i=1}^{n} t_{i} e_{i} .
$$

Note that the cylinder set is of the form

$$
\theta F+\operatorname{span}\left(e_{1}, \cdots, e_{n}\right)^{\perp} .
$$

Thus if $B+M^{\perp}$ is a typical cylinder set, choose an orthonormal basis for $M,\left\{e_{k}\right\}_{k=1}^{n}$ and do the above definition with $F=\theta^{-1} B$.

To see this last claim which is like what was done earlier, let

$$
\left(\left(x, e_{1}\right), \cdots,\left(x, e_{n}\right)\right) \in F
$$

Then $\theta\left(\left(x, e_{1}\right), \cdots,\left(x, e_{n}\right)\right)=\sum_{i}\left(x, e_{i}\right) e_{i}=P x$ and so

$$
\begin{aligned}
x & =x-P x+P x=x-P x+\theta\left(\left(x, e_{1}\right), \cdots,\left(x, e_{n}\right)\right) \\
& \in \theta F+\operatorname{span}\left(e_{1}, \cdots, e_{n}\right)^{\perp}
\end{aligned}
$$

Thus

$$
\left\{x \in H:\left(\left(x, e_{1}\right), \cdots,\left(x, e_{n}\right)\right) \in F\right\} \subseteq \theta F+\operatorname{span}\left(e_{1}, \cdots, e_{n}\right)^{\perp}
$$

To see the other inclusion, if $\mathbf{t} \in F$ and $y \in \operatorname{span}\left(e_{1}, \cdots, e_{n}\right)^{\perp}$, then if $x=\theta \mathbf{t}$, it follows

$$
t_{i}=\left(x, e_{i}\right)
$$

and so $\left(\left(x, e_{1}\right), \cdots,\left(x, e_{n}\right)\right) \in F$. But $\left(y, e_{k}\right)=0$ for all $k$ and so $x+y$ is in

$$
\left\{x \in H:\left(\left(x, e_{1}\right), \cdots,\left(x, e_{n}\right)\right) \in F\right\}
$$

Lemma 61.9.6 The above definition is well defined.
Proof: Let $\left\{f_{k}\right\}$ be another orthonormal set such that for $F, G$ Borel sets in $\mathbb{R}^{n}$,

$$
\begin{aligned}
& A=\left\{x \in H:\left(\left(x, e_{1}\right), \cdots,\left(x, e_{n}\right)\right) \in F\right\} \\
= & \left\{x \in H:\left(\left(x, f_{1}\right), \cdots,\left(x, f_{n}\right)\right) \in G\right\}
\end{aligned}
$$

I need to verify $\boldsymbol{v}(A)$ is the same using either $\left\{f_{k}\right\}$ or $\left\{e_{k}\right\}$. Let $\mathbf{a} \in G$. Then

$$
x \equiv \sum_{i=1}^{n} a_{i} f_{i} \in A
$$

because $\left(x, f_{k}\right)=a_{k}$. Therefore, for this $x$ it is also true that $\left(\left(x, e_{1}\right), \cdots\left(x, e_{n}\right)\right) \in F$. In other words for $\mathbf{a} \in G$,

$$
\left(\sum_{i=1}^{n}\left(e_{1}, f_{i}\right) a_{i}, \cdots, \sum_{i=1}^{n}\left(e_{n}, f_{i}\right) a_{i}\right) \in F
$$

Let $L \in \mathscr{L}\left(\mathbb{R}^{n}, \mathbb{R}^{n}\right)$ be defined by

$$
L \mathbf{a} \equiv \sum_{i} L_{j i} a_{i}, L_{j i} \equiv\left(e_{j}, f_{i}\right)
$$

Since the $\left\{e_{j}\right\}$ and $\left\{f_{k}\right\}$ are orthonormal, this mapping is unitary. Also this has shown that

$$
L G \subseteq F
$$

Similarly

$$
L^{*} F \subseteq G
$$

where $L^{*}$ has the $i j^{t h}$ entry $L_{i j}^{*}=\left(f_{i}, e_{j}\right)$ as above and $L^{*}$ is the inverse of $L$ because $L$ is unitary. Thus

$$
F=L\left(L^{*}(F)\right) \subseteq L(G) \subseteq F
$$

showing that $L G=F$ and $L^{*} F=G$.
Now let $\theta_{e} \mathbf{t} \equiv \sum_{i} t_{i} e_{i}$ with $\theta_{f}$ defined similarly. Then the definition of $v(A)$ corresponding to $\left\{e_{i}\right\}$ is

$$
v(A) \equiv \frac{1}{(2 \pi)^{n / 2}\left(\operatorname{det}\left(\theta_{e}^{*} Q \theta_{e}\right)\right)^{1 / 2}} \int_{F} e^{-\frac{1}{2} \mathbf{t}^{*} \theta_{e}^{*} Q^{-1} \theta_{e} \mathbf{t}} d t
$$

Now change the variables letting $\mathbf{t}=L \mathbf{s}$ where $\mathbf{s} \in G$.
From the definition,

$$
\begin{gathered}
\theta_{e} L \mathbf{s}=\sum_{j} \sum_{i}\left(e_{j}, f_{i}\right) s_{i} e_{j} \\
=\sum_{j}\left(e_{j}, \sum_{i} f_{i} s_{i}\right) e_{j}=\sum_{j}\left(e_{j}, \theta_{f} \mathbf{s}\right) e_{j}
\end{gathered}
$$

and so

$$
L \mathbf{s}=\left(\left(e_{1}, \theta_{f} \mathbf{s}\right), \cdots,\left(e_{n}, \theta_{f} \mathbf{s}\right)\right)=\theta_{e}^{*} \theta_{f} \mathbf{s}
$$

where from the definition,

$$
\begin{aligned}
\left(\theta_{e}^{*} \theta_{e} \mathbf{s}, \mathbf{t}\right) & =\sum_{i} t_{i}\left(\sum_{j} s_{j} e_{j}, e_{i}\right) \\
& =\sum_{i} t_{i} s_{i}=(\mathbf{s}, \mathbf{t})
\end{aligned}
$$

and so $\theta_{e}^{*} \theta_{e}$ is the identity on $\mathbb{R}^{n}$ and similar reasoning yields $\theta_{e} \theta_{e}^{*}$ is the identity on $\theta_{e}\left(\mathbb{R}^{n}\right)$. Then using the change of variables formula and the fact $|\operatorname{det}(L)|=1$,

$$
\begin{aligned}
& \frac{1}{(2 \pi)^{n / 2}\left(\operatorname{det}\left(\theta_{e}^{*} Q \theta_{e}\right)\right)^{1 / 2}} \int_{F} e^{-\frac{1}{2} \mathbf{t}^{*} \theta_{e}^{*} Q^{-1} \theta_{e} \mathbf{t}} d t \\
= & \frac{1}{(2 \pi)^{n / 2}\left(\operatorname{det}\left(\theta_{e}^{*} Q \theta_{e}\right)\right)^{1 / 2}} \int_{G} e^{-\frac{1}{2} s^{*} L^{*} \theta_{e}^{*} Q^{-1} \theta_{e} L s} d s \\
= & \frac{1}{(2 \pi)^{n / 2}\left(\operatorname{det}\left(\theta_{f}^{*} Q \theta_{f}\right)\right)^{1 / 2}} \int_{G} e^{-\frac{1}{2} s^{*} \theta_{f}^{*} \theta_{e} \theta_{e}^{*} Q^{-1} \theta_{e} \theta_{e}^{*} \theta_{f} \mathbf{s}} d s \\
= & \frac{1}{(2 \pi)^{n / 2}\left(\operatorname{det}\left(\theta_{f}^{*} Q \theta_{f}\right)\right)^{1 / 2}} \int_{G} e^{-\frac{1}{2} s^{*} \theta_{f}^{*} Q^{-1} \theta_{f} \mathbf{s}} d s
\end{aligned}
$$

where part of the justification is as follows.

$$
\begin{aligned}
& \operatorname{det}\left(\theta_{f}^{*} Q \theta_{f}\right)=\operatorname{det}\left(\theta_{f}^{*} \theta_{e} \theta_{e}^{*} Q \theta_{e} \theta_{e}^{*} \theta_{f}\right) \\
& \quad=\operatorname{det}\left(\theta_{f}^{*} \theta_{e}\right) \operatorname{det}\left(\theta_{e}^{*} Q \theta_{e}\right) \operatorname{det}\left(\theta_{e}^{*} \theta_{f}\right) \\
& =\operatorname{det}\left(\theta_{e}^{*} Q \theta_{e}\right)
\end{aligned}
$$

because

$$
\operatorname{det}\left(\theta_{f}^{*} \boldsymbol{\theta}_{e}\right) \operatorname{det}\left(\theta_{e}^{*} \theta_{f}\right)=\operatorname{det}\left(\theta_{f}^{*} \theta_{e} \theta_{e}^{*} \theta_{f}\right)=\operatorname{det}\left(\theta_{f}^{*} \theta_{f}\right)=1
$$

This proves the lemma.
It would be natural to try to extend $v$ to the $\sigma$ algebra determined by $\mathscr{C}$ and obtain a measure defined on this $\sigma$ algebra. However, this is always impossible in the case where $Q=I$.

Proposition 61.9.7 For $Q=I$, v cannot be extended to a measure defined on $\sigma(\mathscr{C})$ whenever $H$ is infinite dimensional.

Proof: Let $\left\{e_{n}\right\}$ be a complete orthonormal set of vectors in $H$. Then first note that $H$ is a cylinder set.

$$
H=\left\{x \in H:\left(x, e_{1}\right) \in \mathbb{R}\right\}
$$

and so

$$
v(H)=\frac{1}{\sqrt{2 \pi}} \int_{\mathbb{R}} e^{-\frac{1}{2} t^{2}} d t=1
$$

However, $H$ is also equal to the countable union of the sets,

$$
A_{n} \equiv\left\{x \in H:\left(\left(x, e_{1}\right)_{H}, \cdots,\left(x, e_{a_{n}}\right)_{H}\right) \in B(\mathbf{0}, n)\right\}
$$

where $a_{n} \rightarrow \infty$.

$$
\begin{aligned}
v\left(A_{n}\right) & \equiv \frac{1}{(\sqrt{2 \pi})^{a_{n}}} \int_{B(\mathbf{0}, n)} e^{-\frac{1}{2}|\mathbf{t}|^{2}} d t \\
& \leq \frac{1}{(\sqrt{2 \pi})^{a_{n}}} \int_{-n}^{n} \cdots \int_{-n}^{n} e^{-|\mathbf{t}|^{2} / 2} d t_{1} \cdots d t_{a_{n}} \\
& =\left(\frac{\int_{-n}^{n} e^{-x^{2} / 2} d x}{\sqrt{2 \pi}}\right)^{a_{n}}
\end{aligned}
$$

Now pick $a_{n}$ so large that the above is smaller than $1 / 2^{n+1}$. This can be done because for no matter what choice of $n$,

$$
\frac{\int_{-n}^{n} e^{-x^{2} / 2} d x}{\sqrt{2 \pi}}<1
$$

Then

$$
\sum_{n=1}^{\infty} v\left(A_{n}\right) \leq \sum_{n=1}^{\infty} \frac{1}{2^{n+1}}=\frac{1}{2}
$$

This proves the proposition and shows something else must be done to get a countably additive measure from $v$.

However, let $\mu(C) \equiv v_{M}(C)$ where $C$ is a cylinder set of the form $C=B+M^{\perp}$ for $M$ a finite dimensional subspace.

Proposition 61.9.8 $\mu$ is finitely additive on $\mathscr{C}$ the algebra of cylinder sets.
Proof: Let

$$
\begin{aligned}
A & \equiv\left\{x \in H:\left(\left(x, e_{1}\right), \cdots,\left(x, e_{n}\right)\right) \in E\right\} \\
B & \equiv\left\{x \in H:\left(\left(x, f_{1}\right), \cdots,\left(x, f_{m}\right)\right) \in F\right\}
\end{aligned}
$$

be two disjoint cylinder sets. Then writing them differently as was done earlier they are

$$
\left\{x \in H:\left(\left(x, e_{1}\right), \cdots,\left(x, e_{n}\right),\left(x, f_{1}\right), \cdots,\left(x, f_{m}\right)\right) \in E \times \mathbb{R}^{m}\right\}
$$

and

$$
\left\{x \in H:\left(\left(x, e_{1}\right), \cdots,\left(x, e_{n}\right),\left(x, f_{1}\right), \cdots,\left(x, f_{m}\right)\right) \in \mathbb{R}^{n} \times F\right\}
$$

respectively. Hence the two sets $E \times \mathbb{R}^{m}, \mathbb{R}^{n} \times F$ must be disjoint. Then the definition yields $\mu(A \cup B)=\mu(A)+\mu(B)$. This proves the proposition.

Definition 61.9.9 Let $H$ be a separable Hilbert space and let $\|\cdot\|$ be a norm defined on $H$ which has the following property. Whenever $\left\{e_{n}\right\}$ is an orthonormal sequence of vectors in $H$ and $\mathscr{F}\left(\left\{e_{n}\right\}\right)$ consists of the set of all orthogonal projections onto the span of finitely
many of the $e_{k}$ the following condition holds. For every $\varepsilon>0$ there exists $P_{\varepsilon} \in \mathscr{F}\left(\left\{e_{n}\right\}\right)$ such that if $P \in \mathscr{F}\left(\left\{e_{n}\right\}\right)$ and $P P_{\varepsilon}=0$, then

$$
v(\{x \in H:\|P x\|>\varepsilon\})<\varepsilon .
$$

Then $\|\cdot\|$ is called Gross measurable.
The following lemma is a fundamental result about Gross measurable norms. It is about the continuity of $\|\cdot\|$. It is obvious that with respect to the topology determined by $\|\cdot\|$ that $x \rightarrow\|x\|$ is continuous. However, it would be interesting if this were the case with respect to the topology determined by the norm on $H,|\cdot|$. This lemma shows this is the case and so the funny condition above implies $x \rightarrow\|x\|$ is a continuous, hence Borel measurable function.

Lemma 61.9.10 Let $\|\cdot\|$ be Gross measurable. Then there exists $c>0$ such that

$$
\|x\| \leq c|x|
$$

for all $x \in H$. Furthermore, the above definition is well defined.
Proof: First it is important to consider the question whether the above definition is well defined. To do this note that on $P H$, the two norms are equivalent because $P H$ is a finite dimensional space. Let $G=\{y \in P H:\|y\|>\varepsilon\}$ so $G$ is an open set in $P H$. Then

$$
\{x \in H:\|P x\|>\varepsilon\}
$$

equals

$$
\{x \in H: P x \in G\}
$$

which equals a set of the form

$$
\left\{x \in H:\left(\left(x, e_{i_{1}}\right)_{H}, \cdots,\left(x, e_{i_{m}}\right)_{H}\right) \in G^{\prime}\right\}
$$

for $G^{\prime}$ an open set in $\mathbb{R}^{m}$ and so everything makes sense in the above definition.
Now it is necessary to verify $\|\cdot\| \leq c|\cdot|$. If it is not so, there exists $e_{1}$ such that

$$
\left\|e_{1}\right\| \geq 1,\left|e_{1}\right|=1
$$

Suppose $\left\{e_{k}\right\}_{k=1}^{n}$ have been chosen such that each is a unit vector in $H$ and $\left\|e_{k}\right\| \geq k$. Then considering span $\left(e_{1}, \cdots, e_{n}\right)^{\perp}$ if for every $x \in \operatorname{span}\left(e_{1}, \cdots, e_{n}\right)^{\perp},\|x\| \leq c|x|$, then if $z \in H$ is arbitrary, $z=x+y$ where $y \in \operatorname{span}\left(e_{1}, \cdots, e_{n}\right)$ and so since the two norms are equivalent on a finite dimensional subspace, there exists $c^{\prime}$ corresponding to $\operatorname{span}\left(e_{1}, \cdots, e_{n}\right)$ such that

$$
\begin{aligned}
\|z\|^{2} & \leq(\|x\|+\|y\|)^{2} \leq 2\|x\|^{2}+2\|y\|^{2} \\
& \leq 2 c^{2}|x|^{2}+2 c^{\prime}|y|^{2} \\
& \leq\left(2 c^{2}+2 c^{\prime 2}\right)\left(|x|^{2}+|y|^{2}\right) \\
& =\left(2 c^{2}+2 c^{\prime 2}\right)|z|^{2}
\end{aligned}
$$

and the lemma is proved. Therefore it can be assumed, there exists

$$
e_{n+1} \in \operatorname{span}\left(e_{1}, \cdots, e_{n}\right)^{\perp}
$$

such that $\left|e_{n+1}\right|=1$ and $\left\|e_{n+1}\right\| \geq n+1$.
This constructs an orthonormal set of vectors, $\left\{e_{k}\right\}$. Letting $0<\varepsilon<\frac{1}{2}$, it follows since $\|\cdot\|$ is measurable, there exists $P_{\varepsilon} \in \mathscr{F}\left(\left\{e_{n}\right\}\right)$ such that if $P P_{\varepsilon}=0$ where $P \in \mathscr{F}\left(\left\{e_{n}\right\}\right)$, then

$$
v(\{x \in H:\|P x\|>\varepsilon\})<\varepsilon .
$$

Say $P_{\varepsilon}$ is the projection onto the span of finitely many of the $e_{k}$, the last one being $e_{N}$. Then for $n>N$ and $P_{n}$ the projection onto $e_{n}$, it follows $P_{\varepsilon} P_{n}=0$ and from the definition of $v$,

$$
\begin{aligned}
\varepsilon & >v\left(\left\{x \in H:\left|\left|P_{n} x\right|\right|>\varepsilon\right\}\right) \\
& =v\left(\left\{x \in H:\left|\left(x, e_{n}\right)\right|\left\|e_{n+1}\right\|>\varepsilon\right\}\right) \\
& =v\left(\left\{x \in H:\left|\left(x, e_{n}\right)\right|>\varepsilon /\left\|e_{n+1}\right\|\right\}\right) \\
& \geq v\left(\left\{x \in H:\left|\left(x, e_{n}\right)\right|>\varepsilon /(n+1)\right\}\right) \\
& >\frac{1}{\sqrt{2 \pi}} \int_{\varepsilon /(n+1)}^{\infty} e^{-x^{2} / 2} d x
\end{aligned}
$$

which yields a contradiction for all $n$ large enough. This proves the lemma.
What are examples of Gross measurable norms defined on a separable Hilbert space, $H$ ? The following lemma gives an important example.

Lemma 61.9.11 Let $H$ be a separable Hilbert space and let $A \in \mathscr{L}_{2}(H, H)$, a Hilbert Schmidt operator. Thus A is a continuous linear operator with the property that for any orthonormal set, $\left\{e_{k}\right\}$,

$$
\sum_{k=1}^{\infty}\left|A e_{k}\right|^{2}<\infty .
$$

Then define ||•\| by

$$
\| x| | \equiv|A x|_{H}
$$

Then if $\|\cdot\|$ is a norm, it is measurable ${ }^{1}$.
Proof: Let $\left\{e_{k}\right\}$ be an orthonormal sequence. Let $P_{n}$ denote the orthogonal projection onto span $\left(e_{1}, \cdots, e_{n}\right)$. Let $\varepsilon>0$ be given. Since $A$ is a Hilbert Schmidt operator, there exists $N$ such that

$$
\sum_{k=N}^{\infty}\left|A e_{k}\right|^{2}<\alpha
$$

where $\alpha$ is chosen very small. In fact, $\alpha$ is chosen such that $\alpha<\varepsilon^{2} / r^{2}$ where $r$ is sufficiently large that

$$
\begin{equation*}
\frac{2}{\sqrt{2 \pi}} \int_{r}^{\infty} e^{-t^{2} / 2} d t<\varepsilon \tag{61.9.42}
\end{equation*}
$$

[^42]Let $P$ denote an orthogonal projection in $\mathscr{F}\left(\left\{e_{k}\right\}\right)$ such that $P P_{N}=0$. Thus $P$ is the projection on to span $\left(e_{i_{1}}, \cdots, e_{i_{m}}\right)$ where each $i_{k}>N$. Then

$$
\begin{aligned}
& v(\{x \in H:||P x||>\varepsilon\}) \\
= & v(\{x \in H:|A P x|>\varepsilon\})
\end{aligned}
$$

Now $P x=\sum_{j=1}^{m}\left(x, e_{i_{j}}\right) e_{i_{j}}$ and the above reduces to

$$
\begin{aligned}
& v\left(\left\{x \in H:\left|\sum_{j=1}^{m}\left(x, e_{i_{j}}\right) A e_{i_{j}}\right|>\varepsilon\right\}\right) \leq \\
& v\left(\left\{x \in H:\left(\sum_{j=1}^{m}\left|\left(x, e_{i_{j}}\right)\right|^{2}\right)^{1 / 2}\left(\sum_{j=1}^{m}\left|A e_{i_{j}}\right|^{2}\right)^{1 / 2}>\varepsilon\right\}\right) \\
\leq & v\left(\left\{x \in H:\left(\sum_{j=1}^{m}\left|\left(x, e_{i_{j}}\right)\right|^{2}\right)^{1 / 2} \alpha^{1 / 2}>\varepsilon\right\}\right) \\
= & v\left(\left\{x \in H:\left(\sum_{j=1}^{m}\left|\left(x, e_{i_{j}}\right)\right|^{2}\right)^{1 / 2}>\frac{\varepsilon}{\alpha^{1 / 2}}\right\}\right) \\
= & v\left(\left\{x \in H:\left(\left(x, e_{i_{1}}\right), \cdots,\left(x, e_{i_{m}}\right)\right) \in B\left(\mathbf{0}, \frac{\varepsilon}{\alpha^{1 / 2}}\right)^{C}\right\}\right) \\
\leq & v\left(\left\{x \in H: \max \left\{\left|\left(x, e_{i_{j}}\right)\right|\right\}>\frac{\varepsilon}{\sqrt{m} \alpha^{1 / 2}}\right\}\right)
\end{aligned}
$$

This is no larger than

$$
\begin{gathered}
\frac{1}{(\sqrt{2 \pi})^{m}} \int_{\left|t_{1}\right|>\frac{\varepsilon}{\sqrt{m} \sqrt{\alpha}}} \int_{\left|t_{2}\right|>\frac{\varepsilon}{\sqrt{m} \sqrt{\alpha}}} \cdots \int_{\left|t_{m}\right|>\frac{\varepsilon}{\sqrt{m} \sqrt{\alpha}}} e^{-|t|^{2} / 2} d t_{m} \cdots d t_{1} \\
=\left(\frac{2 \int_{\varepsilon /\left(\sqrt{m} \alpha^{1 / 2}\right)} e^{-t^{2} / 2 d t}}{\sqrt{2 \pi}}\right)^{m}
\end{gathered}
$$

which by Jensen's inequality is no larger than

$$
\begin{aligned}
\frac{2 \int_{\varepsilon /\left(\sqrt{m} \alpha^{1 / 2}\right)}^{\infty} e^{-m t^{2} / 2} d t}{\sqrt{2 \pi}} & =\frac{2 \frac{1}{\sqrt{m}} \int_{\varepsilon /\left(\alpha^{1 / 2}\right)}^{\infty} e^{-t^{2} / 2} d t}{\sqrt{2 \pi}} \\
& \leq \frac{2 \int_{\varepsilon /(\varepsilon / r)}^{\infty} e^{-t^{2} / 2} d t}{\sqrt{2 \pi}} \\
& =\frac{2 \int_{r}^{\infty} e^{-t^{2} / 2} d t}{\sqrt{2 \pi}}<\varepsilon
\end{aligned}
$$

By 61.9.42. This proves the lemma.

Definition 61.9.12 A triple, $(i, H, B)$ is called an abstract Wiener space if $B$ is a separable Banach space and $H$ is a separable Hilbert space such that $H$ is dense and continuously embedded in $B$ and the norm $\|\cdot\|$ on $B$ is Gross measurable.

Next consider a weaker norm for $H$ which comes from the inner product

$$
(x, y)_{E} \equiv \sum_{k=1}^{\infty} \frac{1}{k^{2}}\left(x, e_{k}\right)_{H}\left(y, e_{k}\right)_{H}
$$

Then let $E$ be the completion of $H$ with respect to this new norm. Thus $\left\{k e_{k}\right\}$ is a complete orthonormal basis for $E$. This follows from the density of $H$ in $E$ along with the obvious observation that in the above inner product, $\left\{k e_{k}\right\}$ is an orthonormal set of vectors.

Lemma 61.9.13 There exists a countably additive Gaussian measure, $\lambda$ defined on $\mathscr{B}(E)$. This measure is the law of the random variable,

$$
X(\omega) \equiv \sum_{k=1}^{\infty} \xi_{k}(\omega) e_{k}
$$

where $\left\{\xi_{k}\right\}$ denotes a sequence of independent normally distributed random variables having mean 0 and variance 1, the series converging pointwise a.e. in E. Also

$$
k^{2}\left(X(\omega), e_{k}\right)_{E}=\xi_{k}(\omega) \text { a.e. }
$$

Proof: Observe that $\sum_{k=1}^{\infty} \frac{1}{k^{2}}\left(k e_{k}\right) \otimes\left(k e_{k}\right)$ is a nuclear operator on the Hilbert space, $E$. Letting $\left\{\xi_{k}\right\}$ be a sequence of independent random variables each normally distributed with mean 0 and variance 1 , that

$$
\begin{equation*}
X(\omega) \equiv \sum_{k=1}^{\infty} \frac{1}{k} \xi_{k}(\omega) k e_{k}=\sum_{k=1}^{\infty} \xi_{k}(\omega) e_{k} \tag{61.9.43}
\end{equation*}
$$

is a random variable with values in $E$ and $\mathscr{L}(X)$ is a Gaussian measure on $\mathscr{B}(E)$, the series converging pointwise a.e. in $E$. Let $\lambda$ be the name of this Gaussian measure and denote the probability space on which the $\xi_{k}$ are defined as $(\Omega, \mathscr{F}, P)$. Thus for $F \in \mathscr{B}(E)$,

$$
\lambda(F) \equiv P(\{\omega \in \Omega: X(\omega) \in F\})
$$

Finally, denoting by $X_{N}$, the partial sum,

$$
X_{N}(\omega) \equiv \sum_{k=1}^{N} \xi_{k}(\omega) e_{k}
$$

the definition of $(\cdot, \cdot)_{E}$ on $H$ and a simple computation yields

$$
\begin{align*}
\xi_{k}(\omega) & =\lim _{N \rightarrow \infty} k^{2}\left(X_{N}(\omega), e_{k}\right)_{E} \\
& =k^{2}\left(X(\omega), e_{k}\right)_{E} \tag{61.9.44}
\end{align*}
$$

One can pass to the limit because $X_{N}(\omega)$ converges to $X(\omega)$ in $E$. This proves the lemma.

Theorem 61.9.14 Let $(i, H, B)$ be an abstract Wiener space. Then there exists a Gaussian measure on the Borel sets of $B$.

Proof: Let $E$ be defined above as the completion of $H$ with respect to that weaker norm. Then from Lemma 61.9.13 and $X(\omega)$ given above in 61.9.43,

$$
k^{2}\left(X(\omega), e_{k}\right)_{E}=\xi_{k}(\omega) \text { a.e. } \omega
$$

Let $\left\{e_{n}\right\}$ be a complete orthonormal set for $H$. There exists an increasing sequence of projections, $\left\{Q_{n}\right\} \subseteq \mathscr{F}\left(\left\{e_{n}\right\}\right)$ such that $Q_{n} x \rightarrow x$ in $H$ for each $x \in H$. Say $Q_{n}$ is the orthogonal projection onto span $\left(e_{1}, \cdots, e_{p_{n}}\right)$. Then since $\|\cdot\|$ is measurable, these can be chosen such that if $Q$ is the orthogonal projection onto span $\left(e_{1}, \cdots, e_{k}\right)$ for some $k>p_{n}$ then

$$
v\left(\left\{x:\left\|Q x-Q_{n} x\right\|>2^{-n}\right\}\right)<2^{-n}
$$

In particular,

$$
v\left(\left\{x:\left\|Q_{n} x-Q_{m} x\right\|>2^{-m}\right\}\right)<2^{-m}
$$

whenever $n \geq m$.
I would like to consider the infinite series,

$$
S(\omega) \equiv \sum_{k=1}^{\infty} k^{2}\left(X(\omega), e_{k}\right)_{E} e_{k} \in B
$$

converging in $B$ but of course this might make no sense because the series might not converge. It was shown above that the series converges in $E$ but it has not been shown to converge in $B$.

Suppose the series did converge a.e. Then let $f \in B^{\prime}$ and consider the random variable $f \circ S$ which maps $\Omega$ to $\mathbb{R}$. I would like to verify this is normally distributed. First note that the following finite sum is weakly measurable and separably valued so it is strongly measurable with values in $B$.

$$
S_{p_{n}}(\omega) \equiv \sum_{k=1}^{p_{n}} k^{2}\left(X(\omega), e_{k}\right)_{E} e_{k}
$$

Since $f \in B^{\prime}$ which is a subset of $H^{\prime}$ due to the assumption that $H$ is dense in $B$, there exists a unique $v \in H$ such that $f(x)=(x, v)$ for all $x \in H$. Then from the above sum,

$$
f\left(S_{p_{n}}(\omega)\right)=\left(S_{p_{n}}(\omega), v\right)=\sum_{k=1}^{p_{n}} k^{2}\left(X(\omega), e_{k}\right)_{E}\left(e_{k}, v\right)
$$

which by Lemma 61.9.13 equals

$$
\sum_{k=1}^{p_{n}}\left(e_{k}, v\right)_{H} \xi_{k}(\omega)
$$

a finite linear combination of the independent $N(0,1)$ random variables, $\xi_{k}(\omega)$. Then it follows

$$
\omega \rightarrow f\left(S_{p_{n}}(\omega)\right)
$$

is also normally distributed and has mean 0 and variance equal to

$$
\sum_{k=1}^{p_{n}}\left(e_{k}, v\right)_{H}^{2}
$$

Then it seems reasonable to suppose

$$
\begin{align*}
E\left(e^{i t f \circ S}\right) & =\lim _{n \rightarrow \infty} E\left(e^{i t f \circ S_{p_{n}}}\right) \\
& =\lim _{n \rightarrow \infty} e^{-t^{2} \sum_{k=1}^{p_{n}}\left(e_{k}, v\right)_{H}^{2}} \\
& =e^{-t^{2} \sum_{k=1}^{\infty}\left(e_{k}, v\right)_{H}^{2}} \\
& =e^{-t^{2}|v|_{H}^{2}} \tag{61.9.45}
\end{align*}
$$

the characteristic function of a random variable which is $N\left(0,|v|_{H}^{2}\right)$. Thus at least formally, this would imply for all $f \in B^{\prime}, f \circ S$ is normally distributed and so if $\mu=\mathscr{L}(S)$, then by Lemma 61.7.3 it follows $\mu$ is a Gaussian measure.

What is missing to make the above a proof? First of all, there is the issue of the sum. Next there is the problem of passing to the limit in the little argument above in which the characteristic function is used.

First consider the sum. Note that $Q_{n} X(\omega) \in H$. Then for any $n>p_{m}$,

$$
\begin{align*}
& P\left(\left\{\omega \in \Omega:\left\|S_{n}(\omega)-S_{p_{m}}(\omega)\right\|>2^{-m}\right\}\right) \\
= & P\left(\left\{\omega \in \Omega:\left\|\sum_{k=p_{m}+1}^{n} k^{2}\left(X(\omega), e_{k}\right)_{E} e_{k}\right\|>2^{-m}\right\}\right) \\
= & P\left(\left\{\omega \in \Omega:\left\|\sum_{k=p_{m}+1}^{n} \xi_{k}(\omega) e_{k}\right\|>2^{-m}\right\}\right) \tag{61.9.46}
\end{align*}
$$

Let $Q$ be the orthogonal projection onto span $\left(e_{1}, \cdots, e_{n}\right)$. Define

$$
F \equiv\left\{x \in\left(Q-Q_{m}\right) H:\|x\|>2^{-m}\right\}
$$

Then continuing the chain of equalities ending with 61.9.46,

$$
\begin{aligned}
& =P\left(\left\{\omega \in \Omega: \sum_{k=p_{m}+1}^{n} \xi_{k}(\omega) e_{k} \in F\right\}\right) \\
= & P\left(\left\{\omega \in \Omega:\left(\xi_{n}(\omega), \cdots, \xi_{p_{m}+1}(\omega)\right) \in F^{\prime}\right\}\right) \\
= & v\left(\left\{x \in H:\left(\left(x, e_{n}\right)_{H}, \cdots,\left(x, e_{p_{m}+1}\right)_{H}\right) \in F^{\prime}\right\}\right) \\
= & v\left(\left\{x \in H: Q(x)-Q_{m}(x) \in F\right\}\right) \\
= & v\left(\left\{x \in H:\left\|Q(x)-Q_{m}(x)\right\|>2^{-m}\right\}\right)<2^{-m} .
\end{aligned}
$$

This has shown that

$$
\begin{equation*}
P\left(\left\{\omega \in \Omega: \| S_{n}(\omega)-S_{p_{m}}(\omega)| |>2^{-m}\right\}\right)<2^{-m} \tag{61.9.47}
\end{equation*}
$$

for all $n>p_{m}$. In particular, the above is true if $n=p_{n}$ for $n>m$.
If $\left\{S_{p_{n}}(\omega)\right\}$ fails to converge, then $\omega$ must be contained in the set,

$$
\begin{equation*}
A \equiv \cap_{m=1}^{\infty} \cup_{n=m}^{\infty}\left\{\omega \in \Omega:\left\|S_{p_{n}}(\omega)-S_{p_{m}}(\omega)\right\|>2^{-m}\right\} \tag{61.9.48}
\end{equation*}
$$

because if $\omega$ is in the complement of this set,

$$
\cup_{m=1}^{\infty} \cap_{n=m}^{\infty}\left\{\omega \in \Omega:| | S_{p_{n}}(\omega)-S_{p_{m}}(\omega) \| \leq 2^{-m}\right\}
$$

it follows $\left\{S_{p_{n}}(\omega)\right\}_{n=1}^{\infty}$ is a Cauchy sequence and so it must converge. However, the set in 61.9.48 is a set of measure 0 because of 61.9.47 and the observation that for all $m$,

$$
\begin{aligned}
P(A) & \leq \sum_{n=m}^{\infty} P\left(\left\{\omega \in \Omega:\left\|S_{p_{n}}(\omega)-S_{p_{m}}(\omega)\right\|>2^{-m}\right\}\right) \\
& \leq \sum_{n=m}^{\infty} \frac{1}{2^{m}}
\end{aligned}
$$

Thus the subsequence $\left\{S_{p_{n}}\right\}$ of the sequence of partial sums of the above series does converge pointwise in $B$ and so the dominated convergence theorem also verifies that the computations involving the characteristic function in 61.9.45 are correct.

The random variable obtained as the limit of the partial sums, $\left\{S_{p_{n}}(\omega)\right\}$ described above is strongly measurable because each $S_{p_{n}}(\omega)$ is strongly measurable due to each of these being weakly measurable and separably valued. Thus the measure given as the law of $S$ defined as

$$
S(\omega) \equiv \lim _{n \rightarrow \infty} S_{p_{n}}(\omega)
$$

is defined on the Borel sets of $B$.This proves the theorem.
Also, there is an important observation from the proof which I will state as the following corollary.

Corollary 61.9.15 Let $(i, H, B)$ be an abstract Wiener space. Then there exists a Gaussian measure on the Borel sets of B. This Gaussian measure equals $\mathscr{L}(S)$ where $S(\omega)$ is the a.e. limit of a subsequence of the sequence of partial sums,

$$
S_{p_{n}}(\omega) \equiv \sum_{k=1}^{p_{n}} \xi_{k}(\omega) e_{k}
$$

for $\left\{\xi_{k}\right\}$ a sequence of independent random variables which are normal with mean 0 and variance 1 which are defined on a probability space, $(\Omega, \mathscr{F}, P)$. Furthermore, for any $k>p_{n}$,

$$
P\left(\left\{\omega \in \Omega:\left\|S_{k}(\omega)-S_{p_{n}}(\omega)\right\|>2^{-n}\right\}\right)<2^{-n}
$$

### 61.10 White Noise

In an abstract Wiener space as discussed above there is a Gaussian measure, $\mu$ defined on the Borel sets of $B$. This measure is the law of a random variable having values in $B$ which is the limit of a subsequence of a sequence of partial sums. I will show here that the sequence of partial sums also converges pointwise a.e.

Now with this preparation, here is the theorem about white noise.
Theorem 61.10.1 Let $(i, H, B)$ be an abstract Wiener space and $\left\{e_{k}\right\}$ is a complete orthonormal sequence in $H$. Then there exists a Gaussian measure on the Borel sets of B. This Gaussian measure equals $\mathscr{L}(S)$ where $S(\omega)$ is the a.e. limit of the sequence of partial sums,

$$
S_{n}(\omega) \equiv \sum_{k=1}^{n} \xi_{k}(\omega) e_{k}
$$

for $\left\{\xi_{k}\right\}$ a sequence of independent random variables which are normal with mean 0 and variance 1, defined on a probability space, $(\Omega, \mathscr{F}, P)$

Proof: By Corollary 61.9 .15 there is a subsequence, $\left\{S_{p_{n}}\right\}$ of these partial sums which converge pointwise a.e. to $S(\omega)$. However, this corollary also states that

$$
P\left(\left\{\omega \in \Omega:\left\|S_{k}(\omega)-S_{p_{n}}(\omega)\right\|>2^{-n}\right\}\right)<2^{-n}
$$

whenever $k>p_{n}$ and so by Lemma 59.15.6 the original sequence of partial sums also converges uniformly of a set of measure zero. The reason this lemma applies is that $\xi_{k}(\omega) e_{k}$ has symmetric distribution. This proves the theorem.

### 61.11 Existence Of Abstract Wiener Spaces

It turns out that if $E$ is a separable Banach space, then it is the top third of an abstract Wiener space. This is what will be shown in this section. Therefore, it follows from the above that there exists a Gaussian measure on $E$ which is the law of an a.e. convergent series as discussed above. First recall Lemma 17.4.2 on Page 458.

Lemma 61.11.1 Let $E$ be a separable Banach space. Then there exists an increasing sequence of subspaces, $\left\{F_{n}\right\}$ such that $\operatorname{dim}\left(F_{n+1}\right)-\operatorname{dim}\left(F_{n}\right) \leq 1$ and equals 1 for all $n$ if the dimension of $E$ is infinite. Also $\cup_{n=1}^{\infty} F_{n}$ is dense in $E$.

Lemma 61.11.2 Let $E$ be a separable Banach space. Then there exists a sequence $\left\{e_{n}\right\}$ of points of $E$ such that whenever $|\beta| \leq 1$ for $\beta \in \mathbb{F}^{n}$,

$$
\sum_{k=1}^{n} \beta_{k} e_{k} \in B(0,1)
$$

the unit ball in $E$.

Proof: By Lemma 61.11.1, let $\left\{z_{1}, \cdots, z_{n}\right\}$ be a basis for $F_{n}$ where $\cup_{n=1}^{\infty} F_{n}$ is dense in $E$. Then let $\alpha_{1}$ be such that $e_{1} \equiv \alpha_{1} z_{1} \in B(0,1)$. Thus $\beta_{1} e_{1} \in B(0,1)$ whenever $\left|\beta_{1}\right| \leq 1$. Suppose $\alpha_{i}$ has been chosen for $i=1,2, \cdots, n$ such that for all $\beta \in D_{n} \equiv\left\{\alpha \in \mathbb{F}^{n}:|\alpha| \leq 1\right\}$, it follows

$$
\sum_{k=1}^{n} \beta_{k} \alpha_{k} z_{k} \in B(0,1)
$$

Then

$$
C_{n} \equiv\left\{\sum_{k=1}^{n} \beta_{k} \alpha_{k} z_{k}: \beta \in D_{n}\right\}
$$

is a compact subset of $B(0,1)$ and so it is at a positive distance from the complement of $B(0,1), \delta$. Now let $0<\alpha_{n+1}<\delta /\left\|z_{n+1}\right\|$. Then for $\beta \in D_{n+1}$,

$$
\sum_{k=1}^{n} \beta_{k} \alpha_{k} z_{k} \in C_{n}
$$

and so

$$
\begin{aligned}
\left\|\sum_{k=1}^{n+1} \beta_{k} \alpha_{k} z_{k}-\sum_{k=1}^{n} \beta_{k} \alpha_{k} z_{k}\right\| & =\left\|\beta_{n+1} \alpha_{n+1} z_{n+1}\right\| \\
& <\left\|\alpha_{n+1} z_{n+1}\right\|<\delta
\end{aligned}
$$

which shows

$$
\sum_{k=1}^{n+1} \beta_{k} \alpha_{k} z_{k} \in B(0,1)
$$

This proves the lemma. Let $e_{k} \equiv \alpha_{k} z_{k}$.
Now the main result is the following. It says that any separable Banach space is the upper third of some abstract Wiener space.

Theorem 61.11.3 Let $E$ be a real separable Banach space with norm $\|\cdot\|$. Then there exists a separable Hilbert space, $H$ such that $H$ is dense in $E$ and the inclusion map is continuous. Furthermore, if $v$ is the Gaussian measure defined earlier on the cylinder sets of $H,\|\cdot\|$ is Gross measurable.

Proof: Let $\left\{e_{k}\right\}$ be the points of $E$ described in Lemma 61.11.2. Then let $H_{0}$ denote the subspace of all finite linear combinations of the $\left\{e_{k}\right\}$. It follows $H_{0}$ is dense in $E$. Next decree that $\left\{e_{k}\right\}$ is an orthonormal basis for $H_{0}$. Thus for

$$
\begin{gathered}
\sum_{k=1}^{n} c_{k} e_{k}, \sum_{j=1}^{n} d_{j} e_{k} \in H_{0}, \\
\left(\sum_{k=1}^{n} c_{k} e_{k}, \sum_{j=1}^{n} d_{j} e_{j}\right)_{H_{0}} \equiv \sum_{k=1}^{n} c_{k} d_{k}
\end{gathered}
$$

this being well defined because the $\left\{e_{k}\right\}$ are independent. Let the norm on $H_{0}$ be denoted by $|\cdot|_{H_{0}}$. Let $H_{1}$ be the completion of $H_{0}$ with respect to this norm.

I want to show that $|\cdot|_{H_{0}}$ is stronger than $\|\cdot\|$. Suppose then that

$$
\left|\sum_{k=1}^{n} \beta_{k} e_{k}\right|_{H_{0}} \leq 1
$$

It follows then from the definition of $|\cdot|_{H_{0}}$ that

$$
\left|\sum_{k=1}^{n} \beta_{k} e_{k}\right|_{H_{0}}^{2}=\sum_{k=1}^{n} \beta_{k}^{2} \leq 1
$$

and so from the construction of the $e_{k}$, it follows that

$$
\left\|\sum_{k=1}^{n} \beta_{k} e_{k}\right\|<1
$$

Stated more simply, this has just shown that if $h \in H_{0}$ then since $\left|h /|h|_{H_{0}}\right|_{H_{0}} \leq 1$, it follows that

$$
\|h\| /|h|_{H_{0}}<1
$$

and so

$$
\left|\left|h \|<|h|_{H_{0}}\right.\right.
$$

It follows that the completion of $H_{0}$ must lie in $E$ because this shows that every Cauchy sequence in $H_{0}$ is a Cauchy sequence in $E$. Thus $H_{1}$ embedds continuously into $E$ and is dense in $E$. Denote its norm by $|\cdot|_{H_{1}}$.

Now consider the nuclear operator,

$$
A=\sum_{k=1}^{\infty} \lambda_{k} e_{k} \otimes e_{k}
$$

where each $\lambda_{k}>0$ and $\sum_{k} \lambda_{k}<\infty$. This operator is clearly one to one. Also it is clear the operator is Hilbert Schmidt because $\sum_{k} \lambda_{k}^{2}<\infty$. Let

$$
H \equiv A H_{1}
$$

and for $x \in H$, define

$$
|x|_{H} \equiv\left|A^{-1} x\right|_{H_{1}}
$$

Since each $e_{k}$ is in $H$ it follows that $H$ is dense in $E$. Note also that $H \subseteq H_{1}$ because $A$ maps $H_{1}$ to $H_{1}$.

$$
A x \equiv \sum_{k=1}^{\infty} \lambda_{k}\left(x, e_{k}\right) e_{k}
$$

and the series converges in $H_{1}$ because

$$
\sum_{k=1}^{\infty} \lambda_{k}\left|\left(x, e_{k}\right)\right| \leq\left(\sum_{k=1}^{\infty} \lambda_{k}^{2}\right)^{1 / 2}\left(\sum_{k=1}^{\infty}\left|\left(x, e_{k}\right)\right|^{2}\right)^{1 / 2}<\infty
$$

Also $H$ is a Hilbert space with inner product given by

$$
(x, y)_{H} \equiv\left(A^{-1} x, A^{-1} y\right)_{H_{1}}
$$

$H$ is complete because if $\left\{x_{n}\right\}$ is a Cauchy sequence in $H$, this is the same as $\left\{A^{-1} x_{n}\right\}$ being a Cauchy sequence in $H_{1}$ which implies $A^{-1} x_{n} \rightarrow y$ for some $y \in H_{1}$. Then it follows $x_{n}=A\left(A^{-1} x_{n}\right) \rightarrow A y$ in $H$.

For $x \in H \subseteq H_{1}$,

$$
\|x\| \leq|x|_{H_{1}}=\left|A A^{-1} x\right|_{H_{1}} \leq\|A\|\left|A^{-1} x\right|_{H_{1}} \equiv\|A\||x|_{H}
$$

and so the embedding of $H$ into $E$ is continuous. Why is $\|\cdot\|$ a measurable norm on $H$ ? Note first that for $x \in H \subseteq H_{1}$,

$$
\begin{equation*}
|A x|_{H} \equiv\left|A^{-1} A x\right|_{H_{1}}=|x|_{H_{1}} \geq \|\left. x\right|_{E} \tag{61.11.49}
\end{equation*}
$$

Therefore, if it can be shown $A$ is a Hilbert Schmidt operator on $H$, the desired measurability will follow from Lemma 61.9.11 on Page 2035.

Claim: $A$ is a Hilbert Schmidt operator on $H$.
Proof of the claim: From the definition of the inner product in $H$, it follows an orthonormal basis for $H$ is $\left\{\lambda_{k} e_{k}\right\}$. This is because

$$
\left(\lambda_{k} e_{k}, \lambda_{j} e_{j}\right)_{H} \equiv\left(\lambda_{k} A^{-1} e_{k}, \lambda_{j} A^{-1} e_{j}\right)_{H_{1}}=\left(e_{k}, e_{j}\right)_{H_{1}}=\delta_{j k} .
$$

To show that $A$ is Hilbert Schmidt, it suffices to show that

$$
\sum_{k}\left|A\left(\lambda_{k} e_{k}\right)\right|_{H}^{2}<\infty
$$

because this is the definition of an operator being Hilbert Schmidt. However, the above equals

$$
\sum_{k}\left|A^{-1} A\left(\lambda_{k} e_{k}\right)\right|_{H_{1}}^{2}=\sum_{k} \lambda_{k}^{2}<\infty .
$$

This proves the claim.
Now consider 61.11.49. By Lemma 61.9.11, it follows the norm $\|x\|^{\prime} \equiv|A x|_{H}$ is Gross measurable on $H$. Therefore, $\|\cdot\|_{E}$ is also Gross measurable because it is smaller. This proves the theorem.

Using Theorem 61.11.3 and Theorem 61.10.1 this proves most of the following important corollary.

Corollary 61.11.4 Let E be any real separable Banach space. Then there exists a sequence, $\left\{e_{k}\right\} \subseteq E$ such that for any $\left\{\xi_{k}\right\}$ a sequence of independent random variables such that $\mathscr{L}\left(\xi_{k}\right)=N(0,1)$, it follows

$$
X(\omega) \equiv \sum_{k=1}^{\infty} \xi_{k}(\omega) e_{k}
$$

converges a.e. and its law is a Gaussian measure defined on $\mathscr{B}(E)$. Furthermore, $\left\|e_{k}\right\|_{E} \leq$ $\lambda_{k}$ where $\sum_{k} \lambda_{k}<\infty$.

Proof: From the proof of Theorem 61.11.3 a basis for $H$ is $\left\{\lambda_{k} e_{k}\right\}$. Therefore, by Theorem 61.10.1, if $\left\{\xi_{k}\right\}$ is a sequence of independent $N(0,1)$ random variables, then $\sum_{k=1}^{\infty} \xi_{k}(\omega) \lambda_{k} e_{k}$ converges $a . e$. to a random variable whose law is Gaussian. Also from the proof of Theorem 61.10.1, each $e_{k}$ in that proof has the property that $\left\|e_{k}\right\| \leq 1$ because if $\left\|e_{k}\right\|>1$, then you could consider $\beta \equiv(0,0, \cdots, 1)$ and from the construction of the $e_{k}$, you would need $1 e_{k} \in B(0,1)$ which is a contradiction. Thus $\left\|\lambda_{k} e_{k}\right\| \leq \lambda_{k}$ and changing the notation, replacing $\lambda_{k} e_{k}$ with $e_{k}$, this proves the corollary.

## Chapter 62

## Stochastic Processes

### 62.1 Fundamental Definitions And Properties

Here $E$ will be a separable Banach space and $\mathscr{B}(E)$ will be the Borel sets of $E$. Let $(\Omega, \mathscr{F}, P)$ be a probability space and $I$ will be an interval of $\mathbb{R}$. A set of $E$ valued random variables, one for each $t \in I,\{X(t): t \in I\}$ is called a stochastic process. Thus for each $t$, $X(t)$ is a measurable function of $\omega \in \Omega$. Set $X(t, \omega) \equiv X(t)(\omega)$. Functions $t \rightarrow X(t, \omega)$ are called trajectories. Thus there is a trajectory for each $\omega \in \Omega$. A stochastic process, $Y$ is called a version or a modification of a stochastic process, $X$ if for all $t \in I$,

$$
X(t, \omega)=Y(t, \omega) \text { a.e. } \omega
$$

There are several descriptions of stochastic processes.

1. $X$ is measurable if $X(\cdot, \cdot): I \times \Omega \rightarrow E$ is $B(I) \times \mathscr{F}$ measurable. Note that a stochastic process, $X$ is not necessarily measurable.
2. $X$ is stochastically continuous at $t_{0} \in I$ means: for all $\varepsilon>0$ and $\delta>0$ there exists $\rho>0$ such that

$$
P\left(\left[\left|\mid X(t)-X\left(t_{0}\right) \| \geq \varepsilon\right]\right) \leq \delta \text { whenever }\left|t-t_{0}\right|<\rho, t \in I\right.
$$

Note the above condition says that for each $\varepsilon>0$,

$$
\lim _{t \rightarrow t_{0}} P\left(\left[\left\|X(t)-X\left(t_{0}\right)\right\| \geq \varepsilon\right]\right)=0
$$

3. $X$ is stochastically continuous if it is stochastically continuous at every $t \in I$.
4. $X$ is stochastically uniformly continuous if for every $\varepsilon, \delta>0$ there exists $\rho>0$ such that whenever $s, t \in I$ with $|s-t|<\rho$, it follows

$$
P([\|X(t)-X(s)\| \geq \varepsilon]) \leq \delta
$$

5. $X$ is mean square continuous at $t_{0} \in I$ if

$$
\lim _{t \rightarrow t_{0}} E\left(\left\|X(t)-X\left(t_{0}\right)\right\|^{2}\right) \equiv \lim _{t \rightarrow t_{0}} \int_{\Omega}\left\|X(t)(\omega)-X\left(t_{0}\right)(\omega)\right\|^{2} d P=0
$$

6. $X$ is mean square continuous in $I$ if it is mean square continuous at every point of $I$.
7. $X$ is continuous with probability 1 or continuous if $t \rightarrow X(t, \omega)$ is continuous for all $\omega$ outside some set of measure 0 .
8. $X$ is Hölder continuous if $t \rightarrow X(t, \omega)$ is Hölder continuous for a.e. $\omega$.

Lemma 62.1.1 A stochastically continuous process on $[a, b] \equiv I$ is uniformly stochastically continuous on $[a, b] \equiv I$.

Proof: If this is not so, there exists $\varepsilon, \delta>0$ and points of $I, s_{n}, t_{n}$ such that even though

$$
\begin{gather*}
\left|t_{n}-s_{n}\right|<\frac{1}{n} \\
P\left(\left[\left|\left|X\left(s_{n}\right)-X\left(t_{n}\right)\right|\right| \geq \varepsilon\right]\right)>\delta . \tag{62.1.1}
\end{gather*}
$$

Taking a subsequence, still denoted by $s_{n}$ and $t_{n}$ there exists $t \in I$ such that the above hold and

$$
\lim _{n \rightarrow \infty} s_{n}=\lim _{n \rightarrow \infty} t_{n}=t
$$

Then

$$
\begin{aligned}
& P\left(\left[\left\|X\left(s_{n}\right)-X\left(t_{n}\right)\right\| \geq \varepsilon\right]\right) \\
\leq & P\left(\left[\left\|X\left(s_{n}\right)-X(t)\right\| \geq \varepsilon / 2\right]\right)+P\left(\left[\left\|X(t)-X\left(t_{n}\right)\right\| \geq \varepsilon / 2\right]\right)
\end{aligned}
$$

But the sum of the last two terms converges to 0 as $n \rightarrow \infty$ by stochastic continuity of $X$ at $t$, violating 62.1.1 for all $n$ large enough. This proves the lemma.

For a stochastically continuous process defined on a closed and bounded interval, there always exists a measurable version. This is significant because then you can do things with product measure and iterated integrals.

Proposition 62.1.2 Let $X$ be a stochastically continuous process defined on a closed interval, $I \equiv[a, b]$. Then there exists a measurable version of $X$.

Proof: By Lemma 62.1.1 $X$ is uniformly stochastically continuous and so there exists a sequence of positive numbers, $\left\{\rho_{n}\right\}$ such that if $|s-t|<\rho_{n}$, then

$$
\begin{equation*}
P\left(\left[\|X(t)-X(s)\| \geq \frac{1}{2^{n}}\right]\right) \leq \frac{1}{2^{n}} \tag{62.1.2}
\end{equation*}
$$

Then let $\left\{t_{0}^{n}, t_{1}^{n}, \cdots, t_{m_{n}}^{n}\right\}$ be a partition of $[a, b]$ in which $\left|t_{i}^{n}-t_{i-1}^{n}\right|<\rho_{n}$. Now define $X_{n}$ as follows:

$$
\begin{aligned}
X_{n}(t) & \equiv \sum_{i=1}^{m_{n}} X\left(t_{i-1}^{n}\right) \mathscr{X}_{\left[t_{i-1}^{n}, t_{i}^{n}\right)}(t) \\
X_{n}(b) & \equiv X(b)
\end{aligned}
$$

Then $X_{n}$ is obviously $B(I) \times \mathscr{F}$ measurable because it is the sum of functions which are. Consider the set, $A$ on which $\left\{X_{n}(t, \omega)\right\}$ is a Cauchy sequence. This set is of the form

$$
A=\cap_{n=1}^{\infty} \cup_{m=1}^{\infty} \cap_{p, q \geq m}\left[\left\|X_{p}-X_{q}\right\|<\frac{1}{n}\right]
$$

and so it is a $B(I) \times \mathscr{F}$ measurable set. Now define

$$
Y(t, \omega) \equiv\left\{\begin{array}{l}
\lim _{n \rightarrow \infty} X_{n}(t, \omega) \text { if }(t, \omega) \in A \\
0 \text { if }(t, \omega) \notin A
\end{array}\right.
$$

I claim $Y(t, \omega)=X(t, \omega)$ for a.e. $\omega$. To see this, consider 62.1.2. From the construction of $X_{n}$, it follows that for each $t$,

$$
P\left(\left[\left\|X_{n}(t)-X(t)\right\| \geq \frac{1}{2^{n}}\right]\right) \leq \frac{1}{2^{n}}
$$

Also, for a fixed $t$, if $X_{n}(t, \omega)$ fails to converge to $X(t, \omega)$, then $\omega$ must be in infinitely many of the sets,

$$
B_{n} \equiv\left[\left\|X_{n}(t)-X(t)\right\| \geq \frac{1}{2^{n}}\right]
$$

which is a set of measure zero by the Borel Cantelli lemma. Recall why this is so.

$$
P\left(\cap_{k=1}^{\infty} \cup_{n=k}^{\infty} B_{n}\right) \leq \sum_{n=k}^{\infty} P\left(B_{n}\right)<\frac{1}{2^{k-1}}
$$

Therefore, for each $t,(t, \omega) \in A$ for a.e. $\omega$. Hence $X(t)=Y(t)$ a.e. and so $Y$ is a measurable version of $X$.

Lemma 62.1.3 Let $D$ be a dense subset of an interval, $I=[0, T]$ and suppose $X: D \rightarrow E$ satisfies

$$
\left\|X(d)-X\left(d^{\prime}\right)\right\| \leq C\left|d-d^{\prime}\right|^{\gamma}
$$

for all $d^{\prime}, d \in D$. Then $X$ extends uniquely to a continuous $Y$ defined on $[0, T]$ such that

$$
\left\|Y(t)-Y\left(t^{\prime}\right)\right\| \leq C\left|t-t^{\prime}\right|^{\gamma}
$$

Proof: Let $t \in I$ and let $d_{k} \rightarrow t$ where $d_{k} \in D$. Then $\left\{X\left(d_{k}\right)\right\}$ is a Cauchy sequence because $\left\|X\left(d_{k}\right)-X\left(d_{m}\right)\right\| \leq C\left|d_{k}-d_{m}\right|^{\gamma}$. Therefore, $X\left(d_{k}\right)$ converges. The thing it converges to will be called $Y(t)$. Note this is well defined, giving $X(t)$ if $t \in D$. Also, if $d_{k} \rightarrow t$ and $d_{k}^{\prime} \rightarrow t$, then $\left|\left|X\left(d_{k}\right)-X\left(d_{k}^{\prime}\right) \| \leq C\right| d_{k}-d_{k}^{\prime}\right|^{\gamma}$ and so $X\left(d_{k}\right)$ and $X\left(d_{k}^{\prime}\right)$ converge to the same thing. Therefore, it makes sense to define $Y(t) \equiv \lim _{d \rightarrow t} X(d)$. It only remains to verify the estimate. But letting $|d-t|$ and $\left|d^{\prime}-t^{\prime}\right|$ be small enough,

$$
\begin{aligned}
\left\|Y(t)-Y\left(t^{\prime}\right)\right\| & =\left\|X(d)-X\left(d^{\prime}\right)\right\|+\varepsilon \\
& \leq C\left|d^{\prime}-d\right|+\varepsilon \leq C\left|t-t^{\prime}\right|+2 \varepsilon
\end{aligned}
$$

Since $\varepsilon$ is arbitrary, this proves the existence part of the lemma. Uniqueness follows from observing that $Y(t)$ must equal $\lim _{d \rightarrow t} X(d)$. This proves the lemma.

### 62.2 Kolmogorov Čentsov Continuity Theorem

Lemma 62.2.1 Let $r_{j}^{m}$ denote $j\left(\frac{T}{2^{m}}\right)$ where $j \in\left\{0,1, \cdots, 2^{m}\right\}$. Also let $D_{m}=\left\{r_{j}^{m}\right\}_{j=1}^{2^{m}}$ and $D=\cup_{m=1}^{\infty} D_{m}$. Suppose $X(t)$ satisfies

$$
\begin{equation*}
\left\|X\left(r_{j+1}^{k}\right)-X\left(r_{j}^{k}\right)\right\| \leq 2^{-\gamma k} \tag{62.2.3}
\end{equation*}
$$

for all $k \geq M$. Then if $d, d^{\prime} \in D_{m}$ for $m>n \geq M$ such that $\left|d-d^{\prime}\right| \leq T 2^{-n}$, then

$$
\left\|X\left(d^{\prime}\right)-X(d)\right\| \leq 2 \sum_{j=n+1}^{m} 2^{-\gamma j}
$$

Also, there exists a constant $C$ depending on $M$ such that for all $d, d^{\prime} \in D$,

$$
\left\|X(d)-X\left(d^{\prime}\right)\right\| \leq C\left|d-d^{\prime}\right|^{\gamma}
$$

Proof: Suppose $d^{\prime}<d$. Suppose first $m=n+1$. Then $d=(k+1) T 2^{-(n+1)}$ and $d^{\prime}=$ $k T 2^{-(n+1)}$. Then from 62.2.3

$$
\left\|X\left(d^{\prime}\right)-X(d)\right\| \leq 2^{-\gamma(n+1)} \leq 2 \sum_{j=n+1}^{n+1} 2^{-\gamma j}
$$

Suppose the claim is true for some $m>n$ and let $d, d^{\prime} \in D_{m+1}$ with $\left|d-d^{\prime}\right|<T 2^{-n}$. If there is no point of $D_{m}$ between these, then $d^{\prime}, d$ are adjacent points either in $D_{m}$ or in $D_{m+1}$. Consequently,

$$
\left\|X\left(d^{\prime}\right)-X(d)\right\| \leq 2^{-\gamma m}<2 \sum_{j=n+1}^{m+1} 2^{-\gamma j}
$$

Assume therefore, there exist points of $D_{m}$ between $d^{\prime}$ and $d$. Let $d^{\prime} \leq d_{1}^{\prime} \leq d_{1} \leq d$ where $d_{1}, d_{1}^{\prime}$ are in $D_{m}$ and $d_{1}^{\prime}$ is the smallest element of $D_{m}$ which is at least as large as $d^{\prime}$ and $d_{1}$ is the largest element of $D_{m}$ which is no larger than $d$. Then $\left|d^{\prime}-d_{1}^{\prime}\right| \leq T 2^{-(m+1)}$ and $\left|d_{1}-d\right| \leq T 2^{-(m+1)}$ while all of these points are in $D_{m+1}$ which contains $D_{m}$. Therefore, from 62.2.3 and induction,

$$
\begin{align*}
& \left\|X\left(d^{\prime}\right)-X(d)\right\| \\
\leq & \left\|X\left(d^{\prime}\right)-X\left(d_{1}^{\prime}\right)\right\|+\left\|X\left(d_{1}^{\prime}\right)-X\left(d_{1}\right)\right\| \\
& +\left\|X\left(d_{1}\right)-X(d)\right\| \\
\leq & 2 \times 2^{-\gamma(m+1)}+2 \sum_{j=n+1}^{m} 2^{-\gamma j}=2 \sum_{j=n+1}^{m+1} 2^{-\gamma j} \\
\leq & 2\left(\frac{2^{-\gamma(n+1)}}{1-2^{-\gamma}}\right)=\left(\frac{2 T^{-\gamma}}{1-2^{-\gamma}}\right)\left(T 2^{-(n+1)}\right)^{\gamma} \tag{62.2.4}
\end{align*}
$$

It follows the above holds for any $d, d^{\prime} \in D$ such that $\left|d-d^{\prime}\right| \leq T 2^{-n}$ because they are both in some $D_{m}$ for $m>n$.

Consider the last claim. Let $d, d^{\prime} \in D,\left|d-d^{\prime}\right| \leq T 2^{-M}$. Then $d, d^{\prime}$ are both in some $D_{m}$ for $m>M$. The number $\left|d-d^{\prime}\right|$ satisfies

$$
T 2^{-(n+1)}<\left|d-d^{\prime}\right| \leq T 2^{-n}
$$

for large enough $n \geq M$. Just pick the first $n$ such that $T 2^{-(n+1)}<\left|d-d^{\prime}\right|$. Then from 62.2.4,

$$
\begin{aligned}
\left\|X\left(d^{\prime}\right)-X(d)\right\| & \leq\left(\frac{2 T^{-\gamma}}{1-2^{-\gamma}}\right)\left(T 2^{-(n+1)}\right)^{\gamma} \\
& \leq\left(\frac{2 T^{-\gamma}}{1-2^{-\gamma}}\right)\left(\left|d-d^{\prime}\right|\right)^{\gamma}
\end{aligned}
$$

Now $[0, T]$ is covered by $2^{M}$ intervals of length $T 2^{-M}$ and so for any pair $d, d^{\prime} \in D$,

$$
\left\|X(d)-X\left(d^{\prime}\right)\right\| \leq C\left|d-d^{\prime}\right|^{\gamma}
$$

where $C$ is a suitable constant depending on $2^{M}$.
For $\gamma \leq 1$, you can show, using convexity arguments, that it suffices to have $C=$ $\left(\frac{2 T^{-\gamma}}{1-2^{-\gamma}}\right)^{1 / \gamma}\left(2^{M}\right)^{1-\gamma}$. Of course the case where $\gamma>1$ is not interesting because it would result in $X$ being a constant.

The following is the amazing Kolmogorov Čentsov continuity theorem [78].
Theorem 62.2.2 Suppose $X$ is a stochastic process on $[0, T]$. Suppose also that there exists a constant, $C$ and positive numbers, $\alpha, \beta$ such that

$$
\begin{equation*}
E\left(\|X(t)-X(s)\|^{\alpha}\right) \leq C|t-s|^{1+\beta} \tag{62.2.5}
\end{equation*}
$$

Then there exists a stochastic process $Y$ such that for a.e. $\omega, t \rightarrow Y(t)(\omega)$ is Hölder continuous with exponent $\gamma<\frac{\beta}{\alpha}$ and for each $t, P([\|X(t)-Y(t)\|>0])=0$. (Y is a version of $X$.)

Proof: Let $r_{j}^{m}$ denote $j\left(\frac{T}{2^{m}}\right)$ where $j \in\left\{0,1, \cdots, 2^{m}\right\}$. Also let $D_{m}=\left\{r_{j}^{m}\right\}_{j=1}^{2^{m}}$ and $D=\cup_{m=1}^{\infty} D_{m}$. Consider the set,

$$
[\|X(t)-X(s)\|>\delta]
$$

By 62.2.5,

$$
\begin{align*}
P([\|X(t)-X(s)\|>\delta]) \delta^{\alpha} & \leq \int_{[\|X(t)-X(s)\|>\delta]}\|X(t)-X(s)\|^{\alpha} d P \\
& \leq C|t-s|^{1+\beta} \tag{62.2.6}
\end{align*}
$$

Letting $t=r_{j+1}^{k}, s=r_{j}^{k}$, and $\delta=2^{-\gamma k}$ where

$$
\gamma \in\left(0, \frac{\beta}{\alpha}\right)
$$

this yields

$$
P\left(\left[\left\|X\left(r_{j+1}^{k}\right)-X\left(r_{j}^{k}\right)\right\|>2^{-\gamma k}\right]\right) \leq C 2^{\alpha \gamma k}\left(T 2^{-k}\right)^{1+\beta}
$$

$$
=C T^{1+\beta} 2^{k(\alpha \gamma-(1+\beta))}
$$

There are $2^{k}$ of these differences and so letting

$$
N_{k}=\cup_{j=1}^{2^{k}}\left[\left\|X\left(r_{j+1}^{k}\right)-X\left(r_{j}^{k}\right)\right\|>2^{-\gamma k}\right]
$$

it follows

$$
P\left(N_{k}\right) \leq C 2^{\alpha \gamma k}\left(T 2^{-k}\right)^{1+\beta} 2^{k}=C 2^{k(\alpha \gamma-\beta)} T^{1+\beta}
$$

Since $\gamma<\beta / \alpha$,

$$
\sum_{k=1}^{\infty} P\left(N_{k}\right) \leq C T^{1+\beta} \sum_{k=1}^{\infty} 2^{k(\alpha \gamma-\beta)}<\infty
$$

and so by the Borel Cantelli lemma, Lemma 59.1.2, there exists a set of measure zero $N$, such that if $\omega \notin N$, then $\omega$ is in only finitely many $N_{k}$. In other words, for $\omega \notin N$, there exists $M(\omega)$ such that if $k \geq M(\omega)$, then for each $j$,

$$
\begin{equation*}
\left\|X\left(r_{j+1}^{k}\right)(\omega)-X\left(r_{j}^{k}\right)(\omega)\right\| \leq 2^{-\gamma k} \tag{62.2.7}
\end{equation*}
$$

It follows from Lemma 62.2 .1 that $t \rightarrow X(t)(\omega)$ is Holder continuous on $D$ with Holder exponent $\gamma$. Note the constant is a measurable function of $\omega$, depending on how many measurable $N_{k}$ which contain $\omega$.

By Lemma 62.1.3, one can define $Y(t)(\omega)$ to be the unique function which extends $d \rightarrow X(d)(\omega)$ off $D$ for $\omega \notin N$ and let $Y(t)(\omega)=0$ if $\omega \in N$. Thus by Lemma 62.1.3 $t \rightarrow Y(t)(\omega)$ is Holder continuous. Also, $\omega \rightarrow Y(t)(\omega)$ is measurable because it is the pointwise limit of measurable functions

$$
\begin{equation*}
Y(t)(\omega)=\lim _{d \rightarrow t} X(d)(\omega) \mathscr{X}_{N^{C}}(\omega) \tag{62.2.8}
\end{equation*}
$$

It remains to verify the claim that $Y(t)(\omega)=X(t)(\omega)$ a.e.

$$
\mathscr{X}_{[\| Y(t)-X(t)| |>\varepsilon] \cap N^{C}}(\omega) \leq \lim _{d \rightarrow t} \mathscr{X}_{[\|X(d)-X(t)\|>\varepsilon] \cap N^{C}}(\omega)
$$

because if $\omega \in N$ both sides are 0 and if $\omega \in N^{C}$ then the above limit in 62.2 .8 holds and so if $\|Y(t)(\omega)-X(t)(\omega)\|>\varepsilon$, the same is true of $\|X(d)(\omega)-X(t)(\omega)\|$ whenever $d$ is close enough to $t$ and so by Fatou's lemma,

$$
\begin{aligned}
P([\|Y(t)-X(t)\|>\varepsilon]) & =\int \mathscr{X}_{[\|Y(t)-X(t)\|>\varepsilon] \cap N^{C}}(\omega) d P \\
& \leq \int \liminf _{d \rightarrow t} \mathscr{X}_{[\|X(d)-X(t)\|>\varepsilon]}(\omega) d P \\
& \leq \liminf _{d \rightarrow t} \int \mathscr{X}_{[\|X(d)-X(t)\|>\varepsilon]}(\omega) d P \\
& \leq \liminf _{d \rightarrow t} P\left(\left[\|X(d)-X(t)\|^{\alpha}>\varepsilon^{\alpha}\right]\right) \\
& \leq \liminf _{d \rightarrow t} \varepsilon^{-\alpha} \int_{\left[\|X(d)-X(t)\|^{\alpha}>\varepsilon^{\alpha}\right]}\|X(d)-X(t)\|^{\alpha} d P \\
& \leq \liminf _{d \rightarrow t} \frac{C}{\varepsilon^{\alpha}}|d-t|^{1+\beta}=0 .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& P([\|Y(t)-X(t)\|>0]) \\
= & P\left(\cup_{k=1}^{\infty}\left[\|Y(t)-X(t)\|>\frac{1}{k}\right]\right) \\
\leq & \sum_{k=1}^{\infty} P\left(\left[\|Y(t)-X(t)\|>\frac{1}{k}\right]\right)=0 .
\end{aligned}
$$

A few observations are interesting. In the proof, the following inequality was obtained.

$$
\begin{aligned}
\left\|X\left(d^{\prime}\right)(\omega)-X(d)(\omega)\right\| & \leq \frac{2}{T^{\gamma}\left(1-2^{-\gamma}\right)}\left(T 2^{-(n+1)}\right)^{\gamma} \\
& \leq \frac{2}{T^{\gamma}\left(1-2^{-\gamma}\right)}\left(\left|d-d^{\prime}\right|\right)^{\gamma}
\end{aligned}
$$

which was so for any $d^{\prime}, d \in D$ with $\left|d^{\prime}-d\right|<T 2^{-(M(\omega)+1)}$. Thus the Holder continuous version of $X$ will satisfy

$$
\|Y(t)(\omega)-Y(s)(\omega)\| \leq \frac{2}{T^{\gamma}\left(1-2^{-\gamma}\right)}(|t-s|)^{\gamma}
$$

provided $|t-s|<T 2^{-(M(\omega)+1)}$. Does this translate into an inequality of the form

$$
\|Y(t)(\omega)-Y(s)(\omega)\| \leq \frac{2}{T^{\gamma}\left(1-2^{-\gamma}\right)}(|t-s|)^{\gamma}
$$

for any pair of points $t, s \in[0, T]$ ? It seems it does not for any $\gamma<1$ although it does yield

$$
\|Y(t)(\omega)-Y(s)(\omega)\| \leq C(|t-s|)^{\gamma}
$$

where $C$ depends on the number of intervals having length less than $T 2^{-(M(\omega)+1)}$ which it takes to cover $[0, T]$. First note that if $\gamma>1$, then the Holder continuity will imply $t \rightarrow$ $Y(t)(\omega)$ is a constant. Therefore, the only case of interest is $\gamma<1$. Let $s, t$ be any pair of points and let $s=x_{0}<\cdots<x_{n}=t$ where $\left|x_{i}-x_{i-1}\right|<T 2^{-(M(\omega)+1)}$. Then

$$
\begin{align*}
\| Y(t)(\omega)- & Y(s)(\omega)\left\|\leq \sum_{i=1}^{n}\right\| Y\left(x_{i}\right)(\omega)-Y\left(x_{i-1}\right)(\omega) \| \\
& \leq \frac{2}{T^{\gamma}\left(1-2^{-\gamma}\right)} \sum_{i=1}^{n}\left(\left|x_{i}-x_{i-1}\right|\right)^{\gamma} \tag{62.2.9}
\end{align*}
$$

How does this compare to

$$
\left(\sum_{i=1}^{n}\left|x_{i}-x_{i-1}\right|\right)^{\gamma}=|t-s|^{\gamma_{?}}
$$

This last expression is smaller than the right side of 62.2 .9 for any $\gamma<1$. Thus for $\gamma<1$, the constant in the conclusion of the theorem depends on both $T$ and $\omega \notin N$.

In the case where $\alpha \geq 1$, here is another proof of this theorem. It is based on the one in the book by Stroock [121].

Theorem 62.2.3 Suppose $X$ is a stochastic process on $[0, T]$ having values in the Banach space $E$. Suppose also that there exists a constant, $C$ and positive numbers $\alpha, \beta, \alpha \geq 1$, such that

$$
\begin{equation*}
E\left(\|X(t)-X(s)\|^{\alpha}\right) \leq C|t-s|^{1+\beta} \tag{62.2.10}
\end{equation*}
$$

Then there exists a stochastic process $Y$ such that for a.e. $\omega, t \rightarrow Y(t)(\omega)$ is Hölder continuous with exponent $\gamma<\frac{\beta}{\alpha}$ and for each $t, P([\|X(t)-Y(t)\|>0])=0$. $(Y$ is a version of X.) Also

$$
E\left(\sup _{0 \leq s<t \leq T} \frac{\|Y(t)-Y(s)\|}{(t-s)^{\gamma}}\right) \leq C
$$

where $C$ depends on $\alpha, \beta, T, \gamma$.
Proof: The proof considers piecewise linear approximations of $X$ which are automatically continuous. These are shown to converge to $Y$ in $L^{\alpha}(\Omega ; C([0, T], E))$ so it follows that $Y$ must be continuous for a.e. $\omega$. Finally, it is shown that $Y$ is a version of $X$ and is Holder continuous. In the proof, I will use $C$ to denote a constant which depends on the quantities $\gamma, \alpha, \beta, T$. Let $\left\{t_{k}^{n}\right\}_{k=0}^{2^{n}}$ be a uniform partition of the interval $[0, T]$ so that $t_{k+1}^{n}-t_{k}^{n}=T 2^{-n}$. Now let

$$
M_{n} \equiv \max _{k \leq 2^{n}}\left\|X\left(t_{k}^{n}\right)-X\left(t_{k-1}^{n}\right)\right\|
$$

Then it follows that

$$
M_{n}^{\alpha} \leq \sum_{k=1}^{2^{n}}\left\|X\left(t_{k}^{n}\right)-X\left(t_{k-1}^{n}\right)\right\|^{\alpha}
$$

and so

$$
\begin{equation*}
E\left(M_{n}^{\alpha}\right) \leq \sum_{k=1}^{2^{n}} C\left(T 2^{-n}\right)^{1+\beta}=C 2^{n} 2^{-n(1+\beta)}=C 2^{-n \beta} \tag{62.2.11}
\end{equation*}
$$

Next denote by $X_{n}$ the piecewise linear function which results from the values of $X$ at the points $t_{k}^{n}$. Consider the following picture which illustrates a part of the graphs of $X_{n}$ and $X_{n+1}$.


Then

$$
\max _{t \in[0, T]}\left\|X_{n+1}(t)-X_{n}(t)\right\| \leq \max _{1 \leq k \leq 2^{n+1}}\left\|X\left(t_{2 k-1}^{n+1}\right)-\frac{X\left(t_{k}^{n}\right)+X\left(t_{k-1}^{n}\right)}{2}\right\|
$$

$$
\leq \max _{k \leq 2^{n+1}}\left(\frac{1}{2}\left\|X\left(t_{2 k-1}^{n+1}\right)-X\left(t_{2 k}^{n+1}\right)\right\|+\frac{1}{2}\left\|X\left(t_{2 k-1}^{n+1}\right)-X\left(t_{2 k-2}^{n+1}\right)\right\|\right) \leq M_{n+1}
$$

Denote by $\|\cdot\|_{\infty}$ the usual norm in $C([0, T], E)$,

$$
\max _{t \in[0, T]}\|Z(t)\| \equiv\|Z\|_{\infty}
$$

Then from what was just established,

$$
E\left(\left\|X_{n+1}-X_{n}\right\|_{\infty}^{\alpha}\right)=\int_{\Omega}\left\|X_{n+1}-X_{n}\right\|_{\infty}^{\alpha} d P \leq E\left(M_{n+1}^{\alpha}\right)=C 2^{-n \beta}
$$

which shows that

$$
\left\|X_{n+1}-X_{n}\right\|_{L^{\alpha}(\Omega ; C([0, T], E))}=\left(\int_{\Omega}\left\|X_{n+1}-X_{n}\right\|_{\infty}^{\alpha} d P\right)^{1 / \alpha} \leq C\left(2^{(\beta / \alpha)}\right)^{-n}
$$

Also, for $m>n$, it follows from the assumption that $\alpha \geq 1$,

$$
\begin{gather*}
\left\|X_{m}-X_{n}\right\|_{L^{\alpha}(\Omega ; C([0, T], E))} \leq \\
\sum_{k=n}^{\infty} C\left(2^{(\beta / \alpha)}\right)^{-k} \leq C \frac{\left(2^{(\beta / \alpha)}\right)^{-n}}{1-2^{(-\beta / \alpha)}}=C\left(2^{(\beta / \alpha)}\right)^{-n} \tag{62.2.12}
\end{gather*}
$$

Thus $\left\{X_{n}\right\}$ is a Cauchy sequence in $L^{\alpha}(\Omega ; C([0, T], E))$ and so it converges to some $Y$ in this space, a subsequence converging pointwise. Then from Fatou's lemma,

$$
\begin{equation*}
\left\|Y-X_{n}\right\|_{L^{\alpha}(\Omega ; C([0, T], E))} \leq C\left(2^{(\beta / \alpha)}\right)^{-n} \tag{62.2.13}
\end{equation*}
$$

Also, for a.e. $\omega, t \rightarrow Y(t)$ is in $C([0, T], E)$. It remains to verify that $Y(t)=X(t)$ a.e.
From the construction, it follows that for any $n$ and $m \geq n$

$$
Y\left(t_{k}^{n}\right)=X_{m}\left(t_{k}^{n}\right)=X\left(t_{k}^{n}\right)
$$

Thus

$$
\begin{aligned}
\|Y(t)-X(t)\| & \leq\left\|Y(t)-Y\left(t_{k}^{n}\right)\right\|+\left\|Y\left(t_{k}^{n}\right)-X(t)\right\| \\
& =\left\|Y(t)-Y\left(t_{k}^{n}\right)\right\|+\left\|X\left(t_{k}^{n}\right)-X(t)\right\|
\end{aligned}
$$

Now from the hypotheses of the theorem,

$$
P\left(\left\|X\left(t_{k}^{n}\right)-X(t)\right\|^{\alpha}>\varepsilon\right) \leq \frac{1}{\varepsilon} E\left(\left\|X\left(t_{k}^{n}\right)-X(t)\right\|^{\alpha}\right) \leq \frac{C}{\varepsilon}\left|t_{k}^{n}-t\right|^{1+\beta}
$$

Thus, there exists a sequence of mesh points $\left\{s_{n}\right\}$ converging to $t$ such that

$$
P\left(\left\|X\left(s_{n}\right)-X(t)\right\|^{\alpha}>2^{-n}\right) \leq 2^{-n}
$$

Then by the Borel Cantelli lemma, there is a set of measure zero $N$ such that for $\omega \notin N$,

$$
\left\|X\left(s_{n}\right)-X(t)\right\|^{\alpha} \leq 2^{-n}
$$

for all $n$ large enough. Then

$$
\|Y(t)-X(t)\| \leq\left\|Y(t)-Y\left(s_{n}\right)\right\|+\left\|X\left(s_{n}\right)-X(t)\right\|
$$

which shows that, by continuity of $Y$, for $\omega$ not in an exceptional set of measure zero, $\|Y(t)-X(t)\|=0$.

It remains to verify the assertion about Holder continuity of $Y$. Let $0 \leq s<t \leq T$. Then for some $n$,

$$
\begin{equation*}
2^{-(n+1)} T \leq t-s \leq 2^{-n} T \tag{62.2.14}
\end{equation*}
$$

Thus

$$
\begin{align*}
\|Y(t)-Y(s)\| & \leq\left\|Y(t)-X_{n}(t)\right\|+\left\|X_{n}(t)-X_{n}(s)\right\|+\left\|X_{n}(s)-Y(s)\right\| \\
& \leq 2 \sup _{\tau \in[0, T]}\left\|Y(\tau)-X_{n}(\tau)\right\|+\left\|X_{n}(t)-X_{n}(s)\right\| \tag{62.2.15}
\end{align*}
$$

Now

$$
\frac{\left\|X_{n}(t)-X_{n}(s)\right\|}{t-s} \leq \frac{\left\|X_{n}(t)-X_{n}(s)\right\|}{2^{-(n+1)} T}
$$

From 62.2 .14 a picture like the following must hold.

| - |  |  |  | + |
| :---: | :---: | :---: | :---: | :---: |
| $t_{k-1}^{n+1}$ | $s$ | $t_{k}^{n+1}$ | $t$ | $t_{k+1}^{n+1}$ |

Therefore, from the above,

$$
\begin{aligned}
\frac{\left\|X_{n}(t)-X_{n}(s)\right\|}{t-s} & \leq \frac{\left\|X\left(t_{k-1}^{n+1}\right)-X\left(t_{k}^{n+1}\right)\right\|+\left\|X\left(t_{k+1}^{n+1}\right)-X\left(t_{k}^{n+1}\right)\right\|}{2^{-(n+1)} T} \\
& \leq C 2^{n} M_{n+1}
\end{aligned}
$$

It follows from 62.2.15,

$$
\|Y(t)-Y(s)\| \leq 2\left\|Y-X_{n}\right\|_{\infty}+C 2^{n} M_{n+1}(t-s)
$$

Next, letting $\gamma<\beta / \alpha$, and using 62.2.14,

$$
\begin{aligned}
\frac{\|Y(t)-Y(s)\|}{(t-s)^{\gamma}} & \leq 2\left(T^{-1} 2^{n+1}\right)^{\gamma}\left\|Y-X_{n}\right\|_{\infty}+C 2^{n}\left(2^{-n}\right)^{1-\gamma} M_{n+1} \\
& =C 2^{n \gamma}\left(\left\|Y-X_{n}\right\|_{\infty}+M_{n+1}\right)
\end{aligned}
$$

The above holds for any $s, t$ satisfying 62.2.14. Then

$$
\sup \left\{\frac{\|Y(t)-Y(s)\|}{(t-s)^{\gamma}}, 0 \leq s<t \leq T,|t-s| \in\left[2^{-(n+1)} T, 2^{-n} T\right]\right\}
$$

$$
\leq C 2^{n \gamma}\left(\left\|Y-X_{n}\right\|_{\infty}+M_{n+1}\right)
$$

Denote by $P_{n}$ the ordered pairs $(s, t)$ satisfying the above condition that

$$
\begin{gathered}
0 \leq s<t \leq T,|t-s| \in\left[2^{-(n+1)} T, 2^{-n} T\right] \\
\sup _{(s, t) \in P_{n}} \frac{\|Y(t)-Y(s)\|}{(t-s)^{\gamma}} \leq C 2^{n \gamma}\left(\left\|Y-X_{n}\right\|_{\infty}+M_{n+1}\right)
\end{gathered}
$$

Thus for a.e. $\omega$, and for all $n$,

$$
\left(\sup _{(s, t) \in P_{n}} \frac{\|Y(t)-Y(s)\|}{(t-s)^{\gamma}}\right)^{\alpha} \leq C \sum_{k=0}^{\infty} 2^{k \alpha \gamma}\left(\left\|Y-X_{k}\right\|_{\infty}^{\alpha}+M_{k+1}^{\alpha}\right)
$$

Note that $n$ is arbitrary. Hence

$$
\begin{gathered}
\sup _{0 \leq s<t \leq T}\left(\frac{\|Y(t)-Y(s)\|}{(t-s)^{\gamma}}\right)^{\alpha} \leq \\
\sup _{n} \sup _{(s, t) \in P_{n}}\left(\frac{\|Y(t)-Y(s)\|}{(t-s)^{\gamma}}\right)^{\alpha} \leq \sup _{n}\left(\sup _{(s, t) \in P_{n}} \frac{\|Y(t)-Y(s)\|}{(t-s)^{\gamma}}\right)^{\alpha} \\
\leq \sum_{k=0}^{\infty} C 2^{k \alpha \gamma}\left(\left\|Y-X_{k}\right\|_{\infty}^{\alpha}+M_{k+1}^{\alpha}\right)
\end{gathered}
$$

By continuity of $Y$, the result on the left is unchanged if the ordered pairs are restricted to lie in $\mathbb{Q} \cap[0, T] \times \mathbb{Q} \cap[0, T]$, a countable set. Thus the left side is measurable. It follows from 62.2.11 and 62.2.13 which say

$$
\left\|Y-X_{k}\right\|_{L^{\alpha}(\Omega ; C([0, T], E))} \leq C\left(2^{(\beta / \alpha)}\right)^{-k}, E\left(M_{k}^{\alpha}\right) \leq C 2^{-k \beta}
$$

that

$$
E\left(\sup _{0 \leq s<t \leq T}\left(\frac{\|Y(t)-Y(s)\|}{(t-s)^{\gamma}}\right)^{\alpha}\right) \leq \sum_{k=0}^{\infty} C 2^{k \alpha \gamma} 2^{-\beta k} \equiv C<\infty
$$

because $\alpha \gamma-\beta<0$. By continuity of $Y$, there are no measurability concerns in taking the above expectation. Note that this implies, since $\alpha \geq 1$,

$$
E\left(\sup _{0 \leq s<t \leq T} \frac{\|Y(t)-Y(s)\|}{(t-s)^{\gamma}}\right) \leq\left(E\left(\sup _{0 \leq s<t \leq T}\left(\frac{\|Y(t)-Y(s)\|}{(t-s)^{\gamma}}\right)^{\alpha}\right)\right)^{1 / \alpha} \leq C^{1 / \alpha} \equiv C
$$

Now

$$
P\left(\sup _{0 \leq s<t \leq T}\left(\frac{\|Y(t)-Y(s)\|}{(t-s)^{\gamma}}\right)^{\alpha}>2^{\alpha k}\right) \leq \frac{1}{2^{\alpha k}} C
$$

and so there exists a set of measure zero $N$ such that for $\omega \notin N$,

$$
\sup _{0 \leq s<t \leq T}\left(\frac{\|Y(t)-Y(s)\|}{(t-s)^{\gamma}}\right)^{\alpha} \leq 2^{\alpha k}
$$

for all $k$ large enough. Pick such a $k$, depending on $\omega \notin N$. Then for any $s, t$,

$$
\frac{\|Y(t)-Y(s)\|}{(t-s)^{\gamma}} \leq 2^{k}
$$

and so, this has shown that for $\omega \notin N$,

$$
\|Y(t)-Y(s)\| \leq C(\omega)(t-s)^{\gamma}
$$

Note that if $X(t)$ is known to be continuous off a set of measure zero, then the piecewise linear approximations converge to $X(t)$ in $C([0, T], E)$ off this set of measure zero. Therefore, it must be that off a set of measure zero, $Y(t)=X(t)$ and so in fact $X(t)$ is Holder continuous off a set of measure zero and the condition on expectation also must hold, that is

$$
E\left(\sup _{0 \leq s<t \leq T} \frac{\|X(t)-X(s)\|}{(t-s)^{\gamma}}\right) \leq C
$$

### 62.3 Filtrations

Instead of having a sequence of $\sigma$ algebras, one can consider an increasing collection of $\sigma$ algebras indexed by $t \in \mathbb{R}$. This is called a filtration.

Definition 62.3.1 Let $X$ be a stochastic process defined on an interval, $I=[0, T]$ or $[0, \infty)$. Suppose the probability space, $(\Omega, \mathscr{F}, P)$ has an increasing family of $\sigma$ algebras, $\left\{\mathscr{F}_{t}\right\}$. This is called a filtration. If for arbitrary $t \in I$ the random variable $X(t)$ is $\mathscr{F}_{t}$ measurable, then $X$ is said to be adapted to the filtration $\left\{\mathscr{F}_{t}\right\}$. Denote by $\mathscr{F}_{t+}$ the intersection of all $\mathscr{F}_{s}$ for $s>t$. The filtration is normal if

1. $\mathscr{F}_{0}$ contains all $A \in \mathscr{F}$ such that $P(A)=0$
2. $\mathscr{F}_{t}=\mathscr{F}_{t+}$ for all $t \in I$.
$X$ is called progressively measurable if for every $t \in I$, the mapping

$$
(s, \omega) \in[0, t] \times \Omega, \quad(s, \omega) \rightarrow X(s, \omega)
$$

is $B([0, t]) \times \mathscr{F}_{t}$ measurable.
Thus $X$ is progressively measurable means

$$
(s, \omega) \rightarrow \mathscr{X}_{[0, t]}(s) X(s, \omega)
$$

is $B([0, t]) \times \mathscr{F}_{t}$ measurable. As an example of a normal filtration, here is an example.
Example 62.3.2 For example, you could have a stochastic process, $X(t)$ and you could define

$$
\mathscr{G}_{t} \equiv \overline{\sigma(X(s): s \leq t)}
$$

the completion of the smallest $\sigma$ algebra such that each $X(s)$ is measurable for all $s \leq t$. This gives an example of a filtration to which $X(t)$ is adapted which satisfies 1. More generally, suppose $X(t)$ is adapted to a filtration, $\mathscr{G}_{t}$. Define

$$
\mathscr{F}_{t} \equiv \cap_{s>t} \mathscr{G}_{s}
$$

Then

$$
\mathscr{F}_{t+} \equiv \cap_{s>t} \mathscr{F}_{s}=\cap_{s>t} \cap_{r>s} \mathscr{G}_{r}=\cap_{s>t} \mathscr{F}_{s} \equiv \mathscr{F}_{t} .
$$

and each $X(t)$ is measurable with respect to $\mathscr{F}_{t}$. Thus there is no harm in assuming a stochastic process adapted to a filtration can be modified so the filtration is normal. Also note that $\mathscr{F}_{t}$ defined this way will be complete so if $A \in \mathscr{F}_{t}$ has $P(A)=0$ and if $B \subseteq A$, then $B \in \mathscr{F}_{t}$ also. This is because this relation between the sets and the probability of $A$ being zero, holds for this pair of sets when considered as elements of each $\mathscr{G}_{s}$ for $s>t$. Hence $B \in \mathscr{G}_{s}$ for each $s>t$ and is therefore one of the sets in $\mathscr{F}_{t}$.

What is the description of a progressively measurable set?


It means that for $Q$ progressively measurable, $Q \cap[0, t] \times \Omega$ as shown in the above picture is $\mathscr{B}([0, t]) \times \mathscr{F}_{t}$ measurable. It is like saying a little more descriptively that the function is progressively product measurable.

I shall generally assume the filtration is normal.
Observation 62.3.3 If $X$ is progressively measurable, then it is adapted. Furthermore the progressively measurable sets, those $E \cap[0, T] \times \Omega$ for which $\mathscr{X}_{E}$ is progressively measurable form a $\sigma$ algebra.

To see why this is, consider $X$ progressively measurable and fix $t$. Then $(s, \omega) \rightarrow$ $X(s, \omega)$ for $(s, \omega) \in[0, t] \times \Omega$ is given to be $\mathscr{B}([0, t]) \times \mathscr{F}_{t}$ measurable, the ordinary product measure and so fixing any $s \in[0, t]$, it follows the resulting function of $\omega$ is $\mathscr{F}_{t}$ measurable. In particular, this is true upon fixing $s=t$. Thus $\omega \rightarrow X(t, \omega)$ is $\mathscr{F}_{t}$ measurable and so $X(t)$ is adapted.

A set $E \subseteq[0, T] \times \Omega$ is progressively measurable means that $\mathscr{X}_{E}$ is progressively measurable. That is $\mathscr{X}_{E}$ restricted to $[0, t] \times \Omega$ is $\mathscr{B}([0, t]) \times \mathscr{F}_{t}$ measurable. In other words, $E$ is progressively measurable if

$$
E \cap([0, t] \times \Omega) \in \mathscr{B}([0, t]) \times \mathscr{F}_{t} .
$$

If $E_{i}$ is progressively measurable, does it follow that $E \equiv \cup_{i=1}^{\infty} E_{i}$ is also progressively measurable? Yes.

$$
E \cap([0, t] \times \Omega)=\cup_{i=1}^{\infty} E_{i} \cap([0, t] \times \Omega) \in \mathscr{B}([0, t]) \times \mathscr{F}_{t}
$$

because each set in the union is in $\mathscr{B}([0, t]) \times \mathscr{F}_{t}$. If $E$ is progressively measurable, is $E^{C}$ ?

$$
E^{C} \cap([0, t] \times \Omega) \cup \overbrace{(E \cap([0, t] \times \Omega))}^{\in \mathscr{B}([0, t]) \times \mathscr{F}_{t}}=\overbrace{[0, t] \times \Omega}^{\in \mathscr{B}([0, t]) \times \mathscr{F}_{t}}
$$

and so $E^{C} \cap([0, t] \times \Omega) \in \mathscr{B}([0, t]) \times \mathscr{F}_{t}$. Thus the progressively measurable sets are a $\sigma$ algebra.

Another observation of interest is in the following lemma.
Lemma 62.3.4 Suppose $Q$ is in $\mathscr{B}([0, a]) \times \mathscr{F}_{r}$. Then if $b \geq a$ and $t \geq r$, then $Q$ is also in $\mathscr{B}([0, b]) \times \mathscr{F}_{t}$.

Proof: Consider a measurable rectangle $A \times B$ where $A \in \mathscr{B}([0, a])$ and $B \in \mathscr{F}_{r}$. Is it true that $A \times B \in \mathscr{B}([0, b]) \times \mathscr{F}_{t}$ ? This reduces to the question whether $A \in \mathscr{B}([0, b])$. If $A$ is an interval, it is clear that $A \in \mathscr{B}([0, b])$. Consider the $\pi$ system of intervals and let $\mathscr{G}$ denote those Borel sets $A \in \mathscr{B}([0, a])$ such that $A \in \mathscr{B}([0, b])$. If $A \in \mathscr{G}$, then $[0, b] \backslash A \in \mathscr{B}([0, b])$ by assumption (the difference of Borel sets is surely Borel). However, this set equals

$$
([0, a] \backslash A) \cup(a, b]
$$

and so

$$
[0, b]=([0, a] \backslash A) \cup(a, b] \cup A
$$

The set on the left is in $\mathscr{B}([0, b])$ and the sets on the right are disjoint and two of them are also in $\mathscr{B}([0, b])$. Therefore, the third, $([0, a] \backslash A)$ is in $\mathscr{B}([0, b])$. It is obvious that $\mathscr{G}$ is closed with respect to countable disjoint unions. Therefore, by Lemma 12.12.3, Dynkin's lemma, $\mathscr{G} \supseteq \sigma$ (Intervals) $=\mathscr{B}([0, a])$.

Therefore, such a measurable rectangle $A \times B$ where $A \in \mathscr{B}([0, a])$ and $B \in \mathscr{F}_{r}$ is in $\mathscr{B}([0, b]) \times \mathscr{F}_{t}$ and in fact it is a measurable rectangle in $\mathscr{B}([0, b]) \times \mathscr{F}_{t}$. Now let $\mathscr{K}$ denote all these measurable rectangles $A \times B$ where $A \in \mathscr{B}([0, a])$ and $B \in \mathscr{F} r$. Let $\mathscr{G}$ (new $\mathscr{G})$ denote those sets $Q$ of $\mathscr{B}([0, a]) \times \mathscr{F}_{r}$ which are in $\mathscr{B}([0, b]) \times \mathscr{F}_{t}$. Then if $Q \in \mathscr{G}$,

$$
Q \cup([0, a] \times \Omega \backslash Q) \cup(a, b] \times \Omega=[a, b] \times \Omega
$$

Then the sets are disjoint and all but $[0, a] \times \Omega \backslash Q$ are in $\mathscr{B}([0, b]) \times \mathscr{F}_{t}$. Therefore, this one is also in $\mathscr{B}([0, b]) \times \mathscr{F}_{t}$. If $Q_{i} \in \mathscr{G}$ and the $Q_{i}$ are disjoint, then $\cup_{i} Q_{i}$ is also in $\mathscr{B}([0, b]) \times$ $\mathscr{F}_{t}$ and so $\mathscr{G}$ is closed with respect to countable disjoint unions and complements. Hence $\mathscr{G} \supseteq \sigma(\mathscr{K})=\mathscr{B}([0, a]) \times \mathscr{F}_{r}$ which shows

$$
\mathscr{B}([0, a]) \times \mathscr{F}_{r} \subseteq \mathscr{B}([0, b]) \times \mathscr{F}_{t}
$$

A significant observation is the following which states that the integral of a progressively measurable function is progressively measurable.

Proposition 62.3.5 Suppose $X:[0, T] \times \Omega \rightarrow E$ where $E$ is a separable Banach space. Also suppose that $X(\cdot, \omega) \in L^{1}([0, T], E)$ for each $\omega$. Here $\mathscr{F}_{t}$ is a filtration and with respect to this filtration, $X$ is progressively measurable. Then

$$
(t, \omega) \rightarrow \int_{0}^{t} X(s, \omega) d s
$$

is also progressively measurable.

Proof: Suppose $Q \in[0, T] \times \Omega$ is progressively measurable. This means for each $t$,

$$
Q \cap[0, t] \times \Omega \in \mathscr{B}([0, t]) \times \mathscr{F}_{t}
$$

What about

$$
(s, \omega) \in[0, t] \times \Omega,(s, \omega) \rightarrow \int_{0}^{s} \mathscr{X}_{Q} d r ?
$$

Is that function on the right $\mathscr{B}([0, t]) \times \mathscr{F}_{t}$ measurable? We know that $Q \cap[0, s] \times \Omega$ is $\mathscr{B}([0, s]) \times \mathscr{F}_{s}$ measurable and hence $\mathscr{B}([0, t]) \times \mathscr{F}_{t}$ measurable. When you integrate a product measurable function, you do get one which is product measurable. Therefore, this function must be $\mathscr{B}([0, t]) \times \mathscr{F}_{t}$ measurable. This shows that $(t, \omega) \rightarrow \int_{0}^{t} \mathscr{X}_{Q}(s, \omega) d s$ is progressively measurable. Here is a claim which was just used.

Claim: If $Q$ is $\mathscr{B}([0, t]) \times \mathscr{F}_{t}$ measurable, then $(s, \omega) \rightarrow \int_{0}^{s} \mathscr{X}_{Q} d r$ is also $\mathscr{B}([0, t]) \times \mathscr{F}_{t}$ measurable.

Proof of claim: First consider $A \times B$ where $A \in \mathscr{B}([0, t])$ and $B \in \mathscr{F}_{t}$. Then

$$
\int_{0}^{s} \mathscr{X}_{A \times B} d r=\int_{0}^{s} \mathscr{X}_{A} \mathscr{X}_{B} d r=\mathscr{X}_{B}(\omega) \int_{0}^{s} \mathscr{X}_{A}(s) d r
$$

This is clearly $\mathscr{B}([0, t]) \times \mathscr{F}_{t}$ measurable. It is the product of a continuous function of $s$ with the indicator function of a set in $\mathscr{F}_{t}$. Now let

$$
\mathscr{G} \equiv\left\{Q \in \mathscr{B}([0, t]) \times \mathscr{F}_{t}:(s, \omega) \rightarrow \int_{0}^{s} \mathscr{X}_{Q}(r, \omega) d r \text { is } \mathscr{B}([0, t]) \times \mathscr{F}_{t} \text { measurable }\right\}
$$

Then it was just shown that $\mathscr{G}$ contains the measurable rectangles. It is also clear that $\mathscr{G}$ is closed with respect to countable disjoint unions and complements. Therefore, $\mathscr{G} \supseteq$ $\sigma\left(\mathscr{K}_{t}\right)=\mathscr{B}([0, t]) \times \mathscr{F}_{t}$ where $\mathscr{K}_{t}$ denotes the measurable rectangles $A \times B$ where $B \in \overline{\mathscr{F}}_{t}$ and $A \in \mathscr{B}([0, t])=\mathscr{B}([0, T]) \cap[0, t]$. This proves the claim.

Thus if $Q$ is progressively measurable, it follows that $(s, \omega) \rightarrow \int_{0}^{s} \mathscr{X}_{Q}(r, \omega) d r \equiv f(s, \omega)$ is progressively measurable because for $(s, \omega) \in[0, t] \times \Omega,(s, \omega) \rightarrow f(s, \omega)$ is $\mathscr{B}([0, t]) \times$ $\mathscr{F}_{t}$ measurable. This is what was to be proved in this special case.

Now consider the conclusion of the proposition. By considering the positive and negative parts of $\phi(X)$ for $\phi \in E^{\prime}$, and using Pettis theorem, it suffices to consider the case where $X \geq 0$. Then there exists an increasing sequence of progressively measurable simple functions $\left\{X_{n}\right\}$ converging pointwise to $X$. From what was just shown,

$$
(t, \omega) \rightarrow \int_{0}^{t} X_{n} d s
$$

is progressively measurable. Hence, by the monotone convergence theorem, $(t, \omega) \rightarrow$ $\int_{0}^{t} X d s$ is also progressively measurable.

What else can you do to something which is progressively measurable and obtain something which is progressively measurable? It turns out that shifts in time can preserve progressive measurability. Let $\mathscr{F}_{t}$ be a filtration on $[0, T]$ and extend the filtration to be equal to $\mathscr{F}_{0}$ and $\mathscr{F}_{T}$ for $t<0$ and $t>T$, respectively. Recall the following definition of progressively measurable sets.

Definition 62.3.6 Denote by $\mathscr{P}$ those sets $Q$ in $\mathscr{F}_{T} \times \mathscr{B}([0, T])$ such that for $t \in[-\infty, T]$

$$
\Omega \times(-\infty, t] \cap Q \in \mathscr{F}_{t} \times \mathscr{B}((-\infty, t])
$$

Lemma 62.3.7 Define $Q+h$ as

$$
Q+h \equiv\{(t+h, \omega):(t, \omega) \in Q\}
$$

Then if $Q \in \mathscr{P}$, it follows that $Q+h \in \mathscr{P}$.
Proof: This is most easily seen through the use of the following diagram. In this diagram, $Q$ is in $\mathscr{P}$ so it is progressively measurable.


By definition, $S$ in the picture is $\mathscr{B}((-\infty, t-h]) \times \mathscr{F}_{t-h}$ measurable. Hence $S+h \equiv$ $Q+h \cap \Omega \times(-\infty, t]$ is $\mathscr{B}((-\infty, t]) \times \mathscr{F}_{t-h}$ measurable. To see this, note that if $B \times A \in$ $\mathscr{B}((-\infty, t-h]) \times \mathscr{F}_{t-h}$, then translating it by $h$ gives a set in $\mathscr{B}((-\infty, t]) \times \mathscr{F}_{t-h}$. Then if $\mathscr{G}$ consists of sets $S$ in $\mathscr{B}((-\infty, t-h]) \times \mathscr{F}_{t-h}$ for which $S+h$ is in $\mathscr{B}((-\infty, t]) \times \mathscr{F}_{t-h}$, $\mathscr{G}$ is closed with respect to countable disjoint unions and complements. Thus, $\mathscr{G}$ equals $\mathscr{B}((-\infty, t-h]) \times \mathscr{F}_{t-h}$. In particular, it contains the set $S$ just described.

Now for $h>0$,

$$
\tau_{h} f(t) \equiv\left\{\begin{array}{l}
f(t-h) \text { if } t \geq h \\
0 \text { if } t<h
\end{array}\right.
$$

Lemma 62.3.8 Let $Q \in \mathscr{P}$. Then $\tau_{h} \mathscr{X}_{Q}$ is $\mathscr{P}$ measurable.
Proof: If $\tau_{h} \mathscr{X}_{Q}(t, \omega)=1$, then you need to have $(t-h, \omega) \in Q$ and so $(t, \omega) \in Q+h$. Thus

$$
\tau_{h} \mathscr{X}_{Q}=\mathscr{X}_{Q+h}
$$

which is $\mathscr{P}$ measurable since $Q \in \mathscr{P}$. In general,

$$
\tau_{h} \mathscr{X}_{Q}=\mathscr{X}_{[h, T] \times \Omega} \mathscr{X}_{Q+h},
$$

which is $\mathscr{P}$ measurable.
This lemma implies the following.
Lemma 62.3.9 Let $f(t, \omega)$ have values in a separable Banach space and suppose $f$ is $\mathscr{P}$ measurable. Then $\tau_{h} f$ is $\mathscr{P}$ measurable.

Proof: Taking values in a separable Banach space and being $\mathscr{P}$ measurable, $f$ is the pointwise limit of $\mathscr{P}$ measurable simple functions. If $s_{n}$ is one of these, then from the above lemmas, $\tau_{h} s_{n}$ is $\mathscr{P}$ measurable. Then, letting $n \rightarrow \infty$, it follows that $\tau_{h} f$ is $\mathscr{P}$ measurable.

The following is similar to Proposition 62.1.2. It shows that under pretty weak conditions, an adapted process has a progressively measurable adapted version.

Proposition 62.3.10 Let $X$ be a stochastically continuous adapted process for a normal filtration defined on a closed interval, $I \equiv[0, T]$. Then $X$ has a progressively measurable adapted version.

Proof: By Lemma 62.1.1 $X$ is uniformly stochastically continuous and so there exists a sequence of positive numbers, $\left\{\rho_{n}\right\}$ such that if $|s-t|<\rho_{n}$, then

$$
\begin{equation*}
P\left(\left[\|X(t)-X(s)\| \geq \frac{1}{2^{n}}\right]\right) \leq \frac{1}{2^{n}} \tag{62.3.16}
\end{equation*}
$$

Then let $\left\{t_{0}^{n}, t_{1}^{n}, \cdots, t_{m_{n}}^{n}\right\}$ be a partition of $[0, T]$ in which $\left|t_{i}^{n}-t_{i-1}^{n}\right|<\rho_{n}$. Now define $X_{n}$ as follows:

$$
\begin{aligned}
X_{n}(t)(\omega) & \equiv \sum_{i=1}^{m_{n}} X\left(t_{i-1}^{n}\right)(\omega) \mathscr{X}_{\left[t_{i-1}^{n}, t_{i}^{n}\right)}(t) \\
X_{n}(T) & \equiv X(T)
\end{aligned}
$$

Then $(s, \omega) \rightarrow X_{n}(s, \omega)$ for $(s, \omega) \in[0, t] \times \Omega$ is obviously $B([0, t]) \times \mathscr{F}_{t}$ measurable. Consider the set, $A$ on which $\left\{X_{n}(t, \omega)\right\}$ is a Cauchy sequence. This set is of the form

$$
A=\cap_{n=1}^{\infty} \cup_{m=1}^{\infty} \cap_{p, q \geq m}\left[\left\|X_{p}-X_{q}\right\|<\frac{1}{n}\right]
$$

and so it is a $B(I) \times \mathscr{F}$ measurable set and $A \cap[0, t] \times \Omega$ is $B([0, t]) \times \mathscr{F}_{t}$ measurable for each $t \leq T$ because each $X_{q}$ in the above has the property that its restriction to $[0, t] \times \Omega$ is $B([0, t]) \times \mathscr{F}_{t}$ measurable. Now define

$$
Y(t, \omega) \equiv\left\{\begin{array}{l}
\lim _{n \rightarrow \infty} X_{n}(t, \omega) \text { if }(t, \omega) \in A \\
0 \text { if }(t, \omega) \notin A
\end{array}\right.
$$

I claim that for each $t, Y(t, \omega)=X(t, \omega)$ for a.e. $\omega$. To see this, consider 62.3.16. From the construction of $X_{n}$, it follows that for each $t$,

$$
P\left(\left[\left\|X_{n}(t)-X(t)\right\| \geq \frac{1}{2^{n}}\right]\right) \leq \frac{1}{2^{n}}
$$

Also, for a fixed $t$, if $X_{n}(t, \omega)$ fails to converge to $X(t, \omega)$, then $\omega$ must be in infinitely many of the sets,

$$
B_{n} \equiv\left[\left\|X_{n}(t)-X(t)\right\| \geq \frac{1}{2^{n}}\right]
$$

which is a set of measure zero by the Borel Cantelli lemma. Recall why this is so.

$$
P\left(\cap_{k=1}^{\infty} \cup_{n=k}^{\infty} B_{n}\right) \leq \sum_{n=k}^{\infty} P\left(B_{n}\right)<\frac{1}{2^{k-1}}
$$

Therefore, for each $t,(t, \omega) \in A$ for a.e. $\omega$. Hence $X(t)=Y(t)$ a.e. and so $Y$ is a measurable version of $X . Y$ is adapted because the filtration is normal and hence $\mathscr{F}_{t}$ contains all sets of measure zero. Therefore, $Y(t)$ differs from $X(t)$ on a set which is $\mathscr{F}_{t}$ measurable.

There is a more specialized situation in which the measurability of a stochastic process automatically implies it is adapted. Furthermore, this can be defined easily in terms of a $\pi$ system of sets.

Definition 62.3.11 Let $\mathscr{F}_{t}$ be a filtration on $(\Omega, \mathscr{F}, P)$ and denote by $\mathscr{P}_{\infty}$ the smallest $\sigma$ algebra of sets of $[0, \infty) \times \Omega$ containing the sets

$$
(s, t] \times F, F \in \mathscr{F}_{s} \quad\{0\} \times F, F \in \mathscr{F}_{0} .
$$

This is called the predictable $\sigma$ algebra. and the sets in this $\sigma$ algebra are called the predictable sets. Denote by $\mathscr{P}_{T}$ the intersection of $\mathscr{P}_{\infty}$ to $[0, T] \times \Omega$. A stochastic process $X$ which maps either $[0, T] \times \Omega$ or $[0, \infty) \times \Omega$ to $E$, a separable real Banach space is called predictable if for every Borel set $A \in \mathscr{B}(E)$, it follows $X^{-1}(A) \in \mathscr{P}_{T}$ or $\mathscr{P}_{\infty}$.

This is a lot like product measure except one of the $\sigma$ algebras is changing.
Proposition 62.3.12 Let $\mathscr{F}_{t}$ be a filtration as above and let $X$ be a predictable stochastic process. Then $X$ is $\mathscr{F}_{t}$ adapted.

Proof: Let $s_{0}>0$ and define

$$
\mathscr{G}_{s_{0}} \equiv\left\{S \in \mathscr{P}_{\infty}: S_{s_{0}} \in \mathscr{F}_{s_{0}}\right\}
$$

where

$$
S_{s_{0}} \equiv\left\{\omega \in \Omega:\left(s_{0}, \omega\right) \in S\right\}
$$



It is clear $\mathscr{G}_{s_{0}}$ is a $\sigma$ algebra. The next step is to show $\mathscr{G}_{s_{0}}$ contains the sets

$$
\begin{equation*}
(s, t] \times F, F \in \mathscr{F}_{s} \tag{62.3.17}
\end{equation*}
$$

and

$$
\begin{equation*}
\{0\} \times F, F \in \mathscr{F}_{0} . \tag{62.3.18}
\end{equation*}
$$

It is clear $\{0\} \times F$ is contained in $\mathscr{G}_{s_{0}}$ because $(\{0\} \times F)_{s_{0}}=\emptyset \in \mathscr{F}_{s_{0}}$. Similarly, if $s \geq s_{0}$ or if $s, t<s_{0}$ then $((s, t] \times F)_{s_{0}}=\emptyset \in \mathscr{F}_{s_{0}}$. The only case left is for $s<s_{0}$ and $t \geq s_{0}$. In this case, letting $A_{s} \in \mathscr{F}_{s},\left((s, t] \times A_{s}\right)_{s_{0}}=A_{s} \in \mathscr{F}_{s} \subseteq \mathscr{F}_{s_{0}}$. Therefore, $\mathscr{G}_{s_{0}}$ contains all the sets of the form given in 62.3.17 and 62.3 .18 and so since $\mathscr{P}_{\infty}$ is the smallest $\sigma$ algebra containing these sets, it follows $\mathscr{P}_{\infty}=\mathscr{G}_{s_{0}}$. The case where $s_{0}=0$ is entirely similar but shorter.

Therefore, if $X$ is predictable, letting $A \in \mathscr{B}(E), X^{-1}(A) \in \mathscr{P}_{\infty}$ or $\mathscr{P}_{T}$ and so

$$
\left(X^{-1}(A)\right)_{s} \equiv\{\omega \in \Omega: X(s, \omega) \in A\}=X(s)^{-1}(A) \in \mathscr{F}_{s}
$$

showing $X(t)$ is $\mathscr{F}_{t}$ adapted. This proves the proposition.
Another way to see this is to recall the progressively measurable functions are adapted. Then show the predictable sets are progressively measurable.

Proposition 62.3.13 Let $\mathscr{P}$ denote the predictable $\sigma$ algebra and let $\mathscr{R}$ denote the progressively measurable $\sigma$ algebra. Then $\mathscr{P} \subseteq \mathscr{R}$.

Proof: Let $\mathscr{G}$ denote those sets of $\mathscr{P}$ such that they are also in $\mathscr{R}$. Then $\mathscr{G}$ clearly contains the $\pi$ system of sets $\{0\} \times A, A \in \mathscr{F}_{0}$, and $(s, t] \times A, A \in \mathscr{F}_{s}$. Furthermore, $\mathscr{G}$ is closed with respect to countable disjoint unions and complements. It follows $\mathscr{G}$ contains the $\sigma$ algebra generated by this $\pi$ systems which is $\mathscr{P}$. This proves the proposition.

Proposition 62.3.14 Let $X(t)$ be a stochastic process having values in $E$ a complete metric space and let it be $\mathscr{F}_{t}$ adapted and left continuous. Then it is predictable. Also, if $X(t)$ is stochastically continuous and adapted on $[0, T]$, then it has a predictable version.

Proof:Define $I_{m, k} \equiv\left((k-1) 2^{-m} T, k 2^{-m} T\right]$ if $k \geq 1$ and $I_{m, 0}=\{0\}$ if $k=1$. Then define

$$
\begin{aligned}
X_{m}(t) \equiv & \sum_{k=1}^{2^{m}} X\left(T(k-1) 2^{-m}\right) \mathscr{X}_{\left((k-1) 2^{-m} T, k 2^{-m} T\right]}(t) \\
& +X(0) \mathscr{X}_{[0,0]}(t)
\end{aligned}
$$

Here the sum means that $X_{m}(t)$ has value $X\left(T(k-1) 2^{-m}\right)$ on the interval

$$
\left((k-1) 2^{-m} T, k 2^{-m} T\right] .
$$

Thus $X_{m}$ is predictable because each term in the sum is. Thus

$$
\begin{aligned}
& X_{m}^{-1}(U)=\cup_{k=1}^{2^{m}}\left(X\left(T(k-1) 2^{-m}\right) \mathscr{X}_{\left((k-1) 2^{-m} T, k 2^{-m} T\right]}\right)^{-1}(U) \\
& =\cup_{k=1}^{2^{m}}\left((k-1) 2^{-m} T, k 2^{-m} T\right] \times\left(X\left(T(k-1) 2^{-m}\right)\right)^{-1}(U),
\end{aligned}
$$

a finite union of predictable sets. Since $X$ is left continuous,

$$
X(t, \omega)=\lim _{m \rightarrow \infty} X_{m}(t, \omega)
$$

and so $X$ is predictable.
Next consider the other claim. Since $X$ is stochastically continuous on $[0, T]$, it is uniformly stochastically continuous on this interval by Lemma 62.1.1. Therefore, there exists a sequence of partitions of $[0, T]$, the $m^{t h}$ being

$$
0=t_{m, 0}<t_{m, 1}<\cdots<t_{m, n(m)}=T
$$

such that for $X_{m}$ defined as above, then for each $t$

$$
\begin{equation*}
P\left(\left[d\left(X_{m}(t), X(t)\right) \geq 2^{-m}\right]\right) \leq 2^{-m} \tag{62.3.19}
\end{equation*}
$$

Then as above, $X_{m}$ is predictable. Let $A$ denote those points of $\mathscr{P}_{T}$ at which $X_{m}(t, \omega)$ converges. Thus $A$ is a predictable set because it is just the set where $X_{m}(t, \omega)$ is a Cauchy sequence. Now define the predictable function $Y$

$$
Y(t, \omega) \equiv\left\{\begin{array}{l}
\lim _{m \rightarrow \infty} X_{m}(t, \omega) \text { if }(t, \omega) \in A \\
0 \text { if }(t, \omega) \notin A
\end{array}\right.
$$

From 62.3 .19 it follows from the Borel Cantelli lemma that for fixed $t$, the set of $\omega$ which are in infinitely many of the sets,

$$
\left[d\left(X_{m}(t), X(t)\right) \geq 2^{-m}\right]
$$

has measure zero. Therefore, for each $t$, there exists a set of measure zero, $N(t)$ such that for $\omega \notin N(t)$ and all $m$ large enough

$$
d\left(X_{m}(t, \omega), X(t, \omega)\right)<2^{-m}
$$

Hence for $\omega \notin N(t),(t, \omega) \in A$ and so $X_{m}(t, \omega) \rightarrow Y(t, \omega)$ which shows

$$
d(Y(t, \omega), X(t, \omega))=0 \text { if } \omega \notin N(t) .
$$

The predictable version of $X(t)$ is $Y(t)$.
Here is a summary of what has been shown above.

> adapted and left continuous
> $\Downarrow$
> predictable
> $\Downarrow$
> progressively measurable
> $\Downarrow$
> adapted

Also
stochastically continuous and adapted $\Longrightarrow$ progressively measurable version

### 62.4 Martingales

Definition 62.4.1 Let $X$ be a stochastic process defined on an interval, I with values in a separable Banach space, $E$. It is called integrable if $E(\|X(t)\|)<\infty$ for each $t \in I$. Also let $\mathscr{F}_{t}$ be a filtration. An integrable and adapted stochastic process $X$ is called a martingale if for $s \leq t$

$$
E\left(X(t) \mid \mathscr{F}_{s}\right)=X(s) P \text { a.e. } \omega .
$$

Recalling the definition of conditional expectation this says that for $F \in \mathscr{F}_{s}$

$$
\int_{F} X(t) d P=\int_{F} E\left(X(t) \mid \mathscr{F}_{s}\right) d P=\int_{F} X(s) d P
$$

for all $F \in \mathscr{F}_{s}$. A real valued stochastic process is called a submartingale if whenever $s \leq t$,

$$
E\left(X(t) \mid \mathscr{F}_{s}\right) \geq X(s) \text { a.e. }
$$

and a supermartingale if

$$
E\left(X(t) \mid \mathscr{F}_{s}\right) \leq X(s) \text { a.e. }
$$

Example 62.4.2 Let $\mathscr{F}_{t}$ be a filtration and let $Z$ be in $L^{1}\left(\Omega, \mathscr{F}_{T}, P\right)$. Then let $X(t)=$ $E\left(Z \mid \mathscr{F}_{t}\right)$.

This works because for $s<t, E\left(X(t) \mid \mathscr{F}_{s}\right)=E\left(E\left(Z \mid \mathscr{F}_{t}\right) \mid \mathscr{F}_{s}\right)=E\left(Z \mid \mathscr{F}_{s}\right)=X(s)$.
Proposition 62.4.3 The following statements hold for a stochastic process defined on the product $[0, T] \times \Omega$ having values in a real separable Banach space, $E$.

1. If $X(t)$ is a martingale then $\|X(t)\|, t \in[0, T]$ is a submartingale.
2. If $g$ is an increasing convex function from $[0, \infty)$ to $[0, \infty)$ and $E(g(\|X(t)\|))<\infty$ for all $t \in[0, T]$ then then $g(\|X(t)\|), t \in[0, T]$ is a submartingale.

Proof:Let $s \leq t$

$$
\begin{aligned}
\|X(s)\| & =\left\|E\left(X(s)-X(t) \mid \mathscr{F}_{s}\right)+E\left(X(t) \mid \mathscr{F}_{s}\right)\right\| \\
& \leq \overbrace{\left\|E\left(X(s)-X(t) \mid \mathscr{F}_{s}\right)\right\|}^{=0 \text { a.e. }}+\left\|E\left(X(t) \mid \mathscr{F}_{s}\right)\right\| \\
& \leq\left\|E\left(X(t) \mid \mathscr{F}_{s}\right)\right\| .
\end{aligned}
$$

Now by Theorem 61.1.1 on Page 1983

$$
\left\|E\left(X(t) \mid \mathscr{F}_{s}\right)\right\| \leq E\left(\|X(t)\| \mid \mathscr{F}_{s}\right) .
$$

Thus $\|X(s)\| \leq E\left(\|X(t)\| \mid \mathscr{F}_{s}\right)$ which shows $\|X\|$ is a submartingale as claimed.
Consider the second claim. Recall Jensen's inequality for submartingales, Theorem 60.1.6 on Page 1946. From the first part

$$
\|X(s)\| \leq E\left(\|X(t)\| \mathscr{F}_{s}\right) \text { a.e. }
$$

and so from Jensen's inequality,

$$
g(\|X(s)\|) \leq g\left(E\left(\|X(t)\| \mid \mathscr{F}_{s}\right)\right) \leq E\left(g(\|X(t)\|) \mid \mathscr{F}_{s}\right) \text { a.e. }
$$

showing that $g(\|X(t)\|)$ is also a submartingale. This proves the proposition.

### 62.5 Some Maximal Estimates

Martingales and submartingales have some very interesting maximal estimates. I will present some of these here. The proofs are fairly general and do not require the filtration to be normal.

Lemma 62.5.1 Let $\left\{\mathscr{F}_{t}\right\}$ be a filtration and let $\{X(t)\}$ be a nonnegative valued submartingale for $t \in[S, T]$. Then for $\lambda>0$ and any $p \geq 1$, if $A_{t}$ is a $\mathscr{F}_{t}$ measurable subset of $[X(t) \geq \lambda]$, then

$$
P\left(A_{t}\right) \leq \frac{1}{\lambda^{p}} \int_{A_{t}} X(T)^{p} d P
$$

Proof: From Jensen's inequality,

$$
\begin{aligned}
\lambda^{p} P\left(A_{t}\right) & \leq \int_{A_{t}} X(t)^{p} d P \leq \int_{A_{t}} E\left(X(T) \mid \mathscr{F}_{t}\right)^{p} d P \\
& \leq \int_{A_{t}} E\left(X(T)^{p} \mid \mathscr{F}_{t}\right) d P=\int_{A_{t}} X(T)^{p} d P
\end{aligned}
$$

and this proves the lemma.
The following theorem is the main result.
Theorem 62.5.2 Let $\left\{\mathscr{F}_{t}\right\}$ be a filtration and let $\{X(t)\}$ be a nonnegative valued right continuous ${ }^{1}$ submartingale for $t \in[S, T]$. Then for all $\lambda>0$ and $p \geq 1$, for

$$
\begin{gathered}
X^{*} \equiv \sup _{t \in[S, T]} X(t), \\
P\left(\left[X^{*} \geq \lambda\right]\right) \leq \frac{1}{\lambda^{p}} \int_{\Omega} \mathscr{X}_{\left[X^{*} \geq \lambda\right]} X(T)^{p} d P
\end{gathered}
$$

In the case that $p>1$, it is also true that

$$
E\left(\left(X^{*}\right)^{p}\right) \leq\left(\frac{p}{p-1}\right) E\left(X(T)^{p}\right)^{1 / p}\left(E\left(\left(X^{*}\right)^{p}\right)\right)^{1 / p^{\prime}}
$$

Also there are no measurability issues related to the above $\sup _{t \in[S, T]} X(t) \equiv X^{*}$
Proof: Let $S \leq t_{0}^{m}<t_{1}^{m}<\cdots<t_{N_{m}}^{m}=T$ where $t_{j+1}^{m}-t_{j}^{m}=(T-S) 2^{-m}$. First consider $m=1$.

$$
\begin{gathered}
A_{t_{0}^{1}} \equiv\left\{\omega \in \Omega: X\left(t_{0}^{1}\right)(\omega) \geq \lambda\right\}, A_{t_{1}^{1}} \equiv\left\{\omega \in \Omega: X\left(t_{1}^{1}\right)(\omega) \geq \lambda\right\} \backslash A_{t_{0}^{1}} \\
A_{t_{2}^{1}} \equiv\left\{\omega \in \Omega: X\left(t_{2}^{1}\right)(\omega) \geq \lambda\right\} \backslash\left(A_{t_{0}^{1}} \cup A_{t_{0}^{1}}\right)
\end{gathered}
$$

Do this type of construction for $m=2,3,4, \cdots$ yielding disjoint sets, $\left\{A_{t_{j}^{m}}\right\}_{j=0}^{2^{m}}$ whose union equals

$$
\cup_{t \in D_{m}}[X(t) \geq \lambda]
$$

where $D_{m}=\left\{t_{j}^{m}\right\}_{j=0}^{2^{m}}$. Thus $D_{m} \subseteq D_{m+1}$. Then also, $D \equiv \cup_{m=1}^{\infty} D_{m}$ is dense and countable. From Lemma 62.5.1,

$$
\begin{aligned}
P\left(\cup_{t \in D_{m}}[X(t) \geq \lambda]\right) & =P\left(\sup _{t \in D_{m}} X(t) \geq \lambda\right)=\sum_{j=0}^{2^{m}} P\left(A_{t_{j}^{m}}\right) \\
& \leq \frac{1}{\lambda^{p}} \sum_{j=0}^{2^{m}} \int_{A_{t_{j}^{m}}} \mathscr{X}_{\left[\sup _{t \in D_{m}} X(t) \geq \lambda\right]} X(T)^{p} d P \\
& \leq \frac{1}{\lambda^{p}} \int_{\Omega} \mathscr{X}_{\left[\sup _{t \in D} X(t) \geq \lambda\right]} X(T)^{p} d P .
\end{aligned}
$$

${ }^{1} t \rightarrow M(t)(\omega)$ is continuous from the right for a.e. $\omega$.

Let $m \rightarrow \infty$ in the above to obtain

$$
\begin{equation*}
P\left(\cup_{t \in D}[X(t) \geq \lambda]\right)=P\left(\left[\sup _{t \in D} X(t) \geq \lambda\right]\right) \leq \frac{1}{\lambda^{p}} \int_{\Omega} \mathscr{X}_{\left[\sup _{t \in D} X(t) \geq \lambda\right]} X(T)^{p} d P \tag{62.5.20}
\end{equation*}
$$

From now on, assume that for a.e. $\omega \in \Omega, t \rightarrow X(t)(\omega)$ is right continuous. Then with this assumption, the following claim holds.

$$
\sup _{t \in[S, T]} X(t) \equiv X^{*}=\sup _{t \in D} X(t)
$$

which verifies that $X^{*}$ is measurable. Then from 62.5.20,

$$
\begin{aligned}
P\left(\left[X^{*} \geq \lambda\right]\right) & =P\left(\left[\sup _{t \in D} X(t) \geq \lambda\right]\right) \\
& \leq \frac{1}{\lambda^{p}} \int_{\Omega} \mathscr{X}_{\left[\sup _{t \in D} X(t) \geq \lambda\right]} X(T)^{p} d P \\
& =\frac{1}{\lambda^{p}} \int_{\Omega} \mathscr{X}_{\left[X^{*} \geq \lambda\right]} X(T)^{p} d P
\end{aligned}
$$

Now consider the other inequality. Using the distribution function technique and the above estimate obtained in the first part,

$$
\begin{aligned}
& E\left(\left(X^{*}\right)^{p}\right)=\int_{0}^{\infty} p \alpha^{p-1} P\left(\left[X^{*}>\alpha\right]\right) d \alpha \\
& \leq \int_{0}^{\infty} p \alpha^{p-1} P\left(\left[X^{*} \geq \alpha\right]\right) d \alpha \\
& \leq \int_{0}^{\infty} p \alpha^{p-1} \frac{1}{\alpha} \int_{\Omega} \mathscr{X}_{\left[X^{*} \geq \alpha\right]} X(T) d P d \alpha \\
& =p \int_{\Omega} \int_{0}^{X^{*}} \alpha^{p-2} d \alpha X(T) d P \\
& =\frac{p}{p-1} \int_{\Omega}\left(X^{*}\right)^{p-1} X(T) d P \\
& \leq \frac{p}{p-1}\left(\int_{\Omega}\left(X^{*}\right)^{p}\right)^{1 / p^{\prime}}\left(\int_{\Omega} X(T)^{p}\right)^{1 / p} \\
& =\frac{p}{p-1} E\left(X(T)^{p}\right)^{1 / p} E\left(\left(X^{*}\right)^{p}\right)^{1 / p^{\prime}} . \square
\end{aligned}
$$

Of course it would be nice to divide both sides by $E\left(\left(X^{*}\right)^{p}\right)^{1 / p^{\prime}}$ but we don't know that this is finite. One can use a stopped submartingale which will have $X(t)$ bounded, divide, and then let the stopping time increase to $\infty$. However, this is discussed later.

With Theorem 62.5.2, here is an important maximal estimate for martingales having values in $E$, a real separable Banach space.

Theorem 62.5.3 Let $X(t)$ for $t \in I=[0, T]$ be an $E$ valued right continuous martingale with respect to a filtration, $\mathscr{F}_{t}$. Then for $p \geq 1$,

$$
\begin{equation*}
P\left(\left[\sup _{t \in I}\|X(t)\| \geq \lambda\right]\right) \leq \frac{1}{\lambda^{p}} E\left(\|X(T)\|^{p}\right) \tag{62.5.21}
\end{equation*}
$$

If $p>1$,

$$
\begin{equation*}
E\left(\left(\sup _{t \in[S, T]}\|X(t)\|\right)^{p}\right) \leq\left(\frac{p}{p-1}\right) E\left(\|X(T)\|^{p}\right)^{1 / p} E\left(\left(\sup _{t \in[S, T]}\|X(t)\|\right)^{p}\right)^{1 / p^{\prime}} \tag{62.5.22}
\end{equation*}
$$

Proof: By Proposition 62.4.3 $\|X(t)\|, t \in I$ is a submartingale and so from Theorem 62.5.2, it follows 62.5.21 and 62.5.22 hold.

Definition 62.5.4 Let $K$ be a set of functions of $L^{1}(\Omega, \mathscr{F}, P)$. Then $K$ is called equi integrable if

$$
\limsup _{\lambda \rightarrow \infty} \int_{f \in K} \int_{[|f| \geq \lambda]}|f| d P=0
$$

Recall that from Corollary 20.9.6 on Page 640 such an equi integrable set of functions is weakly sequentially precompact in $L^{1}(\Omega, \mathscr{F}, P)$ in the sense that if $\left\{f_{n}\right\} \subseteq K$, there exists a subsequence, $\left\{f_{n_{k}}\right\}$ and a function, $f \in L^{1}(\Omega, \mathscr{F}, P)$ such that for all $g \in L^{1}(\Omega, \mathscr{F}, P)^{\prime}$,

$$
g\left(f_{n_{k}}\right) \rightarrow g(f)
$$

### 62.6 Optional Sampling Theorems

### 62.6.1 Stopping Times And Their Properties

The optional sampling theorem is very useful in the study of martingales and submartingales as will be shown.

First it is necessary to define the notion of a stopping time.
Definition 62.6.1 Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $\left\{\mathscr{F}_{n}\right\}_{n=1}^{\infty}$ be an increasing sequence of $\sigma$ algebras each contained in $\mathscr{F}$, called a discrete filtration. A stopping time is a measurable function, $\tau$ which maps $\Omega$ to $\mathbb{N}$,

$$
\tau^{-1}(A) \in \mathscr{F} \text { for all } A \in \mathscr{P}(\mathbb{N})
$$

such that for all $n \in \mathbb{N}$,

$$
[\tau \leq n] \in \mathscr{F}_{n}
$$

Note this is equivalent to saying

$$
[\tau=n] \in \mathscr{F}_{n}
$$

because

$$
[\tau=n]=[\tau \leq n] \backslash[\tau \leq n-1] .
$$

For $\tau$ a stopping time define $\mathscr{F}_{\tau}$ as follows.

$$
\mathscr{F}_{\tau} \equiv\left\{A \in \mathscr{F}: A \cap[\tau \leq n] \in \mathscr{F}_{n} \text { for all } n \in \mathbb{N}\right\}
$$

These sets in $\mathscr{F}_{\tau}$ are referred to as "prior" to $\tau$.
The most important example of a stopping time is the first hitting time.
Example 62.6.2 The first hitting time of an adapted process $X(n)$ of a Borel set $G$ is a stopping time. This is defined as

$$
\tau \equiv \min \{k: X(k) \in G\}
$$

To see this, note that

$$
[\tau=n]=\cap_{k<n}\left[X(k) \in G^{C}\right] \cap[X(n) \in G] \in \mathscr{F}_{n}
$$

Proposition 62.6.3 For $\tau$ a stopping time, $\mathscr{F}_{\tau}$ is a $\sigma$ algebra and if $Y(k)$ is $\mathscr{F}_{k}$ measurable for all $k, Y(k)$ having values in a separable Banach space $E$, then

$$
\omega \rightarrow Y(\tau(\omega))
$$

is $\mathscr{F}_{\tau}$ measurable.
Proof: Let $A_{n} \in \mathscr{F}_{\tau}$. I need to show $\cup_{n} A_{n} \in \mathscr{F}_{\tau}$. In other words, I need to show that

$$
\cup_{n} A_{n} \cap[\tau \leq k] \in \mathscr{F}_{k}
$$

The left side equals

$$
\cup_{n}\left(A_{n} \cap[\tau \leq k]\right)
$$

which is a countable union of sets of $\mathscr{F}_{k}$ and so $\mathscr{F}_{\tau}$ is closed with respect to countable unions. Next suppose $A \in \mathscr{F} \tau$.

$$
\left(A^{C} \cap[\tau \leq k]\right) \cup(A \cap[\tau \leq k])=\Omega \cap[\tau \leq k]
$$

and $\Omega \cap[\tau \leq k] \in \mathscr{F}_{k}$. Therefore, so is $A^{C} \cap[\tau \leq k]$.
It remains to verify the last claim. Let $B$ be an open set in $E$. Is

$$
[Y(\tau) \in B] \in \mathscr{F}_{\tau} ?
$$

Is

$$
[Y(\tau) \in B] \cap[\tau \leq k] \in \mathscr{F}_{k} \text { for all } k ?
$$

This equals

$$
\cup_{i=1}^{k}[Y(\tau) \in B] \cap[\tau=i]=\cup_{i=1}^{k}[Y(i) \in B] \cap[\tau=i] \in \mathscr{F}_{k}
$$

Therefore, $Y(\tau)$ must be $\mathscr{F}_{\tau}$ measurable.
The following lemma contains the fundamental properties of stopping times for discrete filtrations.

Lemma 62.6.4 In the situation of Definition 62.6.1, let $\sigma, \tau$ be two stopping times. Then

1. $\tau$ is $\mathscr{F}_{\tau}$ measurable
2. $\mathscr{F}_{\sigma} \cap[\sigma \leq \tau] \subseteq \mathscr{F}_{\sigma \wedge \tau}=\mathscr{F}_{\sigma} \cap \mathscr{F}_{\tau}$
3. $\mathscr{F}_{\tau}=\mathscr{F}_{k}$ on $[\tau=k]$ for all $k$. That is if $A \in \mathscr{F}_{k}$, then $A \cap[\tau=k] \in \mathscr{F}_{\tau}$ and if $A \in \mathscr{F}_{\tau}$, then $A \cap[\tau=k] \in \mathscr{F}_{k}$. In other words, the two $\sigma$ algebras

$$
[\tau=k] \cap \mathscr{F}_{\tau},[\tau=k] \cap \mathscr{F}_{k}
$$

are equal. Letting $\mathscr{G}$ denote this $\sigma$ algebra, if $g$ is either $\mathscr{F}_{\tau}$ or $\mathscr{F}_{k}$ measurable then its restriction to $[\tau=k]$ is $\mathscr{G}$ measurable. Also if $A \in \mathscr{F}_{\tau}$, and $Y \in L^{1}(\Omega ; E)$,

$$
\int_{A \cap[\tau=k]} E\left(Y \mid \mathscr{F}_{\tau}\right) d P=\int_{A \cap[\tau=k]} E\left(Y \mid \mathscr{F}_{k}\right) d P
$$

and

$$
E\left(Y \mid \mathscr{F}_{\tau}\right)=E\left(Y \mid \mathscr{F}_{k}\right) \text { a.e. }
$$

on $[\tau=k]$.
Proof: Consider the first claim. $[\tau \leq l] \cap[\tau \leq m]=[\tau \leq\lfloor l\rfloor \wedge m] \in \mathscr{F}_{[l] \wedge m} \subseteq \mathscr{F}_{m}$ and so $\tau$ is $\mathscr{F} \tau$ measurable. Here $\lfloor l\rfloor$ is the greatest integer less than or equal to $l$. Next note that $\sigma \wedge \tau$ is a stopping time because

$$
[\sigma \wedge \tau \leq k]=[\sigma \leq k] \cup[\tau \leq k] \in \mathscr{F}_{k}
$$

Next consider the second claim. Let $A \in \mathscr{F} \sigma$. I want to show

$$
\begin{equation*}
A \cap[\sigma \leq \tau] \in \mathscr{F} \tau \wedge \sigma \tag{62.6.23}
\end{equation*}
$$

In other words, I want to show

$$
\begin{equation*}
A \cap[\sigma \leq \tau] \cap[\tau \wedge \sigma \leq k] \in \mathscr{F}_{k} \tag{62.6.24}
\end{equation*}
$$

for all $k$. However, the set on the left equals

$$
\begin{aligned}
& A \cap[\sigma \leq \tau] \cap[\sigma \leq k] \\
= & \cup_{j=1}^{k} A \cap[\sigma=j] \cap[\tau \geq j] \cap[\sigma \leq k] \\
= & \cup_{j=1}^{k} A \cap[\sigma=j] \cap[\tau \leq j-1]^{C} \cap[\sigma \leq k] \in \mathscr{F}_{k}
\end{aligned}
$$

Now let $A \in \mathscr{F}_{\sigma \wedge \tau}$. I want to show it is in both $\mathscr{F}_{\tau}$ and $\mathscr{F}_{\sigma}$. To show it is in $\mathscr{F}_{\tau}$ I need to show that

$$
A \cap[\tau \leq k] \in \mathscr{F}_{k}
$$

for all $k$. However,

$$
A \cap[\tau=k]=\cup_{i=1}^{\infty} A \cap[\sigma=i] \cap[\tau=k]
$$

$$
\begin{aligned}
& =\cup_{i=1}^{k-1} \overbrace{A \cap[\sigma \wedge \tau=i]}^{\in \mathscr{F}_{i}} \cap \overbrace{[\tau=k]}^{\in \mathscr{F}_{k}} \cup \cup_{i=k}^{\infty} A \cap[\sigma=i] \cap[\tau=k] \\
& =\cup_{i=1}^{k-1} \overbrace{A \cap[\sigma \wedge \tau=i]}^{\in \mathscr{F}_{i}} \cap \overbrace{[\tau=k]}^{\in \mathscr{F}_{k}} \cup \overbrace{A \cap[\sigma \wedge \tau=k]}^{\in \mathscr{F}_{k}} \cap\left[\begin{array}{c}
\in=k]
\end{array}\right.
\end{aligned}
$$

and so this is in $\mathscr{F}_{k}$. Thus $A \cap[\tau \leq k] \in \mathscr{F}_{k}$ being the finite union of sets which are. Similarly $A \cap[\sigma \leq k] \in \mathscr{F}_{k}$ for all $k$ and so $A \in \mathscr{F}_{\tau} \cap \mathscr{F}_{\sigma}$.

Next let $A \in \mathscr{F}_{\tau} \cap \mathscr{F}_{\sigma}$. Then is it in $\mathscr{F}_{\sigma \wedge \tau}$ ? Is $A \cap[\sigma \wedge \tau \leq k] \in \mathscr{F}_{k}$ ? Of course this is so because

$$
\begin{aligned}
& A \cap[\sigma \wedge \tau \leq k]=A \cap([\sigma \leq k] \cup[\tau \leq k]) \\
& \quad=(A \cap[\sigma \leq k]) \cup(A \cap[\tau \leq k]) \in \mathscr{F}_{k}
\end{aligned}
$$

since both $\sigma, \tau$ are stopping times. This proves part 2.).
Now consider part 3.). Note that $[\tau=k]$ is in both $\mathscr{F}_{k}$ and $\mathscr{F}_{\tau}$. First consider the claim it is in $\mathscr{F}_{\tau}$.

$$
[\tau=k] \cap[\tau \leq l]=\emptyset \text { if } l<k
$$

which is in $\mathscr{F}_{l}$. If $l \geq k$, it reduces to $[\tau=k] \in \mathscr{F}_{k} \subseteq \mathscr{F}_{l}$ so it is in $\mathscr{F}_{\tau} .[\tau=k]$ is obviously in $\mathscr{F}_{k}$.

I need to show

$$
\mathscr{F}_{\tau} \cap[\tau=k]=\mathscr{F}_{k} \cap[\tau=k]
$$

where $\mathscr{H} \cap[\tau=k]$ means all sets of the form $A \cap[\tau=k]$ where $A \in \mathscr{H}$. Let $A \in \mathscr{F} \tau$. Then

$$
A \cap[\tau=k]=(A \cap[\tau \leq k]) \backslash(A \cap[\tau \leq k-1]) \in \mathscr{F}_{k}
$$

Therefore, there exists $B \in \mathscr{F}_{k}$ such that $B=A \cap[\tau=k]$ and so

$$
B \cap[\tau=k]=A \cap[\tau=k]
$$

which shows $\mathscr{F}_{\tau} \cap[\tau=k] \subseteq \mathscr{F}_{k} \cap[\tau=k]$. Now let $A \in \mathscr{F}_{k}$ so that

$$
A \cap[\tau=k] \in \mathscr{F}_{k} \cap[\tau=k]
$$

Then

$$
A \cap[\tau=k] \cap[\tau \leq j] \in \mathscr{F}_{j}
$$

because in case $j<k$, the set on the left is $\emptyset$ and if $j \geq k$ it reduces to $A \cap[\tau=k]$ and both $A$ and $[\tau=k]$ are in $\mathscr{F}_{k} \subseteq \mathscr{F}_{j}$. Thus $A \cap[\tau=k]=B \in \mathscr{F}_{\tau}$ and so

$$
A \cap[\tau=k]=B \cap[\tau=k] \in \mathscr{F}_{\tau} \cap[\tau=k] .
$$

Therefore, the two $\sigma$ algebras of subsets of $[\tau=k]$,

$$
\mathscr{F}_{\tau} \cap[\tau=k], \mathscr{F}_{k} \cap[\tau=k]
$$

are equal. Thus for $A$ in either $\mathscr{F}_{\tau}$ or $\mathscr{F}_{k}, A \cap[\tau=k]$ is a set of both $\mathscr{F}_{\tau}$ and $\mathscr{F}_{k}$ because if $A \in \mathscr{F}_{k}$, then from the above, there exists $B \in \mathscr{F}_{\tau}$ such that

$$
A \cap[\tau=k]=B \cap \overbrace{[\tau=k]}^{\in \mathscr{F}_{\tau} \cap \mathscr{F}_{k}} \in \mathscr{F}_{\tau}
$$

with similar reasoning holding if $A \in \mathscr{F}_{\tau}$. In other words, if $g$ is $\mathscr{F}_{\tau}$ or $\mathscr{F}_{k}$ measurable, then the restriction of $g$ to $[\tau=k]$ is measurable with respect to $\mathscr{F}_{\tau} \cap[\tau=k]$ and $\mathscr{F}_{k} \cap[\tau=k]$. Let $Y$ be an arbitrary random variable in $L^{1}(\Omega, \mathscr{F})$. It follows, since $A \cap[\tau=k]$ is in both $\mathscr{F}_{\tau}$ and $\mathscr{F}_{k}$,

$$
\begin{aligned}
\int_{A \cap[\tau=k]} E\left(Y \mid \mathscr{F}_{\tau}\right) d P & \equiv \int_{A \cap[\tau=k]} Y d P \\
& \equiv \int_{A \cap[\tau=k]} E\left(Y \mid \mathscr{F}_{k}\right) d P
\end{aligned}
$$

Since this holds for an arbitrary set in $\mathscr{F}_{\tau} \cap[\tau=k]=\mathscr{F}_{k} \cap[\tau=k]$, it follows

$$
E\left(Y \mid \mathscr{F}_{\tau}\right)=E\left(Y \mid \mathscr{F}_{k}\right) \text { a.e. on }[\tau=k]
$$

The assertion that

$$
E\left(Y \mid \mathscr{F}_{\tau}\right)=E\left(Y \mid \mathscr{F}_{k}\right) \text { a.e. }
$$

on $[\tau=k]$ and that a function $g$ which is $\mathscr{F}_{\tau}$ or $\mathscr{F}_{k}$ measurable when restricted to $[\tau=k]$ is $\mathscr{G}$ measurable for

$$
\mathscr{G}=[\tau=k] \cap \mathscr{F}_{\tau}=[\tau=k] \cap \mathscr{F}_{k}
$$

is the main result in the above lemma and this fact leads to the amazing Doob optional sampling theorem below. Also note that if $Y(k)$ is any process defined on the positive integers $k$, then by definition, $Y(k)(\omega)=Y(\tau(\omega))(\omega)$ on the set $[\tau=k]$ because $\tau$ is constant on this set.

### 62.6.2 Doob Optional Sampling Theorem

With this lemma, here is a major theorem, the optional sampling theorem of Doob. This one is for martingales having values in a Banach space. To begin with, consider the case of a martingale defined on a countable set.

Theorem 62.6.5 Let $\{M(k)\}$ be a martingale having values in E a separable real Banach space with respect to the increasing sequence of $\sigma$ algebras, $\left\{\mathscr{F}_{k}\right\}$ and let $\sigma, \tau$ be two stopping times such that $\tau$ is bounded. Then $M(\tau)$ defined as

$$
\omega \rightarrow M(\tau(\omega))
$$

is integrable and

$$
M(\sigma \wedge \tau)=E\left(M(\tau) \mid \mathscr{F}_{\sigma}\right)
$$

Proof: By Proposition 62.6.3 $M(\tau)$ is $\mathscr{F}_{\tau}$ measurable.
Next note that since $\tau$ is bounded by some $l$,

$$
\int_{\Omega}\|M(\tau(\omega))\| d P \leq \sum_{i=1}^{l} \int_{[\tau=i]}\|M(i)\| d P<\infty .
$$

This proves the first assertion and makes possible the consideration of conditional expectation.

Let $l \geq \tau$ as described above. Then for $k \leq l$, by Lemma 62.6.4,

$$
\mathscr{F}_{k} \cap[\tau=k]=\mathscr{F}_{\tau} \cap[\tau=k] \equiv \mathscr{G}
$$

implying that if $g$ is either $\mathscr{F}_{k}$ measurable or $\mathscr{F}_{\tau}$ measurable, then its restriction to $[\tau=k]$ is $\mathscr{G}$ measurable and so if $A \in \mathscr{F}_{k} \cap[\tau=k]=\mathscr{F}_{\tau} \cap[\tau=k]$,

$$
\begin{aligned}
\int_{A} E\left(M(l) \mid \mathscr{F}_{\tau}\right) d P & \equiv \int_{A} M(l) d P \\
& =\int_{A} E\left(M(l) \mid \mathscr{F}_{k}\right) d P \\
& =\int_{A} M(k) d P \\
& =\int_{A} M(\tau) d P(\text { on } A, \tau=k)
\end{aligned}
$$

Therefore, since $A$ was arbitrary,

$$
E\left(M(l) \mid \mathscr{F}_{\tau}\right)=M(\tau) \text { a.e. }
$$

on $[\tau=k]$ for every $k \leq l$. It follows

$$
E\left(M(l) \mid \mathscr{F}_{\tau}\right)=M(\tau) \text { a.e. }
$$

since it is true on each $[\tau=k]$ for all $k \leq l$.
Now consider $E\left(M(\tau) \mid \mathscr{F}_{\sigma}\right)$ on the set $[\sigma=i] \cap[\tau=j]$. By Lemma 62.6.4, on this set,

$$
E\left(M(\tau) \mid \mathscr{F}_{\sigma}\right)=E\left(M(\tau) \mid \mathscr{F}_{i}\right)=E\left(E\left(M(l) \mid \mathscr{F}_{\tau}\right) \mid \mathscr{F}_{i}\right)=E\left(E\left(M(l) \mid \mathscr{F}_{j}\right) \mid \mathscr{F}_{i}\right)
$$

If $j \leq i$, this reduces to

$$
E\left(M(l) \mid \mathscr{F}_{j}\right)=M(j)=M(\sigma \wedge \tau)
$$

If $j>i$, this reduces to

$$
E\left(M(l) \mid \mathscr{F}_{i}\right)=M(i)=M(\sigma \wedge \tau)
$$

and since this exhausts all possibilities for values of $\sigma$ and $\tau$, it follows

$$
E\left(M(\tau) \mid \mathscr{F}_{\sigma}\right)=M(\sigma \wedge \tau) \text { a.e. }
$$

You can also give a version of the above to submartingales. This requires the following very interesting decomposition of a submartingale into the sum of an increasing stochastic process and a martingale.

Theorem 62.6.6 Let $\left\{X_{n}\right\}$ be a submartingale. Then there exists a unique stochastic process, $\left\{A_{n}\right\}$ and martingale, $\left\{M_{n}\right\}$ such that

1. $A_{n}(\omega) \leq A_{n+1}(\omega), A_{1}(\omega)=0$,
2. $A_{n}$ is $\mathscr{F}_{n-1}$ adapted for all $n \geq 1$ where $\mathscr{F}_{0} \equiv \mathscr{F}_{1}$.
and also $X_{n}=M_{n}+A_{n}$.
Proof: Let $A_{1} \equiv 0$ and define

$$
A_{n} \equiv \sum_{k=2}^{n} E\left(X_{k}-X_{k-1} \mid \mathscr{F}_{k-1}\right) .
$$

It follows $A_{n}$ is $\mathscr{F}_{n-1}$ measurable. Since $\left\{X_{k}\right\}$ is a submartingale, $A_{n}$ is increasing because

$$
\begin{equation*}
A_{n+1}-A_{n}=E\left(X_{n+1}-X_{n} \mid \mathscr{F}_{n}\right) \geq 0 \tag{62.6.25}
\end{equation*}
$$

It is a submartingale because

$$
\begin{aligned}
E\left(A_{n} \mid \mathscr{F}_{n-1}\right) & =E\left(\sum_{k=2}^{n} E\left(X_{k}-X_{k-1} \mid \mathscr{F}_{k-1}\right) \mid \mathscr{F}_{n-1}\right) \\
& =\sum_{k=2}^{n} E\left(X_{k}-X_{k-1} \mid \mathscr{F}_{k-1}\right) \equiv A_{n} \geq A_{n-1}
\end{aligned}
$$

Now let $M_{n}$ be defined by

$$
X_{n}=M_{n}+A_{n} .
$$

Then from 62.6.25,

$$
\begin{aligned}
& E\left(M_{n+1} \mid \mathscr{F}_{n}\right)=E\left(X_{n+1} \mid \mathscr{F}_{n}\right)-E\left(A_{n+1} \mid \mathscr{F}_{n}\right) \\
= & E\left(X_{n+1} \mid \mathscr{F}_{n}\right)-E\left(A_{n+1}-A_{n} \mid \mathscr{F}_{n}\right)-A_{n} \\
= & E\left(X_{n+1} \mid \mathscr{F}_{n}\right)-E\left(E\left(X_{n+1}-X_{n} \mid \mathscr{F}_{n}\right) \mid \mathscr{F}_{n}\right)-A_{n} \\
= & E\left(X_{n+1} \mid \mathscr{F}_{n}\right)-E\left(X_{n+1}-X_{n} \mid \mathscr{F}_{n}\right)-A_{n} \\
= & E\left(X_{n} \mid \mathscr{F}_{n}\right)-A_{n} \\
= & X_{n}-A_{n} \equiv M_{n}
\end{aligned}
$$

This proves the existence part.
It remains to verify uniqueness. Suppose then that

$$
X_{n}=M_{n}+A_{n}=M_{n}^{\prime}+A_{n}^{\prime}
$$

where $\left\{A_{n}\right\}$ and $\left\{A_{n}^{\prime}\right\}$ both satisfy the conditions of the theorem and $\left\{M_{n}\right\}$ and $\left\{M_{n}^{\prime}\right\}$ are both martingales. Then

$$
M_{n}-M_{n}^{\prime}=A_{n}^{\prime}-A_{n}
$$

and so, since $A_{n}^{\prime}-A_{n}$ is $\mathscr{F}_{n-1}$ measurable and $\left\{M_{n}-M_{n}^{\prime}\right\}$ is a martingale,

$$
\begin{aligned}
M_{n-1}-M_{n-1}^{\prime} & =E\left(M_{n}-M_{n}^{\prime} \mid \mathscr{F}_{n-1}\right) \\
& =E\left(A_{n}^{\prime}-A_{n} \mid \mathscr{F}_{n-1}\right) \\
& =A_{n}^{\prime}-A_{n}=M_{n}-M_{n}^{\prime} .
\end{aligned}
$$

Continuing this way shows $M_{n}-M_{n}^{\prime}$ is a constant. However, since $A_{1}^{\prime}-A_{1}=0=M_{1}-M_{1}^{\prime}$, it follows $M_{n}=M_{n}^{\prime}$ and this proves uniqueness.

Now here is a version of the optional sampling theorem for submartingales.

Theorem 62.6.7 Let $\{X(k)\}$ be a real valued submartingale with respect to the increasing sequence of $\sigma$ algebras, $\left\{\mathscr{F}_{k}\right\}$ and let $\sigma \leq \tau$ be two stopping times such that $\tau$ is bounded. Then $M(\tau)$ defined as

$$
\omega \rightarrow X(\tau(\omega))
$$

is integrable and

$$
X(\sigma) \leq E\left(X(\tau) \mid \mathscr{F}_{\sigma}\right)
$$

Without assuming $\sigma \leq \tau$, one can write

$$
X(\sigma \wedge \tau) \leq E\left(X(\tau) \mid \mathscr{F}_{\sigma}\right)
$$

Proof: That $\omega \rightarrow X(\tau(\omega))$ is integrable follows right away as in the optional sampling theorem for martingales. You just consider the finitely many values of $\tau$.

Use Theorem 62.6.6 above to write

$$
X(n)=M(n)+A(n)
$$

where $M$ is a martingale and $A$ is increasing with $A(n)$ being $\mathscr{F}_{n-1}$ measurable and $A(0)=$ 0 as discussed in Theorem 62.6.6. Then

$$
E\left(X(\tau) \mid \mathscr{F}_{\tau}\right)=E\left(M(\tau)+A(\tau) \mid \mathscr{F}_{\sigma}\right)
$$

Now since $A$ is increasing, you can use the optional sampling theorem for martingales, Theorem 62.6.5 to conclude that, since $\mathscr{F}_{\sigma \wedge \tau} \subseteq \mathscr{F}_{\sigma}$ and $A(\sigma \wedge \tau)$ is $\mathscr{F}_{\sigma \wedge \tau}$ measurable,

$$
\begin{aligned}
& \geq E\left(M(\tau)+A(\sigma \wedge \tau) \mid \mathscr{F}_{\sigma}\right)=E\left(M(\tau) \mid \mathscr{F}_{\sigma}\right)+A(\sigma \wedge \tau) \\
& =M(\sigma \wedge \tau)+A(\sigma \wedge \tau)=X(\sigma \wedge \tau) .
\end{aligned}
$$

In summary, the main results for stopping times for discrete filtrations are the following definitions and theorems.

$$
[\tau \leq m] \in \mathscr{F}_{m}
$$

$$
\begin{gathered}
A \in \mathscr{F}_{\tau} \text { means } A \cap[\tau \leq m] \in \mathscr{F}_{m} \text { for any } m \\
X \text { adapted implies } X(\tau) \text { is } \mathscr{F}_{\tau} \text { measurable } \\
\mathscr{F}_{\sigma \wedge \tau}=\mathscr{F}_{\sigma} \cap \mathscr{F}_{\tau} \\
{[\tau=k] \cap \mathscr{F}_{k}=[\tau=k] \cap \mathscr{F}_{\tau}}
\end{gathered}
$$

This last theorem implies the following amazing result. From these fundamental properties, we obtain the optional sampling theorem for martingales and submartingales.

$$
E\left(Y \mid \mathscr{F}_{\tau}\right)=E\left(Y \mid \mathscr{F}_{k}\right) \text { a.e. on }[\tau=k]
$$

### 62.7 Doob Optional Sampling Continuous Case

### 62.7.1 Stopping Times

Let $X(t)$ be a stochastic process adapted to a filtration $\left\{\mathscr{F}_{t}\right\}$ for $t \in[0, T]$. We will assume two things. The stochastic process is right continuous and the filtration is normal.

Definition 62.7.1 A normal filtration is one which satisfies the following :

1. $\mathscr{F}_{0}$ contains all $A \in \mathscr{F}$ such that $P(A)=0$. Here $\mathscr{F}$ is the $\sigma$ algebra which contains all $\mathscr{F}_{t}$.
2. $\mathscr{F}_{t}=\mathscr{F}_{t+}$ for all $t \in I$ where $\mathscr{F}_{t+} \equiv \cap_{s>t} \mathscr{F}_{s}$.

For an $\mathscr{F}$ measurable $[0, \infty)$ valued function $\tau$ to be a stopping time, we want to have the stopped process $X^{\tau}$ defined by $X^{\tau}(t)(\omega) \equiv X(t \wedge \tau(\omega))(\omega)$ to be adapted whenever $X$ is right continuous and adapted. Thus a stopping time is a measurable function which can be used to stop the process while retaining the property of being adapted. We want to find a simple criterion which will ensure that this happens.

Let $X(t)$ be adapted. Let $O$ be an open set in some metric space where $X$ has its values. This could probably be generalized. Then we need to have

$$
X^{\tau}(t)^{-1}(O) \in \mathscr{F}_{t}
$$

Thus we need to have

$$
[\tau \leq t] \cap\left[X(\tau)^{-1}(O)\right] \cup[\tau>t] \cap\left[X(t)^{-1}(O)\right] \in \mathscr{F}_{t}
$$

How does this happen? Consider $\tau_{k}(\omega) \equiv \sum_{n=0}^{\infty} \mathscr{X}_{\tau^{-1}\left(\left(n 2^{-k},(n+1) 2^{-k}\right]\right)}(\omega)(n+1) 2^{-k}$. Thus for each $\omega, \tau_{k}(\omega) \downarrow \tau(\omega)$. Since $X$ is right continuous for each $\omega$, it follows that, since $O$ is open,

$$
\begin{gathered}
{[\tau \leq t] \cap\left[X(\tau)^{-1}(O)\right]=[\tau \leq t] \cap\left(\cup_{m} \cap_{k \geq m}\left[X\left(\tau_{k}\right)^{-1}(O)\right]\right)} \\
=\cup_{m} \cap_{k \geq m} \cup_{n=0}^{\infty} \tau^{-1}\left(\left(n 2^{-k},(n+1) 2^{-k} \wedge t\right]\right) \cap\left[X\left((n+1) 2^{-k}\right)^{-1}(O)\right]
\end{gathered}
$$

the last union being a disjoint union. Now

$$
\tau^{-1}\left(\left(n 2^{-k},(n+1) 2^{-k} \wedge t\right]\right) \cap\left[X\left((n+1) 2^{-k}\right)^{-1}(O)\right]
$$

is a set of $\mathscr{F}_{(n+1) 2^{-k}}$ intersected with $\left[\tau \in\left(n 2^{-k},(n+1) 2^{-k}\right]\right] \cap[\tau \in(0, t]]$. If we assume $[\tau \leq t] \in \mathscr{F}_{t}$, for all $t$, then this shows that the above expression is a set of $\mathscr{F}_{t+2^{-k}}$. Since this is true for each $k$, and the filtration is normal, this implies $[\tau \leq t] \cap\left[X(\tau)^{-1}(O)\right] \in \mathscr{F}_{t}$. Also with this assumption, $[\tau>t]=[\tau \leq t]^{C} \in \mathscr{F}_{t}$ and so we get $X^{\tau}(t)^{-1}(O) \in \mathscr{F}_{t}$. This is why we define stopping times this way. It is so that when you have a right continuous adapted process, then the stopped process is also adapted.

Definition 62.7.2 $\tau$ an $\mathscr{F}$ measurable function is a stopping time if $[\tau \leq t] \in \mathscr{F}_{t}$.
What follows will be more discussion of this simple idea of preserving the process of being adapted when the process is stopped.

Then the above discussion shows the following proposition.
Proposition 62.7.3 Let $\left\{\mathscr{F}_{t}\right\}$ be a normal filtration and let $X(t)$ be a right continuous process adapted to $\left\{\mathscr{F}_{t}\right\}$. Then if $\tau$ is a stopping time, it follows that the stopped process $X^{\tau}$ defined by $X^{\tau}(t) \equiv X(\tau \wedge t)$ is also adapted.

Definition 62.7.4 Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $\mathscr{F}_{t}$ be a filtration. A measurable function, $\tau: \Omega \rightarrow[0, \infty]$ is called a stopping time if

$$
[\tau \leq t] \in \mathscr{F}_{t}
$$

for all $t \geq 0$. Associated with a stopping time is the $\sigma$ algebra, $\mathscr{F} \tau$ defined by

$$
\mathscr{F}_{\tau} \equiv\left\{A \in \mathscr{F}: A \cap[\tau \leq t] \in \mathscr{F}_{t} \text { for all } t\right\} .
$$

These sets are also called those "prior" to $\tau$.
Note that $\mathscr{F}_{\tau}$ is obviously closed with respect to countable unions. If $A \in \mathscr{F} \tau$, then

$$
A^{C} \cap[\tau \leq t]=[\tau \leq t] \backslash(A \cap[\tau \leq t]) \in \mathscr{F}_{t}
$$

Thus $\mathscr{F}_{\tau}$ is a $\sigma$ algebra.
Proposition 62.7.5 Let B be an open subset of topological space $E$ and let $X(t)$ be a right continuous $\mathscr{F}_{t}$ adapted stochastic process such that $\mathscr{F}_{t}$ is normal. Then define

$$
\tau(\omega) \equiv \inf \{t>0: X(t)(\omega) \in B\}
$$

This is called the first hitting time. Then $\tau$ is a stopping time. If $X(t)$ is continuous and adapted to $\mathscr{F}_{t}$, a normal filtration, then if $H$ is a nonempty closed set such that $H=\cap_{n=1}^{\infty} B_{n}$ for $B_{n}$ open, $B_{n} \supseteq B_{n+1}$,

$$
\tau(\omega) \equiv \inf \{t>0: X(t)(\omega) \in H\}
$$

is also a stopping time.
Proof: Consider the first claim. $\omega \in[\tau=a]$ implies that for each $n \in \mathbb{N}$, there exists $t \in\left[a, a+\frac{1}{n}\right]$ such that $X(t) \in B$. Also for $t<a$, you would need $X(t) \notin B$. By right continuity, this is the same as saying that $X(d) \notin B$ for all rational $d<a$. (If $t<a$, you could let $d_{n} \downarrow t$ where $X\left(d_{n}\right) \in B^{C}$, a closed set. Then it follows that $X(t)$ is also in the closed set $B^{C}$.) Thus, aside from a set of measure zero, for each $m \in \mathbb{N}$,

$$
[\tau=a]=\left(\cap_{n=m}^{\infty} \cup_{t \in\left[a, a+\frac{1}{n}\right]}[X(t) \in B]\right) \cap\left(\cap_{t \in[0, a)}\left[X(t) \in B^{C}\right]\right)
$$

Since $X(t)$ is right continuous, this is the same as

$$
\left(\cap_{n=m}^{\infty} \cup_{d \in \mathbb{Q} \cap\left[a, a+\frac{1}{n}\right]}[X(d) \in B]\right) \cap\left(\cap_{d \in \mathbb{Q} \cap[0, a)}\left[X(d) \in B^{C}\right]\right) \in \mathscr{F}_{a+\frac{1}{m}}
$$

Thus, since the filtration is normal,

$$
[\tau=a] \in \cap_{m=1}^{\infty} \mathscr{F}_{a+\frac{1}{m}}=\mathscr{F}_{a+}=\mathscr{F}_{a}
$$

Now what of $[\tau<a]$ ? This is equivalent to saying that $X(t) \in B$ for some $t<a$. Since $X$ is right continuous, this is the same as saying that $X(t) \in B$ for some $t \in \mathbb{Q}, t<a$. Thus

$$
[\tau<a]=\cup_{d \in \mathbb{Q}, d<a}[X(d) \in B] \in \mathscr{F}_{a}
$$

It follows that $[\tau \leq a]=[\tau<a] \cup[\tau=a] \in \mathscr{F}_{a}$.
Now consider the claim involving the additional assumption that $X(t)$ is continuous and it is desired to hit a closed set $H=\cap_{n=1}^{\infty} B_{n}$ where $B_{n}$ is open, $B_{n} \supseteq B_{n+1}$. (Note that if the topological space is a metric space, this is always possible so this is not a big restriction.) Then let $\tau_{n}$ be the first hitting time of $B_{n}$ by $X(t)$. Then it can be shown that

$$
[\tau \leq a]=\cap_{n}\left[\tau_{n} \leq a\right] \in \mathscr{F}_{a}
$$

To show this, first note that $\omega \in[\tau \leq a]$ if and only if there exists $t \leq a$ such that $X(t)(\omega) \in$ $H$. This follows from continuity and the fact that $H$ is closed. Thus $\tau_{n} \leq a$ for all $n$ because for some $t \leq a, X(t) \in H \subseteq B_{n}$ for all $n$. Next suppose $\omega \in\left[\tau_{n} \leq a\right]$ for all $n$. Then for $\delta_{n} \downarrow 0$, there exists $t_{n} \in\left[0, a+\delta_{n}\right]$ such that $X\left(t_{n}\right)(\omega) \in B_{n}$. It follows there is a subsequence, still denoted by $t_{n}$ such that $t_{n} \rightarrow t \in[0, a]$. By continuity of $X$, it must be the case that $X(t)(\omega) \in H$ and so $\omega \in[\tau \leq a]$. This shows the above formula. Now by the first part, each $\left[\tau_{n} \leq a\right] \in \mathscr{F}_{a}$ and so $[\tau \leq a] \in \mathscr{F}_{a}$ also.

Another useful result for real valued stochastic process is the following.
Proposition 62.7.6 Let $X(t)$ be a real valued stochastic process which is $\mathscr{F}_{t}$ adapted for a normal filtration $\mathscr{F}_{t}$, with the property that off a set of measure zero in $\Omega, t \rightarrow X(t)$ is lower semicontinuous. Then

$$
\tau \equiv \inf \{t: X(t)>\alpha\}
$$

is a stopping time.
Proof: As above, for each $m>0$,

$$
[\tau=a]=\left(\cap_{n=m}^{\infty} \cup_{t \in\left[a, a+\frac{1}{n}\right]}[X(t)>\alpha]\right) \cap\left(\cap_{t \in[0, a)}[X(t) \leq \alpha]\right)
$$

Now

$$
\cap_{t \in[0, a)}[X(t) \leq \alpha] \subseteq \cap_{t \in[0, a), t \in \mathbb{Q}}[X(t) \leq \alpha]
$$

If $\omega$ is in the right side, then for arbitrary $t<a$, let $t_{n} \downarrow t$ where $t_{n} \in \mathbb{Q}$ and $t<a$. Then $X(t) \leq \liminf _{n \rightarrow \infty} X\left(t_{n}\right) \leq \alpha$ and so $\omega$ is in the left side also. Thus

$$
\cap_{t \in[0, a)}[X(t) \leq \alpha]=\cap_{t \in[0, a), t \in \mathbb{Q}}[X(t) \leq \alpha]
$$

$$
\cup_{t \in\left[a, a+\frac{1}{n}\right]}[X(t)>\alpha] \supseteq \cup_{t \in\left[a, a+\frac{1}{n}\right], t \in \mathbb{Q}}[X(t)>\alpha]
$$

If $\omega$ is in the left side, then for some $t$ in the given interval, $X(t)>\alpha$. If for all $s \in$ $\left[a, a+\frac{1}{n}\right] \cap \mathbb{Q}$ you have $X(s) \leq \alpha$, then you could take $s_{n} \rightarrow t$ where $X\left(s_{n}\right) \leq \alpha$ and conclude from lower semicontinuity that $X(t) \leq \alpha$ also. Thus there is some rational $s$ where $X(s)>\alpha$ and so the two sides are equal. Hence,

$$
[\tau=a]=\left(\cap_{n=m}^{\infty} \cup_{t \in\left[a, a+\frac{1}{n}\right], t \in \mathbb{Q}}[X(t)>\alpha]\right) \cap\left(\cap_{t \in[0, a), t \in \mathbb{Q}}[X(t) \leq \alpha]\right)
$$

The first set on the right is in $\mathscr{F}_{a+(1 / m)}$ and so is the next set on the right. Hence $[\tau=a] \in$ $\cap_{m} \mathscr{F}_{a+(1 / m)}=\mathscr{F}_{a}$. To be a stopping time, one needs $[\tau \leq a] \in \mathscr{F}_{a}$. What of $[\tau<a]$ ? This equals $\cup_{t \in[0, a)}[X(t)>\alpha]=\cup_{t \in[0, a) \cap \mathbb{Q}}[X(t)>\alpha] \in \mathscr{F}_{a}$, the equality following from lower semicontinuity. Thus $[\tau \leq a]=[\tau=a] \cup[\tau<a] \in \mathscr{F} a$.

Thus there do exist stopping times, the first hitting time above being an example. When dealing with continuous stopping times on a normal filtration, one uses the following discrete stopping times

$$
\tau_{n} \equiv \sum_{k=1}^{\infty} \mathscr{X}_{\left[\tau \in\left(t_{k}^{n}, t_{k+1}^{n}\right]\right]_{k+1}} t_{k}^{n}
$$

where here $\left|t_{k}^{n}-t_{k+1}^{n}\right|=r_{n}$ for all $k$ where $r_{n} \rightarrow 0$. Then here is an important lemma.
Lemma 62.7.7 $\tau_{n}$ is a stopping time $\left(\left[\tau_{n} \leq t\right] \in \mathscr{F}_{t}\right.$.) Also the inclusion $\mathscr{F}_{\tau} \subseteq \mathscr{F}_{\tau_{n}}$ holds and for each $\omega, \tau_{n}(\omega) \downarrow \tau(\omega)$.

Proof: Say $t \in\left(t_{k-1}^{n}, t_{k}^{n}\right]$. Then $\left[\tau_{n} \leq t\right]=\left[\tau \leq t_{k-1}^{n}\right]$ if $t<t_{k}^{n}$ and it equals $\left[\tau \leq t_{k}^{n}\right]$ if $t=t_{k}^{n}$. Either way $\left[\tau_{n} \leq t\right] \in \mathscr{F}_{t}$ so it is a stopping time. Also from the definition, it follows that $\tau_{n} \geq \tau$ and $\left|\tau_{n}(\omega)-\tau(\omega)\right| \leq r_{n}$ which is given to converge to 0 . Now suppose $A \in \mathscr{F}_{\tau}$ and say $t \in\left(t_{k-1}^{n}, t_{k}^{n}\right]$ as above. Then

$$
A \cap\left[\tau_{n} \leq t\right]=A \cap\left[\tau \leq t_{k-1}^{n}\right] \in \mathscr{F}_{t_{k-1}^{n}}^{n} \subseteq \mathscr{F}_{t} \text { if } t<t_{k}^{n}
$$

and

$$
A \cap\left[\tau_{n} \leq t\right]=A \cap\left[\tau \leq t_{k}^{n}\right] \in \mathscr{F}_{t_{k}^{n}}^{n}=\mathscr{F}_{t} \text { if } t=t_{k}^{n}
$$

Thus $\mathscr{F} \tau \subseteq \mathscr{F}_{\tau_{n}}$ as claimed.
Next is the claim that if $X(t)$ is adapted to $\mathscr{F}_{t}$, then $X(\tau)$ is adapted to $\mathscr{F}_{\tau}$ just like the discrete case.

Proposition 62.7.8 Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $\sigma \leq \tau$ be two stopping times with respect to a filtration, $\mathscr{F}_{\text {f }}$. Then $\mathscr{F}_{\sigma} \subseteq \mathscr{F}_{\tau}$. If $X(t)$ is a right continuous stochastic process adapted to a normal filtration $\mathscr{F}_{t}$ and $\tau$ is a stopping time, $\omega \rightarrow X(\tau(\omega))$ is $\mathscr{F}_{\tau}$ measurable.

Proof: Let $A \in \mathscr{F}_{\sigma}$. Then $A \cap[\sigma \leq t] \in \mathscr{F}_{t}$ for all $t \geq 0$. Since $\sigma \leq \tau$,

$$
A \cap[\tau \leq t]=\overbrace{A \cap[\sigma \leq t]}^{\in \mathscr{F}_{t}} \cap[\tau \leq t] \in \mathscr{F}_{t}
$$

Thus $A \in \mathscr{F}_{\tau}$ and so $\mathscr{F}_{\sigma} \subseteq \mathscr{F}_{\tau}$.
Consider the following approximation of $\tau$ in which $t_{k}^{n}=k 2^{-n}$.

$$
\left.\tau_{n} \equiv \sum_{k=1}^{\infty} \mathscr{X}_{\left[\tau \in\left(t_{k}^{n}, t_{k+1}^{n}\right]\right.}\right]_{k+1}^{n}
$$

Thus $\tau_{n} \downarrow \tau$. Consider for $U$ an open set, $X\left(\tau_{n}\right)^{-1}(U) \cap\left[\tau_{n}<t\right]$. Say $t \in\left(t_{k}^{n}, t_{k+1}^{n}\right]$. Then from the above definition of $\tau_{n}$,

$$
\left[\tau_{n}<t\right]=\left[\tau \leq t_{k}^{n}\right] \in \mathscr{F}_{t}^{n} \subseteq \mathscr{F}_{t}
$$

It follows that

$$
X\left(\tau_{n}\right)^{-1}(U) \cap\left[\tau_{n}<t\right]=\cup_{j=1}^{k} X\left(t_{j}^{n}\right)^{-1}(U) \cap\left[\begin{array}{c}
\in \mathscr{F}_{t}^{n} \\
\tau_{j}^{n} \\
=t_{j}^{n}
\end{array}\right]
$$

and so this set is in $\mathscr{F}_{t_{k}^{n}} \subseteq \mathscr{F}_{t}$. The reason $\left[\tau_{n}=t_{j}^{n}\right] \in \mathscr{F}_{t_{j}^{n}}$ is that it equals $\left[\tau \in\left(t_{j-1}^{n}, t_{j}^{n}\right]\right] \in$ $\mathscr{F}_{t_{j}^{n}}$ by assumption that $\tau$ is a stopping time.

By right continuity of $X$, it follows that

$$
X(\tau)^{-1}(U) \cap[\tau<t]=\cup_{m=1}^{\infty} \cap_{n \geq m} X\left(\tau_{n}\right)^{-1}(U) \cap\left[\tau_{n}<t\right] \in \mathscr{F}_{t}
$$

It follows that for every $m$,

$$
X(\tau)^{-1}(U) \cap[\tau \leq t]=\cap_{n=m}^{\infty} X(\tau)^{-1}(U) \cap\left[\tau<t+\frac{1}{n}\right] \in \mathscr{F}_{t+\frac{1}{m}}
$$

Since the filtration is normal, it follows that

$$
X(\tau)^{-1}(U) \cap[\tau \leq t] \in \mathscr{F}_{t+}=\mathscr{F}_{t} .
$$

Now consider an increasing family of stopping times, $\tau(t)(\omega \rightarrow \tau(t)(\omega))$. It turns out this is a submartingale.

Example 62.7.9 Let $\{\tau(t)\}$ be an increasing family of stopping times. Then $\tau(t)$ is adapted to the $\sigma$ algebras $\mathscr{F}_{\tau(t)}$ and $\{\tau(t)\}$ is a submartingale adapted to these $\sigma$ algebras.

First I need to show that a stopping time, $\tau$ is $\mathscr{F} \tau$ measurable. Consider $[\tau \leq s]$. Is this in $\mathscr{F}_{\tau}$ ? Is $[\tau \leq s] \cap[\tau \leq r] \in \mathscr{F}_{r}$ for each $r$ ? This is obviously so if $s \leq r$ because the intersection reduces to $[\tau \leq s] \in \mathscr{F}_{s} \subseteq \mathscr{F}_{r}$. On the other hand, if $s>r$ then the intersection reduces to $[\tau \leq r] \in \mathscr{F}_{r}$ and so it is clear that $\tau$ is $\mathscr{F}_{\tau}$ measurable. It remains to verify it is a submartingale.

Let $s<t$ and let $A \in \mathscr{F}_{\tau(s)}$

$$
\int_{A} E(\tau(t) \mid \mathscr{F} \tau(s)) d P \equiv \int_{A} \tau(t) d P \geq \int_{A} \tau(s) d P
$$

and this shows $E\left(\tau(t) \mid \mathscr{F}_{\tau(s)}\right) \geq \tau(s)$.
Now here is an important example. First note that for $\tau$ a stopping time, so is $t \vee \tau$. Here is why.

$$
[t \vee \tau \leq s]=[t \leq s] \cap[\tau \leq s] \in \mathscr{F}_{s}
$$

Example 62.7.10 Let $\tau$ be a stopping time and let $X$ be continuous and adapted to the filtration $\mathscr{F}_{t}$. Then for $a>0$, define $\sigma$ as

$$
\sigma(\omega) \equiv \inf \{t>\tau(\omega):\|X(t)(\omega)-X(\tau(\omega))\|=a\}
$$

Then $\sigma$ is also a stopping time.
To see this is so, let

$$
Y(t)(\omega)=\|X(t \vee \tau)(\omega)-X(\tau(\omega))\|
$$

Then $Y(t)$ is $\mathscr{F}_{t \vee \tau}$ measurable. It is desired to show that $Y$ is $\mathscr{F}_{t}$ adapted. Hence if $U$ is open in $\mathbb{R}$, then

$$
Y(t)^{-1}(U)=\left(Y(t)^{-1}(U) \cap[\tau \leq t]\right) \cup\left(Y(t)^{-1}(U) \cap[\tau>t]\right)
$$

The second set in the above union on the right equals either $\emptyset$ or $[\tau>t]$ depending on whether $0 \in U$. If $\tau>t$, then $Y(t)=0$ and so the second set equals $[\tau>t]$ if $0 \in U$. If $0 \notin U$, then the second set equals $\emptyset$. Thus the second set above is in $\mathscr{F}_{t}$. It is necessary to show the first set is also in $\mathscr{F}_{t}$. The first set equals

$$
Y(t)^{-1}(U) \cap[\tau \leq t]=Y(t)^{-1}(U) \cap[\tau \vee t \leq t]
$$

because $[\tau \vee t \leq t]=[\tau \leq t]$. However, $Y(t)^{-1}(U) \in \mathscr{F}_{t \vee \tau}$ and so the set on the right in the above is in $\mathscr{F}_{t}$. Therefore, $Y(t)$ is adapted. Then $\sigma$ is just the first hitting time for $Y(t)$ to equal the closed set $a$. Therefore, $\sigma$ is a stopping time by Proposition 62.7.5.

### 62.7.2 The Optional Sampling Theorem Continuous Case

Next I want a version of the Doob optional sampling theorem which applies to martingales defined on $[0, L], L \leq \infty$. First recall Theorem 61.1.2 part of which is stated as the following lemma.

Lemma 62.7.11 Let $f \in L^{1}(\Omega ; E, \mathscr{F})$ where $E$ is a separable Banach space. Then if $\mathscr{G}$ is a $\sigma$ algebra $\mathscr{G} \subseteq \mathscr{F}$,

$$
\|E(f \mid \mathscr{G})\| \leq E(\|f\| \mid \mathscr{G})
$$

Here is a lemma which is the main idea for the proofs of the optional sampling theorem for the continuous case.

Lemma 62.7.12 Let $X(t)$ be a right continuous nonnegative submartingale such that the filtration $\left\{\mathscr{F}_{t}\right\}$ is normal. Recall this includes

$$
\mathscr{F}_{t}=\cap_{s>t} \mathscr{F}_{s}
$$

Also let $\tau$ be a stopping time with values in $[0, T]$. Let $\mathscr{P}_{n}=\left\{t_{k}^{n}\right\}_{k=1}^{m_{n}+1}$ be a sequence of partitions of $[0, T]$ which have the property that

$$
\mathscr{P}_{n} \subseteq \mathscr{P}_{n+1}, \lim _{n \rightarrow \infty}\left\|\mathscr{P}_{n}\right\|=0
$$

where

$$
\left\|\mathscr{P}_{n}\right\| \equiv \sup \left\{\left|t_{k}^{n}-t_{k+1}^{n}\right|: k=1,2, \cdots, m_{n}\right\}
$$

Then let

$$
\tau_{n}(\omega) \equiv \sum_{k=0}^{m_{n}} t_{k+1}^{n} \mathscr{X}_{\tau^{-1}\left(\left(t_{k}^{n}, t_{k+1}^{n}\right]\right)}(\omega)
$$

It follows that $\tau_{n}$ is a stopping time and also the functions $\left|X\left(\tau_{n}\right)\right|$ are uniformly integrable. Furthermore, $|X(\tau)|$ is integrable.

Proof: First of all, say $t \in\left(t_{k}^{n}, t_{k+1}^{n}\right]$. If $t<t_{k+1}^{n}$, then

$$
\left[\tau_{n} \leq t\right]=\left[\tau \leq t_{k}^{n}\right] \in \mathscr{F}_{t_{k}^{n}} \subseteq \mathscr{F}_{t}
$$

and if $t=t_{k+1}^{n}$, then

$$
\left[\tau_{n} \leq t_{k+1}^{n}\right]=\left[\tau \leq t_{k+1}^{n}\right] \in \mathscr{F}_{k+1}^{n}=\mathscr{F}_{t}
$$

and so $\tau_{n}$ is a stopping time. It follows from Proposition 62.7 .8 that $X\left(\tau_{n}\right)$ is $\mathscr{F}_{\tau_{n}}$ measurable.

Now from Lemma 60.4.3 or Theorem 62.6.7, $X(0), X\left(\tau_{n}\right), X(T)$ is a submartingale. Then

$$
\begin{aligned}
\int_{\left[X\left(\tau_{n}\right) \geq \lambda\right]} X\left(\tau_{n}\right) d P & \leq \int_{\left[X\left(\tau_{n}\right) \geq \lambda\right]} E\left(X(T) \mid \mathscr{F}_{\tau_{n}}\right) d P \\
& =\int_{\Omega} E\left(\mathscr{X}_{\left[X\left(\tau_{n}\right) \geq \lambda\right]} X(T) \mid \mathscr{F}_{\tau_{n}}\right) d P \\
& =\int_{\left[X\left(\tau_{n}\right) \geq \lambda\right]} X(T) d P
\end{aligned}
$$

From maximal estimates, for example Theorem 60.2.8,

$$
P\left(\left[X\left(\tau_{n}\right) \geq \lambda\right]\right) \leq \frac{1}{\lambda} \int_{\Omega} X(T)^{+} d P=\frac{1}{\lambda} \int_{\Omega} X(T) d P
$$

and now it follows from the above that the random variables $X\left(\tau_{n}\right)$ are equiintegrable. Recall this means that

$$
\lim _{\lambda \rightarrow \infty} \sup _{n} \int_{\left[X\left(\tau_{n}\right) \geq \lambda\right]} X\left(\tau_{n}\right) d P=0
$$

Hence they are uniformly integrable.
To verify that $|X(\tau)|$ is integrable, note that by right continuity, $X\left(\tau_{n}\right) \rightarrow X(\tau)$ pointwise. Apply the Vitali convergence theorem to obtain

$$
\int_{\Omega}|X(\tau)| d P=\lim _{n \rightarrow \infty} \int_{\Omega}\left|X\left(\tau_{n}\right)\right| d P \leq \int_{\Omega} X(T) d P<\infty
$$

In fact, you do not need to assume $X$ is nonnegative.

Lemma 62.7.13 Let $X(t)$ be a right continuous submartingale such that the filtration $\left\{\mathscr{F}_{t}\right\}$ is normal. Recall this includes

$$
\mathscr{F}_{t}=\cap_{s>t} \mathscr{F}_{s}
$$

Also let $\tau$ be a stopping time with values in $[0, T]$. Let $\mathscr{P}_{n}=\left\{t_{k}^{n}\right\}_{k=1}^{m_{n}+1}$ be a sequence of partitions of $[0, T]$ which have the property that

$$
\mathscr{P}_{n} \subseteq \mathscr{P}_{n+1}, \lim _{n \rightarrow \infty}\left\|\mathscr{P}_{n}\right\|=0
$$

where

$$
\left\|\mathscr{P}_{n}\right\| \equiv \sup \left\{\left|t_{k}^{n}-t_{k+1}^{n}\right|: k=1,2, \cdots, m_{n}\right\}
$$

Then let

$$
\tau_{n}(\omega) \equiv \sum_{k=0}^{m_{n}} t_{k+1}^{n} \mathscr{X}_{\tau^{-1}\left(\left(t_{k}^{n}, t_{k+1}^{n}\right]\right)}(\omega)
$$

It follows that $\tau_{n}$ is a stopping time and also the functions $\left|X\left(\tau_{n}\right)\right|$ are uniformly integrable. Furthermore, $|X(\tau)|$ is integrable.

Proof: It was shown above that $\tau_{n}$ is a stopping time. Also, $t_{k}^{n} \rightarrow X\left(t_{k}^{n}\right)$ is a discrete submartingale. Then by Theorem 62.6 .6 there is a martingale $t_{k}^{n} \rightarrow M\left(t_{k}^{n}\right)$ and an increasing submartingale $t_{k}^{n} \rightarrow A\left(t_{k}^{n}\right)$ such that $A \geq 0$ and is increasing

$$
X\left(t_{k}^{n}\right)=M\left(t_{k}^{n}\right)+A\left(t_{k}^{n}\right)
$$

You define $A\left(t_{0}^{m}\right) \equiv 0$ and for $n \geq 1$,

$$
A\left(t_{n}^{m}\right) \equiv \sum_{k=1}^{n} E\left(X\left(t_{k}^{m}\right)-X\left(t_{k-1}^{m}\right) \mid \mathscr{F}_{t_{k-1}^{m}}^{m}\right)
$$

and repeat the arguments in that theorem. You know that $A(0), A\left(\tau_{n}\right), A(T)$ is a submartingale by the optional sampling theorem given earlier, Theorem 62.6.7, and so

$$
P\left(A\left(\tau_{n}\right)>\lambda\right) \leq \frac{1}{\lambda} \int_{\left[A\left(\tau_{n}\right)>\lambda\right]} A\left(\tau_{n}\right) d P \leq \frac{1}{\lambda} \int_{\left[A\left(\tau_{n}\right)>\lambda\right]} A(T) d P \leq \frac{\|A(T)\|_{L^{1}}}{\lambda}
$$

It also follows from the definition of $A$ that

$$
\|A(T)\|_{L^{1}}=\int_{\Omega} X(T)-X(0) d P<\infty
$$

Hence

$$
\lim _{\lambda \rightarrow \infty} \int_{\left[A\left(\tau_{n}\right)>\lambda\right]} A\left(\tau_{n}\right) d P \leq \lim _{\lambda \rightarrow \infty} \int_{\left[A\left(\tau_{n}\right)>\lambda\right]} A(T) d P=0
$$

Because $P\left(A\left(\tau_{n}\right)>\lambda\right) \rightarrow 0$ and a single function in $L^{1}$ is uniformly integrable. Thus these functions $A\left(\tau_{n}\right)$ are equi-integrable. Hence they are uniformly integrable. Now $t_{k}^{n} \rightarrow$ $\left|M\left(t_{k}^{n}\right)\right|$ is also a nonnegative submartingale. Thus

$$
|M(0)|,\left|M\left(\tau_{n}\right)\right|,|M(T)|
$$

is a submartingale by the optional sampling theorem for discrete submartingales given earlier. Therefore,

$$
P\left(\left|M\left(\tau_{n}\right)\right|>\lambda\right) \leq \frac{1}{\lambda} \int_{\left[\left|M\left(\tau_{n}\right)\right|>\lambda\right]}\left|M\left(\tau_{n}\right)\right| d P \leq \frac{1}{\lambda} \int_{\left[\left|M\left(\tau_{n}\right)\right|>\lambda\right]}|M(T)| d P \leq \frac{\|M(T)\|_{L^{1}}}{\lambda}
$$

Of course $\|M(T)\|_{L^{1}}$ is finite because it is dominated by

$$
\int_{\Omega} A(T)+|X(T)| d P<\infty
$$

Hence

$$
\lim _{\lambda \rightarrow \infty} \sup _{n} \int_{\left[\left|M\left(\tau_{n}\right)\right|>\lambda\right]}\left|M\left(\tau_{n}\right)\right| d P \leq \lim _{\lambda \rightarrow \infty} \sup _{n} \int_{\left[\left|M\left(\tau_{n}\right)\right|>\lambda\right]}|M(T)| d P=0
$$

because a single function in $L^{1}$ is uniformly integrable and the above estimate shows that $P\left(\left[\left|M\left(\tau_{n}\right)\right|>\lambda\right]\right) \rightarrow 0$ uniformly in $n$. Thus, in fact $X\left(\tau_{n}\right)$ must be uniformly integrable since it is the sum of two which are.

Theorem 62.7.14 Let $\{M(t)\}$ be a right continuous martingale having values in E a separable real Banach space with respect to the increasing sequence of $\sigma$ algebras, $\left\{\mathscr{F}_{t}\right\}$ which is assumed to be a normal filtration satisfying,

$$
\mathscr{F}_{t}=\cap_{s>t} \mathscr{F}_{s}
$$

for $t \in[0, L], L \leq \infty$ and let $\sigma, \tau$ be two stopping times with $\tau$ bounded. Then $M(\tau)$ defined as

$$
\omega \rightarrow M(\tau(\omega))
$$

is integrable and

$$
M(\sigma \wedge \tau)=E\left(M(\tau) \mid \mathscr{F}_{\sigma}\right)
$$

Proof: Since $M(t)$ is a martingale, $\|M(t)\|$ is a submartingale. Let

$$
\tau_{n}(\omega) \equiv \sum_{k=0}^{\infty} 2^{-n}(k+1) T \mathscr{X}_{\tau^{-1}\left(\left(k 2^{-n} T,(k+1) T 2^{-n}\right]\right)}(\omega)
$$

By Lemma 62.7.13, $\tau_{n}$ is a stopping time and the functions $\left\|M\left(\tau_{n}\right)\right\|$ are uniformly integrable. Also $\|M(\tau)\|$ is integrable. Similarly $\left\|M\left(\tau_{n} \wedge \sigma_{n}\right)\right\|$ are uniformly integrable where $\sigma_{n}$ is defined similarly to $\tau_{n}$.

Consider the main claim now. Letting $\sigma, \tau$ be stopping times with $\tau$ bounded, it follows that for $\sigma_{n}$ and $\tau_{n}$ as above, it follows from Theorem 62.6.5

$$
M\left(\sigma_{n} \wedge \tau_{n}\right)=E\left(M\left(\tau_{n}\right) \mid \mathscr{F}_{\sigma_{n}}\right)
$$

Thus, taking $A \in \mathscr{F}_{\sigma}$ and recalling $\sigma \leq \sigma_{n}$ so that by Proposition 62.7.8, $\mathscr{F}_{\sigma} \subseteq \mathscr{F}_{\sigma_{n}}$,

$$
\int_{A} M\left(\sigma_{n} \wedge \tau_{n}\right) d P=\int_{A} E\left(M\left(\tau_{n}\right) \mid \mathscr{F}_{\sigma_{n}}\right) d P=\int_{A} M\left(\tau_{n}\right) d P
$$

Now passing to a limit as $n \rightarrow \infty$, the Vitali convergence theorem, Theorem 11.5.3 on Page 257 and the right continuity of $M$ implies one can pass to the limit in the above and conclude

$$
\int_{A} M(\sigma \wedge \tau) d P=\int_{A} M(\tau) d P
$$

By Proposition 62.7.8, $M(\sigma \wedge \tau)$ is $\mathscr{F}_{\sigma \wedge \tau} \subseteq \mathscr{F}_{\sigma}$ measurable showing

$$
E\left(M(\tau) \mid \mathscr{F}_{\sigma}\right)=M(\sigma \wedge \tau)
$$

A similar theorem is available for submartingales defined on $[0, L], L \leq \infty$.
Theorem 62.7.15 Let $\{X(t)\}$ be a right continuous submartingale with respect to the increasing sequence of $\sigma$ algebras, $\left\{\mathscr{F}_{t}\right\}$ which is assumed to be a normal filtration,

$$
\mathscr{F}_{t}=\cap_{s>t} \mathscr{F}_{s}
$$

for $t \in[0, L], L \leq \infty$ and let $\sigma, \tau$ be two stopping times with $\tau$ bounded. Then $X(\tau)$ defined as

$$
\omega \rightarrow X(\tau(\omega))
$$

is integrable and

$$
X(\sigma \wedge \tau) \leq E\left(X(\tau) \mid \mathscr{F}_{\sigma}\right)
$$

Proof: Let

$$
\tau_{n}(\omega) \equiv \sum_{k \geq 0} 2^{-n}(k+1) T \mathscr{X}_{\tau^{-1}\left(\left(k 2^{-n} T,(k+1) T 2^{-n}\right]\right)}(\omega)
$$

Then by Lemma 62.7.13 $\tau_{n}$ is a stopping time, the functions $\left|X\left(\tau_{n}\right)\right|$ are uniformly integrable, and $|X(\tau)|$ is also integrable. For $\sigma_{n}$ defined similarly to $\tau_{n}$, it also follows $\left|X\left(\tau_{n} \wedge \sigma_{n}\right)\right|$ are uniformly integrable.

Let $A \in \mathscr{F}_{\sigma}$. Since $\sigma \leq \sigma_{n}$, it follows that $\mathscr{F}_{\sigma} \subseteq \mathscr{F}_{\sigma_{n}}$. By the discrete optional sampling theorem for submartingales, Theorem 62.6.7,

$$
X\left(\sigma_{n} \wedge \tau_{n}\right) \leq E\left(X\left(\tau_{n}\right) \mid \mathscr{F}_{\sigma_{n}}\right)
$$

and so

$$
\int_{A} X\left(\sigma_{n} \wedge \tau_{n}\right) d P \leq \int_{A} E\left(X\left(\tau_{n}\right) \mid \mathscr{F}_{\sigma_{n}}\right) d P=\int_{A} X\left(\tau_{n}\right) d P
$$

and now taking $\lim _{n \rightarrow \infty}$ of both sides and using the Vitali convergence theorem along with the right continuity of $X$, it follows

$$
\int_{A} X(\sigma \wedge \tau) d P \leq \int_{A} X(\tau) d P \equiv \int_{A} E\left(X(\tau) \mid \mathscr{F}_{\sigma}\right) d P
$$

By Proposition 62.7.8, $\mathscr{F}_{\sigma \wedge \tau} \subseteq \mathscr{F}_{\sigma}$, and so since $A \in \mathscr{F} \sigma$ was arbitrary,

$$
E\left(X(\tau) \mid \mathscr{F}_{\sigma}\right) \geq X(\sigma \wedge \tau) \text { a.e. }
$$

Note that a function defined on a countable ordered set such as the integers or equally spaced points is right continuous.

Here is an interesting lemma.

Lemma 62.7.16 Suppose $E\left(\left|X_{n}\right|\right)<\infty$ for all $n, X_{n}$ is $\mathscr{F}_{n}$ measurable, $\mathscr{F}_{n+1} \subseteq \mathscr{F}_{n}$ for all $n \in \mathbb{N}$, and there exist $X_{\infty} \mathscr{F}_{\infty}$ measurable such that $\mathscr{F}_{\infty} \subseteq \mathscr{F}_{n}$ for all $n$ and $X_{0} \mathscr{F}_{0}$ measurable such that $\mathscr{F}_{0} \supseteq \mathscr{F}_{n}$ for all $n$ such that for all $n \in\{0,1, \cdots\}$,

$$
E\left(X_{n} \mid \mathscr{F}_{n+1}\right) \geq X_{n+1}, E\left(X_{n} \mid \mathscr{F}_{\infty}\right) \geq X_{\infty}
$$

Then $\left\{X_{n}: n \in \mathbb{N}\right\}$ is uniformly integrable.
Proof:

$$
E\left(X_{n+1}\right) \leq E\left(E\left(X_{n} \mid \mathscr{F}_{n+1}\right)\right)=E\left(X_{n}\right)
$$

Therefore, the sequence $E\left(X_{n}\right)$ is a decreasing sequence bounded below by $E\left(X_{\infty}\right)$ so it has a limit. Let $k$ be large enough that

$$
\begin{equation*}
\left|E\left(X_{k}\right)-\lim _{m \rightarrow \infty} E\left(X_{m}\right)\right|<\varepsilon \tag{62.7.26}
\end{equation*}
$$

and suppose $n>k$. Then if $\lambda>0$,

$$
\begin{aligned}
& \int_{\left[\left|X_{n}\right| \geq \lambda\right]}\left|X_{n}\right| d P=\int_{\left[X_{n} \geq \lambda\right]} X_{n} d P+\int_{\left[X_{n} \leq-\lambda\right]}\left(-X_{n}\right) d P \\
& =\int_{\left[X_{n} \geq \lambda\right]} X_{n} d P+\int_{\Omega}\left(-X_{n}\right) d P-\int_{\left[-X_{n}<\lambda\right]}\left(-X_{n}\right) d P \\
& =\int_{\left[X_{n} \geq \lambda\right]} X_{n} d P-\int_{\Omega} X_{n} d P+\int_{\left[-X_{n}<\lambda\right]} X_{n} d P
\end{aligned}
$$

From 62.7.26,

$$
\leq \int_{\left[X_{n} \geq \lambda\right]} X_{n} d P-\int_{\Omega} X_{k} d P+\varepsilon+\int_{\left[-X_{n}<\lambda\right]} X_{n} d P
$$

By assumption,

$$
E\left(X_{k} \mid \mathscr{F}_{n}\right) \geq X_{n}
$$

and so

$$
\begin{aligned}
& \leq \int_{\left[X_{n} \geq \lambda\right]} E\left(X_{k} \mid \mathscr{F}_{n}\right) d P-\int_{\Omega} X_{k} d P+\varepsilon+\int_{\left[-X_{n}<\lambda\right]} E\left(X_{k} \mid \mathscr{F}_{n}\right) d P \\
& =\int_{\left[X_{n} \geq \lambda\right]} X_{k} d P-\int_{\Omega} X_{k} d P+\varepsilon+\int_{\left[-X_{n}<\lambda\right]} X_{k} d P \\
& =\int_{\left[X_{n} \geq \lambda\right]} X_{k} d P-\int_{\Omega} X_{k} d P+\varepsilon+\int_{\left[X_{n}>-\lambda\right]} X_{k} d P \\
& =\int_{\left[X_{n} \geq \lambda\right]} X_{k} d P+\left(\int_{\Omega}\left(-X_{k}\right) d P-\int_{\left[X_{n}>-\lambda\right]}\left(-X_{k}\right) d P\right)+\varepsilon \\
& =\int_{\left[X_{n} \geq \lambda\right]} X_{k} d P+\int_{\left[X_{n} \leq-\lambda\right]}\left(-X_{k}\right) d P+\varepsilon=\int_{\left[\left|X_{n}\right| \geq \lambda\right]}\left|X_{k}\right| d P+\varepsilon
\end{aligned}
$$

Applying the maximal inequality for submartingales, Theorem 60.6.4,

$$
P\left(\max \left\{\left|X_{j}\right|: j=n, \cdots, 1\right\} \geq \lambda\right) \leq \frac{1}{\lambda}\left(E\left(\left|X_{0}\right|\right)+E\left(\left|X_{\infty}\right|\right)\right) \leq \frac{C}{\lambda}
$$

and taking sup for all $n$,

$$
P\left(\sup \left\{\left|X_{j}\right|\right\} \geq \lambda\right) \leq \frac{C}{\lambda}
$$

It follows that for all $\lambda$ large enough,

$$
\int_{\left[\left|X_{n}\right| \geq \lambda\right]}\left|X_{n}\right| d P \leq 2 \varepsilon
$$

and since $\varepsilon$ is arbitrary, this shows $\left\{X_{n}\right\}$ for $n>k$ is equiintegrable. Since there are only finitely many $X_{j}$ for $j \leq k$, this shows $\left\{X_{n}\right\}$ is equiintegrable. Hence $\left\{X_{n}\right\}$ is uniformly integrable.

### 62.8 Right Continuity Of Submartingales

The following theorem is an attempt to consider the question of right continuity. It turns out that you can always assume right continuity of a submartingale by going to a suitable version and this theorem is a first step in this direction.

Theorem 62.8.1 Let $\{X(t)\}$ be a real valued submartingale adapted to the filtration $\mathscr{F}_{t}$. Then there exists a set of measure zero $N, P(N)=0$, such that if $\omega \notin N$ then,

$$
\lim _{r \rightarrow t+, r \in \mathbb{Q}} X(r, \omega), \lim _{r \rightarrow t-, r \in \mathbb{Q}} X(r, \omega)
$$

both exist. There also exists a set of measure zero $N$ such that for $\mathbb{Q}^{+}$the nonnegative rationals and $\omega \notin N$,

$$
\sup _{t \in \mathbb{Q}^{+} \cap[0, M]}|X(t, \omega)|<\infty
$$

is bounded for each $M \in \mathbb{N}$. $\mathbb{Q}$ can be replaced with any countable dense subset of $\mathbb{R}$.
Proof: Let $\left\{r_{k}\right\}_{k=1}^{\infty}$ be an enumeration of the nonnegative rationals. Let $t>0$ be given. Then let $\left\{s_{1}, s_{2}, \cdots, s_{n}\right\}$ be such that these are in order and $\left\{s_{2}, \cdots, s_{n-1}\right\}$ are the first $n-2$ rationals less than $t$ listed in order and $s_{1}=0$ while $s_{n}=t$. Then let $Y_{k} \equiv X\left(s_{k}\right)$. It follows $\left\{Y_{k}\right\}$ is a submartingale and so from the maximal inequality in Theorem 60.6.4,

$$
\begin{aligned}
P\left(\left[\max _{1 \leq k \leq n}\left|Y_{k}\right| \geq 2^{m}\right]\right) & \leq \frac{1}{2^{m}}\left(2 E\left(\left|Y_{n}\right|+\left|Y_{1}\right|\right)\right) \\
& =2^{-m}(2 E(|X(t)|+|X(0)|))
\end{aligned}
$$

Then letting $n \rightarrow \infty$, it follows upon letting $C_{t}=2 E(|X(t)|+|X(0)|)$,

$$
P\left(\left[\sup _{r \in \mathbb{Q}^{+} \cap[0, t]}|X(r)| \geq 2^{m}\right]\right) \leq 2^{-m} C_{t}
$$

By the Borel Cantelli lemma, there exists a set of measure $0, N_{t}$ such that for $\omega \notin N_{t}, \omega$ is contained in only finitely many of the sets

$$
\left[\sup _{r \in \mathbb{Q}^{+} \cap[0, t]}|X(r)| \geq 2^{m}\right]
$$

which shows that for $\omega \notin N_{t}, \sup _{r \in \mathbb{Q}^{+} \cap[0, t]}|X(r)|$ is bounded. Now let $N=\cup_{j=1}^{\infty} N_{j}$. This proves the second claim.

Next consider the first claim. By the upcrossing estimate, Theorem 60.6.9 or Lemma 60.2.6, and letting $a<b$ and $U_{[a, b]}^{n}[c, d]$ the upcrossings of $Y_{k}$ from $a$ to $b$ on $[c, d]$ for $d \leq t$ and $c \geq 0$,

$$
\begin{aligned}
E\left(U_{[a, b]}^{n}[0, t]\right) & \leq \frac{1}{b-a} E\left(\left(Y_{n}-a\right)^{+}\right) \\
& =\frac{1}{b-a} E\left((X(t)-a)^{+}\right)
\end{aligned}
$$

Hence

$$
\begin{equation*}
P\left(\left[U_{[a, b]}^{n}[0, t] \geq M\right]\right) \leq \frac{1}{M}\left(\frac{1}{b-a} E\left((X(t)-a)^{+}\right)\right) \tag{62.8.27}
\end{equation*}
$$

Suppose for some $s<t$,

$$
\begin{equation*}
\lim \sup _{r \rightarrow s+, r \in \mathbb{Q}} X(r, \omega)>b>a>\lim \inf _{r \rightarrow s+, r \in \mathbb{Q}} X(r, \omega) \tag{62.8.28}
\end{equation*}
$$

If this is so, then in $(s, t) \cap \mathbb{Q}$ there must be infinitely many values of $r \in \mathbb{Q}$ such that $X(r, \omega) \geq b$ as well as infinitely many values of $r \in \mathbb{Q}$ such that $X(r, \omega) \leq a$. Note this involves the consideration of a limit from one side. Thus, since it is a limit from one side only, there are an arbitrarily large number of upcrossings between $s$ and $t$. Therefore, letting $M$ be a large positive number, it follows that for all $n$ sufficiently large,

$$
U_{[a, b]}^{n}[0, t](\omega) \geq M
$$

which implies

$$
\omega \in\left[U_{[a, b]}^{n}[0, t] \geq M\right]
$$

which from 62.8 .27 is a set of measure no more than

$$
\frac{1}{M}\left(\frac{1}{b-a} E\left((X(t)-a)^{+}\right)\right)
$$

This has shown that the set of $\omega$ such that for some $s \in[0, t) 62.8 .28$ holds is contained in the set

$$
N_{[a, b]} \equiv \cap_{M=1}^{\infty} \cup_{n=1}^{\infty}\left[U_{[a, b]}^{n}[0, t] \geq M\right]
$$

Now the sets,

$$
\left[U_{[a, b]}^{n}[0, t] \geq M\right]
$$

are increasing in $n$ and each has measure less than

$$
\frac{1}{M}\left(\frac{1}{b-a} E\left((X(t)-a)^{+}\right)\right)
$$

and so

$$
P\left(\cup_{n=1}^{\infty}\left[U_{[a, b]}^{n}[0, t] \geq M\right]\right) \leq \frac{1}{M}\left(\frac{1}{b-a} E\left((X(t)-a)^{+}\right)\right)
$$

which shows that

$$
P\left(N_{[a, b]}\right) \leq \frac{1}{M}\left(\frac{1}{b-a} E\left((X(t)-a)^{+}\right)\right)
$$

for every $M$ and therefore, $P\left(N_{[a, b]}\right)=0$.
Therefore, corresponding to $a<b$, there exists a set of measure $0, N_{[a, b]}$ such that for $\omega \notin N_{[a, b]} 62.8 .28$ is not true for any $s \in[0, t)$. Let $N \equiv \cup_{a, b \in \mathbb{Q}} N_{[a, b]}$, a set of measure 0 with the property that if $\omega \notin N$, then 62.8 .28 fails to hold for any pair of rational numbers, $a<b$ for any $s \in[0, t)$. Thus for $\omega \notin N$,

$$
\lim _{r \rightarrow s+, r \in \mathbb{Q}} X(r, \omega)
$$

exists for all $s \in[0, t)$. Similar reasoning applies to show the existence of the limit

$$
\lim _{r \rightarrow s-, r \in \mathbb{Q}} X(r, \omega) .
$$

for all $s \in(0, t]$ whenever $\omega$ is outside of a set of measure zero. Of course, this exceptional set depends on $t$. However, if this exceptional set is denoted as $N_{t}$, one could consider $N \equiv \cup_{n=1}^{\infty} N_{n}$. It is obvious there is no change if $\mathbb{Q}$ is replaced with any countable dense subset. This proves the theorem.

Of course the above theorem does not say the left and right limits are equal, just that they exist in some way for $\omega$ not in some set of measure zero. Also it has not been shown that $\lim _{r \rightarrow s+, r \in \mathbb{Q}} X(r, \omega)=X(r, \omega)$ for a.e. $\omega$.

Corollary 62.8.2 In the situation of Theorem 62.8.1, let $s>0$ and let $D_{1}$ and $D_{2}$ be two countable dense subsets of $\mathbb{R}$. Then

$$
\begin{aligned}
\lim _{r \rightarrow s-, r \in D_{1}} X(r, \omega) & =\lim _{r \rightarrow s-, r \in D_{2}} X(r, \omega) \text { a.e. } \omega \\
\lim _{r \rightarrow s+, r \in D_{1}} X(r, \omega) & =\lim _{r \rightarrow s+, r \in D_{2}} X(r, \omega) \text { a.e. } \omega
\end{aligned}
$$

Proof: Let $\left\{r_{n}^{i}\right\}$ be an increasing sequence from $D_{i}$ converging to $s$ and let $N$ be the exceptional set corresponding to the countable dense set $D_{1} \cup D_{2}$. Then for $\omega \notin N$, and $i=1,2$,

$$
\lim _{r \rightarrow s-, r \in D_{1} \cup D_{2}} X(r, \omega)=\lim _{n \rightarrow \infty} X\left(r_{n}^{i}, \omega\right)=\lim _{r \rightarrow s-, r \in D_{i}} X(r, \omega)
$$

The other claim is similar. This proves the corollary.
Now here is an impressive lemma about submartingales and uniform integrability.
Lemma 62.8.3 Let $X(t)$ be a submartingale adapted to a filtration $\mathscr{F}_{t}$. Let $\left\{r_{k}\right\} \subseteq[s, t)$ be a decreasing sequence converging to $s$. Then $\left\{X\left(r_{j}\right)\right\}_{j=1}^{\infty}$ is uniformly integrable.

Proof: First I will show the sequence is equiintegrable. I need to show that for all $\varepsilon>0$ there exists $\lambda$ large enough that for all $n$

$$
\int_{\left[\left|X\left(r_{n}\right)\right| \geq \lambda\right]}\left|X\left(r_{n}\right)\right| d P<\varepsilon
$$

Let $\varepsilon>0$ be given. Since $\{X(r)\}_{r>0}$ is a submartingale, $E\left(X\left(r_{n}\right)\right)$ is a decreasing sequence bounded below by $E(X(s))$. This is because for $r_{n}<r_{k}$,

$$
E\left(X\left(r_{n}\right)\right) \leq E\left(E\left(X\left(r_{k}\right) \mid \mathscr{F}_{n}\right)\right)=E\left(X\left(r_{k}\right)\right)
$$

Pick $k$ such that

$$
\begin{aligned}
& E\left(X\left(r_{k}\right)\right)-\lim _{n \rightarrow \infty} E\left(X\left(r_{n}\right)\right) \\
= & \left|E\left(X\left(r_{k}\right)\right)-\lim _{n \rightarrow \infty} E\left(X\left(r_{n}\right)\right)\right|<\varepsilon / 2
\end{aligned}
$$

Then for $n>k$,

$$
\begin{align*}
& \int_{\left[\left|X\left(r_{n}\right)\right| \geq \lambda\right]}\left|X\left(r_{n}\right)\right| d P=\int_{\left[X\left(r_{n}\right) \geq \lambda\right]} X\left(r_{n}\right) d P+\int_{\left[X\left(r_{n}\right) \leq-\lambda\right]}-X\left(r_{n}\right) d P \\
& =\int_{\left[X\left(r_{n}\right) \geq \lambda\right]} X\left(r_{n}\right) d P+\int_{\left[X\left(r_{n}\right)>-\lambda\right]} X\left(r_{n}\right) d P-\int_{\Omega} X\left(r_{n}\right) d P \\
& \leq \int_{\left[X\left(r_{n}\right) \geq \lambda\right]} X\left(r_{n}\right) d P+\int_{\left[X\left(r_{n}\right)>-\lambda\right]} E\left(X\left(r_{k}\right) \mid \mathscr{F}_{n}\right) d P-\int_{\Omega} X\left(r_{n}\right) d P \\
& \leq \int_{\left[X\left(r_{n}\right) \geq \lambda\right]} X\left(r_{k}\right) d P+\int_{\left[X\left(r_{n}\right)>-\lambda\right]} X\left(r_{k}\right) d P-\int_{\Omega} X\left(r_{k}\right) d P+\varepsilon / 2 \\
& =\int_{\left[X\left(r_{n}\right) \geq \lambda\right]} X\left(r_{k}\right) d P+\int_{\left[X\left(r_{n}\right) \leq-\lambda\right]}\left(-X\left(r_{k}\right)\right) d P+\varepsilon / 2 \\
& =\int_{\left[\left|X\left(r_{n}\right)\right| \geq \lambda\right]}\left|X\left(r_{k}\right)\right| d P+\varepsilon / 2 \\
& \leq \int_{\left[\sup \left\{|X(r)| \geq \lambda: r \in\left\{r_{j}\right\}_{j=1}^{\infty}\right\}\right]}\left|X\left(r_{k}\right)\right| d P+\varepsilon / 2 \tag{62.8.29}
\end{align*}
$$

From maximal inequalities of Theorem 60.6.4

$$
P\left(\left[\sup _{r \in\left\{r_{n}, r_{n-1}, \cdots, r_{1}\right\}}|X(r)| \geq \lambda\right]\right) \leq \frac{2 E(|X(t)|+|X(0)|)}{\lambda} \equiv \frac{C}{\lambda}
$$

and so, letting $n \rightarrow \infty$,

$$
P\left(\left[\sup _{r \in\left\{r_{n}\right\}_{n=1}^{\infty}}|X(r)| \geq \lambda\right]\right) \leq \frac{C}{\lambda}
$$

It follows that for $\lambda$ sufficiently large the first term in 62.8 .29 is smaller than $\varepsilon / 2$ because $k$ is fixed. Now this shows there is a choice of $\lambda$ such that for all $n>k$,

$$
\int_{\left[\left|X\left(r_{n}\right)\right| \geq \lambda\right]}\left|X\left(r_{n}\right)\right| d P<\varepsilon
$$

There are only finitely many $r_{n}$ for $n \leq k$ and by choosing $\lambda$ sufficiently large the above formula can be made to hold for these also, thus showing $\left\{X\left(r_{n}\right)\right\}$ is equi integrable.

Now this implies the sequence of random variables is uniformly integrable as well. Let $\varepsilon>0$ be given and choose $\lambda$ large enough that for all $n$,

$$
\int_{\left[\left|X\left(r_{n}\right)\right| \geq \lambda\right]}\left|X\left(r_{n}\right)\right| d P<\varepsilon / 2
$$

Then let $A$ be a measurable set.

$$
\begin{aligned}
\int_{A}\left|X\left(r_{n}\right)\right| d P & =\int_{A \cap\left[\left|X\left(r_{n}\right)\right| \geq \lambda\right]}\left|X\left(r_{n}\right)\right| d P+\int_{A \cap\left[\left|X\left(r_{n}\right)\right|<\lambda\right]}\left|X\left(r_{n}\right)\right| d P \\
& <\varepsilon / 2+\int_{A \cap\left[\left|X\left(r_{n}\right)\right|<\lambda\right]}\left|X\left(r_{n}\right)\right| d P \leq \frac{\varepsilon}{2}+\lambda P(A)
\end{aligned}
$$

and now you see that if $P(A)$ is sufficiently small then for all $n$,

$$
\int_{A}\left|X\left(r_{n}\right)\right| d P<\varepsilon
$$

which shows the set of functions is uniformly integrable as claimed. This proves the lemma.
You can often consider a submartingale to be right continuous. This is the importance of the following theorem.

Theorem 62.8.4 Let $\{X(t)\}$ be a submartingale adapted to a normal filtration $\mathscr{F}_{t}$. There exists a right continuous submartingale having left limits, $\{Y(t)\}$ such that $Y(t)=X(t)$ a.e. for every $t \in \mathbb{Q}^{+}$. Furthermore $\{X(t)\}$ has a right continuous left limits version if and only if

$$
t \rightarrow E(X(t))
$$

is right continuous. More generally, $Y(t)=X(t)$ a.e. at every point where the above function is right continuous.

Proof: From Theorem 62.8.1, there exists a set of measure zero, $N$ such that for $\omega \notin N$, left and right limits of the following form exist.

$$
\lim _{r \rightarrow t+, r \in \mathbb{Q}} X(r, \omega), \lim _{r \rightarrow t-, r \in \mathbb{Q}} X(r, \omega) .
$$

Then define for each $t$ and $\omega \notin N$,

$$
Y(t, \omega) \equiv \lim _{r \rightarrow t+, r \in \mathbb{Q}} X(r, \omega) .
$$

and for $\omega \in N$,

$$
Y(t, \omega) \equiv 0
$$

Thus $Y(t)(\omega)=X(t)(\omega)$ a.e. for $t \in \mathbb{Q}^{+}$. For each $\omega \notin N$, there exists $\delta>0$ such that if $r \in \mathbb{Q}, t<r<t+2 \delta$, then

$$
|Y(t, \omega)-X(r, \omega)|<\varepsilon / 2
$$

Now suppose $s \in(t, t+\boldsymbol{\delta})$. Then there exists $\delta_{1}<\boldsymbol{\delta}$ such that if $s<r<s+\delta_{1}$ then

$$
|Y(s, \omega)-X(r, \omega)|<\varepsilon / 2
$$

pick $r \in \mathbb{Q} \cap(s, t+\boldsymbol{\delta})$. Then both of the above two inequalities hold and so it follows

$$
\begin{aligned}
|Y(t, \omega)-Y(s, \omega)| & <|Y(t, \omega)-X(r, \omega)|+|X(r, \omega)-Y(s, \omega)| \\
& <\varepsilon / 2+\varepsilon / 2=\varepsilon
\end{aligned}
$$

Therefore, $t \rightarrow Y(t, \omega)$ is right continuous as claimed.
From the definition of $Y(t, \omega)$, it follows $\omega \rightarrow Y(t, \omega)$ is measurable in $\mathscr{F}_{t+}$ because it is the limit of a sequence, $X\left(r_{n}, \omega\right) \mathscr{X}_{N^{C}}$ where $r_{n} \rightarrow t+$. Now each $X\left(r_{n}, \cdot\right)$ is $\mathscr{F}_{r_{n}}$ measurable and so $Y(t, \cdot)$ is $\mathscr{F}_{r_{n}}$ measurable also for each $r_{n}$. Thus $Y(t, \cdot)$ is $\mathscr{F}_{t+}$ measurable. Since the filtration is normal, $\mathscr{F}_{t}=\mathscr{F}_{t+}$ and it follows $Y(t, \cdot)$ is $\mathscr{F}_{t}$ measurable. Why is $Y(t, \cdot) \in L^{1}(\Omega)$ ?

From Lemma 62.8.3, the collection $\left\{X\left(r_{n}\right)\right\}$ is uniformly integrable. Therefore, from the Vitali convergence theorem, Theorem 11.5.3 on Page 257,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \int_{\Omega}\left|Y(s)-X\left(r_{n}\right)\right| d P=0 \tag{62.8.30}
\end{equation*}
$$

and $Y(s) \in L^{1}(\Omega)$.
It remains to verify $\{Y(s)\}$ is a submartingale. For $s<t$, is it true that

$$
E\left(Y(t) \mid \mathscr{F}_{s}\right) \geq Y(s) ?
$$

Fix $A \in \mathscr{F}_{s}$. From the above construction, there exists $w \in \mathbb{Q}$ and $w \geq t$ such that

$$
\int_{A} Y(t) d P \geq \int_{A} X(w) d P-\varepsilon
$$

Then also, there exists $r \in \mathbb{Q} \cap(s, t)$ such that

$$
\int_{A} X(r) d P \geq \int_{A} Y(s) d P-\varepsilon
$$

Now

$$
\begin{aligned}
\int_{A} E\left(Y(t) \mid \mathscr{F}_{s}\right) d P & =\int_{A} Y(t) d P \geq \int_{A} X(w) d P-\varepsilon \\
& =\int_{A} E\left(X(w) \mid \mathscr{F}_{r}\right) d P-\varepsilon \\
& \geq \int_{A} X(r) d P-\varepsilon \geq \int_{A} Y(s) d P-2 \varepsilon
\end{aligned}
$$

Since $\varepsilon$ was arbitrary, this shows

$$
\int_{A} E\left(Y(t) \mid \mathscr{F}_{s}\right) d P \geq \int_{A} Y(s) d P
$$

for any $A \in \mathscr{F}_{s}$ and so this verifies since $A$ is arbitrary that

$$
E\left(Y(t) \mid \mathscr{F}_{s}\right) \geq Y(s)
$$

so $Y$ is a submartingale.
By Theorem 62.8.1 there exists a set of measure $0, N$ such that the left limits of $Y(r, \omega)$ exist through rational numbers if $\omega \notin N$.

Is $\{Y(t)\}$ a version of $\{X(t)\}$ ? This is where the assumption $t \rightarrow E(X(t))$ is continuous is used. We know $E\left(X\left(r_{n}\right) \mid \mathscr{F}_{s}\right) \geq X(s)$ and also for $A \in \mathscr{F}_{s}$

$$
\int_{A} X\left(r_{n}\right) d P=\int_{A} E\left(X\left(r_{n}\right) \mid \mathscr{F}_{s}\right) d P \geq \int_{A} X(s) d P
$$

Hence taking a limit yields

$$
\int_{A} Y(s) d P \geq \int_{A} X(s) d P
$$

and since $A$ is arbitrary, $Y(s) \geq X(s)$. Now since $t \rightarrow E(X(t))$ is continuous,

$$
\begin{aligned}
\int_{\Omega}|Y(s)-X(s)| d P & =E(Y(s))-E(X(s)) \\
& =\lim _{n \rightarrow \infty}\left(E\left(X\left(r_{n}\right)\right)-E\left(X\left(r_{n}\right)\right)\right)=0
\end{aligned}
$$

It only remains to verify the only way $X(t)$ has a right continuous version is for $t \rightarrow E(X(t))$ to be continuous. Suppose then that $\{X(t)\}$ has a right continuous version, $\{Y(t)\}$. Letting $r_{n} \downarrow s$, Lemma 62.8.3 implies $\left\{Y\left(r_{n}\right)\right\}$ is uniformly integrable. Also $Y(s)(\omega)=\lim _{n \rightarrow \infty} Y\left(r_{n}\right)(\omega)$ a.e. and so by the Vitali convergence theorem,

$$
\lim _{n \rightarrow \infty} \int_{\Omega}\left|X\left(r_{n}\right)-X(s)\right| d P=\lim _{n \rightarrow \infty} \int_{\Omega}\left|Y\left(r_{n}\right)-Y(s)\right| d P=0
$$

This proves the theorem.
Note that the condition $t \rightarrow E(X(t))$ being continuous holds for any martingale. Therefore, every martingale has a right continuous version. The condition that $t \rightarrow E(X(t))$ is right continuous is not a very stringent assumption. For $\{X(t)\}$ a submartingale, this is an increasing function. Therefore, the only points where the condition might not hold comprise a countable set.

### 62.9 Some Maximal Inequalities

As in the case of discrete martingales and submartingales, there are maximal inequalities available.

Lemma 62.9.1 Let $X$ be right continuous and adapted such that the given filtration is complete in the sense that $\mathscr{F}_{0}$ contains all sets $A$ of $\mathscr{F}$ such that $P(A)=0$. Then there exists a set of measure zero $N$ and a $\mathscr{F} \times \mathscr{B}(\mathbb{R})$ measurable function $Y$ such that if $\omega \notin N$, then $Y(t)(\omega)=X(t)(\omega)$. Also, if $f$ is $\mathscr{F}$ measurable and nonnegative then $(\lambda, \omega) \rightarrow \mathscr{X}_{[f>\lambda]}$ is $\mathscr{F} \times \mathscr{B}(\mathbb{R})$ measurable.

Proof: Let $\left\{t_{0}^{n}, t_{1}^{n}, \cdots, t_{m_{n}}^{n}\right\}$ be a partition of $[0, T]$ in which $\left|t_{i}^{n}-t_{i-1}^{n}\right|<\rho_{n}$ where $\rho_{n} \rightarrow 0$. Now define $X_{n}$ as follows:

$$
\begin{aligned}
X_{n}(t)(\omega) & \equiv \sum_{i=1}^{m_{n}} X\left(t_{i}^{n}\right)(\omega) \mathscr{X}_{\left(t_{i-1}^{n}, t_{i}^{n}\right]}(t) \\
X_{n}(0) & \equiv X(0)
\end{aligned}
$$

then each $X_{n}$ is obviously product measurable because it is the sum of functions which are. By right continuity, $X_{n}$ converges pointwise to $X$ for $\omega \notin N$ where $N$ is a set of measure zero and so if $Y(t)(\omega) \equiv X(t)(\omega)$ for all $\omega \notin N$ and $Y(t)(\omega)=0$ for all $\omega \in N$, this is the desired product measurable function.

To see the last claim, let $s$ be a nonnegative simple function, $s(\omega)=\sum_{k=1}^{n} c_{k} \mathscr{X}_{E_{k}}(\omega)$ where the $c_{k}$ are strictly increasing in $k$. Also let $F_{k}=\cup_{i=k}^{n} E_{i}$. Then

$$
\mathscr{X}_{[s>\lambda]}=\sum_{k=1}^{n} \mathscr{X}_{\left[c_{k-1}, c_{k}\right)}(\lambda) \mathscr{X}_{F_{k}}(\omega)
$$

which is clearly product measurable. For arbitrary $f \geq 0$ and measurable, there is an increasing sequence of simple functions $s_{n}$ converging pointwise to $f$. Therefore,

$$
\lim _{n \rightarrow \infty} \mathscr{X}_{\left[s_{n}>\lambda\right]}=\mathscr{X}_{[f>\lambda]}
$$

and so $\mathscr{X}_{[f>\lambda]}$ is product measurable.
Definition 62.9.2 Let $X(t)$ be a right continuous submartingale for $t \in I$ and let $\left\{\tau_{n}\right\}$ be a sequence of stopping times such that $\lim _{n \rightarrow \infty} \tau_{n}=\infty$. Then $X^{\tau_{n}}$ is called the stopped submartingale and it is defined by

$$
X^{\tau_{n}}(t) \equiv X\left(t \wedge \tau_{n}\right)
$$

Proposition 62.9.3 The stopped submartingale just defined is a submartingale.
Proof: By the optional sampling theorem for submartingales, Theorem 62.7.15, it follows that for $s<t$,

$$
\begin{aligned}
E\left(X^{\tau_{n}}(t) \mid \mathscr{F}_{s}\right) & \equiv E\left(X\left(t \wedge \tau_{n}\right) \mid \mathscr{F}_{s}\right) \geq X\left(t \wedge \tau_{n} \wedge s\right) \\
& =X\left(\tau_{n} \wedge s\right) \equiv X^{\tau_{n}}(s)
\end{aligned}
$$

Theorem 62.9.4 Let $\{X(t)\}$ be a right continuous nonnegative submartingale adapted to the normal filtration $\mathscr{F}_{t}$ for $t \in[0, T]$. Let $p \geq 1$. Define

$$
X^{*}(t) \equiv \sup \{X(s): 0<s<t\}, X^{*}(0) \equiv 0
$$

Then for $\lambda>0$, if $X(t)^{p}$ is in $L^{1}(\Omega)$ for each $t$,

$$
\begin{equation*}
P\left(\left[X^{*}(T)>\lambda\right]\right) \leq \frac{1}{\lambda^{p}} \int \mathscr{X}_{\left[X^{*}(T)>\lambda\right]} X(T)^{p} d P \tag{62.9.31}
\end{equation*}
$$

If $X(t)$ is continuous, the above inequality holds without this assumption. In case $p>1$, and $X(t)$ continuous, then for each $t \leq T$,

$$
\begin{equation*}
\left(\int_{\Omega}\left|X^{*}(t)\right|^{p} d P\right)^{1 / p} \leq \frac{p}{p-1}\left(\int_{\Omega} X(T)^{p} d P\right)^{1 / p} \tag{62.9.32}
\end{equation*}
$$

Proof: The first inequality follows from Theorem 62.5.2. However, it can also be obtained a different way using stopping times.

Define the stopping time

$$
\tau \equiv \inf \{t>0: X(t)>\lambda\} \wedge T
$$

(The infimum over an empty set will equal $\infty$.) This is a stopping time by 62.7 .5 because it is just a continuous function of the first hitting time of an open set. Also from the definition of $X^{*}$ in which the supremum is taken over an open interval,

$$
[\tau<t]=\left[X^{*}(t)>\lambda\right]
$$

Note this also shows $X^{*}(t)$ is $\mathscr{F}_{t}$ measurable. Then it follows that $X^{p}(t)$ is also a submartingale since $r^{p}$ is increasing and convex. By the optional sampling theorem, the sequence given by $X(0)^{p}, X(\tau)^{p}, X(T)^{p}$ is a submartingale. Also $[\tau<T] \in \mathscr{F} \tau$ and so

$$
\int_{[\tau<T]} X(\tau)^{p} d P \leq \int_{[\tau<T]} E\left(X(T)^{p} \mid \mathscr{F}_{\tau}\right) d P=\int_{[\tau<T]} X(T)^{p} d P
$$

By right continuity, on $[\tau<T], X(\tau) \geq \lambda$. Therefore,

$$
\begin{aligned}
\lambda^{p} P\left(\left[X^{*}(T)>\lambda\right]\right) & =\lambda^{p} P([\tau<T]) \\
& \leq \int_{[\tau<T]} X(\tau)^{p} d P \leq \int_{\left[X^{*}(T)>\lambda\right]} X(T)^{p} d P
\end{aligned}
$$

Next suppose $X(t)$ is continuous and let $\left\{\tau_{n}\right\}$ be a localizing sequence,

$$
\tau_{n} \equiv \inf \{t: X(t)>n\}
$$

Then by continuity, $X^{\tau_{n}}$ is bounded because $X\left(\tau_{n} \wedge t\right) \leq n$, and so from what was just shown,

$$
\lambda^{p} P\left(\left[\left(X^{\tau_{n}}\right)^{*}(T)>\lambda\right]\right) \leq \int_{\left[\left(X^{\tau_{n}}\right)^{*}(T)>\lambda\right]}\left(X^{\tau_{n}}\right)(T)^{p} d P
$$

Then $\left(X^{\tau_{n}}\right)(T)$ is increasing as $\tau_{n} \rightarrow \infty$ so the result follows from the monotone convergence theorem. This proves the first part.

Let $X^{\tau_{n}}$ be as just defined. Thus it is a bounded submartingale. To save on notation, the $X$ in the following argument is really $X^{\tau_{n}}$. This is done so that all the integrals are finite. If $p>1$, then from the first part using the case of $p=1$,

$$
\int_{\Omega}\left|X^{*}(t)\right|^{p} d P \leq \int_{\Omega}\left|X^{*}(T)\right|^{p} d P=\int_{0}^{\infty} p \lambda^{p-1} \overbrace{P\left(\left[X^{*}(T)>\lambda\right]\right)}^{\leq \frac{1}{\lambda} \int \mathscr{X}_{\left[X^{*}(T)>\lambda\right]} X(T) d P} d \lambda
$$

$$
\begin{aligned}
& \leq p \int_{0}^{\infty} \lambda^{p-1} \frac{1}{\lambda} \int \mathscr{X}_{\left[X^{*}(T)>\lambda\right]} X(T) d P d \lambda \\
& =p \int_{\Omega} X(T) \int_{0}^{X^{*}(T)} \lambda^{p-2} d \lambda d P \\
& =p \int_{\Omega} X(T) \frac{X^{*}(T)^{p-1}}{p-1} d P \\
& \leq \frac{p}{p-1}\left(\int_{\Omega} X^{*}(T)^{p} d P\right)^{1 / p^{\prime}}\left(\int_{\Omega} X(T)^{p} d P\right)^{1 / p}
\end{aligned}
$$

Now divide both sides by $\left(\int_{\Omega} X^{*}(T)^{p} d P\right)^{1 / p^{\prime}}$. Substituting $X^{\tau_{n}}$ for $X$

$$
\left(\int_{\Omega}\left|X^{\tau_{n} *}(t)\right|^{p} d P\right)^{1 / p} \leq\left(\int_{\Omega} X^{\tau_{n} *}(T)^{p} d P\right)^{1 / p} \leq \frac{p}{p-1}\left(\int_{\Omega} X^{\tau_{n}}(T)^{p} d P\right)^{1 / p}
$$

Now let $n \rightarrow \infty$ and use the monotone convergence theorem to obtain the inequality of the theorem. This establishes 62.9.32. The use of Fubini's theorem follows from Lemma 62.9.1.

Here is another sort of maximal inequality in which $X(t)$ is not assumed nonnegative.
Theorem 62.9.5 Let $\{X(t)\}$ be a right continuous submartingale adapted to the normal filtration $\mathscr{F}_{t}$ for $t \in[0, T]$ and $X^{*}(t)$ defined as in Theorem 62.9.4

$$
\begin{gather*}
X^{*}(t) \equiv \sup \{X(s): 0<s<t\}, X^{*}(0) \equiv 0 \\
P\left(\left[X^{*}(T)>\lambda\right]\right) \leq \frac{1}{\lambda} E(|X(T)|) \tag{62.9.33}
\end{gather*}
$$

For $t>0$, let

$$
X_{*}(t)=\inf \{X(s): s<t\}
$$

Then

$$
\begin{equation*}
P\left(\left[X_{*}(T)<-\lambda\right]\right) \leq \frac{1}{\lambda} E(|X(T)|+|X(0)|) \tag{62.9.34}
\end{equation*}
$$

Also

$$
\begin{align*}
& P([\sup \{|X(s)|: s<T\}>\lambda]) \\
& \quad \leq \frac{2}{\lambda} E(|X(T)|+|X(0)|) \tag{62.9.35}
\end{align*}
$$

Proof: The function $f(r)=r^{+} \equiv \frac{1}{2}(|r|+r)$ is convex and increasing. Therefore, $X^{+}(t)$ is also a submartingale but this one is nonnegative. Also

$$
\left[X^{*}(T)>\lambda\right]=\left[\left(X^{+}\right)^{*}(T)>\lambda\right]
$$

and so from Theorem 62.9.4,

$$
P\left(\left[X^{*}(T)>\lambda\right]\right)=P\left(\left[\left(X^{+}\right)^{*}(T)>\lambda\right]\right) \leq \frac{1}{\lambda} E\left(X^{+}(T)\right) \leq \frac{1}{\lambda} E(|X(T)|) .
$$

Next let

$$
\tau=\min (\inf \{t: X(t)<-\lambda\}, T)
$$

then as before, $X(0), X(\tau), X(T)$ is a submartingale and so

$$
\int_{[\tau<T]} X(\tau) d P+\int_{[\tau=T]} X(\tau) d P=\int_{\Omega} X(\tau) d P \geq \int_{\Omega} X(0) d P
$$

Now for $\omega \in[\tau<T], X(t)(\omega)<-\lambda$ for some $t<T$ and so by right continuity, $X(\tau)(\omega) \leq$ $-\lambda$. therefore,

$$
-\lambda \int_{[\tau<T]} d P \geq-\int_{[\tau=T]} X(T) d P+\int_{\Omega} X(0) d P
$$

If $X_{*}(T)<-\lambda$, then from the definition given above, there exists $t<T$ such that $X(t)<$ $-\lambda$ and so $\tau<T$. If $\tau<T$, then by definition, there exists $t<T$ such that $X(t)<-\lambda$ and so $X_{*}(T)<-\lambda$. Hence $[\tau<T]=\left[X_{*}(T)<-\lambda\right]$. It follows that

$$
\begin{aligned}
P\left(\left[X_{*}(T)<-\lambda\right]\right) & =P([\tau<T]) \\
& \leq \frac{1}{\lambda} \int_{[\tau=T]} X(T) d P-\frac{1}{\lambda} \int_{\Omega} X(0) d P \\
& \leq \frac{1}{\lambda} E(|X(T)|+|X(0)|)
\end{aligned}
$$

and this proves 62.9.34.
Finally, combining the above two inequalities,

$$
\begin{gathered}
P([\sup \{|X(s)|: s<T\}>\lambda]) \\
=P\left(\left[X_{*}(T)<-\lambda\right]\right)+P\left(\left[X^{*}(T)>\lambda\right]\right) \\
\leq \frac{2}{\lambda} E(|X(T)|+|X(0)|) .
\end{gathered}
$$

### 62.10 Continuous Submartingale Convergence Theorem

In this section, $\{Y(t)\}$ will be a continuous submartingale and $a<b$. Let

$$
X(t) \equiv(Y(t)-a)_{+}+a
$$

so $X(0) \geq a$. Then $X$ is also a submartingale. It is an increasing convex function of one. If $Y(t)$ has an upcrossing of $[a, b]$, then $X(t)$ starts off at $a$ and ends up at least as large as $b$. If $X(t)$ has an upcrossing of $[a, b]$ then it must start off at $a$ since it cannot be smaller and it ends up at least as large as $b$. Thus we can count the upcrossings of $Y(t)$ by considering the upcrossings of $X(t)$.

The next task is to consider an upcrossing estimate as was done before for discrete submartingales.

$$
\begin{aligned}
\tau_{0} & \equiv \min (\inf \{t>0: X(t)=a\}, M), \\
\tau_{1} & \equiv \min \left(\inf \left\{t>0:\left(X\left(t \vee \tau_{0}\right)-X\left(\tau_{0}\right)\right)_{+}=b-a\right\}, M\right), \\
\tau_{2} & \equiv \min \left(\inf \left\{t>0:\left(X\left(\tau_{1}\right)-X\left(t \vee \tau_{1}\right)\right)_{+}=b-a\right\}, M\right), \\
\tau_{3} & \equiv \min \left(\inf \left\{t>0:\left(X\left(t \vee \tau_{2}\right)-X\left(\tau_{2}\right)\right)_{+}=b-a\right\}, M\right), \\
\tau_{4} & \equiv \min \left(\inf \left\{t>0:\left(X\left(\tau_{3}\right)-X\left(t \vee \tau_{3}\right)\right)_{+}=b-a\right\}, M\right),
\end{aligned}
$$

If $X(t)$ is never $a$, then $\tau_{0}=M$ and there are no upcrossings. It is obvious $\tau_{1} \geq \tau_{0}$ since otherwise, the inequality could not hold. Thus the evens have $X\left(\tau_{2 k}\right)=a$ and $X\left(\tau_{2 k+1}\right)=$ $b$.

Lemma 62.10.1 The above $\tau_{i}$ are stopping times for $t \in[0, M]$.
Proof: It is obvious that $\tau_{0}$ is a stopping time because it is the minimum of $M$ and the first hitting time of a closed set by a continuous adapted process. Consider a stopping time $\eta \leq M$ and let

$$
\sigma \equiv \inf \left\{t>0:(X(t \vee \eta)-X(\eta))_{+}=b-a\right\}
$$

I claim that $t \rightarrow X(t \vee \eta)-X(\eta)$ is adapted to $\mathscr{F}_{t}$. Suppose $\alpha \geq 0$ and consider

$$
\begin{equation*}
\left[(X(t \vee \eta)-X(\eta))_{+}>\alpha\right] \tag{62.10.36}
\end{equation*}
$$

The above set equals

$$
\left(\left[(X(t \vee \eta)-X(\eta))_{+}>\alpha\right] \cap[\eta \leq t]\right) \cap\left(\left[(X(t \vee \eta)-X(\eta))_{+}>\alpha\right] \cap[\eta>t]\right)
$$

Consider the second of the above two sets. Since $\alpha \geq 0$, this set is $\emptyset$. This is because for $\eta>t, X(t \vee \eta)-X(\eta)=0$. Now consider the first. It equals

$$
\left[(X(t \vee \eta)-X(\eta))_{+}>\alpha\right] \cap[\eta \vee t \leq t]
$$

a set of $\mathscr{F}_{t \vee \eta}$ intersected with $[\eta \vee t \leq t]$ and so it is in $\mathscr{F}_{t}$ from properties of stopping times.

If $\alpha<0$, then 62.10 .36 reduces to $\Omega$, also in $\mathscr{F}_{t}$. Therefore, by Proposition 62.7.5, $\sigma$ is a stopping time because it is the first hitting time of a closed set of a continuous adapted process. It follows that $\sigma \wedge M$ is also a stopping time. Similarly $t \rightarrow X(\eta)-X(t \vee \eta)$ is adapted and

$$
\sigma \equiv \inf \left\{t>0:(X(\eta)-X(t \vee \eta))_{+}=b-a\right\}
$$

is also a stopping time from the same reasoning. It follows that the $\tau_{i}$ defined above are all stopping times.

Note that in the above, if $\eta=M$, then $\sigma=M$ also. Thus in the definition of the $\tau_{i}$, if any $\tau_{i}=M$, it follows that also $\tau_{i+1}=M$ and so there is no change in the stopping times. Also note that these stopping times $\tau_{i}$ are increasing as $i$ increases.

Let

$$
U_{[a, b]}^{n M} \equiv \lim _{\varepsilon \rightarrow 0} \sum_{k=0}^{n} \frac{X\left(\tau_{2 k+1}\right)-X\left(\tau_{2 k}\right)}{\varepsilon+X\left(\tau_{2 k+1}\right)-X\left(\tau_{2 k}\right)}
$$

Note that if an upcrossing occurs after $\tau_{2 k}$ on $[0, M]$, then $\tau_{2 k+1}>\tau_{2 k}$ because there exists $t$ such that

$$
\left(X\left(t \vee \tau_{2 k}\right)-X\left(\tau_{2 k}\right)\right)_{+}=b-a
$$

However, you could have $\tau_{2 k+1}>\tau_{2 k}$ without an upcrossing occuring. This happens when $\tau_{2 k}<M$ and $\tau_{2 k+1}=M$ which may mean that $X(t)$ never again climbs to $b$. You break the sum into those terms where $X\left(\tau_{2 k+1}\right)-X\left(\tau_{2 k}\right)=b-a$ and those where this is less than $b-a$. Suppose for a fixed $\omega$, the terms where the difference is $b-a$ are for $k \leq m$. Then there might be a last term for which $X\left(\tau_{2 k+1}\right)-X\left(\tau_{2 k}\right)<b-a$ because it fails to complete the up crossing. There is only one of these at $k=m+1$. Then the above sum is

$$
\begin{aligned}
& \leq \frac{1}{b-a} \sum_{k=0}^{m} X\left(\tau_{2 k+1}\right)-X\left(\tau_{2 k}\right)+\frac{X(M)-a}{\varepsilon+X(M)-a} \\
& \leq \frac{1}{b-a} \sum_{k=0}^{n} X\left(\tau_{2 k+1}\right)-X\left(\tau_{2 k}\right)+\frac{X(M)-a}{\varepsilon+X(M)-a} \\
& \leq \frac{1}{b-a} \sum_{k=0}^{n} X\left(\tau_{2 k+1}\right)-X\left(\tau_{2 k}\right)+1
\end{aligned}
$$

Then $U_{[a, b]}^{n M}$ is clearly a random variable which is at least as large as the number of upcrossings occurring for $t \leq M$ using only $2 n+1$ of the stopping times. From the optional sampling theorem,

$$
\begin{aligned}
E\left(X\left(\tau_{2 k}\right)\right)-E\left(X\left(\tau_{2 k-1}\right)\right) & =\int_{\Omega} X\left(\tau_{2 k}\right)-X\left(\tau_{2 k-1}\right) d P \\
& =\int_{\Omega} E\left(X\left(\tau_{2 k}\right) \mid \mathscr{F}_{2 k-1}\right)-X\left(\tau_{2 k-1}\right) d P \\
& \geq \int_{\Omega} X\left(\tau_{2 k-1}\right)-X\left(\tau_{2 k-1}\right) d P=0
\end{aligned}
$$

Note that, $X\left(\tau_{2 k}\right)=a$ while $X\left(\tau_{2 k-1}\right)=b$ so the above may seem surprising. However, the two stopping times can both equal $M$ so this is actually possible. For example, it could happen that $X(t)=a$ for all $t \in[0, M]$.

Next, take the expectation of both sides,

$$
\begin{gathered}
E\left(U_{[a, b]}^{n M}\right) \leq \frac{1}{b-a} \sum_{k=0}^{n} E\left(X\left(\tau_{2 k+1}\right)\right)-E\left(X\left(\tau_{2 k}\right)\right)+1 \\
\leq \frac{1}{b-a} \sum_{k=0}^{n} E\left(X\left(\tau_{2 k+1}\right)\right)-E\left(X\left(\tau_{2 k}\right)\right)+\frac{1}{b-a} \sum_{k=1}^{n} E\left(X\left(\tau_{2 k}\right)\right)-E\left(X\left(\tau_{2 k-1}\right)\right)+1 \\
=\frac{1}{b-a}\left(E\left(X\left(\tau_{1}\right)\right)-E\left(X\left(\tau_{0}\right)\right)\right)+\frac{1}{b-a} \sum_{k=1}^{n} E\left(X\left(\tau_{2 k+1}\right)\right)-E\left(X\left(\tau_{2 k-1}\right)\right)+1
\end{gathered}
$$

$$
\begin{aligned}
& \leq \frac{1}{b-a}\left(E\left(X\left(\tau_{2 n+1}\right)\right)-E\left(X\left(\tau_{0}\right)\right)\right)+1 \\
& \leq \frac{1}{b-a}(E(X(M))-a)+1
\end{aligned}
$$

which does not depend on $n$. The last inequality follows because $0 \leq \tau_{2 n+1} \leq M$ and $X(t)$ is a submartingale. Let $n \rightarrow \infty$ to obtain

$$
E\left(U_{[a, b]}^{M}\right) \leq \frac{1}{b-a}(E(X(M))-a)+1
$$

where $U_{[a, b]}^{M}$ is an upper bound to the number of upcrossings of $\{X(t)\}$ on $[0, M]$. This proves the following interesting upcrossing estimate.

Lemma 62.10.2 Let $\{Y(t)\}$ be a continuous submartingale adapted to a normal filtration $\mathscr{F}_{t}$ for $t \in[0, M]$. Then if $U_{[a, b]}^{M}$ is defined as the above upper bound to the number of upcrossings of $\{Y(t)\}$ for $t \in[0, M]$, then this is a random variable and

$$
\begin{aligned}
E\left(U_{[a, b]}^{M}\right) & \leq \frac{1}{b-a}\left(E(Y(M)-a)_{+}+a-a\right)+1 \\
& =\frac{1}{b-a} E|Y(M)|+\frac{1}{b-a}|a|+1
\end{aligned}
$$

With this it is easy to prove a continuous submartingale convergence theorem.
Theorem 62.10.3 Let $\{X(t)\}$ be a continuous submartingale adapted to a normal filtration such that

$$
\sup _{t}\{E(|X(t)|)\}=C<\infty
$$

Then there exists $X_{\infty} \in L^{1}(\Omega)$ such that

$$
\lim _{t \rightarrow \infty} X(t)(\omega)=X_{\infty}(\omega) \text { a.e. } \omega
$$

Proof: Let $U_{[a, b]}$ be defined by

$$
U_{[a, b]}=\lim _{M \rightarrow \infty} U_{[a, b]}^{M} .
$$

Thus the random variable $U_{[a, b]}$ is an upper bound for the number of upcrossings. From Lemma 62.10.2 and the assumption of this theorem, there exists a constant $C$ independent of $M$ such that

$$
E\left(U_{[a, b]}^{M}\right) \leq \frac{C}{b-a}+1
$$

Letting $M \rightarrow \infty$, it follows from monotone convergence theorem that

$$
E\left(U_{[a, b]}\right) \leq \frac{C}{b-a}+1
$$

also. Therefore, there exists a set of measure $0 N_{a b}$ such that if $\omega \notin N_{a b}$, then $U_{[a, b]}(\omega)<\infty$. That is, there are only finitely many upcrossings. Now let

$$
N=\cup\left\{N_{a b}: a, b \in \mathbb{Q}\right\} .
$$

It follows that for $\omega \notin N$, it cannot happen that

$$
\lim \sup _{t \rightarrow \infty} X(t)(\omega)-\lim \inf _{t \rightarrow \infty} X(t)(\omega)>0
$$

because if this expression is positive, there would be arbitrarily large values of $t$ where $X(t)(\omega)>b$ and arbitrarily large values of $t$ where $X(t)(\omega)<a$ where $a, b$ are rational numbers chosen such that

$$
\lim \sup _{t \rightarrow \infty} X(t)(\omega)>b>a>\lim _{\inf _{t \rightarrow \infty}} X(t)(\omega)
$$

Thus there would be infinitely many upcrossings which is not allowed for $\omega \notin N$. Therefore, the limit $\lim _{t \rightarrow \infty} X(t)(\omega)$ exists for a.e. $\omega$. Let $X_{\infty}(\omega)$ equal this limit for $\omega \notin N$ and let $X_{\infty}(\omega)=0$ for $\omega \in N$. Then $X_{\infty}$ is measurable and by Fatou's lemma,

$$
\int_{\Omega}\left|X_{\infty}(\omega)\right| d P \leq \lim \inf _{n \rightarrow \infty} \int_{\Omega}|X(n)(\omega)| d P<C
$$

Now here is an interesting result due to Doob.
Theorem 62.10.4 Let $\{M(t)\}$ be a continuous real martingale adapted to the normal filtration $\mathscr{F}_{t}$. Then the following are equivalent.

1. The random variables $M(t)$ are equiintegrable.
2. There exists $M(\infty) \in L^{1}(\Omega)$ such that $\lim _{t \rightarrow \infty}\|M(\infty)-M(t)\|_{L^{1}(\Omega)}=0$.

In this case, $M(t)=E\left(M(\infty) \mid \mathscr{F}_{t}\right)$ and convergence also takes place pointwise.
Proof: Suppose the equiintegrable condition. Then there exists $\lambda$ large enough that for all $t$,

$$
\int_{[|M(t)| \geq \lambda]}|M(t)| d t<1
$$

It follows that for all $t$,

$$
\begin{aligned}
\int_{\Omega}|M(t)| d P & =\int_{[|M(t)| \geq \lambda]}|M(t)| d P+\int_{[|M(t)|<\lambda]}|M(t)| d P \\
& \leq 1+\lambda
\end{aligned}
$$

Since the martingale is bounded in $L^{1}$, by Theorem 62.10 .3 there exists $M(\infty) \in L^{1}(\Omega)$ such that $\lim _{t \rightarrow \infty} M(t)(\omega)=M(\infty)(\omega)$ pointwise a.e. By the assumption $\{M(t)\}$ are equiintegrable, it follows these functions are uniformly integrable. Letting $\delta>0$ be such that if $P(E)<\delta$, then

$$
\int_{E}|M(t)| d P<\frac{\varepsilon}{5}
$$

and $t_{n} \rightarrow \infty$, Egoroff's theorem implies that there exists a set $E$ of measure less than $\delta$ such that on $E^{C}$, the convergence of the $M\left(t_{n}\right)$ is uniform. Thus

$$
\begin{aligned}
\int_{\Omega}\left|M\left(t_{m}\right)-M\left(t_{n}\right)\right| d P & =\int_{E}\left|M\left(t_{m}\right)-M\left(t_{n}\right)\right| d P+\int_{E^{C}}\left|M\left(t_{m}\right)-M\left(t_{n}\right)\right| d P \\
& \leq \frac{2 \varepsilon}{5}+\int_{E^{C}}\left|M\left(t_{m}\right)-M\left(t_{n}\right)\right| d P<\varepsilon
\end{aligned}
$$

whenever $m, n$ are large enough. Therefore, the sequence $\left\{M\left(t_{n}\right)\right\}$ is Cauchy in $L^{1}(\Omega)$ which implies it converges to something in $L^{1}(\Omega)$ which must equal $M(\infty)$ a.e.

Next suppose there is a function $M(\infty)$ to which $M(t)$ converges in $L^{1}(\Omega)$. Then for $t$ fixed and $A \in \mathscr{F}_{t}$, then as $s \rightarrow \infty, s>t$

$$
\begin{aligned}
\int_{A} M(t) d P & =\int_{A} E\left(M(s) \mid \mathscr{F}_{t}\right) d P \equiv \int_{A} M(s) d P \\
& \rightarrow \int_{A} M(\infty) d P=\int_{A} E\left(M(\infty) \mid \mathscr{F}_{t}\right)
\end{aligned}
$$

which shows $E\left(M(\infty) \mid \mathscr{F}_{t}\right)=M(t)$ a.e. since $A \in \mathscr{F}_{t}$ is arbitrary. By Lemma 62.7.11,

$$
\begin{align*}
\int_{[|M(t)| \geq \lambda]}|M(t)| d P & =\int_{[|M(t)| \geq \lambda]}\left|E\left(M(\infty) \mid \mathscr{F}_{t}\right)\right| d P \\
& \leq \int_{[|M(t)| \geq \lambda]} E\left(|M(\infty)| \mid \mathscr{F}_{t}\right) d P \\
& =\int_{[|M(t)| \geq \lambda]}|M(\infty)| d P \tag{62.10.37}
\end{align*}
$$

Now from this,

$$
\begin{aligned}
\lambda P([|M(t)| \geq \lambda]) & \leq \int_{[|M(t)| \geq \lambda]}|M(t)| d P \leq \int_{\Omega}\left|E\left(M(\infty) \mid \mathscr{F}_{t}\right)\right| d P \\
& \leq \int_{\Omega} E\left(|M(\infty)| \mid \mathscr{F}_{t}\right) d P=\int_{\Omega}|M(\infty)| d P
\end{aligned}
$$

and so

$$
P([|M(t)| \geq \lambda]) \leq \frac{C}{\lambda}
$$

From 62.10.37, this shows $\{M(t)\}$ is uniformly integrable because this is true of the single function $|M(\infty)|$. By the submartingale convergence theorem, the convergence to $M(\infty)$ also takes place pointwise.

### 62.11 Hitting This Before That

Let $\{M(t)\}$ be a real valued martingale for $t \in[0, T]$ where $T \leq \infty$ and $M(0)=0$. In case $T=\infty$, assume the conditions of Theorem 62.10 .4 are satisfied. Thus there exists $M(\infty)$ and the $M(t)$ are equiintegrable. With the Doob optional sampling theorem it is possible to estimate the probability that $M(t)$ hits $a$ before it hits $b$ where $a<0<b$. There is no loss
of generality in assuming $T=\infty$ since if it is less than $\infty$, you could just let $M(t) \equiv M(T)$ for all $t>T$. In this case, the equiintegrability of the $M(t)$ follows because for $t<T$,

$$
\begin{aligned}
\int_{[|M(t)|>\lambda]}|M(t)| d P & =\int_{[|M(t)|>\lambda]}\left|E\left(M(T) \mid \mathscr{F}_{t}\right)\right| d P \\
& \leq \int_{[|M(t)|>\lambda]}|M(T)| d P
\end{aligned}
$$

and from Theorem 62.9.5,

$$
P(|M(t)|>\lambda) \leq P\left(\left[M^{*}(t)>\lambda\right]\right) \leq \frac{1}{\lambda} \int_{\Omega}|M(T)| d P
$$

Definition 62.11.1 Let $M$ be a process adapted to the filtration $\mathscr{F}_{t}$ and let $\tau$ be a stopping time. Then $M^{\tau}$, called the stopped process is defined by

$$
M^{\tau}(t) \equiv M(\tau \wedge t)
$$

With this definition, here is a simple lemma.
Lemma 62.11.2 Let $M$ be a right continuous martingale adapted to the normal filtration $\mathscr{F}_{t}$ and let $\tau$ be a stopping time. Then $M^{\tau}$ is also a martingale adapted to the filtration $\mathscr{F}_{t}$.

Proof:Let $s<t$. By the Doob optional sampling theorem,

$$
E\left(M^{\tau}(t) \mid \mathscr{F}_{s}\right) \equiv E\left(M(\tau \wedge t) \mid \mathscr{F}_{s}\right)=M(\tau \wedge t \wedge s)=M^{\tau}(s)
$$

Theorem 62.11.3 Let $\{M(t)\}$ be a continuous real valued martingale adapted to the normal filtration $\mathscr{F}_{t}$ and let

$$
M^{*} \equiv \sup \{|M(t)|: t \geq 0\}
$$

and $M(0)=0$. Letting

$$
\tau_{x} \equiv \inf \{t>0: M(t)=x\}
$$

Then if $a<0<b$ the following inequalities hold.

$$
(b-a) P\left(\left[\tau_{b} \leq \tau_{a}\right]\right) \geq-a P\left(\left[M^{*}>0\right]\right) \geq(b-a) P\left(\left[\tau_{b}<\tau_{a}\right]\right)
$$

and

$$
(b-a) P\left(\left[\tau_{a}<\tau_{b}\right]\right) \leq b P\left(\left[M^{*}>0\right]\right) \leq(b-a) P\left(\left[\tau_{a} \leq \tau_{b}\right]\right)
$$

In words, $P\left(\left[\tau_{b} \leq \tau_{a}\right]\right)$ is the probability that $M(t)$ hits $b$ no later than when it hits $a$. (Note that if $\tau_{a}=\infty=\tau_{b}$ then you would have $\left[\tau_{a}=\tau_{b}\right]$.)

Proof: For $x \in \mathbb{R}$, define

$$
\tau_{x} \equiv \inf \{t \in \mathbb{R} \text { such that } M(t)=x\}
$$

with the usual convention that $\inf (\emptyset)=\infty$. Let $a<0<b$ and let

$$
\tau=\tau_{a} \wedge \tau_{b}
$$

Then the following claim will be important.
Claim: $E(M(\tau))=0$.
Proof of the claim: Let $t>0$. Then by the Doob optional sampling theorem,

$$
\begin{align*}
E(M(\tau \wedge t)) & =E\left(E\left(M(t) \mid \mathscr{F}_{\tau}\right)\right)=E(M(t))  \tag{62.11.38}\\
& =E\left(E\left(M(t) \mid \mathscr{F}_{0}\right)\right)=E(M(0))=0 \tag{62.11.39}
\end{align*}
$$

Observe the martingale $M^{\tau}$ must be bounded because it is stopped when $M(t)$ equals either $a$ or $b$. There are two cases according to whether $\tau=\infty$. If $\tau=\infty$, then $M(t)$ never hits $a$ or $b$ so $M(t)$ has values between $a$ and $b$. In this case $M^{\tau}(t)=M(t) \in[a, b]$. On the other hand, you could have $\tau<\infty$. Then in this case $M^{\tau}(t)$ is eventually equal to either $a$ or $b$ depending on which it hits first. In either case, the martingale $M^{\tau}$ is bounded and by the martingale convergence theorem, Theorem 62.10.3, there exists $M^{\tau}(\infty)$ such that

$$
\lim _{t \rightarrow \infty} M^{\tau}(t)(\omega)=M^{\tau}(\infty)(\omega)=M(\tau)(\omega)
$$

and since the $M^{\tau}(t)$ are bounded, the dominated convergence theorem implies

$$
E(M(\tau))=\lim _{t \rightarrow \infty} E(M(\tau \wedge t))=0
$$

This proves the claim.
Recall

$$
M^{*}(\omega) \equiv \sup \{|M(t)(\omega)|: t \in[0, \infty]\}
$$

Also note that $\left[\tau_{a}=\tau_{b}\right]=[\tau=\infty]$. Now from the claim,

$$
\begin{align*}
0= & E(M(\tau))=\int_{\left[\tau_{a}<\tau_{b}\right]} M(\tau) d P \\
& +\int_{\left[\tau_{b}<\tau_{a}\right]} M(\tau) d P+\int_{\left[\tau_{a}=\tau_{b}\right] \cap\left[M^{*}>0\right]} M(\infty) d P  \tag{62.11.40}\\
& +\int_{\left[\tau_{a}=\tau_{b}\right] \cap\left[M^{*}=0\right]} M(\infty) d P
\end{align*}
$$

The last term equals 0 . By continuity, $M(\tau)$ is either equal to $a$ or $b$ depending on whether $\tau_{a}<\tau_{b}$ or $\tau_{b}<\tau_{a}$. Thus

$$
\begin{gather*}
0=E(M(\tau))=a P\left(\left[\tau_{a}<\tau_{b}\right]\right) \\
+b P\left(\left[\tau_{b}<\tau_{a}\right]\right)+\int_{\left[\tau_{a}=\tau_{b}\right] \cap\left[M^{*}>0\right]} M(\infty) d P \tag{62.11.41}
\end{gather*}
$$

Consider this last term. By the definition, $\left[\tau_{a}=\tau_{b}\right]$ corresponds to $M(t)$ never hitting either $a$ or $b$. Since $M(0)=0$, this can only happen if $M(t)$ has values in $[a, b]$. Therefore, this last term satisfies

$$
\begin{align*}
& a P\left(\left[\tau_{a}=\tau_{b}\right] \cap\left[M^{*}>0\right]\right) \\
& \leq \int_{\left[\tau_{a}=\tau_{b}\right] \cap\left[M^{*}>0\right]} M(\infty) d P \\
& \leq b P\left(\left[\tau_{a}=\tau_{b}\right] \cap\left[M^{*}>0\right]\right) \tag{62.11.42}
\end{align*}
$$

It follows

$$
\begin{gather*}
a P\left(\left[\tau_{a}=\tau_{b}\right] \cap\left[M^{*}>0\right]\right)+a P\left(\left[\tau_{a}<\tau_{b}\right]\right)+b P\left(\left[\tau_{b}<\tau_{a}\right]\right) \leq \\
0 \leq b P\left(\left[\tau_{a}=\tau_{b}\right] \cap\left[M^{*}>0\right]\right)+a P\left(\left[\tau_{a}<\tau_{b}\right]\right)+b P\left(\left[\tau_{b}<\tau_{a}\right]\right) \tag{62.11.43}
\end{gather*}
$$

Note that $\left[\tau_{b}<\tau_{a}\right],\left[\tau_{a}<\tau_{b}\right] \subseteq\left[M^{*}>0\right]$ and so

$$
\begin{equation*}
\left[\tau_{b}<\tau_{a}\right] \cup\left[\tau_{a}<\tau_{b}\right] \cup\left(\left[\tau_{a}=\tau_{b}\right] \cap\left[M^{*}>0\right]\right)=\left[M^{*}>0\right] \tag{62.11.44}
\end{equation*}
$$

The following diagram may help in keeping track of the various substitutions.

| $\left[M^{*}>0\right]$ |  |  |
| :--- | :--- | :--- |
| $\left[\tau_{a}<\tau_{b}\right]$ | $\left[\tau_{b}<\tau_{a}\right]$ | $\left[\tau_{b}=\tau_{a}\right] \cap\left[M^{*}>0\right]$ |

## Left side of 62.11.43

From 62.11.44, this yields on substituting for $P\left(\left[\tau_{a}<\tau_{b}\right]\right)$

$$
\begin{aligned}
0 \geq & a P\left(\left[\tau_{a}=\tau_{b}\right] \cap\left[M^{*}>0\right]\right)+a\left[P\left(\left[M^{*}>0\right]\right)-P\left(\left[\tau_{a} \geq \tau_{b}\right] \cap\left[M^{*}>0\right]\right)\right] \\
& +b P\left(\left[\tau_{b}<\tau_{a}\right]\right)
\end{aligned}
$$

and so since $\left[\tau_{a} \neq \tau_{b}\right] \subseteq\left[M^{*}>0\right]$,

$$
\begin{gather*}
0 \geq a\left[P\left(\left[M^{*}>0\right]\right)-P\left(\left[\tau_{a}>\tau_{b}\right]\right)\right]+b P\left(\left[\tau_{b}<\tau_{a}\right]\right) \\
-a P\left(\left[M^{*}>0\right]\right) \geq(b-a) P\left(\left[\tau_{b}<\tau_{a}\right]\right) \tag{62.11.45}
\end{gather*}
$$

Next use 62.11.44 to substitute for $P\left(\left[\tau_{b}<\tau_{a}\right]\right)$

$$
\begin{gathered}
0 \geq a P\left(\left[\tau_{a}=\tau_{b}\right] \cap\left[M^{*}>0\right]\right)+a P\left(\left[\tau_{a}<\tau_{b}\right]\right)+b P\left(\left[\tau_{b}<\tau_{a}\right]\right) \\
=\quad a P\left(\left[\tau_{a}=\tau_{b}\right] \cap\left[M^{*}>0\right]\right)+a P\left(\left[\tau_{a}<\tau_{b}\right]\right) \\
\\
+b\left[P\left(\left[M^{*}>0\right]\right)-P\left(\left[\tau_{a} \leq \tau_{b}\right] \cap\left[M^{*}>0\right]\right)\right] \\
=a P\left(\left[\tau_{a} \leq \tau_{b}\right] \cap\left[M^{*}>0\right]\right)+b\left[P\left(\left[M^{*}>0\right]\right)-P\left(\left[\tau_{a} \leq \tau_{b}\right] \cap\left[M^{*}>0\right]\right)\right]
\end{gathered}
$$

and so

$$
\begin{equation*}
(b-a) P\left(\left[\tau_{a} \leq \tau_{b}\right]\right) \geq b P\left(\left[M^{*}>0\right]\right) \tag{62.11.46}
\end{equation*}
$$

## Right side of 62.11.43

From 62.11.44, used to substitute for $P\left(\left[\tau_{a}<\tau_{b}\right]\right)$ this yields

$$
\begin{aligned}
& 0 \leq b P\left(\left[\tau_{a}=\tau_{b}\right] \cap\left[M^{*}>0\right]\right)+a P\left(\left[\tau_{a}<\tau_{b}\right]\right)+b P\left(\left[\tau_{b}<\tau_{a}\right]\right) \\
= & b P\left(\left[\tau_{a}=\tau_{b}\right] \cap\left[M^{*}>0\right]\right)+a\left[P\left(\left[M^{*}>0\right]\right)-P\left(\left[\tau_{a} \geq \tau_{b}\right] \cap\left[M^{*}>0\right]\right)\right] \\
& +b P\left(\left[\tau_{b}<\tau_{a}\right]\right)
\end{aligned}
$$

$$
=b P\left(\left[\tau_{a} \geq \tau_{b}\right] \cap\left[M^{*}>0\right]\right)+a\left[P\left(\left[M^{*}>0\right]\right)-P\left(\left[\tau_{a} \geq \tau_{b}\right] \cap\left[M^{*}>0\right]\right)\right]
$$

and so

$$
\begin{equation*}
(b-a) P\left(\left[\tau_{a} \geq \tau_{b}\right]\right) \geq-a P\left(\left[M^{*}>0\right]\right) \tag{62.11.47}
\end{equation*}
$$

Next use 62.11 .44 to substitute for the term $P\left(\left[\tau_{b}<\tau_{a}\right]\right)$ and write

$$
\begin{gathered}
0 \leq b P\left(\left[\tau_{a}=\tau_{b}\right] \cap\left[M^{*}>0\right]\right)+a P\left(\left[\tau_{a}<\tau_{b}\right]\right)+b P\left(\left[\tau_{b}<\tau_{a}\right]\right) \\
=\quad b P\left(\left[\tau_{a}=\tau_{b}\right] \cap\left[M^{*}>0\right]\right)+a P\left(\left[\tau_{a}<\tau_{b}\right]\right) \\
+b\left[P\left(\left[M^{*}>0\right]\right)-P\left(\left[\tau_{a} \leq \tau_{b}\right] \cap\left[M^{*}>0\right]\right)\right] \\
=a P\left(\left[\tau_{a}<\tau_{b}\right]\right)+b P\left(\left[M^{*}>0\right]\right)-b P\left(\left[\tau_{a}<\tau_{b}\right] \cap\left[M^{*}>0\right]\right) \\
=a P\left(\left[\tau_{a}<\tau_{b}\right]\right)+b P\left(\left[M^{*}>0\right]\right)-b P\left(\left[\tau_{a}<\tau_{b}\right]\right)
\end{gathered}
$$

and so

$$
\begin{equation*}
(b-a) P\left(\left[\tau_{a}<\tau_{b}\right]\right) \leq b P\left(\left[M^{*}>0\right]\right) \tag{62.11.48}
\end{equation*}
$$

Now the boxed in formulas in 62.11.45-62.11.48 yield the conclusion of the theorem. This proves the theorem.

Note $P\left(\left[\tau_{a}<\tau_{b}\right]\right)$ means $M(t)$ hits $a$ before it hits $b$ with other occurrences of similar expressions being defined similarly.

### 62.12 The Space $\mathscr{M}_{T}^{p}(E)$

Here $p \geq 1$.
Definition 62.12.1 Let $M$ be an $E$ valued martingale. Then $M \in \mathscr{M}_{T}^{p}(E)$ if $t \rightarrow M(t)(\omega)$ is continuous for a.e. $\omega$ and

$$
E\left(\sup _{t \in[0, T]}\|M(t)\|^{p}\right)<\infty
$$

Here $E$ is a separable Banach space.
Proposition 62.12.2 Define a norm on $\mathscr{M}_{T}^{p}(E)$ by

$$
\|M\|_{\mathscr{M}_{T}^{p}(E)} \equiv E\left(\sup _{t \in[0, T]}\|M(t)\|^{p}\right)^{1 / p}
$$

Then with this norm, $\mathscr{M}_{T}^{p}(E)$ is a Banach space.
Proof: First it is good to observe that $\sup _{t \in[0, T]}\|M(t)\|^{p}$ is measurable. This follows because of the continuity of $t \rightarrow M(t)$. Let $D$ be a dense countable set in $[0, T]$. Then by continuity,

$$
\sup _{t \in[0, T]}\|M(t)\|^{p}=\sup _{t \in D}\|M(t)\|^{p}
$$

and the expression on the right is measurable because $D$ is countable.
Next it is necessary to show this is a norm. It is clear that

$$
\|M\|_{\mathscr{M}_{T}^{p}(E)} \geq 0
$$

and equals 0 only if

$$
0=E\left(\sup _{t \in[0, T]}\|M(t)\|^{p}\right)
$$

which requires $M(t)=0$ for all $t$ for $\omega$ off a set of measure zero so that $M=0$. It is also clear that

$$
\|\alpha M\|_{\mathscr{M}_{T}^{p}(E)}=|\alpha|\|M\|_{\mathscr{M}_{T}^{p}(E)}
$$

It remains to check the triangle inequality. Let $M, N \in \mathscr{M}_{T}^{p}(E)$.

$$
\begin{aligned}
& \|M+N\|_{\mathscr{M}_{T}^{p}(E)} \equiv E\left(\sup _{t \in[0, T]}\|M(t)+N(t)\|^{p}\right)^{1 / p} \\
\leq & E\left(\sup _{t \in[0, T]}(\|M(t)\|+\|N(t)\|)^{p}\right)^{1 / p} \\
\leq & E\left(\left(\sup _{t \in[0, T]}\|M(t)\|+\sup _{t \in[0, T]}\|N(t)\|\right)^{p}\right)^{1 / p} \\
\equiv & \left(\int_{\Omega}\left(\sup _{t \in[0, T]}\|M(t)\|+\sup _{t \in[0, T]}\|N(t)\|\right)^{p} d P\right)^{1 / p} \\
\leq & \left(\int_{\Omega}\left(\sup _{t \in[0, T]}\|M(t)\|\right)^{p} d P\right)^{1 / p}+\left(\int_{\Omega}\left(\sup _{t \in[0, T]}\|N(t)\|\right)^{p} d P\right)^{1 / p} \\
\equiv & \|M\|\left\|_{\mathscr{M}_{T}^{p}(E)}+\right\| N \|_{\mathscr{M}_{T}^{p}(E)}
\end{aligned}
$$

Next consider the claim that $\mathscr{M}_{T}^{p}(E)$ is a Banach space. Let $\left\{M_{n}\right\}$ be a Cauchy sequence. Then

$$
\begin{equation*}
E\left(\sup _{t \in[0, T]}\left\|M_{n}(t)-M_{m}(t)\right\|^{p}\right) \rightarrow 0 \tag{62.12.49}
\end{equation*}
$$

as $m, n \rightarrow \infty$. From continuity,

$$
\sup _{t \in[0, T]}\left\|M_{n}(t)-M_{m}(t)\right\|=\sup _{t \in(0, T)}\left\|M_{n}(t)-M_{m}(t)\right\|
$$

Then from theorem 62.5.3 or 62.9.4,

$$
P\left(\sup _{t \in[0, T]}\left\|M_{n}(t)-M_{m}(t)\right\|>\lambda\right) \leq \frac{1}{\lambda^{p}} E\left(\left\|M_{n}(T)-M_{m}(T)\right\|^{p}\right)
$$

Therefore, one can extract a subsequence $\left\{M_{n_{k}}\right\}$ such that

$$
P\left(\sup _{t \in[0, T]}\left\|M_{n_{k}}(t)-M_{n_{k+1}}(t)\right\|>2^{-k}\right) \leq 2^{-k}
$$

By the Borel Cantelli lemma, it follows $\left\{M_{n_{k}}(t)(\omega)\right\}$ converges uniformly on $[0, T]$ for a.e. $\omega$. Denote by $M(t)(\omega)$ the thing to which it converges, a continuous process because of the uniform convergence. Also, because it is the pointwise limit off a set of measure zero, $\omega \rightarrow M(t)(\omega)$ is $\mathscr{F}_{t}$ measurable. Also, from 62.12.49 and Fatou's lemma

$$
\begin{aligned}
& \int_{\Omega} \sup _{t \in[0, T]}\left\|M_{n}(t)-M(t)\right\|^{p} d P \\
\leq & \lim \inf _{k \rightarrow \infty} \int_{\Omega_{t \in[0, T]}} \sup _{t}\left\|M_{n}(t)-M_{n_{k}}(t)\right\|^{p} d P \leq \varepsilon
\end{aligned}
$$

whenever $n$ is large enough, this from the assumption that $\left\{M_{n}\right\}$ is Cauchy. Thus

$$
\lim _{n \rightarrow \infty} E\left(\sup _{t \in[0, T]}\left\|M_{n}(t)-M(t)\right\|^{p}\right)=0
$$

and so for each $t, M_{n}(t) \rightarrow M(t)$ in $L^{p}(\Omega)$. This also shows that for large, $n$

$$
\begin{aligned}
& E\left(\sup _{t \in[0, T]}\|M(t)\|^{p}\right) \leq E\left(\sup _{t \in[0, T]}\left(\left\|M(t)-M_{n}(t)\right\|+\left\|M_{n}(t)\right\|\right)^{p}\right) \\
& \quad \leq 2^{p-1} E\left(\sup _{t \in[0, T]}\left\|M(t)-M_{n}(t)\right\|^{p}+\sup _{t \in[0, T]}\left(\left\|M_{n}(t)\right\|\right)^{p}\right)<\infty
\end{aligned}
$$

It only remains to verify $M$ is a martingale. Let $s \leq t$ and let $B \in \mathscr{F}_{s}$. For each $s, M_{n}(s) \rightarrow$ $M(s)$ in $L^{p}(\Omega)$. Then from the above, $\omega \rightarrow M(s)(\omega)$ is $\mathscr{F}_{s}$ measurable. Then it follows that

$$
\begin{aligned}
\int_{B} M(s) d P & =\lim _{n \rightarrow \infty} \int_{B} M_{n}(s) d P=\lim _{n \rightarrow \infty} \int_{B} E\left(M_{n}(t) \mid \mathscr{F}_{s}\right) d P \\
& =\lim _{n \rightarrow \infty} \int_{B} M_{n}(t) d P=\int_{B} M(t) d P
\end{aligned}
$$

and so by definition, $E\left(M(t) \mid \mathscr{F}_{s}\right)=M(s)$ which shows $M$ is a martingale.
Proposition 62.12.3 The functions $M(t)$ for each $M \in \mathscr{M}_{T}^{p}(E)$ are equi integrable.
Proof: This follows because

$$
\begin{equation*}
\int_{[\|M(t)\| \geq \lambda]}\|M(t)\|^{p} d P \leq \int_{\left[\sup _{t \in[0, T]}| | M(t) \| \geq \lambda\right]}\left(\sup _{t \in[0, T]}\|M(t)\|^{p}\right) d P \tag{62.12.50}
\end{equation*}
$$

which converges to 0 due to the definition of $\mathscr{M}_{T}^{p}(E)$ which requires that

$$
\sup _{t \in[0, T]}\|M(t)\|^{p} \in L^{1}(\Omega, \mathscr{F}, P)
$$

Since the sets $\left[\sup _{t \in[0, T]}\|M(t)\| \geq \lambda\right]$ decrease to $\emptyset$ as $\lambda \rightarrow \infty$, the dominated convergence theorem implies the integral on the right in 62.12 .50 converges to 0 .

## Chapter 63

## The Quadratic Variation Of A Martingale

### 63.1 How To Recognize A Martingale

The main ideas are most easily understood in the special case where it is assumed the martingale is bounded. Then one can extend to more general situations using a localizing sequence of stopping times.

Let $\{M(t)\}$ be a continuous martingale having values in a separable Hilbert space. The idea is to consider the submartingale, $\left\{\|M(t)\|^{2}\right\}$ and write it as the sum of a martingale and a submartingale. An important part of the argument is the following lemma which gives a checkable criterion for a stochastic process to be a martingale.

Lemma 63.1.1 Let $\{X(t)\}$ be a stochastic process adapted to the filtration $\left\{\mathscr{F}_{t}\right\}$ for $t \geq 0$. Then it is a martingale for the given filtration if for every stopping time $\sigma$ it follows

$$
E(X(t))=E(X(\sigma))
$$

In fact, it suffices to check this on stopping times which have two values.
Proof: Let $s<t$ and $A \in \mathscr{F}_{s}$. Define a stopping time

$$
\sigma(\omega) \equiv s \mathscr{X}_{A}(\omega)+t \mathscr{X}_{A^{C}}(\omega)
$$

This is a stopping time because $[\sigma \leq l]=\Omega$ if $l \geq t$. Also $[\sigma \leq l]=A \in \mathscr{F}_{s}$ if $l \in[s, t)$ and $[\sigma \leq l]=\emptyset$ if $l<s$. Then by assumption,

$$
\int_{A} X(t) d P+\int_{A^{C}} X(t) d P=
$$

$$
\overbrace{\int X(t) d P=\int X(\sigma) d P}^{\text {by assumption }}=\int_{A} X(s) d P+\int_{A^{C}} X(t) d P
$$

Therefore,

$$
\int_{A} X(t) d P=\int_{A} X(s) d P
$$

and since $X(s)$ is $\mathscr{F}_{s}$ measurable, it follows $E\left(X(t) \mid \mathscr{F}_{s}\right)=X(s)$ a.e. and this shows $\{X(t)\}$ is a martingale.

Note that if $t \in[0, T]$, it suffices to check the expectation condition for stopping times which have two values no larger than $T$.

The following lemma will be useful.
Lemma 63.1.2 Suppose $X_{n} \rightarrow X$ in $L^{1}(\Omega, \mathscr{F}, P ; E)$ where $E$ is a separable Banach space. Then letting $\mathscr{G}$ be a $\sigma$ algebra contained in $\mathscr{F}$,

$$
E\left(X_{n} \mid \mathscr{G}\right) \rightarrow E(X \mid \mathscr{G})
$$

in $L^{1}(\Omega)$.

Proof: This follows from the definitions and Theorem 61.1.1 on Page 1983.

$$
\begin{aligned}
\int_{\Omega}\left\|E(X \mid \mathscr{G})-E\left(X_{n} \mid \mathscr{G}\right)\right\| d P & =\int_{\Omega}\left\|E\left(X_{n}-X \mid \mathscr{G}\right)\right\| d P \\
& \leq \int_{\Omega} E\left(\left\|X_{n}-X\right\| \mid \mathscr{G}\right) d P \\
& =\int_{\Omega}\left\|X_{n}-X\right\| d P \square
\end{aligned}
$$

Corollary 63.1.3 Let $X, Y$ be in $L^{2}(\Omega, \mathscr{F}, P ; H)$ where $H$ is a separable Hilbert space and let $X$ be $\mathscr{G}$ measurable where $\mathscr{G} \subseteq \mathscr{F}$. Then

$$
E((X, Y) \mid \mathscr{G})=(X, E(Y \mid \mathscr{G})) \text { a.e. }
$$

Proof: First let $X=a \mathscr{X}_{B}$ where $B \in \mathscr{G}$. Then for $A \in \mathscr{G}$,

$$
\begin{aligned}
\int_{A} E\left(\left(a \mathscr{X}_{B}, Y\right) \mid \mathscr{G}\right) d P & =\int_{A} \mathscr{X}_{B} E((a, Y) \mid \mathscr{G}) d P=\int_{A} \mathscr{X}_{B}(a, Y) d P \\
& =\int_{A \cap B}(a, Y) d P=\left(a, \int_{A \cap \mathscr{B}} Y d P\right) \\
\int_{A}\left(a \mathscr{X}_{B}, E(Y \mid \mathscr{G})\right) d P & =\int_{A} \mathscr{X}_{B}(a, E(Y \mid \mathscr{G})) d P \\
& =\left(a, \int_{A} \mathscr{X}_{B} E(Y \mid \mathscr{G}) d P\right)=\left(a, \int_{A \cap \mathscr{B}} Y d P\right)
\end{aligned}
$$

It follows that the formula holds for $X$ simple.
Therefore, letting $X_{n}$ be a sequence of $\mathscr{G}$ measurable simple functions converging pointwise to $X$ and also in $L^{2}(\Omega)$,

$$
E\left(\left(X_{n}, Y\right) \mid \mathscr{G}\right)=\left(X_{n}, E(Y \mid \mathscr{G})\right)
$$

Now the desired formula holds from Lemma 63.1.2.
The following is related to something called a martingale transform. It is a lot like what will happen later with the Ito integral.

Proposition 63.1.4 Let $\left\{\tau_{k}\right\}$ be an increasing sequence of stopping times for the normal filtration $\left\{\mathscr{F}_{t}\right\}$ such that

$$
\lim _{k \rightarrow \infty} \tau_{k}=\infty, \tau_{0}=0
$$

Also let $\xi_{k}$ be $\mathscr{F}_{\tau_{k}}$ measurable with values in $H$, a separable Hilbert space and let $M(t)$ be a right continuous martingale adapted to the normal filtration $\mathscr{F}_{t}$ which has the property that $M(t) \in L^{2}(\Omega ; H)$ for all $t, M(0)=0$. Then if $\left|\xi_{k}\right| \leq C$,

$$
\begin{gather*}
E\left(\left(\sum_{k \geq 0}\left(\xi_{k},\left(M\left(\tau_{k+1} \wedge t\right)-M\left(\tau_{k} \wedge t\right)\right)\right)\right)^{2}\right) \\
\leq C^{2} E\left(\|M(t)\|^{2}\right) \tag{63.1.1}
\end{gather*}
$$

Proof: First of all, the sum converges because eventually $\tau_{k} \wedge t=t$. Therefore, for large enough $k, M\left(\tau_{k+1} \wedge t\right)-M\left(\tau_{k} \wedge t\right) \equiv \Delta M_{k}=0$. Consider first the finite sum, $k \leq q$.

$$
\begin{equation*}
E\left(\left(\sum_{k=0}^{q}\left(\xi_{k}, \Delta M_{k}\right)\right)^{2}\right) \tag{63.1.2}
\end{equation*}
$$

When the sum is multiplied out, you get mixed terms. Consider one of these mixed terms, $j<k$

$$
E\left(\left(\xi_{k}, \Delta M_{k}\right)\left(\xi_{j}, \Delta M_{j}\right)\right)
$$

Using Corollary 63.1.3 and Doob's optional sampling theorem, Theorem 60.5.4, this equals

$$
\begin{gathered}
E\left(E\left(\left(\xi_{k}, \Delta M_{k}\right)\left(\xi_{j}, \Delta M_{j}\right) \mid \mathscr{F}_{\tau_{k}}\right)\right)=E\left(\left(\xi_{j}, \Delta M_{j}\right) E\left(\left(\xi_{k}, \Delta M_{k}\right) \mid \mathscr{F}_{\tau_{k}}\right)\right) \\
=E\left(\left(\xi_{j}, \Delta M_{j}\right)\left(\xi_{k}, E\left(M\left(\tau_{k+1} \wedge t\right)-M\left(\tau_{k} \wedge t\right) \mid \mathscr{F}_{\tau_{k}}\right)\right)\right)=E\left(\left(\xi_{j}, \Delta M_{j}\right)\left(\xi_{k}, 0\right)\right)=0
\end{gathered}
$$

Note that in using the optional sampling theorem, the stopping time $\tau_{k+1} \wedge t$ is bounded.
Therefore, the only terms which survive in 63.1.2 are the non mixed terms and so this expression reduces to

$$
\begin{align*}
& \sum_{k=0}^{q} E\left(\xi_{k}, \Delta M_{k}\right)^{2} \leq C^{2} \sum_{k=0}^{q} E\left(\left\|\Delta M_{k}\right\|^{2}\right) \\
= & C^{2} \sum_{k=0}^{q} E\left(\left\|M\left(\tau_{k+1} \wedge t\right)-M\left(\tau_{k} \wedge t\right)\right\|^{2}\right) \\
= & C^{2} \sum_{k=0}^{q} E\left(\left\|M\left(\tau_{k+1} \wedge t\right)\right\|^{2}\right)+E\left(\left\|M\left(\tau_{k} \wedge t\right)\right\|^{2}\right) \\
& -2 E\left(\left(M\left(\tau_{k} \wedge t\right), M\left(\tau_{k+1} \wedge t\right)\right)\right) \tag{63.1.3}
\end{align*}
$$

Consider the term $E\left(\left(M\left(\tau_{k} \wedge t\right), M\left(\tau_{k+1} \wedge t\right)\right)\right)$. By Doob's optional sampling theorem for martingales and Corollary 63.1.3 again, this equals

$$
\begin{aligned}
& E\left(E\left(\left(M\left(\tau_{k} \wedge t\right), M\left(\tau_{k+1} \wedge t\right)\right) \mid \mathscr{F}_{\tau_{k}}\right)\right) \\
= & E\left(\left(M\left(\tau_{k} \wedge t\right), E\left(M\left(\tau_{k+1} \wedge t\right) \mid \mathscr{F} \tau_{k}\right)\right)\right) \\
= & E\left(\left(M\left(\tau_{k} \wedge t\right), M\left(\tau_{k+1} \wedge t \wedge \tau_{k}\right)\right)\right) \\
= & E\left(\left\|M\left(\tau_{k} \wedge t\right)\right\|^{2}\right)
\end{aligned}
$$

It follows 63.1.3 equals

$$
C^{2} \sum_{k=0}^{q} E\left(\left\|M\left(\tau_{k+1} \wedge t\right)\right\|^{2}\right)-E\left(\left\|M\left(\tau_{k} \wedge t\right)\right\|^{2}\right) \leq C^{2} E\left(\|M(t)\|^{2}\right)
$$

Then from Fatou's lemma,

$$
\begin{aligned}
& E\left(\left(\sum_{k \geq 0}\left(\xi_{k},\left(M\left(\tau_{k+1} \wedge t\right)-M\left(\tau_{k} \wedge t\right)\right)\right)\right)^{2}\right) \leq \\
& \lim _{q \rightarrow \infty} \inf E\left(\left(\sum_{k=0}^{q}\left(\xi_{k},\left(M\left(\tau_{k+1} \wedge t\right)-M\left(\tau_{k} \wedge t\right)\right)\right)\right)^{2}\right) \\
\leq & C^{2} E\left(\|M(t)\|^{2}\right) \square
\end{aligned}
$$

Now here is an interesting lemma which will be used to prove uniqueness in the main result.

Lemma 63.1.5 Let $\mathscr{F}_{t}$ be a normal filtration and let $A(t), B(t)$ be adapted to $\mathscr{F}_{t}$, continuous, and increasing with $A(0)=B(0)=0$ and suppose $A(t)-B(t)$ is a martingale. Then $A(t)-B(t)=0$ for all $t$.

Proof: I shall show $A(l)=B(l)$ where $l$ is arbitrary. Let $M(t)$ be the name of the martingale. Define a stopping time

$$
\begin{aligned}
\tau \equiv & \inf \{t>0:|M(t)|>C\} \wedge l \wedge \inf \{t>0: A(t)>C\} \\
& \wedge \inf \{t>0: B(t)>C\}
\end{aligned}
$$

where $\inf (\emptyset) \equiv \infty$ and denote the stopped martingale

$$
M^{\tau}(t) \equiv M(t \wedge \tau)
$$

Then I claim this is also a martingale with respect to the filtration $\mathscr{F}_{t}$ because by Doob's optional sampling theorem for martingales, if $s<t$,

$$
E\left(M^{\tau}(t) \mid \mathscr{F}_{s}\right) \equiv E\left(M(\tau \wedge t) \mid \mathscr{F}_{s}\right)=M(\tau \wedge t \wedge s)=M(\tau \wedge s)=M^{\tau}(s)
$$

Note the bounded stopping time is $\tau \wedge t$ and the other one is $\sigma=s$ in this theorem. Then $M^{\tau}$ is a continuous martingale which is also uniformly bounded. It equals $A^{\tau}-B^{\tau}$. The stopping time ensures $A^{\tau}$ and $B^{\tau}$ are uniformly bounded by $C$. Thus all of $\left|M^{\tau}(t)\right|, B^{\tau}(t), A^{\tau}(t)$ are bounded by $C$ on $[0, l]$. Now let $\mathscr{P}_{n} \equiv\left\{t_{k}\right\}_{k=1}^{n}$ be a uniform partition of $[0, l]$ and let $M^{\tau}\left(\mathscr{P}_{n}\right)$ denote

$$
M^{\tau}\left(\mathscr{P}_{n}\right) \equiv \max \left\{\left|M^{\tau}\left(t_{i+1}\right)-M^{\tau}\left(t_{i}\right)\right|\right\}_{i=1}^{n}
$$

Then

$$
E\left(M^{\tau}(l)^{2}\right)=E\left(\left(\sum_{k=0}^{n-1} M^{\tau}\left(t_{k+1}\right)-M^{\tau}\left(t_{k}\right)\right)^{2}\right)
$$

Now consider a mixed term in the sum where $j<k$.

$$
E\left(\left(M^{\tau}\left(t_{k+1}\right)-M^{\tau}\left(t_{k}\right)\right)\left(M^{\tau}\left(t_{j+1}\right)-M^{\tau}\left(t_{j}\right)\right)\right)
$$

$$
\begin{aligned}
& =E\left(E\left(\left(M^{\tau}\left(t_{k+1}\right)-M^{\tau}\left(t_{k}\right)\right)\left(M^{\tau}\left(t_{j+1}\right)-M^{\tau}\left(t_{j}\right)\right) \mid \mathscr{F}_{t_{k}}\right)\right) \\
& =E\left(\left(M^{\tau}\left(t_{j+1}\right)-M^{\tau}\left(t_{j}\right)\right) E\left(\left(M^{\tau}\left(t_{k+1}\right)-M^{\tau}\left(t_{k}\right)\right) \mid \mathscr{F}_{t_{k}}\right)\right) \\
& =E\left(\left(M^{\tau}\left(t_{j+1}\right)-M^{\tau}\left(t_{j}\right)\right)\left(M^{\tau}\left(t_{k}\right)-M^{\tau}\left(t_{k}\right)\right)\right)=0
\end{aligned}
$$

It follows

$$
\begin{aligned}
& E\left(M^{\tau}(l)^{2}\right)=E\left(\sum_{k=0}^{n-1}\left(M^{\tau}\left(t_{k+1}\right)-M^{\tau}\left(t_{k}\right)\right)^{2}\right) \\
& \leq E\left(\sum_{k=0}^{n-1} M^{\tau}\left(\mathscr{P}_{n}\right)\left|M^{\tau}\left(t_{k+1}\right)-M^{\tau}\left(t_{k}\right)\right|\right) \\
& \leq E\left(\sum_{k=0}^{n-1} M^{\tau}\left(\mathscr{P}_{n}\right)\left(\left|A^{\tau}\left(t_{k+1}\right)-A^{\tau}\left(t_{k}\right)\right|+\left|B^{\tau}\left(t_{k+1}\right)-B^{\tau}\left(t_{k}\right)\right|\right)\right) \\
& \leq E\left(M^{\tau}\left(\mathscr{P}_{n}\right) \sum_{k=0}^{n-1}\left(\left|A^{\tau}\left(t_{k+1}\right)-A^{\tau}\left(t_{k}\right)\right|+\left|B^{\tau}\left(t_{k+1}\right)-B^{\tau}\left(t_{k}\right)\right|\right)\right) \\
& \leq E\left(M^{\tau}\left(\mathscr{P}_{n}\right) 2 C\right)
\end{aligned}
$$

the last step holding because $A$ and $B$ are increasing. Now letting $n \rightarrow \infty$, the right side converges to 0 by the dominated convergence theorem and the observation that for a.e. $\omega$,

$$
\lim _{n \rightarrow \infty} M^{\tau}\left(\mathscr{P}_{n}\right)(\omega)=0
$$

because of continuity of $M$. Thus for $\tau=\tau_{C}$ given above,

$$
M\left(l \wedge \tau_{C}\right)=0 \text { a.e. }
$$

Now let $C \in \mathbb{N}$ and let $N_{C}$ be the exceptional set off which $M\left(l \wedge \tau_{C}\right)=0$. Then letting $N_{l}$ denote the union of all these exceptional sets for $C \in \mathbb{N}$, it is also a set of measure zero and for $\omega$ not in this set, $M\left(l \wedge \tau_{C}\right)=0$ for all $C$. Since the martingale is continuous, it follows for each such $\omega$, eventually $\tau_{C}>l$ and so $M(l)=0$. Thus for $\omega \notin N_{l}$,

$$
M(l)(\omega)=0
$$

Now let $N=\cup_{l \in \mathbb{Q} \cap[0, \infty)} N_{l}$. Then for $\omega \notin N, M(l)(\omega)=0$ for all $l \in \mathbb{Q} \cap[0, \infty)$ and so by continuity, this is true for all positive $l$.

Note this shows a continuous martingale is not of bounded variation unless it is a constant.

### 63.2 The Quadratic Variation

This section is on the quadratic variation of a martingale. Actually, you can also consider the quadratic variation of a local martingale which is more general. Therefore, this concept is defined first. We will generally assume $M(0)=0$ since there is no real loss of generality in doing so. One can simply subtract $M(0)$ otherwise.

Definition 63.2.1 Let $\{M(t)\}$ be adapted to the normal filtration $\mathscr{F}_{t}$ for $t>0$. Then $\{M(t)\}$ is a local martingale (submartingale) if there exist stopping times $\tau_{n}$ increasing to infinity such that for each $n$, the process $M^{\tau_{n}}(t) \equiv M\left(t \wedge \tau_{n}\right)$ is a martingale (submartingale) with respect to the given filtration. The sequence of stopping times is called a localizing sequence. The martingale $M^{\tau_{n}}$ is called the stopped martingale. Exactly the same convention applies to a localized submartingale.

Proposition 63.2.2 If $M(t)$ is a continuous local martingale (submartingale) for a normal filtration as above, $M(0)=0$, then there exists a localizing sequence $\tau_{n}$ such that for each $n$ the stopped martingale(submartingale) $M^{\tau_{n}}$ is uniformly bounded. Also if $M$ is a martingale, then $M^{\tau}$ is also a martingale (submartingale). If $\tau_{n}$ is an increasing sequence of stopping times such that $\lim _{n \rightarrow \infty} \tau_{n}=\infty$, and for each $\tau_{n}$ and real valued stopping time $\delta$, there exists a function $X$ of $\tau_{n} \wedge \delta$ such that $X\left(\tau_{n} \wedge \boldsymbol{\delta}\right)$ is $\mathscr{F}_{\tau_{n} \wedge \delta}$ measurable, then $\lim _{n \rightarrow \infty} X\left(\tau_{n} \wedge \boldsymbol{\delta}\right) \equiv X(\boldsymbol{\delta})$ exists for each $\omega$ and $X(\boldsymbol{\delta})$ is $\mathscr{F}_{\delta}$ measurable.

Proof: First consider the claim about $M^{\tau}$ being a martingale (submartingale) when $M$ is. By optional sampling theorem,

$$
E\left(M^{\tau}(t) \mid \mathscr{F}_{s}\right)=E\left(M(\tau \wedge t) \mid \mathscr{F}_{s}\right)=M(\tau \wedge t \wedge s)=M^{\tau}(s)
$$

The case where $M$ is a submartingale is similar.
Next suppose $\sigma_{n}$ is a localizing sequence for the local martingale(submartingale) $M$. Then define

$$
\eta_{n} \equiv \inf \{t>0:\|M(t)\|>n\}
$$

Therefore, by continuity of $M,\left\|M\left(\eta_{n}\right)\right\| \leq n$. Now consider $\tau_{n} \equiv \eta_{n} \wedge \sigma_{n}$. This is an increasing sequence of stopping times. By continuity of $M$, it must be the case that $\eta_{n} \rightarrow \infty$. Hence $\sigma_{n} \wedge \eta_{n} \rightarrow \infty$.

Finally, consider the last claim. Pick $\omega$. Then $X\left(\tau_{n}(\omega) \wedge \delta(\omega)\right)(\omega)$ is eventually constant as $n \rightarrow \infty$ because for all $n$ large enough, $\tau_{n}(\omega)>\delta(\omega)$ and so this sequence of functions converges pointwise. That which it converges to, denoted by $X(\delta)$, is $\mathscr{F}_{\delta}$ measurable because each function $\omega \rightarrow X\left(\tau_{n}(\omega) \wedge \delta(\omega)\right)(\omega)$ is $\mathscr{F}_{\delta \wedge \tau_{n}} \subseteq \mathscr{F}_{\delta}$ measurable.

One can also give a generalization of Lemma 63.1.5 to conclude a local martingale must be constant or else they must fail to be of bounded variation.

Corollary 63.2.3 Let $\mathscr{F}_{t}$ be a normal filtration and let $A(t), B(t)$ be adapted to $\mathscr{F}_{t}$, continuous, and increasing with $A(0)=B(0)=0$ and suppose $A(t)-B(t) \equiv M(t)$ is a local martingale. Then $M(t)=A(t)-B(t)=0$ a.e. for all $t$.

Proof: Let $\left\{\tau_{n}\right\}$ be a localizing sequence for $M$. For given $n$, consider the martingale,

$$
M^{\tau_{n}}(t)=A^{\tau_{n}}(t)-B^{\tau_{n}}(t)
$$

Then from Lemma 63.1.5, it follows $M^{\tau_{n}}(t)=0$ for all $t$ for all $\omega \notin N_{n}$, a set of measure 0 . Let $N=\cup_{n} N_{n}$. Then for $\omega \notin N, M\left(\tau_{n}(\omega) \wedge t\right)(\omega)=0$. Let $n \rightarrow \infty$ to conclude that $M(t)(\omega)=0$. Therefore, $M(t)(\omega)=0$ for all $t$.

Recall Example 62.7.10 on Page 2083. For convenience, here is a version of what it says.

Lemma 63.2.4 Let $X(t)$ be continuous and adapted to a normal filtration $\mathscr{F}_{t}$ and let $\eta$ be a stopping time. Then if $K$ is a closed set,

$$
\tau \equiv \inf \{t>\eta: X(t) \in K\}
$$

is also a stopping time.
Proof: First consider $Y(t)=X(t \vee \eta)-X(\eta)$. I claim that $Y(t)$ is adapted to $\mathscr{F}_{t}$. Consider $U$ and open set and $[Y(t) \in U]$. Is it in $\mathscr{F}_{t}$ ? We know it is in $\mathscr{F}_{t \vee \eta}$. It equals

$$
([Y(t) \in U] \cap[\eta \leq t]) \cup([Y(t) \in U] \cap[\eta>t])
$$

Consider the second of these sets. It equals

$$
([X(\eta)-X(\eta) \in U] \cap[\eta>t])
$$

If $0 \in U$, then it reduces to $[\eta>t] \in \mathscr{F}_{t}$. If $0 \notin U$, then it reduces to $\emptyset$ still in $\mathscr{F}_{t}$. Next consider the first set. It equals

$$
\begin{aligned}
& {[X(t \vee \eta)-X(\eta) \in U] \cap[\eta \leq t] } \\
= & {[X(t \vee \eta)-X(\eta) \in U] \cap[t \vee \eta \leq t] \in \mathscr{F}_{t} }
\end{aligned}
$$

from the definition of $\mathscr{F}_{t \vee \eta}$. (You know that $[X(t \vee \eta)-X(\eta) \in U] \in \mathscr{F}_{t \vee \eta}$ and so when this is intersected with $[t \vee \eta \leq t]$ one obtains a set in $\mathscr{F}_{t}$. This is what it means to be in $\mathscr{F}_{t \vee \eta}$.) Now $\tau$ is just the first hitting time of $Y(t)$ of the closed set.

Proposition 63.2.5 Let $M(t)$ be a continuous local martingale for $t \in[0, T]$ having values in $H$ a separable Hilbert space adapted to the normal filtration $\left\{\mathscr{F}_{t}\right\}$ such that $M(0)=0$. Then there exists a unique continuous, increasing, nonnegative, local submartingale $[M](t)$ called the quadratic variation such that

$$
\|M(t)\|^{2}-[M](t)
$$

is a real local martingale and $[M](0)=0$. Here $t \in[0, T]$. If $\delta$ is any stopping time

$$
\left[M^{\delta}\right]=[M]^{\delta}
$$

Proof: First it is necessary to define some stopping times. Define stopping times $\tau_{0}^{n} \equiv \eta_{0}^{n} \equiv 0$.

$$
\begin{aligned}
\eta_{k+1}^{n} & \equiv \inf \left\{s>\eta_{k}^{n}:\left\|M(s)-M\left(\eta_{k}^{n}\right)\right\|=2^{-n}\right\} \\
\tau_{k}^{n} & \equiv \eta_{k}^{n} \wedge T
\end{aligned}
$$

where $\inf \emptyset \equiv \infty$. These are stopping times by Example 62.7.10 on Page 2083. See also Lemma 63.2.4. Then for $t>0$ and $\delta$ any stopping time, and fixed $\omega$, for some $k$,

$$
t \wedge \delta \in I_{k}(\omega), I_{0}(\omega) \equiv\left[\tau_{0}^{n}(\omega), \tau_{1}^{n}(\omega)\right], I_{k}(\omega) \equiv\left(\tau_{k}^{n}(\omega), \tau_{k+1}^{n}(\omega)\right] \text { some } k
$$

Here is why. The sequence $\left\{\tau_{k}^{n}(\omega)\right\}_{k=1}^{\infty}$ eventually equals $T$ for all $n$ sufficiently large. This is because if it did not, it would converge, being bounded above by $T$ and then by continuity of $M,\left\{M\left(\tau_{k}^{n}(\omega)\right)\right\}_{k=1}^{\infty}$ would be a Cauchy sequence contrary to the requirement that

$$
\begin{aligned}
& \left\|M\left(\tau_{k+1}^{n}(\omega)\right)-M\left(\tau_{k}^{n}(\omega)\right)\right\| \\
= & \left\|M\left(\eta_{k+1}^{n}(\omega)\right)-M\left(\eta_{k}^{n}(\omega)\right)\right\|=2^{-n} .
\end{aligned}
$$

Note that if $\delta$ is any stopping time, then

$$
\begin{aligned}
& \left\|M\left(t \wedge \delta \wedge \tau_{k+1}^{n}\right)-M\left(t \wedge \delta \wedge \tau_{k}^{n}\right)\right\| \\
= & \left\|M^{\delta}\left(t \wedge \tau_{k+1}^{n}\right)-M^{\delta}\left(t \wedge \tau_{k}^{n}\right)\right\| \leq 2^{-n}
\end{aligned}
$$

You can see this is the case by considering the cases, $t \wedge \delta \geq \tau_{k+1}^{n}, t \wedge \delta \in\left[\tau_{k}^{n}, \tau_{k+1}^{n}\right)$, and $t \wedge \delta<\tau_{k}^{n}$. It is only this approximation property and the fact that the $\tau_{k}^{n}$ partition $[0, T]$ which is important in the following argument.

Now let $\alpha_{n}$ be a localizing sequence such that $M^{\alpha_{n}}$ is bounded as in Proposition 63.2.2. Thus $M^{\alpha_{n}}(t) \in L^{2}(\Omega)$ and this is all that is needed. In what follows, let $\delta$ be a stopping time and denote $M^{\alpha_{p} \wedge \delta}$ by $M$ to save notation. Thus $M$ will be uniformly bounded and from the definition of the stopping times $\tau_{k}^{n}$, for $t \in[0, T]$,

$$
\begin{equation*}
M(t) \equiv \sum_{k \geq 0} M\left(t \wedge \tau_{k+1}^{n}\right)-M\left(t \wedge \tau_{k}^{n}\right) \tag{63.2.4}
\end{equation*}
$$

and the terms of the series are eventually 0 , as soon as $\eta_{k}^{n}=\infty$.
Therefore,

$$
\|M(t)\|^{2}=\left\|\sum_{k \geq 0} M\left(t \wedge \tau_{k+1}^{n}\right)-M\left(t \wedge \tau_{k}^{n}\right)\right\|^{2}
$$

Then this equals

$$
\begin{gather*}
=\sum_{k \geq 0}\left\|M\left(t \wedge \tau_{k+1}^{n}\right)-M\left(t \wedge \tau_{k}^{n}\right)\right\|^{2} \\
+\sum_{j \neq k}\left(\left(M\left(t \wedge \tau_{k+1}^{n}\right)-M\left(t \wedge \tau_{k}^{n}\right)\right),\left(M\left(t \wedge \tau_{j+1}^{n}\right)-M\left(t \wedge \tau_{j}^{n}\right)\right)\right) \tag{63.2.5}
\end{gather*}
$$

Consider the second sum. It equals

$$
\begin{aligned}
& 2 \sum_{k \geq 0} \sum_{j=0}^{k-1}\left(\left(M\left(t \wedge \tau_{k+1}^{n}\right)-M\left(t \wedge \tau_{k}^{n}\right)\right),\left(M\left(t \wedge \tau_{j+1}^{n}\right)-M\left(t \wedge \tau_{j}^{n}\right)\right)\right) \\
= & 2 \sum_{k \geq 0}\left(\left(M\left(t \wedge \tau_{k+1}^{n}\right)-M\left(t \wedge \tau_{k}^{n}\right)\right), \sum_{j=0}^{k-1}\left(M\left(t \wedge \tau_{j+1}^{n}\right)-M\left(t \wedge \tau_{j}^{n}\right)\right)\right) \\
= & 2 \sum_{k \geq 0}\left(\left(M\left(t \wedge \tau_{k+1}^{n}\right)-M\left(t \wedge \tau_{k}^{n}\right)\right), M\left(t \wedge \tau_{k}^{n}\right)\right)
\end{aligned}
$$

This last sum equals $P_{n}(t)$ defined as

$$
\begin{equation*}
2 \sum_{k \geq 0}\left(M\left(\tau_{k}^{n}\right),\left(M\left(t \wedge \tau_{k+1}^{n}\right)-M\left(t \wedge \tau_{k}^{n}\right)\right)\right) \equiv P_{n}(t) \tag{63.2.6}
\end{equation*}
$$

This is because in the $k^{t h}$ term, if $t \geq \tau_{k}^{n}$, then it reduces to

$$
\left(M\left(\tau_{k}^{n}\right),\left(M\left(t \wedge \tau_{k+1}^{n}\right)-M\left(t \wedge \tau_{k}^{n}\right)\right)\right)
$$

while if $t<\tau_{k}^{n}$, then the term reduces to 0 which is also the same as

$$
\left(M\left(\tau_{k}^{n}\right),\left(M\left(t \wedge \tau_{k+1}^{n}\right)-M\left(t \wedge \tau_{k}^{n}\right)\right)\right)
$$

This is a finite sum because eventually, for large enough $k, \tau_{k}^{n}=T$. However the number of nonzero terms depends on $\omega$. This is not a good thing. However, a little more can be said. In fact the sum also converges in $L^{2}(\Omega)$. Say $\|M(t, \omega)\| \leq C$.

$$
\begin{align*}
& E\left(\left(\sum_{k \geq p}^{q}\left(M\left(\tau_{k}^{n}\right),\left(M\left(t \wedge \tau_{k+1}^{n}\right)-M\left(t \wedge \tau_{k}^{n}\right)\right)\right)\right)^{2}\right) \\
= & \sum_{k \geq p}^{q} E\left(\left(M\left(\tau_{k}^{n}\right),\left(M\left(t \wedge \tau_{k+1}^{n}\right)-M\left(t \wedge \tau_{k}^{n}\right)\right)\right)^{2}\right)+\text { mixed terms } \tag{63.2.7}
\end{align*}
$$

Consider one of these mixed terms for $j<k$.

$$
\begin{aligned}
& E((M\left(\tau_{j}^{n}\right),(\overbrace{M\left(t \wedge \tau_{j+1}^{n}\right)-M\left(t \wedge \tau_{j}^{n}\right)}^{\Delta_{j}})) \\
& (M\left(\tau_{k}^{n}\right),(\overbrace{M\left(t \wedge \tau_{k+1}^{n}\right)-M\left(t \wedge \tau_{k}^{n}\right)}^{\Delta_{k}})))
\end{aligned}
$$

Then it equals

$$
\begin{aligned}
& E\left(E\left(\left(M\left(\tau_{j}^{n}\right), \Delta_{j}\right)\left(M\left(\tau_{j}^{n}\right), \Delta_{k}\right) \mid \mathscr{F}_{\tau_{k}}\right)\right) \\
= & E\left(\left(M\left(\tau_{j}^{n}\right), \Delta_{j}\right) E\left(\left(M\left(\tau_{j}^{n}\right), \Delta_{k}\right) \mid \mathscr{F}_{\tau_{k}}\right)\right) \\
= & E\left(\left(M\left(\tau_{j}^{n}\right), \Delta_{j}\right)\left(M\left(\tau_{j}^{n}\right), E\left(\Delta_{k} \mid \mathscr{F}_{\tau_{k}}\right)\right)\right)=0
\end{aligned}
$$

Now since the mixed terms equal 0 , it follows from 63.2.7, that expression is dominated by

$$
C^{2} \sum_{k \geq p}^{q} E\left(\left\|M\left(t \wedge \tau_{k+1}^{n}\right)-M\left(t \wedge \tau_{k}^{n}\right)\right\|^{2}\right)
$$

Using a similar manipulation to what was just done to show the mixed terms equal 0 , this equals

$$
\begin{aligned}
& C^{2} \sum_{k=p}^{q} E\left(\left\|M\left(t \wedge \tau_{k+1}^{n}\right)\right\|^{2}\right)-E\left(\left\|M\left(t \wedge \tau_{k}^{n}\right)\right\|^{2}\right) \\
\leq & C^{2} E\left(\left\|M\left(t \wedge \tau_{q+1}^{n}\right)\right\|^{2}-\left\|M\left(t \wedge \tau_{p}^{n}\right)\right\|^{2}\right)
\end{aligned}
$$

The integrand converges to 0 as $p, q \rightarrow \infty$ and the uniform bound on $M$ allows a use of the dominated convergence theorem. Thus the partial sums of the series of 63.2 .6 converge in $L^{2}(\Omega)$ as claimed.

By adding in the values of $\left\{\tau_{k}^{n+1}\right\} P_{n}(t)$ can be written in the form

$$
2 \sum_{k \geq 0}\left(M\left(\tau_{k}^{n+1 \prime}\right),\left(M\left(t \wedge \tau_{k+1}^{n+1}\right)-M\left(t \wedge \tau_{k}^{n+1}\right)\right)\right)
$$

where $\tau_{k}^{n+1 \prime}$ has some repeats. From the construction,

$$
\left\|M\left(\tau_{k}^{n+1 \prime}\right)-M\left(\tau_{k}^{n+1}\right)\right\| \leq 2^{-(n+1)}
$$

Thus

$$
P_{n}(t)-P_{n+1}(t)=2 \sum_{k \geq 0}\left(M\left(\tau_{k}^{n+1 \prime}\right)-M\left(\tau_{k}^{n+1}\right),\left(M\left(t \wedge \tau_{k+1}^{n+1}\right)-M\left(t \wedge \tau_{k}^{n+1}\right)\right)\right)
$$

and so from Proposition 63.1.4 applied to $\xi_{k} \equiv M\left(\tau_{k}^{n+1 \prime}\right)-M\left(\tau_{k}^{n+1}\right)$,

$$
\begin{equation*}
E\left(\left\|P_{n}(t)-P_{n+1}(t)\right\|^{2}\right) \leq\left(2^{-2 n} E\left(\|M(t)\|^{2}\right)\right) \tag{63.2.8}
\end{equation*}
$$

Now $t \rightarrow P_{n}(t)$ is continuous because it is a finite sum of continuous functions. It is also the case that $\left\{P_{n}(t)\right\}$ is a martingale. To see this use Lemma 63.1.1. Let $\sigma$ be a stopping time having two values. Then using Corollary 63.1.3 and the Doob optional sampling theorem, Theorem 62.7.14

$$
\begin{aligned}
& E\left(\sum_{k=0}^{q}\left(M\left(\tau_{k}^{n}\right),\left(M\left(\sigma \wedge \tau_{k+1}^{n}\right)-M\left(\sigma \wedge \tau_{k}^{n}\right)\right)\right)\right) \\
= & \sum_{k=0}^{q} E\left(\left(M\left(\tau_{k}^{n}\right),\left(M\left(\sigma \wedge \tau_{k+1}^{n}\right)-M\left(\sigma \wedge \tau_{k}^{n}\right)\right)\right)\right) \\
= & \sum_{k=0}^{q} E\left(\left(E\left(M\left(\tau_{k}^{n}\right),\left(M\left(\sigma \wedge \tau_{k+1}^{n}\right)-M\left(\sigma \wedge \tau_{k}^{n}\right)\right)\right) \mid \mathscr{F}_{\tau_{k}^{n}}\right)\right) \\
= & \sum_{k=0}^{q} E\left(\left(M\left(\tau_{k}^{n}\right), E\left(M\left(\sigma \wedge \tau_{k+1}^{n}\right)-M\left(\sigma \wedge \tau_{k}^{n}\right)\right) \mid \mathscr{F}_{\tau_{k}^{n}}\right)\right) \\
= & \sum_{k=0}^{q} E\left(\left(M\left(\tau_{k}^{n}\right), E\left(M\left(\sigma \wedge \tau_{k+1}^{n} \wedge \tau_{k}^{n}\right)-M\left(\sigma \wedge \tau_{k}^{n}\right)\right)\right)\right)=0
\end{aligned}
$$

Note the Doob theorem applies because $\sigma \wedge \tau_{k+1}^{n}$ is a bounded stopping time due to the fact $\sigma$ has only two values. Similarly

$$
\begin{aligned}
& E\left(\sum_{k=0}^{q}\left(M\left(\tau_{k}^{n}\right),\left(M\left(t \wedge \tau_{k+1}^{n}\right)-M\left(t \wedge \tau_{k}^{n}\right)\right)\right)\right) \\
= & \sum_{k=0}^{q} E\left(\left(M\left(\tau_{k}^{n}\right),\left(M\left(t \wedge \tau_{k+1}^{n}\right)-M\left(t \wedge \tau_{k}^{n}\right)\right)\right)\right) \\
= & \sum_{k=0}^{q} E\left(\left(E\left(M\left(\tau_{k}^{n}\right),\left(M\left(t \wedge \tau_{k+1}^{n}\right)-M\left(t \wedge \tau_{k}^{n}\right)\right)\right) \mid \mathscr{F}_{\tau_{k}^{n}}\right)\right) \\
= & \sum_{k=0}^{q} E\left(\left(M\left(\tau_{k}^{n}\right), E\left(M\left(t \wedge \tau_{k+1}^{n}\right)-M\left(t \wedge \tau_{k}^{n}\right)\right) \mid \mathscr{F}_{\tau_{k}^{n}}\right)\right) \\
= & \sum_{k=0}^{q} E\left(\left(M\left(\tau_{k}^{n}\right), E\left(M\left(t \wedge \tau_{k+1}^{n} \wedge \tau_{k}^{n}\right)-M\left(t \wedge \tau_{k}^{n}\right)\right)\right)\right)=0
\end{aligned}
$$

It follows each partial sum for $P_{n}(t)$ is a martingale. As shown above, these partial sums converge in $L^{2}(\Omega)$ and so it follows that $P_{n}(t)$ is also a martingale. Note the Doob theorem applies because $t \wedge \tau_{k+1}^{n}$ is a bounded stopping time.

I want to argue that $P_{n}$ is a Cauchy sequence in $\mathscr{M}_{T}^{2}(\mathbb{R})$. By Theorem 62.9.4 and continuity of $P_{n}$

$$
E\left(\left(\sup _{t \leq T}\left|P_{n}(t)-P_{n+1}(t)\right|\right)^{2}\right)^{1 / 2} \leq 2 E\left(\left|P_{n}(T)-P_{n+1}(T)\right|^{2}\right)^{1 / 2}
$$

By 63.2.8,

$$
\leq 2^{-n} E\left(\|M(T)\|^{2}\right)^{1 / 2}
$$

which shows $\left\{P_{n}\right\}$ is indeed a Cauchy sequence in $\mathscr{M}_{T}^{2}(\mathbb{R})$.
Therefore, by Proposition 62.12.2, there exists $\{N(t)\} \in \mathscr{M}_{T}^{2}(\mathbb{R})$ such that $P_{n} \rightarrow N$ in $\mathscr{M}_{T}^{2}(H)$. That is

$$
\lim _{n \rightarrow \infty} E\left(\sup _{t \in[0, T]}\left|P_{n}(t)-N(t)\right|^{2}\right)^{1 / 2}=0
$$

Since $\{N(t)\} \in \mathscr{M}_{T}^{2}(\mathbb{R})$, it is a continuous martingale and $N(t) \in L^{2}(\Omega)$, and $N(0)=0$ because this is true of each $P_{n}(0)$. From the above 63.2.5,

$$
\begin{equation*}
\|M(t)\|^{2}=Q_{n}(t)+P_{n}(t) \tag{63.2.9}
\end{equation*}
$$

where

$$
Q_{n}(t)=\sum_{k \geq 0}\left\|M\left(t \wedge \tau_{k+1}^{n}\right)-M\left(t \wedge \tau_{k}^{n}\right)\right\|^{2}
$$

and $P_{n}(t)$ is a martingale. Then from 63.2.9, $Q_{n}(t)$ is a submartingale and converges for each $t$ to something, denoted as $[M](t)$ in $L^{1}(\Omega)$ uniformly in $t \in[0, T]$. This is because $P_{n}(t)$ converges uniformly on $[0, T]$ to $N(t)$ in $L^{2}(\Omega)$ and $\|M(t)\|^{2}$ does not depend on $n$. Then also $[M]$ is a submartingale which equals 0 at 0 because this is true of $Q_{n}$ and because if $A \in \mathscr{F}_{s}$ where $s<t$,

$$
\begin{gathered}
\int_{A} E\left([M](t) \mid \mathscr{F}_{s}\right) d P \equiv \int_{A}[M](t) d P=\lim _{n \rightarrow \infty} \int_{A}\left(\|M(t)\|^{2}-P_{n}(t)\right) d P \\
=\lim _{n \rightarrow \infty} \int_{A} E\left(\|M(t)\|^{2}-P_{n}(t) \mid \mathscr{F}_{s}\right) d P \geq \lim _{n \rightarrow \infty} \inf _{A}\|M(s)\|^{2}-P_{n}(s) d P \\
=\lim _{n \rightarrow \infty} \inf _{A} Q_{n}(s) d P=\int_{A}[M](s) d P .
\end{gathered}
$$

Note that $Q_{n}(t)$ is increasing because as $t$ increases, the definition allows for the possibility of more nonzero terms in the sum. Therefore, $[M](t)$ is also increasing in $t$. The function $t \rightarrow[M](t)$ is continuous because $\|M(t)\|^{2}=[M](t)+N(t)$ and $t \rightarrow N(t)$ is continuous as is $t \rightarrow\|M(t)\|^{2}$. That is, off a set of measure zero, these are both continuous functions of $t$ and so the same is true of $[M]$.

Now put back in $M^{\alpha_{p} \wedge \delta}$ in place of $M$. From the above, this has shown

$$
\left\|M^{\alpha_{p} \wedge \delta}(t)\right\|^{2}=\left[M^{\alpha_{p} \wedge \delta}\right](t)+N_{p}(t)
$$

where $N_{p}$ is a martingale and

$$
\begin{array}{r}
{\left[M^{\alpha_{p} \wedge \delta}\right](t)=\lim _{n \rightarrow \infty} \sum_{k \geq 0}\left\|M^{\alpha_{p} \wedge \delta}\left(t \wedge \tau_{k+1}^{n}\right)-M^{\alpha_{p} \wedge \delta}\left(t \wedge \tau_{k}^{n}\right)\right\|^{2}} \\
=\lim _{n \rightarrow \infty} \sum_{k \geq 0}\left\|M\left(t \wedge \tau_{k+1}^{n} \wedge \alpha_{p} \wedge \delta\right)-M\left(t \wedge \tau_{k}^{n} \wedge \alpha_{p} \wedge \delta\right)\right\|^{2} \text { in } L^{1}(\Omega), \tag{63.2.10}
\end{array}
$$

the convergence being uniform on $[0, T]$. The above formula shows that $\left[M^{\alpha_{p} \wedge \delta}\right](t)$ is a $\mathscr{F}_{t \wedge \delta \wedge \alpha_{p}}$ measurable random variable which depends on $t \wedge \delta \wedge \alpha_{p}$. (Note that $t \wedge \delta$ is a real valued stopping time even if $\delta=\infty$.) Therefore, by Proposition 63.2.2, there exists a random variable, denoted as $\left[M^{\delta}\right](t)$ which is the pointwise limit as $p \rightarrow \infty$ of these random variables which is $\mathscr{F}_{t \wedge \delta}$ measurable because, for a given $\omega$, when $\alpha_{p}$ becomes larger than $t$, the sum in 63.2.10 loses its dependence on $p$. Thus from pointwise convergence in 63.2.10,

$$
\left[M^{\delta}\right](t) \equiv \lim _{n \rightarrow \infty} \sum_{k \geq 0}\left\|M\left(t \wedge \delta \wedge \tau_{k+1}^{n}\right)-M\left(t \wedge \delta \wedge \tau_{k}^{n}\right)\right\|^{2}
$$

In case $\delta=\infty$, the above gives an $\mathscr{F}_{t}$ measurable random variable denoted by $[M](t)$ such that

$$
[M](t) \equiv \lim _{n \rightarrow \infty} \sum_{k \geq 0}\left\|M\left(t \wedge \tau_{k+1}^{n}\right)-M\left(t \wedge \tau_{k}^{n}\right)\right\|^{2}
$$

Now stopping with the stopping time $\delta$, this shows that

$$
\left[M^{\delta}\right](t) \equiv \lim _{n \rightarrow \infty} \sum_{k \geq 0}\left\|M\left(t \wedge \delta \wedge \tau_{k+1}^{n}\right)-M\left(t \wedge \delta \wedge \tau_{k}^{n}\right)\right\|^{2}=[M]^{\delta}(t)
$$

That is, the quadratic variation of the stopped local martingale makes sense a.e. and equals the stopped quadratic variation of the local martingale.

This has now shown that

$$
\begin{aligned}
\left\|M^{\alpha_{n}}(t)\right\|^{2}-[M]^{\alpha_{n}}(t) & =\left\|M^{\alpha_{n}}(t)\right\|^{2}-\left[M^{\alpha_{n}}\right](t) \\
& =N_{n}(t), N_{n}(t) \text { a martingale }
\end{aligned}
$$

and both of the random variables on the left converge pointwise as $n \rightarrow \infty$ to a function which is $\mathscr{F}_{t}$ measurable. Hence so does $N_{n}(t)$. Of course $N_{n}(t)$ is likewise a function of $\alpha_{n} \wedge t$ and so by Proposition 63.2.2 again, it converges pointwise to a $\mathscr{F}_{t}$ measurable function called $N(t)$ and $N(t)$ is a continuous local martingale.

It remains to consider the claim about the uniqueness. Suppose then there are two which work, $[M]$, and $[M]_{1}$. Then $[M]-[M]_{1}$ equals a local martingale $G$ which is 0 when $t=0$. Thus the uniqueness assertion follows from Corollary 63.2.3.

Here is a corollary which tells how to manipulate stopping times. It is contained in the above proposition, but it is worth emphasizing it from a different point of view.

Corollary 63.2.6 In the situation of Proposition 63.2 .5 let $\tau$ be a stopping time. Then

$$
\left[M^{\tau}\right]=[M]^{\tau} .
$$

Proof:

$$
[M]^{\tau}(t)+N_{1}(t)=\left(\|M\|^{2}\right)^{\tau}(t)=\left\|M^{\tau}\right\|^{2}(t)=\left[M^{\tau}\right](t)+N_{2}(t)
$$

where $N_{i}$ is a local martingale. Therefore,

$$
[M]^{\tau}(t)-\left[M^{\tau}\right](t)=N_{2}(t)-N_{1}(t),
$$

a local martingale. Therefore, by Corollary 63.2.3, this shows $[M]^{\tau}(t)-\left[M^{\tau}\right](t)=0$.

### 63.3 The Covariation

Definition 63.3.1 The covariation of two continuous $H$ valued local martingales for $H$ a separable Hilbert space $M, N, M(0)=0=N(0)$, is defined as follows.

$$
[M, N] \equiv \frac{1}{4}([M+N]-[M-N])
$$

Lemma 63.3.2 The following hold for the covariation.

$$
\begin{aligned}
& {[M]=[M, M] } \\
& {[M, N]=} \text { local martingale }+\frac{1}{4}\left(\|M+N\|^{2}-\|M-N\|^{2}\right) \\
&=(M, N)+\text { local martingale. }
\end{aligned}
$$

Proof: From the definition of covariation,

$$
\begin{gathered}
{[M]=\|M\|^{2}-\mathscr{N}_{1}} \\
{[M, M]=\frac{1}{4}([M+M]-[M-M])=\frac{1}{4}\left(\|M+M\|^{2}-\mathscr{N}_{2}\right)} \\
=\|M\|^{2}-\frac{1}{4} \mathscr{N}_{2}
\end{gathered}
$$

where $\mathscr{N}_{i}$ is a local martingale. Thus $[M]-[M, M]$ is equal to the difference of two increasing continuous adapted processes and it also equals a local martingale. By Corollary 63.2 .3 , this process must equal 0 . Now consider the second claim.

$$
\begin{aligned}
{[M, N] } & =\frac{1}{4}([M+N]-[M-N])=\frac{1}{4}\left(\|M+N\|^{2}-\|M-N\|^{2}+\mathscr{N}\right) \\
& =(M, N)+\frac{1}{4} \mathscr{N}
\end{aligned}
$$

where $\mathscr{N}$ is a local martingale.
Corollary 63.3.3 Let $M, N$ be two continuous local martingales, $M(0)=N(0)=0$, as in Proposition 63.2.5. Then $[M, N]$ is of bounded variation and

$$
(M, N)_{H}-[M, N]
$$

is a local martingale. Also for $\tau$ a stopping time,

$$
[M, N]^{\tau}=\left[M^{\tau}, N^{\tau}\right]=\left[M^{\tau}, N\right]=\left[M, N^{\tau}\right] .
$$

In addition to this,

$$
\left[M-M^{\tau}\right]=[M]-\left[M^{\tau}\right] \leq[M]
$$

and also

$$
(M, N) \rightarrow[M, N]
$$

is bilinear and symmetric.
Proof: Since $[M, N]$ is the difference of increasing functions, it is of bounded variation.

$$
\begin{aligned}
(M, N)_{H}-[M, N]= & \overbrace{\frac{1}{4}\left(\|M+N\|^{2}-\|M-N\|^{2}\right)}^{(M, N)_{H}} \\
& \overbrace{-\frac{1}{4}([M+N]-[M-N])}^{[M, N]}
\end{aligned}
$$

which equals a local martingale from the definition of $[M+N]$ and $[M-N]$. It remains to verify the claim about the stopping time. Using Corollary 63.2.6

$$
[M, N]^{\tau}=\frac{1}{4}([M+N]-[M-N])^{\tau}
$$

$$
\begin{aligned}
& =\frac{1}{4}\left([M+N]^{\tau}-[M-N]^{\tau}\right) \\
& =\frac{1}{4}\left(\left[M^{\tau}+N^{\tau}\right]-\left[M^{\tau}-N^{\tau}\right]\right) \equiv\left[M^{\tau}, N^{\tau}\right] .
\end{aligned}
$$

The really interesting part is the next equality. This will involve Corollary 63.2.3.

$$
\begin{gather*}
{[M, N]^{\tau}-\left[M^{\tau}, N\right]=\left[M^{\tau}, N^{\tau}\right]-\left[M^{\tau}, N\right]} \\
\equiv \frac{1}{4}\left(\left[M^{\tau}+N^{\tau}\right]-\left[M^{\tau}-N^{\tau}\right]\right)-\frac{1}{4}\left(\left[M^{\tau}+N\right]-\left[M^{\tau}-N\right]\right) \\
=\frac{1}{4}\left(\left[M^{\tau}+N^{\tau}\right]+\left[M^{\tau}-N\right]\right)-\frac{1}{4}\left(\left[M^{\tau}+N\right]+\left[M^{\tau}-N^{\tau}\right]\right), \tag{63.3.11}
\end{gather*}
$$

the difference of two increasing adapted processes. Also, this equals

$$
\text { local martingale }-\left(M^{\tau}, N\right)+\left(M^{\tau}, N^{\tau}\right)
$$

Claim: $\left(M^{\tau}, N\right)-\left(M^{\tau}, N^{\tau}\right)=\left(M^{\tau}, N-N^{\tau}\right)$ is a local martingale. Let $\sigma_{n}$ be a localizing sequence for both $M$ and $M$. Such a localizing sequence is of the form $\tau_{n}^{M} \wedge \tau_{n}^{N}$ where these are localizing sequences for the indicated local submartingale. Then obviously,

$$
\left(-\left(M^{\tau}, N\right)+\left(M^{\tau}, N^{\tau}\right)\right)^{\sigma_{n}}=-\left(M^{\sigma_{n} \wedge \tau}, N^{\sigma_{n}}\right)+\left(M^{\sigma_{n} \wedge \tau}, N^{\sigma_{n} \wedge \tau}\right)
$$

where $N^{\sigma_{n}}$ and $M^{\sigma_{n}}$ are martingales. To save notation, denote these by $M$ and $N$ respectively. Now use Lemma 63.1.1. Let $\sigma$ be a stopping time with two values.

$$
E\left(\left(M^{\tau}(\sigma), N(\sigma)-N^{\tau}(\sigma)\right)\right)=E\left(E\left(\left(M^{\tau}(\sigma), N(\sigma)-N^{\tau}(\sigma)\right) \mid \mathscr{F}_{\tau}\right)\right)
$$

Now $M^{\tau}(\sigma)$ is $M(\sigma \wedge \tau)$ which is $\mathscr{F}_{\tau}$ measurable and so by the Doob optional sampling theorem,

$$
\begin{gathered}
=E\left(M^{\tau}(\sigma), E\left(N(\sigma)-N^{\tau}(\sigma) \mid \mathscr{F}_{\tau}\right)\right) \\
=E\left(M^{\tau}(\sigma), N(\sigma \wedge \tau)-N(\tau \wedge \sigma)\right)=0
\end{gathered}
$$

while

$$
E\left(\left(M^{\tau}(t), N(t)-N^{\tau}(t)\right)\right)=E\left(E\left(\left(M^{\tau}(t), N(t)-N^{\tau}(t)\right) \mid \mathscr{F}_{\tau}\right)\right)
$$

Since $M^{\tau}(t)$ is $\mathscr{F}_{\tau}$ measurable,

$$
\begin{aligned}
& =E\left(\left(M^{\tau}(t), E\left(N(t)-N^{\tau}(t) \mid \mathscr{F}_{\tau}\right)\right)\right) \\
& =E\left(\left(M^{\tau}(t), E(N(t \wedge \tau)-N(t \wedge \tau))\right)\right)=0
\end{aligned}
$$

This shows the claim is true.
Now from 63.3.11 and Corollary 63.3.3,

$$
[M, N]^{\tau}-\left[M^{\tau}, N\right]=0 .
$$

Similarly

$$
[M, N]^{\tau}-\left[M, N^{\tau}\right]=0
$$

Now consider the next claim that $\left[M-M^{\tau}\right]=[M]-\left[M^{\tau}\right]$. From the definition, it follows

$$
\begin{aligned}
& {\left[M-M^{\tau}\right]-\left([M]+\left[M^{\tau}\right]-2\left[M, M^{\tau}\right]\right) } \\
= & \left\|M-M^{\tau}\right\|^{2}-\left(\|M\|^{2}+\left\|M^{\tau}\right\|^{2}-2\left(M, M^{\tau}\right)\right)+\text { local martingale } \\
= & \text { local martingale. }
\end{aligned}
$$

By the first part of the corollary which ensures $\left[M, M^{\tau}\right]$ is of bounded variation, the left side is the difference of two increasing adapted processes and so by Corollary 63.2.3 again, the left side equals 0 . Thus from the above,

$$
\begin{aligned}
{\left[M-M^{\tau}\right] } & =[M]+\left[M^{\tau}\right]-2\left[M, M^{\tau}\right] \\
& =[M]+\left[M^{\tau}\right]-2\left[M^{\tau}, M^{\tau}\right] \\
& =[M]+\left[M^{\tau}\right]-2\left[M^{\tau}\right] \\
& =[M]-\left[M^{\tau}\right] \leq[M]
\end{aligned}
$$

Finally consider the claim that $[M, N]$ is bilinear. From the definition, letting $M_{1}, M_{2}, N$ be $H$ valued local martingales,

$$
\begin{aligned}
\left(a M_{1}+b M_{2}, N\right)_{H} & =\left[a M_{1}+b M_{2}, N\right]+\text { local martingale } \\
a\left(M_{1}, N\right)+b\left(M_{2}, N\right)_{H} & =a\left[M_{1}, N\right]+b\left[M_{2}, N\right]+\text { local martingale }
\end{aligned}
$$

Hence

$$
\left[a M_{1}+b M_{2}, N\right]-\left(a\left[M_{1}, N\right]+b\left[M_{2}, N\right]\right)=\text { local martingale. }
$$

The left side can be written as the difference of two increasing functions thanks to $[M, N]$ of bounded variation and so by Lemma 63.1.5 it equals $0 .[M, N]$ is obviously symmetric from the definition.

### 63.4 The Burkholder Davis Gundy Inequality

Define

$$
M^{*}(\omega) \equiv \sup \{\|M(t)(\omega)\|: t \in[0, T]\}
$$

The Burkholder Davis Gundy inequality is an amazing inequality which involves $M^{*}$ and $[M](T)$.

Before presenting this, here is the good lambda inequality, Theorem 12.7.1 on Page 299 listed here for convenience.

Theorem 63.4.1 Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space and let $F$ be a continuous increasing function defined on $[0, \infty)$ such that $F(0)=0$. Suppose also that for all $\alpha>1$, there exists a constant $C_{\alpha}$ such that for all $x \in[0, \infty)$,

$$
F(\alpha x) \leq C_{\alpha} F(x)
$$

Also suppose $f, g$ are nonnegative measurable functions and there exists $\beta>1,0<r \leq 1$, such that for all $\lambda>0$ and $1>\delta>0$,

$$
\begin{equation*}
\mu([f>\beta \lambda] \cap[g \leq r \delta \lambda]) \leq \phi(\delta) \mu([f>\lambda]) \tag{63.4.12}
\end{equation*}
$$

where $\lim _{\delta \rightarrow 0+} \phi(\delta)=0$ and $\phi$ is increasing. Under these conditions, there exists a constant $C$ depending only on $\beta, \phi, r$ such that

$$
\int_{\Omega} F(f(\omega)) d \mu(\omega) \leq C \int_{\Omega} F(g(\omega)) d \mu(\omega)
$$

The proof of this important inequality also will depend on the hitting this before that theorem which is listed next for convenience.

Theorem 63.4.2 Let $\{M(t)\}$ be a continuous real valued martingale adapted to the normal filtration $\mathscr{F}_{t}$ and let

$$
\begin{aligned}
& M^{*} \equiv \sup \{|M(t)|: t \geq 0\} \\
& \tau_{x} \equiv \inf \{t>0: M(t)=x\}
\end{aligned}
$$

and $M(0)=0$. Letting

Then if $a<0<b$ the following inequalities hold.

$$
(b-a) P\left(\left[\tau_{b} \leq \tau_{a}\right]\right) \geq-a P\left(\left[M^{*}>0\right]\right) \geq(b-a) P\left(\left[\tau_{b}<\tau_{a}\right]\right)
$$

and

$$
(b-a) P\left(\left[\tau_{a}<\tau_{b}\right]\right) \leq b P\left(\left[M^{*}>0\right]\right) \leq(b-a) P\left(\left[\tau_{a} \leq \tau_{b}\right]\right)
$$

In words, $P\left(\left[\tau_{b} \leq \tau_{a}\right]\right)$ is the probability that $M(t)$ hits $b$ no later than when it hits $a$. (Note that if $\tau_{a}=\infty=\tau_{b}$ then you would have $\left[\tau_{a}=\tau_{b}\right]$.)

Then the Burkholder Davis Gundy inequality is as follows. Generalizations will be presented later.

Theorem 63.4.3 Let $\{M(t)\}$ be a continuous $H$ valued martingale which is uniformly bounded, $M(0)=0$, where $H$ is a separable Hilbert space and $t \in[0, T]$. Then if $F$ is a function of the sort described in the good lambda inequality above, there are constants, $C$ and $c$ independent of such martingales $M$ such that

$$
c \int_{\Omega} F\left(([M](T))^{1 / 2}\right) d P \leq \int_{\Omega} F\left(M^{*}\right) d P \leq C \int_{\Omega} F\left(([M](T))^{1 / 2}\right) d P
$$

where

$$
M^{*}(\omega) \equiv \sup \{\|M(t)(\omega)\|: t \in[0, T]\}
$$

Proof: Using Corollary 63.3.3, let

$$
\begin{aligned}
N(t) & \equiv\left\|M(t)-M^{\tau}(t)\right\|^{2}-\left[M-M^{\tau}\right](t) \\
& =\left\|M(t)-M^{\tau}(t)\right\|^{2}-[M](t)+[M]^{\tau}(t)
\end{aligned}
$$

where

$$
\tau \equiv \inf \{t \in[0, T]:\|M(t)\|>\lambda\}
$$

Thus $N$ is a martingale and $N(0)=0$. In fact $N(t)=0$ as long as $t \leq \tau$. As usual $\inf (\emptyset) \equiv \infty$. Note

$$
[\tau<\infty]=\left[M^{*}>\lambda\right] \supseteq\left[N^{*}>0\right] .
$$

This is because to say $\tau<\infty$ is to say there exists $t<T$ such that $\|M(t)\|>\lambda$ which is the same as saying $M^{*}>\lambda$. Thus the first two sets are equal. If $\tau=\infty$, then from the formula for $N(t)$ above, $N(t)=0$ for all $t \in[0, T]$ and so it can't happen that $N^{*}>0$. Thus the third set is contained in $[\tau<\infty]$ as claimed.

Let $\beta>2$ and let $\delta \in(0,1)$. Then

$$
\beta-1>1>\delta>0
$$

Consider the following which is set up to use the good lambda inequality.

$$
S_{r} \equiv\left[M^{*}>\beta \lambda\right] \cap\left[([M](T))^{1 / 2} \leq r \delta \lambda\right]
$$

where $0<r<1$.It is shown that $S_{r}$ corresponds to hitting "this before that" and there is an estimate for this which involves $P\left(\left[N^{*}>0\right]\right)$ which is bounded above by $P\left(\left[M^{*}>\lambda\right]\right)$ as discussed above. This will satisfy the hypotheses of the good lambda inequality.

Claim: For $\omega \in S_{r}, N(t)$ hits $\lambda^{2}\left(1-\delta^{2}\right)$.
Proof of claim: For $\omega \in S_{r}$, there exists a $t<T$ such that $\|M(t)\|>\beta \lambda$ and so using Corollary 63.3.3,

$$
\begin{aligned}
N(t) & \geq\|M(t)\|-\left\|M^{\tau}(t)\right\| \|^{2}-\left[M-M^{\tau}\right](t) \geq|\beta \lambda-\lambda|^{2}-[M](t) \\
& \geq(\beta-1)^{2} \lambda^{2}-\delta^{2} \lambda^{2}
\end{aligned}
$$

which shows that $N(t)$ hits $(\beta-1)^{2} \lambda^{2}-\delta^{2} \lambda^{2}$ for $\omega \in S_{r}$. By the intermediate value theorem, it also hits $\lambda^{2}\left(1-\delta^{2}\right)$. This proves the claim.

Claim: $N(t)(\omega)$ never hits $-\delta^{2} \lambda^{2}$ for $\omega \in S_{r}$.
Proof of claim: Suppose $t$ is the first time $N(t)$ reaches $-\delta^{2} \lambda^{2}$. Then $t>\tau$ and so

$$
\begin{aligned}
N(t) & =-\delta^{2} \lambda^{2} \geq\| \| M(t) \|-\left.\lambda\right|^{2}-[M](t)+\left[M^{\tau}\right](t) \\
& \geq-r^{2} \lambda^{2} \delta^{2}
\end{aligned}
$$

a contradiction since $r<1$. This proves the claim.
Therefore, for all $\omega \in S_{r}, N(t)(\omega)$ reaches $\lambda^{2}\left(1-\delta^{2}\right)$ before it reaches $-\delta^{2} \lambda^{2}$. It follows

$$
P\left(S_{r}\right) \leq P\left(N(t) \text { reaches } \lambda^{2}\left(1-\delta^{2}\right) \text { before }-\delta^{2} \lambda^{2}\right)
$$

and because of Theorem 62.11 .3 this is no larger than

$$
P\left(\left[N^{*}>0\right]\right) \frac{\delta^{2} \lambda^{2}}{\lambda^{2}\left(1-\delta^{2}\right)-\left(-\delta^{2} \lambda^{2}\right)}=P\left(\left[N^{*}>0\right]\right) \delta^{2} \leq \delta^{2} P\left(\left[M^{*}>\lambda\right]\right)
$$

Thus

$$
P\left(\left[M^{*}>\beta \lambda\right] \cap\left[([M](T))^{1 / 2} \leq r \delta \lambda\right]\right) \leq P\left(\left[M^{*}>\lambda\right]\right) \delta^{2}
$$

By the good lambda inequality,

$$
\int_{\Omega} F\left(M^{*}\right) d P \leq C \int_{\Omega} F\left(([M](T))^{1 / 2}\right) d P
$$

which is one half the inequality.
Now consider the other half. This time define the stopping time $\tau$ by

$$
\tau \equiv \inf \left\{t \in[0, T]:([M](t))^{1 / 2}>\lambda\right\}
$$

and let

$$
S_{r} \equiv\left[([M](T))^{1 / 2}>\beta \lambda\right] \cap\left[2 M^{*} \leq r \delta \lambda\right]
$$

Then there exists $t<T$ such that $[M](t)>\beta^{2} \lambda^{2}$. This time, let

$$
N(t) \equiv[M](t)-\left[M^{\tau}\right](t)-\left\|M(t)-M^{\tau}(t)\right\|^{2}
$$

This is still a martingale since by Corollary 63.3.3

$$
[M](t)-\left[M^{\tau}\right](t)=\left[M-M^{\tau}\right](t)
$$

Claim: $N(t)(\omega)$ hits $\lambda^{2}\left(1-\delta^{2}\right)$ for some $t<T$ for $\omega \in S_{r}$.
Proof of claim: Fix such a $\omega \in S_{r}$. Let $t<T$ be such that $[M](t)>\beta^{2} \lambda^{2}$. Then $t>\tau$ and so for that $\omega$,

$$
\begin{aligned}
N(t) & >\beta^{2} \lambda^{2}-\lambda^{2}-\|M(t)-M(\tau)\|^{2} \\
& \geq(\beta-1)^{2} \lambda^{2}-(\|M(t)\|+\|M(\tau)\|)^{2} \\
& \geq(\beta-1)^{2} \lambda^{2}-r^{2} \delta^{2} \lambda^{2} \geq \lambda^{2}-\delta^{2} \lambda^{2}
\end{aligned}
$$

By the intermediate value theorem, it hits $\lambda^{2}\left(1-\delta^{2}\right)$. This proves the claim.
Claim: $N(t)(\omega)$ never hits $-\delta^{2} \lambda^{2}$ for $\omega \in S_{r}$.
Proof of claim: By Corollary 63.3.3, if it did at $t$, then $t>\tau$ because $N(t)=0$ for $t \leq \tau$, and so

$$
\begin{aligned}
0 & \leq[M](t)-\left[M^{\tau}\right](t)=\|M(t)-M(\tau)\|^{2}-\delta^{2} \lambda^{2} \\
& \leq(\|M(t)\|+\|M(\tau)\|)^{2}-\delta^{2} \lambda^{2} \leq r^{2} \delta^{2} \lambda^{2}-\delta^{2} \lambda^{2}<0
\end{aligned}
$$

a contradiction. This proves the claim.
It follows that for each $r \in(0,1)$,

$$
P\left(S_{r}\right) \leq P\left(N(t) \text { hits } \lambda^{2}\left(1-\delta^{2}\right) \text { before }-\delta^{2} \lambda^{2}\right)
$$

By Theorem 62.11.3 this is no larger than

$$
\begin{aligned}
& P\left(\left[N^{*}>0\right]\right) \frac{\delta^{2} \lambda^{2}}{\lambda^{2}\left(1-\delta^{2}\right)+\delta^{2} \lambda^{2}}=P\left(\left[N^{*}>0\right]\right) \delta^{2} \\
& \quad \leq P([\tau<\infty]) \delta^{2}=P\left(\left[([M](T))^{1 / 2}>\lambda\right]\right) \delta^{2}
\end{aligned}
$$

Now by the good lambda inequality, there is a constant $k$ independent of $M$ such that

$$
\int_{\Omega} F\left(([M](T))^{1 / 2}\right) d P \leq k \int_{\Omega} F\left(2 M^{*}\right) d P \leq k C_{2} \int_{\Omega} F\left(M^{*}\right) d P
$$

by the assumptions about $F$. Therefore, combining this result with the first part,

$$
\begin{aligned}
\left(k C_{2}\right)^{-1} \int_{\Omega} F\left(([M](T))^{1 / 2}\right) d P & \leq \int_{\Omega} F\left(M^{*}\right) d P \\
& \leq C \int_{\Omega} F\left(([M](T))^{1 / 2}\right) d P
\end{aligned}
$$

Of course, everything holds for local martingales in place of martingales.
Theorem 63.4.4 Let $\{M(t)\}$ be a continuous $H$ valued local martingale, $M(0)=0$, where $H$ is a separable Hilbert space and $t \in[0, T]$. Then if $F$ is a function of the sort described in the good lambda inequality, that is,

$$
F(0)=0, F \text { continuous, } F \text { increasing, }
$$

$$
F(\alpha x) \leq c_{\alpha} F(x)
$$

there are constants, $C$ and $c$ independent of such local martingales $M$ such that

$$
c \int_{\Omega} F\left([M](T)^{1 / 2}\right) d P \leq \int_{\Omega} F\left(M^{*}\right) d P \leq C \int_{\Omega} F\left([M](T)^{1 / 2}\right) d P
$$

where

$$
M^{*}(\omega) \equiv \sup \{\|M(t)(\omega)\|: t \in[0, T]\}
$$

Proof: Let $\left\{\tau_{n}\right\}$ be an increasing localizing sequence for $M$ such that $M^{\tau_{n}}$ is uniformly bounded. Such a localizing sequence exists from Proposition 63.2.2. Then from Theorem 63.4.3 there exist constants $c, C$ independent of $\tau_{n}$ such that

$$
\begin{aligned}
c \int_{\Omega} F\left(\left[M^{\tau_{n}}\right](T)^{1 / 2}\right) d P & \leq \int_{\Omega} F\left(\left(M^{\tau_{n}}\right)^{*}\right) d P \\
& \leq C \int_{\Omega} F\left(\left[M^{\tau_{n}}\right](T)^{1 / 2}\right) d P
\end{aligned}
$$

By Corollary 63.3.3, this implies

$$
\begin{aligned}
c \int_{\Omega} F\left(\left([M]^{\tau_{n}}\right)(T)^{1 / 2}\right) d P & \leq \int_{\Omega} F\left(\left(M^{\tau_{n}}\right)^{*}\right) d P \\
& \leq C \int_{\Omega} F\left(\left([M]^{\tau_{n}}\right)(T)^{1 / 2}\right) d P
\end{aligned}
$$

and now note that $\left([M]^{\tau_{n}}\right)(T)^{1 / 2}$ and $\left(M^{\tau_{n}}\right)^{*}$ increase in $n$ to $[M](T)^{1 / 2}$ and $M^{*}$ respectively. Then the result follows from the monotone convergence theorem.

Here is a corollary [108].
Corollary 63.4.5 Let $\{M(t)\}$ be a continuous $H$ valued local martingale and let $\varepsilon, \delta \in$ $(0, \infty)$. Then there is a constant $C$, independent of $\varepsilon, \delta$ such that

$$
P([\overbrace{\sup _{t \in[0, T]}\|M(t)\|}^{M^{*}(T)} \geq \varepsilon]) \leq \frac{C}{\varepsilon} E\left([M]^{1 / 2}(T) \wedge \delta\right)+P\left([M]^{1 / 2}(T)>\delta\right)
$$

Proof: Let the stopping time $\tau$ be defined by

$$
\tau \equiv \inf \left\{t>0:[M]^{1 / 2}(t)>\delta\right\}
$$

Then

$$
P\left(\left[M^{*} \geq \varepsilon\right]\right)=P\left(\left[M^{*} \geq \varepsilon\right] \cap[\tau=\infty]\right)+P\left(\left[M^{*} \geq \varepsilon\right] \cap[\tau<\infty]\right)
$$

On the set where $[\tau=\infty], M^{\tau}=M$ and so $P\left(\left[M^{*} \geq \varepsilon\right]\right) \leq$

$$
\leq \frac{1}{\varepsilon} \int_{\Omega}\left(M^{\tau}\right)^{*} d P+P\left(\left[M^{*} \geq \varepsilon\right] \cap\left[[M]^{1 / 2}(T)>\delta\right]\right)
$$

By Theorem 63.4.4 and Corollary 63.3.3,

$$
\begin{aligned}
\leq & \frac{C}{\varepsilon} \int_{\Omega}\left[M^{\tau}\right]^{1 / 2}(T) d P+P\left(\left[M^{*} \geq \varepsilon\right] \cap\left[[M]^{1 / 2}(T)>\delta\right]\right) \\
= & \frac{C}{\varepsilon} \int_{\Omega}\left([M]^{\tau}\right)^{1 / 2}(T) d P+P\left(\left[M^{*} \geq \varepsilon\right] \cap\left[[M]^{1 / 2}(T)>\delta\right]\right) \\
\leq & \frac{C}{\varepsilon} \int_{\Omega}[M]^{1 / 2}(T) \wedge \delta d P+P\left(\left[M^{*} \geq \varepsilon\right] \cap\left[[M]^{1 / 2}(T)>\delta\right]\right) \\
& \leq \frac{C}{\varepsilon} \int_{\Omega}[M]^{1 / 2}(T) \wedge \delta d P+P\left(\left[[M]^{1 / 2}(T)>\delta\right]\right)
\end{aligned}
$$

The Burkholder Davis Gundy inequality along with the properties of the covariation implies the following amazing proposition.

Proposition 63.4.6 The space $M_{T}^{2}(H)$ is a Hilbert space. Here $H$ is a separable Hilbert space.

Proof: We already know from Proposition 62.12 .2 that this space is a Banach space. It is only necessary to exhibit an equivalent norm which makes it a Hilbert space. However, you can let $F(\lambda)=\lambda^{2}$ in the Burkholder Davis Gundy theorem and obtain for $M \in M_{T}^{2}(H)$, the two norms

$$
\left(\int_{\Omega}[M](T) d P\right)^{1 / 2}=\left(\int_{\Omega}[M, M](T) d P\right)^{1 / 2}
$$

and

$$
\left(\int_{\Omega}\left(M^{*}\right)^{2} d P\right)^{1 / 2}
$$

are equivalent. The first comes from an inner product since from Corollary 63.3.3, $[\cdot, \cdot]$ is bilinear and symmetric and nonnegative. If $[M, M](T)=[M](T)=0$ in $L^{1}(\Omega)$, then from the Burkholder Davis Gundy inequality, $M^{*}=0$ in $L^{2}(\Omega)$ and so $M=0$. Hence

$$
\int_{\Omega}[M, N](T) d P
$$

is an inner product which yields the equivalent norm.
Example 63.4.7 An example of a real martingale is the Wiener process, $W(t)$. It has the property that whenever $t_{1}<t_{2}<\cdots<t_{n}$, the increments $\left\{W\left(t_{i}\right)-W\left(t_{i-1}\right)\right\}$ are independent and whenever $s<t, W(t)-W(s)$ is normally distributed with mean 0 and variance $(t-s)$. For the Wiener process, we let

$$
\mathscr{F}_{t} \equiv \cap_{u>t} \overline{\sigma(W(s)-W(r): r<s \leq u)}
$$

and it is with respect to this normal filtration that $W$ is a continuous martingale. What is the quadratic variation of such a process?

The quadratic variation of the Wiener process is just $t$. This is because if $A \in \mathscr{F}_{s}, s<t$,

$$
\begin{gathered}
E\left(\mathscr{X}_{A}\left(|W(t)|^{2}-t\right)\right)= \\
E\left(\mathscr{X}_{A}\left(|W(t)-W(s)|^{2}+|W(s)|^{2}+2(W(s), W(t)-W(s))-(t-s+s)\right)\right)
\end{gathered}
$$

Now

$$
E\left(\mathscr{X}_{A}(2(W(s), W(t)-W(s)))\right)=P(A) E(2 W(s)) E(W(t)-W(s))=0
$$

by the independence of the increments. Thus the above reduces to

$$
\begin{aligned}
& E\left(\mathscr{X}_{A}\left(|W(t)-W(s)|^{2}+|W(s)|^{2}-(t-s+s)\right)\right) \\
= & E\left(\mathscr{X}_{A}\left(|W(t)-W(s)|^{2}-(t-s)\right)\right)+E\left(\mathscr{X}_{A}\left(|W(s)|^{2}-s\right)\right) \\
= & P(A) E\left(|W(t)-W(s)|^{2}-(t-s)\right)+E\left(\mathscr{X}_{A}\left(|W(s)|^{2}-s\right)\right) \\
= & E\left(\mathscr{X}_{A}\left(|W(s)|^{2}-s\right)\right)
\end{aligned}
$$

and so $E\left(|W(t)|^{2}-t \mid \mathscr{F}_{s}\right)=|W(s)|^{2}-s$ showing that $t \rightarrow|W(t)|^{2}-t$ is a martingale. Hence, by uniqueness, $[W](t)=t$.

### 63.5 The Quadratic Variation And Stochastic Integration

Let $\mathscr{F}_{t}$ be a normal filtration and let $\{M(t)\}$ be a continuous local martingale adapted to $\mathscr{F}_{t}$ having values in $U$ a separable real Hilbert space.

Definition 63.5.1 Let $\mathscr{F}_{t}$ be a normal filtration and let

$$
f(t) \equiv \sum_{k=0}^{n-1} f_{k} \mathscr{X}_{\left(t_{k}, t_{k+1}\right]}(t)
$$

where $\left\{t_{k}\right\}_{k=0}^{n}$ is a partition of $[0, T]$ and each $f_{k}$ is $\mathscr{F}_{t_{k}}$ measurable, $f_{k} M^{*} \in L^{2}(\Omega)$ where

$$
M^{*}(\omega) \equiv \sup _{t \in[0, T]}\|M(t)(\omega)\|
$$

Such a function is called an elementary function. Also let $\{M(t)\}$ be a local martingale adapted to $\mathscr{F}_{t}$ which has values in a separable real Hilbert space $U$ such that $M(0)=0$. For such an elementary real valued function define

$$
\int_{0}^{t} f d M \equiv \sum_{k=0}^{n-1} f_{k}\left(M\left(t \wedge t_{k+1}\right)-M\left(t \wedge t_{k}\right)\right)
$$

Then with this definition, here is a wonderful lemma.
Lemma 63.5.2 For $f$ an elementary function as above, $\left\{\int_{0}^{t} f d M\right\}$ is a continuous local martingale and

$$
\begin{equation*}
E\left(\left\|\int_{0}^{t} f d M\right\|_{U}^{2}\right)=\int_{\Omega} \int_{0}^{t} f(s)^{2} d[M](s) d P \tag{63.5.13}
\end{equation*}
$$

If $N$ is another continuous local martingale adapted to $\mathscr{F}_{t}$ and both $f, g$ are elementary functions such that for each $k$,

$$
f_{k} M^{*}, g_{k} N^{*} \in L^{2}(\Omega),
$$

then

$$
\begin{equation*}
E\left(\left(\int_{0}^{t} f d M, \int_{0}^{t} g d N\right)_{U}\right)=\int_{\Omega} \int_{0}^{t} f g d[M, N] \tag{63.5.14}
\end{equation*}
$$

and both sides make sense.
Proof: Let $\left\{\tau_{l}\right\}$ be a localizing sequence for $M$ such that $M^{\tau_{l}}$ is a bounded martingale. Then from the definition, for each $\omega$

$$
\int_{0}^{t} f d M=\lim _{l \rightarrow \infty} \int_{0}^{t} f d M^{\tau_{l}}=\lim _{l \rightarrow \infty}\left(\int_{0}^{t} f d M\right)^{\tau_{l}}
$$

and it is clear that $\left\{\int_{0}^{t} f d M^{\tau_{l}}\right\}$ is a martingale because it is just the sum of some martingales. Thus $\left\{\tau_{l}\right\}$ is a localizing sequence for $\int_{0}^{t} f d M$. It is also clear $\int_{0}^{t} f d M$ is continuous because it is a finite sum of continuous random variables.

Next consider the formula which is really a version of the Ito isometry. There is no loss of generality in assuming the mesh points are the same for the two elementary functions because if not, one can simply add in points to make this happen. It suffices to consider 63.5.14 because the other formula is a special case. To begin with, let $\left\{\tau_{l}\right\}$ be a localizing sequence which makes both $M^{\tau_{l}}$ and $N^{\tau_{l}}$ into bounded martingales. Consider the stopped process.

$$
\begin{aligned}
& E\left(\left(\int_{0}^{t} f d M^{\tau_{l}}, \int_{0}^{t} g d N^{\tau_{l}}\right)_{U}\right) \\
= & E\left(\left(\sum_{k=0}^{n-1} f_{k}\left(M^{\tau_{l}}\left(t \wedge t_{k+1}\right)-M^{\tau_{l}}\left(t \wedge t_{k}\right)\right)\right.\right. \\
& \left.\left.\sum_{k=0}^{n-1} g_{k}\left(N^{\tau_{l}}\left(t \wedge t_{k+1}\right)-N^{\tau_{l}}\left(t \wedge t_{k}\right)\right)\right)\right)
\end{aligned}
$$

To save on notation, write $M^{\tau_{l}}\left(t \wedge t_{k+1}\right)-M^{\tau_{l}}\left(t \wedge t_{k}\right) \equiv \Delta M_{k}(t)$, similar for $\Delta N_{k}$. Thus

$$
\Delta M_{k}=M^{\tau_{l} \wedge t_{k+1}}-M^{\tau_{l} \wedge t_{k}},
$$

similar for $\Delta N_{k}$. Then the above equals

$$
E\left(\sum_{k=0}^{n-1}\left(f_{k} \Delta M_{k}, \sum_{k=0}^{n-1} g_{k} \Delta N_{k}\right)\right)=E\left(\sum_{k, j} f_{k} g_{j}\left(\Delta M_{k}, \Delta N_{j}\right)\right)
$$

Now consider one of the mixed terms with $j<k$.

$$
\begin{gathered}
E\left(\left(f_{k} \Delta M_{k}, g_{j} \Delta N_{j}\right)\right)=E\left(E\left(\left(f_{k} \Delta M_{k}, g_{j} \Delta N_{j}\right) \mid \mathscr{F}_{t_{k}}\right)\right) \\
=E\left(g_{j} \Delta N_{j}, f_{k} E\left(\Delta M_{k} \mid \mathscr{F}_{t_{k}}\right)\right)=0
\end{gathered}
$$

since $E\left(\Delta M_{k} \mid \mathscr{F}_{t_{k}}\right)=E\left(\left(M^{\tau_{l}}\left(t \wedge t_{k+1}\right)-M^{\tau_{l}}\left(t \wedge t_{k}\right)\right) \mid \mathscr{F}_{t_{k}}\right)=0$ by the Doob optional sampling theorem. Thus

$$
\begin{gather*}
E\left(\left(\int_{0}^{t} f d M^{\tau_{l}}, \int_{0}^{t} g d N^{\tau_{l}}\right)_{U}\right)=  \tag{63.5.15}\\
=\sum_{k=0}^{n-1} E\left(f_{k} g_{k}\left(\Delta M_{k}, \Delta N_{k}\right)\right)=\sum_{k=0}^{n-1} E\left(f_{k} g_{k}\left(\left[\Delta M_{k}, \Delta N_{k}\right]+\mathscr{N}_{k}\right)\right) \tag{63.5.16}
\end{gather*}
$$

where $\mathscr{N}_{k}$ is a martingale such that $\mathscr{N}_{k}(t)=0$ for all $t \leq t_{k}$. This is because the martingale $\left(N^{\tau_{l}}\right)^{t_{k+1}}-\left(N^{\tau_{l}}\right)^{t_{k}}=\Delta N_{k}$ equals 0 for such $t$; and so $E\left(\mathscr{N}_{k}(t)\right)=0$. Thus $f_{k} g_{k} \mathscr{N}_{k}$ is a martingale which equals zero when $t=0$. Therefore, its expectation also equals 0 . Consequently the above reduces to

$$
\sum_{k=0}^{n-1} E\left(f_{k} g_{k}\left[\Delta M_{k}, \Delta N_{k}\right]\right) .
$$

At this point, recall the definition of the covariation. The above equals

$$
\frac{1}{4} \sum_{k=0}^{n-1} E\left(f_{k} g_{k}\left(\left[\Delta M_{k}+\Delta N_{k}\right]-\left[\Delta M_{k}-\Delta N_{k}\right]\right)\right)
$$

Rewriting this yields

$$
\begin{aligned}
= & \frac{1}{4} \sum_{k=0}^{n-1} E\left(f _ { k } g _ { k } \left(\left[\left(M^{\tau_{l}}\right)^{t_{k+1}}+\left(N^{\tau_{l}}\right)^{t_{k+1}}-\left(\left(M^{\tau_{l}}\right)^{t_{k}}+\left(N^{\tau_{l}}\right)^{t_{k}}\right)\right]\right.\right. \\
& \left.\left.-\left[\left(M^{\tau_{l}}\right)^{t_{k+1}}-\left(N^{\tau_{l}}\right)^{t_{k+1}}-\left(\left(M^{\tau_{l}}\right)^{t_{k}}-\left(N^{\tau_{l}}\right)^{t_{k}}\right)\right]\right)\right)
\end{aligned}
$$

To save on notation, denote

$$
\begin{aligned}
\left(M^{\tau_{l}}\right)^{t_{k+1}}+\left(N^{\tau_{l}}\right)^{t_{k+1}}-\left(\left(M^{\tau_{l}}\right)^{t_{k}}+\left(N^{\tau_{l}}\right)^{t_{k}}\right) & \equiv \Delta_{k}\left(M^{\tau_{l}}+N^{\tau_{l}}\right) \\
\left(M^{\tau_{l}}\right)^{t_{k+1}}-\left(N^{\tau_{l}}\right)^{t_{k+1}}-\left(\left(M^{\tau_{l}}\right)^{t_{k}}-\left(N^{\tau_{l}}\right)^{t_{k}}\right) & \equiv \Delta_{k}\left(M^{\tau_{l}}-N^{\tau_{l}}\right)
\end{aligned}
$$

Thus the above equals

$$
\frac{1}{4} \sum_{k=0}^{n-1} E\left(f_{k} g_{k}\left(\left[\Delta_{k}\left(M^{\tau_{l}}+N^{\tau_{l}}\right)\right]-\left[\Delta_{k}\left(M^{\tau_{l}}-N^{\tau_{l}}\right)\right]\right)\right)
$$

Now from Corollary 63.3.3,

$$
=\frac{1}{4} \sum_{k=0}^{n-1} E\left(f_{k} g_{k}\left(\left[\Delta_{k}(M+N)\right]^{\tau_{l}}-\left[\Delta_{k}(M-N)\right]^{\tau_{l}}\right)\right)
$$

Letting $l \rightarrow \infty$, this reduces to

$$
\begin{aligned}
& =\frac{1}{4} \sum_{k=0}^{n-1} E\left(f_{k} g_{k}\left(\left[\Delta_{k}(M+N)\right]-\left[\Delta_{k}(M-N)\right]\right)\right) \\
& =\frac{1}{4}\left(\int_{\Omega} \int_{0}^{t} f g(d[M+N]-d[M-N])\right) \\
& =\int_{\Omega} \int_{0}^{t} f g d[M, N]
\end{aligned}
$$

Now consider the left side of 63.5.16.

$$
\begin{aligned}
& E\left(\left(\int_{0}^{t} f d M^{\tau_{l}}, \int_{0}^{t} g d N^{\tau_{l}}\right)_{U}\right) \\
\equiv & \int_{\Omega} \sum_{k, j} f_{k} g_{j}\left(\left(M^{\tau_{l}}\left(t \wedge t_{k+1}\right)-M^{\tau_{l}}\left(t \wedge t_{k}\right)\right)\right. \\
& \left.\left(N^{\tau_{l}}\left(t \wedge t_{j+1}\right)-N^{\tau_{l}}\left(t \wedge t_{j}\right)\right)\right) d P
\end{aligned}
$$

Then for each $\omega$, the integrand converges as $l \rightarrow \infty$ to

$$
\sum_{k, j} f_{k} g_{j}\left(\left(M\left(t \wedge t_{k+1}\right)-M\left(t \wedge t_{k}\right)\right),\left(N\left(t \wedge t_{j+1}\right)-N\left(t \wedge t_{j}\right)\right)\right)
$$

But also you can do a sloppy estimate which will allow the use of the dominated convergence theorem.

$$
\begin{gathered}
\left\|\sum_{k, j} f_{k} g_{j}\left(M^{\tau_{l}}\left(t \wedge t_{k+1}\right)-M^{\tau_{l}}\left(t \wedge t_{k}\right)\right),\left(N^{\tau_{l}}\left(t \wedge t_{j+1}\right)-N^{\tau_{l}}\left(t \wedge t_{j}\right)\right)\right\| \\
\leq \sum_{k, j}\left|f_{k}\right|\left|g_{j}\right| 4 M^{*} N^{*} \in L^{1}(\Omega)
\end{gathered}
$$

by assumption. Thus the left side of 63.5 .16 converges as $l \rightarrow \infty$ to

$$
\begin{aligned}
\int_{\Omega} \sum_{k, j} f_{k} g_{j}((M(t & \left.\left.\left.\wedge t_{k+1}\right)-M\left(t \wedge t_{k}\right)\right),\left(N\left(t \wedge t_{j+1}\right)-N\left(t \wedge t_{j}\right)\right)\right) d P \\
& =\int_{\Omega}\left(\int_{0}^{t} f d M, \int_{0}^{t} g d N\right)_{U} d P \square
\end{aligned}
$$

Note for each $\omega$, the inside integral in 63.5 .13 is just a Stieltjes integral taken with respect to the increasing integrating function $[M]$.

Of course, with this estimate it is obvious how to extend the integral to a larger class of functions.

Definition 63.5.3 Let $v(\omega)$ denote the Radon measure representing the functional

$$
\Lambda(\omega)(g) \equiv \int_{0}^{T} g d[M](t)(\omega)
$$

$(t \rightarrow[M](t)(\omega)$ is a continuous increasing function and $v(\omega)$ is the measure representing the Stieltjes integral, one for each $\omega$.) Then let $\mathscr{G}_{M}$ denote functions $f(s, \omega)$ which are the limit of such elementary functions in the space $L^{2}\left(\Omega ; L^{2}([0, T], v(\cdot))\right)$, the norm of such functions being

$$
\|f\|_{\mathscr{G}}^{2} \equiv \int_{\Omega} \int_{0}^{T} f(s)^{2} d[M](s) d P
$$

For $f \in \mathscr{G}$ just defined,

$$
\int_{0}^{t} f d M \equiv \lim _{n \rightarrow \infty} \int_{0}^{t} f_{n} d M
$$

where $\left\{f_{n}\right\}$ is a sequence of elementary functions converging to $f$ in

$$
L^{2}\left(\Omega ; L^{2}([0, T], v(\cdot))\right)
$$

Now here is an interesting lemma.

Lemma 63.5.4 Let $M$, $N$ be continuous local martingales, $M(0)=N(0)=0$ having values in a separable Hilbert space, $U$. Then

$$
\begin{gather*}
{[M+N]^{1 / 2} \leq\left([M]^{1 / 2}+[N]^{1 / 2}\right)}  \tag{63.5.17}\\
{[M+N] \leq 2([M]+[N])} \tag{63.5.18}
\end{gather*}
$$

Also, letting $v_{M+N}$ denote the measure obtained from the increasing function $[M+N]$ and $v_{N}, v_{M}$ defined similarly,

$$
\begin{equation*}
v_{M+N} \leq 2\left(v_{M}+v_{N}\right) \tag{63.5.19}
\end{equation*}
$$

on all Borel sets.
Proof: Since $(M, N) \rightarrow[M, N]$ is bilinear and satisfies

$$
\begin{aligned}
{[M, N] } & =[N, M] \\
{\left[a M+b M_{1}, N\right] } & =a[M, N]+b\left[M_{1}, N\right] \\
{[M, M] } & \geq 0
\end{aligned}
$$

which follows from Corollary 63.3.3, the usual Cauchy Schwarz inequality holds and so

$$
|[M, N]| \leq[M]^{1 / 2}[N]^{1 / 2}
$$

Thus

$$
\begin{aligned}
{[M+N] } & \equiv[M+N, M+N]=[M, M]+[N, N]+2[M, N] \\
& \leq[M]+[N]+2[M]^{1 / 2}[N]^{1 / 2}=\left([M]^{1 / 2}+[N]^{1 / 2}\right)^{2}
\end{aligned}
$$

This proves 63.5.17. Now square both sides. Then the right side is no larger than

$$
2([M]+[N])
$$

and this shows 63.5.18.
Now consider the claim about the measures. It was just shown that

$$
\left[(M+N)-(M+N)^{s}\right] \leq 2\left(\left[M-M^{s}\right]+\left[N-N^{s}\right]\right)
$$

and from Corollary 63.3.3 this implies that for $t>s$

$$
\begin{aligned}
& {[M+N](t)-[M+N](s \wedge t) } \\
&= {[M+N](t)-[M+N]^{s}(t) } \\
&= {\left[M+N-\left(M^{s}+N^{s}\right)\right](t) } \\
&= {\left[M-M^{s}+\left(N-N^{s}\right)\right](t) } \\
& \leq 2\left[M-M^{s}\right](t)+2\left[N-N^{s}\right](t) \\
& \leq 2([M](t)-[M](s))+2([N](t)-[N](s))
\end{aligned}
$$

Thus

$$
v_{M+N}([s, t]) \leq 2\left(v_{M}([s, t])+v_{N}([s, t])\right)
$$

By regularity of the measures, this continues to hold with any Borel set $F$ in place of $[s, t]$.

Theorem 63.5.5 The integral is well defined and has a continuous version which is a local martingale. Furthermore it satisfies the Ito isometry,

$$
E\left(\left\|\int_{0}^{t} f d M\right\|_{U}^{2}\right)=\int_{\Omega} \int_{0}^{t} f(s)^{2} d[M](s) d P
$$

Let the norm on $\mathscr{G}_{N} \cap \mathscr{G}_{M}$ be the maximum of the norms on $\mathscr{G}_{N}$ and $\mathscr{G}_{M}$ and denote by $\mathscr{E}_{N}$ and $\mathscr{E}_{M}$ the elementary functions corresponding to the martingales $N$ and $M$ respectively. Define $\mathscr{G}_{N M}$ as the closure in $\mathscr{G}_{N} \cap \mathscr{G}_{M}$ of $\mathscr{E}_{N} \cap \mathscr{E}_{M}$. Then for $f, g \in \mathscr{G}_{N M}$,

$$
\begin{equation*}
E\left(\left(\int_{0}^{t} f d M, \int_{0}^{t} g d N\right)\right)=\int_{\Omega} \int_{0}^{t} f g d[M, N] \tag{63.5.20}
\end{equation*}
$$

Proof: It is clear the definition is well defined because if $\left\{f_{n}\right\}$ and $\left\{g_{n}\right\}$ are two sequences of elementary functions converging to $f$ in $L^{2}\left(\Omega ; L^{2}([0, T], v(\cdot))\right)$ and if $\int_{0}^{1 t} f d M$ is the integral which comes from $\left\{g_{n}\right\}$,

$$
\begin{aligned}
& \int_{\Omega}\left\|\int_{0}^{1 t} f d M-\int_{0}^{t} f d M\right\|^{2} d P \\
= & \lim _{n \rightarrow \infty} \int_{\Omega}\left\|\int_{0}^{t} g_{n} d M-\int_{0}^{t} f_{n} d M\right\|^{2} d P \\
\leq & \lim _{n \rightarrow \infty} \int_{\Omega} \int_{0}^{T}\left\|g_{n}-f_{n}\right\|^{2} d v d P=0 .
\end{aligned}
$$

Consider the claim the integral has a continuous version. Recall Theorem 62.9.4, part of which is listed here for convenience.

Theorem 63.5.6 Let $\{X(t)\}$ be a right continuous nonnegative submartingale adapted to the normal filtration $\mathscr{F}_{t}$ for $t \in[0, T]$. Let $p \geq 1$. Define

$$
X^{*}(t) \equiv \sup \{X(s): 0<s<t\}, X^{*}(0) \equiv 0
$$

Then for $\lambda>0$

$$
\begin{equation*}
P\left(\left[X^{*}(T)>\lambda\right]\right) \leq \frac{1}{\lambda^{p}} \int_{\Omega} X(T)^{p} d P \tag{63.5.21}
\end{equation*}
$$

Let $\left\{f_{n}\right\}$ be a sequence of elementary functions converging to $f$ in

$$
L^{2}\left(\Omega ; L^{2}([0, T], v(\cdot))\right) .
$$

Then letting

$$
\begin{aligned}
X_{n, m}^{\tau_{l}}(t) & =\left\|\int_{0}^{t}\left(f_{n}-f_{m}\right) d M^{\tau_{l}}\right\|_{U} \\
X_{n, m}(t) & =\left\|\int_{0}^{t}\left(f_{n}-f_{m}\right) d M\right\|_{U} \\
& =\left\|\int_{0}^{t} f_{n} d M-\int_{0}^{t} f_{m} d M\right\| \|_{U}
\end{aligned}
$$

It follows $X_{n, m}^{\tau_{l}}$ is a continuous nonnegative submartingale and from Theorem 62.9.4 just listed,

$$
\begin{aligned}
& P\left(\left[X_{n, m}^{\tau_{l} *}(T)>\lambda\right]\right) \leq \frac{1}{\lambda^{2}} \int_{\Omega} X_{n, m}^{\tau_{l}}(T)^{2} d P \\
& \quad \leq \frac{1}{\lambda^{2}} \int_{\Omega} \int_{0}^{T}\left|f_{n}-f_{m}\right|^{2} d\left[M^{\tau_{l}}\right] d P \\
& \quad \leq \frac{1}{\lambda^{2}} \int_{\Omega} \int_{0}^{T}\left|f_{n}-f_{m}\right|^{2} d[M] d P
\end{aligned}
$$

Letting $l \rightarrow \infty$,

$$
P\left(\left[X_{n, m}^{*}(T)>\lambda\right]\right) \leq \frac{1}{\lambda^{2}} \int_{\Omega} \int_{0}^{T}\left|f_{n}-f_{m}\right|^{2} d[M] d P
$$

Therefore, there exists a subsequence, still denoted by $\left\{f_{n}\right\}$ such that

$$
P\left(\left[X_{n, n+1}^{*}(T)>2^{-n}\right]\right)<2^{-n}
$$

Then by the Borel Cantelli lemma, the $\omega$ in infinitely many of the sets

$$
\left[X_{n, n+1}^{*}(T)>2^{-n}\right]
$$

has measure 0 . Denoting this exceptional set as $N$, it follows that for $\omega \notin N$, there exists $n(\omega)$ such that for $n>n(\omega)$,

$$
\sup _{t \in[0, T]}| | \int_{0}^{t} f_{n} d M-\int_{0}^{t} f_{n+1} d M \| \leq 2^{-n}
$$

and this implies uniform convergence of $\left\{\int_{0}^{t} f_{n} d M\right\}$. Letting

$$
G(t)=\lim _{n \rightarrow \infty} \int_{0}^{t} f_{n} d M
$$

for $\omega \notin N$ and $G(t)=0$ for $\omega \in N$, it follows that for each $t$, the continuous adapted process $G(t)$ equals $\int_{0}^{t} f d M$ a.e. Thus $\left\{\int_{0}^{t} f d M\right\}$ has a continuous version.

It suffices to verify 63.5.20. Let $\left\{f_{n}\right\}$ and $\left\{g_{n}\right\}$ be sequences of elementary functions converging to $f$ and $g$ in $\mathscr{G}_{M} \cap \mathscr{G}_{N}$. By Lemma 63.5.2,

$$
E\left(\left(\int_{0}^{t} f_{n} d M, \int_{0}^{t} g_{n} d N\right)_{U}\right)=\int_{\Omega} \int_{0}^{t} f_{n} g_{n} d[M, N]
$$

Then by the Holder inequality and the above definition,

$$
\lim _{n \rightarrow \infty} E\left(\left(\int_{0}^{t} f_{n} d M, \int_{0}^{t} g_{n} d N\right)_{U}\right)=E\left(\left(\int_{0}^{t} f d M, \int_{0}^{t} g d N\right)_{U}\right)
$$

Consider the right side which equals

$$
\frac{1}{4} \int_{\Omega} \int_{0}^{t} f_{n} g_{n} d[M+N] d P-\frac{1}{4} \int_{\Omega} \int_{0}^{t} f_{n} g_{n} d[M-N] d P
$$

Now from Lemma 63.5.4,

$$
\begin{aligned}
& \left|\int_{\Omega} \int_{0}^{t} f_{n} g_{n} d[M+N] d P-\int_{\Omega} \int_{0}^{t} f g d[M+N] d P\right| \\
= & \left|\int_{\Omega} \int_{0}^{t} f_{n} g_{n} d v_{M+N} d P-\int_{\Omega} \int_{0}^{t} f g d v_{M+N} d P\right| \\
\leq 2 & \left(\int_{\Omega} \int_{0}^{t}\left|f_{n} g_{n}-f g\right| d v_{M} d P+\int_{\Omega} \int_{0}^{t}\left|f_{n} g_{n}-f g\right| d v_{N} d P\right)
\end{aligned}
$$

and by the choice of the $f_{n}$ and $g_{n}$, these both converge to 0 . Similar considerations apply to

$$
\left|\int_{\Omega} \int_{0}^{t} f_{n} g_{n} d[M-N] d P-\int_{\Omega} \int_{0}^{t} f g d[M-N] d P\right|
$$

and show

$$
\lim _{n \rightarrow \infty} \int_{\Omega} \int_{0}^{t} f_{n} g_{n} d[M, N]=\int_{\Omega} \int_{0}^{t} f g d[M, N]
$$

### 63.6 Another Limit For Quadratic Variation

The problem to consider first is to define an integral

$$
\int_{0}^{t} f d M
$$

where $f$ has values in $H^{\prime}$ and $M$ is a continuous martingale having values in $H$. For the sake of simplicity assume $M(0)=0$. The process of definition is the same as before. First consider an elementary function

$$
\begin{equation*}
f(t) \equiv \sum_{k=0}^{m-1} f_{k} \mathscr{X}_{\left(t_{k}, t_{k+1}\right]}(t) \tag{63.6.22}
\end{equation*}
$$

where $f_{k}$ is measurable into $H^{\prime}$ with respect to $\mathscr{F}_{t_{k}}$. Then define

$$
\begin{equation*}
\int_{0}^{t} f d M \equiv \sum_{k=0}^{m-1} f_{k}\left(M\left(t \wedge t_{k+1}\right)-M\left(t \wedge t_{k}\right)\right) \in \mathbb{R} \tag{63.6.23}
\end{equation*}
$$

Lemma 63.6.1 The $k^{\text {th }}$ term in the above sum is a martingale and the integral is also a martingale.

Proof: Let $\sigma$ be a stopping time with two values. Then

$$
\begin{aligned}
& E\left(f_{k}\left(M\left(\sigma \wedge t_{k+1}\right)-M\left(\sigma \wedge t_{k}\right)\right)\right) \\
= & E\left(E\left(f_{k}\left(M\left(\sigma \wedge t_{k+1}\right)-M\left(\sigma \wedge t_{k}\right)\right) \mid \mathscr{F}_{t_{k}}\right)\right) \\
= & E\left(f_{k} E\left(\left(M\left(\sigma \wedge t_{k+1}\right)-M\left(\sigma \wedge t_{k}\right)\right) \mid \mathscr{F}_{t_{k}}\right)\right)=0
\end{aligned}
$$

and it works the same with $\sigma$ replaced with $t$. Hence by the lemma about recognizing martingales, Lemma 63.1.1, each term is a martingale and so it follows that the integral $\int_{0}^{t} f d M$ is also a martingale.

Note also that, since $M$ is continuous, this is a continuous martingale.
As before, it is important to estimate this.

$$
E\left(\left|\int_{0}^{t} f d M\right|^{2}\right) \leq ?
$$

Consider a mixed term. For $j<k$, it follows from measurability considerations that

$$
\begin{gathered}
E\left(\left(f_{k}\left(M\left(t \wedge t_{k+1}\right)-M\left(t \wedge t_{k}\right)\right)\right)\left(f_{j}\left(M\left(t \wedge t_{j+1}\right)-M\left(t \wedge t_{j}\right)\right)\right)\right) \\
=E\left(E\left[\left(f_{k}\left(M\left(t \wedge t_{k+1}\right)-M\left(t \wedge t_{k}\right)\right)\right)\left(f_{j}\left(M\left(t \wedge t_{j+1}\right)-M\left(t \wedge t_{j}\right)\right)\right) \mid \mathscr{F}_{t_{k}}\right]\right) \\
=E\left(\left(f_{j}\left(M\left(t \wedge t_{j+1}\right)-M\left(t \wedge t_{j}\right)\right)\right) f_{k} E\left[\left(M\left(t \wedge t_{k+1}\right)-M\left(t \wedge t_{k}\right)\right) \mid \mathscr{F}_{t_{k}}\right]\right)=0
\end{gathered}
$$

Therefore,

$$
\begin{gathered}
E\left(\left|\int_{0}^{t} f d M\right|^{2}\right)=E\left(\sum_{k=0}^{m-1}\left|f_{k}\left(M\left(t \wedge t_{k+1}\right)-M\left(t \wedge t_{k}\right)\right)\right|^{2}\right) \\
\leq E\left(\sum_{k=0}^{m-1}\left\|f_{k}\right\|^{2}\left|M\left(t \wedge t_{k+1}\right)-M\left(t \wedge t_{k}\right)\right|^{2}\right) \\
=E\left(\sum_{k=0}^{m-1}\left\|f_{k}\right\|^{2}\left(\left[M^{t_{k+1}}-M^{t_{k}}\right](t)+N_{k}(t)\right)\right) \\
=E\left(\sum_{k=0}^{m-1}\left\|f_{k}\right\|^{2}\left(\left[M^{t_{k+1}}\right](t)-\left[M^{t_{k}}\right](t)+N_{k}(t)\right)\right) \\
=E\left(\sum_{k=0}^{m-1}\left\|f_{k}\right\|^{2}\left([M]\left(t \wedge t_{k+1}\right)-[M]\left(t \wedge t_{k}\right)+N_{k}(t)\right)\right) \\
=E\left(\sum_{k=0}^{m-1}\left\|f_{k}\right\|^{2}\left([M]\left(t \wedge t_{k+1}\right)-[M]\left(t \wedge t_{k}\right)\right)\right)
\end{gathered}
$$

where $N_{k}$ is a martingale which equals 0 for $t \leq t_{k}$. The above equals

$$
E\left(\int_{0}^{t}\|f\|^{2} d[M]\right) \equiv E\left(\int_{0}^{t}\|f\|^{2} d v\right)
$$

the integral inside being the ordinary Lebesgue Stieltjes integral for the step function where $v$ is the measure determined by the positive linear functional

$$
\Lambda g=\int_{0}^{T} g d[M]
$$

where the integral on the right is the ordinary Stieltjes integral. Thus, the following inequality is obtained.

$$
\begin{equation*}
E\left(\left|\int_{0}^{t} f d M\right|^{2}\right) \leq E\left(\int_{0}^{t}\|f\|^{2} d[M],\right) \tag{63.6.24}
\end{equation*}
$$

Now what would it take for

$$
\begin{equation*}
E\left(\left|\int_{0}^{t} f d M\right|^{2}\right) \tag{63.6.25}
\end{equation*}
$$

to be well defined? A convenient condition would be to insist that each $\left\|f_{k}\right\| M^{*}$ is in $L^{2}(\Omega)$ where

$$
M^{*}(\omega) \equiv \sup _{t \in[0, T]}|M(t)(\omega)|_{H}
$$

Is this condition also sufficient for the above integral 63.6 .25 to be finite? From the above, that integral equals

$$
\begin{gathered}
E\left(\sum_{k=0}^{m-1}\left\|f_{k}\right\|^{2}\left|M\left(t \wedge t_{k+1}\right)-M\left(t \wedge t_{k}\right)\right|^{2}\right) \\
\leq E\left(4 \sum_{k=0}^{m-1}\left\|f_{k}\right\|^{2}\left(M^{*}\right)^{2}\right)
\end{gathered}
$$

Thus the condition that for each $k,\left\|f_{k}\right\| M^{*} \in L^{2}(\Omega)$ is sufficient for all of the above to consist of real numbers and be well defined.

Definition 63.6.2 A function $f$ is called an elementary function if it is a step function of the form given in 63.6 .22 where each $f_{k}$ is $\mathscr{F}_{t_{k}}$ measurable and for each $k,\left\|f_{k}\right\| M^{*} \in L^{2}(\Omega)$. Define $\mathscr{G}_{M}$ to be the collection of functions $f$ having values in $H^{\prime}$ which have the property that there exists a sequence of elementary functions $\left\{f_{n}\right\}$ with $f_{n} \rightarrow f$ in the space

$$
L^{2}\left(\Omega ; L^{2}([0, T], v)\right)
$$

Then picking such an approximating sequence,

$$
\int_{0}^{t} f d M \equiv \lim _{n \rightarrow \infty} \int_{0}^{t} f_{n} d M
$$

the convergence happening in $L^{2}(\Omega)$.

The inequality 63.6 .24 shows that this definition is well defined. So what are the properties of the integral just defined? Each $\int_{0}^{t} f_{n} d M$ is a continuous martingale because it is the sum of continuous martingales. Since convergence happens in $L^{2}(\Omega)$, it follows that $\int_{0}^{t} f d M$ is also a martingale. Is it continuous? By the maximal inequality Theorem 62.9.4, it follows that

$$
\left.\left.\left.\begin{array}{c}
P\left(\left[\sup _{t \in[0, T]} \mid \int_{0}^{t} f_{m} d M\right.\right.
\end{array}-\int_{0}^{t} f_{n} d M \right\rvert\,>\lambda\right]\right) \leq \frac{1}{\lambda^{2}} E\left(\left|\int_{0}^{T}\left(f_{m}-f_{n}\right) d M\right|^{2}\right)
$$

and it follows that there exists a subsequence, still called $n$ such that for all $p$ positive,

$$
P\left(\left[\sup _{t \in[0, T]}\left|\int_{0}^{t} f_{n+p} d M-\int_{0}^{t} f_{n} d M\right|>\frac{1}{n}\right]\right)<2^{-n}
$$

By the Borel Cantelli lemma, there exists a set of measure zero $N$ such that for $\omega \notin N$, $\left\{\int_{0}^{t} f_{n} d M\right\}$ is a Cauchy sequence. Thus, what it converges to is continuous in $t$ for each $\omega \notin N$ and for each $t$, it equals $\int_{0}^{t} f d M$ a.e. Hence we can regard $\int_{0}^{t} f d M$ as this continuous version.

What is an example of such a function in $\mathscr{G}_{M}$ ?
Lemma 63.6.3 Let $R: H \rightarrow H^{\prime}$ be the Riesz map.

$$
\langle R f, g\rangle \equiv(f, g)_{H}
$$

Also suppose $M$ is a uniformly bounded continuous martingale with values in $H$. Then $R M \in \mathscr{G}_{M}$.

Proof: I need to exhibit an approximating sequence of elementary functions as described above. Consider

$$
M_{n}(t) \equiv \sum_{i=0}^{m_{n}-1} M\left(t_{i}\right) \mathscr{X}_{\left[t_{i}^{n}, t_{i+1}^{n}\right]}(t)
$$

Then clearly $R M_{n}\left(t_{i}\right) M^{*} \in L^{\infty}(\Omega)$ and so in particular it is in $L^{2}(\Omega)$. Here

$$
\lim _{n \rightarrow \infty} \max \left\{\left|t_{i}^{n}-t_{i+1}^{n}\right|, i=0, \cdots, m_{n}\right\}=0
$$

Say $M^{*}(\omega) \leq C$. Furthermore, I claim that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} E\left(\int_{0}^{T}\left\|R M_{n}-R M\right\|^{2} d[M]\right)=0 \tag{63.6.26}
\end{equation*}
$$

This requires a little proof. Recall the description of $[M](t)$. It was as follows. You considered

$$
P_{n}(t) \equiv 2 \sum_{k \geq 0}\left(\left(M\left(t \wedge \tau_{k+1}^{n}\right)-M\left(t \wedge \tau_{k}^{n}\right)\right), M\left(t \wedge \tau_{k}^{n}\right)\right)
$$

where the stopping times were defined such that $\tau_{k+1}^{n}$ is the first time $t>\tau_{k}^{n}$ such that $\left|M(t)-M\left(\tau_{k}^{n}\right)\right|^{2}=2^{-n}$ and $\tau_{0}^{n}=0$. Recall that $\lim _{k \rightarrow \infty} \tau_{k}^{n}=\infty$ or $T$ in the way it was formulated earlier. Then it was shown that $P_{n}(t)$ converged to a martingale $P(t)$ in $L^{1}(\Omega)$. Then by the usual procedure using the Borel Cantelli lemma, a subsequence converges to $P(t)$ uniformly off a set of measure zero. It is easy to estimate $P_{n}(t)$.

$$
\left|P_{n}(t)\right| \leq \sum_{k \geq 0}\left|M\left(t \wedge \tau_{k+1}^{n}\right)\right|^{2}-\left|M\left(t \wedge \tau_{k}^{n}\right)\right|^{2}=|M(t)|^{2} \leq M^{*}
$$

This follows from the observation that

$$
\left(M\left(t \wedge \tau_{k+1}^{n}\right), M\left(t \wedge \tau_{k}^{n}\right)\right) \leq \frac{1}{2}\left(\left|M\left(t \wedge \tau_{k+1}^{n}\right)\right|^{2}+\left|M\left(t \wedge \tau_{k}^{n}\right)\right|^{2}\right)
$$

Then it follows that $\sup _{t \in[0, T]}|P(t)(\omega)| \leq M^{*}(\omega) \leq C$ for a.e. $\omega$. The quadratic variation $[M]$ was defined as

$$
|M(t)|^{2}=P(t)+[M](t)
$$

Thus $[M](t) \leq 2\left(M^{*}\right)^{2}$. Now consider the above limit in 63.6.26. From the assumption that $M$ is uniformly bounded,

$$
\int_{0}^{T}\left\|R M_{n}-R M\right\|^{2} d[M] \leq \int_{0}^{T} 4 C^{2} d[M]=4 C^{2}[M](T) \leq 4 C^{2}\left(2 C^{2}\right)<\infty
$$

Also, by the continuity of the martingale, for each $\omega$,

$$
\lim _{n \rightarrow \infty}\left\|R M_{n}-R M\right\|^{2}=0
$$

By the dominated convergence theorem, and the fact that the integrand is bounded,

$$
\lim _{n \rightarrow \infty} \int_{0}^{T}\left\|R M_{n}-R M\right\|^{2} d[M]=0
$$

Then from the above estimate and the dominated convergence theorem again, 63.6.26 follows. Thus $R M \in \mathscr{G}_{M}$.

From the above lemma, it makes sense to speak of

$$
\int_{0}^{t}(R M) d M
$$

and this is a continuous martingale having values in $\mathbb{R}$. Also from the above argument, if $\left\{t_{k}^{n}\right\}_{k=0}^{m_{n}}$ is a sequence of partitions such that

$$
\lim _{n \rightarrow \infty} \max \left\{\left|t_{i}^{n}-t_{i+1}^{n}\right|, i=0, \cdots, m_{n}\right\}=0
$$

then it follows that

$$
\sum_{i=0}^{m_{n}-1} R M\left(t_{i}\right)\left(M\left(t \wedge t_{k+1}\right)-M\left(t \wedge t_{k}\right)\right) \rightarrow \int_{0}^{t}(R M) d M
$$

in $L^{2}(\Omega)$, this for each $t \in[0, T]$.
Now here is the main result.

Theorem 63.6.4 Let $H$ be a Hilbert space and suppose $\left(M, \mathscr{F}_{t}\right), t \in[0, T]$ is a uniformly bounded continuous martingale with values in $H$. Also let $\left\{t_{k}^{n}\right\}_{k=1}^{m_{n}}$ be a sequence of partitions satisfying

$$
\lim _{n \rightarrow \infty} \max \left\{\left|t_{i}^{n}-t_{i+1}^{n}\right|, i=0, \cdots, m_{n}\right\}=0,\left\{t_{k}^{n}\right\}_{k=1}^{m_{n}} \subseteq\left\{t_{k}^{n+1}\right\}_{k=1}^{m_{n+1}}
$$

Then

$$
[M](t)=\lim _{n \rightarrow \infty} \sum_{k=0}^{m_{n}-1}\left|M\left(t \wedge t_{k+1}^{n}\right)-M\left(t \wedge t_{k}^{n}\right)\right|_{H}^{2}
$$

the limit taking place in $L^{2}(\Omega)$. In case $M$ is just a continuous local martingale, the above limit happens in probability.

Proof: First suppose $M$ is uniformly bounded.

$$
\begin{gathered}
\sum_{k=0}^{m_{n}-1}\left|M\left(t \wedge t_{k+1}^{n}\right)-M\left(t \wedge t_{k}^{n}\right)\right|_{H}^{2} \\
=\sum_{k=0}^{m_{n}-1}\left|M\left(t \wedge t_{k+1}^{n}\right)\right|^{2}-\left|M\left(t \wedge t_{k}^{n}\right)\right|^{2}-2 \sum_{k=0}^{m_{n}-1}\left(M\left(t \wedge t_{k}^{n}\right), M\left(t \wedge t_{k+1}^{n}\right)-M\left(t \wedge t_{k}^{n}\right)\right) \\
=|M(t)|_{H}^{2}-2 \sum_{k=0}^{m_{n}-1}\left(M\left(t \wedge t_{k}^{n}\right), M\left(t \wedge t_{k+1}^{n}\right)-M\left(t \wedge t_{k}^{n}\right)\right) \\
=|M(t)|_{H}^{2}-2 \sum_{k=0}^{m_{n}-1} R M\left(t \wedge t_{k}^{n}\right)\left(M\left(t \wedge t_{k+1}^{n}\right)-M\left(t \wedge t_{k}^{n}\right)\right) \\
=|M(t)|_{H}^{2}-2 \sum_{k=0}^{m_{n}-1} R M\left(t_{k}^{n}\right)\left(M\left(t \wedge t_{k+1}^{n}\right)-M\left(t \wedge t_{k}^{n}\right)\right)
\end{gathered}
$$

Then by Lemma 63.6.3, the right side converges to

$$
|M(t)|_{H}^{2}-2 \int_{0}^{t}(R M) d M
$$

Therefore, in $L^{2}(\Omega)$,

$$
\lim _{n \rightarrow \infty} \sum_{k=0}^{m_{n}-1}\left|M\left(t \wedge t_{k+1}^{n}\right)-M\left(t \wedge t_{k}^{n}\right)\right|_{H}^{2}+2 \int_{0}^{t}(R M) d M=|M(t)|_{H}^{2}
$$

That term on the left involving the limit is increasing and equal to 0 when $t=0$. Therefore, it must equal $[M](t)$.

Next suppose $M$ is only a continuous local martingale. By Proposition 63.2.2 there exists an increasing localizing sequence $\left\{\tau_{k}\right\}$ such that $M^{\tau_{k}}$ is a uniformly bounded martingale. Then

$$
P\left(\cup_{k=1}^{\infty}\left[\tau_{k}=\infty\right]\right)=1
$$

To save notation, let

$$
Q_{n}(t) \equiv \sum_{k=0}^{m_{n}-1}\left|M\left(t \wedge t_{k+1}^{n}\right)-M\left(t \wedge t_{k}^{n}\right)\right|_{H}^{2}
$$

Let $\eta, \varepsilon>0$ be given. Then there exists $k$ large enough that $P\left(\left[\tau_{k}=\infty\right]\right)>1-\eta / 2$. This is because the sets $\left[\tau_{k}=\infty\right]$ increase to $\Omega$ other than a set of measure zero. Then,

$$
\left[\left|Q_{n}^{\tau_{k}}-[M]^{\tau_{k}}(t)\right|>\varepsilon\right] \cap\left[\tau_{k}=\infty\right]=\left[\left|Q_{n}-[M](t)\right|>\varepsilon\right] \cap\left[\tau_{k}=\infty\right]
$$

Thus

$$
\begin{aligned}
& P\left(\left[\left|Q_{n}-[M](t)\right|>\varepsilon\right]\right) \leq P\left(\left[\left|Q_{n}-[M](t)\right|>\varepsilon\right] \cap\left[\tau_{k}=\infty\right]\right) \\
&+P\left(\left[\tau_{k}<\infty\right]\right) \\
& \leq P\left(\left[\left|Q_{n}^{\tau_{k}}-[M]^{\tau_{k}}(t)\right|>\varepsilon\right]\right)+\eta / 2
\end{aligned}
$$

From the first part, the convergence in probability of $Q_{n}^{\tau_{k}}(t)$ to $[M]^{\tau_{k}}(t)$ follows from the convergence in $L^{2}(\Omega)$ and so if $n$ is large enough, the right side of the above inequality is less than $\eta / 2+\eta / 2=\eta$. Since $\eta$ was arbitrary, this proves convergence in probability.

### 63.7 Doob Meyer Decomposition

This section is on the Doob Meyer decomposition which is a way of starting with a submartingale and writing it as the sum of a martingale and an increasing adapted stochastic process of a certain form. This is more general than what was done above with the submartingales $\|M(t)\|^{2}$ for $M(t) \in \mathscr{M}_{T}^{2}(H)$ where $M$ is a continuous martingale. There are two forms for this theorem, one for discrete martingales and one for martingales defined on an interval of the real line which is much harder. According to [74], this material is found in [78] however, I am following [74] for the continuous version of this theorem.

Theorem 63.7.1 Let $\left\{X_{n}\right\}$ be a submartingale. Then there exists a unique stochastic process, $\left\{A_{n}\right\}$ and martingale, $\left\{M_{n}\right\}$ such that

1. $A_{n}(\omega) \leq A_{n+1}(\omega), A_{1}(\omega)=0$,
2. $A_{n}$ is $\mathscr{F}_{n-1}$ adapted for all $n \geq 1$ where $\mathscr{F}_{0} \equiv \mathscr{F}_{1}$.
and also $X_{n}=M_{n}+A_{n}$.
Proof: Let $A_{1} \equiv 0$ and define

$$
A_{n} \equiv \sum_{k=2}^{n} E\left(X_{k}-X_{k-1} \mid \mathscr{F}_{k-1}\right)
$$

It follows $A_{n}$ is $\mathscr{F}_{n-1}$ measurable. Since $\left\{X_{k}\right\}$ is a submartingale, $A_{n}$ is increasing because

$$
\begin{equation*}
A_{n+1}-A_{n}=E\left(X_{n+1}-X_{n} \mid \mathscr{F}_{n}\right) \geq 0 \tag{63.7.27}
\end{equation*}
$$

It is a submartingale because

$$
E\left(A_{n} \mid \mathscr{F}_{n-1}\right)=A_{n} \geq A_{n-1}
$$

Now let $M_{n}$ be defined by

$$
X_{n}=M_{n}+A_{n}
$$

Then from 63.7.27,

$$
\begin{aligned}
& E\left(M_{n+1} \mid \mathscr{F}_{n}\right)=E\left(X_{n+1} \mid \mathscr{F}_{n}\right)-E\left(A_{n+1} \mid \mathscr{F}_{n}\right) \\
= & E\left(X_{n+1} \mid \mathscr{F}_{n}\right)-E\left(A_{n+1}-A_{n} \mid \mathscr{F}_{n}\right)-A_{n} \\
= & E\left(X_{n+1} \mid \mathscr{F}_{n}\right)-E\left(E\left(X_{n+1}-X_{n} \mid \mathscr{F}_{n}\right) \mid \mathscr{F}_{n}\right)-A_{n} \\
= & E\left(X_{n+1} \mid \mathscr{F}_{n}\right)-E\left(X_{n+1}-X_{n} \mid \mathscr{F}_{n}\right)-A_{n} \\
= & E\left(X_{n} \mid \mathscr{F}_{n}\right)-A_{n} \\
= & X_{n}-A_{n} \equiv M_{n}
\end{aligned}
$$

This proves the existence part.
It remains to verify uniqueness. Suppose then that

$$
X_{n}=M_{n}+A_{n}=M_{n}^{\prime}+A_{n}^{\prime}
$$

where $\left\{A_{n}\right\}$ and $\left\{A_{n}^{\prime}\right\}$ both satisfy the conditions of the theorem and $\left\{M_{n}\right\}$ and $\left\{M_{n}^{\prime}\right\}$ are both martingales. Then

$$
M_{n}-M_{n}^{\prime}=A_{n}^{\prime}-A_{n}
$$

and so, since $A_{n}^{\prime}-A_{n}$ is $\mathscr{F}_{n-1}$ measurable and $\left\{M_{n}-M_{n}^{\prime}\right\}$ is a martingale,

$$
\begin{aligned}
M_{n-1}-M_{n-1}^{\prime} & =E\left(M_{n}-M_{n}^{\prime} \mid \mathscr{F}_{n-1}\right) \\
& =E\left(A_{n}^{\prime}-A_{n} \mid \mathscr{F}_{n-1}\right) \\
& =A_{n}^{\prime}-A_{n}=M_{n}-M_{n}^{\prime} .
\end{aligned}
$$

Continuing this way shows $M_{n}-M_{n}^{\prime}$ is a constant. However, since $A_{1}^{\prime}-A_{1}=0=M_{1}-M_{1}^{\prime}$, it follows $M_{n}=M_{n}^{\prime}$ and this proves uniqueness. This proves the theorem.

Definition 63.7.2 A stochastic process, $\left\{A_{n}\right\}$ which satisfies the conditions of Theorem 63.7.1,

$$
A_{n}(\omega) \leq A_{n+1}(\omega)
$$

and

$$
A_{n} \text { is } \mathscr{F}_{n-1} \text { adapted for all } n \geq 1
$$

where $\mathscr{F}_{0} \equiv \mathscr{F}_{1}$ is said to be natural.
The Doob Meyer theorem needs to be extended to continuous submartingales and this will require another description of what it means for a stochastic process to be natural. To get an idea of what this condition should be, here is a lemma.

Lemma 63.7.3 Let a stochastic process, $\left\{A_{n}\right\}$ be natural. Then for every martingale, $\left\{M_{n}\right\}$,

$$
E\left(M_{n} A_{n}\right)=E\left(\sum_{j=1}^{n-1} M_{j}\left(A_{j+1}-A_{j}\right)\right)
$$

Proof: Start with the right side.

$$
\begin{aligned}
E\left(\sum_{j=1}^{n-1} M_{j}\left(A_{j+1}-A_{j}\right)\right) & =E\left(\sum_{j=2}^{n} M_{j-1} A_{j}-\sum_{j=1}^{n-1} M_{j} A_{j}\right) \\
& =E\left(\sum_{j=2}^{n-1} A_{j}\left(M_{j-1}-M_{j}\right)\right)+E\left(M_{n-1} A_{n}\right)
\end{aligned}
$$

Then the first term equals zero because since $A_{j}$ is $\mathscr{F}_{j-1}$ measurable,

$$
\begin{aligned}
\int_{\Omega} A_{j} M_{j-1} d P-\int_{\Omega} A_{j} M_{j} & =\int_{\Omega} A_{j} E\left(M_{j} \mid \mathscr{F}_{j-1}\right) d P-\int_{\Omega} A_{j} M_{j} d P \\
& =\int_{\Omega} E\left(A_{j} M_{j} \mid \mathscr{F}_{j-1}\right) d P-\int_{\Omega} A_{j} M_{j} d P \\
& =\int_{\Omega} A_{j} M_{j} d P-\int_{\Omega} A_{j} M_{j} d P=0
\end{aligned}
$$

The last term equals

$$
\begin{aligned}
\int_{\Omega} M_{n-1} A_{n} d P & =\int_{\Omega} E\left(M_{n} \mid \mathscr{F}_{n-1}\right) A_{n} d P \\
& =\int_{\Omega} E\left(M_{n} A_{n} \mid \mathscr{F}_{n-1}\right) d P=E\left(M_{n} A_{n}\right)
\end{aligned}
$$

This proves the lemma.
Definition 63.7.4 Let A be an increasing function defined on $\mathbb{R}$. By Theorem 4.3.4 on Page 50 there exists a positive linear functional, $L$ defined on $C_{c}(\mathbb{R})$ given by

$$
L f \equiv \int_{a}^{b} f d A \text { where } \operatorname{spt}(f) \subseteq[a, b]
$$

where the integral is just the Riemann Stieltjes integral. Then by the Riesz representation theorem, Theorem 12.3.2 on Page 288, there exists a unique Radon measure, $\mu$ which extends this functional, as described in the Riesz representation theorem. Then for $B$ a measurable set, I will write either

$$
\int_{B} f d \mu \text { or } \int_{B} f d A
$$

to denote the Lebesgue integral,

$$
\int \mathscr{X}_{B} f d \mu
$$

Lemma 63.7.5 Let $f$ be right continuous. Then $f$ is Borel measurable. Also, if the limit from the left exists, then $f_{-}(x) \equiv f(x)_{-} \equiv \lim _{y \rightarrow x-} f(y)$ is also Borel measurable. If $A$ is an increasing right continuous function and $f$ is right continuous and $f_{-}$, the left limit function exists, then if $f$ is bounded, on $[a, b]$, and if

$$
\left\{x_{0}^{p}, \cdots, x_{n_{p}}^{p}\right\}_{p=1}^{\infty}
$$

is a sequence of partitions of $[a, b]$ such that

$$
\begin{equation*}
\lim _{p \rightarrow \infty} \max \left\{\left|x_{k}^{p}-x_{k-1}^{p}\right|: k=1,2, \cdots, n_{p}\right\}=0 \tag{63.7.28}
\end{equation*}
$$

then

$$
\begin{equation*}
\int_{(a, b]} f_{-} d A=\lim _{p \rightarrow \infty} \sum_{k=1}^{n_{p}} f\left(x_{k-1}^{p}\right)\left(A\left(x_{k}^{p}\right)-A\left(x_{k-1}^{p}\right)\right) \tag{63.7.29}
\end{equation*}
$$

More generally, let

$$
D \equiv \cup_{p=1}^{\infty}\left\{x_{0}^{p}, \cdots, x_{n_{p}}^{p}\right\}_{p=1}^{\infty}
$$

and

$$
f_{-}(t)=\lim _{s \rightarrow t-, s \in D} f(s)
$$

Then 63.7.29 holds.
Proof: For $x \in f^{-1}((a, \infty))$, denote by $I_{x}$ the union of all intervals containing $x$ such that $f(y)$ is larger than $a$ for all $y$ in the interval. Since $f$ is right continuous, each $I_{x}$ has positive length. Now if $I_{x}$ and $I_{y}$ are two of these intervals, then either they must have empty intersection or they are the same interval. Thus $f^{-1}((a, \infty))$ is of the form $\cup_{x \in f^{-1}((a, \infty))} I_{x}$ and there can only be countably many distinct intervals because each has positive length and $\mathbb{R}$ is separable. Hence $f^{-1}((a, \infty))$ equals the countable union of intervals and is therefore, Borel measurable. Now

$$
f_{-}(x)=\lim _{n \rightarrow \infty} f\left(x-r_{n}\right) \equiv \lim _{n \rightarrow \infty} f_{r_{n}}(x)
$$

where $r_{n}$ is a decreasing sequence converging to 0 . Now each $f_{r_{n}}$ is Borel measurable by the first part of the proof because it is right continuous and so it follows the same is true of $f_{-}$.

Finally consider the claim about the integral. Since $A$ is right continuous, a simple argument involving the dominated convergence theorem and approximating $(c, d]$ with a piecewise linear continuous function nonzero only on $(c, d+h)$ which approximates $\mathscr{X}_{(c, d]}$ will show that for $\mu$ the measure of Definition 63.7.4

$$
\mu((c, d])=A(d)-A(c) .
$$

Therefore, the sum in 63.7.29 is of the form

$$
\sum_{k=1}^{n_{p}} f\left(x_{k-1}^{p}\right) \mu\left(\left(x_{k-1}, x_{k}\right]\right)=\int_{(a, b]} \sum_{k=1}^{n_{p}} f\left(x_{k-1}^{p}\right) \mathscr{X}_{\left(x_{k-1}, x_{k}\right]} d \mu
$$

and by 63.7.28

$$
\lim _{p \rightarrow \infty} \sum_{k=1}^{n_{p}} f\left(x_{k-1}^{p}\right) \mathscr{X}_{\left(x_{k-1}, x_{k}\right]}(x)=f_{-}(x)
$$

for each $x \in(a, b]$. Therefore, since $f$ is bounded, 63.7.29 follows from the dominated convergence theorem. The last claim follows the same way. This proves the lemma.

Definition 63.7.6 An increasing stochastic process, $\{A(t)\}$ which is right continuous is said to be natural if $A(0)=0$ and whenever $\{\xi(t)\}$ is a bounded right continuous martingale,

$$
\begin{equation*}
E(A(t) \xi(t))=E\left(\int_{(0, t]} \xi_{-}(s) d A(s)\right) \tag{63.7.30}
\end{equation*}
$$

Here

$$
\xi_{-}(s, \omega) \equiv \lim _{r \rightarrow s-, r \in D} \xi(r, \omega)
$$

a.e. where $D$ is a countable dense subset of $[0, t]$. By Corollary 62.8.2 the right side of 63.7.30 is not dependent on the choice of $D$ since if $\xi_{-}$is computed using two different dense subsets, the two random variables are equal a.e.

Some discussion is in order for this definition. Pick $\omega \in \Omega$. Then since $A$ is right continuous, the function $t \rightarrow A(t, \omega)$ is increasing and right continuous. Therefore, one can do the Lebesgue Stieltjes integral defined in Definition 63.7.4 for each $\omega$ whenever $f$ is Borel measurable and bounded. Now it is assumed $\{\xi(t)\}$ is bounded and right continuous. By Lemma 63.7.5 $\xi_{-}(t) \equiv \lim _{r \rightarrow t-, r \in D} \xi(r)$ is measurable and by this lemma,

$$
\int_{(0, t]} \xi_{-}(s) d A(s)=\lim _{p \rightarrow \infty} \sum_{k=1}^{n_{p}} \xi\left(t_{k-1}^{p}\right)\left(A\left(t_{k}^{p}\right)-A\left(t_{k-1}^{p}\right)\right)
$$

where $\left\{t_{k}^{p}\right\}_{k=1}^{n_{p}}$ is a sequence of partitions of $[0, t]$ such that

$$
\begin{equation*}
\lim _{p \rightarrow \infty} \max \left\{\left|t_{k}^{p}-t_{k-1}^{p}\right|: k=1,2, \cdots, n_{p}\right\}=0 \tag{63.7.31}
\end{equation*}
$$

and $D \equiv \cup_{p=1}^{\infty} \cup_{k=1}^{n_{p}}\left\{t_{k}^{p}\right\}_{k=1}^{n_{p}}$.
Also, if $t \rightarrow A(t, \omega)$ is right continuous, hence Borel measurable, then for $\xi(t)$ the above bounded right continuous martingale, it follows it makes sense to write

$$
\int_{(0, t]} \xi(s) d A(s)
$$

Consider the right sum,

$$
\sum_{k=1}^{n_{p}} \xi\left(t_{k}^{p}\right)\left(A\left(t_{k}^{p}\right)-A\left(t_{k-1}^{p}\right)\right)
$$

This equals

$$
\int_{(0, t]} \sum_{k=1}^{n_{p}} \xi\left(t_{k}^{p}\right) \mathscr{X}_{\left(t_{k-1}^{p}, t_{k}^{p}\right]}(s) d A(s)
$$

and by right continuity, it follows

$$
\lim _{p \rightarrow \infty} \sum_{k=1}^{n_{p}} \xi\left(t_{k}^{p}\right) \mathscr{X}_{\left(t_{k-1}^{p}, t_{k}^{p}\right]}(s)=\xi(s)
$$

and so the dominated convergence theorem applies and it follows

$$
\lim _{p \rightarrow \infty} \sum_{k=1}^{n_{p}} \xi\left(t_{k}^{p}\right)\left(A\left(t_{k}^{p}\right)-A\left(t_{k-1}^{p}\right)\right)=\int_{(0, t]} \xi(s) d A(s)
$$

where this is a random variable. Thus

$$
\begin{equation*}
E\left(\int_{(0, t]} \xi(s) d A(s)\right)=\int_{\Omega}\left(\lim _{p \rightarrow \infty} \int_{(0, t]} \sum_{k=1}^{n_{p}} \xi\left(t_{k}^{p}\right) \mathscr{X}_{\left(t_{k-1}^{p}, t_{k}^{p}\right]}(s) d A(s)\right) d P \tag{63.7.32}
\end{equation*}
$$

Now as mentioned above,

$$
\int_{(0, t]} \sum_{k=1}^{n_{p}} \xi\left(t_{k}^{p}\right) \mathscr{X}_{\left(t_{k-1}^{p}, t_{k}^{p}\right]}(s) d A(s)=\sum_{k=1}^{n_{p}} \xi\left(t_{k}^{p}\right)\left(A\left(t_{k}^{p}\right)-A\left(t_{k-1}^{p}\right)\right)
$$

and since $A$ is increasing, this is bounded above by an expression of the form $C A(t)$, a function in $L^{1}$. Therefore, by the dominated convergence theorem, 63.7.32 reduces to

$$
\begin{align*}
& \lim _{p \rightarrow \infty} \int_{\Omega} \int_{(0, t]} \sum_{k=1}^{n_{p}} \xi\left(t_{k}^{p}\right) \mathscr{X}_{\left(t_{k-1}^{p}, t_{k}^{p}\right]}(s) d A(s) d P \\
= & \lim _{p \rightarrow \infty} \int_{\Omega} \sum_{k=1}^{n_{p}} \xi\left(t_{k}^{p}\right)\left(A\left(t_{k}^{p}\right)-A\left(t_{k-1}^{p}\right)\right) d P \\
= & \lim _{p \rightarrow \infty} \int_{\Omega}\left(\sum_{k=1}^{n_{p}} \xi\left(t_{k}^{p}\right) A\left(t_{k}^{p}\right)-\sum_{k=0}^{n_{p}-1} \xi\left(t_{k+1}^{p}\right) A\left(t_{k}^{p}\right)\right) d P \\
= & \lim _{p \rightarrow \infty} \sum_{k=1}^{n_{p}-1} \int_{\Omega}\left(\xi\left(t_{k}^{p}\right)-\xi\left(t_{k+1}^{p}\right)\right) A\left(t_{k}^{p}\right) d P+\int_{\Omega} \xi(t) A(t) d P . \tag{63.7.33}
\end{align*}
$$

Since $\xi$ is a martingale,

$$
\begin{aligned}
\int_{\Omega} \xi\left(t_{k+1}^{p}\right) A\left(t_{k}^{p}\right) d P & =\int_{\Omega} E\left(\xi\left(t_{k+1}^{p}\right) A\left(t_{k}^{p}\right) \mid \mathscr{F}_{t_{k}^{p}}\right) d P \\
& =\int_{\Omega} A\left(t_{k}^{p}\right) E\left(\xi\left(t_{k+1}^{p}\right) \mid \mathscr{F}_{t_{k}^{p}}\right) d P \\
& =\int_{\Omega} A\left(t_{k}^{p}\right) \xi\left(t_{k}^{p}\right) d P
\end{aligned}
$$

and so in 63.7.33 the term with the sum equals 0 and it reduces to

$$
E(\xi(t) A(t))
$$

This is sufficiently interesting to state as a lemma.

Lemma 63.7.7 Let A be an increasing adapted stochastic process which is right continuous. Also let $\xi(t)$ be a bounded right continuous martingale. Then

$$
E(\xi(t) A(t))=E\left(\int_{(0, t]} \xi(s) d A(s)\right)
$$

and $A$ is natural, if and only if for all such bounded right continuous martingales,

$$
E(\xi(t) A(t))=E\left(\int_{(0, t]} \xi(s) d A(s)\right)=E\left(\int_{(0, t]} \xi_{-}(s) d A(s)\right)
$$

Lemma 63.7.8 Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $\mathscr{G}$ be a $\sigma$ algebra contained in $\mathscr{F}$. Suppose also that $\left\{f_{n}\right\}$ is a sequence in $L^{1}(\Omega)$ which converges weakly to $f$ in $L^{1}(\Omega)$. That is, for every $h \in L^{\infty}(\Omega)$,

$$
\int_{\Omega} f_{n} h d P \rightarrow \int_{\Omega} f h d P
$$

Then $E\left(f_{n} \mid \mathscr{G}\right)$ converges weakly in $L^{1}(\Omega)$ to $E(f \mid \mathscr{G})$.

Proof:First note that if $h \in L^{\infty}(\Omega, \mathscr{F})$, then $E(h \mid \mathscr{G}) \in L^{\infty}(\Omega, \mathscr{G})$ because if $A \in \mathscr{G}$,

$$
\int_{A}|E(h \mid \mathscr{G})| d P \leq \int_{A} E(|h| \mid \mathscr{G}) d P=\int_{A}|h| d P
$$

and so if $A=\left[|E(h \mid \mathscr{G})|>\|h\|_{\infty}\right]$, then if $P(A)>0$,

$$
\|h\|_{\infty} P(A)<\int_{A}|E(h \mid \mathscr{G})| d P \leq \int_{A}|h| d P \leq\|h\|_{\infty} P(A)
$$

a contradiction. Hence $P(A)=0$ and so $E(h \mid \mathscr{G}) \in L^{\infty}(\Omega, \mathscr{G})$ as claimed. Let $h \in L^{\infty}(\Omega, \mathscr{G})$.

$$
\begin{aligned}
\int_{\Omega} E\left(f_{n} \mid \mathscr{G}\right) h d P & =\int_{\Omega} E\left(E\left(f_{n} \mid \mathscr{G}\right) h \mid \mathscr{G}\right) d P \\
& =\int_{\Omega} E\left(f_{n} \mid \mathscr{G}\right) E(h \mid \mathscr{G}) d P \\
& =\int_{\Omega} E\left(f_{n} E(h \mid \mathscr{G}) \mid \mathscr{G}\right) d P \\
& =\int_{\Omega} f_{n} E(h \mid \mathscr{G}) d P
\end{aligned}
$$

and so

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \int_{\Omega} E\left(f_{n} \mid \mathscr{G}\right) h d P & =\lim _{n \rightarrow \infty} \int_{\Omega} f_{n} E(h \mid \mathscr{G}) d P \\
& =\int_{\Omega} f E(h \mid \mathscr{G}) d P \\
& =\int_{\Omega} E(f E(h \mid \mathscr{G}) \mid \mathscr{G}) d P \\
& =\int_{\Omega} E(h \mid \mathscr{G}) E(f \mid \mathscr{G}) d P \\
& =\int_{\Omega} E(E(f \mid \mathscr{G}) h \mid \mathscr{G}) d P \\
& =\int_{\Omega} E(f \mid \mathscr{G}) h d P
\end{aligned}
$$

and this proves the lemma.
Next suppose $\{X(t)\}$ is a real submartingale and suppose $X(t)=M(t)+A(t)$ where $A(t)$ is an increasing stochastic process adapted to $\mathscr{F}_{t}$ such that $A(0)=0$ and $\{M(t)\}$ is a martingale adapted to $\mathscr{F}_{t}$. Also let $T$ be a stopping time bounded above by $a$. Then by the optional sampling theorem, and the observation that $\{|M(t)|\}$ is a submartingale

$$
\begin{aligned}
& \int_{[|X(T)| \geq \lambda]}|X(T)| d P \\
\leq & \int_{[|X(T)| \geq \lambda]}|M(T)| d P+\int_{[|X(T)| \geq \lambda]} A(T) d P \\
\leq & \int_{[|X(T)| \geq \lambda]} E\left(|M(a)| \mid \mathscr{F}_{T}\right) d P+\int_{[|X(T)| \geq \lambda]} E\left(A(a) \mid \mathscr{F}_{T}\right) d P \\
\leq & \int_{[|X(T)| \geq \lambda]}|M(a)| d P+\int_{[|X(T)| \geq \lambda]} A(a) d P
\end{aligned}
$$

Now by Theorem 60.6.4,

$$
P\left(\left[\left|X_{T}\right| \geq \lambda\right]\right) \leq \frac{2}{\lambda} E(|X(0)|+|X(a)|)
$$

and so $P([|X(T)| \geq \lambda]) \rightarrow 0$ uniformly for $T$ a stopping time bounded by $a$ as $\lambda \rightarrow \infty$ and so this shows equi integrability of $\{X(T)\}$ because $A(t, \omega) \geq 0$.

This motivates the following definition.
Definition 63.7.9 A stochastic process, $\{X(t)\}$ is called DL if for all $a>0$, the set of random variables, $\{X(T)\}$ for $T$ a stopping time bounded by a is equi integrable.

Example 63.7.10 Let $\{M(t)\}$ be a continuous martingale. Then $\{M(t)\}$ is of class DL.
To show this, let $a>0$ be given and let $T$ be a stopping time bounded by $a$. Then by the optional sampling theorem, $M(0), M(T), M(a)$ is a martingale and so

$$
E\left(M(a) \mid \mathscr{F}_{T}\right)=M(T)
$$

and so by Jensen's inequality, $|M(T)| \leq E\left(|M(a)| \mid \mathscr{F}_{T}\right)$. Therefore,

$$
\begin{align*}
\int_{[|M(T)| \geq \lambda]}|M(T)| d P & \leq \int_{[|M(T)| \geq \lambda]} E\left(|M(a)| \mid \mathscr{F}_{T}\right) d P \\
& =\int_{[|M(T)| \geq \lambda]}|M(a)| d P \tag{63.7.34}
\end{align*}
$$

Now by Theorem 62.5.3,

$$
P([|M(T)| \geq \lambda]) \leq \frac{1}{\lambda} E(|M(a)|)
$$

and so since a given $L^{1}$ function is uniformly integrable, there exists $\delta$ such that if $P(A)<\delta$ then

$$
\int_{A}|M(a)| d P<\varepsilon
$$

Now choose $\lambda$ large enough that

$$
\frac{1}{\lambda} E(|M(a)|)<\delta
$$

Then for such $\lambda$, it follows from 63.7.34 that for any stopping time bounded by $a$,

$$
\int_{[|M(T)| \geq \lambda]}|M(T)| d P<\varepsilon
$$

This shows $M$ is $D L$.
Example 63.7.11 Let $\{X(t)\}$ be a nonnegative submartingale with $t \rightarrow E(X(t))$ right continuous so $\{X(t)\}$ can be considered right continuous. Then $\{X(t)\}$ is $D L$.

To show this, let $T$ be a stopping time bounded by $a>0$. Then by the optional sampling theorem,

$$
\int_{[X(T) \geq \lambda]} X(T) d P \leq \int_{[X(T) \geq \lambda]} X(a) d P
$$

and now by Theorem 60.6.4 on Page 1967

$$
P([X(T) \geq \lambda]) \leq \frac{1}{\lambda} E\left(X_{a}^{+}\right)
$$

Thus if $\varepsilon>0$ is given, there exists $\lambda$ large enough that for any stopping time, $T \leq a$,

$$
\int_{[X(T) \geq \lambda]} X(T) d P \leq \varepsilon
$$

Thus the submartingale is $D L$.
Now with this preparation, here is the Doob Meyer decomposition.

Theorem 63.7.12 Let $\{X(t)\}$ be a submartingale of class $D L$. Then there exists a martingale, $\{M(t)\}$ and an increasing submartingale, $\{A(t)\}$ such that for each $t$,

$$
X(t)=M(t)+A(t)
$$

If $\{A(t)\}$ is chosen to be natural and $A(0)=0$, then with this condition, $\{M(t)\}$ and $\{A(t)\}$ are unique.

Proof: First I will show uniqueness. Suppose then that

$$
X(t)=M(t)+A(t)=M^{\prime}(t)+A^{\prime}(t)
$$

where $M, M^{\prime}$ and $A, A^{\prime}$ satisfy the given conditions. Let $t>0$ and consider $s \in[0, t]$. Then

$$
A(s)-A^{\prime}(s)=M^{\prime}(s)-M(s)
$$

Since $A, A^{\prime}$ are natural, it follows that for $\xi(t)$ a right continuous bounded martingale,

$$
\begin{aligned}
& E\left(\xi(t)\left(A(t)-A^{\prime}(t)\right)\right)=E\left(\int_{(0, t]} \xi_{-}(s) d A(s)\right)-E\left(\int_{(0, t]} \xi_{-}(s) d A^{\prime}(s)\right) \\
& =E\left(\lim _{n \rightarrow \infty} \sum_{k=1}^{m_{n}} \xi\left(t_{k-1}^{n}\right)\left(A\left(t_{k}^{n}\right)-A\left(t_{k-1}^{n}\right)\right)-\sum_{k=1}^{m_{n}} \xi\left(t_{k-1}^{n}\right)\left(A^{\prime}\left(t_{k}^{n}\right)-A^{\prime}\left(t_{k-1}^{n}\right)\right)\right)
\end{aligned}
$$

where $\left\{t_{k}^{n}\right\}_{k=0}^{m_{n}}$ is a sequence of partitions of $[0, t]$ such that these are equally spaced points, $\lim _{n \rightarrow \infty} t_{k+1}^{n}-t_{k}^{n}=0$, and $\left\{t_{k}^{n}\right\}_{k=0}^{m_{n}} \subseteq\left\{t_{k}^{n+1}\right\}_{k=0}^{m_{n+1}}$. Then since $A(t)$ and $A^{\prime}(t)$ are increasing, the absolute value of each sum is bounded above by an expression of the form

$$
C A(t) \text { or } C A^{\prime}(t)
$$

and so the dominated convergence theorem can be applied to get the above expression to equal

$$
\lim _{n \rightarrow \infty} E\left(\sum_{k=1}^{m_{n}} \xi\left(t_{k-1}^{n}\right)\left(A\left(t_{k}^{n}\right)-A\left(t_{k-1}^{n}\right)\right)-\sum_{k=1}^{m_{n}} \xi\left(t_{k-1}^{n}\right)\left(A^{\prime}\left(t_{k}^{n}\right)-A^{\prime}\left(t_{k-1}^{n}\right)\right)\right)
$$

Now using $X=A+M$ and $X=A^{\prime}+M^{\prime}$

$$
=\lim _{n \rightarrow \infty} E\left(\sum_{k=1}^{m_{n}} \xi\left(t_{k-1}^{n}\right)\left(M\left(t_{k}^{n}\right)-M\left(t_{k-1}^{n}\right)\right)-\sum_{k=1}^{m_{n}} \xi\left(t_{k-1}^{n}\right)\left(M^{\prime}\left(t_{k}^{n}\right)-M^{\prime}\left(t_{k-1}^{n}\right)\right)\right) .
$$

Both terms in the above equal 0 . Here is why.

$$
\begin{aligned}
E\left(\xi\left(t_{k-1}^{n}\right) M\left(t_{k}^{n}\right)\right) & =E\left(E\left(\xi\left(t_{k-1}^{n}\right) M\left(t_{k}^{n}\right) \mid \mathscr{F}_{t_{k-1}^{n}}^{n}\right)\right) \\
& =E\left(\xi\left(t_{k-1}^{n}\right) E\left(M\left(t_{k}^{n}\right) \mid \mathscr{F}_{t_{k-1}^{n}}^{n}\right)\right) \\
& =E\left(\xi\left(t_{k-1}^{n}\right) M\left(t_{k-1}^{n}\right)\right) .
\end{aligned}
$$

Thus the expected value of the first sum equals 0 . Similarly, the expected value of the second sum equals 0 . Hence this has shown that for any bounded right continuous martingale, $\{\xi(s)\}$ and $t>0$,

$$
E\left(\xi(t)\left(A(t)-A^{\prime}(t)\right)\right)=0 .
$$

Now let $\xi$ be a bounded random variable and let $\xi(t)$ be a right continuous version of the martingale $E\left(\xi \mid \mathscr{F}_{t}\right)$. Then

$$
\begin{aligned}
0 & =E\left(E\left(\xi \mid \mathscr{F}_{t}\right)\left(A(t)-A^{\prime}(t)\right)\right)=E\left(E\left(\xi\left(A(t)-A^{\prime}(t)\right) \mid \mathscr{F}_{t}\right)\right) \\
& =E\left(\xi\left(A(t)-A^{\prime}(t)\right)\right)
\end{aligned}
$$

and since $\xi$ is arbitrary, it follows that $A(t)=A^{\prime}(t)$ a.e. which proves uniqueness.
Because of the uniqueness assertion, it suffices to prove the theorem on an arbitrary interval, $[0, a]$.

Without loss of generality, it can be assumed $X(0)=0$ since otherwise, you could simply consider $X(t)-X(0)$ in its place and then at the end, add $X(0)$ to $M(t)$. Let $\left\{t_{k}^{n}\right\}_{k=0}^{m_{n}}$ be a sequence of partitions of $[0, a]$ such that these are equally spaced points, $\lim _{n \rightarrow \infty} t_{k+1}^{n}-$ $t_{k}^{n}=0$, and $\left\{t_{k}^{n}\right\}_{k=0}^{m_{n}} \subseteq\left\{t_{k}^{n+1}\right\}_{k=0}^{m_{n+1}}$. Then consider the submartingale, $\left\{X\left(t_{k}^{n}\right)\right\}_{k=0}^{m_{n}}$. Theorem 63.7.1 implies there exists a unique martingale, and increasing submartingale,

$$
\left\{M\left(t_{k}^{n}\right)\right\}_{k=0}^{m_{n}} \text { and }\left\{A\left(t_{k}^{n}\right)\right\}_{k=0}^{m_{n}}
$$

respectively such that $M(0)=0=A(0)$,

$$
X\left(t_{k}^{n}\right)=M^{n}\left(t_{k}^{n}\right)+A^{n}\left(t_{k}^{n}\right)
$$

and $A^{n}\left(t_{k}^{n}\right)$ is $\mathscr{F}_{k-1}^{n}$ measurable. Recall how these were defined.

$$
\begin{gathered}
A^{n}\left(t_{k}^{n}\right)=\sum_{j=1}^{k} E\left(X\left(t_{j}^{n}\right)-X\left(t_{j-1}^{n}\right) \mid \mathscr{F}_{j-1}^{n}\right), A^{n}(0)=0 \\
M^{n}\left(t_{k}^{n}\right)=X\left(t_{k}^{n}\right)-A^{n}\left(t_{k}^{n}\right)
\end{gathered}
$$

I want to show that $\left\{A^{n}(a)\right\}$ is equi integrable. From this there will be a weakly convergent subsequence and nice things will happen. Define $T^{n}(\omega)$ to equal $t_{j-1}^{n}$ where $t_{j}^{n}$ is the first time where $A^{n}\left(t_{j}^{n}, \omega\right) \geq \lambda$ or $T^{n}(\omega)=a$ if this never happens. I want to say that $T^{n}$ is a stopping time and so I need to verify that $\left[T^{n} \leq t_{j}^{n}\right] \in \mathscr{F}_{t_{j}^{n}}$ for each $j$. If $\omega \in\left[T^{n} \leq t_{j}^{n}\right]$, then this means the first time, $t_{k}^{n}$, where $A^{n}\left(t_{k}^{n}, \omega\right) \geq \lambda$ is such that $t_{k}^{n} \leq t_{j+1}^{n}$. Since $A_{k}^{n}$ is increasing in $k$,

$$
\begin{aligned}
{\left[T^{n} \leq t_{j}^{n}\right] } & =\cup_{k=0}^{j+1}\left[A^{n}\left(t_{k}^{n}\right) \geq \lambda\right] \\
& =\left[A^{n}\left(t_{j+1}^{n}\right) \geq \lambda\right] \in \mathscr{F}_{t_{j}^{n}}
\end{aligned}
$$

Note $T^{n}$ only has the values $t_{k}^{n}$. Thus for $t \in\left[t_{j-1}^{n}, t_{j}^{n}\right)$,

$$
\left[T^{n} \leq t\right]=\left[T^{n} \leq t_{j-1}^{n}\right] \in \mathscr{F}_{t_{j-1}^{n}}^{n} \subseteq \mathscr{F}_{t}
$$

Thus $T^{n}$ is one of those stopping times bounded by $a$. Since $\{X(t)\}$ is $D L$, this shows $\left\{X\left(T^{n}\right)\right\}$ is equi integrable. Now from the definition of $T^{n}$, it follows

$$
A^{n}\left(T^{n}\right) \leq \lambda
$$

Recall $T^{n}(\omega)=t_{j-1}^{n}$ where $t_{j}^{n}$ is the first time where $A^{n}\left(t_{j}^{n}, \omega\right) \geq \lambda$ or $T^{n}(\omega)=a$ if this never happens. Thus $T^{n}$ is such that it is before $A^{n}$ gets larger than $\lambda$. Thus,

$$
\begin{gathered}
\int_{\left[A^{n}(a) \geq 2 \lambda\right]} \frac{1}{2} A^{n}(a) d P \leq \int_{\left[A^{n}(a) \geq 2 \lambda\right]}\left(A^{n}(a)-\lambda\right) d P \\
\leq \int_{\left[A^{n}(a) \geq 2 \lambda\right]}\left(A^{n}(a)-A^{n}\left(T^{n}\right)\right) d P \\
\leq \int_{\Omega}\left(A^{n}(a)-A^{n}\left(T^{n}\right)\right) d P \\
=\int_{\Omega}\left(X(a)-M^{n}(a)-\left(X\left(T^{n}\right)\right)-M^{n}\left(T^{n}\right)\right) d P \\
=\int_{\Omega}\left(X(a)-X\left(T^{n}\right)\right) d P
\end{gathered}
$$

Because by the discrete optional sampling theorem,

$$
\int_{\Omega}\left(M^{n}(a)-M^{n}\left(T^{n}\right)\right) d P=0
$$

Remember $\left\{M^{n}\left(t_{k}^{n}\right)\right\}_{k=0}^{m_{n}}$ was a martingale.

$$
\begin{aligned}
\int_{\Omega}\left(X(a)-X\left(T^{n}\right)\right) d P= & \int_{\left[A^{n}(a) \geq \lambda\right]}\left(X(a)-X\left(T^{n}\right)\right) d P \\
& +\int_{\left[A^{n}(a)<\lambda\right]}\left(X(a)-X\left(T^{n}\right)\right) d P .
\end{aligned}
$$

The second of the integrals on the right is such that for $\omega$ in this set, $T^{n}(\omega)=a$ and so the second integral equals 0 . Hence from the above,

$$
\int_{\left[A^{n}(a) \geq 2 \lambda\right]} \frac{1}{2} A^{n}(a) d P \leq \int_{\left[A^{n}(a) \geq \lambda\right]}\left(X(a)-X\left(T^{n}\right)\right) d P
$$

and since $\{X(t)\}$ is $D L$, this shows $\left\{A^{n}(a)\right\}_{n=1}^{\infty}$ is equi integrable.
By Corollary 20.9.6 on Page 640 there exists a subsequence $\left\{A^{n_{k}}(a)\right\}_{k=1}^{\infty}$ which converges weakly in $L^{1}(\Omega)$ to $A(a)$. By Lemma 63.7 .8 it also follows that $E\left(A^{n_{k}}(a) \mid \mathscr{F}_{t}\right)$ converges weakly to $E\left(A(a) \mid \mathscr{F}_{t}\right)$ in $L^{1}(\Omega)$. Now define

$$
M(t) \equiv E\left(X(a)-A(a) \mid \mathscr{F}_{t}\right)
$$

Thus it is obvious from properties of conditional expectation that $\{M(t)\}$ is a martingale adapted to $\mathscr{F}_{t}$ and without loss of generality, it is a right continuous version. Let

$$
A(t) \equiv X(t)-M(t)
$$

Then since $\{X(t)\}$ is a submartingale, it follows $\{A(t)\}$ is also a submartingale.
It remains to show several things. First, it is necessary to show $A(t)$ is increasing in $t$ and $A(0)=0$. To see this, let $s<t, s, t \in \cup_{n=1}^{\infty} \cup_{k=0}^{m_{n}} t_{k}^{n}$. Then letting $n$ large enough both $s, t$ are in $\cup_{k=0}^{m_{n}} t_{k}^{n}$. Only consider such $n$. Let $t=t_{k(t)}^{n}, s=t_{k(s)}^{n}$ and let $h \in L^{\infty}(\Omega), h \geq 0$. Then

$$
\begin{array}{r}
\int_{\Omega}(A(t)-A(s)) h d P=\int_{\Omega}(X(t)-M(t)-(X(s)-M(s))) h d P \\
\int_{\Omega}\left(X(t)-E\left(X(a)-A(a) \mid \mathscr{F}_{t}\right)-\left(X(s)-E\left(X(a)-A(a) \mid \mathscr{F}_{s}\right)\right)\right) h d P \tag{63.7.35}
\end{array}
$$

Now by Lemma 63.7.8, the following weak limit holds.

$$
\begin{aligned}
E\left(X(a)-A(a) \mid \mathscr{F}_{t}\right) & =\lim _{k \rightarrow \infty} E(\overbrace{X(a)-A^{n_{k}}(a)}^{M^{n_{k}}(a)} \mathscr{F}_{t}) \\
& =\lim _{k \rightarrow \infty} M^{n_{k}}(t)
\end{aligned}
$$

A similar formula holds for $s$ in place of $t$. Then the expression in 63.7.35 equals

$$
\begin{aligned}
& =\lim _{k \rightarrow \infty} \int_{\Omega}\left(X(t)-M^{n_{k}}(t)-\left(X(s)-M^{n_{k}}(s)\right)\right) h d P \\
& =\lim _{k \rightarrow \infty} \int_{\Omega}\left(A^{n_{k}}(t)-A^{n_{k}}(s)\right) h d P \geq 0
\end{aligned}
$$

Since $h \geq 0$ is arbitrary, this shows $A(t)-A(s) \geq 0$ a.e. Not requiring $h \geq 0$, the above argument also shows that for $s, t \in \cup_{n=1}^{\infty} \cup_{k=0}^{m_{n}} t_{k}^{n}$,

$$
\begin{equation*}
A(t)-A(s)=\text { weak } \lim _{p \rightarrow \infty} A^{n_{p}}(t)-A^{n_{p}}(s) \tag{63.7.36}
\end{equation*}
$$

Now consider the claim that $A(0)=0$. Recall

$$
A(0) \equiv X(0)-E\left(X(a)-A(a) \mid \mathscr{F}_{0}\right)=-E\left(X(a)-A(a) \mid \mathscr{F}_{0}\right)
$$

and so

$$
\begin{aligned}
A(0) & =\lim _{k \rightarrow \infty}-E\left(X(a)-A^{n_{k}}(a) \mid \mathscr{F}_{0}\right) \\
& =\lim _{k \rightarrow \infty}-E\left(M^{n_{k}}(a) \mid \mathscr{F}_{0}\right)=\lim _{k \rightarrow \infty}-M^{n_{k}}(0)=0 .
\end{aligned}
$$

This proves the theorem except for the claim that $A(t)$ is natural. Let $\xi(t)$ be a bounded right continuous martingale. I need to consider

$$
E\left(\int_{(0, t]} \xi_{-}(s) d A(s)\right)
$$

and show it equals $\xi(t) A(t)$. First consider the case $t=a$. By Lemma 63.7.5,

$$
\begin{equation*}
E\left(\int_{(0, a]} \xi_{-}(s) d A(s)\right) \equiv E\left(\lim _{k \rightarrow \infty} \sum_{j=1}^{m_{n_{k}}} \xi\left(t_{j-1}^{n_{k}}\right)\left(A\left(t_{j}^{n_{k}}\right)-A\left(t_{j-1}^{n_{k}}\right)\right)\right) \tag{63.7.37}
\end{equation*}
$$

Since $\xi$ is bounded, you can take the limit outside. This follows from the dominated convergence theorem and the fact, shown above that $A$ is increasing and nonnegative. Here is why.

$$
0 \leq\left|\xi\left(t_{j-1}^{n_{k}}\right)\right| A\left(t_{j}^{n_{k}}\right) \leq A(a) C
$$

where $C$ is a constant larger than the values of $\xi$. Thus the above equals

$$
\begin{gather*}
\lim _{k \rightarrow \infty} E\left(\sum_{j=1}^{m_{n_{k}}} \xi\left(t_{j-1}^{n_{k}}\right)\left(A\left(t_{j}^{n_{k}}\right)-A\left(t_{j-1}^{n_{k}}\right)\right)\right) \\
=\lim _{k \rightarrow \infty} E\left(\sum_{j=1}^{m_{n_{k}}} \xi\left(t_{j-1}^{n_{k}}\right)\left(X\left(t_{j}^{n_{k}}\right)-M\left(t_{j}^{n_{k}}\right)-\left(X\left(t_{j-1}^{n_{k}}\right)-M\left(t_{j-1}^{n_{k}}\right)\right)\right)\right) \\
=\lim _{k \rightarrow \infty} E\left(\sum_{j=1}^{m_{n_{k}}} \xi\left(t_{j-1}^{n_{k}}\right)\left(X\left(t_{j}^{n_{k}}\right)-X\left(t_{j-1}^{n_{k}}\right)\right)\right) \tag{63.7.38}
\end{gather*}
$$

because

$$
\begin{aligned}
E\left(\xi\left(t_{j-1}^{n_{k}}\right) M\left(t_{j}^{n_{k}}\right)\right) & =E\left(E\left(\xi\left(t_{j-1}^{n_{k}}\right) M\left(t_{j}^{n_{k}}\right) \mid \mathscr{F}_{t_{j-1}^{n_{k}}}\right)\right) \\
& =E\left(\xi\left(t_{j-1}^{n_{k}}\right) E\left(M\left(t_{j}^{n_{k}}\right) \mid \mathscr{F}_{t_{j-1}^{n_{k}}}\right)\right) \\
& =E\left(\xi\left(t_{j-1}^{n_{k}}\right) M\left(t_{j-1}^{n_{k}}\right)\right)
\end{aligned}
$$

since $M$ is a martingale. Now by a similar trick, this time using that $\left\{M^{n_{k}}\left(t_{j}^{n_{k}}\right)\right\}_{j=0}^{m_{n_{k}}}$ is a martingale, 63.7.38 equals

$$
\begin{equation*}
\lim _{k \rightarrow \infty} E\left(\sum_{j=1}^{m_{n_{k}}} \xi\left(t_{j-1}^{n_{k}}\right)\left(A^{n_{k}}\left(t_{j}^{n_{k}}\right)-A^{n_{k}}\left(t_{j-1}^{n_{k}}\right)\right)\right) \tag{63.7.39}
\end{equation*}
$$

and now recall that $A^{n_{k}}\left(t_{j}^{n_{k}}\right)$ is $\mathscr{F}_{t_{j-1}}^{n_{k}}$ measurable. This will now be used to change the subscript of $t_{j-1}^{n_{k}}$ in $\xi\left(t_{j-1}^{n_{k}}\right)$ to a $j$. 63.7.39 equals

$$
\begin{aligned}
& =\lim _{k \rightarrow \infty} \sum_{j=1}^{m_{n_{k}}} E\left(E\left(\xi\left(t_{j}^{n_{k}}\right) \mid \mathscr{F}_{t_{j-1}^{n_{k}}}\right)\left(A^{n_{k}}\left(t_{j}^{n_{k}}\right)-A^{n_{k}}\left(t_{j-1}^{n_{k}}\right)\right)\right) \\
& =\lim _{k \rightarrow \infty} \sum_{j=1}^{m_{n_{k}}} E\left(E\left(\xi\left(t_{j}^{n_{k}}\right)\left(A^{n_{k}}\left(t_{j}^{n_{k}}\right)-A^{n_{k}}\left(t_{j-1}^{n_{k}}\right)\right) \mid \mathscr{F}_{t_{j-1}^{n_{k}}}\right)\right) \\
& =\lim _{k \rightarrow \infty} \sum_{j=1}^{m_{n_{k}}} E\left(\xi\left(t_{j}^{n_{k}}\right)\left(A^{n_{k}}\left(t_{j}^{n_{k}}\right)-A^{n_{k}}\left(t_{j-1}^{n_{k}}\right)\right)\right) \\
& =\lim _{k \rightarrow \infty} E\left(\sum_{j=1}^{m_{n_{k}}} \xi\left(t_{j}^{n_{k}}\right)\left(A^{n_{k}}\left(t_{j}^{n_{k}}\right)-A^{n_{k}}\left(t_{j-1}^{n_{k}}\right)\right)\right)
\end{aligned}
$$

From this all that remains is to write the above as

$$
\begin{aligned}
& \lim _{k \rightarrow \infty} E\left(\sum_{j=1}^{m_{n_{k}}} \xi\left(t_{j}^{n_{k}}\right) A^{n_{k}}\left(t_{j}^{n_{k}}\right)-\sum_{j=0}^{m_{n_{k}}-1} \xi\left(t_{j+1}^{n_{k}}\right) A^{n_{k}}\left(t_{j}^{n_{k}}\right)\right) \\
= & \lim _{k \rightarrow \infty}\left(E\left(\xi(a) A^{n_{k}}(a)\right)+E\left(\sum_{j=1}^{m_{n_{k}-1}}\left(\xi\left(t_{j}^{n_{k}}\right)-\xi\left(t_{j+1}^{n_{k}}\right)\right) A^{n_{k}}\left(t_{j}^{n_{k}}\right)\right)\right)
\end{aligned}
$$

Now using the fact $\xi$ is a martingale, this last term equals 0 . Here is why.

$$
\begin{aligned}
E\left(\xi\left(t_{j+1}^{n_{k}}\right) A^{n_{k}}\left(t_{j}^{n_{k}}\right)\right) & =E\left(E\left(\xi\left(t_{j+1}^{n_{k}}\right) A^{n_{k}}\left(t_{j}^{n_{k}}\right) \mid \mathscr{F}_{t_{j}^{n_{k}}}\right)\right) \\
& =E\left(A^{n_{k}}\left(t_{j}^{n_{k}}\right) E\left(\xi\left(t_{j+1}^{n_{k}}\right) \mid \mathscr{F}_{t_{j}^{n_{k}}}\right) \mid \mathscr{F}_{t_{j}^{n_{k}}}\right) \\
& =E\left(A^{n_{k}}\left(t_{j}^{n_{k}}\right) \xi\left(t_{j}^{n_{k}}\right) \mid \mathscr{F}_{t_{j}^{n_{k}}}\right)
\end{aligned}
$$

The first term converges to $E(\xi(a) A(a))$ because this was how $A(a)$ was obtained, as a weak limit in $L^{1}(\Omega)$ of $A^{n_{k}}(a)$. Also by Lemma 63.7.7,

$$
E(\xi(a) A(a))=E\left(\int_{(0, a]} \xi(s) d A(s)\right)
$$

From 63.7.37 this has now shown that

$$
E(\xi(a) A(a))=E\left(\int_{(0, a]} \xi_{-}(s) d A(s)\right)
$$

To get the desired result on $(0, t]$, apply what was just shown to a "stopped martingale",

$$
\begin{gathered}
\xi^{t}(s) \equiv \begin{cases}\xi(s) \text { if } s \leq t \\
\xi(t) \text { if } s>t\end{cases} \\
E\left(\int_{(0, t]} \xi(s) d A(s)\right)+(A(a)-A(t)) E(\xi(t)) \\
=E\left(\int_{(0, a]} \xi^{t}(s) d A(s)\right)
\end{gathered}
$$

From what was shown above,

$$
\begin{aligned}
& =E\left(\int_{(0, a]} \xi_{-}^{t}(s) d A(s)\right) \\
& =E\left(\int_{(0, t]} \xi_{-}(s) d A(s)+\int_{(t, a]} \xi(t) s A(s)\right) \\
& =E\left(\int_{(0, t]} \xi_{-}(s) d A(s)\right)+(A(a)-A(t)) E(\xi(t))
\end{aligned}
$$

and so

$$
E\left(\int_{(0, t]} \xi(s) d A(s)\right)=E\left(\int_{(0, t]} \xi_{-}(s) d A(s)\right)
$$

which shows $A$ is natural by Lemma 63.7.7. This proves the theorem.
There is another interesting variation of the above theorem. It involves the following definition.

Definition 63.7.13 A submartingale, $\{X(t)\}$ is said to be $D$ if

$$
\left\{X_{T}: T<\infty \text { is a stopping time }\right\}
$$

is equi integrable.
In this case, you can consider partitions of the entire positive real line and the martingales, $\left\{M\left(t_{k}^{n}\right)\right\}_{k=0}^{\infty}$ and $\left\{A\left(t_{k}^{n}\right)\right\}_{k=0}^{\infty}$ as before. This time you don't stop at $m_{n}$. By the submartingale convergence theorem, you can argue there exists $A_{\infty}^{n}=\lim _{k \rightarrow \infty} A\left(t_{k}^{n}\right)$. Then repeat the above argument using $A_{\infty}^{n}$ in place of $A^{n}(a)$. This time you get $\{A(t)\}$ equi integrable. Thus the following corollary is obtained.

Corollary 63.7.14 Let $\{X(t)\}$ be a right continuous submartingale of class $D$. Then there exists a right continuous martingale, $\{M(t)\}$ and a right continuous increasing submartingale, $\{A(t)\}$ such that for each $t$,

$$
X(t)=M(t)+A(t)
$$

If $\{A(t)\}$ is chosen to be natural and $A(0)=0$, then with this condition, $\{M(t)\}$ and $\{A(t)\}$ are unique. Furthermore $\{M(t)\}$ and $\{A(t)\}$ are equi integrable on $[0, \infty)$.

In the above theorem, $\{X(t)\}$ was a submartingale and so it has a right continuous version. What if $\{X(t)\}$ is actually continuous? Can one conclude that $A(t)$ and $M(t)$ are also continuous? The answer is yes.

Theorem 63.7.15 Let $\{X(t)\}$ be a right continuous submartingale of class DL. Then there exists a right continuous martingale, $\{M(t)\}$ and a right continuous increasing submartingale, $\{A(t)\}$ such that for each $t$,

$$
X(t)=M(t)+A(t) .
$$

If $\{A(t)\}$ is chosen to be natural and $A(0)=0$, then with this condition, $\{M(t)\}$ and $\{A(t)\}$ are unique. Also, if $\{X(t)\}$ is continuous, $(t \rightarrow X(t, \omega)$ is continuous for a.e. $\omega)$ then the same is true of $\{A(t)\}$ and $\{M(t)\}$.

Proof: The first part is done above. Let $\{X(t)\}$ be continuous. As before, let $\left\{t_{k}^{n}\right\}_{k=0}^{m_{n}}$ be a sequence of partitions of $[0, a]$ such that these are equally spaced points, $\lim _{n \rightarrow \infty} t_{k+1}^{n}-$ $t_{k}^{n}=0$, and $\left\{t_{k}^{n}\right\}_{k=0}^{m_{n}} \subseteq\left\{t_{k}^{n+1}\right\}_{k=0}^{m_{n+1}}$ where here $a>0$ is an arbitrary positive number and let $\lambda>0$ be an arbitrary positive number. Define

$$
\xi^{n}(t) \equiv E\left(\min \left(\lambda, A\left(t_{j}^{n}\right)\right) \mid \mathscr{F}_{t}\right) \text { for } t_{j-1}^{n}<t \leq t_{j}^{n}
$$

Thus on $\left(t_{j-1}^{n}, t_{j}^{n}\right] \xi^{n}(t)$ is a bounded martingale. Assuming we are dealing with a right continuous version of this martingale so there are no measurability questions, it follows since $A$ is natural,

$$
E\left(\int_{\left(t_{j-1}^{n}, t_{j}^{n}\right]} \xi^{n}(s) d A(s)\right)=E\left(\int_{\left(t_{j-1}^{n}, t_{j}^{n}\right]} \xi_{-}^{n}(s) d A(s)\right)
$$

where here

$$
\xi_{-}^{n}(s, \omega) \equiv \lim _{r \rightarrow s-, r \in D} \xi^{n}(s, \omega) \text { a.e. }
$$

for $D \equiv \cup_{n=1}^{\infty} \cup_{k=1}^{m_{n}}\left\{t_{k}^{n}\right\}_{k=0}^{m_{n}}$. Thus, adding these up for all the intervals, $\left(t_{j-1}^{n}, t_{j}^{n}\right]$ yields

$$
E\left(\int_{(0, a]} \xi^{n}(s) d A(s)\right)=E\left(\int_{(0, a]} \xi_{-}^{n}(s) d A(s)\right)
$$

I want to show that for a.e. $\omega, \xi^{n_{k}}(t, \omega)$ converges uniformly to

$$
\min (\lambda, A(t, \omega)) \equiv \lambda \wedge A(t, \omega)
$$

on $(0, a]$. From this it will follow

$$
E\left(\int_{(0, a]} \lambda \wedge A(s, \omega) d A(s)\right)=E\left(\int_{(0, a]} \lambda \wedge A_{-}(s, \omega) d A(s)\right)
$$

Now since $s \rightarrow A(s, \omega)$ is increasing, there is no problem in writing $A_{-}(s, \omega)$ and the above equation will suffice to show with simple considerations that for a.e. $\omega, s \rightarrow A(s, \omega)$ is left continuous. Since $\{A(s)\}$ is a submartingale already, it has a right continuous version which we are using in the above. Thus for a.e. $\omega$ it must be the case that $s \rightarrow A(s, \omega)$ is continuous. Let $t \in\left(t_{j-1}^{n}, t_{j}^{n}\right.$ ]. Then since $\lambda \wedge A(t)$ is $\mathscr{F}_{t}$ measurable,

$$
\xi^{n}(t)-\lambda \wedge A(t) \equiv E\left(\lambda \wedge A\left(t_{j}^{n}\right)-\lambda \wedge A(t) \mid \mathscr{F}_{t}\right) \geq 0
$$

because $A(t)$ is increasing.
Now define a stopping time, $T^{n}(\varepsilon)$ for $\varepsilon>0$ by letting $T^{n}(\varepsilon)$ be the infimum of all $t \in[0, a]$ with the property that

$$
\xi^{n}(t)-\lambda \wedge A(t)>\varepsilon
$$

or if this does not happen, then $T^{n}(\varepsilon)=a$. Thus

$$
T^{n}(\varepsilon)(\omega)=a \wedge \inf \left\{t \in[0, a]: \xi^{n}(t, \omega)-\lambda \wedge A(t, \omega)>\varepsilon\right\}
$$

I need to verify $T^{n}(\varepsilon)$ really is a stopping time. Letting $s<a$, it follows that if $\omega \in$ $\left[T^{n}(\varepsilon) \leq s\right]$, then for each $N$, there exists $t \in\left[s, s+\frac{1}{N}\right)$ such that $\xi^{n}(t, \omega)-\lambda \wedge A(t, \omega)>\varepsilon$. Then by right continuity it follows there exists $r \in D \cap\left[s, s+\frac{1}{N}\right)$ such that

$$
\xi^{n}(r, \omega)-\lambda \wedge A(r, \omega)>\varepsilon
$$

Thus

$$
\left[T^{n}(\varepsilon) \leq s\right]=\cap_{N=1}^{\infty} \cup_{r \in D \cap\left[s, s+\frac{1}{N}\right)}\left[\xi^{n}(r, \omega)-\lambda \wedge A(r, \omega)>\varepsilon\right]
$$

and each $\cup_{r \in D \cap\left[s, s+\frac{1}{N}\right)}\left[\xi^{n}(r, \omega)-\lambda \wedge A(r, \omega)>\varepsilon\right] \in \mathscr{F}_{s+1 / N}$ and so

$$
\left[T^{n}(\varepsilon) \leq s\right] \in \cap_{r \in D, r \geq s} \mathscr{F}_{r}=\mathscr{F}_{s+}=\mathscr{F}_{s}
$$

due to the assumption that the filtration is normal. What if $s \geq a$ ? Then from the definition, $\left[T^{n}(\varepsilon) \leq a\right]=\Omega \in \mathscr{F}_{a}$. Thus this really is a stopping time.

Now let $B_{j} \equiv\left[t_{j-1}^{n}<T^{n}(\varepsilon) \leq t_{j}^{n}\right]$. Note that $T^{n}(\varepsilon) \wedge t_{j}^{n}$ is also a stopping time.

$$
\begin{gathered}
\int_{\Omega} \xi_{T^{n}(\varepsilon)}^{n} d P=\sum_{j=1}^{m_{n}} \int_{B_{j}} \xi_{T^{n}(\varepsilon)}^{n} d P=\sum_{j=1}^{m_{n}} \int_{B_{j}} \xi_{T^{n}(\varepsilon) \wedge t_{j}^{n}}^{n} d P \\
=\sum_{j=1}^{m_{n}} \int_{B_{j}} E\left(\xi_{T^{n}(\varepsilon) \wedge t_{j}^{n}}^{n} \mid \mathscr{F}_{T^{n}(\varepsilon) \wedge t_{j}^{n}}\right) d P
\end{gathered}
$$

This is because $B_{j} \in \mathscr{F}_{T^{n}(\varepsilon) \wedge t_{j}^{n}}$. Thus from the definition, the above equals

$$
\begin{align*}
& =\sum_{j=1}^{m_{n}} \int_{B_{j}} E\left(E\left(\lambda \wedge A\left(t_{j}^{n}\right) \mid \mathscr{F}_{T^{n}(\varepsilon) \wedge t_{j}^{n}}\right) \mid \mathscr{F}_{T^{n}(\varepsilon) \wedge t_{j}^{n}}\right) d P \\
& =\sum_{j=1}^{m_{n}} \int_{B_{j}} E\left(\lambda \wedge A\left(t_{j}^{n}\right) \mid \mathscr{F}_{T^{n}(\varepsilon) \wedge t_{j}^{n}}\right) d P \\
& =\sum_{j=1}^{m_{n}} \int_{B_{j}} \lambda \wedge A\left(t_{j}^{n}\right) d P=\int_{\Omega} \lambda \wedge A\left(\left\lceil T^{n}(\varepsilon)\right\rceil\right) d P \tag{63.7.40}
\end{align*}
$$

where on $\left(t_{j-1}^{n}, t_{j}^{n}\right],\left\lceil T^{n}(\varepsilon)\right\rceil \equiv t_{j}^{n}$. Now $\left\lceil T^{n}(\varepsilon)\right\rceil$ is also a bounded stopping time. Here is why. Suppose $s \in\left(t_{j-1}^{n}, t_{j}^{n}\right]$. Then

$$
\left[\left\lceil T^{n}(\varepsilon)\right] \leq s\right]=\left[T^{n}(\varepsilon) \leq t_{j-1}^{n}\right] \in \mathscr{F}_{t_{j-1}^{n}} \subseteq \mathscr{F}_{s}
$$

Now let

$$
Q_{n} \equiv \sup _{t \in[0, a]}\left|\xi^{n}(t)-\lambda \wedge A(t)\right|
$$

Then first note that

$$
\left[Q_{n}>\varepsilon\right]=\left[\sup _{t \in[0, a)}\left|\xi^{n}(t)-\lambda \wedge A(t)\right|>\varepsilon\right]
$$

because $Q_{n}(a)=0$ follows from the definition of $\xi^{n}(t)$ as

$$
E\left(\lambda \wedge A\left(t_{j}^{n}\right) \mid \mathscr{F}_{t}\right) \text { for } t_{j-1}^{n}<t \leq t_{j}^{n}
$$

and so

$$
\xi^{n}(a)=E\left(\lambda \wedge A(a) \mid \mathscr{F}_{a}\right)=\lambda \wedge A(a) .
$$

Thus it suffices to take the supremum over the half open interval, $[0, a)$. It follows

$$
\left[Q_{n}>\varepsilon\right]=\left[T^{n}(\varepsilon)<a\right]
$$

By right continuity,

$$
\xi^{n}\left(T^{n}(\varepsilon)\right)-\lambda \wedge A\left(T^{n}(\varepsilon)\right) \geq \varepsilon
$$

on $\left[Q_{n}>\varepsilon\right]$.

$$
\begin{aligned}
\varepsilon P\left(\left[Q_{n}>\varepsilon\right]\right) & =\varepsilon P\left(\left[T^{n}(\varepsilon)<a\right]\right) \\
& \leq \int_{\left[Q_{n}>\varepsilon\right]}\left(\xi^{n}\left(T^{n}(\varepsilon)\right)-\lambda \wedge A\left(T^{n}(\varepsilon)\right)\right) d P \\
& \leq \int_{\Omega}\left(\xi^{n}\left(T^{n}(\varepsilon)\right)-\lambda \wedge A\left(T^{n}(\varepsilon)\right)\right) d P
\end{aligned}
$$

Therefore, from 63.7.40,

$$
\begin{align*}
P\left(\left[Q_{n}>\varepsilon\right]\right) & \leq \frac{1}{\varepsilon} \int_{\Omega}\left(\lambda \wedge A\left(\left\lceil T^{n}(\varepsilon)\right\rceil\right)-\lambda \wedge A\left(T^{n}(\varepsilon)\right)\right) d P \\
& \leq \frac{1}{\varepsilon} \int_{\Omega}\left(A\left(\left\lceil T^{n}(\varepsilon)\right\rceil\right)-A\left(T^{n}(\varepsilon)\right)\right) d P \tag{63.7.41}
\end{align*}
$$

By optional sampling theorem,

$$
E\left(M\left(T^{n}(\varepsilon)\right)\right)=E(M(0))=0
$$

and also

$$
E\left(M\left(\left\lceil T^{n}(\varepsilon)\right\rceil\right)\right)=E(M(0))=0 .
$$

Therefore, 63.7.41 reduces to

$$
P\left(\left[Q_{n}>\varepsilon\right]\right) \leq \frac{1}{\varepsilon} \int_{\Omega}\left(X\left(\left\lceil T^{n}(\varepsilon)\right\rceil\right)-X\left(T^{n}(\varepsilon)\right)\right) d P
$$

By the assumption that $\{X(t)\}$ is $D L$, it follows the functions in the above integrand are equi integrable and so since $\lim _{n \rightarrow \infty} X\left(\left\lceil T^{n}(\varepsilon)\right\rceil\right)-X\left(T^{n}(\varepsilon)\right)=0$, the above integral converges to 0 as $n \rightarrow \infty$ by Vitali's convergence theorem, Theorem 11.5.3 on Page 257. It follows that there is a subsequence, $n_{k}$ such that

$$
P\left(\left[Q_{n_{k}}>2^{-k}\right]\right) \leq 2^{-k}
$$

and so from the definition of $Q_{n}$,

$$
\lim _{k \rightarrow \infty} \sup _{t \in[0, a]}\left|\xi^{n_{k}}(t)-\lambda \wedge A(t)\right|
$$

giving uniform convergence. Now recall that

$$
E\left(\int_{(0, a]} \xi^{n_{k}}(s) d A(s)\right)=E\left(\int_{(0, a]} \xi_{-}^{n_{k}}(s) d A(s)\right)
$$

and so passing to the limit as $k \rightarrow \infty$ with the uniform convergence yields

$$
E\left(\int_{(0, a]} \lambda \wedge A(s) d A(s)\right)=E\left(\int_{(0, a]} \lambda \wedge A_{-}(s) d A(s)\right)
$$

Now let $\lambda \rightarrow \infty$. Then from the monotone convergence theorem,

$$
E\left(\int_{(0, a]} A(s) d A(s)\right)=E\left(\int_{(0, a]} A_{-}(s) d A(s)\right)
$$

and so for a.e. $\omega$,

$$
\int_{(0, a]}\left(A(s)-A_{-}(s)\right) d A(s)=0
$$

Thus letting the measure associated with this Lebesgue integral be denoted by $\mu$,

$$
A(s)-A_{-}(s)=0 \mu \text { a.e. }
$$

Suppose then that $A(s)-A_{-}(s)>0$. Then $\mu(\{s\})=0=A(s)-A(s-)$, a contradiction. Hence $A(s)-A_{-}(s)=0$ for all $s$. It is already the case that $s \rightarrow A(s)$ is right continuous. Therefore, this proves the theorem.

Example 63.7.16 Suppose $\{M(t)\}$ is a continuous martingale. Assume

$$
\sup _{t \in[0, a]}\|M(t)\|_{L^{2}(\Omega)}<\infty
$$

Then $\{\|M(t)\|\}$ is a submartingale and so is $\left\{\|M(t)\|^{2}\right\}$. By Example 63.7.11, this is DL. Then there exists a unique Doob Meyer decomposition,

$$
\|M(t)\|^{2}=Y(t)+\langle\|M(t)\|\rangle
$$

where $Y(t)$ is a martingale and $\{\langle\|M(t)\|\rangle\}$ is a submartingale which is continuous, natural, increasing and equal to 0 when $t=0$. This submartingale is called the quadratic variation.

### 63.8 Levy's Theorem

This remarkable theorem has to do with when a martingale is a Wiener process. The proof I am giving here follows [44].

Definition 63.8.1 Let $W(t)$ be a stochastic process which has the properties that whenever $t_{1}<t_{2}<\cdots<t_{m}$, the increments $\left\{W\left(t_{i}\right)-W\left(t_{i-1}\right)\right\}$ are independent and whenever $s<t$, it follows $W(t)-W(s)$ is normally distributed with variance $t-s$ and mean 0 . Also $t \rightarrow W(t)$ is Holder continuous with every exponent $\gamma<1 / 2$ and $W(0)=0$. This is called a Wiener process.

First here is a lemma.

Lemma 63.8.2 Let $\{X(t)\}$ be a real martingale adapted to the filtration $\mathscr{F}_{t}$ for $t \in[a, b]$ some interval such that for all $t \in[a, b], E\left(X(t)^{2}\right)<\infty$. Then $\left\{X(t)^{2}-t\right\}$ is also a martingale if and only if whenever $s<t$,

$$
E\left((X(t)-X(s))^{2} \mid \mathscr{F}_{s}\right)=t-s
$$

Proof: Suppose first $\left\{X(t)^{2}-t\right\}$ is a real martingale. Then since $\{X(t)\}$ is a martingale,

$$
\begin{aligned}
E((X(t) & \left.-X(s))^{2} \mid \mathscr{F}_{s}\right)=E\left(X(t)^{2}-2 X(t) X(s)+X(s)^{2} \mid \mathscr{F}_{s}\right) \\
& =E\left(X(t)^{2} \mid \mathscr{F}_{s}\right)-2 E\left(X(t) X(s) \mid \mathscr{F}_{s}\right)+X(s)^{2} \\
& =E\left(X(t)^{2} \mid \mathscr{F}_{s}\right)-2 X(s) E\left(X(t) \mid \mathscr{F}_{s}\right)+X(s)^{2} \\
& =E\left(X(t)^{2} \mid \mathscr{F}_{s}\right)-2 X(s)^{2}+X(s)^{2} \\
& =E\left(X(t)^{2}-t \mid \mathscr{F}_{s}\right)+t-X(s)^{2} \\
& =X(s)^{2}-s+t-X(s)^{2}=t-s
\end{aligned}
$$

Next suppose $E\left((X(t)-X(s))^{2} \mid \mathscr{F}_{s}\right)=t-s$. Then since $\{X(t)\}$ is a martingale,

$$
\begin{aligned}
t-s & =E\left(X(t)^{2}-X(s)^{2} \mid \mathscr{F}_{s}\right) \\
& =E\left(X(t)^{2}-t \mid \mathscr{F}_{s}\right)+t-X(s)^{2}
\end{aligned}
$$

and so

$$
0=E\left(X(t)^{2}-t \mid \mathscr{F}_{s}\right)-\left(X(s)^{2}-s\right)
$$

which proves the converse.
Theorem 63.8.3 Suppose $\{X(t)\}$ is a real stochastic process which satisfies all the conditions of a real Wiener process except the requirement that it be continuous. Then both $\{X(t)\}$ and $\left\{X(t)^{2}-t\right\}$ are martingales.

Proof: First define the filtration to be

$$
\mathscr{F}_{t} \equiv \sigma(X(s)-X(r): r \leq s \leq t) .
$$

Claim: If $A \in \mathscr{F}_{s}$, then

$$
\int_{\Omega} \mathscr{X}_{A}(X(t)-X(s)) d P=P(A) \int_{\Omega}(X(t)-X(s)) d P
$$

Proof of claim: Let $\mathscr{G}$ denote those sets of $\mathscr{F}_{s}$ for which the above formula holds. Then it is clear that $\mathscr{G}$ is closed with respect to countable unions of disjoint sets and
complements. Let $\mathscr{K}$ denote those sets which are finite intersections of sets of the form $(X(u)-X(r))^{-1}(B)$ where $B$ is a Borel set and $r \leq u \leq s$. Say a set, $A$ of $\mathscr{K}$ is of the form

$$
\cap_{i=1}^{m}\left(X\left(u_{i}\right)-X\left(r_{i}\right)\right)^{-1}\left(B_{i}\right)
$$

Then since disjoint increments are independent, linear combinations of the random variables, $X\left(u_{i}\right)-X\left(r_{i}\right)$ are normally distributed. Consequently,

$$
\left(X\left(u_{1}\right)-X\left(r_{1}\right), \cdots, X\left(u_{m}\right)-X\left(r_{m}\right), X(t)-X(s)\right)
$$

is multivariate normal. The covariance matrix is of the form

$$
\left(\begin{array}{cc}
A & 0 \\
0 & t-s
\end{array}\right)
$$

and so the random vector, $\left(X\left(u_{1}\right)-X\left(r_{1}\right), \cdots, X\left(u_{m}\right)-X\left(r_{m}\right)\right)$ and the random variable $X(t)-X(s)$ are independent. Consequently, $\mathscr{X}_{A}$ is independent of $X(t)-X(s)$ for any $A \in \mathscr{K}$. Then by the lemma on $\pi$ systems, Lemma 12.12.3 on Page $329, \mathscr{F} s \supseteq \mathscr{G} \supseteq$ $\sigma(\mathscr{K})=\mathscr{F}_{s}$. This proves the claim.

Thus

$$
\begin{aligned}
\int_{A}(X(t)-X(s)) d P & =\int_{\Omega}(X(t)-X(s)) \mathscr{X}_{A} d P \\
& =P(A) \int_{\Omega}(X(t)-X(s)) d P=0
\end{aligned}
$$

which shows that since $A \in \mathscr{F}_{s}$ was arbitrary,

$$
E\left(X(t) \mid \mathscr{F}_{s}\right)=X(s)
$$

and $\{X(t)\}$ is a martingale.
Now consider whether $\left\{X(t)^{2}-t\right\}$ is a martingale. By assumption,

$$
\mathscr{L}(X(t)-X(s))=\mathscr{L}(X(t-s))=N(0, t-s)
$$

Then for $A \in \mathscr{F}_{s}$, the independence of $\mathscr{X}_{A}$ and $X(t)-X(s)$ shows

$$
\begin{aligned}
\int_{A} E\left((X(t)-X(s))^{2} \mid \mathscr{F}_{s}\right) d P & =\int_{A}(X(t)-X(s))^{2} d P \\
& =P(A)(t-s)=\int_{A}(t-s) d P
\end{aligned}
$$

and since $A \in \mathscr{F}_{s}$ is arbitrary,

$$
E\left((X(t)-X(s))^{2} \mid \mathscr{F}_{s}\right)=t-s
$$

and so the result follows from Lemma 63.8.2. This proves the theorem.
The next lemma is the main result from which Levy's theorem will be established.

Lemma 63.8.4 Let $\{X(t)\}$ be a real continuous martingale adapted to the filtration $\mathscr{F}_{t}$ for $t \in[a, b]$ some interval such that for all $t \in[a, b], E\left(X(t)^{2}\right)<\infty$. Suppose also that $\left\{X(t)^{2}-t\right\}$ is a martingale. Then for $\lambda$ real,

$$
E\left(e^{i \lambda X(b)}\right)=E\left(e^{i \lambda X(a)}\right) e^{-(b-a) \frac{\lambda^{2}}{2}}
$$

Proof: Let $\lambda \in[-p, p]$ where for most of the proof, $p$ is fixed but arbitrary. Let $\left\{t_{k}^{n}\right\}_{k=0}^{2^{n}}$ be uniform partitions such that $t_{k}^{n}-t_{k-1}^{n}=\delta_{n} \equiv(b-a) / 2^{n}$. Now for $\varepsilon>0$ define a stopping time $\tau_{\varepsilon, n}$ to be the first time, $t$ such that there exist $s_{1}, s_{2} \in[a, t]$ with $\left|s_{1}-s_{2}\right|<\boldsymbol{\delta}_{n}$ but

$$
\left|X\left(s_{1}\right)-X\left(s_{2}\right)\right|=\varepsilon
$$

If no such time exists, then $\tau_{\varepsilon, n} \equiv b$.
Then $\tau_{\varepsilon, n}$ really is a stopping time because from continuity of $X(t)$ and denoting by $r, r_{1}$ elements of $\mathbb{Q}$, then

$$
\left[\tau_{\varepsilon, n}>t\right]=\bigcup_{m=1}^{\infty} \bigcap_{0 \leq r_{1}, r_{2} \leq t,\left|r_{1}-r_{2}\right| \leq \delta_{n}}\left[\left|X\left(r_{1}\right)-X\left(r_{2}\right)\right| \leq \varepsilon-\frac{1}{m}\right] \in \mathscr{F}_{t}
$$

because to be in $\left[\tau_{\varepsilon, n}>t\right]$ it means that by $t$ the absolute value of the differences must always be less than $\varepsilon$. Hence $\left[\tau_{\varepsilon, n} \leq t\right]=\Omega \backslash\left[\tau_{\varepsilon, n}>t\right] \in \mathscr{F}_{t}$.

Now consider $\left[\tau_{\varepsilon, n}=b\right]$ for various $n$. By continuity, it follows that for each $\omega \in \Omega$,

$$
\tau_{\varepsilon, n}(\omega)=b
$$

for all $n$ large enough. Thus

$$
\emptyset=\cap_{n=1}^{\infty}\left[\tau_{\varepsilon, n}<b\right]
$$

the sets in the intersection decreasing. Thus there exists $n(\varepsilon)$ such that

$$
\begin{equation*}
P\left(\left[\tau_{\varepsilon, n(\varepsilon)}<b\right]\right)<\varepsilon . \tag{63.8.42}
\end{equation*}
$$

Denote $\tau_{\varepsilon, n(\varepsilon)}$ as $\tau_{\varepsilon}$ for short and it will always be assumed that $n(\varepsilon)$ is at least this large and that $\lim _{\varepsilon \rightarrow 0+} n(\varepsilon)=\infty$. In addition to this, $n(\varepsilon)$ will also be large enough that

$$
1-\frac{\lambda^{2}}{2} \delta_{n(\varepsilon)}>0
$$

for all $\lambda \in[-p, p]$. To save on notation, $t_{j}$ will take the place of $t_{j}^{n}$. Then consider the stopping times $\tau_{\varepsilon} \wedge t_{j}$ for $j=0,1, \cdots, 2^{n(\varepsilon)}$.

Let $y_{j} \equiv X\left(\tau_{\varepsilon} \wedge t_{j}\right)-X\left(\tau_{\varepsilon} \wedge t_{j-1}\right)$, it follows from the definition of the stopping time that

$$
\begin{equation*}
\left|y_{j}\right| \leq \varepsilon \tag{63.8.43}
\end{equation*}
$$

because both $\tau_{\varepsilon} \wedge t_{j}$ and $\tau_{\varepsilon} \wedge t_{j-1}$ are less than $\tau_{\varepsilon}$ and closer together than $\delta_{n(\varepsilon)}$ and so if $\left|y_{j}\right|>\varepsilon$, then $\tau_{\varepsilon} \leq t_{j}, t_{j-1}$ and so $y_{j}$ would need to equal 0 .

By the optional stopping theorem, $\left\{X\left(\tau_{\varepsilon} \wedge t_{j}\right)\right\}_{j}$ is a martingale as is also

$$
\left\{X\left(\tau_{\varepsilon} \wedge t_{j}\right)-\tau_{\varepsilon} \wedge t_{j}\right\}_{j}
$$

Thus for $A \in \mathscr{F} \tau_{\varepsilon} \wedge t_{j-1}$,

$$
\begin{gathered}
\int_{A} E\left(y_{j}^{2} \mid \mathscr{F}_{\tau_{\varepsilon} \wedge t_{j-1}}\right) d P=\int_{A} E\left(\left(X\left(\tau_{\varepsilon} \wedge t_{j}\right)-X\left(\tau_{\varepsilon} \wedge t_{j-1}\right)\right)^{2} \mid \mathscr{F}_{\tau_{\varepsilon} \wedge t_{j-1}}\right) d P \\
=\int_{A} E\left(X\left(\tau_{\varepsilon} \wedge t_{j}\right)^{2} \mid \mathscr{F}_{\tau_{\varepsilon} \wedge t_{j-1}}\right)+X\left(\tau_{\varepsilon} \wedge t_{j-1}\right)^{2} \\
-2 X\left(\tau_{\varepsilon} \wedge t_{j-1}\right) E\left(X\left(\tau_{\varepsilon} \wedge t_{j}\right) \mid \mathscr{F}_{\tau_{\varepsilon} \wedge t_{j-1}}\right) d P \\
=\int_{A} E\left(X\left(\tau_{\varepsilon} \wedge t_{j}\right)^{2}-\tau_{\varepsilon} \wedge t_{j} \mid \mathscr{F}_{\tau_{\varepsilon} \wedge t_{j-1}}\right) d P+\int_{A} E\left(\tau_{\varepsilon} \wedge t_{j} \mid \mathscr{F}_{\tau_{\varepsilon} \wedge t_{j-1}}\right) d P \\
+\int_{A} X\left(\tau_{\varepsilon} \wedge t_{j-1}\right)^{2} d P-2 \int_{A} X\left(\tau_{\varepsilon} \wedge t_{j-1}\right)^{2} d P \\
=\int_{A} X\left(\tau_{\varepsilon} \wedge t_{j-1}\right)^{2} d P-\int_{A} \tau_{\varepsilon} \wedge t_{j-1} d P+\int_{A} E\left(\tau_{\varepsilon} \wedge t_{j} \mid \mathscr{F}_{\tau_{\varepsilon} \wedge t_{j-1}}\right) d P \\
+\int_{A} X\left(\tau_{\varepsilon} \wedge t_{j-1}\right)^{2} d P-2 \int_{A} X\left(\tau_{\varepsilon} \wedge t_{j-1}\right)^{2} d P \\
=\int_{A} E\left(\tau_{\varepsilon} \wedge t_{j} \mid \mathscr{F}_{\tau_{\varepsilon} \wedge t_{j-1}}\right) d P-\int_{A} \tau_{\varepsilon} \wedge t_{j-1} d P \\
=\int_{A}\left(\tau_{\varepsilon} \wedge t_{j}-\tau_{\varepsilon} \wedge t_{j-1}\right) d P \leq \int_{A} t_{j}-t_{j-1} d P
\end{gathered}
$$

Thus, since $A$ is arbitrary,

$$
\begin{gather*}
\sigma_{j}^{2} \equiv \int_{A} E\left(y_{j}^{2} \mid \mathscr{F}_{\varepsilon} \wedge t_{j-1}\right) d P= \\
E\left(\left(X\left(\tau_{\varepsilon} \wedge t_{j}\right)-X\left(\tau_{\varepsilon} \wedge t_{j-1}\right)\right)^{2} \mid \mathscr{F}_{\tau_{\varepsilon} \wedge t_{j-1}}\right) \leq t_{j}-t_{j-1}=\delta_{n(\varepsilon)} \tag{63.8.44}
\end{gather*}
$$

Also,

$$
\begin{equation*}
E\left(y_{j} \mid \mathscr{F}_{\tau_{\varepsilon} \wedge t_{j-1}}\right)=E\left(X\left(\tau_{\varepsilon} \wedge t_{j}\right)-X\left(\tau_{\varepsilon} \wedge t_{j-1}\right) \mid \mathscr{F}_{\tau_{\varepsilon} \wedge t_{j-1}}\right)=0 \tag{63.8.45}
\end{equation*}
$$

Now it is time to find $E\left(e^{i \lambda X\left(\tau_{\varepsilon} \wedge t_{j}\right)}\right)$.

$$
\begin{align*}
& E\left(e^{i \lambda X\left(\tau_{\varepsilon} \wedge t_{j}\right)}\right)=E\left(e^{i \lambda\left(X\left(\tau_{\varepsilon} \wedge t_{j-1}\right)+y_{j}\right)}\right) \\
& =E\left(e^{i \lambda X\left(\tau_{\varepsilon} \wedge t_{j-1}\right)} E\left(e^{\left.i \lambda y_{j} \mid \mathscr{F}_{\tau_{\varepsilon} \wedge t_{j-1}}\right)}\right)\right. \tag{63.8.46}
\end{align*}
$$

Now let $o(1)$ denote any quantity which converges to 0 as $\varepsilon \rightarrow 0$ for all $\lambda \in[-p, p]$ and $O(1)$ is a quantity which is bounded as $\varepsilon \rightarrow 0$. Then from 63.8.45 and 63.8.46 you can consider the power series for $e^{i \lambda y_{j}}$ which converges uniformly due to 63.8 .43 and write 63.8.46 as

$$
E\left(e^{i \lambda X\left(\tau_{\varepsilon} \wedge t_{j-1}\right)}\left(1-\frac{\lambda^{2}}{2} \sigma_{j}^{2}(1+o(1))\right)\right)
$$

then noting that from 63.8 .44 which shows $\sigma_{j}^{2}$ is $o(1)$,it is routine to verify

$$
1-\frac{\lambda^{2}}{2} \sigma_{j}^{2}(1+o(1))=e^{-\frac{\lambda^{2}}{2} \sigma_{j}^{2}(1+o(1))}
$$

Now this shows

$$
E\left(e^{i \lambda X\left(\tau_{\varepsilon} \wedge t_{j}\right)}\right)=E\left(e^{i \lambda X\left(\tau_{\varepsilon} \wedge t_{j-1}\right)} e^{-\frac{\lambda^{2}}{2} \sigma_{j}^{2}(1+o(1))}\right)
$$

Recall that $\sigma_{j}^{2} \leq \delta_{n}=t_{j}-t_{j-1}$. Consider

$$
\begin{gathered}
\left|E\left(e^{i \lambda X\left(\tau_{\varepsilon} \wedge t_{j}\right)}\right)-E\left(e^{i \lambda X\left(\tau_{\varepsilon} \wedge t_{j-1}\right)} e^{-\frac{\lambda^{2}}{2} \delta_{n}}\right)\right| \\
=\left|E\left(e^{i \lambda X\left(\tau_{\varepsilon} \wedge t_{j-1}\right)} e^{-\frac{\lambda^{2}}{2} \sigma_{j}^{2}(1+o(1))}\right)-E\left(e^{i \lambda X\left(\tau_{\varepsilon} \wedge t_{j-1}\right)} e^{-\frac{\lambda^{2}}{2} \delta_{n}}\right)\right| \\
=\left|E\left(e^{i \lambda X\left(\tau_{\varepsilon} \wedge t_{j-1}\right)}\left(e^{-\frac{\lambda^{2}}{2} \sigma_{j}^{2}(1+o(1))}-e^{-\frac{\lambda^{2}}{2} \delta_{n}}\right)\right)\right| \\
=\left|E\left(e^{i \lambda X\left(\tau_{\varepsilon} \wedge t_{j-1}\right)} e^{-\frac{\lambda^{2}}{2} \delta_{n}}\left(e^{-\frac{\lambda^{2}}{2} \sigma_{j}^{2}(1+o(1))+\frac{\lambda^{2}}{2} \delta_{n}}-1\right)\right)\right| \\
\leq E\left(\left|e^{\frac{\lambda^{2}}{2}\left(\delta_{n}-\sigma_{j}^{2}\right)+\sigma_{j}^{2} o(1)}-1\right|\right)
\end{gathered}
$$

Everything in the exponent is $o(1)$ and so the above expression is bounded by

$$
\begin{align*}
& O(1) E\left(\left|\frac{\lambda^{2}}{2}\left(\delta_{n}-\sigma_{j}^{2}\right)+\sigma_{j}^{2} o(1)\right|\right) \\
\leq & O(1) E\left(\left(\delta_{n}-\sigma_{j}^{2}\right)+\delta_{n}|o(1)|\right) \\
= & O(1)\left[\delta_{n}-E\left(y_{j}^{2}\right)+\delta_{n} o(1)\right] \tag{63.8.47}
\end{align*}
$$

Therefore,

$$
\begin{aligned}
& \left|E\left(e^{i \lambda X\left(\tau_{\varepsilon} \wedge t_{j}\right)}\right)-E\left(e^{i \lambda X\left(\tau_{\varepsilon} \wedge t_{j-1}\right)} e^{-\frac{\lambda^{2}}{2} \delta_{n}}\right)\right| \\
\leq & O(1)\left[\delta_{n}-E\left(y_{j}^{2}\right)+\delta_{n} o(1)\right]
\end{aligned}
$$

and so it also follows

$$
\begin{aligned}
& \left|E\left(e^{i \lambda X\left(\tau_{\varepsilon} \wedge t_{j}\right)}\right) e^{\frac{\lambda^{2}}{2} t_{j}}-E\left(e^{i \lambda X\left(\tau_{\varepsilon} \wedge t_{j-1}\right)} e^{\frac{\lambda^{2}}{2} t_{j-1}}\right)\right| \\
\leq & O(1)\left[\delta_{n}-E\left(y_{j}^{2}\right)+\delta_{n} o(1)\right]
\end{aligned}
$$

Now also remember

$$
y_{j}=X\left(\tau_{\varepsilon} \wedge t_{j}\right)-X\left(\tau_{\varepsilon} \wedge t_{j-1}\right)
$$

and that $\left\{X\left(\tau_{\varepsilon} \wedge t_{j}\right)\right\}_{j}$ is a martingale. Therefore it is routine to show,

$$
E\left(y_{j}^{2}\right)=E\left(X\left(\tau_{\varepsilon} \wedge t_{j}\right)^{2}\right)-E\left(X\left(\tau_{\varepsilon} \wedge t_{j-1}\right)^{2}\right)
$$

and so

$$
\begin{aligned}
& \left|E\left(e^{i \lambda X\left(\tau_{\varepsilon} \wedge t_{j}\right)}\right) e^{\frac{\lambda^{2}}{2} t_{j}}-E\left(e^{i \lambda X\left(\tau_{\varepsilon} \wedge t_{j-1}\right)} e^{\frac{\lambda^{2}}{2} t_{j-1}}\right)\right| \\
\leq & O(1)\left[\delta_{n}-\left(E\left(X\left(\tau_{\varepsilon} \wedge t_{j}\right)^{2}\right)-E\left(X\left(\tau_{\varepsilon} \wedge t_{j-1}\right)^{2}\right)\right)+\delta_{n} o(1)\right]
\end{aligned}
$$

and so, summing over all $j=1, \cdots, 2^{n(\varepsilon)}$,

$$
\begin{align*}
& \left|E\left(e^{i \lambda X\left(\tau_{\varepsilon} \wedge b\right)}\right) e^{\frac{\lambda^{2}}{2} b}-E\left(e^{i \lambda X(a)} e^{\frac{\lambda^{2}}{2} a}\right)\right| \\
\leq & O(1)\left((1+o(1))(b-a)-\left(E\left(X\left(\tau_{\varepsilon} \wedge b\right)^{2}\right)-E\left(X(a)^{2}\right)\right)\right) . \tag{63.8.48}
\end{align*}
$$

Now recall 63.8.42 which said

$$
P\left(\left[\tau_{\varepsilon}<b\right]\right)<\varepsilon
$$

Let $\varepsilon_{k} \equiv 2^{-k}$ and then by the Borel Cantelli lemma,

$$
X\left(\tau_{\varepsilon} \wedge b\right) \rightarrow X(b)
$$

a.e. since if $\omega$ is such that convergence does not take place, $\omega$ must be in infinitely many of the sets, $\left[\tau_{\varepsilon_{k}}<b\right]$, a set of measure 0 . Also since $\left\{X\left(\tau_{\varepsilon} \wedge t_{j}\right)\right\}_{j}$ is a martingale, it follows from optional sampling theorem that $\left\{X(a)^{2}, X\left(\tau_{\varepsilon} \wedge b\right)^{2}, X(b)^{2}\right\}$ is a submartingale and so

$$
\int_{\left[X\left(\tau_{\varepsilon} \wedge b\right)^{2} \geq \alpha\right]} X\left(\tau_{\varepsilon} \wedge b\right)^{2} d P \leq \int_{\left[X\left(\tau_{\varepsilon} \wedge b\right)^{2} \geq \alpha\right]} X(b)^{2} d P
$$

and also from the maximal inequalities, Theorem 60.6.4 on Page 1967 it follows

$$
P\left(\left[X\left(\tau_{\varepsilon} \wedge b\right)^{2} \geq \alpha\right]\right) \leq \frac{1}{\alpha} E\left(X(b)^{2}\right)
$$

and so the functions, $\left\{X\left(\tau_{\varepsilon_{k}} \wedge b\right)^{2}\right\}_{\varepsilon_{k}}$ are uniformly integrable which implies by the Vitali convergence theorem, Theorem 11.5.3 on Page 257, that you can pass to the limit as $\varepsilon_{k} \rightarrow 0$
in the inequality, 63.8.48 and conclude

$$
\begin{aligned}
& \left|E\left(e^{i \lambda X(b)}\right) e^{\frac{\lambda^{2}}{2} b}-E\left(e^{i \lambda X(a)} e^{\frac{\lambda^{2}}{2} a}\right)\right| \\
\leq & O(1)\left((b-a)-\left(E\left(X(b)^{2}\right)-E\left(X(a)^{2}\right)\right)\right)=0
\end{aligned}
$$

Therefore,

$$
E\left(e^{i \lambda X(b)}\right)=E\left(e^{i \lambda X(a)}\right) e^{-\frac{\lambda^{2}}{2}(b-a)}
$$

This proves the lemma because $p$ was arbitrary.
Now from this lemma, it is not hard to establish Levy's theorem.
Theorem 63.8.5 Let $\{X(t)\}$ be a real continuous martingale adapted to the filtration $\mathscr{F}_{t}$ for $t \in[0, a]$ some interval such that for all $t \in[0, a], E\left(X(t)^{2}\right)<\infty$. Suppose also that $\left\{X(t)^{2}-t\right\}$ is a martingale. Then for $s<t, X(t)-X(s)$ is normally distributed with mean 0 and variance $t-s$. Also if $0 \leq t_{0}<t_{1}<\cdots<t_{m} \leq b$, then the increments $\left\{X\left(t_{j}\right)-X\left(t_{j-1}\right)\right\}$ are independent.

Proof: Let the $t_{j}$ be as described above and consider the interval $\left[t_{m-1}, t_{m}\right]$ in place of [ $a, b]$ in Lemma 63.8.4. Also let $\lambda_{k}$ for $k=1,2, \cdots, m$ be given. For $t \in\left[t_{m-1}, t_{m}\right]$, and $\lambda_{m} \neq 0$,

$$
Z_{\lambda_{m}}(t)=\frac{1}{\lambda_{m}} \sum_{j=1}^{m-1} \lambda_{j}\left(X\left(t_{j}\right)-X\left(t_{j-1}\right)\right)+\left(X(t)-X\left(t_{m-1}\right)\right)
$$

Then it is clear that $\left\{Z_{\lambda_{m}}(t)\right\}$ is a martingale on $\left[t_{m-1}, t_{m}\right]$. What is possibly less clear is that $\left\{Z_{\lambda_{m}}(t)^{2}-t\right\}$ is also a martingale. Note that $Z_{\lambda_{m}}(t)=X(t)+Y$ where $Y$ is measurable in $\mathscr{F}_{t_{m-1}}$. Therefore, for $s<t, s \in\left[t_{m-1}, t_{m}\right]$,

$$
\begin{gathered}
E\left(Z_{\lambda_{m}}(t)^{2}-t \mid \mathscr{F}_{s}\right)=E\left(X(t)^{2}+2 X(t) Y+Y^{2}-t \mid \mathscr{F}_{s}\right) \\
=X(s)^{2}-s+2 E\left(X(t) Y \mid \mathscr{F}_{s}\right)+Y^{2} \\
=X(s)^{2}-s+2 Y X(s)+Y^{2}=Z_{\lambda_{m}}(s)^{2}-s
\end{gathered}
$$

and so Lemma 63.8.4 can be applied to conclude

$$
E\left(e^{i \lambda Z_{\lambda_{m}}\left(t_{m}\right)}\right)=E\left(e^{i \lambda Z_{\lambda_{m}}\left(t_{m-1}\right)}\right) e^{-\frac{\lambda^{2}}{2}\left(t_{m}-t_{m-1}\right)}
$$

Now letting $\lambda=\lambda_{m}$,

$$
E\left(e^{i \sum_{j=1}^{m} \lambda_{j}\left(X\left(t_{j}\right)-X\left(t_{j-1}\right)\right)}\right)=E\left(e^{i \sum_{j=1}^{m-1} \lambda_{j}\left(X\left(t_{j}\right)-X\left(t_{j-1}\right)\right)}\right) e^{-\frac{\lambda_{m}^{2}}{2}\left(t_{m}-t_{m-1}\right)}
$$

By continuity, this equation continues to hold for $\lambda_{m}=0$. Then iterate this, using a similar argument on the first factor of the right side to eventually obtain

$$
E\left(e^{i \sum_{j=1}^{m} \lambda_{j}\left(X\left(t_{j}\right)-X\left(t_{j-1}\right)\right)}\right)=\prod_{j=1}^{m} e^{-\frac{\lambda_{j}^{2}}{2}\left(t_{j}-t_{j-1}\right)}
$$

Then letting all but one $\lambda_{j}$ equal zero, this shows the increment, $X\left(t_{j}\right)-X\left(t_{j-1}\right)$ is a random variable which is normally distributed having variance $t_{j}-t_{j-1}$ and mean 0 . The above formula also shows from Proposition 59.11.1 on Page 1889 that the increments are independent. This proves the theorem.

## Chapter 64

## Wiener Processes

A real valued random variable $X$ is normally distributed with mean 0 and variance $\sigma^{2}$ if

$$
P(X \in A)=\frac{1}{\sqrt{2 \pi} \sigma} \int_{A} e^{-\frac{1}{2} \frac{x^{2}}{\sigma^{2}}} d x
$$

Consider the characteristic function. By definition it is

$$
\phi_{X}(\lambda) \equiv \int_{\mathbb{R}} e^{i \lambda x} d \lambda_{X}(x)
$$

where $\lambda_{X}$ is the distribution measure for this random variable. Thus the characteristic function of this random variable is

$$
\frac{1}{\sqrt{2 \pi} \sigma} \int_{-\infty}^{\infty} e^{i \lambda x} e^{-\frac{1}{2 \sigma^{2}} x^{2}} d x
$$

One can then show through routine arguments that this equals $\exp \left(-\frac{1}{2} \sigma \lambda^{2}\right)$.

### 64.1 Real Wiener Processes

Here is the definition of a Wiener process.
Definition 64.1.1 Let $W(t)$ be a stochastic process which has the properties that whenever $t_{1}<t_{2}<\cdots<t_{m}$, the increments $\left\{W\left(t_{i}\right)-W\left(t_{i-1}\right)\right\}$ are independent and whenever $s<t$, it follows $W(t)-W(s)$ is normally distributed with variance $t-s$ and mean 0 . Also $t \rightarrow W(t)$ is Holder continuous with every exponent $\gamma<1 / 2$ and $W(0)=0$. This is called a Wiener process.

Do Wiener processes exist? Yes, they do. First here is a simple lemma which has really been done before. It depends on the Kolmogorov extension theorem, Theorem 59.2.3 on Page 1860.

Lemma 64.1.2 There exists a sequence, $\left\{\xi_{k}\right\}_{k=1}^{\infty}$ of random variables such that

$$
\mathscr{L}\left(\xi_{k}\right)=N(0,1)
$$

and $\left\{\xi_{k}\right\}_{k=1}^{\infty}$ is independent.
Proof: Let $i_{1}<i_{2} \cdots<i_{n}$ be positive integers and define

$$
\mu_{i_{1} \cdots i_{n}}\left(F_{1} \times \cdots \times F_{n}\right) \equiv \frac{1}{(\sqrt{2 \pi})^{n}} \int_{F_{1} \times \cdots \times F_{n}} e^{-|\mathbf{x}|^{2} / 2} d x
$$

Then for the index set equal to $\mathbb{N}$ the measures satisfy the necessary consistency condition for the Kolmogorov theorem. Therefore, there exists a probability space, $(\Omega, P, \mathscr{F})$ and
measurable functions, $\xi_{k}: \Omega \rightarrow \mathbb{R}$ such that

$$
\begin{aligned}
& P\left(\left[\xi_{i_{1}} \in F_{i_{1}}\right] \cap\left[\xi_{i_{2}} \in F_{i_{2}}\right] \cdots \cap\left[\xi_{i_{n}} \in F_{i_{n}}\right]\right) \\
= & \mu_{i_{1} \cdots i_{n}}\left(F_{1} \times \cdots \times F_{n}\right) \\
= & P\left(\left[\xi_{i_{1}} \in F_{i_{1}}\right]\right) \cdots P\left(\left[\xi_{i_{n}} \in F_{i_{n}}\right]\right)
\end{aligned}
$$

which shows the random variables are independent as well as normal with mean 0 and variance 1 .

Recall that the sum of independent normal random variables is normal. The Wiener process is just an infinite weighted sum of the above independent normal random variables, the weights depending on $t$. Therefore, if the sum converges, it is not too surprising that the result will be normally distributed and the variance will depend on $t$. This is the idea behind the following theorem.

Theorem 64.1.3 There exists a real Wiener process as defined in Definition 64.1.1. Furthermore, the distribution of $W(t)-W(s)$ is the same as the distribution of $W(t-s)$ and $W$ is Holder continuous with exponent $\gamma$ for any $\gamma<1 / 2$. Also for each $\alpha>1$,

$$
E\left(|W(t)-W(s)|^{\alpha}\right) \leq C_{\alpha}|t-s|^{\alpha / 2} E\left(|W(1)|^{\alpha}\right)
$$

Proof: Let $\left\{g_{m}\right\}_{m=1}^{\infty}$ be a complete orthonormal set in $L^{2}(0, \infty)$. Thus, if $f \in L^{2}(0, \infty)$,

$$
f=\sum_{i=1}^{\infty}\left(f, g_{i}\right)_{L^{2}} g_{i}
$$

The Wiener process is defined as

$$
W(t, \omega) \equiv \sum_{i=1}^{\infty}\left(\mathscr{X}_{(0, t)}, g_{i}\right)_{L^{2}} \xi_{i}(\omega)
$$

where the random variables, $\left\{\xi_{i}\right\}$ are as described in Lemma 64.1.2. The series converges in $L^{2}(\Omega)$ where $(\Omega, \mathscr{F}, P)$ is the probability space on which the random variables, $\xi_{i}$ are defined. This will first be shown. Note first that from the independence of the $\xi_{i}$,

$$
\int_{\Omega} \xi_{i} \xi_{j} d P=0
$$

Therefore,

$$
\begin{aligned}
\int_{\Omega}\left|\sum_{i=m}^{n}\left(\mathscr{X}_{(0, t)}, g_{i}\right)_{L^{2}} \xi_{i}(\omega)\right|^{2} d P & =\sum_{i=m}^{n}\left(\mathscr{X}_{(0, t)}, g_{i}\right)_{L^{2}}^{2} \int_{\Omega}\left|\xi_{i}\right|^{2} d P \\
& =\sum_{i=m}^{n}\left(\mathscr{X}_{(0, t)}, g_{i}\right)_{L^{2}}^{2}
\end{aligned}
$$

which converges to 0 as $m, n \rightarrow \infty$. Thus the partial sums are a Cauchy sequence in $L^{2}(\Omega, P)$.

It just remains to verify this definition satisfies the desired conditions. First I will show that $\omega \rightarrow W(t, \omega)$ is normally distributed with mean 0 and variance $t$. That it should be normally distributed is not surprising since it is just a sum of independent random variables which are this way. Selecting a suitable subsequence, $\left\{n_{k}\right\}$ it can be assumed

$$
W(t, \omega)=\lim _{k \rightarrow \infty} \sum_{i=1}^{n_{k}}\left(\mathscr{X}_{(0, t)}, g_{i}\right)_{L^{2}} \xi_{i}(\omega) \text { a.e. }
$$

and so from the dominated convergence theorem and the independence of the $\xi_{i}$,

$$
\begin{aligned}
E(\exp (i \lambda W(t))) & =\lim _{k \rightarrow \infty} E\left(\exp \left(i \lambda \sum_{j=1}^{n_{k}}\left(\mathscr{X}_{(0, t)}, g_{j}\right)_{L^{2}} \xi_{j}(\omega)\right)\right) \\
& =\lim _{k \rightarrow \infty} E\left(\prod_{j=1}^{n_{k}} \exp \left(i \lambda\left(\mathscr{X}_{(0, t)}, g_{j}\right)_{L^{2}} \xi_{j}(\omega)\right)\right) \\
& =\lim _{k \rightarrow \infty} \prod_{j=1}^{n_{k}} E\left(\exp \left(i \lambda\left(\mathscr{X}_{(0, t)}, g_{j}\right)_{L^{2}} \xi_{j}(\omega)\right)\right) \\
& =\lim _{k \rightarrow \infty} \prod_{j=1}^{n_{k}} e^{-\frac{1}{2} \lambda^{2}\left(\mathscr{X}_{\left.(0, t), g_{j}\right)_{L^{2}}^{2}}^{2}\right.} \\
& =\lim _{k \rightarrow \infty} \exp \left(\sum_{j=1}^{n_{k}}-\frac{1}{2} \lambda^{2}\left(\mathscr{X}_{(0, t)}, g_{j}\right)_{L^{2}}^{2}\right) \\
& =\exp \left(-\frac{1}{2} \lambda^{2}\left\|\mathscr{X}_{(0, t)}\right\|_{L^{2}}^{2}\right)=\exp \left(-\frac{1}{2} \lambda^{2} t\right)
\end{aligned}
$$

the characteristic function of a normally distributed random variable having variance $t$ and mean 0 .

It is clear $W(0)=0$. It remains to verify the increments are independent. To do this, consider

$$
\begin{equation*}
E(\exp (i[\lambda(W(t)-W(s))+\mu(W(s)-W(r))])) \tag{64.1.1}
\end{equation*}
$$

Is this equal to

$$
\begin{equation*}
E(\exp (i[\lambda(W(t)-W(s))])) E(\exp (i[\mu(W(s)-W(r))])) ? \tag{64.1.2}
\end{equation*}
$$

Letting $n_{k} \rightarrow \infty$ such that convergence happens pointwise for each function of interest, and using the independence of the $\xi_{i}$, and the dominated convergence theorem as needed,

$$
\begin{aligned}
& E\left(\exp \left(i\left[\sum_{i=1}^{\infty} \lambda\left(\mathscr{X}_{(s, t)}, g_{i}\right)_{L^{2}} \xi_{i}+\sum_{i=1}^{\infty} \mu\left(\mathscr{X}_{(r, s)}, g_{i}\right)_{L^{2}} \xi_{i}\right]\right)\right) \\
= & \lim _{k \rightarrow \infty} E\left(\exp \left(i\left[\sum_{j=1}^{n_{k}}\left(\lambda\left(\mathscr{X}_{(s, t)}, g_{j}\right)_{L^{2}}+\mu\left(\mathscr{X}_{(r, s)}, g_{j}\right)_{L^{2}}\right) \xi_{j}\right]\right)\right)
\end{aligned}
$$

$$
\begin{gathered}
=\lim _{k \rightarrow \infty} E\left(\prod_{j=1}^{n_{k}} \exp \left(i\left(\lambda\left(\mathscr{X}_{(s, t)}, g_{j}\right)_{L^{2}}+\mu\left(\mathscr{X}_{(r, s)}, g_{j}\right)_{L^{2}}\right) \xi_{j}\right)\right) \\
=\lim _{k \rightarrow \infty} \prod_{j=1}^{n_{k}} E\left(\exp \left(i\left(\lambda\left(\mathscr{X}_{(s, t)}, g_{j}\right)_{L^{2}}+\mu\left(\mathscr{X}_{(r, s)}, g_{j}\right)_{L^{2}}\right) \xi_{j}\right)\right) \\
=\lim _{k \rightarrow \infty} \prod_{j=1}^{n_{k}} \exp \left(-\frac{1}{2}\left(\lambda \mathscr{X}_{(s, t)}+\mu \mathscr{X}_{(r, s)}, g_{j}\right)_{L^{2}}^{2}\right) \\
=\lim _{k \rightarrow \infty} \exp \left(-\frac{1}{2} \sum_{j=1}^{n_{k}}\left(\lambda \mathscr{X}_{(s, t)}+\mu \mathscr{X}_{(r, s)}, g_{j}\right)_{L^{2}}^{2}\right) \\
=\exp \left(-\frac{1}{2} \sum_{j=1}^{\infty}\left(\lambda \mathscr{X}_{(s, t)}+\mu \mathscr{X}_{(r, s)}, g_{j}\right)_{L^{2}}^{2}\right)=\exp \left(-\frac{1}{2}\left\|\lambda \mathscr{X}_{(s, t)}+\mu \mathscr{X}_{(r, s)}\right\|_{L^{2}}^{2}\right) \\
=\exp \left(-\frac{1}{2}\left[\lambda^{2}\left\|\mathscr{X}_{(s, t)}\right\|_{L^{2}}^{2}+\mu^{2}\left\|\mathscr{X}_{(r, s)}\right\|_{L^{2}}^{2}\right]\right)
\end{gathered}
$$

because the functions $\lambda \mathscr{X}_{(s, t)}, \mu \mathscr{X}_{(r, s)}$ are orthogonal. Then this equals

$$
\begin{gathered}
=\exp \left(-\frac{1}{2}\left[\lambda^{2}(t-s)+\mu^{2}(s-r)\right]\right) \\
=\exp \left(-\frac{1}{2}(t-s) \lambda^{2}\right) \exp \left(-\frac{1}{2}(s-r) \mu^{2}\right)
\end{gathered}
$$

which equals 64.1.2 and this shows the increments are independent. Obviously, this same argument shows this holds for any finite set of disjoint increments.

From the definition, if $t>s$

$$
W(t-s)=\sum_{k=1}^{\infty}\left(\mathscr{X}_{(0, t-s)}, g_{k}\right)_{L^{2}} \xi_{k}
$$

while

$$
W(t)-W(s)=\sum_{k=1}^{\infty}\left(\mathscr{X}_{(s, t)}, g_{k}\right)_{L^{2}} \xi_{k} .
$$

Then the same argument given above involving the characteristic function to show $W(t)$ is normally distributed shows both of these random variables are normally distributed with mean 0 and variance $t-s$ because they have the same characteristic function.

For example, ignoring the limit questions and proceding formally,

$$
\begin{aligned}
E(\exp (i \lambda(W(t)-W(s)))) & =E\left(\exp \left(i \lambda\left(\sum_{k=1}^{\infty}\left(\mathscr{X}_{(s, t)}, g_{k}\right)_{L^{2}} \xi_{k}\right)\right)\right) \\
& =E\left(\prod_{k=1}^{\infty} \exp \left(i \lambda\left(\mathscr{X}_{(s, t)}, g_{k}\right)_{L^{2}} \xi_{k}\right)\right) \\
& =\prod_{k=1}^{\infty} E\left(\exp \left(i \lambda\left(\mathscr{X}_{(s, t)}, g_{k}\right)_{L^{2}} \xi_{k}\right)\right) \\
& =\prod_{k=1}^{\infty} e^{-\frac{1}{2} \lambda^{2}\left(\mathscr{X}_{(s, t)}, g_{k}\right)_{L^{2}}^{2}} \\
& =\exp \left(-\frac{1}{2} \lambda^{2} \sum_{k=1}^{\infty}\left(\mathscr{X}_{(s, t)}, g_{k}\right)_{L^{2}}^{2}\right) \\
& =\exp \left(-\frac{1}{2} \lambda^{2}(t-s)\right)
\end{aligned}
$$

which is the characteristic function of a random variable having mean 0 and variance $t-s$.
Finally note the distribution of $W(t-s)$ is the same as the distribution of

$$
W(1)(t-s)^{1 / 2}=\sum_{k=1}^{\infty}\left(\mathscr{X}_{(0,1)}, g_{k}\right)_{L^{2}} \xi_{k}(t-s)^{1 / 2}
$$

because the characteristic function of this last random variable is the same as the characteristic function of $W(t-s)$ which is $e^{-\frac{1}{2} \lambda^{2}(t-s)}$ which follows from a simple computation. Since $W(1)$ is a normally distrubuted random variable with mean 0 and variance 1 ,

$$
E\left(\exp \left(i \lambda W(1)(t-s)^{1 / 2}\right)\right)=e^{-\frac{1}{2} \lambda^{2}(t-s)}
$$

which is the same as the characteristic function of $W(t-s)$.
Hence for any positive $\alpha$,

$$
\begin{align*}
E\left(|W(t)-W(s)|^{\alpha}\right) & =E\left(|W(t-s)|^{\alpha}\right) \\
& =E\left(\left|(t-s)^{1 / 2} W(1)\right|^{\alpha}\right) \\
& =|t-s|^{\alpha / 2} E\left(|W(1)|^{\alpha}\right) \tag{64.1.3}
\end{align*}
$$

It follows from Theorem 62.2.2 that $W(t)$ is Holder continuous with exponent $\gamma$ where $\gamma$ is any positive number less than $\beta / \alpha$ where $\alpha / 2=1+\beta$. Thus $\gamma$ is any constant less than

$$
\frac{\frac{\alpha}{2}-1}{\alpha}=\frac{1}{2} \frac{\alpha-2}{\alpha}
$$

Thus $\gamma$ is any constant less than $\frac{1}{2}$.
The proof of the theorem, which only depended on $\left\{\xi_{i}\right\}_{i=1}^{\infty}$ being independent random variables each normal with mean 0 and variance 1 , implies the following corollary.

Corollary 64.1.4 Let $\left\{\xi_{i}\right\}_{i=1}^{\infty}$ be independent random variables each normal with mean 0 and variance 1. Then

$$
W(t, \omega) \equiv \sum_{i=1}^{\infty}\left(\mathscr{X}_{[0, t]}, g_{i}\right)_{L^{2}} \xi_{i}(\omega)
$$

is a real Wiener process. Furthermore, the distribution of $W(t)-W(s)$ is the same as the distribution of $W(t-s)$ and $W$ is Holder continuous with exponent $\gamma$ for any $\gamma<1 / 2$. Also for each $\alpha>1$,

$$
E\left(|W(t)-W(s)|^{\alpha}\right) \leq C_{\alpha}|t-s|^{\alpha / 2} E\left(|W(1)|^{\alpha}\right)
$$

### 64.2 Nowhere Differentiability of Wiener Processes

If $W(t)$ is a Wiener process, it turns out that $t \rightarrow W(t, \omega)$ is nowhere differentiable for a.e. $\omega$. This fact is based on the independence of the increments and the fact that these increments are normally distributed.

First note that $W(t)-W(s)$ has the same distribution as $(t-s)^{1 / 2} W(1)$. This is because they have the same characteristic function. Next it follows that because of the independence of the increments and what was just noted that,

$$
\begin{align*}
& P\left(\cap_{r=1}^{5}[|W(t+r \delta)-W(t+(r-1) \boldsymbol{\delta})| \leq K \boldsymbol{\delta}]\right) \\
= & \prod_{r=1}^{5} P([|W(t+r \delta)-W(t+(r-1) \delta)| \leq K \boldsymbol{\delta}]) \\
= & \prod_{r=1}^{5} P\left(\left[\left|\delta^{1 / 2} W(1)\right| \leq K \delta\right]\right)=\left(\frac{1}{\sqrt{2 \pi}} \int_{-K \sqrt{\delta}}^{K \sqrt{\delta}} e^{-\frac{1}{2} t^{2}} d t\right)^{5} \\
\leq & C \delta^{5 / 2} \tag{64.2.4}
\end{align*}
$$

With this observation, here is the proof which follows [120] and according to this reference is due to Payley, Wiener and Zygmund and the proof is like one given by Dvoretsky, Erdös and Kakutani.

Theorem 64.2.1 Let $W(t)$ be a Wiener process. Then there exists a set of measure $0, N$ such that for all $\omega \notin N$,

$$
t \rightarrow W(t, \omega)
$$

is nowhere differentiable.
Proof: Let $[0, a]$ be an interval. If for some $\omega, t \rightarrow W(t, \omega)$ is differentiable at some $s$, then for some $n, p>0$,

$$
\left|\frac{W(t, \omega)-W(s, \omega)}{t-s}\right| \leq p
$$

whenever $|t-s|<5 a 2^{-n} \equiv 5 \delta_{n}$. Define $C_{n p}$ by

$$
\begin{equation*}
\left\{\omega: \text { for some } s \in[0, a),\left|\frac{W(t, \omega)-W(s, \omega)}{t-s}\right| \leq p \text { if }|t-s| \leq 5 \delta_{n}\right\} \tag{64.2.5}
\end{equation*}
$$

Thus $\cup_{n, p \in \mathbb{N}} C_{n p}$ contains the set of $\omega$ such that $t \rightarrow W(t, \omega)$ is differentiable for some $s \in[0, a)$.

Now define uniform partitions of $[0, a),\left\{t_{k}^{n}\right\}_{k=0}^{2^{n}}$ such that

$$
\left|t_{k}^{n}-t_{k-1}^{n}\right|=a 2^{-n} \equiv \delta_{n}
$$

Let

$$
D_{n p} \equiv \cup_{i=0}^{2^{n}-1}\left(\cap_{r=1}^{5}\left[\left|W\left(t_{i}^{n}+r \delta_{n}\right)-W\left(t_{i}^{n}+(r-1) \delta_{n}\right)\right| \leq 10 p \delta_{n}\right]\right)
$$

If $\omega \in C_{n p}$, then for some $s \in[0, a)$, the condition of 64.2 .5 holds. Suppose $k$ is the number such that $s \in\left[t_{k-1}^{n}, t_{k}^{n}\right)$. Then for $r \in\{1,2,3,4,5\}$,

$$
\begin{gathered}
\left|W\left(t_{k-1}^{n}+r \delta_{n}, \omega\right)-W\left(t_{k-1}^{n}+(r-1) \delta_{n}, \omega\right)\right| \\
\leq\left|W\left(t_{k-1}^{n}+r \delta_{n}, \omega\right)-W(s, \omega)\right|+\left|W(s, \omega)-W\left(t_{k-1}^{n}+(r-1) \delta_{n}, \omega\right)\right| \\
\leq 5 p \delta_{n}+5 p \delta_{n}=10 p \delta_{n}
\end{gathered}
$$

Thus $C_{n p} \subseteq D_{n p}$. Now from 64.2.4,

$$
\begin{equation*}
P\left(D_{n p}\right) \leq 2^{n} C \delta_{n}^{5 / 2}=C a^{5 / 2} 2^{n}\left(2^{-n}\right)^{5 / 2}=C(\sqrt{a})^{5} 2^{-\frac{3}{2} n} \tag{64.2.6}
\end{equation*}
$$

Let

$$
C_{p}=\cup_{n=1}^{\infty} \cap_{k=n}^{\infty} C_{k p} \subseteq \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} D_{k p} .
$$

It was just shown in 64.2 .6 that $P\left(\cap_{k=n}^{\infty} D_{k p}\right)=0$ and so $C_{p}$ has measure 0 . Thus $\cup_{p=1}^{\infty} C_{p}$, the set of points, $\omega$ where $t \rightarrow W(t, \omega)$ could have a derivative has measure 0 . Taking the union of the exceptional sets corresponding to intervals $[0, n)$ for $n \in \mathbb{N}$, this proves the theorem.

This theorem on nowhere differentiability is very important because it shows it is doubtful one can define an integral $\int f(s) d W(s)$ by simply fixing $\omega$ and then doing some sort of Stieltjes integral in time. The reason for this is that the nowhere differentiability of $W$ implies it is also not of bounded variation on any interval since if it were, it would equal the difference of two increasing functions and would therefore have a derivative at a.e. point.

I have presented the theorem on nowhere differentiability for one dimensional Wiener processes but the same proof holds with minor modifications if you have defined the Wiener process in $\mathbb{R}^{n}$ or you could simply consider the components and apply the above result.

### 64.3 Wiener Processes In Separable Banach Space

Here is an important lemma on which the existence of Wiener processes will be based.
Lemma 64.3.1 There exists a sequence of real Wiener processes, $\left\{\psi_{k}(t)\right\}_{k=1}^{\infty}$ which have the following properties. Let $t_{0}<t_{1}<\cdots<t_{n}$ be an arbitrary sequence. Then the random variables

$$
\begin{equation*}
\left\{\psi_{k}\left(t_{q}\right)-\psi_{k}\left(t_{q-1}\right):(q, k) \in(1,2, \cdots, n) \times\left(k_{1}, \cdots, k_{m}\right)\right\} \tag{64.3.7}
\end{equation*}
$$

are independent. Also each $\psi_{k}$ is Holder continuous with exponent $\gamma$ for any $\gamma<1 / 2$ and for each $m \in \mathbb{N}$ there exists a constant $C_{m}$ independent of $k$ such that

$$
\begin{equation*}
\int_{\Omega}\left|\psi_{k}(t)-\psi_{k}(s)\right|^{2 m} d P \leq C_{m}|t-s|^{m} \tag{64.3.8}
\end{equation*}
$$

Proof: First, there exists a sequence $\left\{\xi_{i j}\right\}_{(i, j) \in \mathbb{N} \times \mathbb{N}}$ such that the $\left\{\xi_{i j}\right\}$ are independent and each normally distributed with mean 0 and variance 1 . This follows from Lemma 64.1.2. Let $\left\{\xi_{i}\right\}_{i=1}^{\infty}$ be independent and normally distributed with mean 0 and variance 1. (Let $\theta$ be a one to one and onto map from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$. Then define $\xi_{i j} \equiv \xi_{\theta^{-1}(i, j)}$.)

Let

$$
\begin{equation*}
\psi_{k}(t)=\sum_{j=1}^{\infty}\left(\mathscr{X}_{[0, t]}, g_{j}\right)_{L^{2}} \xi_{k j} \tag{64.3.9}
\end{equation*}
$$

where $\left\{g_{j}\right\}$ is a orthonormal basis for $L^{2}(0, \infty)$. By Corollary 64.1.4, this defines a real Wiener process satisfying 64.3.8. It remains to show that the random variables

$$
\begin{equation*}
\psi_{k_{r}}\left(t_{q}\right)-\psi_{k_{r}}\left(t_{q-1}\right) \tag{64.3.10}
\end{equation*}
$$

are independent.
Let

$$
P=\sum_{q=1}^{n} \sum_{r=1}^{m} s_{q r}\left(\psi_{k_{r}}\left(t_{q}\right)-\psi_{k_{r}}\left(t_{q-1}\right)\right)
$$

and consider $E\left(e^{i P}\right)$. I want to use Proposition 59.11.1 on Page 1889. To do this I need to show $E\left(e^{i P}\right)$ equals

$$
\prod_{q=1}^{n} \prod_{r=1}^{m} E\left(\exp \left(i s_{q r}\left(\psi_{k_{r}}\left(t_{q}\right)-\psi_{k_{r}}\left(t_{q-1}\right)\right)\right)\right)
$$

Using 64.3.9, $E\left(e^{i P}\right)$ equals

$$
\begin{aligned}
& E\left(\exp \left(i \sum_{q=1}^{n} \sum_{r=1}^{m} s_{q r} \sum_{j=1}^{\infty}\left(\mathscr{X}_{\left[t_{q-1}, t_{q}\right]}, g_{j}\right)_{L^{2}} \xi_{k_{r} j}\right)\right) \\
= & \lim _{N \rightarrow \infty} E\left(\exp \left(i \sum_{q=1}^{n} \sum_{r=1}^{m} s_{q r} \sum_{j=1}^{N}\left(\mathscr{X}_{\left[t_{q-1}, t_{q}\right]}, g_{j}\right)_{L^{2}} \xi_{k_{r} j}\right)\right)
\end{aligned}
$$

Now the $\xi_{k_{r} j}$ are independent by construction. Therefore, the above equals

$$
\begin{aligned}
= & \lim _{N \rightarrow \infty} \prod_{q=1}^{n} \prod_{r=1}^{m} \prod_{j=1}^{N} E\left(\exp \left(i s_{q r}\left(\mathscr{X}_{\left[t_{q-1}, t_{q}\right]}, g_{j}\right)_{L^{2}} \xi_{k_{r} j}\right)\right) \\
& =\lim _{N \rightarrow \infty} \prod_{q=1}^{n} \prod_{r=1}^{m} \prod_{j=1}^{N} \exp \left(-\frac{1}{2} s_{q r}^{2}\left(\mathscr{X}_{\left[t_{q-1}, t_{q}\right]}, g_{j}\right)_{L^{2}}^{2}\right) \\
& =\prod_{q=1}^{n} \prod_{r=1}^{m} \lim _{N \rightarrow \infty} \exp \left(-\frac{1}{2} s_{q r}^{2} \sum_{j=1}^{N}\left(\mathscr{X}_{\left[t_{q-1}, t_{q}\right]}, g_{j}\right)_{L^{2}}^{2}\right) \\
& =\prod_{q=1}^{n} \prod_{r=1}^{m} \exp \left(-\frac{1}{2} s_{q r}^{2}\left(t_{q}-t_{q-1}\right)\right)
\end{aligned}
$$

$$
=\prod_{q=1}^{n} \prod_{r=1}^{m} E\left(\exp \left(i s_{q r}\left(\psi_{k_{r}}\left(t_{q}\right)-\psi_{k_{r}}\left(t_{q-1}\right)\right)\right)\right)
$$

because $\psi_{k_{r}}\left(t_{q}\right)-\psi_{k_{r}}\left(t_{q-1}\right)$ is normally distributed with variance $t_{q}-t_{q-1}$ and mean 0 . By Proposition 59.11.1 on Page 1889, it follows the random variables of 64.3.10 are independent. Note that as a special case, this also shows the random variables, $\left\{\psi_{k}(t)\right\}_{k=1}^{\infty}$ are independent due to the fact $\psi_{k}(0)=0$.

Recall Corollary 61.11 .4 which is stated here for convenience.

Corollary 64.3.2 Let $E$ be any real separable Banach space. Then there exists a sequence, $\left\{e_{k}\right\} \subseteq E$ such that for any $\left\{\xi_{k}\right\}$ a sequence of independent random variables such that $\mathscr{L}\left(\xi_{k}\right)=N(0,1)$, it follows

$$
X(\omega) \equiv \sum_{k=1}^{\infty} \xi_{k}(\omega) e_{k}
$$

converges a.e. and its law is a Gaussian measure defined on $\mathscr{B}(E)$. Furthermore, $\left\|e_{k}\right\|_{E} \leq$ $\lambda_{k}$ where $\sum_{k} \lambda_{k}<\infty$.

Now let $\left\{\psi_{k}(t)\right\}$ be the sequence of Wiener processes described in Lemma 64.3.1. Then define a process with values in $E$ by

$$
\begin{equation*}
W(t) \equiv \sum_{k=1}^{\infty} \psi_{k}(t) e_{k} \tag{64.3.11}
\end{equation*}
$$

Then $\psi_{k}(t) / \sqrt{t}$ is $N(0,1)$ and so by Corollary 61.11.4 the law of

$$
W(t) / \sqrt{t}=\sum_{k=1}^{\infty}\left(\psi_{k}(t) / \sqrt{t}\right) e_{k}
$$

is a Gaussian measure. Therefore, the same is true of $W(t)$. Similar reasoning applies to the increments, $W(t)-W(s)$ to conclude the law of each of these is Gaussian. Consider the question whether the increments are independent. Let $0 \leq t_{0}<t_{1}<\cdots<t_{m}$ and let $\phi_{j} \in E^{\prime}$. Then by the dominated convergence theorem and the properties of the $\left\{\psi_{k}\right\}$,

$$
\begin{aligned}
& E\left(\exp \left(i \sum_{j=1}^{m} \phi_{j}\left(W\left(t_{j}\right)-W\left(t_{j-1}\right)\right)\right)\right) \\
= & E\left(\exp \left(i \sum_{j=1}^{m}\left(\sum_{k=1}^{\infty}\left(\psi_{k}\left(t_{j}\right)-\psi_{k}\left(t_{j-1}\right)\right) \phi_{j}\left(e_{k}\right)\right)\right)\right) \\
= & E\left(\prod_{j=1}^{m} \exp \left(i \sum_{k=1}^{\infty}\left(\psi_{k}\left(t_{j}\right)-\psi_{k}\left(t_{j-1}\right)\right) \phi_{j}\left(e_{k}\right)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
& =\lim _{n \rightarrow \infty} E\left(\prod_{j=1}^{m} \exp \left(i \sum_{k=1}^{n}\left(\psi_{k}\left(t_{j}\right)-\psi_{k}\left(t_{j-1}\right)\right) \phi_{j}\left(e_{k}\right)\right)\right) \\
& =\lim _{n \rightarrow \infty} \prod_{j=1}^{m} \prod_{k=1}^{n} E\left(\exp \left(i\left(\psi_{k}\left(t_{j}\right)-\psi_{k}\left(t_{j-1}\right)\right) \phi_{j}\left(e_{k}\right)\right)\right) \\
& =\lim _{n \rightarrow \infty} \prod_{j=1}^{m} E\left(\exp \left(i \sum_{k=1}^{n}\left(\psi_{k}\left(t_{j}\right)-\psi_{k}\left(t_{j-1}\right)\right) \phi_{j}\left(e_{k}\right)\right)\right) \\
& =\lim _{n \rightarrow \infty} E\left(\prod_{j=1}^{m} \exp \left(i \sum_{k=1}^{n}\left(\psi_{k}\left(t_{j}\right)-\psi_{k}\left(t_{j-1}\right)\right) \phi_{j}\left(e_{k}\right)\right)\right) \\
& =\lim _{n \rightarrow \infty} \prod_{j=1}^{m} \prod_{k=1}^{n} E\left(\exp \left(i\left(\psi_{k}\left(t_{j}\right)-\psi_{k}\left(t_{j-1}\right)\right) \phi_{j}\left(e_{k}\right)\right)\right) \\
& =\lim _{n \rightarrow \infty} \prod_{j=1}^{m} E\left(\exp \left(i \sum_{k=1}^{n}\left(\psi_{k}\left(t_{j}\right)-\psi_{k}\left(t_{j-1}\right)\right) \phi_{j}\left(e_{k}\right)\right)\right) \\
& =\prod_{j=1}^{m} E\left(\exp \left(i \sum_{k=1}^{\infty}\left(\psi_{k}\left(t_{j}\right)-\psi_{k}\left(t_{j-1}\right)\right) \phi_{j}\left(e_{k}\right)\right)\right) \\
& =\prod_{j=1}^{m} E\left(\exp \left(i \phi_{j}\left(\sum_{k=1}^{\infty}\left(\psi_{k}\left(t_{j}\right)-\psi_{k}\left(t_{j-1}\right)\right) e_{k}\right)\right)\right) \\
& =\prod_{j=1}^{m} E\left(\exp \left(i \phi_{j}\left(W\left(t_{j}\right)-W\left(t_{j-1}\right)\right)\right)\right)
\end{aligned}
$$

which shows by Theorem 59.13.3 on Page 1896 that the random vectors,

$$
\left\{W\left(t_{j}\right)-W\left(t_{j-1}\right)\right\}_{j=1}^{m}
$$

are independent.
It is also routine to verify using properties of the $\psi_{k}$ and characteristic functions that $\mathscr{L}(W(t)-W(s))=\mathscr{L}(W(t-s))$. To see this, let $\phi \in E^{\prime}$

$$
\begin{gathered}
E(\exp (i \phi(W(t)-W(s)))) \\
=E\left(\left(\exp \left(i \phi \sum_{k=1}^{\infty}\left(\psi_{k}(t)-\psi_{k}(s)\right) e_{k}\right)\right)\right) \\
=\lim _{n \rightarrow \infty} E\left(\left(\exp \left(i \phi \sum_{k=1}^{n}\left(\psi_{k}(t)-\psi_{k}(s)\right) e_{k}\right)\right)\right)
\end{gathered}
$$

$$
\begin{aligned}
& =\lim _{n \rightarrow \infty} \prod_{k=1}^{n} E\left(\exp \left(i \phi\left(e_{k}\right)\left(\psi_{k}(t)-\psi_{k}(s)\right)\right)\right) \\
& =\lim _{n \rightarrow \infty} \prod_{k=1}^{n} E\left(\exp \left(-\frac{1}{2} \phi\left(e_{k}\right)^{2}(t-s)\right)\right) \\
& =\lim _{n \rightarrow \infty} E\left(\exp \sum_{k=1}^{n}\left(-\frac{1}{2} \phi\left(e_{k}\right)^{2}(t-s)\right)\right)
\end{aligned}
$$

which is the same as the result for

$$
E(\exp (i \phi(W(t-s))))
$$

and

$$
E(\exp (i \phi(\sqrt{t-s} W(1)))) .
$$

This has proved the following lemma.
Lemma 64.3.3 Let $E$ be a real separable Banach space. Then there exists an $E$ valued stochastic process, $W(t)$ such that $\mathscr{L}(W(t))$ and $\mathscr{L}(W(t)-W(s))$ are Gaussian measures and the increments, $\{W(t)-W(s)\}$ are independent. Furthermore, the increment $W(t)-W(s)$ has the same distribution as $W(t-s)$ and $W(t)$ has the same distribution as $\sqrt{t} W(1)$.

Now I want to consider the question of Holder continuity of the functions, $t \rightarrow W(t, \omega)$.

$$
\begin{aligned}
\int_{\Omega}\|W(t)-W(s)\|^{\alpha} d P & =\int_{E}\|x\|^{\alpha} d \mu_{W(t)-W(s)} \\
& =\int_{E}\|x\|^{\alpha} d \mu_{W(t-s)}=\int_{E}\|x\|^{\alpha} d \mu_{\sqrt{t-s} W(1)} \\
& =\int_{\Omega}\|\sqrt{t-s} W(1)\|^{\alpha} d P \\
& =|t-s|^{\alpha / 2} \int_{\Omega}\|W(1)\|^{\alpha} d P=C_{\alpha}|t-s|^{\alpha / 2}
\end{aligned}
$$

by Fernique's theorem, Theorem 61.7.5. From the Kolmogorov Čentsov theorem, Theorem 62.2.2, it follows $\{W(t)\}$ is Holder continuous with exponent $\gamma<\left(\frac{\alpha}{2}-1\right) / \alpha$.

This completes the proof of the following theorem.
Theorem 64.3.4 Let $E$ be a separable real Banach space. Then there exists a stochastic process, $\{W(t)\}$ such that the distribution of $W(t)$ and every increment, $W(t)-W(s)$ is Gaussian. Furthermore, the increments corresponding to disjoint intervals are independent, $\mathscr{L}(W(t)-W(s))=\mathscr{L}(W(t-s))=\mathscr{L}(\sqrt{t-s} W(1))$. Also for a.e. $\omega, t \rightarrow W(t, \omega)$ is Holder continuous with exponent $\gamma<1 / 2$.

### 64.4 Independent Increments and Martingales

Here is an interesting lemma.

Lemma 64.4.1 Let $\left(W(t), \mathscr{F}_{t}\right)$ be a stochastic process which has independent increments having values in E a real separable Banach space. Let

$$
A \in \mathscr{F}_{s} \equiv \sigma(W(u)-W(r): 0 \leq r<u \leq s)
$$

Suppose $g(W(t)-W(s)) \in L^{1}(\Omega ; E)$. Then the following formula holds.

$$
\begin{equation*}
\int_{\Omega} \mathscr{X}_{A} g(W(t)-W(s)) d P=P(A) \int_{\Omega} g(W(t)-W(s)) d P \tag{64.4.12}
\end{equation*}
$$

Proof: Let $\mathscr{G}$ denote the set, of all $A \in \mathscr{F}_{s}$ such that 64.4.12 holds. Then it is obvious $\mathscr{G}$ is closed with respect to complements and countable disjoint unions. Let $\mathscr{K}$ denote those sets which are finite intersections of the form

$$
A=\cap_{i=1}^{m} A_{i}
$$

where each $A_{i}$ is in a set of $\sigma\left(W\left(u_{i}\right)-W\left(r_{i}\right)\right)$ for some $0 \leq r_{i}<u_{i} \leq s$. For such $A$, it follows

$$
A \in \sigma\left(W\left(u_{i}\right)-W\left(r_{i}\right), i=1, \cdots, m\right)
$$

Now consider the random vector having values in $E^{m+1}$,

$$
\left(W\left(u_{1}\right)-W\left(r_{1}\right), \cdots, W\left(u_{m}\right)-W\left(r_{m}\right), g(W(t)-W(s))\right)
$$

Let $\mathbf{t}^{*} \in\left(E^{\prime}\right)^{m}$ and $s^{*} \in E^{\prime}$.

$$
\mathbf{t}^{*} \cdot\left(W\left(u_{1}\right)-W\left(r_{1}\right), \cdots, W\left(u_{m}\right)-W\left(r_{m}\right)\right)
$$

can be written in the form $\mathbf{g}^{*} \cdot\left(W\left(\tau_{1}\right)-W\left(\eta_{1}\right), \cdots, W\left(\tau_{l}\right)-W\left(\eta_{l}\right)\right)$ where the intervals, $\left(\eta_{j}, \tau_{j}\right)$ are disjoint and each $\tau_{j} \leq s$. For example, suppose you have

$$
a(W(2)-W(1))+b(W(2)-W(0))+c(W(3)-W(1))
$$

where obviously the increments are not disjoint. Then you would write the above expression as

$$
\begin{aligned}
& a(W(2)-W(1))+b(W(2)-W(1))+b(W(1)-W(0)) \\
& +c(W(3)-W(2))+c(W(2)-W(1))
\end{aligned}
$$

and then you would collect the terms to obtain

$$
b(W(1)-W(0))+(a+b+c)(W(2)-W(1))+c(W(3)-W(2))
$$

and now these increments are disjoint.
Therefore, by independence of the increments,

$$
\begin{aligned}
& E\left(\exp i\left(\mathbf{t}^{*} \cdot\left(W\left(u_{1}\right)-W\left(r_{1}\right), \cdots, W\left(u_{m}\right)-W\left(r_{m}\right)\right)+s^{*}(g(W(t)-W(s)))\right)\right) \\
= & E\left(\exp i\left(\mathbf{g}^{*} \cdot\left(W\left(\tau_{1}\right)-W\left(\eta_{1}\right), \cdots, W\left(\tau_{l}\right)-W\left(\eta_{l}\right)\right)+s^{*}(g(W(t)-W(s)))\right)\right)
\end{aligned}
$$

$$
\begin{gathered}
=\prod_{j=1}^{l} E\left(\exp \left(i g_{j}\left(W\left(\tau_{j}\right)-W\left(\eta_{j}\right)\right)\right)\right) E\left(\exp \left(i s^{*}(g(W(t)-W(s)))\right)\right) \\
=E\left(\exp \left(i\left(\mathbf{t}^{*} \cdot\left(W\left(u_{1}\right)-W\left(r_{1}\right), \cdots, W\left(u_{m}\right)-W\left(r_{m}\right)\right)\right)\right)\right) . \\
E\left(\exp \left(i s^{*}(g(W(t)-W(s)))\right) .\right.
\end{gathered}
$$

By Theorem 59.13.3, it follows the vector $\left(W\left(u_{1}\right)-W\left(r_{1}\right), \cdots, W\left(u_{m}\right)-W\left(r_{m}\right)\right)$ is independent of the random variable $g(W(t)-W(s))$ which shows that for $A \in \mathscr{K}, \mathscr{X}_{A}$, measurable in $\sigma\left(W\left(u_{1}\right)-W\left(r_{1}\right), \cdots, W\left(u_{m}\right)-W\left(r_{m}\right)\right)$ is independent of $g(W(t)-W(s))$. Therefore,

$$
\begin{aligned}
\int_{\Omega} \mathscr{X}_{A} g(W(t)-W(s)) d P & =\int_{\Omega} \mathscr{X}_{A} d P \int_{\Omega} g(W(t)-W(s)) d P \\
& =P(A) \int_{\Omega} g(W(t)-W(s)) d P
\end{aligned}
$$

Thus $\mathscr{K} \subseteq \mathscr{G}$ and so by the lemma on $\pi$ systems, Lemma 12.12.3 on Page 329 , it follows $\mathscr{G} \supseteq \sigma(\mathscr{K}) \supseteq \mathscr{F}_{s} \supseteq \mathscr{G}$.

Lemma 64.4.2 Let $\{W(t)\}$ be a stochastic process having values in a separable Banach space which has the property that if $t_{1}<t_{2} \cdots<t_{n}$, then the increments,

$$
\left\{W\left(t_{k}\right)-W\left(t_{k-1}\right)\right\}
$$

are independent and integrable and $E(W(t)-W(s))=0$. Suppose also that $W(t)$ is right continuous, meaning that for $\omega$ off a set of measure zero, $t \rightarrow W(t)(\omega)$ is right continuous. Also suppose that for some $q>1$

$$
\|W(t)-W(s)\|_{L^{q}(\Omega)}
$$

is bounded independent of $s \leq t$. Then $\{W(t)\}$ is also a martingale with respect to the normal filtration defined by

$$
\mathscr{F}_{s} \equiv \cap_{t>s} \overline{\sigma(W(u)-W(r): 0 \leq r<u \leq t)}
$$

where this denotes the intersection of the completions of the $\sigma$ algebras

$$
\sigma(W(u)-W(r): 0 \leq r<u \leq t)
$$

Also, in the same situation but without the assumption that $E(W(t)-W(s))=0$, if $t>s$ and $A \in \mathscr{F}_{s}$ it follows that if $g$ is a continuous function such that

$$
\begin{equation*}
\|g(W(t)-W(s))\|_{L^{q}(\Omega)} \tag{64.4.13}
\end{equation*}
$$

is bounded independent of $s \leq t$ for some $q>1$ then for $t>s$,

$$
\begin{equation*}
\int_{\Omega} \mathscr{X}_{A} g(W(t)-W(s)) d P=P(A) \int_{\Omega} g(W(t)-W(s)) d P . \tag{64.4.14}
\end{equation*}
$$

Proof: Consider first the claim, 64.4.14. To begin with I show that if $A \in \mathscr{F}_{s}$ then for all $\varepsilon$ small enough that $t>s+\varepsilon,{ }^{1}$

$$
\begin{equation*}
\int_{\Omega} \mathscr{X}_{A} g(W(t)-W(s+\varepsilon)) d P=P(A) \int_{\Omega} g(W(t)-W(s+\varepsilon)) d P \tag{64.4.15}
\end{equation*}
$$

This will happen if $\mathscr{X}_{A}$ and $g(W(t)-W(s+\varepsilon))$ are independent. First note that from the definition

$$
A \in \overline{\sigma(W(u)-W(r): 0 \leq r<u \leq s+\varepsilon)}
$$

and so from the process of completion of a measure space, there exists

$$
B \in \sigma(W(u)-W(r): 0 \leq r<u \leq s+\varepsilon)
$$

such that $B \supseteq A$ and $P(B \backslash A)=0$. Therefore, letting $\phi \in E^{\prime}$,

$$
\begin{gathered}
E\left(\exp \left(i t \mathscr{X}_{A}+i \phi(g(W(t)-W(s+\varepsilon)))\right)\right) \\
=E\left(\exp \left(i t \mathscr{X}_{B}+i \phi(g(W(t)-W(s+\varepsilon)))\right)\right) \\
=E\left(\exp \left(i t \mathscr{X}_{B}\right)\right) E(\exp (i \phi(g(W(t)-W(s+\varepsilon)))))
\end{gathered}
$$

because $\mathscr{X}_{\boldsymbol{B}}$ is independent of $g(W(t)-W(s+\varepsilon))$ by Lemma 64.4.1 above. Then the above equals

$$
=E\left(\exp \left(i t \mathscr{X}_{A}\right)\right) E(\exp (i \phi(g(W(t)-W(s+\varepsilon)))))
$$

Now by Theorem 59.13.3, 64.4 .15 follows. Next pass to the limit in both sides of 64.4.15 as $\varepsilon \rightarrow 0$. One can do this because of 64.4 .13 which implies the functions in the integrands are uniformly integrable and Vitali's convergence theorem, Theorem 21.5.7. This yields 64.4.14.

Now consider the part about the stochastic process being a martingale. Let $g$ be the identity map. If $A \in \mathscr{F}_{s}$, the above implies

$$
\begin{aligned}
\int_{A} E\left(W(t) \mid \mathscr{F}_{s}\right) d P & =\int_{A} W(t) d P=\int_{A}(W(t)-W(s)) d P+\int_{A} W(s) d P \\
& =P(A) \int_{\Omega}(W(t)-W(s)) d P+\int_{A} W(s) d P=\int_{A} W(s) d P
\end{aligned}
$$

and so since $A$ is arbitrary, $E\left(W(t) \mid \mathscr{F}_{s}\right)=W(s)$.
Note this implies immediately from Lemma 63.1.5 that Wiener process is not of bounded variation on any interval. This is because this lemma implies if it were of bounded variation, then it would be constant which is not the case due to

$$
\mathscr{L}(W(t)-W(s))=\mathscr{L}(W(t-s))=\mathscr{L}(\sqrt{t-s} W(1)) .
$$

Here is an interesting theorem about approximation.

[^43]Theorem 64.4.3 Let $\{W(t)\}$ be a Wiener process having values in a separable Banach space as described in Theorem 64.3.4. There exists a set of measure $0, N$ such that for $\omega \notin N$, the sum in 64.3 .11 converges uniformly to $W(t, \omega)$ on any interval, $[0, T]$. That is, for each $\omega$ not in a set of measure zero, the partial sums of the sum in that formula converge uniformly to $t \rightarrow W(t, \omega)$ on $[0, T]$.

Proof: By Lemma 64.4.2 the independence of the increments imply

$$
\sum_{k=m}^{n} \psi_{k}(t) e_{k}
$$

is a martingale and so by Theorem 62.5.3,

$$
P\left(\left[\sup _{t \in[0, T]}\left\|\sum_{k=m}^{n} \psi_{k}(t) e_{k}\right\| \geq \alpha\right]\right) \leq \frac{1}{\alpha} \int_{\Omega}\left\|\sum_{k=m}^{n} \psi_{k}(T) e_{k}\right\| d P
$$

From Corollary 64.3.2

$$
\begin{aligned}
\int_{\Omega}\left\|\sum_{k=m}^{n} \psi_{k}(T) e_{k}\right\| d P & \leq \sum_{k=m}^{n} \int_{\Omega}\left|\psi_{k}(T)\right| d P \lambda_{k} \\
& \leq \sum_{k=m}^{n} \lambda_{k}
\end{aligned}
$$

which shows that there exists a subsequence, $m_{l}$ such that whenever $n>m_{l}$,

$$
P\left(\left[\sup _{t \in[0, T]}\left\|\sum_{k=m_{l}}^{n} \psi_{k}(t) e_{k}\right\| \geq 2^{-k}\right]\right) \leq 2^{-k}
$$

Recall Lemma 59.15.6 stated below for convenience.
Lemma 64.4.4 Let $\left\{\zeta_{k}\right\}$ be a sequence of random variables having values in a separable real Banach space, $E$ whose distributions are symmetric. Letting $S_{k} \equiv \sum_{i=1}^{k} \zeta_{i}$, suppose $\left\{S_{n_{k}}\right\}$ converges a.e. Also suppose that for every $m>n_{k}$,

$$
\begin{equation*}
P\left(\left[\left|\mid S_{m}-S_{n_{k}} \|_{E}>2^{-k}\right]\right)<2^{-k}\right. \tag{64.4.16}
\end{equation*}
$$

Then in fact,

$$
\begin{equation*}
S_{k}(\omega) \rightarrow S(\omega) \text { a.e. } \omega \tag{64.4.17}
\end{equation*}
$$

Apply this lemma to the situation in which the Banach space, $E$ is $C([0, T] ; E)$ and $\zeta_{k}=\psi_{k} e_{k}$. Then you can conclude uniform convergence of the partial sums,

$$
\sum_{k=1}^{m} \psi_{k}(t) e_{k}
$$

This proves the theorem.

Why is $C([0, T] ; E)$ separable? You can assume without loss of generality that the interval is $[0,1]$ and consider the Bernstein polynomials

$$
p_{n}(t) \equiv \sum_{k=0}^{n}\binom{n}{k} f\left(\frac{k}{n}\right) t^{k}(1-t)^{n-k}
$$

These converge uniformly to $f$ Now look at all polynomials of the form

$$
\sum_{k=0}^{n} a_{k} t^{k}\left(1-t^{k}\right)
$$

where the $a_{k}$ is one of the countable dense set and $n \in \mathbb{N}$. Each Bernstein polynomial uniformly close to one of these and also uniformly close to $f$. Hence polynomials of this sort are countable and dense in $C([0, T] ; E)$.

### 64.5 Hilbert Space Valued Wiener Processes

Next I will consider the case of Hilbert space valued Wiener processes. This will include the case of $\mathbb{R}^{n}$ valued Wiener processes. I will present this material independent of the more general case of $E$ valued Wiener processes.

Definition 64.5.1 Let $W(t)$ be a stochastic process with values in $H$, a real separable Hilbert space which has the properties that $t \rightarrow W(t, \omega)$ is continuous, whenever $t_{1}<$ $t_{2}<\cdots<t_{m}$, the increments $\left\{W\left(t_{i}\right)-W\left(t_{i-1}\right)\right\}$ are independent, $W(0)=0$, and whenever $s<t$,

$$
\mathscr{L}(W(t)-W(s))=N(0,(t-s) Q)
$$

which means that whenever $h \in H$,

$$
\mathscr{L}((h, W(t)-W(s)))=N(0,(t-s)(Q h, h))
$$

Also

$$
E\left(\left(h_{1}, W(t)-W(s)\right)\left(h_{2}, W(t)-W(s)\right)\right)=\left(Q h_{1}, h_{2}\right)(t-s) .
$$

Here $Q$ is a nonnegative trace class operator. Recall this means

$$
Q=\sum_{i=1}^{\infty} \lambda_{i} e_{i} \otimes e_{i}
$$

where $\left\{e_{i}\right\}$ is a complete orthonormal basis, $\lambda_{i} \geq 0$, and

$$
\sum_{i=1}^{\infty} \lambda_{i}<\infty
$$

Such a stochastic process is called a $Q$ Wiener process. In the case where these have values in $\mathbb{R}^{n}$ tQ ends up being the covariance matrix of $W(t)$.

Note the characteristic function of a $Q$ Wiener process is

$$
\begin{equation*}
E\left(e^{i(h, W(t))}\right)=e^{-\frac{1}{2} t^{2}(Q h, h)} \tag{64.5.18}
\end{equation*}
$$

Note that by Theorem 61.8.5 if you simply say that the distribution measure of $W(t)$ is Gaussian, then it follows there exists a trace class operator $Q_{t}$ and $m_{t} \in H$ such that this measure is $N\left(m_{t}, Q_{t}\right)$. Thus for $W(t)$ a Wiener process, $Q_{t}=t Q$ and $m_{t}=0$. In addition, the increments are independent so this is much more specific than the earlier definition of a Gaussian measure.

What is a $Q$ Wiener process if the Hilbert space is $\mathbb{R}^{n}$ ? In particular, what is $Q$ ? It is given that

$$
\mathscr{L}((h, W(t)-W(s)))=N(0,(t-s)(Q h, h))
$$

In this case everything is a vector in $\mathbb{R}^{n}$ and so for $h \in \mathbb{R}^{n}$,

$$
E\left(e^{i \lambda(h, W(t)-W(s))}\right)=e^{-\frac{1}{2} \lambda^{2}(t-s)(Q h, h)}
$$

In particular, letting $\lambda=1$ this shows $W(t)-W(s)$ is normally distributed with covariance $(t-s) Q$ because its characteristic function is $e^{-\frac{1}{2} h^{*}(t-s) Q h}$.

With this and definition, one can describe Hilbert space valued Wiener processes in a fairly general setting.

Theorem 64.5.2 Let $U$ be a real separable Hilbert space and let $J: U_{0} \rightarrow U$ be a Hilbert Schmidt operator where $U_{0}$ is a real separable Hilbert space. Then let $\left\{g_{k}\right\}$ be a complete orthonormal basis for $U_{0}$ and define for $t \in[0, T]$

$$
W(t) \equiv \sum_{k=1}^{\infty} \psi_{k}(t) J g_{k}
$$

Then $W(t)$ is a $Q$ Wiener process for $Q=J J^{*}$ as in Definition 64.5.1. Furthermore, the distribution of $W(t)-W(s)$ is the same as the distribution of $W(t-s)$, and $W$ is Holder continuous with exponent $\gamma$ for any $\gamma<1 / 2$. There also is a subsequence denoted by $N$ such that the convergence of the series

$$
\sum_{k=1}^{N} \psi_{k}(t) J g_{k}
$$

is uniform for all $\omega$ not in some set of measure zero.
Proof: First it is necessary to show the series converges in $L^{2}(\Omega ; U)$ for each $t$. For convenience I will consider the series for $W(t)-W(s)$. (Always, it is assumed $t>s$.) Then since $\psi_{k}(t)-\psi_{k}(s)$ is normal with mean 0 and variance $(t-s)$ and $\psi_{k}(t)-\psi_{k}(s)$ and $\psi_{l}(t)-\psi_{l}(s)$ are independent,

$$
\begin{aligned}
& \int_{\Omega}\left|\sum_{k=m}^{n}\left(\psi_{k}(t)-\psi_{k}(s)\right) J g_{k}\right|_{U}^{2} d P \\
= & \int_{\Omega_{k, l=m}} \sum_{k}^{n}\left(\left(\psi_{k}(t)-\psi_{k}(s)\right) J g_{k},\left(\psi_{l}(t)-\psi_{l}(s)\right) J g_{l}\right)
\end{aligned}
$$

$$
=(t-s) \sum_{k=m}^{n}\left(J g_{k}, J g_{k}\right)=(t-s) \sum_{k=m}^{n}\left\|J g_{k}\right\|_{U}^{2}
$$

which converges to 0 as $m, n \rightarrow \infty$ thanks to the assumption that $J$ is Hilbert Schmidt. It follows the above sum converges in $L^{2}(\Omega ; U)$. Now letting $m<n$, it follows by the maximal estimate, Theorem 62.5.3, and the above

$$
\begin{aligned}
& P\left(\left[\sup _{t \in[0, T]}\left|\sum_{k=1}^{m} \psi_{k}(t) J g_{k}-\sum_{k=1}^{n} \psi_{k}(t) J g_{k}\right|_{U} \geq \lambda\right]\right) \\
& \leq \frac{1}{\lambda^{2}} E\left(\left|\sum_{k=m+1}^{n} \psi_{k}(T) J g_{k}\right|_{U}^{2}\right) \leq \frac{1}{\lambda^{2}} T \sum_{k=m}^{n}\left\|J g_{k}\right\|_{U}^{2}
\end{aligned}
$$

and so there exists a subsequence $n_{l}$ such that for all $p \geq 0$,

$$
P\left(\left[\sup _{t \in[0, T]}\left|\sum_{k=1}^{n_{l}} \psi_{k}(t) J g_{k}-\sum_{k=1}^{n_{l}+p} \psi_{k}(t) J g_{k}\right|_{U} \geq 2^{-l}\right]\right)<2^{-l}
$$

Therefore, by Borel Cantelli lemma, there is a set of measure zero such that for $\omega$ not in this set,

$$
\lim _{l \rightarrow \infty} \sum_{k=1}^{n_{l}} \psi_{k}(t) J g_{k}=\sum_{k=1}^{\infty} \psi_{k}(t) J g_{k}
$$

is uniform on $[0, T]$. From now on denote this subsequence by $N$ to save on notation.
I need to consider the characteristic function of $(h, W(t)-W(s))_{U}$ for $h \in U$. Then

$$
\begin{gathered}
E\left(\exp \left(\operatorname{ir}(h,(W(t)-W(s)))_{U}\right)\right) \\
=\lim _{N \rightarrow \infty} E\left(\exp \left(\operatorname{ir}\left(\sum_{j=1}^{N}\left(\psi_{j}(t)-\psi_{j}(s)\right)\left(h, J g_{j}\right)\right)\right)\right) \\
=\lim _{N \rightarrow \infty} E\left(\prod_{j=1}^{N} e^{i r\left(\psi_{j}(t)-\psi_{j}(s)\right)\left(h, J g_{j}\right)}\right)
\end{gathered}
$$

Since the random variables $\psi_{j}(t)-\psi_{j}(s)$ are independent,

$$
=\lim _{N \rightarrow \infty} \prod_{j=1}^{N} E\left(e^{i r\left(h, J g_{j}\right)\left(\psi_{j}(t)-\psi_{j}(s)\right)}\right)
$$

Since $\psi_{j}(t)-\psi_{j}(s)$ is a Gaussian random variable having mean 0 and variance $(t-s)$, the above equals

$$
=\lim _{N \rightarrow \infty} \prod_{j=1}^{N} e^{-\frac{1}{2} r^{2}\left(h, J g_{j}\right)^{2}(t-s)}
$$

$$
\begin{align*}
& =\lim _{N \rightarrow \infty} \exp \left(\sum_{j=1}^{N}-\frac{1}{2} r^{2}\left(h, J g_{j}\right)^{2}(t-s)\right) \\
& =\exp \left(-\frac{1}{2} r^{2}(t-s) \sum_{j=1}^{\infty}\left(h, J g_{j}\right)_{U}^{2}\right) \\
& =\exp \left(-\frac{1}{2} r^{2}(t-s) \sum_{j=1}^{\infty}\left(J^{*} h, g_{j}\right)_{U_{0}}^{2}\right) \\
& =\exp \left(-\frac{1}{2} r^{2}(t-s)\left\|J^{*} h\right\|_{U_{0}}^{2}\right)=\exp \left(-\frac{1}{2} r^{2}(t-s)\left(J J^{*} h, h\right)_{U}\right) \\
& =\exp \left(-\frac{1}{2} r^{2}(t-s)(Q h, h)_{U}\right) \tag{64.5.19}
\end{align*}
$$

which shows $(h, W(t)-W(s))_{U}$ is normally distributed with mean 0 and variance

$$
(t-s)(Q h, h)
$$

where $Q \equiv J J^{*}$. It is obvious from the definition that $W(0)=0$. Note that $Q$ is of trace class because if $\left\{e_{k}\right\}$ is an orthonormal basis for $U$,

$$
\begin{aligned}
\sum_{k}\left(Q e_{k}, e_{k}\right)_{U} & =\sum_{k}\left\|J^{*} e_{k}\right\|_{U_{0}}^{2}=\sum_{k} \sum_{l}\left(J^{*} e_{k}, g_{l}\right)_{U_{0}}^{2} \\
& =\sum_{l} \sum_{k}\left(e_{k}, J g_{l}\right)_{U}^{2}=\sum_{l}\left\|J g_{l}\right\|_{U}^{2}<\infty
\end{aligned}
$$

To find the covariance, consider

$$
E\left(\left(h_{1}, W(t)-W(s)\right)\left(h_{2}, W(t)-W(s)\right)\right)
$$

This equals

$$
E\left(\sum_{k=1}^{\infty}\left(\psi_{k}(t)-\psi_{k}(s)\right)\left(h_{1}, J g_{k}\right) \sum_{j=1}^{\infty}\left(\psi_{j}(t)-\psi_{j}(s)\right)\left(h_{2}, J g_{j}\right)\right)
$$

Since the series converge in $L^{2}(\Omega ; U)$, the independence of the $\psi_{k}(t)-\psi_{k}(s)$ implies the above equals

$$
\begin{aligned}
=\lim _{n \rightarrow \infty} E\left(\sum_{k=1}^{n}\right. & \left.\left(\psi_{k}(t)-\psi_{k}(s)\right)\left(h_{1}, J g_{k}\right) \sum_{j=1}^{n}\left(\psi_{j}(t)-\psi_{j}(s)\right)\left(h_{2}, J g_{j}\right)\right) \\
& =\lim _{n \rightarrow \infty}(t-s) \sum_{k=1}^{n}\left(h_{1}, J g_{k}\right)\left(h_{2}, J g_{k}\right) \\
& =\lim _{n \rightarrow \infty}(t-s) \sum_{k=1}^{n}\left(J^{*} h_{1}, g_{k}\right)_{U_{0}}\left(J^{*} h_{2}, g_{k}\right)_{U_{0}} \\
& =(t-s) \sum_{k=1}^{\infty}\left(J^{*} h_{1}, g_{k}\right)_{U_{0}}\left(J^{*} h_{2}, g_{k}\right)_{U_{0}} \\
= & (t-s)\left(J^{*} h_{1}, J^{*} h_{2}\right)=(t-s)\left(Q h_{1}, h_{2}\right)
\end{aligned}
$$

Next consider the claim that the increments are independent. Let $W^{N}(t)$ be given by the appropriate partial sum and let $\left\{h_{j}\right\}_{j=1}^{m}$ be a finite list of vectors of $U$. Then from the independence properties of $\psi_{j}$ explained above,

$$
\begin{aligned}
& E\left(\exp \sum_{j=1}^{m} i\left(h_{j}, W^{N}\left(t_{j}\right)-W^{N}\left(t_{j-1}\right)\right)_{U}\right) \\
& E\left(\exp \sum_{j=1}^{m} i\left(h_{j}, \sum_{k=1}^{N} J g_{k}\left(\psi_{k}\left(t_{j}\right)-\psi_{k}\left(t_{j-1}\right)\right)\right)_{U}\right) \\
= & E\left(\exp \sum_{j=1}^{m} \sum_{k=1}^{N} i\left(h_{j}, J g_{k}\right)_{U}\left(\psi_{k}\left(t_{j}\right)-\psi_{k}\left(t_{j-1}\right)\right)\right) \\
= & E\left(\prod_{j, k} \exp \left(i\left(h_{j}, J g_{k}\right)_{U}\left(\psi_{k}\left(t_{j}\right)-\psi_{k}\left(t_{j-1}\right)\right)\right)\right) \\
= & \prod_{j, k} E\left(\exp \left(i\left(h_{j}, J g_{k}\right)_{U}\left(\psi_{k}\left(t_{j}\right)-\psi_{k}\left(t_{j-1}\right)\right)\right)\right)
\end{aligned}
$$

This can be done because of the independence of the random variables

$$
\left\{\psi_{k}\left(t_{j}\right)-\psi_{k}\left(t_{j-1}\right)\right\}_{j, k}
$$

Thus the above equals

$$
\begin{aligned}
& \prod_{j, k} \exp \left(-\frac{1}{2}\left(h_{j}, J g_{k}\right)_{U}^{2}\left(t_{j}-t_{j-1}\right)\right) \\
= & \prod_{j=1}^{m} \exp \left(-\frac{1}{2} \sum_{k=1}^{N}\left(h_{j}, J g_{k}\right)_{U}^{2}\left(t_{j}-t_{j-1}\right)\right)
\end{aligned}
$$

because $\psi_{k}\left(t_{j}\right)-\psi_{k}\left(t_{j-1}\right)$ is normally distributed having variance $t_{j}-t_{j-1}$. Now letting
$N \rightarrow \infty$, this implies

$$
\begin{align*}
& E\left(\exp \sum_{j=1}^{m} i\left(h_{j}, W\left(t_{j}\right)-W\left(t_{j-1}\right)\right)_{U}\right) \\
= & \prod_{j=1}^{m} \exp \left(-\frac{1}{2} \sum_{k=1}^{\infty}\left(h_{j}, J g_{k}\right)_{U}^{2}\left(t_{j}-t_{j-1}\right)\right) \\
= & \prod_{j=1}^{m} \exp \left(-\frac{1}{2}\left(t_{j}-t_{j-1}\right) \sum_{k=1}^{\infty}\left(J^{*} h_{j}, g_{k}\right)_{U_{0}}^{2}\right) \\
= & \prod_{j=1}^{m} \exp \left(-\frac{1}{2}\left(t_{j}-t_{j-1}\right)\left\|J^{*} h_{j}\right\|_{U_{0}}^{2}\right) \\
= & \prod_{j=1}^{m} \exp \left(-\frac{1}{2}\left(t_{j}-t_{j-1}\right)\left(Q h_{j}, h_{j}\right)_{U}\right) \\
= & \prod_{j=1}^{m} \exp \left(i\left(h_{j}, W\left(t_{j}\right)-W\left(t_{j-1}\right)\right)_{U}\right) \tag{64.5.20}
\end{align*}
$$

from 64.5.19, letting $r=1$. By Theorem 59.13.3 on Page 1896, this shows the increments are independent.

It remains to verify the Holder continuity. Recall

$$
W(t)=\sum_{k=1}^{\infty} J g_{k} \psi_{k}(t)
$$

where $\psi_{k}$ is a real Wiener process.
Next consider the claim about Holder continuity. It was shown above that

$$
E\left(\exp \left(\operatorname{ir}(h,(W(t)-W(s)))_{U}\right)\right)=\exp \left(-\frac{1}{2} r^{2}(t-s)(Q h, h)_{U}\right)
$$

Therefore, taking a derivative with respect to $r$ two times yields

$$
\begin{aligned}
& E\left(\left(-(h,(W(t)-W(s)))_{U}^{2}\right) \exp \left(\operatorname{ir}(h,(W(t)-W(s)))_{U}\right)\right) \\
= & -(t-s)(Q h, h) \exp \left(-\frac{1}{2} r^{2}(t-s)(Q h, h)_{U}\right)+ \\
& r^{2}(t-s)^{2}(Q h, h)_{U}^{2} \exp \left(-\frac{1}{2} r^{2}(t-s)(Q h, h)_{U}\right)
\end{aligned}
$$

Now plug in $r=0$ to obtain

$$
E\left((h,(W(t)-W(s)))_{U}^{2}\right)=(t-s)(Q h, h)
$$

Similarly, taking 4 derivatives, it follows that an expression of the following form holds.

$$
E\left((h,(W(t)-W(s)))_{U}^{4}\right)=C_{2}(Q h, h)^{2}(t-s)^{2}
$$

and in general,

$$
E\left((h,(W(t)-W(s)))_{U}^{2 m}\right)=C_{m}(Q h, h)^{m}(t-s)^{m}
$$

Now it follows from Minkowsky's inequality applied to the two integrals $\sum_{i=1}^{\infty}$ and $\int_{\Omega}$ that

$$
\begin{aligned}
{\left[E\left(|W(t)-W(s)|^{2 m}\right)\right]^{1 / m} } & =\left[E\left(\left(\sum_{k=1}^{\infty}\left(e_{k}, W(t)-W(s)\right)^{2}\right)^{m}\right)\right]^{1 / m} \\
& \leq \sum_{k=1}^{\infty}\left[E\left(\left(e_{k}, W(t)-W(s)\right)^{2 m}\right)\right]^{1 / m} \\
& =\sum_{k=1}^{\infty}\left[C_{m}\left(Q e_{k}, e_{k}\right)^{m}(t-s)^{m}\right]^{1 / m} \\
& =C_{m}^{1 / m}|t-s|\left(\sum_{k=1}^{\infty}\left(Q e_{k}, e_{k}\right)\right) \equiv C_{m}^{\prime}|t-s|
\end{aligned}
$$

Hence there exists a constant $C_{m}$ such that

$$
E\left(|W(t)-W(s)|^{2 m}\right) \leq C_{m}|t-s|^{m}
$$

By the Kolmogorov Čentsov Theorem, Theorem 62.2.2, it follows that off a set of measure $0, t \rightarrow W(t, \omega)$ is Holder continuous with exponent $\gamma$ such that

$$
\gamma<\frac{m-1}{2 m}, m>2 .
$$

Finally, from 64.5 .19 with $r=1$,

$$
E\left(\exp i(h, W(t)-W(s))_{U}\right)=\exp \left(-\frac{1}{2}(t-s)(Q h, h)\right)
$$

which is the same as $E\left(\exp i(h, W(t-s))_{U}\right)$ due to the fact $W(0)=0$.
The above has shown that $W(t)$ satisfies the conditions of Lemma 64.4.2 and so it is a martingale with respect to the filtration given there. What is its quadratic variation?

$$
E\left(\|W(t)\|^{2}\right)=\sum_{k=1}^{\infty} E\left(\left(W(t), e_{k}\right)\left(W(t), e_{k}\right)\right)=\sum_{k=1}^{\infty}\left(Q e_{k}, e_{k}\right) t=\operatorname{trace}(Q) t
$$

Is it the case that $[W](t)=\operatorname{trace}(Q) t$ ? Let the filtration be as in Lemma 64.4.2 and let $A \in \mathscr{F}_{s}$. Then using the result of that lemma,

$$
\begin{gathered}
\int_{A}\left(\|W(t)\|^{2}-t \operatorname{trace}(Q) \mid \mathscr{F}_{s}\right) d P \\
=\quad \int_{A}\left(\|W(t)-W(s)\|^{2}+2(W(t), W(s))-\|W(s)\|^{2}\right. \\
\left.-(t-s) \operatorname{trace} Q-\operatorname{trace} Q s \mid \mathscr{F}_{s}\right) d P
\end{gathered}
$$

$$
\begin{aligned}
= & P(A) \int_{\Omega}\|W(t)-W(s)\|^{2}-(t-s) \operatorname{trace} Q d P \\
& \quad+\int_{A}\left(2(W(t), W(s))-\|W(s)\|^{2}-\operatorname{trace}(Q) s \mid \mathscr{F}_{s}\right) d P \\
= & \int_{A} 2\left(W(s), E\left(W(t) \mid \mathscr{F}_{s}\right)\right) d P-\int_{A}\|W(s)\|^{2} d P-\int_{A} s \operatorname{trace} Q d P \\
= & \int_{A}\left(\|W(s)\|^{2}-s \operatorname{trace} Q\right) d P
\end{aligned}
$$

and this shows that the quadratic variation $[W](t)=t$ trace $(Q)$ by uniqueness of the quadratic variation.

Now suppose you start with a nonnegative trace class operator $Q$. Then in this case also one can define a $Q$ Wiener process. It is possible to get this theorem from Theorem 64.5.2 but this will not be done here.

Theorem 64.5.3 Let $U$ be a real separable Hilbert space and let $Q$ be a nonnegative trace class operator defined on $U$. Then there exists a $Q$ Wiener process as defined in Definition 64.5.1. Furthermore, the distribution of $W(t)-W(s)$ is the same as the distribution of $W(t-s)$ and $W$ is Holder continuous with exponent $\gamma$ for any $\gamma<1 / 2$.

Proof: One can obtain this theorem as a corollary of Theorem 64.5.2 but this will not be done here.

Let

$$
Q=\sum_{i=1}^{\infty} \lambda_{i} e_{i} \otimes e_{i}
$$

where $\left\{e_{i}\right\}$ is a complete orthonormal set and $\lambda_{i} \geq 0$ and $\sum \lambda_{i}<\infty$. Now the definition of the $Q$ Wiener process is

$$
\begin{equation*}
W(t) \equiv \sum_{k=1}^{\infty} \sqrt{\lambda_{k}} e_{k} \psi_{k}(t) \tag{64.5.21}
\end{equation*}
$$

where $\left\{\psi_{k}(t)\right\}$ are the real Wiener processes defined in Lemma 64.3.1.
Now consider 64.5.21. From this formula, if $s<t$

$$
\begin{equation*}
W(t)-W(s)=\sum_{k=1}^{\infty} \sqrt{\lambda_{k}} e_{k}\left(\psi_{k}(t)-\psi_{k}(s)\right) \tag{64.5.22}
\end{equation*}
$$

First it is necessary to show this sum converges. Since $\psi_{j}(t)$ is a Wiener process,

$$
\begin{aligned}
& \int_{\Omega}\left|\sum_{j=m}^{n} \sqrt{\lambda_{j}}\left(\psi_{j}(t)-\psi_{j}(s)\right) e_{j}\right|_{U}^{2} d P \\
= & \int_{\Omega} \sum_{j=m}^{n} \lambda_{j}\left(\psi_{j}(t)-\psi_{j}(s)\right)^{2} d P \\
= & (t-s) \sum_{j=m}^{n} \lambda_{j}
\end{aligned}
$$

and this converges to 0 as $m, n \rightarrow \infty$ because it was given that

$$
\sum_{j=1}^{\infty} \lambda_{j}<\infty
$$

so the series in 64.5 .22 converges in $L^{2}(\Omega ; U)$.
Therefore, there exists a subsequence

$$
\left\{\sum_{k=1}^{N} \sqrt{\lambda_{k}} e_{k}\left(\psi_{k}(t)-\psi_{k}(s)\right)\right\}
$$

which converges pointwise a.e. to $W(t)-W(s)$ as well as in $L^{2}(\Omega ; U)$ as $N \rightarrow \infty$. Then letting $h \in U$,

$$
\begin{equation*}
(h, W(t)-W(s))_{U}=\sum_{k=1}^{\infty} \sqrt{\lambda_{k}}\left(\psi_{k}(t)-\psi_{k}(s)\right)\left(h, e_{k}\right) \tag{64.5.23}
\end{equation*}
$$

Then by the dominated convergence theorem,

$$
\begin{gathered}
E\left(\exp \left(\operatorname{ir}(h,(W(t)-W(s)))_{U}\right)\right) \\
=\lim _{N \rightarrow \infty} E\left(\exp \left(i r\left(\sum_{j=1}^{N} \sqrt{\lambda_{j}}\left(\psi_{j}(t)-\psi_{j}(s)\right)\left(h, e_{j}\right)\right)\right)\right) \\
=\lim _{N \rightarrow \infty} E\left(\prod_{j=1}^{N} e^{i r \sqrt{\lambda_{j}}\left(\psi_{j}(t)-\psi_{j}(s)\right)\left(h, e_{j}\right)}\right)
\end{gathered}
$$

Since the random variables $\psi_{j}(t)-\psi_{j}(s)$ are independent,

$$
=\lim _{N \rightarrow \infty} \prod_{j=1}^{N} E\left(e^{i r \sqrt{\lambda_{j}}\left(\psi_{j}(t)-\psi_{j}(s)\right)\left(h, e_{j}\right)}\right)
$$

Since $\psi_{j}(t)$ is a real Wiener process,

$$
\begin{align*}
& =\lim _{N \rightarrow \infty} \prod_{j=1}^{N} e^{-\frac{1}{2} r^{2} \lambda_{j}(t-s)\left(h, e_{j}\right)^{2}} \\
= & \lim _{N \rightarrow \infty} \exp \left(\sum_{j=1}^{N}-\frac{1}{2} r^{2} \lambda_{j}(t-s)\left(h, e_{j}\right)^{2}\right) \\
= & \exp \left(-\frac{1}{2} r^{2}(t-s) \sum_{j=1}^{\infty} \lambda_{j}\left(h, e_{j}\right)^{2}\right) \\
= & \exp \left(-\frac{1}{2} r^{2}(t-s)(Q h, h)\right) \tag{64.5.24}
\end{align*}
$$

Thus $(h, W(t)-W(s))$ is normally distributed with mean 0 and variance $(t-s)(Q h, h)$. It is obvious from the definition that $W(0)=0$. Also to find the covariance, consider

$$
E\left(\left(h_{1}, W(t)-W(s)\right)\left(h_{2}, W(t)-W(s)\right)\right)
$$

and use 64.5 .23 to obtain this is equal to

$$
\begin{gathered}
E\left(\sum_{k=1}^{\infty} \sqrt{\lambda_{k}}\left(\psi_{k}(t)-\psi_{k}(s)\right)\left(h_{1}, e_{k}\right) \sum_{j=1}^{\infty} \sqrt{\lambda_{j}}\left(\psi_{j}(t)-\psi_{j}(s)\right)\left(h_{2}, e_{j}\right)\right) \\
=\lim _{n \rightarrow \infty} E\left(\sum_{k=1}^{n} \sqrt{\lambda_{k}}\left(\psi_{k}(t)-\psi_{k}(s)\right)\left(h_{1}, e_{k}\right) \sum_{j=1}^{n} \sqrt{\lambda_{j}}\left(\psi_{j}(t)-\psi_{j}(s)\right)\left(h_{2}, e_{j}\right)\right) \\
=\lim _{n \rightarrow \infty}(t-s) \sum_{k=1}^{n} \lambda_{k}\left(h_{1}, e_{k}\right)\left(h_{2}, e_{j}\right)=(t-s)\left(Q h_{1}, h_{2}\right)
\end{gathered}
$$

(Recall $\left.Q \equiv \sum_{k} \lambda_{k} e_{k} \otimes e_{k}.\right)$
Next I show the increments are independent. Let $N$ be the subsequence defined above and let $W^{N}(t)$ be given by the appropriate partial sum and let $\left\{h_{j}\right\}_{j=1}^{m}$ be a finite list of vectors of $U$. Then from the independence properties of $\psi_{j}$ explained above,

$$
\begin{aligned}
& E\left(\exp \sum_{j=1}^{m} i\left(h_{j}, W^{N}\left(t_{j}\right)-W^{N}\left(t_{j-1}\right)\right)_{U}\right) \\
& E\left(\exp \sum_{j=1}^{m} i\left(h_{j}, \sum_{k=1}^{N} \sqrt{\lambda_{k}} e_{k}\left(\psi_{k}\left(t_{j}\right)-\psi_{k}\left(t_{j-1}\right)\right)\right)_{U}\right) \\
= & E\left(\exp \sum_{j=1}^{m} \sum_{k=1}^{N} i \sqrt{\lambda_{k}}\left(h_{j}, e_{k}\right)_{U}\left(\psi_{k}\left(t_{j}\right)-\psi_{k}\left(t_{j-1}\right)\right)\right) \\
= & E\left(\prod_{j, k} \exp \left(i \sqrt{\lambda_{k}}\left(h_{j}, e_{k}\right)_{U}\left(\psi_{k}\left(t_{j}\right)-\psi_{k}\left(t_{j-1}\right)\right)\right)\right) \\
= & \prod_{j, k} E\left(\exp \left(i \sqrt{\lambda_{k}}\left(h_{j}, e_{k}\right)_{U}\left(\psi_{k}\left(t_{j}\right)-\psi_{k}\left(t_{j-1}\right)\right)\right)\right)
\end{aligned}
$$

This can be done because of the independence of the random variables

$$
\left\{\psi_{k}\left(t_{j}\right)-\psi_{k}\left(t_{j-1}\right)\right\}_{j, k}
$$

Thus the above equals

$$
=\prod_{j=1}^{m} \exp \left(-\frac{1}{2} \sum_{k=1}^{N} \lambda_{k}\left(h_{j}, e_{k}\right)_{U}^{2}\left(t_{j}-t_{j-1}\right)\right)
$$

because $\psi_{k}\left(t_{j}\right)-\psi_{k}\left(t_{j-1}\right)$ is normally distributed having variance $t_{j}-t_{j-1}$ and mean 0 . Now letting $N \rightarrow \infty$, this implies

$$
\begin{align*}
& E\left(\exp \sum_{j=1}^{m} i\left(h_{j}, W\left(t_{j}\right)-W\left(t_{j-1}\right)\right)_{U}\right) \\
= & \prod_{j=1}^{m} \exp \left(-\frac{1}{2}\left(t_{j}-t_{j-1}\right) \sum_{k=1}^{\infty} \lambda_{k}\left(h_{j}, e_{k}\right)_{U}^{2}\right) \\
= & \prod_{j=1}^{m} \exp \left(-\frac{1}{2}\left(t_{j}-t_{j-1}\right)(Q h, h)_{U}\right) \\
= & \prod_{j=1}^{m} \exp \left(i\left(h_{j}, W\left(t_{j}\right)-W\left(t_{j-1}\right)\right)_{U}\right) \tag{64.5.25}
\end{align*}
$$

because of the fact shown above that $(h, W(t)-W(s))$ is normally distributed with mean 0 and variance $(t-s)(Q h, h)$. By Theorem 59.13.3 on Page 1896, this shows the increments are independent.

Next consider the continuity assertion. Recall

$$
W(t)=\sum_{k=1}^{\infty} \sqrt{\lambda_{k}} e_{k} \psi_{k}(t)
$$

where $\psi_{k}$ is a real Wiener process. Therefore, letting $2 m>2, m \in \mathbb{N}$ and using 64.1.3 for $\psi_{k}$ and Jensen's inequality along with Lemma 64.3.1,

$$
\begin{align*}
E\left(|W(t)-W(s)|^{2 m}\right) & =E\left(\left|\sum_{k=1}^{\infty} \sqrt{\lambda_{k}} e_{k}\left(\psi_{k}(t)-\psi_{k}(s)\right)\right|^{2 m}\right) \\
& =E\left(\left(\sum_{k=1}^{\infty} \lambda_{k}\left|\psi_{k}(t)-\psi_{k}(s)\right|^{2}\right)^{m}\right) \\
& \leq E\left(\left(\sum_{k=1}^{\infty} \lambda_{k}\right)^{m-1} \sum_{k=1}^{\infty} \lambda_{k}\left|\psi_{k}(t)-\psi_{k}(s)\right|^{2 m}\right) \\
& \leq C_{m} \sum_{k=1}^{\infty} \lambda_{k} E\left(\left|\psi_{k}(t)-\psi_{k}(s)\right|^{2 m}\right)  \tag{64.5.26}\\
& \leq C_{m}|t-s|^{m} \tag{64.5.27}
\end{align*}
$$

By the Kolmogorov Čentsov Theorem, Theorem 62.2.2, it follows that off a set of measure $0, t \rightarrow W(t, \omega)$ is Holder continuous with exponent $\gamma$ such that

$$
\gamma<\frac{m-1}{2 m}
$$

Finally, from 64.5.24 taking $r=1$,

$$
E\left(\exp i(h, W(t)-W(s))_{U}\right)=\exp \left(-\frac{1}{2}(t-s)(Q h, h)\right)
$$

which is the same as $E\left(\exp i(h, W(t-s))_{U}\right)$ due to the fact $W(0)=0$. This proves the theorem.

The above shows there exists $Q$ Wiener processes in any separable Hilbert space. Next I will show the way described above is the only way it can happen.

Theorem 64.5.4 Suppose $\{W(t)\}$ is a $Q$ Wiener process in $U$, a real separable Hilbert space. Then letting

$$
Q=\sum_{k=1}^{\infty} \lambda_{k} e_{k} \otimes e_{k}
$$

where the $\left\{e_{k}\right\}$ are orthonormal, $\lambda_{k} \geq 0$, and $\sum_{k=1}^{\infty} \lambda_{k}<\infty$, it follows

$$
\begin{equation*}
W(t)=\sum_{k=1}^{\infty} \sqrt{\lambda_{k}} \psi_{k}(t) e_{k} \tag{64.5.28}
\end{equation*}
$$

where

$$
\psi_{k}(t) \equiv\left\{\begin{array}{l}
\frac{1}{\sqrt{\lambda_{k}}}\left(W(t), e_{k}\right)_{U} \text { if } \lambda_{k} \neq 0 \\
0 \text { if } \lambda_{k}=0
\end{array}\right.
$$

then $\left\{\psi_{k}(t)\right\}$ is a Wiener process and for $t_{0}<t_{1}<\cdots<t_{n}$ the random variables

$$
\left\{\psi_{k}\left(t_{q}\right)-\psi_{k}\left(t_{q-1}\right):(q, k) \in(1,2, \cdots, n) \times\left(k_{1}, \cdots, k_{m}\right)\right\}
$$

are independent. Furthermore, the sum in 64.5 .28 converges uniformly for a.e. $\omega$ on any closed interval, $[0, T]$.

Proof: First of all, the fact that $W(t)$ has values in $U$ and that $\left\{e_{k}\right\}$ is an orthonormal basis implies the sum in 64.5 .28 converges for each $\omega$. Consider

$$
E\left(\exp \left(i r \psi_{k}(t)\right)\right)=E\left(\exp \left(\operatorname{ir} \frac{1}{\sqrt{\lambda_{k}}}\left(W(t), e_{k}\right)_{U}\right)\right)
$$

Since $W(t)$ is given to be a $Q$ Wiener process, $\left(W(t), e_{k}\right)_{U}$ is normally distributed with mean 0 and variance $t(Q h, h)$. Therefore, the above equals

$$
=e^{-\frac{1}{2} r^{2} \frac{1}{\lambda_{k}} t\left(Q e_{k}, e_{k}\right)}=e^{-\frac{1}{2} r^{2} \frac{1}{\lambda_{k}} t \lambda_{k}}=e^{-\frac{1}{2} r^{2} t}
$$

the characteristic function for a random variable which is $N(0, t)$. The independence of the increments for a given $\psi_{k}(t)$ follows right away from the independence of the increments of $W(t)$ and the distribution of the increments being $N(0,(t-s))$ follows similarly to the above.

For $t_{1}<t_{2}<\cdots<t_{n}$, why are the random variables,

$$
\begin{equation*}
\left\{\left(W\left(t_{q}\right), e_{k}\right)_{U}-\left(W\left(t_{q-1}\right), e_{k}\right)_{U}:(q, k) \in(1,2, \cdots, n) \times\left(k_{1}, \cdots, k_{m}\right)\right\} \tag{64.5.29}
\end{equation*}
$$

independent? Let

$$
P=\sum_{q=1}^{n} \sum_{j=1}^{m} s_{q j}\left(\left(W\left(t_{q}\right), e_{k_{j}}\right)_{U}-\left(W\left(t_{q-1}\right), e_{k_{j}}\right)_{U}\right)
$$

and consider $E\left(e^{i P}\right)$. This equals

$$
\begin{gather*}
e^{i P}=E\left(\exp \left(i \sum_{q=1}^{n} \sum_{j=1}^{m} s_{q j}\left(\left(W\left(t_{q}\right), e_{k_{j}}\right)_{U}-\left(W\left(t_{q-1}\right), e_{k_{j}}\right)_{U}\right)\right)\right)  \tag{64.5.30}\\
=E\left(\exp \left(i \sum_{q=1}^{n}\left(\left(W\left(t_{q}\right), \sum_{j=1}^{m} s_{q j} e_{k_{j}}\right)_{U}-\left(W\left(t_{q-1}\right), \sum_{j=1}^{m} s_{q j} e_{k_{j}}\right)_{U}\right)\right)\right) \\
=E\left(\prod_{q=1}^{n} \exp \left(i\left(W\left(t_{q}\right)-W\left(t_{q-1}\right), \sum_{j=1}^{m} s_{q j} e_{k_{j}}\right)_{U}\right)\right)
\end{gather*}
$$

Now recall that by assumption the increments $W(t)-W(s)$ are independent. Therefore, the above equals

$$
\prod_{q=1}^{n} E\left(\exp \left(i\left(W\left(t_{q}\right)-W\left(t_{q-1}\right), \sum_{j=1}^{m} s_{q j} e_{k_{j}}\right)_{U}\right)\right)
$$

Recall that by assumption $(W(t)-W(s), h)_{U}$ is normally distributed with variance given by the expresson $(t-s)(Q h, h)$ and mean 0 . Therefore, the above equals

$$
\begin{gather*}
\prod_{q=1}^{n} \exp \left(-\frac{1}{2}\left(t_{q}-t_{q-1}\right)\left(Q \sum_{j=1}^{m} s_{q j} e_{k_{j}}, \sum_{j=1}^{m} s_{q j} e_{k_{j}}\right)\right) \\
=\prod_{q=1}^{n} \exp \left(-\frac{1}{2}\left(t_{q}-t_{q-1}\right) \sum_{j=1}^{m} s_{q j}^{2} \lambda_{k_{j}}\right) \\
\quad \exp \left(-\frac{1}{2} \sum_{q=1}^{n} \sum_{j=1}^{m}\left(t_{q}-t_{q-1}\right) s_{q j}^{2} \lambda_{k_{j}}\right) \tag{64.5.31}
\end{gather*}
$$

Also

$$
\begin{array}{r}
\prod_{q=1}^{n} \prod_{j=1}^{n} E\left(\exp \left(i s_{q j}\left(\left(W\left(t_{q}\right), e_{k_{j}}\right)_{U}-\left(W\left(t_{q-1}\right), e_{k_{j}}\right)_{U}\right)\right)\right)  \tag{64.5.32}\\
=\prod_{q=1}^{n} \prod_{j=1}^{n} E\left(\exp \left(i s_{q j}\left(\left(W\left(t_{q}\right)-W\left(t_{q-1}\right), e_{k_{j}}\right)_{U}\right)\right)\right) \\
=\prod_{q=1}^{n} \prod_{j=1}^{n} \exp \left(-\frac{1}{2}\left(t_{q}-t_{q-1}\right) s_{q j}^{2}\left(Q e_{k_{j}}, e_{k_{j}}\right)\right) \\
=\prod_{q=1}^{n} \prod_{j=1}^{n} \exp \left(-\frac{1}{2}\left(t_{q}-t_{q-1}\right) s_{q j}^{2} \lambda_{k_{j}}\right) \\
=\exp \left(-\frac{1}{2} \sum_{q=1}^{n} \sum_{j=1}^{m}\left(t_{q}-t_{q-1}\right) s_{q j}^{2} \lambda_{k_{j}}\right)
\end{array}
$$

Therefore, $e^{i P}$ equals the expression in 64.5.32 because both equal the expression in 64.5.31 and it follows from Proposition 59.11.1 on Page 1889 that the random variables of 64.5.29 are independent.

What about the claim of uniform convergence? By the independence of the increments, it follows from Lemma 64.4.2 that $\{W(t)\}$ is a martingale and each real valued function, $\left(W(t), e_{k}\right)_{U}$ is also a martingale. Therefore, Theorem 62.5.3 can be applied to conclude

$$
\begin{aligned}
& P\left(\left[\sup _{t \in[0, T]}\left|\sum_{k=m}^{n}\left(W(t), e_{k}\right)_{U} e_{k}\right| \geq \alpha\right]\right) \leq \frac{1}{\alpha} \int_{\Omega}\left|\sum_{k=m}^{n}\left(W(T), e_{k}\right)_{U} e_{k}\right| d P \\
& \quad \leq \frac{1}{\alpha} \int_{\Omega}\left|\sum_{k=m}^{n}\left(W(T), e_{k}\right)_{U} e_{k}\right|^{2} d P=\frac{1}{\alpha} \sum_{k=m}^{n} \int_{\Omega}\left(W(T), e_{k}\right)_{U}^{2} d P \\
& \quad=\frac{1}{\alpha} \sum_{k=m}^{n}\left(Q e_{k}, e_{k}\right) T=\frac{T}{\alpha} \sum_{k=m}^{n} \lambda_{k} \leq \frac{T}{\alpha} \sum_{k=m}^{\infty} \lambda_{k}
\end{aligned}
$$

Since $\sum_{k=1}^{\infty} \lambda_{k}<\infty$, there exists a sequence, $\left\{m_{l}\right\}$ such that if $n>m_{l}$

$$
P\left(\left[\sup _{t \in[0, T]}\left|\sum_{k=m_{l}}^{n}\left(W(t), e_{k}\right)_{U} e_{k}\right|>2^{-k}\right]\right)<2^{-k}
$$

and so by the Borel Cantelli lemma, off a set of measure 0 the partial sums

$$
\left\{\sum_{k=1}^{m_{l}}\left(W(t), e_{k}\right)_{U} e_{k}\right\}
$$

converge uniformly on $[0, T]$. This is very interesting but more can be said. In fact the original partial sums converge.

Recall Lemma 59.15.6 stated below for convenience.
Lemma 64.5.5 Let $\left\{\zeta_{k}\right\}$ be a sequence of random variables having values in a separable real Banach space, $E$ whose distributions are symmetric. Letting $S_{k} \equiv \sum_{i=1}^{k} \zeta_{i}$, suppose $\left\{S_{n_{k}}\right\}$ converges a.e. Also suppose that for every $m>n_{k}$,

$$
\begin{equation*}
P\left(\left[\left\|S_{m}-S_{n_{k}}\right\|_{E}>2^{-k}\right]\right)<2^{-k} \tag{64.5.33}
\end{equation*}
$$

Then in fact,

$$
\begin{equation*}
S_{k}(\omega) \rightarrow S(\omega) \text { a.e. } \omega \tag{64.5.34}
\end{equation*}
$$

Apply this lemma to the situation in which the Banach space, $E$ is $C([0, T] ; U)$. Then you can conclude uniform convergence of the partial sums,

$$
\sum_{k=1}^{m}\left(W(t), e_{k}\right)_{U} e_{k}
$$

This proves the theorem.

### 64.6 Wiener Processes, Another Approach

### 64.6.1 Lots Of Independent Normally Distributed Random Variables

You can use the Kolmogorov extension theorem to prove the following corollary. It is Corollary 59.20.3 on Page 1934.

Corollary 64.6.1 Let $H$ be a real Hilbert space. Then there exist real valued random variables $W(h)$ for $h \in H$ such that each is normally distributed with mean 0 and for every $h, g,(W(f), W(g))$ is normally distributed and

$$
E(W(h) W(g))=(h, g)_{H}
$$

Furthermore, if $\left\{e_{i}\right\}$ is an orthogonal set of vectors of $H$, then $\left\{W\left(e_{i}\right)\right\}$ are independent random variables. Also for any finite set $\left\{f_{1}, f_{2}, \cdots, f_{n}\right\}$,

$$
\left(W\left(f_{1}\right), W\left(f_{2}\right), \cdots, W\left(f_{n}\right)\right)
$$

is normally distributed.
Corollary 64.6.2 The map $h \rightarrow W(h)$ is linear. Also, $\{W(h): h \in H\}$ is a closed subspace of $L^{2}(\Omega, \mathscr{F}, P)$ where $\mathscr{F}=\sigma(W(h): h \in H)$.

Proof: This follows from the above description.

$$
\begin{gathered}
E\left([W(g+h)-(W(g)+W(h))]^{2}\right)=E\left(W(g+h)^{2}\right) \\
+E\left((W(g)+W(h))^{2}\right)-2 E(W(g+h)(W(g)+W(h))) \\
=|g+h|^{2}+|g|^{2}+|h|^{2}+2(g, h)-2(g+h, g)-2(g+h, h) \\
=|g|^{2}+|h|^{2}+2(g, h)++2(g, h)+|g|^{2} \\
+|h|^{2}-2|g|^{2}-2(g, h)-2(g, h)-2|h|^{2}=0
\end{gathered}
$$

Hence $W(h+g)=W(g)+W(h)$.

$$
\begin{gathered}
E\left((W(\alpha f)-\alpha W(f))^{2}\right)=E\left(W(\alpha f)^{2}\right)+E\left(\alpha^{2} W(f)^{2}\right)-2 E(W(\alpha f) \alpha W(f)) \\
=\alpha^{2}|f|^{2}+\alpha^{2}|f|^{2}-2 \alpha(\alpha f, f)=0
\end{gathered}
$$

Why is $\{W(h): h \in H\}$ a subspace? This is obvious because $W$ is linear. Why is it closed? Say $W\left(h_{n}\right) \rightarrow f \in L^{2}(\Omega)$. This requires that $\left\{h_{n}\right\}$ is a Cauchy sequence. Thus $h_{n} \rightarrow h$ and so

$$
\begin{aligned}
E\left(|f-W(h)|^{2}\right) & \leq 2\left[\lim _{n \rightarrow \infty} E\left(\left|f-W\left(h_{n}\right)\right|^{2}\right)+E\left(\left|W\left(h_{n}\right)-W(h)\right|^{2}\right)\right] \\
& =2 \lim _{n \rightarrow \infty} E\left(\left|W\left(h_{n}\right)-W(h)\right|^{2}\right)=2 \lim _{n \rightarrow \infty}\left|h_{n}-h\right|_{H}^{2}=0
\end{aligned}
$$

and so $f=W(h)$ showing that this is indeed a closed subspace.
Next is a technical lemma which will be of considerable use.

Lemma 64.6.3 Let $X \geq 0$ and measurable. Also define a finite measure on $\mathscr{B}\left(\mathbb{R}^{p}\right)$

$$
v(B) \equiv \int_{\Omega} X \mathscr{X}_{B}(\mathbf{Y}) d P
$$

Then let $f: \mathbb{R}^{p} \rightarrow[0, \infty)$ be Borel measurable. Then

$$
\int_{\Omega} f(\mathbf{Y}) X d P=\int_{\mathbb{R}^{p}} f(\mathbf{y}) d v(y)
$$

where here $\mathbf{Y}$ is a given measurable function with values in $\mathbb{R}^{p}$. Formally, $X d P=d v$.
Note that $\mathbf{Y}$ is given and $X$ is just some random variable which here has nonnegative values. Of course similar things will work without this stipulation.

Proof: First say $X=\mathscr{X}_{D}$ and replace $f(\mathbf{Y})$ with $\mathscr{X}_{\mathbf{Y}^{-1}(B)}$. Then

$$
\begin{aligned}
\int_{\Omega} \mathscr{X}_{D} & \mathscr{X}_{\mathbf{Y}^{-1}(B)} d P=P\left(D \cap \mathbf{Y}^{-1}(B)\right) \\
\int_{\mathbb{R}^{p}} \mathscr{X}_{B}(\mathbf{y}) d v(y) & \equiv v(B) \equiv \int_{\Omega} \mathscr{X}_{D} \mathscr{X}_{B}(\mathbf{Y}) d P \\
& =\int_{\Omega} \mathscr{X}_{D} \mathscr{X}_{\mathbf{Y}^{-1}(B)} d P=P\left(D \cap \mathbf{Y}^{-1}(B)\right)
\end{aligned}
$$

Thus

$$
\int_{\Omega} \mathscr{X}_{D} \mathscr{X}_{\mathbf{Y}^{-1}(B)} d P=\int_{\Omega} \mathscr{X}_{D} \mathscr{X}_{B}(\mathbf{Y}) d P=\int_{\mathbb{R}^{p}} \mathscr{X}_{B}(\mathbf{y}) d v(y)
$$

Now let $s_{n}(\mathbf{y}) \uparrow f(\mathbf{y})$, and let $s_{n}(\mathbf{y})=\sum_{k=1}^{m} c_{k} \mathscr{X}_{B_{k}}(\mathbf{y})$ where $B_{k}$ is a Borel set. Then

$$
\begin{gathered}
\int_{\mathbb{R}^{p}} s_{n}(\mathbf{y}) d v(y)=\int_{\mathbb{R}^{p}} \sum_{k=1}^{m} c_{k} \mathscr{X}_{B_{k}}(\mathbf{y}) d v(\mathbf{y})=\sum_{k=1}^{m} c_{k} \int_{\mathbb{R}^{p}} \mathscr{X}_{B_{k}}(\mathbf{y}) d v(\mathbf{y}) \\
=\sum_{k=1}^{m} c_{k} P\left(D \cap \mathbf{Y}^{-1}\left(B_{k}\right)\right) \\
\int_{\Omega} s_{n}(\mathbf{Y}) \mathscr{X}_{D} d P=\sum_{k=1}^{m} c_{k} \int_{\Omega} \mathscr{X}_{D} \mathscr{X}_{B_{k}}(\mathbf{Y}) d P=\sum_{k=1}^{m} c_{k} P\left(D \cap \mathbf{Y}^{-1}\left(B_{k}\right)\right)
\end{gathered}
$$

which is the same thing. Therefore,

$$
\int_{\Omega} s_{n}(\mathbf{Y}) \mathscr{X}_{D} d P=\int_{\mathbb{R}^{p}} s_{n}(\mathbf{y}) d v(y)
$$

Now pass to a limit using the monotone convergence theorem to obtain

$$
\int_{\Omega} f(\mathbf{Y}) \mathscr{X}_{D} d P=\int_{\mathbb{R}^{p}} f(\mathbf{y}) d v(y)
$$

Next replace $\mathscr{X}_{D}$ with $\sum_{k=1}^{m} d_{k} \mathscr{X}_{D_{k}} \equiv s_{n}(\omega)$, a simple function.

$$
\int_{\Omega} f(\mathbf{Y}) \sum_{k=1}^{m} d_{k} \mathscr{X}_{D_{k}} d P=\sum_{k=1}^{m} d_{k} \int_{\Omega} f(\mathbf{Y}) \mathscr{X}_{D_{k}} d P
$$

$$
=\sum_{k=1}^{m} d_{k} \int_{\mathbb{R}^{p}} f(\mathbf{Y}) d v_{k}
$$

where $v_{k}(B) \equiv \int_{\Omega} \mathscr{X}_{D_{k}} \mathscr{X}_{B}(\mathbf{Y}) d P$. Now let

$$
v_{n}(B) \equiv \int_{\Omega} \sum_{k=1}^{m} d_{k} \mathscr{X}_{D_{k}} \mathscr{X}_{B}(\mathbf{Y})=\int_{\Omega} s_{n} \mathscr{X}_{B}(\mathbf{Y}) d P
$$

It is indexed with $n$ thanks to $s_{n}$. Then

$$
v_{n}(B)=\sum_{k=1}^{m} d_{k} \int_{\Omega} \mathscr{X}_{D_{k}} \mathscr{X}_{B}(\mathbf{Y}) d P=\sum_{k=1}^{m} d_{k} v_{k}(B)
$$

Hence

$$
\begin{aligned}
\int_{\Omega} f(\mathbf{Y}) s_{n} d P & =\int_{\Omega} f(\mathbf{Y}) \sum_{k=1}^{m} d_{k} \mathscr{X}_{D_{k}} d P=\sum_{k=1}^{m} d_{k} \int_{\mathbb{R}^{p}} f(\mathbf{y}) d v_{k} \\
& =\int_{\mathbb{R}^{p}} f(\mathbf{y}) \sum_{k=1}^{m} d_{k} d v_{k}=\int_{\mathbb{R}^{p}} f(\mathbf{y}) d v_{n}
\end{aligned}
$$

$\left(s_{n} d P=d v_{n}\right.$ so to speak.) Then let $s_{n}(\omega) \uparrow X(\omega)$. Clearly $v_{n} \ll v$ and so by the Radon Nikodym theorem $d v_{n}=h_{n} d v$ where $h_{n} \uparrow$. It follows from the monotone convergence theorem that one can pass to a limit in the above and obtain

$$
\int_{\Omega} f(\mathbf{Y}) X d P=\int_{\mathbb{R}^{p}} f(\mathbf{y}) d v
$$

The interest here is to let $f(\mathbf{Y}) \equiv e^{\lambda \cdot \mathbf{Y}}$ so $f(\mathbf{y})=e^{\lambda \cdot \mathbf{y}}$. To remember this, $X d P=d \nu$ in a sort of sloppy way then the above formula holds.

Lemma 64.6.4 Each $e^{W(h)}$ is in $L^{p}(\Omega)$ for every $h \in H$ and for every $p \geq 1$. In fact,

$$
\int_{\Omega}\left(e^{W(h)}\right)^{p} d P=\int_{\Omega} e^{W(p h)} d P=e^{\frac{1}{2}|p h|_{H}^{2}}
$$

In addition to this,

$$
\sum_{k=0}^{n} \frac{W(h)^{k}}{k!} \rightarrow e^{W(h)} \text { in } L^{p}(\Omega, \mathscr{F}, P), p>1
$$

Proof: It suffices to verify this for all positive integers $p$. Let $p$ be such an integer. Note that from the linearity of $W,\left(e^{W(h)}\right)^{p}=e^{p W(h)}=e^{W(p h)}$ and so it suffices to verify that for each $h \in H, e^{W(h)}$ is in $L^{1}(\Omega)$. From Lemma 64.6.3,

$$
\int_{\Omega} e^{W(h)} d P=\int_{\mathbb{R}} e^{y} d v(y)
$$

where $v(B) \equiv \int_{\Omega} \mathscr{X}_{B}(W(h)) d P=\int_{\mathbb{R}} \mathscr{X}_{B}(y) d v(y)$. In using this lemma, $\mathbf{Y}=W(h), X=$ 1. Thus

$$
\int_{\Omega} e^{W(h)} d P=\int_{0}^{\infty} v\left(e^{y}>\lambda\right) d \lambda=\int_{0}^{\infty} \frac{1}{\sqrt{2 \pi}|h|} \int_{[y>\ln (\lambda)]} e^{-\frac{1}{2} \frac{y^{2}}{|h|^{2}}} d y d \lambda, u=\ln (\lambda)
$$

$$
\begin{aligned}
& =\frac{1}{\sqrt{2 \pi}|h|} \int_{-\infty}^{\infty} e^{u} \int_{u}^{\infty} e^{-\frac{1}{2} \frac{y^{2}}{|h|^{2}}} d y d u=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} \int_{u /|h|}^{\infty} e^{u} e^{-\frac{1}{2} v^{2}} d v d u \\
& \quad=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{|h| v} e^{u} e^{-\frac{1}{2} v^{2}} d u d v=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} v^{2}} e^{|h| v} d v \\
& \quad=\frac{1}{\sqrt{2 \pi}} \sqrt{2} \sqrt{\pi} e^{|h|^{2} / 2}=e^{\frac{1}{2}|h|^{2}}<\infty
\end{aligned}
$$

If $h=0, W(h)$ would be 0 because by the construction, $E\left(W(0)^{2}\right)=(0,0)_{H}=0$. Then

$$
\int_{\Omega} e^{W(h)} d P=\int_{\Omega} e^{0} d P=1
$$

Consider the last claim. It is enough to assume $p$ is an integer.

$$
\begin{aligned}
\left|\sum_{k=0}^{n} \frac{W(h)^{k}}{k!}-e^{W(h)}\right| & =\left|\sum_{k=n+1}^{\infty} \frac{W(h)^{k}}{k!}\right|=\left|W(h)^{n+1}\right|\left|\sum_{k=0}^{\infty} \frac{W(h)^{k}}{(n+1+k)!}\right| \\
& =\left|W(h)^{n+1}\right|\left|\sum_{k=0}^{\infty} \frac{W(h)^{k}}{k!} \frac{k!}{(n+1+k)!}\right| \\
& \leq\left|W(h)^{n+1}\right| \frac{1}{(n+1)!}\left|\sum_{k=0}^{\infty} \frac{W(h)^{k}}{k!}\right|=\left|\frac{W(h)^{n+1}}{(n+1)!}\right| e^{W(h)}
\end{aligned}
$$

This converges to 0 for each $\omega$ because it says nothing more than that the $n^{t h}$ term of a convergent sequence converges to 0 .

$$
\begin{gathered}
\int_{\Omega}\left(\left|\frac{W(h)^{n+1}}{(n+1)!}\right| e^{W(h)}\right)^{2 p} d P=\int_{\Omega}\left(\frac{W(h)^{n+1}}{(n+1)!}\right)^{2 p}\left(e^{W(h)}\right)^{2 p} d P \\
=\left(\frac{1}{(n+1)!}\right)^{2 p} \frac{1}{\sqrt{2 \pi}|h|} \int_{\mathbb{R}} e^{-\frac{1}{2} \frac{x^{2}}{|h|^{2}}} e^{2 p x} x^{2 p(n+1)} d x \\
=\left(\frac{1}{(n+1)!}\right)^{2 p} \frac{1}{\sqrt{2 \pi}|h|} e^{2 p|h|^{2}} \int_{\mathbb{R}} e^{-\frac{1}{2|h|^{2}}\left(x-2 p|h|^{2}\right)^{2}} x^{2 p(n+1)} d x \\
\leq\left(\frac{1}{(n+1)!}\right)^{2 p} \frac{2^{2 p(n+1)}}{\sqrt{2 \pi}|h|} e^{2 p|h|^{2}} \int_{\mathbb{R}} e^{-\frac{1}{2|h|^{2}}\left(x-2 p|h|^{2}\right)^{2}}\left(x-2 p|h|^{2}\right)^{2 p(n+1)} d x \\
+\left(\frac{1}{(n+1)!}\right)^{2 p} \frac{2^{2 p(n+1)}}{\sqrt{2 \pi}|h|} e^{2 p|h|^{2}} \int_{\mathbb{R}} e^{-\frac{1}{2|h|^{2}}\left(x-2 p|h|^{2}\right)^{2}}\left(2 p|h|^{2}\right)^{2 p(n+1)} d x
\end{gathered}
$$

The second term clearly converges to 0 as $n \rightarrow \infty$. Consider the first term. To simplify, let $t=\frac{x-2 p|h|}{|h|}$. Then this term reduces to

$$
\begin{aligned}
& \left(\frac{1}{(n+1)!}\right)^{2 p} \frac{2^{2 p(n+1)}|h|^{2 p(n+1)}|h|}{\sqrt{2 \pi}} e^{2 p|h|^{2}} \int_{\mathbb{R}} e^{-t^{2}} t^{2 p(n+1)} d t \\
= & 2\left(\frac{1}{(n+1)!}\right)^{2 p} \frac{2^{2 p(n+1)}|h|^{2 p(n+1)}|h|}{\sqrt{2 \pi}} e^{2 p|h|^{2}} \int_{0}^{\infty} e^{-t^{2}} t^{2 p(n+1)} d t
\end{aligned}
$$

Now let $t^{2}=u$. Then this becomes

$$
\begin{aligned}
& 2\left(\frac{1}{(n+1)!}\right)^{2 p} \frac{2^{2 p(n+1)}|h|^{2 p(n+1)}|h|}{\sqrt{2 \pi}} e^{2 p|h|^{2}} \int_{0}^{\infty} e^{-u} u^{p(n+1)} u^{-(1 / 2)} \frac{1}{2} d u \\
= & \left(\frac{1}{(n+1)!}\right)^{2 p} \frac{2^{2 p(n+1)}|h|^{2 p(n+1)}|h|}{\sqrt{2 \pi}} e^{2 p|h|^{2}} \int_{0}^{\infty} e^{-u} u^{p+n p-\frac{1}{2}} d u \\
\leq & C(h)(2|h|)^{2 p(n+1)} \frac{1}{(n+1)!} \frac{1}{((n+1)!)^{2 p-1}} \Gamma\left(p(n+1)-\frac{1}{2}\right) \\
= & C(h) \frac{(2|h|)^{2 p(n+1)}}{(n+1)!} \frac{\Gamma\left(p(n+1)-\frac{1}{2}\right)}{((n+1)!)^{2 p-1}} \\
\leq & C(h) \frac{\left(2^{2}|h|^{2}\right)^{p(n+1)}}{(n+1)!} \frac{\Gamma\left(p(n+1)-\frac{1}{2}\right)}{((n+1)!)^{2 p-1}} \\
= & C(h) \frac{\left(2^{2}|h|^{2}\right)^{p(n+1)}}{(n+1)!} \frac{(p(n+1))!}{((n+1)!)^{2 p-1}}
\end{aligned}
$$

this converges to 0 as $n \rightarrow \infty$. This is obvious for $\frac{\left(2^{2}|h|^{2}\right)^{p(n+1)}}{(n+1)!}$. Consider $\frac{(p(n+1))!}{((n+1)!)^{2 p-1}}$. By the ratio test, $\sum_{n} \frac{(p(n+1))!}{((n+1)!)^{2 p-1}}<\infty$ so this also converges to 0 . The details of this ratio test argument are as follows. The ratio, after simplifying is

$$
\overbrace{\frac{(p n+2 p)(p n+2 p-1) \cdots(p n+p+1)}{(n+2)^{2 p-1}}}^{p \text { factors }} \leq \frac{p^{p}(n+p)^{p}}{(n+2)^{2 p-1}}
$$

which clearly converges to 0 since $2 p-1>p$ since $p$ is an integer larger than 1 .
Therefore, $\left\{\left|\frac{W(h)^{n+1}}{(n+1)!}\right| e^{W(h)}\right\}_{n=1}^{\infty}$ is bounded in $L^{2 p}(\Omega)$. Then

$$
\int_{\Omega}\left|\sum_{k=0}^{n} \frac{W(h)^{k}}{k!}-e^{W(h)}\right|^{p} d P \rightarrow 0
$$

because the integrand is bounded by $\left(\left|\frac{W(h)^{n+1}}{(n+1)!}\right| e^{W(h)}\right)^{p}$ and it was just shown that these functions are bounded in $L^{2}(\Omega)$. Therefore, the claimed convergence follows from the Vitali convergence theorem.

The following lemma shows that the functions $e^{W(h)}$ are dense in $L^{p}(\Omega)$ for every $p>1$.
Lemma 64.6.5 Let $\mathscr{F}$ be the $\sigma$ algebra determined by the random variables $W(h)$. If $X \in L^{p}(\Omega, \mathscr{F}, P), p>1$ and $\int_{\Omega} X e^{W(h)} d P=0$ for every $h \in H$, then $X=0$.

Proof: Let $h_{1}, \cdots, h_{p}$ be given. Then for $t_{i} \in \mathbb{R}$,

$$
\sum_{i} t_{i} h_{i} \in H
$$

and so since $W$ is linear,

$$
\int_{\Omega} X e^{\mathbf{t} \cdot \mathbf{W}(\mathbf{h})} d P=0, \mathbf{W}(\mathbf{h}) \equiv\left(W\left(h_{1}\right), \cdots, W\left(h_{p}\right)\right)
$$

Now by Lemma 64.6.3,

$$
\int_{\Omega} X^{+} e^{\mathbf{t} \cdot\left(W\left(h_{1}\right), \cdots, W\left(h_{p}\right)\right)} d P=\int_{\mathbb{R}^{p}} e^{\mathbf{t} \cdot \mathbf{y}} d v_{+}(y)
$$

where $v_{+}(B)=E\left(X^{+} \mathscr{X}_{B}(\mathbf{W}(\mathbf{h}))\right)$. From Lemma 64.6.4, this function of $\mathbf{t}$ is finite for all $\mathbf{t} \in \mathbb{R}^{p}$. Similarly,

$$
\int_{\Omega} X^{-} e^{\mathbf{t} \cdot\left(W\left(h_{1}\right), \cdots, W\left(h_{p}\right)\right)} d P=\int_{\mathbb{R}^{p}} e^{\mathbf{t} \cdot \mathbf{y}} d v_{-}(y)
$$

where $v_{-}(B)=E\left(X^{-} \mathscr{X}_{B}(\mathbf{W}(\mathbf{h}))\right)$. Thus for $v$ equal to the signed measure $v \equiv v_{+}-v_{-}$,

$$
f(\mathbf{t}) \equiv \int_{\mathbb{R}^{p}} e^{\mathbf{t} \cdot \mathbf{y}} d v(y)=0
$$

for $\mathbf{t} \in \mathbb{R}^{p}$. Also

$$
\int_{\Omega} X^{+} e^{i \mathbf{t} \cdot(\mathbf{W}(\mathbf{h}))} d P=\int_{\mathbb{R}^{p}} e^{i \mathbf{t} \cdot \mathbf{y}} d v_{+}(y)
$$

with a similar formula holding for $X^{-}$. Thus

$$
f(\mathbf{t}) \equiv \int_{\mathbb{R}^{p}} e^{\mathbf{t} \cdot \mathbf{y}} d v(y) \in \mathbb{C}
$$

is well defined for all $\mathbf{t} \in \mathbb{C}^{p}$. Consider

$$
\int_{\mathbb{R}^{p}} e^{\mathbf{t} \cdot \mathbf{y}} d v_{+}(y)
$$

Is this function analytic in each $t_{k}$ ? Take a difference quotient. It equals for $h \in \mathbb{C}$,

$$
\int_{\Omega} X^{+} \frac{\left(e^{\left(\mathbf{t}+h \mathbf{e}_{k}\right) \cdot(\mathbf{W}(\mathbf{h}))}-e^{\mathbf{t} \cdot(\mathbf{W}(\mathbf{h}))}\right)}{h} d P=\int_{\Omega} X^{+} e^{\mathbf{t} \cdot \mathbf{W}(\mathbf{h})} \frac{\left(e^{h \mathbf{e}_{k} \cdot(\mathbf{W}(\mathbf{h}))}-1\right)}{h} d P
$$

In case $\mathbf{e}_{k} \cdot \mathbf{W}(\mathbf{h})=0$ there is nothing to show. Assume then that this is not 0 . Then this equals

$$
\int_{\Omega} X^{+} \mathbf{e}_{k} \cdot(\mathbf{W}(\mathbf{h})) e^{\mathbf{t} \cdot \mathbf{W}(\mathbf{h})} \frac{\left(e^{h \mathbf{e}_{k} \cdot(\mathbf{W}(\mathbf{h}))}-1\right)}{h\left(\mathbf{e}_{k} \cdot(\mathbf{W}(\mathbf{h}))\right)} d P
$$

Now

$$
\left|\frac{e^{z}-1}{z}\right|=\left|\frac{1}{z} \sum_{k=1}^{\infty} \frac{z^{k}}{k!}\right| \leq \sum_{k=0}^{\infty} \leq\left|e^{z}\right|
$$

and so the integrand is dominated by

$$
\begin{aligned}
\left|X^{+} \mathbf{e}_{k} \cdot(\mathbf{W}(\mathbf{h})) e^{\mathbf{t} \cdot \mathbf{W}(\mathbf{h})} \frac{\left(e^{h \mathbf{e}_{k} \cdot(\mathbf{W}(\mathbf{h}))}-1\right)}{h\left(\mathbf{e}_{k} \cdot(\mathbf{W}(\mathbf{h}))\right)}\right| & \leq X^{+}\left|\mathbf{e}_{k} \cdot(\mathbf{W}(\mathbf{h})) e^{\mathbf{t} \cdot \mathbf{W}(\mathbf{h})} e^{h\left(\mathbf{e}_{k} \cdot(\mathbf{W}(\mathbf{h}))\right)}\right| \\
& =X^{+}\left|\mathbf{e}_{k} \cdot(\mathbf{W}(\mathbf{h})) e^{\left(\mathbf{t}+h \mathbf{e}_{k}\right) \cdot \mathbf{W}(\mathbf{h})}\right|
\end{aligned}
$$

From Lemma 64.6 .4 which says that $e^{W(h)}$ is in $L^{q}(\Omega)$ for each $q>1$, this is in particular true for $q=m p$ where $m$ is an arbitrary positive integer satisfying

$$
p>\frac{m+1}{m}
$$

Then the integrand is of the form $f g_{h}$ where $f \in L^{p}$ and $g_{h}$ is bounded in $L^{m p}$. Therefore,

$$
\alpha \equiv(p m) /(m+1)>1
$$

and

$$
\int_{\Omega}\left|f g_{h}\right|^{\alpha} d P=\int_{\Omega}|f|^{\alpha}\left|g_{h}\right|^{\alpha} d P \leq\left(\int_{\Omega}|f|^{p} d P\right)^{m /(m+1)}\left(\int_{\Omega}\left|g_{h}\right|^{p m} d P\right)^{1 /(m+1)}
$$

which is bounded. By the Vitali convergence theorem,

$$
\lim _{h \rightarrow 0} \int_{\Omega} X^{+} \frac{\left(e^{\left(\mathbf{t}+h \mathbf{e}_{k}\right) \cdot(\mathbf{W}(\mathbf{h}))}-e^{\mathbf{t} \cdot(\mathbf{W}(\mathbf{h}))}\right)}{h} d P=\int_{\Omega} X^{+} \mathbf{e}_{k} \cdot(\mathbf{W}(\mathbf{h})) e^{\mathbf{t} \cdot \mathbf{W}(\mathbf{h})} d P
$$

and so this function of $t_{k}$ is analytic. Similarly one can do the same thing for the integral involving $X^{-}$. Thus

$$
0=\int_{\mathbb{R}^{p}} e^{\boldsymbol{t} \cdot \mathbf{y}} d v(y)
$$

whenever $t_{j} \in \mathbb{R}$ for all $j$ and $t_{1} \rightarrow \int_{\mathbb{R}^{p}} e^{\mathbf{t} \cdot \mathbf{y}} d \boldsymbol{v}(y)$ is analytic on $\mathbb{C}$. Thus this analytic function of $t_{1}$ is zero for all $t_{1} \in \mathbb{C}$ since it is zero on a set which has a limit point, and in particular

$$
\int_{\mathbb{R}^{p}} e^{i t_{1} y_{1}+t_{2} y_{2}+\cdots+t_{p} y_{p}} d v(y)=0
$$

where each $t_{j}$ is real. Now repeat the argument with respect to $t_{2}$ and conclude that

$$
\int_{\mathbb{R}^{p}} e^{i t_{1} y_{1}+i t_{2} y_{2}+\cdots+t_{p} y_{p}} d v(y)=0
$$

and continue this way to conclude that

$$
0=\int_{\mathbb{R}^{p}} e^{i \cdot \mathbf{y}} d v(y)
$$

which shows that the inverse Fourier transform of $v$ is 0 . Thus $v=0$. To see this, let $\psi \in \mathfrak{S}$, the Schwartz class. Then neglecting troublesome constants in the Fourier transform,

$$
0=\int_{\mathbb{R}^{p}} \psi(\mathbf{t}) \int_{\mathbb{R}^{p}} e^{i t \cdot \mathbf{y}} d v(y) d t=\int_{\mathbb{R}^{p}} \int_{\mathbb{R}^{p}} \psi(\mathbf{t}) e^{i \mathbf{t} \cdot \mathbf{y}} d t d v(y)=v\left(F^{-1} \psi\right)
$$

Now $F^{-1}$ maps $\mathfrak{S}$ onto $\mathfrak{S}$ and so this reduces to

$$
\int_{\mathbb{R}^{p}} \psi d v=0
$$

for all $\psi \in \mathfrak{S}$. By density of $\mathfrak{S}$ in $C_{0}\left(\mathbb{R}^{p}\right)$, it follows that the above holds for all $\psi \in C_{0}\left(\mathbb{R}^{p}\right)$ and so $v=0$.

It follows that for every $B$ Borel and for every such description of $\mathbf{W}(\mathbf{h})$.

$$
0=\int_{\Omega} X \mathscr{X}_{B}(\mathbf{W}(\mathbf{h})) d P=\int_{\Omega} X \mathscr{X}_{\mathbf{W}(\mathbf{h})^{-1}(B)} d P
$$

Let $\mathscr{K}$ be sets of the form $\mathbf{W}(\mathbf{h})^{-1}(B)$ where $B$ is of the form $B_{1} \times \cdots \times B_{p}, B_{i}$ open, this for some $p$. Then this is clearly a $\pi$ system because the intersection of any two of them is another one and

$$
\emptyset, \Omega=\mathbf{W}(\mathbf{h})^{-1}\left(\mathbb{R}^{p}\right)
$$

are both in $\mathscr{K}$. Also $\sigma(\mathscr{K})=\mathscr{F}$. Let $\mathscr{G}$ be those sets $F$ of $\mathscr{F}$ such that

$$
\begin{equation*}
0=\int_{\Omega} X \mathscr{X}_{F} d P \tag{64.6.35}
\end{equation*}
$$

This is true for $F \in \mathscr{K}$. Now it is clear that $\mathscr{G}$ is closed with respect to complements and countable disjoint unions. It is closed with respect to complements because

$$
\int_{\Omega} X \mathscr{X}_{F^{C}} d P=\int_{\Omega} X\left(1-\mathscr{X}_{F}\right) d P=\int_{\Omega} X d P-\int_{\Omega} X \mathscr{X}_{F} d P=0
$$

By Dynkin's lemma, $\mathscr{G}=\mathscr{F}$ and so 64.6 .35 holds for all $F \in \mathscr{F}$ which requires $X=0$.

### 64.6.2 The Wiener Processes

Recall the definition of the Wiener process.
Definition 64.6.6 Let $W(t)$ be a stochastic process which has the properties that whenever $t_{1}<t_{2}<\cdots<t_{m}$, the increments $\left\{W\left(t_{i}\right)-W\left(t_{i-1}\right)\right\}$ are independent and whenever $s<t$, it follows $W(t)-W(s)$ is normally distributed with variance $t-s$ and mean 0 . Also $t \rightarrow W(t)$ is Holder continuous with every exponent $\gamma<1 / 2, W(0)=0$. This is called a Wiener process.

Now in the definition of $W$ above, you begin with a Hilbert space $H$. There exists a probability space $(\Omega, \widehat{\mathscr{F}}, P)$ and a linear mapping $W$ such that

$$
E(W(f) W(g))=(f, g)
$$

and $\left(W\left(f_{1}\right), W\left(f_{2}\right), \cdots, W\left(f_{n}\right)\right)$ is normally distributed with mean $\mathbf{0}$. Next define $\mathscr{F}=$ $\sigma(W(h): h \in H)$.

Consider the special example where $H=L^{2}(0, \infty ; \mathbb{R})$, real valued functions which are square integrable with respect to Lebesgue measure. Note that for each $t \in[0, \infty), \mathscr{X}_{[0, t)} \in$ $H$. Let

$$
W(t) \equiv W\left(\mathscr{X}_{(0, t)}\right)
$$

Then from definition, if $t_{1}<t_{2}<\cdots<t_{m}$, the increments $\left\{W\left(t_{i}\right)-W\left(t_{i-1}\right)\right\}$ are independent. This is because, due to the linearity of $W$, each of these equals

$$
W\left(\mathscr{X}_{\left(0, t_{i}\right)}-\mathscr{X}_{\left(0, t_{i-1}\right)}\right)=W\left(\mathscr{X}_{\left(t_{i-1}, t_{i}\right)}\right)
$$

and from Corollary 64.6.1, the random vector $\left(W\left(\mathscr{X}_{\left(t_{1}, t_{2}\right)}\right), \cdots, W\left(\mathscr{X}_{\left(t_{m}, t_{m-1}\right)}\right)\right)$ is normally distributed with covariance equal to a diagonal matrix. Also

$$
E\left(W(t)^{2}\right)=E\left(W\left(\mathscr{X}_{(0, t)}\right)^{2}\right)=\int_{0}^{\infty} \mathscr{X}_{(0, t)}^{2} d s=t
$$

More generally,

$$
\begin{gathered}
W(t)-W(s)=W\left(\mathscr{X}_{(0, t)}\right)-W\left(\mathscr{X}_{(0, s)}\right)=W\left(\mathscr{X}_{(s, t)}\right) \\
W(t-s)=W\left(\mathscr{X}_{(0, t-s)}\right)
\end{gathered}
$$

so both $W(t)-W(s)$ and $W(t-s)$ are normally distrubuted with mean 0 and variance $t-s$. What about the Holder continuity? The characteristic function of $W(t)-W(s)$ is

$$
E\left(e^{i \lambda(W(t-s))}\right)=e^{\frac{1}{2} \lambda^{2}|t-s|}
$$

Consider a few derivatives of the right side with respect to $\lambda$ and then let $\lambda=0$. This will yield $E\left((W(t)-W(s))^{n}\right)$ for $n=1,2,3,4$.

$$
0,|s-t|, 0,3|s-t|^{2}
$$

You see the pattern. By induction, you can show that $E\left((W(t)-W(s))^{2 m}\right)=C_{m}|t-s|^{m}$. By the Kolmogorov Centsov theorem, Theorem 62.2.3,

$$
E\left(\sup _{0 \leq s<t \leq T} \frac{\|W(t)-W(s)\|}{(t-s)^{\gamma}}\right) \leq C_{m}
$$

whenever $\gamma<\beta / \alpha=\frac{m-1}{2 m}$. Thus the above is true whenever $\gamma<1 / 2$. It follows that there exists a set of measure zero off which $t \rightarrow W(t)$ is Holder continuous with exponent $\gamma<$ 1/2.

Thus this gives a construction of the real Wiener process. Now consider the normal filtration

$$
\mathscr{F}_{s} \equiv \cap_{t>s} \overline{\sigma(W(u)-W(r): 0 \leq r<u \leq t)}
$$

By Lemma 64.4.2, $\{W(t)\}$ is a martingale with respect to this filtration, because of the independence of the increments.

You could also take an arbitrary $f \in L^{2}(0, \infty)$ and consider $W(t) \equiv W\left(\mathscr{X}_{(0, t)} f\right)$. You could consider this as an integral and write it in the notation

$$
W(t) \equiv \int_{0}^{t} f d W \equiv W\left(f \mathscr{X}_{(0, t)}\right)
$$

Then from the construction,

$$
E\left(\left(\int_{0}^{t} f d W\right)^{2}\right)=E\left(W\left(f \mathscr{X}_{(0, t)}\right)^{2}\right)=\int_{0}^{T} f^{2} \mathscr{X}_{(0, t)} d s=\int_{0}^{t}|f|^{2} d s=E\left(\int_{0}^{t}|f|^{2} d s\right)
$$

because $f$ does not depend on $\omega$. This of course is formally the Ito isometry.

### 64.6.3 $Q$ Wiener Processes In Hilbert Space

Now let $U$ be a real separable Hilbert space. Let an orthonormal basis for $U$ be $\left\{g_{i}\right\}$. Now let $L^{2}(0, \infty, U)$ be $H$ in the above construction. For $h, g \in L^{2}(0, \infty, U)$.

$$
E(W(h) W(g))=(h, g)_{L^{2}(0, \infty, U)} \equiv(h, g)_{H}
$$

Here each $W(g)$ will be a real valued normal random variable, the variance of $W(g)$ is $|g|_{L^{2}(0, \infty, U)}^{2}$ and its mean is 0 , every vector $\left(W\left(h_{1}\right), \cdots, W\left(h_{n}\right)\right)$ being generalized multivariate normal. Let

$$
\psi_{k}(t)=W\left(\mathscr{X}_{(0, t)} g_{k}\right)
$$

Then this is a real valued random variable. Disjoint increments are obviously independent in the same way as before. Also

$$
\begin{equation*}
E\left(\psi_{k}(t) \psi_{j}(s)\right)=E\left(W\left(\mathscr{X}_{(0, t)} g_{k}\right) W\left(\mathscr{X}_{(0, s)} g_{j}\right)\right) \equiv \int_{0}^{\infty} \mathscr{X}_{(0, t \wedge s)}\left(g_{k}, g_{j}\right)_{U} d t=0 \tag{64.6.36}
\end{equation*}
$$

if $j \neq k$. Thus the random variables $\psi_{k}(t)$ and $\psi_{j}(s)$ are independent. This is because, from the construction, $\left(\psi_{k}(t), \psi_{j}(s)\right)$ is normally distributed and the covariance is a diagonal matrix. Also

$$
\begin{gathered}
\psi_{k}(t)-\psi_{k}(s)=W\left(\mathscr{X}_{(0, t)} J g_{k}\right)-W\left(\mathscr{X}_{(0, s)} J g_{k}\right)=W\left(\mathscr{X}_{(s, t)} J g_{k}\right) \\
\psi_{k}(t-s) \equiv W\left(\mathscr{X}_{(0, t-s)} J g_{k}\right)
\end{gathered}
$$

so $\psi_{k}(t-s)$ has the same mean, 0 and variance, $|t-s|$, as $\psi_{k}(t)-\psi_{s}(s)$. Thus these have the same distribution because both are normally distributed.

Now let $J$ be a Hilbert Schmidt map from $U$ to $H$. Then consider

$$
\begin{equation*}
W(t)=\sum_{k} \psi_{k}(t) J g_{k} . \tag{64.6.37}
\end{equation*}
$$

This has values in $H$. It is shown below that the series converges in $L^{2}(\Omega ; H)$. Recall the definition of a $Q$ Wiener process.

Definition 64.6.7 Let $W(t)$ be a stochastic process with values in $H$, a real separable Hilbert space which has the properties that $t \rightarrow W(t, \omega)$ is continuous, whenever $t_{1}<$ $t_{2}<\cdots<t_{m}$, the increments $\left\{W\left(t_{i}\right)-W\left(t_{i-1}\right)\right\}$ are independent, $W(0)=0$, and whenever $s<t$,

$$
\mathscr{L}(W(t)-W(s))=N(0,(t-s) Q)
$$

which means that whenever $h \in H$,

$$
\mathscr{L}((h, W(t)-W(s)))=N(0,(t-s)(Q h, h))
$$

Also

$$
E\left(\left(h_{1}, W(t)-W(s)\right)\left(h_{2}, W(t)-W(s)\right)\right)=\left(Q h_{1}, h_{2}\right)(t-s) .
$$

Here $Q$ is a nonnegative trace class operator. Recall this means

$$
Q=\sum_{i=1}^{\infty} \lambda_{i} e_{i} \otimes e_{i}
$$

where $\left\{e_{i}\right\}$ is a complete orthonormal basis, $\lambda_{i} \geq 0$, and

$$
\sum_{i=1}^{\infty} \lambda_{i}<\infty
$$

Such a stochastic process is called a $Q$ Wiener process. In the case where these have values in $\mathbb{R}^{n}$, tQ ends up being the covariance matrix of $W(t)$.

Proposition 64.6.8 The process defined in 64.6 .37 is a $Q$ Wiener process in $H$ where $Q=$ $J J^{*}$.

Proof: First, why does the sum converge? Consider the sum for an increment in time. Let $t_{i-1}=0$ to obtain the convergence of the sum for a given $t$. Consider the difference of two partial sums.

$$
\begin{aligned}
& E\left(\sum_{k, l=m}^{n}\left(\psi_{k}\left(t_{i}\right)-\psi_{k}\left(t_{i-1}\right)\right) J g_{k},\left(\psi_{l}\left(t_{i}\right)-\psi_{l}\left(t_{i-1}\right)\right) J g_{k}\right) \\
= & E\left(\sum_{k, l=m}^{n}\left(J^{*} J g_{k}, g_{l}\right)\left(\psi_{k}\left(t_{i}\right)-\psi_{k}\left(t_{i-1}\right)\right)\left(\psi_{l}\left(t_{i}\right)-\psi_{l}\left(t_{i-1}\right)\right)\right) \\
= & \sum_{k, l=m}^{n}\left(J^{*} J g_{k}, g_{l}\right) E\left(\left(\psi_{k}\left(t_{i}\right)-\psi_{k}\left(t_{i-1}\right)\right)\left(\psi_{l}\left(t_{i}\right)-\psi_{l}\left(t_{i-1}\right)\right)\right) \\
= & \sum_{k=m}^{n}\left(J^{*} J g_{k}, g_{k}\right) E\left(\psi_{k}\left(t_{i}\right)-\psi_{k}\left(t_{i-1}\right)^{2}\right)=\sum_{k=m}^{n}\left(J^{*} J g_{k}, g_{k}\right)\left(t_{i}-t_{i-1}\right) \\
= & \sum_{k=m}^{n}\left|J g_{k}\right|_{H}^{2}\left(t_{i}-t_{i-1}\right)
\end{aligned}
$$

and this converges to 0 as $m, n \rightarrow \infty$ since $J$ is Hilbert Schmidt. Thus the sum converges in $L^{2}(\Omega, H)$. Why are the disjoint increments independent?

Let $\lambda_{k} \in H$. Consider $t_{0}<t_{1}<\cdots<t_{n}$.

$$
\begin{equation*}
E\left(\exp i \sum_{k=1}^{n}\left(\lambda_{k}, W\left(t_{k}\right)-W\left(t_{k-1}\right)\right)\right)=\prod_{k=1}^{n} E\left(\exp \left(i\left(\lambda_{k}, W\left(t_{k}\right)-W\left(t_{k-1}\right)\right)\right)\right) ? \tag{64.6.38}
\end{equation*}
$$

Start with the left. There are finitely many increments concerned and so it can be assumed that for each $k$ one can have $m \rightarrow \infty$ such that the partial sums up to $m$ in the definition of $W\left(t_{k}\right)-W\left(t_{k-1}\right)$ converge pointwise a.e. Thus

$$
\begin{aligned}
& E\left(\exp i \sum_{k=1}^{n}\left(\lambda_{k}, W\left(t_{k}\right)-W\left(t_{k-1}\right)\right)\right) \\
= & \lim _{m \rightarrow \infty} E\left(\exp i \sum_{k=1}^{n}\left(\lambda_{k}, \sum_{j=1}^{m}\left(\psi_{j}\left(t_{k}\right)-\psi_{j}\left(t_{k-1}\right)\right) J g_{j}\right)\right) \\
= & \lim _{m \rightarrow \infty} E\left(\exp \sum_{k=1}^{n} \sum_{j=1}^{m} i\left(\lambda_{k},\left(\psi_{j}\left(t_{k}\right)-\psi_{j}\left(t_{k-1}\right)\right) J g_{j}\right)\right) \\
= & \lim _{m \rightarrow \infty} E\left(\prod_{j=1}^{m} \exp \left(\sum_{k=1}^{n} i\left(\lambda_{k},\left(\psi_{j}\left(t_{k}\right)-\psi_{j}\left(t_{k-1}\right)\right) J g_{j}\right)\right)\right)
\end{aligned}
$$

Now from 64.6.36, $\left\{\sum_{k=1}^{n} i\left(\lambda_{k},\left(\psi_{j}\left(t_{k}\right)-\psi_{j}\left(t_{k-1}\right)\right) J g_{j}\right)\right\}_{j=1}^{m}$ are independent. Hence the above equals

$$
\begin{aligned}
& =\lim _{m \rightarrow \infty} \prod_{j=1}^{m} E\left(\exp \left(\sum_{k=1}^{n} i\left(\lambda_{k},\left(\psi_{j}\left(t_{k}\right)-\psi_{j}\left(t_{k-1}\right)\right) J g_{j}\right)\right)\right) \\
& =\lim _{m \rightarrow \infty} \prod_{j=1}^{m} E\left(\prod_{k=1}^{n} \exp \left(i\left(\lambda_{k},\left(\psi_{j}\left(t_{k}\right)-\psi_{j}\left(t_{k-1}\right)\right) J g_{j}\right)\right)\right)
\end{aligned}
$$

Now from independence of the increments for the $\psi_{j}$, this equals

$$
\begin{align*}
& =\lim _{m \rightarrow \infty} \prod_{j=1}^{m} \prod_{k=1}^{n} E\left(\exp \left(i\left(\lambda_{k},\left(\psi_{j}\left(t_{k}\right)-\psi_{j}\left(t_{k-1}\right)\right) J g_{j}\right)\right)\right) \\
= & \lim _{m \rightarrow \infty} \prod_{j=1}^{m} \prod_{k=1}^{n} E\left(\exp \left(i\left(\lambda_{k}, J g_{j}\right)\left(\psi_{j}\left(t_{k}\right)-\psi_{j}\left(t_{k-1}\right)\right)\right)\right) \\
= & \lim _{m \rightarrow \infty} \prod_{j=1}^{m} \prod_{k=1}^{n} e^{-\frac{1}{2}\left(\lambda_{k}, J g_{j}\right)^{2}\left(t_{k}-t_{k-1}\right)}=\lim _{m \rightarrow \infty} \prod_{j=1}^{m} e^{-\frac{1}{2} \sum_{k=1}^{n}\left(\lambda_{k}, J g_{j}\right)^{2}\left(t_{k}-t_{k-1}\right)} \\
= & \lim _{m \rightarrow \infty} \exp \left(-\frac{1}{2} \sum_{j=1}^{m} \sum_{k=1}^{n}\left(\lambda_{k}, J g_{j}\right)^{2}\left(t_{k}-t_{k-1}\right)\right) \\
= & \exp \left(-\frac{1}{2} \sum_{k=1}^{n} \sum_{j=1}^{\infty}\left(J^{*} \lambda_{k}, g_{j}\right)^{2}\left(t_{k}-t_{k-1}\right)\right) \tag{64.6.39}
\end{align*}
$$

What is the right side of 64.6 .38 .

$$
\begin{aligned}
& \prod_{k=1}^{n} E\left(\exp \left(i\left(\lambda_{k}, W\left(t_{k}\right)-W\left(t_{k-1}\right)\right)\right)\right) \\
= & \prod_{k=1}^{n} E\left[\exp \left(i\left(\lambda_{k}, \sum_{j=1}^{\infty}\left(\psi_{j}\left(t_{k}\right)-\psi_{j}\left(t_{k-1}\right)\right) J g_{j}\right)\right)\right] \\
= & \lim _{m \rightarrow \infty} \prod_{k=1}^{n} E\left[\exp \left(i\left(\lambda_{k}, \sum_{j=1}^{m}\left(\psi_{j}\left(t_{k}\right)-\psi_{j}\left(t_{k-1}\right)\right) J g_{j}\right)\right)\right] \\
= & \lim _{m \rightarrow \infty} \prod_{k=1}^{n} E\left[\exp \left(i \sum_{j=1}^{m}\left(J g_{j}, \lambda_{k}\right)\left(\psi_{j}\left(t_{k}\right)-\psi_{j}\left(t_{k-1}\right)\right)\right)\right] \\
= & \lim _{m \rightarrow \infty} \prod_{k=1}^{n} E\left(\prod_{j=1}^{m} i\left(J^{*} \lambda_{k}, g_{j}\right)\left(\psi_{j}\left(t_{k}\right)-\psi_{j}\left(t_{k-1}\right)\right)\right)
\end{aligned}
$$

and by independence, 64.6.36,

$$
\begin{aligned}
& =\lim _{m \rightarrow \infty} \prod_{k=1}^{n} \prod_{j=1}^{m} E\left[i\left(J^{*} \lambda_{k}, g_{j}\right)\left(\psi_{j}\left(t_{k}\right)-\psi_{j}\left(t_{k-1}\right)\right)\right] \\
& =\lim _{m \rightarrow \infty} \prod_{k=1}^{n} \prod_{j=1}^{m} e^{-\frac{1}{2}\left(J^{*} \lambda_{k}, g_{j}\right)^{2}\left(t_{k}-t_{k-1}\right)}=\lim _{m \rightarrow \infty} \prod_{k=1}^{n} \exp \left(-\frac{1}{2} \sum_{j=1}^{m}\left(J^{*} \lambda_{k}, g_{j}\right)^{2}\left(t_{k}-t_{k-1}\right)\right) \\
& =\prod_{k=1}^{n} \exp \left(-\frac{1}{2} \sum_{j=1}^{\infty}\left(J^{*} \lambda_{k}, g_{j}\right)^{2}\left(t_{k}-t_{k-1}\right)\right) \\
& =\exp \left(-\frac{1}{2} \sum_{k=1}^{n} \sum_{j=1}^{\infty}\left(J^{*} \lambda_{k}, g_{j}\right)^{2}\left(t_{k}-t_{k-1}\right)\right)
\end{aligned}
$$

which is exactly the same thing as 64.6 .39 . Thus the disjoint increments are independent.
You could also do something like the following. Let $W_{m}(t)$ denote the partial sum for $W(t)$ and since there are only finitely many increments, we can assume the partial sums converge a.e. Then we need to consider the random variables

$$
\left\{\left(W_{m}\left(t_{k}\right)-W_{m}\left(t_{k-1}\right)\right)\right\}_{k=1}^{m}=\left\{\left(\sum_{i=1}^{m}\left(\psi_{i}\left(t_{k}\right)-\psi_{i}\left(t_{k-1}\right)\right) J g_{i}\right)\right\}_{k=1}^{m}
$$

Then for any $h \in H$, you could consider

$$
\left\{\left(\sum_{i=1}^{m}\left(\psi_{i}\left(t_{k}\right)-\psi_{i}\left(t_{k-1}\right)\right)\left(J g_{i}, h\right)_{H}\right)\right\}_{k=1}^{m}
$$

and the vector whose $k^{t h}$ component is $\sum_{i=1}^{m}\left(\psi_{i}\left(t_{k}\right)-\psi_{i}\left(t_{k-1}\right)\right)\left(J g_{i}, h\right)_{H}$ for $k=1,2, \cdots, n$ is normally distributed and the covariance is a diagonal matrix. Hence these are independent random variables as hoped. Now you can pass to a limit as $m \rightarrow \infty$. Since this is true
for any $h \in H$ that the random variables $\left(W\left(t_{k}\right)-W\left(t_{k-1}\right), h\right)_{H}$ are independent, it follows that the random variables $W\left(t_{k}\right)-W\left(t_{k-1}\right)$ are also.

What of the Holder continuity? In the above computation for independence, as a special case, for $\lambda \in H$,

$$
\begin{equation*}
E(\exp i(\lambda, W(t)-W(s)))=\exp \left(-\frac{1}{2}\left|J^{*} \lambda\right|_{U}^{2}(t-s)\right) \tag{64.6.40}
\end{equation*}
$$

In particular, replacing $\lambda$ with $\lambda r$ for $r$ real,

$$
E(\exp \operatorname{ir}(\lambda, W(t)-W(s)))=\exp \left(-\frac{1}{2} r^{2}\left|J^{*} \lambda\right|_{U}^{2}(t-s)\right)
$$

Now we differentiate with respect to $r$ and then take $r=0$ as before to obtain finally that

$$
E\left((\lambda, W(t)-W(s))^{2 m}\right) \leq C_{m}\left|J^{*} \lambda\right|^{2 m}|t-s|^{m}=C_{m}(Q \lambda, \lambda)^{m}|t-s|^{m}
$$

Then letting $\left\{h_{k}\right\}$ be an orthonormal basis for $H$, and using the above inequality with Minkowski’s inequalitiy,

$$
\begin{aligned}
& \left(E\left(|W(t)-W(s)|^{2 m}\right)\right)^{1 / m}=\left(E\left(\left[\sum_{k=1}^{\infty}\left(W(t)-W(s), h_{k}\right)^{2}\right]^{m}\right)\right)^{1 / m} \\
& \leq \sum_{k=1}^{\infty}\left[E\left(\left(W(t)-W(s), h_{k}\right)^{2 m}\right)\right]^{1 / m} \leq \sum_{k=1}^{\infty}\left(C_{m}(t-s)^{m}\left|J^{*} h_{k}\right|_{U}^{2 m}\right)^{1 / m} \\
& =C_{m}^{1 / m}|t-s| \sum_{k=1}^{\infty}\left|J^{*} h_{k}\right|_{U}^{2}=C_{m}^{1 / m}|t-s| \sum_{k=1}^{\infty} \sum_{j=1}^{\infty}\left(J^{*} h_{k}, g_{j}\right)^{2} \\
& =C_{m}^{1 / m}|t-s| \sum_{j=1}^{\infty} \sum_{k=1}^{\infty}\left(h_{k}, J g_{j}\right)^{2}=|t-s| C_{m}^{1 / m} \sum_{j=1}^{\infty}\left|J g_{j}\right|_{H}^{2}
\end{aligned}
$$

and since $J$ is Hilbert Schmidt, modifying the constant yields

$$
E\left(|W(t)-W(s)|^{2 m}\right) \leq C_{m}|t-s|^{m}
$$

By the Kolmogorov Centsov theorem, Theorem 62.2.3,

$$
E\left(\sup _{0 \leq s<t \leq T} \frac{\|W(t)-W(s)\|}{(t-s)^{\gamma}}\right) \leq C_{m}
$$

whenever $\gamma<\beta / \alpha=\frac{m-1}{2 m}$. Thus the above is true whenever $\gamma<1 / 2$. Hence off a set of measure zero, $t \rightarrow W(t)$ is Holder continuous.

What of the covariance condition? From 64.6.40, letting $f, g$ be two elements of $H$,

$$
E(\exp i(\alpha f+\beta g, W(t)-W(s)))=\exp \left(-\frac{1}{2}(Q(\alpha f+\beta g), \alpha f+\beta g)(t-s)\right)
$$

Differentiate with respect to $\alpha$

$$
\begin{aligned}
& E(i(f, W(t)-W(s)) \exp i(\alpha f+\beta g, W(t)-W(s))) \\
= & -[\alpha(Q f, f)+(Q f, \beta g)](t-s) \exp \left(-\frac{1}{2}(Q(\alpha f+\beta g), \alpha f+\beta g)(t-s)\right)
\end{aligned}
$$

Let $\alpha=0$.

$$
\begin{aligned}
& E(i(f, W(t)-W(s)) \exp i(\beta g, W(t)-W(s))) \\
= & -[(Q f, \beta g)](t-s) \exp \left(-\frac{1}{2}(Q(\beta g), \beta g)(t-s)\right)
\end{aligned}
$$

Now differentiate with respect to $\beta$

$$
\begin{gathered}
E(-(f, W(t)-W(s))(g, W(t)-W(s)) \exp i(\beta g, W(t)-W(s))) \\
=-[(Q f, g)](t-s) \exp \left(-\frac{1}{2}(Q(\beta g), \beta g)(t-s)\right)+-[(Q f, \beta g)](t-s)(\text { something })
\end{gathered}
$$

Now let $\beta=0$.

$$
E((f, W(t)-W(s))(g, W(t)-W(s)))=(Q f, g)(t-s)
$$

Finally, $Q=J J^{*}$. It is self adjoint and nonnegative and so there is a complete orthonormal basis $\left\{e_{i}\right\}$ such that $Q e_{i}=\lambda_{i} e_{i}$. Then $\lambda_{i}=\left(Q e_{i}, e_{i}\right)_{H}$ and so

$$
\sum_{i} \lambda_{i}=\sum_{i}\left(Q e_{i}, e_{i}\right)=\sum_{i}\left|J^{*} e_{i}\right|_{U}^{2}<\infty
$$

because $J$ and hence $J^{*}$ are both Hilbert Schmidt operators.
Recall the notion of the Hilbert space $L U$ in Definition 19.2.1.
What if you have a given $Q \in \mathscr{L}(H, H)$ which is trace class, $Q=Q^{*}$, and nonnegative. Does there exist a $Q$ Wiener process of the sort just described? It appears this amounts to obtaining a Hilbert Schmidt map $J$ from some Hilbert space $U$ to $H$ such that $Q=J J^{*}$.

Since $Q$ is trace class and is self adjoint, it follows that there is an orthonormal basis $\left\{e_{i}\right\}, Q e_{i}=\lambda_{i} e_{i}$, where $\lambda_{i}$ is positive for $i \leq L$ or positive for all $i$. Then

$$
Q^{1 / 2}=\sum_{i=1}^{L} \sqrt{\lambda_{i}} e_{i} \otimes e_{i}
$$

and

$$
Q^{1 / 2} e_{i}=\sqrt{\lambda_{i}} e_{i}
$$

Then also on $Q^{1 / 2} H$,

$$
\left(Q^{1 / 2} e_{i}, Q^{1 / 2} e_{j}\right)_{Q^{1 / 2} H} \equiv\left(e_{i}, e_{j}\right)_{H}
$$

and so an orthonormal basis in $Q^{1 / 2} H$ is $\left\{\sqrt{\lambda_{i}} e_{i}\right\}_{i=1}^{L}$. Then define $J: Q^{1 / 2} H \rightarrow H$

$$
J x \equiv \sum_{k=1}^{L}\left(x, \sqrt{\lambda_{k}} e_{k}\right)_{Q^{1 / 2} H} \sqrt{\lambda_{k}} e_{k}
$$

It follows from the above that

$$
J e_{j}=\sum_{k=1}^{L} \frac{1}{\sqrt{\lambda_{j}}} \overbrace{\left(\sqrt{\lambda_{j}} e_{j}, \sqrt{\lambda_{k}} e_{k}\right)_{Q^{1 / 2} H}}^{\delta_{i j}} \sqrt{\lambda_{k}} e_{k}=e_{j}
$$

Then

$$
\begin{aligned}
\sum_{i=1}^{L}\left|J \sqrt{\lambda_{i}} e_{i}\right|_{H}^{2} & =\sum_{i=1}^{L}\left|\sum_{k=1}^{L}\left(\sqrt{\lambda_{i}} e_{i}, \sqrt{\lambda_{k}} e_{k}\right)_{Q^{1 / 2} H} \sqrt{\lambda_{k}} e_{k}\right|_{H}^{2} \\
& =\sum_{i=1}^{L}\left|\sqrt{\lambda_{i}} e_{i}\right|_{H}^{2}=\sum_{i=1}^{L} \lambda_{i}<\infty
\end{aligned}
$$

Thus it is clear that $J$ is Hilbert Schmidt. Is $J J^{*}=Q$ ? For $y \in Q^{1 / 2} H, x \in H$,

$$
\begin{aligned}
\left(J^{*} x, y\right)_{Q^{1 / 2} H} & \equiv(x, J(y))_{H}=\left(x, \sum_{k=1}^{L}\left(y, \sqrt{\lambda_{k}} e_{k}\right)_{Q^{1 / 2} H} \sqrt{\lambda_{k}} e_{k}\right)_{H} \\
& =\sum_{k=1}^{L}\left(x, \sqrt{\lambda_{k}} e_{k}\right)_{H}\left(y, \sqrt{\lambda_{k}} e_{k}\right)_{Q^{1 / 2} H}
\end{aligned}
$$

Thus for $y \in H, x \in H$,

$$
\begin{aligned}
\left(J^{*} x, J^{*} y\right)_{Q^{1 / 2} H} & =\sum_{k=1}^{L}\left(x, \sqrt{\lambda_{k}} e_{k}\right)_{H}\left(J^{*} y, \sqrt{\lambda_{k}} e_{k}\right)_{Q^{1 / 2} H} \\
& \equiv \sum_{k=1}^{L}\left(x, \sqrt{\lambda_{k}} e_{k}\right)_{H}\left(y, \sqrt{\lambda_{k}} J e_{k}\right)_{H} \\
& =\sum_{k=1}^{L} \lambda_{k}\left(x, e_{k}\right)_{H}\left(y, e_{k}\right)_{H}=(Q x, y)
\end{aligned}
$$

and so $\left(J J^{*} x, y\right)=(Q x, y)$ showing that $J J^{*}=Q$. This shows the following.
Proposition 64.6.9 Let $Q \in \mathscr{L}(H, H)$ where $H$ is a real separable Hilbert space and $(Q x, x) \geq 0$ and is trace class. Then there exists a one to one Hilbert Schmidt map J : $Q^{1 / 2} H \rightarrow H$ such that $J J^{*}=Q$. Then the $Q$ Wiener process is $W(t)=\sum_{k=1}^{\infty} \psi_{k}(t) J g_{k}$ where $\left\{g_{k}\right\}$ is a complete orthonormal basis for the Hilbert space $Q^{1 / 2} H$.

Note that in case $H$ is $\mathbb{R}^{p}$ and $Q$ is any symmetric $p \times p$ matrix, having nonnegative eigenvalues, this is automatically trace class and so the above conclusion holds. In particular, the covariance condition says in this case that

$$
\begin{aligned}
& E\left(\left(\mathbf{e}_{i}, \mathbf{W}(t)-\mathbf{W}(s)\right)\left(\mathbf{e}_{j}, \mathbf{W}(t)-\mathbf{W}(s)\right)\right) \\
= & E\left(\left(W_{i}(t)-W_{i}(s)\right)\left(W_{j}(t)-W_{j}(s)\right)\right)=\left(Q e_{i}, e_{j}\right)=Q_{i j}
\end{aligned}
$$

This is a $p$ dimensional Wiener process.

### 64.6.4 Levy's Theorem In Hilbert Space

Recall the concept of quadratic variation. Let $W(t)$ be a $Q$ Wiener process. Does it follow $\{W(t)\} \in \mathscr{M}_{T}^{2}(H)$ ? The Wiener process is continuous. Furthermore,

$$
E\left(|W(t)|_{H}^{2}\right)<\infty
$$

for each $t \in[0, T]$. Since $\{W(t)\}$ is a martingale, Theorem 62.5.3 can be applied to conclude

$$
E\left(|W(t)|_{H}^{2}\right)^{1 / 2} \leq E\left(\left(\sup _{t \in[0, T]}|W(t)|\right)^{2}\right)^{1 / 2} \leq 2 E\left(|W(T)|_{H}^{2}\right)^{1 / 2}
$$

and so $\{W(t)\} \in \mathscr{M}_{T}^{2}(H)$. Therefore, by the Doob Meyer decomposition, Theorem 63.7.15, there exists an increasing natural process, $A(t)$ and a martingale, $Y(t)$ such that

$$
|W(t)|_{H}^{2}=Y(t)+A(t)
$$

What is $A(t)$ ? Consider the process

$$
|W(t)|^{2}
$$

From Theorem 64.5.4 this equals

$$
\sum_{k=1}^{\infty} \lambda_{k} \psi_{k}(t)^{2}
$$

where $\psi_{k}(t)$ is a one dimensional Wiener process and

$$
Q=\sum_{k=1}^{\infty} \lambda_{k} e_{k} \otimes e_{k}, \sum_{k=1}^{\infty} \lambda_{k}<\infty .
$$

By Lemma 64.4.2, $\{W(t)\}$ is a martingale. Therefore, for $s<t$ and $A \in \mathscr{F}_{s}$, it follows since $\mathscr{X}_{A}$ is independent of $W(t)-W(s)$ as in the proof of Lemma 64.4.2 that the following holds.

$$
\begin{gathered}
\int_{A} E\left(|W(t)|^{2} \mid \mathscr{F}_{s}\right)-|W(s)|^{2} d P \\
=\int_{A} E\left(|W(t)|^{2}+|W(s)|^{2}-2 W(t) \cdot W(s) \mid \mathscr{F}_{s}\right) d P \\
=\int_{A} E\left(|W(t)-W(s)|^{2} \mid \mathscr{F}_{s}\right) d P=\int_{A}|W(t)-W(s)|^{2} d P \\
=P(A) \int_{\Omega}|W(t)-W(s)|^{2} d P \\
=P(A) \sum_{k=1}^{\infty} \lambda_{k} E\left(\left(\psi_{k}(t)-\psi_{k}(s)\right)^{2}\right) \\
=P(A)(t-s) \sum_{k=1}^{\infty} \lambda_{k}=P(A)(t-s) \operatorname{tr}(Q) .
\end{gathered}
$$

Therefore,

$$
\int_{A} E\left(|W(t)|^{2}-t \operatorname{tr}(Q) \mid \mathscr{F}_{s}\right)-\left(|W(s)|^{2}-s \operatorname{tr}(Q)\right) d P=0
$$

and since $A \in \mathscr{F}_{s}$ is arbitrary, this shows $\left\{|W(t)|^{2}-t \operatorname{tr}(Q)\right\}$ is a martingale. Hence the Doob Meyer decomposition for $|W(t)|^{2}$ is

$$
|W(t)|^{2}=Y(t)+t \operatorname{tr}(Q)
$$

where $Y(t)$ is a martingale.
There is a generalization of Levy's theorem to Hilbert space valued Wiener processes.
Theorem 64.6.10 Let $\{W(t)\} \in \mathscr{M}_{T}^{2}(H), E(W(t))=0$, where $H$ is a real separable Hilbert space. Then for $Q$ a nonnegative symmetric trace class operator, $\{W(t)\}$ is a $Q$ Wiener process if and only if both $\{W(t)\}$ and $\left\{(W(t), h)^{2}-t(Q h, h)\right\}$ are martingales for every $h \in H$.

Proof: First suppose $\{W(t)\}$ is a $Q$ Wiener process. Then defining the filtration to be

$$
\mathscr{F}_{t} \equiv \sigma(W(s)-W(u): u \leq s \leq t)
$$

it follows from Lemma 64.4.2 that $\{W(t)\}$ is a martingale. Consider

$$
\left\{(W(t), h)^{2}-t(Q h, h)\right\} .
$$

Let $A \in \mathscr{F}_{s}$ where $s \leq t$. Then using the fact $\{W(t)\}$ is a martingale,

$$
\begin{gathered}
\int_{A} E\left((W(t)-W(s), h)^{2} \mid \mathscr{F}_{s}\right) d P \\
=\int_{A} E\left((W(t), h)^{2}+(W(s), h)^{2}-2(W(t), h)(W(s), h) \mid \mathscr{F}_{s}\right) d P \\
=\int_{A} E\left((W(t), h)^{2} \mid \mathscr{F}_{s}\right)+(W(s), h)^{2}-E\left(2(W(t), h)(W(s), h) \mid \mathscr{F}_{s}\right) d P \\
=\int_{A} E\left((W(t), h)^{2} \mid \mathscr{F}_{s}\right) d P+\int_{A}(W(s), h)^{2} d P \\
-\int_{A}(W(s), h) E\left(2(W(t), h) \mid \mathscr{F}_{s}\right) d P \\
=\int_{A} E\left((W(t), h)^{2} \mid \mathscr{F}_{s}\right) d P-\int_{A}(W(s), h)^{2} d P .
\end{gathered}
$$

Also since $\mathscr{X}_{A}$ is independent of $(W(t)-W(s), h)^{2}$ as in the proof of Lemma 64.4.2, and $\{W(t)\}$ is a $Q$ Wiener process,

$$
\begin{aligned}
& \int_{A} E\left((W(t)-W(s), h)^{2} \mid \mathscr{F}_{s}\right) d P \\
= & \int_{A}(W(t)-W(s), h)^{2} d P \\
= & P(A) \int_{\Omega}(W(t)-W(s), h)^{2} d P \\
= & P(A)(t-s)(Q h, h)
\end{aligned}
$$

Thus, this has shown that for all $A \in \mathscr{F}_{s}$,

$$
\begin{aligned}
& \int_{A} E\left((W(t), h)^{2} \mid \mathscr{F}_{s}\right) d P-\int_{A}(W(s), h)^{2} d P \\
= & P(A)(t-s)(Q h, h)=\int_{A}(t-s)(Q h, h) d P
\end{aligned}
$$

and since $A \in \mathscr{F}_{s}$ is arbitrary, this proves

$$
E\left((W(t), h)^{2}-t(Q h, h) \mid \mathscr{F}_{s}\right)=(W(s), h)^{2}-s(Q h, h)
$$

This proves one half of the theorem.
Next suppose both $\{W(t)\}$ and $\left\{(W(t), h)^{2}-t(Q h, h)\right\}$ are martingales for any $h \in H$. It follows that both $\{(W(t), h)\}$ and $\left\{(W(t), h)^{2}-t(Q h, h)\right\}$ are martingales also. Therefore, by Levy's theorem, Theorem 63.8.5, $\{(W(t), h)\}$ is a Wiener process with the property that its variance at $t$ equals $(Q h, h) t$ instead of $t$. Thus the time increments are normal and independent. I need to verify that $\{W(t)\}$ is a $Q$ Wiener process. One of the things which needs to be shown is that

$$
\begin{equation*}
E\left(\left(W(t)-W(s), h_{1}\right)\left(W(t)-W(s), h_{2}\right)\right)=\left(Q h_{1}, h_{2}\right)(t-s) . \tag{64.6.41}
\end{equation*}
$$

I have just shown

$$
\begin{equation*}
E\left((W(t)-W(s), h)^{2}\right)=(t-s)(Q h, h) \tag{64.6.42}
\end{equation*}
$$

which follows from Levy's theorem which concludes $\{(W(t), h)\}$ is a Wiener process. Therefore,

$$
\begin{aligned}
& E\left(\left(W(t)-W(s), h_{1}+h_{2}\right)\left(W(t)-W(s), h_{2}+h_{1}\right)\right) \\
= & \left(Q\left(h_{1}+h_{2}\right),\left(h_{1}+h_{2}\right)\right)(t-s)
\end{aligned}
$$

Now using 64.6.42, it follows from this that

$$
E\left(\left(W(t)-W(s), h_{1}\right)\left(W(t)-W(s), h_{2}\right)\right)=\left(Q h_{1}, h_{2}\right)(t-s)
$$

which shows 64.6.41. This completes the proof.

## Chapter 65

## Stochastic Integration

### 65.1 Integrals Of Elementary Processes

Stochastic integration starts with a $Q$ Wiener process having values in a separable Hilbert space $U$. Thus it satisfies the following definition.

Definition 65.1.1 Let $W(t)$ be a stochastic process with values in $U$, a real separable Hilbert space which has the properties that $t \rightarrow W(t, \omega)$ is continuous. Whenever $t_{1}<$ $t_{2}<\cdots<t_{m}$, the increments $\left\{W\left(t_{i}\right)-W\left(t_{i-1}\right)\right\}$ are independent, $W(0)=0$, and whenever $s<t$,

$$
\mathscr{L}(W(t)-W(s))=N(0,(t-s) Q)
$$

which means that whenever $h \in H$,

$$
\mathscr{L}((h, W(t)-W(s)))=N(0,(t-s)(Q h, h))
$$

Also

$$
E\left(\left(h_{1}, W(t)-W(s)\right)\left(h_{2}, W(t)-W(s)\right)\right)=\left(Q h_{1}, h_{2}\right)(t-s) .
$$

Here $Q$ is a nonnegative trace class operator. Recall this means

$$
Q=\sum_{i=1}^{\infty} \lambda_{i} e_{i} \otimes e_{i}
$$

where $\left\{e_{i}\right\}$ is a complete orthonormal basis, $\lambda_{i} \geq 0$, and

$$
\sum_{i=1}^{\infty} \lambda_{i}<\infty
$$

Such a stochastic process is called a $Q$ Wiener process.
Recall that such Wiener processes are always of the form

$$
\sum_{k=1}^{\infty} \psi_{k}(t) J g_{k}
$$

where $J$ is a Hilbert Schmidt operator from a suitable space $U_{0}$ to $U$ and the $\psi_{k}$ are real independent Wiener processes described earlier. This follows from Theorem 64.5.4 where you let $U_{0} \subseteq U$ be such that for $J$ the inclusion map, $J e_{k}=\sqrt{\lambda_{k}} e_{k}$ for $Q=\sum_{k} \lambda_{k} e_{k} \otimes e_{k}$, the $e_{k}$ an orthonormal set in $U$. Thus

$$
\begin{aligned}
(Q x, y) & =\left(\sum_{k} \lambda_{k} e_{k}\left(x, e_{k}\right), y\right)=\sum_{k}\left(x, \sqrt{\lambda_{k}} e_{k}\right)\left(y, \sqrt{\lambda_{k}} e_{k}\right) \\
& =\sum_{k}\left(x, J e_{k}\right)\left(y, J e_{k}\right)=\sum_{k}\left(J^{*} x, e_{k}\right)\left(J^{*} y, e_{k}\right)=\left(J^{*} x, J^{*} y\right)=\left(J J^{*} x, y\right)
\end{aligned}
$$

so it follows that $Q=J J^{*}$. Of course in finite dimensions, there is no issue because the identity map is Hilbert Schmidt.

Recall the definition of $\mathscr{L}_{2}(U, H) \equiv \mathscr{L}_{2}$ the space of Hilbert Schmidt operators. $\Psi \in$ $\mathscr{L}_{2}(U, H)$ means $\Psi$ has the property that for some (equivalently all) orthonormal basis of $U\left\{e_{k}\right\}$, it follows

$$
\sum_{k=1}^{\infty}\left\|\Psi\left(e_{k}\right)\right\|^{2}<\infty
$$

and the inner product for two of these, $\Psi, \Phi$ is given by

$$
(\Psi, \Phi)_{\mathscr{L}_{2}} \equiv \sum_{k}\left(\Psi\left(e_{k}\right), \Phi\left(e_{k}\right)\right)
$$

Then for such a Hilbert Schmidt operator, the norm in $\mathscr{L}_{2}$ is given by

$$
\left(\sum_{k=1}^{\infty}\left\|\Psi\left(e_{k}\right)\right\|^{2}\right)^{1 / 2} \equiv\|\Psi\|_{\mathscr{L}_{2}}
$$

Note this is the same as

$$
\begin{equation*}
\left(\sum_{k=1}^{\infty} \sum_{j=1}^{\infty}\left(\Psi\left(e_{k}\right), f_{j}\right)^{2}\right)^{1 / 2} \tag{65.1.1}
\end{equation*}
$$

where $\left\{f_{j}\right\}$ is an orthonormal basis for $H$. This is the analog of the Frobenius norm for matrices obtained as

$$
\operatorname{trace}\left(M M^{*}\right)^{1 / 2}=\left(\sum_{i}\left(M M^{*}\right)_{i i}\right)^{1 / 2}=\left(\sum_{i, j} M_{i j}^{2}\right)^{1 / 2}
$$

Also 65.1 .1 shows right away that if $\Psi \in \mathscr{L}_{2}(U, H)$, then

$$
\begin{aligned}
\|\Psi\|_{\mathscr{L}_{2}(U, H)}^{2} & =\sum_{k=1}^{\infty} \sum_{j=1}^{\infty}\left(\Psi e_{k}, f_{j}\right)_{H}^{2} \\
& =\sum_{k=1}^{\infty} \sum_{j=1}^{\infty}\left(e_{k}, \Psi^{*} f_{j}\right)_{U}^{2}=\left\|\Psi^{*}\right\|_{\mathscr{L}_{2}(H, U)}^{2}
\end{aligned}
$$

and that $\Psi$ and $\Psi^{*}$ are Hilbert Schmidt together.
The filtration will continue to be denoted by $\mathscr{F}_{t}$. It will be defined as the following normal filtration in which

$$
\overline{\sigma(W(s)-W(r): 0 \leq r<s \leq u)}
$$

is the completion of $\sigma(W(s)-W(r): 0 \leq r<s \leq u)$.

$$
\begin{equation*}
\mathscr{F}_{t} \equiv \cap_{u>t} \overline{\sigma(W(s)-W(r): 0 \leq r<s \leq u)} \tag{65.1.2}
\end{equation*}
$$

and $\sigma(W(s)-W(r): 0 \leq r<s \leq u)$ denotes the $\sigma$ algebra of all sets of the form

$$
(W(s)-W(r))^{-1}(\text { Borel })
$$

where $0 \leq r<s \leq u$.

Definition 65.1.2 Let $\Phi(t) \in \mathscr{L}(U, H)$ be constant on each interval, $\left(t_{m}, t_{m+1}\right]$ determined by a partition of $[a, T], 0 \leq a=t_{0}<t_{1} \cdots<t_{n}=T$. Then $\Phi(t)$ is said to be elementary if also $\Phi\left(t_{m}\right)$ is $\mathscr{F}_{t_{m}}$ measurable and $\Phi\left(t_{m}\right)$ equals a sum of the form

$$
\Phi\left(t_{m}\right)(\omega)=\sum_{j=1}^{m} \Phi_{j} \mathscr{X}_{A_{j}}
$$

where $\Phi_{j} \in \mathscr{L}(U, H), A_{j} \in \mathscr{F}_{t_{m}}$. What does the measurability assertion mean? It means that if $O$ is an open (Borel) set in the topological space $\mathscr{L}(U, H), \Phi\left(t_{m}\right)^{-1}(O) \in \mathscr{F}_{t_{m}}$. Thus an elementary function is of the form

$$
\Phi(t)=\sum_{k=0}^{n-1} \Phi\left(t_{k}\right) \mathscr{X}_{\left(t_{k}, t_{k+1}\right]}(t)
$$

Then for $\Phi$ elementary, the stochastic integral is defined by

$$
\int_{a}^{t} \Phi(s) d W(s) \equiv \sum_{k=0}^{n-1} \Phi\left(t_{k}\right)\left(W\left(t \wedge t_{k+1}\right)-W\left(t \wedge t_{k}\right)\right)
$$

It is also sometimes denoted by $\Phi \cdot W(t)$.
The above definition is the same as saying that for $t \in\left(t_{m}, t_{m+1}\right]$,

$$
\begin{align*}
\int_{a}^{t} \Phi(s) d W(s)= & \sum_{k=0}^{m-1} \Phi\left(t_{k}\right)\left(W\left(t_{k+1}\right)-W\left(t_{k}\right)\right) \\
& +\Phi\left(t_{m}\right)\left(W(t)-W\left(t_{m}\right)\right) \tag{65.1.3}
\end{align*}
$$

The following lemma will be useful.
Lemma 65.1.3 Let $f, g \in L^{2}(\Omega ; H)$ and suppose $g$ is $\mathscr{G}$ measurable and $f$ is $\mathscr{F}$ measurable where $\mathscr{F} \supseteq \mathscr{G}$. Then

$$
E\left((f, g)_{H} \mid \mathscr{G}\right)=(E(f \mid \mathscr{G}), g)_{H} \text { a.e. }
$$

Similarly if $\Phi$ is $\mathscr{G}$ measurable as a map into $\mathscr{L}(U, H)$ with

$$
\int_{\Omega}\|\Phi\|^{2} d P<\infty
$$

and $f$ is $\mathscr{F}$ measurable as a map into $U$ such that $f \in L^{2}(\Omega ; H)$, then

$$
E(\Phi f \mid \mathscr{G})=\Phi E(f \mid \mathscr{G})
$$

Proof: Let $A \in \mathscr{G}$. Let $\left\{g_{n}\right\}$ be a sequence of simple functions, measurable with respect to $\mathscr{G}$,

$$
g_{n}(\omega) \equiv \sum_{k=1}^{m_{n}} a_{k}^{n} \mathscr{X}_{E_{k}^{n}}(\omega)
$$

which converges in $L^{2}(\Omega ; H)$ and pointwise to $g$.Then

$$
\begin{gathered}
\int_{A}(E(f \mid \mathscr{G}), g)_{H} d P=\lim _{n \rightarrow \infty} \int_{A}\left(E(f \mid \mathscr{G}), g_{n}\right)_{H} d P \\
=\lim _{n \rightarrow \infty} \int_{A} \sum_{k=1}^{m_{n}}\left(E(f \mid \mathscr{G}), a_{k}^{n} \mathscr{X}_{E_{k}^{n}}\right)_{H} d P=\lim _{n \rightarrow \infty} \int_{A} \sum_{k=1}^{m_{n}} E\left(\left(f, a_{k}^{n}\right)_{H} \mid \mathscr{G}\right) \mathscr{X}_{E_{k}^{n}} d P \\
=\lim _{n \rightarrow \infty} \int_{A} \sum_{k=1}^{m_{n}} E\left(\left(f, a_{k}^{n} \mathscr{X}_{E_{k}^{n}}\right)_{H} \mid \mathscr{G}\right) d P=\lim _{n \rightarrow \infty} \int_{A} E\left(\left(f, \sum_{k=1}^{m_{n}} a_{k}^{n} \mathscr{X}_{E_{k}^{n}}\right)_{H} \mid \mathscr{G}\right) d P \\
=\lim _{n \rightarrow \infty} \int_{A} E\left(\left(f, g_{n}\right)_{H} \mid \mathscr{G}\right) d P=\lim _{n \rightarrow \infty} \int_{A}\left(f, g_{n}\right)_{H} d P=\int_{A}(f, g)_{H} d P
\end{gathered}
$$

which shows

$$
(E(f \mid \mathscr{G}), g)_{H}=E\left((f, g)_{H} \mid \mathscr{G}\right)
$$

as claimed.
Consider the other claim. Let

$$
\Phi_{n}(\omega)=\sum_{k=1}^{m_{n}} \Phi_{k}^{n} \mathscr{X}_{E_{k}^{n}}(\omega), E_{k}^{n} \in \mathscr{G}
$$

where $\Phi_{k}^{n} \in \mathscr{L}(U, H)$ be such that $\Phi_{n}$ converges to $\Phi$ pointwise in $\mathscr{L}(U, H)$ and also

$$
\int_{\Omega}\left\|\Phi_{n}-\Phi\right\|^{2} d P \rightarrow 0
$$

Then letting $A \in \mathscr{G}$ and using Corollary 21.2.6 as needed,

$$
\begin{aligned}
& \int_{A} \Phi E(f \mid \mathscr{G}) d P \\
= & \lim _{n \rightarrow \infty} \int_{A} \Phi_{n} E(f \mid \mathscr{G}) d P=\lim _{n \rightarrow \infty} \int_{A} \sum_{k=1}^{m_{n}} \Phi_{k}^{n} E(f \mid \mathscr{G}) \mathscr{X}_{E_{k}^{n}} d P \\
= & \lim _{n \rightarrow \infty} \sum_{k=1}^{m_{n}} \Phi_{k}^{n} \int_{A} E(f \mid \mathscr{G}) \mathscr{X}_{E_{k}^{n}} d P=\lim _{n \rightarrow \infty} \sum_{k=1}^{m_{n}} \Phi_{k}^{n} \int_{A} E\left(\mathscr{X}_{E_{k}^{n}} f \mid \mathscr{G}\right) d P \\
= & \lim _{n \rightarrow \infty} \sum_{k=1}^{m_{n}} \Phi_{k}^{n} \int_{A} \mathscr{X}_{E_{k}^{n}} f d P=\lim _{n \rightarrow \infty} \int_{A} \sum_{k=1}^{m_{n}} \Phi_{k}^{n} \mathscr{X}_{E_{k}^{n}} f d P \\
= & \lim _{n \rightarrow \infty} \int_{A} \Phi_{n} f d P=\lim _{n \rightarrow \infty} \int_{A} \Phi f d P \equiv \int_{A} E(\Phi f \mid \mathscr{G}) d P
\end{aligned}
$$

Since $A \in \mathscr{G}$ is arbitrary, this proves the lemma.
Lemma 65.1.4 Let $J: U_{0} \rightarrow U$ be a Hilbert Schmidt operator and let $W(t)$ be the resulting Wiener process

$$
W(t)=\sum_{k=1}^{\infty} \psi_{k}(t) J g_{k}
$$

where $\left\{g_{k}\right\}$ is an orthonormal basis for $U_{0}$. Let $f \in H$. Then considering one of the terms of the integral defined above,

$$
\begin{aligned}
& E\left(\left(\Phi\left(t_{k}\right)\left(W\left(t \wedge t_{k+1}\right)-W\left(t \wedge t_{k}\right)\right), f\right)^{2}\right) \\
= & E\left(\left(\left(W\left(t \wedge t_{k+1}\right)-W\left(t \wedge t_{k}\right)\right), \Phi\left(t_{k}\right)^{*} f\right)^{2}\right) \\
= & \left(t \wedge t_{k+1}-t \wedge t_{k}\right) E\left(\left\|J^{*} \Phi\left(t_{k}\right)^{*} f\right\|_{U_{0}}^{2}\right) .
\end{aligned}
$$

Proof: For simplicity, write $\Delta W_{k}(t)$ for $W\left(t \wedge t_{k+1}\right)-W\left(t \wedge t_{k}\right)$ and $\Delta_{k}(t)=\left(t \wedge t_{k+1}\right)-$ $\left(t \wedge t_{k}\right)$. If $\Phi\left(t_{k}\right)$ were a constant, then the result would follow right away from the fact that $W(t)$ is a Wiener process. Therefore, suppose for disjoint $E_{i}$,

$$
\Phi\left(t_{k}\right)(\omega)=\sum_{i=1}^{m} \Phi_{i} \mathscr{X}_{E_{i}}(\omega)
$$

where $\Phi_{i} \in \mathscr{L}(U, H)$ and $E_{i} \in \mathscr{F}_{t}$. Then, since the $E_{i}$ are disjoint,

$$
\begin{gathered}
E\left(\left(\Phi\left(t_{k}\right)\left(W\left(t \wedge t_{k+1}\right)-W\left(t \wedge t_{k}\right)\right), f\right)^{2}\right) \\
=\sum_{i=1}^{m} E\left(\left(\left(\Delta_{k} W(t)\right), \Phi_{i}^{*} f \mathscr{X}_{E_{i}}\right)^{2}\right)=\sum_{i=1}^{m} \int_{\Omega}\left(\mathscr{X}_{E_{i}}\left(\left(\Delta_{k} W(t)\right), \Phi_{i}^{*} f\right)^{2}\right) d P
\end{gathered}
$$

Each $E_{i}$ is $\mathscr{F}_{t_{k}}$ measurable. By Lemma 64.4.2, and the properties of the Wiener process, this equals

$$
\sum_{i=1}^{m} P\left(E_{i}\right) \int_{\Omega}\left(\left(\left(\Delta_{k} W(t)\right), \Phi_{i}^{*} f\right)^{2}\right) d P=\sum_{i=1}^{m} P\left(E_{i}\right) \Delta_{k} t\left(Q \Phi_{i}^{*} f, \Phi_{i}^{*} f\right)_{U}
$$

where $Q=J J^{*}$. Then the above reduces to

$$
\left(t \wedge t_{k+1}-t \wedge t_{k}\right) E\left(\left\|J^{*} \Phi\left(t_{k}\right)^{*} f\right\|_{U_{0}}^{2}\right)
$$

Now here is a major result on the integral of elementary functions. The last assertion in the following proposition is called the Ito isometry.

Proposition 65.1.5 Let $\Phi(t)$ be an elementary process as defined in Definition 65.1.2 and let $W(t)$ be a Wiener process.

$$
W(t)=\sum_{k=1}^{\infty} \psi_{k}(t) J g_{k}
$$

where $J: U_{0} \rightarrow U$ is Hilbert Schmidt and the $\psi_{k}$ are real independent Wiener processes as described above.

$$
\underset{\left\{g_{k}\right\}}{U_{0}} \xrightarrow{J} \underset{W(t)}{U} \xrightarrow{\Phi} H
$$

Then $\int_{a}^{t} \Phi(s) d W$ is a continuous square integrable $H$ valued martingale with respect to the $\sigma$ algebras of 65.1.2 on $[0, T]$ and

$$
E\left(\left|\int_{a}^{t} \Phi(s) d W\right|_{H}^{2}\right)=\int_{a}^{t} E\left(\|\Phi \circ J\|_{\mathscr{L}_{2}\left(U_{0}, H\right)}^{2}\right) d s
$$

Proof: Start with the left side. Denote by $\Delta_{k} W(t) \equiv W\left(t \wedge t_{k+1}\right)-W\left(t \wedge t_{k}\right)$. Then

$$
E\left(\left|\int_{a}^{t} \Phi(s) d W\right|_{H}^{2}\right)=E\left(\left|\sum_{k=0}^{n-1} \Phi\left(t_{k}\right) \Delta_{k} W(t)\right|_{H}^{2}\right)
$$

Consider a mixed term for $j<k$. Using Lemma 65.1.3 and the fact that $W(t)$ is a martingale,

$$
\begin{aligned}
& E\left(\left(\Phi\left(t_{k}\right) \Delta_{k} W(t), \Phi\left(t_{j}\right) \Delta_{j} W(t)\right)_{H}\right) \\
= & E\left(E\left(\left(\Phi\left(t_{k}\right) \Delta_{k} W(t), \Phi\left(t_{j}\right) \Delta_{j} W(t)\right)_{H}\right) \mid \mathscr{F}_{t_{k}}\right) \\
= & E\left(\left(\Phi\left(t_{j}\right) \Delta_{j} W(t), E\left(\Phi\left(t_{k}\right) \Delta_{k} W(t) \mid \mathscr{F}_{t_{k}}\right)\right)\right) \\
= & E\left(\left(\Phi\left(t_{j}\right) \Delta_{j} W(t), \Phi\left(t_{k}\right) E\left(\Delta_{k} W(t) \mid \mathscr{F}_{t_{k}}\right)\right)\right) \\
= & E\left(\left(\Phi\left(t_{j}\right) \Delta_{j} W(t), \Phi\left(t_{k}\right) 0\right)\right)=0 .
\end{aligned}
$$

Therefore, from Lemma 65.1.4, and letting $\left\{f_{j}\right\}$ be an orthonormal basis for $H$, it follows that since the mixed terms disappeared,

$$
\begin{gathered}
E\left(\left|\int_{a}^{t} \Phi(s) d W\right|_{H}^{2}\right)=\sum_{k=0}^{n-1} E\left(\left(\Phi\left(t_{k}\right) \Delta_{k} W(t), \Phi\left(t_{k}\right) \Delta_{k} W(t)\right)\right) \\
=\sum_{k=0}^{n-1} E\left(\sum_{j=1}^{\infty}\left(\Phi\left(t_{k}\right) \Delta_{k} W(t), f_{j}\right)^{2}\right)=\sum_{k=0}^{n-1} \sum_{j=1}^{\infty} E\left(\left(\Phi\left(t_{k}\right) \Delta_{k} W(t), f_{j}\right)^{2}\right) \\
=\sum_{k=0}^{n-1} \sum_{j=1}^{\infty}\left(t \wedge t_{k+1}-t \wedge t_{k}\right) E\left(\left\|J^{*} \Phi\left(t_{k}\right)^{*} f_{j}\right\|_{U_{0}}^{2}\right) \\
=\sum_{k=0}^{n-1}\left(t \wedge t_{k+1}-t \wedge t_{k}\right) E\left(\left\|J^{*} \Phi\left(t_{k}\right)^{*}\right\|_{\mathscr{L}_{2}\left(H, U_{0}\right)}^{2}\right) \\
=\sum_{k=0}^{n-1}\left(t \wedge t_{k+1}-t \wedge t_{k}\right) E\left(\left\|\Phi\left(t_{k}\right) J\right\|_{\mathscr{L}_{2}\left(U_{0}, H\right)}^{2}\right) \\
=\int_{a}^{t} E\left(\|\Phi \circ J\|_{\mathscr{L}_{2}\left(U_{0}, H\right)}^{2}\right) d s
\end{gathered}
$$

It is obvious that $\int_{a}^{t} \Phi(s) d W$ is a continuous square integrable martingale from the definition, because it is just a finite sum of such things.

Of course this is a version of the Ito isometry. The presence of the $J$ is troublesome but it is hidden in the definition of $W$ on the left side of the conclusion of the proposition. In finite dimensions one could just let $J=I$ and this fussy detail would not be there to cause confusion. The next task is to generalize the above integral to a more general class of functions and obtain a process which is not explicitly dependent on $J$.

### 65.2 Different Definition Of Elementary Functions

What if elementary functions had been defined in terms of $\mathscr{X}_{\left[t_{k}, t_{k+1}\right)}$ ? That is, what if the elementary functions had been of the form

$$
\Phi(t)=\sum_{k=0}^{n-1} \Phi\left(t_{k}\right) \mathscr{X}_{\left[t_{k}, t_{k+1}\right)}(t) ?
$$

Would anything change? If you go over the arguments given, it is clear that nothing would change at all. Furthermore, this elementary function equals the one described above off a finite set of mesh points so the convergence properties in $L^{2}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$, which will be important in what follows are exactly the same. Thus it does not matter whether we give elementary functions in this form or in the form described above. However, some arguments given later about localization depend on it being in the earlier form.

### 65.3 Approximating With Elementary Functions

Here is a really surprising result about approximating with step functions which is due to Doob. See [78] which is where I found this lemma. This is based on continuity of translation in the $L^{p}(\mathbb{R} ; E)$.

Lemma 65.3.1 Let $\Phi:[0, T] \times \Omega \rightarrow E$, be $\mathscr{B}([0, T]) \times \mathscr{F}$ measurable and suppose

$$
\Phi \in K \equiv L^{p}([0, T] \times \Omega ; E), p \geq 1
$$

Then there exists a sequence of nested partitions, $\mathscr{P}_{k} \subseteq \mathscr{P}_{k+1}$,

$$
\mathscr{P}_{k} \equiv\left\{t_{0}^{k}, \cdots, t_{m_{k}}^{k}\right\}
$$

such that the step functions given by

$$
\begin{aligned}
\Phi_{k}^{r}(t) & \equiv \sum_{j=1}^{m_{k}} \Phi\left(t_{j}^{k}\right) \mathscr{X}_{\left[t_{j-1}^{k}, t_{j}^{k}\right)}(t) \\
\Phi_{k}^{l}(t) & \equiv \sum_{j=1}^{m_{k}} \Phi\left(t_{j-1}^{k}\right) \mathscr{X}_{\left[t_{j-1}^{k}, t_{j}^{k}\right)}(t)
\end{aligned}
$$

both converge to $\Phi$ in $K$ as $k \rightarrow \infty$ and

$$
\lim _{k \rightarrow \infty} \max \left\{\left|t_{j}^{k}-t_{j+1}^{k}\right|: j \in\left\{0, \cdots, m_{k}\right\}\right\}=0
$$

Also, each $\Phi\left(t_{j}^{k}\right), \Phi\left(t_{j-1}^{k}\right)$ is in $L^{p}(\Omega ; E)$. One can also assume that $\Phi(0)=0$. The mesh points $\left\{t_{j}^{k}\right\}_{j=0}^{m_{k}}$ can be chosen to miss a given set of measure zero. In addition to this, we can assume that

$$
\left|t_{j}^{k}-t_{j-1}^{k}\right|=2^{-n_{k}}
$$

except for the case where $j=1$ or $j=m_{n_{k}}$ when this is so, you could have $\left|t_{j}^{k}-t_{j-1}^{k}\right|<2^{-n_{k}}$.

Note that it would make no difference in terms of the conclusion of this lemma if you defined

$$
\Phi_{k}^{l}(t) \equiv \sum_{j=1}^{m_{k}} \Phi\left(t_{j-1}^{k}\right) \mathscr{X}_{\left(t_{j-1}^{k}, t_{j}^{k}\right]}(t)
$$

because the modified function equals the one given above off a countable subset of $[0, T]$, the union of the mesh points. One could change $\Phi_{k}^{r}$ similarly with no change in the conclusion.

Proof: For $t \in \mathbb{R}$ let $\gamma_{n}(t) \equiv k / 2^{n}, \delta_{n}(t) \equiv(k+1) / 2^{n}$, where $t \in\left(k / 2^{n},(k+1) / 2^{n}\right]$, and $2^{-n}<T / 4$. Also suppose $\Phi$ is defined to equal 0 on $[0, T]^{C} \times \Omega$. There exists a set of measure zero $N$ such that for $\omega \notin N, t \rightarrow\|\Phi(t, \omega)\|$ is in $L^{p}(\mathbb{R})$. Therefore by continuity of translation, as $n \rightarrow \infty$ it follows that for $\omega \notin N$, and $t \in[0, T]$,

$$
\int_{\mathbb{R}}\left\|\Phi\left(\gamma_{n}(t)+s\right)-\Phi(t+s)\right\|_{E}^{p} d s \rightarrow 0
$$

The above is dominated by

$$
\begin{aligned}
& \int_{\mathbb{R}} 2^{p-1}\left(\|\Phi(s)\|^{p}+\|\Phi(s)\|^{p}\right) \mathscr{X}_{[-2 T, 2 T]}(s) d s \\
= & \int_{-2 T}^{2 T} 2^{p-1}\left(\|\Phi(s)\|^{p}+\|\Phi(s)\|^{p}\right) d s<\infty
\end{aligned}
$$

Consider

$$
\int_{\Omega} \int_{-2 T}^{2 T}\left(\int_{\mathbb{R}}\left\|\Phi\left(\gamma_{n}(t)+s\right)-\Phi(t+s)\right\|_{E}^{p} d s\right) d t d P
$$

By the dominated convergence theorem, this converges to 0 as $n \rightarrow \infty$. This is because the integrand with respect to $\omega$ is dominated by

$$
\int_{-2 T}^{2 T}\left(\int_{\mathbb{R}} 2^{p-1}\left(\|\Phi(s)\|^{p}+\|\Phi(s)\|^{p}\right) \mathscr{X}_{[-2 T, 2 T]}(s) d s\right) d t
$$

and this is in $L^{1}(\Omega)$ by assumption that $\Phi \in K$. Now Fubini. This yields

$$
\int_{\Omega} \int_{\mathbb{R}} \int_{-2 T}^{2 T}\left\|\Phi\left(\gamma_{n}(t)+s\right)-\Phi(t+s)\right\|_{E}^{p} d t d s d P
$$

Change the variables on the inside.

$$
\int_{\Omega} \int_{\mathbb{R}} \int_{-2 T+s}^{2 T+s}\left\|\Phi\left(\gamma_{n}(t-s)+s\right)-\Phi(t)\right\|_{E}^{p} d t d s d P
$$

Now by definition, $\Phi(t)$ vanishes if $t \notin[0, T]$, thus the above reduces to

$$
\begin{aligned}
& \int_{\Omega} \int_{\mathbb{R}} \int_{0}^{T}\left\|\Phi\left(\gamma_{n}(t-s)+s\right)-\Phi(t)\right\|_{E}^{p} d t d s d P \\
& +\int_{\Omega} \int_{\mathbb{R}} \int_{-2 T+s}^{2 T+s} \mathscr{X}_{[0, T]^{c}}\left\|\Phi\left(\gamma_{n}(t-s)+s\right)\right\|_{E}^{p} d t d s d P
\end{aligned}
$$

$$
\begin{aligned}
= & \int_{\Omega} \int_{\mathbb{R}} \int_{0}^{T}\left\|\Phi\left(\gamma_{n}(t-s)+s\right)-\Phi(t)\right\|_{E}^{p} d t d s d P \\
& +\int_{\Omega} \int_{\mathbb{R}} \int_{-2 T+s}^{2 T+s} \mathscr{X}_{[0, T]}{ }^{C}\left\|\Phi\left(\gamma_{n}(t-s)+s\right)-\Phi(t)\right\|_{E}^{p} d t d s d P
\end{aligned}
$$

Also by definition, $\gamma_{n}(t-s)+s$ is within $2^{-n}$ of $t$ and so the integrand in the integral on the right equals 0 unless $t \in\left[-2^{-n}-T, T+2^{-n}\right] \subseteq[-2 T, 2 T]$. Thus the above reduces to

$$
\int_{\Omega} \int_{\mathbb{R}} \int_{-2 T}^{2 T}\left\|\Phi\left(\gamma_{n}(t-s)+s\right)-\Phi(t)\right\|_{E}^{p} d t d s d P
$$

Now Fubini again.

$$
\int_{\mathbb{R}} \int_{\Omega} \int_{-2 T}^{2 T}\left\|\Phi\left(\gamma_{n}(t-s)+s\right)-\Phi(t)\right\|_{E}^{p} d t d P d s
$$

This converges to 0 as $n \rightarrow \infty$ as was shown above. Therefore,

$$
\int_{0}^{T} \int_{\Omega} \int_{0}^{T}\left\|\Phi\left(\gamma_{n}(t-s)+s\right)-\Phi(t)\right\|_{E}^{p} d t d P d s
$$

also converges to 0 as $n \rightarrow \infty$. The only problem is that $\gamma_{n}(t-s)+s \geq t-2^{-n}$ and so $\gamma_{n}(t-s)+s$ could be less than 0 for $t \in\left[0,2^{-n}\right]$. Since this is an interval whose measure converges to 0 it follows

$$
\int_{0}^{T} \int_{\Omega} \int_{0}^{T}\left\|\Phi\left(\left(\gamma_{n}(t-s)+s\right)^{+}\right)-\Phi(t)\right\|_{E}^{p} d t d P d s
$$

converges to 0 as $n \rightarrow \infty$. Let

$$
m_{n}(s)=\int_{\Omega} \int_{0}^{T}\left\|\Phi\left(\left(\gamma_{n}(t-s)+s\right)^{+}\right)-\Phi(t)\right\|_{E}^{p} d t d P
$$

Then letting $\mu$ denote Lebesgue measure,

$$
\mu\left(\left[m_{n}(s)>\lambda\right]\right) \leq \frac{1}{\lambda} \int_{0}^{T} m_{n}(s) d s
$$

It follows there exists a subsequence $n_{k}$ such that

$$
\mu\left(\left[m_{n_{k}}(s)>\frac{1}{k}\right]\right)<2^{-k}
$$

Hence by the Borel Cantelli lemma, there exists a set of measure zero $N$ such that for $s \notin N$,

$$
m_{n_{k}}(s) \leq 1 / k
$$

for all $k$ sufficiently large. Pick such an $s$. Then consider $t \rightarrow \Phi\left(\left(\gamma_{n_{k}}(t-s)+s\right)^{+}\right)$. For $n_{k}, t \rightarrow\left(\gamma_{n_{k}}(t-s)+s\right)^{+}$has jumps at points of the form $0, s+l 2^{-n_{k}}$ where $l$ is an integer.

Thus $\mathscr{P}_{n_{k}}$ consists of points of $[0, T]$ which are of this form and these partitions are nested. Define $\Phi_{k}^{l}(0) \equiv 0, \Phi_{k}^{l}(t) \equiv \Phi\left(\left(\gamma_{n_{k}}(t-s)+s\right)^{+}\right)$. Now suppose $N_{1}$ is a set of measure zero. Can $s$ be chosen such that all jumps for all partitions occur off $N_{1}$ ? Let $(a, b)$ be an interval contained in $[0, T]$. Let $S_{j}$ be the points of $(a, b)$ which are translations of the measure zero set $N_{1}$ by $t_{j}^{l}$ for some $j$. Thus $S_{j}$ has measure 0 . Now pick $s \in(a, b) \backslash \cup_{j} S_{j}$.

It will be assumed that all these mesh points miss the set of all $t$ such that $\omega \rightarrow \Phi(t, \omega)$ is not in $L^{p}(\Omega ; E)$. To get the other sequence of step functions, the right step functions, just use a similar argument with $\delta_{n}$ in place of $\gamma_{n}$. Just apply the argument to a subsequence of $n_{k}$ so that the same $s$ can hold for both.

The following proposition says that elementary functions can be used to approximate progressively measurable functions under certain conditions.

Proposition 65.3.2 Let $\Phi \in L^{p}([0, T] \times \Omega, E)$, $p \geq 1$, be progressively measurable. Then there exists a sequence of elementary functions which converges to $\Phi$ in

$$
L^{p}([0, T] \times \Omega, E) .
$$

These elementary functions have values in $E_{0}$, a dense subset of $E$. If $\varepsilon_{n} \rightarrow 0$, and

$$
\Phi_{n}(t)=\sum_{k=1}^{m_{n}} \Psi_{k}^{n} \mathscr{X}_{\left(t_{k}, t_{k+1}\right]}(t)
$$

$\Psi_{k}^{n}$ having values in $E_{0}$, it can be assumed that

$$
\begin{equation*}
\sum_{k=1}^{m_{n}}\left\|\Psi_{k}^{n}-\Phi\left(t_{k}\right)\right\|_{L^{p}(\Omega ; E)}<\varepsilon_{n} \tag{65.3.4}
\end{equation*}
$$

Proof: By Lemma 65.3.1 there exists a sequence of step functions

$$
\Phi_{k}^{l}(t)=\sum_{j=1}^{m_{k}} \Phi\left(t_{j-1}^{k}\right) \mathscr{X}_{\left(t_{j-1}^{k}, t_{j}^{k}\right]}(t)
$$

which converges to $\Phi$ in $L^{p}([0, T] \times \Omega, E)$ where at the left endpoint $\Phi(0)$ can be modified as described above. Now each $\Phi\left(t_{j-1}^{k}\right)$ is in $L^{p}(\Omega, E)$ and is $\mathscr{F}\left(t_{j-1}^{k}\right)$ measurable and so it can be approximated as closely as desired in $L^{p}(\Omega)$ with a simple function

$$
s\left(t_{j-1}^{k}\right) \equiv \sum_{i=1}^{m_{k}} c_{i}^{j} \mathscr{X}_{F_{i}}(\omega), F_{i} \in \mathscr{F}\left(t_{j-1}^{k}\right)
$$

Furthermore, by density of $E_{0}$ in $E$, it can be assumed each $c_{i}^{j} \in E_{0}$ and the condition 65.3.4 holds. Replacing each $\Phi\left(t_{j-1}^{k}\right)$ with $s\left(t_{j-1}^{k}\right)$, the result is an elementary function which approximates $\Phi_{k}^{l}$.

Of course everything in the above holds with obvious modifications replacing $[0, T]$ with $[a, T]$ where $a<T$.

Here is another interesting proposition about the time integral being adapted.

Proposition 65.3.3 Suppose $f \geq 0$ is progressively measureable and $\mathscr{F}_{t}$ is a filtration. Then

$$
\omega \rightarrow \int_{a}^{t} f(s, \omega) d s
$$

is $\mathscr{F}_{t}$ adapted.
Proof: This follows right away from the fact $f$ is $\mathscr{B}([a, t]) \times \mathscr{F}_{t}$ measurable. This is just product measure and so the integral from $a$ to $t$ is $\mathscr{F}_{t}$ measurable. See also Proposition 62.3.5.

### 65.4 Some Hilbert Space Theory

Recall the following definition which makes $L U$ into a Hilbert space where $L \in \mathscr{L}(U, H)$.
Definition 65.4.1 Let $L \in \mathscr{L}(U, H)$, the bounded linear maps from $U$ to $H$ for $U, H$ Hilbert spaces. For $y \in L(U)$, let $L^{-1} y$ denote the unique vector in

$$
\{x: L x=y\} \equiv M_{y}
$$

which is closest in $U$ to 0 .


Note this is a good definition because $\{x: L x=y\}$ is closed thanks to the continuity of $L$ and it is obviously convex. Thus Theorem 19.1.8 applies. With this definition define an inner product on $L(U)$ as follows. For $y, z \in L(U)$,

$$
(y, z)_{L(U)} \equiv\left(L^{-1} y, L^{-1} z\right)_{U}
$$

Thus it is obvious that $L^{-1}: L U \rightarrow U$ is continuous. The notation is abominable because $L^{-1}(y)$ is the normal notation for $M_{y}$.

With this definition, here is one of the main results. It is Theorem 19.2.3 proved earlier.
Theorem 65.4.2 Let $U, H$ be Hilbert spaces and let $L \in \mathscr{L}(U, H)$. Then Definition 65.4.1 makes $L(U)$ into a Hilbert space. Also $L: U \rightarrow L(U)$ is continuous and $L^{-1}: L(U) \rightarrow U$ is continuous. Furthermore there is a constant $C$ independent of $x \in U$ such that

$$
\begin{equation*}
\|L\|_{\mathscr{L}(U, H)}\|L x\|_{L(U)} \geq\|L x\|_{H} \tag{65.4.5}
\end{equation*}
$$

If $U$ is separable, so is $L(U)$. Also $\left(L^{-1}(y), x\right)=0$ for all $x \in \operatorname{ker}(L)$, and $L^{-1}: L(U) \rightarrow U$ is linear. Also, in case that $L$ is one to one, both $L$ and $L^{-1}$ preserve norms.

Let $U$ be a separable Hilbert space and let $Q$ be a positive self adjoint operator. Then consider

$$
J: Q^{1 / 2} U \rightarrow U_{1}
$$

a one to one Hilbert Schmidt operator, where $U_{1}$ is a separable real Hilbert space. First of all, there is the obvious question whether there are any examples.

Lemma 65.4.3 Let $A \in \mathscr{L}(U, U)$ be a bounded linear transformation defined on $U$ a separable real Hilbert space. There exists a one to one Hilbert Schmidt operator $J$ : $A U \rightarrow U_{1}$ where $U_{1}$ is a separable real Hilbert space. In fact you can take $U_{1}=U$.

Proof: Let $\alpha_{k}>0$ and $\sum_{k=1}^{\infty} \alpha_{k}^{2}<\infty$. Then let $\left\{g_{k}\right\}_{k=1}^{L}$ be an orthonormal basis for $A U$, the inner product and norm given in Definition 65.4.1 above, and let

$$
J x \equiv \sum_{k=1}^{L}\left(x, g_{k}\right)_{A U} \alpha_{k} g_{k}
$$

Then it is clear that $J \in \mathscr{L}(A U, U)$. This is because,

$$
\begin{aligned}
\|J x\|_{U} & \leq \sum_{k=1}^{L}\left|\left(x, g_{k}\right)_{A U}\right| \alpha_{k}| | g_{k} \|_{U} \\
& \leq C \sum_{k=1}^{L}\left|\left(x, g_{k}\right)_{A U}\right| \alpha_{k} \mid \overbrace{g_{k} \|_{A U}}^{=1} \\
& \leq C\left(\sum_{k=1}^{L}\left|\left(x, g_{k}\right)_{A U}\right|^{2}\right)^{1 / 2}\left(\sum_{k=1}^{L} \alpha_{k}^{2}\right)^{1 / 2} \\
& =C\left(\sum_{k=1}^{L} \alpha_{k}^{2}\right)^{1 / 2}\|x\|_{A U}
\end{aligned}
$$

Also, from the definition, $J g_{j}=\alpha_{j} g_{j}$. Say $g_{j}=A f_{j}$ where $f_{j} \in U$ and $1=\left\|g_{j}\right\|_{A U}=$ $\left\|f_{j}\right\|_{U}$. Since $A$ is continuous,

$$
\left\|g_{j}\right\|_{U}=\left\|A f_{j}\right\|_{U} \leq\|A\|\left\|f_{j}\right\|_{U}=\|A\|\left\|g_{j}\right\|_{A U}=\|A\| \equiv C^{1 / 2}
$$

Thus

$$
\sum_{j=1}^{L}\left\|J g_{j}\right\|_{U}^{2}=\sum_{j=1}^{L} \alpha_{j}^{2}\left\|g_{j}\right\|_{U}^{2} \leq C \sum_{j=1}^{L} \alpha_{j}^{2}<\infty
$$

and so $J$ is also a Hilbert Schmidt operator which maps $A U$ to $U$. It is clear that $J$ is one to one because each $\alpha_{k}>0$. If $A U$ is finite dimensional, $L<\infty$ and so the above sum is finite.

Definition 65.4.4 Let $U_{1}, U, H$ be real separable Hilbert spaces and let $Q$ be a nonnegative self adjoint operator, $Q \in \mathscr{L}(U, U)$. Let $Q^{1 / 2} U$ be the Hilbert space described in Definition 65.4.1. Let $J$ be a one to one Hilbert Schmidt map from $Q^{1 / 2} U$ to $U_{1}$.

$$
U_{1} \stackrel{J}{\leftarrow} Q^{1 / 2} U \xrightarrow{\Phi} H
$$

Then denote by $\mathscr{L}\left(U_{1}, H\right)_{0}$ the space of restrictions of elements of $\mathscr{L}\left(U_{1}, H\right)$ to the Hilbert space $J Q^{1 / 2} U \subseteq U_{1}$.

Here is a diagram to keep this straight.


Lemma 65.4.5 In the context of the above definition, $\mathscr{L}\left(U_{1}, H\right)_{0}$ is dense in

$$
\mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)
$$

the Hilbert Schmidt operators from $J Q^{1 / 2} U$ to $H$. That is, if $f \in \mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)$, there exists

$$
g \in \mathscr{L}\left(U_{1}, H\right)_{0},\|g-f\|_{\mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)}<\varepsilon
$$

Proof: The operator $J J^{*} \equiv Q_{1}: U_{1} \rightarrow U_{1}$ is self adjoint and nonnegative. It is also compact because $J$ is Hilbert Schmidt. Therefore, by Theorem 21.3.9 on Page 663,

$$
Q_{1}=\sum_{k=1}^{L} \lambda_{k} e_{k} \otimes e_{k}
$$

where the $\lambda_{k}$ are decreasing and positive, the $\left\{e_{k}\right\}$ are an orthonormal basis for $U_{1}$, and $\lambda_{L}$ is the last positive $\lambda_{j}$. (This is a lot like the singular value matrix in linear algebra.) Thus also

$$
Q_{1} e_{k}=\lambda_{k} e_{k}
$$

If the $\lambda_{k}$ are all positive, then $L \equiv \infty$. Then for $k \leq L$ if $L<\infty, k<\infty$ otherwise,

$$
\left(\frac{J^{*} e_{k}}{\sqrt{\lambda_{k}}}, \frac{J^{*} e_{j}}{\sqrt{\lambda_{j}}}\right)_{Q^{1 / 2}(U)}=\left(\frac{J J^{*} e_{k}}{\sqrt{\lambda_{k}}}, \frac{e_{j}}{\sqrt{\lambda_{j}}}\right)_{U_{1}}=\left(\frac{\lambda_{k} e_{k}}{\sqrt{\lambda_{k}}}, \frac{e_{j}}{\sqrt{\lambda_{j}}}\right)_{U_{1}}=\frac{\sqrt{\lambda_{k}}}{\sqrt{\lambda_{j}}} \delta_{k j}=\delta_{j k}
$$

Now in case $L<\infty, J\left(Q^{1 / 2}(U)\right) \subseteq \operatorname{span}\left(e_{1}, \cdots, e_{L}\right)$. Here is why. First note that $Q_{1}$ is one to one on span $\left(e_{1}, \cdots, e_{L}\right)$ and maps this space onto itself because $Q_{1}$ maps $e_{k}$ to a nonzero multiple of $e_{k}$. Hence its restriction to this subspace has an inverse which does the same. It also maps all of $U_{1}$ to span $\left(e_{1}, \cdots, e_{L}\right)$. This follows from the definition of $Q_{1}$ given in the above sum. For $x \in Q^{1 / 2}(U), J x \in U_{1}$ and so

$$
J J^{*} J x=Q_{1}(J x) \in \operatorname{span}\left(e_{1}, \cdots, e_{L}\right)
$$

Hence $J x \in Q_{1}^{-1}\left(\operatorname{span}\left(e_{1}, \cdots, e_{L}\right)\right) \in \operatorname{span}\left(e_{1}, \cdots, e_{L}\right)$. Recall that $J$ is one to one so there is only one element of $J^{-1} x$.

Then for $x \in Q^{1 / 2} U$,

$$
\begin{aligned}
& \sum_{j=1}^{L} \sqrt{\lambda_{j}} e_{j} \otimes_{Q^{1 / 2} U} \frac{J^{*} e_{j}}{\sqrt{\lambda_{j}}}(x)=\sum_{j=1}^{L} \sqrt{\lambda_{j}} e_{j}\left(\frac{J^{*} e_{j}}{\sqrt{\lambda_{j}}}, x\right)_{Q^{1 / 2}(U)} \\
= & \sum_{j=1}^{L} \sqrt{\lambda_{j}} e_{j}\left(\frac{e_{j}}{\sqrt{\lambda_{j}}}, J x\right)_{U_{1}}=\sum_{j=1}^{L} e_{j}\left(e_{j}, J x\right)_{U_{1}} \\
= & \sum_{j=1}^{\infty} e_{j}\left(e_{j}, J x\right)_{U_{1}}=J x .\left(J\left(Q^{1 / 2}(U)\right) \subseteq \operatorname{span}\left(e_{1}, \cdots, e_{L}\right) \text { if } L<\infty\right)
\end{aligned}
$$

Thus,

$$
J=\sum_{j=1}^{L} \sqrt{\lambda_{j}} e_{j} \otimes_{Q^{1 / 2} U} \frac{J^{*} e_{j}}{\sqrt{\lambda_{j}}}
$$

It follows that an orthonormal basis in $J Q^{1 / 2} U$ is $\left\{\frac{J J^{*} e_{j}}{\sqrt{\lambda_{j}}}\right\}_{j=1}^{L}$. This is because an orthonormal basis for $Q^{1 / 2} U$ is $\left\{\frac{J^{*} e_{k}}{\sqrt{\lambda_{k}}}\right\}$. Since $J$ is one to one, it preserves norms between $Q^{1 / 2} U$ and $J Q^{1 / 2} U$. Let $\Phi \in \mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)$. Then by the discussion of Hilbert Schmidt operators given earlier, in particular the demonstration that these operators are compact,

$$
\Phi=\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \phi_{i j} f_{i} \otimes_{J Q^{1 / 2} U} \frac{J J^{*} e_{j}}{\sqrt{\lambda_{j}}}
$$

where $\left\{f_{i}\right\}$ is an orthonormal basis for $H$. In fact, $\left\{f_{i} \otimes \frac{J J^{*} e_{j}}{\sqrt{\lambda_{j}}}\right\}_{i, j}$ is an orthonormal basis for $\mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)$ and $\sum_{i} \sum_{j} \phi_{i j}^{2}<\infty$, the $\phi_{i j}$ being the Fourier coefficients of $\Phi$. Then consider

$$
\begin{equation*}
\Phi_{n}=\sum_{i=1}^{n} \sum_{j=1}^{n} \phi_{i j} f_{i} \otimes_{J Q^{1 / 2} U} \frac{J J^{*} e_{j}}{\sqrt{\lambda_{j}}} \tag{65.4.6}
\end{equation*}
$$

Consider one of the finitely many operators in this sum. For $x \in J Q^{1 / 2} U$, since $J$ preserves norms,

$$
\begin{gathered}
f_{i} \otimes_{J Q^{1 / 2} U} \frac{J J^{*} e_{j}}{\sqrt{\lambda_{j}}}(x) \equiv f_{i}\left(\frac{J J^{*} e_{j}}{\sqrt{\lambda_{j}}}, x\right)_{J Q^{1 / 2} U}=f_{i}\left(\frac{J^{*} e_{j}}{\sqrt{\lambda_{j}}}, J^{-1} x\right)_{Q^{1 / 2} U} \\
=f_{i}\left(\frac{e_{j}}{\sqrt{\lambda_{j}}}, J J^{-1} x\right)_{U_{1}}=f_{i}\left(\frac{e_{j}}{\sqrt{\lambda_{j}}}, x\right)_{U_{1}} \equiv \Lambda_{i j}(x)
\end{gathered}
$$

Recall how, since $J$ is one to one, it preserves norms and inner products. Now $\Lambda_{i j}$ makes sense from the above formula for all $x \in U_{1}$ and is also a continuous linear map from $U_{1}$ to
$H$ because

$$
\left\|f_{i}\left(\frac{e_{j}}{\sqrt{\lambda_{j}}}, x\right)_{U_{1}}\right\|_{H} \leq\left\|f_{i}\right\|_{H} \frac{1}{\sqrt{\lambda_{j}}}\|x\|_{U_{1}}
$$

Thus each term in the finite sum of 65.4.6 is in $\mathscr{L}\left(U_{1}, H\right)_{0}$ and this proves the lemma.
It is interesting to note that $Q_{1}^{1 / 2} U_{1}=J\left(Q^{1 / 2}(U)\right)$.

$$
\sum_{j=1}^{L} \sqrt{\lambda_{j}} e_{j}\left(\frac{J^{*} e_{j}}{\sqrt{\lambda_{j}}}, x\right)_{Q^{1 / 2}(U)}=J x
$$

and $\left\{\frac{J^{*} e_{j}}{\sqrt{\lambda_{j}}}\right\}$ are an orthonormal set in $Q^{1 / 2}(U)$. Therefore, the sum of the squares of $\left(\frac{J^{*} e_{j}}{\sqrt{\lambda_{j}}}, x\right)_{Q^{1 / 2}(U)}$ is finite. Hence you can define $y \in U_{1}$ by

$$
y \equiv \sum_{j=1}^{L}\left(\frac{J^{*} e_{j}}{\sqrt{\lambda_{j}}}, x\right)_{Q^{1 / 2}(U)} e_{j}
$$

Also

$$
\begin{aligned}
\sum_{i=1}^{L} \sqrt{\lambda_{i}} e_{i} \otimes e_{i}(y) & =\sum_{i=1}^{L} \sqrt{\lambda_{i}} e_{i}\left(\frac{J^{*} e_{i}}{\sqrt{\lambda_{i}}}, x\right)_{Q^{1 / 2}(U)} \\
& =\sum_{i=1}^{L} e_{i}\left(e_{i}, J x\right)_{U_{1}}=J x
\end{aligned}
$$

Now you can show that $Q_{1}^{1 / 2}=\sum_{i=1}^{L} \sqrt{\lambda_{i}} e_{i} \otimes e_{i}$. You do this by showing that it works and commutes with every operator which commutes with $Q_{1}$. Thus $J x=Q_{1}^{1 / 2} y$. This shows that $J\left(Q^{1 / 2}(U)\right) \subseteq Q_{1}^{1 / 2}\left(U_{1}\right)$. However, you can also turn the inclusion around. Thus if you start with $y \in U_{1}$ and form

$$
Q_{1}^{1 / 2} y=\sum_{i=1}^{L} \sqrt{\lambda_{i}} e_{i} \otimes e_{i}(y)=\sum_{i=1}^{L} \sqrt{\lambda_{i}} e_{i}\left(y, e_{i}\right)
$$

then the $\left(y, e_{i}\right)_{U_{1}}^{2}$ has a finite sum because the $\left\{e_{i}\right\}$ are orthonormal. Thus you can form

$$
x \equiv \sum_{i=1}^{L}\left(y, e_{i}\right)_{U_{1}} \frac{J^{*} e_{i}}{\sqrt{\lambda_{i}}} \in Q^{1 / 2}(U)
$$

Then since the $\left\{\frac{J^{*} e_{j}}{\sqrt{\lambda_{j}}}\right\}$ are orthonormal,

$$
\begin{aligned}
J(x) & =\sum_{j=1}^{L} \sqrt{\lambda_{j}} e_{j}\left(\frac{J^{*} e_{j}}{\sqrt{\lambda_{j}}}, x\right)_{Q^{1 / 2}(U)}=\sum_{j=1}^{L} \sqrt{\lambda_{j}} e_{j}\left(y, e_{j}\right)_{U_{1}} \\
& =\sum_{j=1}^{L} \sqrt{\lambda_{j}} e_{j} \otimes e_{j}(y)=Q_{1}^{1 / 2}(y)
\end{aligned}
$$

It follows that $Q_{1}^{1 / 2}\left(U_{1}\right) \subseteq J\left(Q^{1 / 2}(U)\right)$.
One can also show that $W(t) \equiv \sum_{k=1}^{L} \psi_{k}(t) J g_{k}$ where the $\psi_{k}(t)$ are the real Wiener processes described earlier and $\left\{g_{k}\right\}$ is an orthonormal basis for $Q^{1 / 2}(U)$, is a $Q_{1}$ Wiener process. To see this, recall the above definition of a Wiener process in terms of Hilbert Schmidt operators, the convergence happening in $U_{1}$ in this case. Then by independence of the $\psi_{j}$,

$$
\begin{aligned}
& \quad E\left(\left(h, \sum_{k=1}^{L} \psi_{k}(t-s) J g_{k}\right)\left(l, \sum_{j=1}^{L} \psi_{j}(t-s) J g_{j}\right)\right) \\
& =E\left(\sum_{k}\left(h, J g_{k}\right)\left(l, J g_{j}\right) \psi_{k}(t-s) \psi_{j}(t-s)\right) \\
& =\sum_{k}\left(h, J g_{k}\right)\left(l, J g_{k}\right) E\left(\psi_{k}^{2}(t-s)\right)=(t-s) \sum_{k}\left(h, J g_{k}\right)\left(l, J g_{k}\right) \\
& =(t-s) \sum_{k}\left(J^{*} h, g_{k}\right)_{Q^{1 / 2}(U)}\left(J^{*} l, g_{k}\right)_{Q^{1 / 2}(U)}=(t-s)\left(J^{*} h, J^{*} l\right)_{Q^{1 / 2}(U)} \\
& =(t-s)\left(J J^{*} h, l\right)_{U_{1}} \equiv(t-s)\left(Q_{1} h, l\right)_{U_{1}}
\end{aligned}
$$

### 65.5 The General Integral

It is time to generalize the integral. The following diagram illustrates the ingredients of the next lemma.

$$
\begin{array}{ccc}
W(t) \in U_{1} \stackrel{J}{\leftarrow} Q^{1 / 2} U & \xrightarrow{\Phi} H \\
& & \\
& \\
\supseteq J & \\
& \downarrow & Q^{1 / 2} U \\
& \underset{1-1}{J} & Q^{1 / 2} U \\
\Phi_{n} & \searrow & \\
& & \\
& & \\
& &
\end{array}
$$

Lemma 65.5.1 Let $\Phi \in L^{2}\left([a, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$ and suppose also that $\Phi$ is progressively measurable with respect to the usual filtration associated with the Wiener process

$$
W(t)=\sum_{k=1}^{L} \psi_{k}(t) J g_{k}
$$

which has values in $U_{1}$ for $U_{1}$ a separable real Hilbert space such that $J: Q^{1 / 2} U \rightarrow U_{1}$ is Hilbert Schmidt and one to one, $\left\{g_{k}\right\}$ an orthonormal basis in $Q^{1 / 2} U$. Then letting $J^{-1}: J Q^{1 / 2} U \rightarrow Q^{1 / 2} U$ be the map described in Definition 65.4.1, it follows that

$$
\Phi \circ J^{-1} \in L^{2}\left([a, T] \times \Omega ; \mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)\right) .
$$

Also there exists a sequence of elementary functions $\left\{\Phi_{n}\right\}$ having values in $\mathscr{L}\left(U_{1}, H\right)_{0}$ which converges to $\Phi \circ J^{-1}$ in $L^{2}\left([a, T] \times \Omega ; \mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)\right)$.

Proof: First, why is $\Phi \circ J^{-1} \in L^{2}\left([a, T] \times \Omega ; \mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)\right)$ ? This follows from the observation that $A$ is Hilbert Schmidt if and only if $A^{*}$ is Hilbert Schmidt. In fact, the Hilbert Schmidt norms of $A$ and $A^{*}$ are the same. Now since $\Phi$ is Hilbert Schmidt, it follows that $\Phi^{*}$ is and since $J^{-1}$ is continuous, it follows $\left(J^{-1}\right)^{*} \Phi^{*}=\left(\Phi \circ J^{-1}\right)^{*}$ is Hilbert Schmidt. Also letting $\mathscr{L}_{2}$ be the appropriate space of Hilbert Schmidt operators,

$$
\left\|\left(J^{-1}\right)^{*}\right\|\|\Phi\|_{\mathscr{L}_{2}}=\left\|\left(J^{-1}\right)^{*}\right\|\left\|\Phi^{*}\right\|_{\mathscr{L}_{2}} \geq\left\|\left(\Phi \circ J^{-1}\right)^{*}\right\|_{\mathscr{L}_{2}}=\left\|\Phi \circ J^{-1}\right\|_{\mathscr{L}_{2}}
$$

Thus $\Phi \circ J^{-1}$ has values in $\mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)$. This also shows that

$$
\Phi \circ J^{-1} \in L^{2}\left([a, T] \times \Omega ; \mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)\right)
$$

Since $\Phi$ is given to be progressively measurable, so is $\Phi \circ J^{-1}$. Therefore, the existence of the desired sequence of elementary functions follows from Proposition 65.3.2 and Lemma 65.4.5.

Definition 65.5.2 Let $\Phi \in L^{2}\left([a, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$ and be progressively measurable where $Q$ is a self adjoint nonnegative operator defined on $U$. Let $J: Q^{1 / 2} U \rightarrow U_{1}$ be Hilbert Schmidt. Then the stochastic integral

$$
\begin{equation*}
\int_{a}^{t} \Phi d W \tag{65.5.7}
\end{equation*}
$$

is defined as

$$
\lim _{n \rightarrow \infty} \int_{a}^{t} \Phi_{n} d W \text { in } L^{2}(\Omega ; H)
$$

where $W(t)$ is a Wiener process

$$
\sum_{k=1}^{\infty} \psi_{k}(t) J g_{k},\left\{g_{k}\right\} \text { orthonormal basis in } Q^{1 / 2} U
$$

and $\Phi_{n}$ is an elementary function which has values in $\mathscr{L}\left(U_{1}, H\right)$ and converges to $\Phi \circ J^{-1}$ in

$$
L^{2}\left([a, T] \times \Omega ; \mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)\right)
$$

such a sequence exists by Lemma 65.4.5 and Proposition 65.3.2.


It is necessary to show that this is well defined and does not depend on the choice of $U_{1}$ and $J$.

Theorem 65.5.3 The stochastic integral 65.5 .7 is well defined. It also is a continuous martingale and does not depend on the choice of $J$ and $U_{1}$. Furthermore,

$$
E\left(\left|\int_{a}^{t} \Phi(s) d W\right|_{H}^{2}\right)=\int_{a}^{t} E\left(\|\Phi\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2}\right) d s
$$

Proof: First of all, it is obvious that it is well defined in the sense that the same stochastic process is obtained from two different sequences of elementary functions. This follows from the isometry of Proposition 65.1 .5 with $U_{1}$ in place of $U$ and $Q^{1 / 2} U$ in place of $U_{0}$. Thus if $\left\{\Psi_{n}\right\}$ and $\left\{\Phi_{n}\right\}$ are two sequences of elementary functions converging to $\Phi \circ J^{-1}$ in $L^{2}\left([a, T] \times \Omega ; \mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)\right)$,

$$
\begin{equation*}
E\left(\left|\int_{a}^{T}\left(\Phi_{n}(s)-\Psi_{n}(s)\right) d W\right|_{H}^{2}\right)=\int_{a}^{T} E\left(\left\|\left(\Phi_{n}-\Psi_{n}\right) \circ J\right\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2}\right) d s \tag{65.5.8}
\end{equation*}
$$

Now for $\Phi \in \mathscr{L}_{2}\left(U_{1}, H\right)$ and $\left\{g_{k}\right\}$ an orthonormal basis for $Q^{1 / 2} U$,

$$
\|\Phi \circ J\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} \equiv \sum_{k=1}^{\infty}\left|\Phi\left(J\left(g_{k}\right)\right)\right|_{H}^{2}=\|\Phi\|_{\mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)}^{2}
$$

because, by definition, $\left\{J g_{k}\right\}$ is an orthonormal basis in $J Q^{1 / 2} U$. Hence 65.5 .8 reduces to

$$
\int_{a}^{T} E\left(\left\|\left(\Phi_{n}-\Psi_{n}\right)\right\|_{\mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)}^{2}\right) d s
$$

which is given to converge to 0 . This reasoning also shows that the sequence $\left\{\int_{a}^{t} \Phi_{n} d W\right\}$ is indeed a Cauchy sequence in $L^{2}(\Omega, H)$.

Why is $\int_{a}^{t} \Phi d W$ a continuous martingale? The integrals $\int_{a}^{t} \Phi_{n} d W$ are martingales and so, by the maximal estimate of Theorem 62.5.3,

$$
\begin{gather*}
P\left(\left[\sup _{t \in[a, T]}\left|\int_{a}^{t} \Phi_{n} d W-\int_{a}^{t} \Phi_{m} d W\right|_{H} \geq \lambda\right]\right) \leq \frac{1}{\lambda^{2}} E\left(\left|\int_{a}^{T}\left(\Phi_{n}-\Phi_{m}\right) d W\right|^{2}\right) \\
=\frac{1}{\lambda^{2}} \int_{a}^{T} E\left(\left\|\left(\Phi_{n}-\Phi_{m}\right) \circ J\right\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2}\right) d s \\
=\frac{1}{\lambda^{2}} \int_{a}^{T} E\left(\left\|\left(\Phi_{n}-\Phi_{m}\right)\right\|_{\mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)}^{2}\right) d s \tag{65.5.9}
\end{gather*}
$$

which is given to converge to 0 as $m, n \rightarrow \infty$. Therefore, there exists a subsequence $\left\{n_{k}\right\}$ such that

$$
P\left(\left[\sup _{t \in[a, T]}\left|\int_{a}^{t} \Phi_{n_{k}} d W-\int_{a}^{t} \Phi_{n_{k+1}} d W\right|_{H} \geq 2^{-k}\right]\right) \leq 2^{-k}
$$

Consequently, by the Borel Cantelli lemma, there is a set of measure zero $N$ such that if $\omega \notin$ $N$, then the convergence of $\int_{a}^{t} \Phi_{n_{k}} d W$ to $\int_{a}^{t} \Phi d W$ is uniform on $[a, T]$. Hence $t \rightarrow \int_{a}^{t} \Phi d W$ is continuous as claimed.

Why is it a martingale? Let $s<t$ and $A \in \mathscr{F}_{s}$. Then

$$
\begin{aligned}
\int_{A}\left(\int_{a}^{t} \Phi d W\right) d P & =\lim _{n \rightarrow \infty} \int_{A}\left(\int_{a}^{t} \Phi_{n} d W\right) d P=\lim _{n \rightarrow \infty} \int_{A} E\left(\left(\int_{a}^{t} \Phi_{n} d W\right) \mid \mathscr{F}_{s}\right) d P \\
= & \lim _{n \rightarrow \infty} \int_{A}\left(\int_{a}^{s} \Phi_{n} d W\right) d P=\int_{A}\left(\int_{a}^{s} \Phi d W\right) d P
\end{aligned}
$$

Hence this is a martingale as claimed.
It remains to verify that the stochastic process does not depend on $J$ and $U_{1}$. Let the approximating sequence of elementary functions be

$$
\Phi_{n}(t)=\sum_{j=0}^{m_{n}} f_{j}^{n} \mathscr{X}_{\left[t_{j}^{n}, t_{j+1}^{n}\right]}(t)
$$

where $f_{j}^{n}$ is $\mathscr{F}_{t_{j}^{n}}$ measurable and has finitely many values in $\mathscr{L}\left(U_{1}, H\right)_{0}$, the restrictions of things in $\mathscr{L}\left(U_{1}, H\right)$ to $J Q^{1 / 2} U$. These are the elementary functions which converge to $\Phi \circ J^{-1}$. Also let the partitions be such that

$$
\begin{equation*}
\Phi^{n} \circ J^{-1} \equiv \sum_{j=0}^{m_{n}} \Phi\left(t_{j}^{n}\right) \circ J^{-1} \mathscr{X}_{\left(t_{j}^{n}, t_{j+1}^{n}\right]} \tag{65.5.10}
\end{equation*}
$$

converges to $\Phi \circ J^{-1}$ in $L^{2}\left([a, T] \times \Omega ; \mathscr{L}_{2}\left(J Q^{1 / 2}(U), H\right)\right)$. Then by definition,

$$
\begin{aligned}
& \int_{a}^{t} \Phi_{n} d W=\sum_{j=0}^{m_{n}} f_{j}^{n}\left(W\left(t \wedge t_{j+1}^{n}\right)-W\left(t \wedge t_{j}^{n}\right)\right) \\
& =\sum_{j=0}^{m_{n}} f_{j}^{n} \sum_{k=1}^{\infty}\left(\psi_{k}\left(t \wedge t_{j+1}^{n}\right)-\psi_{k}\left(t \wedge t_{j}^{n}\right)\right) J g_{k}
\end{aligned}
$$

where $\left\{g_{k}\right\}$ is an orthonormal basis for $Q^{1 / 2} U$. The infinite sum converges in $L^{2}\left(\Omega ; U_{1}\right)$ and $f_{j}^{n}$ is continuous on $U_{1}$. Therefore, $f_{j}^{n}$ can go inside the infinite sum, and this last expression equals

$$
\begin{equation*}
=\sum_{j=0}^{m_{n}} \sum_{k=1}^{\infty}\left(\psi_{k}\left(t \wedge t_{k+1}\right)-\psi_{k}\left(t \wedge t_{k}\right)\right) f_{j}^{n} J g_{k}, \tag{65.5.11}
\end{equation*}
$$

the infinite sum converging in $L^{2}(\Omega, H)$.
Now consider the left sum 65.5.10. Since $\Phi\left(t_{j}^{n}\right) \in \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)$, it follows that the sum

$$
\begin{align*}
& \sum_{k=1}^{\infty}\left(\psi_{k}\left(t \wedge t_{j+1}^{n}\right)-\psi_{k}\left(t \wedge t_{j}^{n}\right)\right) \Phi\left(t_{j}^{n}\right) g_{k} \\
= & \sum_{k=1}^{\infty}\left(\psi_{k}\left(t \wedge t_{j+1}^{n}\right)-\psi_{k}\left(t \wedge t_{j}^{n}\right)\right) \Phi\left(t_{j}^{n}\right) \circ J^{-1}\left(J g_{k}\right) \tag{65.5.12}
\end{align*}
$$

must converge in $L^{2}(\Omega, H)$. Lets review why this is.

Diversion The reason the series converges goes as follows. Estimate

$$
E\left(\left|\sum_{k=p}^{q}\left(\psi_{k}\left(t \wedge t_{j+1}^{n}\right)-\psi_{k}\left(t \wedge t_{j}^{n}\right)\right) \Phi\left(t_{j}^{n}\right) g_{k}\right|_{H}^{2}\right)
$$

First consider the mixed terms. Let $\Delta \psi_{k}=\psi_{k}\left(t \wedge t_{j+1}^{n}\right)-\psi_{k}\left(t \wedge t_{j}^{n}\right)$. For $l<k$,

$$
\begin{aligned}
& E\left(\left(\Delta \psi_{k} \Phi\left(t_{j}^{n}\right) g_{k}, \Delta \psi_{l} \Phi\left(t_{j}^{n}\right) g_{l}\right)\right) \\
= & E\left(\Delta \psi_{k} \Delta \psi_{l}\left(\Phi\left(t_{j}^{n}\right) g_{k}, \Phi\left(t_{j}^{n}\right) g_{l}\right)\right)
\end{aligned}
$$

Now by independence, this equals

$$
\begin{aligned}
& E\left(\Delta \psi_{k} \Delta \psi_{l}\right) E\left(\left(\Phi\left(t_{j}^{n}\right) g_{k}, \Phi\left(t_{j}^{n}\right) g_{l}\right)\right) \\
= & E\left(\Delta \psi_{k}\right) E\left(\Delta \psi_{l}\right) E\left(\left(\Phi\left(t_{j}^{n}\right) g_{k}, \Phi\left(t_{j}^{n}\right) g_{l}\right)\right)=0
\end{aligned}
$$

Thus you only need to consider the non mixed terms, and the thing you want to estimate is of the form

$$
\sum_{k=p}^{q} E\left(\left|\left(\psi_{k}\left(t \wedge t_{j+1}^{n}\right)-\psi_{k}\left(t \wedge t_{j}^{n}\right)\right) \Phi\left(t_{j}^{n}\right) g_{k}\right|^{2}\right)
$$

Now by independence again, this equals

$$
\begin{aligned}
& \sum_{k=p}^{q} E\left(\left(\Delta \psi_{k} \Phi\left(t_{j}^{n}\right) g_{k}, \Delta \psi_{k} \Phi\left(t_{j}^{n}\right) g_{k}\right)\right) \\
= & \sum_{k=p}^{q} E\left(\Delta \psi_{k}^{2}\left(\Phi\left(t_{j}^{n}\right) g_{k}, \Phi\left(t_{j}^{n}\right) g_{k}\right)\right) \\
= & \sum_{k=p}^{q} E\left(\Delta \psi_{k}^{2}\right) E\left(\Phi\left(t_{j}^{n}\right) g_{k}, \Phi\left(t_{j}^{n}\right) g_{k}\right) \\
= & \left(\left(t \wedge t_{j+1}^{n}\right)-\left(t \wedge t_{j}^{n}\right)\right) \sum_{k=p}^{q} E\left(\left|\Phi\left(t_{j}^{n}\right) g_{k}\right|_{H}^{2}\right)
\end{aligned}
$$

and this sum is just a part of the convergent infinite sum for

$$
\int_{\Omega}\left\|\Phi\left(t_{j}^{n}\right)\right\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d P<\infty
$$

Therefore, this converges to 0 as $p, q \rightarrow \infty$ and so the sum converges in $L^{2}(\Omega, H)$ as claimed.

End of diversion
The $J$ and the $J^{-1}$ cancel in 65.5 .12 because $J$ is one to one. It follows that 65.5.11 equals

$$
\sum_{j=0}^{m_{n}} \sum_{k=1}^{\infty}\left(\psi_{k}\left(t \wedge t_{j+1}^{n}\right)-\psi_{k}\left(t \wedge t_{j}^{n}\right)\right) \Phi\left(t_{j}^{n}\right) g_{k}+
$$

$$
\sum_{j=0}^{m_{n}} \sum_{k=1}^{\infty}\left(\psi_{k}\left(t \wedge t_{j+1}^{n}\right)-\psi_{k}\left(t \wedge t_{j}^{n}\right)\right)\left(\left(f_{j}^{n}-\Phi\left(t_{j}^{n}\right) \circ J^{-1}\right)\left(J g_{k}\right)\right)
$$

The first expression does not depend on $J$ or $U_{1}$. I need only argue that the second expression converges to 0 as $n \rightarrow \infty$. The infinite sum converges in $L^{2}(\Omega ; H)$ and also, as in the above diversion, the independence of the $\psi_{k}$ implies that

$$
\begin{aligned}
& E\left(\left|\sum_{j=0}^{m_{n}} \sum_{k=1}^{\infty}\left(\psi_{k}\left(t \wedge t_{j+1}^{n}\right)-\psi_{k}\left(t \wedge t_{j}^{n}\right)\right)\left(\left(f_{j}^{n}-\Phi\left(t_{j}^{n}\right) \circ J^{-1}\right)\left(J g_{k}\right)\right)\right|_{H}^{2}\right) \\
& \quad=\sum_{j=0}^{m_{n}}\left(t \wedge t_{j+1}^{n}-t \wedge t_{j}^{n}\right) \sum_{k=1}^{\infty} E\left(\left|\left(f_{j}^{n}-\Phi\left(t_{j}^{n}\right) \circ J^{-1}\right)\left(J g_{k}\right)\right|_{H}^{2}\right) \\
& \quad=\sum_{j=0}^{m_{n}}\left(t \wedge t_{j+1}^{n}-t \wedge t_{j}^{n}\right) E\left(\left\|f_{j}^{n}-\Phi\left(t_{j}^{n}\right) \circ J^{-1}\right\|_{\mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)}^{2}\right) \\
& \quad=\int_{a}^{t} E\left(\left\|\Phi_{n}-\Phi^{n} \circ J^{-1}\right\|_{\mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)}^{2}\right) d s
\end{aligned}
$$

which is given to converge to 0 since both converge to $\Phi \circ J^{-1}$. Consequently, the stochastic integral defined above does not depend on $J$ or $U_{1}$.

It is interesting to note that in the above definition, the approximate problems do appear to depend on $J$ and $U_{1}$ but the limiting stochastic process does not. Since it is the case that the stochastic integral is independent of $U_{1}$ and $J$, it can only be dependent on $Q^{1 / 2} U$ and $U$, and so we refer to $W(t)$ as a cylindrical process on $U$. By Lemma 65.4.3 you can take $U_{1}=U$ and so you can consider the finite sums defining the Wiener process to be in $U$ itself. From the proof of this lemma, you can even have $J$ being the identity on the span of the first $n$ vectors in the orthonormal basis for $Q^{1 / 2} U$. The case where $Q$ is trace class follows in the next section. In this case, $W$ is an actual $Q$ Wiener process on $U$.

The following corollary follows right away from the above theorem.
Corollary 65.5.4 Let $\Phi, \Psi \in L^{2}\left([a, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$ and suppose they are both progressively measurable. Then

$$
E\left(\left(\int_{a}^{t} \Phi d W, \int_{a}^{t} \Psi d W\right)_{H}\right)=E\left(\int_{a}^{t}(\Phi, \Psi)_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)} d s\right)
$$

Also if $L$ is in $L^{\infty}(\Omega, \mathscr{L}(H, H))$ and is $\mathscr{F}_{a}$ measurable, then

$$
\begin{equation*}
L \int_{a}^{t} \Phi d W=\int_{a}^{t} L \Phi d W \tag{65.5.13}
\end{equation*}
$$

and

$$
\begin{equation*}
E\left(\left(L \int_{a}^{t} \Phi d W, \int_{a}^{t} \Psi d W\right)_{H}\right)=E\left(\int_{a}^{t}(L \Phi, \Psi)_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)} d s\right) \tag{65.5.14}
\end{equation*}
$$

Proof: First note that

$$
\left(\int_{a}^{t} \Phi d W, \int_{a}^{t} \Psi d W\right)_{H}=\frac{1}{4}\left[\left|\int_{a}^{t}(\Phi+\Psi) d W\right|_{H}^{2}-\left|\int_{a}^{t}(\Phi-\Psi) d W\right|_{H}^{2}\right]
$$

and so from the above theorem,

$$
\begin{gathered}
E\left(\left(\int_{a}^{t} \Phi d W, \int_{a}^{t} \Psi d W\right)_{H}\right)= \\
=E\left(\frac{1}{4}\left[\left|\int_{a}^{t}(\Phi+\Psi) d W\right|_{H}^{2}-\left|\int_{a}^{t}(\Phi-\Psi) d W\right|_{H}^{2}\right]\right) \\
\frac{1}{4} E\left(\int_{a}^{t}\|\Phi+\Psi\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d s\right)+\frac{1}{4} E\left(\int_{a}^{t}\|\Phi-\Psi\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d s\right) \\
=E\left(\int_{a}^{t} \frac{1}{4}\left[\|\Phi+\Psi\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2}+\|\Phi-\Psi\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2}\right] d s\right) \\
=E\left(\int_{a}^{t}(\Phi, \Psi)_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)} d s\right)
\end{gathered}
$$

Now consider the last claim. First suppose $L=l \mathscr{X}_{A}$ where $A \in \mathscr{F}_{a}$, and $l \in \mathscr{L}(H, H)$. Also suppose $\Phi$ is an elementary function

$$
\Phi=\sum_{i=0}^{n} \psi_{i} \mathscr{X}_{\left(s_{i}, s_{i+1}\right]}
$$

Then

$$
\begin{aligned}
L \int_{a}^{t} \Phi d W & =l \mathscr{X}_{A} \sum_{i=0}^{n} \psi_{i}\left(W\left(t \wedge s_{i+1}\right)-W\left(t \wedge s_{i}\right)\right) \\
& =\sum_{i=0}^{n} l \mathscr{X}_{A} \psi_{i}\left(W\left(t \wedge s_{i+1}\right)-W\left(t \wedge s_{i}\right)\right)
\end{aligned}
$$

Thus 65.5.13 also holds for $L$ a simple function which is $\mathscr{F}_{a}$ measurable. For general $L \in L^{\infty}(\Omega, \mathscr{L}(H, H))$, approximating with a sequence of such simple functions $L_{n}$ yields

$$
L \int_{a}^{t} \Phi d W=\lim _{n \rightarrow \infty} L_{n} \int_{a}^{t} \Phi d W=\lim _{n \rightarrow \infty} \int_{a}^{t} L_{n} \Phi d W=\int_{a}^{t} L \Phi d W
$$

because $L_{n} \Phi \rightarrow L \Phi$ in $L^{2}\left([a, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$. Now what about general $\Phi$ ? Let $\left\{\Phi_{n}\right\}$ be elementary functions converging to $\Phi \circ J^{-1}$ in $L^{2}\left([a, T] \times \Omega ; \mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)\right)$. Then by definition of the integral,

$$
L \int_{a}^{t} \Phi d W=\lim _{n \rightarrow \infty} L \int_{a}^{t} \Phi_{n} d W=\lim _{n \rightarrow \infty} \int_{a}^{t} L \Phi_{n} d W=\int_{a}^{t} L \Phi d W
$$

The remaining claim now follows from the first part of the proof.
The above has discussed the integral of $\Phi \in L^{2}\left([a, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$. An obvious case to consider is when

$$
\Phi=\sum_{k=0}^{n-1} \Phi_{k} \mathscr{X}_{\left(t_{k}, t_{k+1}\right]}(t)
$$

and $\Phi_{k} \in L^{2}\left(\Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$ with $\Phi_{k}$ measurable with respect to $\mathscr{F}_{t_{k}}$. What is $\int_{0}^{t} \Phi d W$ ? First note that $\Phi_{k} \circ J^{-1} \in L^{2}\left(\Omega ; \mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)\right)$. Let

$$
\lim _{m \rightarrow \infty} \Phi_{k}^{m} \rightarrow \Phi_{k} \circ J^{-1}
$$

in $L^{2}\left(\Omega ; \mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)\right)$ where $\Phi_{k}^{m}$ is $\mathscr{F}_{t_{k}}$ measurable and is a simple function having values in $\mathscr{L}\left(U_{1}, H\right)$. Thus

$$
\Phi_{m} \equiv \sum_{k=0}^{n-1} \Phi_{k}^{m} \mathscr{X}_{\left(t_{k}, t_{k+1}\right]}(t)
$$

is an elementary function and it converges to $\Phi \circ J^{-1}$ in $L^{2}\left([a, T] \times \Omega ; \mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)\right)$. It follows that

$$
\begin{aligned}
\int_{a}^{t} \Phi d W & \equiv \lim _{m \rightarrow \infty} \int_{a}^{t} \Phi_{m} d W \equiv \lim _{m \rightarrow \infty} \sum_{k=0}^{n-1} \Phi_{k}^{m}\left(W\left(t \wedge t_{k+1}\right)-W\left(t \wedge t_{k}\right)\right) \\
& =\sum_{k=0}^{n-1} \Phi_{k} \circ J^{-1}\left(W\left(t \wedge t_{k+1}\right)-W\left(t \wedge t_{k}\right)\right)
\end{aligned}
$$

Note again how it appears to depend on $J$ but really doesn't because there is a $J$ in the definition of $W$.

### 65.6 The Case That $Q$ Is Trace Class

In this special case, you have a $Q$ Wiener process with values in $U$ and still you have

$$
\Phi \in L^{2}\left([a, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)
$$

with $\Phi$ progressively measurable. The difference here is that in fact, $Q$ is trace class.

$$
Q=\sum_{i=1}^{L} \lambda_{i} e_{i} \otimes e_{i}
$$

where $\lambda_{i}>0, \sum_{i} \lambda_{i}<\infty$, and the $e_{i}$ form an orthonormal set of vectors. $L$ is either a positive integer or $\infty$. Then let $U_{0}=Q^{1 / 2} U$. Then $Q^{1 / 2}=\sum_{i=1}^{L} \sqrt{\lambda_{i}} e_{i} \otimes e_{i}$ because this works, and the square root is unique. Hence $Q^{1 / 2} e_{i}=\sqrt{\lambda_{i}} e_{i}$ and so an orthonormal basis for $U_{0}=Q^{1 / 2} U$ is $\left\{\sqrt{\lambda_{i}} e_{i}\right\}_{i=1}^{L}$. Now consider $J=\sum_{i=1}^{L} \sqrt{\lambda_{i}}\left(e_{i} \otimes \sqrt{\lambda_{i}} e_{i}\right), J: U_{0} \rightarrow U$, where the tensor product is defined in the usual way,

$$
u \otimes v(w) \equiv u(w, v)_{U_{0}}
$$

Then $J^{*}=\sum_{i=1}^{L} \sqrt{\lambda_{i}}\left(\sqrt{\lambda_{i}} e_{i} \otimes e_{i}\right)$ and $J J^{*}=\sum_{i=1}^{L} \lambda_{i} e_{i} \otimes e_{i}=Q$. Also, $J$ is a Hilbert Schmidt map into $U$ from $U_{0}$.

$$
\sum_{i=1}^{L}\left\|J\left(\sqrt{\lambda_{i}} e_{i}\right)\right\|_{U}^{2}=\sum_{i=1}^{L}\left\|\sqrt{\lambda_{i}} e_{i}\right\|_{U}^{2}=\sum_{i=1}^{L} \lambda_{i}<\infty
$$

and so $J$ is a Hilbert Schmidt mapping. In addition to this, from the construction, the span of $\left\{e_{i}\right\}_{i=1}^{L}$ is dense in $U_{0}$ and $J e_{i}=e_{i}$ because

$$
J e_{k}=\sum_{i=1}^{L} \sqrt{\lambda_{i}}\left(e_{i} \otimes \sqrt{\lambda_{i}} e_{i}\right)\left(e_{k}\right)=e_{k} \sqrt{\lambda_{k}}\left(e_{k}, \sqrt{\lambda_{k}} e_{k}\right)_{U_{0}}=e_{k}
$$

so in fact $J$ is just the injection map of $U_{0}$ into $U$. Hence $J^{-1}$ must also be the identity map. Now we can let $U_{1}=U$ with $J$ the injection map. Thus, in this case, the elementary functions $\Phi_{n}$ simply converge to $\Phi$ in

$$
L^{2}\left([a, T] \times \Omega ; \mathscr{L}_{2}\left(J U_{0}, H\right)\right)
$$

Note that $\left\|J \sqrt{\lambda_{i}} e_{i}\right\|_{U}=\sqrt{\lambda_{i}}$ whereas $\left\|\sqrt{\lambda_{i}} e_{i}\right\|_{Q^{1 / 2} U}=1$, and so $J$ definitely does not preserve norms. That is, the norm in $U_{0}$ is not the same as the norm in $U$. Then everything else is the same. In particular

$$
E\left(\left|\int_{a}^{t} \Phi d W\right|_{H}^{2}\right)=\int_{a}^{t} E\left(\|\Phi\|_{\mathscr{L}_{2}\left(U_{0}, H\right)}^{2}\right) d s
$$

### 65.7 A Short Comment On Measurability

It will also be important to consider the composition of functions. The following is the main result. With the explanation of progressively measurable given, it says the composition of progressively measurable functions is progressively measurable.

Proposition 65.7.1 Let $A:[a, T] \times V \times \Omega \rightarrow U$ where $V, U$ are topological spaces and suppose $A$ satisfies its restriction to $[a, t] \times V \times \Omega$ is $\mathscr{B}([a, t]) \times \mathscr{B}(V) \times \mathscr{F}_{t}$ measurable . This will be referred to as $A$ is progressively measurable. Then if $X:[a, T] \times \Omega \rightarrow V$ is progressively measurable, then so is the map

$$
(t, \omega) \rightarrow A(t, X(t, \omega), \omega)
$$

Proof: Consider the restriction of this map to $\left[a, t_{0}\right] \times \Omega$. For such $(t, \omega)$, to say

$$
A(t, X(t, \omega), \omega) \in O
$$

for $O$ a Borel set in $U$ is to say that

$$
X(t, \omega) \in\left\{v:(t, v, \omega) \in A^{-1}(O), t \leq t_{0}\right\} \equiv A^{-1}(O)_{t \omega}
$$

Consider the set

$$
\left\{(t, \omega) \in\left[a, t_{0}\right] \times \Omega: X(t, \omega) \in A^{-1}(O)_{t \omega}\right\}
$$

Is this in $\mathscr{B}\left(\left[a, t_{0}\right]\right) \times \mathscr{F}_{t_{0}}$ ? This is what needs to be checked. Since $A$ is progressively measurable,

$$
A^{-1}(O) \cap\left[a, t_{0}\right] \times V \times \Omega \in \mathscr{B}\left(\left[a, t_{0}\right]\right) \times \mathscr{B}(V) \times \mathscr{F}_{t_{0}} \equiv \mathscr{P}_{t_{0}}
$$

because $A^{-1}(O)$ is a progressively measurable set. So let

$$
\mathscr{G} \equiv\left\{S \in \mathscr{P}_{t_{0}}:\left\{(t, \omega) \in\left[a, t_{0}\right] \times \Omega: X(t, \omega) \in S_{t \omega}\right\} \in \mathscr{B}\left(\left[a, t_{0}\right]\right) \times \mathscr{F}_{t_{0}}\right\}
$$

It is clear that $\mathscr{G}$ contains the $\pi$ system composed of sets of the form $I \times B \times W$ where $I$ is an interval in $\left[a, t_{0}\right], B$ is Borel, and $W \in \mathscr{F}_{t_{0}}$. This is because for $S$ of this form, $S_{t \omega}=B$ or $\emptyset$. Thus if not empty,

$$
\begin{aligned}
& \left\{(t, \omega) \in\left[a, t_{0}\right] \times \Omega: X(t, \omega) \in S_{t \omega}\right\} \\
= & X^{-1}(B) \cap\left[0, t_{0}\right] \times \Omega \in \mathscr{B}\left(\left[a, t_{0}\right]\right) \times \mathscr{F}_{t_{0}}
\end{aligned}
$$

because $X$ is given to be progressively measurable. Now if $S \in \mathscr{G}$, what about $S^{C}$ ? You have $\left(S^{C}\right)_{t \omega}=\left(S_{t \omega}\right)^{C}$ thus

$$
\begin{aligned}
& \left\{(t, \omega) \in\left[a, t_{0}\right] \times \Omega: X(t, \omega) \in\left(S^{C}\right)_{t \omega}\right\} \\
= & \left\{(t, \omega) \in\left[a, t_{0}\right] \times \Omega: X(t, \omega) \in\left(S_{t \omega}\right)^{C}\right\}
\end{aligned}
$$

which is the complement with respect to $\left[a, t_{0}\right] \times \Omega$ of a set in $\mathscr{B}\left(\left[a, t_{0}\right]\right) \times \mathscr{F}_{t_{0}}$. Therefore, $\mathscr{G}$ is closed with respect to complements. It is clearly closed with respect to countable disjoint unions. It follows, $\mathscr{G}=\mathscr{P}_{t_{0}}$. Thus

$$
\left\{(t, \omega) \in\left[a, t_{0}\right] \times \Omega: X(t, \omega) \in S_{t \omega}\right\} \in \mathscr{B}\left(\left[a, t_{0}\right]\right) \times \mathscr{F}_{t_{0}}
$$

where $S=A^{-1}(O) \cap\left[a, t_{0}\right] \times V \times \Omega$. In other words,

$$
\left\{(t, \omega), t \leq t_{0}: A(t, X(t, \omega), \omega) \in O\right\} \in \mathscr{B}\left(\left[0, t_{0}\right]\right) \times \mathscr{F}_{t_{0}}
$$

and so $(t, \omega) \rightarrow A(t, X(t, \omega), \omega)$ is progressively measurable.

### 65.8 Localization For Elementary Functions

It is desirable to extend everything to stochastically square integrable functions. This will involve localization using a suitable stopping time. First it is necessary to understand localization for elementary functions. As above, we are in the situation described by the following diagram.

$$
\begin{aligned}
& U \\
& \begin{array}{l}
\downarrow Q^{1 / 2} \\
Q^{1 / 2} U
\end{array} \\
& \Phi_{n} \searrow \\
& \downarrow \Phi
\end{aligned}
$$

The elementary functions $\left\{\Phi_{n}\right\}$ have values in $\mathscr{L}\left(U_{1}, H\right)_{0}$ meaning they are restrictions of functions in $\mathscr{L}\left(U_{1}, H\right)$ to $J Q^{1 / 2} U$ and converge to $\Phi \circ J^{-1}$ in

$$
L^{2}\left([a, T] \times \Omega ; \mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)\right)
$$

where $\Phi \in L^{2}\left([a, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$ is given. Let

$$
\Phi(t) \equiv \sum_{k=0}^{n-1} \Phi\left(t_{k}\right) \mathscr{X}_{\left(t_{k}, t_{k+1}\right]}(t)
$$

be an elementary function. In particular, let $\Phi\left(t_{k}\right)$ be $\mathscr{F}_{t_{k}}$ measurable as a map into $\mathscr{L}\left(U_{1}, H\right)$, and has finitely many values. As just mentioned, the topic of interest is the elementary functions $\Phi_{n}$ in the above diagram. Thus $\Phi$ will be one of these elementary functions.

Let $\tau$ be a stopping time having values from the set of mesh points $\left\{t_{k}\right\}$ for the elementary function. Then from the definition of the integral for elementary functions,

$$
\int_{a}^{t \wedge \tau} \Phi d W \equiv \sum_{k=0}^{n-1} \Phi\left(t_{k}\right)\left(W\left(t \wedge \tau \wedge t_{k+1}\right)-W\left(t \wedge \tau \wedge t_{k}\right)\right)
$$

If $\omega$ is such that $\tau(\omega)=t_{j}$, then to get something nonzero, you must have $t_{j}>t_{k}$ so $k \leq j-1$. Thus the above on the right reduces to

$$
\sum_{k=0}^{j-1} \Phi\left(t_{k}\right)\left(W\left(t \wedge t_{k+1}\right)-W\left(t \wedge t_{k}\right)\right)
$$

It clearly is 0 if $j=0$. Define $\sum_{k=0}^{-1} \equiv 0$. Thus the integral equals

$$
\sum_{j=0}^{n} \mathscr{X}_{\left[\tau=t_{j}\right]} \sum_{k=0}^{j-1} \Phi\left(t_{k}\right)\left(W\left(t \wedge t_{k+1}\right)-W\left(t \wedge t_{k}\right)\right)
$$

Interchanging the order of summation, $k \leq j-1$ so $j \geq k+1$ and this equals

$$
\begin{aligned}
& \sum_{k=0}^{n-1} \sum_{j=k+1}^{n} \mathscr{X}_{\left[\tau=t_{j}\right]} \Phi\left(t_{k}\right)\left(W\left(t \wedge t_{k+1}\right)-W\left(t \wedge t_{k}\right)\right) \\
&= \sum_{k=0}^{n-1} \overbrace{\mathscr{X}_{\left[\tau>t_{k}\right.} \text { measurable }} \Phi\left(t_{k}\right) \\
& \mathscr{F}_{k} \\
&\left.W\left(t \wedge t_{k+1}\right)-W\left(t \wedge t_{k}\right)\right)
\end{aligned}
$$

Therefore

$$
\begin{equation*}
\int_{a}^{t \wedge \tau} \Phi d W=\int_{a}^{t} \sum_{k=0}^{t-1} \mathscr{X}_{\left[\tau>t_{k}\right]} \Phi\left(t_{k}\right) \mathscr{X}_{\left(t_{k}, t_{k+1}\right]} d W \tag{65.8.15}
\end{equation*}
$$

Now observe

$$
\begin{align*}
\mathscr{X}_{[a, \tau]}(t) \Phi(t) & =\sum_{k=0}^{n-1} \mathscr{X}_{[a, \tau]}(t) \Phi\left(t_{k}\right) \mathscr{X}_{\left[t_{k}, t_{k+1}\right]}(t) \\
& =\sum_{k=0}^{n-1} \mathscr{X}_{[\tau \geq t]} \Phi\left(t_{k}\right) \mathscr{X}_{\left(t_{k}, t_{k+1}\right]}(t) \\
& =\sum_{k=0}^{n-1} \mathscr{X}_{\left[\tau>t_{k}\right]} \Phi\left(t_{k}\right) \mathscr{X}_{\left(t_{k}, t_{k+1}\right]}(t) \tag{65.8.16}
\end{align*}
$$

The last step occurs because of the following reasoning. The $k^{t h}$ term of the sum in the middle expression above equals $\Phi\left(t_{k}\right)$ if and only if $t>t_{k}$ and $\tau \geq t$. If the two conditions do not hold, then the $k^{t h}$ term equals 0 . As to the third line, if $\tau>t_{k}$ and $t \in\left(t_{k}, t_{k+1}\right]$, then $\tau \geq t_{k+1} \geq t$ which is the same as the situation in the second line. The term equals $\Phi\left(t_{k}\right)$. Note that $\mathscr{X}_{\left[\tau>t_{k}\right]}(\omega)$ is $\mathscr{F}_{t_{k}}$ measurable, because $\left[\tau>t_{k}\right]$ is the complement of $\left[\tau \leq t_{k}\right]$. Therefore, this is an elementary function. Thus, from 65.8.15-65.8.16, $\mathscr{X}_{[a, \tau]}(t) \Phi(t)$ is an elementary function and

$$
\int_{a}^{t \wedge \tau} \Phi d W=\int_{a}^{t} \sum_{k=0}^{t-1} \mathscr{X}_{\left[\tau>t_{k}\right]} \Phi\left(t_{k}\right) \mathscr{X}_{\left(t_{k}, t_{k+1}\right]}(t) d W=\int_{a}^{t} \mathscr{X}_{[a, \tau]}(t) \Phi(t) d W
$$

From Proposition 65.1.5, if you have $\Phi, \Psi$ two of these elementary functions

$$
\begin{gather*}
E\left(\left\|\int_{a}^{t} \mathscr{X}_{[a, \tau]}(t) \Phi(t) d W-\int_{a}^{t} \mathscr{X}_{[a, \tau]}(t) \Psi(t) d W\right\|_{H}^{2}\right)= \\
\quad \int_{a}^{t} \int_{\Omega} \mathscr{X}_{[a, \tau]}(t)\|(\Phi(s)-\Psi(s)) \circ J\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d P d s \\
\leq \int_{a}^{t} \int_{\Omega}\|(\Phi(s)-\Psi(s)) \circ J\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d P d s \tag{65.8.17}
\end{gather*}
$$

### 65.9 Localization In General

Next, what about the general case where $\Phi \in L^{2}\left([a, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$ and is progressively measurable? Is it the case that for an arbitrary stopping time $\tau$,

$$
\int_{a}^{t \wedge \tau} \Phi d W=\int_{a}^{t} \mathscr{X}_{[a, \tau]} \Phi d W ?
$$

This is the sort of thing which would be expected for an ordinary Stieltjes integral which of course this isn't. Let

$$
L^{2}\left([a, T] \times \Omega ; \mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)\right)=K
$$

From Doob's result Proposition 65.3.2 and Lemma 65.3.1, there exists a sequence of elementary functions $\left\{\Phi_{k}\right\}$

$$
\Phi_{k}(t)=\sum_{j=0}^{m_{k}-1} \Phi\left(t_{j}^{k}\right) \mathscr{X}_{\left(t_{j}^{k}, t_{j+1}^{k}\right]}(t)
$$

which converges to $\Phi \circ J^{-1}$ in $K$ where also the lengths of the sub intervals converge uniformly to 0 as $k \rightarrow \infty$.

Now let $\tau$ be an arbitrary stopping time. The partition points corresponding to $\Phi_{k}$ are $\left\{t_{j}^{k}\right\}_{j=0}^{m_{k}}$. Let $\tau_{k}=t_{j+1}^{k}$ on $\tau^{-1}\left(t_{j}^{k}, t_{j+1}^{k}\right]$. Then $\tau_{k}$ is a stopping time because

$$
\left[\tau_{k} \leq t\right] \in \mathscr{F}_{t}
$$

Here is why. If $t \in\left(t_{j}^{k}, t_{j+1}^{k}\right]$, then if $t=t_{j+1}^{k}$, it would follow that $\tau_{k}(\omega) \leq t$ would be the same as saying $\omega \in\left[\tau \leq t_{j+1}^{k}\right]=[\tau \leq t] \in \mathscr{F}_{t}$. On the other hand, if $t<t_{j+1}^{k}$, then $\left[\tau_{k} \leq t\right]=\left[\tau \leq t_{j}^{k}\right] \in \mathscr{F}_{t_{j}^{k}} \subseteq \mathscr{F}_{t}$ because $\tau_{k}$ can only take the values $t_{j}^{k}$.

Consider $\mathscr{X}_{\left[a, \tau_{k}\right]} \Phi_{k}$. It is given that $\Phi_{k} \rightarrow \Phi \circ J^{-1}$ in $K$. Does it follow that $\mathscr{X}_{\left[a, \tau_{k}\right]} \Phi_{k} \rightarrow$ $\mathscr{X}_{[a, \tau]} \Phi \circ J^{-1}$ in $K$ ? Consider first the indicator function. Let $\tau(\omega) \in\left(t_{j}^{k}, t_{j+1}^{k}\right]$. Fixing $t$, if $\mathscr{X}_{[a, \tau]}(t)=1$, then also $\mathscr{X}_{\left[a, \tau_{k}\right]}(t)=1$ because $\tau_{k} \geq \tau$. Therefore, in this case

$$
\lim _{k \rightarrow \infty} \mathscr{X}_{\left[a, \tau_{k}\right]}(t)=\mathscr{X}_{[a, \tau]}(t) .
$$

Next suppose $\mathscr{X}_{[a, \tau]}(t)=0$ so that $\tau(\omega)<t$. Since the intervals defined by the partition points have lengths which converge to 0 , it follows that for all $k$ large enough, $\tau_{k}(\omega)<t$ also and so $\mathscr{X}_{\left[a, \tau_{k}\right]}(t)=0$. Therefore,

$$
\lim _{k \rightarrow \infty} \mathscr{X}_{\left[a, \tau_{k}(\omega)\right]}(t)=\mathscr{X}_{[a, \tau(\omega)]}(t) .
$$

It follows that $\mathscr{X}_{\left[a, \tau_{k}\right]} \Phi_{k} \rightarrow \mathscr{X}_{[a, \tau]} \Phi \circ J^{-1}$ in $K$. Now from 65.8.16, the function $\mathscr{X}_{\left[a, \tau_{k}\right]} \Phi_{k}$ is progressively measurable. Therefore, the same is true of $\mathscr{X}_{[a, \tau]} \Phi \circ J^{-1}$.

From the proof of Theorem 65.5.3, the part depending on maximal estimates and the fact that $\int_{a}^{t} \mathscr{X}_{\left[a, \tau_{k}\right]} \Phi_{k} d W$ is a continuous martingale, there is a set of measure zero $N$, such that off this set, a suitable subsequence satisfies

$$
\int_{a}^{t} \mathscr{X}_{\left[a, \tau_{k}\right]} \Phi_{k} d W \rightarrow \int_{a}^{t} \mathscr{X}_{[a, \tau]} \Phi d W
$$

uniformly on $[a, T]$. But also, since $\Phi_{k} \rightarrow \Phi \circ J^{-1}$ in $K$, a suitable subsequence satisfies,

$$
\int_{a}^{t} \Phi_{k} d W \rightarrow \int_{a}^{t} \Phi d W
$$

uniformly on $[a, T]$ a.e. $\omega$. In particular, $\int_{a}^{t \wedge \tau_{k}} \Phi_{k} d W \rightarrow \int_{a}^{t \wedge \tau} \Phi d W$. Therefore,

$$
\begin{aligned}
\int_{a}^{t} \mathscr{X}_{[a, \tau]} \Phi d W & =\lim _{k \rightarrow \infty} \int_{a}^{t} \mathscr{X}_{\left[a, \tau_{k}\right]} \Phi_{k} d W \\
& =\lim _{k \rightarrow \infty} \int_{a}^{t \wedge \tau_{k}} \Phi_{k} d W \\
& =\int_{a}^{t \wedge \tau} \Phi d W
\end{aligned}
$$

This has proved the following major localization lemma. This is a marvelous result. It says that the stochastic integral acts algebraically like an ordinary Stieltjes integral, one for each $\omega$ off a set of measure zero.

Lemma 65.9.1 Let $\Phi$ be progressively measurable and in

$$
L^{2}\left([a, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)
$$

Let $W(t)$ be a cylindrical Wiener process as described above. Then for $\tau$ a stopping time, $\mathscr{X}_{[a, \tau]} \Phi$ is progressively measurable, in $K$, and

$$
\int_{a}^{t \wedge \tau} \Phi d W=\int_{a}^{t} \mathscr{X}_{[a, \tau]} \Phi d W
$$

### 65.10 The Stochastic Integral As A Local Martingale

With Lemma 65.9.1, it becomes possible to define the stochastic integral on functions which are only stochastically square integrable.

Definition 65.10.1 $\Phi$ is stochastically square integrable in $\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)$ if $\Phi$ is progressively measurable and

$$
P\left(\left[\int_{a}^{T}\|\Phi(s)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d s<\infty\right]\right)=1
$$

Thus equivalently, there exists $N$ such that $P(N)=0$ and for $\omega \notin N$,

$$
\int_{a}^{T}\|\Phi(s, \omega)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d s<\infty
$$

Lemma 65.10.2 Suppose $\Phi$ is $\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)$ progressively measurable and

$$
P\left(\left[\int_{a}^{T}\|\Phi\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d s<\infty\right]\right)=1 .
$$

Define

$$
\tau_{n}(\omega) \equiv \inf \left\{t \in[a, T]: \int_{a}^{t}\|\Phi\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d s \geq n\right\}
$$

By convention, let $\inf \emptyset=\infty$. Then $\tau_{n}$ is a stopping time. Furthermore, $\tau_{n}$ has the following properties.

1. $\left\{\tau_{n}\right\}$ is an increasing sequence and for $\omega$ outside a set of measure zero $N$, for every $t \in[a, T]$ there exists $n$ such that $\tau_{n}(\omega)>t$. (It is a localizing sequence of stopping times.)
2. For each $n, \mathscr{X}_{\left[a, \tau_{n}\right]} \Phi$ is progressively measurable and

$$
E\left(\int_{a}^{T}\left\|\mathscr{X}_{\left[a, \tau_{n}\right]} \Phi\right\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d t\right)<\infty
$$

Proof: It follows from Proposition 62.7.5 that $\tau_{n}$ is a stopping time because it is the first hitting time of a closed set by an adapted continuous process.

It remains to verify the two claims. There exists a set of measure $0, N$ such that for $\omega \notin N$

$$
\int_{a}^{T}\|\Phi\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d t<\infty
$$

Therefore, for such $\omega$, there exists $n$ large enough that

$$
\int_{a}^{t}\|\Phi\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d s<n
$$

and so $\tau_{n}(\omega) \geq t$. Now consider the second claim.

$$
\begin{aligned}
& E\left(\int_{a}^{T}\left\|\mathscr{X}_{\left[a, \tau_{n}\right]} \Phi\right\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d t\right) \\
= & E\left(\int_{a}^{\tau_{n}(\omega) \wedge T}\|\Phi\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d t\right) \leq E(n)=n .
\end{aligned}
$$

With this lemma, it is possible to give the following definition.
Definition 65.10.3 Suppose $\Phi$ is $\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)$ progressively measurable and

$$
\begin{equation*}
P\left(\left[\int_{a}^{T}\|\Phi\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d s<\infty\right]\right)=1 \tag{65.10.18}
\end{equation*}
$$

More generally, suppose there exists a localizing sequence of stopping times $\tau_{n}$ having the two properties of Lemma 65.10.2. Then for all $\omega$ not in the exceptional set $N$.

$$
\int_{a}^{t} \Phi d W \equiv \lim _{n \rightarrow \infty} \int_{a}^{t} \mathscr{X}_{\left[a, \tau_{n}\right]} \Phi d W
$$

Lemma 65.10.4 The above definition is well defined. For all $\omega$ not in a set of measure zero,

$$
\int_{a}^{t} \Phi d W(\omega) \equiv \lim _{n \rightarrow \infty} \int_{a}^{t} \mathscr{X}_{\left[a, \tau_{n}\right]} \Phi d W(\omega)
$$

the function on the right being constant for all n large enough for a given $\omega$. The random variable $\int_{a}^{t} \Phi d W$ is also $\mathscr{F}_{t}$ adapted.

Proof: Let $\left\{\tau_{n}\right\}$ be a sequence of stopping times as described in 1 and 2 of Lemma 65.10.2. Such a sequence exists by Lemma 65.10.2. It makes sense to define the random variable

$$
\int_{a}^{t} \mathscr{X}_{\left[a, \tau_{n}\right]} \Phi d W
$$

Now what if both $\tau_{m}$ and $\tau_{n}$ are at least as large as $t$ for some $\omega$ ? Do the two random variables coincide at that value of $\omega$ ? Say $m>n$ so that $\tau_{m}(\omega) \geq \tau_{n}(\omega)>t$. For the given $\omega$,

$$
\int_{a}^{t} \mathscr{X}_{\left[a, \tau_{m}\right]} \Phi d W=\int_{a}^{t \wedge \tau_{m}} \mathscr{X}_{\left[a, \tau_{m}\right]} \Phi d W
$$

For the particular $\omega$ of interest,

$$
=\int_{a}^{t \wedge \tau_{n}} \mathscr{X}_{\left[a, \tau_{m}\right]} \Phi d W
$$

and this equals

$$
=\int_{a}^{t} \mathscr{X}_{\left[a, \tau_{n}\right]} \mathscr{X}_{\left[a, \tau_{m}\right]} \Phi d W=\int_{a}^{t} \mathscr{X}_{\left[a, \tau_{n}\right]} \Phi d W
$$

for all $\omega$, in particular for the given $\omega$. Therefore, for the particular $\omega$ of interest,

$$
\int_{a}^{t} \mathscr{X}_{\left[a, \tau_{n}\right]} \Phi d W=\int_{a}^{t} \mathscr{X}_{\left[a, \tau_{m}\right]} \Phi d W
$$

Thus the limit exists because for all $n$ large enough, the integral is eventually constant. Then $\int_{a}^{t} \Phi d W$ is $\mathscr{F}_{t}$ adapted because for $U$ an open set in $H$,

$$
\left(\int_{a}^{t} \Phi d W\right)^{-1}(U)=\cup_{n=1}^{\infty}\left(\left(\int_{a}^{t} \mathscr{X}_{\left[a, \tau_{n}\right]} \Phi d W\right)^{-1}(U) \cap\left[\tau_{n}>t\right]\right) \in \mathscr{F}_{t}
$$

The next lemma says that even when $\int_{a}^{t} \Phi(s) d W(s)$ is only a local martingale relative to a suitable localizing sequence, it is still the case that

$$
\int_{a}^{t \wedge \sigma} \Phi d W=\int_{a}^{t} \mathscr{X}_{[a, \sigma]} \Phi d W
$$

Lemma 65.10.5 Let $\Phi$ be progressively measurable and suppose there exists the localizing sequence described above. Then if $\sigma$ is a stopping time,

$$
\int_{a}^{t \wedge \sigma} \Phi d W(s)=\int_{a}^{t} \mathscr{X}_{[a, \sigma]} \Phi d W(s)
$$

Proof: Let $\left\{\tau_{n}\right\}$ be the localizing sequence described above for which, when the local martingale is stopped, it results in a martingale, (satisfying 1 and 2 on Page 2253). Then by definition,

$$
\begin{aligned}
\int_{a}^{t \wedge \sigma} \Phi d W(s) & \equiv \lim _{n \rightarrow \infty} \int_{a}^{t \wedge \tau_{n} \wedge \sigma} \Phi d W(s) \\
& =\lim _{n \rightarrow \infty} \int_{a}^{t \wedge \tau_{n}} \mathscr{X}_{[a, \sigma]} \Phi d W(s)=\int_{a}^{t} \mathscr{X}_{[a, \sigma]} \Phi d W(s)
\end{aligned}
$$

since $t \wedge \tau_{n}=t$ for all $n$ large enough.

### 65.11 The Quadratic Variation Of The Stochastic Integral

An important corollary of Lemma 65.9.1 concerns the quadratic variation of $\int_{a}^{t} \Phi d W$. It is convenient here to use the notation $\int_{a}^{t} \Phi d W \equiv \Phi \cdot W(t)$. Recall this is a local submartingale $[\Phi \cdot W]$ such that

$$
\|\Phi \cdot W(t)\|_{H}^{2}=[\Phi \cdot W](t)+N(t)
$$

where $N$ is a local martingale. Recall the quadratic variation is unique so that if it acts like the quadratic variation, then it is the quadratic variation. Recall also why this was so. If you have a local martingale equal to the difference of increasing adapted processes which equals 0 when $t=0$, then the local martingale was equal to 0 . Of course you can substitute $a$ for 0 .

Corollary 65.11.1 Suppose $\Phi$ is $\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)$ progressively measurable and has the localizing sequence with the two properties in Lemma 65.10.2. Then the quadratic variation, $[\Phi \cdot W]$ is given by the formula

$$
[\Phi \cdot W](t)=\int_{a}^{t}\|\Phi(s)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d s
$$

Proof: By the above discussion, $\int_{a}^{t} \Phi d W$ is a local martingale. Let $\left\{\tau_{n}\right\}$ be a localizing sequence for which the stopped local martingale is a martingale and $\Phi \mathscr{X}_{\left[a, \tau_{n}\right]}$ is in $L^{2}\left([a, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$. Also let $\sigma$ be a stopping time with two values no larger than $T$. Then from Lemma 65.10.5,

$$
\begin{gathered}
E\left(\left|\int_{a}^{\tau_{n} \wedge \sigma} \Phi d W\right|_{H}^{2}-\int_{a}^{\tau_{n} \wedge \sigma}\|\Phi(s)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d s\right) \\
E\left(\left|\int_{a}^{T \wedge \tau_{n} \wedge \sigma} \Phi d W\right|_{H}^{2}-\int_{a}^{T \wedge \tau_{n} \wedge \sigma}\|\Phi(s)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d s\right) \\
=E\left(\left|\int_{a}^{T} \mathscr{X}_{\left[a, \tau_{n}\right]} \mathscr{X}_{[0, \sigma]} \Phi d W\right|_{H}^{2}-\int_{a}^{T} \mathscr{X}_{\left[a, \tau_{n}\right]} \mathscr{X}_{[0, \sigma]}\|\Phi(s)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d s\right) \\
= \\
E\left(\int_{a}^{T}\left\|\mathscr{X}_{\left[a, \tau_{n}\right]} \mathscr{X}_{[0, \sigma]} \Phi\right\|_{\mathscr{L}_{2}}^{2} d t\right)-E\left(\int_{a}^{T}\left\|\mathscr{X}_{\left[a, \tau_{n}\right]} \mathscr{X}_{[0, \sigma]} \Phi(s)\right\|_{\mathscr{L}_{2}}^{2} d s\right)=0
\end{gathered}
$$

thanks to the Ito isometry. There is also no change in letting $\sigma=t$. You still get 0 . It follows from Lemma 63.1.1, the lemma about recognizing a martingale when you see one, that

$$
t \rightarrow\left|\int_{a}^{t \wedge \tau_{n}} \Phi d W\right|_{H}^{2}-\int_{a}^{t \wedge \tau_{n}}\|\Phi(s)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d s
$$

is a martingale. Therefore,

$$
\left|\int_{a}^{t} \Phi d W\right|_{H}^{2}-\int_{a}^{t}\|\Phi(s)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d s
$$

is a local martingale and so, by uniqueness of the quadratic variation,

$$
[\Phi \cdot W](t)=\int_{a}^{t}\|\Phi(s)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d s
$$

Here is an interesting little lemma which seems to be true.

Lemma 65.11.2 Let $\Phi, \Phi_{n}$ all be in $L^{2}\left([0, T], \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$ off some set of measure zero. These are all progressively measurable. Thus there are all stochastically square integrable.

$$
P\left(\int_{0}^{T}\|\Phi\|^{2} d s\right)=1
$$

Suppose also that for each $\omega \notin N$, the exceptional set,

$$
\int_{0}^{T}\left\|\Phi_{n}-\Phi\right\|_{\mathscr{L}_{2}}^{2} d t \rightarrow 0
$$

Then there exists a set of measure zero, still denoted as $N$ and a subsequence, still denoted as $n$ such that for each $\omega \notin N$,

$$
\lim _{n \rightarrow \infty} \int_{0}^{T} \Phi_{n} d W=\int_{0}^{T} \Phi d W
$$

Proof: Define stopping times

$$
\tau_{n p} \equiv \inf \left\{t \in[0, T]: \int_{0}^{t}\left\|\Phi_{n}\right\|^{2} d s>p\right\}
$$

Let $\tau_{p}$ be similar but defined with reference to $\Phi$. Then by Ito isometry,

$$
\begin{align*}
& E\left(\left|\int_{0}^{T} \mathscr{X}_{\left[0, \tau_{n p}\right]} \Phi_{n} d W-\int_{0}^{T} \mathscr{X}_{\left[0, \tau_{p}\right]} \Phi d W\right|^{2}\right) \\
= & E\left(\int_{0}^{T}\left\|\mathscr{X}_{\left[0, \tau_{n p}\right]} \Phi_{n}-\mathscr{X}_{\left[0, \tau_{p}\right]} \Phi\right\|_{\mathscr{L}_{2}}^{2} d t\right) \tag{65.11.19}
\end{align*}
$$

The integrand in the right side is bounded by $2 p^{2}$. Also this integrand converges to 0 for each $\omega$ as $n \rightarrow \infty$. This is shown next.

$$
\begin{aligned}
& \int_{0}^{T}\left\|\mathscr{X}_{\left[0, \tau_{n p}\right]} \Phi_{n}-\mathscr{X}_{\left[0, \tau_{p}\right]} \Phi\right\|_{\mathscr{L}_{2}}^{2} d t \\
\leq & 2 \int_{0}^{T}\left(\left\|\mathscr{X}_{\left[0, \tau_{n p}\right]} \Phi_{n}-\mathscr{X}_{\left[0, \tau_{n p}\right]} \Phi\right\|_{\mathscr{L}_{2}}^{2}+\left\|\mathscr{X}_{\left[0, \tau_{n p}\right]} \Phi-\mathscr{X}_{\left[0, \tau_{p}\right]} \Phi\right\|_{\mathscr{L}_{2}}^{2}\right) d t \\
\leq & 2 \int_{0}^{T}\left\|\Phi_{n}-\Phi\right\|_{\mathscr{L}_{2}}^{2} d t+2 \int_{0}^{T}\left|\mathscr{X}_{\left[0, \tau_{n p}\right]}(t)-\mathscr{X}_{\left[0, \tau_{p}\right]}(t)\right|^{2}\|\Phi\|_{\mathscr{L}_{2}}^{2} d t
\end{aligned}
$$

The first converges to 0 by assumption. Problem is, it does not look like this second integral converges to 0 . We do know that $\int_{0}^{t}\left\|\Phi_{n}\right\|^{2} d s \rightarrow \int_{0}^{t}\|\Phi\|^{2} d s$ uniformly so $\tau_{n p} \rightarrow \tau_{p}$ is likely. However, this does not imply $\mathscr{X}_{\left[0, \tau_{n p}\right]} \rightarrow \mathscr{X}_{\left[0, \tau_{p}\right]}$. However, it would converge in $L^{2}(0, T)$ and so there is a subsequence such that convergence takes place a.e. $t$. Then restricting to this subsequence, the second integral converges to 0 . Actually, it may be easier than this. $\mathscr{X}_{\left[0, \tau_{p}\right]}$ has a single point of discontinuity and convergence takes place at every other point. Thus it appears that the integrand in 65.11 .19 converges to 0 for each $\omega$. Thus, by dominated convergence theorem the whole expectation converges to 0 .

Now consider

$$
\begin{aligned}
& P\left(\left|\int_{0}^{T} \mathscr{X}_{\left[0, \tau_{n p}\right]} \Phi_{n} d W-\int_{0}^{T} \mathscr{X}_{\left[0, \tau_{p}\right]} \Phi d W\right|^{2}>\lambda\right) \\
\leq & \frac{E\left(\left|\int_{0}^{T} \mathscr{X}_{\left[0, \tau_{n p}\right]} \Phi_{n} d W-\int_{0}^{T} \mathscr{X}_{\left[0, \tau_{p}\right]} \Phi d W\right|^{2}\right)}{\lambda}
\end{aligned}
$$

and so, there exists a subsequence, still denoted as $n$ such that

$$
P\left(\left|\int_{0}^{T} \mathscr{X}_{\left[0, \tau_{n p}\right]} \Phi_{n} d W-\int_{0}^{T} \mathscr{X}_{\left[0, \tau_{p}\right]} \Phi d W\right|^{2}>\frac{1}{n}\right)<2^{-k}
$$

It follows that $N$ can be enlarged so that for $\omega \notin N_{p}$

$$
\left|\int_{0}^{T} \mathscr{X}_{\left[0, \tau_{n p}\right]} \Phi_{n} d W-\int_{0}^{T} \mathscr{X}_{\left[0, \tau_{p}\right]} \Phi d W\right|^{2} \leq \frac{1}{n}
$$

for all $n$ large enough. Now obtain a succession of subsequences for $p=1,2, \cdots$, each a subsequence of the preceeding one such that the above convergence takes place and let $N$ include $\cup_{p} N_{p}$. Then for $\omega \notin N$, and letting $n$ denote the diagonal sequence, it follows that for all $p$,

$$
\lim _{n \rightarrow \infty}\left|\int_{0}^{T} \mathscr{X}_{\left[0, \tau_{n p}\right]} \Phi_{n} d W-\int_{0}^{T} \mathscr{X}_{\left[0, \tau_{p}\right]} \Phi d W\right|=0
$$

For $\omega \notin N$, there is a $p$ such that $\tau_{p}=\infty$. Then this means $\int_{0}^{T}\|\Phi\|^{2} d s<p$. It follows that the same is true for $\Phi_{n}$ for all $n$ large enough. Hence $\tau_{n p}=\infty$ also. Thus, for large enough $n$,

$$
\left|\int_{0}^{T} \Phi_{n} d W-\int_{0}^{T} \Phi d W\right|=\left|\int_{0}^{T} \mathscr{X}_{\left[0, \tau_{n p}\right]} \Phi_{n} d W-\int_{0}^{T} \mathscr{X}_{\left[0, \tau_{p}\right]} \Phi d W\right|
$$

and the latter was just shown to converge to 0 .

### 65.12 The Holder Continuity Of The Integral

Let $\Phi \in L^{2}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$. Then you can consider the stochastic integral as described above and it yields a continuous function off a set of measure zero. What if $\Phi \in L^{\infty}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$ ? Can you say more? The short answer is yes. You obtain a Holder condition in addition to continuity. This is a consequence of the Burkolder Davis Gundy inequality and Corollary 65.11 .1 above. Let $\alpha>2$. Let $\|\Phi\|_{\infty}$ denote the norm in $L^{\infty}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$. By the Burkholder Davis Gundy inequality,

$$
\begin{gathered}
\int_{\Omega}\left(\left|\int_{s}^{t} \Phi d W\right|\right)^{\alpha} d P \leq \\
\int_{\Omega}\left(\sup _{r \in[s, t]}\left|\int_{s}^{r} \Phi d W\right|\right)^{\alpha} d P \leq C \int_{\Omega}\left(\int_{s}^{t}\|\Phi\|^{2} d \tau\right)^{\alpha / 2} d P
\end{gathered}
$$

$$
\leq C\|\Phi\|_{\infty}^{\alpha} \int_{\Omega}\left(\int_{s}^{t} d \tau\right)^{\alpha / 2} d P=C\|\Phi\|_{\infty}^{\alpha}|t-s|^{\alpha / 2}
$$

By the Kolmogorov Čentsov theorem, Theorem 62.2.2, this shows that $t \rightarrow \int_{0}^{t} \Phi d W$ is Holder continuous with exponent

$$
\gamma<\frac{(\alpha / 2)-1}{\alpha}=\frac{1}{2}-\frac{1}{\alpha}
$$

Since $\alpha>2$ is arbitrary, this shows that for any $\gamma<1 / 2$, the stochastic integral is Holder continuous with exponent $\gamma$. This is exactly the same kind of continuity possessed by the Wiener process.

Theorem 65.12.1 Suppose $\Phi \in L^{\infty}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$ and is progressively measurable. Then if $\gamma<1 / 2$, there exists a set of measure zero such that off this set,

$$
t \rightarrow \int_{0}^{t} \Phi d W
$$

is Holder continuous with exponent $\gamma$.

### 65.13 Taking Out A Linear Transformation

When is

$$
L \int_{a}^{T} \Phi d W=\int_{a}^{T} L \Phi d W ?
$$

It is assumed $L \in \mathscr{L}\left(H, H_{1}\right)$ where $H_{1}$ is another separable real Hilbert space. First of all, here is a lemma which shows $\int_{a}^{t} L \Phi d W$ at least makes sense.

Proposition 65.13.1 Suppose $\Phi$ is $\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)$ progressively measurable and

$$
P\left(\left[\int_{a}^{T}\|\Phi\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d s<\infty\right]\right)=1 .
$$

Then the same is true of $L \Phi$. Furthermore, for each $t \in[a, T]$

$$
\int_{a}^{t} L \Phi d W=L \int_{a}^{t} \Phi d W
$$

Proof: First note that if $\Phi \in \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)$, then $L \Phi \in \mathscr{L}_{2}\left(Q^{1 / 2} U, H_{1}\right)$ and that the map $\Phi \rightarrow L \Phi$ is continuous. It follows $L \Phi$ is $\mathscr{L}_{2}\left(Q^{1 / 2} U, H_{1}\right)$ progressively measurable. All that remains is to check the appropriate integral.

$$
\int_{a}^{T}\|L \Phi\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H_{1}\right)}^{2} d t \leq \int_{a}^{T}\|L\|^{2}\|\Phi\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d t
$$

and so this proves $L \Phi$ satisfies the same conditions as $\Phi$, being stochastically square integrable.

It follows one can consider

$$
\int_{a}^{T} L \Phi d W
$$

Assume to begin with that $\Phi \in L^{2}\left([a, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$. Next recall the situation in which the definition of the integral is considered.

$$
\begin{array}{ccc} 
& & \\
& & \\
& & \\
& \\
Q^{1 / 2} U & \stackrel{J}{\leftarrow} & Q^{1 / 2} \\
& & Q^{1 / 2} U \\
\Phi_{n} & \searrow & \\
& & \downarrow \\
& & H
\end{array}
$$

Letting $\left\{\Phi_{n}\right\}$ be an approximating sequence of elementary functions satisfying

$$
E\left(\int_{a}^{T}\left\|\Phi_{n}-\Phi \circ J^{-1}\right\|_{\mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)}^{2} d t\right) \rightarrow 0
$$

it is also the case that

$$
E\left(\int_{a}^{T}\left\|L \Phi_{n}-L \Phi \circ J^{-1}\right\|_{\mathscr{L}_{2}\left(J Q^{1 / 2} U, H_{1}\right)}^{2} d t\right) \rightarrow 0
$$

By the definition of the integral, for each $t$

$$
\begin{aligned}
\int_{a}^{t} L \Phi d W & =\lim _{n \rightarrow \infty} \int_{a}^{t} L \Phi_{n} d W=\lim _{n \rightarrow \infty} L \int_{a}^{t} \Phi_{n} d W \\
& =L \lim _{n \rightarrow \infty} \int_{a}^{t} \Phi_{n} d W=L \int_{a}^{t} \Phi d W
\end{aligned}
$$

The second equality is obvious for elementary functions.
Now consider the case where $\Phi$ is only stochastically square integrable so that all is known is that

$$
P\left(\left[\int_{a}^{T}\|\Phi\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d t<\infty\right]\right)=1 .
$$

Then define $\tau_{n}$ as above

$$
\tau_{n} \equiv \inf \left\{t: \int_{a}^{t}\|\Phi\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d t \geq n\right\}
$$

This sequence of stopping times works for $L \Phi$ also. Recall there were two conditions the sequence of stopping times needed to satisfy. The first is obvious. Here is why the second holds.

$$
\begin{aligned}
\int_{a}^{T}\left\|\mathscr{X}_{\left[a, \tau_{n}\right]} L \Phi\right\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H_{1}\right)}^{2} d t & \leq\|L\|^{2} \int_{a}^{T}\left\|\mathscr{X}_{\left[a, \tau_{n}\right]} \Phi\right\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d t \\
& =\|L\|^{2} \int_{a}^{\tau_{n}}\|\Phi\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d t \leq\|L\|^{2} n
\end{aligned}
$$

Then let $t$ be given and pick $n$ such that $\tau_{n}(\omega) \geq t$. Then from the first part, for that $\omega$,

$$
\begin{aligned}
L \int_{a}^{t} \Phi d W & \equiv L \int_{a}^{t} \mathscr{X}_{\left[a, \tau_{n}\right]} \Phi d W \\
& =\int_{a}^{t} L \mathscr{X}_{\left[a, \tau_{n}\right]} \Phi d W \\
& =\int_{a}^{t} \mathscr{X}_{\left[a, \tau_{n}\right]} L \Phi d W \equiv \int_{a}^{t} L \Phi d W
\end{aligned}
$$

### 65.14 A Technical Integration By Parts Result

Let $Z \in L^{2}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$ where this has reference to the usual diagram

$$
\begin{array}{cccc} 
& & & \\
U_{1} & \supseteq J Q^{1 / 2} U & \stackrel{J}{亡} & \downarrow \\
& & Q^{1 / 2} & Q^{1 / 2} \\
& \Phi_{n} \searrow & & \downarrow \\
& & & \\
& &
\end{array}
$$

Also suppose $X \in L^{2}([0, T] \times \Omega, H)$, both $X$ and $Z$ being progressively measurable. Let $\left\{t_{j}^{n}\right\}_{j=1}^{m_{n}}$ denote a sequence of partitions of the sort discussed earlier where

$$
X_{n}(t) \equiv \sum_{j=0}^{m_{n}-1} X\left(t_{j}^{n}\right) \mathscr{X}_{\left[t_{j}^{n}, t_{j+1}^{n}\right)}(t)
$$

converges to $X$ in $L^{2}([0, T] \times \Omega, H)$. Thus $X_{n}(t)$ is right continuous. Let

$$
\tau_{p}^{n}=\inf \left\{t:\left|X_{n}(t)\right|_{H}>p\right\}
$$

This is the first hitting time of a right continuous adapted process so it is a stopping time. Also there exists a set of measure zero $N$ such that for $\omega \notin N$, then given $t$,

$$
\tau_{p}^{n} \geq t
$$

if $p$ is large enough because of the assumption on $X$. Here is why. There exists a set of measure $0 N$ such that if $\omega \notin N$, then

$$
\int_{0}^{T}\left|X_{n}(t)\right|_{H}^{2} d t=\sum_{j=0}^{m_{n}-1}\left|X\left(t_{k}^{n}\right)\right|_{H}^{2}\left(t_{j+1}^{n}-t_{j}^{n}\right)<\infty
$$

It follows that there exists an upper bound, depending on $\omega$ which dominates each of the values $\left|X\left(t_{k}^{n}\right)\right|_{H}^{2}$. Then if $p$ is larger than this upper bound, $\tau_{p}^{n}=\infty>t$.

Next consider the expression

$$
\begin{equation*}
\sum_{j=0}^{m-1}\left(\int_{t_{j}^{n} \wedge t}^{t_{j+1}^{n} \wedge t} Z(u) d W, X\left(t_{j}^{n}\right)\right)_{H} \tag{65.14.20}
\end{equation*}
$$

This expression is a function of $\omega$.
I want to write this in the form of a stochastic integral. To begin with, consider one of the terms. For simplicity of notation, consider

$$
\left(\int_{a}^{b} Z(u) d W, X(a)\right)_{H}
$$

where $Z \in L^{2}\left([a, b] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$ and $X(a) \in L^{2}(\Omega, H)$. Also assume the function of $\omega,|X(a)|_{H}$, is bounded. There is an Ito integral involved in the above. Let $Z_{n}$ be a sequence of elementary functions defined on $[a, b]$ which converges to $Z \circ J^{-1}$ in $L^{2}\left([a, b] \times \Omega, \mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)\right)$. Then by the definition of the integral,

$$
\left\|\int_{a}^{t} Z(u) d W-\int_{a}^{t} Z_{n}(u) d W\right\|_{L^{2}(\Omega, H)} \rightarrow 0
$$

Also, by the use of a maximal inequality and the fact that the two integrals above are martingales, there is a subsequence, still called $n$ and a set of measure zero $N$ such that for $\omega \notin N$, the convergence

$$
\int_{a}^{t} Z_{n}(u) d W(\omega) \rightarrow \int_{a}^{t} Z(u) d W(\omega)
$$

is uniform for $t \in[a, b]$. Therefore, for such $\omega$,

$$
\left(\int_{a}^{t} Z(u) d W, X(a)\right)_{H}=\lim _{n \rightarrow \infty}\left(\int_{a}^{t} Z_{n}(u) d W, X(a)\right)_{H}
$$

Say $\left.Z_{n}(u)=\sum_{k=0}^{m_{n}-1} Z_{k}^{n} \mathscr{X}_{\left[t_{k}^{n}, t_{k+1}^{n}\right.}\right)(u)$ where $Z_{k}^{n}$ has finitely many values in $\mathscr{L}\left(U_{1}, H\right)_{0}$, the restrictions of $\mathscr{L}\left(U_{1}, H\right)$ to $J Q^{1 / 2} U$. Then the inner product in the above formula on the right is of the form

$$
\begin{aligned}
& \sum_{k=0}^{m_{n}-1}\left(Z_{k}^{n}\left(W\left(t \wedge t_{k+1}^{n}\right)-W\left(t \wedge t_{k}^{n}\right)\right), X(a)\right)_{H} \\
= & \sum_{k=0}^{m_{n}-1}\left(\left(W\left(t \wedge t_{k+1}^{n}\right)-W\left(t \wedge t_{k}^{n}\right)\right),\left(Z_{k}^{n}\right)^{*} X(a)\right)_{U_{1}} \\
= & \sum_{k=0}^{m_{n}-1} \mathscr{R}\left(\left(Z_{k}^{n}\right)^{*} X(a)\right)\left(W\left(t \wedge t_{k+1}^{n}\right)-W\left(t \wedge t_{k}^{n}\right)\right) \\
\equiv & \int_{a}^{t} \mathscr{R}\left(Z_{n}^{*} X(a)\right) d W
\end{aligned}
$$

where $\mathscr{R}$ is the Riesz map from $U_{1}$ to $U_{1}^{\prime}$. Note that $\mathscr{R}\left(Z_{n}^{*} X(a)\right)$ has values in

$$
\mathscr{L}\left(U_{1}, \mathbb{R}\right)_{0} \subseteq \mathscr{L}_{2}\left(J Q^{1 / 2} U, \mathbb{R}\right)
$$

Now let $\left\{g_{i}\right\}$ be an orthonormal basis for $Q^{1 / 2} U$, so it follows that $\left\{J g_{i}\right\}$ is an orthonormal basis for $J Q^{1 / 2} U$. Then

$$
\begin{gathered}
\sum_{i}\left|\mathscr{R}\left(Z_{n}^{*} X(a)-\left(Z \circ J^{-1}\right)^{*} X(a)\right)\left(J g_{i}\right)\right|^{2} \\
\equiv \sum_{i}\left|\left(Z_{n}^{*} X(a)-\left(Z \circ J^{-1}\right)^{*} X(a), J g_{i}\right)_{U_{1}}\right|^{2}=\sum_{i}\left|\left(X(a),\left(Z_{n}-Z \circ J^{-1}\right) J g_{i}\right)_{H}\right|^{2} \\
\leq \sum_{i}|X(a)|_{H}^{2}\left|\left(Z_{n}-Z \circ J^{-1}\right) J g_{i}\right|_{H}^{2}=|X(a)|_{H}^{2}\left\|Z_{n}-Z \circ J^{-1}\right\|_{\mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)}^{2}
\end{gathered}
$$

When integrated over $[a, b] \times \Omega$, it is given that this converges to 0 . This has shown that

$$
\mathscr{R}\left(Z_{n}^{*} X(a)\right) \rightarrow \mathscr{R}\left(\left(Z \circ J^{-1}\right)^{*} X(a)\right)
$$

in $\mathscr{L}_{2}\left(J Q^{1 / 2} U, \mathbb{R}\right)$. In other words

$$
\mathscr{R}\left(Z_{n}^{*} X(a)\right) \rightarrow\left(\mathscr{R}\left(\left(Z \circ J^{-1}\right)^{*} X(a)\right) \circ J\right) \circ J^{-1}
$$

It follows that

$$
\left(\int_{a}^{t} Z(u) d W, X(a)\right)_{H}=\int_{a}^{t} \mathscr{R}\left(\left(Z \circ J^{-1}\right)^{*} X(a)\right) \circ J d W
$$

From localization,

$$
\begin{aligned}
\left(\int_{a \wedge \tau_{p}^{n}}^{b \wedge \tau_{p}^{n}} Z(u) d W, X(a)\right)_{H} & =\left(\int_{a}^{b} \mathscr{X}_{\left[0, \tau_{p}^{n}\right]} Z(u) d W, X(a)\right)_{H} \\
& =\int_{a}^{b} \mathscr{X}_{\left[0, \tau_{p}^{n}\right]} \mathscr{R}\left(\left(Z \circ J^{-1}\right)^{*} X(a)\right) \circ J d W \\
& =\int_{a \wedge \tau_{p}^{n}}^{b \wedge \tau_{p}^{n}} \mathscr{R}\left(\left(Z \circ J^{-1}\right)^{*} X(a)\right) \circ J d W
\end{aligned}
$$

Then it follows that, using the stopping time,

$$
\begin{gathered}
\sum_{j=0}^{m-1}\left(\int_{t_{j}^{n} \wedge \tau_{p}^{n} \wedge t}^{t_{j+1}^{n} \wedge \tau_{\wedge}^{n} \wedge t} Z(u) d W, X\left(t_{j}^{n}\right)\right)_{H}=\sum_{j=0}^{m-1} \int_{t_{j}^{n} \wedge \tau_{p}^{n} \wedge t}^{t_{j+1}^{n} \wedge \tau_{p}^{n} \wedge t} \mathscr{R}\left(\left(Z \circ J^{-1}\right)^{*} X_{n}\left(t_{j}^{n}\right)\right) \circ J d W \\
=\int_{0}^{t_{m}^{n} \wedge \tau_{p}^{n} \wedge t} \mathscr{R}\left(\left(Z \circ J^{-1}\right)^{*}\left(X_{n}^{l}\right)\right) \circ J d W
\end{gathered}
$$

where $X_{n}^{l}$ is the step function

$$
X_{n}^{l}(t) \equiv \sum_{k=0}^{m_{n}-1} X\left(t_{k}^{n}\right) \mathscr{X}_{\left[t_{k}^{n}, t_{k+1}^{n}\right)}(t)
$$

By localization, this is

$$
\int_{0}^{t} \mathscr{X}_{\left[0, \tau_{p}^{n}\right]} \mathscr{R}\left(\left(Z \circ J^{-1}\right)^{*}\left(X_{n}^{l}\right)\right) \circ J d W
$$

If $\omega$ is not in a suitable set of measure zero, then $\tau_{p}^{n}(\omega) \geq t$ provided $p$ is large enough. Thus, for such $\omega$, if $p$ is large enough,

$$
\begin{aligned}
\sum_{j=0}^{m_{n}-1}\left(\int_{t_{j}^{n} \wedge \tau_{p}^{n} \wedge t}^{t_{j+1}^{n} \wedge \tau_{p}^{n} \wedge t} Z(u) d W, X\left(t_{j}^{n}\right)\right)_{H} & =\int_{0}^{t} \mathscr{X}_{\left[0, \tau_{p}^{n}\right]} \mathscr{R}\left(\left(Z \circ J^{-1}\right)^{*}\left(X_{n}^{l}\right)\right) \circ J d W \\
& =\int_{0}^{t} \mathscr{R}\left(\left(Z \circ J^{-1}\right)^{*}\left(X_{n}^{l}\right)\right) \circ J d W
\end{aligned}
$$

This shows that the expression is a local martingale. Also note that the expression on the left does not depend on $J$ or $U_{1}$ so the same must be true of the expression on the right although it does not look that way. This has proved the following important theorem.

Theorem 65.14.1 Let $Z \in L^{2}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$ and let $X \in L^{2}([0, T] \times \Omega, H)$, both $X, Z$ progressively measurable. Also let $\left\{t_{j}^{n}\right\}_{j=1}^{m_{n}}$ be a sequence of partitions of $[0, T]$ such that each $X\left(t_{j}^{n}\right)$ is in $L^{2}(\Omega, H)$. Then

$$
\begin{equation*}
\sum_{j=0}^{m-1}\left(\int_{t_{j}^{n} \wedge t}^{t_{j+1}^{n} \wedge t} Z(u) d W, X\left(t_{j}^{n}\right)\right)_{H} \tag{65.14.21}
\end{equation*}
$$

is a stochastic integral of the form

$$
\int_{0}^{t} \mathscr{R}\left(\left(Z \circ J^{-1}\right)^{*}\left(X_{n}^{l}\right)\right) \circ J d W
$$

where $\left\{\tau_{p}^{n}\right\}_{p=1}^{\infty}$ is a localizing sequence used to define the above integral whose integrand is only stochastically square integrable. Here $X_{n}^{l}$ is the step function defined by

$$
X_{n}^{l}(t) \equiv \sum_{k=0}^{m_{n}-1} X\left(t_{k}^{n}\right) \mathscr{X}_{\left[t_{k}^{n}, t_{k+1}^{n}\right)}(t)
$$

In particular, 65.14 .21 is a local martingale.
Of course it would be very interesting to see what happens in the case where $X_{n}^{l} \rightarrow X$ in $L^{2}([0, T] \times \Omega, H)$. Is it the case that convergence to

$$
\begin{equation*}
\int_{0}^{t} \mathscr{R}\left(\left(Z \circ J^{-1}\right)^{*}(X)\right) \circ J d W \tag{65.14.22}
\end{equation*}
$$

happens in some sense? Also, does the above stochastic integral even make sense? First of all, consider the question whether it makes sense. It would be nice to define a stopping time

$$
\tau_{n} \equiv \inf \left\{t:|X(t)|_{H}>n\right\}
$$

because then $\mathscr{X}_{\left[0, \tau_{n}\right]} \mathscr{R}\left(\left(Z \circ J^{-1}\right)^{*}(X)\right) \circ J$ would end up being integrable in the right way and you could define the stochastic integral provided $\tau_{n}>t$ whenever $n$ is large enough. However, this is problematic because $t \rightarrow X(t)$ is not known to be continuous. Therefore, some other condition must be assumed.

Lemma 65.14.2 Suppose $t \rightarrow X(t)$ is weakly continuous into $H$ for a.e. $\omega$, and that $X$ is adapted. Then the $\tau_{n}$ described above is a stopping time.

Proof: Let $B \equiv\{x \in H:|x|>n\}$. Then the complement of $B$ is a closed convex set. It follows that $B^{C}$ is also weakly closed. Hence $B$ must be weakly open. Now $t \rightarrow X(t)$ is adapted as a function mapping into the topological space consisting of $H$ with the weak topology because it is in fact adapted into the strong topolgy. Therefore, the above $\tau_{n}$ is just the first hitting time of an open set by a continuous process so $\tau_{n}$ is a stopping time by Proposition 62.7.5. Also, by the assumption that $t \rightarrow X(t)$ is weakly continuous, it follows that $X(t)$ for $t \in[0, T]$ is weakly bounded. Hence, for each $\omega$ off a set of measure zero, $|X(t)|$ is bounded for $t \in[0, T]$. This follows from the uniform boundedness theorem. It follows that $\tau_{n}=\infty$ for $n$ large enough.

Hence the weak continuity of $t \rightarrow X(t)$ suffices to define the stochastic integral in 65.14.22. It remains to verify some sort of convergence in the case that

$$
\lim _{n \rightarrow \infty}\left[\max _{j \leq m_{n}-1}\left(t_{j+1}^{n}-t_{j}^{n}\right)\right]=0
$$

Lemma 65.14.3 Let $X(s)-X_{k}^{l}(s) \equiv \Delta_{k}(s)$. Here $Z \in L^{2}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$ and let $X \in L^{2}([0, T] \times \Omega, H)$ with both $X$ and $Z$ progressively measurable, $t \rightarrow X(t)$ being weakly continuous into $H$,

$$
\lim _{k \rightarrow \infty}\left\|X-X_{k}^{l}\right\|_{L^{2}([0, T] \times \Omega, H)}=0
$$

Then the integral

$$
\int_{0}^{t} \mathscr{R}\left(\left(Z \circ J^{-1}\right)^{*}(X)\right) \circ J d W
$$

exists as a local martingale and the following limit occurs for a suitable subsequence, still called $k$.

$$
\begin{equation*}
\lim _{k \rightarrow \infty} P\left(\left[\sup _{t \in[0, T]}\left|\int_{0}^{t} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*} \Delta_{k}(s)\right) \circ J d W(s)\right| \geq \varepsilon\right]\right)=0 \tag{65.14.23}
\end{equation*}
$$

That is,

$$
\sup _{t \in[0, T]}\left|\int_{0}^{t} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*}\left(X(s)-X_{k}^{l}(s)\right)\right) \circ J d W(s)\right|
$$

converges to 0 in probability.
Proof: Let $k$ denote a subsequence for which $X_{k}^{l}$ also converges pointwise to $X$.

The existence of the integral follows from Lemma 65.14.2. From the assumption of weak continuity, $\sup _{t \in[0, T]}|X(t)| \leq C(\omega)$ for a.e. $\omega$. For the first part of the argument, assume $C$ does not depend on $\omega$ off a set of measure zero. Let

$$
M(t) \equiv \int_{0}^{t} Z d W
$$

Let $\left\{e_{k}\right\}$ be an orthonormal basis for $H$ and let $P_{n}$ be the orthogonal projection onto $\operatorname{span}\left(e_{1}, \cdots, e_{n}\right)$. For each $e_{i}$

$$
\lim _{k \rightarrow \infty}\left|\left(X(s)-X_{k}^{l}(s), e_{i}\right)\right|=0
$$

and so, by weak continuity,

$$
\lim _{k \rightarrow \infty} P_{n}\left(X(s)-X_{k}^{l}(s)\right)=0 \text { for a.e. } \omega
$$

Then

$$
\lim _{k \rightarrow \infty} \int_{\Omega} \int_{0}^{T}\left|P_{n}\left(X(s)-X_{k}^{l}(s)\right)\right|^{2}\|Z(s)\|_{\mathscr{L}_{2}}^{2} d s d P=0
$$

because you can apply the dominated convergence theorem with respect to the measure $\|Z(s)\|_{\mathscr{L}_{2}}^{2} d s d P$.

Therefore,

$$
\begin{equation*}
\lim _{k \rightarrow \infty} P\left(\left[\sup _{t \in[0, T]}\left|\int_{0}^{t} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*} P_{n} \Delta_{k}(s)\right) \circ J d W(s)\right| \geq \varepsilon / 2\right]\right)=0 \tag{65.14.24}
\end{equation*}
$$

Here is why. By the Burkholder Davis Gundy theorem, Theorem 63.4.4 and Corollary 65.11.1 which describes the quadratic variation of the stochastic integral,

$$
\begin{aligned}
& \int_{\Omega}\left(\sup _{t \in[0, T]}\left|\int_{0}^{t} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*} P_{n} \Delta_{k}(s)\right) d W(s)\right|\right) d P \\
& \leq C \int_{\Omega}\left(\int_{0}^{T}\left|P_{n}\left(X(s)-X_{k}^{l}(s)\right)\right|^{2}\|Z(s)\|_{\mathscr{L}_{2}}^{2} d s\right)^{1 / 2} d P
\end{aligned}
$$

Consider the following two probabilities.

$$
\begin{align*}
& P\left(\left[\sup _{t \in[0, T]}\left|\int_{0}^{t} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*}\left(I-P_{n}\right) X(s)\right) \circ J d W(s)\right| \geq \varepsilon / 2\right]\right)  \tag{65.14.25}\\
& P\left(\left[\sup _{t \in[0, T]}\left|\int_{0}^{t} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*}\left(I-P_{n}\right) X_{k}^{l}(s)\right) \circ J d W(s)\right| \geq \varepsilon / 2\right]\right) \tag{65.14.26}
\end{align*}
$$

By Corollary 63.4.5 which depends on the Burkholder Davis Gundy inequality and Corollary 65.11 .1 which describes the quadratic variation of the stochastic integral, 65.14.25
is dominated by

$$
\begin{align*}
& \frac{C}{\varepsilon} E\left(\left(\int_{0}^{T}\|Z(s)\|^{2}\left|\left(I-P_{n}\right) X(s)\right|^{2} d s\right)^{1 / 2} \wedge \delta\right) \\
&+P\left(\left[\left(\int_{0}^{T}\|Z(s)\|^{2}\left|\left(I-P_{n}\right) X(s)\right|^{2} d s\right)^{1 / 2}>\delta\right]\right) \\
& \leq \frac{C \delta}{\varepsilon}+P\left(\left[\left(\int_{0}^{T}\|Z(s)\|^{2}\left|\left(I-P_{n}\right) X(s)\right|^{2} d s\right)^{1 / 2}>\delta\right]\right) \tag{65.14.27}
\end{align*}
$$

Let $\eta>0$ be given. Then let $\delta$ be small enough that the first term is less than $\eta$. Fix such a $\delta$.

Consider the second of the above terms.

$$
\begin{aligned}
& P\left(\left[\left(\int_{0}^{T}\|Z(s)\|^{2}\left|\left(I-P_{n}\right) X(s)\right|^{2} d s\right)^{1 / 2}>\delta\right]\right) \\
\leq & \frac{1}{\delta}\left(E\left(\int_{0}^{T}\|Z(s)\|^{2}\left|\left(I-P_{n}\right) X(s)\right|^{2} d s\right)\right)^{1 / 2}
\end{aligned}
$$

and this converges to 0 because $\left(I-P_{n}\right) X(s)$ is assumed to be bounded and converges to 0 . Next consider 65.14.26. By similar reasoning, we end up with having to estimate

$$
\frac{1}{\delta}\left(E\left(\int_{0}^{T}\|Z(s)\|^{2}\left|\left(I-P_{n}\right) X_{k}^{l}(s)\right|^{2} d s\right)\right)^{1 / 2}
$$

But this is dominated by

$$
\begin{aligned}
& \frac{2}{\delta}\left(E\left(\int_{0}^{T}\|Z(s)\|^{2}\left|X_{k}^{l}(s)-X(s)\right|^{2} d s\right)\right)^{1 / 2} \\
& +\frac{2}{\delta}\left(E\left(\int_{0}^{T}\|Z(s)\|^{2}\left|\left(I-P_{n}\right) X(s)\right|^{2} d s\right)\right)^{1 / 2}
\end{aligned}
$$

The first term is no larger than $\eta$ provided $k$ is large enough, independent of $n$ thanks to the pointwise convergence and the assumption that $X$ is bounded. Thus, there exists $K$ such that if $k>K$, then the term in 65.14 .26 is dominated by

$$
2 \eta+\frac{2}{\delta}\left(E\left(\int_{0}^{T}\|Z(s)\|^{2}\left|\left(I-P_{n}\right) X(s)\right|^{2} d s\right)\right)^{1 / 2}
$$

It follows that for $k>K$, the sum of 65.14 .25 and 65.14 .26 is dominated by

$$
3 \eta+\frac{3}{\delta}\left(E\left(\int_{0}^{T}\|Z(s)\|^{2}\left|\left(I-P_{n}\right) X(s)\right|^{2} d s\right)\right)^{1 / 2}
$$

This is then no larger than $4 \eta$ provided $n$ is large enough. Pick such an $n$. Then for all $k>K$, this has shown that

$$
\begin{aligned}
& P\left(\left[\sup _{t \in[0, T]}\left|\int_{0}^{t} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*} \Delta_{k}(s)\right) \circ J d W(s)\right| \geq \varepsilon\right]\right) \\
& \leq P\left(\left[\sup _{t \in[0, T]}\left|\int_{0}^{t} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*} P_{n} \Delta_{k}(s)\right) \circ J d W(s)\right| \geq \varepsilon / 2\right]\right)+ \\
& P\left(\left[\sup _{t \in[0, T]}\left|\int_{0}^{t} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*}\left(I-P_{n}\right) \Delta_{k}(s)\right) \circ J d W(s)\right| \geq \varepsilon / 2\right]\right) \\
& \leq P\left(\left[\sup _{t \in[0, T]}\left|\int_{0}^{t} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*} P_{n} \Delta_{k}(s)\right) \circ J d W(s)\right| \geq \varepsilon / 2\right]\right)+4 \eta
\end{aligned}
$$

By 65.14 .24 this whole thing is less than $5 \eta$ provided $k$ is large enough. This has proved that under the assumption that $X$ is bounded uniformly off a set of measure zero,

$$
\lim _{k \rightarrow \infty} P\left(\left[\sup _{t \in[0, T]}\left|\int_{0}^{t} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*} \Delta_{k}(s)\right) \circ J d W(s)\right| \geq \varepsilon\right]\right)=0
$$

This is what was desired to show. It remains to remove the extra assumption that $X$ is bounded.

Now to finish the argument, define the stopping time

$$
\tau_{m} \equiv \inf \left\{t>0:|X(t)|_{H}>m\right\}
$$

As observed in Lemma 65.14.2, this is a valid stopping time. Also define $\Delta_{k}^{\tau_{m}} \equiv X^{\tau_{m}}-$ $\left(X_{k}^{l}\right)^{\tau_{m}}$. Using this stopping time on $X$ and $X_{k}^{l}$ does not affect the pointwise convergence to 0 as $k \rightarrow \infty$ of $\Delta_{k}^{\tau_{m}}$ on which the above argument depends.

Consider

$$
A_{k \varepsilon} \equiv\left[\sup _{t \in[0, T]}\left|\int_{0}^{t} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*} \Delta_{k}(s)\right) \circ J d W(s)\right| \geq \varepsilon\right]
$$

Then

$$
P\left(A_{k \varepsilon} \cap\left[\tau_{m}=\infty\right]\right) \leq P\left(\left[\sup _{t \in[0, T]}\left|\int_{0}^{t} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*} \Delta_{k}^{\tau_{m}}(s)\right) \circ J d W(s)\right| \geq \varepsilon\right]\right)
$$

which converges to 0 as $k \rightarrow \infty$ by the first part of the argument. This is because $\left|X^{\tau_{m}}\right|$ and $\left|\left(X_{k}^{l}\right)^{\tau_{m}}\right|$ are both bounded by $m$ and the same pointwise convergence condition still holds. Now

$$
A_{k \varepsilon}=\cup_{m=1}^{\infty} A_{k \varepsilon} \cap\left(\left[\tau_{m}=\infty\right] \backslash\left[\tau_{m-1}<\infty\right]\right)
$$

Thus

$$
\begin{equation*}
P\left(A_{k \varepsilon}\right)=\sum_{m=1}^{\infty} P\left(A_{k \varepsilon} \cap\left(\left[\tau_{m}=\infty\right] \backslash\left[\tau_{m-1}<\infty\right]\right)\right) \tag{65.14.28}
\end{equation*}
$$

Also

$$
P\left(A_{k \varepsilon} \cap\left(\left[\tau_{m}=\infty\right] \backslash\left[\tau_{m-1}<\infty\right]\right)\right) \leq P\left(\left[\tau_{m}=\infty\right] \backslash\left[\tau_{m-1}<\infty\right]\right)
$$

which is summable because these are disjoint sets. Hence one can apply the dominated convergence theorem in 65.14.28 and conclude

$$
\lim _{k \rightarrow \infty} P\left(A_{k \varepsilon}\right)=\sum_{m=1}^{\infty} \lim _{k \rightarrow \infty} P\left(A_{k \varepsilon} \cap\left(\left[\tau_{m}=\infty\right] \backslash\left[\tau_{m-1}<\infty\right]\right)\right)=0 \square
$$

## Chapter 66

## The Integral $\int_{0}^{t}(Y, d M)_{H}$

First the integral is defined for elementary functions.
Definition 66.0.1 Let an elementary function be one which is of the form

$$
\sum_{i=0}^{m-1} Y_{i} \mathscr{X}_{\left(t_{i}, t_{i+1}\right]}(t)
$$

where $Y_{i}$ is $\mathscr{F}_{t_{i}}$ measurable with values in $H$ a separable real Hilbert space for $0=t_{0}<$ $t_{1}<\cdots<t_{m}=T$.

Definition 66.0.2 Now let $M$ be a $H$ valued continuous local martingale, $M(0)=0$. Then for $Y$ a simple function as above,

$$
\int_{0}^{t}(Y, d M) \equiv \sum_{i=0}^{m-1}\left(Y_{i}, M\left(t \wedge t_{i+1}\right)-M\left(t \wedge t_{i}\right)\right)_{H}
$$

Assumption 66.0.3 We will always assume that $d[M]$ is absolutely continuous with respect to Lebesgue measure. Thus $d[M]=k d t$ where $k \geq 0$ and is in $L^{1}([0, T] \times \Omega)$. This is done to avoid technical questions related to whether $t \rightarrow \int_{0}^{t} d[M]$ is continuous and also to make it easier to get examples of a certain class of functions.

This includes the usual stochastic integral $M(t)=\int_{0}^{t} \Phi d W$ where $[M](t)=\int_{0}^{t}\|\Phi\|_{\mathscr{L}_{2}}^{2} d s$ so $d[M]=\|\Phi\|^{2} d t$.

Next is to consider how this relates to stopping times which have values in the $\left\{t_{i}\right\}$. Let $\tau$ be a stopping time which takes the values $\left\{t_{i}\right\}_{i=0}^{m}$. Then

$$
\begin{equation*}
\int_{0}^{t \wedge \tau}(Y, d M) \equiv \sum_{i=0}^{m-1}\left(Y_{i}, M\left(t \wedge t_{i+1} \wedge \tau\right)-M\left(t \wedge t_{i} \wedge \tau\right)\right)_{H} \tag{66.0.1}
\end{equation*}
$$

Now consider $\mathscr{X}_{[0, \tau]} Y$. Is it also an elementary function?

$$
\mathscr{X}_{[0, \tau]} Y=\sum_{i=0}^{m-1} \mathscr{X}_{[0, \tau]}(t) Y_{i} \mathscr{X}_{\left(t_{i}, t_{i+1}\right]}(t)
$$

To get the $i^{t h}$ term to be non zero, you must have $\tau \geq t$ and $t \in\left(t_{i}, t_{i+1}\right]$. Thus it must be the case that $\tau>t_{i}$. Also, if $\tau>t_{i}$ and $t \in\left(t_{i}, t_{i+1}\right]$, then $\tau \geq t_{i+1}$ because $\tau$ has only the values $t_{i}$. Hence also $\tau \geq t$. Thus the above sum reduces to

$$
\sum_{i=0}^{m-1} \mathscr{X}_{\left[\tau>t_{i}\right]}(\omega) Y_{i} \mathscr{X}_{\left(t_{i}, t_{i+1}\right]}(t)
$$

This shows that $\mathscr{X}_{[0, \tau]} Y$ is of the right sort, the sum of $\mathscr{F}_{t_{i}}$ measurable functions times $\mathscr{X}_{\left(t_{i}, t_{i+1}\right]}(t)$. Thus from the definition of this funny integral,

$$
\begin{equation*}
\int_{0}^{t}\left(\mathscr{X}_{[0, \tau]} Y, d M\right) \equiv \sum_{i=0}^{m-1}(\overbrace{\mathscr{X}_{\left[\tau>t_{i}\right]}(\omega) Y_{i}}^{\mathscr{F}_{i}}, M\left(t \wedge t_{i+1}\right)-M\left(t \wedge t_{i}\right))_{H} \tag{66.0.2}
\end{equation*}
$$

Are the right sides of 66.0.1 and 66.0.2 equal?
Begin with the right side of 66.0 .1 and consider $\tau=t_{j}$. Then to get something nonzero in the terms of the sum in 66.0.1, you would need to have $t_{j} \geq t_{i+1}$. Otherwise, $t_{j} \leq t_{i}$ and the difference involving $M$ would give 0 . Hence, for such $\omega$ you would need to have the sum in 66.0.1 equal to

$$
\sum_{i=0}^{j-1}\left(Y_{i}, M\left(t \wedge t_{i+1}\right)-M\left(t \wedge t_{i}\right)\right)_{H}
$$

Thus this sum in 66.0.1 equals

$$
\sum_{j=0}^{m} \mathscr{X}_{\left[\tau=t_{j}\right]} \sum_{i=0}^{j-1}\left(Y_{i}, M\left(t \wedge t_{i+1}\right)-M\left(t \wedge t_{i}\right)\right)_{H}
$$

Of course when $j=0$ the term in the sum in 66.0.1 equals 0 so there is no harm in defining $\sum_{i=0}^{-1} \equiv 0$. Then from the sum, you have $i \leq j-1$ and so when you interchange the order, you get that $\int_{0}^{t \wedge \tau}(Y, d M)=$

$$
\begin{aligned}
& \sum_{i=0}^{m-1} \sum_{j=i+1}^{m} \mathscr{X}_{\left[\tau=t_{j}\right]}\left(Y_{i}, M\left(t \wedge t_{i+1}\right)-M\left(t \wedge t_{i}\right)\right)_{H} \\
& =\sum_{i=0}^{m-1}\left(\mathscr{X}_{\left[\tau>t_{i}\right]}(\omega) Y_{i}, M\left(t \wedge t_{i+1}\right)-M\left(t \wedge t_{i}\right)\right)_{H}
\end{aligned}
$$

Thus the right side of 66.0.1 equals the right side of 66.0.2.

$$
\int_{0}^{t}\left(\mathscr{X}_{[0, \tau]} Y, d M\right)=\sum_{i=0}^{m-1}\left(\mathscr{X}_{\left[\tau>t_{i}\right]}(\omega) Y_{i}, M\left(t \wedge t_{i+1}\right)-M\left(t \wedge t_{i}\right)\right)=\int_{0}^{t \wedge \tau}(Y, d M)
$$

This has proved the first part of the following lemma.
Lemma 66.0.4 For an elementary function $Y$, and a stopping time $\tau$ having values in the $\left\{t_{i}\right\}$, the points of discontinuity of $Y$, it follows that $\mathscr{X}_{[0, \tau]} Y$ is also an elementary function and

$$
\int_{0}^{t \wedge \tau}(Y, d M)=\int_{0}^{t}\left(\mathscr{X}_{[0, \tau]} Y, d M\right)=\int_{0}^{t}\left(Y, d M^{\tau}\right)
$$

Proof: Consider the second equal sign. By definition,

$$
\begin{aligned}
\int_{0}^{t \wedge \tau}(Y, d M) & =\sum_{i=0}^{m-1}\left(Y_{i}, M\left(t \wedge t_{i+1} \wedge \tau\right)-M\left(t \wedge t_{i} \wedge \tau\right)\right)_{H} \\
& =\sum_{i=0}^{m-1}\left(Y_{i}, M^{\tau}\left(t \wedge t_{i+1}\right)-M^{\tau}\left(t \wedge t_{i}\right)\right)_{H} \equiv \int_{0}^{t}\left(Y, d M^{\tau}\right)
\end{aligned}
$$

Next is another lemma about these integrals of elementary functions. First recall the following definition

$$
M^{*} \equiv \sup \{\|M(t)\|: t \in[0, T]\}
$$

Lemma 66.0.5 Let $M$ be a local martingale on $[0, T]$ where $M(0)=0$ and $M$ is continuous. Let $0<r<s<T$ and consider $\left(Y,\left(M^{\tau_{p} s}-M^{\tau_{p} r}\right)(t)\right)$ where $Y\left(M^{\tau_{p}}\right)^{*} \in L^{2}(\Omega)$ and $Y$ is $\mathscr{F}_{r}$ measurable and $\tau_{p}$ is a localizing sequence of stopping times for which $M^{\tau_{p}}$ is a $L^{2}$ martingale. Then this is a martingale on $[0, T]$ which equals 0 at $t=0$ and

$$
\begin{aligned}
{\left[\left(Y,\left(M^{\tau_{p} s}-M^{\tau_{p} r}\right)\right)\right](t) } & \leq\|Y\|^{2}\left[M^{\tau_{p} s}-M^{\tau_{p} r}\right](t) \\
& =\|Y\|^{2}\left(\left[M^{\tau_{p}}\right]^{s}(t)-\left[M^{\tau_{p}}\right]^{r}(t)\right) \\
& =\|Y\|^{2}\left(\left[M^{\tau_{p}}\right](t \wedge s)-\left[M^{\tau_{p}}\right](t \wedge r)\right)
\end{aligned}
$$

It follows that for $Y$ an elementary function where each $Y_{i}\left(M^{\tau_{p}}\right)^{*}$ is in $L^{2}(\Omega)$,

$$
\int_{0}^{t}(Y, d M)
$$

is a local martingale.
Proof: To save notation, $M$ is written in place of $M^{\tau_{p}}$. It is clear that $\left(Y,\left(M^{s}-M^{r}\right)(t)\right)=$ 0 if $t \leq r$. Is it a martingale?

$$
\begin{aligned}
E\left(\left(Y,\left(M^{s}-M^{r}\right)(t)\right)\right) & =E\left(E\left(\left(Y,\left(M^{s}-M^{r}\right)(t)\right) \mid \mathscr{F}_{r}\right)\right) \\
& =E\left(\left(Y, E\left((M(s \wedge t)-M(r \wedge t)) \mid \mathscr{F}_{r}\right)\right)\right)=0
\end{aligned}
$$

because $M$ is a martingale. Now let $\sigma$ be a bounded stopping time with two values. Then using the optional sampling theorem where needed,

$$
\begin{aligned}
E\left(\left(Y,\left(M^{s}-M^{r}\right)(\sigma)\right)\right) & =E\left(E\left(\left(Y,\left(M^{s}-M^{r}\right)(\sigma)\right) \mid \mathscr{F}_{r}\right)\right) \\
& =E\left(\left(Y, E\left((M(s \wedge \sigma)-M(r \wedge \sigma)) \mid \mathscr{F}_{r}\right)\right)\right) \\
& =E((Y, M(s \wedge \sigma \wedge r)-M(r \wedge \sigma))) \\
& =E((Y, M(\sigma \wedge r)-M(r \wedge \sigma)))=0
\end{aligned}
$$

It follows that this is indeed a martingale as claimed.
By the definition of the quadratic variation,

$$
\begin{gathered}
\left|\left(Y,\left(M^{s}-M^{r}\right)(t)\right)\right|^{2} \leq\|Y\|^{2}\left\|\left(M^{s}-M^{r}\right)(t)\right\|^{2} \\
=\|Y\|^{2}\left[\left(M^{s}-M^{r}\right)\right](t)+\|Y\|^{2} \hat{N}(t)
\end{gathered}
$$

where $\hat{N}(t)$ is a martingale. It equals 0 if $t \leq r$. By similar reasoning to the above, $\|Y\|^{2} \hat{N}(t)$ is a martingale. To see this,

$$
\begin{aligned}
E\left(\|Y\|^{2} \hat{N}(\sigma)\right) & =E\left(E\left(\|Y\|^{2} \hat{N}(\sigma) \mid \mathscr{F}_{r}\right)\right) \\
& =E\left(\|Y\|^{2} E\left(\hat{N}(\sigma) \mid \mathscr{F}_{r}\right)\right) \\
& =E\left(\|Y\|^{2} N(\sigma \wedge r)\right)=0
\end{aligned}
$$

One also sees that $E\left(\|Y\|^{2} \hat{N}(t)\right)=0$.

Now it follows from Corollary 63.3.3 that

$$
\left[\left(M^{s}-M^{r}\right)\right]=\left[M^{s}\right]-\left[M^{r}\right]=[M]^{s}-[M]^{r}
$$

Hence

$$
\left[\left(Y,\left(M^{s}-M^{r}\right)\right)\right](t) \leq\|Y\|^{2}\left[M^{s}-M^{r}\right](t)=\|Y\|^{2}\left([M]^{s}(t)-[M]^{r}(t)\right)
$$

as claimed.
The last claim is easy. Let $\tau_{p}$ be a localizing sequence for which $M^{\tau_{p}}$ is a martingale. Then

$$
\begin{aligned}
\int_{0}^{t \wedge \tau_{p}}(Y, d M) & \equiv \sum_{i=0}^{m-1}\left(Y_{i}, M\left(t \wedge t_{i+1} \wedge \tau_{p}\right)-M\left(t \wedge t_{i} \wedge \tau_{p}\right)\right)_{H} \\
& =\sum_{i=0}^{m-1}\left(Y_{i}, M^{\tau_{p}}\left(t \wedge t_{i+1}\right)-M^{\tau_{p}}\left(t \wedge t_{i}\right)\right)_{H}
\end{aligned}
$$

a finite sum of martingales.
Note that this is just a definition and did not use the above localization lemma. In particular, $\tau_{p}$ is not restricted to having only the partition points as values.

Next one needs to generalize past the elementary functions.
Continue writing $M$ in place of $M^{\tau_{p}}$ in what follows. Consider an elementary function

$$
Y \equiv \sum_{k=0}^{m_{n}-1} Y_{k} \mathscr{X}_{\left(t_{k}, t_{k+1}\right]}(t)
$$

where $Y_{k} M^{*} \in L^{2}(\Omega)$. Consider

$$
\begin{equation*}
\int_{0}^{t}(Y, d M) \equiv \sum_{k=0}^{m_{n}-1}\left(Y_{k}, M\left(t \wedge t_{k+1}\right)-M\left(t \wedge t_{k}\right)\right) \tag{66.0.3}
\end{equation*}
$$

Then it is routine to verify that

$$
\begin{gather*}
E\left(\left(\sum_{k=0}^{m_{n}-1}\left(Y_{k}, M\left(t \wedge t_{k+1}\right)-M\left(t \wedge t_{k}\right)\right)_{H}\right)^{2}\right) \\
=\sum_{k=0}^{m_{n}-1} E\left(\left(Y_{k}, M\left(t \wedge t_{k+1}\right)-M\left(t \wedge t_{k}\right)\right)_{H}^{2}\right) \tag{66.0.4}
\end{gather*}
$$

This is because the mixed terms all vanish. This follows from the following reasoning. Let $t_{j}<t_{k}$

$$
\begin{gathered}
E\left(\left(Y_{k}, M\left(t \wedge t_{k+1}\right)-M\left(t \wedge t_{k}\right)\right)_{H}\left(Y_{j}, M\left(t \wedge t_{j+1}\right)-M\left(t \wedge t_{j+1}\right)\right)_{H}\right) \\
=E\left(E\left(\left(Y_{k}, \Delta_{k} M(t)\right)_{H}\left(Y_{j}, \Delta_{j} M(t)\right)_{H} \mid \mathscr{F}_{t_{k}}\right)\right)
\end{gathered}
$$

$$
\begin{gathered}
=E\left(\left(Y_{j}, \Delta_{j} M(t)\right)_{H} E\left(\left(Y_{k}, \Delta_{k} M(t)\right)_{H} \mid \mathscr{F}_{t_{k}}\right)\right) \\
=E\left(\left(Y_{j}, \Delta_{j} M(t)\right)_{H}\left(Y_{k}, E\left(\Delta_{k} M(t) \mid \mathscr{F}_{t_{k}}\right)\right)_{H}\right) \\
\quad=E\left(\left(Y_{j}, \Delta_{j} M(t)\right)_{H}\left(Y_{k}, 0\right)_{H}\right)=0
\end{gathered}
$$

Now

$$
\sum_{k=0}^{m_{n}-1} E\left(\left(Y_{k}, M\left(t \wedge t_{k+1}\right)-M\left(t \wedge t_{k}\right)\right)_{H}^{2}\right)=\sum_{k=0}^{m_{n}-1} E\left(\left(Y_{k},\left(M^{t_{k+1}}-M^{t_{k}}\right)(t)\right)_{H}^{2}\right)
$$

It follows from 66.0.4

$$
\begin{gathered}
E\left(\left(\sum_{k=0}^{m_{n}-1}\left(Y_{k}, M\left(t \wedge t_{k+1}\right)-M\left(t \wedge t_{k}\right)\right)_{H}\right)^{2}\right)=\sum_{k=0}^{m_{n}-1} E\left(\left(Y_{k},\left(M^{t_{k+1}}-M^{t_{k}}\right)(t)\right)_{H}^{2}\right) \\
=\sum_{k=0}^{m_{n}-1} E\left(\left[\left(Y_{k},\left(M^{t_{k+1}}-M^{t_{k}}\right)(t)\right)\right]+N_{k}(t)\right)
\end{gathered}
$$

where $N_{k}$ is a martingale equal to 0 for $t \leq t_{k}$. Then this equals

$$
\sum_{k=0}^{m_{n}-1} E\left(\left[\left(Y_{k},\left(M^{t_{k+1}}-M^{t_{k}}\right)(t)\right)\right]\right)
$$

From Lemma 66.0.5

$$
\begin{align*}
& \leq E\left(\sum_{k=0}^{m_{n}-1}\left\|Y_{k}\right\|_{H}^{2}\left([M]^{t_{k+1}}(t)-[M]^{t_{k}}(t)\right)\right) \\
= & E\left(\sum_{k=0}^{m_{n}-1}\left\|Y_{k}\right\|_{H}^{2}\left([M]\left(t_{k+1}^{n} \wedge t\right)-[M]\left(t_{k}^{n} \wedge t\right)\right)\right)  \tag{66.0.5}\\
= & E\left(\int_{0}^{t}\|Y\|_{H}^{2} d\left[M^{\tau_{p}}\right]\right)=E\left(\int_{0}^{t}\|Y\|_{H}^{2} d[M]^{\tau_{p}}\right)
\end{align*}
$$

Note that everything makes sense because it is assumed that $\left\|Y_{k}\right\| M^{*} \in L^{2}(\Omega)$. This proves the following lemma.

Lemma 66.0.6 Let $\|Y(t)\|\left(M^{\tau_{p}}\right)^{*} \in L^{2}(\Omega)$ for each $t$, where $Y$ is an elementary function and let $\tau_{p}$ be a stopping time for which $M^{\tau_{p}}$ is a $L^{2}$ martingale. Then

$$
E\left(\left|\int_{0}^{t}\left(Y, d M^{\tau_{p}}\right)\right|^{2}\right) \leq E\left(\int_{0}^{t}\|Y\|_{H}^{2} d[M]^{\tau_{p}}\right)
$$

The condition that $\|Y(t)\|\left(M^{\tau_{p}}\right)^{*} \in L^{2}(\Omega)$ ensures that

$$
E\left(\left(Y_{k}, M^{\tau_{p}}\left(t \wedge t_{k+1}\right)-M^{\tau_{p}}\left(t \wedge t_{k+1}\right)\right)_{H}^{2}\right)
$$

always is finite.

Definition 66.0.7 Let $\mathscr{G}$ denote those functions $Y$ which are adapted and have the property that for each $p$,

$$
\lim _{n \rightarrow \infty} E\left(\int_{0}^{T}\left\|Y-Y^{n}\right\|_{H}^{2} d[M]^{\tau_{p}}\right)=0
$$

for some sequence $Y^{n}$ of elementary functions for which $\left\|Y^{n}(t)\right\| M^{*} \in L^{2}(\Omega)$ for each $t$. Here $d[M]^{\tau_{p}}$ signifies the Lebesgue Stieltjes measure determined by the increasing function $t \rightarrow\left[M^{\tau_{p}}\right](t)$. Let $M^{\tau_{p}}$ be an $L^{2}$ martingale. Recall that $\tau_{p}$ is just a localizing sequence for the local martingale $M$.

It is not known whether this increasing function is absolutely continuous.
Definition 66.0.8 Let $Y \in \mathscr{G}$. Then

$$
\int_{0}^{t}\left(Y, d M^{\tau_{p}}\right) \equiv \lim _{n \rightarrow \infty} \int_{0}^{t}\left(Y^{n}, d M^{\tau_{p}}\right) \text { in } L^{2}(\Omega)
$$

For example, suppose $Y$ is a bounded continuous process having values in $H$. Then you could look at the left step functions

$$
Y^{n}(t) \equiv \sum_{i=0}^{m_{n}-1} Y\left(t_{i}\right) \mathscr{X}_{\left[t_{i}, t_{i+1}\right)}(t)
$$

The $Y^{n}$ would converge to $Y$ pointwise on $[0, T]$ for each $\omega$ and these $Y^{n}$ are bounded. In fact, in this case, these converge uniformly to $Y$ on $[0, T]$. Thus this is an example of the situation in the above definition. In this case, the integrand would be bounded by $C$ for some $C$ and

$$
E\left(\int_{0}^{T} C d[M]^{\tau_{p}}\right)=E\left([M]^{\tau_{p}}(T)\right)=E\left(\left\|M^{\tau_{p}}(T)\right\|^{2}\right)<\infty
$$

by assumption. Hence, by the dominated convergence theorem,

$$
\lim _{n \rightarrow \infty} E\left(\int_{0}^{T}\left\|Y-Y^{n}\right\|_{H}^{2} d[M]^{\tau_{p}}\right)=0
$$

What if $[M]^{\tau_{p}}$ were bounded and absolutely continuous with respect to Lebesgue measure? This could be the case if you had $\tau_{p}$ a stopping time of the form

$$
\tau_{p}=\inf \{t:[M](t)>p\}
$$

Then if $Y \in L^{2}([0, T] \times \Omega, H)$, and progressively measurable there are left step functions which converge to $Y$ in $L^{2}([0, T] \times \Omega, H)$. Say $d\left[M^{\tau_{p}}\right]=k(t, \omega) d m$ where $k$ is bounded. Then

$$
E\left(\int_{0}^{T}\left\|Y-Y^{n}\right\|_{H}^{2} d[M]^{\tau_{p}}\right)=E\left(\int_{0}^{T}\left\|Y-Y^{n}\right\|_{H}^{2} k d t\right) \rightarrow 0
$$

Lemma 66.0.9 The above definition is well defined. Also, $\int_{0}^{t}\left(Y, d M^{\tau_{p}}\right)$ is a continuous martingale. The inequality

$$
E\left(\left|\int_{0}^{t}\left(Y, d M^{\tau_{p}}\right)\right|^{2}\right) \leq E\left(\int_{0}^{t}\|Y\|_{H}^{2} d[M]^{\tau_{p}}\right)
$$

is also valid. For any sequence of elementary functions $\left\{Y^{n}\right\},\left\|Y^{n}(t)\right\| M^{*} \in L^{2}(\Omega)$,

$$
\left\|Y^{n}-Y\right\|_{L^{2}\left(\Omega ; L^{2}\left([0, T] ; H, d\left[M^{\left.\left.\left.\tau_{p}\right]\right)\right)}\right.\right.\right.} \rightarrow 0
$$

there exists a subsequence, still denoted as $\left\{Y^{n}\right\}$ of elementary functions for which

$$
\int_{0}^{t}\left(Y^{n}, d M^{\tau_{p}}\right)
$$

converges uniformly to $\int_{0}^{t}\left(Y, d M^{\tau_{p}}\right)$ on $[0, T]$ for $\omega$ off some set of measure zero.
Proof: First of all, why does the limit even exist? From Lemma 66.0.6,

$$
E\left(\left|\int_{0}^{t}\left(Y^{n}, d M^{\tau_{p}}\right)-\int_{0}^{t}\left(Y^{m}, d M^{\tau_{p}}\right)\right|^{2}\right) \leq E\left(\int_{0}^{T}\left\|Y^{n}-Y^{m}\right\|_{H}^{2} d[M]^{\tau_{p}}\right)
$$

which converges to 0 as $n, m \rightarrow \infty$ by definition of $Y \in \mathscr{G}$. This also shows that the definition is well defined and that the same thing is obtained from any other sequence converging to $Y$. $\left\{\int_{0}^{t}\left(Y^{n}, d M^{\tau_{p}}\right)\right\}$ is a Cauchy sequence in $L^{2}(\Omega)$. Hence it converges to something $N(t) \in L^{2}(\Omega)$. This is a martingale because if $A \in \mathscr{F}_{s}, s<t$

$$
\begin{aligned}
\int_{A} N(t) d P & =\lim _{n \rightarrow \infty} \int_{A} \int_{0}^{t}\left(Y^{n}, d M^{\tau_{p}}\right) d P \\
& =\lim _{n \rightarrow \infty} \int_{A} \int_{0}^{s}\left(Y^{n}, d M^{\tau_{p}}\right) d P=\int_{A} N(s) d P
\end{aligned}
$$

Since $A$ is arbitrary, this shows that $E\left(N(t) \mid \mathscr{F}_{s}\right)=N(s)$. Then

$$
N(t) \equiv \int_{0}^{t}\left(Y^{n}, d M^{\tau_{p}}\right)
$$

In fact, this has a continuous version off a set of measure zero.
These are martingales and so actually, by maximal theorems,

$$
\begin{gathered}
P\left(\sup _{t \in[0, T]}\left|\int_{0}^{t}\left(Y^{n}, d M^{\tau_{p}}\right)-\int_{0}^{t}\left(Y^{m}, d M^{\tau_{p}}\right)\right|^{2}>\lambda\right) \\
\leq \frac{1}{\lambda} E\left(\left|\int_{0}^{T}\left(Y^{n}, d M^{\tau_{p}}\right)-\int_{0}^{T}\left(Y^{m}, d M^{\tau_{p}}\right)\right|^{2}\right) \\
\leq \frac{1}{\lambda} E\left(\int_{0}^{T}\left\|Y^{n}-Y^{m}\right\|_{H}^{2} d[M]^{\tau_{p}}\right)
\end{gathered}
$$

which converges to 0 . Thus there is a subsequence still denoted with index $k$ such that

$$
P\left(\sup _{t \in[0, T]}\left|\int_{0}^{t}\left(Y^{k}, d M^{\tau_{p}}\right)-\int_{0}^{t}\left(Y^{k+1}, d M^{\tau_{p}}\right)\right|^{2}>2^{-k}\right)<2^{-k}
$$

and so there exists a set of measure zero $N$ such that for $\omega \notin N$,

$$
\sup _{t \in[0, T]}\left|\int_{0}^{t}\left(Y^{k}, d M^{\tau_{p}}\right)-\int_{0}^{t}\left(Y^{k+1}, d M^{\tau_{p}}\right)\right|^{2} \leq 2^{-k}
$$

for all $k$ large enough and so for this subsequence, the convergence is uniform. Hence $t \rightarrow \int_{0}^{t}\left(Y^{n}, d M^{\tau_{p}}\right)$ has a continuous version obtained from the uniform limit of these.

Finally,

$$
\begin{aligned}
E\left(\left|\int_{0}^{t}\left(Y, d M^{\tau_{p}}\right)\right|^{2}\right) & =\lim _{n \rightarrow \infty} E\left(\left|\int_{0}^{t}\left(Y^{n}, d M^{\tau_{p}}\right)\right|^{2}\right) \\
& \leq \lim _{n \rightarrow \infty} E\left(\int_{0}^{t}\left\|Y^{n}\right\|_{H}^{2} d[M]^{\tau_{p}}\right)=E\left(\int_{0}^{t}\|Y\|_{H}^{2} d[M]^{\tau_{p}}\right)
\end{aligned}
$$

What is the quadratic variation of the martingale in the above lemma? I am not going to give it exactly but it is easy to give an estimate for it. Recall the following result. It is Theorem 63.6.4.

Theorem 66.0.10 Let $H$ be a Hilbert space and suppose $\left(M, \mathscr{F}_{t}\right), t \in[0, T]$ is a uniformly bounded continuous martingale with values in $H$. Also let $\left\{t_{k}^{n}\right\}_{k=1}^{m_{n}}$ be a sequence of partitions satisfying

$$
\lim _{n \rightarrow \infty} \max \left\{\left|t_{i}^{n}-t_{i+1}^{n}\right|, i=0, \cdots, m_{n}\right\}=0,\left\{t_{k}^{n}\right\}_{k=1}^{m_{n}} \subseteq\left\{t_{k}^{n+1}\right\}_{k=1}^{m_{n+1}}
$$

Then

$$
[M](t)=\lim _{n \rightarrow \infty} \sum_{k=0}^{m_{n}-1}\left|M\left(t \wedge t_{k+1}^{n}\right)-M\left(t \wedge t_{k}^{n}\right)\right|_{H}^{2}
$$

the limit taking place in $L^{2}(\Omega)$. In case $M$ is just a continuous local martingale, the above limit happens in probability.

In the above Lemma, you would find the quadratic variation according to this theorem as follows.

$$
\left[\int_{0}^{(\cdot)}\left(Y, d M^{\tau_{p}}\right)\right](t)=\lim _{n \rightarrow \infty} \sum_{k=0}^{m_{n}-1}\left|\int_{t \wedge t_{k}^{n}}^{t \wedge t_{k+1}^{n}}\left(Y, d M^{\tau_{p}}\right)\right|_{H}^{2}
$$

where the limit is in probability. Thus

$$
\left.\lim _{n \rightarrow \infty} P\left(| | \int_{0}^{(\cdot)}\left(Y, d M^{\tau_{p}}\right)\right](t)-\sum_{k=0}^{m_{n}-1}\left|\int_{t \wedge t_{k}^{n}}^{t \wedge t_{k+1}^{n}}\left(Y, d M^{\tau_{p}}\right)\right|^{2} \mid \geq \varepsilon\right)=0
$$

Then you can obtain from this and the usual appeal to the Borel Cantelli lemma a set of measure zero $N_{t}$ and a subsequence still denoted with $n$ satisfying that for all $\omega \notin N_{t}$ and $n$ large enough,

$$
\left.\left|\left[\int_{0}^{(\cdot)}\left(Y, d M^{\tau_{p}}\right)\right](t)-\sum_{k=0}^{m_{n}-1}\right| \int_{t \wedge t_{k}^{n}}^{t \wedge t_{k+1}^{n}}\left(Y, d M^{\tau_{p}}\right)\right|^{2} \left\lvert\, \leq \frac{1}{n}\right.
$$

Hence

$$
\begin{gathered}
{\left[\int_{0}^{(\cdot)}\left(Y, d M^{\tau_{p}}\right)\right](t) \leq \frac{1}{n}+\sum_{k=0}^{m_{n}-1} \int_{t \wedge t_{k}^{n}}^{t \wedge t_{k+1}^{n}}\|Y\|_{H}^{2} d[M]^{\tau_{p}}} \\
=\frac{1}{n}+\int_{0}^{t}\|Y\|_{H}^{2} d[M]^{\tau_{p}}
\end{gathered}
$$

Then for that $t$, you have on taking a limit as $n \rightarrow \infty$,

$$
\left[\int_{0}^{(\cdot)}\left(Y, d M^{\tau_{p}}\right)\right](t) \leq \int_{0}^{t}\|Y\|_{H}^{2} d[M]^{\tau_{p}}
$$

Now take the union of $N_{t}$ for $t \in \mathbb{Q} \cap[0, T]$. Denote this as $N$. Then if $\omega \notin N$, the above shows that for such $t$,

$$
\left[\int_{0}^{(\cdot)}\left(Y, d M^{\tau_{p}}\right)\right](t) \leq \int_{0}^{t}\|Y\|_{H}^{2} d[M]^{\tau_{p}}
$$

But both sides are continuous in $t$ and so this inequality holds for all $t \in[0, T]$. Thus the following corollary is obtained.

Corollary 66.0.11 Let $M$ be a continuous local martingale and $\tau_{p}$ a localizing sequence which makes $M^{\tau_{p}}$ an $L^{2}$ martingale and assume that $Y \in \mathscr{G}$. Then the quadratic variation of this martingale satisfies

$$
\left[\int_{0}^{(\cdot)}\left(Y, d M^{\tau_{p}}\right)\right](t) \leq \int_{0}^{t}\|Y\|_{H}^{2} d[M]^{\tau_{p}} \leq \int_{0}^{t}\|Y\|_{H}^{2} d[M]
$$

for $\omega$ off a set of measure zero.
Does the localization stuff hold for an arbitrary stopping time? Let $\left\{t_{i}^{k}\right\}$ denote the $k^{t h}$ partition of a sequence of nested partitions whose maximum length between successive points converges to 0 . Let $\tau$ be a stopping time and let $\tau_{k}=t_{j+1}^{k}$ on $\tau^{-1}\left(t_{j}^{k}, t_{j+1}^{k}\right]$. Then $\tau_{k}$ is a stopping time because

$$
\left[\tau_{k} \leq t\right] \in \mathscr{F}_{t}
$$

Here is why. If $t \in\left(t_{j}^{k}, t_{j+1}^{k}\right]$, then if $t=t_{j+1}^{k}$, it would follow that $\tau_{k}(\omega) \leq t$ would be the same as saying $\omega \in\left[\tau \leq t_{j+1}^{k}\right]=[\tau \leq t] \in \mathscr{F}_{t}$. On the other hand, if $t<t_{j+1}^{k}$, then $\left[\tau_{k} \leq t\right]=\left[\tau \leq t_{j}^{k}\right] \in \mathscr{F}_{t_{j}^{k}} \subseteq \mathscr{F}_{t}$ because $\tau_{k}$ can only take the values $t_{j}^{k}$.

Let $Y$ be one of those elementary functions which is in $\mathscr{G},\|Y(t)\| M^{*} \in L^{2}(\Omega)$.

$$
Y(t)=\sum_{i=0}^{m_{k}-1} Y_{i} \mathscr{X}_{\left(t_{i}^{k}, t_{i+1}^{k}\right]}(t)
$$

and consider $\mathscr{X}_{\left[0, \tau_{k}\right]} Y$. Here $Y$ will be always the same for the different partitions. It is just that some of the $Y_{i}$ will be repeated on smaller and smaller intervals. Does it follow that $\mathscr{X}_{\left[0, \tau_{k}\right]} Y \rightarrow \mathscr{X}_{[0, \tau]} Y$ for each fixed $\omega$ ? This depends only on the indicator function. Let $\tau(\omega) \in\left(t_{j}^{k}, t_{j+1}^{k}\right]$. Fixing $t$, if $\mathscr{X}_{[0, \tau]}(t)=1$, then also $\mathscr{X}_{\left[0, \tau_{k}\right]}(t)=1$ because $\tau_{k} \geq \tau$. Therefore, in this case $\lim _{k \rightarrow \infty} \mathscr{X}_{\left[0, \tau_{k}\right]}(t)=\mathscr{X}_{[0, \tau]}(t)$. Next suppose $\mathscr{X}_{[0, \tau]}(t)=0$ so that $\tau(\omega)<t$. Since the intervals defined by the partition points have lengths which converge to 0 , it follows that for all $k$ large enough, $\tau_{k}(\omega)<t$ also and so $\mathscr{X}_{\left[0, \tau_{k}\right]}(t)=0$. Therefore,

$$
\lim _{k \rightarrow \infty} \mathscr{X}_{\left[0, \tau_{k}(\omega)\right]}(t)=\mathscr{X}_{[0, \tau(\omega)]}(t) .
$$

It follows that $\mathscr{X}_{\left[0, \tau_{k}\right]} Y \rightarrow \mathscr{X}_{[0, \tau]} Y$. Also it is clear from the dominated convergence theorem,

$$
\left\|\mathscr{X}_{\left[0, \tau_{k}\right]} Y-\mathscr{X}_{[0, \tau]} Y\right\|_{H}^{2} \leq 4\|Y\|_{H}^{2}
$$

that

$$
\lim _{k \rightarrow \infty} E\left(\int_{0}^{T}\left\|\mathscr{X}_{\left[0, \tau_{k}\right]} Y-\mathscr{X}_{[0, \tau]} Y\right\|_{H}^{2} d\left[M^{\tau_{p}}\right]\right)=0
$$

Thus $\mathscr{X}_{[0, \tau]} Y \in \mathscr{G}$. By Lemma 66.0.9, there is a subsequence, still denoted as $\mathscr{X}_{\left[0, \tau_{k}\right]} Y$ such that off a set of measure zero,

$$
\int_{0}^{t}\left(\mathscr{X}_{\left[0, \tau_{k}\right]} Y, d M^{\tau_{p}}\right) \rightarrow \int_{0}^{t}\left(\mathscr{X}_{[0, \tau]} Y, d M^{\tau_{p}}\right)
$$

uniformly on $[0, T]$. Therefore, from the localization for elementary functions and this uniform convergence,

$$
\int_{0}^{t}\left(\mathscr{X}_{[0, \tau]} Y, d M^{\tau_{p}}\right)=\lim _{n \rightarrow \infty} \int_{0}^{t}\left(\mathscr{X}_{\left[0, \tau_{n}\right]} Y, d M^{\tau_{p}}\right)=\lim _{n \rightarrow \infty} \int_{0}^{t \wedge \tau_{n}}\left(Y, d M^{\tau_{p}}\right)=\int_{0}^{t \wedge \tau}\left(Y, d M^{\tau_{p}}\right)
$$

This proves most of the following lemma.
Lemma 66.0.12 Let $Y$ be an elementary function. Then if $\tau$ is any stopping time, then off a set of measure zero,

$$
\int_{0}^{t \wedge \tau}\left(Y, d M^{\tau_{p}}\right)=\int_{0}^{t}\left(\mathscr{X}_{[0, \tau]} Y, d M^{\tau_{p}}\right)=\int_{0}^{t}\left(Y, d M^{\tau \wedge \tau_{p}}\right)
$$

Proof: It remains to prove the second equation.

$$
\begin{aligned}
\int_{0}^{t \wedge \tau}\left(Y, d M^{\tau_{p}}\right) & \equiv \sum_{i=0}^{m-1}\left(Y_{i}, M^{\tau_{p}}\left(t \wedge t_{i+1} \wedge \tau\right)-M^{\tau_{p}}\left(t \wedge t_{i} \wedge \tau\right)\right) \\
& \equiv \sum_{i=0}^{m-1}\left(Y_{i}, M^{\tau \wedge \tau_{p}}\left(t \wedge t_{i+1}\right)-M^{\tau_{p}}\left(t \wedge t_{i}\right)\right) \\
& \equiv \int_{0}^{t}\left(Y, d M^{\tau \wedge \tau_{p}}\right) \square
\end{aligned}
$$

Lemma 66.0.13 Let $Y \in \mathscr{G}$. Then for any stopping time $\tau$,

$$
\int_{0}^{t \wedge \tau}\left(Y, d M^{\tau_{p}}\right)=\int_{0}^{t}\left(\mathscr{X}_{[0, \tau]} Y, d M^{\tau_{p}}\right)=\int_{0}^{t}\left(Y, d M^{\tau \wedge \tau_{p}}\right)
$$

for $\omega$ off some set of measure zero.
Proof: From Lemma 66.0.9, there exists a sequence of elementary functions $Y^{n}$ such that $t \rightarrow \int_{0}^{t}\left(Y^{n}, d M\right)$ converges uniformly to $t \rightarrow \int_{0}^{t}\left(Y, d M^{\tau_{p}}\right)$ on $[0, T]$ for each $\omega \notin N$, a set of measure zero. Then

$$
\begin{aligned}
\int_{0}^{t \wedge \tau}\left(Y, d M^{\tau_{p}}\right) & =\lim _{n \rightarrow \infty} \int_{0}^{t \wedge \tau}\left(Y^{n}, d M^{\tau_{p}}\right) \\
& =\lim _{n \rightarrow \infty} \int_{0}^{t}\left(\mathscr{X}_{[0, \tau]} Y^{n}, d M^{\tau_{p}}\right)=\int_{0}^{t}\left(\mathscr{X}_{[0, \tau]} Y, d M^{\tau_{p}}\right)
\end{aligned}
$$

The last claim needs a little clarification. As shown in the above discussion proving Lemma 66.0.12, while $\mathscr{X}_{[0, \tau]} Y^{n}$ is no longer obviously an elementary function due to the fact that $\tau$ has values which are not partition points, it is still the limit of a sequence of elementary functions $\mathscr{X}_{\left[0, \tau_{k}\right]} Y^{n}$ and so the integral makes sense. Then from the inequality of Lemma 66.0.9,

$$
E\left(\left|\int_{0}^{t}\left(\mathscr{X}_{[0, \tau]} Y^{n}, d M^{\tau_{p}}\right)-\int_{0}^{t}\left(\mathscr{X}_{[0, \tau]} Y, d M^{\tau_{p}}\right)\right|^{2}\right) \leq E\left(\int_{0}^{T}\left\|Y^{n}-Y\right\|_{H}^{2} d[M]^{\tau_{p}}\right)
$$

and so by the same Borel Canteli argument of that lemma, there is a further subsequence for which the convergence is uniform off a set of measure zero as $n \rightarrow \infty$. (Actually, the same subsequence as in the first part of the argument works.) Therefore, the conclusion follows.

What of the second equation? Let $\left\{Y^{n}\right\}$ be as above where uniform convergence takes place for the stochastic integrals. Then from Lemma 66.0.12

$$
\int_{0}^{t}\left(\mathscr{X}_{[0, \tau]} Y^{n}, d M^{\tau_{p}}\right)=\int_{0}^{t}\left(Y^{n}, d M^{\tau \wedge \tau_{p}}\right)
$$

Hence

$$
E\left(\left|\int_{0}^{t}\left(Y^{n}, d M^{\tau \wedge \tau_{p}}\right)-\int_{0}^{t}\left(\mathscr{X}_{[0, \tau]} Y, d M^{\tau_{p}}\right)\right|^{2}\right) \leq E\left(\int_{0}^{T}\left\|Y^{n}-Y\right\|_{H}^{2} d[M]^{\tau_{p}}\right)
$$

Now by the usual application of the Borel Canelli lemma, there is a subsequence and a set of measure zero off which $\int_{0}^{t}\left(Y^{n}, d M^{\tau \wedge \tau_{p}}\right)$ converges uniformly to $\int_{0}^{t}\left(Y, d M^{\tau \wedge \tau_{p}}\right)$ on $[0, T]$ and as $n \rightarrow \infty$, and also

$$
\int_{0}^{t}\left(Y^{n}, d M^{\tau \wedge \tau_{p}}\right) \rightarrow \int_{0}^{t}\left(\mathscr{X}_{[0, \tau]} Y, d M^{\tau_{p}}\right)
$$

uniformly on $t \in[0, T]$. Then from the above,

$$
\int_{0}^{t}\left(Y^{n}, d M^{\tau \wedge \tau_{p}}\right) \rightarrow \int_{0}^{t}\left(\mathscr{X}_{[0, \tau]} Y, d M^{\tau_{p}}\right)=\int_{0}^{t \wedge \tau}\left(Y, d M^{\tau_{p}}\right)
$$

uniformly. Thus $\int_{0}^{t}\left(Y, d M^{\tau \wedge \tau_{p}}\right)=\int_{0}^{t \wedge \tau}\left(Y, d M^{\tau_{p}}\right)$.

Definition 66.0.14 Let $\tau_{p}$ be an increasing sequence of stopping times for which

$$
\lim _{p \rightarrow \infty} \tau_{p}=\infty
$$

and such that $M^{\tau_{p}}$ is a $L^{2}$ martingale and $\mathscr{X}_{\left[0, \tau_{p}\right]} Y \in \mathscr{G}$. Then the definition of $\int_{0}^{t}(Y, d M)$ is as follows. For each $\omega$,

$$
\int_{0}^{t}(Y, d M) \equiv \lim _{p \rightarrow \infty} \int_{0}^{t}\left(\mathscr{X}_{\left[0, \tau_{p}\right]} Y, d M^{\tau_{p}}\right)
$$

In fact, this is well defined.
Theorem 66.0.15 The above definition is well defined. Also this makes $\int_{0}^{t}(Y, d M)$ a local martingale. In particular,

$$
\int_{0}^{t \wedge \tau_{p}}(Y, d M)=\int_{0}^{t}\left(\mathscr{X}_{\left[0, \tau_{p}\right]} Y, d M^{\tau_{p}}\right)
$$

In addition to this, if $\sigma$ is any stopping time,

$$
\int_{0}^{t \wedge \sigma}(Y, d M)=\int_{0}^{t}\left(\mathscr{X}_{[0, \sigma]} Y, d M\right)
$$

In this last formula, $\mathscr{X}_{[0, \sigma]} \mathscr{X}_{\left[0, \tau_{p}\right]} Y \in \mathscr{G}$. In addition, the following estimate holds for the quadratic variation.

$$
\left[\int_{0}^{(\cdot)}(Y, d M)\right](t) \leq \int_{0}^{t}\|Y\|^{2} d[M]
$$

Proof: Suppose for some $\omega, t<\tau_{p}<\tau_{q}$. Let $\omega$ be such that both $\tau_{p}, \tau_{q}$ are larger than $t$. Then for all $\omega$, and $\tau$ a stopping time,

$$
\int_{0}^{t \wedge \tau}\left(\mathscr{X}_{\left[0, \tau_{q}\right]} Y, d M^{\tau_{q}}\right)=\int_{0}^{t}\left(\mathscr{X}_{\left[0, \tau_{q}\right]} Y, d\left(\left(M^{\tau_{q}}\right)^{\tau}\right)\right)
$$

In particular, for the given $\omega$,

$$
\int_{0}^{t}\left(\mathscr{X}_{\left[0, \tau_{q}\right]} Y, d\left(M^{\tau_{q}}\right)^{\tau_{q}}\right)=\int_{0}^{t}\left(\mathscr{X}_{\left[0, \tau_{q}\right]} Y, d M^{\tau_{q}}\right)=\int_{0}^{t \wedge \tau_{q}}\left(\mathscr{X}_{\left[0, \tau_{q}\right]} Y, d M^{\tau_{q}}\right)
$$

For the particular $\omega$, this equals

$$
\int_{0}^{t \wedge \tau_{p}}\left(\mathscr{X}_{\left[0, \tau_{q}\right]} Y, d M^{\tau_{q}}\right)
$$

Now for all $\omega$ including the particular one, this equals

$$
\int_{0}^{t}\left(\mathscr{X}_{\left[0, \tau_{q}\right]} Y, d\left(\left(M^{\tau_{q}}\right)^{\tau_{p}}\right)\right)=\int_{0}^{t}\left(\mathscr{X}_{\left[0, \tau_{q}\right]} Y, d M^{\tau_{p}}\right)
$$

For the $\omega$ of interest, this is

$$
\int_{0}^{t \wedge \tau_{p}}\left(\mathscr{X}_{\left[0, \tau_{q}\right]} Y, d M^{\tau_{p}}\right)
$$

and for all $\omega$, including the one of interest, the above equals

$$
\int_{0}^{t}\left(\mathscr{X}_{\left[0, \tau_{p}\right]} \mathscr{X}_{\left[0, \tau_{q}\right]} Y, d M^{\tau_{p}}\right)=\int_{0}^{t}\left(\mathscr{X}_{\left[0, \tau_{p}\right]} Y, d M^{\tau_{p}}\right)
$$

thus for this particular $\omega$, you get the same for both $p$ and $q$. Thus the definition is well defined because for a given $\omega, \int_{0}^{t}\left(\mathscr{X}_{\left[0, \tau_{p}\right]} Y, d M^{\tau_{p}}\right)$ is constant for all $p$ large enough.

Next consider the claim about this process being a local martingale. Is

$$
\int_{0}^{t \wedge \tau_{p}}(Y, d M)
$$

is a martingale? From the definition,

$$
\begin{gather*}
\int_{0}^{t \wedge \tau_{p}}(Y, d M)=\lim _{q \rightarrow \infty} \int_{0}^{t \wedge \tau_{p}}\left(\mathscr{X}_{\left[0, \tau_{q}\right]} Y, d M^{\tau_{q}}\right) \\
=\lim _{q \rightarrow \infty} \int_{0}^{t}\left(\mathscr{X}_{\left[0, \tau_{q}\right]} Y, d\left(M^{\tau_{q}}\right)^{\tau_{p}}\right)=\lim _{q \rightarrow \infty} \int_{0}^{t \wedge \tau_{p}}\left(\mathscr{X}_{\left[0, \tau_{q}\right]} Y, d M^{\tau_{p}}\right) \\
=\lim _{q \rightarrow \infty} \int_{0}^{t}\left(\mathscr{X}_{\left[0, \tau_{p}\right]} \mathscr{X}_{\left[0, \tau_{q}\right]} Y, d M^{\tau_{p}}\right)=\int_{0}^{t}\left(\mathscr{X}_{\left[0, \tau_{p}\right]} Y, d M^{\tau_{p}}\right) \tag{66.0.6}
\end{gather*}
$$

which is known to be a martingale since $\mathscr{X}_{\left[0, \tau_{p}\right]} Y \in \mathscr{G}$. This is what it means to be a local martingale. You localize and get a martingale.

Next consider the claim about an arbitrary stopping time. Why is $\mathscr{X}_{[0, \sigma]} \mathscr{X}_{\left[0, \tau_{p}\right]} Y \in \mathscr{G}$ ? This is part of a more general question. Suppose $\hat{Y} \in \mathscr{G}$. Then why is $\mathscr{X}_{[0, \sigma]} \hat{Y} \in \mathscr{G}$. It suffices to show this. Let $\left\{Y^{n}\right\}$ be the sequence of elementary functions which converge to $\hat{Y}$ as in the definition. Also let $\sigma_{n}$ be the stopping time with discreet values which equals $t_{k+1}^{n}$ when $\sigma \in\left(t_{k}^{n}, t_{k+1}^{n}\right],\left\{t_{k}^{n}\right\}_{k=0}^{m_{n}}$ being the partition associated with $Y^{n}$. Then, as explained earlier, $\mathscr{X}_{\left[0, \sigma_{n}\right]} Y^{n}$ is an acceptable elementary function and also

$$
\begin{gathered}
\left\{E\left(\int_{0}^{T}\left\|\mathscr{X}_{\left[0, \sigma_{n}\right]} Y^{n}-\mathscr{X}_{[0, \sigma]} \hat{Y}\right\|^{2} d[M]\right)\right\}^{1 / 2} \\
\leq \\
+\left\{E\left(\int_{0}^{T}\left\|\mathscr{X}_{\left[0, \sigma_{n}\right]} Y^{n}-\mathscr{X}_{\left[0, \sigma_{n}\right]} \hat{Y}\right\|^{2} d[M]\right)\right\}^{1 / 2} \\
\leq\left\{E\left(\int_{0}^{T}\left\|\mathscr{X}_{\left[\sigma, \sigma_{n}\right]} \hat{Y}\right\|^{2} d[M]\right)\right\}^{1 / 2} \\
\left.=\left\{Y_{0}^{n}-\hat{Y} \|^{2} d[M]\right)\right\}^{1 / 2}+\left\{E\left(\int_{0}^{T}\left\|\mathscr{X}_{\left[\sigma, \sigma_{n}\right]} \hat{Y}\right\|^{2} d[M]\right)\right\}^{1 / 2}
\end{gathered}
$$

which converges to 0 from the definition of $\hat{Y} \in \mathscr{G}$ and the dominated convergence theorem. Thus $\mathscr{X}_{[0, \sigma]} \hat{Y} \in \mathscr{G}$.

From the above definition, for each $\omega$ off a suitable set of measure zero, from Lemma 66.0.13,

$$
\begin{aligned}
\int_{0}^{t \wedge \sigma}(Y, d M) & \equiv \lim _{p \rightarrow \infty} \int_{0}^{t \wedge \sigma}\left(\mathscr{X}_{\left[0, \tau_{p}\right]} Y, d M^{\tau_{p}}\right) \\
& =\lim _{p \rightarrow \infty} \int_{0}^{t}\left(\mathscr{X}_{\left[0, \tau_{p}\right]} \mathscr{X}_{[0, \sigma]} Y, d M^{\tau_{p}}\right) \equiv \int_{0}^{t}\left(\mathscr{X}_{[0, \sigma]} Y, d M\right)
\end{aligned}
$$

Finally, consider the claim about the quadratic variation. Using 66.0.6,

$$
\begin{gathered}
{\left[\left(\int_{0}^{(\cdot)}(Y, d M)\right)\right]^{\tau_{p}}(t)=\left[\left(\int_{0}^{(\cdot)}(Y, d M)\right)^{\tau_{p}}\right](t)=\left[\int_{0}^{(\cdot)}\left(\mathscr{X}_{\left[0, \tau_{p}\right]} Y, d M^{\tau_{p}}\right)\right](t)} \\
\leq \int_{0}^{t}\left\|\mathscr{X}_{\left[0, \tau_{p}\right]} Y\right\|^{2} d[M]^{\tau_{p}} \leq \int_{0}^{t}\|Y\|^{2} d[M]
\end{gathered}
$$

Now letting $\tau_{p} \rightarrow \infty$,

$$
\left[\left(\int_{0}^{(\cdot)}(Y, d M)\right)\right](t) \leq \int_{0}^{t}\|Y\|^{2} d[M]
$$

Next is the case in which $Y$ is continuous in $t$ but not necessarily bounded nor assumed to be in any kind of $L^{2}$ space either.

Definition 66.0.16 Let $Y$ be continuous in $t$ and adapted. Let $M$ be a continuous local martingale $M(0)=0$. Then the definition of a local martingale $\int_{0}^{t}(Y, d M)$ is as follows. Let $\tau_{p}$ be an increasing sequence of stopping times for which $[M]^{\tau_{p}},\left\|M^{\tau_{p}}\right\|,\left\|\mathscr{X}_{\left[0, \tau_{p}\right]} Y\right\|$ are all bounded by $p$. Then

$$
\int_{0}^{t}(Y, d M) \equiv \lim _{p \rightarrow \infty} \int_{0}^{t}\left(\mathscr{X}_{\left[0, \tau_{p}\right]} Y, d M^{\tau_{p}}\right)
$$

Then it is clear that $\mathscr{X}_{\left[0, \tau_{p}\right]} Y \in \mathscr{G}$. Therefore, the above Theorem yields the following corollary.
Corollary 66.0.17 The above definition is well defined. Also this makes $\int_{0}^{t}(Y, d M)$ a local martingale. In particular,

$$
\int_{0}^{t \wedge \tau_{p}}(Y, d M)=\int_{0}^{t}\left(\mathscr{X}_{\left[0, \tau_{p}\right]} Y, d M^{\tau_{p}}\right)
$$

In addition to this, if $\sigma$ is any stopping time,

$$
\int_{0}^{t \wedge \sigma}(Y, d M)=\int_{0}^{t}\left(\mathscr{X}_{[0, \sigma]} Y, d M\right)
$$

In this last formula, $\mathscr{X}_{[0, \sigma]} Y$ has the same properties as $Y$, being the pointwise limit on $[0, T]$ of a bounded seuqence of elementary functions for each $\omega$. In addition to this, there is an estimate for the quadratic variation

$$
\left[\int_{0}^{(\cdot)}(Y, d M)\right](t) \leq \int_{0}^{t}\|Y\|^{2} d[M]
$$

Of course there is no change in anything if $M$ has its values in a Hilbert space $W$ while $Y$ has its values in its dual space. Then one defines $\int_{0}^{t}\langle Y, d M\rangle_{W^{\prime}, W}$ by analogy to the above for $Y$ an elementary function, step function which is adapted.

We use the following definition.
Definition 66.0.18 Let $\tau_{p}$ be an increasing sequence of stopping times for which $M^{\tau_{p}}$ is a $L^{2}$ martingale. If $M$ is already an $L^{2}$ martingale, simply let $\tau_{p} \equiv \infty$. Let $\mathscr{G}$ denote those functions $Y$ which are adapted and for which there is a sequence of elementary functions $\left\{Y^{n}\right\}$ satisfying $\left\|Y^{n}(t)\right\|_{W^{\prime}} M^{*} \in L^{2}(\Omega)$ for each $t$ with

$$
\lim _{n \rightarrow \infty} E\left(\int_{0}^{T}\left\|Y-Y^{n}\right\|_{W^{\prime}}^{2} d[M]^{\tau_{p}}\right)=0
$$

for each $\tau_{p}$.
Then exactly the same arguments given above yield the following simple generalizations.

Definition 66.0.19 Let $Y \in \mathscr{G}$. Then

$$
\int_{0}^{t}\left\langle Y, d M^{\tau_{p}}\right\rangle_{W^{\prime}, W} \equiv \lim _{n \rightarrow \infty} \int_{0}^{t}\left\langle Y^{n}, d M^{\tau_{p}}\right\rangle_{W^{\prime}, W} \text { in } L^{2}(\Omega)
$$

Lemma 66.0.20 The above definition is well defined. Also, $\int_{0}^{t}\left\langle Y, d M^{\tau_{p}}\right\rangle_{W^{\prime}, W}$ is a continuous martingale. The inequality

$$
E\left(\left|\int_{0}^{t}\left\langle Y, d M^{\tau_{p}}\right\rangle_{W^{\prime}, W}\right|^{2}\right) \leq E\left(\int_{0}^{t}\|Y\|_{W^{\prime}}^{2} d[M]^{\tau_{p}}\right)
$$

is also valid. For any sequence of elementary functions $\left\{Y^{n}\right\},\left\|Y^{n}(t)\right\|_{W^{\prime}} M^{*} \in L^{2}(\Omega)$,

$$
\left\|Y^{n}-Y\right\|_{L^{2}\left(\Omega ; L^{2}\left([0, T] ; W^{\prime}, d\left[M^{\tau_{p}}\right]\right)\right)} \rightarrow 0
$$

there exists a subsequence, still denoted as $\left\{Y^{n}\right\}$ of elementary functions for which

$$
\int_{0}^{t}\left\langle Y^{n}, d M^{\tau_{p}}\right\rangle_{W^{\prime}, W}
$$

converges uniformly to $\int_{0}^{t}\left\langle Y, d M^{\tau_{p}}\right\rangle_{W^{\prime}, W}$ on $[0, T]$ for $\omega$ off some set of measure zero. In addition, the quadratic variation satisfies the following inequality.

$$
\left[\int_{0}^{(\cdot)}\left\langle Y, d M^{\tau_{p}}\right\rangle_{W^{\prime}, W}\right](t) \leq \int_{0}^{t}\|Y\|_{W^{\prime}}^{2} d[M]^{\tau_{p}} \leq \int_{0}^{t}\|Y\|_{W^{\prime}}^{2} d[M]
$$

As before, you can consider the case where you only know $\mathscr{X}_{\left[0, \tau_{p}\right]} Y \in \mathscr{G}$. This yields a local martingale as before.

Definition 66.0.21 Let $\tau_{p}$ be an increasing sequence of stopping times for which

$$
\lim _{p \rightarrow \infty} \tau_{p}=\infty
$$

and such that $M^{\tau_{p}}$ is a martingale and $\mathscr{X}_{\left[0, \tau_{p}\right]} Y \in \mathscr{G}$. Then the definition of $\int_{0}^{t}\langle Y, d M\rangle_{W^{\prime}, W}$ is as follows. For each $\omega$ off a set of measure zero,

$$
\int_{0}^{t}\langle Y, d M\rangle_{W^{\prime}, W} \equiv \lim _{p \rightarrow \infty} \int_{0}^{t}\left\langle\mathscr{X}_{\left[0, \tau_{p}\right]} Y, d M^{\tau_{p}}\right\rangle_{W^{\prime}, W}
$$

where $\int_{0}^{t}\left\langle\mathscr{X}_{\left[0, \tau_{p}\right]} Y, d M^{\tau_{p}}\right\rangle_{W^{\prime}, W}$ is a martingale.
In fact, this is well defined.
Theorem 66.0.22 The above definition is well defined. Also this makes $\int_{0}^{t}\langle Y, d M\rangle_{W^{\prime}, W} a$ local martingale. In particular,

$$
\int_{0}^{t \wedge \tau_{p}}\langle Y, d M\rangle_{W^{\prime}, W}=\int_{0}^{t}\left\langle\mathscr{X}_{\left[0, \tau_{p}\right]} Y, d M^{\tau_{p}}\right\rangle_{W^{\prime}, W}
$$

In addition to this, if $\sigma$ is any stopping time,

$$
\int_{0}^{t \wedge \sigma}\langle Y, d M\rangle_{W^{\prime}, W}=\int_{0}^{t}\left\langle\mathscr{X}_{[0, \sigma]} Y, d M\right\rangle_{W^{\prime}, W}
$$

In this last formula, $\mathscr{X}_{[0, \sigma]} \mathscr{X}_{\left[0, \tau_{p}\right]} Y \in \mathscr{G}$. In addition, the following estimate holds for the quadratic variation.

$$
\left[\int_{0}^{(\cdot)}\langle Y, d M\rangle_{W^{\prime}, W}\right](t) \leq \int_{0}^{t}\|Y\|_{W^{\prime}}^{2} d[M]
$$

Note that from Definition 66.0.21 it is also true that

$$
\int_{0}^{t}\langle Y, d M\rangle_{W^{\prime}, W} \equiv \lim _{p \rightarrow \infty} \int_{0}^{t}\left\langle\mathscr{X}_{\left[0, \tau_{p}\right]} Y, d M^{\tau_{p}}\right\rangle_{W^{\prime}, W}
$$

in probability. In addition, since $\tau_{p} \rightarrow \infty$, it follows that for each $\omega$, eventually $\tau_{p}>T$. Therefore, $t \rightarrow \int_{0}^{t}\langle Y, d M\rangle_{W^{\prime}, W}$ is continuous, being equal to $\int_{0}^{t}\left\langle\mathscr{X}_{\left[0, \tau_{p}\right]} Y, d M^{\tau_{p}}\right\rangle_{W^{\prime}, W}$ for that $\omega$.

## Chapter 67

## The Easy Ito Formula

First recall 64.5 .26 where it is shown that for every $\alpha$

$$
E\left(|W(t)-W(s)|^{\alpha}\right) \leq C_{\alpha}|t-s|^{\alpha / 2}
$$

and so by Kolmogorov Čentsov continuity theorem

$$
\begin{equation*}
|W(t)-W(s)| \leq C_{\gamma}|t-s|^{\gamma} \tag{67.0.1}
\end{equation*}
$$

for every $\gamma<1 / 2$.

### 67.1 The Situation

The idea is as follows. You have a sufficiently smooth function $F:[0, T] \times H \rightarrow \mathbb{R}$ where $H$ is a separable Hilbert space. You also have the random variable

$$
X(t)=X_{0}+\int_{0}^{t} \phi(s) d s+\int_{0}^{t} \Phi d W
$$

where $\Phi$ is progressively measurable and in $L^{2}\left([0, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$ where $Q: U \rightarrow$ $U$ is a positive self adjoint operator. Also assume $X_{0}$ is $\mathscr{F}_{0}$ measurable with values in $H$. Recall the descriptive diagram.

\[

\]

Here the Wiener process is in $U_{1}$ and the filtration with respect to which $\Phi$ is progressively measurable is the usual filtration determined by this Wiener process. Then the Ito formula is about writing the random variable $F(t, X(t))$ in terms of various integrals and derivatives of $F$.

### 67.2 Assumptions And A Lemma

Assume $F:[0, T] \times H \times \Omega \rightarrow \mathbb{R}^{1}$ has continuous partial derivatives $F_{t}, F_{X}$, and $F_{X X}$ which are uniformly continuous and bounded on bounded subsets of $[0, T] \times H$ independent of $\omega \in \Omega$. Also assume $F_{X X}$ is uniformly bounded and that $F_{X X X}$ exists. Let $\phi:[0, T] \times \Omega \rightarrow H$ be progressively measurable and Bochner integrable for each $\omega$. Assume $\Phi$ is progressively measurable, and is in $L^{2}\left([0, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$.

Now here is the important lemma which makes the Ito formula possible.

Lemma 67.2.1 Suppose $\eta_{j}$ are real random variables $E\left(\eta_{j}^{2}\right)<\infty$, such that $\eta_{k}$ is measurable with respect to $\mathscr{G}_{j}$ for all $j>k$ where $\left\{\mathscr{G}_{k}\right\}$ is increasing. Then

$$
\begin{align*}
& E\left(\left[\sum_{k=0}^{m-1} \eta_{k}-\sum_{k=0}^{m-1} E\left(\eta_{k} \mid \mathscr{G}_{k}\right)\right]^{2}\right)  \tag{67.2.2}\\
& \quad=E\left(\sum_{k=0}^{m-1} \eta_{k}^{2}-E\left(\eta_{k} \mid \mathscr{G}_{k}\right)^{2}\right)
\end{align*}
$$

Proof: First consider a mixed term $i<k$.

$$
E\left(\left(\eta_{i}-E\left(\eta_{i} \mid \mathscr{G}_{i}\right)\right)\left(\eta_{k}-E\left(\eta_{k} \mid \mathscr{G}_{k}\right)\right)\right)
$$

This equals

$$
\begin{gathered}
E\left(\eta_{i} \eta_{k}\right)-E\left(\eta_{i} E\left(\eta_{k} \mid \mathscr{G}_{k}\right)\right)-E\left(\eta_{k} E\left(\eta_{i} \mid \mathscr{G}_{i}\right)\right)+E\left(E\left(\eta_{i} \mid \mathscr{G}_{i}\right) E\left(\eta_{k} \mid \mathscr{G}_{k}\right)\right) \\
=E\left(\eta_{i} \eta_{k}\right)-E\left(E\left(\eta_{i} \eta_{k} \mid \mathscr{G}_{k}\right)\right)-E\left(\eta_{k} E\left(\eta_{i} \mid \mathscr{G}_{i}\right)\right)+E\left(E\left(\eta_{k} E\left(\eta_{i} \mid \mathscr{G}_{i}\right) \mid \mathscr{G}_{k}\right)\right) \\
=E\left(\eta_{i} \eta_{k}\right)-E\left(E\left(\eta_{i} \eta_{k} \mid \mathscr{G}_{k}\right)\right)-E\left(\eta_{k} E\left(\eta_{i} \mid \mathscr{G}_{i}\right)\right)+E\left(\eta_{k} E\left(\eta_{i} \mid \mathscr{G}_{i}\right)\right) \\
=E\left(\eta_{i} \eta_{k}\right)-E\left(\eta_{i} \eta_{k}\right)-E\left(\eta_{k} E\left(\eta_{i} \mid \mathscr{G}_{i}\right)\right)+E\left(\eta_{k} E\left(\eta_{i} \mid \mathscr{G}_{i}\right)\right)=0
\end{gathered}
$$

Thus 67.2.2 equals

$$
\sum_{k=0}^{m-1} E\left(\left(\eta_{k}-E\left(\eta_{k} \mid \mathscr{G}_{k}\right)\right)^{2}\right)
$$

which equals

$$
\begin{aligned}
& \sum_{k=0}^{m-1} E\left(\eta_{k}^{2}\right)-2 E\left(\eta_{k} E\left(\eta_{k} \mid \mathscr{G}_{k}\right)\right)+E\left(E\left(\eta_{k} \mid \mathscr{G}_{k}\right)^{2}\right) \\
= & \sum_{k=0}^{m-1} E\left(\eta_{k}^{2}\right)-2 E\left(E\left(\eta_{k} E\left(\eta_{k} \mid \mathscr{G}_{k}\right)\right) \mid \mathscr{G}_{k}\right)+E\left(E\left(\eta_{k} \mid \mathscr{G}_{k}\right)^{2}\right) \\
= & \sum_{k=0}^{m-1} E\left(\eta_{k}^{2}\right)-2 E\left(E\left(\eta_{k} \mid \mathscr{G}_{k}\right) E\left(\eta_{k} \mid \mathscr{G}_{k}\right)\right)+E\left(E\left(\eta_{k} \mid \mathscr{G}_{k}\right)^{2}\right) \\
= & \sum_{k=0}^{m-1} E\left(\eta_{k}^{2}\right)-2 E\left(E\left(\eta_{k} \mid \mathscr{G}_{k}\right)^{2}\right)+E\left(E\left(\eta_{k} \mid \mathscr{G}_{k}\right)^{2}\right) \\
= & \sum_{k=0}^{m-1} E\left(\eta_{k}^{2}\right)-E\left(E\left(\eta_{k} \mid \mathscr{G}_{k}\right)^{2}\right)
\end{aligned}
$$

### 67.3 A Special Case

To make it simpler, first consider the situation in which $\Phi=\Phi_{0}$ where $\Phi_{0}$ is $\mathscr{F}_{0}$ measurable and has finitely many values in $\mathscr{L}\left(U_{1}, H\right)$, and $\phi=\phi_{0}$ where $\phi_{0}$ is $\mathscr{F}_{0}$ measurable and a simple function with values in $H$. Thus

$$
X(t)=X_{0}+\int_{0}^{t} \phi_{0} d s+\int_{0}^{t} \Phi_{0} d W
$$

Now let $\left\{t_{k}^{n}\right\}_{k=0}^{m_{n}}$ denote the $n^{\text {th }}$ partition of $[0, T]$, referred to as $\mathscr{P}_{n}$ such that

$$
\lim _{n \rightarrow \infty}\left(\max \left\{\left|t_{k}^{n}-t_{k-1}^{n}\right|, k=0,1,2, \cdots, m_{n}\right\}\right) \equiv \lim _{n \rightarrow \infty}\left\|\mathscr{P}_{n}\right\|=0
$$

The superscript $n$ will be suppressed to save notation. Then

$$
\begin{aligned}
F(T, X(T))- & F\left(0, X_{0}\right)=\sum_{k=0}^{m_{n}-1}\left(F\left(t_{k+1}, X\left(t_{k+1}\right)\right)-F\left(t_{k}, X\left(t_{k}\right)\right)\right) \\
= & \sum_{k=0}^{m_{n}-1}\left(F\left(t_{k+1}, X\left(t_{k+1}\right)\right)-F\left(t_{k}, X\left(t_{k+1}\right)\right)\right) \\
& +\sum_{k=0}^{m_{n}-1}\left(F\left(t_{k}, X\left(t_{k+1}\right)\right)-F\left(t_{k}, X\left(t_{k}\right)\right)\right)
\end{aligned}
$$

This equals

$$
\begin{gather*}
\sum_{k=0}^{m_{n}-1} F_{t}\left(t_{k}, X\left(t_{k+1}\right)\right)\left(t_{k+1}-t_{k}\right)+o\left(\left|t_{k+1}-t_{k}\right|\right)  \tag{67.3.3}\\
+\sum_{k=0}^{m_{n}-1} F_{X}\left(t_{k}, X\left(t_{k}\right)\right)\left(X\left(t_{k+1}\right)-X\left(t_{k}\right)\right)  \tag{67.3.4}\\
+\frac{1}{2} \sum_{k=0}^{m_{n}-1}\left(F_{X X}\left(t_{k}, X\left(t_{k}\right)\right)\left(X\left(t_{k+1}\right)-X\left(t_{k}\right)\right),\left(X\left(t_{k+1}\right)-X\left(t_{k}\right)\right)\right)_{H}  \tag{67.3.5}\\
+\sum_{k=0}^{m_{n}-1} O\left(\left|X\left(t_{k+1}\right)-X\left(t_{k}\right)\right|_{H}^{3}\right) \tag{67.3.6}
\end{gather*}
$$

Recall

$$
X(t)=X_{0}+\int_{0}^{t} \phi_{0} d s+\int_{0}^{t} \Phi_{0} d W
$$

From the properties of the Wiener process in 67.0.1, the term in 67.3 .6 converges to 0 as $n \rightarrow \infty$ since these properties of the Wiener process imply $X$ is Holder continuous with exponent $2 / 5$.

Now consider the term of 67.3.5. All terms converge to 0 except

$$
\begin{equation*}
\frac{1}{2} \sum_{k=0}^{m_{n}-1}\left(F_{X X}\left(t_{k}, X\left(t_{k}\right)\right) \int_{t_{k}}^{t_{k+1}} \Phi_{0} d W, \int_{t_{k}}^{t_{k+1}} \Phi_{0} d W\right)_{H} \tag{67.3.7}
\end{equation*}
$$

Consider one of the terms in 67.3.7. Let $A \in \mathscr{F}_{t_{k}}$.By Corollary 65.5.4,

$$
\begin{aligned}
& \int_{A} \frac{1}{2}\left(F_{X X}\left(t_{k}, X\left(t_{k}\right)\right) \int_{t_{k}}^{t_{k+1}} \Phi_{0} d W, \int_{t_{k}}^{t_{k+1}} \Phi_{0} d W\right)_{H} d P \\
= & \int_{A} \frac{1}{2}\left(\int_{t_{k}}^{t_{k+1}} F_{X X}\left(t_{k}, X\left(t_{k}\right)\right) \Phi_{0} d W, \int_{t_{k}}^{t_{k+1}} \Phi_{0} d W\right)_{H} d P
\end{aligned}
$$

By independence,

$$
=P(A) \frac{1}{2} \int_{\Omega}\left(\int_{t_{k}}^{t_{k+1}} F_{X X}\left(t_{k}, X\left(t_{k}\right)\right) \Phi_{0} d W, \int_{t_{k}}^{t_{k+1}} \Phi_{0} d W\right)_{H} d P
$$

By the Ito isometry results presented earlier,

$$
\begin{aligned}
& =\int_{\Omega} \mathscr{X}_{A} \frac{1}{2} \int_{t_{k}}^{t_{k+1}}\left(F_{X X}\left(t_{k}, X\left(t_{k}\right)\right) \Phi_{0}, \Phi_{0}\right)_{\mathscr{L}_{2}} d s d P \\
& =\int_{A} \frac{1}{2} \int_{\int_{t_{k}} t_{k+1}}^{t_{k}}\left(F_{X X}\left(t_{k}, X\left(t_{k}\right)\right) \Phi_{0}, \Phi_{0}\right)_{\mathscr{L}_{2}}
\end{aligned} d s d P \text { measurable } \quad \begin{aligned}
& =\int_{A} \frac{1}{2}\left(F_{X X}\left(t_{k}, X\left(t_{k}\right)\right) \Phi_{0}, \Phi_{0}\right)_{\mathscr{L}_{2}}\left(t_{k+1}-t_{k}\right) d P
\end{aligned}
$$

Since $A \in \mathscr{F}_{t_{k}}$ was arbitrary,

$$
\begin{aligned}
& E\left(\left.\frac{1}{2}\left(F_{X X}\left(t_{k}, X\left(t_{k}\right)\right) \int_{t_{k}}^{t_{k+1}} \Phi_{0} d W, \int_{t_{k}}^{t_{k+1}} \Phi_{0} d W\right)_{H} \right\rvert\, \mathscr{F}_{t_{k}}\right) \\
= & \frac{1}{2}\left(F_{X X}\left(t_{k}, X\left(t_{k}\right)\right) \Phi_{0}, \Phi_{0}\right)_{\mathscr{L}_{2}}\left(t_{k+1}-t_{k}\right)
\end{aligned}
$$

From what was just shown, and Lemma 67.2.1,

$$
\begin{align*}
& E\left(\left[\frac{1}{2} \sum_{k=0}^{m_{n}-1}\left(F_{X X}\left(t_{k}, X\left(t_{k}\right)\right) \Phi_{0} \Delta W\left(t_{k}\right), \Phi_{0} \Delta W\left(t_{k}\right)\right)_{H}-\right.\right. \\
& \left.\left.\sum_{k=0}^{m_{n}-1} \frac{1}{2}\left(F_{X X}\left(t_{k}, X\left(t_{k}\right)\right) \Phi_{0}, \Phi_{0}\right)_{\mathscr{L}_{2}}\left(t_{k+1}-t_{k}\right)\right]^{2}\right)  \tag{67.3.8}\\
& =\frac{1}{4} E\left(\sum_{k=0}^{m_{n}-1}\left(F_{X X}\left(t_{k}, X\left(t_{k}\right)\right) \Phi_{0} \Delta W\left(t_{k}\right), \Phi_{0} \Delta W\left(t_{k}\right)\right)_{H}^{2}\right. \\
& \left.\quad-\sum_{k=0}^{m_{n}-1}\left(F_{X X}\left(t_{k}, X\left(t_{k}\right)\right) \Phi_{0}, \Phi_{0}\right)_{\mathscr{L}_{2}}^{2}\left(t_{k+1}-t_{k}\right)^{2}\right)
\end{align*}
$$

Now $F_{X X}$ is bounded and so there exists a constant $M$ independent of $k$ and $n$,

$$
M \geq\left\|\Phi_{0}^{*} F_{X X}\left(t_{k}, X\left(t_{k}\right)\right) \Phi_{0}\right\|,\left|\left(F_{X X}\left(t_{k}, X\left(t_{k}\right)\right) \Phi_{0}, \Phi_{0}\right)_{\mathscr{L}_{2}}\right|
$$

Hence the above is dominated by

$$
\begin{aligned}
& \leq \frac{1}{4} M^{2} \sum_{k=0}^{m_{n}-1} E\left\|\Delta W\left(t_{k}\right)\right\|_{U_{1}}^{4}+\frac{1}{4} M^{2} \sum_{k=0}^{m_{n}-1}\left(t_{k+1}-t_{k}\right)^{2} \\
& \leq \frac{M^{2}}{4}\left(\sum_{k=0}^{m_{n}-1}\left(C_{4}+1\right)\left(t_{k+1}-t_{k}\right)^{2}\right)
\end{aligned}
$$

which converges to 0 as $n \rightarrow \infty$. Then from 67.3.8, and referring to 67.3.5,

$$
\begin{gather*}
\lim _{n \rightarrow \infty} \frac{1}{2} \sum_{k=0}^{m_{n}-1}\left(F_{X X}\left(t_{k}, X\left(t_{k}\right)\right)\left(X\left(t_{k+1}\right)-X\left(t_{k}\right)\right),\left(X\left(t_{k+1}\right)-X\left(t_{k}\right)\right)\right)_{H}  \tag{67.3.9}\\
=\lim _{n \rightarrow \infty} \frac{1}{2} \sum_{k=0}^{m_{n}-1}\left(F_{X X}\left(t_{k}, X\left(t_{k}\right)\right) \int_{t_{k}}^{t_{k+1}} \Phi_{0} d W, \int_{t_{k}}^{t_{k+1}} \Phi_{0} d W\right)_{H} \\
=\lim _{n \rightarrow \infty} \frac{1}{2} \sum_{k=0}^{m_{n}-1}\left(F_{X X}\left(t_{k}, X\left(t_{k}\right)\right) \Phi_{0}, \Phi_{0}\right)_{\mathscr{L}_{2}}\left(t_{k+1}-t_{k}\right)
\end{gather*}
$$

if this last limit exists in $L^{2}(\Omega)$. However, since $F_{X X}$ is bounded, this limit certainly exists for a.e. $\omega$ and equals

$$
=\frac{1}{2} \int_{0}^{T}\left(F_{X X}(t, X(t)) \Phi_{0}, \Phi_{0}\right)_{\mathscr{L}_{2}} d t
$$

The limit also exists in $L^{2}(\Omega)$ obviously, since $F_{X X}$ is assumed bounded. Therefore, a subsequence of 67.3.9, still denoted as $n$ must converge for a.e. $\omega$ to the above integral as $n \rightarrow \infty$.

Next consider 67.3.4.

$$
\begin{gather*}
\sum_{k=0}^{m_{n}-1} F_{X}\left(t_{k}, X\left(t_{k}\right)\right)\left(X\left(t_{k+1}\right)-X\left(t_{k}\right)\right)=\sum_{k=0}^{m_{n}-1} F_{X}\left(t_{k}, X\left(t_{k}\right)\right)\left(\int_{t_{k}}^{t_{k+1}} \phi_{0} d s\right) \\
+\sum_{k=0}^{m_{n}-1} F_{X}\left(t_{k}, X\left(t_{k}\right)\right) \int_{t_{k}}^{t_{k+1}} \Phi_{0} d W \tag{67.3.10}
\end{gather*}
$$

Consider the second of these in 67.3.10. From Corollary 65.5.4, it equals

$$
\begin{aligned}
& \sum_{k=0}^{m_{n}-1} \int_{t_{k}}^{t_{k+1}} F_{X}\left(t_{k}, X\left(t_{k}\right)\right) \Phi_{0} d W \\
= & \int_{0}^{T}\left(\sum_{k=0}^{m_{n}-1} \mathscr{X}_{\left(t_{k}, t_{k+1}\right]}(t) F_{X}\left(t_{k}, X\left(t_{k}\right)\right)\right) \Phi_{0} d W
\end{aligned}
$$

which converges as $n \rightarrow \infty$ to

$$
\int_{0}^{T} F_{X}(t, X(t)) \Phi_{0} d W
$$

because

$$
\lim _{n \rightarrow \infty}\left(\sum_{k=0}^{m_{n}-1} \mathscr{X}_{\left(t_{k}, t_{k+1}\right]}(t) F_{X}\left(t_{k}, X\left(t_{k}\right)\right)\right) \Phi_{0}=F_{X}(t, X(t)) \Phi_{0}
$$

in $L^{2}\left([0, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$. Next consider the first on the right in 67.3.10. It equals

$$
\sum_{k=0}^{m_{n}-1}\left(F_{X}\left(t_{k}, X\left(t_{k}\right)\right) \phi_{0}\left(t_{k+1}-t_{k}\right)\right)
$$

and converges to

$$
\int_{0}^{T} F_{X}(t, X(t)) \phi_{0} d t
$$

Finally, it is obviously the case that 67.3 .3 converges to

$$
\int_{0}^{T} F_{t}(t, X(t)) d t
$$

This has shown

$$
\begin{gathered}
F(T, X(T))=F\left(0, X_{0}\right)+\int_{0}^{T} F_{t}(t, X(t))+F_{X}(t, X(t)) \phi_{0} d t \\
+\int_{0}^{T} F_{X}(t, X(t)) \Phi_{0} d W+\frac{1}{2} \int_{0}^{T}\left(F_{X X}(t, X(t)) \Phi_{0}, \Phi_{0}\right)_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)} d t
\end{gathered}
$$

when

$$
X(t)=X_{0}+\int_{0}^{t} \phi_{0} d s+\int_{0}^{t} \Phi_{0} d W
$$

$\phi_{0}, \Phi_{0} \mathscr{F}_{0}$ measurable as described above. This is the first version of the Ito formula.

### 67.4 The Case Of Elementary Functions

Of course there was nothing special about the interval $[0, T]$. It follows that for $[a, b] \subseteq$ $[0, T], \Phi_{a} \in \mathscr{L}\left(U_{1}, U\right)$ and $\mathscr{F}_{a}$ measurable, having finitely many values, $\phi_{a}$ a simple function which is $\mathscr{F}_{a}$ measurable,

$$
\begin{gathered}
X(t)=X(a)+\int_{a}^{t} \phi_{a} d t+\int_{a}^{t} \Phi_{a} d W \\
F(b, X(b))=F(a, X(a))+\int_{a}^{b}\left(F_{t}(t, X(t))+F_{X}(t, X(t)) \phi_{a}\right) d t \\
+\int_{a}^{b} F_{X}(t, X(t)) \Phi_{a} d W+\frac{1}{2} \int_{a}^{b}\left(F_{X X}(t, X(t)) \Phi_{a}, \Phi_{a}\right)_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)} d t .
\end{gathered}
$$

Therefore, if $\Phi$ is any elementary function, being a sum of functions like $\Phi_{a} \mathscr{X}_{(a, b]}$, and $\phi$ a similar sort of elementary fuction with

$$
X(t)=X_{0}+\int_{0}^{t} \phi d s+\int_{0}^{t} \Phi d W
$$

then

$$
\begin{array}{r}
F(T, X(T))=F\left(0, X_{0}\right)+\int_{0}^{T} F_{t}(t, X(t))+F_{X}(t, X(t)) \phi(t) d t \\
+\int_{0}^{T} F_{X}(t, X(t)) \Phi d W+\frac{1}{2} \int_{0}^{T}\left(F_{X X}(t, X(t)) \Phi, \Phi\right)_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)} d t \tag{67.4.11}
\end{array}
$$

This has proved the following lemma.
Lemma 67.4.1 Let $\Phi, \phi$ be elementary functions as described and let

$$
X(t)=X_{0}+\int_{0}^{t} \phi(s) d s+\int_{0}^{t} \Phi d W
$$

Then 67.4.11 holds.

### 67.5 The Integrable Case

Now let $\Phi \in L^{2}\left([0, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right), \phi \in L^{1}([0, T] \times \Omega ; H)$ and be progressively measurable. Let $\phi$ be as above, and let

$$
\begin{equation*}
X(t)=X_{0}+\int_{0}^{t} \phi(t) d t+\int_{0}^{t} \Phi d W \tag{67.5.12}
\end{equation*}
$$

Suppose also the additional condition that for some $M$,

$$
|X(t, \omega)|<M \text { for all }(t, \omega) \in[0, T] \times N^{C}, P(N)=0
$$

Does it follow that 67.4.11 holds?
There exists a sequence of elementary functions $\left\{\Phi_{n}\right\}$ converging to $\Phi \circ J^{-1}$ in

$$
L^{2}\left([0, T] \times \Omega ; \mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)\right)
$$

Similarly let $\left\{\phi_{n}\right\}$ converge to $\phi$ in $L^{1}([0, T] \times \Omega ; H)$ where $\phi_{n}$ is also an elementary function, $\left|\phi_{n}\right| \leq|\phi|$ at the mesh points. You could use that theorem about approximating with left and right step functions if desired, Lemma 65.3.1. Let

$$
X_{n}(t)=X_{0}+\int_{0}^{t} \phi_{n}(s) d s+\int_{0}^{t} \Phi_{n} d W
$$

Also let $\tau_{n}$ be the stopping times

$$
\tau_{n} \equiv \inf \left\{t>0:\left|X_{n}(t)\right|>M\right\}
$$

Since $X_{n}$ is continuous, this is a well defined stopping time. Thus

$$
X_{n}^{\tau_{n}}(t)=X_{0}+\int_{0}^{t} \mathscr{X}_{\left[0, \tau_{n}\right]} \phi_{n}(t) d t+\int_{0}^{t} \mathscr{X}_{\left[0, \tau_{n}\right]} \Phi_{n} d W
$$

and as noted in the discussion of localization for elementary functions, $\mathscr{X}_{\left[0, \tau_{n}\right]} \Phi_{n}$ is an elementary function.

Claim: $\lim _{n \rightarrow \infty} \mathscr{X}_{\left[0, \tau_{n}\right]}=1$.
Proof of claim: From maximal estimates as in the construction of the stochastic integral and the Borel Cantelli lemma, it follows that there exists a subsequence still denoted by $n$ and a set of measure zero $N$ such that for $\omega \notin N_{1}$,

$$
\int_{0}^{t} \Phi_{n} d W \rightarrow \int_{0}^{t} \Phi d W
$$

uniformly on $[0, T]$. Also one can show that off a set of measure zero, there is a subsequence still called $n$ such that $\int_{0}^{t} \phi_{n}(s) d s \rightarrow \int_{0}^{t} \phi(s) d s$ uniformly on $[0, T]$. Here is why.

$$
E\left(\left|\int_{0}^{t} \phi_{n}(s) d s-\int_{0}^{t} \phi(s) d s\right|\right) \leq \int_{\Omega} \int_{0}^{T}\left|\phi_{n}-\phi\right| d t d P
$$

which is given to converge to 0 . Thus

$$
\begin{gathered}
P\left(\max _{t \in[0, T]}\left|\int_{0}^{t} \phi_{n}(s) d s-\int_{0}^{t} \phi(s) d s\right|>\lambda\right) \leq P\left(\int_{0}^{T}\left|\phi_{n}(s)-\phi(s)\right| d s>\lambda\right) \\
\leq \frac{1}{\lambda} \int_{\left[\int_{0}^{T}\left|\phi_{n}(s)-\phi(s)\right| d s>\lambda\right]} \int_{0}^{T}\left|\phi_{n}(s)-\phi(s)\right| d s d P \\
\leq \frac{1}{\lambda} \int_{\Omega} \int_{0}^{T}\left|\phi_{n}(s)-\phi(s)\right| d s d P
\end{gathered}
$$

Thus

$$
P\left(\max _{t \in[0, T]}\left|\int_{0}^{t} \phi_{n}(s) d s-\int_{0}^{t} \phi(s) d s\right|>2^{-k}\right) \leq 2^{k} \int_{\Omega} \int_{0}^{T}\left|\phi_{n}(s)-\phi(s)\right| d s d P
$$

If $n>n_{k}$, the right side is less than $2^{-k}$. Use $\phi_{n_{k}}$. Then there exists a set of measure zero $N_{2}$ such that for $\omega \notin N_{2}$,

$$
\left|\int_{0}^{t} \phi_{n}(s) d s-\int_{0}^{t} \phi(s) d s\right| \rightarrow 0
$$

uniformly. Hence, you can take a couple of subsequences and assert that there exists a subsequence still called $n$ and a set of measure zero $N$ such that $X_{n}(t) \rightarrow X(t)$ uniformly on $[0, T]$ for each $\omega \notin N$. Since $|X(t, \omega)|<M$, it follows that for each $\omega \notin N$, when $n$ is large enough, $\tau_{n}=\infty$ and this proves the claim.

From the claim, it follows that $\mathscr{X}_{\left[0, \tau_{n}\right]} \Phi_{n} \rightarrow \Phi \circ J^{-1}$ in $L^{2}\left([0, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$ and $\mathscr{X}_{\left[0, \tau_{n}\right]} \phi_{n} \rightarrow \phi$ in $L^{1}([0, T] \times \Omega ; \mathscr{H})$. Thus you can replace $\Phi_{n}$ in the above with $\mathscr{X}_{\left[0, \tau_{n}\right]} \Phi_{n}$ and $\phi_{n}$ with $\mathscr{X}_{\left[0, \tau_{n}\right]} \phi_{n}$. Thus there exists a subsequence, still called $n$ and a set of measure zero $N$ such that for $\omega \notin N$,

$$
\int_{0}^{t} \mathscr{X}_{\left[0, \tau_{n}\right]} \Phi_{n} d W \rightarrow \int_{0}^{t} \Phi d W
$$

uniformly and

$$
\int_{0}^{t} \mathscr{X}_{\left[0, \tau_{n}\right]} \phi_{n} d s \rightarrow \int_{0}^{t} \phi d s
$$

uniformly. Hence also $X_{n}^{\tau_{n}}(t) \rightarrow X(t)$ uniformly on $[0, T]$ whenever $\omega \notin N$. Of course $\left|X_{n}^{\tau_{n}}(t)\right|_{H}$ has the advantage of being bounded by $M$.

From the above,

$$
\begin{gathered}
F\left(T, X_{n}^{\tau_{n}}(T)\right)=F\left(0, X_{0}\right)+\int_{0}^{T} F_{t}\left(t, X_{n}^{\tau_{n}}(t)\right)+F_{X}\left(t, X_{n}^{\tau_{n}}(t)\right) \mathscr{X}_{\left[0, \tau_{n}\right]} \phi_{n}(t) d t \\
+\int_{0}^{T} F_{X}\left(t, X_{n}^{\tau_{n}}(t)\right) \Phi_{n} d W+\frac{1}{2} \int_{0}^{T}\left(F_{X X}\left(t, X_{n}^{\tau_{n}}(t)\right) \mathscr{X}_{\left[0, \tau_{n}\right]} \Phi_{n}, \mathscr{X}_{\left[0, \tau_{n}\right]} \Phi_{n}\right)_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)} d t
\end{gathered}
$$

Then it is obvious that one can pass to the limit in each of the non stochastic integrals in the above. It is necessary to consider the other one.

From the above claim, $\mathscr{X}_{\left[0, \tau_{n}\right]} \Phi_{n} \rightarrow \Phi \circ J^{-1}$ in $L^{2}\left([0, T] \times \Omega ; \mathscr{L}_{2}\left(J Q^{1 / 2} U, H\right)\right)$ and also, from the stopping times $\tau_{n}, F_{X}\left(t, X_{n}^{\tau_{n}}(t)\right)$ is bounded and converges to $F_{X}(t, X(t))$. Hence the dominated convergence theorem applies, and letting $n \rightarrow \infty$, the following is obtained for a.e. $\omega$

$$
\begin{array}{r}
F(T, X(T))=F\left(0, X_{0}\right)+\int_{0}^{T} F_{t}(t, X(t))+F_{X}(t, X(t)) \phi(t) d t \\
+\int_{0}^{T} F_{X}(t, X(t)) \Phi d W+\frac{1}{2} \int_{0}^{T}\left(F_{X X}(t, X(t)) \Phi, \Phi\right)_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)} d t \tag{67.5.13}
\end{array}
$$

This is the Ito formula in case that $\Phi \in L^{2}\left([0, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$ and $|X|$ is bounded above by $M$.

It is easy to remove this assumption on $|X|$. Let $X$ be given in 67.5.12. Let $\tau_{n}$ be the stopping time

$$
\tau_{n} \equiv \inf \{t>0:|X|>n\}
$$

Then 67.5.13 holds for the stopped process $X^{\tau_{n}}$ and $\Phi$ and $\phi$ replaced with $\Phi \mathscr{X}_{\left[0, \tau_{n}\right]}$ and $\phi \mathscr{X}_{\left[0, \tau_{n}\right]}$ respectively. Then let $n \rightarrow \infty$ in this expression, using the continuity of $X$ and the fact that $\tau_{n} \rightarrow \infty$ to to recover 67.5.13 without the restriction on $|X|$.

### 67.6 The General Stochastically Integrable Case

Now suppose that $\Phi$ is only progressively measurable and stochastically integrable

$$
P\left(\left[\int_{0}^{T}\|\Phi\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d t<\infty\right]\right)=1
$$

Also $\phi$ is only progressively measurable and Bochner integrable in $t$. Define a stopping time

$$
\tau(\omega)=\inf \left\{t \geq 0:|X(t, \omega)|_{H}+\int_{0}^{t}\|\Phi\|^{2} d s+\int_{0}^{t}|\phi| d s>C\right\}
$$

This is just the first hitting time of an open set so it is a stopping time. For $t \leq \tau$, all of the above quantities must be no larger than $C$. In particular,

$$
\mathscr{X}_{[0, \tau]} \Phi \in L^{2}\left([0, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right) .
$$

Then

$$
X^{\tau}(t)=X_{0}+\int_{0}^{t} \mathscr{X}_{[0, \tau]} \phi d s+\int_{0}^{t} \mathscr{X}_{[0, \tau]} \Phi d W
$$

and so 67.5.13 holds with $X \rightarrow X^{\tau}, \Phi \rightarrow \mathscr{X}_{[0, \tau]} \Phi$ and $\phi \rightarrow \mathscr{X}_{[0, \tau]} \phi$. Now simply let $C \rightarrow \infty$ and exploit the continuity of $X$ given by the formula 67.5.12 to obtain the validity of 67.5.13 without any reference to the stopping time. Of course arbitrary $t$ can replace $T$. This leads to the main result.

Theorem 67.6.1 Let $\Phi$ be a progressively measurable with values in $\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)$ which is stochastically integrable in $[0, T]$ because

$$
P\left(\left[\int_{0}^{T}\|\Phi\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d t<\infty\right]\right)=1
$$

and let $\phi:[0, T] \times \Omega \rightarrow H$ be progressively measurable and Bochner integrable on $[0, T]$ for a.e. $\omega$, and let $X_{0}$ be $\mathscr{F}_{0}$ measurable and $H$ valued. Let

$$
X(t) \equiv X_{0}+\int_{0}^{t} \phi(s) d s+\int_{0}^{t} \Phi d W
$$

Let $F:[0, T] \times H \times \Omega \rightarrow \mathbb{R}^{1}$ be progressively measurable, have continuous partial derivatives $F_{t}, F_{X}, F_{X X}$ which are uniformly continuous on bounded subsets of $[0, T] \times H$ independent of $\omega \in \Omega$. Also assume $F_{X X}$ is bounded and let $F_{X X X}$ exist and be bounded. Then the following formula holds for a.e. $\omega$.

$$
\begin{gathered}
F(t, X(t))=F\left(0, X_{0}\right)+\int_{0}^{t} F_{X}(\cdot, X(\cdot)) \Phi d W+ \\
\int_{0}^{t} F_{t}(s, X(s))+F_{X}(s, X(s)) \phi(s) d s+\frac{1}{2} \int_{0}^{t}\left(F_{X X}(s, X(s)) \Phi, \Phi\right)_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)} d s
\end{gathered}
$$

The dependence of $F$ on $\omega$ is suppressed.
That last term is interesting and can be written differently. Let $\left\{g_{j}\right\}$ be an orthonormal basis for $Q^{1 / 2} U$. Then this integrand equals

$$
\sum_{i=1}^{L}\left(F_{X X}(s, X(s)) \Phi g_{i}, \Phi g_{i}\right)_{H}=\sum_{i=1}^{L}\left(\Phi^{*} F_{X X}(s, X(s)) \Phi g_{i}, g_{i}\right)_{Q^{1 / 2} H}
$$

and we write this as

$$
\operatorname{trace}\left(\Phi^{*}(s) F_{X X}(s, X(s)) \Phi(s)\right)
$$

A simple special case is where $Q=I$ and then $Q^{1 / 2} U=U$. Thus it is only required that $\Phi$ have values in $\mathscr{L}_{2}(U, H)$.

### 67.7 Remembering The Formula

I find it almost impossible to remember this formula. Here is a way to do it. Recall that $|\Delta W|^{2}$ is like $\Delta t$. Therefore, in what follows, neglect all terms which are like $d W d t, d t^{2}$, but keep terms which are $d W, d t, d W^{2}$. Then you start with $d X=\phi d t+\Phi d W$. Thus for $F(t, X)$,

$$
d F=F_{t} d t+F_{X} d X+\frac{1}{2}\left(F_{X X} d X, d X\right)
$$

other terms from Taylor's formula are neglected because they involve $d t d W$ or $d t^{2}$. Now the above equals

$$
d F=F_{t} d t+F_{X}(\phi d t+\Phi d W)+\frac{1}{2}\left(F_{X X} \Phi d W, \Phi d W\right)
$$

Since the $d W$ occurs twice, in that inner product, you get a $d t$ out of it. Hence you get

$$
d F=\left(F_{t}+F_{X} \phi\right) d t+\frac{1}{2}\left(F_{X X} \Phi, \Phi\right) d t+F_{X} \Phi d W
$$

Now place an $\int_{0}^{t}$ in front of everything and you have the Ito formula.

### 67.8 An Interesting Formula

Suppose everything is real valued and $\phi$ is progressively measurable and in

$$
L^{2}([0, T] \times \Omega)
$$

Let

$$
X(t)=\int_{0}^{t} \phi d W-\frac{1}{2} \int_{0}^{t} \phi^{2} d s
$$

and consider $F(X)=e^{X}$. Then from the Ito formula,

$$
\begin{gathered}
d F=-\left(e^{X} \phi^{2} \frac{1}{2}\right) d t+\frac{1}{2} e^{X} \phi^{2} d t+e^{X} \phi d W \\
d F=e^{X} \phi d W
\end{gathered}
$$

and then do an integral

$$
e^{X(t)}-1=\int_{0}^{t} e^{X} \phi d W
$$

Thus

$$
e^{X(t)}=1+\int_{0}^{t} e^{X(s)} \phi d W
$$

That expression on the right is obviously a local martingale and so the expression on the left is also. To see this, you can use a localizing sequence of stopping times which depend on the size of $X(t)$. This will work fine because $X(t)$ is continuous.

### 67.9 Some Representation Theorems

In this section is a very interesting representation theorem which comes from the Ito formula. In all of this, $\mathbf{W}$ will be a $Q$ Wiener process having values in $\mathbb{R}^{n}$ for which $Q=I$. Recall that, letting

$$
\mathscr{G}_{t} \equiv \sigma(\mathbf{W}(s): s \leq t)
$$

the normal filtration determined by the Wiener process is given by

$$
\mathscr{F}_{t} \equiv \cap_{s>t} \overline{\mathscr{G}}_{s}
$$

where $\overline{\mathscr{G}_{s}}$ is the completion of $\mathscr{G}_{s}$. In this section, the theorems will all feature the smaller filtration $\mathscr{G}_{t}$, not the filtration $\mathscr{F}_{t}$. First here are some simple observations which tie this specialized material to what was presented earlier.

When you have $\mathbf{f}$ an $\mathscr{G}_{t}$ adapted function in $L^{2}\left(\Omega, \mathbb{R}^{n}\right)$, you can consider

$$
\mathbf{f}^{T} \in L^{2}\left([0, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} \mathbb{R}^{n}, \mathbb{R}\right)\right)
$$

as follows. Letting $\left\{\mathbf{g}_{i}\right\}$ be an orthonormal basis for the subspace $Q^{1 / 2} \mathbb{R}^{n}$ in the norm of $Q^{1 / 2} \mathbb{R}^{n}$,

$$
\|\mathbf{f}\|_{\mathscr{L}_{2}\left(Q^{1 / 2} \mathbb{R}^{n}, \mathbb{R}\right)}^{2} \equiv \sum_{i}\left(\mathbf{f}^{T} \mathbf{g}_{i}\right)^{2}<\infty
$$

For simplicity, let $Q=I$. Then you have the simple situation that

$$
\left\|\mathbf{f}^{T}\right\|_{\mathscr{L}_{2}\left(Q^{1 / 2} \mathbb{R}^{n}, \mathbb{R}\right)}=\|\mathbf{f}\|_{\mathbb{R}^{n}}^{2}
$$

In what follows $\mathbf{W}_{t}$ will be the $Q$ Wiener process on $\mathbb{R}^{n}$ where $Q=I$. Then the Ito isometry is nothing more than the following lemma.

Lemma 67.9.1 Let $\mathbf{f}$ be $\mathscr{F}_{t}$ adapted in the sense that every component is $\mathscr{F}_{t}$ adapted and $\mathbf{f} \in L^{2}\left(\Omega ; \mathbb{R}^{n}\right)$. Here $\mathscr{F}_{t}$ is the normal filtration coming from the Wiener process. Then

$$
\left\|\int_{0}^{T} \mathbf{f}(s)^{T} d \mathbf{W}\right\|_{L^{2}(\Omega)}=\|\mathbf{f}\|_{L^{2}\left(\Omega \times[0, T] ; \mathbb{R}^{n}\right)}
$$

Lemma 67.9.2 Let $X \geq 0$ and measurable and integrable. Also define a finite measure $v$ on $\mathscr{B}\left(\mathbb{R}^{p}\right)$ by

$$
v(B) \equiv \int_{\Omega} X \mathscr{X}_{B}(\mathbf{Y}) d P
$$

Then

$$
\int_{\Omega} g(\mathbf{Y}) X d P=\int_{\mathbb{R}^{p}} g(\mathbf{y}) d v(y)
$$

where here $\mathbf{Y}$ is a measurable function with values in $\mathbb{R}^{p}$ and $g \geq 0$ is Borel measurable. Formally, $X d P=d \nu$.

Proof: First say $X=\mathscr{X}_{D}$ and replace $g(\mathbf{Y})$ with $\mathscr{X}_{\mathbf{Y}^{-1}(B)}$. Let

$$
\mu(B) \equiv \int_{\Omega} \mathscr{X}_{D} \mathscr{X}_{B}(\mathbf{Y}) d P
$$

Then

$$
\begin{aligned}
& \int_{\Omega} \mathscr{X}_{D} \mathscr{X}_{\mathbf{Y}^{-1}(B)} d P=P\left(D \cap \mathbf{Y}^{-1}(B)\right) \\
& \int_{\mathbb{R}^{p}} \mathscr{X}_{B}(\mathbf{y}) d \mu(y)=\mu(B) \equiv \int_{\Omega} \mathscr{X}_{D} \mathscr{X}_{B}(\mathbf{Y}) d P \\
&=\int_{\Omega} \mathscr{X}_{D} \mathscr{X}_{\mathbf{Y}^{-1}(B)} d P=P\left(D \cap \mathbf{Y}^{-1}(B)\right)
\end{aligned}
$$

Thus

$$
\int_{\Omega} \mathscr{X}_{D} \mathscr{X}_{\mathbf{Y}^{-1}(B)} d P=\int_{\Omega} \mathscr{X}_{D} \mathscr{X}_{B}(\mathbf{Y}) d P=\int_{\mathbb{R}^{p}} \mathscr{X}_{B}(\mathbf{y}) d \mu(y)
$$

Now let $s_{n}(\mathbf{y}) \uparrow g(\mathbf{y})$, and let $s_{n}(\mathbf{y})=\sum_{k=1}^{m} c_{k} \mathscr{X}_{B_{k}}(\mathbf{y})$ where $B_{k}$ is a Borel set. Then

$$
\begin{gathered}
\int_{\mathbb{R}^{p}} s_{n}(\mathbf{y}) d \mu(y)=\int_{\mathbb{R}^{p}} \sum_{k=1}^{m} c_{k} \mathscr{X}_{B_{k}}(\mathbf{y}) d \mu(\mathbf{y})=\sum_{k=1}^{m} c_{k} \int_{\mathbb{R}^{p}} \mathscr{X}_{B_{k}}(\mathbf{y}) d \mu(\mathbf{y}) \\
=\sum_{k=1}^{m} c_{k} P\left(D \cap \mathbf{Y}^{-1}\left(B_{k}\right)\right) \\
\int_{\Omega} s_{n}(\mathbf{Y}) \mathscr{X}_{D} d P=\sum_{k=1}^{m} c_{k} \int_{\Omega} \mathscr{X}_{D} \mathscr{X}_{B_{k}}(\mathbf{Y}) d P=\sum_{k=1}^{m} c_{k} P\left(D \cap \mathbf{Y}^{-1}\left(B_{k}\right)\right)
\end{gathered}
$$

which is the same thing. Therefore,

$$
\int_{\Omega} s_{n}(\mathbf{Y}) \mathscr{X}_{D} d P=\int_{\mathbb{R}^{p}} s_{n}(\mathbf{y}) d \mu(y)
$$

Now pass to a limit using the monotone convergence theorem to obtain

$$
\int_{\Omega} g(\mathbf{Y}) \mathscr{X}_{D} d P=\int_{\mathbb{R}^{p}} g(\mathbf{y}) d \mu(y)
$$

Next replace $\mathscr{X}_{D}$ with $\sum_{k=1}^{m} d_{k} \mathscr{X}_{D_{k}}(\omega) \equiv s_{n}(\omega)$, a simple function. Then from what was just shown,

$$
\begin{gathered}
\int_{\Omega} g(\mathbf{Y}) \sum_{k=1}^{m} d_{k} \mathscr{X}_{D_{k}} d P=\sum_{k=1}^{m} d_{k} \int_{\Omega} g(\mathbf{Y}) \mathscr{X}_{D_{k}} d P \\
=\sum_{k=1}^{m} d_{k} \int_{\mathbb{R}^{p}} g(\mathbf{y}) d \mu_{k}
\end{gathered}
$$

where $\mu_{k}(B) \equiv \int_{\Omega} \mathscr{X}_{D_{k}} \mathscr{X}_{B}(\mathbf{Y}) d P$. Now let

$$
v_{n}(B) \equiv \int_{\Omega} \sum_{k=1}^{m} d_{k} \mathscr{X}_{D_{k}} \mathscr{X}_{B}(\mathbf{Y})=\int_{\Omega} s_{n} \mathscr{X}_{B}(\mathbf{Y}) d P
$$

Then

$$
v_{n}(B)=\sum_{k=1}^{m} d_{k} \int_{\Omega} \mathscr{X}_{D_{k}} \mathscr{X}_{B}(\mathbf{Y}) d P=\sum_{k=1}^{m} d_{k} \mu_{k}(B)
$$

Hence

$$
\begin{aligned}
\int_{\Omega} g(\mathbf{Y}) s_{n} d P & =\int_{\Omega} g(\mathbf{Y}) \sum_{k=1}^{m} d_{k} \mathscr{X}_{D_{k}} d P=\sum_{k=1}^{m} d_{k} \int_{\mathbb{R}^{p}} g(\mathbf{y}) d \mu_{k} \\
& =\int_{\mathbb{R}^{p}} g(\mathbf{y}) \sum_{k=1}^{m} d_{k} d \mu_{k}=\int_{\mathbb{R}^{p}} g(\mathbf{y}) d v_{n}
\end{aligned}
$$

Then let $s_{n}(\omega) \uparrow X(\omega)$. Clearly $v_{n} \ll v$ and so by the Radon Nikodym theorem $d v_{n}=$ $h_{n} d v$. Then by the monotone convergence theorem, for any $B$ Borel in $\mathbb{R}^{p}$,

$$
\int_{B} h_{n} d v=v_{n}(B) \equiv \int_{\Omega} s_{n}(\omega) \mathscr{X}_{B}(\mathbf{Y}(\omega)) d P \uparrow \int_{\Omega} X(\omega) \mathscr{X}_{B}(\mathbf{Y}(\omega)) d P \equiv v(B)
$$

Thus for each $B$ Borel, $0 \leq h_{n} \leq 1$ and

$$
\int_{B} h_{n} d v \rightarrow v(B)
$$

and so $h_{n} \uparrow 1 v$ a.e. Thus, from the above,

$$
\int_{\Omega} g(\mathbf{Y}) s_{n} d P=\int_{\mathbb{R}^{p}} g(\mathbf{y}) d v_{n}=\int_{\mathbb{R}^{p}} g(\mathbf{y}) h_{n}(\mathbf{y}) d v
$$

It follows from the monotone convergence theorem that one can pass to a limit in the above and obtain

$$
\int_{\Omega} g(\mathbf{Y}) X d P=\int_{\mathbb{R}^{p}} g(\mathbf{y}) d v
$$

Note that the same conclusion will hold if the functions are suitably integrable without any restriction on the sign. In particular, this will hold if $g(\mathbf{y})$ is bounded. One just considers positive and negative parts of real and imaginary parts of $g$ and applies the above lemma.

Let

$$
\mathscr{G}_{t} \equiv \sigma(\mathbf{W}(s): s \leq t)
$$

thus the normal filtration for the Wiener process and the Ito integral and so forth is

$$
\mathscr{F}_{t}=\cap_{s>t} \overline{\mathscr{G}_{s}}
$$

Lemma 67.9.3 Let $\mathbf{h}$ be a deterministic step function of the form

$$
\mathbf{h}=\sum_{i=0}^{m-1} \mathbf{a}_{i} \mathscr{X}_{\left[t_{i}, t_{i+1}\right)}, t_{m}=t
$$

Then for $\mathbf{h}$ of this form, linear combinations of functions of the form

$$
\begin{equation*}
\exp \left(\int_{0}^{t} \mathbf{h}^{T} d \mathbf{W}-\frac{1}{2} \int_{0}^{t} \mathbf{h} \cdot \mathbf{h} d \tau\right) \tag{67.9.14}
\end{equation*}
$$

are dense in $L^{2}\left(\Omega, \mathscr{G}_{t}, P\right)$ for each $t$.

Proof: I will show in the process of the proof that functions of the form 67.9.14 are in $L^{2}(\Omega, P)$. Let $g \in L^{2}\left(\Omega, \mathscr{G}_{t}, P\right)$ be such that

$$
\begin{aligned}
& \int_{\Omega} g(\omega) \exp \left(\int_{0}^{t} \mathbf{h}^{T} d \mathbf{W}-\frac{1}{2} \int_{0}^{t} \mathbf{h} \cdot \mathbf{h} d \tau\right) d P \\
= & \exp \left(-\frac{1}{2} \int_{0}^{t} \mathbf{h} \cdot \mathbf{h} d \tau\right) \int_{\Omega} g(\omega) \exp \left(\int_{0}^{t} \mathbf{h}^{T} d \mathbf{W}\right) d P=0
\end{aligned}
$$

for all such $\mathbf{h}$. It is required to show that whenever this happens for all such functions $\exp \left(\int_{0}^{t} \mathbf{h}^{T} d \mathbf{W}-\frac{1}{2} \int_{0}^{t} \mathbf{h} \cdot \mathbf{h} d t\right)$ then $g=0$.

Letting $\mathbf{h}$ be given as above, $\int_{0}^{t} \mathbf{h}^{T} d \mathbf{W}$

$$
\begin{align*}
& =\sum_{i=0}^{m-1} \mathbf{a}_{i}^{T}\left(\mathbf{W}\left(t_{i+1}\right)-\mathbf{W}\left(t_{i}\right)\right)  \tag{67.9.15}\\
& =\sum_{i=1}^{m} \mathbf{a}_{i-1}^{T} \mathbf{W}\left(t_{i}\right)-\sum_{i=0}^{m-1} \mathbf{a}_{i}^{T} \mathbf{W}\left(t_{i}\right) \\
& =\sum_{i=1}^{m-1}\left(\mathbf{a}_{i-1}^{T}-\mathbf{a}_{i}^{T}\right) \mathbf{W}\left(t_{i}\right)+\mathbf{a}_{0}^{T} \mathbf{W}\left(t_{0}\right)+\mathbf{a}_{n-1}^{T} \mathbf{W}\left(t_{n}\right) . \tag{67.9.16}
\end{align*}
$$

Also 67.9.15 shows $\exp \left(\int_{0}^{t} \mathbf{h}^{T} d \mathbf{W}\right)$ is in $L^{2}(\Omega, P)$. To see this recall the $\mathbf{W}\left(t_{i+1}\right)-\mathbf{W}\left(t_{i}\right)$ are independent and the density of $\mathbf{W}\left(t_{i+1}\right)-\mathbf{W}\left(t_{i}\right)$ is

$$
C\left(n, \Delta t_{i}\right) \exp \left(-\frac{1}{2} \frac{|\mathbf{x}|^{2}}{\left(t_{i+1}-t_{i}\right)}\right), \Delta t_{i} \equiv t_{i+1}-t_{i}
$$

so

$$
\begin{aligned}
& \int_{\Omega}\left(\exp \left(\int_{0}^{t} \mathbf{h}^{T} d \mathbf{W}\right)\right)^{2} d P=\int_{\Omega} \exp \left(2 \int_{0}^{t} \mathbf{h}^{T} d \mathbf{W}\right) d P \\
& =\int_{\Omega} \exp \left(\sum_{i=0}^{m-1} 2 \mathbf{a}_{i}^{T}\left(\mathbf{W}\left(t_{i+1}\right)-\mathbf{W}\left(t_{i}\right)\right)\right) d P \\
& =\int_{\Omega}^{m-1} \prod_{i=0}^{m} \exp \left(2 \mathbf{a}_{i}^{T}\left(\mathbf{W}\left(t_{i+1}\right)-\mathbf{W}\left(t_{i}\right)\right)\right) d P \\
& =\prod_{i=0}^{m-1} \int_{\Omega} \exp \left(2 \mathbf{a}_{i}^{T}\left(\mathbf{W}\left(t_{i+1}\right)-\mathbf{W}\left(t_{i}\right)\right)\right) d P \\
& =\prod_{i=0}^{m-1} \int_{\mathbb{R}^{n}} C\left(n, \Delta t_{i}\right) \exp \left(2 \mathbf{a}_{i}^{T} \mathbf{x}\right) \exp \left(-\frac{1}{2} \frac{|\mathbf{x}|^{2}}{\Delta t_{i}}\right) d x<\infty
\end{aligned}
$$

Choosing the $\mathbf{a}_{i}$ appropriately in 67.9.16, the formula in 67.9.16 is of the form

$$
\sum_{i=0}^{m} \mathbf{y}_{i}^{T} \mathbf{W}_{t_{i}}
$$

where $\mathbf{y}_{i}$ is an arbitrary vector in $\mathbb{R}^{n}$. It follows that for all choices of $\mathbf{y}_{j} \in \mathbb{R}^{n}$,

$$
\int_{\Omega} g(\omega) \exp \left(\sum_{j=0}^{m} \mathbf{y}_{j}^{T} \mathbf{W}_{t_{j}}(\omega)\right) d P=0
$$

Now the mapping

$$
\mathbf{y}=\left(\mathbf{y}_{0}, \cdots, \mathbf{y}_{m}\right) \rightarrow \int_{\Omega} g(\omega) \exp \left(\sum_{j=0}^{m} \mathbf{y}_{j}^{T} \mathbf{W}_{t_{j}}(\omega)\right) d P
$$

is analytic on $\mathbb{C}^{(m+1) n}$ and equals zero on $\mathbb{R}^{(m+1) n}$ so from standard complex variable theory, this analytic function must equal zero on $\mathbb{C}^{(m+1) n}$, not just on $\mathbb{R}^{(m+1) n}$. In particular, for all $\mathbf{y}=\left(\mathbf{y}_{0}, \cdots, \mathbf{y}_{m}\right) \in \mathbb{R}^{n(m+1)}$,

$$
\begin{equation*}
\int_{\Omega} g(\omega) \exp \left(\sum_{j=0}^{m} i \mathbf{y}_{j}^{T} \mathbf{W}_{t_{j}}(\omega)\right) d P=0 \tag{67.9.17}
\end{equation*}
$$

This left side equals

$$
\int_{\Omega} g_{+}(\omega) \exp \left(\sum_{j=0}^{m} i \mathbf{y}_{j}^{T} \mathbf{W}_{t_{j}}(\omega)\right) d P-\int_{\Omega} g_{-}(\omega) \exp \left(\sum_{j=0}^{m} i \mathbf{y}_{j}^{T} \mathbf{W}_{t_{j}}(\omega)\right) d P
$$

where $g_{+}$and $g_{-}$are the positive and negative parts of $g$. By the Lemma 67.9.2 and the observation at the end, this equals

$$
\int_{\mathbb{R}^{n m}} \exp \left(\sum_{j=0}^{m} i \mathbf{y}_{j}^{T} \mathbf{x}_{j}\right) d v_{+}-\int_{\mathbb{R}^{n m}} \exp \left(\sum_{j=0}^{m} i \mathbf{y}_{j}^{T} \mathbf{x}_{j}\right) d v_{-}
$$

where $v_{+}(B) \equiv \int_{\Omega} g_{+}(\omega) \mathscr{X}_{B}\left(\mathbf{W}_{t_{1}}(\omega), \cdots, \mathbf{W}_{t_{m}}(\omega)\right) d P$ and $v_{-}$is defined similarly. Then letting $v$ be the measure $v_{+}-v_{-}$, it follows that

$$
0=\int_{\mathbb{R}^{n m}} \exp \left(\sum_{j=0}^{m} i \mathbf{y}_{j}^{T} \mathbf{x}_{j}\right) d v(y)
$$

and this just says that the inverse Fourier transform of $v$ is 0 . It follows that $v=0$. Thus

$$
\begin{aligned}
& \int_{\Omega} g(\omega) \mathscr{X}_{B}\left(\mathbf{W}_{t_{1}}(\omega), \cdots, \mathbf{W}_{t_{m}}(\omega)\right) d P \\
= & \int_{\Omega} g(\omega) \mathscr{X}_{\mathbf{W}_{m}^{-1}(B)}(\omega) d P=0
\end{aligned}
$$

for every $B$ Borel in $\mathbb{R}^{n m}$ where

$$
\mathbf{W}_{m}(\omega) \equiv\left(\mathbf{W}_{t_{1}}(\omega), \cdots, \mathbf{W}_{t_{m}}(\omega)\right)
$$

Let $\mathscr{K}$ be the $\pi$ system defined as $\mathbf{W}_{m}^{-1}(B)$ for $B$ of the form $\prod_{i=1}^{m} U_{i}$ where $U_{i}$ is open in $\mathbb{R}^{n}$, this for some $m$ a positive integer. This is indeed a $\pi$ system because it includes
$\mathbf{W}_{1}^{-1}\left(\mathbb{R}^{n}\right)=\Omega$ and the empty set. Also it is closed with respect to intersections because, in the situation where each $s_{i}$ is larger than every $t_{i}$,

$$
\begin{aligned}
& \left(\mathbf{W}_{t_{1}}, \cdots, \mathbf{W}_{t_{m_{1}}}\right)^{-1}\left(\prod_{i=1}^{m_{1}} U_{i}\right) \cap\left(\mathbf{W}_{s_{1}}, \cdots, \mathbf{W}_{s_{m_{2}}}\right)^{-1}\left(\prod_{i=1}^{m_{2}} V_{i}\right)= \\
& \left(\mathbf{W}_{t_{1}}, \cdots, \mathbf{W}_{t_{m_{1}}}, \mathbf{W}_{s_{1}}, \cdots, \mathbf{W}_{s_{m_{2}}}\right)^{-1}\left(\prod_{i=1}^{m_{1}} U_{i} \times \prod_{k=1}^{m_{2}} \mathbb{R}^{n}\right) \\
& \cap\left(\left(\mathbf{W}_{t_{1}}, \cdots, \mathbf{W}_{t_{m_{1}}}, \mathbf{W}_{s_{1}}, \cdots, \mathbf{W}_{s_{m_{2}}}\right)^{-1}\left(\prod_{i=1}^{m_{1}} \mathbb{R}^{n} \times \prod_{k=1}^{m_{2}} V_{i}\right)\right) \\
& =\left(\mathbf{W}_{t_{1}}, \cdots, \mathbf{W}_{t_{m_{1}}}, \mathbf{W}_{s_{1}}, \cdots, \mathbf{W}_{s_{m_{2}}}\right)^{-1}\left(\prod_{i=1}^{m_{1}} U_{i} \times \prod_{k=1}^{m_{2}} V_{k}\right)
\end{aligned}
$$

In general, you would just make the obvious modification where you insert a copy of $\mathbb{R}^{n}$ in the appropriate position after rearranging so that the indices are increasing. It was just shown that $\mathscr{K} \subseteq \mathscr{G}$ where

$$
\mathscr{G} \equiv\left\{U \in \mathscr{G}_{t}: \int_{\Omega} g \mathscr{X}_{U} d P=0\right\}
$$

Now it is clear that $\mathscr{G}$ is closed with respect to countable disjoint unions and complements. The case of complements goes as follows. $\Omega \in \mathscr{K}$ and so if $U \in \mathscr{G}$,

$$
\int_{\Omega} g \mathscr{X}_{U}{ }^{C} d P+\int_{\Omega} g \mathscr{X}_{U} d P=\int_{\Omega} g d P
$$

The last on the left and the integral on the right are both 0 so it follows that $\int_{\Omega} g \mathscr{X}_{U}{ }^{c} d P=0$ also. It follows from Dynkin's lemma that $\mathscr{G} \supseteq \sigma(\mathscr{K})$. Now $\sigma(\mathscr{K})$ is $\sigma(\mathbf{W}(u): u \leq t) \equiv$ $\mathscr{G}_{t}$. Hence, $\mathscr{G}=\mathscr{G}_{t}$ and so $g$ is in $L^{2}\left(\Omega, \mathscr{G}_{t}\right)$ and for every $U \in \mathscr{G}_{t}$,

$$
\int_{\Omega} g \mathscr{X}_{U} d P=0
$$

which requires $g=0$. Thus functions of the above form are indeed dense in $L^{2}\left(\Omega, \mathscr{G}_{t}\right)$.
Note that this involves $g$ being $\mathscr{G}_{t}$ measurable, not $\mathscr{F}_{t}$ measurable. It is not clear to me whether it suffices to assume only that $g$ is $\mathscr{F}_{t}$ measurable. If true, this above has not proved it. The problem is the argument at the end using Dynkin's lemma to conclude that $g=0$.

Why such a funny lemma? It is because of the following computation which depends on Itô's formula. Let

$$
X=\int_{0}^{t} \mathbf{h}^{T} d \mathbf{W}-\frac{1}{2} \int_{0}^{t} \mathbf{h} \cdot \mathbf{h} d \tau
$$

and $g(x)=e^{x}$ and consider $g(X)=Y$. Recall the Ito formula. Formally,

$$
d Y=g^{\prime}(X) d X+\frac{1}{2} g^{\prime \prime}(X)(d X)^{2}
$$

$$
\begin{aligned}
& d Y= g(X)\left(\mathbf{h}^{T} d \mathbf{W}-\frac{1}{2}|\mathbf{h}|^{2} d t\right) \\
&+\frac{1}{2} g(X)\left(\mathbf{h}^{T} d \mathbf{W}-\frac{1}{2}|\mathbf{h}|^{2} d t\right)\left(\mathbf{h}^{T} d \mathbf{W}-\frac{1}{2}|\mathbf{h}|^{2} d t\right) \\
&=Y\left(\mathbf{h}^{T} d \mathbf{W}-\frac{1}{2}|\mathbf{h}|^{2} d t\right)+\frac{1}{2} Y\left[\left(\mathbf{h}^{T} d \mathbf{W}\right)\left(\mathbf{h}^{T} d \mathbf{W}\right)-\mathbf{h}^{T} d \mathbf{W}|\mathbf{h}|^{2} d t+\frac{1}{4}|\mathbf{h}|^{2} d t^{2}\right]
\end{aligned}
$$

Then neglecting the terms of the form $d \mathbf{W} d t, d t^{2}$ and so forth,

$$
d Y=Y \mathbf{h}^{T} d \mathbf{W}-\frac{1}{2} Y|\mathbf{h}|^{2} d t+\frac{1}{2} Y\left(\mathbf{h}^{T} d \mathbf{W}\right)\left(\mathbf{h}^{T} d \mathbf{W}\right)
$$

Now the $d \mathbf{W}$ occurs twice in the last term so it leads to a $d t$ and you get

$$
\begin{aligned}
d Y & =Y \mathbf{h}^{T} d \mathbf{W}-\frac{1}{2} Y|\mathbf{h}|^{2} d t+\frac{1}{2}\left(Y \mathbf{h}^{T}, \mathbf{h}^{T}\right) d t \\
d Y & =Y \mathbf{h}^{T} d \mathbf{W}-\frac{1}{2} Y|\mathbf{h}|^{2} d t+\frac{1}{2} Y|\mathbf{h}|^{2} d t \\
d Y & =Y \mathbf{h}^{T} d \mathbf{W}
\end{aligned}
$$

Note that $\left\|\mathbf{h}^{T}\right\|_{\mathscr{L}_{2}\left(\mathbb{R}^{n}, \mathbb{R}\right)} \equiv \sum_{k=1}^{n}\left(\mathbf{h}^{T} \mathbf{e}_{k}\right)^{2}=|\mathbf{h}|_{\mathbb{R}^{n}}^{2}$. Place an $\int_{0}^{t}$ in place of both sides to obtain

$$
\begin{align*}
Y(t)-Y(0) & =\int_{0}^{t} Y \mathbf{h}^{T} d \mathbf{W} \\
Y(t) & =1+\int_{0}^{t} Y \mathbf{h}^{T} d \mathbf{W} \tag{67.9.18}
\end{align*}
$$

Now here is the interesting part of this formula.

$$
E\left(\int_{0}^{t} Y \mathbf{h}^{T} d \mathbf{W}\right)=0
$$

because the stochastic integral is a martingale and equals 0 at $t=0$.

$$
E\left(\int_{0}^{t} Y \mathbf{h}^{T} d \mathbf{W}\right)=E\left(E\left(\int_{0}^{t} Y \mathbf{h}^{T} d \mathbf{W} \mid \mathscr{F}_{0}\right)\right)=0
$$

Thus

$$
E(Y(t))=1
$$

and for $Y$ one obtains

$$
\begin{aligned}
Y(t) & =E(Y(t))+\int_{0}^{t} Y \mathbf{h}^{T} d \mathbf{W} \\
& \equiv E(Y(t))+\int_{0}^{t} \mathbf{f}^{T} d \mathbf{W}
\end{aligned}
$$

where $\mathbf{f}^{T}$ is adapted and square integrable. It is just $Y \mathbf{h}^{T}$ where $\mathbf{h}$ does not depend on $\omega$ and $Y$ is a function of an adapted function.

Does such a function $\mathbf{f}$ exist for all $F \in L^{2}\left(\Omega, \mathscr{G}_{t}, P\right)$ ? The answer is yes and this is the content of the next theorem which is called the Itô representation theorem.

Theorem 67.9.4 Let $F \in L^{2}\left(\Omega, \mathscr{G}_{t}, P\right)$. Then there exists a unique $\mathscr{G}_{t}$ adapted

$$
\mathbf{f} \in L^{2}\left(\Omega \times[0, t] ; \mathbb{R}^{n}\right)
$$

such that $F=E(F)+\int_{0}^{t} \mathbf{f}(s, \omega)^{T} d \mathbf{W}$.
Proof: By Lemma 67.9.3, the span of functions of the form

$$
\exp \left(\int_{0}^{t} \mathbf{h}^{T} d \mathbf{W}-\frac{1}{2} \int_{0}^{t} \mathbf{h} \cdot \mathbf{h} d t\right)
$$

where $\mathbf{h}$ is a vector valued deterministic step function of the sort described in this lemma, are dense in $L^{2}\left(\Omega, \mathscr{G}_{t}, P\right)$. Given $F \in L^{2}\left(\Omega, \mathscr{G}_{t}, P\right),\left\{G_{k}\right\}_{k=1}^{\infty}$ be functions in the subspace of linear combinations of the above functions which converge to $F$ in $L^{2}\left(\Omega, \mathscr{G}_{t}, P\right)$. For each of these functions there exists $\mathbf{f}_{k}$ an adapted step function such that

$$
G_{k}=E\left(G_{k}\right)+\int_{0}^{t} \mathbf{f}_{k}(s, \omega)^{T} d \mathbf{W}
$$

Then from the Itô isometry, and the observation that $E\left(G_{k}-G_{l}\right)^{2} \rightarrow 0$ as $k, l \rightarrow \infty$ by the above definition of $G_{k}$ in which the $G_{k}$ converge to $F$ in $L^{2}(\Omega)$,

$$
\begin{align*}
& 0= \lim _{k, l \rightarrow \infty} E\left(\left(G_{k}-G_{l}\right)^{2}\right) \\
&= \lim _{k, l \rightarrow \infty} E\left(\left(E\left(G_{k}\right)+\int_{0}^{t} \mathbf{f}_{k}(s, \omega)^{T} d \mathbf{W}-\left(E\left(G_{l}\right)+\int_{0}^{t} \mathbf{f}_{l}(s, \omega)^{T} d \mathbf{W}\right)\right)^{2}\right) \\
&= \lim _{k, l \rightarrow \infty}\left\{E\left(G_{k}-G_{l}\right)^{2}+2 E\left(G_{k}-G_{l}\right) \int_{\Omega} \int_{0}^{t}\left(\mathbf{f}_{k}-\mathbf{f}_{l}\right)^{T} d \mathbf{W} d P\right. \\
&\left.+\int_{\Omega}\left(\int_{0}^{t}\left(\mathbf{f}_{k}-\mathbf{f}_{l}\right)^{T} d \mathbf{W}\right)^{2} d P\right\} \\
&= \lim _{k, l \rightarrow \infty}\left\{E\left(G_{k}-G_{l}\right)^{2}+\int_{\Omega}\left(\int_{0}^{t}\left(\mathbf{f}_{k}-\mathbf{f}_{l}\right)^{T} d \mathbf{W}\right)^{2} d P\right\}= \\
& \lim _{k, l \rightarrow \infty} \int_{\Omega}\left(\int_{0}^{t}\left(\mathbf{f}_{k}-\mathbf{f}_{l}\right)^{T} d \mathbf{W}\right)^{2} d P=\lim _{k, l \rightarrow \infty}\left\|\mathbf{f}_{k}-\mathbf{f}_{l}\right\|_{L^{2}\left(\Omega \times[0, T] ; \mathbb{R}^{n}\right)} \tag{67.9.19}
\end{align*}
$$

Going from the third to the fourth equations, is justified because

$$
\int_{\Omega} \int_{0}^{t}\left(\mathbf{f}_{k}-\mathbf{f}_{l}\right)^{T} d \mathbf{W} d P=0
$$

thanks to the fact that the Ito integral is a martingale which equals 0 at $t=0$.
This shows $\left\{\mathbf{f}_{k}\right\}_{k=1}^{\infty}$ is a Cauchy sequence in $L^{2}\left(\Omega \times[0, t] ; \mathbb{R}^{n}, \mathscr{P}\right)$, where $\mathscr{P}$ denotes the progressively measurable sets. It follows there exists a subsequence and

$$
\mathbf{f} \in L^{2}\left(\Omega \times[0, t] ; \mathbb{R}^{n}\right)
$$

such that $\mathbf{f}_{k}$ converges to $\mathbf{f}$ in $L^{2}\left(\Omega \times[0, t] ; \mathbb{R}^{n}, \mathscr{P}\right)$ with $\mathbf{f}$ being progressively measurable. Then by the Itô isometry and the equation

$$
G_{k}=E\left(G_{k}\right)+\int_{0}^{t} \mathbf{f}_{k}(s, \omega)^{T} d \mathbf{W}
$$

you can pass to the limit as $k \rightarrow \infty$ and obtain

$$
F=E(F)+\int_{0}^{t} \mathbf{f}(s, \omega)^{T} d \mathbf{W}
$$

Now $E\left(G_{k}\right) \rightarrow E(F)$. Consider the stochastic integrals. By the maximal estimate, Theorem 62.9.4, and the Ito isometry,

$$
\begin{aligned}
& P(\sup _{s \in[0, t]} \overbrace{\left|\int_{0}^{s} \mathbf{f}_{k}(\cdot, \omega)^{T} d \mathbf{W}-\int_{0}^{s} \mathbf{f}(\cdot, \omega)^{T} d \mathbf{W}\right|}^{\text {nonnegative submartingale }}>\delta) \\
< & \frac{E\left(\left|\int_{0}^{t} \mathbf{f}_{k}(\cdot, \omega)^{T} d \mathbf{W}-\int_{0}^{t} \mathbf{f}(\cdot, \omega)^{T} d \mathbf{W}\right|^{2}\right)}{\delta^{2}} \\
= & \frac{E\left(\int_{0}^{t}\left\|\mathbf{f}_{k}-\mathbf{f}\right\|_{\mathbb{R}^{n}}^{2} d s\right)}{\delta^{2}}
\end{aligned}
$$

From the above convergence result and an application of the Borel Cantelli lemma, there is a set of measure zero $N$ and a subsequence, still denoted as $\mathbf{f}_{k}$ such that for $\omega \notin N$, the convergence of the stochastic integrals for this subsequence is uniform. Thus for $\omega \notin N$,

$$
F=E(F)+\int_{0}^{t} \mathbf{f}(s, \omega)^{T} d \mathbf{W}
$$

This proves the existence part of this theorem.
It remains to consider the uniqueness. Suppose then that

$$
F=E(F)+\int_{0}^{T} \mathbf{f}(t, \omega)^{T} d \mathbf{W}=E(F)+\int_{0}^{T} \mathbf{f}_{1}(t, \omega)^{T} d \mathbf{W}
$$

Then

$$
\int_{0}^{T} \mathbf{f}(t, \omega)^{T} d \mathbf{W}=\int_{0}^{T} \mathbf{f}_{1}(t, \omega)^{T} d \mathbf{W}
$$

and so

$$
\int_{0}^{T}\left(\mathbf{f}(t, \omega)^{T}-\mathbf{f}_{1}(t, \omega)^{T}\right) d \mathbf{W}=0
$$

and by the Itô isometry,

$$
0=\left\|\int_{0}^{T}\left(\mathbf{f}(t, \omega)^{T}-\mathbf{f}_{1}(t, \omega)^{T}\right) d \mathbf{W}\right\|_{L^{2}(\Omega)}=\left\|\mathbf{f}-\mathbf{f}_{1}\right\|_{L^{2}\left(\Omega \times[0, T] ; \mathbb{R}^{n}\right)}
$$

which proves uniqueness.
With the above major result, here is another interesting representation theorem. Recall that if you have an $\mathscr{F}_{t}$ adapted function $\mathbf{f}$ and $\mathbf{f} \in L^{2}\left(\Omega \times[0, T] ; \mathbb{R}^{n}\right)$, then $\int_{0}^{t} \mathbf{f}^{T} d \mathbf{W}$ is a martingale. The next theorem is sort of a converse. It starts with a $\mathscr{G}_{t}$ martingale and represents it as an Itô integral. In this theorem, $\mathscr{G}_{t}$ continues to be the filtration determined by $n$ dimensional Wiener process.

Theorem 67.9.5 Let $M$ be an $\mathscr{G}_{t}$ martingale and suppose $M(t) \in L^{2}(\Omega)$ for all $t \geq 0$. Then there exists a unique stochastic process, $\mathbf{g}(s, \omega)$ such that $\mathbf{g}$ is $\mathscr{G}_{t}$ adapted and in $L^{2}(\Omega \times[0, t])$ for each $t>0$, and for all $t \geq 0$,

$$
M(t)=E(M(0))+\int_{0}^{t} \mathbf{g}^{T} d \mathbf{W}
$$

Proof: First suppose $\mathbf{f}$ is an adapted function of the sort that $\mathbf{g}$ is. Then the following claim is the first step in the proof.

Claim: Let $t_{1}<t_{2}$. Then

$$
E\left(\int_{t_{1}}^{t_{2}} \mathbf{f}^{T} d \mathbf{W} \mid \mathscr{G}_{t_{1}}\right)=0
$$

Proof of claim: This follows from the fact that the Ito integral is a martingale adapted to $\mathscr{G}_{t}$. Hence the above reduces to

$$
E\left(\int_{0}^{t_{2}} \mathbf{f}^{T} d \mathbf{W}-\int_{0}^{t_{1}} \mathbf{f}^{T} d \mathbf{W} \mid \mathscr{G}_{t_{1}}\right)=\int_{0}^{t_{1}} \mathbf{f}^{T} d \mathbf{W}-\int_{0}^{t_{1}} \mathbf{f}^{T} d \mathbf{W}=0
$$

Now to prove the theorem, it follows from Theorem 67.9.4 and the assumption that $M$ is a martingale that for $t>0$ there exists $\mathbf{f}^{t} \in L^{2}\left(\Omega \times[0, T] ; \mathbb{R}^{n}\right)$ such that

$$
\begin{aligned}
M(t) & =E(M(t))+\int_{0}^{t} \mathbf{f}^{t}(s, \cdot)^{T} d \mathbf{W} \\
& =E(M(0))+\int_{0}^{t} \mathbf{f}^{t}(s, \cdot)^{T} d \mathbf{W}
\end{aligned}
$$

Now let $t_{1}<t_{2}$. Then since $M$ is a martingale and so is the Ito integral,

$$
\begin{gathered}
M\left(t_{1}\right)=E\left(M\left(t_{2}\right) \mid \mathscr{G}_{t_{1}}\right)=E\left(E(M(0))+\int_{0}^{t_{2}} \mathbf{f}^{t_{2}}(s, \cdot)^{T} d \mathbf{W} \mid \mathscr{G}_{t_{1}}\right) \\
=E(M(0))+E\left(\int_{0}^{t_{1}} \mathbf{f}^{t_{2}}(s, \cdot)^{T} d \mathbf{W}\right)
\end{gathered}
$$

Thus

$$
M\left(t_{1}\right)=E(M(0))+\int_{0}^{t_{1}} \mathbf{f}^{t_{2}}(s, \cdot)^{T} d \mathbf{W}=E(M(0))+\int_{0}^{t_{1}} \mathbf{f}^{t_{1}}(s, \cdot)^{T} d \mathbf{W}
$$

and so

$$
0=\int_{0}^{t_{1}} \mathbf{f}^{t_{1}}(s, \cdot)^{T} d \mathbf{W}-\int_{0}^{t_{1}} \mathbf{f}^{t_{2}}(s, \cdot)^{T} d \mathbf{W}
$$

and so by the Itô isometry,

$$
\left\|\mathbf{f}^{t_{1}}-\mathbf{f}^{t_{2}}\right\|_{L^{2}\left(\Omega \times\left[0, t_{1}\right] ; \mathbb{R}^{n}\right)}=0
$$

Letting $N \in \mathbb{N}$, it follows that

$$
M(t)=E(M(0))+\int_{0}^{t} \mathbf{f}^{N}(s, \cdot)^{T} d \mathbf{W}
$$

for all $t \leq N$. Let $\mathbf{g}=\mathbf{f}^{N}$ for $t \in[0, N]$. Then asside from a set of measure zero, this is well defined and for all $t \geq 0$

$$
M(t)=E(M(0))+\int_{0}^{t} \mathbf{g}(s, \cdot)^{T} d \mathbf{W}
$$

Surely this is an incredible theorem. Note that it implies all the martingales adapted to $\mathscr{G}_{t}$ which are in $L^{2}$ for each $t$ must be continuous a.e. and are obtained from an Ito integral. Also, any such martingale satisfies $M(0)=E(M(0))$. Isn't that amazing? Also note that this featured $\mathbb{R}^{n}$ as where $\mathbf{W}$ has its values and $n$ was arbitrary. One could have $n=1$ if desired.

The above theorems can also be obtained from another approach. It involves showing that random variables of the form

$$
\phi\left(\mathbf{W}\left(t_{1}\right), \cdots, \mathbf{W}\left(t_{k}\right)\right)
$$

are dense in $L^{2}\left(\Omega, \mathscr{G}_{T}\right)$. This theorem is interesting for its own sake and it involves interesting results discussed earlier. Recall the Doob Dynkin lemma, Lemma 59.3.6 on Page 1866 which is listed here.

Lemma 67.9.6 Suppose $\mathbf{X}, \mathbf{Y}_{1}, \mathbf{Y}_{2}, \cdots, \mathbf{Y}_{k}$ are random vectors, $\mathbf{X}$ having values in $\mathbb{R}^{n}$ and $\mathbf{Y}_{j}$ having values in $\mathbb{R}^{p_{j}}$ and

$$
\mathbf{X}, \mathbf{Y}_{j} \in L^{1}(\Omega)
$$

Suppose $\mathbf{X}$ is $\sigma\left(\mathbf{Y}_{1}, \cdots, \mathbf{Y}_{k}\right)$ measurable. Thus

$$
\left\{\mathbf{X}^{-1}(E): E \text { Borel }\right\} \subseteq\left\{\left(\mathbf{Y}_{1}, \cdots, \mathbf{Y}_{k}\right)^{-1}(F): F \text { is Borel in } \prod_{j=1}^{k} \mathbb{R}^{p_{j}}\right\}
$$

Then there exists a Borel function, $\mathbf{g}: \prod_{j=1}^{k} \mathbb{R}^{p_{j}} \rightarrow \mathbb{R}^{n}$ such that

$$
\mathbf{X}=\mathbf{g}(\mathbf{Y}) .
$$

Recall also the submartingale convergence theorem.
Theorem 67.9.7 (submartingale convergence theorem) Let

$$
\left\{\left(X_{i}, \mathscr{S}_{i}\right)\right\}_{i=1}^{\infty}
$$

be a submartingale with $K \equiv \sup E\left(\left|X_{n}\right|\right)<\infty$. Then there exists a random variable $X$, such that $E(|X|) \leq K$ and

$$
\lim _{n \rightarrow \infty} X_{n}(\omega)=X(\omega) \text { a.e. }
$$

Recall

$$
\mathscr{G}_{t} \equiv \sigma(\mathbf{W}(u)-\mathbf{W}(r): 0 \leq r<u \leq t)
$$

It suffices to consider only $t$ and $u, r$ in a countable dense subset of $\mathbb{R}$ denoted as $D$. This follows from continuity of the Wiener process. To see this, let $0 \leq r<u \leq t U$ be open and $U_{n} \uparrow U$ where each $U_{n}$ is open and $\overline{U_{n}} \subseteq U_{n+1}, \cup_{n} U_{n}=U$. Then letting $u_{n} \uparrow u$ and $r_{n} \uparrow r, u_{n} r_{n}$ being in the countable dense set,

$$
\begin{aligned}
(\mathbf{W}(u)-\mathbf{W}(r))^{-1}\left(U_{n}\right) & \subseteq \cup_{k=1}^{\infty} \cap_{j \geq k}\left(\mathbf{W}\left(u_{j}\right)-\mathbf{W}\left(r_{j}\right)\right)^{-1}\left(U_{n}\right) \\
& \subseteq(\mathbf{W}(u)-\mathbf{W}(r))^{-1}\left(\overline{U_{n}}\right)
\end{aligned}
$$

and so

$$
\begin{aligned}
(\mathbf{W}(u)-\mathbf{W}(r))^{-1}(U) & =\cup_{n}(\mathbf{W}(u)-\mathbf{W}(r))^{-1}\left(U_{n}\right) \\
& \subseteq \cup_{n=1}^{\infty} \cup_{k=1}^{\infty} \cap_{j \geq k}\left(\mathbf{W}\left(u_{j}\right)-\mathbf{W}\left(r_{j}\right)\right)^{-1}\left(U_{n}\right) \\
& \subseteq \cup_{n}(\mathbf{W}(u)-\mathbf{W}(r))^{-1}\left(\overline{U_{n}}\right)=(\mathbf{W}(u)-\mathbf{W}(r))^{-1}(U)
\end{aligned}
$$

Now the set in the middle which has two countable unions and a countable intersection is in

$$
\sigma(\mathbf{W}(u)-\mathbf{W}(r): 0 \leq r<u \leq t, r, u \in D)
$$

Thus in particular, one would get the same filtration from

$$
\mathscr{G}_{t}=\sigma(\mathbf{W}(u)-\mathbf{W}(r): 0 \leq r<u \leq t, r, u \in D)
$$

Since $\mathbf{W}(0)=\mathbf{0}$, this is the same as

$$
\mathscr{G}_{t}=\sigma(\mathbf{W}(u): 0 \leq u \leq t, u \in D)
$$

Lemma 67.9.8 Random variables of the form

$$
\phi\left(\mathbf{W}\left(t_{1}\right), \cdots, \mathbf{W}\left(t_{k}\right)\right), \phi \in C_{c}^{\infty}\left(\mathbb{R}^{k}\right)
$$

are dense in $L^{2}\left(\Omega, \mathscr{G}_{T}, P\right)$ where $t_{1}<t_{2} \cdots<t_{k}$ is a finite increasing sequence of

$$
(\mathbb{Q} \cup\{T\}) \cap[0, T] .
$$

Proof: Let $g \in L^{2}\left(\Omega, \mathscr{G}_{T}, P\right)$. Also let $\left\{t_{j}\right\}_{j=1}^{\infty}$ be the points of $(\mathbb{Q} \cup\{T\}) \cap[0, T]$. Let

$$
\mathscr{G}_{m} \equiv \sigma\left(\mathbf{W}\left(t_{k}\right): k \leq m\right)
$$

Thus the $\mathscr{G}_{m}$ are increasing but each is generated by finitely many $\mathbf{W}\left(t_{k}\right)$. Also as explained above,

$$
\begin{aligned}
\mathscr{G}_{T} & =\sigma(\mathbf{W}(u): 0 \leq u \leq T, u \in(\mathbb{Q} \cup\{T\}) \cap[0, T]) \\
& =\sigma\left(\mathscr{G}_{m}, m<\infty\right)
\end{aligned}
$$

Now consider the martingale, $\left\{E\left(g_{M} \mid \mathscr{G}_{m}\right)\right\}_{m=1}^{\infty}$. where here

$$
g_{M}(\omega) \equiv\left\{\begin{array}{l}
g(\omega) \text { if } g(\omega) \in[-M, M] \\
M \text { if } g(\omega)>M \\
-M \text { if } g(\omega)<-M
\end{array}\right.
$$

and $M$ is chosen large enough that

$$
\begin{equation*}
\left\|g-g_{M}\right\|_{L^{2}(\Omega)}<\varepsilon / 4 \tag{67.9.20}
\end{equation*}
$$

Now the terms of this martingale are uniformly bounded by $M$ because

$$
\left|E\left(g_{M} \mid \mathscr{G}_{m}\right)\right| \leq E\left(\left|g_{M}\right| \mid \mathscr{G}_{m}\right) \leq E\left(M \mid \mathscr{G}_{m}\right)=M
$$

It follows the martingale is certainly bounded in $L^{1}$ and so the martingale convergence theorem stated above can be applied, and so there exists $f$ measurable in $\sigma\left(\mathscr{G}_{m}, m<\infty\right)$ such that $\lim _{m \rightarrow \infty} E\left(g_{M} \mid \mathscr{G}_{m}\right)(\omega)=f(\omega)$ a.e. Also $|f(\omega)| \leq M$ a.e. Since all functions are bounded, it follows that this convergence is also in $L^{2}(\Omega)$.

Now letting $A \in \sigma\left(\mathscr{G}_{m}, m<\infty\right)$, it follows from the dominated convergence theorem that

$$
\int_{A} f d P=\lim _{m \rightarrow \infty} \int_{A} E\left(g_{M} \mid \mathscr{G}_{m}\right) d P=\int_{A} g_{M} d P
$$

Now $\mathscr{G}_{T}=\sigma\left(\mathbf{W}\left(t_{k}\right), t_{k} \leq T\right)=\sigma\left(\mathscr{G}_{m}, m \geq 1\right)$ and so the above equation implies that $f=$ $g_{M}$ a.e.

By the Doob Dynkin lemma listed above, there exists a Borel measurable $h: \mathbb{R}^{n m} \rightarrow \mathbb{R}$ such that

$$
E\left(g_{M} \mid \mathscr{G}_{m}\right)=h\left(\mathbf{W}_{t_{1}}, \cdots, \mathbf{W}_{t_{m}}\right) \text { a.e. }
$$

Of course $h$ is not in $C_{c}^{\infty}\left(\mathbb{R}^{n m}\right)$. Let $m$ be large enough that

$$
\begin{equation*}
\left\|g_{M}-E\left(g_{M} \mid \mathscr{G}_{m}\right)\right\|_{L^{2}}=\left\|f-E\left(g_{M} \mid \mathscr{G}_{m}\right)\right\|_{L^{2}}<\frac{\varepsilon}{4} . \tag{67.9.21}
\end{equation*}
$$

Let $\lambda_{\left(\mathbf{W}_{t_{1}}, \cdots, \mathbf{W}_{t_{m}}\right)}$ be the distribution measure of the random vector $\left(\mathbf{W}_{t_{1}}, \cdots, \mathbf{W}_{t_{m}}\right)$. Thus $\lambda_{\left(\mathbf{W}_{t_{1}}, \cdots, \mathbf{W}_{t m}\right)}$ is a Radon measure and so there exists $\phi \in C_{c}\left(\mathbb{R}^{n m}\right)$ such that

$$
\begin{gathered}
\left(\int_{\Omega}\left|E\left(g_{M} \mid \mathscr{G}_{m}\right)-\phi\left(\mathbf{W}_{t_{1}}, \cdots, \mathbf{W}_{t_{m}}\right)\right|^{2} d P\right)^{1 / 2} \\
=\left(\int_{\Omega}\left|h\left(\mathbf{W}_{t_{1}}, \cdots, \mathbf{W}_{t_{m}}\right)-\phi\left(\mathbf{W}_{t_{1}}, \cdots, \mathbf{W}_{t_{m}}\right)\right|^{2} d P\right)^{1 / 2} \\
=\left(\int_{\mathbb{R}^{n m}}\left|h\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{m}\right)-\phi\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{m}\right)\right|^{2} d \lambda\left(\mathbf{W}_{\left.t_{1}, \cdots, \mathbf{W}_{t_{m}}\right)}\right)^{1 / 2}<\varepsilon / 4 .\right.
\end{gathered}
$$

By convolving with a mollifier, one can assume that $\phi \in C_{c}^{\infty}\left(\mathbb{R}^{n m}\right)$ also. It follows from 67.9.20 and 67.9.21 that

$$
\begin{aligned}
& \left\|g-\phi\left(\mathbf{W}_{t_{1}}, \cdots, \mathbf{W}_{t_{m}}\right)\right\|_{L^{2}} \\
\leq & \left\|g-g_{M}\right\|_{L^{2}}+\left\|g_{M}-E\left(g_{M} \mid \mathscr{G}_{m}\right)\right\|_{L^{2}} \\
& +\left\|E\left(g_{M} \mid \mathscr{G}_{m}\right)-\phi\left(\mathbf{W}_{t_{1}}, \cdots, \mathbf{W}_{t_{m}}\right)\right\|_{L^{2}} \\
\leq & 3\left(\frac{\varepsilon}{4}\right)<\varepsilon
\end{aligned}
$$

## Chapter 68

## A Different Kind Of Stochastic Integration

For more on this material, see [102] which is what this is based on. Recall the following corollary. It is Corollary 59.20.3 on Page 1934.

Corollary 68.0.1 Let $H$ be a real Hilbert space. Then there exist random variables $W(h)$ for $h \in H$ such that each is normally distributed with mean 0 and for every $h, g$, it follows that $(W(h), W(g))$ is normally distributed and

$$
E(W(h) W(g))=(h, g)_{H}
$$

Furthermore, if $\left\{e_{i}\right\}$ is an orthogonal set of vectors of $H$, then $\left\{W\left(e_{i}\right)\right\}$ are independent random variables. Also for any finite set $\left\{f_{1}, f_{2}, \cdots, f_{n}\right\}$,

$$
\left(W\left(f_{1}\right), W\left(f_{2}\right), \cdots, W\left(f_{n}\right)\right)
$$

is normally distributed.
Here are some simple examples.
Example 68.0.2 Let $H=L^{2}([0, T])$. For $f \in H$, let

$$
W(f) \equiv \int_{0}^{T} f(u) d W
$$

where $W(t)$ is the one dimensional Wiener process.
First of all, note that the integrand is adapted to the usual filtration determined by the Wiener process. This is because $f$ does not depend on $\omega$. That $W(f)$ is normally distributed can be seen from the approximation of the Ito integral with the integral of elementary functions. These are clearly normally distributed because they are just linear combinations of increments of the Wiener process. Recall these increments were independent. Thus the integrals of these elementary functions are all normally distributed. If $I_{n}(\omega)$ is one of these, then $I_{n} \rightarrow I$ in $L^{2}(\Omega)$ where $I$ is the above Ito integral. It follows that

$$
E\left(e^{i I t}\right)=\lim _{n \rightarrow \infty} E\left(e^{i I_{n} t}\right)=\lim _{n \rightarrow \infty} e^{-(1 / 2)\left\|f_{n}\right\|_{L^{2}}^{2}}=e^{-(1 / 2)\|f\|_{L^{2}}^{2}}
$$

so in fact, $W(f)$ is normally distributed with mean 0 and variance $\|f\|_{L^{2}}^{2}$. As to the other condition, the Ito isometry implies that

$$
E(W(f) W(g))=\int_{0}^{T} f(u) g(u) d u=(f, g)_{H}
$$

One can verify this by considering $E\left(W(f+g)^{2}\right), E\left(W(f-g)^{2}\right)$.
This example is called the isonormal Gaussian process. There is a measure space $(\Omega, \mathscr{F}, P)$ where $\overline{\sigma(W(s): s \leq T)} \equiv \mathscr{F}$. There must be an underlying measure space which comes from having to define the Wiener process.

Example 68.0.3 You could let $H=\mathbb{R}$ and let $\xi$ be normally distributed density with mean 0 and variance 1. Then let $W(a) \equiv \xi$ a. Does it work?

$$
E\left(a b \xi^{2}\right)=a b E\left(\xi^{2}\right)=a b
$$

Is $W(a)$ normally distributed? $E\left(e^{i a \xi t}\right)=\int_{\mathbb{R}} e^{i a x t} \xi d x=e^{-(1 / 2) a^{2} t^{2}}$ which is the characteristic function of a normally distributed random variable having mean 0 and variance $a^{2}$. It is clear that any linear combination of $a_{i} \xi$ is normally distributed and so the vector $\left(a_{1} \xi, \cdots, a_{n} \xi\right)$ is normally distributed. This is by Theorem 59.16.4.

The above implies $W$ is actually linear.

$$
\begin{gathered}
E\left((W(f+g)-(W(f)+W(g)))^{2}\right) \\
=E\binom{(W(f+g))^{2}+\left[W(f)^{2}+W(g)^{2}+2 W(f) W(g)\right]}{-2[W(f+g) W(f)+W(f+g) W(g)]} \\
=E\left((W(f+g))^{2}\right)+E\left(W(f)^{2}\right)+E\left(W(g)^{2}\right)+E(W(f) W(g)) \\
-2(E(W(f+g) W(f))+E(W(f+g) W(g)))
\end{gathered}
$$

which from the above equals

$$
\begin{aligned}
& |f+g|^{2}+|f|^{2}+|g|^{2}+2(f, g)-2[(f+g, f)+(f+g), g] \\
= & 2|f|^{2}+2|g|^{2}+4(f, g)-2\left[|f|^{2}+|g|^{2}+2(f, g)\right]=0
\end{aligned}
$$

Thus $W(f+g)-(W(f)+W(g))=0$. Is it true that

$$
(W(a f))=a W(f) ?
$$

This is easier to show.

$$
\begin{aligned}
E((W(a f) & \left.-a W(f))^{2}\right)=E\left(W(a f)^{2}-2 W(a f) a W(f)+a^{2} W(f)^{2}\right) \\
& =|a f|^{2}-2 a E(W(a f) W(f))+a^{2} E\left(W(f)^{2}\right) \\
& =a^{2}|f|^{2}-2 a^{2}|f|^{2}+a^{2}|f|^{2}=0
\end{aligned}
$$

Thus $W$ is indeed linear.

### 68.1 Hermite Polynomials

Consider

$$
\exp \left(t x-\frac{t^{2}}{2}\right)=\exp \left(\frac{x^{2}}{2}-\frac{1}{2}(x-t)^{2}\right)
$$

Now the Hermite polynomials are the coefficients of the power series of this function expanded in powers of $t$. Thus the $n^{t h}$ one of these is

$$
\begin{equation*}
H_{n}(x)=\left.\exp \left(\frac{x^{2}}{2}\right) \frac{1}{n!} \frac{d^{n}}{d t^{n}}\left(\exp \left(-\frac{1}{2}(x-t)^{2}\right)\right)\right|_{t=0} \tag{68.1.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\exp \left(t x-\frac{t^{2}}{2}\right)=\sum_{n=0}^{\infty} H_{n}(x) t^{n} \tag{68.1.2}
\end{equation*}
$$

Note that $H_{0}(x)=1$,

$$
\begin{aligned}
H_{1}(x) & =\left.\exp \left(\frac{x^{2}}{2}\right) \frac{d}{d t}\left(\exp \left(-\frac{1}{2}(x-t)^{2}\right)\right)\right|_{t=0} \\
& =-\left.e^{-\frac{1}{2}(t-x)^{2}} e^{\frac{1}{2} x^{2}}(t-x)\right|_{t=0}=x
\end{aligned}
$$

From 68.1.2, differentiating both sides formally with respect to $x$,

$$
t \exp \left(t x-\frac{t^{2}}{2}\right)=\sum_{n=1}^{\infty} H_{n}^{\prime}(x) t^{n}
$$

and so

$$
\sum_{n=0}^{\infty} H_{n}(x) t^{n}=\exp \left(t x-\frac{t^{2}}{2}\right)=\sum_{n=1}^{\infty} H_{n}^{\prime}(x) t^{n-1}=\sum_{n=0}^{\infty} H_{n+1}^{\prime}(x) t^{n}
$$

showing that

$$
H_{n}^{\prime}(x)=H_{n-1}(x), n \geq 1, \quad H_{0}(x)=0, H_{1}(x)=x
$$

which could have been obtained with more work from 68.1.1. Also, differentiating both sides of 68.1.2 with respect to $t$,

$$
-\exp \left(t x-\frac{t^{2}}{2}\right)(t-x)=\sum_{n=0}^{\infty} n H_{n}(x) t^{n-1}
$$

Thus

$$
(x-t) \sum_{n=0}^{\infty} H_{n}(x) t^{n}=\sum_{n=0}^{\infty} n H_{n}(x) t^{n-1}=\sum_{n=0}^{\infty}(n+1) H_{n+1}(x) t^{n}
$$

and so

$$
\sum_{n=0}^{\infty} x H_{n}(x) t^{n}-\sum_{n=0}^{\infty} H_{n}(x) t^{n+1}=\sum_{n=0}^{\infty}(n+1) H_{n+1}(x) t^{n}
$$

and so

$$
\sum_{n=0}^{\infty} x H_{n}(x) t^{n}-\sum_{n=1}^{\infty} H_{n-1}(x) t^{n}=\sum_{n=0}^{\infty}(n+1) H_{n+1}(x) t^{n}
$$

Thus for $n \geq 1$,

$$
x H_{n}(x)-H_{n-1}(x)=(n+1) H_{n+1}(x)
$$

Now also

$$
\exp \left(t(-x)-\frac{t^{2}}{2}\right)=\sum_{n=0}^{\infty} H_{n}(-x) t^{n}
$$

and taking successive derivatives with respect to $t$ of the left side and evaluating at $t=0$ yields

$$
H_{n}(-x)=(-1)^{n} H_{n}(x)
$$

Summarizing these as in [102],

$$
\begin{align*}
& H_{n}^{\prime}(x)=H_{n-1}(x), n \geq 1, \quad H_{0}(x)=0, H_{1}(x)=x \\
& x H_{n}(x)-H_{n-1}(x)=(n+1) H_{n+1}(x), \quad n \geq 1  \tag{68.1.3}\\
& H_{n}(-x)=(-1)^{n} H_{n}(x)
\end{align*}
$$

Clearly, these relations show that all of these $H_{n}$ are polynomials. Also the degree of $H_{n}(x)$ is $n$ and the coefficient of $x^{n}$ is $1 / n!$.

Definition 68.1.1 You can also consider Hermite polynomials which depend on $\lambda$. These are defined as follows:

$$
H_{n}(x, \lambda) \equiv \frac{(-\lambda)^{n}}{n!} e^{\frac{1}{2 \lambda} x^{2}} \frac{\partial^{n}}{\partial x^{n}}\left(e^{-\frac{1}{2 \lambda} x^{2}}\right)
$$

You can see clearly that these are polynomials in $x$. For example, let $n=2$. Then you would have from the above definition.

$$
\begin{gathered}
H_{0}(x, \lambda)=1, H_{1}(x, \lambda)=\frac{(-\lambda)^{1}}{1!} e^{\frac{1}{2 \lambda} x^{2}} \frac{\partial}{\partial x}\left(e^{-\frac{1}{2} \frac{x^{2}}{\lambda}}\right)=x \\
H_{2}(x, \lambda) \equiv \frac{(-\lambda)^{2}}{2!} e^{\frac{1}{2 \lambda} x^{2}} \frac{\partial^{2}}{\partial x^{2}}\left(e^{-\frac{1}{2 \lambda} x^{2}}\right)=\frac{1}{2} x^{2}-\frac{1}{2} \lambda
\end{gathered}
$$

The idea is you end up with polynomials of degree $n$ times $e^{-x^{2} / 2 \lambda}$ in the derivative part and then this cancels with $e^{x^{2} / 2 \lambda}$ to leave you with a polynomial of degree $n$. Also the leading term will always be $\frac{x^{n}}{n!}$ which is easily seen from the above. Then there are some relationships satisfied by these.

Say $n>1$ in what follows.

$$
\begin{gathered}
\frac{\partial}{\partial x} H_{n}(x, \lambda)=\frac{x}{\lambda} \frac{(-\lambda)^{n} e^{\frac{1}{2} \frac{x^{2}}{\lambda}}}{n!} \frac{\partial^{n}}{\partial x^{n}}\left(e^{-\frac{1}{2 \lambda} x^{2}}\right)+\frac{(-\lambda)^{n}}{n!} e^{\frac{1}{2 \lambda} x^{2}} \frac{\partial^{n}}{\partial x^{n}}\left(\frac{\partial}{\partial x} e^{-\frac{1}{2 \lambda} x^{2}}\right) \\
=\frac{x}{\lambda} \frac{(-\lambda)^{n} e^{\frac{1}{2} \frac{x^{2}}{\lambda}}}{n!} \frac{\partial^{n}}{\partial x^{n}}\left(e^{-\frac{1}{2 \lambda} x^{2}}\right)+\frac{(-\lambda)^{n}}{n!} e^{\frac{1}{2 \lambda} x^{2}} \frac{\partial^{n}}{\partial x^{n}}\left(-\frac{x}{\lambda} e^{-\frac{1}{2} \frac{x^{2}}{\lambda}}\right)
\end{gathered}
$$

Now since $n>1$, that last term reduces to

$$
\frac{(-\lambda)^{n}}{n!} e^{\frac{1}{2 \lambda} x^{2}}\left[-\frac{x}{\lambda} \frac{\partial^{n}}{\partial x^{n}}\left(e^{-\frac{1}{2} \frac{x^{2}}{\lambda}}\right)+n \frac{\partial}{\partial x}\left(-\frac{x}{\lambda}\right) \frac{\partial^{n-1}}{\partial x^{n-1}}\left(e^{-\frac{1}{2} \frac{x^{2}}{\lambda}}\right)\right]
$$

,this by Leibniz formula. Thus this cancels with the first term to give

$$
\begin{aligned}
\frac{\partial}{\partial x} H_{n}(x, \lambda) & =\frac{(-\lambda)^{n} n}{n!}\left(-\frac{1}{\lambda}\right) e^{\frac{1}{2 \lambda} x^{2}} \frac{\partial^{n-1}}{\partial x^{n-1}}\left(e^{-\frac{1}{2} \frac{x^{2}}{\lambda}}\right) \\
& =\frac{(-\lambda)^{n-1}}{(n-1)!} e^{\frac{1}{2 \lambda} x^{2}} \frac{\partial^{n-1}}{\partial x^{n-1}}\left(e^{-\frac{1}{2} \frac{x^{2}}{\lambda}}\right) \equiv H_{n-1}(x, \lambda)
\end{aligned}
$$

In case of $n=1$, this appears to also work. $\frac{\partial}{\partial x} H_{1}(x, \lambda)=1=H_{0}(x, \lambda)$ from the above computations. This shows that

$$
\frac{\partial}{\partial x} H_{n}(x, \lambda)=H_{n-1}(x, \lambda)
$$

Next, is the claim that

$$
(n+1) H_{n+1}(x, \lambda)=x H_{n}(x, \lambda)-\lambda H_{n-1}(x, \lambda)
$$

If $n=1$, this says that

$$
\begin{aligned}
2 H_{2}(x, \lambda) & =x H_{1}(x, \lambda)-\lambda H_{0}(x, \lambda) \\
& =x^{2}-\lambda
\end{aligned}
$$

and so the formula does indeed give the correct description of $H_{2}(x, \lambda)$ when $n=1$. Thus assume $n>1$ in what follows. The left side equals

$$
\frac{(-\lambda)^{n+1}}{n!} e^{\frac{1}{2 \lambda} x^{2}} \frac{\partial^{n+1}}{\partial x^{n+1}}\left(e^{-\frac{1}{2 \lambda} x^{2}}\right)
$$

This equals

$$
\frac{(-\lambda)^{n+1}}{n!} e^{\frac{1}{2 \lambda} x^{2}} \frac{\partial^{n}}{\partial x^{n}}\left(-\frac{x}{\lambda} e^{-\frac{1}{2} \frac{x^{2}}{\lambda}}\right)
$$

Now by Liebniz formula,

$$
\begin{aligned}
& =\frac{(-\lambda)^{n+1}}{n!} e^{\frac{1}{2 \lambda} x^{2}}\left[-\frac{x}{\lambda} \frac{\partial^{n}}{\partial x^{n}} e^{-\frac{1}{2} \frac{x^{2}}{\lambda}}+n\left(\frac{-1}{\lambda}\right) \frac{\partial^{n-1}}{\partial x^{n-1}}\left(e^{-\frac{1}{2} \frac{x^{2}}{\lambda}}\right)\right] \\
& =\frac{(-\lambda)^{n+1}}{n!} e^{\frac{1}{2 \lambda} x^{2}}\left(-\frac{x}{\lambda} \frac{\partial^{n}}{\partial x^{n}} e^{-\frac{1}{2} \frac{x^{2}}{\lambda}}\right)+\frac{(-\lambda)^{n+1}}{n!} e^{\frac{1}{2 \lambda} x^{2}} n\left(\frac{-1}{\lambda}\right) \frac{\partial^{n-1}}{\partial x^{n-1}}\left(e^{-\frac{1}{2} \frac{x^{2}}{\lambda}}\right) \\
& =x \frac{(-\lambda)^{n}}{n!} e^{\frac{1}{2 \lambda} x^{2}} \frac{\partial^{n}}{\partial x^{n}} e^{-\frac{1}{2} \frac{x^{2}}{\lambda}}+\frac{(-\lambda)^{n}}{(n-1)!} e^{\frac{1}{2 \lambda} x^{2}} \frac{\partial^{n-1}}{\partial x^{n-1}}\left(e^{-\frac{1}{2} \frac{x^{2}}{\lambda}}\right) \\
& =x H_{n}(x, \lambda)-\lambda H_{n-1}(x, \lambda)
\end{aligned}
$$

which shows the formula is valid for all $n \geq 1$.
Next is the claim that

$$
H_{n}(-x, \lambda)=(-1)^{n} H_{n}(x, \lambda)
$$

This is easy to see from the observation that

$$
\frac{\partial}{\partial x}=\frac{\partial}{\partial(-x)}(-1)
$$

Thus if it involves $n$ derivatives, you end up multiplying by $(-1)^{n}$.
Finally is the claim that

$$
\frac{\partial}{\partial \lambda} H_{n}(x, \lambda)=-\frac{1}{2} \frac{\partial^{2}}{\partial x^{2}} H_{n}(x, \lambda)
$$

It is certainly true for $n=0,1,2$. So suppose it is true for all $k \leq n$. Then from earlier claims and induction,

$$
\begin{gathered}
(n+1) H_{(n+1) \lambda}(x, \lambda)=x H_{n \lambda}(x, \lambda)-H_{(n-1)}(x, \lambda)-\lambda H_{(n-1) \lambda}(x, \lambda) \\
=x\left(\frac{-1}{2}\right) H_{n x x}-H_{n-1}+\lambda \frac{1}{2} H_{(n-1) x x}=x\left(\frac{-1}{2}\right) H_{n-2}-H_{n-1}+\lambda \frac{1}{2} H_{(n-3)} \\
=-\frac{1}{2}\left(x H_{n-2}-\lambda H_{n-3}+2 H_{n-1}\right)=-\frac{1}{2}\left((n-1) H_{n-1}+2 H_{n-1}\right)=-\frac{1}{2}\left((n+1) H_{n-1}\right)
\end{gathered}
$$

comparing the ends,

$$
H_{(n+1) \lambda}=-\frac{1}{2} H_{n-1}=-\frac{1}{2} H_{(n+1) x x}
$$

This proves the following theorem.
Theorem 68.1.2 Let $H_{n}(x, \lambda)$ be defined by

$$
H_{n}(x, \lambda) \equiv \frac{(-\lambda)^{n}}{n!} e^{\frac{1}{2 \lambda} x^{2}} \frac{\partial^{n}}{\partial x^{n}}\left(e^{-\frac{1}{2 \lambda} x^{2}}\right)
$$

for $\lambda>0$. Then the following properties are valid.

$$
\begin{gather*}
\frac{\partial}{\partial x} H_{n}(x, \lambda)=H_{n-1}(x, \lambda)  \tag{68.1.4}\\
(n+1) H_{n+1}(x, \lambda)=x H_{n}(x, \lambda)-\lambda H_{n-1}(x, \lambda)  \tag{68.1.5}\\
H_{n}(-x, \lambda)=(-1)^{n} H_{n}(x, \lambda)  \tag{68.1.6}\\
\frac{\partial}{\partial \lambda} H_{n}(x, \lambda)=-\frac{1}{2} \frac{\partial^{2}}{\partial x^{2}} H_{n}(x, \lambda) \tag{68.1.7}
\end{gather*}
$$

With this theorem, one can also prove the following.
Theorem 68.1.3 The Hermite polynomials are the coefficients of a certain power series. Specifically,

$$
\exp \left(t x-\frac{1}{2} t^{2} \lambda\right)=\sum_{n=0}^{\infty} H_{n}(x, \lambda) t^{n}
$$

Proof: Replace $H_{n}$ with $K_{n}$ which really are the coefficients of the power series and then show $K_{n}=H_{n}$. Thus

$$
\exp \left(t x-\frac{1}{2} t^{2} \lambda\right)=\sum_{n=0}^{\infty} K_{n}(x, \lambda) t^{n}
$$

Then $K_{0}=1=H_{0}(x)$. Also $K_{1}(x)=x=H_{1}(x)$.

$$
\begin{gathered}
\frac{\partial}{\partial t}\left(\exp \left(t x-\frac{1}{2} t^{2} \lambda\right)\right)=\exp \left(t x-\frac{1}{2} t^{2} \lambda\right)(x-t \lambda) \\
=\sum_{n=0}^{\infty} x K_{n}(x, \lambda) t^{n}-\sum_{n=0}^{\infty} \lambda K_{n}(x, \lambda) t^{n+1}=\sum_{n=0}^{\infty} x K_{n}(x, \lambda) t^{n}-\sum_{n=1}^{\infty} \lambda K_{n-1}(x, \lambda) t^{n}
\end{gathered}
$$

Also,

$$
\frac{\partial}{\partial t}\left(\exp \left(t x-\frac{1}{2} t^{2} \lambda\right)\right)=\sum_{n=1}^{\infty} n K_{n}(x, \lambda) t^{n-1}=\sum_{n=0}^{\infty}(n+1) K_{n+1}(x, \lambda) t^{n}
$$

It follows that for $n \geq 1$,

$$
(n+1) K_{n+1}(x, \lambda)=x K_{n}(x, \lambda)-\lambda K_{n-1}(x, \lambda)
$$

Thus the first two $K_{0}, K_{1}$ coincide with $H_{0}$ and $H_{1}$ respectively. Then since both $K_{n}$ and $H_{n}$ satisfy the recursion relation 68.1.5, it follows that $K_{n}=H_{n}$ for all $n$.

The first version is just letting $\lambda=1$ in the second version.
There is something very interesting about these Hermite polynomials $H_{n}(x, \lambda)$. Let $W$ be the real Wiener process. Consider the stochastic process $H_{n}(W(t), t), n \geq 1$. This ends up being a martingale. Using Ito's formula, the easy to remember version of it presented above, and the above properties of the Hermite polynomials,

$$
\begin{aligned}
& d H_{n}=H_{n x}(W(t), t) d W+H_{n t}(W(t), t) d t+\frac{1}{2} H_{n x x}(W(t), t) d W^{2} \\
& \quad=H_{n-1}(W(t), t) d W-\frac{1}{2} H_{n x x}(W(t), t) d t+\frac{1}{2} H_{n x x}(W(t), t) d t
\end{aligned}
$$

Note that if $n<2$, both of the last two terms are 0 . In general, they cancel and so

$$
d H_{n}=H_{n-1}(W(t), t) d W
$$

and so

$$
H_{n}(W(t), t)=H_{n}(W(0), 0)+\int_{0}^{t} H_{n-1}(W(t), t) d W
$$

now the constant term in the above equation is $\mathscr{F}_{0}$ measurable and the stochastic integral is a martingale. Thus this is indeed a martingale assuming everything is suitably integrable. However, this is not hard to see because these $H_{n}$ are just polynomials. It was shown in Theorem 64.1.3 that $W(t) \in L^{q}(\Omega)$ for all $q$. Hence there is no integrability issue in doing
these things. Actually, $H_{n}(W(0), 0)=0$ To see this, note that $E\left(W(0)^{2}\right)=(0,0)_{H}=0$ and so $W(0)=0$. Now it is not hard to see that $H_{n}(0,0)=0$. Indeed,

$$
\exp \left(t x-\frac{1}{2} t^{2} \lambda\right)=\sum_{n=0}^{\infty} H_{n}(x, \lambda) t^{n}
$$

Thus $H_{n}(x, 0)=\sum_{n=0}^{\infty} H_{n}(x, 0) t^{n}=\exp (t x)=\sum_{n=0}^{\infty} \frac{(t x)^{n}}{n!}=\sum_{n=0}^{\infty} x^{n} \frac{t^{n}}{n!}$ and so for all $n \geq$ $1, H_{n}(0,0)=0$. Thus in fact, for $n \geq 1, t \rightarrow H_{n}(W(t), t)$ is a martingale which equals 0 when $t=0$.

### 68.2 A Remarkable Theorem, Hermite Polynomials

Lemma 68.2.1 Say $(X, Y)$ is generalized normally distributed and

$$
E(X)=E(Y)=0, E\left(X^{2}\right)=E\left(Y^{2}\right)=1
$$

Then for $m, n \geq 0$,

$$
E\left(H_{n}(X) H_{m}(Y)\right)=\left\{\begin{array}{c}
0 \text { if } n \neq m \\
\frac{1}{n!}(E(X Y))^{n} \text { if } n=m
\end{array}\right.
$$

Proof: By assumption, $s X+t Y$ is normal distributed with mean 0 . This follows from Theorem 59.16.4. Also

$$
\sigma^{2} \equiv E\left((s X+t Y)^{2}\right)=s^{2}+t^{2}+2 E(X Y) s t
$$

and so its characteristic function is

$$
E(\exp (i \lambda(s X+t Y)))=\phi_{s X+t Y}(\lambda)=e^{-\frac{1}{2} \sigma^{2} \lambda^{2}}=e^{-\frac{1}{2}\left(s^{2}+t^{2}\right) \lambda^{2}} e^{-E(X Y) s t \lambda^{2}}
$$

So let $\lambda=-i$. You can do this because both sides are analytic in $\lambda \in \mathbb{C}$ and they are equal for real $\lambda$, a set with a limit point. This leads to

$$
E(\exp (s X+t Y))=e^{\frac{1}{2}\left(s^{2}+t^{2}\right)} e^{E(X Y) s t}
$$

Hence, multiplying both sides by $e^{-\frac{1}{2}\left(s^{2}+t^{2}\right)}$,

$$
\begin{aligned}
e^{-\frac{1}{2}\left(s^{2}+t^{2}\right)} E(\exp (s X+t Y)) & =E\left(\exp \left(s X-\frac{s^{2}}{2}\right) \exp \left(t Y-\frac{t^{2}}{2}\right)\right) \\
& =\exp (s t E(X Y))
\end{aligned}
$$

Now take $\frac{\partial^{n+m}}{\partial^{n} s \partial^{m} t}$ of both sides. Recall the description of the Hermite polynomials given above

$$
n!H_{n}(x)=\left.\frac{d^{n}}{d t^{n}} \exp \left(t x-\frac{t^{2}}{2}\right)\right|_{t=0}
$$

Thus

$$
E\left(n!H_{n}(X) m!H_{m}(Y)\right)=\left.\frac{\partial^{n+m}}{\partial^{n} s \partial^{m} t} \exp (s t E(X Y))\right|_{s=t=0}
$$

Consider $m<n$

$$
\frac{\partial^{n+m}}{\partial^{n} s \partial^{m} t} \exp (s t E(X Y))=\frac{\partial^{m}}{\partial t^{m}}\left((t E(X Y))^{n} \exp (s t E(X Y))\right)
$$

You have something like

$$
\frac{\partial^{m}}{\partial t^{m}}\left[t^{n}\left((E(X Y))^{n} \exp (s t E(X Y))\right)\right]
$$

and $m<n$ so when you take partial derivatives with respect to $t, m$ times and set $s, t=0$, you must have 0 . Hence, if $n>m$,

$$
E\left(n!H_{n}(X) m!H_{m}(Y)\right)=0
$$

Similarly this equals 0 if $m>n$. So assume $m=n$. Then you will go through the same process just described but this time at the end you will have something of the form

$$
n!E(X Y)^{n}+\text { terms multiplied by } s \text { or } t
$$

Hence, in this case,

$$
E\left(n!H_{n}(X) n!H_{n}(Y)\right)=n!E(X Y)^{n}
$$

and so

$$
E\left(H_{n}(X) H_{n}(Y)\right)=\frac{1}{n!} E(X Y)^{n}
$$

Let $W$ be the function defined above, $W(h)$ is normally distributed with mean 0 and variance $|h|^{2}$ and $E(W(h) W(g))=(h, g)_{H}$. Then from Lemma 68.2.1,

$$
\begin{aligned}
E\left(H_{n}(W(h)) H_{m}(W(g))\right) & =\left\{\begin{array}{c}
0 \text { if } n \neq m \\
\frac{1}{n!}(E(W(h) W(g)))^{n} \text { if } n=m
\end{array}\right. \\
& =\left\{\begin{array}{c}
0 \text { if } n \neq m \\
\frac{1}{n!}(h, g)_{H}^{n} \text { if } n=m
\end{array}\right.
\end{aligned}
$$

This is a really neat result. From definition of $W$,

$$
E\left((W(h) W(g))^{1}\right)=(h, g)_{H}
$$

Note this is a special case of the above result because $H_{1}(x)=x$. However, we don't know that $E\left((W(h) W(g))^{n}\right)$ is equal to something times $(h, g)_{H}^{n}$ but we know that this is true of some $n^{\text {th }}$ degree polynomials in $W(h)$ and $W(g)$.
Definition 68.2.2 Let $\mathscr{H}_{n} \equiv \overline{\operatorname{span}\left\{H_{n}(W(h)): h \in H,|h|_{H}=1\right\}}$.
Thus $\mathscr{H}_{n}$ is a closed subspace of $L^{2}(\Omega, \mathscr{F})$. Recall $\mathscr{F} \equiv \sigma(W(h): h \in H)$. This subspace $\mathscr{H}_{n}$ is called the Wiener chaos of order $n$.

Theorem 68.2.3 $L^{2}(\Omega, \mathscr{F}, P)=\oplus_{n=0}^{\infty} \mathscr{H}_{n}$. The symbol denotes the infinite orthogonal sum of the closed subspaces $\mathscr{H}_{n}$. That is, if $f \in L^{2}(\Omega)$, there exists $f_{n} \in \mathscr{H}_{n}$ and constants such that $f=\sum_{n} c_{n} f_{n}$ and if $f \in \mathscr{H}_{n}, g \in \mathscr{H}_{n}$, then $(f, g)_{L^{2}(\Omega)}=0$.

Proof: Clearly each $\mathscr{H}_{n}$ is a closed subspace. Also, if $f \in \mathscr{H}_{n}$ and $g \in \mathscr{H}_{m}$ for $n \neq m$, what about $(f, g)_{L^{2}(\Omega)}$ ?

$$
\begin{gathered}
(f, g)_{L^{2}(\Omega)}=\lim _{l \rightarrow \infty} E\left(\sum_{k=1}^{M_{l}} a_{k}^{l} H_{n}\left(W\left(h_{k}^{l}\right)\right), \sum_{j=1}^{M_{p}} a_{j}^{l} H_{m}\left(W\left(h_{j}^{l}\right)\right)\right) \\
=\lim _{l \rightarrow \infty} \sum_{k, j} a_{k}^{l} a_{j}^{l} E\left(H_{n}\left(W\left(h_{k}^{l}\right)\right) H_{m}\left(W\left(h_{j}^{l}\right)\right)\right)=0
\end{gathered}
$$

Thus these are orthogonal subspaces. Clearly $L^{2}(\Omega) \supseteq \oplus \mathscr{H}_{n}$. Suppose $X$ is orthogonal to each $\mathscr{H}_{n}$. Is $X=0$ ? Each $x^{n}$ can be obtained as a linear combination of the $H_{k}(x)$ for $k \leq n$. This is clear because the space of polynomials of degree $n$ is of dimension $n+1$ and $\left\{H_{0}(x), H_{1}(x), \cdots, H_{n}(x)\right\}$ is independent on $\mathbb{R}$.

This is easily seen as follows. Suppose

$$
\sum_{k=0}^{n} c_{k} H_{k}(x)=0
$$

and that not all $c_{k}=0$. Let $m$ be the smallest index such that

$$
\sum_{k=0}^{m} c_{k} H_{k}(x)=0
$$

with $c_{m} \neq 0$. Then just differentiate both sides and obtain

$$
\sum_{k=1}^{m} c_{k} H_{k-1}(x)=0
$$

contradicting the choice of $m$.
Therefore, each $x^{n}$ is really a unique linear combination of the $H_{k}$ as claimed. Say

$$
x^{n}=\sum_{k=0}^{n} c_{k} H_{k}(x)
$$

Then for $|h|=1$,

$$
W(h)^{n}=\sum_{k=0}^{n} c_{k} H_{k}(W(h)) \in \mathscr{H}_{n}
$$

Hence $\left(X, W(h)^{n}\right)_{L^{2}(\Omega)}=0$ whenever $|h|=1$. It follows that for $h \in H$ arbitrary, and the fact that $W$ is linear,

$$
\left(X, W(h)^{n}\right)_{L^{2}}=\left(X,\left(|h| W\left(\frac{h}{|h|}\right)\right)^{n}\right)=|h|^{n}\left(X, W\left(\frac{h}{|h|}\right)^{n}\right)=0
$$

Therefore, $X$ is perpendicular to $e^{W(h)}$ for every $h \in H$ and so from Lemma 64.6.5, $X=0$. Thus $\oplus \mathscr{H}_{n}$ is dense in $L^{2}(\Omega)$.

Note that from Lemma 64.6.4, every polynomial in $W(h)$ is in $L^{p}(\Omega)$ for all $p>1$. Now what is next is really tricky.

Corollary 68.2.4 Let $\mathscr{P}_{n}^{0}$ denote all polynomials of the form

$$
p\left(W\left(h_{1}\right), \cdots, W\left(h_{k}\right)\right), \text { degree of } p \leq n, \text { some } h_{1}, \cdots, h_{k}
$$

Also let $\mathscr{P}_{n}$ denote the closure in $L^{2}(\Omega, \mathscr{F}, P)$ of $\mathscr{P}_{n}^{0}$. Then

$$
\mathscr{P}_{n}=\oplus_{i=0}^{n} \mathscr{H}_{i}
$$

Proof: It is obvious that $\mathscr{P}_{n} \supseteq \oplus_{i=0}^{n} \mathscr{H}_{i}$ because the thing on the right is just the closure of a set of polynomials of degree no more than $n$, a possibly smaller set than the polynomials used to determine $\mathscr{P}_{n}^{0}$ and hence $\mathscr{P}_{n}$. If $\mathscr{P}_{n}$ is orthogonal to $\mathscr{H}_{m}$ for all $m>n$, then from the above Theorem 68.2.3, you must have $\mathscr{P}_{n} \subseteq \oplus_{i=0}^{n} \mathscr{H}_{i}$. So consider $H_{m}(W(h))$. Recall that $\mathscr{H}_{m}$ is the closure of the span of things like this for $|h|_{H}=1$. Thus we need to consider

$$
E\left(p\left(W\left(h_{1}\right), \cdots, W\left(h_{k}\right)\right) H_{m}(W(h))\right),|h|_{H}=1
$$

and show that this is 0 . Now here is the tricky part. Let $\left\{e_{1}, \cdots, e_{s}, h\right\}$ be an orthonormal basis for

$$
\operatorname{span}\left(h_{1}, \cdots, h_{k}, h\right)
$$

Then since $W$ is linear, there is a polynomial $q$ of degree no more than $n$ such that

$$
p\left(W\left(h_{1}\right), \cdots, W\left(h_{k}\right)\right)=q\left(W\left(e_{1}\right), \cdots, W\left(e_{s}\right), W(h)\right)
$$

Then consider a term of $q\left(W\left(e_{1}\right), \cdots, W\left(e_{s}\right), W(h)\right) H_{m}(W(h))$

$$
a W\left(e_{1}\right)^{r_{1}} \cdots W\left(e_{s}\right)^{r_{s}} W(h)^{r} H_{m}(W(h))
$$

Now from Corollary 64.6 .1 these random variables $\left\{W\left(e_{1}\right), \cdots, W\left(e_{s}\right), W(h)\right\}$ are independent due to the fact that the vector $\left(W\left(e_{1}\right), \cdots, W\left(e_{s}\right), W(h)\right)$ is multivariate normally distributed and the covariance is diagonal. Therefore,

$$
\begin{gathered}
E\left(a W\left(e_{1}\right)^{r_{1}} \cdots W\left(e_{s}\right)^{r_{s}} W(h)^{r} H_{m}(W(h))\right) \\
=a E\left(W\left(e_{1}\right)^{r_{1}}\right) \cdots E\left(W\left(e_{s}\right)^{r_{s}}\right) E\left(W(h)^{r} H_{m}(W(h))\right)
\end{gathered}
$$

Now since $r \leq n, W(h)^{r}=\sum_{k=1}^{r} c_{k} H_{k}(W(h))$ for some choice of scalars $c_{k}$. By Lemma 68.2.1, this last term,

$$
E\left(W(h)^{r} H_{m}(W(h))\right)=\sum_{k} c_{k} E\left(H_{k}(W(h)) H_{m}(W(h))\right)=0
$$

since each $k<m$.
Note how remarkable this is. $\mathscr{P}_{n}^{0}$ includes all polynomials in $W\left(h_{1}\right), \cdots, W\left(h_{k}\right)$ some $h_{1}, \cdots, h_{k}$, of degree no more than $n$, including those which have mixed terms but a typical thing in $\oplus_{i=0}^{n} \mathscr{H}_{i}$ is a sum of Hermite polynomials in $W\left(h_{k}\right)$. It is not the case that you would have terms like $W\left(h_{1}\right) W\left(h_{2}\right)$ as could happen in the case of $\mathscr{P}_{n}$.

Obviously it would be a good idea to obtain an orthonormal basis for $L^{2}(\Omega, \mathscr{F}, P)$. This is done next. Let $\Lambda$ be the multiindices, $\left(a_{1}, a_{2}, \cdots\right)$ each $a_{k}$ a nonnegative integer. Also in the description of $\Lambda$ assume that $a_{k}=0$ for all $k$ large enough. For such a multiindex $\mathbf{a} \in \Lambda$,

$$
\mathbf{a}!\equiv \prod_{i=1}^{\infty} a_{i}!, \quad|\mathbf{a}| \equiv \sum_{i} a_{i}
$$

Also for $\mathbf{a} \in \Lambda$, define

$$
H_{\mathbf{a}}(x) \equiv \prod_{j=1}^{\infty} H_{a_{i}}(x)
$$

This is well defined because $H_{0}(x)=1$ and all but finitely many terms of this infinite product are therefore equal to 1 . Now let $\left\{e_{i}\right\}$ be an orthonormal basis for $H$. For $\mathbf{a} \in \Lambda$,

$$
\Phi_{\mathbf{a}} \equiv \sqrt{\mathbf{a}!} \prod_{i=1}^{\infty} H_{a_{i}}\left(W\left(e_{i}\right)\right) \in L^{2}(\Omega)
$$

Suppose $\mathbf{a}, \mathbf{b} \in \Lambda$.

$$
\int_{\Omega} \Phi_{\mathbf{a}} \Phi_{\mathbf{b}} d P=\sqrt{\mathbf{a}!} \sqrt{\mathbf{b}!} \int_{\Omega} \prod_{i=1}^{\infty} H_{a_{i}}\left(W\left(e_{i}\right)\right) H_{b_{i}}\left(W\left(e_{i}\right)\right) d P
$$

Now recall from Corollary 64.6 .1 the random variables $\left\{W\left(e_{i}\right)\right\}$ are independent. Therefore, the above equals

$$
\sqrt{\mathbf{a}!} \sqrt{\mathbf{b}}!\prod_{i=1}^{\infty} \int_{\Omega} H_{a_{i}}\left(W\left(e_{i}\right)\right) H_{b_{i}}\left(W\left(e_{i}\right)\right) d P=\left\{\begin{array}{l}
1 \text { if } \mathbf{a}=\mathbf{b} \\
0 \text { if } \mathbf{a} \neq \mathbf{b}
\end{array}\right.
$$

Thus $\left\{\Phi_{\mathbf{a}}: \mathbf{a} \in \Lambda\right\}$ is an orthonormal set in $L^{2}(\Omega)$.
Lemma 68.2.5 If $s_{k} \rightarrow h$, then for $n \in \mathbb{N}$, there is a subsequence, still called $s_{k}$ for which $W\left(s_{k}\right)^{n} \rightarrow W(h)^{n}$ in $L^{2}(\Omega)$.

Proof: If $s_{k} \rightarrow h$, does $W\left(s_{k}\right)^{n} \rightarrow W(h)^{n}$ in $L^{2}(\Omega)$ for some subsequence? First of all,

$$
\left\|W(h)-W\left(s_{k}\right)\right\|_{L^{2}(\Omega)}^{2}=\left|s_{k}-h\right|_{H}^{2} \rightarrow 0
$$

and so there is a subsequence, still called $k$ such that $W\left(s_{k}\right)(\omega) \rightarrow W(h)(\omega)$ for a.e. $\omega$. Consider

$$
\begin{equation*}
\int_{\Omega}\left|W(h)^{n}-W\left(s_{k}\right)^{n}\right|^{2} d P \tag{68.2.8}
\end{equation*}
$$

Does this converge to 0 ? The integrand is bounded by $2\left(W(h)^{2 n}+W\left(s_{k}\right)^{2 n}\right)$. Since $W(h), W\left(s_{k}\right)$ are symmetric,

$$
\begin{aligned}
& \int_{\Omega}\left(2\left(W(h)^{2 n}+W\left(s_{k}\right)^{2 n}\right)\right)^{2} d P \leq 8 \int_{\Omega}\left(W(h)^{4 n}+W\left(s_{k}\right)^{4 n}\right) d P \\
& \quad=16 \int_{\Omega \cap[W(h) \geq 0]} e^{4 n W(h)} d P+16 \int_{\Omega \cap\left[W\left(s_{k}\right) \geq 0\right]} e^{4 n W\left(s_{k}\right)} d P \\
& \quad \leq 16 \int_{\Omega} e^{4 n W(h)} d P+16 \int_{\Omega} e^{4 n W\left(s_{k}\right)} d P \\
& \quad \leq 16 e^{\frac{1}{2}|4 n h|}+16 e^{\frac{1}{2}\left|4 n s_{k}\right|}
\end{aligned}
$$

which is bounded independent of $k$, the last step following from Lemma 64.6.4. Therefore, the Vitali convergence theorem applies in 68.2.8.

Given an $h \in H$, let $s_{k}=\sum_{j=1}^{k}\left(h, e_{j}\right) e_{j}$, the $k^{t h}$ partial sum in the Fourier series for $h$.

$$
W\left(s_{k}\right)^{m}=\left(\sum_{j=1}^{k}\left(h, e_{j}\right) W\left(e_{j}\right)\right)^{m}=p\left(W\left(e_{1}\right), \cdots, W\left(e_{k}\right)\right)
$$

where $p$ is a homogeneous polynomial of degree $m$. Now this equals

$$
q\left(H_{0}\left(W\left(e_{1}\right)\right), \cdots, H_{0}\left(W\left(e_{k}\right)\right) \cdots H_{m}\left(W\left(e_{1}\right)\right), \cdots, H_{m}\left(W\left(e_{k}\right)\right)\right)
$$

where $q$ is a polynomial. This is because each $W\left(e_{j}\right)^{r}$ is a linear combination of $H_{s}\left(W\left(e_{j}\right)\right)$ for $s \leq r$. Now you look at terms of this polynomial. They are all of the form $c \Phi_{\mathbf{a}}$ for some constant $c$ and $\mathbf{a} \in \Lambda$. Therefore, if $X \in L^{2}(\Omega)$, there is a subsequence, still denoted as $\left\{s_{k}\right\}$ such that

$$
E\left(W(h)^{n} X\right)=\lim _{k \rightarrow \infty} E\left(W\left(s_{k}\right)^{n} X\right)
$$

Now if $X$ is orthogonal to each $\Phi_{\mathbf{a}}$, then for any $h$ and $n$, there is a subsequence still denoted with $k$ such that

$$
E\left(W(h)^{n} X\right)=\lim _{k \rightarrow \infty} E\left(W\left(s_{k}\right)^{n} X\right)=0
$$

It follows from Lemma 64.6.4, the part about the convergence of the partial sums to $e^{W(h)}$ that $X$ is orthogonal to $e^{W(h)}$ for any $h$. Here are the details. From the lemma, for large $n$,

$$
\left|E\left(e^{W(h)} X\right)-E\left(\sum_{j=0}^{n} \frac{W(h)^{j}}{j!} X\right)\right|<\varepsilon
$$

Also for large $k$,

$$
\left|E\left(\sum_{j=0}^{n} \frac{W(h)^{j}}{j!} X\right)-E\left(\sum_{j=0}^{n} \frac{W\left(s_{k}\right)^{j}}{j!} X\right)\right|=\left|E\left(\sum_{j=0}^{n} \frac{W(h)^{j}}{j!} X\right)\right|<\varepsilon
$$

Therefore,

$$
\left|E\left(e^{W(h)} X\right)\right|<2 \varepsilon
$$

Since $\varepsilon$ is arbitrary, this proves the desired result. By Lemma 64.6.5, $X=0$ and this shows that $\left\{\Phi_{\mathbf{a}}, \mathbf{a} \in \Lambda\right\}$ is complete.

Proposition 68.2.6 $\left\{\Phi_{\mathbf{a}}: \mathbf{a} \in \Lambda\right\}$ is a complete orthonormal set for $L^{2}(\Omega, \mathscr{F}, P)$.

### 68.3 A Multiple Integral

Consider trying to find

$$
\int_{0}^{1} \int_{0}^{1} d B_{s} d B_{t}
$$

Here $B_{t}$ is just one dimensional Wiener process. You would want it to equal

$$
2 \int_{0}^{1} \int_{0}^{t} d B_{s} d B_{t}=2 \int_{0}^{1} B_{t} d B_{t}
$$

So what should this equal? Let $F(x)=x^{2}$ so $F^{\prime}(x)=2 x, F^{\prime \prime}(x)=2$. Consider $F\left(B_{t}\right)$. Then using the formalism for the Ito formula,

$$
\begin{aligned}
d F\left(B_{t}\right) & =d\left(B_{t}^{2}\right)=2 B_{t} d B_{t}+\frac{1}{2}(2)\left(d B_{t}\right)^{2} \\
& =2 B_{t} d B_{t}+d t
\end{aligned}
$$

Therefore,

$$
B_{t}^{2}=2 \int_{0}^{t} B_{s} d B_{s}+t
$$

and letting $t=1$,

$$
\frac{1}{2} B_{1}^{2}-\frac{1}{2}=\int_{0}^{1} B_{s} d B_{s}=\int_{0}^{1} \int_{0}^{s} d B_{r} d B_{s}
$$

and so we would want to have

$$
B_{1}^{2}-1=2 \int_{0}^{1} \int_{0}^{s} d B_{r} d B_{s}
$$

and we want this to equal $\int_{0}^{1} \int_{0}^{1} d B_{s} d B_{t}$ so we need to be defining this in a way such that this will result. Of course, this is just the simplest example of an iterated integral with respect to these one dimensional Wiener processes.

Now partition $[0,1)$ as $0=t_{0}<t_{1}<\cdots, t_{n}=1$. Then sum over all $\left[t_{i-1}, t_{i}\right) \times\left[t_{j-1}, t_{j}\right)$ but leave out those which are on the "diagonal". These would be of the form $\left[t_{i-1}, t_{i}\right) \times\left[t_{i-1}, t_{i}\right)$. Here you would have in the sum products of the form $\left(B_{t_{i}}-B_{t_{i-1}}\right)\left(B_{t_{j}}-B_{t_{j-1}}\right)$. Thus you would have

$$
\begin{gathered}
\sum_{i, j}\left(B_{t_{i}}-B_{t_{i-1}}\right)\left(B_{t_{j}}-B_{t_{j-1}}\right)-\sum_{i=1}^{n}\left(B_{t_{i}}-B_{t_{i-1}}\right)^{2} \\
=\left(\sum_{i} B_{t_{i}}-B_{t_{i-1}}\right)^{2}-\sum_{i=1}^{n}\left(B_{t_{i}}-B_{t_{i-1}}\right)^{2} \\
=\left(B_{1}-B_{0}\right)^{2}-\sum_{i=1}^{n}\left(B_{t_{i}}-B_{t_{i-1}}\right)^{2}
\end{gathered}
$$

Then of course you take a limit as the norm of the partition goes to 0 . This yields in the limit

$$
B_{1}^{2}-1
$$

which is the thing which is wanted. Thus the idea is to consider only functions which are equal to 0 on the "diagonal" and define an integral for these. Then hopefully these will be dense in $L^{2}\left([0, T]^{n}\right)$ and the multiple integral can then be defined as some sort of limit.

Now in the above construction, from now on, unless indicated otherwise, $H=L^{2}(T)$ where the measure is ordinary Lebesgue measure on $T=[0, T]$ or $[0, \infty)$ or some other interval of time. However, it could be more general, but for the sake of simplicity let it be Lebesgue measure. Generalities appear to be nothing but identifying that which works in the case of Lebesgue measure. If $\mu \ll m$ everything would work also. A careful description of what kind of measures work is in [102]. Also, for $A$ a Borel set having finite Lebesgue measure,

$$
W(A) \equiv W\left(\mathscr{X}_{A}\right)
$$

This is a random variable, and as explained earlier, since any finite set of these is normally distributed, if all the sets are pairwise disjoint, the random variables are independent because the covariance is a diagonal matrix.

Definition 68.3.1 Let $D \equiv\left\{\left(t_{1}, \cdots, t_{m}\right): t_{i}=t_{j}\right.$ for some $\left.i \neq j\right\}$. This is called the diagonal set. Here $D \subseteq T^{m}$ where $T$ is an interval $[0, T)$. Assume $T<\infty$ here. Let

$$
0=\tau_{0}<\tau_{1}<\cdots<\tau_{k}=T
$$

Then this can be used to partition $T^{m}$ into sets of the form

$$
\left[\tau_{i_{1}-1}, \tau_{i_{1}}\right) \times \cdots \times\left[\tau_{i_{m}-1}, \tau_{i_{m}}\right)
$$

such that $T^{m}$ is the disjoint union of these. An off diagonal step function $f$ is one which is of the form

$$
f\left(t_{1}, \cdots, t_{m}\right)=\sum_{i_{1} \cdots, i_{m}}^{k} a_{i_{1}, \cdots, i_{m}} \mathscr{X}_{\left[\tau_{i_{1}-1}, \tau_{i_{1}}\right) \times \cdots \times\left[\tau_{i_{m}-1}, \tau_{i_{m}}\right)}\left(t_{1}, \cdots, t_{m}\right)
$$

where $a_{i_{1}, \cdots, i_{m}}=0$ if $i_{p}=i_{q}$. This would correspond to a diagonal term because it would result in a repeated half open interval. Thus we assume all these are equal to 0 . The collection of all such off diagonal step functions will be denoted as $\mathscr{E}_{m}$. The $m$ corresponds to the dimension.

Definition 68.3.2 Let $I_{m}: \mathscr{E}_{m} \rightarrow L^{2}(\Omega)$ be defined in the obvious way.

$$
\begin{gathered}
I_{m}\left(\sum_{i_{1} \cdots, i_{m}}^{k} a_{i_{1}, \cdots, i_{m}} \mathscr{X}_{\left[\tau_{i_{1}-1}, \tau_{i_{1}}\right) \times \cdots \times\left[\tau_{i_{m}-1}, \tau_{i_{m}}\right)}\left(t_{1}, \cdots, t_{m}\right)\right) \\
\equiv \sum_{i_{1} \cdots, i_{m}}^{k} a_{i_{1}, \cdots, i_{m}} \prod_{p=1}^{m}\left(B_{\tau_{i_{p}}}-B_{\tau_{i_{p}-1}}\right)
\end{gathered}
$$

Then $I_{m}$ is linear. If you had two different partitions, you could take the union of them both and by letting coefficients be repeated on the smaller boxes, one can assume that a single partition is being used. This is why it is clear that $I_{m}$ is linear.

Definition 68.3.3 Let $f \in L^{2}\left(T^{m}\right)$. The symetrization of $f$ is given by

$$
\tilde{f}\left(t_{1}, \cdots, t_{m}\right) \equiv \frac{1}{m!} \sum_{\sigma \in S_{m}} f\left(t_{\sigma_{1}}, \cdots, t_{\sigma_{m}}\right)
$$

Lemma 68.3.4 The following holds for $f \in \mathscr{E}_{m}$

$$
\|\tilde{f}\|_{L^{2}\left(T^{m}\right)} \leq\|f\|_{L^{2}\left(T^{m}\right)}
$$

also

$$
I_{m}(f)=I_{m}(\tilde{f})
$$

Proof: This follows because, thanks to the properties of Lebesgue measure,

$$
\begin{aligned}
\int_{T^{m}}\left|f\left(t_{1}, \cdots, t_{m}\right)\right|^{2} d t_{1} \cdots d t_{m} & =\int_{T^{m}}\left|f\left(t_{\sigma_{1}}, \cdots, t_{\sigma_{m}}\right)\right|^{2} d t_{1} \cdots d t_{m} \\
& \equiv \int_{T^{m}}\left|f_{\sigma}\right|^{2} d t_{1} \cdots d t_{m}
\end{aligned}
$$

therefore,

$$
\|\tilde{f}\|_{L^{2}\left(T^{m}\right)} \leq \frac{1}{m!} \sum_{\sigma \in S_{m}}\left\|f_{\sigma}\right\|_{L^{2}\left(T^{m}\right)}=\frac{1}{m!} \sum_{\sigma \in S_{m}}\|f\|_{L^{2}\left(T^{m}\right)}=\|f\|_{L^{2}\left(T^{m}\right)}
$$

The next claim follows because on the right, the terms making up the sum just happen in a different order for each $\sigma$.

More generally, here is a lemma about off diagonal things. It uses sets $A_{i}$ rather than intervals $[a, b)$.

Lemma 68.3.5 Let $\left\{A_{1}, \cdots, A_{m}\right\}$ be pairwise disjoint sets in $\mathscr{B}(T)$ each having finite measure. Then the products $A_{i_{1}} \times \cdots \times A_{i_{n}}$ are pairwise disjoint. Also to say that the function

$$
\left(t_{1}, \cdots, t_{n}\right) \rightarrow \sum_{\mathbf{i}} c_{\mathbf{i}} \mathscr{X}_{A_{i_{1}} \times \cdots \times A_{i_{n}}}\left(t_{1}, \cdots, t_{n}\right)
$$

equals 0 whenever some $t_{j}=t_{i}, i \neq j$ is to say that $c_{\mathbf{i}}=0$ whenever there is a repeated index in $\mathbf{i}$.

Proof: Suppose the condition that the $A_{k}$ are pairwise disjoint holds and consider two of these products, $A_{i_{1}} \times \cdots \times A_{i_{n}}$ and $A_{j_{1}} \times \cdots \times A_{j_{n}}$. If the two ordered lists ( $i_{1}, \cdots, i_{n}$ ) and $\left(j_{1}, \cdots, j_{n}\right)$ are different, then since the $A_{k}$ are disjoint the two products have empty intersection because they differ in some position.

Now suppose that $c_{\mathbf{i}}=0$ whenever there is a repeated index. Then the sum is taken over all permutations of $n$ things taken from $\{1, \cdots, m\}$ and so if some $t_{r}=t_{s}$ for $r \neq s$, all terms of the sum equal zero because $\mathscr{X}_{A_{i_{1}} \times \cdots \times A_{i_{n}}} \neq 0$ only if $\mathbf{t} \in A_{i_{1}} \times \cdots \times A_{i_{n}}$ and since $t_{r}=t_{s}$ and the sets $\left\{A_{k}\right\}$ are disjoint, there must be the same set in positions $r$ and $s$ so $c_{\mathbf{i}}=0$. Hence the function equals 0 .

Conversely, suppose the sum $\sum_{\mathbf{i}} c_{\mathbf{i}} \mathscr{X}_{A_{i_{1} \times \cdots \times A_{i_{n}}}}$ equals zero whenever some $t_{r}=t_{s}$ for $s \neq r$. Does it follow that $c_{\mathbf{i}}=0$ whenever some $t_{r}=t_{s}$ ? The value of this function at $\mathbf{t} \in A_{i_{1}} \times \cdots \times A_{i_{n}}$ is $c_{\mathbf{i}}$ because for any other ordered list of indices, the resulting product has empty intersecton with $A_{i_{1}} \times \cdots \times A_{i_{n}}$. Thus, since $t_{r}=t_{s}$, it is given that this function equals 0 which equals $c_{i}$.

This says that when you consider such a function $\sum_{\mathbf{i}} c_{\mathbf{i}} \mathscr{X}_{A_{i_{1}} \times \cdots \times A_{i_{n}}}$ with the $A_{k}$ pairwise disjoint, then to say that it equals 0 whenever some $t_{i}=t_{j}$ is to say that it is really a sum over all permutations of $n$ indices taken from $\{1, \cdots, m\}$. Thus there are $\binom{m}{n} n!=P(m, n)$ possible non zero terms in this sum.

Lemma 68.3.6 Consider the set of all ordered lists of $n$ indices from $\{1,2, \cdots, m\}$. Thus two lists are the same if they consist of the same numbers in the same positions. We denote by $\mathbf{i}$ or $\mathbf{j}$ such an index, $\mathbf{i}$ from $\{1, \cdots, m\}$ and $\mathbf{j}$ from $\{1, \cdots, q\}$. Also let

$$
\left\{A_{1}, \cdots, A_{m}\right\},\left\{B_{1}, \cdots, B_{q}\right\}
$$

are two lists of pairwise disjoint Borel sets from T having finite Lebesgue measure. Also suppose

$$
\sum_{\mathbf{i}} c_{\mathbf{i}} \mathscr{X}_{A_{i_{1}} \times \cdots \times A_{i_{n}}}=\sum_{\mathbf{j}} d_{\mathbf{j}} \mathscr{X}_{B_{j_{1}} \times \cdots \times B_{j_{n}}}
$$

Then

$$
\sum_{\mathbf{i}} c_{\mathbf{i}} \prod_{k=1}^{n} W\left(A_{i_{k}}\right)=\sum_{\mathbf{j}} d_{\mathbf{j}} \prod_{k=1}^{n} W\left(B_{j_{k}}\right)
$$

Proof: Suppose that $n=1$ first. Then you have

$$
\begin{equation*}
\sum_{i} c_{i} \mathscr{X}_{A_{i}}=\sum_{j} d_{j} \mathscr{X}_{B_{j}} \tag{68.3.9}
\end{equation*}
$$

where the sets $\left\{A_{i}\right\}$ and $\left\{B_{j}\right\}$ are disjoint. Clearly

$$
\begin{equation*}
A_{i} \supseteq \cup_{j} A_{i} \cap B_{j} \tag{68.3.10}
\end{equation*}
$$

Consider

$$
\begin{equation*}
c_{i} \mathscr{X}_{A_{i}}, \sum_{j} c_{i} \mathscr{X}_{A_{i} \cap B_{j}} \tag{68.3.11}
\end{equation*}
$$

If strict inequality holds in 68.3.10, then you must have a point in $A_{i} \backslash \cup_{j} A_{i} \cap B_{j}$ where the left side of $\sum_{i} c_{i} \mathscr{X}_{A_{i}}$ equals $c_{i}$ but the right side would equal 0 . Hence $c_{i}=0$ and so $\sum_{j} c_{i} \mathscr{X}_{A_{i} \cap B_{j}}=0$ which shows that the two expressions in 68.3.11 are equal. If $A_{i}=$ $\cup_{j} A_{i} \cap B_{j}$, it is also true that the two expressions in 68.3.11 are equal. Thus

$$
\sum_{i} c_{i} \mathscr{X}_{A_{i}}=\sum_{i} \sum_{j} c_{i} \mathscr{X}_{A_{i} \cap B_{j}}
$$

Similar considerations apply to the right side. Thus

$$
\begin{gathered}
\sum_{i} \sum_{j} c_{i} \mathscr{X}_{A_{i} \cap B_{j}}=\sum_{j} \sum_{i} d_{j} \mathscr{X}_{A_{i} \cap B_{j}} \\
\sum_{i, j}\left(c_{i}-d_{j}\right) \mathscr{X}_{A_{i} \cap B_{j}}=0
\end{gathered}
$$

hence if $W\left(A_{i} \cap B_{j}\right) \neq 0$, then, since these sets are disjoint, $c_{i}-d_{j}=0$. It follows that

$$
\sum_{i, j}\left(c_{i}-d_{j}\right) W\left(A_{i} \cap B_{j}\right)=0
$$

and so

$$
\sum_{i} c_{i} W\left(A_{i}\right)=\sum_{i} \sum_{j} c_{i} W\left(A_{i} \cap B_{j}\right)=\sum_{j} \sum_{i} d_{j} W\left(A_{i} \cap B_{j}\right)=\sum_{j} d_{j} W\left(B_{j}\right)
$$

This proves the theorem if $n=1$. Consider the general case. Let $\mathbf{i}^{\prime}$ be

$$
\begin{gathered}
\left(i_{1}, \cdots, i_{n-1}\right), i_{k} \leq m \\
\sum_{i_{n}=1}^{m} \sum_{\mathbf{i}^{\prime}} c_{\left(\mathbf{i}^{\prime}, i_{n}\right)} \mathscr{X}_{A_{i_{n}}}\left(t_{n}\right) \mathscr{X}_{A_{i_{1}} \times \cdots \times A_{i_{n-1}}}=\sum_{\mathbf{i}} c_{\mathbf{i}} \mathscr{X}_{A_{i_{1}} \times \cdots \times A_{i_{n}}} \\
=\sum_{\mathbf{j}} d_{\mathbf{j}} \mathscr{X}_{B_{j_{1}} \times \cdots \times B_{j_{n}}}=\sum_{j_{n}=1}^{m} \sum_{\mathbf{j}^{\prime}} d_{\left(\mathbf{j}^{\prime}, j_{n}\right)} \mathscr{X}_{B_{j_{n}}}\left(t_{n}\right) \mathscr{X}_{B_{j_{1}} \times \cdots \times B_{j_{n-1}}}
\end{gathered}
$$

Now pick $\left(t_{1}, \cdots, t_{n-1}\right)$. The above is then

$$
\begin{aligned}
& \sum_{i_{n}=1}^{m}\left(\sum_{\mathbf{i}^{\prime}} c_{\left(\mathbf{i}^{\prime}, i_{n}\right)} \mathscr{X}_{A_{i_{1}} \times \cdots \times A_{i_{n-1}}}\left(t_{1}, \cdots, t_{n-1}\right)\right) \mathscr{X}_{A_{i_{n}}}\left(t_{n}\right) \\
= & \sum_{j_{n}=1}^{m}\left(\sum_{\mathbf{j}^{\prime}} d_{\left(\mathbf{j}^{\prime}, j_{n}\right)} \mathscr{X}_{B_{j_{1}} \times \cdots \times B_{j_{n-1}}}\left(t_{1}, \cdots, t_{n-1}\right)\right) \mathscr{X}_{B_{j_{n}}}\left(t_{n}\right)
\end{aligned}
$$

and by what was just shown for $n=1$, for each such choice,

$$
\begin{aligned}
& \sum_{i_{n}}\left(\sum_{\mathbf{i}^{\prime}} c_{\left(\mathbf{i}^{\prime}, i_{n}\right)} \mathscr{X}_{A_{i_{1}} \times \cdots \times A_{i_{n-1}}}\right) W\left(A_{i_{n}}\right) \\
= & \sum_{j_{n}}\left(\sum_{\mathbf{j}^{\prime}} d_{\left(\mathbf{j}^{\prime}, j_{n}\right)} \mathscr{X}_{B_{j_{1}} \times \cdots \times B_{j_{n-1}}}\right) W\left(B_{j_{n}}\right)
\end{aligned}
$$

Then

$$
\begin{aligned}
& \sum_{\mathbf{i}^{\prime}}(\overbrace{\sum_{i_{n}} W\left(A_{i_{n}}\right) c_{\left(\mathbf{i}^{\prime}, i_{n}\right)}}^{\text {function of } \omega}) \overbrace{\mathscr{X}_{A_{i_{1}} \times \cdots \times A_{i_{n-1}}}}^{\text {not a function of } \omega}= \\
& \sum_{\mathbf{j}^{\prime}}\left(\sum_{j_{n}} W\left(B_{j_{n}}\right) d_{\left(\mathbf{j}^{\prime}, j_{n}\right)}\right) \mathscr{X}_{B_{j_{1}} \times \cdots \times B_{j_{n-1}}}
\end{aligned}
$$

Pick $\omega=\omega_{0}$. Then by induction,

$$
\begin{aligned}
& \sum_{\mathbf{i}^{\prime}}\left(\sum_{i_{n}} W\left(A_{i_{n}}\right)\left(\omega_{0}\right) c_{\left(\mathbf{i}^{\prime}, i_{n}\right)}\right) W\left(A_{i_{1}}\right) \cdots W\left(A_{i_{n-1}}\right) \\
= & \sum_{\mathbf{j}^{\prime}}\left(\sum_{j_{n}} W\left(B_{j_{n}}\right)\left(\omega_{0}\right) d_{\left(\mathbf{j}^{\prime}, j_{n}\right)}\right) W\left(B_{j_{1}}\right) \cdots W\left(B_{j_{n-1}}\right)
\end{aligned}
$$

and this reduces to what was to be shown because $\omega_{0}$ was arbitrary.
In what follows it will be assumed $c_{\mathbf{i}}=0$ if any two of the $i_{k}$ are equal. That is

$$
\sum_{\mathbf{i}} c_{\mathbf{i}} \mathscr{X}_{A_{i_{1} \times \cdots \times A_{i_{n}}}}\left(t_{1}, \cdots, t_{n}\right)=0
$$

if any $t_{i}=t_{j}$.
Definition 68.3.7 Let $\mathscr{E}_{n}$ be functions of the form

$$
f\left(t_{1}, \cdots, t_{n}\right) \equiv \sum_{\mathbf{i}} c_{\mathbf{i}} \mathscr{X}_{A_{i_{1}} \times \cdots \times A_{i_{n}}}\left(t_{1}, \cdots, t_{n}\right)
$$

where the $A_{k}$ come from some list of the form $\left\{A_{1}, A_{2}, \cdots, A_{m}\right\}$ where this list of sets is pairwise disjoint, each $A_{k} \neq \emptyset$ and $c_{\mathbf{i}}=0$ whenever two indices are equal. By Lemma 68.3.5 this is the same as saying that $f=0$ if $t_{i}=t_{j}$ for some $i \neq j$. A function of $n$ variables $f$ is symmetric means that for $\sigma$ a permutation,

$$
f\left(t_{1}, \cdots, t_{n}\right)=f\left(t_{\sigma(1)}, \cdots, t_{\sigma(n)}\right)
$$

Lemma 68.3.8 Let $f\left(t_{1}, \cdots, t_{n}\right) \equiv \sum_{\mathbf{i}} c_{\mathbf{i}} \mathscr{X}_{A_{i_{1} \times \cdots \times A_{i_{n}}}}\left(t_{1}, \cdots, t_{n}\right)$. Then $f$ is symmetric if and only iffor all $\left\{c_{i_{1}}, \cdots, i_{n}\right\}$

$$
c_{i_{1}, \cdots, i_{n}}=c_{i_{\sigma(1)}, \cdots, i_{\sigma(n)}}
$$

Proof: First of all, every $c_{\mathbf{i}}=0$ if there are repeated indices so it suffices to consider only the case where all indices are distinct.

Consider all the terms associated with a particular set of indices $\left\{i_{1}, \cdots, i_{n}\right\}$. Then, since these sets $A_{i_{k}}$ are disjoint, the function $f$ is symmetric if and only if the part of the sum in the definition of $f$ associated with each such set of indices is symmetric. To save on notation, denote such a list by $\{1,2, \cdots, n\}$. It suffices then to show that

$$
f\left(t_{1}, \cdots, t_{n}\right)=\sum_{\sigma \in S_{n}} c_{\sigma(1) \cdots \sigma(n)} \mathscr{X}_{A_{\sigma(1)} \times \cdots \times A_{\sigma(n)}}\left(t_{1}, \cdots, t_{n}\right)
$$

is symmetric if and only if for all $\sigma, c_{\sigma(1) \cdots \sigma(n)}=c_{1 \cdots n}$. Suppose then that $f$ is symmetric. Then

$$
\begin{aligned}
f\left(t_{\beta(1)}, \cdots, t_{\beta(n)}\right) & =\sum_{\sigma \in S_{n}} c_{\sigma(1) \cdots \sigma(n)} \mathscr{X}_{A_{\sigma(1)} \times \cdots \times A_{\sigma(n)}}\left(t_{\beta(1)}, \cdots, t_{\beta(n)}\right) \\
& =\sum_{\sigma \in S_{n}} c_{\sigma(1) \cdots \sigma(n)} \mathscr{X}_{A_{\sigma(1)} \times \cdots \times A_{\sigma(n)}}\left(t_{1}, \cdots, t_{n}\right)=f\left(t_{1}, \cdots, t_{n}\right)
\end{aligned}
$$

However,

$$
\begin{align*}
& \sum_{\sigma \in S_{n}} c_{\sigma(1) \cdots \sigma(n)} \mathscr{X}_{A_{\sigma(1)} \times \cdots \times A_{\sigma(n)}}\left(t_{\beta(1)}, \cdots, t_{\beta(n)}\right) \\
= & \sum_{\sigma \in S_{n}} c_{\sigma(1) \cdots \sigma(n)} \mathscr{X}_{A_{\beta^{-1} \sigma(1)} \times \cdots \times A_{\beta^{-1} \sigma(n)}}\left(t_{1}, \cdots, t_{n}\right) \tag{68.3.12}
\end{align*}
$$

It is supposed to equal

$$
\begin{align*}
& \sum_{\sigma \in S_{n}} c_{\sigma(1) \cdots \sigma(n)} \mathscr{X}_{A_{\sigma(1)} \times \cdots \times A_{\sigma(n)}}\left(t_{1}, \cdots, t_{n}\right) \\
= & \sum_{\sigma \in S_{n}} c_{\beta^{-1} \sigma(1) \cdots \beta^{-1} \sigma(n)} \mathscr{X}_{A_{\beta^{-1} \sigma(1)} \times \cdots \times A_{\beta^{-1} \sigma(n)}}\left(t_{1}, \cdots, t_{n}\right) \tag{68.3.13}
\end{align*}
$$

Thus

$$
\begin{align*}
& \sum_{\sigma \in S_{n}} c_{\sigma(1) \cdots \sigma(n)} \mathscr{X}_{A_{\beta^{-1} \sigma(1)} \times \cdots \times A_{\beta^{-1} \sigma(n)}}\left(t_{1}, \cdots, t_{n}\right) \\
= & \sum_{\sigma \in S_{n}} c_{\beta^{-1} \sigma(1) \cdots \beta^{-1} \sigma(n)} \mathscr{X}_{A_{\beta^{-1} \sigma(1)} \times \cdots \times A_{\beta^{-1} \sigma(n)}}\left(t_{1}, \cdots, t_{n}\right) \tag{68.3.14}
\end{align*}
$$

Since the sets $A_{k}$ are distinct, as explained above, this requires that

$$
\mathscr{X}_{A_{\sigma(1)} \times \cdots \times A_{\sigma(n)}} \neq \mathscr{X}_{A_{\alpha(1)} \times \cdots \times A_{\alpha(n)}}
$$

if $\alpha \neq \sigma$. Therefore, 68.3 .14 requires that for all $\beta$ and each $\sigma$,

$$
c_{\sigma(1) \cdots \sigma(n)}=c_{\beta^{-1} \sigma(1) \cdots \beta^{-1} \sigma(n)}
$$

In particular, this is true if $\beta=\sigma$ and so $c_{\sigma(1) \cdots \sigma(n)}=c_{1 \cdots n}$.
The converse of this is also clear. If $c_{\sigma(1) \cdots \sigma(n)}=c_{1 \cdots n}$ for each $\sigma$, then

$$
\begin{aligned}
f\left(t_{\beta(1)}, \cdots, t_{\beta(n)}\right) & =\sum_{\sigma \in S_{n}} c_{\sigma(1) \cdots \sigma(n)} \mathscr{X}_{A_{\sigma(1)} \times \cdots \times A_{\sigma(n)}}\left(t_{\beta(1)}, \cdots, t_{\beta(n)}\right) \\
& =\sum_{\sigma \in S_{n}} c_{\sigma(1) \cdots \sigma(n)} \mathscr{X}_{A_{\beta^{-1} \sigma(1)} \times \cdots \times A_{\beta^{-1} \sigma(n)}}\left(t_{1}, \cdots, t_{n}\right) \\
& =\sum_{\sigma \in S_{n}} c_{1 \cdots n} \mathscr{X}_{A_{\beta^{-1} \sigma(1)} \times \cdots \times A_{\beta^{-1} \sigma(n)}}\left(t_{1}, \cdots, t_{n}\right) \\
& =\sum_{\alpha \in S_{n}} c_{1 \cdots n} \mathscr{X}_{A_{\alpha(1)} \times \cdots \times A_{\alpha(n)}\left(t_{1}, \cdots, t_{n}\right)} \\
& =\sum_{\alpha \in S_{n}} c_{\alpha(1) \cdots \alpha(n)} \mathscr{X}_{A_{\alpha(1)} \times \cdots \times A_{\alpha(n)}}\left(t_{1}, \cdots, t_{n}\right) \\
& =f\left(t_{1}, \cdots, t_{n}\right) \boldsymbol{\square}
\end{aligned}
$$

Observe that $\mathscr{E}_{n}$ is a vector space because if you have two such functions

$$
\sum_{\mathbf{i}} c_{\mathbf{i}} \mathscr{X}_{A_{i_{1}} \times \cdots \times A_{i_{n}}}\left(t_{1}, \cdots, t_{n}\right), \sum_{\mathbf{i}} d_{\mathbf{i}} \mathscr{X}_{B_{i_{1}} \times \cdots \times B_{i_{n}}}\left(t_{1}, \cdots, t_{n}\right)
$$

where the $A_{i_{k}}$ are from $\left\{A_{1}, A_{2}, \cdots, A_{m}\right\}$ and the $B_{i_{k}}$ are from $\left\{B_{1}, B_{2}, \cdots, B_{q}\right\}$. Then consider the single list consisting of the sets of the form $A_{k} \cap B_{j}$. You could write each of these functions in terms of indicator functions of products of these disjoint sets. Thus the sum of the two functions can be written in the desired form. Since each are equal to 0 when some $t_{j}=t_{k}$, the same is true of their sum. Thus $\mathscr{E}_{n}$ is closed with respect to sums. It is obviously
closed with respect to scalar multiplication. Hence it is a subspace of the vector space of all functions and it is therefore, a vector space.

Following [102], for $f$ one of these elementary functions,

$$
f\left(t_{1}, \cdots, t_{n}\right) \equiv \sum_{\mathbf{i}} c_{\mathbf{i}} \mathscr{X}_{A_{i_{1}} \times \cdots \times A_{i_{n}}}\left(t_{1}, \cdots, t_{n}\right)
$$

where if any two indices are repeated, then $c_{\mathbf{i}}=0$, and the $A_{k}$ are all disjoint,

$$
I_{n}(f) \equiv \sum_{\mathbf{i}} c_{\mathbf{i}} W\left(A_{i_{1}}\right) \cdots W\left(A_{i_{n}}\right)
$$

Lemma 68.3.9 $I_{n}$ is linear on $\mathscr{E}_{n}$. If $f \in \mathscr{E}_{n}$ and $\sigma$ is a permutation of $(1, \cdots, n)$ and

$$
f_{\sigma}\left(t_{1}, \cdots, t_{n}\right) \equiv f\left(t_{\sigma(1)}, \cdots, t_{\sigma\left(t_{n}\right)}\right),
$$

and $f$ is symmetric, then

$$
I_{n}\left(f_{\sigma}\right)=I_{n}(f)
$$

For $f=\sum_{\mathbf{i}} c_{\mathbf{i}} \mathscr{X}_{A_{i_{1}} \times \cdots \times A_{i_{n}}}$, one can conclude that

$$
\begin{equation*}
I_{n}(f)=n!\sum_{i_{1}<i_{2}<\cdots<i_{n}} c_{i_{1}, \cdots, i_{n}} \prod W\left(A_{i_{n}}\right) \tag{68.3.15}
\end{equation*}
$$

Also, the following holds for the expectation. For $f, g \in \mathscr{E}_{n}, \mathscr{E}_{m}$ respectively,

$$
E\left(I_{n}(f) I_{m}(g)\right)=\left\{\begin{array}{l}
0 \text { if } n \neq m \\
n!(\tilde{f}, \tilde{g})_{L^{2}\left(T^{n}\right)}
\end{array} \text { if } n=m\right.
$$

where $\tilde{f}$ denotes the symetrization of $f$ given by

$$
\tilde{f}\left(t_{1}, \cdots, t_{n}\right) \equiv \frac{1}{n!} \sum_{\sigma \in S_{n}} f\left(t_{\sigma(1)}, \cdots, t_{\sigma(n)}\right)
$$

Proof: It is clear from the definition being well defined that $I_{n}$ is linear. In particular, consider

$$
I_{n}\left(a \sum_{\mathbf{i}} c_{\mathbf{i}} \mathscr{X}_{A_{i_{1}} \times \cdots \times A_{i_{n}}}+b \sum_{\mathbf{j}} d_{\mathbf{j}} \mathscr{X}_{B_{j_{1}} \times \cdots \times B_{j_{n}}}\right) .
$$

As explained above in the observation that $\mathscr{E}_{n}$ is a vector space, it can be assumed that the $A_{i_{k}}$ and $B_{j_{k}}$ are all from a single set of disjoint Borel sets of $T$. Then the above is of the form

$$
\begin{aligned}
& a \sum_{\mathbf{i}} c_{\mathbf{i}} \prod_{k} W\left(A_{i_{k}}\right)+b \sum_{\mathbf{j}} d_{\mathbf{j}} \prod_{k} W\left(B_{j_{k}}\right) \\
= & a I_{n}\left(\sum_{\mathbf{i}} c_{\mathbf{i}} \mathscr{X}_{A_{i_{1}} \times \cdots \times A_{i_{n}}}\right)+b I_{n}\left(\sum_{\mathbf{j}} d_{\mathbf{j}} \mathscr{X}_{B_{j_{1}} \times \cdots \times B_{j_{n}}}\right)
\end{aligned}
$$

Next consider for $\mathbf{i}=\left(i_{1} \cdots i_{n}\right)$,

$$
\begin{aligned}
f_{\sigma}\left(t_{1}, \cdots, t_{n}\right) & =\sum_{\mathbf{i}} c_{\mathbf{i}} \mathscr{X}_{A_{i_{1}} \times \cdots \times A_{i_{n}}}\left(t_{\sigma(1)}, \cdots, t_{\sigma(n)}\right) \\
& =\sum_{\mathbf{i}} c_{\mathbf{i}} \prod_{j=1}^{n} \mathscr{X}_{A_{i_{j}}}\left(t_{\sigma(j)}\right)=\sum_{\mathbf{i}} c_{\mathbf{i}} \prod_{j=1}^{n} \mathscr{X}_{A_{\sigma^{-1}(j)}}\left(t_{j}\right) \\
& =\sum_{\mathbf{i}} c_{\mathbf{i}} \mathscr{X}_{A_{i_{\sigma^{-1}(1)}}} \times \cdots \times A_{i_{\sigma^{-1}(n)}}\left(t_{1}, \cdots, t_{n}\right)
\end{aligned}
$$

Thus, it appears that $f_{\sigma} \neq f$. However,

$$
\begin{equation*}
I_{n}\left(f_{\sigma}\right) \equiv \sum_{\mathbf{i}} c_{\mathbf{i}} \prod_{k=1}^{n} W\left(A_{i_{\sigma^{-1}(k)}}\right)=I_{n}(f) \tag{68.3.16}
\end{equation*}
$$

because one just considers the factors in a different order than the other. The permutation acts on $\left(i_{1} \cdots i_{n}\right)$. Define the symetrization of $f$ by $\tilde{f}$ given by

$$
\tilde{f}\left(t_{1}, \cdots, t_{n}\right) \equiv \frac{1}{n!} \sum_{\sigma} f_{\sigma}\left(t_{1}, \cdots, t_{n}\right)
$$

Then $I_{n}(\tilde{f})=I_{n}(f)$ and

$$
\tilde{f}\left(t_{\sigma(1)}, \cdots, t_{\sigma(n)}\right)=\tilde{f}\left(t_{1}, \cdots, t_{n}\right) .
$$

If $f\left(t_{\sigma(1)}, \cdots, t_{\sigma(n)}\right)=f\left(t_{1}, \cdots, t_{n}\right)$ for all $\sigma$ then $\tilde{f}=f$. From the above, $\tilde{f}$ equals

$$
\frac{1}{n!} \sum_{\sigma} \sum_{\mathbf{i}} c_{\mathbf{i}} \mathscr{X}_{A_{i^{-1}(1)}} \times \cdots \times A_{\sigma^{-1}(n)}\left(t_{1}, \cdots, t_{n}\right)
$$

Note that 68.3.16 implies that

$$
\begin{equation*}
I_{n}(f)=I_{n}(\tilde{f})=n!\sum_{i_{1}<i_{2}<\cdots<i_{n}} c_{i_{1}, \cdots, i_{n}} \prod_{k} W\left(A_{i_{k}}\right) \tag{68.3.17}
\end{equation*}
$$

Now consider

$$
\tilde{f}=\sum_{\mathbf{i}} c_{\mathbf{i}} \mathscr{X}_{A_{i_{1}} \times \cdots \times A_{i_{n}}} \text { and } \tilde{g}=\sum_{\mathbf{i}} d_{\mathbf{i}} \mathscr{X}_{A_{i_{1}} \times \cdots \times A_{i_{n}}}
$$

where without loss of generality, these sets $A_{k}$ come from a single list of disjoint sets. As above, $I_{n}(f)=I_{n}(\tilde{f})$ and so it follows that

$$
E\left(I_{n}(f) I_{n}(g)\right)=E\left(I_{n}(\tilde{f}) I_{n}(\tilde{g})\right)
$$

From the above, $E\left(I_{n}(\tilde{f}) I_{n}(\tilde{g})\right)=$

$$
\begin{align*}
& E\left((n!)^{2} \sum_{i_{1}<\cdots<i_{n}} \sum_{j_{1}<\cdots<j_{n}} c_{i_{1}, \cdots, i_{n}} d_{j_{1}, \cdots, j_{n}} \prod_{k} W\left(A_{i_{k}}\right) \prod_{l} W\left(A_{j_{l}}\right)\right) \\
&=(n!)^{2} \sum_{i_{1}<\cdots<i_{n}} \sum_{j_{1}<\cdots<j_{n}} c_{i_{1}, \cdots, i_{n}} d_{j_{1}, \cdots, j_{n}} E\left(\prod_{k} W\left(A_{i_{k}}\right) \prod_{l} W\left(A_{j_{l}}\right)\right) \\
&=\quad(n!)^{2} \sum_{i_{1}<\cdots<i_{n}} \sum_{j_{1}<\cdots<j_{n}} c_{i_{1}, \cdots, i_{n}} d_{j_{1}, \cdots, j_{n}} E\left(\prod_{k} W\left(A_{i_{k}}\right) W\left(A_{j_{k}}\right)\right) \tag{68.3.18}
\end{align*}
$$

That product is of independent random variables. Recall any collection of the $W\left(A_{k}\right)$ are normally distributed and also the covariance is diagonal and so these will all be independent random variables. If any one of them is not repeated, say $W\left(A_{i_{k}}\right)$, then

$$
E\left(\prod_{k} W\left(A_{i_{k}}\right) W\left(A_{j_{k}}\right)\right)=E\left(W\left(A_{i_{k}}\right)\right)(\text { stuff })=0
$$

It follows that to get something nonzero out of this, all $A_{i_{k}}$ are repeated. That is, you must have $\mathbf{j}=\mathbf{i}$ and 68.3.18 reduces to $E\left(I_{n}(f) I_{n}(g)\right)=$

$$
\begin{align*}
& (n!)^{2} \sum_{i_{1}<\cdots<i_{n}} c_{i_{1}, \cdots, i_{n}} d_{i_{1}, \cdots, i_{n}} E\left(\prod_{k} W\left(A_{i_{k}}\right)^{2}\right) \\
= & (n!)^{2} \sum_{i_{1}<\cdots<i_{n}} c_{i_{1}, \cdots, i_{n}} d_{i_{1}, \cdots, i_{n}} \prod_{k} E\left(W\left(A_{i_{k}}\right)^{2}\right) \\
= & (n!)^{2} \sum_{i_{1}<\cdots<i_{n}} c_{i_{1}, \cdots, i_{n}} d_{i_{1}, \cdots, i_{n}} \prod_{k} m\left(A_{i_{k}}\right) \tag{68.3.19}
\end{align*}
$$

By Lemma 68.3.8, used at the end of the following string of equalities, and the observation that

$$
\mathscr{X}_{A_{i_{1}} \times \cdots \times A_{i_{n}}} \mathscr{X}_{A_{j_{1}} \times \cdots \times A_{j_{n}}}=0
$$

to eliminate mixed terms,

$$
\begin{gathered}
(\tilde{f}, \tilde{g})_{L^{2}\left(T^{n}\right)}= \\
=\int_{0}^{\infty} \cdots \int_{0}^{\infty}\left(\sum_{\mathbf{i}} c_{\mathbf{i}} \mathscr{X}_{A_{i_{1}} \times \cdots \times A_{i_{n}}}\right)\left(\sum_{\mathbf{i}} d_{\mathbf{i}} \mathscr{X}_{A_{i_{1} \times \cdots \times A_{i_{n}}}}\right) d t \cdots d t \\
=\int_{0}^{\infty} \cdots \int_{0}^{\infty}\left(\sum_{\mathbf{i}} c_{\mathbf{i}} d_{\mathbf{i}} \mathscr{X}_{A_{i_{1}} \times \cdots \times A_{i_{n}}}\right) d t \cdots d t \\
=\sum_{\mathbf{i}} c_{\mathbf{i}} d_{\mathbf{i}} \prod_{k} m\left(A_{i_{k}}\right)=n!\sum_{i_{1}<\cdots<i_{n}} c_{i_{1} \cdots i_{n}} d_{i_{1} \cdots i_{n}} \prod_{k} m\left(A_{i_{k}}\right)
\end{gathered}
$$

Now it follows from this and 68.3.19 that

$$
E\left(I_{n}(f) I_{n}(g)\right)=n!(\tilde{f}, \tilde{g})_{L^{2}\left(T^{n}\right)}
$$

What happens if you consider $E\left(I_{n}(f) I_{m}(g)\right)$ where $m<n$ ? You would still get $E\left(I_{n}(f) I_{m}(g)\right)=E\left(I_{n}(\tilde{f}) I_{m}(\tilde{g})\right)$

$$
=E\binom{(n!)(m!) \sum_{i_{1}<i_{2}<\cdots<i_{n}} \sum_{j_{1}<\cdots<j_{j}} c_{i_{1}, \cdots, i_{n}} d_{j_{1}, \cdots, j_{m}} W\left(A_{i_{1}}\right)}{\cdots W\left(A_{i_{n}}\right) W\left(A_{j_{1}}\right) \cdots W\left(A_{j_{m}}\right)}
$$

Then at least one of the $W\left(A_{i_{k}}\right)$ is not repeated. This is because $n>m$. That product is a product of independent random variables at least one of which is of the form $W\left(A_{i_{k}}\right)$.

Hence the expectation of the product it is of the form $E\left(W\left(A_{i_{k}}\right)\right)$ (Other terms) $=0$. Thus if $n \neq m$, the result is 0 as claimed.

An integral has now been defined on the functions of the form

$$
f\left(t_{1}, \cdots, t_{n}\right) \equiv \sum_{\mathbf{i}} c_{\mathbf{i}} \mathscr{X}_{A_{i_{1}} \times \cdots \times A_{i_{n}}}\left(t_{1}, \cdots, t_{n}\right)
$$

where $f=0$ if any $t_{i}=t_{j}$ for $i \neq j$. This integral defined on these elementary functions is interesting because for such functions $f, g$

$$
E\left(I_{n}(f) I_{m}(g)\right)=\left\{\begin{array}{l}
0 \text { if } n \neq m \\
n!(\tilde{f}, \tilde{g})_{L^{2}\left(T^{n}\right)}
\end{array} \text { if } n=m\right.
$$

where $\tilde{f}$ is the symmetrization of $f$. It is desired to extend this integral to $L^{2}\left(T^{n}\right)$. Simple functions are always dense in $L^{2}(T)$. Also, there is an easy lemma which can be concluded for $L^{2}\left(T^{n}\right)$.

Lemma 68.3.10 Let $\mathscr{B}_{0}(T)$ be the Borel sets having finite measure. Linear combinations of functions of the form

$$
\mathscr{X}_{A_{1} \times \cdots \times A_{n}}
$$

where $A_{i} \in \mathscr{B}_{0}(T)$ are dense in $L^{2}\left(T, \mathscr{B}^{n}\right)$ where of course $\mathscr{B}^{n}$ refers to the product $\sigma$ algebra.

Proof: If you have $U=A_{1} \times \cdots \times A_{n}$ in $T^{n}$ one can approximate $\mathscr{X}_{U \cap R_{p}}$ for $R_{p} \equiv$ $(-p, p)^{n}$ in $L^{2}$ with linear combinations of sets of the desired form. In fact, you just consider $\mathscr{X}_{A_{1} \cap(-p, p) \times \cdots \times A_{n}(-p, p)}$ and you get equality. Now let $\mathscr{K}$ denote the $\pi$ system of sets of this sort. Let $\mathscr{G}$ denote those Borel sets $G$ such that there exists a sequence of linear combinations of sets of the form $\mathscr{X}_{\mathbf{A}}, \mathbf{A}=A_{1} \times \cdots \times A_{n}$ which converges to $\mathscr{X}_{G \cap R_{p}}$ in $L^{2}\left(T^{n}\right)$. Thus $\mathscr{G} \supseteq \mathscr{K}$.

Let $\left\{G_{k}\right\}$ be a disjoint sequence of sets of $\mathscr{G}$. Is $G \equiv \cup_{k} G_{k} \in \mathscr{G}$ ? By monotone convergence theorem,

$$
\left\|\mathscr{X}_{G \cap R_{p}}-\sum_{k=1}^{m} \mathscr{X}_{G_{k} \cap R_{p}}\right\|_{L^{2}\left(T^{n}\right)}<\varepsilon
$$

provided $m$ is large enough. Now by definition of $\mathscr{G}$ there exists $L_{k}$ a linear combination of these special sets such that

$$
\left\|\mathscr{X}_{G_{k} \cap R_{p}}-L_{k}\right\|_{L^{2}\left(T^{n}\right)}<\frac{\varepsilon}{m}
$$

It follows that

$$
\begin{aligned}
&\left\|\mathscr{X}_{G \cap R_{p}}-\sum_{k=1}^{m} L_{k}\right\|_{L^{2}} \leq\left\|\mathscr{X}_{G \cap R_{p}}-\sum_{k=1}^{m} \mathscr{X}_{G_{k} \cap R_{p}}\right\|_{L^{2}} \\
&+\left\|\sum_{k=1}^{m} \mathscr{X}_{G_{k} \cap R_{p}}-\sum_{k=1}^{m} L_{k}\right\|<\varepsilon+\varepsilon
\end{aligned}
$$

and so, it follows that $G \in \mathscr{G}$. If $G \in \mathscr{G}$, does it follow that $G^{C}$ is also?

$$
\mathscr{X}_{R_{p}}=\mathscr{X}_{R_{p} \cap G}+\mathscr{X}_{R^{p} \cap G^{C}}
$$

Hence

$$
\mathscr{X}_{R_{p}}-\mathscr{X}_{R_{p} \cap G}=\mathscr{X}_{R^{p} \cap G^{C}}
$$

Both of the functions on the left can be approximated in $L^{2}$ by the desired kind of functions and so the one on the right can also. It follows from Dynkin's lemma that $\mathscr{G} \equiv \sigma(\mathscr{K})$ which is the product measurable sets. Thus if $U$ is any set in $\mathscr{B}^{n}$, it follows that $\mathscr{X}_{U}$ can be approximated in $L^{2}\left(T^{n}\right)$ with linear combinations of sets like $\mathscr{X}_{A_{1} \times \cdots \times A_{n}}$.

Of course nothing is known about whether the sets $A_{i}$ are disjoint. Also it is not known whether these linear combinations of these functions equals 0 if $t_{i}=t_{j}$. Thus there is something which needs to be proved.

Lemma 68.3.11 The functions in $\mathscr{E}_{n}$ mentioned above are dense in $L^{2}\left(T^{n}\right)$.
Proof: From Lemma 68.3.10, it suffices to show that $\mathscr{X}_{A_{1} \times \cdots \times A_{n}}$ can be approximated in $L^{2}\left(T^{n}\right)$ with functions in $\mathscr{E}_{n}$. This is where it will be important that the measure is sufficiently like Lebesgue measure. Let $\left\{B_{k}^{i}\right\}_{k=1}^{m}$ be a partition of $A_{i}$ such that $m\left(B_{k}^{i}\right) \leq$ $\frac{2 m\left(A_{i}\right)}{m}$. Let $\left\{B_{k}\right\}_{k=1}^{p}$ denote all these sets so $p=m n$. They are not necessarily disjoint because it is not known that the $A_{i}$ are disjoint. However, one can say that it is possible to choose $e_{\mathbf{i}}$ equal to either 0 or 1 such that

$$
\mathscr{X}_{A_{1} \times \cdots \times A_{n}}=\sum_{\mathbf{i}} e_{\mathbf{i}} \mathscr{X}_{B_{i_{1}} \times \cdots \times B_{i_{n}}}
$$

where we can have $B_{i_{k}} \subseteq A_{k}$. Let $J$ be those indices $\mathbf{i}$ which involve a repeated set. That is some $B_{i_{j}}=B_{i_{k}}$ for some $j \neq k$. How many possibilities are there? There are no more than $C(n, 2) m$ because there are $C(n, 2)$ possibilities for duplicates among the $A_{k}$ and then there are $m$ sets in the partition of $A_{k}$.

$$
\begin{aligned}
& \int_{T} \cdots \int_{T}\left(\sum_{i \in J} e_{i} \mathscr{X}_{B_{i_{1}} \times \cdots \times B_{i_{n}}}\right)^{2} d t \cdots d t \\
= & \int_{T} \cdots \int_{T} C(n, 2) m \mathscr{X}_{B_{i_{1}} \times \cdots \times B_{i_{n}}} d t \cdots d t \\
\leq & C(n, 2) m \prod_{k=1}^{n} m\left(B_{i_{k}}\right)
\end{aligned}
$$

The mixed terms are 0 because for a fixed $k,\left\{B_{i_{k}}\right\}_{i=1}^{m}$ are disjoint. Now from the description of these, $m\left(B_{i_{k}}\right) m<m\left(A_{k}\right)$ and so

$$
\begin{aligned}
& \int_{T} \cdots \int_{T}\left(\sum_{\mathbf{i} \in J} e_{i} \mathscr{X}_{B_{i_{1}} \times \cdots \times B_{i_{n}}}\right)^{2} d t \cdots d t \\
\leq & C(n, 2) m \prod_{k=1}^{n} \frac{2 m\left(A_{k}\right)}{m}=\frac{C(n, 2) m}{m^{n}} \prod_{k=1}^{n} m\left(A_{k}\right)
\end{aligned}
$$

which clearly converges to 0 as $m \rightarrow \infty$ provided that $n \geq 2$. In case $n=1$, all you have to do is approximate $\mathscr{X}_{A}$ from something in $\mathscr{E}_{1}$ and of course you just use $\mathscr{X}_{A}$.

Let $f, g \in \mathscr{E}_{n}$. Then from Lemma 68.3.9,

$$
\begin{aligned}
& E\left(\left(I_{n}(f-g)\right)^{2}\right)=n!\|\tilde{f}-\tilde{g}\|_{L^{2}\left(T^{n}\right)}^{2} \\
&\|\tilde{f}\|_{L^{2}\left(T^{n}\right)}=\left(\int_{T} \cdots \int_{T}|\tilde{f}(\mathbf{t})|^{2} d \mathbf{t}\right)^{1 / 2} \\
&=\left(\int_{T} \cdots \int_{T}\left|\frac{1}{n!} \sum_{\sigma} f\left(t_{\sigma(1)}, \cdots, t_{\sigma(n)}\right)\right|^{2} d \mathbf{t}\right)^{1 / 2} \\
& \leq \frac{1}{n!} \sum_{\sigma}\left(\int_{T} \cdots \int_{T}\left|f\left(t_{\sigma(1)}, \cdots, t_{\sigma(n)}\right)\right|^{2} d \mathbf{t}\right)^{1 / 2} \\
&= \frac{1}{n!} \sum_{\sigma}\|f\|_{L^{2}\left(T^{n}\right)}=\|f\|_{L^{2}\left(T^{n}\right)}
\end{aligned}
$$

Therefore,

$$
\begin{equation*}
E\left(\left(I_{n}(f-g)\right)^{2}\right)=n!\|\tilde{f}-\tilde{g}\|_{L^{2}\left(T^{n}\right)}^{2} \leq n!\|f-g\|_{L^{2}\left(T^{n}\right)}^{2} \tag{68.3.20}
\end{equation*}
$$

The following theorem comes right away from this and Lemma 68.3.11.
Theorem 68.3.12 The integral $I_{n}$ defined on $\mathscr{E}_{n}$ extends uniquely to an integral $I_{n}$ defined on $L^{2}\left(T^{n}\right)$. This integral satisfies

$$
I_{n}(f) \in L^{2}(\Omega)
$$

Also

$$
E\left(I_{n}(f) I_{n}(g)\right)=n!(\tilde{f}, \tilde{g})_{L^{2}\left(T^{n}\right)}
$$

Proof: This follows right away from the density of $\mathscr{E}_{n}$ in $L^{2}\left(T^{n}\right)$ and the inequality 68.3.20.

Obviously one wonders whether linear combinations $\sum_{n} c_{n} I_{n}\left(f_{n}\right)$ are dense in $L^{2}(\Omega)$. It looks like the important thing to notice is that for $f \in \mathscr{E}_{n}, I_{n}(f)$ is a polynomial in $W\left(A_{i_{k}}\right) \equiv$ $W\left(\mathscr{X}_{A_{i_{k}}}\right)$. Recall the corollary above, Corollary 68.2.4,

Corollary 68.3.13 Let $\mathscr{P}_{n}^{0}$ denote all polynomials of the form

$$
p\left(W\left(h_{1}\right), \cdots, W\left(h_{k}\right)\right), \text { degree of } p \leq n, \text { some } h_{1}, \cdots, h_{k}
$$

Also let $\mathscr{P}_{n}$ denote the closure in $L^{2}(\Omega, \mathscr{F}, P)$ of $\mathscr{P}_{n}^{0}$. Then

$$
\mathscr{P}_{n}=\oplus_{i=0}^{n} \mathscr{H}_{i}
$$

Consider $\cup_{p \leq n}\left\{I_{p}(f): f \in \mathscr{E}_{p}\right\}$. This is a subset of $\mathscr{P}_{n}^{0}$ and so it is a subset of $\oplus_{i=0}^{n} \mathscr{H}_{i}$. Now for $h \in L^{2}\left(T^{n}\right)$, it was shown above that there exists a sequence $g_{k} \rightarrow h$ in $L^{2}\left(T^{n}\right)$ where each $h_{k} \in \mathscr{E}_{n}$. Then $I_{n}\left(g_{k}\right) \rightarrow I_{n}(h)$. In particular, if $h \in L^{2}(T) \equiv H$, then there is a sequence $g_{k} \in \mathscr{E}_{1}$ such that $g_{k} \rightarrow h$ in $L^{2}(T)$. Then clearly

$$
E\left(\left|W\left(g_{k}\right)-W(h)\right|^{2}\right)=E\left(\left|W\left(g_{k}-h\right)\right|^{2}\right)=\left\|g_{k}-h\right\|_{L^{2}(T)}^{2} \rightarrow 0
$$

and so each polynomial $p\left(W\left(h_{1}\right), \cdots, W\left(h_{k}\right)\right)$ can be approximated in $L^{2}(\Omega)$ by one which is of the form $p\left(W\left(g_{1}\right), \cdots, W\left(g_{k}\right)\right)$ where each $g_{j} \in \mathscr{E}_{1}$. Corresponding to each $g_{j}$ there is a list of disjoint sets. Now consider the union of all the sets just described and let $\left\{A_{k}\right\}$ be a partition of this union such that the $A_{k}$ are pairwise disjoint and for each $j$, every set corresponding to $g_{j}$ is partitioned by a subset of the $\left\{A_{k}\right\}$. Thus

$$
g_{j}=\sum_{i} c_{i} \mathscr{X}_{B_{i}}=\sum_{i} c_{i} \sum_{s=1}^{m_{j}} \mathscr{X}_{A_{s}^{i}}
$$

where $B_{i}$ is partitioned by the $A_{s}^{i}$. Then consider $p\left(g_{1}, \cdots, g_{k}\right)$. Then the terms of degree $m$ are of the form

$$
\begin{equation*}
p_{m} \equiv \sum_{\mathbf{i}} c_{\mathbf{i}} \mathscr{X}_{A_{i_{1} \times \cdots \times A_{i_{m}}}} \tag{68.3.21}
\end{equation*}
$$

where the $A_{i_{k}}$ come from the list of disjoint sets $\left\{A_{k}\right\}$. The terms of degree $m$ in

$$
p\left(W\left(g_{1}\right), \cdots, W\left(g_{k}\right)\right)
$$

are also of the form

$$
p_{m}\left(W\left(g_{1}\right), \cdots, W\left(g_{k}\right)\right) \equiv \sum_{\mathbf{i}} c_{\mathbf{i}} \prod_{k} W\left(A_{i_{k}}\right)
$$

The problem is that 68.3 .21 is not in $\mathscr{E}_{m}$ because it is not known whether $c_{\mathbf{i}}=0$ if two indices are repeated. However, as explained in the proof of Lemma 68.3.11 there is a further partition such that the contribution of those terms corresponding to $\mathbf{i}$ in which two indices are repeated can be made as small as desired. Therefore, the terms of order $m$ are approximated in $L^{2}\left(T^{m}\right)$ by $g_{m} \in \mathscr{E}_{m}$. Assume this approximation is good enough that, from the estimates given above in Lemma 68.3.9,

$$
E\left(\left|I_{m}\left(g_{m}\right)-p_{m}\left(W\left(g_{1}\right), \cdots, W\left(g_{k}\right)\right)\right|^{2}\right)^{1 / 2}<\frac{\varepsilon}{n+1}
$$

Thus, taking a succession of partitions if necessary,

$$
\begin{aligned}
& E\left(\left|p\left(W\left(g_{1}\right), \cdots, W\left(g_{k}\right)\right)-\sum_{m=0}^{n} I_{m}\left(g_{m}\right)\right|^{2}\right)^{1 / 2} \\
\leq & \sum_{m=1}^{n} E\left(\left|I_{m}\left(g_{m}\right)-p_{m}\left(W\left(g_{1}\right), \cdots, W\left(g_{k}\right)\right)\right|^{2}\right)^{1 / 2}<\sum_{m=1}^{n} \frac{\varepsilon}{n+1}<\varepsilon .
\end{aligned}
$$

This has proved the following lemma.

Lemma 68.3.14 Let $n$ be given. Then $\cup_{p \leq n}\left\{I_{p}(f): f \in \mathscr{E}_{p}\right\}$ is dense in $\mathscr{P}_{n}=\oplus_{i=0}^{n} \mathscr{H}_{i}$. Consequently, every $f \in L^{2}(\Omega, \mathscr{F})$ may be written as an infinite sum

$$
f=\sum_{k=1}^{\infty} c_{k} I_{k}\left(g_{k}\right)
$$

where $g_{k} \in \mathscr{E}_{k}$ and it can also be assumed that $g_{k}$ is symmetric.
Proof: It only remains to verify that $g_{k}$ can be symmetric. However, this is obvious because if $g_{k}$ is replaced with $\tilde{g}_{k}$ the integral $I_{k}$ is unchanged.

### 68.4 The Skorokhod Integral

This integral allows for one to obtain a stochastic integral of functions which are not adapted. It is a generalization of the Ito integral. There is also a strange sort of derivative which can be defined and the two are related in a natural way.

### 68.4.1 The Derivative

Let $F: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be smooth and have polynomial growth. Then consider

$$
F\left(W\left(h_{1}\right), \cdots, W\left(h_{n}\right)\right)
$$

where $W$ is defined above. Recall that $h \in H$ a separable real Hilbert space and $W(h) \in$ $L^{2}(\Omega, \mathscr{F}, P)$ where $\mathscr{F}=\sigma(W(h), h \in H)$. Also $\left(W\left(h_{1}\right), \cdots, W\left(h_{n}\right)\right)$ is multivariate normal and $E(W(g) W(h))=(h, g)_{H}$.

Definition 68.4.1 In the above situation,

$$
D F \equiv \sum_{k=1}^{n} D_{k} F\left(W\left(h_{1}\right), \cdots, W\left(h_{n}\right)\right) h_{k}
$$

Thus from Lemma 64.6.4, $F, D_{k} F$ are in $L^{p}(\Omega)$ and so $D F$ is in $L^{p}(\Omega ; H)$ for every $p$.
First it is good to consider whether $D F$ is well defined.
Lemma 68.4.2 The derivative is well defined. Also, if $F\left(W\left(h_{1}\right), \cdots, W\left(h_{n}\right)\right)=0$ for $\left\{h_{1}, \cdots, h_{n}\right\}$ independent, then for all $\mathbf{x}, F(\mathbf{x})=0$.

Proof: Suppose

$$
F\left(W\left(h_{1}\right), \cdots, W\left(h_{n}\right)\right)=0
$$

Is it true that $D F=0$ ? Let $\lambda$ be the distribution measure of $\left(W\left(h_{1}\right), \cdots, W\left(h_{n}\right)\right) \equiv \mathbf{W}(\mathbf{h})$. Then the above requires that for any ball $B$ in $\mathbb{R}^{n}$,

$$
E\left(\mathscr{X}_{B}(\mathbf{W}(\mathbf{h})) F^{2}(\mathbf{W}(\mathbf{h}))\right)=\int_{B} F^{2}(\mathbf{x}) d \lambda(\mathbf{x})=0
$$

If $\left\{h_{1}, \cdots, h_{n}\right\}$ is independent, then $\lambda$ has a normal density function and $\lambda \ll m_{n}$ and so $F^{2}(\mathbf{x})=0$ for a.e. $\mathbf{x}$. Since $F$ is smooth, this means that $F=0$ everywhere. Hence $D_{k} F=0$ and so $D F=0$. Thus the case where the $h_{i}$ are independent is easy.

Next suppose without loss of generality that a basis for

$$
\operatorname{span}\left(h_{1}, \cdots, h_{n}\right)
$$

is

$$
\left\{h_{1}, \cdots, h_{r}\right\}
$$

where $r<n$. Say $h_{k}=\sum_{i=1}^{r} c_{i}^{k} h_{i}$ for $k>r$. Then

$$
\begin{aligned}
0 & =F\left(W\left(h_{1}\right), \cdots W\left(h_{r}\right), W\left(\sum_{i=1}^{r} c_{i}^{r+1} h_{i}\right) \cdots W\left(\sum_{i=1}^{r} c_{i}^{n} h_{i}\right)\right) \\
& =F\left(W\left(h_{1}\right), \cdots W\left(h_{r}\right), \sum_{j=1}^{r} c_{j}^{r+1} W\left(h_{j}\right) \cdots \sum_{j=1}^{r} c_{j}^{n} W\left(h_{j}\right)\right) \\
& \equiv G\left(W\left(h_{1}\right), \cdots W\left(h_{r}\right)\right)
\end{aligned}
$$

and so in terms of $\left\{h_{1}, \cdots, h_{r}\right\}$,

$$
\begin{aligned}
D F & =\sum_{i=1}^{r}\left(D_{i} F\right) h_{i}+\sum_{i=r+1}^{n}\left(D_{i} F\right) \overbrace{\sum_{j=1}^{r} c_{j}^{i} h_{j}}^{=h_{i}} \\
& =\sum_{j=1}^{r}\left(D_{j} F\right) h_{j}+\sum_{j=1}^{r} \sum_{i=r+1}^{n}\left(D_{i} F\right) c_{j}^{i} h_{j} \\
& =\sum_{j=1}^{r}\left(D_{j} F+\sum_{i=r+1}^{n}\left(D_{i} F\right) c_{j}^{i}\right) h_{j}
\end{aligned}
$$

Now it was just shown that $G(\mathbf{x})$ is identically 0 and so $D_{j} G=0, j \leq r$. So what is $D_{j} G$ ? From the above, it equals

$$
D_{j} F+\sum_{i=r+1}^{n}\left(D_{i} F\right) c_{j}^{i}=0
$$

Hence $D F=0$. Now if $F\left(W\left(h_{1}\right), \cdots, W\left(h_{n}\right)\right)=G\left(W\left(k_{1}\right), \cdots, W\left(k_{m}\right)\right)$, then $F-G=0$ and so from what was just shown, $D(F-G)=D F-D G=0$. Thus the derivative is well defined.

Lemma 68.4.3 Let $\mathscr{P}$ denote the set of all polynomials in $W(h)$ for $h \in H$. Then $\mathscr{P}$ is dense in $L^{p}(\Omega)$.

Proof: Let $g \in L^{p^{\prime}}(\Omega)$ and suppose that for every $f \in D, \int_{\Omega} g f d P=0$. Does it follow that $g=0$ ? If so, then by the Riesz representation theorem, $\mathscr{P}$ is dense in $L^{p}(\Omega)$. From

Lemma 64.6.4, for a given $h$, there is a sequence of functions of $\mathscr{P},\left\{f_{n}\right\}$ which converges to $e^{W(h)}$ in $L^{p}(\Omega)$. It follows that

$$
\int_{\Omega} g e^{W(h)} d P=\lim _{n \rightarrow \infty} \int_{\Omega} g f_{n} d P=0
$$

Hence by Lemma 64.6 .5 it follows that $g=0$. Hence $\mathscr{P}$ is dense in $L^{p}(\Omega)$.
Let $D^{1, p}$ denote the closure in $L^{p}(\Omega)$ of functions in $\mathscr{P}$ with respect to the seminorm

$$
\|f\|_{1, p} \equiv\left(\|f\|_{L^{p}(\Omega)}^{p}+\|D f\|_{L^{p}(\Omega, H)}^{p}\right)^{1 / p}
$$

By this we mean the following. The above $\|f\|_{1, p}$ makes perfect sense for every $f \in \mathscr{P}$ and is algebraically like a norm. Thus it makes $\mathscr{P}$ into a normed linear space. $D^{1, p}$ is just the completion of this normed linear space. Then for $f \in D^{1, p}$, we define $D f \equiv \lim _{n \rightarrow \infty} D f_{n}$ in $L^{p}(\Omega, H)$ where $f_{n} \in \mathscr{P}$.

### 68.4.2 The Integral

The derivative has been defined above. Now here is the definition of the integral defined on functions in $L^{p^{\prime}}(\Omega, H)$, possibly not all of them.

Definition 68.4.4 We say a random variable $F$ is "smooth" if it is of the form $F(\omega)=$ $F\left(W\left(h_{1}\right), \cdots W\left(h_{r}\right)\right)$ where $\mathbf{x} \rightarrow F(\mathbf{x})$ is a smooth function of the real variables $x_{i}$. It has polynomial growth if

$$
\frac{|F(\mathbf{x})|}{\left(1+|\mathbf{x}|^{2}\right)^{m}}
$$

is bounded for some positive integer $m$. Let $u \in L^{p^{\prime}}(\Omega, H)$. Then $u \in D(\delta)$ if for all $F$ smooth having polynomial growth in the $W(h)$,

$$
|E\langle D F, u\rangle| \leq C(u)\|F\|_{L^{p}(\Omega)}
$$

Then $\delta u \in L^{p^{\prime}}(\Omega)$ is defined by

$$
E\langle D F, u\rangle \equiv E(F \delta u)
$$

Thus you have $\delta$ is the adjoint of $D$.

$$
\begin{array}{ccc}
L^{p^{\prime}}(\Omega) & \stackrel{\delta}{\leftarrow} & D(\delta) \subseteq L^{p^{\prime}}(\Omega, H) \\
L^{p}(\Omega) \supseteq D(D) & \stackrel{D}{\longrightarrow} & L^{p}(\Omega, H)
\end{array}
$$

Next it is shown that there are functions in $D(\boldsymbol{\delta})$ by giving examples of them. It turns out that functions of the form $\sum_{i} F_{i} h_{i}$ where $F_{i}$ is smooth with polynomial growth are in $D(\delta)$. Consider

$$
E\left\langle D G, F\left(W\left(h_{1}\right), \cdots, W\left(h_{n}\right)\right) h\right\rangle
$$

where $G=G\left(W\left(k_{1}\right), \cdots, W\left(k_{p}\right)\right)$ and for simplicity, $\|h\|_{H}=1$.
Consider the vectors $\left\{h, h_{1}, \cdots, h_{n}, k_{1}, \cdots, k_{p}\right\}$. Starting with the left and moving toward the right, delete vectors which are dependent on the preceding vectors, obtaining a linearly independent set of vectors which includes $h$. Then let $\left\{h, e_{1}, \cdots, e_{q}\right\}$ be an orthonormal basis having the same span as the original vectors $\left\{h, h_{1}, \cdots, h_{n}, k_{1}, \cdots, k_{p}\right\}$. Then from the fact that $W$ is linear, there are smooth functions having polynomial growth $\hat{G}, \hat{F}$ such that

$$
\begin{aligned}
G\left(W\left(k_{1}\right), \cdots, W\left(k_{p}\right)\right) & =\hat{G}\left(W(h), W\left(e_{1}\right), \cdots, W\left(e_{q}\right)\right) \\
F\left(W\left(h_{1}\right), \cdots, W\left(h_{n}\right)\right) & =\hat{F}\left(W(h), W\left(e_{1}\right), \cdots, W\left(e_{q}\right)\right)
\end{aligned}
$$

Note that $h_{i}=\sum_{j=1}^{q}\left(h_{i}, e_{j}\right) e_{j}+\left(h_{i}, h\right) h$. Thus

$$
\begin{gathered}
F\left(W\left(h_{1}\right), \cdots, W\left(h_{n}\right)\right)= \\
=F\left(W\left(\sum_{j=1}^{q}\left(h_{1}, e_{j}\right) e_{j}+\left(h_{1}, h\right) h\right), \cdots, W\left(\sum_{j=1}^{q}\left(h_{n}, e_{j}\right) e_{j}+\left(h_{n}, h\right) h\right)\right) \\
\left.\left.=h_{1}, e_{j}\right) W\left(e_{j}\right)+\left(h_{1}, h\right) W(h), \cdots, \sum_{j=1}^{q}\left(h_{n}, e_{j}\right) W\left(e_{j}\right)+\left(h_{n}, h\right) W(h)\right)
\end{gathered}
$$

and so, $D_{1} \hat{F}$ is given by

$$
D_{1} \hat{F}=\sum_{i=1}^{n} D_{i}\left(F\left(W\left(h_{1}\right), \cdots, W\left(h_{n}\right)\right)\right)\left(h_{i}, h\right)
$$

Then by Lemma 68.4.2

$$
\begin{gathered}
E\left\langle D G, F\left(W\left(h_{1}\right), \cdots, W\left(h_{n}\right)\right) h\right\rangle=E\langle D \hat{G}, \hat{F} h\rangle \\
=E\left\langle D_{1}(\hat{G}) h+\sum_{k=1}^{q} D_{k}(\hat{G}) e_{k}, \hat{F} h\right\rangle=E\left(D_{1}(\hat{G}) \hat{F}\right) \\
=\frac{1}{(\sqrt{2 \pi})^{q+1}} \int_{\mathbb{R}^{q}} \int_{\mathbb{R}} D_{1} \hat{G}(\mathbf{x}) \hat{F}(\mathbf{x}) e^{-\frac{1}{2}|\mathbf{x}|^{2}} d x_{1} d \hat{\mathbf{x}}_{1} \\
=\frac{-1}{(\sqrt{2 \pi})^{q+1}} \int_{\mathbb{R}^{q}} \int_{\mathbb{R}} \hat{G}(\mathbf{x}) D_{1}\left(\hat{F}(\mathbf{x}) e^{-\frac{1}{2}|\mathbf{x}|^{2}}\right) d x_{1} d \hat{\mathbf{x}}_{1} \\
=\frac{-1}{(\sqrt{2 \pi})^{q+1}} \int_{\mathbb{R}^{q}} \int_{\mathbb{R}} \hat{G}(\mathbf{x})\left(\left(D_{1} \hat{F}\right)(\mathbf{x}) e^{-\frac{1}{2}|\mathbf{x}|^{2}}-\hat{F}(\mathbf{x}) x_{1} e^{-\frac{1}{2}|\mathbf{x}|^{2}}\right) d x_{1} d \hat{\mathbf{x}}_{1} \\
=E\left(\left(\hat{F} W(h)-D_{1} \hat{F}\right) \hat{G}\right)=E\left(\left(F W(h)-\sum_{i=1}^{n} D_{i}(F)\left(h_{i}, h\right)\right) G\right)
\end{gathered}
$$

Thus $F h \in D(\delta)$ and

$$
\delta(F h)=F W(h)-\sum_{i=1}^{n} D_{i}(F)\left(h_{i}, h\right)
$$

Since $\delta$ is an adjoint map, it is clearly linear. Hence, if $h$ is arbitrary, $h \neq 0$ of course,

$$
\begin{align*}
\delta(F h) & =\|h\| \delta\left(F\left(\frac{W(h)}{\|h\|}\right)\right)=\|h\| F \frac{W(h)}{\|h\|}-\|h\| \sum_{i=1}^{n} D_{i}(F)\left(h_{i}, \frac{h}{\|h\|}\right) \\
& =F W(h)-\sum_{i=1}^{n} D_{i}(F)\left(h_{i}, h\right)_{H}=F W(h)-\langle D F, h\rangle \tag{68.4.22}
\end{align*}
$$

Note how this looks just like integration by parts. More generally,

$$
\delta\left(\sum_{j=1}^{m} F_{j} h_{j}\right)=\sum_{j=1}^{m} \delta\left(F_{j} h_{j}\right)=\sum_{j=1}^{m} F_{j} W\left(h_{j}\right)-\left\langle D F_{j}, h_{j}\right\rangle
$$

Are functions like $\sum_{j=1}^{m} F_{j} h_{j}$ where $F_{j}$ is a polynomial in variables of the form $W(h)$ dense in $L^{p}(\Omega, H)$ ? It was shown earlier in Lemma 68.4.3 that polynomial functions $F$ in the $W(h)$ are dense in $L^{p}(\Omega)$ for any $p$. Let $s(\omega)=\sum_{k=1}^{n} h_{k} \mathscr{X}_{E_{k}}$ be a simple function. Then $\mathscr{X}_{E_{k}}$ is clearly in $L^{p}(\Omega)$ and so there exists $F_{k}$ a polynomial in the $W(h)$ which is as close as desired to $\mathscr{X}_{E_{k}}$ in $L^{p}$. Hence $\sum_{k=1}^{n} h_{k} F_{k}$ is close to $s$ in $L^{p}(\Omega, H)$ and so since these simple functions are dense, it follows that these kinds of functions are indeed dense in $L^{p}(\Omega, H)$, this for any $p>1$. The above discussion is summarized in the following lemma.

Lemma 68.4.5 Functions of the form $\sum_{k=1}^{n} F_{k} h_{k}$ where $F_{k}$ is a polynomial in the $W(h)$ $\left(F_{j} \in \mathscr{P}\right)$ are dense in $L^{p}(\Omega, H)$ for any $p>1$. Also each function of this form is in $D \delta$ and

$$
\delta\left(\sum_{j=1}^{m} F_{j} h_{j}\right)=\sum_{j=1}^{m} \delta\left(F_{j} h_{j}\right)=\sum_{j=1}^{m} F_{j} W\left(h_{j}\right)-\left\langle D F_{j}, h_{j}\right\rangle
$$

What does $D$ do to $\delta(F h)$ ? It is shown above that $\delta(F h)=F W(h)-\langle D F, h\rangle$. Say $F=F\left(W\left(h_{1}\right), \cdots, W\left(h_{n}\right)\right)$. Then when you do $D$ to $\delta(F h)$, you would get

$$
F h+\sum_{k=1}^{n} D_{k}(F) W(h) h_{k}-\sum_{k=1}^{n} \sum_{j=1}^{n} D_{j}\left(D_{k}(F)\right) h_{j}\left(h_{k}, h\right)
$$

In other words,

$$
F h+W(h) D(F)-D\langle D F, h\rangle
$$

Recall that $D G$ is well defined. This means that we can replace $\left\{h_{1}, \cdots, h_{n}, h\right\}$ with an orthonormal basis $\left\{e_{1}, \cdots, e_{p}, h\right\}$ as in

$$
G\left(W\left(h_{1}\right), \cdots, W\left(h_{n}\right), W(h)\right)=\hat{G}\left(W\left(e_{1}\right), \cdots, W\left(e_{p}\right), W(h)\right)
$$

where we assume $\|h\|=1$ for simplicity. Thus the above equals

$$
D(\delta(F))=D(\delta(\hat{F}))=\hat{F} h+W(h) D(\hat{F})-D\langle D \hat{F}, h\rangle
$$

Now consider $E\left(\delta(F h)^{2}\right)=E\langle D(\delta(F h)), F h\rangle$. Thus the following must be considered.

$$
\begin{equation*}
E\langle\hat{F} h+W(h) D(\hat{F})-D\langle D \hat{F}, h\rangle, \hat{F} h\rangle \tag{68.4.23}
\end{equation*}
$$

Consider the terms involved. The first term is just $E\left(\|\hat{F} h\|_{H}^{2}\right)=E\left(\|F h\|_{H}^{2}\right)$. Now consider the third term. It equals

$$
\begin{gathered}
-E\left(D\left(D_{p+1}(\hat{F})\right), \hat{F} h\right)=-E\left(D_{p+1}^{2}(\hat{F}) \hat{F}\right) \\
=\frac{-1}{(\sqrt{2 \pi})^{p+1}} \int_{\mathbb{R}^{p}} \int_{\mathbb{R}} D_{p+1}^{2}(\hat{F}(\mathbf{x})) \hat{F}(\mathbf{x}) e^{-\frac{1}{2}|\mathbf{x}|^{2}} d x_{p+1} d \hat{\mathbf{x}}_{p+1} \\
=\frac{1}{(\sqrt{2 \pi})^{p+1}} \int_{\mathbb{R}^{p}} \int_{\mathbb{R}} D_{p+1}(\hat{F}(\mathbf{x})) D_{p+1}(\hat{F}(\mathbf{x})) e^{-\frac{1}{2}|\mathbf{x}|^{2}} d x_{p+1} d \hat{\mathbf{x}}_{p+1} \\
-\frac{1}{(\sqrt{2 \pi})^{p+1}} \int_{\mathbb{R}^{p}} \int_{\mathbb{R}} D_{p+1}(\hat{F}(\mathbf{x}))\left(x_{p+1} \hat{F}(\mathbf{x})\right) e^{-\frac{1}{2}|\mathbf{x}|^{2}} d x_{p+1} d \hat{\mathbf{x}}_{p+1} \\
=E\left(\left(D_{p+1} \hat{F}\right)^{2}\right)-E\left(W(h) D_{p+1}(\hat{F}) \hat{F}\right) \\
=E\left(\left(D_{p+1} \hat{F}\right)^{2}\right)-E(W(h) D(\hat{F}), \hat{F} h)
\end{gathered}
$$

Hence 68.4.23 reduces to

$$
\begin{aligned}
E\left(\|\hat{F} h\|^{2}\right)+E\left(\left(D_{p+1} \hat{F}\right)^{2}\right) & =E\left(\|\hat{F} h\|^{2}\right)+E\left(\langle D(\hat{F}), h\rangle^{2}\right) \\
& =E\left(\|F h\|^{2}\right)+E\left(\langle D(F), h\rangle^{2}\right)
\end{aligned}
$$

This assumed that $\|h\|=1$. For arbitrary nonzero $h$,

$$
\begin{aligned}
E\left(\delta(F h)^{2}\right) & =\|h\|^{2} E\left(\delta\left(F \frac{h}{\|h\|}\right)^{2}\right) \\
& =\|h\|^{2}\left(E\left(\left\|F \frac{h}{\|h\|}\right\|^{2}\right)+E\left(\left\langle D(F), \frac{h}{\|h\|}\right\rangle^{2}\right)\right) \\
& =E\left(\|F h\|^{2}\right)+E\left(\langle D(F), h\rangle^{2}\right)
\end{aligned}
$$

Next consider a generalization, $u=\sum_{j=1}^{m} F_{j} h_{j}$ where the $\left\{h_{j}\right\}$ is an orthonormal set of vectors. Say $F_{j}=F_{j}\left(W\left(k_{1}\right), \cdots, W\left(k_{n_{j}}\right)\right)$. Let $\left\{h_{1}, \cdots, h_{m}, e_{1}, \cdots, e_{p}\right\}=\left\{g_{i}\right\}_{i=1}^{m+p}$ be an orthonormal basis for the span of all the $h_{j}$ and $k_{i}$. Thus $g_{i}=h_{i}$ for $i \leq m$. Then let

$$
F_{j}\left(W\left(k_{1}\right), \cdots, W\left(k_{n_{j}}\right)\right)=\hat{F}_{j}\left(W\left(h_{1}\right), \cdots, W\left(h_{m}\right), W\left(e_{1}\right), \cdots, W\left(e_{p}\right)\right)
$$

The computations will be done with respect to this orthonormal set because it will be simpler. Also, the above argument using the density function for the normal distribution will be used without explicitly repeating it.

It is desired to consider $E\left(\delta(u)^{2}\right)$. Recall that

$$
D(\delta(F h))=F h+W(h) D(F)-D\langle D F, h\rangle .
$$

Thus $E\left(\delta(u)^{2}\right)=$

$$
\begin{gathered}
\sum_{j, k=1}^{m} E\left(\delta\left(\hat{F}_{j} h_{j}\right) \delta\left(\hat{F}_{k} h_{k}\right)\right)=\sum_{j, k=1}^{m} E\left(\left\langle D\left(\delta\left(\hat{F}_{j} h_{j}\right)\right),\left(\hat{F}_{k} h_{k}\right)\right\rangle\right) \\
\sum_{j, k=1}^{m} E\left(\left\langle\hat{F}_{j} h_{j}+W\left(h_{j}\right) D\left(\hat{F}_{j}\right)-D\left\langle D \hat{F}_{j}, h_{j}\right\rangle,\left(\hat{F}_{k} h_{k}\right)\right\rangle\right)
\end{gathered}
$$

Separating out the first term this is

$$
\begin{align*}
= & E\left(\sum_{k=1}^{m}\left\|\hat{F}_{k}\right\|^{2}\right)+\sum_{k, k} E\left(\left\langle W\left(h_{j}\right) D\left(\hat{F}_{j}\right), \hat{F}_{k} h_{k}\right\rangle\right) \\
& -\sum_{j, k} E\left(D_{k}\left(D_{j} \hat{F}_{j}\right) F_{k}\right) \\
= & E\left(\sum_{k=1}^{m}\left\|\hat{F}_{k}\right\|^{2}\right)+\sum_{k, k} E\left(\left\langle W\left(h_{j}\right) D\left(\hat{F}_{j}\right), \hat{F}_{k} h_{k}\right\rangle\right) \\
& -\sum_{j, k} E\left(D_{k}\left(D_{j} \hat{F}_{j}\right) \hat{F}_{k}\right) \\
= & E\left(\sum_{k=1}^{m}\left\|\hat{F}_{k}\right\|^{2}\right)+\sum_{k, k} E\left(W\left(h_{j}\right) D_{k}\left(\hat{F}_{j}\right) \hat{F}_{k}\right) \\
& \quad-\sum_{j, k} E\left(D_{k}\left(D_{j} \hat{F}_{j}\right) \hat{F}_{k}\right) \tag{68.4.24}
\end{align*}
$$

By equality of mixed partial derivatives, the third term equals

$$
-\sum_{j, k} E\left(D_{j}\left(D_{k} \hat{F}_{j}\right) \hat{F}_{k}\right)=\sum_{j, k} E\left(\left(D_{k} \hat{F}_{j}\right)\left(D_{j} \hat{F}_{k}\right)\right)-\sum_{j, k} E\left(D_{k}\left(\hat{F}_{j}\right) \hat{F}_{k} W\left(h_{j}\right)\right)
$$

Therefore, 68.4.24 reduces to

$$
\begin{aligned}
E\left(\delta\left(\sum_{j=1}^{m} F_{j} h_{j}\right)^{2}\right) & =E\left(\sum_{k=1}^{m}\left\|\hat{F}_{k}\right\|_{H}^{2}\right)+\sum_{j, k=1}^{m} E\left(\left(D_{k} \hat{F}_{j}\right)\left(D_{j} \hat{F}_{k}\right)\right) \\
& =E\left(\sum_{k=1}^{m}\left\|\hat{F}_{k}\right\|_{H}^{2}\right)+\sum_{j, k=1}^{m} E\left(\left\langle D \hat{F}_{j}, h_{k}\right\rangle\left\langle D \hat{F}_{k}, h_{j}\right\rangle\right) \\
& =E\left(\sum_{k=1}^{m}\left\|F_{k}\right\|_{H}^{2}\right)+\sum_{j, k=1}^{m} E\left(\left\langle D F_{j}, h_{k}\right\rangle\left\langle D F_{k}, h_{j}\right\rangle\right)
\end{aligned}
$$

because the derivative is well defined. All of this assumes the $h_{k}$ form an orthonormal set. Suppose these are just orthogonal but nonzero. Then

$$
E\left(\delta\left(\sum_{j=1}^{m} F_{j} h_{j}\right)^{2}\right)=E\left(\left(\sum_{j=1}^{m} \delta\left(F_{j} h_{j}\right)\right)^{2}\right)=E\left(\sum_{j, k} \delta\left(F_{j} h_{j}\right) \delta\left(F_{k} h_{k}\right)\right)
$$

$$
=E\left(\sum_{j, k}\left\|h_{j}\right\|\left\|h_{k}\right\| \delta\left(F_{j} h_{j} /\left\|h_{j}\right\|\right) \boldsymbol{\delta}\left(F_{k} h_{k} /\left\|h_{k}\right\|\right)\right)
$$

and doing exactly the same steps but keeping the factor $\left\|h_{j}\right\|\left\|h_{k}\right\|$ throughout, this yields

$$
\begin{aligned}
& E\left(\sum_{k=1}^{m}\left\|h_{k}\right\|^{2}\left\|F_{k}\right\|_{H}^{2}\right)+\sum_{j, k=1}^{m} E\left(\left\|h_{j}\right\|\left\|h_{k}\right\|\left\langle D F_{j}, h_{k} /\left\|h_{k}\right\|\right\rangle\left\langle D F_{k}, h_{j} /\left\|h_{j}\right\|\right\rangle\right) \\
= & E\left(\sum_{k=1}^{m}\left\|F_{k} h_{k}\right\|_{H}^{2}\right)+\sum_{j, k=1}^{m} E\left(\left\langle D F_{j}, h_{k}\right\rangle\left\langle D F_{k}, h_{j}\right\rangle\right) \\
= & E\left(\left\|\sum_{k=1}^{m} F_{k} h_{k}\right\|_{H}^{2}\right)+\sum_{j, k=1}^{m} E\left(\left\langle D F_{j}, h_{k}\right\rangle\left\langle D F_{k}, h_{j}\right\rangle\right)
\end{aligned}
$$

It appears from the computations to be correct, but it does not look right. This is because the second term is not clearly nonnegative. It is the expectation of the trace of $A^{2}$ where $A$ is the matrix whose $j k^{t h}$ entry is $\left\langle D F_{j}, h_{k}\right\rangle$. One wonders whether the end result is nonnegative.

### 68.4.3 The Ito And Skorokhod Integrals

If you let $H=L^{2}(0, \infty ; U)$ where $U$ is a separable Hilbert space, and if $f \in D(\boldsymbol{\delta})$, it is very natural to ask whether $f \mathscr{X}_{(0, t)} \in D(\delta)$. This is not so. There is a counter example given in [102]. However, this is true if you change the definition of the integral such that in the definition of $\delta$, it is only necessary for

$$
|\langle D F, G\rangle| \leq C\|F\|_{L^{p}(\Omega)}
$$

where $F$ is in $\mathscr{P}$. When you see why this is so, it will be clear why it is not so for the definition given above.

Lemma 68.4.6 Suppose the definition of the Skorokhod integral $\delta$ is changed so that it is only necessary to have

$$
|\langle D F, G\rangle| \leq C\|F\|_{L^{p}(\Omega)}
$$

for all $F$ in $\mathscr{P}$. Then let $H \equiv L^{2}(0, \infty ; U)$ or $L^{2}([0, T] ; U)$ where $U$ is a separable real Hilbert space. For this modified definition of the integral, if $f \in D(\boldsymbol{\delta})$, it follows that $f \mathscr{X}_{(0, t)} \in D(\boldsymbol{\delta})$.

Proof: The case $L^{2}(0, \infty ; U)$ is considered here. The other case is similar. $\delta$ will be defined on some things in $L^{2}\left(\Omega, L^{2}(0, \infty ; U), \mathscr{F}\right)$ where, as discussed earlier,

$$
\mathscr{F}=\sigma(W(h): h \in H)
$$

Then if you have $f \in D(\boldsymbol{\delta})$ so $f \in L^{2}\left(\Omega, L^{2}(0, \infty ; U)\right)$, does it follow that $f \mathscr{X}_{[0, t]} \in D(\boldsymbol{\delta})$ also? Let $F$ be one of those polynomial functions of some $W(h)$. Assume first that $a_{0}$, the
constant term is 0 and consider

$$
E\left\langle D F, f \mathscr{X}_{[0, t]}\right\rangle \equiv E\left\langle\sum_{k} D_{k}(F) h_{k}, f \mathscr{X}_{[0, t]}\right\rangle
$$

Since $h_{k} \in H=L^{2}(0, \infty ; U)$, so is $h_{k} \mathscr{X}_{[0, t]}$. Thus the above reduces to

$$
=\sum_{k} E\left\langle D_{k}(F) h_{k}, f \mathscr{X}_{[0, t]}\right\rangle=\sum_{k} E\left(\int_{0}^{\infty} D_{k}\left(F\left(W\left(h_{1}\right), \cdots W\left(h_{n}\right)\right)\right) h_{k} \mathscr{X}_{[0, t]} f d t\right)
$$

Since $F$ is just a polynomial and $W$ is linear and $\mathscr{X}_{[0, t]}^{q}=\mathscr{X}_{[0, t]}$, this equals

$$
\sum_{k} E\left(\int_{0}^{\infty} D_{k}\left(F\left(W\left(\mathscr{X}_{[0, t]} h_{1}\right), \cdots W\left(\mathscr{X}_{[0, t]} h_{n}\right)\right)\right) h_{k} \mathscr{X}_{[0, t]} f d t\right)
$$

Let $F_{t}=F\left(W\left(\mathscr{X}_{[0, t]} h_{1}\right), \cdots W\left(\mathscr{X}_{[0, t]} h_{n}\right)\right)$ and so the above is nothing more than

$$
E\left\langle D F, f \mathscr{X}_{[0, t]}\right\rangle=E\left\langle D F_{t}, f\right\rangle
$$

and since $f \in D(\boldsymbol{\delta})$,

$$
\left|E\left\langle D F, f \mathscr{X}_{[0, t]}\right\rangle\right|=\left|E\left\langle D F_{t}, f\right\rangle\right| \leq C(f)\left\|F_{t}\right\|_{L^{2}(\Omega)}
$$

Also

$$
\begin{aligned}
\left\|F_{t}\right\|_{L^{2}(\Omega)}^{2} & =\int_{\Omega} F\left(W\left(\mathscr{X}_{[0, t]} h_{1}\right), \cdots W\left(\mathscr{X}_{[0, t]} h_{n}\right)\right)^{2} d P \\
& =\int_{\Omega} \mathscr{X}_{[0, t]} F\left(W\left(h_{1}\right), \cdots W\left(h_{n}\right)\right)^{2} d P \\
& \leq \int_{\Omega} F\left(W\left(h_{1}\right), \cdots W\left(h_{n}\right)\right)^{2} d P
\end{aligned}
$$

Thus for such $F$ which have zero constant term,

$$
\left|E\left\langle D F, f \mathscr{X}_{[0, t]}\right\rangle\right| \leq C\|F\|_{L^{2}(\Omega)}
$$

Now what if $F$ is a constant $a$ ? In this case, $D F=D a=0$

$$
\left|E\left\langle D a, f \mathscr{X}_{[0, t]}\right\rangle\right|=0 \leq\|a\|_{L^{2}(\Omega)}
$$

It follows that $\mathscr{X}_{[0, t]} f \in D(\boldsymbol{\delta})$ whenever $f$ is.
Note how it was essential in this argument to have $F$ be a polynomial or perhaps more generally an analytic function. However, in the definition of the Skorokhod integral, one must test with functions $F$ which are smooth and have polynomial growth. In particular, this would include functions which are infinitely differentiable with compact support, none of which have valid power series.

How does the Skorokhod integral relate to the Ito integral? What about elementary functions and so forth? Let $0=t_{0}<t_{1}<\cdots<t_{n}=T$. Consider

$$
\sum_{k=0}^{n-1} F_{k} \mathscr{X}_{\left(t_{k}, t_{k+1}\right)}
$$

As shown above, this is one of the things in $D(\delta)$.

$$
\begin{gathered}
\delta\left(\mathscr{X}_{(0, t)} \sum_{k=0}^{n-1} F_{k} \mathscr{X}_{\left(t_{k}, t_{k+1}\right)}\right)=\delta\left(\sum_{k=0}^{n-1} F_{k} \mathscr{X}_{\left[t_{k} \wedge t, t \wedge t_{k+1}\right]}\right) \\
=\sum_{k=0}^{n-1} F_{k} W\left(\mathscr{X}_{\left[t_{k} \wedge t, t \wedge t_{k+1}\right]}\right)-\left\langle D F_{k}, \mathscr{X}_{\left[t_{k} \wedge t, t \wedge t_{k+1}\right]}\right\rangle \\
=\sum_{k=0}^{n-1} F_{k}\left(W\left(\mathscr{X}_{\left(0, t \wedge t_{k+1}\right)}\right)-W\left(\mathscr{X}_{\left(0, t \wedge t_{k}\right)}\right)\right)-\left\langle D F_{k}, \mathscr{X}_{\left[t_{k} \wedge t, t \wedge t_{k+1}\right]}\right\rangle
\end{gathered}
$$

In terms of the Wiener process, this is of the form

$$
=\sum_{k=0}^{n-1} F_{k}\left(W\left(t \wedge t_{k+1}\right)-W\left(t \wedge t_{k}\right)\right)-\left\langle D F_{k}, \mathscr{X}_{\left[0, t \wedge t_{k+1}\right]}-\mathscr{X}_{\left[0, t \wedge t_{k}\right]}\right\rangle_{H}
$$

What if

$$
F_{k}=F_{k}\left(W\left(\mathscr{X}_{\left[0, t_{k}\right]} h_{1}\right), \cdots W\left(\mathscr{X}_{\left[0, t_{k}\right]} h_{n}\right)\right) ?
$$

Let $\mathscr{F}_{t} \equiv \sigma\left(W\left(\mathscr{X}_{[0, t]} h\right): h \in H\right)$. Then this is clearly a filtration. If $F_{k}$ is as just described, then $F_{k}$ is $\mathscr{F}_{t_{k}}$ adapted.

$$
\left\langle D F_{k}, \mathscr{X}_{\left[0, t \wedge t_{k+1}\right]}-\mathscr{X}_{\left[0, t \wedge t_{k}\right]}\right\rangle=\int_{0}^{\infty} \sum_{s} D_{s}\left(F_{k}\right) \mathscr{X}_{\left(0, t_{k}\right)} h_{s} \mathscr{X}_{\left(t \wedge t_{k}, t \wedge t_{k+1}\right)}=0
$$

because the intervals are disjoint. In this case, the troublesome term at the end vanishes and you are left with

$$
\begin{equation*}
\sum_{k=0}^{n-1} F_{k}\left(W\left(t \wedge t_{k+1}\right)-W\left(t \wedge t_{k}\right)\right) \tag{68.4.25}
\end{equation*}
$$

which is similar to the usual definition for the Ito integral.
What if $F \in L^{2}(\Omega \times[0, T])$ and is progressively measurable. Does it have a Skorokhod integral, and if so, is it the same as the Ito integral? Recall the following useful lemma. It is Lemma 65.3.1 on Page 2231.

Lemma 68.4.7 Let $\Phi:[0, T] \times \Omega \rightarrow E$, be $\mathscr{B}([0, T]) \times \mathscr{F}$ measurable and suppose

$$
\Phi \in K \equiv L^{p}([0, T] \times \Omega ; E), p \geq 1
$$

Then there exists a sequence of nested partitions, $\mathscr{P}_{k} \subseteq \mathscr{P}_{k+1}$,

$$
\mathscr{P}_{k} \equiv\left\{t_{0}^{k}, \cdots, t_{m_{k}}^{k}\right\}
$$

such that the step functions given by

$$
\begin{aligned}
\Phi_{k}^{r}(t) & \equiv \sum_{j=1}^{m_{k}} \Phi\left(t_{j}^{k}\right) \mathscr{X}_{\left[t_{j-1}^{k}, t_{j}^{k}\right)}(t) \\
\Phi_{k}^{l}(t) & \equiv \sum_{j=1}^{m_{k}} \Phi\left(t_{j-1}^{k}\right) \mathscr{X}_{\left[t_{j-1}^{k}, t_{j}^{k}\right)}(t)
\end{aligned}
$$

both converge to $\Phi$ in $K$ as $k \rightarrow \infty$ and

$$
\lim _{k \rightarrow \infty} \max \left\{\left|t_{j}^{k}-t_{j+1}^{k}\right|: j \in\left\{0, \cdots, m_{k}\right\}\right\}=0
$$

Also, each $\Phi\left(t_{j}^{k}\right), \Phi\left(t_{j-1}^{k}\right)$ is in $L^{p}(\Omega ; E)$. One can also assume that $\Phi(0)=0$. The mesh points $\left\{t_{j}^{k}\right\}_{j=0}^{m_{k}}$ can be chosen to miss a given set of measure zero. In addition to this, we can assume that

$$
\left|t_{j}^{k}-t_{j-1}^{k}\right|=2^{-n_{k}}
$$

except for the case where $j=1$ or $j=m_{n_{k}}$ when this is so, you could have $\left|t_{j}^{k}-t_{j-1}^{k}\right|<2^{-n_{k}}$.
Theorem 68.4.8 Let $F \in L^{2}(\Omega \times[0, T])$ and is progressively measurable. Then it has $a$ Skorokhod integral which coincides with the Ito integral.

Proof: From Lemma 68.4.7, there is a sequence of left step functions denoted here as $\left\{F_{k}^{l}\right\}_{k=1}^{\infty}$ which converges to $F$ in $L^{2}(\Omega \times[0, T])$ where $F_{k}^{l}\left(t_{j}^{k}\right)=F\left(t_{j}^{k}\right)$. We can take a subsequence if necessary and assume

$$
\left\|F_{k}^{l}-F\right\|_{L^{2}([0, T] \times \Omega)}<2^{-k}
$$

Here the $\left\{t_{j}^{k}\right\}$ are mesh points corresponding to the $k^{t h}$ partition described above. Thus each $F_{k}^{l}\left(t_{j}^{k}\right)$ is in $L^{2}(\Omega)$. By Lemma 68.4.3 there exists a random variable $G_{k}^{l}\left(t_{j}^{k}\right)$ which is a polynomial function of some $W(h)$ for $h \in L^{2}\left(0, t_{j}^{k}\right)$ which can approximate $F_{k}^{l}\left(t_{j}^{k}\right)$ as closely as desired in $L^{2}(\Omega)$. Then choosing these sufficiently close, it can be assumed that the step functions

$$
G_{k}^{l} \equiv \sum_{j=0}^{m_{k}-1} G_{k}^{l}\left(t_{j}^{k}\right) \mathscr{X}_{\left(t_{j}^{k}, t_{j+1}^{k}\right)}
$$

also converge in $L^{2}(\Omega \times[0, T])$ to $F$. Of course, each of these last step functions are in $D(\delta)$.

The idea is to show that $\delta\left(G_{k}^{l}\right)$ is Cauchy in $L^{2}(\Omega)$ as $k \rightarrow \infty$ and then use the fact that, since $\delta$ is an adjoint, it must be a closed operator. This will show that $F \in L^{2}(\Omega \times[0, T])$, considered as a subspace of $L^{2}\left(\Omega ; L^{2}(0, \infty, \mathbb{R})\right)$, is in $D(\delta)$ and $\delta(F)$ is equal to the above
limit. Using 68.4.25 which comes from the fact that the functions are adapted to the given filtration,

$$
\begin{aligned}
& \left\|\delta\left(\mathscr{X}_{[0, T]} G_{k}^{l}\right)-\delta\left(\mathscr{X}_{[0, T]} G_{k+1}^{l}\right)\right\|_{L^{2}(\Omega)}^{2} \\
= & E\left(\sum_{j=0}^{m_{k+1}-1}\left(G_{k}^{l}\left(t_{j+1}^{k+1}\right)-G_{k+1}^{l}\left(t_{j}^{k+1}\right)\right)\left(W\left(t_{j+1}^{k+1}\right)-W\left(t_{j}^{k+1}\right)\right)\right)^{2}
\end{aligned}
$$

Consider a mixed term. To save on space, let $\Delta_{j}=G_{k}^{l}\left(t_{j+1}^{k+1}\right)-G_{k+1}^{l}\left(t_{j}^{k+1}\right)$ and say $i<j$. Then

$$
E\left(\left(\Delta_{j}\right)\left(\Delta_{i}\right)\left(W\left(t_{j+1}^{k+1}\right)-W\left(t_{j}^{k+1}\right)\right)\left(W\left(t_{i+1}^{k+1}\right)-W\left(t_{i}^{k+1}\right)\right)\right)
$$

By independence of the increments for $W$, this is

$$
E\left(W\left(t_{j+1}^{k+1}\right)-W\left(t_{j}^{k+1}\right)\right) E\left(\left(\Delta_{j}\right)\left(\Delta_{i}\right)\left(W\left(t_{i+1}^{k+1}\right)-W\left(t_{i}^{k+1}\right)\right)\right)=0
$$

and so the above reduces to

$$
\begin{gathered}
\sum_{j=0}^{m_{k+1}^{-1}} E\left(\Delta_{j}^{2}\left(W\left(t_{j+1}^{k+1}\right)-W\left(t_{j}^{k+1}\right)\right)^{2}\right) \\
=\sum_{j=0}^{m_{k+1}^{-1}} E\left(\Delta_{j}^{2}\right) E\left(\left(W\left(t_{j+1}^{k+1}\right)-W\left(t_{j}^{k+1}\right)\right)^{2}\right) \\
=\sum_{j=0}^{m_{k+1}-1} E\left(\left(G_{k}^{l}\left(t_{j+1}^{k+1}\right)-G_{k+1}^{l}\left(t_{j}^{k+1}\right)\right)^{2}\right)\left(t_{j+1}^{k+1}-t_{j}^{k+1}\right) \\
=E\left(\int_{0}^{T}\left(G_{k}^{l}-G_{k+1}^{l}\right)^{2} d t\right) \leq 2\left(E \int_{0}^{T}\left(G_{k}^{l}-F\right)^{2} d t+E \int_{0}^{T}\left(F-G_{k+1}^{l}\right)^{2} d t\right)
\end{gathered}
$$

which is given to converge to 0 as $k \rightarrow \infty$. It follows that

$$
\mathscr{X}_{[0, T]} G_{k}^{l} \rightarrow \mathscr{X}_{[0, T]} F
$$

in $L^{2}\left(\Omega, L^{2}(0, \infty, \mathbb{R})\right)$ by construction and $\delta\left(\mathscr{X}_{[0, T]} G_{k}^{l}\right)$ is a Cauchy sequence in $L^{2}(\Omega)$. Therefore, it converges to something in $L^{2}(\Omega)$ and since $\delta$ is a closed operator, that which it converges to is $\delta(F)$.

However, by the definition of the Ito integral, $\delta\left(\mathscr{X}_{[0, T]} G_{k}^{l}\right)$ also converges to the Ito integral $\int_{0}^{T} F d W$.

It follows that the Skorokhod integral is more general than the Ito integral but it gives the Ito integral in the special case where the function is adapted. This also shows that the progressively measurable functions in $L^{2}([0, T] \times \Omega)$ are in $D(\delta)$, but as shown above, there are many other functions which are not progressively measurable but which are still in $D(\boldsymbol{\delta})$. Just consider, for example $\sum_{k=1}^{n} F h_{k}$ where $F$ is just a polynomial in $W(h)$ for $h \in L^{2}(0, \infty ; \mathbb{R})$.

## Chapter 69

## Gelfand Triples

Let $H$ be a separable real Hilbert space and let $V \subseteq H$ be a separable Banach space which is embedded continuously into $H$ and which is also dense in $H$. Then identifying $H$ and $H^{\prime}$ you can write

$$
V \subseteq H=H^{\prime} \subseteq V^{\prime}
$$

This is called a Gelfand triple. If $V$ is reflexive, you could conclude separability of $V$ from the separability of $H$. However, if $V$ is not reflexive, this might not happen. For example, you could take $V=L^{\infty}(0,1)$ and $H=L^{2}(0,1)$.

Proposition 69.0.1 Suppose $V$ is reflexive and a subset of $H$ a separable Hilbert space with the inclusion map continuous. Suppose also that $V$ is dense in $H$. Then identifying $H$ and $H^{\prime}$, it follows that $H$ is dense in $V^{\prime}$ and $V$ is separable.

Proof: If $H$ is not dense in $V^{\prime}$, then by the Hahn Banach theorem, there exists $\phi^{* *} \in V^{\prime \prime}$ such that $\phi^{* *}(H)=0$ but $\phi^{* *}\left(\phi^{*}\right) \neq 0$ for some $\phi^{*} \in V^{\prime} \backslash \bar{H}$. Since $V$ is reflexive there exists $v \in V$ such that $\phi^{* *}=J v$ for $J$ the standard mapping from $V$ to $V^{\prime \prime}$. Thus

$$
\phi^{* *}(h) \equiv\langle h, v\rangle \equiv(v, h)_{H}=0
$$

for all $h \in H$. Therefore, $v=0$ and so $J v=0=\phi^{* *}$ which contradicts $\phi^{* *}\left(\phi^{*}\right) \neq 0$. Therefore, $H$ is dense in $V^{\prime}$. Now by Theorem 21.1.16 which says separability of the dual space implies separability of the space, it follows $V$ is separable as claimed. This proves the proposition.

From now on, it is assumed $V$ and $V^{\prime}$ are both separable and that $H$ is dense in $V^{\prime}$. This is summarized in the following definition.

Definition 69.0.2 $V, H, V^{\prime}$ will be called a Gelfand triple if $V, V^{\prime}$ are separable, $V \subseteq H$ with the inclusion map continuous, $H=H^{\prime}$, and $H=H^{\prime}$ is dense in $V^{\prime}$.

What about the Borel sets on $V$ and $H$ ?
Proposition 69.0.3 Denote by $\mathscr{B}(X)$ the Borel sets of $X$ where $X$ is any separable Banach space. Then

$$
\mathscr{B}(X)=\sigma\left(X^{\prime}\right)
$$

Here $\sigma\left(X^{\prime}\right)$ is the smallest $\sigma$ algebra such that each $\phi \in X^{\prime}$ is measurable. Also in the context of the above definition, $\mathscr{B}(V)=\sigma\left(i^{*} H^{\prime}\right)$ because $H^{\prime}$ is dense in $V^{\prime}$. Here $i^{*}$ is the restriction to $V$ so that $i^{*} h(v) \equiv h(v) \equiv(h, v)_{H}$ for all $v \in V$ and $\sigma\left(i^{*} H^{\prime}\right)$ denotes the smallest $\sigma$ algebra such that $i^{*} h$ is measurable for each $h \in H^{\prime}$.

Proof: By Lemma 21.1.6 there exists a countable subset of the unit ball in $X^{\prime}$

$$
\left\{\phi_{n}\right\}_{n=1}^{\infty}=D^{\prime}
$$

such that

$$
\|v\|_{X}=\sup \left\{|\phi(v)|: \phi \in D^{\prime}\right\} .
$$

Consider a closed ball $\overline{B\left(v_{0}, r\right)}$ in $X$. This equals

$$
\left\{v \in X: \sup _{n}\left|\phi_{n}(v)-\phi_{n}\left(v_{0}\right)\right| \leq r\right\}=\cap_{n=1}^{\infty} \phi_{n}^{-1}\left(\overline{B\left(\phi_{n}\left(v_{0}\right), r\right)}\right)
$$

and this last set is in $\sigma\left(D^{\prime}\right)$. Therefore, every closed ball is in $\sigma\left(D^{\prime}\right)$ which implies every open ball is also in $\sigma\left(D^{\prime}\right)$ since open balls are the countable union of closed balls. Since $X$ is separable, it follows every open set is the countable union of balls and so every open set is in $\sigma\left(D^{\prime}\right)$. It follows $\mathscr{B}(X) \subseteq \sigma\left(D^{\prime}\right) \subseteq \sigma\left(X^{\prime}\right)$. On the other hand, every $\phi \in X^{\prime}$ is continuous and so it is Borel measurable. Hence $\sigma\left(X^{\prime}\right) \subseteq \mathscr{B}(X)$.

Now consider the last claim. From Lemma 21.1.6 and density of $H^{\prime}=H$ in $V^{\prime}$, it can be assumed $D^{\prime} \subseteq H=H^{\prime}$. Therefore, from the first part of the argument

$$
\mathscr{B}(V) \subseteq \sigma\left(D^{\prime}\right) \subseteq \sigma\left(i^{*} H^{\prime}\right)
$$

Also each $i^{*} h$ is continuous on $V$ so in fact, equality holds in the above because $\sigma\left(i^{*} H^{\prime}\right) \subseteq$ $\mathscr{B}(V)$. This proves the proposition.

Next I want to verify that $V$ is in $\mathscr{B}(H)$. This will be true if $V$ is reflexive. More generally, here is an interesting result.

Proposition 69.0.4 Let $X \subseteq Y, X$ dense in $Y$ and suppose $X, Y$ are Banach spaces and that $X$ is reflexive. Then $X \in \mathscr{B}(Y)$.

Proof: Define the functional

$$
\phi(x) \equiv\left\{\begin{array}{l}
\|x\|_{X} \text { if } x \in X \\
\infty \text { if } x \in Y \backslash X
\end{array}\right.
$$

Then $\phi$ is lower semicontinuous on $Y$. Here is why. Suppose $(x, a) \notin \operatorname{epi}(\phi)$ so that $a<$ $\phi(x)$. I need to verify this situation persists for $(x, b)$ near $(x, a)$. If this is not so, there exists $x_{n} \rightarrow x$ and $a_{n} \rightarrow a$ such that $a_{n} \geq \phi\left(x_{n}\right)$. If $\liminf _{n \rightarrow \infty} \phi\left(x_{n}\right)<\infty$, then there exists a subsequence still denoted by $n$ such that $\left\|x_{n}\right\|_{X}$ is bounded. Then by the Eberlein Smulian theorem, there exists a further subsequence such that $x_{n}$ converges weakly in $X$ to some $z$. Now since $X$ is dense in $Y$ it follows $Y^{\prime}$ can be considered a subspace of $X^{\prime}$ and so for $f \in Y^{\prime}$

$$
f\left(x_{n}\right) \rightarrow f(z), f\left(x_{n}\right) \rightarrow f(x)
$$

and so $f(z-x)=0$ for all $f \in Y^{\prime}$ which requires $z=x$. Now $x \rightarrow\|x\|_{X}$ is convex and lower semicontinuous on $X$ so it follows from Corollary 18.2.12

$$
a=\lim \inf _{n \rightarrow \infty} a_{n} \geq \lim \inf _{n \rightarrow \infty} \phi\left(x_{n}\right) \geq \phi(x)>a
$$

which is a contradiction. If $\liminf _{n \rightarrow \infty} \phi\left(x_{n}\right)=\infty$, then

$$
\infty>a=\lim \inf _{n \rightarrow \infty} a_{n}=\infty
$$

another contradiction. Therefore, epi $(\phi)$ is closed and so $\phi$ is lower semicontinuous as claimed. Therefore,

$$
X=Y \backslash\left(\cap_{n=1}^{\infty} \phi^{-1}((n, \infty))\right)
$$

and since $\phi$ is lower semicontinuous, each $\phi^{-1}((n, \infty))$ is open. Hence $X$ is a Borel subset of $Y$. This proves the proposition.

### 69.1 An Unnatural Example

Recall Gelfand triples are of the form

$$
V \subseteq H \subseteq V^{\prime}
$$

where $H$ is a Hilbert space and $V$ is a Banach space contained in $H$ and each of the above inclusions is continuous and each space is dense in the next one. The standard example of a Gelfand triple is $H_{0}^{1}(D) \subseteq L^{2}(D) \subseteq\left(H_{0}^{1}(D)\right)^{\prime}$ with the convention that $L^{2}(D)$ is identified with its dual space. Thus for $f \in L^{2}(D), f$ is considered as something in $\left(H_{0}^{1}(D)\right)^{\prime}$ according to the rule

$$
\langle f, \phi\rangle \equiv(f, \phi)_{L^{2}(D)}
$$

This is a very pleasant thing to contemplate and it is natural and transparent. However, there are other ways to come up with a Gelfand triple which are much more perverse. The following is an example of such a thing along with an application. See [108] and references given there.

First consider the following situation.

$$
X \xrightarrow{\theta} Y
$$

where $\theta$ is continuous, linear and one to one and $X$ is a Banach space. Then $\theta(X) \subseteq Y$ and you could define

$$
\|\theta x\|_{\theta(X)} \equiv\|x\|_{X} .
$$

Then $\theta(X)$ can be considered the same thing as $X$ because $\theta$ preserves distances and all algebraic properties. Thus people write $X \subseteq Y$ to save space. In the above simple example, it is obvious what $\theta$ is. This is because the things in $H_{0}^{1}$ and things in $L^{2}$ are both functions defined on $D$ and we can simply take $\theta$ to be the identity map. However, you might have $H$ be the dual space of something. Thus it consists of bounded linear transformations defined on some Banach space. Then it becomes necessary to specify the manner in which vectors in $V$ can be considered as vectors of $H$.

Let $\infty>p \geq 2$. Then letting $D$ be a bounded open set, $H_{0}^{1}(D)$ embedds continuously into $L^{p^{\prime}}(D)$. That is

$$
\begin{equation*}
\|\phi\|_{L^{p^{\prime}}} \leq C\|\phi\|_{H_{0}^{1}} \tag{69.1.1}
\end{equation*}
$$

Here $\frac{1}{p^{\prime}}+\frac{1}{p}=1$. Also note that an equivalent inner product on $H_{0}^{1}(D)$ is

$$
(f, g)_{H_{0}^{1}} \equiv \int_{D} \nabla f \cdot \nabla g d x
$$

Then with respect to this inner product, the Riesz map is given by $-\Delta$.

$$
-\Delta: H_{0}^{1}(D) \rightarrow\left(H_{0}^{1}(D)\right)^{\prime}
$$

Thus a typical vector of $\left(H_{0}^{1}(D)\right)^{\prime}$ is of the form $-\Delta \phi$ where $\phi \in H_{0}^{1}(D)$ and the following hold.

$$
(\phi, \psi)_{H_{0}^{1}} \equiv\langle-\Delta \phi, \psi\rangle,(-\Delta \phi,-\Delta \psi)_{\left(H_{0}^{1}\right)^{\prime}} \equiv(\phi, \psi)_{H_{0}^{1}}=\langle-\Delta \psi, \phi\rangle
$$

The following is about the Gelfand triple

$$
V=L^{p}(D) \subseteq\left(H_{0}^{1}\right)^{\prime} \subseteq\left(L^{p}(D)\right)^{\prime}
$$

Lemma 69.1.1 It is possible to consider $L^{p}(D) \equiv V$ as a dense subspace of $\left(H_{0}^{1}\right)^{\prime} \equiv H$ as follows. For $f \in L^{p}(D)$ and $\phi \in H_{0}^{1}(D)$,

$$
\langle f, \phi\rangle \equiv \int_{D} f(x) \phi(x) d x
$$

One can also consider $H \equiv\left(H_{0}^{1}\right)^{\prime}$ as a dense subspace of $\left(L^{p}(D)\right)^{\prime} \equiv V^{\prime}$ as follows. For $-\Delta \phi \in H$ and $f \in L^{p}(D)$,

$$
\langle-\Delta \phi, f\rangle \equiv(-\Delta \phi, f)_{H} \equiv\langle f, \phi\rangle
$$

$-\Delta$ maps $H_{0}^{1}(D)$ to $H \equiv\left(H_{0}^{1}\right)^{\prime} \subseteq V^{\prime} .-\Delta$ can be extended to yield a map $-\Delta_{1}$ from $L^{p^{\prime}}(D)$ to $V^{\prime}$.

$$
\begin{aligned}
& H_{0}^{1}(D) \xrightarrow{-\Delta}\left(H_{0}^{1}\right)^{\prime} \\
& L^{p^{\prime}}(D)=V \xrightarrow{-\Delta_{1}} V^{\prime}
\end{aligned}
$$

Proof: First of all, note that by 69.1.1

$$
|\langle f, \phi\rangle| \leq\|f\|_{L^{p}}\|\phi\|_{L^{p^{\prime}}} \leq C\|f\|_{L^{p}}\|\phi\|_{H_{0}^{1}}
$$

and so it is certainly possible to consider $L^{p} \subseteq H \equiv\left(H_{0}^{1}\right)^{\prime}$ as just claimed. Now why can $L^{p}(D)$ be considered dense in $H \equiv\left(H_{0}^{1}\right)^{\prime}$ ? If it isn't dense, then there exists $\psi \in$ $H_{0}^{1}(D), \psi \neq 0$ such that

$$
(-\Delta \psi, f)_{H}=0
$$

for all $f \in L^{p}(D)$. However, the above would say that for all $f \in L^{p}$,

$$
(-\Delta \psi, f)_{H} \equiv\langle f, \psi\rangle \equiv \int_{D} f \psi=0
$$

But $\psi \in L^{p^{\prime}}(D)$ because $H_{0}^{1}(D)$ embedds continuously into $L^{p^{\prime}}(D)$ and so the above holding for all $f \in L^{p}(D)$ implies by the usual Riesz representation theorem that $\psi=0$ contrary to the way $\psi$ was chosen.

Now consider the next claim. For $-\Delta \phi \in H \equiv\left(H_{0}^{1}\right)^{\prime}$ and $f \in L^{p}(D)$ and from the first part

$$
|\langle-\Delta \phi, f\rangle| \equiv\left|(-\Delta \phi, f)_{H}\right| \equiv|\langle f, \phi\rangle| \leq C\|f\|_{L^{p}}\|\phi\|_{H_{0}^{1}(D)}
$$

Thus $-\Delta \phi \in H$ can be considered in $\left(L^{p}(D)\right)^{\prime}$. Why should $H$ be dense in $\left(L^{p}(D)\right)^{\prime}$ ? If it is not dense, then there exists $g^{*} \in\left(L^{p}(D)\right)^{\prime}$ which is not the limit of vectors of $H$. Then
since $L^{p}(D)$ is reflexive, an application of the Hahn Banach theorem shows there exists $f \in L^{p}(D)$ such that

$$
\begin{equation*}
\left\langle g^{*}, f\right\rangle_{\left(L^{p}(D)\right)^{\prime}, L^{p}(D)} \neq 0,\langle-\Delta \phi, f\rangle_{\left(L^{p}(D)\right)^{\prime}, L^{p}(D)}=0 \tag{69.1.2}
\end{equation*}
$$

for all $-\Delta \phi \in H$. However, it was just shown $H$ could be considered a subset of $\left(L^{p}(D)\right)^{\prime}$ in the manner described above. Therefore, the last equation in the above is of the form

$$
0=(-\Delta \phi, f)_{H}=\langle f, \phi\rangle=\int_{D} f \phi d x
$$

and since this holds for all $\phi \in H_{0}^{1}(D)$, it follows by density of $H_{0}^{1}(D)$ in $L^{p^{\prime}}(D)$, that $f=0$ and now this contradicts the inequality in 69.1.2.

Now $\Delta$ is defined on $H_{0}^{1}(D)$ and it delivers something in $\left(H_{0}^{1}\right)^{\prime} \equiv H$. Of course $H_{0}^{1}(D)$ is dense in $L^{p^{\prime}}(D)$. Can $\Delta$ be extended to all of $L^{p^{\prime}}(D)$ ? The answer is yes and it is more of the same given above. For $\phi \in H_{0}^{1}(D),-\Delta \phi \in H \subseteq\left(L^{p}(D)\right)^{\prime}$. Then by the above, for $\phi \in H_{0}^{1}(D)$ and $f \in L^{p}(D)$,

$$
\begin{gathered}
\langle-\Delta \phi, f\rangle \equiv\langle f, \phi\rangle \equiv \int_{D} f \phi d x \\
|\langle-\Delta \phi, f\rangle| \equiv|\langle f, \phi\rangle| \equiv\left|\int_{D} f \phi d s\right| \leq\|\phi\|_{L^{p^{\prime}(D)}}\|f\|_{L^{p}(D)}
\end{gathered}
$$

and so $-\Delta$ is a continuous linear mapping defined on a dense subspace $H_{0}^{1}(D)$ of $L^{p^{\prime}}(D)$ and so this does indeed extend to a continuous linear map defined on all of $L^{p^{\prime}}(D)$ given by the formula

$$
\langle-\Delta g, f\rangle \equiv \int_{D} f g d x
$$

This proves the lemma.
Thus letting $V \equiv L^{p}(D)$, and $H \equiv\left(H_{0}^{1}(D)\right)^{\prime}$, it follows $V \subseteq H \subseteq V^{\prime}$ is a Gelfand triple with the understanding of what it means for one space to be included in another described above. To emphasize the above, for $-\Delta \phi \in H, f \in L^{p}$,

$$
\langle-\Delta \phi, f\rangle \equiv(-\Delta \phi, f)_{H} \equiv\langle f, \phi\rangle \equiv \int_{D} f \phi d x
$$

More generally, for $g \in L^{p^{\prime}}(D),-\Delta g \in\left(L^{p}(D)\right)^{\prime}$ according to the rule

$$
\langle-\Delta g, f\rangle \equiv \int_{D} f g d x
$$

With this example of a Gelfand triple, one can define a "porous medium operator" $A: V \rightarrow V^{\prime}$. Let $\Psi$ be a real valued function defined on $\mathbb{R}$ which satisfies

$$
\begin{equation*}
\Psi \text { is continuous } \tag{69.1.3}
\end{equation*}
$$

$$
\begin{equation*}
(t-s)(\Psi(t)-\Psi(s)) \geq 0 \tag{69.1.4}
\end{equation*}
$$

There exists $p \geq 2, p<\infty$ and $\alpha \in(0, \infty)$ such that for all $s \in \mathbb{R}$

$$
\begin{equation*}
s \Psi(s) \geq \alpha|s|^{p}-c \tag{69.1.5}
\end{equation*}
$$

There exist $c_{3}, c_{4} \in(0, \infty)$ such that for all $s \in \mathbb{R}$

$$
\begin{equation*}
|\Psi(s)| \leq c_{4}+c_{3}|s|^{p-1} \tag{69.1.6}
\end{equation*}
$$

Note that 69.1.6 implies that if $v \in L^{p}(D)$, Then

$$
\int_{D}|\Psi(v)|^{p^{\prime}} d x \leq C \int_{D}\left(1+|v|^{p^{\prime}(p-1)}\right) d x=C \int_{D}\left(1+|v|^{p}\right) d x<\infty .
$$

Thus for $v \in L^{p}(D), \Psi(v)$ is something you can do $\Delta$ to and obtain something in $V^{\prime}$. The porous medium operator $A: V \rightarrow V^{\prime}$ is given as follows.

$$
\langle A v, w\rangle_{V^{\prime}, V} \equiv\langle\Delta \Psi(v), w\rangle_{V^{\prime}, V} \equiv-\int_{D} \Psi(v) w d x
$$

What are the properties of $A$ ?

$$
\langle A(u+\lambda v), w\rangle \equiv-\int_{D} \Psi(u+\lambda v) w d x
$$

and this is easily seen to be a continuous function of $\lambda$ Thus $A$ is Hemicontinuous.

$$
\langle A(u)-A(v), u-v\rangle \equiv-\int_{D} \Psi(u)(u-v) d x+\int_{D} \Psi(v)(u-v) d x \leq 0
$$

Thus $-A$ is monotone. Also there is a coercivity estimate which is routine.

$$
\langle A(v), v\rangle \equiv-\int_{D} \Psi(v) v \leq \int_{D} c-\alpha|v|^{p} d x=C-\alpha\|v\|_{V}^{p}
$$

This operator also has a boundedness estimate.

$$
\begin{aligned}
& \|A(v)\|_{V^{\prime}} \equiv \sup _{\|w\|_{V} \leq 1}|\langle A(v), w\rangle| \equiv \sup _{\|w\|_{V} \leq 1}\left|\int_{D} \Psi(v) w\right| \\
& \leq \sup _{\|w\|_{V} \leq 1}\left(\int_{D}\left(c_{4}+c_{3}|v|^{p-1}\right) w d x\right) \\
& \leq\left(\int_{D} C\left(1+|v|^{p}\right) d x\right)^{1 / p^{\prime}} \leq C+C\left(\int_{D}|v|^{p} d x\right)^{1 / p^{\prime}} \\
& =C+C\|v\|_{V}^{p / p^{\prime}}=C+C\|v\|_{V}^{p-1} .
\end{aligned}
$$

Since $\Psi$ is continuous, it will also follow that $A$ is $\mathscr{B}(V)$ measurable. Consider

$$
u \rightarrow\langle A u, w\rangle \equiv-\int_{D} \Psi(u) w d x
$$

for fixed $w \in V$. Suppose $u_{n} \rightarrow u$ in $V$ and fix $w \in L^{\infty}(D) \subseteq V$. Then it follows from an easy argument using the Vitali convergence theorem and the fact that from the estimates above

$$
\Psi\left(u_{n}\right) w
$$

is uniformly integrable that

$$
u \rightarrow-\int_{D} \Psi(u) w d x
$$

is continuous. For general $w \in L^{p}(D)$, let $w_{n} \rightarrow w$ in $L^{p}(D)$ where each $w_{n}$ is in $L^{\infty}(D)$. Then the function

$$
\begin{equation*}
u \rightarrow-\int_{D} \Psi(u) w d x \equiv\langle A u, w\rangle \tag{69.1.7}
\end{equation*}
$$

is the limit of the continuous functions

$$
u \rightarrow-\int_{D} \Psi(u) w_{n} d x
$$

and so the function 69.1.7 is Borel measurable. Now by the Pettis theorem this shows $A: V \rightarrow V^{\prime}$ is $\mathscr{B}(V)$ measurable. This shows $A$ is an example of an operator which satisfies some conditions which will be considered later.

### 69.2 Standard Techniques In Evolution Equations

In this section, several significant theorems are presented. Unless indicated otherwise, the measure will be Lebesgue measure. First here is a lemma.

Lemma 69.2.1 Suppose $g \in L^{1}([a, b] ; X)$ where $X$ is a Banach space. If $\int_{a}^{b} g(t) \phi(t) d t=0$ for all $\phi \in C_{c}^{\infty}(a, b)$, then $g(t)=0$ a.e.

Proof: Let $E$ be a measurable subset of $(a, b)$ and let $K \subseteq E \subseteq V \subseteq(a, b)$ where $K$ is compact, $V$ is open and $m(V \backslash K)<\varepsilon$. Let $K \prec h \prec V$ as in the proof of the Riesz representation theorem for positive linear functionals. Enlarging $K$ slightly and convolving with a mollifier, it can be assumed $h \in C_{c}^{\infty}(a, b)$. Then

$$
\begin{aligned}
\left|\int_{a}^{b} \mathscr{X}_{E}(t) g(t) d t\right| & =\left|\int_{a}^{b}\left(\mathscr{X}_{E}(t)-h(t)\right) g(t) d t\right| \\
& \leq \int_{a}^{b}\left|\mathscr{X}_{E}(t)-h(t)\right|\|g(t)\| d t \\
& \leq \int_{V \backslash K}\|g(t)\| d t
\end{aligned}
$$

Now let $K_{n} \subseteq E \subseteq V_{n}$ with $m\left(V_{n} \backslash K_{n}\right)<2^{-n}$. Then from the above,

$$
\left|\int_{a}^{b} \mathscr{X}_{E}(t) g(t) d t\right| \leq \int_{a}^{b} \mathscr{X}_{V_{n} \backslash K_{n}}(t)\|g(t)\| d t
$$

and the integrand of the last integral converges to 0 a.e. as $n \rightarrow \infty$ because $\sum_{n} m\left(V_{n} \backslash K_{n}\right)<$ $\infty$. By the dominated convergence theorem, this last integral converges to 0 . Therefore, whenever $E \subseteq(a, b)$,

$$
\int_{a}^{b} \mathscr{X}_{E}(t) g(t) d t=0
$$

Since the endpoints have measure zero, it also follows that for any measurable $E$, the above equation holds.

Now $g \in L^{1}([a, b] ; X)$ and so it is measurable. Therefore, $g([a, b])$ is separable. Let $D$ be a countable dense subset and let $E$ denote the set of linear combinations of the form $\sum_{i} a_{i} d_{i}$ where $a_{i}$ is a rational point of $\mathbb{F}$ and $d_{i} \in D$. Thus $E$ is countable. Denote by $Y$ the closure of $E$ in $X$. Thus $Y$ is a separable closed subspace of $X$ which contains all the values of $g$.

Now let $S_{n} \equiv g^{-1}\left(B\left(y_{n},\left\|y_{n}\right\| / 2\right)\right)$ where $E=\left\{y_{n}\right\}_{n=1}^{\infty}$. Thus, $\cup_{n} S_{n}=g^{-1}(X \backslash\{0\})$. This follows because if $x \in Y$ and $x \neq 0$, then in $B\left(x, \frac{\|x\|}{4}\right)$ there is a point of $E, y_{n}$. Therefore, $\left\|y_{n}\right\|>\frac{3}{4}\|x\|$ and so $\frac{\left\|y_{n}\right\|}{2}>\frac{3\|x\|}{8}>\frac{\|x\|}{4}$ so $x \in B\left(y_{n},\left\|y_{n}\right\| / 2\right)$. It follows that if each $S_{n}$ has measure zero, then $g(t)=0$ for a.e. $t$. Suppose then that for some $n$, the set, $S_{n}$ has positive measure. Then from what was shown above,

$$
\begin{aligned}
\left\|y_{n}\right\| & =\left\|\frac{1}{m\left(S_{n}\right)} \int_{S_{n}} g(t) d t-y_{n}\right\|=\left\|\frac{1}{m\left(S_{n}\right)} \int_{S_{n}} g(t)-y_{n} d t\right\| \\
& \leq \frac{1}{m\left(S_{n}\right)} \int_{S_{n}}\left\|g(t)-y_{n}\right\| d t \leq \frac{1}{m\left(S_{n}\right)} \int_{S_{n}}\left\|y_{n}\right\| / 2 d t=\left\|y_{n}\right\| / 2
\end{aligned}
$$

and so $y_{n}=0$ which implies $S_{n}=\emptyset$, a contradiction to $m\left(S_{n}\right)>0$. This contradiction shows each $S_{n}$ has measure zero and so as just explained, $g(t)=0$ a.e.

Definition 69.2.2 For $f \in L^{1}(a, b ; X)$, define an extension, $\bar{f}$ defined on

$$
[2 a-b, 2 b-a]=[a-(b-a), b+(b-a)]
$$

as follows.

$$
\bar{f}(t) \equiv\left\{\begin{array}{l}
f(t) \text { if } t \in[a, b] \\
f(2 a-t) \text { if } t \in[2 a-b, a] \\
f(2 b-t) \text { if } t \in[b, 2 b-a]
\end{array}\right.
$$

Definition 69.2.3 Also if $f \in L^{p}(a, b ; X)$ and $h>0$, define for $t \in[a, b], f_{h}(t) \equiv \bar{f}(t-h)$ for all $h<b-a$. Thus the map $f \rightarrow f_{h}$ is continuous and linear on $L^{p}(a, b ; X)$. It is continuous because

$$
\begin{aligned}
\int_{a}^{b}\left\|f_{h}(t)\right\|^{p} d t & =\int_{a}^{a+h}\|f(2 a-t+h)\|^{p} d t+\int_{a}^{b-h}\|f(t)\|^{p} d t \\
& =\int_{a}^{a+h}\|f(t)\|^{p} d t+\int_{a}^{b-h}\|f(t)\|^{p} d t \leq 2\|f\|_{p}^{p}
\end{aligned}
$$

The following lemma is on continuity of translation in $L^{p}(a, b ; X)$.

Lemma 69.2.4 Let $\bar{f}$ be as defined in Definition 69.2.2. Then for $f \in L^{p}(a, b ; X)$ for $p \in$ $[1, \infty)$,

$$
\lim _{\delta \rightarrow 0} \int_{a}^{b}\|\bar{f}(t-\delta)-f(t)\|_{X}^{p} d t=0
$$

Proof: Regarding the measure space as $(a, b)$ with Lebesgue measure, by regularity of the measure, there exists $g \in C_{c}(a, b ; X)$ such that $\|f-g\|_{p}<\varepsilon$. Here the norm is the norm in $L^{p}(a, b ; X)$. Therefore,

$$
\begin{aligned}
\left\|f_{h}-f\right\|_{p} & \leq\left\|f_{h}-g_{h}\right\|_{p}+\left\|g_{h}-g\right\|_{p}+\|g-f\|_{p} \\
& \leq\left(2^{1 / p}+1\right)\|f-g\|_{p}+\left\|g_{h}-g\right\|_{p} \\
& <\left(2^{1 / p}+1\right) \varepsilon+\varepsilon
\end{aligned}
$$

whenever $h$ is sufficiently small. This is because of the uniform continuity of $g$. Therefore, since $\varepsilon>0$ is arbitrary, this proves the lemma.

Definition 69.2.5 Let $f \in L^{1}(a, b ; X)$. Then the distributional derivative in the sense of $X$ valued distributions is given by

$$
f^{\prime}(\phi) \equiv-\int_{a}^{b} f(t) \phi^{\prime}(t) d t
$$

Then $f^{\prime} \in L^{1}(a, b ; X)$ if there exists $h \in L^{1}(a, b ; X)$ such that for all $\phi \in C_{c}^{\infty}(a, b)$,

$$
f^{\prime}(\phi)=\int_{a}^{b} h(t) \phi(t) d t
$$

Then $f^{\prime}$ is defined to equal $h$. Here $f$ and $f^{\prime}$ are considered as vector valued distributions in the same way as was done for scalar valued functions.

Lemma 69.2.6 The above definition is well defined.
Proof: Suppose both $h$ and $g$ work in the definition. Then for all $\phi \in C_{c}^{\infty}(a, b)$,

$$
\int_{a}^{b}(h(t)-g(t)) \phi(t) d t=0
$$

Therefore, by Lemma 69.2.1, $h(t)-g(t)=0$ a.e.
The other thing to notice about this is the following lemma. It follows immediately from the definition.

Lemma 69.2.7 Suppose $f, f^{\prime} \in L^{1}(a, b ; X)$. Then if $[c, d] \subseteq[a, b]$, it follows that $\left(\left.f\right|_{[c, d]}\right)^{\prime}=$ $\left.f^{\prime}\right|_{[c, d]}$. This notation means the restriction to $[c, d]$.

Recall that in the case of scalar valued functions, if you had both $f$ and its weak derivative, $f^{\prime}$ in $L^{1}(a, b)$, then you were able to conclude that $f$ is almost everywhere equal to a continuous function, still denoted by $f$ and

$$
f(t)=f(a)+\int_{a}^{t} f^{\prime}(s) d s
$$

In particular, you can define $f(a)$ to be the initial value of this continuous function. It turns out that an identical theorem holds in this case. To begin with here is the same sort of lemma which was used earlier for the case of scalar valued functions. It says that if $f^{\prime}=0$ where the derivative is taken in the sense of $X$ valued distributions, then $f$ equals a constant.

Lemma 69.2.8 Suppose $f \in L^{1}(a, b ; X)$ and for all $\phi \in C_{c}^{\infty}(a, b)$,

$$
\int_{a}^{b} f(t) \phi^{\prime}(t) d t=0
$$

Then there exists a constant, $a \in X$ such that $f(t)=a$ a.e.
Proof: Let $\phi_{0} \in C_{c}^{\infty}(a, b), \int_{a}^{b} \phi_{0}(x) d x=1$ and define for $\phi \in C_{c}^{\infty}(a, b)$

$$
\psi_{\phi}(x) \equiv \int_{a}^{x}\left[\phi(t)-\left(\int_{a}^{b} \phi(y) d y\right) \phi_{0}(t)\right] d t
$$

Then $\psi_{\phi} \in C_{c}^{\infty}(a, b)$ and $\psi_{\phi}^{\prime}=\phi-\left(\int_{a}^{b} \phi(y) d y\right) \phi_{0}$. Then

$$
\begin{aligned}
\int_{a}^{b} f(t)(\phi(t)) d t & =\int_{a}^{b} f(t)\left(\psi_{\phi}^{\prime}(t)+\left(\int_{a}^{b} \phi(y) d y\right) \phi_{0}(t)\right) d t \\
& =\overbrace{\int_{a}^{b} f(t) \psi_{\phi}^{\prime}(t) d t}^{=0 \text { by assumption }}+\left(\int_{a}^{b} \phi(y) d y\right) \int_{a}^{b} f(t) \phi_{0}(t) d t \\
& =\left(\int_{a}^{b}\left(\int_{a}^{b} f(t) \phi_{0}(t) d t\right) \phi(y) d y\right) .
\end{aligned}
$$

It follows that for all $\phi \in C_{c}^{\infty}(a, b)$,

$$
\int_{a}^{b}\left(f(y)-\left(\int_{a}^{b} f(t) \phi_{0}(t) d t\right)\right) \phi(y) d y=0
$$

and so by Lemma 69.2.1,

$$
f(y)-\left(\int_{a}^{b} f(t) \phi_{0}(t) d t\right)=0 \text { a.e. } y
$$

Theorem 69.2.9 Suppose $f, f^{\prime}$ both are in $L^{1}(a, b ; X)$ where the derivative is taken in the sense of $X$ valued distributions. Then there exists a unique point of $X$, denoted by $f(a)$ such that the following formula holds a.e. $t$.

$$
f(t)=f(a)+\int_{a}^{t} f^{\prime}(s) d s
$$

## Proof:

$$
\int_{a}^{b}\left(f(t)-\int_{a}^{t} f^{\prime}(s) d s\right) \phi^{\prime}(t) d t=\int_{a}^{b} f(t) \phi^{\prime}(t) d t-\int_{a}^{b} \int_{a}^{t} f^{\prime}(s) \phi^{\prime}(t) d s d t
$$

Now consider $\int_{a}^{b} \int_{a}^{t} f^{\prime}(s) \phi^{\prime}(t) d s d t$. Let $\Lambda \in X^{\prime}$. Then it is routine from approximating $f^{\prime}$ with simple functions to verify

$$
\Lambda\left(\int_{a}^{b} \int_{a}^{t} f^{\prime}(s) \phi^{\prime}(t) d s d t\right)=\int_{a}^{b} \int_{a}^{t} \Lambda\left(f^{\prime}(s)\right) \phi^{\prime}(t) d s d t
$$

Now the ordinary Fubini theorem can be applied to obtain

$$
=\int_{a}^{b} \int_{s}^{b} \Lambda\left(f^{\prime}(s)\right) \phi^{\prime}(t) d t d s=\Lambda\left(\int_{a}^{b} \int_{s}^{b} f^{\prime}(s) \phi^{\prime}(t) d t d s\right)
$$

Since $X^{\prime}$ separates the points of $X$, it follows

$$
\int_{a}^{b} \int_{a}^{t} f^{\prime}(s) \phi^{\prime}(t) d s d t=\int_{a}^{b} \int_{s}^{b} f^{\prime}(s) \phi^{\prime}(t) d t d s
$$

Therefore,

$$
\begin{aligned}
& \int_{a}^{b}\left(f(t)-\int_{a}^{t} f^{\prime}(s) d s\right) \phi^{\prime}(t) d t \\
= & \int_{a}^{b} f(t) \phi^{\prime}(t) d t-\int_{a}^{b} \int_{s}^{b} f^{\prime}(s) \phi^{\prime}(t) d t d s \\
= & \int_{a}^{b} f(t) \phi^{\prime}(t) d t-\int_{a}^{b} f^{\prime}(s) \int_{s}^{b} \phi^{\prime}(t) d t d s \\
= & \int_{a}^{b} f(t) \phi^{\prime}(t) d t+\int_{a}^{b} f^{\prime}(s) \phi(s) d s=0 .
\end{aligned}
$$

Therefore, by Lemma 69.2.8, there exists a constant, denoted as $f(a)$ such that

$$
f(t)-\int_{a}^{t} f^{\prime}(s) d s=f(a)
$$

The integration by parts formula is also important.

Corollary 69.2.10 Suppose $f, f^{\prime} \in L^{1}(a, b ; X)$ and suppose $\phi \in C^{1}([a, b])$. Then the following integration by parts formula holds.

$$
\int_{a}^{b} f(t) \phi^{\prime}(t) d t=f(b) \phi(b)-f(a) \phi(a)-\int_{a}^{b} f^{\prime}(t) \phi(t) d t
$$

Proof: From Theorem 69.2.9

$$
\begin{aligned}
& \int_{a}^{b} f(t) \phi^{\prime}(t) d t \\
= & \int_{a}^{b}\left(f(a)+\int_{a}^{t} f^{\prime}(s) d s\right) \phi^{\prime}(t) d t \\
= & f(a)(\phi(b)-\phi(a))+\int_{a}^{b} \int_{a}^{t} f^{\prime}(s) d s \phi^{\prime}(t) d t \\
= & f(a)(\phi(b)-\phi(a))+\int_{a}^{b} f^{\prime}(s) \int_{s}^{b} \phi^{\prime}(t) d t d s \\
= & f(a)(\phi(b)-\phi(a))+\int_{a}^{b} f^{\prime}(s)(\phi(b)-\phi(s)) d s \\
= & f(a)(\phi(b)-\phi(a))-\int_{a}^{b} f^{\prime}(s) \phi(s) d s+(f(b)-f(a)) \phi(b) \\
= & f(b) \phi(b)-f(a) \phi(a)-\int_{a}^{b} f^{\prime}(s) \phi(s) d s .
\end{aligned}
$$

The interchange in order of integration is justified as in the proof of Theorem 69.2.9.
With this integration by parts formula, the following interesting lemma is obtained. This lemma shows why it was appropriate to define $\bar{f}$ as in Definition 69.2.2.

Lemma 69.2.11 Let $\bar{f}$ be given in Definition 69.2 .2 and suppose $f, f^{\prime} \in L^{1}(a, b ; X)$. Then $\bar{f}, \bar{f}^{\prime} \in L^{1}(2 a-b, 2 b-a ; X)$ also and

$$
\bar{f}^{\prime}(t) \equiv\left\{\begin{array}{l}
f^{\prime}(t) \text { if } t \in[a, b]  \tag{69.2.8}\\
-f^{\prime}(2 a-t) \text { if } t \in[2 a-b, a] \\
-f^{\prime}(2 b-t) \text { if } t \in[b, 2 b-a]
\end{array}\right.
$$

Proof: It is clear from the definition of $\bar{f}$ that $\bar{f} \in L^{1}(2 a-b, 2 b-a ; X)$ and that in fact

$$
\begin{equation*}
\|\bar{f}\|_{L^{1}(2 a-b, 2 b-a ; X)} \leq 3\|f\|_{L^{1}(a, b ; X)} \tag{69.2.9}
\end{equation*}
$$

Let $\phi \in C_{c}^{\infty}(2 a-b, 2 b-a)$. Then from the integration by parts formula,

$$
\begin{aligned}
& \int_{2 a-b}^{2 b-a} \bar{f}(t) \phi^{\prime}(t) d t \\
= & \int_{a}^{b} f(t) \phi^{\prime}(t) d t+\int_{b}^{2 b-a} f(2 b-t) \phi^{\prime}(t) d t+\int_{2 a-b}^{a} f(2 a-t) \phi^{\prime}(t) d t \\
= & \int_{a}^{b} f(t) \phi^{\prime}(t) d t+\int_{a}^{b} f(u) \phi^{\prime}(2 b-u) d u+\int_{a}^{b} f(u) \phi^{\prime}(2 a-u) d u \\
= & f(b) \phi(b)-f(a) \phi(a)-\int_{a}^{b} f^{\prime}(t) \phi(t) d t-f(b) \phi(b)+f(a) \phi(2 b-a) \\
& +\int_{a}^{b} f^{\prime}(u) \phi(2 b-u) d u-f(b) \phi(2 a-b) \\
& +f(a) \phi(a)+\int_{a}^{b} f^{\prime}(u) \phi(2 a-u) d u \\
= & -\int_{a}^{b} f^{\prime}(t) \phi(t) d t+\int_{a}^{b} f^{\prime}(u) \phi(2 b-u) d u+\int_{a}^{b} f^{\prime}(u) \phi(2 a-u) d u \\
= & -\int_{a}^{b} f^{\prime}(t) \phi(t) d t-\int_{b}^{2 b-a}-f^{\prime}(2 b-t) \phi(t) d t-\int_{2 a-b}^{a}-f^{\prime}(2 a-t) \phi(t) d t \\
= & -\int_{2 a-b}^{2 b-a} \bar{f}^{\prime}(t) \phi(t) d t
\end{aligned}
$$

where $\bar{f}^{\prime}(t)$ is given in 69.2.8.
Definition 69.2.12 Let $V$ be a Banach space and let $H$ be a Hilbert space. (Typically $H=L^{2}(\Omega)$ ) Suppose $V \subseteq H$ is dense in $H$ meaning that the closure in $H$ of $V$ gives $H$. Then it is often the case that $H$ is identified with its dual space, and then because of the density of $V$ in $H$, it is possible to write

$$
V \subseteq H=H^{\prime} \subseteq V^{\prime}
$$

When this is done, $H$ is called a pivot space. Another notation which is often used is $\langle f, g\rangle$ to denote $f(g)$ for $f \in V^{\prime}$ and $g \in V$. This may also be written as $\langle f, g\rangle_{V^{\prime}, V}$. Another term is that $V \subseteq H=H^{\prime} \subseteq V^{\prime}$ is called a Gelfand triple.

The next theorem is an example of a trace theorem. In this theorem, $f \in L^{p}(0, T ; V)$ while $f^{\prime} \in L^{p}\left(0, T ; V^{\prime}\right)$. It makes no sense to consider the initial values of $f$ in $V$ because it is not even continuous with values in $V$. However, because of the derivative of $f$ it will turn out that $f$ is continuous with values in a larger space and so it makes sense to consider initial values of $f$ in this other space. This other space is called a trace space.

Theorem 69.2.13 Let $V$ and $H$ be a Banach space and Hilbert space as described in Definition 69.2.12. Suppose $f \in L^{p}(0, T ; V)$ and $f^{\prime} \in L^{p^{\prime}}\left(0, T ; V^{\prime}\right)$. Then $f$ is a.e. equal to $a$ continuous function mapping $[0, T]$ to $H$. Furthermore, there exists $f(0) \in H$ such that

$$
\begin{equation*}
\frac{1}{2}|f(t)|_{H}^{2}-\frac{1}{2}|f(0)|_{H}^{2}=\int_{0}^{t}\left\langle f^{\prime}(s), f(s)\right\rangle d s \tag{69.2.10}
\end{equation*}
$$

and for all $t \in[0, T]$,

$$
\begin{equation*}
\int_{0}^{t} f^{\prime}(s) d s \in H \tag{69.2.11}
\end{equation*}
$$

and for a.e. $t \in[0, T]$,

$$
\begin{equation*}
f(t)=f(0)+\int_{0}^{t} f^{\prime}(s) d s \text { in } H \tag{69.2.12}
\end{equation*}
$$

Here $f^{\prime}$ is being taken in the sense of $V^{\prime}$ valued distributions and $\frac{1}{p}+\frac{1}{p^{\prime}}=1$ and $p \geq 2$.
Proof: Let $\Psi \in C_{c}^{\infty}(-T, 2 T)$ satisfy $\Psi(t)=1$ if $t \in[-T / 2,3 T / 2]$ and $\Psi(t) \geq 0$. For $t \in \mathbb{R}$, define

$$
\widehat{f}(t) \equiv\left\{\begin{array}{l}
\bar{f}(t) \Psi(t) \text { if } t \in[-T, 2 T] \\
0 \text { if } t \notin[-T, 2 T]
\end{array}\right.
$$

and

$$
\begin{equation*}
f_{n}(t) \equiv \int_{-1 / n}^{1 / n} \widehat{f}(t-s) \phi_{n}(s) d s \tag{69.2.13}
\end{equation*}
$$

where $\phi_{n}$ is a mollifier having support in $(-1 / n, 1 / n)$. Then by Minkowski's inequality

$$
\begin{aligned}
\| f_{n}- & \widehat{f} \|_{L^{p}(\mathbb{R} ; V)}=\left(\int_{\mathbb{R}}\left\|\widehat{f}(t)-\int_{-1 / n}^{1 / n} \widehat{f}(t-s) \phi_{n}(s) d s\right\|_{V}^{p} d t\right)^{1 / p} \\
& =\left(\int_{\mathbb{R}}\left\|\int_{-1 / n}^{1 / n}(\widehat{f}(t)-\widehat{f}(t-s)) \phi_{n}(s) d s\right\|_{V}^{p} d t\right)^{1 / p} \\
& \leq\left(\int_{\mathbb{R}}\left(\int_{-1 / n}^{1 / n}\|\widehat{f}(t)-\widehat{f}(t-s)\|_{V} \phi_{n}(s) d s\right)^{p} d t\right)^{1 / p} \\
& \leq \int_{-1 / n}^{1 / n} \phi_{n}(s)\left(\int_{\mathbb{R}}\|\widehat{f}(t)-\widehat{f}(t-s)\|_{V}^{p} d t\right)^{1 / p} d s \\
& \leq \int_{-1 / n}^{1 / n} \phi_{n}(s) \varepsilon d s=\varepsilon
\end{aligned}
$$

provided $n$ is large enough. This follows from continuity of translation in $L^{p}$ with Lebesgue measure. Since $\varepsilon>0$ is arbitrary, it follows $f_{n} \rightarrow \widehat{f}$ in $L^{p}(\mathbb{R} ; V)$. Similarly, $f_{n} \rightarrow f$ in $L^{2}(\mathbb{R} ; H)$. This follows because $p \geq 2$ and the norm in $V$ and norm in $H$ are related by $|x|_{H} \leq C\|x\|_{V}$ for some constant, $C$. Now

$$
\widehat{f}(t)=\left\{\begin{array}{l}
\Psi(t) f(t) \text { if } t \in[0, T] \\
\Psi(t) f(2 T-t) \text { if } t \in[T, 2 T] \\
\Psi(t) f(-t) \text { if } t \in[0, T] \\
0 \text { if } t \notin[-T, 2 T]
\end{array}\right.
$$

An easy modification of the argument of Lemma 69.2.11 yields

$$
\widehat{f}^{\prime}(t)=\left\{\begin{array}{l}
\Psi^{\prime}(t) f(t)+\Psi(y) f^{\prime}(t) \text { if } t \in[0, T] \\
\Psi^{\prime}(t) f(2 T-t)-\Psi(t) f^{\prime}(2 T-t) \text { if } t \in[T, 2 T] \\
\Psi^{\prime}(t) f(-t)-\Psi(t) f^{\prime}(-t) \text { if } t \in[-T, 0] \\
0 \text { if } t \notin[-T, 2 T]
\end{array}\right.
$$

Recall

$$
\begin{aligned}
f_{n}(t) & =\int_{-1 / n}^{1 / n} \widehat{f}(t-s) \phi_{n}(s) d s=\int_{\mathbb{R}} \widehat{f}(t-s) \phi_{n}(s) d s \\
& =\int_{\mathbb{R}} \widehat{f}(s) \phi_{n}(t-s) d s
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f_{n}^{\prime}(t) & =\int_{\mathbb{R}} \widehat{f}(s) \phi_{n}^{\prime}(t-s) d s=\int_{-T-\frac{1}{n}}^{2 T+\frac{1}{n}} \widehat{f}(s) \phi_{n}^{\prime}(t-s) d s \\
& =\int_{-T-\frac{1}{n}}^{2 T+\frac{1}{n}} \widehat{f}^{\prime}(s) \phi_{n}(t-s) d s=\int_{\mathbb{R}} \widehat{f}^{\prime}(s) \phi_{n}(t-s) d s \\
& =\int_{\mathbb{R}} \widehat{f}^{\prime}(t-s) \phi_{n}(s) d s=\int_{-1 / n}^{1 / n} \widehat{f}^{\prime}(t-s) \phi_{n}(s) d s
\end{aligned}
$$

and it follows from the first line above that $f_{n}^{\prime}$ is continuous with values in $V$ for all $t \in \mathbb{R}$. Also note that both $f_{n}^{\prime}$ and $f_{n}$ equal zero if $t \notin[-T, 2 T]$ whenever $n$ is large enough. Exactly similar reasoning to the above shows that $f_{n}^{\prime} \rightarrow \widehat{f}^{\prime}$ in $L^{p^{\prime}}\left(\mathbb{R} ; V^{\prime}\right)$.

Now let $\phi \in C_{c}^{\infty}(0, T)$.

$$
\begin{align*}
\int_{\mathbb{R}}\left|f_{n}(t)\right|_{H}^{2} \phi^{\prime}(t) d t & =\int_{\mathbb{R}}\left(f_{n}(t), f_{n}(t)\right)_{H} \phi^{\prime}(t) d t  \tag{69.2.14}\\
=-\int_{\mathbb{R}} 2\left(f_{n}^{\prime}(t), f_{n}(t)\right) \phi(t) d t & =-\int_{\mathbb{R}} 2\left\langle f_{n}^{\prime}(t), f_{n}(t)\right\rangle \phi(t) d t
\end{align*}
$$

Now

$$
\begin{aligned}
& \left|\int_{\mathbb{R}}\left\langle f_{n}^{\prime}(t), f_{n}(t)\right\rangle \phi(t) d t-\int_{\mathbb{R}}\left\langle f^{\prime}(t), f(t)\right\rangle \phi(t) d t\right| \\
\leq & \int_{\mathbb{R}}\left(\left|\left\langle f_{n}^{\prime}(t)-f^{\prime}(t), f_{n}(t)\right\rangle\right|+\left|\left\langle f^{\prime}(t), f_{n}(t)-f(t)\right\rangle\right|\right) \phi(t) d t
\end{aligned}
$$

From the first part of this proof which showed that $f_{n} \rightarrow \widehat{f}$ in $L^{p}(\mathbb{R} ; V)$ and $f_{n}^{\prime} \rightarrow \widehat{f}^{\prime}$ in $L^{p^{\prime}}\left(\mathbb{R} ; V^{\prime}\right)$, an application of Holder's inequality shows the above converges to 0 as $n \rightarrow \infty$. Therefore, passing to the limit as $n \rightarrow \infty$ in the 69.2.15,

$$
\int_{\mathbb{R}}|\widehat{f}(t)|_{H}^{2} \phi^{\prime}(t) d t=-\int_{\mathbb{R}} 2\left\langle\widehat{f}^{\prime}(t), \widehat{f}(t)\right\rangle \phi(t) d t
$$

which shows $t \rightarrow|\widehat{f}(t)|_{H}^{2}$ equals a continuous function a.e. and it also has a weak derivative equal to $2\left\langle\widehat{f}^{\prime}, \widehat{f}\right\rangle$.

It remains to verify that $\widehat{f}$ is continuous on $[0, T]$. Of course $\widehat{f}=f$ on this interval. Let
$N$ be large enough that $f_{n}(-T)=0$ for all $n>N$. Then for $m, n>N$ and $t \in[-T, 2 T]$

$$
\begin{aligned}
\left|f_{n}(t)-f_{m}(t)\right|_{H}^{2} & =2 \int_{-T}^{t}\left(f_{n}^{\prime}(s)-f_{m}^{\prime}(s), f_{n}(s)-f_{m}(s)\right) d s \\
& =2 \int_{-T}^{t}\left\langle f_{n}^{\prime}(s)-f_{m}^{\prime}(s), f_{n}(s)-f_{m}(s)\right\rangle_{V^{\prime}, V} d s \\
& \leq 2 \int_{\mathbb{R}}\left\|f_{n}^{\prime}(s)-f_{m}^{\prime}(s)\right\|_{V^{\prime}}\left\|f_{n}(s)-f_{m}(s)\right\|_{V} d s \\
& \leq 2\left\|f_{n}-f_{m}\right\|_{L^{p^{\prime}\left(\mathbb{R} ; V^{\prime}\right)}}\left\|f_{n}-f_{m}\right\|_{L^{p}(\mathbb{R} ; V)}
\end{aligned}
$$

which shows from the above that $\left\{f_{n}\right\}$ is uniformly Cauchy on $[-T, 2 T]$ with values in $H$. Therefore, there exists $g$ a continuous function defined on $[-T, 2 T]$ having values in $H$ such that

$$
\lim _{n \rightarrow \infty} \max \left\{\left|f_{n}(t)-g(t)\right|_{H} ; t \in[-T, 2 T]\right\}=0
$$

However, $g=\widehat{f}$ a.e. because $f_{n}$ converges to $f$ in $L^{p}(0, T ; V)$. Therefore, taking a subsequence, the convergence is a.e. It follows from the fact that $V \subseteq H=H^{\prime} \subseteq V^{\prime}$ and Theorem 69.2.9, there exists $f(0) \in V^{\prime}$ such that for a.e. $t$,

$$
f(t)=f(0)+\int_{0}^{t} f^{\prime}(s) d s \text { in } V^{\prime}
$$

Now $g=f$ a.e. and $g$ is continuous with values in $H$ hence continuous with values in $V^{\prime}$ and so

$$
g(t)=f(0)+\int_{0}^{t} f^{\prime}(s) d s \text { in } V^{\prime}
$$

for all $t$. Since $g$ is continuous with values in $H$ it is continuous with values in $V^{\prime}$. Taking the limit as $t \downarrow 0$ in the above, $g(a)=\lim _{t \rightarrow 0+} g(t)=f(0)$, showing that $f(0) \in H$. Therefore, for a.e. $t$,

$$
f(t)=f(0)+\int_{0}^{t} f^{\prime}(s) d s \text { in } H, \int_{0}^{t} f^{\prime}(s) d s \in H
$$

Note that if $f \in L^{p}(0, T ; V)$ and $f^{\prime} \in L^{p^{\prime}}\left(0, T ; V^{\prime}\right)$, then you can consider the initial value of $f$ and it will be in $H$. What if you start with something in $H$ ? Is it an initial condition for a function $f \in L^{p}(0, T ; V)$ such that $f^{\prime} \in L^{p^{\prime}}\left(0, T ; V^{\prime}\right)$ ? This is worth thinking about. If it is not so, what is the space of initial values? How can you give this space a norm? What are its properties? It turns out that if $V$ is a closed subspace of the Sobolev space, $W^{1, p}(\Omega)$ which contains $W_{0}^{1, p}(\Omega)$ for $p \geq 2$ and $H=L^{2}(\Omega)$ the answer to the above question is yes. Not surprisingly, there are many generalizations of the above ideas.

### 69.3 An Important Formula

It is not necessary to have $p>2$ in order to do the sort of thing just described. Here is a major result which will have a much more difficult stochastic version presented later. First is an approximation theorem of Doob. See Lemma 65.3.1.

Lemma 69.3.1 Let $Y:[0, T] \rightarrow E$, be $\mathscr{B}([0, T])$ measurable and suppose

$$
Y \in L^{p}(0, T ; E) \equiv K, p \geq 1
$$

Then there exists a sequence of nested partitions, $\mathscr{P}_{k} \subseteq \mathscr{P}_{k+1}$,

$$
\mathscr{P}_{k} \equiv\left\{t_{0}^{k}, \cdots, t_{m_{k}}^{k}\right\}
$$

such that the step functions given by

$$
\begin{aligned}
Y_{k}^{r}(t) & \equiv \sum_{j=1}^{m_{k}} Y\left(t_{j}^{k}\right) \mathscr{X}_{\left[t_{j-1}^{k}, t_{j}^{k}\right)}(t) \\
Y_{k}^{l}(t) & \equiv \sum_{j=1}^{m_{k}} Y\left(t_{j-1}^{k}\right) \mathscr{X}_{\left(t_{j-1}^{k}, t_{j}^{k}\right]}(t)
\end{aligned}
$$

both converge to $Y$ in $K$ as $k \rightarrow \infty$ and

$$
\lim _{k \rightarrow \infty} \max \left\{\left|t_{j}^{k}-t_{j+1}^{k}\right|: j \in\left\{0, \cdots, m_{k}\right\}\right\}=0
$$

Also, each $Y\left(t_{j}^{k}\right), Y\left(t_{j-1}^{k}\right)$ is in $E$. One can also assume that $Y(0)=0$. The mesh points $\left\{t_{j}^{k}\right\}_{j=0}^{m_{k}}$ can be chosen to miss a given set of measure zero. In addition to this, we can assume that

$$
\left|t_{j}^{k}-t_{j-1}^{k}\right|=2^{-n_{k}}
$$

except for the case where $j=1$ or $j=m_{n_{k}}$ when this might not be so. In the case of the last subinterval defined by the partition, we can assume

$$
\left|t_{m}^{k}-t_{m-1}^{k}\right|=\left|T-t_{m-1}^{k}\right| \geq 2^{-\left(n_{k}+1\right)}
$$

Theorem 69.3.2 Let $V \subseteq H=H^{\prime} \subseteq V^{\prime}$ be a Gelfand triple and suppose $Y \in L^{p^{\prime}}\left(0, T ; V^{\prime}\right) \equiv$ $K^{\prime}$ and

$$
\begin{equation*}
X(t)=X_{0}+\int_{0}^{t} Y(s) d s \text { in } V^{\prime} \tag{69.3.15}
\end{equation*}
$$

where $X_{0} \in H$, and it is known that $X \in L^{p}(0, T, V) \equiv K$ for $p>1$. Then $t \rightarrow X(t)$ is in $C([0, T], H)$ and also

$$
\frac{1}{2}|X(t)|_{H}^{2}=\frac{1}{2}\left|X_{0}\right|_{H}^{2}+\int_{0}^{t}\langle Y(s), X(s)\rangle d s
$$

Proof: By Lemma 65.3.1, there exists a sequence of uniform partitions $\left\{t_{k}^{n}\right\}_{k=0}^{m_{n}}=$ $\mathscr{P}_{n}, \mathscr{P}_{n} \subseteq \mathscr{P}_{n+1}$, of $[0, T]$ such that the step functions

$$
\begin{aligned}
\sum_{k=0}^{m_{n}-1} X\left(t_{k}^{n}\right) \mathscr{X}_{\left(t_{k}^{n}, t_{k+1}^{n}\right]}(t) & \equiv X^{l}(t) \\
\sum_{k=0}^{m_{n}-1} X\left(t_{k+1}^{n}\right) \mathscr{X}_{\left(t_{k}^{n}, t_{k+1}^{n}\right]}(t) & \equiv X^{r}(t)
\end{aligned}
$$

converge to $X$ in $K$ and in $L^{2}([0, T], H)$.

Lemma 69.3.3 Let $s<t$. Then for $X, Y$ satisfying 69.3.15

$$
\begin{equation*}
|X(t)|^{2}=|X(s)|^{2}+2 \int_{s}^{t}\langle Y(u), X(t)\rangle d u-|X(t)-X(s)|^{2} \tag{69.3.16}
\end{equation*}
$$

Proof: It follows from the following computations

$$
\begin{gathered}
X(t)-X(s)=\int_{s}^{t} Y(u) d u \\
-|X(t)-X(s)|^{2}=-|X(t)|^{2}+2(X(t), X(s))-|X(s)|^{2} \\
=-|X(t)|^{2}+2\left(X(t), X(t)-\int_{s}^{t} Y(u) d u\right)-|X(s)|^{2} \\
=-|X(t)|^{2}+2|X(t)|^{2}-2\left\langle\int_{s}^{t} Y(u) d u, X(t)\right\rangle-|X(s)|^{2}
\end{gathered}
$$

Hence

$$
|X(t)|^{2}=|X(s)|^{2}+2 \int_{s}^{t}\langle Y(u), X(t)\rangle d u-|X(t)-X(s)|^{2}
$$

Lemma 69.3.4 In the above situation,

$$
\sup _{t \in[0, T]}|X(t)|_{H} \leq C\left(\|Y\|_{K^{\prime}},\|X\|_{K}\right)
$$

Also, $t \rightarrow X(t)$ is weakly continuous with values in $H$.
Proof: From the above formula applied to the $k^{t h}$ partition of $[0, T]$ described above,

$$
\begin{aligned}
& \left|X\left(t_{m}\right)\right|^{2}-\left|X_{0}\right|^{2}=\sum_{j=0}^{m-1}\left|X\left(t_{j+1}\right)\right|^{2}-\left|X\left(t_{j}\right)\right|^{2} \\
= & \sum_{j=0}^{m-1} 2 \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X\left(t_{j+1}\right)\right\rangle d u-\left|X\left(t_{j+1}\right)-X\left(t_{j}\right)\right|_{H}^{2} \\
= & \sum_{j=0}^{m-1} 2 \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u-\left|X\left(t_{j+1}\right)-X\left(t_{j}\right)\right|_{H}^{2}
\end{aligned}
$$

Thus, discarding the negative terms and denoting by $\mathscr{P}_{k}$ the $k^{t h}$ of these partitions,

$$
\begin{gathered}
\sup _{t_{j} \in \mathscr{P}_{k}}\left|X\left(t_{j}\right)\right|_{H}^{2} \leq\left|X_{0}\right|^{2}+2 \int_{0}^{T}\left|\left\langle Y(u), X_{k}^{r}(u)\right\rangle\right| d u \\
\quad \leq\left|X_{0}\right|^{2}+2 \int_{0}^{T}\|Y(u)\|_{V^{\prime}}\left\|X_{k}^{r}(u)\right\|_{V} d u
\end{gathered}
$$

$$
\leq\left|X_{0}\right|^{2}+2\left(\int_{0}^{T}\|Y(u)\|_{V^{\prime}}^{p^{\prime}} d u\right)^{1 / p^{\prime}}\left(\int_{0}^{T}\left\|X_{k}^{r}(u)\right\|_{V}^{p} d u\right)^{1 / p} \leq C\left(\|Y\|_{K^{\prime}},\|X\|_{K}\right)
$$

because these partitions are chosen such that

$$
\lim _{k \rightarrow \infty}\left(\int_{0}^{T}\left\|X_{k}^{r}(u)\right\|_{V}^{p}\right)^{1 / p}=\left(\int_{0}^{T}\|X(u)\|_{V}^{p}\right)^{1 / p}
$$

and so these are bounded. This has shown that for the dense subset of $[0, T], D \equiv \cup_{k} \mathscr{P}_{k}$,

$$
\sup _{t \in D}|X(t)|<C\left(\|Y\|_{K^{\prime}},\|X\|_{K}\right)
$$

Now let $\left\{g_{k}\right\}_{k=1}^{\infty}$ be linearly independent vectors of $V$ whose span is dense in $V$. This is possible because $V$ is separable. Then let $\left\{e_{j}\right\}_{j=1}^{\infty}$ be an orthonormal basis for $H$ such that $e_{k} \in \operatorname{span}\left(g_{1}, \ldots, g_{k}\right)$ and each $g_{k} \in \operatorname{span}\left(e_{1}, \ldots, e_{k}\right)$. This is done with the Gram Schmidt process. Then it follows that span $\left(\left\{e_{k}\right\}_{k=1}^{\infty}\right)$ is dense in $V$. I claim

$$
|y|_{H}^{2}=\sum_{j=1}^{\infty}\left|\left\langle y, e_{j}\right\rangle\right|^{2}
$$

This is certainly true if $y \in H$ because

$$
\left\langle y, e_{j}\right\rangle=\left(y, e_{j}\right)_{H}
$$

If $y \notin H$, then the series must diverge since otherwise, you could consider the infinite sum

$$
\sum_{j=1}^{\infty}\left\langle y, e_{j}\right\rangle e_{j} \in H
$$

because

$$
\left|\sum_{j=p}^{q}\left\langle y, e_{j}\right\rangle e_{j}\right|^{2}=\sum_{j=p}^{q}\left|\left\langle y, e_{j}\right\rangle\right|^{2} \rightarrow 0 \text { as } p, q \rightarrow \infty
$$

Letting $z=\sum_{j=1}^{\infty}\left\langle y, e_{j}\right\rangle e_{j}$, it follows that $\left\langle y, e_{j}\right\rangle$ is the $j^{\text {th }}$ Fourier coefficient of $z$ and that

$$
\langle z-y, v\rangle=0
$$

for all $v \in \operatorname{span}\left(\left\{e_{k}\right\}_{k=1}^{\infty}\right)$ which is dense in $V$. Therefore, $z=y$ in $V^{\prime}$ and so $y \in H$.
It follows

$$
|X(t)|^{2}=\sup _{n} \sum_{j=1}^{n}\left|\left\langle X(t), e_{j}\right\rangle\right|^{2}
$$

which is just the sup of continuous functions of $t$. Therefore, $t \rightarrow|X(t)|^{2}$ is lower semicontinuous. It follows that for any $t$, letting $t_{j} \rightarrow t$ for $t_{j} \in D$,

$$
|X(t)|^{2} \leq \lim \inf _{j \rightarrow \infty}\left|X\left(t_{j}\right)\right|^{2} \leq C\left(\|Y\|_{K^{\prime}},\|X\|_{K}\right)
$$

This proves the first claim of the lemma.
Consider now the claim that $t \rightarrow X(t)$ is weakly continuous. Letting $v \in V$,

$$
\lim _{t \rightarrow s}(X(t), v)=\lim _{t \rightarrow s}\langle X(t), v\rangle=\langle X(s), v\rangle=(X(s), v)
$$

Since it was shown that $|X(t)|$ is bounded independent of $t$, and since $V$ is dense in $H$, the claim follows.

Now

$$
\begin{aligned}
-\sum_{j=0}^{m-1}\left|X\left(t_{j+1}\right)-X\left(t_{j}\right)\right|_{H}^{2} & =\left|X\left(t_{m}\right)\right|^{2}-\left|X_{0}\right|^{2}-\sum_{j=0}^{m-1} 2 \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u \\
& =\left|X\left(t_{m}\right)\right|^{2}-\left|X_{0}\right|^{2}-2 \int_{0}^{t_{m}}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u
\end{aligned}
$$

Thus, since the partitions are nested, eventually $\left|X\left(t_{m}\right)\right|^{2}$ is constant for all $k$ large enough and the integral term converges to

$$
\int_{0}^{t_{m}}\langle Y(u), X(u)\rangle d u
$$

It follows that the term on the left does converge to something. It just remains to consider what it does converge to. However, from the equation solved by $X$,

$$
X\left(t_{j+1}\right)-X\left(t_{j}\right)=\int_{t_{j}}^{t_{j+1}} Y(u) d u
$$

Therefore, this term is dominated by an expression of the form

$$
\begin{gathered}
\sum_{j=0}^{m_{k}-1}\left(\int_{t_{j}}^{t_{j+1}} Y(u) d u, X\left(t_{j+1}\right)-X\left(t_{j}\right)\right) \\
=\sum_{j=0}^{m_{k}-1}\left\langle\int_{t_{j}}^{t_{j+1}} Y(u) d u, X\left(t_{j+1}\right)-X\left(t_{j}\right)\right\rangle \\
=\sum_{j=0}^{m_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X\left(t_{j+1}\right)-X\left(t_{j}\right)\right\rangle d u \\
=\sum_{j=0}^{m_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X\left(t_{j+1}\right)\right\rangle-\sum_{j=0}^{m_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X\left(t_{j}\right)\right\rangle \\
=\int_{0}^{T}\left\langle Y(u), X^{r}(u)\right\rangle d u-\int_{0}^{T}\left\langle Y(u), X^{l}(u)\right\rangle d u
\end{gathered}
$$

However, both $X^{r}$ and $X^{l}$ converge to $X$ in $K=L^{p}(0, T, V)$. Therefore, this term must converge to 0 . Passing to a limit, it follows that for all $t \in D$, the desired formula holds. Thus, for such $t$,

$$
|X(t)|^{2}=\left|X_{0}\right|^{2}+2 \int_{0}^{t}\langle Y(u), X(u)\rangle d u
$$

It remains to verify that this holds for all $t$. Let $t \notin D$ and let $t(k) \in \mathscr{P}_{k}$ be the largest point of $\mathscr{P}_{k}$ which is less than $t$. Suppose $t(m) \leq t(k)$ so that $m \leq k$. Then

$$
X(t(m))=X_{0}+\int_{0}^{t(m)} Y(s) d s
$$

a similar formula for $X(t(k))$. Thus for $t>t(m)$,

$$
X(t)-X(t(m))=\int_{t(m)}^{t} Y(s) d s
$$

which is the same sort of thing already looked at except that it starts at $t(m)$ rather than at 0 and $X_{0}=0$. Therefore,

$$
|X(t(k))-X(t(m))|^{2}=2 \int_{t(m)}^{t(k)}\langle Y(s), X(s)-X(t(m))\rangle d s
$$

Thus, for $m \leq k$

$$
\lim _{m, k \rightarrow \infty}|X(t(k))-X(t(m))|^{2}=0
$$

Hence $\{X(t(k))\}_{k=1}^{\infty}$ is a convergent sequence in $H$. Does it converge to $X(t)$ ? Let $\xi(t) \in H$ be what it does converge to. Let $v \in V$. Then

$$
(\xi(t), v)=\lim _{k \rightarrow \infty}(X(t(k)), v)=\lim _{k \rightarrow \infty}\langle X(t(k)), v\rangle=\langle X(t), v\rangle=(X(t), v)
$$

because it is known that $t \rightarrow X(t)$ is continuous into $V^{\prime}$ and it is also known that $X(t) \in H$ and that the $X(t)$ for $t \in[0, T]$ are uniformly bounded. Therefore, since $V$ is dense in $H$, it follows that $\xi(t)=X(t)$.

Now for every $t \in D$, it was shown above that

$$
|X(t)|^{2}=\left|X_{0}\right|^{2}+2 \int_{0}^{t}\langle Y(s), X(s)\rangle d s
$$

Thus, using what was just shown, if $t \notin D$ and $t_{k} \rightarrow t$,

$$
\begin{aligned}
|X(t)|^{2} & =\lim _{k \rightarrow \infty}\left|X\left(t_{k}\right)\right|^{2}=\lim _{k \rightarrow \infty}\left(\left|X_{0}\right|^{2}+2 \int_{0}^{t_{k}}\langle Y(s), X(s)\rangle d s\right) \\
& =\left|X_{0}\right|^{2}+2 \int_{0}^{t}\langle Y(s), X(s)\rangle d s
\end{aligned}
$$

which proves the desired formula. From this it follows right away that $t \rightarrow X(t)$ is continuous into $H$ because it was just shown that $t \rightarrow|X(t)|$ is continuous and $t \rightarrow X(t)$ is weakly continuous. Since Hilbert space is uniformly convex, this implies the $t \rightarrow X(t)$ is continuous. To see this in the special cas of Hilbert space,

$$
|X(t)-X(s)|^{2}=|X(t)|^{2}-2(X(s), X(t))+|X(s)|^{2}
$$

Then $\lim _{t \rightarrow s}\left(|X(t)|^{2}-2(X(s), X(t))+|X(s)|^{2}\right)=0$ by weak convergence of $X(t)$ to $X(s)$ and the convergence of $|X(t)|^{2}$ to $|X(s)|^{2}$.

### 69.4 The Implicit Case

The above theorem can be generalized to the case where the formula is of the form

$$
B X(t)=B X_{0}+\int_{0}^{t} Y(s) d s
$$

This involves an operator $B \in \mathscr{L}\left(W, W^{\prime}\right)$ and $B$ satisfies

$$
\langle B x, x\rangle \geq 0,\langle B x, y\rangle=\langle B y, x\rangle
$$

for

$$
V \subseteq W, W^{\prime} \subseteq V^{\prime}
$$

Where $V$ is dense in the Hilbert space $W$. Before giving the theorem, here is a technical lemma.

Lemma 69.4.1 Suppose $V, W$ are separable Banach spaces, $W$ also a Hilbert space such that $V$ is dense in $W$ and $B \in \mathscr{L}\left(W, W^{\prime}\right)$ satisfies

$$
\langle B x, x\rangle \geq 0,\langle B x, y\rangle=\langle B y, x\rangle, B \neq 0 .
$$

Then there exists a countable set $\left\{e_{i}\right\}$ of vectors in $V$ such that

$$
\left\langle B e_{i}, e_{j}\right\rangle=\delta_{i j}
$$

and for each $x \in W$,

$$
\langle B x, x\rangle=\sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2},
$$

and also

$$
B x=\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i}
$$

the series converging in $W^{\prime}$.
Proof: Let $\left\{g_{k}\right\}_{k=1}^{\infty}$ be linearly independent vectors of $V$ whose span is dense in $V$. This is possible because $V$ is separable. Thus, their span is also dense in $W$. Let $n_{1}$ be the first index such that $\left\langle B g_{n_{1}}, g_{n_{1}}\right\rangle \neq 0$.

Claim: If there is no such index, then $B=0$.
Proof of claim: First note that if there is no such first index, then if $x=\sum_{i=1}^{k} a_{i} g_{i}$

$$
\begin{aligned}
|\langle B x, x\rangle| & =\left|\sum_{i \neq j} a_{i} a_{j}\left\langle B g_{i}, g_{j}\right\rangle\right| \leq \sum_{i \neq j}\left|a_{i}\right|\left|a_{j}\right|\left|\left\langle B g_{i}, g_{j}\right\rangle\right| \\
& \leq \sum_{i \neq j}\left|a_{i}\right|\left|a_{j}\right|\left\langle B g_{i}, g_{i}\right\rangle^{1 / 2}\left\langle B g_{j}, g_{j}\right\rangle^{1 / 2}=0
\end{aligned}
$$

Therefore, if $x$ is given, you could take $x_{k}$ in the span of $\left\{g_{1}, \cdots, g_{k}\right\}$ such that $\left\|x_{k}-x\right\|_{W} \rightarrow$ 0 . Then

$$
|\langle B x, y\rangle|=\lim _{k \rightarrow \infty}\left|\left\langle B x_{k}, y\right\rangle\right| \leq \lim _{k \rightarrow \infty}\left\langle B x_{k}, x_{k}\right\rangle^{1 / 2}\langle B y, y\rangle^{1 / 2}=0
$$

because $\left\langle B x_{k}, x_{k}\right\rangle$ is zero by what was just shown.
Thus assume there is such a first index. Let

$$
e_{1} \equiv \frac{g_{n_{1}}}{\left\langle B g_{n_{1}}, g_{n_{1}}\right\rangle^{1 / 2}}
$$

Then $\left\langle B e_{1}, e_{1}\right\rangle=1$. Now if you have constructed $e_{j}$ for $j \leq k$,

$$
e_{j} \in \operatorname{span}\left(g_{n_{1}}, \cdots, g_{n_{k}}\right),\left\langle B e_{i}, e_{j}\right\rangle=\delta_{i j}
$$

$g_{n_{j+1}}$ being the first for which

$$
\left\langle B g_{n_{j+1}}-\sum_{i=1}^{j}\left\langle B g_{n_{j+1}}, e_{i}\right\rangle B e_{i}, g_{n_{j+1}}-\sum_{i=1}^{j}\left\langle B g_{n j}, e_{i}\right\rangle e_{i}\right\rangle \neq 0,
$$

and

$$
\operatorname{span}\left(g_{n_{1}}, \cdots, g_{n_{k}}\right)=\operatorname{span}\left(e_{1}, \cdots, e_{k}\right)
$$

let $g_{n_{k+1}}$ be such that $g_{n_{k+1}}$ is the first in the list $\left\{g_{n_{k}}\right\}$ such that

$$
\left\langle B g_{n_{k+1}}-\sum_{i=1}^{k}\left\langle B g_{n_{k+1}}, e_{i}\right\rangle B e_{i}, g_{n_{k+1}}-\sum_{i=1}^{k}\left\langle B g_{n_{k+1}}, e_{i}\right\rangle e_{i}\right\rangle \neq 0
$$

Note the difference between this and the Gram Schmidt process. Here you don't necessarily use all of the $g_{k}$ due to the possible degeneracy of $B$.

Claim: If there is no such first $g_{n_{k+1}}$, then $B\left(\operatorname{span}\left(e_{i}, \cdots, e_{k}\right)\right)=B W$ so in this case, $\left\{B e_{i}\right\}_{i=1}^{k}$ is actually a basis for $B W$.

Proof: Let $x \in W$. Let $x_{r} \in \operatorname{span}\left(g_{1}, \cdots, g_{r}\right), r>n_{k}$ such that $\lim _{r \rightarrow \infty} x_{r}=x$ in $W$. Then

$$
\begin{equation*}
x_{r}=\sum_{i=1}^{k} c_{i}^{r} e_{i}+\sum_{i \notin\left\{n_{1}, \cdots, n_{k}\right\}}^{r} d_{i}^{r} g_{i} \equiv y_{r}+z_{r} \tag{69.4.17}
\end{equation*}
$$

If $l \notin\left\{n_{1}, \cdots, n_{k}\right\}$, then by the construction and the above assumption, for some $j \leq k$

$$
\begin{equation*}
\left\langle B g_{l}-\sum_{i=1}^{j}\left\langle B g_{l}, e_{i}\right\rangle B e_{i}, g_{l}-\sum_{i=1}^{j}\left\langle B g_{l}, e_{i}\right\rangle e_{i}\right\rangle=0 \tag{69.4.18}
\end{equation*}
$$

If $l<n_{k}$, this follows from the construction. If the above is nonzero all $j \leq k$, then $l$ would have been chosen but it wasn't. Thus

$$
B g_{l}=\sum_{i=1}^{j}\left\langle B g_{l}, e_{i}\right\rangle B e_{i}
$$

If $l>n_{k}$, then by assumption, 69.4.18 holds for $j=k$. Thus, in any case, it follows that for each $l \notin\left\{n_{1}, \cdots, n_{k}\right\}$,

$$
B g_{l} \in B\left(\operatorname{span}\left(e_{i}, \cdots, e_{k}\right)\right)
$$

Now it follows from 69.4.17 that

$$
\begin{aligned}
B x_{r} & =\sum_{i=1}^{k} c_{i}^{r} B e_{i}+\sum_{i \notin\left\{n_{1}, \cdots, n_{k}\right\}}^{r} d_{i}^{r} B g_{i} \\
& =\sum_{i=1}^{k} c_{i}^{r} B e_{i}+\sum_{i \notin\left\{n_{1}, \cdots, n_{k}\right\}}^{r} d_{i}^{r} \sum_{j=1}^{k} c_{j}^{i} B e_{j}
\end{aligned}
$$

and so $B x_{r} \in B\left(\operatorname{span}\left(e_{i}, \cdots, e_{k}\right)\right)$. Then

$$
B x=\lim _{r \rightarrow \infty} B x_{r}=\lim _{r \rightarrow \infty} B y_{r}
$$

where $y_{r} \in \operatorname{span}\left(e_{i}, \cdots, e_{k}\right)$. Say

$$
B x_{r}=\sum_{i=1}^{k} a_{i}^{r} B e_{i}
$$

It follows easily that $\left\langle B x_{r}, e_{j}\right\rangle=a_{j}^{r}$. (Act on $e_{j}$ by both sides and use $\left\langle B e_{i}, e_{j}\right\rangle=\delta_{i j}$.) Now since $x_{r}$ is bounded, it follows that these $a_{j}^{r}$ are also bounded. Hence, defining $y_{r} \equiv \sum_{i=1}^{k} a_{i}^{r} e_{i}$, it follows that $y_{r}$ is bounded in $\operatorname{span}\left(e_{i}, \cdots, e_{k}\right)$ and so, there exists a subsequence, still denoted by $r$ such that $y_{r} \rightarrow y \in \operatorname{span}\left(e_{i}, \cdots, e_{k}\right)$. Therefore, $B x=$ $\lim _{r \rightarrow \infty} B y_{r}=B y$. In other words, $B W=B\left(\operatorname{span}\left(e_{i}, \cdots, e_{k}\right)\right)$ as claimed. This proves the claim.

If this happens, the process being described stops. You have found what is desired which has only finitely many vectors involved.

As long as the process does not stop, let

$$
e_{k+1} \equiv \frac{g_{n_{k+1}}-\sum_{i=1}^{k}\left\langle B g_{n_{k+1}}, e_{i}\right\rangle e_{i}}{\left\langle B\left(g_{n_{k+1}}-\sum_{i=1}^{k}\left\langle B g_{n_{k+1}}, e_{i}\right\rangle e_{i}\right), g_{n_{k+1}}-\sum_{i=1}^{k}\left\langle B g_{n_{k+1}}, e_{i}\right\rangle e_{i}\right\rangle^{1 / 2}}
$$

Thus, as in the usual argument for the Gram Schmidt process, $\left\langle B e_{i}, e_{j}\right\rangle=\delta_{i j}$ for $i, j \leq k+1$. This is already known for $i, j \leq k$. Letting $l \leq k$, and using the orthogonality already shown,

$$
\begin{aligned}
\left\langle B e_{k+1}, e_{l}\right\rangle & =C\left\langle B\left(g_{n_{k+1}}-\sum_{i=1}^{k}\left\langle B g_{n_{k+1}}, e_{i}\right\rangle e_{i}\right), e_{l}\right\rangle \\
& =C\left(\left\langle B g_{k+1}, e_{l}\right\rangle-\left\langle B g_{n_{k+1}}, e_{l}\right\rangle\right)=0
\end{aligned}
$$

Consider

$$
\left\langle B g_{p}-B\left(\sum_{i=1}^{k}\left\langle B g_{p}, e_{i}\right\rangle e_{i}\right), g_{p}-\sum_{i=1}^{k}\left\langle B g_{p}, e_{i}\right\rangle e_{i}\right\rangle
$$

Either this equals 0 because $p$ is never one of the $n_{k}$ or eventually it equals 0 for some $k$ because $g_{p}=g_{n_{k}}$ for some $n_{k}$ and so, from the construction, $g_{n_{k}}=g_{p} \in \operatorname{span}\left(e_{1}, \cdots, e_{k}\right)$ and therefore,

$$
g_{p}=\sum_{j=1}^{k} a_{j} e_{j}
$$

which requires easily that

$$
B g_{p}=\sum_{i=1}^{k}\left\langle B g_{p}, e_{i}\right\rangle B e_{i}
$$

the above holding for all $k$ large enough. It follows that for any $x \in \operatorname{span}\left(\left\{g_{k}\right\}_{k=1}^{\infty}\right)$, (finite linear combination of vectors in $\left\{g_{k}\right\}_{k=1}^{\infty}$ )

$$
\begin{equation*}
B x=\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i} \tag{69.4.19}
\end{equation*}
$$

because for all $k$ large enough,

$$
B x=\sum_{i=1}^{k}\left\langle B x, e_{i}\right\rangle B e_{i}
$$

Also note that for such $x \in \operatorname{span}\left(\left\{g_{k}\right\}_{k=1}^{\infty}\right)$,

$$
\begin{aligned}
\langle B x, x\rangle & =\left\langle\sum_{i=1}^{k}\left\langle B x, e_{i}\right\rangle B e_{i}, x\right\rangle=\sum_{i=1}^{k}\left\langle B x, e_{i}\right\rangle\left\langle B x, e_{i}\right\rangle \\
& =\sum_{i=1}^{k}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}=\sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}
\end{aligned}
$$

Now for $x$ arbitrary, let $x_{k} \rightarrow x$ in $W$ where $x_{k} \in \operatorname{span}\left(\left\{g_{k}\right\}_{k=1}^{\infty}\right)$. Then by Fatou's lemma,

$$
\begin{align*}
\sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2} & \leq \lim \inf _{k \rightarrow \infty} \sum_{i=1}^{\infty}\left|\left\langle B x_{k}, e_{i}\right\rangle\right|^{2} \\
& =\lim \inf _{k \rightarrow \infty}\left\langle B x_{k}, x_{k}\right\rangle=\langle B x, x\rangle  \tag{69.4.20}\\
& \leq\|B x\|_{W^{\prime}}\|x\|_{W} \leq\|B\|\|x\|_{W}^{2}
\end{align*}
$$

Thus the series on the left converges. Then also, from the above inequality,

$$
\begin{aligned}
& \left|\left\langle\sum_{i=p}^{q}\left\langle B x, e_{i}\right\rangle B e_{i}, y\right\rangle\right| \leq \sum_{i=p}^{q}\left|\left\langle B x, e_{i}\right\rangle\right|\left|\left\langle B e_{i}, y\right\rangle\right| \\
& \leq\left(\sum_{i=p}^{q}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}\right)^{1 / 2}\left(\sum_{i=p}^{q}\left|\left\langle B y, e_{i}\right\rangle\right|^{2}\right)^{1 / 2} \\
& \leq\left(\sum_{i=p}^{q}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}\right)^{1 / 2}\left(\sum_{i=1}^{\infty}\left|\left\langle B y, e_{i}\right\rangle\right|^{2}\right)^{1 / 2}
\end{aligned}
$$

By 69.4.20,

$$
\leq\left(\sum_{i=p}^{q}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}\right)^{1 / 2}\left(\|B\|\|y\|_{W}^{2}\right)^{1 / 2} \leq\left(\sum_{i=p}^{q}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}\right)^{1 / 2}\|B\|^{1 / 2}\|y\|_{W}
$$

It follows that

$$
\begin{equation*}
\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i} \tag{69.4.21}
\end{equation*}
$$

converges in $W^{\prime}$ because it was just shown that

$$
\left\|\sum_{i=p}^{q}\left\langle B x, e_{i}\right\rangle B e_{i}\right\|_{W^{\prime}} \leq\left(\sum_{i=p}^{q}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}\right)^{1 / 2}\|B\|^{1 / 2}
$$

and it was shown above that $\sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}<\infty$, so the partial sums of the series 69.4.21 are a Cauchy sequence in $W^{\prime}$. Also, the above estimate shows that for $\|y\|=1$,

$$
\begin{aligned}
\left|\left\langle\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i}, y\right\rangle\right| & \leq\left(\sum_{i=1}^{\infty}\left|\left\langle B y, e_{i}\right\rangle\right|^{2}\right)^{1 / 2}\left(\sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}\right)^{1 / 2} \\
& \leq\left(\sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}\right)^{1 / 2}\|B\|^{1 / 2}
\end{aligned}
$$

and so

$$
\begin{equation*}
\left\|\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i}\right\|_{W^{\prime}} \leq\left(\sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}\right)^{1 / 2}\|B\|^{1 / 2} \tag{69.4.22}
\end{equation*}
$$

Now for $x$ arbitrary, let $x_{k} \in \operatorname{span}\left(\left\{g_{j}\right\}_{j=1}^{\infty}\right)$ and $x_{k} \rightarrow x$ in $W$. Then for a fixed $k$ large enough,

$$
\begin{gathered}
\left\|B x-\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i}\right\| \leq\left\|B x-B x_{k}\right\| \\
+\left\|B x_{k}-\sum_{i=1}^{\infty}\left\langle B x_{k}, e_{i}\right\rangle B e_{i}\right\|+\left\|\sum_{i=1}^{\infty}\left\langle B x_{k}, e_{i}\right\rangle B e_{i}-\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i}\right\| \\
\leq \varepsilon+\left\|\sum_{i=1}^{\infty}\left\langle B\left(x_{k}-x\right), e_{i}\right\rangle B e_{i}\right\|
\end{gathered}
$$

the term

$$
\left\|B x_{k}-\sum_{i=1}^{\infty}\left\langle B x_{k}, e_{i}\right\rangle B e_{i}\right\|
$$

equaling 0 by 69.4.19. From 69.4.22 and 69.4.20,

$$
\begin{aligned}
& \leq \varepsilon+\|B\|^{1 / 2}\left(\sum_{i=1}^{\infty}\left|\left\langle B\left(x_{k}-x\right), e_{i}\right\rangle\right|^{2}\right)^{1 / 2} \\
& \leq \varepsilon+\|B\|^{1 / 2}\left\langle B\left(x_{k}-x\right), x_{k}-x\right\rangle^{1 / 2}<2 \varepsilon
\end{aligned}
$$

whenever $k$ is large enough. Therefore,

$$
B x=\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i}
$$

in $W^{\prime}$. It follows that

$$
\langle B x, x\rangle=\lim _{k \rightarrow \infty}\left\langle\sum_{i=1}^{k}\left\langle B x, e_{i}\right\rangle B e_{i}, x\right\rangle=\lim _{k \rightarrow \infty} \sum_{i=1}^{k}\left|\left\langle B x, e_{i}\right\rangle\right|^{2} \equiv \sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}
$$

Theorem 69.4.2 Let $V \subseteq W, W^{\prime} \subseteq V^{\prime}$ be separable Banach spaces, $W$ a separable Hilbert space, and let $Y \in L^{p^{\prime}}\left(0, T ; V^{\prime}\right) \equiv K^{\prime}$ and

$$
\begin{equation*}
B X(t)=B X_{0}+\int_{0}^{t} Y(s) d s \text { in } V^{\prime} \tag{69.4.23}
\end{equation*}
$$

where $X_{0} \in W$, and it is known that $X \in L^{p}(0, T, V) \equiv K$ for $p>1$. Then $t \rightarrow B X(t)$ is in $C\left([0, T], W^{\prime}\right)$ and also

$$
\frac{1}{2}\langle B X(t), X(t)\rangle=\frac{1}{2}\left\langle B X_{0}, X_{0}\right\rangle+\int_{0}^{t}\langle Y(s), X(s)\rangle d s
$$

Proof: By Lemma 65.3.1, there exists a sequence of uniform partitions $\left\{t_{k}^{n}\right\}_{k=0}^{m_{n}}=$ $\mathscr{P}_{n}, \mathscr{P}_{n} \subseteq \mathscr{P}_{n+1}$, of $[0, T]$ such that the step functions

$$
\begin{aligned}
\sum_{k=0}^{m_{n}-1} X\left(t_{k}^{n}\right) \mathscr{X}_{\left(t_{k}^{n}, t_{k+1}^{n}\right]}(t) & \equiv X^{l}(t) \\
\sum_{k=0}^{m_{n}-1} X\left(t_{k+1}^{n}\right) \mathscr{X}_{\left(t_{k}^{n}, t_{k+1}^{n}\right]}(t) & \equiv X^{r}(t)
\end{aligned}
$$

converge to $X$ in $K$ and also $B X^{l}, B X^{r} \rightarrow B X$ in $L^{2}\left([0, T], W^{\prime}\right)$.
Lemma 69.4.3 Let $s<t$. Then for $X, Y$ satisfying 69.4.23

$$
\begin{gather*}
\langle B X(t), X(t)\rangle=\langle B X(s), X(s)\rangle \\
+2 \int_{s}^{t}\langle Y(u), X(t)\rangle d u-\langle B(X(t)-X(s)),(X(t)-X(s))\rangle \tag{69.4.24}
\end{gather*}
$$

Proof: It follows from the following computations

$$
B(X(t)-X(s))=\int_{s}^{t} Y(u) d u
$$

and so

$$
\begin{aligned}
& 2 \int_{s}^{t}\langle Y(u), X(t)\rangle d u-\langle B(X(t)-X(s)),(X(t)-X(s))\rangle \\
= & 2\langle B(X(t)-X(s)), X(t)\rangle-\langle B(X(t)-X(s)),(X(t)-X(s))\rangle
\end{aligned}
$$

$$
\begin{gathered}
=\quad 2\langle B X(t), X(t)\rangle-2\langle B X(s), X(t)\rangle-\langle B X(t), X(t)\rangle \\
+2\langle B X(s), X(t)\rangle-\langle B X(s), X(s)\rangle \\
=\langle B X(t), X(t)\rangle-\langle B X(s), X(s)\rangle
\end{gathered}
$$

Thus

$$
\begin{gathered}
\langle B X(t), X(t)\rangle-\langle B X(s), X(s)\rangle \\
=2 \int_{s}^{t}\langle Y(u), X(t)\rangle d u-\langle B(X(t)-X(s)),(X(t)-X(s))\rangle
\end{gathered}
$$

Lemma 69.4.4 In the above situation,

$$
\sup _{t \in[0, T]}\langle B X(t), X(t)\rangle \leq C\left(\|Y\|_{K^{\prime}},\|X\|_{K}\right)
$$

Also, $t \rightarrow B X(t)$ is weakly continuous with values in $W^{\prime}$.
Proof: From the above formula applied to the $k^{t h}$ partition of $[0, T]$ described above,

$$
\begin{aligned}
& \left\langle B X\left(t_{m}\right), X\left(t_{m}\right)\right\rangle-\left\langle B X_{0}, X_{0}\right\rangle=\sum_{j=0}^{m-1}\left\langle B X\left(t_{j+1}\right), X\left(t_{j+1}\right)\right\rangle-\left\langle B X\left(t_{j}\right), X\left(t_{j}\right)\right\rangle \\
& =\sum_{j=0}^{m-1} 2 \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X\left(t_{j+1}\right)\right\rangle d u-\left\langle B\left(X\left(t_{j+1}\right)-X\left(t_{j}\right)\right), X\left(t_{j+1}\right)-X\left(t_{j}\right)\right\rangle \\
& =\sum_{j=0}^{m-1} 2 \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u-\left\langle B\left(X\left(t_{j+1}\right)-X\left(t_{j}\right)\right), X\left(t_{j+1}\right)-X\left(t_{j}\right)\right\rangle
\end{aligned}
$$

Thus, discarding the negative terms and denoting by $\mathscr{P}_{k}$ the $k^{t h}$ of these partitions,

$$
\begin{gathered}
\sup _{t_{j} \in \mathscr{P}_{k}}\left\langle B X\left(t_{j}\right), X\left(t_{j}\right)\right\rangle \leq\left\langle B X_{0}, X_{0}\right\rangle+2 \int_{0}^{T}\left|\left\langle Y(u), X_{k}^{r}(u)\right\rangle\right| d u \\
\leq\left\langle B X_{0}, X_{0}\right\rangle+2 \int_{0}^{T}\|Y(u)\|_{V^{\prime}}\left\|X_{k}^{r}(u)\right\|_{V} d u \\
\leq\left\langle B X_{0}, X_{0}\right\rangle+2\left(\int_{0}^{T}\|Y(u)\|_{V^{\prime}}^{p^{\prime}} d u\right)^{1 / p^{\prime}}\left(\int_{0}^{T}\left\|X_{k}^{r}(u)\right\|_{V}^{p} d u\right)^{1 / p} \leq C\left(\|Y\|_{K^{\prime}},\|X\|_{K}\right)
\end{gathered}
$$

because these partitions are chosen such that

$$
\lim _{k \rightarrow \infty}\left(\int_{0}^{T}\left\|X_{k}^{r}(u)\right\|_{V}^{p}\right)^{1 / p}=\left(\int_{0}^{T}\|X(u)\|_{V}^{p}\right)^{1 / p}
$$

and so these are bounded. This has shown that for the dense subset of $[0, T], D \equiv \cup_{k} \mathscr{P}_{k}$,

$$
\sup _{t \in D}\langle B X(t), X(t)\rangle<C\left(\|Y\|_{K^{\prime}},\|X\|_{K}\right)
$$

From Lemma 69.4.1 above, there exists $\left\{e_{i}\right\} \subseteq V$ such that $\left\langle B e_{i}, e_{j}\right\rangle=\delta_{i j}$ and

$$
\langle B X(t), X(t)\rangle=\sum_{k=1}^{\infty}\left|\left\langle B X(t), e_{i}\right\rangle\right|^{2}=\sup _{m} \sum_{k=1}^{m}\left|\left\langle B X(t), e_{i}\right\rangle\right|^{2}
$$

Since each $e_{i} \in V$, and since $t \rightarrow B X(t)$ is continuous into $V^{\prime}$ thanks to the formula 69.4.23, it follows that $t \rightarrow \sum_{k=1}^{m}\left|\left\langle B X(t), e_{i}\right\rangle\right|$ is continuous and so $t \rightarrow\langle B X(t), X(t)\rangle$ is the sup of continuous functions. Therefore, this function of $t$ is lower semicontinuous. Since $D$ is dense in $[0, T]$, it follows that for all $t$,

$$
\langle B X(t), X(t)\rangle \leq C\left(\|Y\|_{K^{\prime}},\|X\|_{K}\right)
$$

It only remains to verify the claim about weak continuity.
Consider now the claim that $t \rightarrow B X(t)$ is weakly continuous. Letting $v \in V$,

$$
\begin{equation*}
\lim _{t \rightarrow s}\langle B X(t), v\rangle=\langle B X(s), v\rangle=\langle B X(s), v\rangle \tag{69.4.25}
\end{equation*}
$$

The limit follows from the formula 69.4.23 which implies $t \rightarrow B X(t)$ is continuous into $V^{\prime}$. Now

$$
\|B X(t)\|=\sup _{\|v\| \leq 1}|\langle B X(t), v\rangle| \leq\langle B v, v\rangle^{1 / 2}\langle B X(t), X(t)\rangle^{1 / 2}
$$

which was shown to be bounded for $t \in[0, T]$. Now let $w \in W$. Then

$$
|\langle B X(t), w\rangle-\langle B X(s), w\rangle| \leq|\langle B X(t)-B X(s), w-v\rangle|+|\langle B X(t)-B X(s), v\rangle|
$$

Then the first term is less than $\varepsilon$ if $v$ is close enough to $w$ and the second converges to 0 so 69.4.25 holds for all $v \in W$ and so this shows the weak continuity.

Now pick $t \in D$, the union of all the mesh points. Then for all $k$ large enough, $t \in \mathscr{P}_{k}$. Say $t=t_{m}$. From Lemma 69.4.3,

$$
\begin{gathered}
-\sum_{j=0}^{m-1}\left\langle B\left(X\left(t_{j+1}\right)-X\left(t_{j}\right)\right),\left(X\left(t_{j+1}\right)-X\left(t_{j}\right)\right)\right\rangle= \\
\left\langle B X\left(t_{m}\right), X\left(t_{m}\right)\right\rangle-\left\langle B X_{0}, X_{0}\right\rangle-2 \sum_{j=0}^{m-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u
\end{gathered}
$$

Thus, $\left\langle\boldsymbol{B X}\left(t_{m}\right), X\left(t_{m}\right)\right\rangle$ is constant for all $k$ large enough and the integral term converges to

$$
\int_{0}^{t_{m}}\langle Y(u), X(u)\rangle d u
$$

It follows that the term on the left does converge to something as $k \rightarrow \infty$. It just remains to consider what it does converge to. However, from the equation solved by $X$,

$$
B X\left(t_{j+1}\right)-B X\left(t_{j}\right)=\int_{t_{j}}^{t_{j+1}} Y(u) d u
$$

Therefore, this term is dominated by an expression of the form

$$
\begin{gathered}
\sum_{j=0}^{m_{k}-1}\left(\int_{t_{j}}^{t_{j+1}} Y(u) d u, X\left(t_{j+1}\right)-X\left(t_{j}\right)\right) \\
=\sum_{j=0}^{m_{k}-1}\left\langle\int_{t_{j}}^{t_{j+1}} Y(u) d u, X\left(t_{j+1}\right)-X\left(t_{j}\right)\right\rangle \\
=\sum_{j=0}^{m_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X\left(t_{j+1}\right)-X\left(t_{j}\right)\right\rangle d u \\
=\sum_{j=0}^{m_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X\left(t_{j+1}\right)\right\rangle-\sum_{j=0}^{m_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X\left(t_{j}\right)\right\rangle \\
=\int_{0}^{T}\left\langle Y(u), X^{r}(u)\right\rangle d u-\int_{0}^{T}\left\langle Y(u), X^{l}(u)\right\rangle d u
\end{gathered}
$$

However, both $X^{r}$ and $X^{l}$ converge to $X$ in $K=L^{p}(0, T, V)$. Therefore, this term must converge to 0 . Passing to a limit, it follows that for all $t \in D$, the desired formula holds. Thus, for such $t \in D$,

$$
\langle B X(t), X(t)\rangle=\left\langle B X_{0}, X_{0}\right\rangle+2 \int_{0}^{t}\langle Y(u), X(u)\rangle d u
$$

It remains to verify that this holds for all $t$. Let $t \notin D$ and let $t(k) \in \mathscr{P}_{k}$ be the largest point of $\mathscr{P}_{k}$ which is less than $t$. Suppose $t(m) \leq t(k)$ so that $m \leq k$. Then

$$
B X(t(m))=B X_{0}+\int_{0}^{t(m)} Y(s) d s
$$

a similar formula for $X(t(k))$. Thus for $t>t(m)$,

$$
B X(t)-B X(t(m))=\int_{t(m)}^{t} Y(s) d s
$$

which is the same sort of thing already looked at except that it starts at $t(m)$ rather than at 0 and $X_{0}=0$. Therefore,

$$
\begin{aligned}
& \langle B(X(t(k))-X(t(m))), X(t(k))-X(t(m))\rangle \\
= & 2 \int_{t(m)}^{t(k)}\langle Y(s), X(s)-X(t(m))\rangle d s
\end{aligned}
$$

Thus, for $m \leq k$

$$
\begin{equation*}
\lim _{m, k \rightarrow \infty}\langle B(X(t(k))-X(t(m))), X(t(k))-X(t(m))\rangle=0 \tag{69.4.26}
\end{equation*}
$$

Hence $\{B X(t(k))\}_{k=1}^{\infty}$ is a convergent sequence in $W^{\prime}$ because

$$
\begin{aligned}
& |\langle B(X(t(k))-X(t(m))), y\rangle| \\
\leq & \langle B(X(t(k))-X(t(m))), X(t(k))-X(t(m))\rangle^{1 / 2}\langle B y, y\rangle^{1 / 2} \\
\leq & \langle B(X(t(k))-X(t(m))), X(t(k))-X(t(m))\rangle^{1 / 2}\|B\|^{1 / 2}\|y\|_{W}
\end{aligned}
$$

Does it converge to $B X(t)$ ? Let $\xi(t) \in W^{\prime}$ be what it does converge to. Let $v \in V$. Then

$$
\langle\xi(t), v\rangle=\lim _{k \rightarrow \infty}\langle B X(t(k)), v\rangle=\lim _{k \rightarrow \infty}\langle B X(t(k)), v\rangle=\langle B X(t), v\rangle
$$

because it is known that $t \rightarrow B X(t)$ is continuous into $V^{\prime}$. It is also known that $B X(t) \in$ $W^{\prime} \subseteq V^{\prime}$ and that the $B X(t)$ for $t \in[0, T]$ are uniformly bounded in $W^{\prime}$. Therefore, since $V$ is dense in $W$, it follows that $\xi(t)=B X(t)$.

Now for every $t \in D$, it was shown above that

$$
\langle B X(t), X(t)\rangle=\left\langle B X_{0}, X_{0}\right\rangle+2 \int_{0}^{t}\langle Y(u), X(u)\rangle d u
$$

Also it was just shown that $B X(t(k)) \rightarrow B X(t)$. Then

$$
\begin{gathered}
|\langle B X(t(k)), X(t(k))\rangle-\langle B X(t), X(t)\rangle| \\
\leq|\langle B X(t(k)), X(t(k))-X(t)\rangle|+|\langle B X(t(k))-B X(t), X(t)\rangle|
\end{gathered}
$$

Then the second term converges to 0 . The first equals

$$
\begin{aligned}
& |\langle B X(t(k))-B X(t), X(t(k))\rangle| \\
\leq & \langle B(X(t(k))-X(t)), X(t(k))-X(t)\rangle^{1 / 2}\langle B X(t(k)), X(t(k))\rangle^{1 / 2}
\end{aligned}
$$

From the above, this is dominated by an expression of the form

$$
\langle B(X(t(k))-X(t)), X(t(k))-X(t)\rangle^{1 / 2} C
$$

Then using the lower semicontinuity of $t \rightarrow\langle B(X(t(k))-X(t)), X(t(k))-X(t)\rangle$ which follows from the above, this is no larger than

$$
\lim \inf _{m \rightarrow \infty}\langle B(X(t(k))-X(t(m))), X(t(k))-X(t(m))\rangle^{1 / 2} C<\varepsilon
$$

provided $k$ is large enough. This follows from 69.4.26. Since $\varepsilon$ is arbitrary, it follows that

$$
\lim _{k \rightarrow \infty}|\langle B X(t(k)), X(t(k))\rangle-\langle B X(t), X(t)\rangle|=0
$$

Then from the formula,

$$
\langle B X(t), X(t)\rangle=\left\langle B X_{0}, X_{0}\right\rangle+2 \int_{0}^{t}\langle Y(u), X(u)\rangle d u
$$

valid for $t \in D$, it follows that the same formula holds for all $t$. This formula implies $t \rightarrow\langle B X(t), X(t)\rangle$ is continuous. Also recall that $t \rightarrow B X(t)$ was shown to be weakly continuous into $W^{\prime}$. Then

$$
\begin{aligned}
& \langle B(X(t)-X(s)), X(t)-X(s)\rangle \\
= & \langle B X(t), X(t)\rangle-2\langle B X(t), X(s)\rangle+\langle B X(s), X(s)\rangle
\end{aligned}
$$

From this, it follows that $t \rightarrow B X(t)$ is continuous into $W^{\prime}$ because $\lim _{t \rightarrow s}$ of the right side gives 0 and so the same is true of the left. Hence,

$$
\begin{aligned}
|\langle B(X(t)-X(s)), y\rangle| & \leq\langle B y, y\rangle^{1 / 2}\langle B(X(t)-X(s)), X(t)-X(s)\rangle^{1 / 2} \\
& \leq\|B\|^{1 / 2}\langle B(X(t)-X(s)), X(t)-X(s)\rangle^{1 / 2}\|y\|
\end{aligned}
$$

so

$$
\|B(X(t)-X(s))\|_{W^{\prime}} \leq\|B\|^{1 / 2}\langle B(X(t)-X(s)), X(t)-X(s)\rangle^{1 / 2}
$$

which converges to 0 as $t \rightarrow s$.

### 69.5 Some Imbedding Theorems

The next theorem is very useful in getting estimates in partial differential equations. It is called Erling's lemma.

Definition 69.5.1 Let $E, W$ be Banach spaces such that $E \subseteq W$ and the injection map from $E$ into $W$ is continuous. The injection map is said to be compact if every bounded set in $E$ has compact closure in $W$. In other words, if a sequence is bounded in $E$ it has a convergent subsequence converging in $W$. This is also referred to by saying that bounded sets in $E$ are precompact in $W$.

Theorem 69.5.2 Let $E \subseteq W \subseteq X$ where the injection map is continuous from $W$ to $X$ and compact from $E$ to $W$. Then for every $\varepsilon>0$ there exists a constant, $C_{\varepsilon}$ such that for all $u \in E$,

$$
\|u\|_{W} \leq \varepsilon\|u\|_{E}+C_{\varepsilon}\|u\|_{X}
$$

Proof: Suppose not. Then there exists $\varepsilon>0$ and for each $n \in \mathbb{N}, u_{n}$ such that

$$
\left\|u_{n}\right\|_{W}>\varepsilon\left\|u_{n}\right\|_{E}+n\left\|u_{n}\right\|_{X}
$$

Now let $v_{n}=u_{n} /\left\|u_{n}\right\|_{E}$. Therefore, $\left\|v_{n}\right\|_{E}=1$ and

$$
\left\|v_{n}\right\|_{W}>\varepsilon+n\left\|v_{n}\right\|_{X}
$$

It follows there exists a subsequence, still denoted by $v_{n}$ such that $v_{n}$ converges to $v$ in $W$. However, the above inequality shows that $\left\|v_{n}\right\|_{X} \rightarrow 0$. Therefore, $v=0$. But then the above inequality would imply that $\left\|v_{n}\right\|>\varepsilon$ and passing to the limit yields $0>\varepsilon$, a contradiction.

Definition 69.5.3 Define $C([a, b] ; X)$ the space of functions continuous at every point of $[a, b]$ having values in $X$.

You should verify that this is a Banach space with norm

$$
\|u\|_{\infty, X}=\max \left\{\left\|u_{n_{k}}(t)-u(t)\right\|_{X}: t \in[a, b]\right\} .
$$

The following theorem is an infinite dimensional version of the Ascoli Arzela theorem. It is like a well known result due to Simon. It is the appropriate generalization to stochastic problems in which you do not have weak derivatives. See Theorem 65.12.1 on the Holder continuity of the stochastic integral.

Theorem 69.5.4 Let $q>1$ and let $E \subseteq W \subseteq X$ where the injection map is continuous from $W$ to $X$ and compact from $E$ to $W$. Let $S$ be defined by

$$
\left\{u \text { such that }\|u(t)\|_{E} \leq R \text { for all } t \in[a, b], \text { and }\|u(s)-u(t)\|_{X} \leq R|t-s|^{1 / q}\right\} .
$$

Thus $S$ is bounded in $L^{\infty}(0, T, E)$ and in addition, the functions are uniformly Holder continuous into $X$. Then $S \subseteq C([a, b] ; W)$ and if $\left\{u_{n}\right\} \subseteq S$, there exists a subsequence, $\left\{u_{n_{k}}\right\}$ which converges to a function $u \in C([a, b] ; W)$ in the following way.

$$
\lim _{k \rightarrow \infty}\left\|u_{n_{k}}-u\right\|_{\infty, W}=0
$$

Proof: First consider the issue of $S$ being a subset of $C([a, b] ; W)$. Let $\varepsilon>0$ be given. Then by Theorem 69.5.2 there exists a constant, $C_{\varepsilon}$ such that for all $u \in W$

$$
\|u\|_{W} \leq \frac{\varepsilon}{4 R}\|u\|_{E}+C_{\varepsilon}\|u\|_{X}
$$

Therefore, for all $u \in S$,

$$
\begin{align*}
\|u(t)-u(s)\|_{W} & \leq \frac{\varepsilon}{6 R}\|u(t)-u(s)\|_{E}+C_{\varepsilon}\|u(t)-u(s)\|_{X} \\
& \leq \frac{\varepsilon}{6 R}\left(\|u(t)\|_{E}+\|u(s)\|_{E}\right)+C_{\varepsilon}\|u(t)-u(s)\|_{X} \\
& \leq \frac{\varepsilon}{3}+C_{\varepsilon} R|t-s|^{1 / q} \tag{69.5.27}
\end{align*}
$$

Since $\varepsilon$ is arbitrary, it follows $u \in C([a, b] ; W)$.
Let $D=\mathbb{Q} \cap[a, b]$ so $D$ is a countable dense subset of $[a, b]$. Let $D=\left\{t_{n}\right\}_{n=1}^{\infty}$. By compactness of the embedding of $E$ into $W$, there exists a subsequence $u_{(n, 1)}$ such that as $n \rightarrow \infty, u_{(n, 1)}\left(t_{1}\right)$ converges to a point in $W$. Now take a subsequence of this, called $(n, 2)$ such that as $n \rightarrow \infty, u_{(n, 2)}\left(t_{2}\right)$ converges to a point in $W$. It follows that $u_{(n, 2)}\left(t_{1}\right)$ also converges to a point of $W$. Continue this way. Now consider the diagonal sequence, $u_{k} \equiv$ $u_{(k, k)}$ This sequence is a subsequence of $u_{(n, l)}$ whenever $k>l$. Therefore, $u_{k}\left(t_{j}\right)$ converges for all $t_{j} \in D$.

Claim: Let $\left\{u_{k}\right\}$ be as just defined, converging at every point of $D \equiv[a, b] \cap \mathbb{Q}$. Then $\left\{u_{k}\right\}$ converges at every point of $[a, b]$.

Proof of claim: Let $\varepsilon>0$ be given. Let $t \in[a, b]$. Pick $t_{m} \in D \cap[a, b]$ such that in 69.5.27 $C_{\varepsilon} R\left|t-t_{m}\right|<\varepsilon / 3$. There exists $N$ such that if $l, n>N$, then $\left\|u_{l}\left(t_{m}\right)-u_{n}\left(t_{m}\right)\right\|_{X}<$ $\varepsilon / 3$. It follows that for $l, n>N$,

$$
\begin{aligned}
\left\|u_{l}(t)-u_{n}(t)\right\|_{W} \leq & \left\|u_{l}(t)-u_{l}\left(t_{m}\right)\right\|_{W}+\left\|u_{l}\left(t_{m}\right)-u_{n}\left(t_{m}\right)\right\|_{W} \\
& +\left\|u_{n}\left(t_{m}\right)-u_{n}(t)\right\|_{W} \\
\leq & \frac{2 \varepsilon}{3}+\frac{\varepsilon}{3}+\frac{2 \varepsilon}{3}<2 \varepsilon
\end{aligned}
$$

Since $\varepsilon$ was arbitrary, this shows $\left\{u_{k}(t)\right\}_{k=1}^{\infty}$ is a Cauchy sequence. Since $W$ is complete, this shows this sequence converges.

Now for $t \in[a, b]$, it was just shown that if $\varepsilon>0$ there exists $N_{t}$ such that if $n, m>N_{t}$, then

$$
\left\|u_{n}(t)-u_{m}(t)\right\|_{W}<\frac{\varepsilon}{3}
$$

Now let $s \neq t$. Then

$$
\left\|u_{n}(s)-u_{m}(s)\right\|_{W} \leq\left\|u_{n}(s)-u_{n}(t)\right\|_{W}+\left\|u_{n}(t)-u_{m}(t)\right\|_{W}+\left\|u_{m}(t)-u_{m}(s)\right\|_{W}
$$

From 69.5.27

$$
\left\|u_{n}(s)-u_{m}(s)\right\|_{W} \leq 2\left(\frac{\varepsilon}{3}+C_{\varepsilon} R|t-s|^{1 / q}\right)+\left\|u_{n}(t)-u_{m}(t)\right\|_{W}
$$

and so it follows that if $\delta$ is sufficiently small and $s \in B(t, \boldsymbol{\delta})$, then when $n, m>N_{t}$

$$
\left\|u_{n}(s)-u_{m}(s)\right\|<\varepsilon
$$

Since $[a, b]$ is compact, there are finitely many of these balls, $\left\{B\left(t_{i}, \delta\right)\right\}_{i=1}^{p}$, such that for $s \in B\left(t_{i}, \boldsymbol{\delta}\right)$ and $n, m>N_{t_{i}}$, the above inequality holds. Let $N>\max \left\{N_{t_{1}}, \cdots, N_{t_{p}}\right\}$. Then if $m, n>N$ and $s \in[a, b]$ is arbitrary, it follows the above inequality must hold. Therefore, this has shown the following claim.

Claim: Let $\varepsilon>0$ be given. There exists $N$ such that if $m, n>N$, then $\left\|u_{n}-u_{m}\right\|_{\infty, W}<\varepsilon$.
Now let $u(t)=\lim _{k \rightarrow \infty} u_{k}(t)$.

$$
\begin{equation*}
\|u(t)-u(s)\|_{W} \leq\left\|u(t)-u_{n}(t)\right\|_{W}+\left\|u_{n}(t)-u_{n}(s)\right\|_{W}+\left\|u_{n}(s)-u(s)\right\|_{W} \tag{69.5.28}
\end{equation*}
$$

Let $N$ be in the above claim and fix $n>N$. Then

$$
\left\|u(t)-u_{n}(t)\right\|_{W}=\lim _{m \rightarrow \infty}\left\|u_{m}(t)-u_{n}(t)\right\|_{W} \leq \varepsilon
$$

and similarly, $\left\|u_{n}(s)-u(s)\right\|_{W} \leq \varepsilon$. Then if $|t-s|$ is small enough, 69.5 .27 shows the middle term in 69.5.28 is also smaller than $\varepsilon$. Therefore, if $|t-s|$ is small enough,

$$
\|u(t)-u(s)\|_{W}<3 \varepsilon
$$

Thus $u$ is continuous. Finally, let $N$ be as in the above claim. Then letting $m, n>N$, it follows that for all $t \in[a, b]$,

$$
\left\|u_{m}(t)-u_{n}(t)\right\|_{W}<\varepsilon
$$

Therefore, letting $m \rightarrow \infty$, it follows that for all $t \in[a, b]$,

$$
\left\|u(t)-u_{n}(t)\right\|_{W} \leq \varepsilon
$$

and so $\left\|u-u_{n}\right\|_{\infty, W} \leq \varepsilon$.
Here is an interesting corollary. Recall that for $E$ a Banach space $C^{0, \alpha}([0, T], E)$ is the space of continuous functions $u$ from $[0, T]$ to $E$ such that

$$
\|u\|_{\alpha, E} \equiv\|u\|_{\infty, E}+\rho_{\alpha, E}(u)<\infty
$$

where here

$$
\rho_{\alpha, E}(u) \equiv \sup _{t \neq s} \frac{\|u(t)-u(s)\|_{E}}{|t-s|^{\alpha}}
$$

Corollary 69.5.5 Let $E \subseteq W \subseteq X$ where the injection map is continuous from $W$ to $X$ and compact from $E$ to $W$. Then if $\gamma>\alpha$, the embedding of $C^{0, \gamma}([0, T], E)$ into $C^{0, \alpha}([0, T], X)$ is compact.

Proof: Let $\phi \in C^{0, \gamma}([0, T], E)$

$$
\begin{gathered}
\frac{\|\phi(t)-\phi(s)\|_{X}}{|t-s|^{\alpha}} \leq\left(\frac{\|\phi(t)-\phi(s)\|_{W}}{|t-s|^{\gamma}}\right)^{\alpha / \gamma}\|\phi(t)-\phi(s)\|_{W}^{1-(\alpha / \gamma)} \\
\leq\left(\frac{\|\phi(t)-\phi(s)\|_{E}}{|t-s|^{\gamma}}\right)^{\alpha / \gamma}\|\phi(t)-\phi(s)\|_{W}^{1-(\alpha / \gamma)} \leq \rho_{\gamma, E}(\phi)\|\phi(t)-\phi(s)\|_{W}^{1-(\alpha / \gamma)}
\end{gathered}
$$

Now suppose $\left\{u_{n}\right\}$ is a bounded sequence in $C^{0, \gamma}([0, T], E)$. By Theorem 69.5.4 above, there is a subsequence still called $\left\{u_{n}\right\}$ which converges in $C^{0}([0, T], W)$. Thus from the above inequality

$$
\begin{aligned}
& \frac{\left\|u_{n}(t)-u_{m}(t)-\left(u_{n}(s)-u_{m}(s)\right)\right\|_{X}}{|t-s|^{\alpha}} \\
\leq & \rho_{\gamma, E}\left(u_{n}-u_{m}\right)\left\|u_{n}(t)-u_{m}(t)-\left(u_{n}(s)-u_{m}(s)\right)\right\|_{W}^{1-(\alpha / \gamma)} \\
\leq & C\left(\left\{u_{n}\right\}\right)\left(2\left\|u_{n}-u_{m}\right\|_{\infty, W}\right)^{1-(\alpha / \gamma)}
\end{aligned}
$$

which converges to 0 as $n, m \rightarrow \infty$. Thus

$$
\rho_{\alpha, X}\left(u_{n}-u_{m}\right) \rightarrow 0 \text { as } n, m \rightarrow \infty
$$

Also $\left\|u_{n}-u_{m}\right\|_{\infty, X} \rightarrow 0$ as $n, m \rightarrow \infty$ so this is a Cauchy sequence in $C^{0, \alpha}([0, T], X)$.
The next theorem is a well known result probably due to Lions.
Theorem 69.5.6 Let $E \subseteq W \subseteq X$ where the injection map is continuous from $W$ to $X$ and compact from $E$ to $W$. Let $p \geq 1$, let $q>1$, and define

$$
S \equiv\left\{u \in L^{p}([a, b] ; E): \text { for some } C,\|u(t)-u(s)\|_{X} \leq C|t-s|^{1 / q}\right.
$$

$$
\text { and } \left.\|u\|_{L^{p}([a, b] ; E)} \leq R\right\} .
$$

Thus $S$ is bounded in $L^{p}([a, b] ; E)$ and Holder continuous into $X$. Then $S$ is precompact in $L^{p}([a, b] ; W)$. This means that if $\left\{u_{n}\right\}_{n=1}^{\infty} \subseteq S$, it has a subsequence $\left\{u_{n_{k}}\right\}$ which converges in $L^{p}([a, b] ; W)$.

Proof: By Proposition 7.6 .5 on Page 144 it suffices to show that $S$ has an $\eta$ net in $L^{p}([a, b] ; W)$ for each $\eta>0$.

If not, there exists $\eta>0$ and a sequence $\left\{u_{n}\right\} \subseteq S$, such that

$$
\begin{equation*}
\left\|u_{n}-u_{m}\right\| \geq \eta \tag{69.5.29}
\end{equation*}
$$

for all $n \neq m$ and the norm refers to $L^{p}([a, b] ; W)$. Let

$$
a=t_{0}<t_{1}<\cdots<t_{k}=b, t_{i}-t_{i-1}=(b-a) / k
$$

Now define

$$
\bar{u}_{n}(t) \equiv \sum_{i=1}^{k} \bar{u}_{n_{i}} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t), \bar{u}_{n_{i}} \equiv \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} u_{n}(s) d s
$$

The idea is to show that $\bar{u}_{n}$ approximates $u_{n}$ well and then to argue that a subsequence of the $\left\{\bar{u}_{n}\right\}$ is a Cauchy sequence yielding a contradiction to 69.5.29.

Therefore,

$$
\begin{gathered}
u_{n}(t)-\bar{u}_{n}(t)=\sum_{i=1}^{k} u_{n}(t) \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)-\sum_{i=1}^{k} \bar{u}_{n_{i}} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t) \\
=\sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} u_{n}(t) d s \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)-\sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} u_{n}(s) d s \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t) \\
=\sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left(u_{n}(t)-u_{n}(s)\right) d s \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)
\end{gathered}
$$

It follows from Jensen's inequality that

$$
\begin{aligned}
& \left\|u_{n}(t)-\bar{u}_{n}(t)\right\|_{W}^{p} \\
= & \sum_{i=1}^{k}\left\|\frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left(u_{n}(t)-u_{n}(s)\right) d s\right\|_{W}^{p} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t) \\
\leq & \sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left\|u_{n}(t)-u_{n}(s)\right\|_{W}^{p} d s \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)
\end{aligned}
$$

and so

$$
\begin{align*}
& \int_{a}^{b}\left\|\left(u_{n}(t)-\bar{u}_{n}(s)\right)\right\|_{W}^{p} d s \\
\leq & \int_{a}^{b} \sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}}\left\|u_{n}(t)-u_{n}(s)\right\|_{W}^{p} d s \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t) d t \\
= & \sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t_{i}}\left\|u_{n}(t)-u_{n}(s)\right\|_{W}^{p} d s d t . \tag{69.5.30}
\end{align*}
$$

From Theorem 69.5.2 if $\varepsilon>0$, there exists $C_{\varepsilon}$ such that

$$
\begin{gathered}
\left\|u_{n}(t)-u_{n}(s)\right\|_{W}^{p} \leq \varepsilon\left\|u_{n}(t)-u_{n}(s)\right\|_{E}^{p}+C_{\varepsilon}\left\|u_{n}(t)-u_{n}(s)\right\|_{X}^{p} \\
\leq 2^{p-1} \varepsilon\left(\left\|u_{n}(t)\right\|^{p}+\left\|u_{n}(s)\right\|^{p}\right)+C_{\varepsilon}|t-s|^{p / q}
\end{gathered}
$$

This is substituted in to 69.5 .30 to obtain

$$
\begin{gathered}
\int_{a}^{b}\left\|\left(u_{n}(t)-\bar{u}_{n}(s)\right)\right\|_{W}^{p} d s \leq \\
\\
\sum_{i=1}^{k} \frac{1}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t_{i}}\left(2^{p-1} \varepsilon\left(\left\|u_{n}(t)\right\|^{p}+\left\|u_{n}(s)\right\|^{p}\right)+C_{\varepsilon}|t-s|^{p / q}\right) d s d t \\
= \\
\sum_{i=1}^{k} 2^{p} \varepsilon \int_{t_{i-1}}^{t_{i}}\left\|u_{n}(t)\right\|_{W}^{p}+\frac{C_{\varepsilon}}{t_{i}-t_{i-1}} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t_{i}}|t-s|^{p / q} d s d t \\
\leq \quad 2^{p} \varepsilon \int_{a}^{b}\left\|u_{n}(t)\right\|^{p} d t+C_{\varepsilon} \sum_{i=1}^{k} \frac{1}{\left(t_{i}-t_{i-1}\right)}\left(t_{i}-t_{i-1}\right)^{p / q} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t_{i}} d s d t \\
= \\
2^{p} \varepsilon \int_{a}^{b}\left\|u_{n}(t)\right\|^{p} d t+C_{\varepsilon} \sum_{i=1}^{k} \frac{1}{\left(t_{i}-t_{i-1}\right)}\left(t_{i}-t_{i-1}\right)^{p / q}\left(t_{i}-t_{i-1}\right)^{2} \\
\leq \\
2^{p} \varepsilon R^{p}+C_{\varepsilon} \sum_{i=1}^{k}\left(t_{i}-t_{i-1}\right)^{1+p / q}=2^{p} \varepsilon R^{p}+C_{\varepsilon} k\left(\frac{b-a}{k}\right)^{1+p / q} .
\end{gathered}
$$

Taking $\varepsilon$ so small that $2^{p} \varepsilon R^{p}<\eta^{p} / 8^{p}$ and then choosing $k$ sufficiently large, it follows

$$
\left\|u_{n}-\bar{u}_{n}\right\|_{L^{p}([a, b] ; W)}<\frac{\eta}{4} .
$$

Thus $k$ is fixed and $\bar{u}_{n}$ at a step function with $k$ steps having values in $E$. Now use compactness of the embedding of $E$ into $W$ to obtain a subsequence such that $\left\{\bar{u}_{n}\right\}$ is Cauchy in $L^{p}(a, b ; W)$ and use this to contradict 69.5.29. The details follow.

Suppose $\bar{u}_{n}(t)=\sum_{i=1}^{k} u_{i}^{n} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)$. Thus

$$
\left\|\bar{u}_{n}(t)\right\|_{E}=\sum_{i=1}^{k}\left\|u_{i}^{n}\right\|_{E} \mathscr{X}_{\left[t_{i-1}, t_{i}\right)}(t)
$$

and so

$$
R \geq \int_{a}^{b}\left\|\bar{u}_{n}(t)\right\|_{E}^{p} d t=\frac{T}{k} \sum_{i=1}^{k}\left\|u_{i}^{n}\right\|_{E}^{p}
$$

Therefore, the $\left\{u_{i}^{n}\right\}$ are all bounded. It follows that after taking subsequences $k$ times there exists a subsequence $\left\{u_{n_{k}}\right\}$ such that $u_{n_{k}}$ is a Cauchy sequence in $L^{p}(a, b ; W)$. You simply get a subsequence such that $u_{i}^{n_{k}}$ is a Cauchy sequence in $W$ for each $i$. Then denoting this subsequence by $n$,

$$
\begin{aligned}
\left\|u_{n}-u_{m}\right\|_{L^{p}(a, b ; W)} \leq & \left\|u_{n}-\bar{u}_{n}\right\|_{L^{p}(a, b ; W)} \\
& +\left\|\bar{u}_{n}-\bar{u}_{m}\right\|_{L^{p}(a, b ; W)}+\left\|\bar{u}_{m}-u_{m}\right\|_{L^{p}(a, b ; W)} \\
\leq & \frac{\eta}{4}+\left\|\bar{u}_{n}-\bar{u}_{m}\right\|_{L^{p}(a, b ; W)}+\frac{\eta}{4}<\eta
\end{aligned}
$$

provided $m, n$ are large enough, contradicting 69.5.29.

## Chapter 70

## Measurability Without Uniqueness

With the Ito formula which holds for a single space, it is time to consider stochastic ordinary differential equations. First is a general theory which allows one to consider measurable solutions to stochastic equations in which there is no uniqueness available. Unfortunately, it does not include obtaining adapted solutions. Instead, it includes measurability of functions with respect to a single $\sigma$ algebra. Then when path uniqueness is available, one can include the concept of adapted solutions rather easily and this will be done for ordinary differential equations. First is a general result about multifunctions.

### 70.1 Multifunctions And Their Measurability

Let $X$ be a separable complete metric space and let $(\Omega, \mathscr{C}, \mu)$ be a set, a $\sigma$ algebra of subsets of $\Omega$, and a measure $\mu$ such that this is a complete $\sigma$ finite measure space. Also let $\Gamma: \Omega \rightarrow \mathscr{P}_{F}(X)$, the closed subsets of $X$.

Definition 70.1.1 We define $\Gamma^{-}(S) \equiv\{\omega \in \Omega: \Gamma(\omega) \cap S \neq \emptyset\}$
We will consider a theory of measurability of set valued functions. The following theorem is the main result in the subject. In this theorem the third condition is what we will refer to as measurable. The second condition is called strongly measurable. More can be said than what we will prove here.

Theorem 70.1.2 In the following, $1 . \Rightarrow 2 . \Rightarrow 3 . \Rightarrow 4$.

1. For all B a Borel set in $X, \Gamma^{-}(B) \in \mathscr{C}$.
2. For all $F$ closed in $X, \Gamma^{-}(F) \in \mathscr{C}$
3. For all $U$ open in $X, \Gamma^{-}(U) \in \mathscr{C}$
4. There exists a sequence, $\left\{\sigma_{n}\right\}$ of measurable functions satisfying $\sigma_{n}(\omega) \in \Gamma(\omega)$ such that for all $\omega \in \Omega$,

$$
\Gamma(\omega)=\overline{\left\{\sigma_{n}(\omega): n \in \mathbb{N}\right\}}
$$

These functions are called measurable selections.
Also $4 . \Rightarrow 3$. If $\Gamma(\omega)$ is compact for each $\omega$, then also $3 . \Rightarrow 2$.
Proof: It is obvious that 1.) $\Rightarrow 2$.). To see that 2.) $\Rightarrow$ 3.) note that $\Gamma^{-}\left(\cup_{i=1}^{\infty} F_{i}\right)=$ $\cup_{i=1}^{\infty} \Gamma^{-}\left(F_{i}\right)$. Since any open set in $X$ can be obtained as a countable union of closed sets, this implies 2.) $\Rightarrow 3$.).

Now we verify that 3.) $\Rightarrow 4$.). Let $\left\{x_{n}\right\}_{n=1}^{\infty}$ be a countable dense subset of $X$. For $\omega \in \Omega$, let $\psi_{1}(\omega)=x_{n}$ where $n$ is the smallest integer such that $\Gamma(\omega) \cap B\left(x_{n}, 1\right) \neq \emptyset$. Therefore, $\psi_{1}(\omega)$ has countably many values, $x_{n_{1}}, x_{n_{2}}, \cdots$ where $n_{1}<n_{2}<\cdots$. Now

$$
\left\{\omega: \psi_{1}=x_{n}\right\}=
$$

$$
\left\{\omega: \Gamma(\omega) \cap B\left(x_{n}, 1\right) \neq \emptyset\right\} \cap\left[\Omega \backslash \cup_{k<n}\left\{\omega: \Gamma(\omega) \cap B\left(x_{k}, 1\right) \neq \emptyset\right\}\right] \in \mathscr{C}
$$

Thus we see that $\psi_{1}$ is measurable and $\operatorname{dist}\left(\psi_{1}(\omega), \Gamma(\omega)\right)<1$. Let

$$
\Omega_{n} \equiv\left\{\omega \in \Omega: \psi_{1}(\omega)=x_{n}\right\}
$$

Then $\Omega_{n} \in \mathscr{C}$ and $\Omega_{n} \cap \Omega_{m}=\emptyset$ for $n \neq m$ and $\cup_{n=1}^{\infty} \Omega_{n}=\Omega$. Let $D_{n} \equiv\left\{x_{k}: x_{k} \in B\left(x_{n}, 1\right)\right\}$. Now for each $n$, and $\omega \in \Omega_{n}$, let $\psi_{2}(\omega)=x_{k}$ where $k$ is the smallest index such that $x_{k} \in$ $D_{n}$ and $B\left(x_{k}, \frac{1}{2}\right) \cap \Gamma(\omega) \neq \emptyset$. Thus dist $\left(\psi_{2}(\omega), \Gamma(\omega)\right)<\frac{1}{2}$ and $d\left(\psi_{2}(\omega), \psi_{1}(\omega)\right)<1$. Continue this way obtaining $\psi_{k}$ a measurable function such that

$$
\operatorname{dist}\left(\psi_{k}(\omega), \Gamma(\omega)\right)<\frac{1}{2^{k-1}}, d\left(\psi_{k}(\omega), \psi_{k+1}(\omega)\right)<\frac{1}{2^{k-2}}
$$

Then for each $\omega,\left\{\psi_{k}(\omega)\right\}$ is a Cauchy sequence converging to a point, $\sigma(\omega) \in \Gamma(\omega)$. This has shown that if $\Gamma$ is measurable there exists a measurable selection, $\sigma(\omega) \in \Gamma(\omega)$. It remains to show there exists a sequence of these measurable selections, $\sigma_{n}$ such that the conclusion of 4.) holds. To do this we define

$$
\Gamma_{n i}(\omega) \equiv\left\{\begin{array}{l}
\Gamma(\omega) \cap B\left(x_{n}, 2^{-i}\right) \text { if } \Gamma(\omega) \cap B\left(x_{n}, 2^{-i}\right) \neq \emptyset \\
\Gamma(\omega) \text { otherwise }
\end{array}\right.
$$

First we show that $\Gamma_{n i}$ is measurable. Let $U$ be open. Then

$$
\begin{gathered}
\left\{\omega: \Gamma_{n i}(\omega) \cap U \neq \emptyset\right\}=\left\{\omega: \Gamma(\omega) \cap B\left(x_{n}, 2^{-i}\right) \cap U \neq \emptyset\right\} \cup \\
{\left[\left\{\omega: \Gamma(\omega) \cap B\left(x_{n}, 2^{-i}\right)=\emptyset\right\} \cap\{\omega: \Gamma(\omega) \cap U \neq \emptyset\}\right]} \\
=\left\{\omega: \Gamma(\omega) \cap B\left(x_{n}, 2^{-i}\right) \cap U \neq \emptyset\right\} \cup \\
{\left[\left(\Omega \backslash\left\{\omega: \Gamma(\omega) \cap B\left(x_{n}, 2^{-i}\right) \neq \emptyset\right\}\right) \cap\{\omega: \Gamma(\omega) \cap U \neq \emptyset\}\right]}
\end{gathered}
$$

a measurable set. By what was just shown there exists $\sigma_{n i}$, a measurable function such that $\sigma_{n i}(\omega) \in \Gamma_{n i}(\omega) \subseteq \Gamma(\omega)$ for all $\omega \in \Omega$. If $x \in \Gamma(\omega)$, then $x \in B\left(x_{n}, 2^{-i}\right)$ whenever $x_{n}$ is close enough to $x$. Therefore, $\left|\sigma_{n i}(\omega)-x\right|<2^{-i}$. And it follows that condition 4.) holds.

Now we verify that 4.) $\Rightarrow 3$.). Suppose there exist measurable selections $\sigma_{n}(\omega) \in$ $\Gamma(\omega)$ satisfying condition 4.). Let $U$ be open. Then

$$
\{\omega: \Gamma(\omega) \cap U \neq \emptyset\}=\cup_{n=1}^{\infty} \sigma_{n}^{-1}(U) \in \mathscr{C}
$$

Now suppose $\Gamma(\omega)$ is compact for every $\omega$ and that $\Gamma^{-}(U) \in \mathscr{C}$ for every $U$ open. Then let $F$ be a closed set and let $\left\{U_{n}\right\}$ be a decreasing sequence of open sets whose intersection equals $F$ such that also, for all $n, U_{n} \supseteq \overline{U_{n+1}}$. Then

$$
\Gamma(\omega) \cap F=\cap_{n} \Gamma(\omega) \cap U_{n}=\cap_{n} \Gamma(\omega) \cap \overline{U_{n}}
$$

Now because of compactness, the set on the left is nonempty if and only if each set on the right is also nonempty. Thus $\Gamma^{-}(F)=\cap_{n} \Gamma^{-1}\left(U_{n}\right) \in \mathscr{C}$.

Actually these are all equivalent in the case of complete measure spaces but we do not need this and it is much harder to show.

### 70.2 A Measurable Selection

This section deals with the problem of getting product measurable functions in a context of no uniqueness. The following is the main result. It is stated in great generality because it has fairly wide application although it will be used first in finite dimensions.

Theorem 70.2.1 Let $V$ be a reflexive separable Banach space and $V^{\prime}$ its dual and $\frac{1}{p}+\frac{1}{p^{\prime}}=$ 1 where $p>1$ as usual. For $n \in \mathbb{N}$ let the functions $t \rightarrow u_{n}(t, \omega)$ be in $L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)$ and $(t, \omega) \rightarrow u_{n}(t, \omega)$ be $\mathscr{B}([0, T]) \times \mathscr{F} \equiv \mathscr{P}$ measurable into $V^{\prime}$. Suppose there is a set of measure zero $N$ such that if $\omega \notin N$, then for all $n$,

$$
\sup _{t \in[0, T]}\left\|u_{n}(t, \omega)\right\|_{V^{\prime}} \leq C(\omega)
$$

Also suppose for each $\omega \notin N$, each subsequence of $\left\{u_{n}\right\}$ has a further subsequence which converges weakly in $L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)$ to $u(\cdot, \omega) \in L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)$ such that $t \rightarrow u(t, \omega)$ is weakly continuous into $V^{\prime}$. Then there exists u product measurable, with $t \rightarrow u(t, \omega)$ being weakly continuous into $V^{\prime}$. Moreover, there exists, for each $\omega \notin N$, a subsequence $u_{n(\omega)}$ such that $u_{n(\omega)}(\cdot, \omega) \rightarrow u(\cdot, \omega)$ weakly in $L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)$.

Note that the exceptional set is given. It could be the empty set with no change in the conclusion of the theorem.

Let $X=\prod_{k=1}^{\infty} C([0, T])$ with the product topology. One can consider this as a metric space using the metric

$$
d(\mathbf{f}, \mathbf{g}) \equiv \sum_{k=1}^{\infty} 2^{-k} \frac{\left\|f_{k}-g_{k}\right\|}{1+\left\|f_{k}-g_{k}\right\|}
$$

where the norm is the maximum norm in $C([0, T])$. With this metric, $X$ is complete and separable.

Lemma 70.2.2 Let $\left\{\mathbf{f}_{n}\right\}$ be a sequence in $X$ and suppose that the $k^{\text {th }}$ components $f_{n k}$ are bounded in $C^{0,1}([0, T])$. (This refers to the Hölder space with $\gamma=1$.) Then there exists a subsequence converging to some $\mathbf{f} \in X$. Thus if $\left\{\mathbf{f}_{n}\right\}$ has each component bounded in $C^{0,1}([0, T])$, then $\left\{\mathbf{f}_{n}\right\}$ is pre-compact in $X$.

Proof: By the Ascoli-Arzelà theorem, there exists a subsequence $n_{1}$ such that the first component $f_{n_{1} 1}$ converges in $C([0, T])$. Then taking a subsequence, one can obtain $n_{2}$ a subsequence of $n_{1}$ such that both the first and second components of $\mathbf{f}_{n_{2}}$ converge. Continuing this way one obtains a sequence of subsequences, each a subsequence of the previous one such that $\mathbf{f}_{n_{j}}$ has the first $j$ components converging to functions in $C([0, T])$. Therefore, the diagonal subsequence has the property that it has every component converging to a function in $C([0, T])$. The resulting function in $\prod_{k} C([0, T])$ is $\mathbf{f}$.

Now for $m \in \mathbb{N}$ and $\phi \in V^{\prime}$, define $l_{m}(t) \equiv \max (0, t-(1 / m))$ and

$$
\psi_{m, \phi}: L^{p^{\prime}}\left([0, T] ; V^{\prime}\right) \rightarrow C([0, T])
$$

as follows

$$
\psi_{m, \phi} u(t) \equiv \int_{0}^{T}\left\langle m \phi \mathscr{X}_{\left[l_{m}(t), t\right]}(s), u(s)\right\rangle_{V, V^{\prime}} d s=m \int_{l_{m}(t)}^{t}\langle\phi, u(s)\rangle_{V, V^{\prime}} d s
$$

Let $\mathscr{D}=\left\{\phi_{r}\right\}_{r=1}^{\infty}$ denote a countable dense subset of $V$. Then the pairs $(\phi, m)$ for $\phi \in \mathscr{D}$ and $m \in \mathbb{N}$ yield a countable set. Let $\left(m_{k}, \phi_{r_{k}}\right)$ denote an enumeration of these pairs $(m, \phi) \in \mathbb{N} \times \mathscr{D}$. To save notation, we denote

$$
f_{k}(u)(t) \equiv \psi_{m_{k}, \phi_{r_{k}}}(u)(t)=m_{k} \int_{l_{m_{k}}(t)}^{t}\left\langle\phi_{r_{k}}, u(s)\right\rangle_{V, V^{\prime}} d s
$$

For fixed $\omega \notin N$ and $k$, the functions $\left\{t \rightarrow f_{k}\left(u_{j}(\cdot, \omega)\right)(t)\right\}_{j}$ are uniformly bounded and equicontinuous because they are in $C^{0,1}([0, T])$. Indeed,

$$
\left|f_{k}\left(u_{j}(\cdot, \omega)\right)(t)\right|=\left|m_{k} \int_{l_{m_{k}}(t)}^{t}\left\langle\phi_{r_{k}}, u_{j}(s, \omega)\right\rangle_{V, V^{\prime}} d s\right| \leq C(\omega)\left\|\phi_{r_{k}}\right\|_{V}
$$

and for $t \leq t^{\prime}$

$$
\begin{aligned}
& \left|f_{k}\left(u_{j}(\cdot, \omega)\right)(t)-f_{k}\left(u_{j}(\cdot, \omega)\right)\left(t^{\prime}\right)\right| \\
\leq & \left|m_{k} \int_{l_{m_{k}}(t)}^{t}\left\langle\phi_{r_{k}}, u_{j}(s, \omega)\right\rangle_{V, V^{\prime}} d s-m_{k} \int_{l_{m_{k}}\left(t^{\prime}\right)}^{t^{\prime}}\left\langle\phi_{r_{k}}, u_{j}(s, \omega)\right\rangle_{V, V^{\prime}} d s\right| \\
\leq & 2 m_{k}\left|t^{\prime}-t\right|\left\|\phi_{r_{k}}\right\|_{V^{\prime}} C(\omega) .
\end{aligned}
$$

By Lemma 70.2.2, the set of functions $\left\{\mathbf{f}\left(u_{j}(\cdot, \omega)\right)\right\}_{j=n}^{\infty}$ is pre-compact in the space defined as $X=\prod_{k} C([0, T])$. Then define a set valued map $\Gamma^{n}: \Omega \rightarrow X$ as follows.

$$
\Gamma^{n}(\omega) \equiv \overline{\cup_{j \geq n}\left\{\mathbf{f}\left(u_{j}(\cdot, \omega)\right)\right\}}
$$

where the closure is taken in $X$. Then $\Gamma^{n}(\omega)$ is the closure of a pre-compact set in $\prod_{k} C([0, T])$ and so $\Gamma^{n}(\omega)$ is compact in $\prod_{k} C([0, T])$. From the definition, a function $\mathbf{f}$ is in $\Gamma^{n}(\omega)$ if and only if $d\left(\mathbf{f}, \mathbf{f}\left(w_{l}\right)\right) \rightarrow 0$ as $l \rightarrow \infty$, where each $w_{l}$ is one of the $u_{j}(\cdot, \omega)$ for $j \geq n$. From the topology on $X$ this happens if and only if for every $k$,

$$
f_{k}(t)=\lim _{l \rightarrow \infty} m_{k} \int_{l_{m_{k}}(t)}^{t}\left\langle\phi_{r_{k}}, w_{l}(s, \omega)\right\rangle_{V, V^{\prime}} d s
$$

where the limit is the uniform limit in $t$.
Note that in the case of a filtration, instead of a single $\sigma$-algebra $\mathscr{F}$ where each $u_{j}$ is progressively measurable, if the sequence $w_{l}$ does not have the index $l$ dependent on $\omega$, then if such a limit holds for each $\omega$, it follows that $(t, \omega) \rightarrow f_{k}(t, \omega)$ will inherit progressive measurability from the $w_{l}$. This situation will be typical when dealing with stochastic equations with path uniqueness known. Thus this is a reasonable way to attempt to consider measurability and the more difficult question of whether a process is adapted.

Lemma 70.2.3 $\omega \rightarrow \Gamma^{n}(\omega)$ is a $\mathscr{F}$ measurable set valued map with values in $X$. If $\sigma$ is a measurable selection, $\left(\sigma(\omega) \in \Gamma^{n}(\omega)\right.$ so $\sigma=\sigma(\cdot, \omega)$ a continuous function. To have this measurable would mean that $\sigma_{k}^{-1}($ open $) \in \mathscr{F}$ where the open set is in $C([0, T])$.) then for each $t, \omega \rightarrow \sigma(t, \omega)$ is $\mathscr{F}$ measurable and $(t, \omega) \rightarrow \sigma(t, \omega)$ is $\mathscr{B}([0, T]) \times \mathscr{F} \equiv \mathscr{P}$ measurable.

Proof: Let $O$ be a basic open set in $X$. Thus $O=\prod_{k=1}^{\infty} O_{k}$ where $O_{k}$ is a proper open set of $C([0, T])$ only for $k \in\left\{k_{1}, \cdots, k_{r}\right\}$. We need to consider whether

$$
\Gamma^{n-}(O) \equiv\left\{\omega: \Gamma^{n}(\omega) \cap O \neq \emptyset\right\} \in \mathscr{F} .
$$

Now $\Gamma^{n-}(O)$ equals

$$
\cap_{i=1}^{r}\left\{\omega: \Gamma^{n}(\omega)_{k_{i}} \cap O_{k_{i}} \neq \emptyset\right\}
$$

Thus we consider whether

$$
\begin{equation*}
\left\{\omega: \Gamma^{n}(\omega)_{k_{i}} \cap O_{k_{i}} \neq \emptyset\right\} \in \mathscr{F} \tag{70.2.1}
\end{equation*}
$$

From the definition of $\Gamma^{n}(\omega)$, this is equivalent to the condition that for some $j \geq n$,

$$
f_{k_{i}}\left(u_{j}(\cdot, \omega)\right)=\left(\mathbf{f}\left(u_{j}(\cdot, \omega)\right)\right)_{k_{i}} \in O_{k_{i}}
$$

and so the above set in 70.2 .1 is of the form

$$
\cup_{j=n}^{\infty}\left\{\omega:\left(\mathbf{f}\left(u_{j}(\cdot, \omega)\right)\right)_{k_{i}} \in O_{k_{i}}\right\}
$$

Now $\omega \rightarrow\left(\mathbf{f}\left(u_{j}(\cdot, \omega)\right)\right)_{k_{i}}$ is $\mathscr{F}$ measurable into $C([0, T])$ and so the above set is in $\mathscr{F}$. To see this, let $g \in C([0, T])$ and consider the inverse image of the ball $B(g, r)$,

$$
\left\{\omega:\left\|\left(\mathbf{f}\left(u_{j}(\cdot, \omega)\right)\right)_{k_{i}}-g\right\|_{C([0, T])}<r\right\} .
$$

By continuity considerations,

$$
\left\|\left(\mathbf{f}\left(u_{j}(\cdot, \omega)\right)\right)_{k_{i}}-g\right\|_{C([0, T])}=\sup _{t \in \mathbb{Q} \cap[0, T]}\left|\left(\mathbf{f}\left(u_{j}(t, \omega)\right)\right)_{k_{i}}-g(t)\right|
$$

which is the sup of countably many $\mathscr{F}$ measurable functions. Thus it is $\mathscr{F}$ measurable. Since every open set is the countable union of such balls, it follows that the claim about $\mathscr{F}$ measurability is valid. Thus $\Gamma^{n-}(O)$ is $\mathscr{F}$ measurable whenever $O$ is a basic open set.

Now $X$ is a separable metric space and so every open set is a countable union of these basic sets. Let $U$ be an open set in $X$ and let $U=\cup_{l=1}^{\infty} O^{l}$ where $O^{l}$ is a basic open set as above. Then

$$
\Gamma^{n-}(U)=\cup_{l=1}^{\infty} \Gamma^{n-}\left(O^{l}\right) \in \mathscr{F}
$$

That there exists a measurable selection follows from the standard theory of measurable multi-functions [10], [70]. This is proved in Theorem 70.1.2 above. For $\sigma$ one of these measurable selections, the evaluation at $t$ is $\mathscr{F}$ measurable. Thus $\omega \rightarrow \sigma(t, \omega)$ is $\mathscr{F}$ measurable with values in $\mathbb{R}^{\infty}$. Also $t \rightarrow \sigma(t, \omega)$ is continuous, and so it follows that in fact $\sigma$ is product measurable as claimed.

Definition 70.2.4 Let $\Gamma(\omega) \equiv \cap_{n=1}^{\infty} \Gamma^{n}(\omega)$.
Lemma 70.2.5 $\Gamma$ is a nonempty $\mathscr{F}$ measurable set valued function having values in the compact sub-sets of $X$. There exists a measurable selection $\gamma$. For $\gamma$ a $\mathscr{F}$ measurable selection, $(t, \omega) \rightarrow \gamma(t, \omega)$ is $\mathscr{P}$ measurable. Also, for each $\omega$, there exists a subsequence, $u_{n(\omega)}(\cdot, \omega)$ such that for each $k$,

$$
\gamma_{k}(t, \omega)=\lim _{n(\omega) \rightarrow \infty} \mathbf{f}\left(u_{n(\omega)}(t, \omega)\right)_{k}=\lim _{n(\omega) \rightarrow \infty} m_{k} \int_{l_{m_{k}}(t)}^{t}\left\langle\phi_{r_{k}}, u_{n(\omega)}(s, \omega)\right\rangle_{V, V^{\prime}} d s
$$

Proof: Consider $\Gamma(\omega)=\cap_{n=1}^{\infty} \Gamma^{n}(\omega)$. Then $\omega \rightarrow \Gamma(\omega)$ is a compact set valued map in $X$. It is nonempty because each $\Gamma^{n}(\omega)$ is nonempty and compact, and these sets are nested. Is it $\mathscr{F}$ measurable? Each $\Gamma^{n}$ is compact valued and $\mathscr{F}$ measurable. Hence if $F$ is closed,

$$
\Gamma(\omega) \cap F=\cap_{n=1}^{\infty} \Gamma^{n}(\omega) \cap F
$$

and the left is non empty if and only if each $\Gamma^{n}(\omega) \cap F \neq \emptyset$. Hence for $F$ closed,

$$
\{\omega: \Gamma(\omega) \cap F \neq \emptyset\}=\cap_{n}\left\{\omega: \Gamma^{n}(\omega) \cap F \neq \emptyset\right\}
$$

and so

$$
\Gamma^{-}(F)=\cap_{n} \Gamma^{n-}(F) \in \mathscr{F}
$$

The last claim follows from the theory of multi-functions Theorem 70.1.2, [10], [70]. Since $\Gamma^{n}(\omega)$ is compact, the measurability of $\Gamma^{n}$, that $\Gamma^{n-}(U) \in \mathscr{F}$ for $U$ open implies the strong measurability of $\Gamma^{n}$, that $\Gamma^{n-}(F) \in \mathscr{F}$. Thus $\omega \rightarrow \Gamma(\omega)$ is non empty compact valued in $X$ and $\mathscr{F}$ measurable.

From standard theory of measurable multi-functions, Theorem 70.1.2, [10], [70], there exists a $\mathscr{F}$ measurable selection $\omega \rightarrow \gamma(\omega)$ with $\gamma(\omega) \in \Gamma(\omega)$ for each $\omega$. Now it follows that $t \rightarrow \gamma_{k}(t, \omega)$ is continuous. This is what it means for $\gamma(\omega) \in X$. What of the product measurability of $\gamma_{k}$ ? We know that $\omega \rightarrow \gamma_{k}(\omega)$ is $\mathscr{F}$ measurable into $C([0, T])$ and so since pointwise evaluation is continuous, $\omega \rightarrow \gamma_{k}(t, \omega)$ is $\mathscr{F}$ measurable. Then since $t \rightarrow$ $\gamma_{k}(t, \omega)$ is continuous, it follows that $\gamma_{k}$ is a $\mathscr{P}$ measurable real valued function and that $\gamma$ is a $\mathscr{P}$ measurable $\mathbb{R}^{\infty}$ valued function.

Since $\gamma(\omega) \in \Gamma(\omega)$, it follows that for each $n, \gamma(\omega) \in \Gamma^{n}(\omega)$. Therefore, there exists $j_{n} \geq n$ such that for each $\omega$,

$$
d\left(\mathbf{f}\left(u_{j_{n}}(\cdot, \omega)\right), \gamma(\omega)\right)<2^{-n}
$$

It follows that, taking a suitable subsequence, denoted as $\left\{u_{n(\omega)}(\cdot, \omega)\right\}$,

$$
\gamma(\omega)=\lim _{n(\omega) \rightarrow \infty} \mathbf{f}\left(u_{n(\omega)}(\cdot, \omega)\right)
$$

for each $\omega$. In particular, for each $k$

$$
\begin{equation*}
\gamma_{k}(t, \omega)=\lim _{n(\omega) \rightarrow \infty} \mathbf{f}\left(u_{n(\omega)}(t, \omega)\right)_{k}=\lim _{n(\omega) \rightarrow \infty} m_{k} \int_{l_{m_{k}}(t)}^{t}\left\langle\phi_{r_{k}}, u_{n(\omega)}(s, \omega)\right\rangle_{V, V^{\prime}} d s \tag{70.2.2}
\end{equation*}
$$

for each $t$.
Note that it is not clear that $(t, \omega) \rightarrow \mathbf{f}\left(u_{n(\omega)}(t, \omega)\right)$ is $\mathscr{P}$ measurable although $(t, \omega) \rightarrow$ $\gamma(t, \omega)$ is $\mathscr{P}$ measurable.

Proof of the theorem: By assumption, there exists a further subsequence still denoted by $n(\omega)$ such that, in addition to 70.2.2 above, the weak limit

$$
\lim _{n(\omega) \rightarrow \infty} u_{n(\omega)}(\cdot, \omega)=u(\cdot, \omega)
$$

exists in $L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)$ such that $t \rightarrow u(t, \omega)$ is weakly continuous into $V^{\prime}$. Then the above equation 70.2.2 continues to hold for this further subsequence and in addition to this,

$$
m_{k} \int_{l_{m_{k}}(t)}^{t}\left\langle\phi_{r_{k}}, u(s, \omega)\right\rangle_{V, V^{\prime}} d s=\lim _{n(\omega) \rightarrow \infty} m_{k} \int_{l_{m_{k}}(t)}^{t}\left\langle\phi_{r_{k}}, u_{n(\omega)}(s, \omega)\right\rangle_{V, V^{\prime}} d s=\gamma_{k}(t, \omega)
$$

Letting $\phi \in \mathscr{D}$ given, there exists a sub-sequence denoted by $k$ such that $m_{k} \rightarrow \infty$ and $\phi_{r_{k}}=\phi$ for all $k$. Then passing to a limit and using the assumed continuity of $s \rightarrow u(s, \omega)$, the left side of this equation converges to $\langle\phi, u(t, \omega)\rangle_{V, V^{\prime}}$ and so the right side, $\gamma_{k}(t, \omega)$ must also converge, this for each $\omega$. Since the right side is a product measurable function of $(t, \omega)$, it follows that the pointwise limit is also product measurable. Hence $(t, \omega) \rightarrow$ $\langle\phi, u(t, \omega)\rangle_{V, V^{\prime}}$ is product measurable, this for each $\phi \in \mathscr{D}$. Since $\mathscr{D}$ is a dense set, it follows that $(t, \omega) \rightarrow\langle\phi, u(t, \omega)\rangle_{V, V^{\prime}}$ is $\mathscr{P}$ measurable for all $\phi \in V$ and so by the Pettis theorem, [127], $(t, \omega) \rightarrow u(t, \omega)$ is $\mathscr{P}$ measurable into $V^{\prime}$.

One can say more about the measurability of the approximating sequence. In fact, we can obtain one for which $\omega \rightarrow u_{n(\omega)}(t, \omega)$ is also $\mathscr{F}$ measurable.

Lemma 70.2.6 Suppose, $u_{n(\omega)} \rightarrow u$ weakly in $L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)$ where $u$ is product measurable measurable and $\left\{u_{n(\omega)}\right\}$ is a subsequence of $\left\{u_{n}\right\}$ where

$$
\sup _{t \in[0, T]}\left\|u_{n}(t, \omega)\right\|_{V^{\prime}}<C(\omega), \text { for } \omega \notin N \text { a set of measure zero, }
$$

Then for each $\omega \notin N$, there exists a subsequence of $\left\{u_{n}\right\}$ denoted as $\left\{u_{k(\omega)}\right\}$ such that $u_{k(\omega)} \rightarrow u$ weakly in $L^{p^{\prime}}\left([0, T] ; V^{\prime}\right), \omega \rightarrow k(\omega)$ is $\mathscr{F}$ measurable, and $\omega \rightarrow u_{k(\omega)}(t, \omega)$ is also $\mathscr{F}$ measurable, the last assertions holding for all $\omega \notin N$.

Proof: For $f, g \in L^{p^{\prime}}\left([0, T] ; V^{\prime}\right) \equiv \mathscr{V}^{\prime}, L^{p}([0, T] ; V) \equiv \mathscr{V}$, let $\left\{\phi_{k}\right\}$ be a countable dense subset of $L^{p}([0, T] ; V)$. Then a bounded set in $L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)$ with the weak topology can be considered a complete metric space using the following metric.

$$
d(f, g) \equiv \sum_{j=1}^{\infty} 2^{-j} \frac{\left|\left\langle\phi_{k}, f-g\right\rangle_{\mathscr{V}, \mathscr{V}^{\prime}}\right|}{1+\left|\left\langle\phi_{k}, f-g\right\rangle_{\mathscr{V}, \mathscr{V}^{\prime}}\right|}
$$

Now let $k(\omega)$ be the first index from the indices of $\left\{u_{n}\right\}$ at least as large as $k$ such that

$$
d\left(u_{k(\omega)}, u\right) \leq 2^{-k}
$$

Such an index exists because there exists a convergent sequence $u_{n(\omega)}$ which does converge weakly to $u$. This is just picking another one which happens to also retain measurability. In fact,

$$
\{\omega: k(\omega)=l\}=\left\{\omega: d\left(u_{l}, u\right) \leq 2^{-k}\right\} \cap \cap_{j=k}^{k-1}\left\{\omega: d\left(u_{j}, u\right)>2^{-k}\right\}
$$

Since $u$ is product measurable and each $u_{l}$ is also product measurable, these are all measurable sets with respect to $\mathscr{F}$ and so $\omega \rightarrow k(\omega)$ is $\mathscr{F}$ measurable. Now we have $u_{k(\omega)} \rightarrow u$ weakly in $L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)$ for each $\omega$ with each function being $\mathscr{F}$ measurable because

$$
u_{k(\omega)}(t, \omega)=\sum_{j=1}^{\infty} \mathscr{X}_{[k(\omega)=j]} u_{j}(t, \omega)
$$

and every term in the sum is $\mathscr{F}$ measurable.
The following obvious corollary shows the significance of this lemma.
Corollary 70.2.7 Let $V$ be a reflexive separable Banach space and $V^{\prime}$ its dual and $\frac{1}{p}+$ $\frac{1}{p^{\prime}}=1$ where $p>1$ as usual. Let the functions $t \rightarrow u_{n(\omega)}(t, \omega)$ be in $L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)$ and $(t, \omega) \rightarrow u_{n(\omega)}(t, \omega)$ be $\mathscr{B}([0, T]) \times \mathscr{F} \equiv \mathscr{P}$ measurable into $V^{\prime}$. Here $\left\{u_{n(\omega)}\right\}_{n=1}^{\infty}$ is a sequence, one for each $\omega$. Suppose there is a set of measure zero $N$ such that if $\omega \notin N$, then for all $n$,

$$
\sup _{t \in[0, T]}\left\|u_{n(\omega)}(t, \omega)\right\|_{V^{\prime}} \leq C(\omega)
$$

Also suppose for each $\omega \notin N$, each subsequence of $\left\{u_{n(\omega)}\right\}$ has a further subsequence which converges weakly in $L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)$ to $u(\cdot, \omega) \in L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)$ such that $t \rightarrow u(t, \omega)$ is weakly continuous into $V^{\prime}$. Then there exists u product measurable, with $t \rightarrow u(t, \omega)$ being weakly continuous into $V^{\prime}$. Moreover, there exists, for each $\omega \notin N$, a subsequence $u_{n(\omega)}$ such that $u_{n(\omega)}(\cdot, \omega) \rightarrow u(\cdot, \omega)$ weakly in $L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)$.

Proof: It suffices to consider the functions $v_{n}(t, \omega) \equiv u_{n(\omega)}(t, \omega)$ and use the result of Theorem 70.2.1.

Of course when you have all functions having values in $H$ a separable Hilbert space, there is no change in the argument to obtain the following theorem.

Theorem 70.2.8 Let $H$ be a real separable Hilbert space. For $n \in \mathbb{N}$ let the functions $t \rightarrow$ $u_{n}(t, \omega)$ be in $L^{2}([0, T] ; H)$ and $(t, \omega) \rightarrow u_{n}(t, \omega)$ be $\mathscr{B}([0, T]) \times \mathscr{F} \equiv \mathscr{P}$ measurable into $H$. Suppose there is a set of measure zero $N$ such that if $\omega \notin N$, then for all $n$,

$$
\sup _{t \in[0, T]}\left|u_{n}(t, \omega)\right|_{H} \leq C(\omega)
$$

Also suppose for each $\omega \notin N$, each subsequence of $\left\{u_{n}\right\}$ has a further subsequence which converges weakly in $L^{2}([0, T] ; H)$ to $u(\cdot, \omega) \in L^{2}([0, T] ; H)$ such that $t \rightarrow u(t, \omega)$ is weakly continuous into $H$. Then there exists u product measurable, with $t \rightarrow u(t, \omega)$ being weakly continuous into $H$. Moreover, there exists, for each $\omega \notin N$, a subsequence $u_{n(\omega)}$ such that $u_{n(\omega)}(\cdot, \omega) \rightarrow u(\cdot, \omega)$ weakly in $L^{2}([0, T] ; H)$.

### 70.3 Measurability In Finite Dimensional Problems

What follows is like the Peano existence theorem from ordinary differential equations except that it provides a solution which retains product measurability. It is a nice example of the above theory. It will be used in the next section in the Galerkin method.

Lemma 70.3.1 Suppose $\mathbf{N}(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega) \in \mathbb{R}^{d}$ for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^{d}, t \in[0, T]$ and

$$
(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega) \rightarrow \mathbf{N}(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega)
$$

is progressively measurable relative to the filtration consisting of the single $\sigma$ algebra $\mathscr{F}$. Also suppose that $(t, \mathbf{u}, \mathbf{v}, \mathbf{w}) \rightarrow \mathbf{N}(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega)$ is continuous and that also $\mathbf{N}(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega)$ is uniformly bounded in $(t, \mathbf{u}, \mathbf{v}, \mathbf{w})$ by $M(\omega)$. Let $\mathbf{f}$ be $\mathscr{P}$ measurable and $\mathbf{f}(\cdot, \omega) \in$ $L^{2}\left([0, T] ; \mathbb{R}^{d}\right)$. Then for $h>0$, there exists a $\mathscr{P}$ measurable solution $\mathbf{u}$ to the integral equation

$$
\mathbf{u}(t, \omega)-\mathbf{u}_{0}(\omega)+\int_{0}^{t} \mathbf{N}(s, \mathbf{u}(s, \omega), \mathbf{u}(s-h, \omega), \mathbf{w}(s, \omega) \omega) d s=\int_{0}^{t} \mathbf{f}(s, \omega) d s
$$

Here $\mathbf{u}_{0}$ has values in $\mathbb{R}^{d}$ and is $\mathscr{F}$ measurable, $\mathbf{u}(s-h, \omega) \equiv \mathbf{u}_{0}(\omega)$ if $s-h<0$ and for $\mathbf{w}_{0}$ a given $\mathscr{F}$ measurable function,

$$
\mathbf{w}(t, \omega) \equiv \mathbf{w}_{0}(\omega)+\int_{0}^{t} \mathbf{u}(s, \omega) d s
$$

Proof: Let $\mathbf{u}_{n}$ be the solution to the following equation:

$$
\begin{aligned}
& \mathbf{u}_{n}(t, \omega)-\mathbf{u}_{0}(\omega)+\int_{0}^{t} \mathbf{N}\left(s, \tau_{1 / n} \mathbf{u}_{n}(s, \omega), \mathbf{u}_{n}(s-h, \omega), \tau_{1 / n} \mathbf{w}_{n}(s, \omega), \omega\right) d s \\
= & \int_{0}^{t} \mathbf{f}(s, \omega) d s
\end{aligned}
$$

where here $\tau_{1 / n}$ is defined as follows. For $\delta>0$,

$$
\tau_{\delta} \mathbf{u}(s) \equiv\left\{\begin{array}{l}
\mathbf{u}(s-\delta) \text { if } s>\delta \\
\mathbf{0} \text { if } s-\delta \leq 0
\end{array}\right.
$$

It follows that $(t, \omega) \rightarrow \mathbf{u}_{n}(t, \omega)$ is $\mathscr{P}$ measurable. From the assumptions on $\mathbf{N}$, it follows that for fixed $\omega,\left\{\mathbf{u}_{n}(\cdot, \omega)\right\}$ is uniformly bounded:

$$
\sup _{t \in[0, T]}\left|\mathbf{u}_{n}(t, \omega)\right| \leq\left|\mathbf{u}_{0}(\omega)\right|+\int_{0}^{T} M(\omega) d s+\int_{0}^{T}|\mathbf{f}(s, \omega)| d s=: C(\omega),
$$

and is also equicontinuous because for $s<t$,

$$
\begin{aligned}
& \left|\mathbf{u}_{n}(t, \omega)-\mathbf{u}_{n}(s, \omega)\right| \\
\leq & \int_{s}^{t}\left|\mathbf{N}\left(r, \tau_{1 / n} \mathbf{u}_{n}(r, \omega), \mathbf{u}_{n}(r-h, \omega), \tau_{1 / n} \mathbf{w}_{n}(r, \omega), \omega\right)\right| d r
\end{aligned}
$$

$$
+\int_{s}^{t}|\mathbf{f}(r, \omega)| d r \leq C(\omega, \mathbf{f})|t-s|^{1 / 2}
$$

Therefore, by the Ascoli-Arzelà theorem, for each $\omega$, there exists a subsequence $\tilde{n}(\omega)$ depending on $\omega$ and a function $\tilde{\mathbf{u}}(t, \omega)$ such that

$$
\mathbf{u}_{\tilde{n}(\omega)}(t, \omega) \rightarrow \tilde{\mathbf{u}}(t, \omega) \text { uniformly in } C\left([0, T] ; \mathbb{R}^{d}\right)
$$

This verifies the assumptions of Theorem 70.2.8.
It follows that there exists $\overline{\mathbf{u}}$ product measurable and a subsequence $\left\{\mathbf{u}_{n(\omega)}\right\}$ for each $\omega$ such that

$$
\lim _{n(\omega) \rightarrow \infty} \mathbf{u}_{n(\omega)}(\cdot, \omega)=\overline{\mathbf{u}}(\cdot, \omega) \text { weakly in } L^{2}\left([0, T] ; \mathbb{R}^{d}\right)
$$

and that $t \rightarrow \overline{\mathbf{u}}(t, \omega)$ is continuous. (Note that weak continuity is the same as continuity in $\mathbb{R}^{d}$.) The same argument given above applied to the $\mathbf{u}_{n(\omega)}$ for a fixed $\omega$ yields a further subsequence, denoted as $\left\{\mathbf{u}_{\bar{n}(\omega)}(\cdot, \omega)\right\}$ which converges uniformly to a function $\mathbf{u}(\cdot, \omega)$ on $[0, T]$. So $\overline{\mathbf{u}}(t, \omega)=\mathbf{u}(t, \omega)$ in $L^{2}\left([0, T] ; \mathbb{R}^{d}\right)$. Since both of these functions are continuous in $t$, they must be equal for all $t$. Hence, $(t, \omega) \rightarrow \mathbf{u}(t, \omega)$ is product measurable. Passing to the limit in the equation solved by $\left\{\mathbf{u}_{\bar{n}(\omega)}(\cdot, \omega)\right\}$ using the dominated convergence theorem, we obtain

$$
\mathbf{u}(t, \omega)-\mathbf{u}_{0}(\omega)+\int_{0}^{t} \mathbf{N}(s, \mathbf{u}(s, \omega), \mathbf{u}(s-h, \omega), \mathbf{w}(s, \omega), \omega) d s=\int_{0}^{t} \mathbf{f}(s, \omega) d s
$$

Thus $t \rightarrow \mathbf{u}(t, \omega)$ is a product measurable solution to the integral equation.
This lemma gives the existence of the approximate solutions in the following theorem in which the assumption that the integrand is bounded is replaced with an estimate. The following elementary consideration will be used whenever convenient. Note that it holds for all $\omega$.

Remark 70.3.2 When $\mathbf{w}(t) \equiv \mathbf{w}_{0}(\omega)+\int_{0}^{t} \mathbf{u}(s, \omega) d s$,

$$
\mathbf{v}(t)=\left\{\begin{array}{c}
\mathbf{u}(t-h) \text { if } t \geq h \\
\mathbf{u}_{0} \text { if } t<h
\end{array}\right.
$$

and when the estimate

$$
(\mathbf{N}(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega), \mathbf{u}) \geq-C(t, \omega)-\mu\left(|\mathbf{u}|^{2}+|\mathbf{v}|^{2}+|\mathbf{w}|^{2}\right)
$$

holds, it follows that

$$
\int_{0}^{t}(\mathbf{N}(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega), \mathbf{u}) d s \geq-C\left(C(\omega)+\int_{0}^{t}|\mathbf{u}|^{2} d s\right)
$$

for some constant $C$ depending on the initial data but not on $\mathbf{u}$.
To see this,

$$
\int_{0}^{t}|\mathbf{u}(s-h)|^{2} d s=\int_{0}^{h}\left|\mathbf{u}_{0}\right|^{2} d s+\int_{h}^{t}|\mathbf{u}(s-h)|^{2} d s
$$

$$
=\left|\mathbf{u}_{0}\right|^{2} h+\int_{0}^{t-h}|\mathbf{u}(s)|^{2} d s \leq\left|\mathbf{u}_{0}\right|^{2} h+\int_{0}^{t}|\mathbf{u}(s)|^{2} d s
$$

if $t \geq h$ and if $s<h$, this is dominated by

$$
\left|\mathbf{u}_{0}\right|^{2} t \leq\left|\mathbf{u}_{0}\right|^{2} h \leq\left|\mathbf{u}_{0}\right|^{2} h+\int_{0}^{t}|\mathbf{u}(s)|^{2} d s
$$

As to the terms from $\mathbf{w}$,

$$
\begin{aligned}
& \int_{0}^{t}|\mathbf{w}(s)|^{2} d s \\
\leq & \int_{0}^{t}\left|\mathbf{w}_{0}+\int_{0}^{s} \mathbf{u}(r) d r\right|^{2} d s \leq \int_{0}^{t}\left(\left|\mathbf{w}_{0}\right|+\left|\int_{0}^{s} \mathbf{u}(r) d r\right|\right)^{2} d s \\
\leq & \int_{0}^{t}\left(\left|\mathbf{w}_{0}\right|^{2}+2\left|\mathbf{w}_{0}\right|\left|\int_{0}^{s} \mathbf{u}(r) d r\right|+\left|\int_{0}^{s} \mathbf{u}(r) d r\right|^{2}\right) d s \\
\leq & T\left|\mathbf{w}_{0}\right|^{2}+T\left|\mathbf{w}_{0}\right|^{2}+\int_{0}^{t}\left|\int_{0}^{s} \mathbf{u}(r) d r\right|^{2} d s+\int_{0}^{t}\left|\int_{0}^{s} \mathbf{u}(r) d r\right|^{2} d s \\
\leq & 2 T\left|\mathbf{w}_{0}\right|^{2}+2 \int_{0}^{t}\left(\int_{0}^{s}|\mathbf{u}(r)| d r\right)^{2} d s \leq 2 T\left|\mathbf{w}_{0}\right|^{2}+2 \int_{0}^{t} s \int_{0}^{s}|\mathbf{u}(r)|^{2} d r d s \\
\leq & 2 T\left|\mathbf{w}_{0}\right|^{2}+2 T \int_{0}^{t} \int_{0}^{s}|\mathbf{u}(r)|^{2} d r d s \leq 2 T\left|\mathbf{w}_{0}\right|^{2}+2 T^{2} \int_{0}^{t}|\mathbf{u}(r)|^{2} d r
\end{aligned}
$$

From this, the claimed result follows.
Theorem 70.3.3 Suppose $\mathbf{N}(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega) \in \mathbb{R}^{d}$ for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^{d}, t \in[0, T]$ and

$$
(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega) \rightarrow \mathbf{N}(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega)
$$

is progressively measurable with respect to a constant filtration $\mathscr{F}_{t}=\mathscr{F}$. Also suppose $(t, \mathbf{u}, \mathbf{v}, \mathbf{w}) \rightarrow \mathbf{N}(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega)$ is continuous and satisfies $C(\cdot, \omega) \geq 0$ in $L^{1}([0, T])$ and some $\mu>0$ :

$$
(\mathbf{N}(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega), \mathbf{u}) \geq-C(t, \omega)-\mu\left(|\mathbf{u}|^{2}+|\mathbf{v}|^{2}+|\mathbf{w}|^{2}\right)
$$

Also let $\mathbf{f}$ be product measurable and $\mathbf{f}(\cdot, \omega) \in L^{2}\left([0, T] ; \mathbb{R}^{d}\right)$. Then for $h>0$, there exists a product measurable solution $\mathbf{u}$ to the integral equation

$$
\mathbf{u}(t, \omega)-\mathbf{u}_{0}(\omega)+\int_{0}^{t} \mathbf{N}(s, \mathbf{u}(s, \omega), \mathbf{u}(s-h, \omega), \mathbf{w}(s, \omega), \omega) d s=\int_{0}^{t} \mathbf{f}(s, \omega) d s
$$

where $\mathbf{u}_{0}$ has values in $\mathbb{R}^{d}$ and is $\mathscr{F}$ measurable. Here $\mathbf{u}(s-h, \omega) \equiv \mathbf{u}_{0}(\omega)$ for all $s-h \leq 0$ and for $\mathbf{w}_{0}$ a given $\mathscr{F}$ measurable function,

$$
\mathbf{w}(t, \omega) \equiv \mathbf{w}_{0}(\omega)+\int_{0}^{t} \mathbf{u}(s, \omega) d s
$$

Proof: Let $P_{m}$ denote the projection onto the closed ball $\overline{B\left(\mathbf{0}, 9^{m}\right)}$. Then from the above lemma, there exists a product measurable solution $\mathbf{u}_{m}$ to the integral equation

$$
\begin{aligned}
& \mathbf{u}_{m}(t, \omega)-\mathbf{u}_{0}(\omega)+\int_{0}^{t} \mathbf{N}\left(s, P_{m} \mathbf{u}_{m}(s, \omega), P_{m} \mathbf{u}_{m}(s-h, \omega), P_{m} \mathbf{w}_{m}(s, \omega), \omega\right) d s \\
= & \int_{0}^{t} \mathbf{f}(s, \omega) d s
\end{aligned}
$$

Define a stopping time

$$
\tau_{m}(\omega) \equiv \inf \left\{t \in[0, T]:\left|\mathbf{u}_{m}(t, \omega)\right|^{2}+\left|\mathbf{w}_{m}(t, \omega)\right|^{2}>2^{m}\right\}
$$

where $\inf \emptyset \equiv T$. Localizing with the stopping time,

$$
\begin{aligned}
& \mathbf{u}_{m}^{\tau_{m}}(t, \omega)-\mathbf{u}_{0}(\omega)+\int_{0}^{t} \mathscr{X}_{\left[0, \tau_{m}\right]} \mathbf{N}\left(s, \mathbf{u}_{m}^{\tau_{m}}(s, \omega), \mathbf{u}_{m}^{\tau_{m}}(s-h, \omega), \mathbf{w}_{m}^{\tau_{m}}(s, \omega), \omega\right) d s \\
= & \int_{0}^{t} \mathscr{X}_{\left[0, \tau_{m}\right]} \mathbf{f}(s, \omega) d s .
\end{aligned}
$$

Note how the stopping time allowed the elimination of the projection map in the equation. Then we get

$$
\begin{aligned}
& \frac{1}{2}\left|\mathbf{u}_{m}^{\tau_{m}}(t, \omega)\right|^{2}-\frac{1}{2}\left|\mathbf{u}_{0}(\omega)\right|^{2} \\
& +\int_{0}^{t}\left(\mathscr{X}_{\left[0, \tau_{m}\right]} \mathbf{N}\left(s, \mathbf{u}_{m}^{\tau_{m}}(s, \omega), \mathbf{u}_{m}^{\tau_{m}}(s-h, \omega), \mathbf{w}_{m}^{\tau_{m}}(s, \omega), \omega\right), \mathbf{u}_{m}^{\tau_{m}}(s, \omega)\right) d s \\
= & \int_{0}^{t} \mathscr{X}_{\left[0, \tau_{m}\right]}\left(\mathbf{f}(s, \omega), \mathbf{u}_{m}^{\tau_{m}}(s, \omega)\right) d s .
\end{aligned}
$$

From the estimate,

$$
\begin{aligned}
\frac{1}{2}\left|\mathbf{u}_{m}^{\tau_{m}}(t, \omega)\right|^{2}- & \frac{1}{2}\left|\mathbf{u}_{0}(\omega)\right|^{2} \leq \int_{0}^{t}\left(\mu\left(\left|\mathbf{u}_{m}^{\tau_{m}}(s, \omega)\right|^{2}+\left|\mathbf{u}_{m}^{\tau_{m}}(s-h, \omega)\right|^{2}+\left|\mathbf{w}_{m}^{\tau_{m}}(s, \omega)\right|^{2}\right)\right. \\
& \left.+C(s, \omega)+\frac{1}{2}|\mathbf{f}(s, \omega)|^{2}\right) d s+\frac{1}{2} \int_{0}^{t}\left|\mathbf{u}_{m}^{\tau_{m}}(s, \omega)\right|^{2} d s
\end{aligned}
$$

Note that

$$
\left|\mathbf{u}_{0}\right|^{2} h+\int_{0}^{t}\left|\mathbf{u}_{n}^{\tau_{n}}(s)\right|^{2} d s \geq \int_{0}^{t}\left|\mathbf{u}_{n}^{\tau_{n}}(s-h, \omega)\right|^{2} d s
$$

and

$$
\begin{aligned}
& \int_{0}^{t}\left|\mathbf{w}_{n}^{\tau_{n}}(s, \omega)\right|^{2} d s=\int_{0}^{t}\left|\mathbf{w}_{0}+\int_{0}^{s} \mathscr{X}_{\left[0, \tau_{n}\right]} \mathbf{u}_{n}(r) d r\right|^{2} d s \\
&=\int_{0}^{t}\left|\mathbf{w}_{0}+\int_{0}^{s} \mathscr{X}_{\left[0, \tau_{n}\right]} \mathbf{u}_{n}^{\tau_{n}}(r) d r\right|^{2} d s \\
& \leq C\left(\mathbf{w}_{0}(\omega)\right)+C T \int_{0}^{t}\left|\mathbf{u}_{n}^{\tau_{n}}\right|^{2} d s
\end{aligned}
$$

By Gronwall's inequality,

$$
\begin{aligned}
\left|\mathbf{u}_{m}^{\tau_{m}}(t, \omega)\right|^{2} & \leq C\left(\mathbf{u}_{0}(\omega), \mathbf{w}_{0}(\omega), \mu,\|C(\cdot, \omega)\|_{L^{1}\left([0, T] ; \mathbb{R}^{d}\right)}, T,\|\mathbf{f}(\cdot, \omega)\|_{L^{2}\left([0, T] ; \mathbb{R}^{d}\right)}\right) \\
& \equiv C(\omega)
\end{aligned}
$$

Thus, for a.e. $\omega, \tau_{m}=T$ for all $m$ large enough, say for $m \geq M(\omega)$ where

$$
C(\omega) \leq 2^{M(\omega)}
$$

Then define the functions

$$
\mathbf{y}_{n}(t, \omega) \equiv \mathbf{u}_{n}^{\tau_{n}}(t, \omega)
$$

These are product measurable and

$$
\begin{aligned}
& \mathbf{y}_{n}(t, \omega)-\mathbf{u}_{0}(\omega)+ \\
& \int_{0}^{t} \mathscr{X}_{\left[0, \tau_{n}\right]} \mathbf{N}\left(s, \mathbf{y}_{n}(s, \omega), \mathbf{y}_{n}(s-h, \omega), \mathbf{w}_{0}(\omega)+\int_{0}^{s} \mathbf{y}_{n}(r, \omega) d r, \omega\right) d s \\
= & \int_{0}^{t} \mathscr{X}_{\left[0, \tau_{n}\right]} \mathbf{f}(s, \omega) d s .
\end{aligned}
$$

So each is continuous in $t$. For large enough $n, \tau_{n}=T$ and hence

$$
\begin{aligned}
& \mathbf{y}_{n}(t, \omega)-\mathbf{u}_{0}(\omega)+\int_{0}^{t} \mathbf{N}\left(s, \mathbf{y}_{n}(s, \omega), \mathbf{y}_{n}(s-h, \omega), \mathbf{w}_{0}(\omega)+\int_{0}^{s} \mathbf{y}_{n}(r, \omega) d r, \omega\right) d s \\
= & \int_{0}^{t} \mathbf{f}(s, \omega) d s
\end{aligned}
$$

Also these satisfy the inequality

$$
\begin{equation*}
\sup _{t \in[0, T]}\left|\mathbf{y}_{n}(t, \omega)\right|^{2} \leq C(\omega) \leq 2^{M(\omega)}<9^{M(\omega)} \tag{70.3.4}
\end{equation*}
$$

the constant on the right not depending on $n$. Thus for fixed $\omega$, we can regard $\mathbf{N}$ as bounded and the same reasoning used in the above lemma involving the Ascoli-Arzelà theorem implies that every subsequence has a further subsequence which converges to a solution to the integral equation for that $\omega$. Thus it is continuous into $\mathbb{R}^{d}$. It follows from the measurable selection theorem above that there exists $\mathbf{u}$ product measurable and continuous in $t$ such that $\mathbf{u}(\cdot, \omega)=\lim _{n(\omega) \rightarrow \infty} \mathbf{y}_{n(\omega)}(\cdot, \omega)$ in $L^{2}\left([0, T] ; \mathbb{R}^{d}\right)$. By the reasoning of the above lemma, there is a further subsequence, denoted the same way, for which $\lim _{n \rightarrow \infty} \mathbf{y}_{n(\omega)}$ in $C\left([0, T] ; \mathbb{R}^{d}\right)$ solves the integral equation for a fixed $\omega$. Thus $\mathbf{u}$ is a product measurable solution to the integral equation as claimed.

We made use of an estimate in order to get the conclusion of this theorem. However, all that is really needed is the following.

Corollary 70.3.4 Suppose $\mathbf{N}(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega) \in \mathbb{R}^{d}$ for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^{d}, t \in[0, T]$ and

$$
(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega) \rightarrow \mathbf{N}(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega)
$$

is progressively measurable with respect to a constant filtration $\mathscr{F}_{t}=\mathscr{F}$. Also suppose $(t, \mathbf{u}, \mathbf{v}, \mathbf{w}) \rightarrow \mathbf{N}(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega)$ is continuous. Suppose for each $\omega$, there exists an estimate for any solution $\mathbf{u}(\cdot, \omega)$ to the integral equation

$$
\begin{equation*}
\mathbf{u}(t, \omega)-\mathbf{u}_{0}(\omega)+\int_{0}^{t} \mathbf{N}(s, \mathbf{u}(s, \omega), \mathbf{u}(s-h, \omega), \mathbf{w}(s, \omega), \omega) d s=\int_{0}^{t} \mathbf{f}(s, \omega) d s \tag{70.3.5}
\end{equation*}
$$

which is of the form

$$
\sup _{t \in[0, T]}|\mathbf{u}(t, \omega)| \leq C(\omega)<\infty
$$

Also let $\mathbf{f}$ be product measurable and $\mathbf{f}(\cdot, \omega) \in L^{1}\left([0, T] ; \mathbb{R}^{d}\right)$. Here $\mathbf{u}_{0}$ has values in $\mathbb{R}^{d}$ and is $\mathscr{F}$ measurable and $\mathbf{u}(s-h, \omega) \equiv \mathbf{u}_{0}(\omega)$ whenever $s-h \leq 0$ and

$$
\mathbf{w}(t, \omega) \equiv \mathbf{w}_{0}(\omega)+\int_{0}^{t} \mathbf{u}(s, \omega) d s
$$

where $\mathbf{w}_{0}$ is a given $\mathscr{F}$ measurable function. Then for $h>0$, there exists a product measurable solution $\mathbf{u}$ to the integral equation 70.3.5.

Of course the same conclusions apply when there is no dependence in the integral equation on $\mathbf{u}(s-h, \omega)$ or the integral $\mathbf{w}(t, \omega)$. Note that these theorems hold for all $\omega$.

### 70.4 The Navier-Stokes Equations

In this section, we study the stochastic Navier-Stokes equations of arbitrary dimension. We prove there exists a global solution which is product measurable. The main result is Theorem 70.4.6. We use the Galerkin method and Theorem 70.3.3 to get product measurable approximate solutions. Then we take weak limits and get path solutions. After this, we apply Theorem 70.2.8 to get product measurable global solutions.

As in [15], an important part of our argument is the theorem in Lions [91] which follows. See Theorem 69.5.6.

Theorem 70.4.1 Let $W, H$, and $V^{\prime}$ be separable Banach spaces. Suppose $W \subseteq H \subseteq V^{\prime}$ where the injection map is continuous from $H$ to $V^{\prime}$ and compact from $W$ to $H$. Let $q_{1} \geq 1$, $q_{2}>1$, and define

$$
\begin{aligned}
& S \equiv\left\{u \in L^{q_{1}}([a, b] ; W): u^{\prime} \in L^{q_{2}}\left([a, b] ; V^{\prime}\right)\right. \\
& \text { and } \left.\|u\|_{L^{q_{1}}([a, b] ; W)}+\left\|u^{\prime}\right\|_{L^{q_{2}}\left([a, b] ; V^{\prime}\right)} \leq R\right\} .
\end{aligned}
$$

Then $S$ is pre-compact in $L^{q_{1}}([a, b] ; H)$. This means that if $\left\{u_{n}\right\}_{n=1}^{\infty} \subseteq S$, it has a subsequence $\left\{u_{n_{k}}\right\}$ which converges in $L^{q_{1}}([a, b] ; H)$.

A proof of a generalization of this theorem is found on Page 2385. Let $U$ be a bounded open set in $\mathbb{R}^{d}$ and let $S$ denote the functions which are infinitely differentiable having zero divergence and also having compact support in $U$. We have in mind $d=3$, but the approach is not limited by dimension. We use the same Galerkin method found in [15], the details being included in slightly abbreviated form for convenience of the reader. The difference is
that we switch the roles of $V$ and $W$ along with a few other minor modifications. This is the part of the argument which gives a solution for each $\omega$ and it is standard material. Define

$$
V \equiv \bar{S} \text { in }\left(H^{d^{*}}(U)\right)^{d}, W \equiv \bar{S} \text { in }\left(H^{1}(U)\right)^{d}, \text { and } H \equiv \bar{S} \text { in }\left(L^{2}(U)\right)^{d}
$$

where $d^{*}$ is such that for $\mathbf{w} \in\left(H^{d^{*}}(U)\right)^{d}$ then $\|D \mathbf{w}\|_{L^{\infty}(U)}<\infty$. For example, you could take $d^{*}=3$ for $d=3$. In [15], they take $d^{*}=8$ which is large enough to work for all dimensions of interest.

Let $A: W \rightarrow W^{\prime}$ and $N: W \rightarrow V^{\prime}$ be defined by

$$
\langle A \mathbf{u}, \mathbf{v}\rangle \equiv \int_{U} \nabla u_{i} \cdot \nabla v_{i} d x, \quad\langle N \mathbf{u}, \mathbf{v}\rangle \equiv-\int_{U} u_{i} u_{j} v_{j, i} d x
$$

Then $N$ is a continuous function. Indeed, pick $\mathbf{v} \in V$ and suppose $\mathbf{u}_{n} \rightarrow \mathbf{u}$ in $W$, then

$$
\begin{aligned}
& \left|\left\langle N \mathbf{u}-N \mathbf{u}_{n}, \mathbf{v}\right\rangle\right| \\
\leq & \int_{U}\left|\sum_{i, j}\left(u_{n i} u_{n j}-u_{i} u_{j}\right) v_{j, i}\right| d x \leq C\|\mathbf{v}\|_{V} \int_{U}\left(\left|\mathbf{u}_{n}\right|+|\mathbf{u}|\right)\left(\left|\mathbf{u}_{n}-\mathbf{u}\right|\right) d x \\
\leq & C\|\mathbf{v}\|_{V}\left(\int_{U}\left|\mathbf{u}_{n}\right|^{2}+|\mathbf{u}|^{2} d x\right)^{1 / 2}\left(\int_{U}\left|\mathbf{u}_{n}-\mathbf{u}\right|^{2} d x\right)^{1 / 2},
\end{aligned}
$$

where what multiplies $\|\mathbf{v}\|_{V}$ clearly converges to 0 .
An abstract form for the incompressible Navier-Stokes equations is

$$
\mathbf{u}^{\prime}+v A \mathbf{u}+N \mathbf{u}=\mathbf{f}, \mathbf{u}(0)=\mathbf{u}_{0}
$$

where $\mathbf{f} \in L^{2}\left([0, T] ; W^{\prime}\right)$, for some fixed $T>0$. As in [15], we will let $v=1$ to simplify the presentation. A stochastic version of this would be the integral equation in $V^{\prime}$

$$
\mathbf{u}(t, \omega)-\mathbf{u}_{0}(\omega)+\int_{0}^{t} A(\mathbf{u}(s, \omega)) d s+\int_{0}^{t} N(\mathbf{u}(s, \omega)) d s=\int_{0}^{t} \mathbf{f}(s, \omega) d s+\mathbf{q}(t, \omega)
$$

where $\mathbf{q}(\cdot, \omega)$ will be continuous into $V,(t, \omega) \rightarrow \mathbf{q}(t, \omega)$ will be product measurable having values in $V$, and $\mathbf{q}(0, \omega)=\mathbf{0}$. So $\mathbf{q}$ here is a fixed stochastic process, which serves as the random source. Also $(t, \omega) \rightarrow \mathbf{f}(t, \omega)$ will be product measurable into $W^{\prime}$ as well as having $t \rightarrow \mathbf{f}(t, \omega)$ in $L^{2}\left([0, T] ; W^{\prime}\right)$. Our problem is to show the existence of a product measurable solution.

Let $T$ be any fixed positive number and let $\mathbf{q}$ be any fixed process satisfying the above.
Definition 70.4.2 A global solution to the above integral equation is a process $\mathbf{u}(t, \omega)$, for which $\omega \rightarrow \mathbf{u}(t, \omega)$ is $\mathscr{F}$ measurable and satisfies for each $\omega$ outside a set of measure zero and all $t \in[0, T]$,

$$
\mathbf{u}(t, \omega)-\mathbf{u}_{0}(\omega)+\int_{0}^{t} A(\mathbf{u}(s, \omega)) d s+\int_{0}^{t} N(\mathbf{u}(s, \omega)) d s=\int_{0}^{t} \mathbf{f}(s, \omega) d s+\mathbf{q}(t, \omega) .
$$

In order to apply the earlier result, let $\mathbf{w}(t, \omega)=\mathbf{u}(t, \omega)-\mathbf{q}(t, \omega)$ and write the equation in terms of $\mathbf{w}$,

$$
\begin{aligned}
& \mathbf{w}(t, \omega)-\mathbf{u}_{0}(\omega)+\int_{0}^{t} A(\mathbf{w}(\mathbf{s}, \omega)+\mathbf{q}(\mathbf{s}, \omega)) d s+\int_{0}^{t} N(\mathbf{w}(\mathbf{s}, \omega)+\mathbf{q}(\mathbf{s}, \omega)) d s \\
= & \int_{0}^{t} \mathbf{f}(s, \omega) d s
\end{aligned}
$$

It turns out that it is convenient to define

$$
\langle B(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle \equiv-\int_{U} u_{i} v_{j} w_{j, i} d x
$$

and write the equation in the following form:

$$
\mathbf{w}(t, \omega)-\mathbf{u}_{0}(\omega)+\int_{0}^{t} A(\mathbf{w}(s, \omega)) d s+\int_{0}^{t} \hat{N}(\mathbf{w}(s, \omega)) d s=\int_{0}^{t} \hat{\mathbf{f}}(s, \omega) d s
$$

where

$$
\begin{aligned}
\hat{N}(\mathbf{w}(t, \omega)) & \equiv & & N(\mathbf{w}(t, \omega))+B(\mathbf{w}(t, \omega), \mathbf{q}(t, \omega))+B(\mathbf{q}(t, \omega), \mathbf{w}(t, \omega)) \\
\hat{\mathbf{f}}(t, \omega) & \equiv & & \mathbf{f}(t, \omega)-A(\mathbf{q}(t, \omega))-N(\mathbf{q}(t, \omega))
\end{aligned}
$$

This is an equation in $V^{\prime}$. Moreover, we have the following:
Lemma 70.4.3 For fixed $\omega \in \Omega, \hat{\mathbf{f}} \in L^{2}\left([0, T] ; W^{\prime}\right)$, and

$$
(t, \mathbf{w}) \rightarrow B(\mathbf{w}, \mathbf{q}(t, \omega)),(t, \mathbf{w}) \rightarrow B(\mathbf{q}(t, \omega), \mathbf{w})
$$

are continuous functions having values in $W^{\prime}$. For fixed $\mathbf{w} \in W$,

$$
(t, \omega) \rightarrow B(\mathbf{w}, \mathbf{q}(t, \omega)),(t, \omega) \rightarrow B(\mathbf{q}(t, \omega), \mathbf{w})
$$

are product measurable. In addition to this, if $\mathbf{z} \in W$,

$$
\begin{aligned}
|\langle B(\mathbf{w}, \mathbf{q}(t, \omega)), \mathbf{z}\rangle| & \leq C\|\mathbf{q}(t, \omega)\|_{V}\|\mathbf{w}\|_{H}\|\mathbf{z}\|_{H}, \\
|\langle B(\mathbf{q}(t, \omega), \mathbf{w}), \mathbf{z}\rangle| & \leq C\|\mathbf{q}(t, \omega)\|_{V}\|\mathbf{w}\|_{H}\|\mathbf{z}\|_{H} .
\end{aligned}
$$

Proof: The first claim is straightforward to prove from the definition of $A$ and $N$. Consider the next claim about continuity. Let $\mathbf{z} \in W$ be given. Then from the fact that all the functions are divergence free,

$$
\begin{aligned}
& |\langle B(\mathbf{w}, \mathbf{q}(t))-B(\overline{\mathbf{w}}, \mathbf{q}(s)), \mathbf{z}\rangle| \\
\equiv & \left|\int_{U}\left(w_{i} q_{j}(t)-\bar{w}_{i} q_{j}(s)\right) z_{j, i} d x\right|=\left|\int_{U}\left(w_{i} q_{j, i}(t)-\bar{w}_{i} q_{j, i}(s)\right) z_{j} d x\right| \\
\leq & \left|\int_{U}\left(w_{i} q_{j, i}(t)-\bar{w}_{i} q_{j, i}(t)\right) z_{j} d x\right|+\left|\int_{U}\left(\bar{w}_{i} q_{j, i}(t)-\bar{w}_{i} q_{j, i}(s)\right) z_{j} d x\right| \\
\leq & C\left(\|\mathbf{q}(t)\|_{V} \int_{U}|\mathbf{w}-\overline{\mathbf{w}}||\mathbf{z}| d x+\|\mathbf{q}(t)-\mathbf{q}(s)\|_{V} \int_{U}|\overline{\mathbf{w}}||\mathbf{z}| d x\right) \\
\leq & C\left(\|\mathbf{q}(t)\|_{V}|\mathbf{w}-\overline{\mathbf{w}}|_{H}+\|\mathbf{q}(t)-\mathbf{q}(s)\|_{V}|\overline{\mathbf{w}}|_{H}\right)|\mathbf{z}|_{H},
\end{aligned}
$$

where we have suppressed the dependence of $\mathbf{q}$ on $\omega$ to simplify the notation. The other function is similar.

As to the claim about product measurability, this follows from the above definition and assumptions about $\mathbf{q}$ being product measurable. For the estimates,

$$
|\langle B(\mathbf{w}, \mathbf{q}), \mathbf{z}\rangle|=\left|\int_{U} w_{i} q_{j} z_{j, i}\right|=\left|\int_{U} w_{i} q_{j, i} z_{j}\right| \leq C\|\mathbf{q}\|_{V} \int_{U}|\mathbf{w}||\mathbf{z}| d x
$$

and apply Hölder's inequality. The other estimate is similar.
This has shown that it suffices to verify that there exists a global solution $\mathbf{u}$ to the equation

$$
\mathbf{u}(t, \omega)-\mathbf{u}_{0}(\omega)+\int_{0}^{t} A(\mathbf{u}(s, \omega)) d s+\int_{0}^{t} \hat{N}(\mathbf{u}(s, \omega)) d s=\int_{0}^{t} \mathbf{f}(s, \omega) d s
$$

where $\mathbf{f}(\cdot, \omega) \in L^{2}\left([0, T] ; W^{\prime}\right)$ for each $\omega \in \Omega$.
Let $R$ be the Riesz map from $V$ to $V^{\prime}$, so $\left\langle R \mathbf{v}_{1}, \mathbf{v}_{2}\right\rangle_{V^{\prime}, V}=\left(\mathbf{v}_{1}, \mathbf{v}_{2}\right)_{V}$ for any $\mathbf{v}_{1}, \mathbf{v}_{2} \in V$. The compactness of the embeddings imply that $R^{-1}$ is a compact self adjoint operator on $H$ and so there is a complete orthonormal basis $\left\{\mathbf{w}_{k}\right\}$ for $H$ such that $R^{-1} \mathbf{w}_{k}=\mu_{k} \mathbf{w}_{k}$, where $\left\{\mu_{k}\right\}$ is a decreasing sequence of positive numbers which converges to 0 . Thus $R \mathbf{w}_{k}=\lambda_{k} \mathbf{w}_{k}$, where $\lim _{k \rightarrow \infty} \lambda_{k}=\infty$. $\left\{\mathbf{w}_{k}\right\}$ is a special basis. It is orthonormal in $H$ and orthogonal in $V$, since $\left(\mathbf{w}_{k}, \mathbf{w}_{l}\right)_{V}=\left\langle R \mathbf{w}_{k}, \mathbf{w}_{l}\right\rangle=\left(\lambda_{k} \mathbf{w}_{k}, \mathbf{w}_{l}\right)_{H}$.

To use the Galerkin method, let $V_{n}=\operatorname{span}\left(\mathbf{w}_{1}, \cdots, \mathbf{w}_{n}\right)$. Clearly $\cup_{n} V_{n}$ is dense in $H$. This is also dense in $V$. If not, then there exists $\phi \in V^{\prime}$ such that $\phi \neq 0$ but $\cup_{n} V_{n} \subseteq \operatorname{ker} \phi$. Then $\phi=R \mathbf{y}$. Hence for $\mathbf{z} \in \cup_{n=1}^{M} V_{n}$

$$
0=\langle R \mathbf{y}, \mathbf{z}\rangle=\langle R \mathbf{z}, \mathbf{y}\rangle=(R \mathbf{z}, \mathbf{y})_{H}
$$

But $R \mathbf{z} \in \operatorname{span}\left(\mathbf{w}_{1}, \cdots, \mathbf{w}_{M}\right)$ and in fact, $R$ maps $V_{n}$ onto $V_{n}$ and so this shows that $\mathbf{y}$ is perpendicular to $\operatorname{span}\left(\mathbf{w}_{1}, \cdots, \mathbf{w}_{M}\right)$ for each $M$ so $\mathbf{y}=\mathbf{0}$ and $\phi=0$ after all. Thus $\cup_{n} V_{n}$ is also dense in $V$ and hence it is also dense in $W$.

Let $\mathbf{u}_{n}(t, \omega)=\sum_{k=1}^{n} x_{k}(t, \omega) \mathbf{w}_{k}$, where $\mathbf{x}(t, \omega)=\left(x_{1}(t, \omega), \cdots, x_{n}(t, \omega)\right)^{T} \in \mathbb{R}^{n}$. We consider the problem of finding $\mathbf{x}(t, \omega)$ such that for all $\mathbf{w}_{k}, k \leq n$,

$$
\begin{gather*}
\left(\mathbf{u}_{n}(t, \omega), \mathbf{w}_{k}\right)_{H}-\left(\mathbf{u}_{0 n}(\omega), \mathbf{w}_{k}\right)_{H}+\int_{0}^{t}\left\langle A\left(\mathbf{u}_{n}(s, \omega)\right), \mathbf{w}_{k}\right\rangle d s \\
\quad+\int_{0}^{t}\left\langle\hat{N}\left(\mathbf{u}_{n}(s, \omega)\right), \mathbf{w}_{k}\right\rangle d s=\int_{0}^{t}\left\langle\mathbf{f}(s, \omega), \mathbf{w}_{k}\right\rangle d s, \tag{70.4.6}
\end{gather*}
$$

where $\mathbf{u}_{0 n}$ is the orthogonal projection of $\mathbf{u}_{0}$ onto $V_{n}$.
By the continuity of the operators described above, and the orthogonality of the $\mathbf{w}_{k}$, this is nothing but an ordinary differential equation for the vector $\mathbf{x}(t, \omega)$. By Theorem 70.2 .8 , there exists a product measurable solution $\mathbf{x}$ and therefore, $\mathbf{u}_{n}(t, \omega)$ is also product measurable in $H$.

Take the derivative, multiply by $x_{k}(t, \omega)$, add, and integrate again in the usual way to obtain

$$
\begin{aligned}
& \frac{1}{2}\left|\mathbf{u}_{n}(t, \omega)\right|_{H}^{2}-\frac{1}{2}\left|\mathbf{u}_{0 n}(\omega)\right|_{H}^{2} \\
= & -\int_{0}^{t}\left\langle A\left(\mathbf{u}_{n}(s, \omega)\right), \mathbf{u}_{n}(s, \omega)\right\rangle d s-\int_{0}^{t}\left\langle\hat{N}\left(\mathbf{u}_{n}(s, \omega)\right), \mathbf{u}_{n}(s, \omega)\right\rangle d s \\
& +\int_{0}^{t}\left\langle\mathbf{f}(s, \omega), \mathbf{u}_{n}(s, \omega)\right\rangle d s .
\end{aligned}
$$

Recall that $\hat{N}(\mathbf{u}(t, \omega))=N(\mathbf{u}(t, \omega))+B(\mathbf{u}(t, \omega), \mathbf{q}(s, \omega))+B(\mathbf{q}(t, \omega), \mathbf{u}(t, \omega))$. From the above lemma and that all functions are divergence free, we obtain

$$
\int_{0}^{t}\left\langle B\left(\mathbf{q}(s, \omega), \mathbf{u}_{n}(s, \omega)\right), \mathbf{u}_{n}(s, \omega)\right\rangle d s=0
$$

and

$$
\left|\int_{0}^{t}\left\langle B\left(\mathbf{u}_{n}(s, \omega), \mathbf{q}(s, \omega)\right), \mathbf{u}_{n}(s, \omega)\right\rangle d s\right| \leq C \int_{0}^{t}\|\mathbf{q}(s, \omega)\|_{V}\left|\mathbf{u}_{n}(s, \omega)\right|_{H}^{2} d s
$$

Then one can obtain an inequality of the following form

$$
\begin{aligned}
& \frac{1}{2}\left|\mathbf{u}_{n}(t, \omega)\right|_{H}^{2}+\int_{0}^{t}\left\|\mathbf{u}_{n}(s, \omega)\right\|_{W}^{2} d s \\
\leq \quad & \frac{1}{2}\left|\mathbf{u}_{0 n}(\omega)\right|_{H}^{2}+C \int_{0}^{t}\|\mathbf{q}(s, \omega)\|_{V}\left|\mathbf{u}_{n}(s, \omega)\right|_{H}^{2} d s \\
& +C \int_{0}^{t}\|\mathbf{f}(s, \omega)\|_{W^{\prime}}^{2} d s+\frac{1}{2} \int_{0}^{t}\left\|\mathbf{u}_{n}(s, \omega)\right\|_{W}^{2} d s
\end{aligned}
$$

Since $t \rightarrow\|\mathbf{q}(t, \omega)\|_{V}$ is continuous, it follows from Gronwall's inequality that there is an estimate of the form

$$
\begin{equation*}
\left|\mathbf{u}_{n}(t, \omega)\right|_{H}^{2}+\int_{0}^{t}\left\|\mathbf{u}_{n}(s, \omega)\right\|_{W}^{2} d s \leq C\left(\mathbf{u}_{0}, \mathbf{f}, \mathbf{q}, T, \omega\right) \tag{70.4.7}
\end{equation*}
$$

The next task is to estimate $\left\|\mathbf{u}_{n}^{\prime}(\omega)\right\|_{L^{2}\left([0, T] ; V^{\prime}\right)}$ for each fixed $\omega \in \Omega$. We will suppress the dependence on $\omega$ of all functions whenever it is appropriate. With 70.4.6, the fundamental theorem of calculus implies that for each $\mathbf{w} \in V_{n}$,

$$
\left\langle\mathbf{u}_{n}^{\prime}(t), \mathbf{w}\right\rangle_{V^{\prime}, V}+\left\langle A\left(\mathbf{u}_{n}(t)\right), \mathbf{w}\right\rangle_{V^{\prime}, V}+\left\langle\hat{N}\left(\mathbf{u}_{n}(t)\right), \mathbf{w}\right\rangle=\langle\mathbf{f}(t), \mathbf{w}\rangle .
$$

In terms of inner products in $V$,

$$
\left(R^{-1} \mathbf{u}_{n}^{\prime}(t)+R^{-1} A\left(\mathbf{u}_{n}(t)\right)+R^{-1} \hat{N}\left(\mathbf{u}_{n}(t)\right)-R^{-1} \mathbf{f}(t), \mathbf{w}\right)_{V}=0
$$

for all $\mathbf{w} \in V_{n}$. This is equivalent to saying that for $P_{n}$ the orthogonal projection in $V$ onto $V_{n}$,

$$
\left(R^{-1} \mathbf{u}_{n}^{\prime}(t)+R^{-1} A\left(\mathbf{u}_{n}(t)\right)+R^{-1} \hat{N}\left(\mathbf{u}_{n}(t)\right)-R^{-1} \mathbf{f}(t), P_{n} \mathbf{w}\right)_{V}=0
$$

for all $\mathbf{w} \in V$. This is to say that

$$
R^{-1} \mathbf{u}_{n}^{\prime}(t)+P_{n} R^{-1} A\left(\mathbf{u}_{n}(t)\right)+P_{n} R^{-1} \hat{N}\left(\mathbf{u}_{n}(t)\right)=P_{n} R^{-1} \mathbf{f}(t) .
$$

Now the projection map decreases norms and $R^{-1}$ preserves norms. Hence

$$
\left\|\mathbf{u}_{n}^{\prime}(t)\right\|_{V^{\prime}}=\left\|R^{-1} \mathbf{u}_{n}^{\prime}(t)\right\|_{V} \leq\left\|A\left(\mathbf{u}_{n}(t)\right)\right\|_{V^{\prime}}+\left\|\hat{N}\left(\mathbf{u}_{n}(t)\right)\right\|_{V^{\prime}}+\|\mathbf{f}(t)\|_{V^{\prime}}
$$

from which it follows that $\mathbf{u}_{n}^{\prime}$ is bounded in $L^{2}\left([0, T] ; V^{\prime}\right)$. Indeed, this is the case because $A\left(\mathbf{u}_{n}\right)$ and $\hat{N}\left(\mathbf{u}_{n}\right)$ are both bounded in $L^{2}\left([0, T] ; V^{\prime}\right)$. The term $\left\|\hat{N}\left(\mathbf{u}_{n}(t)\right)\right\|_{V^{\prime}}$ can be split further into terms involving $\left\|N\left(\mathbf{u}_{n}\right)\right\|,\left\|B\left(\mathbf{u}_{n}, \mathbf{q}\right)\right\|$, and $\left\|B\left(\mathbf{q}, \mathbf{u}_{n}\right)\right\|$. For example, consider $N\left(\mathbf{u}_{n}\right)$ which is the least obvious. Let $\mathbf{w} \in L^{2}([0, T] ; V)$. From the definitions,

$$
\begin{aligned}
& \left|\left\langle N\left(\mathbf{u}_{n}\right), \mathbf{w}\right\rangle_{L^{2}([0, T], V)}\right|=\left|\int_{0}^{T} \int_{U} u_{n i} u_{n j} w_{j, i} d x d t\right| \\
& \quad \leq C \int_{0}^{T}\|\mathbf{w}(t)\|_{V}\left|\mathbf{u}_{n}\right|_{H}^{2} d t \\
& \quad \leq C\|\mathbf{w}\|_{L^{2}([0, T], V)} C\left(\mathbf{u}_{0}, \mathbf{f}, \mathbf{q}, T, \omega\right)
\end{aligned}
$$

We have now shown that

$$
\begin{equation*}
\sup _{t \in[0, T]}\left|\mathbf{u}_{n}(t, \omega)\right|_{H}^{2}+\int_{0}^{T}\left\|\mathbf{u}_{n}(s, \omega)\right\|_{W}^{2} d s+\left\|\mathbf{u}_{n}^{\prime}(\omega)\right\|_{L^{2}\left([0, T] ; V^{\prime}\right)} \leq C\left(\mathbf{u}_{0}, \mathbf{f}, \mathbf{q}, T, \omega\right) \tag{70.4.8}
\end{equation*}
$$

This condition holds for all $\omega$. Now for each $\omega$, one can take a subsequence such that a solution to the evolution equation is obtained. Then, when this is done, we will apply the measurable selection result to obtain a product measurable solution.

It follows from the above estimate 70.4 .8 that there is a subsequence, still denoted as $n$ and a function $\mathbf{u}(t, \omega)$ such that

$$
\begin{align*}
\mathbf{u}_{n} & \rightarrow \mathbf{u} \text { weak } * \text { in } L^{\infty}([0, T] ; H),  \tag{70.4.9}\\
\mathbf{u}_{n}^{\prime} & \rightarrow \mathbf{u}^{\prime} \text { weakly in } L^{2}\left([0, T] ; V^{\prime}\right), \\
\mathbf{u}_{n} & \rightarrow \mathbf{u} \text { weakly in } L^{2}([0, T] ; W), \\
\mathbf{u}_{n} & \rightarrow \mathbf{u} \text { strongly in } L^{2}([0, T] ; H) . \tag{70.4.10}
\end{align*}
$$

This last convergence follows from Theorem 70.4.1. The sequence is bounded in the space $L^{2}([0, T] ; W)$ and the derivative is bounded in $L^{2}\left([0, T] ; V^{\prime}\right)$ so such a strongly convergent subsequence exists. Since $A$ is linear, we can also assume that

$$
\begin{equation*}
A \mathbf{u}_{n} \rightarrow A \mathbf{u} \text { weakly in } L^{2}\left([0, T] ; W^{\prime}\right) \tag{70.4.11}
\end{equation*}
$$

What happens with the nonlinear operator $\hat{N}$ ? Let $\mathbf{w} \in L^{\infty}([0, T] ; V)$. A computation shows then that

$$
\begin{aligned}
& \left|\int_{0}^{T}\left\langle N \mathbf{u}_{n}(t)-N \mathbf{u}(t), \mathbf{w}(t)\right\rangle d t\right| \\
= & \left|\int_{0}^{T} \int_{U}\left(u_{n i}(t) u_{n j}(t)-u_{i}(t) u_{j}(t)\right) w_{j, i} d x d t\right| \\
\leq & \|\mathbf{w}\|_{L^{\infty}([0, T], V)} \int_{0}^{T} \int_{U}\left(\left|\mathbf{u}_{n}(t)\right|+|\mathbf{u}(t)|\right)\left(\left|\mathbf{u}(t)-\mathbf{u}_{n}(t)\right|\right) d x d t \\
\leq & \|\mathbf{w}\|_{L^{\infty}([0, T], V)}\left(\int_{0}^{T} \int_{U}\left(\left|\mathbf{u}_{n}\right|+|\mathbf{u}|\right)^{2} d x d t\right)^{1 / 2}\left(\int_{0}^{T} \int_{U}\left(\left|\mathbf{u}-\mathbf{u}_{n}\right|\right)^{2} d x d t\right)^{1 / 2}
\end{aligned}
$$

This converges to 0 thanks to the estimates and the strong convergence 70.4.10. Similar convergence holds for the other nonlinear terms $B\left(\mathbf{u}_{n}(t), \mathbf{q}\right), B\left(\mathbf{q}, \mathbf{u}_{n}(t)\right)$.

We have shown that for any $n \geq m$, and $\mathbf{w} \in V_{m}$,

$$
\begin{equation*}
\left\langle\mathbf{u}_{n}^{\prime}(t), \mathbf{w}\right\rangle_{V^{\prime}, V}+\left\langle A\left(\mathbf{u}_{n}(t)\right), \mathbf{w}\right\rangle_{V^{\prime}, V}+\left\langle\hat{N}\left(\mathbf{u}_{n}(t)\right), \mathbf{w}\right\rangle=\langle\mathbf{f}(t), \mathbf{w}\rangle \tag{70.4.12}
\end{equation*}
$$

Let $\zeta \in C^{\infty}([0, T])$ be such that $\zeta(T)=0$. Then

$$
\left\langle\mathbf{u}_{n}^{\prime}(t), \mathbf{w} \zeta(t)\right\rangle_{V^{\prime}, V}+\left\langle A\left(\mathbf{u}_{n}(t)\right), \mathbf{w} \zeta(t)\right\rangle_{V^{\prime}, V}+\left\langle\hat{N}\left(\mathbf{u}_{n}(t)\right), \mathbf{w} \zeta(t)\right\rangle=\langle\mathbf{f}(t), \mathbf{w} \zeta(t)\rangle
$$

Integrating this equation from 0 to $T$ we obtain

$$
\begin{aligned}
& -\left(\mathbf{u}_{0 n}(\omega), \mathbf{w} \zeta(0)\right)_{H}-\int_{0}^{T} \zeta^{\prime}(s)\left(\mathbf{u}_{n}(s, \omega), \mathbf{w}\right)_{H} d s \\
= & -\int_{0}^{T}\left\langle A\left(\mathbf{u}_{n}(s, \omega)\right), \mathbf{w} \zeta(s)\right\rangle d s-\int_{0}^{T}\left\langle\hat{N}\left(\mathbf{u}_{n}(s, \omega)\right), \mathbf{w} \zeta(s)\right\rangle d s \\
& +\int_{0}^{T}\langle\mathbf{f}(s, \omega), \mathbf{w} \zeta(s)\rangle d s
\end{aligned}
$$

Now letting $n \rightarrow \infty$, from the above list of convergent sequences,

$$
\begin{aligned}
& -\left(\mathbf{u}_{0}(\omega), \mathbf{w} \zeta(0)\right)_{H}-\int_{0}^{T} \zeta^{\prime}(s)(\mathbf{u}(s, \omega), \mathbf{w})_{H} d s \\
= & -\int_{0}^{T}\langle A(\mathbf{u}(s, \omega)), \mathbf{w} \zeta(s)\rangle d s-\int_{0}^{T}\langle\hat{N}(\mathbf{u}(s, \omega)), \mathbf{w} \zeta(s)\rangle d s \\
& +\int_{0}^{T}\langle\mathbf{f}(s), \mathbf{w} \zeta(s)\rangle d s .
\end{aligned}
$$

It follows that in the sense of $V^{\prime}$ valued distributions,

$$
\begin{equation*}
\mathbf{u}^{\prime}(\omega)+A(\mathbf{u}(\omega))+\hat{N}(\mathbf{u}(\omega))=\mathbf{f}(\omega) \tag{70.4.13}
\end{equation*}
$$

along with the initial condition

$$
\begin{equation*}
\mathbf{u}(0)=\mathbf{u}_{0} \tag{70.4.14}
\end{equation*}
$$

This has proved most of the following lemma:

Lemma 70.4.4 Let $\mathbf{u}_{0}$ have values in $H$ and be $\mathscr{F}$ measurable, and let $\mathbf{u}_{n}$ be a solution to 70.4.6. Then for each $\omega$, the estimate 70.4 .8 holds. Also there is a subsequence, still called $\mathbf{u}_{n}$ such that the convergence for 70.4.9-70.4.11 are valid. For all $\omega$, the function $\mathbf{u}(\cdot, \omega)$ is a solution to 70.4.13-70.4.14 and satisfies

$$
\mathbf{u}(\cdot, \omega) \in L^{\infty}([0, T] ; H) \cap L^{2}([0, T] ; W), \mathbf{u}^{\prime} \in L^{2}\left([0, T] ; V^{\prime}\right)
$$

This solution is also weakly continuous into $H$ for each $\omega$.
Proof: All that remains to show is the last claim about weak continuity into $H$. The equation 70.4.13 shows that $\mathbf{u}(\cdot, \omega)$ is continuous into $V^{\prime}$. However, the weak convergence and the estimate 70.4.8 show that $\mathbf{u}(\cdot, \omega)$ is bounded in $H$. It follows from density of $V$ in $H$ that $t \rightarrow \mathbf{u}(t, \omega)$ is weakly continuous into $H$.

From 70.4.13, 70.4.14, the following integral equation for a path solution holds:

$$
\mathbf{u}(t, \omega)-\mathbf{u}_{0}(\omega)+\int_{0}^{t} A(\mathbf{u}(s, \omega)) d s+\int_{0}^{t} \hat{N}(\mathbf{u}(s, \omega)) d s=\int_{0}^{t} \mathbf{f}(s, \omega) d s
$$

We apply Theorem 70.2.8 to prove the above solution could be taken product measurable.
Theorem 70.4.5 Let $\mathbf{f}(t, \omega)$, $\mathbf{q}(t, \omega)$ be product measurable and $\mathbf{u}_{0}$ be measurable, such that for each $\omega \in \Omega, \mathbf{f}(\cdot, \omega) \in L^{2}\left([0, T] ; W^{\prime}\right), \mathbf{q}(\cdot, \omega) \in C([0, T] ; V)$ with $\mathbf{q}(0)=\mathbf{0}$, and $\mathbf{u}_{0}(\omega) \in H$. Then there exists a global solution to the integral equation

$$
\mathbf{u}(t, \omega)-\mathbf{u}_{0}(\omega)+\int_{0}^{t} A(\mathbf{u}(s, \omega)) d s+\int_{0}^{t} \hat{N}(\mathbf{u}(s, \omega)) d s=\int_{0}^{t} \mathbf{f}(s, \omega) d s
$$

Proof: Letting $\mathbf{u}_{n}$ be a solution to 70.4.6, we verify the conditions of Theorem 70.2.8 for $\mathbf{u}_{n}$.

The assumption in this theorem that the $\mathbf{u}_{n}$ are bounded follows from the above estimate 70.4.8. Then it was shown in the above lemma that whenever a sequence satisfies the estimate 70.4.8, it has a subsequence which converges as in 70.4.9-70.4.11 to a weakly continuous $\mathbf{u}(\cdot, \omega)$. Therefore, by Theorem 70.2 .8 there is a subsequence $\mathbf{u}_{n(\omega)}(\cdot, \omega)$ converging weakly to $\mathbf{u}(\cdot, \omega)$, such that $(t, \omega) \rightarrow \mathbf{u}(t, \omega)$ is a product measurable function into $H$. Then a further subsequence converges to a path solution to the above integral equation, which must be the same function because when a sequence converges, all subsequences converge to the same thing. In addition to this, $\mathbf{u}$ is also product measurable into $W$. This follows from the above estimate 70.4.8. For $\phi \in H,(t, \omega) \rightarrow(\phi, \mathbf{u}(t, \omega))$ is product measurable. However, $H$ is dense in $W^{\prime}$ and so if $\psi \in W^{\prime}$, there is a sequence $\left\{\phi_{n}\right\}$ in $H$ such that $\phi_{n} \rightarrow \psi$. Then

$$
\langle\psi, \mathbf{u}\rangle=\lim _{n \rightarrow \infty}\left(\phi_{n}, \mathbf{u}\right),
$$

so by the Pettis theorem [127], $\mathbf{u}$ is product measurable into $W$ also.
This shows much of the following theorem which is the main result.
Theorem 70.4.6 Let $\mathbf{f}(t, \omega)$, $\mathbf{q}(t, \omega)$ be product measurable and $\mathbf{u}_{0}$ be measurable, such that for each $\omega \in \Omega, \mathbf{f}(\cdot, \omega) \in L^{2}\left([0, T] ; W^{\prime}\right), \mathbf{q}(\cdot, \omega) \in C([0, T] ; V)$ with $\mathbf{q}(0)=\mathbf{0}$, and $\mathbf{u}_{0}(\omega) \in H$. Then there exists a global solution to the integral equation

$$
\mathbf{u}(t, \omega)-\mathbf{u}_{0}(\omega)+\int_{0}^{t} A(\mathbf{u}(s, \omega)) d s+\int_{0}^{t} N(\mathbf{u}(s, \omega)) d s=\int_{0}^{t} \mathbf{f}(s, \omega) d s+\mathbf{q}(t, \omega)
$$

In addition to this, $t \rightarrow \mathbf{u}(t, \omega)$ is continuous into $H$ and satisfies

$$
\mathbf{u}(\cdot, \omega) \in L^{\infty}([0, T] ; H) \cap L^{2}([0, T] ; W)
$$

If, in addition to the above, $\mathbf{u}_{0} \in L^{2}(\Omega ; H)$ and $\mathbf{f} \in L^{2}\left([0, T] \times \Omega ; W^{\prime}\right)$ and

$$
\mathbf{q} \in L^{2}([0, T] \times \Omega ; V)
$$

then the solution $\mathbf{u}$ is in $L^{2}([0, T] \times \Omega ; H) \cap L^{2}([0, T] \times \Omega ; W)$.
Proof: The last claim follows from the estimates used in the Galerkin method, taking expectations and passing to a limit. To verify the continuity into $H$, one can observe that from the integral equation, $\mathbf{u}$ is continuous into $V^{\prime}$. One has

$$
|\mathbf{u}(t)|_{H}^{2}=\sum_{k=1}^{\infty}\left(\mathbf{u}(t), \mathbf{w}_{k}\right)^{2}=\sum_{k=1}^{\infty}\left\langle\mathbf{u}(t), \mathbf{w}_{k}\right\rangle^{2},
$$

and so $t \rightarrow|\mathbf{u}(t)|_{H}^{2}$ is lower semi-continuous. Since it is in $L^{\infty}$, this implies this function is bounded. Hence the continuity into $V^{\prime}$ and density of $V$ in $H$ implies that $\mathbf{u}(t)$ is weakly continuous into $H$. Then one can use the formulation in Theorem 70.4.5 to verify $t \rightarrow$ $|\mathbf{u}(t)|_{H}$ is continuous and apply uniform convexity of the Hilbert space $H$.

One can replace $\mathbf{q}(t, \omega)$ with $\mathbf{q}(t, \omega, \mathbf{u})$ and $\mathbf{f}(t, \omega)$ with $\mathbf{f}(t, \omega, \mathbf{u})$ in the above with no change in the argument, provided it is assumed that

$$
(t, \omega, \mathbf{u}) \rightarrow \mathbf{q}(t, \omega, \mathbf{u}), \mathbf{f}(t, \omega, \mathbf{u})
$$

are product measurable, continuous in $(t, \mathbf{u})$ and bounded.

### 70.5 A Friction contact problem

In this section we will consider a friction contact problem which has a coefficient of friction which is dependent on the slip speed.

$$
\begin{gather*}
\ddot{u}_{i}=\sigma_{i j, j}(\mathbf{u}, \dot{\mathbf{u}})+f_{i} \text { for }(t, \mathbf{x}) \in(0, T) \times U,  \tag{70.5.15}\\
\mathbf{u}(0, \mathbf{x})=\mathbf{u}_{0}(\mathbf{x}),  \tag{70.5.16}\\
\dot{\mathbf{u}}(0, \mathbf{x})=\mathbf{v}_{0}(\mathbf{x}), \tag{70.5.17}
\end{gather*}
$$

where $U$ is a bounded open subset of $\mathbb{R}^{3}$ having Lipschitz boundary, along with some boundary conditions which pertain to a part of the boundary of $U, \Gamma_{C}$. For $\mathbf{x} \in \Gamma_{C}$,

$$
\begin{equation*}
\sigma_{n}=-p\left(\left(u_{n}-g\right)_{+}\right) C_{n}, \tag{70.5.18}
\end{equation*}
$$

This is the normal compliance boundary condition.

$$
\begin{gather*}
\left|\sigma_{T}\right| \leq F\left(\left(u_{n}-g\right)_{+}\right) \mu\left(\left|\dot{\mathbf{u}}_{T}-\dot{\mathbf{U}}_{T}\right|\right)  \tag{70.5.19}\\
\left|\sigma_{T}\right|<F\left(\left(u_{n}-g\right)_{+}\right) \mu\left(\left|\dot{\mathbf{u}}_{T}-\dot{\mathbf{U}}_{T}\right|\right) \text { implies } \dot{\mathbf{u}}_{T}-\dot{\mathbf{U}}_{T}=\mathbf{0} \tag{70.5.20}
\end{gather*}
$$

$$
\begin{equation*}
\left|\sigma_{T}\right|=F\left(\left(u_{n}-g\right)_{+}\right) \mu\left(\left|\dot{\mathbf{u}}_{T}-\dot{\mathbf{U}}_{T}\right|\right) \text { implies } \dot{\mathbf{u}}_{T}-\dot{\mathbf{U}}_{T}=-\lambda \sigma_{T}(\mathbf{u}, \dot{\mathbf{u}}) \tag{70.5.21}
\end{equation*}
$$

Here $C_{n}$ is a positive function in $L^{\infty}\left(\Gamma_{C}\right)$, which we take equal to 1 to simplify notation, $\dot{\mathbf{U}}_{T}$ is the velocity of the foundation, $\lambda$ is non negative, and $\mu$ is a bounded positive function having a bounded continuous derivative. We could also let $\mu$ depend on $\mathbf{x} \in \Gamma_{C}$ to model the roughness of the contact surface but we will suppress this dependence in the interest of simpler notation. Also, $\mathbf{n}$ is the unit outward normal to $\partial U$ and $u_{n}, \mathbf{u}_{T}, \sigma_{T}$, and $\sigma_{n}$ are defined by the following.

$$
\begin{gathered}
u_{n}=\mathbf{u} \cdot \mathbf{n} \\
\mathbf{u}_{T}=\mathbf{u}-(\mathbf{u} \cdot \mathbf{n}) \mathbf{n} \\
\sigma_{n}=\sigma_{i j} n_{j} n_{i} \\
\sigma_{T i}=\sigma_{i j} n_{j}-\sigma_{n} n_{i}
\end{gathered}
$$

written more simply,

$$
\sigma_{T}=\sigma \mathbf{n}-\sigma_{n} \mathbf{n}
$$

Systems like the above model dynamic friction contact problems [93], [51] [46]. The function $g$ represents the gap between the contact surface of $U, \Gamma_{C}$, and a foundation which is sliding tangent to $\Gamma_{C}$ with tangential velocity $\dot{\mathbf{U}}_{T}$.

The new ingredient in this paper is that we allow

$$
g=g(t, \mathbf{x}, \omega)
$$

where $\omega \in(\Omega, \mathscr{F})$ and we assume $(t, \mathbf{x}, \omega) \rightarrow g(\mathbf{x}, \omega)$ is $\mathscr{B}\left([0, T] \times \Gamma_{C}\right) \times \mathscr{F}$ measurable. Also, we make the reasonable assumption that

$$
0 \leq g(t, \mathbf{x}, \omega) \leq l<\infty
$$

for all $(t, \mathbf{x}, \omega)$. We also assume that the given motion of the foundation $\dot{\mathbf{U}}_{T}$ is a stochastic process

$$
\dot{\mathbf{U}}_{T}=\dot{\mathbf{U}}_{T}(t, \mathbf{x}, \omega)
$$

and is $\mathscr{B}\left([0, T] \times \Gamma_{C}\right) \times \mathscr{F}$ measurable. Here $\mathscr{B}\left([0, T] \times \Gamma_{C}\right)$ denotes the Borel sets of $[0, T] \times \Gamma_{C}$. We make the reasonable assumption that $\dot{\mathbf{U}}_{T}(t, \mathbf{x}, \omega)$ is uniformly bounded. In the interest of notation, we will often suppress the dependence on $t, \mathbf{x}$, and $\omega$.

The condition 70.5 .18 is the contact condition. It says the normal component of the traction force density is dependent on the normal penetration of the body into the foundation surface. Conditions 70.5.19-70.5.21 model friction. They say that the tangential part of the traction force density is bounded by a function determined by the normal force or penetration. No sliding takes place until $\left|\sigma_{T}\right|$ reaches this bound, $F\left(\left(u_{n}-g\right)_{+}\right) \mu(0)$, 70.5.20. When this occurs, the tangential force density has a direction opposite the relative tangential velocity 70.5 .21 . The dependence of the friction coefficient on the magnitude of the slip velocity, $\left|\dot{\mathbf{u}}_{T}-\dot{\mathbf{U}}_{T}\right|$ may be experimentally verified and so it has been included. The new feature in this model is the assumption that the gap is a random variable for each $\mathbf{x} \in \Gamma_{C}$ and we want to consider measurability of the solutions. Thus for a fixed $\omega$, we have a standard friction problem and it is the measurability which is of interest here.

In this paper, we assume the following on $p$ and $F$. The functions $p$ and $F$ are increasing and

$$
\begin{gather*}
\delta^{2} r-K \leq p(r) \leq K(1+r), r \geq 0  \tag{70.5.22}\\
p(r)=0, r<0 \\
F(r) \leq K(1+r) r \geq 0  \tag{70.5.23}\\
F(r)=0 \text { if } r<0 \\
\left|\mu\left(r_{1}\right)-\mu\left(r_{2}\right)\right| \leq \operatorname{Lip}(\mu)\left|r_{1}-r_{2}\right|, \quad\|\mu\|_{\infty} \leq C \tag{70.5.24}
\end{gather*}
$$

and for $a=F, p$, and $r_{1}, r_{2} \geq 0$,

$$
\begin{equation*}
\left|a\left(r_{1}\right)-a\left(r_{2}\right)\right| \leq K\left|r_{1}-r_{2}\right| . \tag{70.5.25}
\end{equation*}
$$

One can consider more general growth conditions than this, but we are keeping this part simple to emphasize the new stochastic considerations.

It will be assumed that

$$
\begin{equation*}
\sigma_{i j}=A_{i j k l} u_{k, l}+C_{i j k l} \dot{u}_{k, l}, \tag{70.5.26}
\end{equation*}
$$

where $A$ and $C$ are in $L^{\infty}(U)$ and for $B=A$ or $C$, we have the following symmetries.

$$
\begin{equation*}
B_{i j k l}=B_{i j l k}, B_{j i k l}=B_{i j k l}, B_{i j k l}=B_{k l i j} \tag{70.5.27}
\end{equation*}
$$

and we also assume for $B=A$ or $C$ that

$$
\begin{equation*}
B_{i j k l} H_{i j} H_{k l} \geq \varepsilon H_{r s} H_{r s} \tag{70.5.28}
\end{equation*}
$$

for all symmetric $H$.
Throughout the paper, $V$ will be a closed subspace of $\left(H^{1}(U)\right)^{3}$ containing the test functions $\left(C_{0}^{\infty}(U)\right)^{3}, \rightharpoonup$ will denote weak or weak $*$ convergence while $\rightarrow$ will mean strong convergence. $\gamma$ will denote the trace map from $W^{12}(U)$ into $L^{2}(\partial U)$. $H$ will denote ( $\left.L^{2}(U)\right)^{3}$ and we will always identify $H$ and $H^{\prime}$ to write

$$
V \subseteq H=H^{\prime} \subseteq V^{\prime}
$$

We define

$$
\mathscr{V}=L^{2}(0, T ; V), \mathscr{H}=L^{2}(0, T, H), \mathscr{V}^{\prime}=L^{2}\left(0, T ; V^{\prime}\right)
$$

### 70.5.1 The Abstract Problem

We shall use two theorems found in Lions [91], and Simon [117] respectively. These theorems apply for fixed $\omega$. Proofs of generalizations of these theorems begin on Page 2383.

Theorem 70.5.1 If $p \geq 1, q>1$, and $W \subseteq U \subseteq Y$ where the inclusion map of $W$ into $U$ is compact and the inclusion map of $U$ into $Y$ is continuous, let

$$
\begin{aligned}
S= & \left\{\mathbf{u} \in L^{p}(0, T ; W): \mathbf{u}^{\prime} \in L^{q}(0, T ; Y)\right. \text { and } \\
& \left.\|\mathbf{u}\|_{L^{p}(0, T ; W)}+\left\|\mathbf{u}^{\prime}\right\|_{L^{q}(0, T ; Y)}<R\right\}
\end{aligned}
$$

Then $S$ is pre compact in $L^{p}(0, T ; U)$.

Theorem 70.5.2 Let $W, U$, and $Y$ be as in Theorem 70.5.1 and let

$$
S=\left\{\mathbf{u}:\|\mathbf{u}(t)\|_{W}+\left\|\mathbf{u}^{\prime}\right\|_{L^{q}(0, T ; Y)} \leq R \text { for } t \in[0, T]\right\}
$$

for $q>1$. Then $S$ is pre compact in $C(0, T ; U)$.
Now we give an abstract formulation of the problem described roughly in 70.5.1570.5.21. We begin by defining several operators. Let $M, A: V \rightarrow V^{\prime}$ be given by

$$
\begin{align*}
\langle M \mathbf{u}, \mathbf{v}\rangle & =\int_{U} C_{i j k l} \mathbf{u}_{k, l} \mathbf{v}_{i, j} d x  \tag{70.5.29}\\
\langle A \mathbf{u}, \mathbf{v}\rangle & =\int_{U} A_{i j k l} \mathbf{u}_{k, l} \mathbf{v}_{i, j} d x \tag{70.5.30}
\end{align*}
$$

Also let the operator $\mathbf{v} \rightarrow P(\mathbf{u})$ map $\mathscr{V}$ to $\mathscr{V}^{\prime}$ be given by

$$
\begin{equation*}
\langle P(\mathbf{u}), \mathbf{w}\rangle=\int_{0}^{T} \int_{\Gamma_{C}} p\left(\left(u_{n}-g\right)_{+}\right) w_{n} d \alpha d t \tag{70.5.31}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathbf{u}(t)=\mathbf{u}_{0}+\int_{0}^{t} \mathbf{v}(s) d s \tag{70.5.32}
\end{equation*}
$$

for $\mathbf{u}_{0} \in V_{q}$. (Technically, $P$ depends on $\mathbf{u}_{0}$ but we suppress this in favor of simpler notation ). Let

$$
\gamma_{T}^{*}: L^{2}\left(0, T ; L^{2}\left(\Gamma_{C}\right)^{3}\right) \rightarrow \mathscr{V}^{\prime}
$$

is defined as

$$
\left\langle\gamma_{T}^{*} \xi, \mathbf{w}\right\rangle \equiv \int_{0}^{T} \int_{\Gamma_{C}} \xi \cdot \mathbf{w}_{T} d \alpha d t
$$

Now the abstract form of the problem, denoted by $\mathscr{P}$, is the following.

$$
\begin{gather*}
\mathbf{v}^{\prime}+M \mathbf{v}+A \mathbf{u}+P \mathbf{u}+\gamma_{T}^{*} \xi=\mathbf{f} \text { in } \mathscr{V}^{\prime}  \tag{70.5.33}\\
\mathbf{v}(0)=\mathbf{v}_{0} \in H \tag{70.5.34}
\end{gather*}
$$

where

$$
\begin{equation*}
\mathbf{u}(t)=\mathbf{u}_{0}+\int_{0}^{t} \mathbf{v}(s) d s, \mathbf{u}_{0} \in V_{p} \tag{70.5.35}
\end{equation*}
$$

and for all $\mathbf{w} \in \mathscr{V}$,

$$
\begin{gather*}
\left\langle\gamma_{T}^{*} \xi, \mathbf{w}\right\rangle \leq \int_{0}^{T} \int_{\Gamma_{C}} F\left(\left(u_{n}-g\right)_{+}\right) \mu\left(\left|\mathbf{v}_{T}-\dot{\mathbf{U}}_{T}\right|\right) \\
{\left[\left|\mathbf{v}_{T}-\dot{\mathbf{U}}_{T}+\mathbf{w}_{T}\right|-\left|\mathbf{v}_{T}-\dot{\mathbf{U}}_{T}\right|\right] d \alpha d t} \tag{70.5.36}
\end{gather*}
$$

Also $\mathbf{f} \in L^{2}\left(0, T ; V^{\prime}\right)$ so $\mathbf{f}$ can include the body force as well as traction forces on various parts of $\partial U$. If $\mathbf{v}$ solves the above abstract problem, then $\mathbf{u}$ can be considered a weak
solution to 70.5.15-70.5.21 along with other variational and stable boundary conditions depending on the choice of $W$ and $\mathbf{f} \in L^{2}\left(0, T ; V^{\prime}\right)$.

In order to carry out our existence and uniqueness proofs, we assume $M$ and $A$ satisfy the following for some $\delta>0, \lambda \geq 0$.

$$
\begin{equation*}
\langle B \mathbf{u}, \mathbf{u}\rangle \geq \delta^{2}\|\mathbf{u}\|_{W}^{2}-\lambda|\mathbf{u}|_{H}^{2},\langle B \mathbf{u}, \mathbf{u}\rangle \geq 0,\langle B \mathbf{u}, \mathbf{v}\rangle=\langle B \mathbf{v}, \mathbf{u}\rangle, \tag{70.5.37}
\end{equation*}
$$

for $B=M$ or $A$. This is the assumption that we use, and we note that 70.5 .37 is a consequence of 70.5.26-70.5.28 and Korn's inequality [104].

### 70.5.2 An Approximate Problem

We will use the Galerkin method. To do this, we will first regularize that subgradient material. Let

$$
\psi_{\varepsilon}(\mathbf{r})=\sqrt{|\mathbf{r}|^{2}+\varepsilon}
$$

Then this is a convex, Lipschitz continuous function having bounded derivative which converges uniformly to $\psi(\mathbf{r})=|\mathbf{r}|$ on $\mathbb{R}$. Also

$$
\left|\psi_{\varepsilon}(\mathbf{x})-\psi_{\varepsilon}(\mathbf{y})\right| \leq|\mathbf{x}-\mathbf{y}|, \quad\left|\psi_{\varepsilon}^{\prime}(\mathbf{t})\right| \leq 1
$$

And finally, $\psi_{\varepsilon}^{\prime}$ is Lipschitz continuous with a Lipschitz constant $C / \sqrt{\varepsilon}$. Here $\psi_{\varepsilon}^{\prime}$ denotes the gradient or Frechet derivative of the scalar valued function.

Our approximate problem for which we will apply the Galerkin method will be $\mathscr{P}_{\varepsilon}$ given by

$$
\begin{gather*}
\mathbf{v}^{\prime}+M \mathbf{v}+A \mathbf{u}+P \mathbf{u}+\gamma_{T}^{*} F\left(\left(u_{n}-g\right)_{+}\right) \mu\left(\left|\mathbf{v}_{T}-\dot{\mathbf{U}}_{T}\right|\right) \psi_{\varepsilon}^{\prime}\left(\mathbf{v}_{T}-\dot{\mathbf{U}}_{T}\right)=\mathbf{f} \text { in } \mathscr{V}^{\prime},  \tag{70.5.38}\\
\mathbf{v}(0)=\mathbf{v}_{0} \in H, \tag{70.5.39}
\end{gather*}
$$

where

$$
\begin{equation*}
\mathbf{u}(t)=\mathbf{u}_{0}+\int_{0}^{t} \mathbf{v}(s) d s, \mathbf{u}_{0} \in V \tag{70.5.40}
\end{equation*}
$$

Here the long operator on the left is defined in the following manner.

$$
\begin{aligned}
& \left\langle\gamma_{T}^{*} F\left(\left(u_{n}-g\right)_{+}\right) \mu\left(\left|\mathbf{v}_{T}-\dot{\mathbf{U}}_{T}\right|\right) \psi_{\varepsilon}^{\prime}\left(\mathbf{v}_{T}-\dot{\mathbf{U}}_{T}\right), \mathbf{w}\right\rangle \\
= & \int_{\Gamma_{C}} F\left(\left(u_{n}-g\right)_{+}\right) \mu\left(\left|\mathbf{v}_{T}-\dot{\mathbf{U}}_{T}\right|\right) \psi_{\varepsilon}^{\prime}\left(\mathbf{v}_{T}-\dot{\mathbf{U}}_{T}\right) \cdot \mathbf{w}_{T} d S
\end{aligned}
$$

Let $R$ denote the Riesz map from $V$ to $V^{\prime}$ defined by $\langle R \mathbf{u}, \mathbf{v}\rangle=(\mathbf{u}, \mathbf{v})_{V}$. Then $R^{-1}: H \rightarrow$ $V$ is a compact self adjoint operator and so there exists a complete orthonormal basis for $H,\left\{\mathbf{e}_{k}\right\} \subseteq V$ such that

$$
R \mathbf{e}_{k}=\lambda_{k} \mathbf{e}_{k}
$$

where $\lambda_{k} \rightarrow \infty$. Let $V_{n}=\operatorname{span}\left(\mathbf{e}_{1}, \cdots, \mathbf{e}_{n}\right)$. Thus $\cup_{n} V_{n}$ is dense in $H$. In addition $\cup_{n} V_{n}$ is dense in $V$ and $\left\{\mathbf{e}_{k}\right\}$ is also orthogonal in $V$. To see first that $\left\{\mathbf{e}_{k}\right\}$ is orthogonal in $V$,

$$
0=\left(\mathbf{e}_{k}, \mathbf{e}_{l}\right)_{H}=\frac{1}{\lambda_{k}}\left(R \mathbf{e}_{k}, \mathbf{e}_{l}\right)_{H}=\frac{1}{\lambda_{k}}\left\langle R \mathbf{e}_{k}, \mathbf{e}_{l}\right\rangle=\frac{1}{\lambda_{k}}\left(\mathbf{e}_{l}, \mathbf{e}_{k}\right)_{V}
$$

Next consider why $\cup_{n} V_{n}$ is dense in $V$. If this is not so, then there exists $f \in V^{\prime}, f \neq 0$ such that $\cup_{n} V_{n}$ is in $\operatorname{ker}(f)$. But $f=R \mathbf{u}$ and so

$$
0=\left\langle R \mathbf{u}, \mathbf{e}_{k}\right\rangle=\left\langle R \mathbf{e}_{k}, \mathbf{u}\right\rangle=\lambda_{k}\left(\mathbf{e}_{k}, \mathbf{u}\right)_{H}
$$

for all $\mathbf{e}_{k}$ and so $\mathbf{u}=\mathbf{0}$ by density of $\cup_{n} V_{n}$ in $H$. Hence $R \mathbf{u}=0=f$ after all, a contradiction. Hence $\cup_{n} V_{n}$ is dense in $V$ as claimed.

Now we set up the Galerkin method for Problem $\mathscr{P}_{\varepsilon}$. Let

$$
\mathbf{v}_{k}(t, \omega)=\sum_{j=1}^{k} x_{j}(t, \omega) \mathbf{e}_{j}, \mathbf{u}_{k}(t)=\mathbf{u}_{0}+\int_{0}^{t} \mathbf{v}_{k}(s) d s
$$

and let $\mathbf{v}_{k}$ be the solution to the following integral equation for each $\omega$ and $j \leq k$. The dependence on $\omega$ is suppressed in most terms in order to save space.

$$
\begin{gather*}
\left\langle\mathbf{v}_{k}(t)-\mathbf{v}_{0 k}+\int_{0}^{t} M \mathbf{v}_{k}+A \mathbf{u}_{k}+P \mathbf{u}_{k}+\gamma_{T}^{*} F\left(\left(u_{k n}-g(\omega)\right)_{+}\right) \cdot\right. \\
\mu\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right) \psi_{\varepsilon}^{\prime}\left(\mathbf{v}_{k T}\right\rangle  \tag{70.5.41}\\
=\int_{0}^{t}\left\langle\mathbf{f}, \mathbf{e}_{j}\right\rangle d s
\end{gather*}
$$

Here $\mathbf{v}_{0 k} \rightarrow \mathbf{v}_{0} \in H$ and the equation holds for each $\mathbf{e}_{j}$ for each $j \leq k$. Then this integral equation reduces to a system of ordinary differential equations for the vector $\mathbf{x}(t, \omega)$ whose $j^{\text {th }}$ component is $x_{j}(t, \omega)$ mentioned above. Differentiate, multiply by $x_{j}$ and add. Then integrate. This will yield some terms which need to be estimated. Here is the one which comes from the long term.

$$
\begin{aligned}
& \int_{0}^{t} \int_{\Gamma_{C}} F\left(\left(u_{k n}-g(\omega)\right)_{+}\right) \mu\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right) \psi_{\varepsilon}^{\prime}\left(\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right) \cdot \mathbf{v}_{k T} d S d s \\
&= \int_{0}^{t} \int_{\Gamma_{C}} F\left(\left(u_{k n}-g(\omega)\right)_{+}\right) \mu\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right) \psi_{\varepsilon}^{\prime}\left(\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right) \\
& \cdot\left(\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right) d S d s \\
& \quad+\int_{0}^{t} \int_{\Gamma_{C}} F\left(\left(u_{k n}-g(\omega)\right)_{+}\right) \mu\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right) \psi_{\varepsilon}^{\prime}\left(\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right) \cdot \dot{\mathbf{U}}_{T} d S d s
\end{aligned}
$$

The first of these is nonnegative and the second is bounded below by an expression of the form

$$
\begin{aligned}
-C \int_{0}^{t} \int_{\Gamma_{C}}\left(1+\left|u_{k n}\right|\right)\left|\dot{\mathbf{U}}_{T}\right| d S d s & \geq-C \int_{0}^{t}\left\|\mathbf{u}_{k}\right\|_{W}\left\|\dot{\mathbf{U}}_{T}\right\|_{L^{2}\left(\Gamma_{C}\right)^{3}} d s-C \\
& \geq-C \int_{0}^{t}\left\|\mathbf{u}_{k}\right\|_{W}-C
\end{aligned}
$$

Where $W$ embedds compactly into $V$ and the trace map from $W$ to $L^{2}\left(\Gamma_{C}\right)^{3}$ is continuous. In the above, $C$ is independent of $\varepsilon, \omega$ and $k$. To estimate the term from $P$ one exploits the linear growth condition of $P$ in 70.5 .22 to obtain a suitable estimate.

It follows from equivalence of norms in finite dimensional spaces, the assumed estimates on $M, A$, and $P$ and standard manipulations depending on compact embeddings that there exists an estimate suitable to apply Theorem 70.3.3 to obtain the existence of a solution such that $(t, \omega) \rightarrow \mathbf{x}(t, \omega)$ is measurable into $\mathbb{R}^{k}$ which implies that $(t, \omega) \rightarrow \mathbf{v}_{k}(t, \omega)$ is product measurable into $V$ and $H$. This yields the measurable Galerkin approximation.

Also, the estimates and compact embedding results for Sobolev spaces imply an inequality of the form

$$
\begin{equation*}
\left|\mathbf{v}_{k}(t)\right|_{H}^{2}+\int_{0}^{T}\left\|\mathbf{v}_{k}\right\|_{V}^{2} d s+\left\|\mathbf{u}_{k}(t)\right\|_{V}^{2} \leq C \tag{70.5.42}
\end{equation*}
$$

where in fact $C$ does not depend on $\varepsilon, \omega$ or $k$. Everything would work if $C$ depended on $\omega$ but because of our simplifying assumptions, we can get a single $C$ as above.

Next we need to estimate the time derivative in $\mathscr{V}^{\prime}$. The integral equation implies that for all $\mathbf{w} \in V_{k}$,

$$
\begin{gather*}
\left\langle\mathbf{v}_{k}^{\prime}(t), \mathbf{w}\right\rangle_{V^{\prime}, V}+\left\langle\begin{array}{c}
M \mathbf{v}_{k}+A \mathbf{u}_{k}+P \mathbf{u}_{k}+\gamma_{T}^{*} F\left(\left(u_{k n}-g(\omega)\right)_{+}\right) \cdot \\
\mu\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right) \psi_{\varepsilon}^{\prime}\left(\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right) \\
=\langle\mathbf{f}, \mathbf{w}\rangle
\end{array}, \quad\right\rangle_{V^{\prime}, V} \\ \tag{70.5.43}
\end{gather*}
$$

where the dependence on $t$ and $\omega$ is suppressed in most terms. In terms of inner products in $V$ this reduces to

$$
\begin{gathered}
\left(R^{-1} \mathbf{v}_{k}^{\prime}(t), \mathbf{w}\right)_{V}+\left(R^{-1}\binom{M \mathbf{v}_{k}+A \mathbf{u}_{k}+P \mathbf{u}_{k}+\gamma_{T}^{*} F\left(\left(u_{n}-g(\omega)\right)_{+}\right) \cdot}{\mu\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right) \psi_{\varepsilon}^{\prime}\left(\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right)}, \mathbf{w}\right)_{V} \\
=\left(R^{-1} \mathbf{f}, \mathbf{w}\right)_{V}
\end{gathered}
$$

In terms of $P_{k}$ the orthogonal projection in $V$ onto $V_{k}$, this takes the form

$$
\begin{gathered}
\left(R^{-1} \mathbf{v}_{k}^{\prime}(t), P_{k} \mathbf{w}\right)_{V}+ \\
\left(R^{-1}\binom{M \mathbf{v}_{k}+A \mathbf{u}_{k}+P \mathbf{u}_{k}+\gamma_{T}^{*} F\left(\left(u_{n}-g(\omega)\right)_{+}\right) \cdot}{\mu\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right) \psi_{\varepsilon}^{\prime}\left(\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right)}, P_{k} \mathbf{w}\right)_{V} \\
=\left(R^{-1} \mathbf{f}, P_{k} \mathbf{w}\right)_{V}
\end{gathered}
$$

for all $\mathbf{w} \in V$. Now $\mathbf{v}_{k}^{\prime}(t) \in V_{k}$ and so the first term can be simplified and we can write

$$
\begin{gathered}
\left(R^{-1} \mathbf{v}_{k}^{\prime}(t), \mathbf{w}\right)_{V}+ \\
\left(R^{-1}\binom{M \mathbf{v}_{k}+A \mathbf{u}_{k}+P \mathbf{u}_{k}+\gamma_{T}^{*} F\left(\left(u_{n}-g(\omega)\right)_{+}\right) \cdot}{\mu\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right) \psi_{\varepsilon}^{\prime}\left(\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right)}, P_{k} \mathbf{w}\right)_{V} \\
=\left(R^{-1} \mathbf{f}, P_{k} \mathbf{w}\right)_{V}
\end{gathered}
$$

for all $\mathbf{w} \in V$. Then it follows that for all $\mathbf{w} \in V$,

$$
\left(R^{-1} \mathbf{v}_{k}^{\prime}(t), \mathbf{w}\right)_{V}+
$$

$$
\left(P_{k} R^{-1}\left(\begin{array}{c}
M \mathbf{v}_{k}+A \mathbf{u}_{k}+P \mathbf{u}_{k}+\gamma_{T}^{*} F\left(\left(u_{n}-g(\omega)\right)_{+}\right) \cdot \\
\mu\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right) \psi_{\varepsilon}^{\prime}\left(\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right) \\
=\left(P_{k} R^{-1} \mathbf{f}, \mathbf{w}\right)_{V}
\end{array}\right), \mathbf{w}\right)_{V}
$$

Thus in $V$ we have

$$
R^{-1} \mathbf{v}_{k}^{\prime}(t)+P_{k} R^{-1}\binom{M \mathbf{v}_{k}+A \mathbf{u}_{k}+P \mathbf{u}_{k}+\gamma_{T}^{*} F\left(\left(u_{n}-g(\omega)\right)_{+}\right) \cdot}{\mu\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right) \psi_{\varepsilon}^{\prime}\left(\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right)}=P_{k} R^{-1} \mathbf{f}
$$

and $R^{-1}$ preserves norms while $P_{k}$ decreases them. Hence the estimate 70.5 .42 implies that $\left\|\mathbf{v}_{k}^{\prime}\right\|_{\mathscr{V}^{\prime}}$ is also bounded independent of $\varepsilon, \omega$ and $k$. Then summarizing this yields

$$
\begin{equation*}
\left|\mathbf{v}_{k}(t, \omega)\right|_{H}+\left\|\mathbf{v}_{k}(\cdot, \omega)\right\|_{\mathscr{V}}+\left\|\mathbf{v}_{k}^{\prime}(\cdot, \omega)\right\|_{\mathscr{V}^{\prime}}+\left\|\mathbf{u}_{k}(t, \omega)\right\|_{V} \leq C(\omega) \tag{70.5.44}
\end{equation*}
$$

where $C$ is some constant which does not depend on $\varepsilon, \omega$, and $k$. Also, integrating 70.5.43, it follows that

$$
\begin{gather*}
i_{k}^{*}\left(\mathbf{v}_{k}(t)-\mathbf{v}_{0 k}+\int_{0}^{t} M \mathbf{v}_{k} d s+\int_{0}^{t} A \mathbf{u}_{k} d s+\int_{0}^{t} P \mathbf{u}_{k} d s+\right. \\
\left.\int_{0}^{t} \gamma_{T}^{*} F\left(\left(u_{k n}-g(\omega)\right)_{+}\right) \mu\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right) \psi_{\varepsilon}^{\prime}\left(\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right) d s\right)=i_{k}^{*} \int_{0}^{t} \mathbf{f} d s \tag{70.5.45}
\end{gather*}
$$

Where $i_{k}^{*}$ is the dual map to the inclusion map $i_{k}: V_{k} \rightarrow V$.
Let

$$
V \subseteq W, V \text { dense in } W
$$

where the embedding is compact and the trace map onto the boundary of $U$ is continuous. Using Theorem 70.5.2 and 70.5.1, it follows that for a fixed $\omega$, there exist the following convergences valid for a suitable subsequence, still denoted as $\left\{\mathbf{v}_{k}\right\}$ which may depend on $\omega$.

$$
\begin{gather*}
\mathbf{v}_{k} \rightharpoonup \mathbf{v} \text { in } \mathscr{V}  \tag{70.5.46}\\
\mathbf{v}_{k}^{\prime} \rightharpoonup \mathbf{v}^{\prime} \text { in } \mathscr{V}^{\prime}  \tag{70.5.47}\\
\mathbf{v}_{k} \rightarrow \mathbf{v} \text { strongly in } C\left([0, T], W^{\prime}\right)  \tag{70.5.48}\\
\mathbf{v}_{k} \rightarrow \mathbf{v} \text { strongly in } L^{2}([0, T] ; W)  \tag{70.5.49}\\
\mathbf{v}_{k}(t) \rightarrow \mathbf{v}(t) \text { in } W \text { for } \text { a.e.t }  \tag{70.5.50}\\
\mathbf{u}_{k} \rightarrow \mathbf{u} \text { strongly in } C([0, T] ; W)  \tag{70.5.51}\\
A \mathbf{u}_{k} \rightharpoonup A \mathbf{u} \text { in } \mathscr{V}^{\prime}  \tag{70.5.52}\\
M \mathbf{v}_{k} \rightharpoonup M \mathbf{v} \text { in } \mathscr{V}^{\prime} \tag{70.5.53}
\end{gather*}
$$

Now from these convergences and the density of $\cup_{n} V_{n}$, it follows on passing to a limit and using dominated convergence theorem and the strong convergences above in the nonlinear terms, we obtain the following equation which holds in $V^{\prime}$.

$$
\mathbf{v}(t)-\mathbf{v}_{0}+\int_{0}^{t} M \mathbf{v} d s+\int_{0}^{t} A \mathbf{u} d s+\int_{0}^{t} P \mathbf{u} d s+
$$

$$
\begin{equation*}
\left.\int_{0}^{t} \gamma_{T}^{*} F\left(\left(u_{n}-g(\omega)\right)_{+}\right) \mu\left(\left|\mathbf{v}_{T}-\dot{\mathbf{U}}_{T}\right|\right) \psi_{\varepsilon}^{\prime}\left(\mathbf{v}_{T}-\dot{\mathbf{U}}_{T}\right) d s\right)=\int_{0}^{t} \mathbf{f} d s \tag{70.5.54}
\end{equation*}
$$

Thus $t \rightarrow \mathbf{v}(t, \omega)$ is continuous into $V^{\prime}$. This along with the estimate 70.5 .44 , implies that the conditions of Theorem 70.2 .1 are satisfied. It follows that there is a function $\overline{\mathbf{v}}$ which is product measurable into $V^{\prime}$ and weakly continuous in $t$ and for each $\omega$, a subsequence $\mathbf{v}_{k(\omega)}$ such that $\mathbf{v}_{k(\omega)}(\cdot, \omega) \rightharpoonup \overline{\mathbf{v}}(\cdot, \omega)$ in $\mathscr{V}^{\prime}$. Then by a repeat of the above argument, for each $\omega$, there exists a further subsequence still denoted as $\mathbf{v}_{k(\omega)}$ which converges in $\mathscr{V}^{\prime}$ to $\mathbf{v}(\cdot, \omega)$ which is a solution to the above integral equation which is continuous into $V^{\prime}$. Hence, $\overline{\mathbf{v}}(\cdot, \omega)=\mathbf{v}(\cdot, \omega)$ and since these are both weakly continuous into $V^{\prime}$ they must be the same function. Hence, there is a product measurable solution $\mathbf{v}$.

Next we pass to a limit as $\varepsilon \rightarrow 0$. Denoting the product measurable solution to the above integral equation as $\mathbf{v}_{k}$, where $\varepsilon=1 / k$. The estimate 70.5.42 is obtained as before. Then we get a subsequence, still denoted as $\mathbf{v}_{k}$ which has the same convergences as in 70.5.46-70.5.53. Thus we obtain these convergences along with the fact that $\mathbf{v}_{k}$ is product measurable and for each $\omega$, it is a solution of

$$
\begin{gather*}
\mathbf{v}_{k}(t)-\mathbf{v}_{0}+\int_{0}^{t} M \mathbf{v}_{k} d s+\int_{0}^{t} A \mathbf{u}_{k} d s+\int_{0}^{t} P \mathbf{u}_{k} d s+ \\
\int_{0}^{t} \gamma_{T}^{*} F\left(\left(u_{k n}-g(\omega)\right)_{+}\right) \mu\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right) \psi_{1 / k}^{\prime}\left(\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right) d s=\int_{0}^{t} \mathbf{f} d s \tag{70.5.55}
\end{gather*}
$$

Now in addition to these convergences, we can also obtain

$$
\psi_{1 / k}^{\prime}\left(\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right) \rightharpoonup \xi \text { in } L^{\infty}\left([0, T] ; L^{\infty}\left(\Gamma_{C}\right)^{3}\right)
$$

We have also

$$
\psi_{1 / k}^{\prime}\left(\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right) \cdot \mathbf{w}_{T} \leq \psi_{1 / k}\left(\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}+\mathbf{w}_{T}\right)-\psi_{1 / k}\left(\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right)
$$

and so, passing to a limit, using the strong convergence of $\mathbf{v}_{k T}$ to $\mathbf{v}_{T}$ in $L^{2}([0, T] ; W)$, uniform convergence of $\psi_{1 / k}$ to $\|\cdot\|$, and pointwise convergence in $W$, we obtain using the dominated convergence theorem that for $\mathbf{w} \in \mathscr{V}$,

$$
\begin{aligned}
& \int_{0}^{t} \int_{\Gamma_{C}} F\left(\left(u_{k n}-g(\omega)\right)_{+}\right) \mu\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right) \psi_{1 / k}^{\prime}\left(\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right) \cdot \mathbf{w}_{T} d x d s \\
\rightarrow & \int_{0}^{t} \int_{\Gamma_{C}} F\left(\left(u_{n}-g(\omega)\right)_{+}\right) \mu\left(\left|\mathbf{v}_{T}-\dot{\mathbf{U}}_{T}\right|\right) \xi \cdot \mathbf{w}_{T} d x d s
\end{aligned}
$$

where

$$
\begin{equation*}
\int_{0}^{t} \int_{\Gamma_{C}} \xi \cdot \mathbf{w}_{T} d \alpha d s \leq \int_{0}^{t} \int_{\Gamma_{C}}\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}+\mathbf{w}_{T}\right|-\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right| d \alpha d s \tag{70.5.56}
\end{equation*}
$$

Then passing to the limit in the integral equation 70.5.55, we obtain that $\mathbf{v}$ is a solution for each $\omega$ to the integral equation

$$
\mathbf{v}(t)-\mathbf{v}_{0}+\int_{0}^{t} M \mathbf{v} d s+\int_{0}^{t} A \mathbf{u} d s+\int_{0}^{t} P \mathbf{u} d s+
$$

$$
\begin{equation*}
\int_{0}^{t} \gamma_{T}^{*} F\left(\left(u_{n}-g(\omega)\right)_{+}\right) \mu\left(\left|\mathbf{v}_{T}-\dot{\mathbf{U}}_{T}\right|\right) \xi d s=\int_{0}^{t} \mathbf{f} d s \tag{70.5.57}
\end{equation*}
$$

where $\xi$ satisfies the inequality 70.5 .56 . In particular, $\mathbf{v}$ is continuous into $V^{\prime}$ and now, the conclusion of the measurable selection theorem applies and yields the existence of a measurable solution to the integral equation just displayed for each $\omega$. Taking a weak derivative, it follows that we have obtained a measurable solution to the system 70.5.3370.5.36.

In this case of Lipschitz $\mu$ one can show that the solution for each $\omega$ to the above integral equation is unique although this it is not an obvious theorem. This follows standard procedures involving Gronwall's inequality and estimates. Therefore, it is possible to obtain the measurability using more elementary methods. In addition, it becomes possible to include a stochastic integral of the form $\int_{0}^{t} \Phi d W$. In this case one must consider a filtration and obtain solutions which are adapted to the filtration. In the next section we consider the case of discontinuous friction coefficient and in this case it is not clear whether there is uniqueness but we have still obtained a measurable solution.

### 70.5.3 Discontinuous coefficient of friction

In this section we consider the case where the coefficient of friction is a discontinuous function of the slip speed. This is the case described in elementary physics courses which state that the coefficient of sliding friction is less than the coefficient of static friction. Specifically, we assume the function $\mu$, has a jump discontinuity at 0 , becoming smaller when the speed is positive.


Fig. 2. The graph of $\mu$ vs. the slip rate $\left|\mathbf{v}_{*}\right|$, and $v$.
We assume the function $\mu_{s}$ of the picture is Lipschitz continuous and decreasing just as shown. The new function $v$ is extended for $r<0$ as shown and is just $\mu_{s}(r)+\eta$ for $r>0$.

Let

$$
\begin{aligned}
& h_{\varepsilon}(r) \equiv\left(\eta^{2} r^{2}+\varepsilon\right)^{1 / 2} \\
& \mu_{\varepsilon}(r)=v(r)-h_{\varepsilon}^{\prime}(r)
\end{aligned}
$$

Thus $\mu_{\varepsilon}$ is bounded, Lipschitz continuous and as $\varepsilon \rightarrow 0, \mu_{\varepsilon}(r) \rightarrow \mu(r)$ for $r>0$. Thus,
for each $\varepsilon=1 / k$, there exists a measurable solution to the integral equation

$$
\begin{array}{r}
\mathbf{v}_{k}(t)-\mathbf{v}_{0}+\int_{0}^{t} M \mathbf{v}_{k} d s+\int_{0}^{t} A \mathbf{u}_{k} d s+\int_{0}^{t} P \mathbf{u}_{k} d s+ \\
\int_{0}^{t} \gamma_{T}^{*} F\left(\left(u_{k n}-g(\omega)\right)_{+}\right) \mu_{1 / k}\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right) \xi_{k} d s=\int_{0}^{t} \mathbf{f} d s \tag{70.5.58}
\end{array}
$$

where

$$
\begin{equation*}
\int_{0}^{t} \int_{\Gamma_{C}} \xi_{k} \cdot \mathbf{w}_{T} d \alpha d s \leq \int_{0}^{t} \int_{\Gamma_{C}}\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}+\mathbf{w}_{T}\right|-\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right| d \alpha d s \tag{70.5.59}
\end{equation*}
$$

Now for a given $\omega$, the same estimate obtained earlier, 70.5.42 is available. Thus

$$
\left|\mathbf{v}_{k}(t)\right|_{H}^{2}+\int_{0}^{T}\left\|\mathbf{v}_{k}\right\|_{V}^{2} d s+\left\|\mathbf{u}_{k}(t)\right\|_{V}^{2} \leq C
$$

where $C$ is not dependent on $k$. Recall also that $\xi_{k}$ is bounded. Hence from 70.5 .58 , and this estimate, it also follows that $\mathbf{v}_{k}^{\prime}$ is bounded in $\mathscr{V}^{\prime}$. Thus

$$
\left|\mathbf{v}_{k}(t)\right|_{H}^{2}+\int_{0}^{T}\left\|\mathbf{v}_{k}\right\|_{V}^{2} d s+\left\|\mathbf{u}_{k}(t)\right\|_{V}^{2}+\left\|\mathbf{v}_{k}^{\prime}\right\|_{\mathscr{V}^{\prime}} \leq C
$$

As earlier, we can take $C$ independent of $k$ and $\omega$ although we do not need this constant to be independent of $\omega$. Now for fixed $\omega$, there exists a subsequence, still denoted as $\left\{\mathbf{v}_{k}\right\}$ such that the convergences obtained earlier all hold, that is 70.5.46-70.5.53. Taking a further subsequence, we may assume also that

$$
\begin{gathered}
\psi-h_{1 / k}^{\prime}\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right) \rightharpoonup 0 \text { in } L^{\infty}\left([0, T], L^{\infty}\left(\Gamma_{C}\right)\right) \\
\xi_{k} \rightharpoonup \xi \text { weak } * \text { in } L^{\infty}\left([0, T], L^{\infty}\left(\Gamma_{C}\right)^{3}\right)
\end{gathered}
$$

That is, $h_{1 / k}^{\prime}\left(\left|\mathbf{v}_{(1 / k) T}-\dot{\mathbf{U}}_{T}\right|\right)$ converges weak $*$ in $L^{\infty}\left([0, T], L^{\infty}\left(\Gamma_{C}\right)\right)$ to some $\psi$. This is because

$$
h_{\varepsilon}^{\prime}(r)=\frac{\eta^{2} r}{\sqrt{r^{2} \eta^{2}+\varepsilon}}
$$

and this is bounded. Letting $w \in L^{1}\left([0, T] ; L^{1}\left(\Gamma_{C}\right)\right)$,

$$
\begin{aligned}
& \int_{0}^{T} \int_{\Gamma_{C}} h_{(1 / k)}^{\prime}\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right) w d \alpha d t \\
\leq & \int_{0}^{T} \int_{\Gamma_{C}} h_{(1 / k)}\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|+w\right)-h_{(1 / k)}\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right) d \alpha d t
\end{aligned}
$$

Thanks to the strong convergences and the uniform convergence of $h_{(1 / k)}(r)$ to $|\eta r|$,

$$
\int_{0}^{T} \int_{\Gamma_{C}} \psi w d \alpha d t \leq \int_{0}^{T} \int_{\Gamma_{C}}\left|\eta\left(\left|\mathbf{v}_{T}-\dot{\mathbf{U}}_{T}\right|+w\right)\right|-\eta\left|\mathbf{v}_{T}-\dot{\mathbf{U}}_{T}\right| d \alpha d t
$$

Therefore, for a.e.t, $\psi(t, \mathbf{x}, \omega)$ is in the subgradient of the function $\phi_{\eta}(r)=|\eta r|$ for a.e. $\mathbf{x} \in$ $\Gamma_{C}$ at the point $r=\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|$. In particular, $\psi \in[-\eta, \eta]$ so that $v\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right)-\psi$ is between $\mu_{s}(0)$ and $\mu_{0}$ if $\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|=0$. If this quantity is positive, then $\psi=\eta$ and $v\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right)-\psi$ reduces to $\mu_{s}\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right)$. Thus

$$
\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|, v\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right)-\psi\right)
$$

is in the graph of $\mu$ a.e. Similar reasoning based on strong convergence and 70.5.59 implies that for a.e.t, $\xi \in \partial \gamma$ where $\gamma(\mathbf{y})=|\mathbf{y}|$ at the point $\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}$ for a.e. $\mathbf{x} \in \Gamma_{C}$.

Consider the friction terms in 70.5.58. Letting $\mathbf{w} \in \mathscr{V}$ and recalling that $\mu_{(1 / k)}(r)=$ $v(r)-h_{(1 / k)}^{\prime}(r)$,

$$
\begin{gather*}
\int_{0}^{T} \int_{\Gamma_{C}} F\left(\left(u_{k n}-g\right)_{+}\right) \mu_{(1 / k)}\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right) \xi_{k} \cdot \mathbf{w}_{T} d \alpha d t \\
=\int_{0}^{T} \int_{\Gamma_{C}} F\left(\left(u_{k n}-g\right)_{+}\right)\left(v\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right)-h_{(1 / k)}^{\prime}\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right)\right) \xi_{k} \cdot \mathbf{w}_{T} d \alpha d t \\
=\int_{0}^{T} \int_{\Gamma_{C}} F\left(\left(u_{k n}-g\right)_{+}\right)\left(v\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right)-\psi\right) \xi_{k} \cdot \mathbf{w}_{T} d \alpha d t  \tag{70.5.60}\\
+\int_{0}^{T} \int_{\Gamma_{C}} F\left(\left(u_{k n}-g\right)_{+}\right)\left(\psi-h_{(1 / k)}^{\prime}\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right)\right) \xi_{k} \cdot \mathbf{w}_{T} d \alpha d t \tag{70.5.61}
\end{gather*}
$$

Now consider the first integral. The strong convergence yields that this integral in 70.5.60 converges to

$$
\int_{0}^{T} \int_{\Gamma_{C}} F\left(\left(u_{n}-g\right)_{+}\right)\left(v\left(\left|\mathbf{v}_{T}-\dot{\mathbf{U}}_{T}\right|\right)-\psi\right) \xi \cdot \mathbf{w}_{T} d \alpha d t
$$

where $v\left(\left|\mathbf{v}_{T}-\dot{\mathbf{U}}_{T}\right|\right)-\psi$ is in the graph of $\mu$ a.e.
Consider the second integral in 70.5.61.

$$
\begin{gathered}
\int_{0}^{T} \int_{\Gamma_{C}} F\left(\left(u_{k n}-g\right)_{+}\right)\left(\psi-h_{(1 / k)}^{\prime}\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right)\right) \xi_{k} \cdot \mathbf{w}_{T} d \alpha d t \\
\leq \int_{0}^{T} \int_{\Gamma_{C}} F\left(\left(u_{k n}-g\right)_{+}\right)\left(\psi-h_{(1 / k)}^{\prime}\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right)\right) \\
\quad\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}+\mathbf{w}_{T}\right|-\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right) d \alpha d t
\end{gathered}
$$

Similarly,

$$
\begin{gathered}
-\int_{0}^{T} \int_{\Gamma_{C}} F\left(\left(u_{k n}-g\right)_{+}\right)\left(\psi-h_{(1 / k)}^{\prime}\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right)\right) \xi_{k} \cdot \mathbf{w}_{T} d \alpha d t \\
\quad \leq \int_{0}^{T} \int_{\Gamma_{C}} F\left(\left(u_{k n}-g\right)_{+}\right)\left(\psi-h_{(1 / k)}^{\prime}\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right)\right) \\
\quad\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}-\mathbf{w}_{T}\right|-\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right) d \alpha d t
\end{gathered}
$$

Each of these integrals on the right side converge to 0 because, from the strong convergence results,

$$
F\left(\left(u_{k n}-g\right)_{+}\right)\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T} \pm \mathbf{w}_{T}\right|-\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right)
$$

converges in $L^{1}\left([0, T], L^{1}\left(\Gamma_{C}\right)\right)$ and so the weak $*$ convergence to 0 of

$$
\psi-h_{(1 / k)}^{\prime}\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right)
$$

implies that these integrals converge to 0 . Thus the integral in 70.5.61

$$
\int_{0}^{T} \int_{\Gamma_{C}} F\left(\left(u_{k n}-g\right)_{+}\right)\left(\psi-h_{(1 / k)}^{\prime}\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|\right)\right) \xi_{k} \cdot \mathbf{w}_{T} d \alpha d t
$$

is between two sequences each of which converges to 0 so it also converges to 0 .
To save space, denote by

$$
\hat{\mu}=v\left(\left|\mathbf{v}_{T}-\dot{\mathbf{U}}_{T}\right|\right)-\psi
$$

Then passing to the limit in this subsequence, we obtain for fixed $\omega$ the existence of a solution to the following integral equation.

$$
\begin{equation*}
\mathbf{v}(t)-\mathbf{v}_{0}+\int_{0}^{t} M \mathbf{v} d s+\int_{0}^{t} A \mathbf{u} d s+\int_{0}^{t} P \mathbf{u} d s+\int_{0}^{t} \gamma_{T}^{*} F\left(\left(u_{n}-g\right)_{+}\right) \hat{\mu} \xi d s=\int_{0}^{t} \mathbf{f} d s \tag{70.5.62}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathbf{u}(t)=\mathbf{u}_{0}+\int_{0}^{t} \mathbf{v}(s) d s \tag{70.5.63}
\end{equation*}
$$

and $\left(\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right|, \hat{\mu}\right)$ is contained in the graph of $\mu$ a.e. Also for each $\mathbf{w} \in \mathscr{V}$,

$$
\begin{equation*}
\int_{0}^{T} \int_{\Gamma_{C}} \xi \cdot \mathbf{w}_{T} d \alpha d s \leq \int_{0}^{T} \int_{\Gamma_{C}}\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}+\mathbf{w}_{T}\right|-\left|\mathbf{v}_{k T}-\dot{\mathbf{U}}_{T}\right| d \alpha d s \tag{70.5.64}
\end{equation*}
$$

The remaining issue concerns the existence of a measurable solution. However, this follows in the same way as before from the measurable selection theorem, Theorem 70.2.1. From the above reasoning, for fixed $\omega$ any sequence has a subsequence which leads to a solution to the integral equation $70.5 .62-70.5 .64$ which is continuous into $V^{\prime}$. There is also an estimate of the right sort for all of the $\mathbf{v}_{k}$. Therefore, from this theorem, there is a function $\mathbf{v}(\cdot, \omega)$ in $\mathscr{V}^{\prime}$ which is weakly continuous into $V^{\prime}$ and a sequence $\mathbf{v}_{k(\omega)}(\cdot, \omega)$ converging to $\mathbf{v}(\cdot, \omega)$. Then from the above argument, a subsequence converges to a solution to the integral equation and since both are weakly continuous into $V^{\prime}$, it follows that the solution to the integral equation equals this measurable function for all $t$, this for each $\omega$. Thus there is a measurable solution to the stochastic friction problem. The result is stated in the following theorem.

Theorem 70.5.3 For each $\omega$ let $\mathbf{u}_{0}(\omega) \in V, \mathbf{v}_{0}(\omega) \in H$. Let $\mathbf{f} \in \mathscr{V}^{\prime}$. Also assume the gap $g$ and sliding velocity $\dot{\mathbf{U}}_{T}$ are $\mathscr{F}$ measurable. Then there exists a solution $\mathbf{v}$, to the problem summarized in 70.5.62-70.5.64 for each $\omega$. This solution $(t, \omega) \rightarrow \mathbf{v}(t, \omega)$ is measurable into $V^{\prime}, H^{\prime}$ and $V^{\prime}$.

It only remains to check the last claim about measurability into the other spaces. By density of $V$ into $H$, it follows that $H^{\prime}$ is dense in $V^{\prime}$ and so a simple Pettis theorem argument implies right away that $\omega \rightarrow \mathbf{v}(t, \omega)$ is $\mathscr{F}$ measurable into both $V$ and $H$.

## Chapter 71

## Stochastic O.D.E. One Space

### 71.1 Adapted Solutions With Uniqueness

Instead of a single $\sigma$ algebra $\mathscr{F}$, one can generalize to the case of a normal filtration $\mathscr{F}_{t}$ and obtain adapted solutions to finite dimensional theorems, provided one also knows path uniqueness of the solutions. Recall that a filtration is normal includes the following condition which is what we will use.

$$
\begin{equation*}
\mathscr{F}_{t}=\cap_{s>t} \mathscr{F}_{s} \tag{71.1.1}
\end{equation*}
$$

Theorem 71.1.1 Suppose $\mathbf{N}(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega) \in \mathbb{R}^{d}$ for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^{d}, t \in[0, T]$ and

$$
(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega) \rightarrow \mathbf{N}(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega)
$$

is progressively measurable with respect to a normal filtration or more generally one which satisfies 71.1.1. Also suppose $(t, \mathbf{u}, \mathbf{v}, \mathbf{w}) \rightarrow \mathbf{N}(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega)$ is continuous. Suppose for each $\omega$, there exists an estimate for any solution $\mathbf{u}(\cdot, \omega)$ to the integral equation

$$
\begin{equation*}
\mathbf{u}(t, \omega)-\mathbf{u}_{0}(\omega)+\int_{0}^{t} \mathbf{N}(s, \mathbf{u}(s, \omega), \mathbf{u}(s-h, \omega), \mathbf{w}(s, \omega), \omega) d s=\int_{0}^{t} \mathbf{f}(s, \omega) d s \tag{71.1.2}
\end{equation*}
$$

which is of the form

$$
\sup _{t \in[0, T]}|\mathbf{u}(t, \omega)| \leq C(\omega)<\infty
$$

Also let $\mathbf{f}$ be progressively measurable and $\mathbf{f}(\cdot, \omega) \in L^{1}\left([0, T] ; \mathbb{R}^{d}\right)$. Here $\mathbf{u}_{0}$ has values in $\mathbb{R}^{d}$ and is $\mathscr{F}_{0}$ measurable and $\mathbf{u}(s-h, \omega) \equiv \mathbf{u}_{0}(\omega)$ whenever $s-h \leq 0$ and

$$
\mathbf{w}(t, \omega) \equiv \mathbf{w}_{0}(\omega)+\int_{0}^{t} \mathbf{u}(s, \omega) d s
$$

where $\mathbf{w}_{0}$ is a given $\mathscr{F}_{0}$ measurable function. Also assume that for each $\omega$ there is at most one solution to the integral equation 71.1.2. Then for $h>0$, there exists a progressively measurable solution $\mathbf{u}$ to the integral equation 71.1.2.

Proof: Let $0=t_{0}<t_{1}<\cdots<t_{n}=T$. From Theorem 70.3.3, there exists a solution to the integral equation $\mathbf{u}$ which has the property that $\mathbf{u}\left(t \wedge t_{j}\right)$ is $\mathscr{F}_{t_{j}}$ measurable. One simply applies this theorem to the succession of intervals determined by the given partition. Now suppose $\mathscr{P}^{n}$ consists of the points $k 2^{-n} T \equiv t_{j}^{n}$ so that these satisfy $\mathscr{P}^{n} \subseteq \mathscr{P}^{n+1}$ and the lengths of the sub-intervals decreases to 0 with increasing $n$. Let $\mathbf{u}_{n}$ denote the solution just described corresponding to $\mathscr{P}^{n}$ such that $\mathbf{u}_{n}\left(t \wedge t_{j}^{n}\right)$ is $\mathscr{F}_{t_{j}^{n}}$ measurable. As before, using the estimate, these $\mathbf{u}_{n}(\cdot, \omega)$ for a fixed $\omega$ are uniformly bounded and equicontinuous. This is because it is a solution to the integral equation for each $\omega$ and so by assumption, there is an estimate. Therefore, for fixed $\omega$, there exists $\mathbf{u}(\cdot, \omega)$ and a subsequence, denoted as $\mathbf{u}_{n}(\cdot, \omega)$ which converges uniformly to $\mathbf{u}(\cdot, \omega)$ on $[0, T]$. Therefore, $\mathbf{u}(\cdot, \omega)$ will be a
solution to the integral equation for that $\omega$. It follows from the uniqueness assumption, that it is not necessary to take a subsequence. Thus

$$
\mathbf{u}(t, \omega)=\lim _{n \rightarrow \infty} \mathbf{u}_{n}(t, \omega)
$$

For $t \in\left(t_{j-1}^{n}, t_{j}^{n}\right]$, it follows that $\omega \rightarrow \mathbf{u}(t, \omega)$ is $\mathscr{F}_{t_{j}^{n}}$ measurable. Since this is true for each $n$ and the filtration is assumed to be a normal filtration, we conclude that $\omega \rightarrow \mathbf{u}(t, \omega)$ is $\mathscr{F}_{t}$ measurable.

Why can't this be generalized to the situation where no uniqueness is known? We have been unable to do this. It appears that the difficulty is related to the need to use theorems about measurable selections and these theorems pertain to a single $\sigma$ algebra. Attempts to use the $\sigma$ - algebra of progressively measurable sets have not been successful either.

### 71.2 Including Stochastic Integrals

It is not surprising that Theorem 71.1.1 is sufficient to allow the inclusion of a stochastic integral. Thus, with the same descriptions of the symbols used in that theorem, one could consider the following integral equation.

$$
\begin{aligned}
\mathbf{u}(t, \omega)-\mathbf{u}_{0}(\omega)+ & \int_{0}^{t} \mathbf{N}(s, \mathbf{u}(s, \omega), \mathbf{u}(s-h, \omega), \mathbf{w}(s, \omega), \omega) d s \\
& =\int_{0}^{t} \mathbf{f}(s, \omega) d s+\int_{0}^{t} \Phi d W
\end{aligned}
$$

where, as usual $\Phi \in L^{2}\left([0, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, \mathbb{R}^{d}\right)\right)$ where $U$ is a Hilbert space. It could be $\mathbb{R}^{d}$ of course. To include a stochastic integral, you define a new variable.

$$
\hat{\mathbf{u}}(t)=\mathbf{u}(t)-\int_{0}^{t} \Phi d W
$$

Then in terms of this new variable, the integral equation is

$$
\begin{gathered}
\hat{\mathbf{u}}(t, \omega)-\mathbf{u}_{0}(\omega)+\int_{0}^{t} \mathbf{N}\left(s, \hat{\mathbf{u}}(s, \omega)+\int_{0}^{s} \Phi d W, \hat{\mathbf{u}}(s-h, \omega)+\int_{0}^{s-h} \Phi d W\right. \\
\left.\int_{0}^{s}\left(\hat{\mathbf{u}}(r)+\int_{0}^{r} \Phi d W\right) d r, \omega\right) d s=\int_{0}^{t} \mathbf{f}(s, \omega) d s
\end{gathered}
$$

This is in the situation of Theorem 71.1.1 provided $\mathbf{N}$ is progressively measurable with respect to the normal filtration $\mathscr{F}_{t}$ determined by the Wiener process and there exists an estimate of the sort in this theorem and for a given $\omega$ there is at most one solution $t \rightarrow$ $\hat{\mathbf{u}}(t, \omega)$ to the above integral equation.

Theorem 71.2.1 Suppose $\mathbf{N}(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega) \in \mathbb{R}^{d}$ for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^{d}, t \in[0, T]$ and

$$
(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega) \rightarrow \mathbf{N}(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega)
$$

is progressively measurable with respect to the normal filtration $\mathscr{F}_{t}$ determined by a given Wiener process $W(t)$. Also suppose

$$
(t, \mathbf{u}, \mathbf{v}, \mathbf{w}) \rightarrow \mathbf{N}(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega)
$$

is continuous and satisfies the following conditions for $C(\cdot, \omega) \geq 0$ in $L^{1}([0, T])$ and some $\mu>0$ :

$$
\begin{equation*}
(\mathbf{N}(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega), \mathbf{u}) \geq-C(t, \omega)-\mu\left(|\mathbf{u}|^{2}+|\mathbf{v}|^{2}+|\mathbf{w}|^{2}\right) . \tag{71.2.3}
\end{equation*}
$$

Also let $\mathbf{f}$ be progressively measurable and $\mathbf{f}(\cdot, \omega) \in L^{2}\left([0, T] ; \mathbb{R}^{d}\right)$. Let

$$
\Phi \in L^{2}\left([0, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, \mathbb{R}^{d}\right)\right)
$$

where $U$ is some Hilbert space, $\mathbb{R}^{d}$, for example. Also suppose path uniqueness. That is, for each $\omega$, there is at most one solution to the integral equation

$$
\begin{align*}
\mathbf{u}(t, \omega)-\mathbf{u}_{0}(\omega) & +\int_{0}^{t} \mathbf{N}(s, \mathbf{u}(s, \omega), \mathbf{u}(s-h, \omega), \mathbf{w}(s, \omega), \omega) d s \\
& =\int_{0}^{t} \mathbf{f}(s, \omega) d s+\int_{0}^{t} \Phi d W \tag{71.2.4}
\end{align*}
$$

Then for $h>0$, there exists a unique progressively measurable solution $\mathbf{u}$ to the integral equation 71.2 .4 where $\mathbf{u}_{0}$ has values in $\mathbb{R}^{d}$ and is $\mathscr{F}_{0}$ measurable. Here $\mathbf{u}(s-h, \omega) \equiv$ $\mathbf{u}_{0}(\omega)$ for all $s-h \leq 0$ and for $\mathbf{w}_{0}$ a given $\mathscr{F}_{0}$ measurable function,

$$
\mathbf{w}(t, \omega) \equiv \mathbf{w}_{0}(\omega)+\int_{0}^{t} \mathbf{u}(s, \omega) d s
$$

Proof: The only thing left is to observe that the given estimate is sufficient to obtain an estimate for the solutions to the integral equation for $\hat{\mathbf{u}}$ defined above. Then from Theorem 71.1.1, there exists a unique progressively measurable solution for $\hat{\mathbf{u}}$ and hence for $\mathbf{u}$.

Note that the integral equation holds for all $t$ for each $\omega$. There is no exceptional set of measure zero which might depend on the initial condition needed.

What is a sufficient condition for path uniqueness? Suppose the following weak monotonicity condition for $\mu=\mu(\omega)$.

$$
\begin{align*}
& \left(\mathbf{N}\left(t, \mathbf{u}_{1}, \mathbf{v}_{1}, \mathbf{w}_{1}, \omega\right)-\mathbf{N}\left(t, \mathbf{u}_{2}, \mathbf{v}_{2}, \mathbf{w}_{2}, \omega\right), \mathbf{u}_{1}-\mathbf{u}_{2}\right) \\
\geq \quad & -\mu\left(\left|\mathbf{u}_{1}-\mathbf{u}_{2}\right|^{2}+\left|\mathbf{v}_{1}-\mathbf{v}_{2}\right|^{2}+\left|\mathbf{w}_{1}-\mathbf{w}_{2}\right|^{2}\right) \tag{71.2.5}
\end{align*}
$$

Then path uniqueness will hold. This follows from subtracting the two integral equations, one for $\mathbf{u}_{1}$ and one for $\mathbf{u}_{2}$, using the estimate and then applying Gronwall's inequality.

Recall the Ito formula

$$
u(t)-u_{0}+\int_{0}^{t} N d s=\int_{0}^{t} f d s+\int_{0}^{t} \Phi d W
$$

where $u(t) \in H$ a Hilbert space. Consider $F(u)=\frac{1}{2}|u|^{2}$. Also let $R$ denote the Riesz map from $H \rightarrow H^{\prime}$ such that $\langle R x, y\rangle \equiv(x, y)_{H}$. Then proceding formally, to see what the Ito formula says,

$$
d F=D F(u) d u+\frac{1}{2} D^{2} F(u)(d u, d u)+O\left(d u^{3}\right)
$$

Recall then that $d u=-N d t+f d t+\Phi d W$ and so recalling $(d W, d W)=d t$,

$$
R(u)(-N d t+f d t+\Phi d W)+\frac{1}{2}\|\Phi\|^{2} d t
$$

Hence

$$
\frac{1}{2}|u(t)|_{H}^{2}-\frac{1}{2}\left|u_{0}\right|_{H}^{2}+\int_{0}^{t}(N, u)_{H} d s-\frac{1}{2} \int_{0}^{t}\|\Phi\|_{\mathscr{L}_{2}}^{2} d s=\int_{0}^{t}(f, u) d s+\int_{0}^{t} R u(\Phi) d W
$$

The last term is a martingale or local martingale $M$ whose quadratic variation is given by

$$
[M](t)=\int_{0}^{t}\|\Phi\|_{\mathscr{L}_{2}}^{2}|u|^{2} d s
$$

This is all that is of importance in what follows. Therefore, this martingale may be simply denoted as $M(t)$ in what follows.

Under the assumption 71.2.5 you can include instead of the term $\int_{0}^{t} \Phi d W$, the more general term $\int_{0}^{t} \sigma(s, \mathbf{u}, \omega) d W$. This will be shown by doing the argument and indicating what extra assumptions are needed as this is done. Let $\mathbf{z}$ be progressively measurable and in $L^{2}\left(\Omega ; C\left([0, T] ; \mathbb{R}^{n}\right)\right)$. Also assume that $\sigma$ has linear growth. That is

$$
\begin{equation*}
\|\sigma(s, \mathbf{u}, \omega)\|_{\mathscr{L}_{2}} \leq a+b|\mathbf{u}|_{\mathbb{R}^{n}} \tag{71.2.6}
\end{equation*}
$$

Then from the above theorem, there exists a unique progressively measurable solution $\mathbf{u}$ to

$$
\begin{align*}
\mathbf{u}(t, \omega)-\mathbf{u}_{0}(\omega) & +\int_{0}^{t} \mathbf{N}(s, \mathbf{u}(s, \omega), \mathbf{u}(s-h, \omega), \mathbf{w}(s, \omega), \omega) d s \\
= & \int_{0}^{t} \mathbf{f}(s, \omega) d s+\int_{0}^{t} \sigma(s, \mathbf{z}) d W \tag{71.2.7}
\end{align*}
$$

This holds for all $\omega$. There is no exceptional set needed. Now assume

$$
\begin{equation*}
\mathbf{u}_{0} \in L^{2}(\Omega) \tag{71.2.8}
\end{equation*}
$$

and also a Lipschitz condition

$$
\begin{equation*}
\|\sigma(s, \mathbf{u}, \omega)-\sigma(s, \hat{\mathbf{u}}, \omega)\|_{\mathscr{L}_{2}} \leq K|\mathbf{u}-\hat{\mathbf{u}}| \tag{71.2.9}
\end{equation*}
$$

Then let $\mathbf{u}$ coincide with $\mathbf{z}$ and $\hat{\mathbf{u}}$ come from $\hat{\mathbf{z}}$. Then applying the Ito formula, one can obtain the following for a constant $C$ which does not depend on $\mathbf{u}, \hat{\mathbf{u}}$.

$$
\frac{1}{2}|\mathbf{u}(t)-\hat{\mathbf{u}}(t)|^{2}-C \int_{0}^{t}|\mathbf{u}(s)-\hat{\mathbf{u}}(s)|^{2} d s-K \int_{0}^{t}|\mathbf{u}(s)-\hat{\mathbf{u}}(s)|^{2} d s=M(t)
$$

where $M(t)$ is a local martingale whose quadratic variation satisfies

$$
[M](t)=\int_{0}^{t}\|\sigma(s, \mathbf{z}, \omega)-\sigma(s, \hat{\mathbf{z}}, \omega)\|_{\mathscr{L}_{2}}^{2}|\mathbf{u}-\hat{\mathbf{u}}|^{2} d s
$$

Thus, simplifying the constants,

$$
\sup _{s \in[0, t]}|\mathbf{u}(s)-\hat{\mathbf{u}}(s)|^{2} \leq C \int_{0}^{t}|\mathbf{u}(s)-\hat{\mathbf{u}}(s)|^{2} d s+M^{*}(t)
$$

where $M^{*}(t)=\sup _{s \in[0, t]}|M(s)|$. Then by Gronwall's inequality,

$$
\sup _{s \in[0, t]}|\mathbf{u}(s)-\hat{\mathbf{u}}(s)|^{2} \leq C M^{*}(t)
$$

Then take the expectation of both sides. Using the Burkholder Davis Gundy inequality,

$$
E\left(\sup _{s \in[0, t]}|\mathbf{u}(s)-\hat{\mathbf{u}}(s)|^{2}\right) \leq C E\left(\left(\int_{0}^{t} K|\mathbf{z}-\hat{\mathbf{z}}|^{2}|\mathbf{u}-\hat{\mathbf{u}}|^{2} d s\right)^{1 / 2}\right)
$$

Then adjusting the constant again,

$$
\leq \frac{1}{2} E\left(\sup _{s \in[0, t]}|\mathbf{u}(s)-\hat{\mathbf{u}}(s)|^{2}\right)+C E\left(\int_{0}^{t} K|\mathbf{z}-\hat{\mathbf{z}}|^{2} d s\right)
$$

and so,

$$
E\left(\sup _{s \in[0, t]}|\mathbf{u}(s)-\hat{\mathbf{u}}(s)|^{2}\right) \leq C \int_{0}^{t} E\left(\sup _{r \in[0, s]}|\mathbf{z}(s)-\hat{\mathbf{z}}(s)|^{2}\right) d s
$$

Letting $\mathscr{T} \mathbf{z}=\mathbf{u}$ where $\mathbf{u}$ is defined from $\mathbf{z}$ in the integral equation 71.2.7, the above inequality implies that

$$
\begin{gathered}
E\left(\sup _{s \in[0, t]}\left|\mathscr{T}^{n} \mathbf{z}_{1}(s)-\mathscr{T}^{n} \mathbf{z}_{2}(s)\right|_{H}^{2}\right) \leq C \int_{0}^{t} E\left(\sup _{r \in[0, s]}\left|\mathscr{T}^{n-1} \mathbf{z}_{1}(r)-\mathscr{T}^{n-1} \mathbf{z}_{2}(r)\right|^{2}\right) d s \\
\leq C^{2} \int_{0}^{t} \int_{0}^{s} E\left(\sup _{r_{1} \in[0, r]}\left|\mathscr{T}^{n-2} \mathbf{z}_{1}\left(r_{1}\right)-\mathscr{T}^{n-2} \mathbf{z}_{2}\left(r_{1}\right)\right|^{2}\right) d r d s
\end{gathered}
$$

One can iterate this, eventually finding that

$$
\begin{gathered}
E\left(\sup _{s \in[0, t]}\left|\mathscr{T}^{n} \mathbf{z}_{1}(s)-\mathscr{T}^{n} \mathbf{z}_{2}(s)\right|_{H}^{2}\right) \\
\leq C^{n} \int_{0}^{t} \int_{0}^{t_{1}} \cdots \int_{0}^{t_{n-1}} d t_{n-1} \cdots d t E\left(\sup _{s \in[0, t]}\left|\mathbf{z}_{1}(s)-\mathbf{z}_{2}(s)\right|_{H}^{2}\right) \\
=\frac{C^{n} T^{n}}{(n!)} E\left(\sup _{s \in[0, t]}\left|\mathbf{z}_{1}(s)-\mathbf{z}_{2}(s)\right|_{H}^{2}\right)
\end{gathered}
$$

In particular, this holds for $t=T$ and so, letting $\mathbf{z} \in L^{2}\left(\Omega, C\left([0, T], \mathbb{R}^{n}\right)\right),\left\{\mathscr{T}^{n} \mathbf{z}\right\}$ is a Cauchy sequence in this space because a high enough power is a contraction map, so it converges to a unique fixed point $\mathbf{u}$. Each $\mathscr{T}^{n} \mathbf{z}$ is progressively measurable and so the fixed point is also. In $L^{2}\left(\Omega, C\left([0, T], \mathbb{R}^{n}\right)\right)$, you get the integral equation

$$
\begin{align*}
\mathbf{u}(t, \omega)-\mathbf{u}_{0}(\omega) & +\int_{0}^{t} \mathbf{N}(s, \mathbf{u}(s, \omega), \mathbf{u}(s-h, \omega), \mathbf{w}(s, \omega), \omega) d s \\
= & \int_{0}^{t} \mathbf{f}(s, \omega) d s+\int_{0}^{t} \sigma(s, \mathbf{u}) d W \tag{71.2.10}
\end{align*}
$$

Thus off a set of measure zero, the equation holds for all $t$ and $\mathbf{u}$ is progressively measurable. The place where $\mathbf{u}_{0} \in L^{2}(\Omega)$ is needed is in having $\mathscr{T} \mathbf{z} \in L^{2}\left(\Omega, C\left([0, T] ; \mathbb{R}^{n}\right)\right)$. One uses a similar procedure involving the Ito formula, the growth condition

$$
(\mathbf{N}(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega), \mathbf{u}) \geq-C(t, \omega)-\mu\left(|\mathbf{u}|^{2}+|\mathbf{v}|^{2}+|\mathbf{w}|^{2}\right)
$$

and the Burkholder Davis Gundy inequality to verify this. Since $\mathscr{T}$ depends on $\mathbf{u}_{0}$, it appears that the set of measure zero, off which the integral equation holds, will also depend on $\mathbf{u}_{0}$. It appears that this ultimately results from the need to take an expectation in order to deal with the stochastic integral. If this integral could be generalized in such a way that it made sense for each $\omega$ as in the usual Riemann Stieltjes integral, then likely this restriction could be removed. It is a problem because the Wiener process is not of bounded variation.

Theorem 71.2.2 Suppose the weak monotonicity condition 71.2.5 and the growth estimate 71.2.3. Also assume $\mathbf{N}(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega) \in \mathbb{R}^{d}$ for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^{d}, t \in[0, T]$ and $(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega) \rightarrow$ $\mathbf{N}(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega)$ is progressively measurable with respect to the normal filtration $\mathscr{F}_{t}$ determined by a given Wiener process $W(t)$. Also suppose $(t, \mathbf{u}, \mathbf{v}, \mathbf{w}) \rightarrow \mathbf{N}(t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \omega)$ is continuous. Let $\mathbf{f} \in L^{2}\left(\Omega, C\left([0, T], \mathbb{R}^{n}\right)\right)$ and $(t, \mathbf{u}, \omega) \rightarrow \sigma(t, \mathbf{u}, \omega)$ is progressively measurable and satisfies the linear growth condition 71.2.6 and the Lipschitz condition 71.2.9. Also suppose $\mathbf{u}_{0}$ is $\mathscr{F}_{0}$ measurable and in $L^{2}\left(\Omega, \mathbb{R}^{n}\right)$. Then there exists a progressively measurable solution $\mathbf{u}$ to 71.2.10. If $\hat{\mathbf{u}}$ is another such solution, then there is a set of measure zero $N$ such that for $\omega \notin N, \hat{\mathbf{u}}(t)=\mathbf{u}(t)$ for all $t$.

Proof: It only remains to verify the uniqueness assertion. This happens because the fixed point is unique in $L^{2}\left(\Omega, C\left([0, T], \mathbb{R}^{n}\right)\right)$. Therefore, off a set of measure zero the two solutions are equal for all $t$.

### 71.3 Stochastic Differential Equations

In this section, ordinary differential equations in Hilbert space which are of the form

$$
d u+N(u) d t=f d t+\sigma(u) d W
$$

are considered under Lipschitz assumptions on $N$ and $\sigma$. A very satisfactory theorem can be proved.

The assumptions made are as follows.

$$
\begin{equation*}
\|\sigma(t, u, \omega)-\sigma(t, \hat{u}, \omega)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)} \leq K|u-\hat{u}|_{H} \tag{71.3.11}
\end{equation*}
$$

$$
\begin{equation*}
\left|N\left(t, u_{1}, v_{1}, w_{1}, \omega\right)-N\left(t, u_{2}, v_{2}, w_{2}, \omega\right)\right| \leq K\left(\left|u_{1}-u_{2}\right|+\left|v_{1}-v_{2}\right|+\left|w_{1}-w_{2}\right|\right) \tag{71.3.12}
\end{equation*}
$$

where the norms $|\cdot|$ refer here to the Hilbert space $H$. Assume $N, \sigma$ are both progressively measurable. From the Lipschitz condition given above,

$$
|N(t, u, v, w, \omega)-N(t, 0,0,0, \omega)| \leq K(|u|+|v|+|w|)
$$

and it is assumed that

$$
\begin{equation*}
t \rightarrow N(t, 0,0,0, \omega) \tag{71.3.13}
\end{equation*}
$$

is in $L^{2}(\Omega, C([0, T] ; H))$. Also consider the growth condition which is implied by the above condition and the Lipschitz assumption.

$$
\begin{equation*}
(N(t, u, v, w, \omega), u) \geq-C(t, \omega)-\mu\left(|u|^{2}+|v|^{2}+|w|^{2}\right) \tag{71.3.14}
\end{equation*}
$$

where $C \in L^{1}([0, T] \times \Omega)$ and the linear growth condition for $\sigma$,

$$
\begin{equation*}
\|\sigma(t, u, \omega)\| \leq a+b|u|_{H} \tag{71.3.15}
\end{equation*}
$$

### 71.3.1 The Lipschitz Case

Theorem 71.3.1 Suppose 71.3.11, 71.3.13, 71.3.15, 71.3.12 and let

$$
w(t)=w_{0}+\int_{0}^{t} u(s) d s, w_{0} \in L^{2}(\Omega), w_{0} \text { is } \mathscr{F}_{0} \text { measurable. }
$$

Then there exists a unique progressively measurable solution $u$ to the integral equation

$$
\begin{gather*}
u(t, \omega)-u_{0}(\omega)+\int_{0}^{t} N(s, u(s, \omega), u(s-h, \omega), w(s, \omega), \omega) d s \\
=\int_{0}^{t} f(s, \omega) d s+\int_{0}^{t} \sigma(s, u, \omega) d W \tag{71.3.16}
\end{gather*}
$$

71.3.16 where $u \in L^{2}(\Omega, C([0, T] ; H)), u_{0} \in L^{2}(\Omega), u_{0}$ is $\mathscr{F}_{0}$ measurable, $f$ is progressively measurable and in $L^{2}([0, T] \times \Omega ; H)$. Here there is a set of measure zero such that if $\omega$ is not in this set, then $u(\cdot, \omega)$ solves the above integral equation 71.3 .16 and furthermore, if $\hat{u}(\cdot, \omega)$ is another solution to it, then $u(t, \omega)=\hat{u}(t, \omega)$ for all $t$ if $\omega$ is off some set of measure zero.

Proof: Let $v \in L^{2}(\Omega ; C([0, T] ; H))$ where $v$ is also progressively measurable. Then let $u$ be given by

$$
\begin{gather*}
u(t, \omega)-u_{0}(\omega)+\int_{0}^{t} N(s, v(s, \omega), v(s-h, \omega), w(s, \omega), \omega) d s \\
=\int_{0}^{t} f(s, \omega) d s+\int_{0}^{t} \sigma(s, v, \omega) d W \tag{71.3.17}
\end{gather*}
$$

The Lipschitz condition 71.3.12, the assumption 71.3.13, and the linear growth assertion 71.3.15, implies that $u$ is also in $L^{2}(\Omega ; C([0, T] ; H))$. The proof of this involves the same
arguments about to be given in order to show that this determines a mapping which has a sufficiently high power a contraction map. They are also the same arguments to be used in the following theorem to establish estimates which imply a stopping time is eventually infinity.

Let $v_{1}, v_{2}$ be two given functions of this sort and let the corresponding $u$ be denoted by $u_{1}, u_{2}$ respectively. Then

$$
\begin{gathered}
u_{1}(t)-u_{2}(t)+\int_{0}^{t} N\left(s, v_{1}(s), v_{1}(s-h), w_{1}(s)\right)-N\left(s, v_{2}(s), v_{2}(s-h), w_{2}(s)\right) d s \\
=\int_{0}^{t} \sigma\left(s, v_{1}, \omega\right)-\sigma\left(s, v_{2}, \omega\right) d W
\end{gathered}
$$

Use the Ito formula and the Lipschitz condition on $N$ to obtain an expression of the form

$$
\begin{gathered}
\frac{1}{2}\left|u_{1}(t)-u_{2}(t)\right|^{2}-C \int_{0}^{t}\left|v_{1}-v_{2}\right|^{2} d s-C \int_{0}^{t}\left|u_{1}-u_{2}\right|^{2} d s \\
\quad-\frac{1}{2} \int_{0}^{t}\left\|\sigma\left(s, v_{1}, \omega\right)-\sigma\left(s, v_{2}, \omega\right)\right\|^{2} d s \leq|M(t)|
\end{gathered}
$$

where $M(t)$ is a martingale whose quadratic variation is dominated by

$$
C \int_{0}^{t}\left\|\sigma\left(s, v_{1}, \omega\right)-\sigma\left(s, v_{2}, \omega\right)\right\|^{2}\left|u_{1}-u_{2}\right|^{2} d s
$$

Therefore, using the Lipschitz condition on $\sigma$ and the Burkholder-Davis-Gundy inequality, the above implies

$$
\begin{aligned}
E\left(\sup _{s \in[0, t]}\left|u_{1}(s)-u_{2}(s)\right|^{2}\right) \leq & C E \int_{0}^{t} \sup _{r \in[0, s]}\left|u_{1}(r)-u_{2}(r)\right|^{2} d s \\
& +C E \int_{0}^{t} \sup _{r \in[0, s]}\left|v_{1}(r)-v_{2}(r)\right|^{2} d s \\
\quad+ & C E\left(\left(\int_{0}^{t}\left\|\sigma\left(s, v_{1}, \omega\right)-\sigma\left(s, v_{2}, \omega\right)\right\|^{2}\left|u_{1}-u_{2}\right|^{2} d s\right)^{1 / 2}\right)
\end{aligned}
$$

Then a use of Gronwall's inequality allows this to be simplified to an expression of the form

$$
\begin{gathered}
E\left(\sup _{s \in[0, t]}\left|u_{1}(s)-u_{2}(s)\right|^{2}\right) \leq C E \int_{0}^{t} \sup _{r \in[0, s]}\left|v_{1}(r)-v_{2}(r)\right|^{2} d s \\
+C E\left(\left(\int_{0}^{t}\left\|\sigma\left(s, v_{1}, \omega\right)-\sigma\left(s, v_{2}, \omega\right)\right\|^{2}\left|u_{1}-u_{2}\right|^{2} d s\right)^{1 / 2}\right) \\
\leq C \int_{0}^{t} E\left(\sup _{r \in[0, s]}\left|v_{1}(r)-v_{2}(r)\right|^{2}\right) d s+\frac{1}{2} E\left(\sup _{s \in[0, t]}\left|u_{1}(s)-u_{2}(s)\right|^{2}\right)
\end{gathered}
$$

$$
+C E\left(\int_{0}^{t}\left\|\sigma\left(s, v_{1}, \omega\right)-\sigma\left(s, v_{2}, \omega\right)\right\|^{2} d s\right)
$$

Now using the Lipschitz condition on $\sigma$, this simplifies further to give an inequality of the form

$$
E\left(\sup _{s \in[0, t]}\left|u_{1}(s)-u_{2}(s)\right|^{2}\right) \leq C \int_{0}^{t} E\left(\sup _{r \in[0, s]}\left|v_{1}(r)-v_{2}(r)\right|^{2}\right) d s
$$

Letting $\mathscr{T} v=u$ where $u$ is defined from $v$ in the integral equation 71.3.17, the above inequality implies that

$$
\begin{gathered}
E\left(\sup _{s \in[0, t]}\left|\mathscr{T}^{n} v_{1}(s)-\mathscr{T}^{n} v_{2}(s)\right|_{H}^{2}\right) \leq C \int_{0}^{t} E\left(\sup _{r \in[0, s]}\left|\mathscr{T}^{n-1} v_{1}(r)-\mathscr{T}^{n-1} v_{2}(r)\right|^{2}\right) d s \\
\leq C^{2} \int_{0}^{t} \int_{0}^{s} E\left(\sup _{r_{1} \in[0, r]}\left|\mathscr{T}^{n-2} v_{1}\left(r_{1}\right)-\mathscr{T}^{n-2} v_{2}\left(r_{1}\right)\right|^{2}\right) d r d s
\end{gathered}
$$

One can iterate this, eventually finding that

$$
\begin{gathered}
E\left(\sup _{s \in[0, t]}\left|\mathscr{T}^{n} v_{1}(s)-\mathscr{T}^{n} v_{2}(s)\right|_{H}^{2}\right) \\
\leq C^{n} \int_{0}^{t} \int_{0}^{t_{1}} \cdots \int_{0}^{t_{n-1}} d t_{n-1} \cdots d t E\left(\sup _{s \in[0, t]}\left|v_{1}(s)-v_{2}(s)\right|_{H}^{2}\right) \\
=\frac{C^{n} T^{n}}{(n!)} E\left(\sup _{s \in[0, t]}\left|v_{1}(s)-v_{2}(s)\right|_{H}^{2}\right)
\end{gathered}
$$

In particular, one could take $t=T$. This shows that for all $n$ large enough, $\mathscr{T}^{n}$ is a contraction map on $L^{2}(\Omega, C([0, T] ; H))$. Therefore, picking $v \in L^{2}(\Omega, C([0, T] ; H))$, such that $v$ is also progressively measurable, $\left\{\mathscr{T}^{k} v\right\}_{k=1}^{\infty}$ converges in $L^{2}(\Omega, C([0, T] ; H))$ to the unique fixed point of $\mathscr{T}$ denoted as $u$. Thus $\mathscr{T} u=u$ in $L^{2}(\Omega ; C([0, T] ; H))$. That is,

$$
\int_{\Omega} \sup _{t}|\mathscr{T} u-u|^{2} d P=0
$$

It follows that there is a set of measure zero such that for $\omega$ not in this set,

$$
\begin{gather*}
u(t, \omega)-u_{0}(\omega)+\int_{0}^{t} N(s, u(s, \omega), u(s-h, \omega), w(s, \omega), \omega) d s \\
=\int_{0}^{t} f(s, \omega) d s+\int_{0}^{t} \sigma(s, u, \omega) d W \tag{71.3.18}
\end{gather*}
$$

The function $u$ is progressively measurable because each $\mathscr{T}^{n} v$ is progressively measurable and there exists a subsequence still indexed with $n$ such that for $\omega$ off a set of measure zero, $\mathscr{T}^{n} v(\cdot, \omega) \rightarrow u(\cdot, \omega)$ in $C([0, T] ; H)$.

Note that the fixed point of $\mathscr{T}$ is unique in the space $L^{2}(\Omega ; C([0, T] ; H))$ and so any solution to the integral equation in this space must equal this one. Hence, there exists a set of measure zero such that for $\omega$ off this set, the two solutions are equal for all $t$.

### 71.3.2 The Locally Lipschitz Case

Now replace the Lipschitz assumpton 71.3 .12 with the locally Lipschitz assumption which says that if $\max (|u|,|v|,|w|)<R$, then there is a constant $K_{R}$ such that

$$
\begin{equation*}
\left|N\left(t, u_{1}, v_{1}, w_{1}, \omega\right)-N\left(t, u_{2}, v_{2}, w_{2}, \omega\right)\right| \leq K(R)\left(\left|u_{1}-u_{2}\right|+\left|v_{1}-v_{2}\right|+\left|w_{1}-w_{2}\right|\right) \tag{71.3.19}
\end{equation*}
$$

Also assume the growth condition

$$
\begin{equation*}
(N(t, u, v, w, \omega), u) \geq-C(t, \omega)-\mu\left(|u|^{2}+|v|^{2}+|w|^{2}\right) \tag{71.3.20}
\end{equation*}
$$

and the linear growth condition on $\sigma$

$$
\|\sigma(t, u, \omega)\| \leq a+b|u|_{H}
$$

and the Lipschitz condition on $\sigma$ 71.3.11. This can likely be relaxed as in the case of the Lipschitz condition for $N$ but for simplicity, we keep it.

Theorem 71.3.2 Suppose 71.3.11, 71.3.14, 71.3.15,71.3.13, 71.3.19 and let

$$
w(t)=w_{0}+\int_{0}^{t} u(s) d s, w_{0} \in L^{2}(\Omega), w_{0} \text { is } \mathscr{F}_{0} \text { measurable. }
$$

Then there exists a unique progressively measurable solution $u$ to the integral equation

$$
\begin{gather*}
u(t, \omega)-u_{0}(\omega)+\int_{0}^{t} N(s, u(s, \omega), u(s-h, \omega), w(s, \omega), \omega) d s \\
=\int_{0}^{t} f(s, \omega) d s+\int_{0}^{t} \sigma(s, u, \omega) d W \tag{71.3.21}
\end{gather*}
$$

71.3.16 where $u \in L^{2}(\Omega, C([0, T] ; H)), u_{0} \in L^{2}(\Omega, H)$, $u_{0}$ is $\mathscr{F}_{0}$ measurable, $f$ is progressively measurable and in $L^{2}([0, T] \times \Omega ; H)$. Here there is a set of measure zero such that if $\omega$ is not in this set, then $u(\cdot, \omega)$ solves the above integral equation 71.3.16 and furthermore, if $\hat{u}(\cdot, \omega)$ is another solution to it, then $u(t, \omega)=\hat{u}(t, \omega)$ for all $t$ if $\omega$ is off some set of measure zero.

Proof: Let $u_{n}$ be the unique solution to the integral equation

$$
\begin{align*}
u_{n}(t, \omega)-u_{0}(\omega) & +\int_{0}^{t} N\left(s, P_{n} u_{n}(s, \omega), P_{n} u_{n}(s-h, \omega), P_{n} w_{n}(s, \omega), \omega\right) d s \\
& =\int_{0}^{t} f(s, \omega) d s+\int_{0}^{t} \sigma\left(s, u_{n}, \omega\right) d W \tag{71.3.22}
\end{align*}
$$

where $P_{n}$ is the projection onto $\overline{B\left(0,9^{n}\right)}$. Thus the modified problem is in the situation of Theorem 71.3.1 so there exists such a solution. Then let

$$
\tau_{n}=\inf \left\{t:\left|u_{n}(t)\right|+\left|w_{n}(t)\right|>2^{n}\right\}
$$

Then stopping the equation with this stopping time, we can write

$$
\begin{gather*}
u_{n}^{\tau_{n}}(t, \omega)-u_{0}(\omega)+\int_{0}^{t} \mathscr{X}_{\left[0, \tau_{n}\right]} N\left(s, u_{n}^{\tau_{n}}(s, \omega), u_{n}^{\tau_{n}}(s-h, \omega), w_{n}^{\tau_{n}}(s, \omega), \omega\right) d s \\
=\int_{0}^{t} \mathscr{X}_{\left[0, \tau_{n}\right]} f(s, \omega) d s+\int_{0}^{t} \mathscr{X}_{\left[0, \tau_{n}\right]} \sigma\left(s, u_{n}^{\tau_{n}}, \omega\right) d W \tag{71.3.23}
\end{gather*}
$$

Then using the growth condition 71.3.20 and the Ito formula,

$$
\frac{1}{2}\left|u_{n}^{\tau_{n}}(t)\right|_{H}^{2} \leq C\left(u_{0}, w_{0}, f\right)+C \int_{0}^{t}\left|u_{n}^{\tau_{n}}\right|_{H}^{2} d s+\sup _{s \in[0, t]}|M(t)|
$$

where $M(t)$ is a martingale whose quadratic variation is dominated by

$$
\int_{0}^{t}\left\|\sigma\left(s, u_{n}^{\tau_{n}}\right)\right\|^{2}\left|u_{n}^{\tau_{n}}\right|^{2} d s
$$

Then it follows by the Burkholder-Davis-Gundy inequality

$$
\begin{gathered}
E\left(\sup _{s \in[0, t]}\left|u_{n}^{\tau_{n}}(s)\right|_{H}^{2}\right) \leq E\left(C\left(u_{0}, w_{0}, f\right)\right)+C \int_{0}^{t} E\left(\sup _{r \in[0, s]}\left|u_{n}^{\tau_{n}}(r)\right|^{2} d r\right) d s \\
+C E\left(\left(\int_{0}^{t}\left\|\sigma\left(s, u_{n}^{\tau_{n}}\right)\right\|^{2}\left|u_{n}^{\tau_{n}}\right|^{2} d s\right)^{1 / 2}\right)
\end{gathered}
$$

Now apply Gronwall's inequality and modify the constants so that

$$
\begin{gathered}
E\left(\sup _{s \in[0, t]}\left|u_{n}^{\tau_{n}}(s)\right|_{H}^{2}\right) \leq E\left(C\left(u_{0}, w_{0}, f\right)\right)+C E\left(\left(\int_{0}^{t}\left\|\sigma\left(s, u_{n}^{\tau_{n}}\right)\right\|^{2}\left|u_{n}^{\tau_{n}}\right|^{2} d s\right)^{1 / 2}\right) \\
\quad \leq E\left(C\left(u_{0}, w_{0}, f\right)\right)+\frac{1}{2} E\left(\sup _{s \in[0, t]}\left|u_{n}^{\tau_{n}}(s)\right|_{H}^{2}\right)+C E\left(\int_{0}^{t}\left\|\sigma\left(s, u_{n}^{\tau_{n}}\right)\right\|^{2} d s\right)
\end{gathered}
$$

Then, using the linear growth condition on $\sigma$, it follows on modification of the constants again that

$$
\begin{aligned}
E\left(\sup _{s \in[0, t]}\left|u_{n}^{\tau_{n}}(s)\right|_{H}^{2}\right) & \leq E\left(C\left(u_{0}, w_{0}, f\right)\right)+C E\left(\int_{0}^{t}\left|u_{n}^{\tau_{n}}\right|^{2} d s\right) \\
& \leq E\left(C\left(u_{0}, w_{0}, f\right)\right)+C E\left(\int_{0}^{t} \sup _{r \in[0, s]}\left|u_{n}^{\tau_{n}}\right|^{2} d r\right)
\end{aligned}
$$

and so, another application of Gronwall's inequality implies that

$$
E\left(\sup _{s \in[0, T]}\left|u_{n}^{\tau_{n}}(s)\right|_{H}^{2}\right) \leq E\left(C\left(u_{0}, w_{0}, f\right)\right)<\infty
$$

Then

$$
P\left(\sup _{s \in[0, T]}\left|u_{n}^{\tau_{n}}(s)\right|_{H}^{2}>\left(\frac{3}{2}\right)^{n}\right) \leq E\left(C\left(u_{0}, w_{0}, f\right)\right)\left(\frac{2}{3}\right)^{n}
$$

Now an application of the Borel Cantelli lemma shows that there exists a set of measure zero $\hat{N}$ such that for $\omega \notin \hat{N}$, it follows that for all $n$ large enough,

$$
\sup _{s \in[0, T]}\left|u_{n}^{\tau_{n}}(s)\right|_{H}^{2}<(3 / 2)^{n}
$$

and so $\tau_{n}=\infty$ for all $n$ large enough.
Claim: For $m<n$, there is a set of measure zero $N_{m n}$ such that if $\omega \notin N_{m n}$, then $u_{n}^{\tau_{n}}(s)=$ $u_{m}^{\tau_{m}}(s)$ on $\left[0, T \wedge \tau_{m}\right]$.

Proof of the claim: Note that $\tau_{m} \leq \tau_{n}$. Therefore, these are both progressively measurable solutions to the integral equation

$$
\begin{align*}
u\left(t \wedge \tau_{m}, \omega\right) & -u_{0}(\omega)+\int_{0}^{t} \mathscr{X}_{\left[0, \tau_{m}\right]} N\left(s, u(s, \omega), u(s-h, \omega), w_{u}(s, \omega), \omega\right) d s \\
& =\int_{0}^{t} \mathscr{X}_{\left[0, \tau_{m}\right]} f(s, \omega) d s+\int_{0}^{t} \mathscr{X}_{\left[0, \tau_{m}\right]} \sigma(s, u, \omega) d W \tag{71.3.24}
\end{align*}
$$

where

$$
w_{u}(t)=w_{0}+\int_{0}^{t} u(s) d s
$$

To save notation, refer to these functions as $u, v$ and let $\tau_{m}=\tau$. Subtract and use the Ito formula to obtain

$$
\begin{aligned}
& \frac{1}{2}|u(t \wedge \tau)-v(t \wedge \tau)|_{H}^{2} \leq \\
& \int_{0}^{t} \mathscr{X}_{\left[0, \tau_{m}\right]}\left(N\left(s, u(s), u(s-h), w_{u}(s)\right)-N\left(s, v(s), v(s-h), w_{u-v}(s)\right), u-v\right) d s \\
& +\sup _{s \in[0, t]}|M(t)|
\end{aligned}
$$

where the quadratic variation of the martingale $M(t)$ is dominated by

$$
\int_{0}^{t} \mathscr{X}_{[0, \tau]}\|\sigma(s, u, \omega)-\sigma(s, v, \omega)\|^{2}|u-v|^{2} d s
$$

Then from the assumption that $N$ is locally Lipschitz and routine manipulations,

$$
\frac{1}{2}|u(t \wedge \tau)-v(t \wedge \tau)|_{H}^{2} \leq C_{m} \int_{0}^{t} \mathscr{X}_{[0, \tau]}|u-v|^{2} d s+\sup _{s \in[0, t]}|M(s)|
$$

and so, adjusting the constants yields

$$
\begin{aligned}
& \sup _{s \in[0, t]}|u(s \wedge \tau)-v(s \wedge \tau)|_{H}^{2} \\
\leq & C_{m} \int_{0}^{t} \mathscr{X}_{[0, \tau]} \sup _{r \in[0, s]}|u(r \wedge \tau)-v(r \wedge \tau)|^{2} d s+\sup _{s \in[0, t]}|M(s)|
\end{aligned}
$$

and so, by Gronwall's inequality followed by the Burkholder-Davis-Gundy inequality,

$$
\begin{gathered}
E\left(\sup _{s \in[0, t]}|u(s \wedge \tau)-v(s \wedge \tau)|_{H}^{2}\right) \leq \\
C E\left(\left(\int_{0}^{t} \mathscr{X}_{[0, \tau]}\|\sigma(s, u, \omega)-\sigma(s, v, \omega)\|^{2}|u(s \wedge \tau)-v(s \wedge \tau)|^{2} d s\right)^{1 / 2}\right) \\
\leq \frac{1}{2} E\left(\sup _{s \in[0, t]}|u(s \wedge \tau)-v(s \wedge \tau)|_{H}^{2}\right)+C E\left(\int_{0}^{t} \mathscr{X}_{[0, \tau]}\|\sigma(s, u)-\sigma(s, v)\|^{2} d s\right)
\end{gathered}
$$

and so, adjusting the constant again,

$$
\begin{aligned}
& E\left(\sup _{s \in[0, t]}|u(s \wedge \tau)-v(s \wedge \tau)|_{H}^{2}\right) \\
\leq & C E\left(\int_{0}^{t} \mathscr{X}_{[0, \tau]}\|\sigma(s, u(s \wedge \tau))-\sigma(s, v(s \wedge \tau))\|^{2} d s\right) \\
\leq & C E\left(\int_{0}^{t} \mathscr{X}_{[0, \tau]} K|u(s \wedge \tau)-v(s \wedge \tau)|^{2} d s\right) \\
\leq & C \int_{0}^{t} E\left(\sup _{r \in[0, s]}|u(r \wedge \tau)-v(r \wedge \tau)|^{2}\right) d s
\end{aligned}
$$

and so, Gronwall's inequality shows that for every $t$,

$$
E\left(\sup _{s \in[0, t]}|u(s \wedge \tau)-v(s \wedge \tau)|_{H}^{2}\right)=0
$$

In particular, for $t=T$ this holds. Hence

$$
E\left(\sup _{s \in[0, T]}|u(s \wedge \tau)-v(s \wedge \tau)|_{H}^{2}\right)=0
$$

It follows that

$$
E\left(\sup _{s \in[0, \tau \wedge T]}|u(s)-v(s)|_{H}^{2}\right)=0
$$

so that off a set of measure zero, $u(s)=v(s)$ for all $s \in[0, \tau]$. This proves the claim.
Now let the set of measure zero $N$ be given by $N \equiv \cup_{m<n} N_{m n} \cup \hat{N}$ where $\hat{N}$ is the set of measure zero off which $\tau_{m}=\infty$ for all $m$ large enough. Then for $\omega \notin N$, it follows that $u_{n}^{\tau_{n}}(s)=u_{m}^{\tau_{m}}(s)$ on $\left[0, \tau_{m} \wedge T\right]$ and, for all $m$ large enough, $\tau_{m}=\infty$. Hence for all $m$ large enough, and such $\omega, u_{n}(s, \omega)=u_{m}(s, \omega)$ for all $s \in[0, T]$. Thus, for $\omega$ off $N$, it follows that $\lim _{m \rightarrow \infty} u_{m}^{\tau_{m}}(s, \omega) \equiv u(s, \omega)$ exists, this for each $s \in[0, T]$ and $\omega$ off a fixed set of measure zero. In fact, this convergence is uniform on $[0, T]$ because for all $n$ sufficiently large
and for such a fixed $\omega \notin N$, there is no change in increasing $m$. Hence, $u$ is progressively measurable and satisfies the integral equation 71.3.21.

It remains to verify uniqueness. Suppose there are two solutions $u, v$ each progressively measurable solutions of the given integral equation. Then let $\tau_{n}$ be a stopping time

$$
\tau_{n}=\inf \left\{t:|u(t)|+|v(t)|>2^{n}\right\}
$$

Then a repeat of the arguments given in the above claim shows that on $\left[0, \tau_{n} \wedge T\right]$ the two functions $u^{\tau_{n}}, v^{\tau_{n}}$ are equal on $\left[0, \tau_{n} \wedge T\right]$ off a set of measure zero $N_{n}$. Let $N$ be the union of the exceptional sets. Then for $\omega \notin N, u(t, \omega)=v(t, \omega)$ for all $t \in\left[0, \tau_{n} \wedge T\right]$. However, $\tau_{n}(\omega)=\infty$ for all $n$ large enough because each of these functions is continuous. Hence, the two functions are equal on $[0, T]$ for such $\omega$. This shows uniqueness.

## Chapter 72

## The Hard Ito Formula

Recall the following definition of stochastically continuous.
$X$ is stochastically continuous at $t_{0} \in I$ means: for all $\varepsilon>0$ and $\delta>0$ there exists $\rho>0$ such that

$$
P\left(\left[\left|\mid X(t)-X\left(t_{0}\right) \| \geq \varepsilon\right]\right) \leq \delta \text { whenever }\left|t-t_{0}\right|<\rho, t \in I\right.
$$

Note the above condition says that for each $\varepsilon>0$,

$$
\lim _{t \rightarrow t_{0}} P\left(\left[\left\|X(t)-X\left(t_{0}\right)\right\| \geq \varepsilon\right]\right)=0
$$

### 72.1 Predictable And Stochastic Continuity

Definition 72.1.1 Let $\mathscr{F}_{t}$ be a filtration. The predictable sets consists of those sets which are in the smallest $\sigma$ algebra which contains the sets $E \times\{0\}$ for $E \in \mathscr{F}_{0}$ and $E \times(a, b]$ where $E \in \mathscr{F}_{a}$. Thus every predictable set is a progressively measurable set.

First of all, here is an important observation.
Proposition 72.1.2 Let $X(t)$ be a stochastic process having values in $E$ a complete metric space and let it be $\mathscr{F}_{t}$ adapted and left continuous where $\mathscr{F}_{t}$ is a normal filtration. Then it is predictable. If $t \rightarrow X(t, \omega)$ is continuous for all $\omega \notin N, P(N)=0$, then $(t, \omega) \rightarrow$ $X(t, \omega) \mathscr{X}_{N^{C}}(\omega)$ is predictable. Also, if $X(t)$ is stochastically continuous and adapted on $[0, T]$, then it has a predictable version. If $X \in C\left([0, T] ; L^{p}(\Omega ; F)\right), p \geq 1$ for $F$ a Banach space, then $X$ is stochastically continuous.

Proof: First suppose $X$ is continuous for all $\omega \in \Omega$. Define

$$
I_{m, k} \equiv\left((k-1) 2^{-m} T, k 2^{-m} T\right]
$$

if $k \geq 1$ and $I_{m, 0}=\{0\}$ if $k=1$. Then define

$$
\begin{aligned}
X_{m}(t) \equiv & \sum_{k=1}^{2^{m}} X\left(T(k-1) 2^{-m}\right) \mathscr{X}_{\left((k-1) 2^{-m} T, k 2^{-m} T\right]}(t) \\
& +X(0) \mathscr{X}_{[0,0]}(t)
\end{aligned}
$$

Here the sum means that $X_{m}(t)$ has value $X\left(T(k-1) 2^{-m}\right)$ on the interval

$$
\left((k-1) 2^{-m} T, k 2^{-m} T\right]
$$

Thus $X_{m}$ is predictable because each term in the formal sum is. Thus

$$
\begin{gathered}
X_{m}^{-1}(U)=\cup_{k=1}^{2^{m}}\left(X\left(T(k-1) 2^{-m}\right) \mathscr{X}_{\left((k-1) 2^{-m} T, k 2^{-m} T\right]}\right)^{-1}(U) \\
=\cup_{k=1}^{2^{m}}\left((k-1) 2^{-m} T, k 2^{-m} T\right] \times\left(X\left(T(k-1) 2^{-m}\right)\right)^{-1}(U),
\end{gathered}
$$

a finite union of predictable sets. Since $X$ is left continuous,

$$
X(t, \omega)=\lim _{m \rightarrow \infty} X_{m}(t, \omega)
$$

and so $X$ is predictable.
Now suppose that for $\omega \notin N, P(N)=0, t \rightarrow X(t, \omega)$ is continuous. Then applying the above argument to $X(t) \mathscr{X}_{N^{C}}$ it follows $X(t) \mathscr{X}_{N^{C}}$ is predictable by completeness of $\mathscr{F}_{t}$, $X(t) \mathscr{X}_{N^{C}}$ is $\mathscr{F}_{t}$ measurable.

Next consider the other claim. Since $X$ is stochastically continuous on $[0, T]$ it is uniformly stochastically continuous on this interval by Lemma 62.1.1. Therefore, there exists a sequence of partitions of $[0, T]$, the $m^{\text {th }}$ being

$$
0=t_{m, 0}<t_{m, 1}<\cdots<t_{m, n(m)}=T
$$

such that for $X_{m}$ defined as above, then for each $t$

$$
\begin{equation*}
P\left(\left[d\left(X_{m}(t), X(t)\right) \geq 2^{-m}\right]\right) \leq 2^{-m} \tag{72.1.1}
\end{equation*}
$$

Then as above, $X_{m}$ is predictable. Let $A$ denote those points of $\mathscr{P}_{T}$ at which $X_{m}(t, \omega)$ converges. Thus $A$ is a predictable set because it is just the set where $X_{m}(t, \omega)$ is a Cauchy sequence. Now define the predictable function $Y$

$$
Y(t, \omega) \equiv\left\{\begin{array}{l}
\lim _{m \rightarrow \infty} X_{m}(t, \omega) \text { if }(t, \omega) \in A \\
0 \text { if }(t, \omega) \notin A
\end{array}\right.
$$

From 72.1.1 it follows from the Borel Cantelli lemma that for fixed $t$, the set of $\omega$ which are in infinitely many of the sets,

$$
\left[d\left(X_{m}(t), X(t)\right) \geq 2^{-m}\right]
$$

has measure zero. Therefore, for each $t$, there exists a set of measure zero, $N(t)$ such that for $\omega \notin N(t)$ and all $m$ large enough

$$
\left[d\left(X_{m}(t, \omega), X(t, \omega)\right)<2^{-m}\right]
$$

Hence for $\omega \notin N(t),(t, \omega) \in A$ and so $X_{m}(t, \omega) \rightarrow Y(t, \omega)$ which shows

$$
d(Y(t, \omega), X(t, \omega))=0 \text { if } \omega \notin N(t)
$$

The predictable version of $X(t)$ is $Y(t)$.
Finally consider the claim about the specific example where

$$
\begin{gathered}
X \in C\left([0, T] ; L^{p}(\Omega ; F)\right) . \\
P\left(\left[\|X(t)-X(s)\|_{F} \geq \varepsilon\right]\right) \varepsilon^{p} \leq \int_{\Omega}\|X(t)-X(s)\|_{F}^{p} d P \leq \varepsilon^{p} \delta
\end{gathered}
$$

provided $|s-t|$ sufficiently small. Thus

$$
P\left(\left[\|X(t)-X(s)\|_{F} \geq \varepsilon\right]\right)<\delta
$$

when $|s-t|$ is small enough.

### 72.2 Approximating With Step Functions

This Ito formula seems to be the fundamental idea which allows one to obtain solutions to stochastic partial differential equations using a variational point of view. I am following the treatment found in [108]. The following lemma is fundamental to the presentation. It approximates a function with a sequence of two step functions $X_{k}^{r}, X_{k}^{l}$ where $X_{k}^{r}$ has the value of $X$ at the right end of each interval and $X_{k}^{l}$ gives the value $X$ at the left end of the interval. The lemma is very interesting for its own sake. You can obviously do this sort of thing for a continuous function but here the function is not continuous and in addition, it is a stochastic process depending on $\omega$ also. This lemma was proved earlier Lemma 65.3.1.

Lemma 72.2.1 Let $\Phi:[0, T] \times \Omega \rightarrow V$, be $\mathscr{B}([0, T]) \times \mathscr{F}$ measurable and suppose

$$
\Phi \in K \equiv L^{p}([0, T] \times \Omega ; E), p \geq 1
$$

Then there exists a sequence of nested partitions, $\mathscr{P}_{k} \subseteq \mathscr{P}_{k+1}$,

$$
\mathscr{P}_{k} \equiv\left\{t_{0}^{k}, \cdots, t_{m_{k}}^{k}\right\}
$$

such that the step functions given by

$$
\begin{aligned}
\Phi_{k}^{r}(t) & \equiv \sum_{j=1}^{m_{k}} \Phi\left(t_{j}^{k}\right) \mathscr{X}_{\left(t_{j-1}^{k}, t_{j}^{k}\right]}(t) \\
\Phi_{k}^{l}(t) & \equiv \sum_{j=1}^{m_{k}} \Phi\left(t_{j-1}^{k}\right) \mathscr{X}_{\left[t_{j-1}^{k}, t_{j}^{k}\right)}(t)
\end{aligned}
$$

both converge to $\Phi$ in $K$ as $k \rightarrow \infty$ and

$$
\lim _{k \rightarrow \infty} \max \left\{\left|t_{j}^{k}-t_{j+1}^{k}\right|: j \in\left\{0, \cdots, m_{k}\right\}\right\}=0
$$

Also, each $\Phi\left(t_{j}^{k}\right), \Phi\left(t_{j-1}^{k}\right)$ is in $L^{p}(\Omega ; E)$. One can also assume that $\Phi(0)=0$. The mesh points $\left\{t_{j}^{k}\right\}_{j=0}^{m_{k}}$ can be chosen to miss a given set of measure zero. In addition to this, we can assume that

$$
\left|t_{j}^{k}-t_{j-1}^{k}\right|=2^{-n_{k}}
$$

except for the case where $j=1$ or $j=m_{n_{k}}$ when this might not be so. In the case of the last subinterval defined by the partition, we can assume

$$
\left|t_{m}^{k}-t_{m-1}^{k}\right|=\left|T-t_{m-1}^{k}\right| \geq 2^{-\left(n_{k}+1\right)}
$$

The following lemma is convenient.
Lemma 72.2.2 Let $f_{n} \rightarrow f$ in $L^{p}([0, T] \times \Omega, E)$. Then there exists a subsequence $n_{k}$ and a set of measure zero $N$ such that if $\omega \notin N$, then

$$
f_{n_{k}}(\cdot, \omega) \rightarrow f(\cdot, \omega)
$$

in $L^{p}([0, T], E)$ and for a.e. $t$.

Proof: We have

$$
\begin{aligned}
P\left(\left[\left\|f_{n}-f\right\|_{L^{p}([0, T], E)}>\lambda\right]\right) & \leq \frac{1}{\lambda} \int_{\Omega}\left\|f_{n}-f\right\|_{L^{p}([0, T], E)} d P \\
& \leq \frac{1}{\lambda}\left\|f_{n}-f\right\|_{L^{p}([0, T] \times \Omega, E)}
\end{aligned}
$$

Hence there exists a subsequence $n_{k}$ such that

$$
P\left(\left[\left\|f_{n_{k}}-f\right\|_{L^{p}([0, T], E)}>2^{-k}\right]\right) \leq 2^{-k}
$$

Then by the Borel Cantelli lemma, it follows that there exists a set of measure zero $N$ such that for all $k$ large enough and $\omega \notin N$,

$$
\left\|f_{n_{k}}-f\right\|_{L^{p}([0, T], E)} \leq 2^{-k}
$$

Because of this lemma, it can also be assumed that for a.e. $\omega$ pointwise convergence is obtained on $[0, T]$ as well as convergence in $L^{p}([0, T])$. This kind of assumption will be tacitly made whenever convenient in the context of the above lemma.

Also recall the diagram for the definition of the integral.

$$
\begin{aligned}
& U \\
& \begin{array}{l}
\downarrow Q^{1 / 2} \\
Q^{1 / 2} U
\end{array} \\
& \Phi_{n} \searrow \\
& \downarrow \Phi \\
& \text { H }
\end{aligned}
$$

The idea was to get $\int_{0}^{t} \Phi d W$ where $\Phi \in L^{2}\left([0, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$. Here $W(t)$ was a cylindrical Wiener process. This meant that it was a $Q_{1}$ Wiener process on $U_{1}$ for $Q_{1}=J J^{*}$ and $J$ was a Hilbert Schmidt operator mapping $Q^{1 / 2} U$ to $U_{1}$.

### 72.3 The Situation

Now consider the following situation.
Situation 72.3.1 Let $X$ satisfy the following.

$$
\begin{equation*}
X(t)=X_{0}+\int_{0}^{t} Y(s) d s+\int_{0}^{t} Z(s) d W(s) \tag{72.3.2}
\end{equation*}
$$

$X_{0} \in L^{2}(\Omega ; H)$ and is $\mathscr{F}_{0}$ measurable, where $Z$ is $\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)$ progressively measurable and

$$
\int_{0}^{T} \int_{\Omega}\|Z(s)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d P d t<\infty
$$

so that the stochastic integral makes sense. Also $X$ has a measurable representative $\bar{X}$ which has values in $V$. (For a.e.t, $\bar{X}(t)=X(t)$ for $P$ a.e. $\omega$ ). This representative satisfies

$$
\bar{X} \in L^{2}([0, T] \times \Omega, \mathscr{B}([0, T] \times \mathscr{F}, H)) \cap L^{p}([0, T] \times \Omega, \mathscr{B}([0, T]) \times \mathscr{F}, V)
$$

Assume $Y(s)$ satisfies

$$
Y \in K^{\prime}=L^{p^{\prime}}\left([0, T] \times \Omega ; V^{\prime}\right)
$$

where $1 / p^{\prime}+1 / p=1$ and $Y$ is $V^{\prime}$ progressively measurable. The situation in which the equation holds is as follows. For a.e. $\omega$, the equation holds for all $t \in[0, T]$ in $V^{\prime}$. Thus it follows that $X(t)$ is automatically progressively measurable into $V^{\prime}$ from Proposition 72.1.2. Also $W(t)$ is a Wiener process on $U_{1}$ in the above diagram. Thus $X$ is continuous into $V^{\prime}$ off a set of measure zero, and it is also $V^{\prime}$ predictable.

The goal is to prove the following Ito formula.

$$
\begin{align*}
|X(t)|^{2}= & \left|X_{0}\right|^{2}+\int_{0}^{t}\left(2\langle Y(s), \bar{X}(s)\rangle+\|Z(s)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2}\right) d s \\
& +2 \int_{0}^{t} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*} \bar{X}(s)\right) \circ J d W(s) \tag{72.3.3}
\end{align*}
$$

where $\mathscr{R}$ is the Riesz map which takes $U_{1}$ to $U_{1}^{\prime}$. The main thing is that the last term above be a local martingale.

In all that follows, the mesh points $t_{j}$ will be points where $\bar{X}\left(t_{j}\right)=X\left(t_{j}\right)$ a.e. $\omega$.
Lemma 72.3.2 Let $X$ be as in Situation 72.3.1 and let $X_{k}^{l}$ be as in Lemma 72.2.1 corresponding to $\bar{X}$ above. Say

$$
X_{k}^{l}(t)=\sum_{j=0}^{m_{k}} \bar{X}\left(t_{j}\right) \mathscr{X}_{\left[t_{j}, t_{j+1)}\right.}(t), X_{k}^{l}(0) \equiv 0
$$

Then each term in the above sum for which $t_{j}>0$ is predictable into $H$. As mentioned earlier, we can take $X(0) \equiv 0$ in the definition of the "left step function". Since, at the mesh points, $\bar{X}=X$ a.e., it makes no difference off a set of measure zero whether we use $\bar{X}\left(t_{j}\right)$ or $X\left(t_{j}\right)$ at the left end point.

Proof: This is a step function and a typical term is of the form $X(a) \mathscr{X}_{[a, b)}(t)$. I will try and show this is predictable. Let $a_{n}$ be an increasing sequence converging to $a$ and let $b_{n}$ be an increasing sequence converging to $b$. Then for a.e. $\omega$,

$$
X\left(a_{n}\right) \mathscr{X}_{\left(a_{n}, b_{n}\right]}(t) \rightarrow X(a) \mathscr{X}_{[a, b)}(t)
$$

in $V^{\prime}$ due to the fact that $t \rightarrow X(t)$ is continuous into $V^{\prime}$ for a.e. $\omega$. Therefore, letting $v \in V$ be given, it follows that for a.e. $\omega$

$$
\left\langle X\left(a_{n}\right) \mathscr{X}_{\left(a_{n}, b_{n}\right]}(t), v\right\rangle \rightarrow\left\langle X(a) \mathscr{X}_{[a, b)}(t), v\right\rangle,
$$

and since the filtration is a normal filtration in which all sets of measure zero from $\mathscr{F}_{T}$ are in $\mathscr{F}_{0}$, this shows

$$
(t, \omega) \rightarrow\left\langle X(a) \mathscr{X}_{[a, b)}(t), v\right\rangle
$$

is real predictable because it is the pointwise limit of real predictable functions, those in the sequence being real predictable because of the continuity of $X(t)$ into $V^{\prime}$ and Propositon 72.1.2. Now since $H \subseteq V^{\prime}$ it follows that for all $v \in V$,

$$
(t, \omega) \rightarrow\left(X(a) \mathscr{X}_{[a, b)}(t), v\right)
$$

is real predictable. This holds for $h \in H$ replacing $v$ in the above because $V$ is dense in $H$. By the Pettis theorem, this proves the lemma.

Lemma 72.3.3 In Situation 72.3.1 the following formula holds for a.e. $\omega$ for $0<s<t$ where $M(t) \equiv \int_{0}^{t} Z(u) d W(u)$. Here and elsewhere, $|\cdot|$ denotes the norm in $H$ and $\langle\cdot, \cdot\rangle$ denotes the duality pairing between $V, V^{\prime}$. Also $X=\bar{X}$ for a.e. $\omega$ at $t, s$ so that it makes no difference off a set of measure zero whether we write $\langle Y(u), X(t)\rangle$ or $\langle Y(u), \bar{X}(t)\rangle$

$$
\begin{align*}
|X(t)|^{2} & =|X(s)|^{2}+2 \int_{s}^{t}\langle Y(u), X(t)\rangle d u+2(X(s), M(t)-M(s)) \\
& +|M(t)-M(s)|^{2}-|X(t)-X(s)-(M(t)-M(s))|^{2} \tag{72.3.4}
\end{align*}
$$

Also for $t>0$

$$
\begin{align*}
|X(t)|^{2}= & \left|X_{0}\right|^{2}+2 \int_{0}^{t}\langle Y(u), X(t)\rangle d u+2\left(X_{0}, M(t)\right) \\
& +|M(t)|^{2}-\left|X(t)-X_{0}-M(t)\right|^{2} \tag{72.3.5}
\end{align*}
$$

Proof: The formula is a straight forward computation which holds a.e. $\omega$.

$$
\begin{aligned}
& \mid M(t)-\left.M(s)\right|^{2}-|X(t)-X(s)-(M(t)-M(s))|^{2}+2(X(s), M(t)-M(s)) \\
&=|M(t)-M(s)|^{2}-|X(t)-X(s)|^{2}-|M(t)-M(s)|^{2} \\
&+2(X(t)-X(s), M(t)-M(s))+2(X(s), M(t)-M(s)) \\
&=-|X(t)-X(s)|^{2}+2(X(t), M(t)-M(s)) \\
&=-|X(t)-X(s)|^{2}+2(X(t), X(t)-X(s))-2\left\langle\int_{s}^{t} Y(u) d u, X(t)\right\rangle \\
&=\quad-|X(t)|^{2}-|X(s)|^{2}+2(X(t), X(s))+2|X(t)|^{2}-2(X(t), X(s)) \\
&-2 \int_{s}^{t}\langle Y(u), X(t)\rangle d u \\
& \quad=|X(t)|^{2}-|X(s)|^{2}-2 \int_{s}^{t}\langle Y(u), X(t)\rangle d u
\end{aligned}
$$

Comparing the ends of this string of equations,

$$
\begin{aligned}
|X(t)|^{2}= & |X(s)|^{2}+2 \int_{s}^{t}\langle Y(u), X(t)\rangle d u+2(X(s), M(t)-M(s)) \\
& +|M(t)-M(s)|^{2}-|X(t)-X(s)-(M(t)-M(s))|^{2}
\end{aligned}
$$

which is what was to be shown.
Now it is time to prove the other assertion.

$$
\begin{gathered}
|M(t)|^{2}-\left|X(t)-X_{0}-M(t)\right|^{2}+2\left(X_{0}, M(t)\right) \\
=-\left|X(t)-X_{0}\right|^{2}+2\left(X(t)-X_{0}, M(t)\right)+2\left(X_{0}, M(t)\right) \\
=-\left|X(t)-X_{0}\right|^{2}+2(X(t), M(t)) \\
=-\left|X(t)-X_{0}\right|^{2}+2\left(X(t), X(t)-X_{0}\right)-2\left\langle\int_{0}^{t} Y(s) d s, X(t)\right\rangle \\
=|X(t)|^{2}-\left|X_{0}\right|^{2}-2\left\langle\int_{0}^{t}\langle Y(s), X(t)\rangle d s\right\rangle
\end{gathered}
$$

Noting that $X(0)=X_{0} \in L^{2}(\Omega, H)$ and is $\mathscr{F}_{0}$ measurable, the first formula works in both cases.

### 72.4 The Main Estimate

The following phenomenal estimate holds and it is this estimate which is the main idea in proving the Ito formula. The last assertion about continuity is like the well known result that if $y \in L^{p}(0, T ; V)$ and $y^{\prime} \in L^{p^{\prime}}\left(0, T ; V^{\prime}\right)$, then $y$ is actually continuous with values in $H$. Later, this continuity result is strengthened further to give strong continuity.

Lemma 72.4.1 In the Situation 72.3.1,

$$
E\left(\sup _{t \in[0, T]}|X(t)|_{H}^{2}\right)<C\left(\|Y\|_{K^{\prime}},\|X\|_{K},\|Z\|_{J},\left\|X_{0}\right\|_{L^{2}(\Omega, H)}\right)<\infty
$$

where

$$
\begin{aligned}
J & =L^{2}\left([0, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U ; H\right)\right), K \equiv L^{p}([0, T] \times \Omega ; V), \\
K^{\prime} & \equiv L^{p^{\prime}}\left([0, T] \times \Omega ; V^{\prime}\right) .
\end{aligned}
$$

Also, $C$ is a continuous function of its arguments and $C(0,0,0,0)=0$. Thus for a.e. $\omega$,

$$
\sup _{t \in[0, T]}|X(t, \omega)|_{H} \leq C(\omega)<\infty .
$$

Also for a.e. $\omega, t \rightarrow X(t, \omega)$ is weakly continuous with values in $H$.

Proof: Consider the formula in Lemma 72.3.3.

$$
\begin{align*}
|X(t)|^{2} & =|X(s)|^{2}+2 \int_{s}^{t}\langle Y(u), X(t)\rangle d u+2(X(s), M(t)-M(s)) \\
& +|M(t)-M(s)|^{2}-|X(t)-X(s)-(M(t)-M(s))|^{2} \tag{72.4.6}
\end{align*}
$$

Now let $t_{j}$ denote a point of $\mathscr{P}_{k}$ from Lemma 72.2.1. Then for $t_{j}>0, X\left(t_{k}\right)$ is just the value of $X$ at $t_{k}$ but when $t=0$, the definition of $X(0)$ in this step function is $X(0) \equiv 0$. Thus

$$
\left|X\left(t_{m}\right)\right|^{2}-\left|X_{0}\right|^{2}=\sum_{j=1}^{m-1}\left|X\left(t_{j+1}\right)\right|^{2}-\left|X\left(t_{j}\right)\right|^{2}+\left|X\left(t_{1}\right)\right|^{2}-\left|X_{0}\right|^{2}
$$

Using the formula in Lemma 72.3.3, for $t=t_{m}$ this yields

$$
\begin{gather*}
\left|X\left(t_{m}\right)\right|^{2}-\left|X_{0}\right|^{2}=2 \sum_{j=1}^{m-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u+ \\
+2 \sum_{j=1}^{m-1}\left(\int_{t_{j}}^{t_{j+1}} Z(u) d W, X\left(t_{j}\right)\right)_{H}+\sum_{j=1}^{m-1}\left|M\left(t_{j+1}\right)-M\left(t_{j}\right)\right|^{2} \\
-\sum_{j=1}^{m-1}\left|X\left(t_{j+1}\right)-X\left(t_{j}\right)-\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right|^{2} \\
+2 \int_{0}^{t_{1}}\left\langle Y(u), X\left(t_{1}\right)\right\rangle d u+2\left(X_{0}, \int_{0}^{t_{1}} Z(u) d W\right)+\left|M\left(t_{1}\right)\right|^{2} \\
-\left|X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right|^{2} \tag{72.4.7}
\end{gather*}
$$

Of course

$$
2 \int_{0}^{t_{1}}\left\langle Y(u), X\left(t_{1}\right)\right\rangle d u+2\left(X_{0}, \int_{0}^{t_{1}} Z(u) d W\right)+\left|M\left(t_{1}\right)\right|^{2}
$$

converges to 0 for a.e. $\omega$ as $k \rightarrow \infty$ because the norms of the partitions converge to 0 and the stochastic integral is continuous off a set of measure zero. Actually this is not completely clear for the first of the above terms. This term is dominated by

$$
\begin{aligned}
& \left(\int_{0}^{t_{1}}\|Y(u)\|^{p^{\prime}} d u\right)^{1 / p}\left(\int_{0}^{T}\left\|X_{k}^{r}(u)\right\|^{p} d u\right)^{1 / p} \\
\leq & C(\omega)\left(\int_{0}^{t_{1}}\|Y(u)\|^{p^{\prime}} d u\right)^{1 / p}
\end{aligned}
$$

Hence this converges to 0 for a.e. $\omega$. At this time, not much is known about the last term in 72.4.7, but it is negative and is about to be neglected anyway.The Ito isometry implies the other two terms converge to 0 in $L^{1}(\Omega)$ also, in addition to converging for a.e. $\omega$. At this time, not much is known about the last term in 72.4.7, but it is negative and is about to be neglected anyway.

The term involving the stochastic integral equals

$$
2 \sum_{j=1}^{m-1}\left(\int_{t_{j}}^{t_{j+1}} Z(u) d W, X\left(t_{j}\right)\right)_{H}
$$

By Theorem 65.14.1, this equals

$$
2 \int_{t_{1}}^{t_{m}} \mathscr{R}\left(\left(Z(u) \circ J^{-1}\right)^{*} X_{k}^{l}(u)\right) \circ J d W
$$

$t \rightarrow \int_{0}^{t} \mathscr{R}\left(\left(Z(u) \circ J^{-1}\right)^{*} X_{k}^{l}(u)\right) \circ J d W$ being a local martingale. Therefore, 72.4.7 equals

$$
\begin{gathered}
\left|X\left(t_{m}\right)\right|^{2}-\left|X_{0}\right|^{2}=2 \int_{0}^{t_{m}}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u+e(k) \\
2 \int_{t_{1}}^{t_{m}} \mathscr{R}\left(\left(Z(u) \circ J^{-1}\right)^{*} X_{k}^{l}(u)\right) \circ J d W+\sum_{j=1}^{m-1}\left|M\left(t_{j+1}\right)-M\left(t_{j}\right)\right|^{2} \\
-\sum_{j=1}^{m-1}\left|X\left(t_{j+1}\right)-X\left(t_{j}\right)-\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right|^{2}-\left|X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right|^{2}
\end{gathered}
$$

where $e(k)$ converges to 0 in $L^{1}(\Omega)$ and for a.e. $\omega$. Note that $X_{k}^{l}(u)=0$ on $\left[0, t_{1}\right)$ and so that stochastic integral equals

$$
\int_{0}^{t_{m}} \mathscr{R}\left(\left(Z(u) \circ J^{-1}\right)^{*} X_{k}^{l}(u)\right) \circ J d W .
$$

Therefore, from the above,

$$
\begin{gathered}
\left|X\left(t_{m}\right)\right|^{2}-\left|X_{0}\right|^{2}=2 \int_{0}^{t_{m}}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u+e(k) \\
2 \int_{0}^{t_{m}} \mathscr{R}\left(\left(Z(u) \circ J^{-1}\right)^{*} X_{k}^{l}(u)\right) \circ J d W+\sum_{j=0}^{m-1}\left|M\left(t_{j+1}\right)-M\left(t_{j}\right)\right|^{2}-\left|M\left(t_{1}\right)\right|^{2} \\
-\sum_{j=1}^{m-1}\left|X\left(t_{j+1}\right)-X\left(t_{j}\right)-\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right|^{2}-\left|X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right|^{2}
\end{gathered}
$$

Then since $\left|M\left(t_{1}\right)\right|^{2}$ converges to 0 in $L^{1}(\Omega)$ and for a.e. $\omega$, as discussed above,

$$
\begin{gathered}
\left|X\left(t_{m}\right)\right|^{2}-\left|X_{0}\right|^{2}=2 \int_{0}^{t_{m}}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u+e(k) \\
+2 \int_{0}^{t_{m}} \mathscr{R}\left(\left(Z(u) \circ J^{-1}\right)^{*} X_{k}^{l}(u)\right) \circ J d W+\sum_{j=0}^{m-1}\left|M\left(t_{j+1}\right)-M\left(t_{j}\right)\right|^{2} \\
-\left|X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right|^{2}
\end{gathered}
$$

$$
\begin{equation*}
-\sum_{j=1}^{m-1}\left|X\left(t_{j+1}\right)-X\left(t_{j}\right)-\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right|^{2} \tag{72.4.8}
\end{equation*}
$$

where $e(k) \rightarrow 0$ for a.e. $\omega$ and also in $L^{1}(\Omega)$.
Now it follows on discarding the negative terms,

$$
\begin{gathered}
\sup _{t_{j} \in \mathscr{P}_{k}}\left|X\left(t_{j}\right)\right|^{2} \leq\left|X_{0}\right|^{2}+2 \int_{0}^{T}\left|\left\langle Y(u), X_{k}^{r}(u)\right\rangle\right| d u \\
+2 \sup _{t \in[0, T]}\left|\int_{0}^{t} \mathscr{R}\left(\left(Z(u) \circ J^{-1}\right)^{*} X_{k}^{l}(u)\right) \circ J d W\right|+\sum_{j=0}^{m_{k}-1}\left|\int_{t_{j}}^{t_{j+1}} Z(u) d W\right|^{2}
\end{gathered}
$$

where there are $m_{k}+1$ points in $\mathscr{P}_{k}$.
Do $\int_{\Omega}$ to both sides. Using the Ito isometry, this yields

$$
\begin{gathered}
\int_{\Omega}\left(\sup _{t_{j} \in \mathscr{P}_{k}}\left|X\left(t_{j}\right)\right|^{2}\right) d P \leq \\
\\
+\sum_{j=0}^{m_{k}-1} \int_{t_{j}}^{t_{j+1}} \int_{\Omega}\left\|Z\left(\left|X_{0}\right|^{2}\right)+2\right\| Y\left\|_{K^{\prime}}| |\right\|_{k}^{r} \|_{K} d P d u \\
+2 \int_{\Omega}\left(\sup _{t \in[0, T]}\left|\int_{0}^{T} \mathscr{R}\left(\left(Z(u) \circ J^{-1}\right)^{*} X_{k}^{l}(u)\right) \circ J d W\right|\right) d P+E(|e(k)|) \\
\leq C+\int_{0}^{T} \int_{\Omega}\|Z(u)\|^{2} d P d u+ \\
+2 \int_{\Omega}\left(\sup _{t \in[0, T]}\left|\int_{0}^{T} \mathscr{R}\left(\left(Z(u) \circ J^{-1}\right)^{*} X_{k}^{l}(u)\right) \circ J d W\right|\right) d P \\
\leq C+2 \int_{\Omega}\left(\sup _{t \in[0, T]}\left|\int_{0}^{T} \mathscr{R}\left(\left(Z(u) \circ J^{-1}\right)^{*} X_{k}^{l}(u)\right) \circ J d W\right|\right) d P
\end{gathered}
$$

where the result of Lemma 72.2 .1 that $X_{k}^{r}$ converges to $\bar{X}$ in $K$ shows that the term which is the product $2\|Y\|_{K^{\prime}}\left\|X_{k}^{r}\right\|_{K}$ is bounded. The constant $C$ is a continuous function of

$$
\|Y\|_{K^{\prime}},\|\bar{X}\|_{K},\|Z\|_{J},\left\|X_{0}\right\|_{L^{2}(\Omega, H)}
$$

which equals zero when all are equal to zero. The term involving the stochastic integral is next.

Applying the Burkholder Davis Gundy inequality, Theorem 63.4.4 for $F(r)=r$ along with the description of the quadratic variation of the Ito integral found in Corollary 65.11.1

$$
\begin{aligned}
& \int_{\Omega_{t_{j} \in \mathscr{P}_{k}}} \sup _{k}\left|X\left(t_{k}\right)\right|^{2} d P \\
\leq & C+C \int_{\Omega}\left(\int_{0}^{T}\left\|\mathscr{R}\left(\left(Z(u) \circ J^{-1}\right)^{*} X_{k}^{l}(u)\right) \circ J\right\|^{2} d u\right)^{1 / 2} d P
\end{aligned}
$$

$$
\leq C+C \int_{\Omega}\left(\int_{0}^{T}\|Z(u)\|^{2}\left|X_{k}^{l}(u)\right|^{2} d u\right)^{1 / 2} d P
$$

Now for each $\omega$, there are only finitely many values of $X_{k}^{l}(u)$ and they equal $X\left(t_{j}\right)$ for $t_{j} \in \mathscr{P}_{k}$ with the convention that $X(0)=0$. Therefore, the above is dominated by

$$
\begin{aligned}
& C+C \int_{\Omega}\left(\sup _{t_{j} \in \mathscr{P}_{k}}\left|X\left(t_{j}\right)\right|^{2}\right)^{1 / 2}\left(\int_{0}^{T}\|Z(u)\|^{2} d u\right)^{1 / 2} d P \\
\leq & C+\frac{1}{2} \int_{\Omega_{t_{j} \in \mathscr{P}_{k}}} \sup \left|X\left(t_{j}\right)\right|^{2}+C \int_{\Omega} \int_{0}^{T}\|Z(u)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d u d P
\end{aligned}
$$

and so

$$
\frac{1}{2} \int_{\Omega_{t_{j}} \in \sup _{k}}\left|X\left(t_{k}\right)\right|^{2} d P \leq C
$$

for some constant $C$ independent of $\mathscr{P}_{k}$ dependent on $\int_{\Omega} \int_{0}^{T}\|Z(u)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d u d P$. This constant is dependent on $\|Y\|_{K^{\prime}},\|\bar{X}\|_{K},\|Z\|_{J},\left\|X_{0}\right\|_{L^{2}(\Omega, H)}$ and equals zero when all of these quantities equal 0 .

Let $D$ denote the union of all the $\mathscr{P}_{k}$. Thus $D$ is a dense subset of $[0, T]$ and it has just been shown that for a constant $C$ independent of $\mathscr{P}_{k}$,

$$
E\left(\sup _{t \in D}|X(t)|^{2}\right) \leq C
$$

Let $\left\{e_{j}\right\}$ be an orthonormal basis for $H$ which is also contained in $V$ and has the property that $\operatorname{span}\left(\left\{e_{k}\right\}_{k=1}^{\infty}\right)$ is dense in $V$. I claim that for $y \in V^{\prime}$

$$
|y|_{H}^{2}=\sup _{n} \sum_{j=1}^{n}\left|\left\langle y, e_{j}\right\rangle\right|^{2}
$$

This is certainly true if $y \in H$ because in this case

$$
\left\langle y, e_{j}\right\rangle=\left(y, e_{j}\right)
$$

If $y \notin H$, then the series must diverge. If not, you could consider the infinite sum

$$
z \equiv \sum_{j=1}^{\infty}\left\langle y, e_{j}\right\rangle e_{j} \in H
$$

and argue that $\langle z-y, v\rangle=0$ for all $v \in \operatorname{span}\left(\left\{e_{k}\right\}_{k=1}^{\infty}\right)$ which would also imply that this is true for all $v \in V$. Then since $z=y$ in $V^{\prime}$, it follows that $y \in H$ contrary to the assumption that $y \notin H$.

It follows

$$
|X(t)|^{2}=\sup _{n} \sum_{j=1}^{n}\left|\left\langle X(t), e_{j}\right\rangle\right|^{2}
$$

and for a.e. $\omega$, this is just the sup of continuous functions of $t$. Therefore, for given $\omega$ off a set of measure zero,

$$
t \rightarrow|X(t)|^{2}
$$

is lower semicontinuous. Hence letting $t \in[0, T]$ and $t_{j} \rightarrow t$ where $t_{j} \in D$,

$$
|X(t)|^{2} \leq \lim \inf _{j \rightarrow \infty}\left|X\left(t_{j}\right)\right|^{2}
$$

so it follows for a.e. $\omega$

$$
\sup _{t \in[0, T]}|X(t)|^{2} \leq \sup _{t \in D}|X(t)|^{2} \leq \sup _{t \in[0, T]}|X(t)|^{2}
$$

Hence

$$
\begin{equation*}
E\left(\sup _{t \in[0, T]}|X(t)|^{2}\right) \leq C\left(\|Y\|_{K^{\prime}},\|X\|_{K},\|Z\|_{J},\left\|X_{0}\right\|_{L^{2}(\Omega, H)}\right) \tag{72.4.9}
\end{equation*}
$$

Note the above shows that for a.e. $\omega, \sup _{t \in[0, T]}|X(t)|_{H}<\infty$ so that for such $\omega, X(t)$ has values in $H$. Note that we began by assuming it had a representative with values in $H$ although the equation only held in $V^{\prime}$. Say

$$
|X(t, \omega)| \leq C(\omega)
$$

Hence if $v \in V$, then for a.e. $\omega$

$$
\lim _{t \rightarrow s}(X(t), v)=\lim _{t \rightarrow s}\langle X(t), v\rangle=\langle X(s), v\rangle=(X(s), v)
$$

Therefore, since for such $\omega,|X(t, \omega)|$ is bounded, the above holds for all $h \in H$ in place of $v$ as well. Therefore, for a.e. $\omega, t \rightarrow X(t, \omega)$ is weakly continuous with values in $H$.

Eventually, it is shown that in fact, the function $t \rightarrow X(t, \omega)$ is continuous with values in $H$.

This lemma also provides a way to simplify one of the formulas derived earlier in the case that $X_{0} \in L^{p}(\Omega, V)$. Refer to 72.4.8. One term there is $\left|X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right|^{2}$.

$$
\left|X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right|^{2} \leq 2\left|X\left(t_{1}\right)-X_{0}\right|^{2}+2\left|M\left(t_{1}\right)\right|^{2}
$$

It was shown above that $2\left|M\left(t_{1}\right)\right|^{2} \rightarrow 0$ a.e. and also in $L^{1}(\Omega)$ as $k \rightarrow \infty$. Apply the above lemma to $\left|X(t)-X_{0}\right|^{2}$ using $\left[0, t_{1}\right]$ instead of $[0, T]$. The new $X_{0}$ equals 0 . Then from the estimate 72.4.9, it follows that

$$
E\left(\left|X\left(t_{1}\right)-X_{0}\right|^{2}\right) \rightarrow 0
$$

as $k \rightarrow \infty$. Taking a subsequence, we could also assume that $\left|X\left(t_{1}\right)-X_{0}\right|^{2} \rightarrow 0$ a.e. $\omega$ as $k \rightarrow \infty$. Then, using this subsequence, it would follow from 72.4.8,

$$
\left|X\left(t_{m}\right)\right|^{2}-\left|X_{0}\right|^{2}=2 \int_{0}^{t_{m}}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u+e(k)
$$

$$
\begin{gather*}
+2 \int_{0}^{t_{m}} \mathscr{R}\left(\left(Z(u) \circ J^{-1}\right)^{*} X_{k}^{l}(u)\right) \circ J d W+\sum_{j=0}^{m-1}\left|M\left(t_{j+1}\right)-M\left(t_{j}\right)\right|^{2} \\
\quad-\sum_{j=1}^{m-1}\left|X\left(t_{j+1}\right)-X\left(t_{j}\right)-\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right|^{2} \tag{72.4.10}
\end{gather*}
$$

where $e(k) \rightarrow 0$ in $L^{1}(\Omega)$ and a.e. $\omega$.
Can you obtain something similar even in case $X_{0}$ is not assumed to be in $L^{p}(\Omega, V)$ ? Let $Z_{0 k} \in L^{p}(\Omega, V) \cap L^{2}(\Omega, H), Z_{0 k} \rightarrow X_{0}$ in $L^{2}(\Omega, H)$. Then

$$
\left|X\left(t_{1}\right)-X_{0}\right| \leq\left|X\left(t_{1}\right)-Z_{0 k}\right|+\left|Z_{0 k}-X_{0}\right|
$$

Also, restoring the superscript to identify the parition,

$$
X\left(t_{1}^{k}\right)-Z_{0 k}=X_{0}-Z_{0 k}+\int_{0}^{t_{1}^{k}} Y(s) d s+\int_{0}^{t_{1}^{k}} Z(s) d W
$$

Of course $\left\|\bar{X}-Z_{0 k}\right\|_{K}$ is not bounded but for each $k$ it is at least finite. There is a sequence of partitions $\mathscr{P}_{k},\left\|\mathscr{P}_{k}\right\| \rightarrow 0$ such that all the above holds. In the definitions of $K, K^{\prime}, J$ replace $[0, T]$ with $[0, t]$ and let the resulting spaces be denoted by $K_{t}, K_{t}^{\prime}, J_{t}$. Let $n_{k}$ denote a subsequence of $\{k\}$ such that

$$
\left\|\bar{X}-Z_{0 k}\right\|_{K_{t_{1} n_{k}}}<1 / k
$$

Then from the above lemma,

$$
\begin{aligned}
& E\left(\sup _{t \in\left[0, t_{1}^{n_{k}}\right]}\left|X\left(t_{1}^{n_{k}}\right)-Z_{0 k}\right|_{H}^{2}\right) \\
\leq & C\left(\|Y\|_{\substack{K_{t_{1}}^{\prime} \\
t_{1}}},\left\|\bar{X}-Z_{0 k}\right\|_{K_{t_{1} n_{k}}},\|Z\|_{J_{t_{1} n_{k}}},\left\|X_{0}-Z_{0 k}\right\|_{L^{2}(\Omega, H)}\right) \\
\leq & C\left(\|Y\|_{\substack{K_{n_{1}}^{\prime} \\
t_{1}}}, \frac{1}{k},\|Z\|_{\substack{J_{t_{1} n_{k}} \\
t_{1}}},\left\|X_{0}-Z_{0 k}\right\|_{L^{2}(\Omega, H)}\right)
\end{aligned}
$$

Hence

$$
\begin{aligned}
& E\left(\left|X\left(t_{1}^{n_{k}}\right)-X_{0}\right|^{2}\right) \leq 2 E\left(\left|X\left(t_{1}^{n_{k}}\right)-Z_{0 k}\right|_{H}^{2}\right)+2 E\left(\left|Z_{0 k}-X_{0}\right|_{H}^{2}\right) \\
& \leq 2 C\left(\|Y\|_{\substack{K_{t_{1}}^{\prime}, t_{1}}}, \frac{1}{k},\|Z\|_{\substack{J_{t_{1}} k_{k}}},\left\|X_{0}-Z_{0 k}\right\|_{L^{2}(\Omega, H)}\right)+2\left\|Z_{0 k}-X_{0}\right\|^{2}
\end{aligned}
$$

which converges to 0 as $k \rightarrow \infty$. It follows that there exists a suitable subsequence such that 72.4.10 holds even in the case that $X_{0}$ is only known to be in $L^{2}(\Omega, H)$. From now on, assume this subsequence for the paritions $\mathscr{P}_{k}$. Thus $k$ will really be $n_{k}$.

### 72.5 Converging In Probability

I am working toward the Ito formula 72.3.3. In order to get this, there is a technical result which will be needed.

Lemma 72.5.1 Let $X(s)-X_{k}^{l}(s) \equiv \Delta_{k}(s)$. Then the following limit occurs.

$$
\begin{equation*}
\lim _{k \rightarrow \infty} P\left(\left[\sup _{t \in[0, T]}\left|\int_{0}^{t} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*} \Delta_{k}(s)\right) \circ J d W(s)\right| \geq \varepsilon\right]\right)=0 \tag{72.5.11}
\end{equation*}
$$

That is,

$$
\sup _{t \in[0, T]}\left|\int_{0}^{t} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*}\left(X(s)-X_{k}^{l}(s)\right)\right) \circ J d W(s)\right|
$$

converges to 0 in probability. Also the stochastic integral makes sense because $X$ is $H$ predictable.

Proof: First note that from Lemma 72.4.1, for a.e. $\omega, X(t)$ has values in $H$ for $t \in[0, T]$ and so it makes sense to consider it in the stochastic integral provided it is $H$ progressively measurable. However, as noted in Situation 72.3.1, this function is automatically $V^{\prime}$ predictable. Therefore,

$$
\langle X(t), v\rangle=(X(t), v)
$$

is real predictable for every $v \in V$. Now if $h \in H$, let $v_{n} \rightarrow h$ in $H$ and so for each $\omega$,

$$
\left(X(t, \omega), v_{n}\right) \rightarrow(X(t, \omega), h)
$$

By the Pettis theorem, $X$ is $H$ predictable, hence progressively measurable. Also it was shown above that $t \rightarrow X(t)$ is weakly continuous into $H$. Therefore, the desired result follows from Lemma 65.14.3 on Page 2265.

### 72.6 The Ito Formula

Now at long last, here is the first version of the Ito formula.
Lemma 72.6.1 In Situation 72.3.1, let $D$ be as above, the union of all the positive mesh points for all the $\mathscr{P}_{k}$. Also assume $X_{0} \in L^{2}(\Omega ; H)$. Then for every $t \in D$,

$$
\begin{align*}
|X(t)|^{2}= & \left|X_{0}\right|^{2}+\int_{0}^{t}\left(2\langle Y(s), \bar{X}(s)\rangle+\|Z(s)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2}\right) d s \\
& +2 \int_{0}^{t} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*} X(s)\right) \circ J d W(s) \tag{72.6.12}
\end{align*}
$$

Note that it was shown above that $X(t, \omega)$ has values in $H$ for a.e. $\omega$.
Proof: Let $t \in D$. Then $t \in \mathscr{P}_{k}$ for all $k$ large enough. Consider 72.4.10,

$$
|X(t)|^{2}-\left|X_{0}\right|^{2}=2 \int_{0}^{t}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u
$$

$$
\begin{align*}
& +2 \int_{0}^{t} \mathscr{R}\left(\left(Z(u) \circ J^{-1}\right)^{*} X_{k}^{l}(u)\right) \circ J d W+\sum_{j=0}^{q_{k}-1}\left|M\left(t_{j+1}\right)-M\left(t_{j}\right)\right|^{2} \\
& -\sum_{j=1}^{q_{k}-1}\left|X\left(t_{j+1}\right)-X\left(t_{j}\right)-\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right|^{2}+e(k) \tag{72.6.13}
\end{align*}
$$

where $t_{q_{k}}=t$. By Lemma 72.5 .1 the second term on the right, the stochastic integral, converges to

$$
2 \int_{0}^{t} \mathscr{R}\left(\left(Z(u) \circ J^{-1}\right)^{*} \bar{X}(u)\right) \circ J d W
$$

in probability. The first term on the right converges to

$$
2 \int_{0}^{t}\langle Y(u), \bar{X}(u)\rangle d u
$$

in $L^{1}(\Omega)$ because $X_{k}^{r} \rightarrow X$ in $K$. Therefore, this also happens in probability. Consider the next term.

$$
E\left(\sum_{j=0}^{q_{k}-1}\left|M\left(t_{j+1}\right)-M\left(t_{j}\right)\right|^{2}\right) .
$$

It is known from the theory of the quadratic variation that this term converges in probability to $[M](t)=\int_{0}^{t}\|Z(s)\|^{2} d s$. See Theorem 63.6.4 on Page 2147 and the description of the quadratic variation in Corollary 65.11.1.

Thus all the terms in 72.6 .13 converge in probability except for the last term which also must converge in probability because it equals the sum of terms which do. It remains to find what this last term converges to. Thus

$$
\begin{gathered}
|X(t)|^{2}-\left|X_{0}\right|^{2}=2 \int_{0}^{t}\langle Y(u), \bar{X}(u)\rangle d u \\
+2 \int_{0}^{t} \mathscr{R}\left(\left(Z(u) \circ J^{-1}\right)^{*} X(u)\right) \circ J d W+\int_{0}^{t}\|Z(s)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d s-a
\end{gathered}
$$

where $a$ is the limit in probability of the term

$$
\sum_{j=1}^{q_{k}-1}\left|X\left(t_{j+1}\right)-X\left(t_{j}\right)-\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right|^{2}
$$

Let $P_{n}$ be the projection onto span $\left(e_{1}, \cdots, e_{n}\right)$ as before where $\left\{e_{k}\right\}$ is an orthonormal basis for $H$ with each $e_{k} \in V$. Then using

$$
X\left(t_{j+1}\right)-X\left(t_{j}\right)-\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)=\int_{t_{j}}^{t_{j+1}} Y(s) d s
$$

the troublesome term above is of the form

$$
\begin{equation*}
\sum_{j=1}^{q_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(s), X\left(t_{j+1}\right)-X\left(t_{j}\right)-P_{n}\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right\rangle d s \tag{72.6.14}
\end{equation*}
$$

$$
\begin{equation*}
-\sum_{j=1}^{q_{k}-1}\left(X\left(t_{j+1}\right)-X\left(t_{j}\right)-\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right),\left(I-P_{n}\right)\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right) \tag{72.6.15}
\end{equation*}
$$

The sum in 72.6 .15 is dominated by

$$
\begin{gather*}
\left(\sum_{j=1}^{q_{k}-1}\left|X\left(t_{j+1}\right)-X\left(t_{j}\right)-\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right|^{2}\right)^{1 / 2} \\
\left(\sum_{j=1}^{q_{k}-1}\left|\left(I-P_{n}\right)\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right|^{2}\right)^{1 / 2} \tag{72.6.16}
\end{gather*}
$$

Now it is known that $\sum_{j=1}^{q_{k}-1}\left|X\left(t_{j+1}\right)-X\left(t_{j}\right)-\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right|^{2}$ converges in probability to $a$. If you take the expectation of the other factor it is

$$
\begin{gathered}
E\left(\sum_{j=1}^{q_{k}-1}\left|\left(I-P_{n}\right) \int_{t_{j}}^{t_{j+1}} Z(s) d W(s)\right|^{2}\right) \\
=\sum_{j=1}^{q_{k}-1} E\left(\left|\int_{t_{j}}^{t_{j+1}}\left(I-P_{n}\right) Z(s) d W(s)\right|^{2}\right) \\
=\sum_{j=1}^{q_{k}-1} E\left(\int_{t_{j}}^{t_{j+1}}\left\|\left(I-P_{n}\right) Z(s)\right\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2}\right) d s \\
\leq E\left(\int_{0}^{T}\left\|\left(I-P_{n}\right) Z(s)\right\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d s\right) \\
=\int_{\Omega} \int_{0}^{T} \sum_{i=n+1}^{\infty}\left(Z(s), e_{i}\right)^{2} d s d P
\end{gathered}
$$

The integrand converges to 0 as $n \rightarrow \infty$ and is dominated by $\sum_{i=1}^{\infty}\left(Z(s), e_{i}\right)^{2}$ which is given to be in $L^{1}([0, T] \times \Omega)$. Therefore, it converges to 0 .

Thus the expression in 72.6 .16 is of the form $f_{k} g_{n k}$ where $f_{k}$ converges in probability to $a$ as $k \rightarrow \infty$ and $g_{n k}$ converges in probability to 0 as $n \rightarrow \infty$ independently of $k$. Now this implies $f_{k} g_{n k}$ converges in probability to 0 . Here is why.

$$
\begin{aligned}
P\left(\left[\left|f_{k} g_{n k}\right|>\varepsilon\right]\right) & \leq P\left(2 \delta\left|f_{k}\right|>\varepsilon\right)+P\left(2 C_{\delta}\left|g_{n k}\right|>\varepsilon\right) \\
& \leq P\left(2 \delta\left|f_{k}-a\right|+2 \delta|a|>\varepsilon\right)+P\left(2 C_{\delta}\left|g_{n k}\right|>\varepsilon\right)
\end{aligned}
$$

where $\delta\left|f_{k}\right|+C_{\delta}\left|g_{k n}\right|>\left|f_{k} g_{n k}\right|$ and $\lim _{\delta \rightarrow 0} C_{\delta}=\infty$. Pick $\delta$ small enough that $\varepsilon-2 \delta|a|>$ $\varepsilon / 2$. Then this is dominated by

$$
\leq P\left(2 \delta\left|f_{k}-a\right|>\varepsilon / 2\right)+P\left(2 C_{\delta}\left|g_{n k}\right|>\varepsilon\right)
$$

Fix $n$ large enough that the second term is less than $\eta$. Now taking $k$ large enough, the above is less than $\eta$. It follows the expression in 72.6 .16 and consequently in 72.6 .15 converges to 0 in probability.

Now consider the other term, 72.6 .14 using the $n$ just determined. This term is of the form

$$
\begin{aligned}
& \sum_{j=1}^{q_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(s), X_{k}^{r}(s)-X_{k}^{l}(s)-P_{n}\left(M_{k}^{r}(s)-M_{k}^{l}(s)\right)\right\rangle d s \\
= & \int_{t_{1}}^{t}\left\langle Y(s), X_{k}^{r}(s)-X_{k}^{l}(s)-P_{n}\left(M_{k}^{r}(s)-M_{k}^{l}(s)\right)\right\rangle d s
\end{aligned}
$$

where $M_{k}^{r}$ denotes the step function

$$
M_{k}^{r}(t)=\sum_{i=0}^{m_{k}-1} M\left(t_{i+1}\right) \mathscr{X}_{\left(t_{i}, t_{i+1}\right]}(t)
$$

and $M_{k}^{l}$ is defined similarly. The term

$$
\int_{t_{1}}^{t}\left\langle Y(s), P_{n}\left(M_{k}^{r}(s)-M_{k}^{l}(s)\right)\right\rangle d s
$$

converges to 0 for a.e. $\omega$ as $k \rightarrow \infty$. This is because the integrand converges to 0 thanks to the continuity of $M(t)$ and also since this is a projection onto a finite dimensional subspace of $V$, Therefore, for each $\omega$ off a set of measure zero,

$$
\begin{aligned}
& \int_{t_{1}}^{t}\left\langle Y(s), P_{n}\left(M_{k}^{r}(s)-M_{k}^{l}(s)\right)\right\rangle d s \\
\leq & \int_{t_{1}}^{t}\|Y(s)\|_{V^{\prime}}\left\|P_{n}\left(M_{k}^{r}(s)-M_{k}^{l}(s)\right)\right\|_{V} d s
\end{aligned}
$$

and this last integral converges to 0 as $k \rightarrow \infty$ because $P_{n}(M(s))$ is uniformly bounded in $V$ so there is no problem getting a dominating function for the dominated convergence theorem. Let

$$
A_{k} \equiv\left[\left|\int_{t_{1}}^{t}\|Y(s)\|_{V^{\prime}}\left\|P_{n}\left(M_{k}^{r}(s)-M_{k}^{l}(s)\right)\right\|_{V} d s\right|>\varepsilon\right]
$$

Then since the partitions are increasing, these sets are decreasing as $k$ increases and their intersection has measure zero. Hence $P\left(A_{k}\right) \rightarrow 0$. It follows that

$$
\begin{gathered}
\lim _{k \rightarrow \infty} P\left(\left[\left|\int_{t_{1}}^{t}\left\langle Y(s), P_{n}\left(M_{k}^{r}(s)-M_{k}^{l}(s)\right)\right\rangle d s\right|>\varepsilon\right]\right) \leq \\
\lim _{k \rightarrow \infty} P\left(\left[\left|\int_{t_{1}}^{t}\|Y(s)\|_{V^{\prime}}\left\|P_{n}\left(M_{k}^{r}(s)-M_{k}^{l}(s)\right)\right\|_{V} d s\right|>\varepsilon\right]\right)=0
\end{gathered}
$$

Now consider

$$
\int_{t_{1}}^{t}\left\langle Y(s), X_{k}^{r}(s)-X_{k}^{l}(s)\right\rangle d s
$$

This converges to 0 in $L^{1}(\Omega)$ because it is of the form

$$
\int_{t_{1}}^{t}\left\langle Y(s), X_{k}^{r}(s)\right\rangle d s-\int_{t_{1}}^{t}\left\langle Y(s), X_{k}^{l}(s)\right\rangle d s
$$

and both $X_{k}^{l}$ and $X_{k}^{r}$ converge to $X$ in $K$. Therefore, the expression

$$
\sum_{j=1}^{q_{k}-1}\left|X\left(t_{j+1}\right)-X\left(t_{j}\right)-\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right|^{2}
$$

converges to 0 in probability.
In fact, the formula 72.6 .12 is valid for all $t \in[0, T]$.
Theorem 72.6.2 In Situation 72.3.1, off a set of measure zero, for every $t \in[0, T]$,

$$
\begin{align*}
|X(t)|^{2}= & \left|X_{0}\right|^{2}+\int_{0}^{t}\left(2\langle Y(s), \bar{X}(s)\rangle+\|Z(s)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2}\right) d s \\
& +2 \int_{0}^{t} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*} X(s)\right) \circ J d W(s) \tag{72.6.17}
\end{align*}
$$

Furthermore, for $t \in[0, T], t \rightarrow X(t)$ is continuous as a map into $H$ for a.e. $\omega$. In addition to this,

$$
\begin{equation*}
E\left(|X(t)|^{2}\right)=E\left(\left|X_{0}\right|^{2}\right)+E\left(\int_{0}^{t}\left(2\langle Y(s), \bar{X}(s)\rangle+\|Z(s)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2}\right) d s\right) \tag{72.6.18}
\end{equation*}
$$

Proof: Let $t \notin D$. For $t>0$, let $t(k)$ denote the largest point of $\mathscr{P}_{k}$ which is less than $t$. Suppose $t(m)<t(k)$. Hence $m \leq k$. Then

$$
X(t(m))=X_{0}+\int_{0}^{t(m)} Y(s) d s+\int_{0}^{t(m)} Z(s) d W(s)
$$

a similar formula holding for $X(t(k))$. Thus for $t>t(m)$,

$$
X(t)-X(t(m))=\int_{t(m)}^{t} Y(s) d s+\int_{t(m)}^{t} Z(s) d W(s)
$$

which is the same sort of thing studied so far except that it starts at $t(m)$ rather than at 0 and $X_{0}=0$. Therefore, from Lemma 72.6.1 it follows

$$
\begin{align*}
& |X(t(k))-X(t(m))|^{2}=\int_{t(m)}^{t(k)}\left(2\langle Y(s), X(s)-X(t(m))\rangle+\|Z(s)\|^{2}\right) d s \\
& \quad+2 \int_{t(m)}^{t(k)} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*}(X(s)-X(t(m)))\right) \circ J d W(s) \tag{72.6.19}
\end{align*}
$$

Consider that last term. It equals

$$
\begin{equation*}
2 \int_{t(m)}^{t(k)} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*}\left(X(s)-X_{m}^{l}(s)\right)\right) \circ J d W(s) \tag{72.6.20}
\end{equation*}
$$

This is dominated by

$$
\begin{aligned}
& 2 \mid \int_{0}^{t(k)} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*}\left(X(s)-X_{m}^{l}(s)\right)\right) \circ J d W(s) \\
& -2 \int_{0}^{t(m)} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*}\left(X(s)-X_{m}^{l}(s)\right)\right) \circ J d W(s) \mid
\end{aligned}
$$

$$
\begin{aligned}
& \leq 2\left|\int_{0}^{t(k)} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*}\left(X(s)-X_{m}^{l}(s)\right)\right) \circ J d W(s)\right| \\
& \quad+2\left|\int_{0}^{t(m)} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*}\left(X(s)-X_{m}^{l}(s)\right)\right) \circ J d W(s)\right| \\
& \leq 4 \sup _{t \in[0, T]}\left|\int_{0}^{t} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*}\left(X(s)-X_{m}^{l}(s)\right)\right) \circ J d W(s)\right|
\end{aligned}
$$

In Lemma 72.5.1 the above expression was shown to converge to 0 in probability. Therefore, by the usual appeal to the Borel Canteli lemma, there is a subsequence still referred to as $\{m\}$, such that it converges to 0 pointwise in $\omega$ for all $\omega$ off some set of measure 0 as $m \rightarrow \infty$. It follows there is a set of measure 0 such that for $\omega$ not in that set, 72.6.20 converges to 0 in $\mathbb{R}$. Note that $t>0$ is arbitrary. Similar reasoning shows the first term in the non stochastic integral of 72.6 .19 is dominated by an expression of the form

$$
4 \int_{0}^{T}\left|\left\langle Y(s), \bar{X}(s)-X_{m}^{l}(s)\right\rangle\right| d s
$$

which clearly converges to 0 for $\omega$ not in some set of measure zero because $X_{m}^{l}$ converges in $K$ to $\bar{X}$. Finally, it is obvious that

$$
\lim _{m \rightarrow \infty} \int_{t(m)}^{t(k)}\|Z(s)\|^{2} d s=0 \text { for a.e. } \omega
$$

due to the assumptions on $Z$.
This shows that for $\omega$ off a set of measure 0

$$
\lim _{m, k \rightarrow \infty}|X(t(k))-X(t(m))|^{2}=0
$$

and so $\{X(t(k))\}_{k=1}^{\infty}$ is a convergent sequence in $H$. Does it converge to $X(t)$ ? Let $\xi(t) \in$ $H$ be what it converges to. Let $v \in V$ then

$$
(\xi(t), v)=\lim _{k \rightarrow \infty}(X(t(k)), v)=\lim _{k \rightarrow \infty}\langle X(t(k)), v\rangle=\langle X(t), v\rangle=(X(t), v)
$$

and now, since $V$ is dense in $H$, this implies $\xi(t)=X(t)$.
Now for every $t \in D$,

$$
\begin{aligned}
|X(t)|^{2}= & \left|X_{0}\right|^{2}+\int_{0}^{t}\left(2\langle Y(s), \bar{X}(s)\rangle+\|Z(s)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2}\right) d s \\
& +2 \int_{0}^{t} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*} \bar{X}(s)\right) \circ J d W(s)
\end{aligned}
$$

and so, using what was just shown along with the obvious continuity of the functions of $t$ on the right of the equal sign, it follows the above holds for all $t \in[0, T]$ off a set of measure zero.

It only remains to verify $t \rightarrow X(t)$ is continuous with values in $H$. However, the above shows $t \rightarrow|X(t)|^{2}$ is continuous and it was shown in Lemma 72.4.1 that $t \rightarrow X(t)$ is weakly
continuous into $H$. Therefore, from the uniform convexity of the norm in $H$ it follows $t \rightarrow X(t)$ is continuous.This is very easy to see in Hilbert space. Say $a_{n} \rightharpoonup a$ and $\left|a_{n}\right| \rightarrow|a|$. From the parallelogram identity.

$$
\left|a_{n}-a\right|^{2}+\left|a_{n}+a\right|^{2}=2\left|a_{n}\right|^{2}+2|a|^{2}
$$

so

$$
\left|a_{n}-a\right|^{2}=2\left|a_{n}\right|^{2}+2|a|^{2}-\left(\left|a_{n}\right|^{2}+2\left(a_{n}, a\right)+|a|^{2}\right)
$$

Then taking limsup both sides,

$$
0 \leq \lim \sup _{n \rightarrow \infty}\left|a_{n}-a\right|^{2} \leq 2|a|^{2}+2|a|^{2}-\left(|a|^{2}+2(a, a)+|a|^{2}\right)=0
$$

Of course this fact also holds in any uniformly convex Banach space.
Now consider the last claim. If the last term in 72.6 .17 were a martingale, then there would be nothing to prove. This is because if $M(t)$ is a martingale which equals 0 when $t=0$, then

$$
E(M(t))=E\left(E\left(M(t) \mid \mathscr{F}_{0}\right)\right)=E(M(0))=0
$$

However, that last term is unfortunately only a local martingale. One can obtain a localizing sequence as follows.

$$
\tau_{n}(\omega) \equiv \inf \{t:|X(t, \omega)|>n\}
$$

where as usual $\inf (\emptyset) \equiv \infty$. This is all right because it was shown above that $t \rightarrow X(t, \omega)$ is continuous into $H$ for a.e. $\omega$. Then stopping both processes on the two sides of 72.6.17 with $\tau_{n}$,

$$
\begin{aligned}
\left|X\left(t \wedge \tau_{n}\right)\right|^{2} & =\left|X_{0}\right|^{2}+\int_{0}^{t \wedge \tau_{n}}\left(2\langle Y(s), X(s)\rangle+\|Z(s)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2}\right) d s \\
& +2 \int_{0}^{t \wedge \tau_{n}} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*} X(s)\right) \circ J d W(s)
\end{aligned}
$$

Now from Lemma 65.10.5,

$$
\begin{aligned}
\left|X\left(t \wedge \tau_{n}\right)\right|^{2}= & \left|X_{0}\right|^{2}+\int_{0}^{t} \mathscr{X}_{\left[0, \tau_{n}\right]}(s)\left(2\langle Y(s), X(s)\rangle+\|Z(s)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2}\right) d s \\
& +2 \int_{0}^{t} \mathscr{X}_{\left[0, \tau_{n}\right]}(s) \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*} X(s)\right) \circ J d W(s)
\end{aligned}
$$

That last term is now a martingale and so you can take the expectation of both sides. This gives

$$
\begin{gathered}
E\left(\left|X\left(t \wedge \tau_{n}\right)\right|^{2}\right)=E\left(\left|X_{0}\right|^{2}\right) \\
+E\left(\int_{0}^{t} \mathscr{X}_{\left[0, \tau_{n}\right]}(s)\left(2\langle Y(s), \bar{X}(s)\rangle+\|Z(s)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2}\right) d s\right)
\end{gathered}
$$

Letting $n \rightarrow \infty$ and using the dominated convergence theorem and $\tau_{n} \rightarrow \infty$ yields the desired result.

Notation 72.6.3 The stochastic integrals are unpleasant to look at.

$$
\begin{aligned}
& \int_{0}^{t} \mathscr{R}\left(\left(Z(s) \circ J^{-1}\right)^{*} X(s)\right) \circ J d W(s) \\
\equiv & \int_{0}^{t}(X(s), Z(s) d W(s))
\end{aligned}
$$

## Chapter 73

## The Hard Ito Formula, Implicit Case

### 73.1 Approximating With Step Functions

This Ito formula seems to be the fundamental idea which allows one to obtain solutions to stochastic partial differential equations using a variational point of view. I am following the treatment found in [108]. The following lemma is fundamental to the presentation. It approximates a function with a sequence of two step functions $X_{k}^{r}, X_{k}^{l}$ where $X_{k}^{r}$ has the value of $X$ at the right end of each interval and $X_{k}^{l}$ gives the value $X$ at the left end of the interval. The lemma is very interesting for its own sake. You can obviously do this sort of thing for a continuous function but here the function is not continuous and in addition, it is a stochastic process depending on $\omega$ also. This lemma was proved earlier, Lemma 65.3.1.

Lemma 73.1.1 Let $\Phi:[0, T] \times \Omega \rightarrow V$, be $\mathscr{B}([0, T]) \times \mathscr{F}$ measurable and suppose

$$
\Phi \in K \equiv L^{p}([0, T] \times \Omega ; E), p \geq 1
$$

Then there exists a sequence of nested partitions, $\mathscr{P}_{k} \subseteq \mathscr{P}_{k+1}$,

$$
\mathscr{P}_{k} \equiv\left\{t_{0}^{k}, \cdots, t_{m_{k}}^{k}\right\}
$$

such that the step functions given by

$$
\begin{aligned}
\Phi_{k}^{r}(t) & \equiv \sum_{j=1}^{m_{k}} \Phi\left(t_{j}^{k}\right) \mathscr{X}_{\left(t_{j-1}^{k}, t_{j}^{k}\right]}(t) \\
\Phi_{k}^{l}(t) & \equiv \sum_{j=1}^{m_{k}} \Phi\left(t_{j-1}^{k}\right) \mathscr{X}_{\left[t_{j-1}^{k}, t_{j}^{k}\right)}(t)
\end{aligned}
$$

both converge to $\Phi$ in $K$ as $k \rightarrow \infty$ and

$$
\lim _{k \rightarrow \infty} \max \left\{\left|t_{j}^{k}-t_{j+1}^{k}\right|: j \in\left\{0, \cdots, m_{k}\right\}\right\}=0
$$

Also, each $\Phi\left(t_{j}^{k}\right), \Phi\left(t_{j-1}^{k}\right)$ is in $L^{p}(\Omega ; E)$. One can also assume that $\Phi(0)=0$. The mesh points $\left\{t_{j}^{k}\right\}_{j=0}^{m_{k}}$ can be chosen to miss a given set of measure zero. In addition to this, we can assume that

$$
\left|t_{j}^{k}-t_{j-1}^{k}\right|=2^{-n_{k}}
$$

except for the case where $j=1$ or $j=m_{n_{k}}$ when this might not be so. In the case of the last subinterval defined by the partition, we can assume

$$
\left|t_{m}^{k}-t_{m-1}^{k}\right|=\left|T-t_{m-1}^{k}\right| \geq 2^{-\left(n_{k}+1\right)}
$$

The following lemma is convenient.

Lemma 73.1.2 Let $f_{n} \rightarrow f$ in $L^{p}([0, T] \times \Omega, E)$. Then there exists a subsequence $n_{k}$ and a set of measure zero $N$ such that if $\omega \notin N$, then

$$
f_{n_{k}}(\cdot, \omega) \rightarrow f(\cdot, \omega)
$$

in $L^{p}([0, T], E)$ and for a.e. $t$.
Proof: We have

$$
\begin{aligned}
P\left(\left[\left\|f_{n}-f\right\|_{L^{p}([0, T], E)}>\lambda\right]\right) & \leq \frac{1}{\lambda} \int_{\Omega}\left\|f_{n}-f\right\|_{L^{p}([0, T], E)} d P \\
& \leq \frac{1}{\lambda}\left\|f_{n}-f\right\|_{L^{p}([0, T] \times \Omega, E)}
\end{aligned}
$$

Hence there exists a subsequence $n_{k}$ such that

$$
P\left(\left[\left\|f_{n_{k}}-f\right\|_{L^{p}([0, T], E)}>2^{-k}\right]\right) \leq 2^{-k}
$$

Then by the Borel Cantelli lemma, it follows that there exists a set of measure zero $N$ such that for all $k$ large enough and $\omega \notin N$,

$$
\left\|f_{n_{k}}-f\right\|_{L^{p}([0, T], E)} \leq 2^{-k}
$$

Now by the usual arguments used in proving completeness, $f_{n_{k}}(t) \rightarrow f(t)$ for a.e.t.
Because of this lemma, it can also be assumed that for a.e. $\omega$, pointwise convergence is obtained on $[0, T]$ as well as convergence in $L^{p}([0, T])$. This kind of assumption will be tacitly made whenever convenient.

Also recall the diagram for the definition of the integral which has values in a Hilbert space $W$.

$$
\begin{array}{cccc} 
& & & \\
U_{1} & \supseteq J Q^{1 / 2} U & \stackrel{y}{1-1} & \downarrow \\
& & Q^{1 / 2} U \\
& Z_{n} \quad \searrow & & \downarrow \\
& & & W
\end{array}
$$

The idea was to get $\int_{0}^{t} Z d W$ where $Z \in L^{2}\left([0, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right)$. Here $W(t)$ was a cylindrical Wiener process. This meant that it was a $Q_{1}$ Wiener process on $U_{1}$ for $Q_{1}=J J^{*}$ and $J$ was a Hilbert Schmidt operator mapping $Q^{1 / 2} U$ to $U_{1}$. To get $\int_{0}^{t} Z d W, Z \circ J^{-1}$ was approximated by a sequence of elementary functions $\left\{Z_{n}\right\}$ having values in $\mathscr{L}\left(U_{1}, W\right)$. Then

$$
\int_{0}^{t} Z d W \equiv \lim _{n \rightarrow \infty} \int_{0}^{t} Z_{n} d W
$$

and this limit exists in $L^{2}(\Omega, W)$ and is independent of the choice of $U_{1}$ and $J$. In fact, $U_{1}$ can be assumed to be $U$.

### 73.2 The Situation

Now consider the following situation. There are real separable Banach spaces $V, W$ such that $W$ is a Hilbert space and

$$
V \subseteq W, \quad W^{\prime} \subseteq V^{\prime}
$$

where $V$ is dense in $W$. Also let $B \in \mathscr{L}\left(W, W^{\prime}\right)$ satisfy

$$
\langle B w, w\rangle \geq 0,\langle B u, v\rangle=\langle B v, u\rangle
$$

Note that $B$ does not need to be one to one. Also allowed is the case where $B$ is the Riesz map. It could also happen that $V=W$.

Situation 73.2.1 Let $X$ have values in $V$ and satisfy the following

$$
\begin{equation*}
B X(t)=B X_{0}+\int_{0}^{t} Y(s) d s+B \int_{0}^{t} Z(s) d W(s) \tag{73.2.1}
\end{equation*}
$$

$X_{0} \in L^{2}(\Omega ; W)$ and is $\mathscr{F}_{0}$ measurable, where $Z$ is $\mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)$ progressively measurable and

$$
\|Z\|_{L^{2}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right)}<\infty .
$$

This is what is needed to define the stochastic integral in the above formula. Here $Q$ is a nonnegative self adjoint operator defined on $U$. It could even be $I$.

Assume $X, Y$ satisfy

$$
B X, Y \in K^{\prime} \equiv L^{p^{\prime}}\left([0, T] \times \Omega ; V^{\prime}\right)
$$

the $\sigma$ algebra of measurable sets defining $K^{\prime}$ will be the progressively measurable sets. Here $1 / p^{\prime}+1 / p=1, p>1$.

Also the sense in which the equation holds is as follows. For a.e. $\omega$, the equation holds in $V^{\prime}$ for all $t \in[0, T]$. Thus we are considering a particular representative $X$ of $K$ for which this happens. Also it is only assumed that $B X(t)=B(X(t))$ for a.e. t. Thus $B X$ is the name of a function having values in $V^{\prime}$ for which $B X(t)=B(X(t))$ for a.e. $t$. Assume that $X$ is progressively measurable also and

$$
X \in L^{p}([0, T] \times \Omega, V)
$$

Also $W(t)$ is a $J J^{*}$ Wiener process on $U_{1}$ in the above diagram. $U_{1}$ can be assumed to be $U$.

The goal is to prove the following Ito formula valid for a.e. $t$ for each $\omega$ off a set of measure zero.

$$
\begin{align*}
\langle B X(t), X(t)\rangle= & \left\langle B X_{0}, X_{0}\right\rangle+\int_{0}^{t}\left(2\langle Y(s), X(s)\rangle+\langle B Z, Z\rangle_{\mathscr{L}_{2}}\right) d s \\
& +\int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B X \circ J d W \tag{73.2.2}
\end{align*}
$$

The most significant feature of the last term is that it is a local martingale. The term $\langle B Z, Z\rangle_{\mathscr{L}_{2}}$ will be discussed later, as will the meaning of the stochastic integral.

The idea is that $\left(Z \circ J^{-1}\right)^{*} B X \circ J$ has values in $\mathscr{L}_{2}\left(Q^{1 / 2} U, \mathbb{R}\right)$ and so it makes sense to consider this stochastic integral. To see this, $B X \in W^{\prime}$ and

$$
\left(Z \circ J^{-1}\right)^{*} \in \mathscr{L}_{2}\left(W^{\prime},\left(J Q^{1 / 2} U\right)^{\prime}\right)
$$

and so

$$
\left(Z \circ J^{-1}\right)^{*} B X \in\left(J Q^{1 / 2} U\right)^{\prime}
$$

and so $\left(Z \circ J^{-1}\right)^{*} B X \circ J \in \mathscr{L}_{2}\left(Q^{1 / 2} U, \mathbb{R}\right)=\left(Q^{1 / 2} U\right)^{\prime}$. Note that in general $H^{\prime}=\mathscr{L}_{2}(H, \mathbb{R})$ because if you have $\left\{e_{i}\right\}$ an orthonormal basis in $H$, then for $f \in H^{\prime}$,

$$
\sum_{i}\left|\left(R^{-1} f, e_{i}\right)\right|^{2}=\sum_{i}\left|\left\langle f, e_{i}\right\rangle\right|^{2}=\|f\|_{H^{\prime}}^{2}
$$

The main item of interest relative to this stochastic integral will be a statement about its quadratic variation. It appears to depend on $J$ but this is not the case because the other terms in the formula do not.

### 73.3 Preliminary Results

Here are discussed some preliminary results which will be needed. From the integral equation, if $\phi \in L^{q}(\Omega ; V)$ and $\psi \in C_{c}^{\infty}(0, T)$ for $q=\max (p, 2)$,

$$
\begin{aligned}
& \int_{\Omega} \int_{0}^{T}\left((B X)(t)-B \int_{0}^{t} Z(s) d W(s)-B X_{0}\right) \psi^{\prime} \phi d t d P \\
= & \int_{\Omega} \int_{0}^{T} \int_{0}^{t} Y(s) \psi^{\prime}(t) d s \phi d t d P
\end{aligned}
$$

Then the term on the right equals

$$
\int_{\Omega} \int_{0}^{T} \int_{s}^{T} Y(s) \psi^{\prime}(t) d t d s \phi(\omega) d P=\int_{\Omega}\left(-\int_{0}^{T} Y(s) \psi(s) d s\right) \phi(\omega) d P
$$

It follows that, since $\phi$ is arbitrary,

$$
\int_{0}^{T}\left((B X)(t)-B \int_{0}^{t} Z(s) d W(s)-B X_{0}\right) \psi^{\prime}(t) d t=-\int_{0}^{T} Y(s) \psi(s) d s
$$

in $L^{q^{\prime}}\left(\Omega ; V^{\prime}\right)$ and so the weak time derivative of

$$
t \rightarrow(B X)(t)-B \int_{0}^{t} Z(s) d W(s)-B X_{0}
$$

equals $Y$ in $L^{q^{\prime}}\left([0, T] ; L^{q^{\prime}}\left(\Omega, V^{\prime}\right)\right)$.Thus, by Theorem 34.2.9, for a.e. $t$, say $t \notin \hat{N} \subseteq$ $[0, T], m(\hat{N})=0$,

$$
B\left(X(t)-\int_{0}^{t} Z(s) d W(s)\right)=B X_{0}+\int_{0}^{t} Y(s) d s \text { in } L^{q^{\prime}}\left(\Omega, V^{\prime}\right)
$$

That is,

$$
(B X)(t)=B X_{0}+\int_{0}^{t} Y(s) d s+B \int_{0}^{t} Z(s) d W(s)
$$

holds in $L^{q^{\prime}}\left(\Omega, V^{\prime}\right)$ where $(B X)(t)=B(X(t))$ a.e. $t$, in addition to holding for all $t$ for each $\omega$. Now let $\left\{t_{k}^{n}\right\}_{k=1 n=1}^{m_{n} \infty}$ be partitions for which, from Lemma 73.1.1 there are left and right step functions $X_{k}^{l}, X_{k}^{r}$, which converge in $L^{p}([0, T] \times \Omega ; V)$ to $X$ and such that each $\left\{t_{k}^{n}\right\}_{k=1}^{m_{n}}$ has empty intersection with the set of measure zero $\hat{N}$ where, in $L^{p^{\prime}}\left(\Omega ; V^{\prime}\right)$, $(B X)(t) \neq B(X(t))$ in $L^{q^{\prime}}\left(\Omega ; V^{\prime}\right)$. Thus for $t_{k}$ a generic partition point,

$$
B X\left(t_{k}\right)=B\left(X\left(t_{k}\right)\right) \text { in } L^{q^{\prime}}\left(\Omega ; V^{\prime}\right)
$$

Hence there is an exceptional set of measure zero, $N\left(t_{k}\right) \subseteq \Omega$ such that for

$$
\omega \notin N\left(t_{k}\right), B X\left(t_{k}\right)(\omega)=B\left(X\left(t_{k}, \omega\right)\right) .
$$

We define an exceptional set $N \subseteq \Omega$ to be the union of all these $N\left(t_{k}\right)$. There are countably many and so $N$ is also a set of measure zero. Then for $\omega \notin N$, and $t_{k}$ any mesh point at all, $B X\left(t_{k}\right)(\omega)=B\left(X\left(t_{k}, \omega\right)\right)$. This will be important in what follows. In addition to this, from the integral equation, for each of these $\omega \notin N, B X(t)(\omega)=B(X(t, \omega))$ for all $t \notin N_{\omega} \subseteq[0, T]$ where $N_{\omega}$ is a set of Lebesgue measure zero. Thus the $t_{k}$ from the various partitions are always in $N_{\omega}^{C}$. By Lemma 69.4.1, there exists a countable set $\left\{e_{i}\right\}$ of vectors in $V$ such that

$$
\left\langle B e_{i}, e_{j}\right\rangle=\delta_{i j}
$$

and for each $x \in W$,

$$
\langle B x, x\rangle=\sum_{i=0}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}, B x=\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i}
$$

Thus the conclusion of the above discussion is that at the mesh points, it is valid to write

$$
\begin{aligned}
\left\langle(B X)\left(t_{k}\right), X\left(t_{k}\right)\right\rangle & =\left\langle B\left(X\left(t_{k}\right)\right), X\left(t_{k}\right)\right\rangle \\
& =\sum_{i}\left\langle(B X)\left(t_{k}\right), e_{i}\right\rangle^{2}=\sum_{i}\left\langle B\left(X\left(t_{k}\right)\right), e_{i}\right\rangle^{2}
\end{aligned}
$$

just as would be the case if $(B X)(t)=B(X(t))$ for every $t$. In all which follows, the mesh points will be like this and an appropriate set of measure zero which may be replaced with a larger set of measure zero finitely many times is being neglected. Obviously, one can take a subsequence of the sequence of partitions described above without disturbing the above observations. We will denote these partitions as $\mathscr{P}_{k}$. As a case of this, we obtain the following interesting lemma.

Lemma 73.3.1 In the above situation, there exists a set of measure zero $N \subseteq \Omega$ and a dense subset of $[0, T], D$ such that for $\omega \notin N, B X(t, \omega)=B(X(t, \omega))$ for all $t \in D$.

Theorem 73.3.2 Let $Z$ be progressively measurable and in

$$
L^{2}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right)
$$

Also suppose $X$ is progressively measurable and in $L^{2}([0, T] \times \Omega, W)$. Let $\left\{t_{j}^{n}\right\}_{j=0}^{m_{n}}$ be a sequence of partitions of the sort in Lemma 73.1.1 such that if

$$
X_{n}(t) \equiv \sum_{j=0}^{m_{n}-1} X\left(t_{j}^{n}\right) \mathscr{X}_{\left[t_{j}^{n}, t_{j+1}^{n}\right)}(t) \equiv X_{n}^{l}(t)
$$

then $X_{n} \rightarrow X$ in $L^{p}([0, T] \times \Omega, W)$. Also, it can be assumed that none of these mesh points are in the exceptional set off which $B X(t)=B(X(t))$. (Thus it will make no difference whether we write $B X(t)$ or $B(X(t))$ in what follows for all $t$ one of these mesh points.) Then the expression

$$
\begin{equation*}
\sum_{j=0}^{m_{n}-1}\left\langle B \int_{t_{j}^{n} \wedge t}^{t_{j+1}^{n} \wedge t} Z d W, X\left(t_{j}^{n}\right)\right\rangle=\sum_{j=0}^{m_{n}-1}\left\langle B X\left(t_{j}^{n}\right), \int_{t_{j}^{n} \wedge t}^{t_{j+1}^{n} \wedge t} Z d W\right\rangle \tag{73.3.3}
\end{equation*}
$$

is a local martingale which can be written in the form

$$
\int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B X_{n}^{l} \circ J d W
$$

where

$$
X_{n}^{l}(t)=\sum_{k=0}^{m_{n}-1} X\left(t_{k}^{n}\right) \mathscr{X}_{\left[t_{k}^{n}, t_{k+1}^{n}\right)}(t)
$$

Proof: First suppose that $\left\langle B X\left(t_{k}^{n}\right), X\left(t_{k}^{n}\right)\right\rangle \in L^{\infty}(\Omega)$. Then

$$
\left\langle B X\left(t_{j}^{n}\right), \int_{t_{j}^{n} \wedge t}^{t_{j+1}^{n} \wedge t} Z d W\right\rangle
$$

is in $L^{1}(\Omega)$ for each $t$ since both entries are in $L^{2}(\Omega)$. Why is this a martingale?

$$
\begin{gathered}
E\left(\left\langle B X\left(t_{j}^{n}\right), \int_{t_{j}^{n} \wedge t}^{t_{j+1}^{n} \wedge t} Z d W\right\rangle\right)=E\left(E\left(\left\langle B X\left(t_{j}^{n}\right), \int_{t_{j}^{n} \wedge t}^{t_{j+1}^{n} \wedge t} Z d W\right\rangle \mid \mathscr{F}_{t_{j}^{n}}\right)\right) \\
\quad=E\left(\left\langle B X\left(t_{j}^{n}\right), E\left(\int_{t_{j}^{n} \wedge t}^{t_{j+1}^{n} \wedge t} Z d W \mid \mathscr{F}_{t_{j}^{n}}\right)\right\rangle\right)=E\left(\left\langle B X\left(t_{j}^{n}\right), 0\right\rangle\right)=0
\end{gathered}
$$

because the stochastic integral is a martingale. Now let $\sigma$ be a bounded stopping time.

$$
\begin{gathered}
E\left(\left\langle B X\left(t_{j}^{n}\right), \int_{t_{j}^{n} \wedge \sigma}^{t_{j+1}^{n} \wedge \sigma} Z d W\right\rangle\right)=E\left(E\left(\left\langle B X\left(t_{j}^{n}\right), \int_{t_{j}^{n} \wedge \sigma}^{t_{j+1}^{n} \wedge \sigma} Z d W\right\rangle \mid \mathscr{F}_{t_{j}^{n}}\right)\right) \\
=E\left(\left\langle B X\left(t_{j}^{n}\right),\left(E \int_{t_{j}^{n} \wedge \sigma}^{t_{j+1}^{n} \wedge \sigma} Z d W \mid \mathscr{F}_{t_{j}^{n}}\right)\right\rangle\right)
\end{gathered}
$$

$$
\begin{aligned}
& =E\left(\left\langle B X\left(t_{j}^{n}\right), E\left(\int_{0}^{t_{j+1}^{n} \wedge \sigma} Z d W-\int_{0}^{t_{j}^{n} \wedge \sigma} Z d W \mid \mathscr{F}_{t_{j}^{n}}\right)\right\rangle\right) \\
& =E\left(\left\langle B X\left(t_{j}^{n}\right), 0\right\rangle\right)=0
\end{aligned}
$$

and so this is a martingale by Lemma 63.1.1. I want to write the formula in 73.3 .3 as a stochastic integral. First note that $W$ has values in $U_{1}$.

Consider one of the terms of the sum more simply as

$$
\left\langle B \int_{a}^{b} Z d W, X(a)\right\rangle, a=t_{k}^{n} \wedge t, b=t_{k+1}^{n} \wedge t
$$

Then from the definition of the integral, let $Z_{n}$ be a sequence of elementary functions converging to $Z \circ J^{-1}$ in $L^{2}\left([a, b] \times \Omega, \mathscr{L}_{2}\left(J Q^{1 / 2} U, W\right)\right)$ and

$$
\left\|\int_{a}^{t} Z d W-\int_{a}^{t} Z_{n} d W\right\|_{L^{2}(\Omega, W)} \rightarrow 0
$$

Using a maximal inequality and the fact that the two integrals are martingales along with the Borel Cantelli lemma, there exists a set of measure $0 N$ such that for $\omega \notin N$, the convergence of a suitable subsequence of these integrals, still denoted by $n$, is uniform for $t \in[a, b]$. It follows that for such $\omega$,

$$
\begin{equation*}
\left\langle B \int_{a}^{t} Z d W, X(a)\right\rangle=\lim _{n \rightarrow \infty}\left\langle B \int_{a}^{t} Z_{n} d W, X(a)\right\rangle \tag{73.3.4}
\end{equation*}
$$

Say

$$
Z_{n}(u)=\sum_{k=0}^{m_{n}-1} Z_{k}^{n} \mathscr{X}_{\left[t_{k}^{n}, t_{k+1}^{n}\right)}(u)
$$

where $Z_{k}^{n}$ has finitely many values in $\mathscr{L}\left(U_{1}, W\right)_{0}$, the restrictions of maps in $\mathscr{L}\left(U_{1}, W\right)$ to $J Q^{1 / 2} U$, and the $t_{k}^{n}$ refer to a partition of $[a, b]$. Then the product on the right in 73.3.4 is of the form

$$
\sum_{k=0}^{m_{n}-1}\left\langle B Z_{k}^{n}\left(W\left(t \wedge t_{k+1}^{n}\right)-W\left(t \wedge t_{k}^{n}\right)\right), X(a)\right\rangle_{W^{\prime}, W}
$$

Note that it makes sense because $Z_{k}^{n}$ is the restriction to $J\left(Q^{1 / 2} U\right)$ of a map from $U_{1}$ to $W$ and so $B Z_{k}^{n}$ is a map from $U_{1}$ to $W^{\prime}$. Then the Wiener process has values in $U_{1}$ so when you apply $B Z_{k}^{n}$ to $W\left(t \wedge t_{k+1}^{n}\right)-W\left(t \wedge t_{k}^{n}\right)$, you get something in $W^{\prime}$ and so the duality pairing is between $W^{\prime}$ and $W$ as shown. Also, $Z_{k}^{n}\left(W\left(t \wedge t_{k+1}^{n}\right)-W\left(t \wedge t_{k}^{n}\right)\right)$ gives something in $W$ because the Wiener process has values in $U_{1}$ and $Z_{k}^{n}$ acts on these things to give something in $W$. Thus the above equals

$$
=\sum_{k=0}^{m_{n}-1}\left\langle B X(a), Z_{k}^{n}\left(W\left(t \wedge t_{k+1}^{n}\right)-W\left(t \wedge t_{k}^{n}\right)\right)\right\rangle_{W^{\prime}, W}
$$

$$
\begin{aligned}
& =\sum_{k=0}^{m_{n}-1}\left\langle\left(Z_{k}^{n}\right)^{*} B X(a),\left(W\left(t \wedge t_{k+1}^{n}\right)-W\left(t \wedge t_{k}^{n}\right)\right)\right\rangle_{U_{1}^{\prime}, U_{1}} \\
& =\sum_{k=0}^{m_{n}-1}\left(Z_{k}^{n}\right)^{*} B X(a)\left(W\left(t \wedge t_{k+1}^{n}\right)-W\left(t \wedge t_{k}^{n}\right)\right) \\
& =\int_{a}^{t} Z_{n}^{*} B X(a) d W
\end{aligned}
$$

Note that the restriction of $\left(Z_{n}\right)^{*} B X(a)$ is in

$$
\mathscr{L}\left(U_{1}, \mathbb{R}\right)_{0} \subseteq \mathscr{L}_{2}\left(J Q^{1 / 2} U, \mathbb{R}\right)
$$

Recall also that the space on the left is dense in the one on the right. Now let $\left\{g_{i}\right\}$ be an orthonormal basis for $Q^{1 / 2} U$, so that $\left\{J g_{i}\right\}$ is an orthonormal basis for $J Q^{1 / 2} U$. Then

$$
\begin{gathered}
\sum_{i=1}^{\infty}\left|\left(\left(Z_{n}\right)^{*} B X(a)-\left(Z \circ J^{-1}\right)^{*} B X(a)\right)\left(J g_{i}\right)\right|^{2} \\
=\sum_{i=1}^{\infty}\left|\left\langle B X(a),\left(Z_{n}-Z \circ J^{-1}\right)\left(J g_{i}\right)\right\rangle\right|^{2} \\
\leq\langle B X(a), X(a)\rangle \sum_{i=1}^{\infty}\left\langle B\left(Z_{n}-Z \circ J^{-1}\right)\left(J g_{i}\right),\left(Z_{n}-Z \circ J^{-1}\right)\left(J g_{i}\right)\right\rangle \\
\leq\langle B X(a), X(a)\rangle\|B\| \sum_{i=1}^{\infty}\left\|\left(Z_{n}-Z \circ J^{-1}\right)\left(J g_{i}\right)\right\|_{W}^{2} \\
=\langle B X(a), X(a)\rangle\|B\|\left\|Z_{n}-Z \circ J^{-1}\right\|_{\mathscr{L}_{2}\left(J Q^{1 / 2} U, W\right)}^{2}
\end{gathered}
$$

When integrated over $[a, b] \times \Omega$, it is given that this converges to 0 , if it is assumed that $\langle B X(a), X(a)\rangle \in L^{\infty}(\Omega)$, which is assumed for now.

It follows that, with this assumption,

$$
Z_{n}^{*} B X(a) \rightarrow\left(Z \circ J^{-1}\right)^{*} B X(a)
$$

in $L^{2}\left([a, b] \times \Omega, \mathscr{L}_{2}\left(J Q^{1 / 2} U, \mathbb{R}\right)\right)$. Writing this differently, it says

$$
Z_{n}^{*} B X(a) \rightarrow\left(\left(Z \circ J^{-1}\right)^{*} B X(a) \circ J\right) \circ J^{-1} \text { in } L^{2}\left([a, b] \times \Omega, \mathscr{L}_{2}\left(J Q^{1 / 2} U, \mathbb{R}\right)\right)
$$

It follows from the definition of the integral that the Ito integrals converge. Therefore,

$$
\left\langle B \int_{a}^{t} Z d W, X(a)\right\rangle=\int_{a}^{t}\left(Z \circ J^{-1}\right)^{*} B X(a) \circ J d W
$$

The term on the right is a martingale because the one on the left is.

Next it is necessary to drop the assumption that $\langle B X(a), X(a)\rangle \in L^{\infty}(\Omega)$. Note that $X_{n}^{l}$ is right continuous and $B X_{n}^{l}$ progressively measurable. Thus,

$$
\left\langle B X_{n}^{l}(t), X_{n}^{l}(t)\right\rangle=\sum_{i}\left\langle B X_{n}^{l}(t), e_{i}\right\rangle^{2}
$$

where $\left\{e_{i}\right\}$ is the set defined in Lemma 76.2.1 each in $V$. Thus $\left\langle B X_{n}^{l}, X_{n}^{l}\right\rangle$ is also progressively measurable and right continuous, and one can define the stopping time

$$
\begin{equation*}
\sigma_{q}^{n} \equiv \inf \left\{t:\left\langle B X_{n}^{l}(t), X_{n}^{l}(t)\right\rangle>q\right\} \tag{73.3.5}
\end{equation*}
$$

the first hitting time of an open set. Also, for each $\omega$, there are only finitely many values for $\left\langle B X_{n}^{l}(t), X_{n}^{l}(t)\right\rangle$ and so $\sigma_{q}^{n}=\infty$ for all $q$ large enough.

From localization of the stochastic integral,

$$
\begin{aligned}
\left\langle B \int_{a \wedge \sigma_{q}^{n}}^{t \wedge \sigma_{q}^{n}} Z d W, X(a)\right\rangle & =\left\langle B \int_{a}^{t} \mathscr{X}_{\left[0, \sigma_{q}^{n}\right]} Z d W, X(a)\right\rangle \\
& =\int_{a}^{t}\left(\mathscr{X}_{\left[0, \sigma_{q}^{n}\right]} Z \circ J^{-1}\right)^{*} B X(a) \circ J d W \\
& =\int_{a \wedge \sigma_{q}^{n}}^{t \wedge \sigma_{q}^{n}}\left(Z \circ J^{-1}\right)^{*} B X(a) \circ J d W
\end{aligned}
$$

Then it follows that, using the stopping time,

$$
\sum_{j=0}^{m_{n}-1}\left\langle B \int_{t_{j}^{n} \wedge t \wedge \sigma_{q}^{n}}^{t_{j+1}^{n} \wedge t \wedge \sigma_{q}^{n}} Z d W, X\left(t_{j}^{n}\right)\right\rangle=\int_{0}^{t \wedge \sigma_{q}^{n}}\left(Z \circ J^{-1}\right)^{*} B X_{n}^{l} \circ J d W
$$

where $X_{n}^{l}$ is the step function

$$
X_{n}^{l}(t)=\sum_{k=0}^{m_{n}-1} X\left(t_{k}^{n}\right) \mathscr{X}_{\left[t_{k}^{n}, t_{k+1}^{n}\right)}(t)
$$

Thus the given sum equals the local martingale

$$
\int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B X_{n}^{l} \circ J d W
$$

Note that the sum 73.3 .3 does not depend on $J$ or on $U_{1}$ so the same must be true of what it equals although it does not look that way. The question of convergence as $n \rightarrow \infty$ is considered later.

What follows is the main estimate and discrete formulas.

### 73.4 The Main Estimate

The argument will be based on a formula which follows in the next lemma.

Lemma 73.4.1 In Situation 73.2.1 the following formula holds for a.e. $\omega$ for $0<s<t$ where $M(t) \equiv \int_{0}^{t} Z(u) d W(u)$ which has values in $W$. In the following, $\langle\cdot, \cdot\rangle$ denotes the duality pairing between $V, V^{\prime}$.

$$
\begin{gather*}
\langle B X(t), X(t)\rangle=\langle B X(s), X(s)\rangle+ \\
+2 \int_{s}^{t}\langle Y(u), X(t)\rangle d u+\langle B(M(t)-M(s)), M(t)-M(s)\rangle \\
-\langle B X(t)-B X(s)-(M(t)-M(s)), X(t)-X(s)-(M(t)-M(s))\rangle \\
+2\langle B X(s), M(t)-M(s)\rangle \tag{73.4.6}
\end{gather*}
$$

Also for $t>0$

$$
\begin{align*}
& \langle B X(t), X(t)\rangle=\left\langle B X_{0}, X_{0}\right\rangle+2 \int_{0}^{t}\langle Y(u), X(t)\rangle d u+2\left\langle B X_{0}, M(t)\right\rangle+ \\
& \langle B M(t), M(t)\rangle-\left\langle B X(t)-B X_{0}-B M(t), X(t)-X_{0}-M(t)\right\rangle \tag{73.4.7}
\end{align*}
$$

Proof: From the formula which is assumed to hold,

$$
\begin{aligned}
& B X(t)=B X_{0}+\int_{0}^{t} Y(u) d u+B M(t) \\
& B X(s)=B X_{0}+\int_{0}^{s} Y(u) d u+B M(s)
\end{aligned}
$$

Then

$$
B M(t)-B M(s)+\int_{s}^{t} Y(u) d u=B X(t)-B X(s)
$$

It follows that

$$
\begin{gathered}
\langle B(M(t)-M(s)), M(t)-M(s)\rangle- \\
\langle B X(t)-B X(s)-(M(t)-M(s)), X(t)-X(s)-(M(t)-M(s))\rangle \\
+2\langle B X(s), M(t)-M(s)\rangle \\
=\quad\langle B(M(t)-M(s)), M(t)-M(s)\rangle-\langle B X(t)-B X(s), X(t)-X(s)\rangle \\
+2\langle B X(t)-B X(s), M(t)-M(s)\rangle \\
-\langle B(M(t)-M(s)), M(t)-M(s)\rangle+2\langle B X(s), M(t)-M(s)\rangle
\end{gathered}
$$

Some terms cancel and this yields

$$
\begin{aligned}
=-\langle B X(t)- & B X(s), X(t)-X(s)\rangle+2\langle B X(t), M(t)-M(s)\rangle \\
=-\langle B X(t)- & B X(s), X(t)-X(s)\rangle+2\langle B(M(t)-M(s)), X(t)\rangle \\
= & -\langle B(X(t)-X(s)), X(t)-X(s)\rangle \\
& +2\left\langle B X(t)-B X(s)-\int_{s}^{t} Y(u) d u, X(t)\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&=-\langle B X(t), X(t)\rangle-\langle B X(s), X(s)\rangle \\
&+2\langle B X(t), X(s)\rangle+2\langle B X(t), X(t)\rangle \\
&-2\langle B X(s), X(t)\rangle-2 \int_{s}^{t}\langle Y(u), X(t)\rangle d u \\
&=\langle B X(t), X(t)\rangle-\langle B X(s), X(s)\rangle-2 \int_{s}^{t}\langle Y(u), X(t)\rangle d u
\end{aligned}
$$

Therefore,

$$
\begin{gathered}
\langle B X(t), X(t)\rangle-\langle B X(s), X(s)\rangle \\
=2 \int_{s}^{t}\langle Y(u), X(t)\rangle d u+\langle B(M(t)-M(s)), M(t)-M(s)\rangle \\
-\langle B X(t)-B X(s)-(M(t)-M(s)), X(t)-X(s)-(M(t)-M(s))\rangle \\
+2\langle B X(s), M(t)-M(s)\rangle
\end{gathered}
$$

The case with $X_{0}$ is similar.
The following phenomenal estimate holds and it is this estimate which is the main idea in proving the Ito formula. The last assertion about continuity is like the well known result that if $y \in L^{p}(0, T ; V)$ and $y^{\prime} \in L^{p^{\prime}}\left(0, T ; V^{\prime}\right)$, then $y$ is actually continuous a.e. with values in $H$, for $V, H, V^{\prime}$ a Gelfand triple. Later, this continuity result is strengthened further to give strong continuity. In all of this, $X_{k}^{l}$ and $X_{k}^{r}$ are as described above, converging in $K$ to $X$.

Lemma 73.4.2 In the Situation 73.2.1, the following holds. For a.e. $t$

$$
\begin{align*}
& E(\langle B X(t), X(t)\rangle) \\
< & C\left(\|Y\|_{K^{\prime}},\|X\|_{K},\|Z\|_{J},\left\|\left\langle B X_{0}, X_{0}\right\rangle\right\|_{L^{1}(\Omega)}\right)<\infty . \tag{73.4.8}
\end{align*}
$$

where $K, K^{\prime}$ were defined earlier and

$$
J=L^{2}\left([0, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U ; W\right)\right)
$$

In fact,

$$
E\left(\sup _{t \in[0, T]} \sum_{i}\left\langle B X(t), e_{i}\right\rangle^{2}\right) \leq C\left(\|Y\|_{K^{\prime}},\|X\|_{K},\|Z\|_{J},\left\|\left\langle B X_{0}, X_{0}\right\rangle\right\|_{L^{1}(\Omega)}\right)
$$

Also, $C$ is a continuous function of its arguments, increasing in each one, and $C(0,0,0,0)=$ 0 . Thus for a.e. $\omega$,

$$
\sup _{t \notin N_{\omega}^{C}}\langle B X(t, \omega), X(t, \omega)\rangle \leq C(\omega)<\infty
$$

Also for $\omega$ off a set of measure zero described earlier, $t \rightarrow B X(t)(\omega)$ is weakly continuous with values in $W^{\prime}$ on $[0, T]$. Also $t \rightarrow\langle B X(t), X(t)\rangle$ is lower semicontinuous on $N_{\omega}^{C}$.

Proof: Consider the formula in Lemma 73.4.1.

$$
\begin{gather*}
\langle B X(t), X(t)\rangle=\langle B X(s), X(s)\rangle \\
+2 \int_{s}^{t}\langle Y(u), X(t)\rangle d u+\langle B(M(t)-M(s)), M(t)-M(s)\rangle \\
-\langle B(X(t)-X(s)-(M(t)-M(s))), X(t)-X(s)-(M(t)-M(s))\rangle \\
+2\langle B X(s), M(t)-M(s)\rangle \tag{73.4.9}
\end{gather*}
$$

Now let $t_{j}$ denote a point of $\mathscr{P}_{k}$ from Lemma 73.1.1. Then for $t_{j}>0, X\left(t_{j}\right)$ is just the value of $X$ at $t_{j}$ but when $t=0$, the definition of $X(0)$ in this step function is $X(0) \equiv 0$. Thus

$$
\begin{aligned}
& \sum_{j=1}^{m-1}\left\langle B X\left(t_{j+1}\right), X\left(t_{j+1}\right)\right\rangle-\left\langle B X\left(t_{j}\right), X\left(t_{j}\right)\right\rangle \\
& +\left\langle B X\left(t_{1}\right), X\left(t_{1}\right)\right\rangle-\left\langle B X_{0}, X_{0}\right\rangle \\
& \quad=\left\langle B X\left(t_{m}\right), X\left(t_{m}\right)\right\rangle-\left\langle B X_{0}, X_{0}\right\rangle
\end{aligned}
$$

Using the formula in Lemma 73.4.1, for $t=t_{m}$ this yields

$$
\begin{gather*}
\left\langle B X\left(t_{m}\right), X\left(t_{m}\right)\right\rangle-\left\langle B X_{0}, X_{0}\right\rangle=2 \sum_{j=1}^{m-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u+ \\
+2 \sum_{j=1}^{m-1}\left\langle B \int_{t_{j}}^{t_{j+1}} Z(u) d W, X\left(t_{j}\right)\right\rangle \\
+\sum_{j=1}^{m-1}\left\langle B\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right), M\left(t_{j+1}\right)-M\left(t_{j}\right)\right\rangle \\
-\sum_{j=1}^{m-1}\left\langle B\left(X\left(t_{j+1}\right)-X\left(t_{j}\right)-\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right),\right. \\
\left.X\left(t_{j+1}\right)-X\left(t_{j}\right)-\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right\rangle \\
+2 \int_{0}^{t_{1}}\left\langle Y(u), X\left(t_{1}\right)\right\rangle d u+2\left\langle B X_{0}, \int_{0}^{t_{1}} Z(u) d W\right\rangle+\left\langle B M\left(t_{1}\right), M\left(t_{1}\right)\right\rangle \\
-\left\langle B\left(X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right), X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right\rangle \tag{73.4.10}
\end{gather*}
$$

First consider

$$
2 \int_{0}^{t_{1}}\left\langle Y(u), X\left(t_{1}\right)\right\rangle d u+2\left\langle B X_{0}, \int_{0}^{t_{1}} Z(u) d W\right\rangle+\left\langle B M\left(t_{1}\right), M\left(t_{1}\right)\right\rangle .
$$

Each term of the above converges to 0 for a.e. $\omega$ as $k \rightarrow \infty$ and in $L^{1}(\Omega)$. This follows right away for the second two terms from the Ito isometry and continuity properties of the stochastic integral. Consider the first term. This term is dominated by

$$
\begin{aligned}
& \left(\int_{0}^{t_{1}}\|Y(u)\|^{p^{\prime}} d u\right)^{1 / p^{\prime}}\left(\int_{0}^{T}\left\|X_{k}^{r}(u)\right\|^{p} d u\right)^{1 / p} \\
\leq & C(\omega)\left(\int_{0}^{t_{1}}\|Y(u)\|^{p^{\prime}} d u\right)^{1 / p^{\prime}},\left(\int_{\Omega} C(\omega)^{p} d P\right)^{1 / p}<\infty
\end{aligned}
$$

Hence this converges to 0 for a.e. $\omega$ and also converges to 0 in $L^{1}(\Omega)$.
At this time, not much is known about the last term in 73.4.10, but it is negative and is about to be neglected anyway.

The term involving the stochastic integral equals

$$
2 \sum_{j=1}^{m-1}\left\langle B \int_{t_{j}}^{t_{j+1}} Z(u) d W, X\left(t_{j}\right)\right\rangle
$$

By Theorem 73.3.2 this equals

$$
2 \int_{t_{1}}^{t_{m}}\left(Z \circ J^{-1}\right)^{*} B X_{k}^{l} \circ J d W
$$

Also note that since $\left\langle B M\left(t_{1}\right), M\left(t_{1}\right)\right\rangle$ converges to 0 in $L^{1}(\Omega)$ and for a.e. $\omega$, the sum involving

$$
\left\langle B\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right), M\left(t_{j+1}\right)-M\left(t_{j}\right)\right\rangle
$$

can be started at 0 rather than 1 at the expense of adding in a term which converges to 0 a.e. and in $L^{1}(\Omega)$. Thus 73.4.10 is of the form

$$
\begin{align*}
& \left\langle B X\left(t_{m}\right), X\left(t_{m}\right)\right\rangle-\left\langle B X_{0}, X_{0}\right\rangle=e(k)+2 \int_{0}^{t_{m}}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u+ \\
& \quad+2 \int_{0}^{t_{m}}\left(Z \circ J^{-1}\right)^{*} B X_{k}^{l} \circ J d W \\
& \quad+\sum_{j=0}^{m-1}\left\langle B\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right), M\left(t_{j+1}\right)-M\left(t_{j}\right)\right\rangle \\
& \quad-\sum_{j=1}^{m-1}\left\langle B\left(X\left(t_{j+1}\right)-X\left(t_{j}\right)-\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right)\right. \\
& \left.\quad X\left(t_{j+1}\right)-X\left(t_{j}\right)-\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right\rangle \\
& \quad-\left\langle B\left(X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right), X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right\rangle \tag{73.4.11}
\end{align*}
$$

where $e(k) \rightarrow 0$ for a.e. $\omega$ and also in $L^{1}(\Omega)$.

By definition, $M\left(t_{j+1}\right)-M\left(t_{j}\right)=\int_{t_{j}}^{t_{j+1}} Z d W$. Now it follows, on discarding the negative terms,

$$
\begin{aligned}
& \left\langle B X\left(t_{m}\right), X\left(t_{m}\right)\right\rangle-\left\langle B X_{0}, X_{0}\right\rangle \leq e(k)+2 \int_{0}^{t_{m}}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u+ \\
& +2 \int_{0}^{t_{m}}\left(Z \circ J^{-1}\right)^{*} B X_{k}^{l} \circ J d W+\sum_{j=0}^{m-1}\left\langle B \int_{t_{j}}^{t_{j+1}} Z d W, \int_{t_{j}}^{t_{j+1}} Z d W\right\rangle
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \sup _{t_{m} \in \mathscr{P}_{k}}\left\langle B X\left(t_{m}\right), X\left(t_{m}\right)\right\rangle \leq\left\langle B X_{0}, X_{0}\right\rangle+e(k)+2 \int_{0}^{T}\left|\left\langle Y(u), X_{k}^{r}(u)\right\rangle\right| d u+ \\
&+2 \sup _{t_{m} \in \mathscr{P}_{k}}\left|\int_{0}^{t_{m}}\left(Z \circ J^{-1}\right)^{*} B X_{k}^{l} \circ J d W\right| \\
&+\sum_{j=0}^{m_{k}-1}\left\langle B\left(\int_{t_{j}}^{t_{j+1}} Z(u) d W\right), \int_{t_{j}}^{t_{j+1}} Z(u) d W\right\rangle
\end{aligned}
$$

where there are $m_{k}+1$ points in $\mathscr{P}_{k}$.
The next task is to somehow take the expectation of both sides. However, this is problematic because the stochastic integral is only a local martingale. Let

$$
\tau_{p}=\inf \left\{t:\left\langle B X_{k}^{l}(t), X_{k}^{l}(t)\right\rangle>p\right\}
$$

By right continuity this is a well defined stopping time. Then you obtain the above inequality for $\left(X_{k}^{l}\right)^{\tau_{p}}$ in place of $X_{k}^{l}$. Take the expectation and use the Ito isometry to obtain

$$
\begin{gather*}
\int_{\Omega}\left(\sup _{t_{m} \in \mathscr{P}_{k}}\left\langle B\left(X_{k}^{l}\right)^{\tau_{p}}\left(t_{m}\right),\left(X_{k}^{l}\right)^{\tau_{p}}\left(t_{m}\right)\right\rangle\right) d P \\
\leq E\left(\left\langle B X_{0}, X_{0}\right\rangle\right)+2\|Y\|_{K^{\prime}}\left\|X_{k}^{r}\right\|_{K} \\
+\|B\| \sum_{j=0}^{m_{k}-1} \int_{t_{j}}^{t_{j+1}} \int_{\Omega}\|Z(u)\|^{2} d P d u \\
+2 \int_{\Omega}\left(\sup _{t \in[0, T]}\left|\int_{0}^{t} \mathscr{X}_{\left[0, \tau_{p}\right]}\left(Z \circ J^{-1}\right)^{*} B\left(X_{k}^{l}\right)^{\tau_{p}} \circ J d W\right|\right) d P+E(|e(k)|) \\
\leq C+\|B\| \int_{0}^{T} \int_{\Omega}\|Z(u)\|^{2} d P d u+E(|e(k)|) \\
+2 \int_{\Omega}\left(\sup _{t \in[0, T]}\left|\int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B\left(X_{k}^{l}\right)^{\tau_{p}} \circ J d W\right|\right) d P \leq \\
C+E(|e(k)|)+2 \int_{\Omega}\left(\sup _{t \in[0, T]}\left|\int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B\left(X_{k}^{l}\right)^{\tau_{p}} \circ J d W\right|\right) d P \tag{73.4.12}
\end{gather*}
$$

where the convergence of $X_{k}^{r}$ to $X$ in $K$ shows the term $2\|Y\|_{K^{\prime}}\left\|X_{k}^{r}\right\|_{K}$ is bounded. Thus the constant $C$ can be assumed to be a continuous function of

$$
\|Y\|_{K^{\prime}},\|X\|_{K},\|Z\|_{J},\left\|\left\langle B X_{0}, X_{0}\right\rangle\right\|_{L^{1}(\Omega)}
$$

which equals zero when all are equal to zero and is increasing in each. The term involving the stochastic integral is next.

Let $\mathscr{M}(t)=\int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B\left(X_{k}^{l}\right)^{\tau_{p}} \circ J d W$. Then thanks to Corollary 65.11.1

$$
d[\mathscr{M}]=\left\|\left(Z \circ J^{-1}\right)^{*} B\left(X_{k}^{l}\right)^{\tau_{p}} \circ J\right\|^{2} d s
$$

Applying the Burkholder Davis Gundy inequality, Theorem 63.4.4 for $F(r)=r$ in that stochastic integral,

$$
\begin{array}{r}
2 \int_{\Omega}\left(\sup _{t \in[0, T]}\left|\int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B\left(X_{k}^{l}\right)^{\tau_{p}} \circ J d W\right|\right) d P \\
\leq C \int_{\Omega}\left(\int_{0}^{T}\left\|\left(Z \circ J^{-1}\right)^{*} B\left(X_{k}^{l}\right)^{\tau_{p}} \circ J\right\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, \mathbb{R}\right)}^{2} d s\right)^{1 / 2} d P \tag{73.4.13}
\end{array}
$$

So let $\left\{g_{i}\right\}$ be an orthonormal basis for $Q^{1 / 2} U$ and consider the integrand in the above. It equals

$$
\begin{aligned}
& \sum_{i=1}^{\infty}\left(\left(\left(Z \circ J^{-1}\right)^{*} B\left(X_{k}^{l}\right)^{\tau_{p}}\right)\left(J\left(g_{i}\right)\right)\right)^{2}=\sum_{i=1}^{\infty}\left\langle B\left(X_{k}^{l}\right)^{\tau_{p}}, Z\left(g_{i}\right)\right\rangle^{2} \\
& \quad \leq \sum_{i=1}^{\infty}\left\langle B\left(X_{k}^{l}\right)^{\tau_{p}},\left(X_{k}^{l}\right)^{\tau_{p}}\right\rangle\left\langle B Z\left(g_{i}\right), Z\left(g_{i}\right)\right\rangle \\
& \quad \leq\left(\sup _{t_{m} \in \mathscr{P}_{k}}\left\langle B\left(X_{k}^{l}\right)^{\tau_{p}}\left(t_{m}\right),\left(X_{k}^{l}\right)^{\tau_{p}}\left(t_{m}\right)\right\rangle\right)\|B\|\|Z\|_{\mathscr{L}_{2}}^{2}
\end{aligned}
$$

It follows that the integral in 73.4.13 is dominated by

$$
C \int_{\Omega_{t_{m} \in \mathscr{P}_{k}}} \sup _{k}\left\langle B\left(X_{k}^{l}\right)^{\tau_{p}}\left(t_{m}\right),\left(X_{k}^{l}\right)^{\tau_{p}}\left(t_{m}\right)\right\rangle^{1 / 2}\|B\|^{1 / 2}\left(\int_{0}^{T}\|Z\|_{\mathscr{L}_{2}}^{2} d s\right)^{1 / 2} d P
$$

Now return to 73.4.12. From what was just shown,

$$
\begin{gathered}
E\left(\sup _{t_{m} \in \mathscr{P}_{k}}\left\langle B\left(X_{k}^{l}\right)^{\tau_{p}}\left(t_{m}\right),\left(X_{k}^{l}\right)^{\tau_{p}}\left(t_{m}\right)\right\rangle\right) \\
\leq C+E(|e(k)|)+2 \int_{\Omega}\left(\sup _{t \in[0, T]}\left|\int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B\left(X_{k}^{l}\right)^{\tau_{p}} \circ J d W\right|\right) d P \\
\leq C+C \int_{\Omega_{t_{m} \in \mathscr{P}_{k}}} \sup _{k}\left\langle B\left(X_{k}^{l}\right)^{\tau_{p}}\left(t_{m}\right),\left(X_{k}^{l}\right)^{\tau_{p}}\left(t_{m}\right)\right\rangle^{1 / 2} \\
\|B\|^{1 / 2}\left(\int_{0}^{T}\|Z\|_{\mathscr{L}_{2}}^{2} d s\right)^{1 / 2} d P+E(|e(k)|)
\end{gathered}
$$

$$
\begin{aligned}
\leq & C+\frac{1}{2} E\left(\sup _{t_{m} \in \mathscr{P}_{k}}\left\langle B\left(X_{k}^{l}\right)^{\tau_{p}}\left(t_{m}\right),\left(X_{k}^{l}\right)^{\tau_{p}}\left(t_{m}\right)\right\rangle\right) \\
& +C\|Z\|_{\mathscr{L}^{2}\left([0, T] \times \Omega, \mathscr{L}_{2}\right)}^{2}+E(|e(k)|) .
\end{aligned}
$$

It follows that

$$
\frac{1}{2} E\left(\sup _{t_{m} \in \mathscr{P}_{k}}\left\langle B\left(X_{k}^{l}\right)^{\tau_{p}}\left(t_{m}\right),\left(X_{k}^{l}\right)^{\tau_{p}}\left(t_{m}\right)\right\rangle\right) \leq C+E(|e(k)|)
$$

Now let $p \rightarrow \infty$ and use the monotone convergence theorem to obtain

$$
\begin{equation*}
E\left(\sup _{t_{m} \in \mathscr{P}_{k}}\left\langle B X_{k}^{l}\left(t_{m}\right), X_{k}^{l}\left(t_{m}\right)\right\rangle\right)=E\left(\sup _{t_{m} \in \mathscr{P}_{k}}\left\langle B X\left(t_{m}\right), X\left(t_{m}\right)\right\rangle\right) \leq C+E(|e(k)|) \tag{73.4.14}
\end{equation*}
$$

As mentioned above, this constant $C$ is a continuous function of

$$
\|Y\|_{K^{\prime}},\|X\|_{K},\|Z\|_{J},\left\|\left\langle B X_{0}, X_{0}\right\rangle\right\|_{L^{1}(\Omega, H)}
$$

and equals zero when all of these quantities equal 0 and is increasing with respect to each of the above quantities. Also, for each $\varepsilon>0$,

$$
E\left(\sup _{t_{m} \in \mathscr{P}_{k}}\left\langle B X\left(t_{m}\right), X\left(t_{m}\right)\right\rangle\right) \leq C+\varepsilon
$$

whenever $k$ is large enough.
Let $D$ denote the union of all the $\mathscr{P}_{k}$. Thus $D$ is a dense subset of $[0, T]$ and it has just been shown, since the $\mathscr{P}_{k}$ are nested, that for a constant $C$ dependent only on the above quantities which is independent of $\mathscr{P}_{k}$,

$$
E\left(\sup _{t \in D}\langle B X(t), X(t)\rangle\right) \leq C+\varepsilon
$$

Since $\varepsilon>0$ is arbitrary,

$$
\begin{equation*}
E\left(\sup _{t \in D}\langle B X(t), X(t)\rangle\right) \leq C \tag{73.4.15}
\end{equation*}
$$

Thus, enlarging $N$, for $\omega \notin N$,

$$
\begin{equation*}
\sup _{t \in D}\langle B X(t), X(t)\rangle=C(\omega)<\infty \tag{73.4.16}
\end{equation*}
$$

where $\int_{\Omega} C(\omega) d P<\infty$. By Lemma 69.4.1, there exists a countable set $\left\{e_{i}\right\}$ of vectors in $V$ such that

$$
\left\langle B e_{i}, e_{j}\right\rangle=\delta_{i j}
$$

and for each $x \in W$,

$$
\langle B x, x\rangle=\sum_{i=0}^{\infty}\left\langle B x, e_{i}\right\rangle^{2}, B x=\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i}
$$

Thus for $t$ not in a set of measure zero off which $B X(t)=B(X(t))$,

$$
\langle B X(t), X(t)\rangle=\sum_{i=0}^{\infty}\left\langle B X(t), e_{i}\right\rangle^{2}=\sup _{m} \sum_{k=1}^{m}\left\langle B X(t), e_{i}\right\rangle^{2}
$$

Now from the formula for $B X(t)$, it follows that $B X$ is continuous into $V^{\prime}$. For any $t \notin \hat{N}$ so that $(B X)(t)=B(X(t))$ in $L^{q^{\prime}}\left(\Omega ; V^{\prime}\right)$ and letting $t_{k} \rightarrow t$ where $t_{k} \in D$, Fatou's lemma implies

$$
\begin{gathered}
E(\langle B X(t), X(t)\rangle)=\sum_{i} E\left(\left\langle B X(t), e_{i}\right\rangle^{2}\right)=\sum_{i} \lim _{k \rightarrow \infty} E\left(\left\langle B X\left(t_{k}\right), e_{i}\right\rangle^{2}\right) \\
\quad \leq \quad \lim \inf _{k \rightarrow \infty} \sum_{i} E\left(\left\langle B X\left(t_{k}\right), e_{i}\right\rangle^{2}\right)=\lim \inf _{k \rightarrow \infty} E\left(\left\langle B X\left(t_{k}\right), X\left(t_{k}\right)\right\rangle\right) \\
\quad \leq C\left(\|Y\|_{K^{\prime}},\|X\|_{K},\|Z\|_{J},\left\|\left\langle B X_{0}, X_{0}\right\rangle\right\|_{L^{1}(\Omega)}\right)
\end{gathered}
$$

In addition to this, for arbitrary $t \in[0, T]$, and $t_{k} \rightarrow t$ from $D$,

$$
\sum_{i}\left\langle B X(t), e_{i}\right\rangle^{2} \leq \lim \inf _{k \rightarrow \infty} \sum_{i}\left\langle B X\left(t_{k}\right), e_{i}\right\rangle^{2} \leq \sup _{s \in D}\langle B X(s), X(s)\rangle
$$

Hence

$$
\begin{aligned}
\sup _{t \in[0, T]} \sum_{i}\left\langle B X(t), e_{i}\right\rangle^{2} & \leq \sup _{s \in D}\langle B X(s), X(s)\rangle \\
& =\sup _{s \in D} \sum_{i}\left\langle B X(s), e_{i}\right\rangle^{2} \leq \sup _{t \in[0, T]} \sum_{i}\left\langle B X(t), e_{i}\right\rangle^{2}
\end{aligned}
$$

It follows that $\sup _{t \in[0, T]} \sum_{i}\left\langle B X(t), e_{i}\right\rangle^{2}$ is measurable and

$$
\begin{aligned}
& E\left(\sup _{t \in[0, T]} \sum_{i}\left\langle B X(t), e_{i}\right\rangle^{2}\right) \leq E\left(\sup _{s \in D}\langle B X(s), X(s)\rangle\right) \\
\leq & C\left(\|Y\|_{K^{\prime}},\|X\|_{K},\|Z\|_{J},\left\|\left\langle B X_{0}, X_{0}\right\rangle\right\|_{L^{1}(\Omega)}\right)
\end{aligned}
$$

And so, for $\omega$ off a set of measure zero, $\sup _{t \in[0, T]} \sum_{i}\left\langle B X(t), e_{i}\right\rangle^{2}$ is bounded above.
Also for $t \notin N_{\omega}$ and a given $\omega \notin N$, letting $t_{k} \rightarrow t$ for $t_{k} \in D$,

$$
\begin{aligned}
\langle B X(t), X(t)\rangle & =\sum_{i}\left\langle B X(t), e_{i}\right\rangle^{2} \leq \lim \inf _{k \rightarrow \infty} \sum_{i}\left\langle B X\left(t_{k}\right), e_{i}\right\rangle^{2} \\
& =\lim \inf _{k \rightarrow \infty}\left\langle B X\left(t_{k}\right), X\left(t_{k}\right)\right\rangle \leq \sup _{t \in D}\langle B X(t), X(t)\rangle
\end{aligned}
$$

and so

$$
\sup _{t \notin N_{\omega}}\langle B X(t), X(t)\rangle \leq \sup _{t \in D}\langle B X(t), X(t)\rangle \leq \sup _{t \notin N_{\omega}}\langle B X(t), X(t)\rangle
$$

From 73.4.16,

$$
\sup _{t \notin N_{\omega}}\langle B X(t), X(t)\rangle=C(\omega) \text { a.e. } \omega
$$

where $\int_{\Omega} C(\omega) d P<\infty$. In particular, $\sup _{t \notin N_{\omega}}\langle B X(t), X(t)\rangle$ is bounded for a.e. $\omega$ say for $\omega \notin N$ where $N$ includes the earlier sets of measure zero. This shows that $B X(t)$ is bounded in $W^{\prime}$ for $t \in N_{\omega}^{C}$.

If $v \in V$, then for $\omega \notin N$,

$$
\lim _{t \rightarrow s}\langle B X(t), v\rangle=\langle B X(s), v\rangle, t, s
$$

Therefore, since for such $\omega,\|B X(t)\|_{W^{\prime}}$ is bounded for $t \notin N_{\omega}$, the above holds for all $v \in W$ also. Therefore, for a.e. $\omega, t \rightarrow B X(t, \omega)$ is weakly continuous with values in $W^{\prime}$ for $t \notin N_{\omega}$.

Note also that

$$
\begin{aligned}
\|B X\|_{W^{\prime}}^{2} & \equiv\left(\sup _{\|y\|_{W} \leq 1}\langle B X, y\rangle\right)^{2} \leq \sup _{\|y\| \leq 1}\left(\langle B X, X\rangle^{1 / 2}\langle B y, y\rangle^{1 / 2}\right)^{2} \\
& \leq\langle B X, X\rangle\|B\|
\end{aligned}
$$

and so

$$
\begin{align*}
& \int_{0}^{T} \int_{\Omega}\|B X(t)\|^{2} d P d t \leq \int_{\Omega} \int_{0}^{T}\|B\|\langle B X(t), X(t)\rangle d t d P \\
& \quad \leq C\left(\|Y\|_{K^{\prime}},\|X\|_{K},\|Z\|_{J},\left\|\left\langle B X_{0}, X_{0}\right\rangle\right\|_{L^{1}(\Omega)}\right)\|B\| T \tag{73.4.17}
\end{align*}
$$

Eventually, it is shown that in fact, the function $t \rightarrow B X(t, \omega)$ is continuous with values in $W^{\prime}$. The above shows that $B X \in L^{2}\left([0, T] \times \Omega, W^{\prime}\right)$.

Finally consider the claim of weak continuity of $B X$ into $W^{\prime}$. From the integral equation, $B X$ is continuous into $V^{\prime}$. Also $B X$ is bounded on $N_{\omega}^{C}$. Let $s \in[0, T]$ be arbitrary. I claim that if $t_{n} \rightarrow s, t_{n} \in D$, it follows that $B X\left(t_{n}\right) \rightarrow B X(s)$ weakly in $W^{\prime}$. If not, then there is a subsequence, still denoted as $t_{n}$ such that $B X\left(t_{n}\right) \rightarrow Y$ weakly in $W^{\prime}$ but $Y \neq B X(s)$. However, the continuity into $V^{\prime}$ means that for all $v \in V$,

$$
\langle Y, v\rangle=\lim _{n \rightarrow \infty}\left\langle B X\left(t_{n}\right), v\right\rangle=\langle B X(s), v\rangle
$$

which is a contradiction since $V$ is dense in $W$. This establishes the claim. Also this shows that $B X(s)$ is bounded in $W^{\prime}$.

$$
|\langle B X(s), w\rangle|=\lim _{n \rightarrow \infty}\left|\left\langle B X\left(t_{n}\right), w\right\rangle\right| \leq \lim \inf _{n \rightarrow \infty}\left\|B X\left(t_{n}\right)\right\|_{W^{\prime}}\|w\|_{W} \leq C(\omega)\|w\|_{W}
$$

Now a repeat of the above argument shows that $s \rightarrow B X(s)$ is weakly continuous into $W^{\prime}$.

### 73.5 A Simplification Of The Formula

This lemma also provides a way to simplify one of the formulas derived earlier in the case that $X_{0} \in L^{p}(\Omega, V)$ so that $X-X_{0} \in L^{p}([0, T] \times \Omega, V)$. Refer to 73.4.11. One term there is

$$
\left\langle B\left(X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right), X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right\rangle
$$

Also,

$$
\begin{gathered}
\left\langle B\left(X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right), X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right\rangle \\
\leq \\
2\left\langle B\left(X\left(t_{1}\right)-X_{0}\right), X\left(t_{1}\right)-X_{0}\right\rangle+2\left\langle B M\left(t_{1}\right), M\left(t_{1}\right)\right\rangle
\end{gathered}
$$

It was observed above that $2\left\langle B M\left(t_{1}\right), M\left(t_{1}\right)\right\rangle \rightarrow 0$ a.e. and also in $L^{1}(\Omega)$ as $k \rightarrow \infty$. Apply the above lemma to $\left\langle B\left(X\left(t_{1}\right)-X_{0}\right), X\left(t_{1}\right)-X_{0}\right\rangle$ using $\left[0, t_{1}\right]$ instead of $[0, T]$. The new $X_{0}$ equals 0 . Then from the estimate 73.4.8, it follows that

$$
E\left(\left\langle B\left(X\left(t_{1}\right)-X_{0}\right), X\left(t_{1}\right)-X_{0}\right\rangle\right) \rightarrow 0
$$

as $k \rightarrow \infty$. Taking a subsequence, we could also assume that

$$
\left\langle B\left(X\left(t_{1}\right)-X_{0}\right), X\left(t_{1}\right)-X_{0}\right\rangle \rightarrow 0
$$

a.e. $\omega$ as $k \rightarrow \infty$. Then, using this subsequence, it would follow from 73.4.11,

$$
\begin{align*}
\left\langle B X\left(t_{m}\right), X\left(t_{m}\right)\right\rangle- & \left\langle B X_{0}, X_{0}\right\rangle=e(k)+2 \int_{0}^{t_{m}}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u+ \\
& +2 \int_{0}^{t_{m}}\left(Z \circ J^{-1}\right)^{*} B X_{k}^{l} \circ J d W \\
+ & \sum_{j=0}^{m-1}\left\langle B\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right), M\left(t_{j+1}\right)-M\left(t_{j}\right)\right\rangle \\
- & \sum_{j=1}^{m-1}\left\langle B\left(\Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right), \Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right\rangle \tag{73.5.18}
\end{align*}
$$

where $e(k) \rightarrow 0$ in $L^{1}(\Omega)$ and a.e. $\omega$ and

$$
\Delta X\left(t_{j}\right) \equiv X\left(t_{j+1}\right)-X\left(t_{j}\right)
$$

$\Delta M\left(t_{j}\right)$ being defined similarly. Note how this eliminated the need to consider the term

$$
\left\langle B\left(X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right), X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right\rangle
$$

in passing to a limit. This is a very desirable thing to be able to conclude.
Can you obtain something similar even in case $X_{0}$ is not assumed to be in $L^{p}(\Omega, V)$ ? Let $Z_{0 k} \in L^{p}(\Omega, V) \cap L^{2}(\Omega, W), Z_{0 k} \rightarrow X_{0}$ in $L^{2}(\Omega, W)$. Then from the usual arguments involving the Cauchy Schwarz inequality,

$$
\begin{aligned}
\left\langle B\left(X\left(t_{1}\right)-X_{0}\right), X\left(t_{1}\right)-X_{0}\right\rangle^{1 / 2} \leq & \left\langle B\left(X\left(t_{1}\right)-Z_{0 k}\right), X\left(t_{1}\right)-Z_{0 k}\right\rangle^{1 / 2} \\
& +\left\langle B\left(Z_{0 k}-X_{0}\right), Z_{0 k}-X_{0}\right\rangle^{1 / 2}
\end{aligned}
$$

Also, restoring the superscript to identify the parition,

$$
B\left(X\left(t_{1}^{k}\right)-Z_{0 k}\right)=B\left(X_{0}-Z_{0 k}\right)+\int_{0}^{t_{1}^{k}} Y(s) d s+B \int_{0}^{t_{1}^{k}} Z(s) d W
$$

Of course $\left\|X-Z_{0 k}\right\|_{K}$ is not bounded, but for each $k$ it is finite. There is a sequence of partitions $\mathscr{P}_{k},\left\|\mathscr{P}_{k}\right\| \rightarrow 0$ such that all the above holds. In the definitions of $K, K^{\prime}, J$ replace $[0, T]$ with $[0, t]$ and let the resulting spaces be denoted by $K_{t}, K_{t}^{\prime}, J_{t}$. Let $n_{k}$ denote a subsequence of $\{k\}$ such that

$$
\left\|X-Z_{0 k}\right\|_{\substack{K_{t_{1}} n_{k}}}<1 / k
$$

Then from the above lemma,

$$
\begin{gathered}
E\left(\left\langle B\left(X\left(t_{1}^{n_{k}}\right)-Z_{0 k}\right), X\left(t_{1}^{n_{k}}\right)-Z_{0 k}\right\rangle\right) \\
\leq C\left(\|Y\|_{K_{K_{1}^{\prime}}^{\prime},},\left\|X-Z_{0 k}\right\|_{K_{t_{1} n_{k}}},\|Z\|_{J_{t_{1} n_{k}}},\left\langle B\left(X_{0}-Z_{0 k}\right), X_{0}-Z_{0 k}\right\rangle_{L^{1}(\Omega)}\right) \\
\leq C\left(\|Y\|_{\substack{K_{n_{k}}^{\prime} \\
t_{1}}}, \frac{1}{k},\|Z\|_{J_{t_{1} n_{k}}},\left\langle B\left(X_{0}-Z_{0 k}\right), X_{0}-Z_{0 k}\right\rangle_{L^{1}(\Omega)}\right)
\end{gathered}
$$

Hence

$$
\begin{gathered}
E\left(\left\langle B\left(X\left(t_{1}^{n_{k}}\right)-X_{0}\right), X\left(t_{1}^{n_{k}}\right)-X_{0}\right\rangle\right) \\
\leq 2 E\left(\left\langle B\left(X\left(t_{1}^{n_{k}}\right)-Z_{0 k}\right), X\left(t_{1}^{n_{k}}\right)-Z_{0 k}\right\rangle\right)+2 E\left(\left\langle B\left(Z_{0 k}-X_{0}\right), Z_{0 k}-X_{0}\right\rangle\right) \\
\leq 2 C\left(\|Y\|_{K_{t_{1}^{\prime} n_{k}},}, \frac{1}{k},\|Z\|_{J_{t_{1} n_{k}}},\left\langle B\left(X_{0}-Z_{0 k}\right), X_{0}-Z_{0 k}\right\rangle_{L^{1}(\Omega)}\right) \\
+2\|B\|\left\|Z_{0 k}-X_{0}\right\|_{L^{2}(\Omega, W)}^{2}
\end{gathered}
$$

which converges to 0 as $k \rightarrow \infty$. It follows that there exists a suitable subsequence such that 73.5.18 holds even in the case that $X_{0}$ is only known to be in $L^{2}(\Omega, W)$. From now on, assume this subsequence for the partitions $\mathscr{P}_{k}$. Thus $k$ will really be $n_{k}$ and it suffices to consider the limit as $k \rightarrow \infty$ of the equation of 73.5 .18 . To emphasize this point again, the reason for the above observations is to argue that, even when $X_{0}$ is only in $L^{2}(\Omega, W)$, one can neglect

$$
\left\langle B\left(X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right), X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right\rangle
$$

in passing to the limit as $k \rightarrow \infty$ provided a suitable subsequence is used.

### 73.6 Convergence

The question is whether the above stochastic integral $\int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B X_{n}^{l} \circ J d W$ converges as $n \rightarrow \infty$ in some sense to

$$
\begin{equation*}
\int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B X \circ J d W \tag{73.6.19}
\end{equation*}
$$

and whether the above is also a local martingale. Maybe it is well to pause and consider the integral and and what it means. $Z \circ J^{-1}$ maps $J Q^{1 / 2} U$ to $W$ and so $\left(Z \circ J^{-1}\right)^{*}$ maps $W^{\prime}$ to $\left(J Q^{1 / 2} U\right)^{\prime}$. Thus

$$
\left(Z \circ J^{-1}\right)^{*} B X \in\left(J Q^{1 / 2} U\right)^{\prime}, \text { so }\left(Z \circ J^{-1}\right)^{*} B X \circ J \in Q^{1 / 2}(U)^{\prime}=\mathscr{L}_{2}\left(Q^{1 / 2} U, \mathbb{R}\right)
$$

Thus it has the right values.
Does the stochastic integral just written even make sense? The integrand is Hilbert Schmidt and has values in $\mathbb{R}$ so it seems like we ought to be able to define an integral. The problem is that the integrand is not in $L^{2}\left([0, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, \mathbb{R}\right)\right)$.

By assumption, $t \rightarrow B X(t)$ is continuous into $V^{\prime}$ thanks to the integral equation solved, and also $B X(t)=B(X(t))$ for $t \notin N_{\omega}$ a set of measure zero. For such $t$, it follows from Lemma 69.4.1,

$$
\langle B X(t), X(t)\rangle=\sum_{i}\left\langle B X(t), e_{i}\right\rangle_{V^{\prime}, V}^{2} \text { a.e. } \omega
$$

and so $t \rightarrow \sum_{i}\left\langle B X(t), e_{i}\right\rangle^{2}$ is lower semicontinuous and so it equals $\langle B X(t), X(t)\rangle$ for a.e. $t$, this for each $\omega \notin N$, a single set of measure zero. Also, $t \rightarrow \sum_{i}\left\langle B X(t), e_{i}\right\rangle_{V^{\prime}, V}^{2}$ is progressively measurable and lower semicontinuous in $t$ so by Proposition 62.7.6, one can define a stopping time

$$
\begin{equation*}
\tau_{p} \equiv \inf \left\{t: \sum_{i}\left\langle B X(t), e_{i}\right\rangle_{V^{\prime}, V}^{2}>p\right\}, \tau_{0} \equiv 0 \tag{73.6.20}
\end{equation*}
$$

Instead of referring to this Proposition, you could consider

$$
\tau_{p}^{m} \equiv \inf \left\{t: \sum_{i=1}^{m}\left\langle B X(t), e_{i}\right\rangle_{V^{\prime}, V}^{2}>p\right\}
$$

which is clearly a stopping time because $t \rightarrow \sum_{i=1}^{m}\left\langle B X(t), e_{i}\right\rangle_{V^{\prime}, V}^{2}$ is a continuous process. Then observe that $\tau_{p}=\sup _{m} \tau_{p}^{m}$. Then

$$
\left[\tau_{p} \leq t\right]=\cup_{m}\left[\tau_{p}^{m} \leq t\right] \in \mathscr{F}_{t}
$$

Is it the case that $\tau_{p}=\infty$ for all $p$ large enough? Yes, this follows from Lemma 73.4.2.
Lemma 73.6.1 Suppose $\tau_{p}=\infty$ for all plarge enough off a set of measure zero, then

$$
P\left(\int_{0}^{T}\left|\left(Z \circ J^{-1}\right)^{*} B X \circ J\right|^{2} d t<\infty\right)=1
$$

Also $\int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B X \circ J d W$ can be defined as a local martingale.
Proof: Let

$$
A \equiv\left\{\omega: \int_{0}^{T}\left|\left(Z \circ J^{-1}\right)^{*} B X \circ J\right|^{2} d t=\infty\right\}
$$

Then from the assumption that $\tau_{p}=\infty$ for all $p$ large enough, it follows that

$$
A=\cup_{m=1}^{\infty} A \cap\left(\left[\tau_{m}=\infty\right] \backslash\left[\tau_{m-1}<\infty\right]\right)
$$

Now

$$
\begin{equation*}
P\left(A \cap\left[\tau_{m}=\infty\right]\right) \leq P\left(\omega: \int_{0}^{T} \mathscr{X}_{\left[0, \tau_{m}\right]}\left|\left(Z \circ J^{-1}\right)^{*} B X \circ J\right|^{2} d t=\infty\right) \tag{73.6.21}
\end{equation*}
$$

Consider the integrand. What is the meaning of $\left|\left(Z \circ J^{-1}\right)^{*} B X \circ J\right|^{2}$ ? You have $\left(Z \circ J^{-1}\right)^{*} \in$ $\mathscr{L}_{2}\left(W^{\prime}, J\left(Q^{1 / 2} U\right)^{\prime}\right)$ while $B X \in W^{\prime}$ and so $\left(Z \circ J^{-1}\right)^{*} B X \in \mathscr{L}_{2}\left(J\left(Q^{1 / 2} U\right)^{\prime}, \mathbb{R}\right)$ which is just $\left(J\left(Q^{1 / 2} U\right)\right)^{\prime}$. Thus $\left(Z \circ J^{-1}\right)^{*} B X \circ J$ would be in $\left(Q^{1 / 2} U\right)^{\prime}$ and to get the $\mathscr{L}_{2}$ norm, you would take an orthonormal basis in $Q^{1 / 2} U$ denoted as $\left\{g_{i}\right\}$ and the square of this norm is just

$$
\begin{aligned}
\sum_{i}\left[\left(\left(Z \circ J^{-1}\right)^{*} B X \circ J\right)\left(g_{i}\right)\right]^{2} & \equiv \sum_{i}\left[\left(Z \circ J^{-1}\right)^{*} B X\left(J g_{i}\right)\right]^{2} \\
& \equiv \sum_{i}\left[B X\left(Z \circ J^{-1}\left(J g_{i}\right)\right)\right]^{2} \\
& =\sum_{i}\left[(B X)\left(Z g_{i}\right)\right]^{2} \\
& \leq \sum_{i}\|B X\|^{2}\left\|Z g_{i}\right\|_{W}^{2}
\end{aligned}
$$

Now incorporating the stopping time, you know that for a.e. $t$,

$$
\langle B X, X\rangle(t)=\langle B X(t), X(t)\rangle \leq m
$$

and so $\|B X(t)\|$ can be estimated in terms of $m$ as follows.

$$
\begin{aligned}
|\langle B(X(t)), w\rangle| & \leq\langle B(X(t)), X(t)\rangle^{1 / 2}\|B\|^{1 / 2}\|w\|_{W} \\
& =\left(\sum_{i}\left\langle B X(t), e_{i}\right\rangle_{V^{\prime}, V}^{2}\right)^{1 / 2}\|B\|^{1 / 2}\|w\|_{W} \\
& \leq \sqrt{m}\|B\|^{1 / 2}\|w\|_{W}, \text { so }\|B X(t)\| \leq m\|B\|^{1 / 2}
\end{aligned}
$$

Thus the integrand satisfies for a.e. $t$

$$
\mathscr{X}_{\left[0, \tau_{m}\right]}\left|\left(Z \circ J^{-1}\right)^{*} B X \circ J\right|^{2} \leq m\|B\|\|Z\|_{\mathscr{L}_{2}}^{2}
$$

Hence, from 73.6.21, $P\left(A \cap\left[\tau_{m}=\infty\right]\right)$

$$
\leq P\left(\omega: \int_{0}^{T}\|Z\|_{\mathscr{L}_{2}}^{2} m\|B\| d t=\infty\right)
$$

However,

$$
\int_{\Omega} \int_{0}^{T}\|Z\|_{\mathscr{L}_{2}}^{2} m\|B\| d t d P<\infty
$$

by the assumptions on $Z$. Therefore, $P\left(A \cap\left[\tau_{m}=\infty\right]\right)=0$. It follows that

$$
P(A)=\sum_{m} P\left(A \cap\left(\left[\tau_{m}=\infty\right] \backslash\left[\tau_{m-1}<\infty\right]\right)\right)=\sum_{m} 0=0
$$

It follows that $P\left(\int_{0}^{T}\left|\left(Z \circ J^{-1}\right)^{*} B X \circ J\right|^{2} d t<\infty\right)=1$ and so from Definition 65.10.3, one can define $\int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B X \circ J d W$ as a local martingale.

Convergence will be shown for a subsequence and from now on every sequence will be a subsequence of this one. As part of Lemma 73.4.2, see 73.4.17, it was shown that $B X \in L^{2}\left([0, T] \times \Omega, W^{\prime}\right)$. Therefore, there exist partitions of $[0, T]$ like the above such that

$$
B X_{k}^{r}, B X_{k}^{l} \rightarrow B X \text { in } L^{2}\left([0, T] \times \Omega, W^{\prime}\right)
$$

in addition to the convergence of $X_{k}^{l}, X_{k}^{r}$ to $X$ in $K$. From now on, the argument will involve a subsequence of these.

Lemma 73.6.2 There exists a subsequence still denoted with the subscript $k$ and an enlarged set of measure zero $N$ including the earlier one such that $B X_{k}^{l}(t), B X_{k}^{r}(t)$ also converges pointwise a.e. to $B X(t)$ in $W^{\prime}$ and $X_{k}^{l}(t), X_{k}^{r}(t)$ converge pointwise a.e. in $V$ to $X(t)$ for $\omega \notin N$ as well as having convergence of $X_{k}^{l}(\cdot, \omega)$ to $X(\cdot, \omega)$ in $L^{p}([0, T] ; V)$ and $B X_{k}^{l}(\cdot, \omega)$ to $B X(\cdot, \omega)$ in $L^{2}([0, T] ; W)$.

Proof: To see that such a sequence exists, let $n_{k}$ be such that

$$
\begin{gathered}
\int_{\Omega} \int_{0}^{T}\left\|B X_{n_{k}}^{r}(t)-B X(t)\right\|_{W^{\prime}}^{2} d t d P+\int_{\Omega} \int_{0}^{T}\left\|X_{n_{k}}^{r}(t)-X(t)\right\|_{V}^{p} d t d P+ \\
\int_{\Omega} \int_{0}^{T}\left\|B X_{n_{k}}^{l}(t)-B X(t)\right\|_{W^{\prime}}^{2} d t d P+\int_{\Omega} \int_{0}^{T}\left\|X_{n_{k}}^{l}(t)-X(t)\right\|_{V}^{p} d t d P<4^{-k}
\end{gathered}
$$

Then

$$
\begin{gathered}
P\left(\int_{0}^{T}\left\|B X_{n_{k}}^{l}(t)-B X(t)\right\|_{W^{\prime}}^{2} d t+\int_{0}^{T}\left\|X_{n_{k}}^{r}(t)-X(t)\right\|_{V}^{p} d t+\right. \\
\left.\int_{0}^{T}\left\|B X_{n_{k}}^{l}(t)-B X(t)\right\|_{W^{\prime}}^{2} d t+\int_{0}^{T}\left\|X_{n_{k}}^{l}(t)-X(t)\right\|_{V}^{p} d t>2^{-k}\right) \\
\leq 2^{k}\left(4^{-k}\right)=2^{-k}
\end{gathered}
$$

and so by Borel Cantelli lemma, there is a set of measure zero $N$ such that if $\omega \notin N$,

$$
\begin{gathered}
\int_{0}^{T}\left\|B X_{n_{k}}^{l}(t)-B X(t)\right\|_{W^{\prime}}^{2} d t+\int_{0}^{T}\left\|X_{n_{k}}^{r}(t)-X(t)\right\|_{V}^{p} d t+ \\
\int_{0}^{T}\left\|B X_{n_{k}}^{l}(t)-B X(t)\right\|_{W^{\prime}}^{2} d t+\int_{0}^{T}\left\|X_{n_{k}}^{l}(t)-X(t)\right\|_{V}^{p} d t \leq 2^{-k}
\end{gathered}
$$

for all $k$ large enough. By the usual proof of completeness of $L^{p}$, it follows that $X_{n_{k}}^{l}(t) \rightarrow$ $X(t)$ for a.e. $t$, this for each $\omega \notin N$, a similar assertion holding for $X_{n_{k}}^{r}$. We denote these subsequences as $\left\{X_{k}^{r}\right\}_{k=1}^{\infty},\left\{X_{k}^{l}\right\}_{k=1}^{\infty}$.

Now with this preparation, it is possible to show the desired convergence.

Lemma 73.6.3 In the above context, let $X(s)-X_{k}^{l}(s) \equiv \Delta_{k}(s)$. Then the integral

$$
\int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B X \circ J d W
$$

exists as a local martingale and the following limit is valid for the subsequence of Lemma 73.6.2

$$
\lim _{k \rightarrow \infty} P\left(\left[\sup _{t \in[0, T]}\left|\int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B \Delta_{k} \circ J d W\right| \geq \varepsilon\right]\right)=0
$$

That is,

$$
\sup _{t \in[0, T]}\left|\int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B \Delta_{k} \circ J d W\right|
$$

converges to 0 in probability.
Proof: In the argument $\tau_{m}$ will be defined in 73.6.20. Let

$$
A_{k} \equiv\left\{\omega: \sup _{t \in[0, T]}\left|\int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B \Delta_{k} \circ J d W\right| \geq \varepsilon\right\}
$$

then

$$
A_{k} \cap\left\{\omega: \tau_{m}=\infty\right\} \subseteq\left\{\omega: \sup _{t \in[0, T]}\left|\int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B \Delta_{k}^{\tau_{m}} \circ J d W\right| \geq \varepsilon\right\}
$$

By Burkholder Davis Gundy inequality,

$$
\begin{aligned}
P\left(A_{k} \cap\left\{\omega: \tau_{m}=\infty\right\}\right) & \leq \frac{C}{\varepsilon} \int_{\Omega_{t \in[0, T]}} \sup \left|\int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B \Delta_{k}^{\tau_{m}} \circ J d W\right| d P \\
& \leq \frac{C}{\varepsilon} \int_{\Omega}\left(\int_{0}^{T}\|Z\|_{\mathscr{L}_{2}}^{2}\left\|B \Delta_{k}^{\tau_{m}}\right\|^{2} d t\right)^{1 / 2} d P \\
& \leq \frac{C}{\varepsilon}\left(\int_{\Omega} \int_{0}^{T}\|Z\|_{\mathscr{L}_{2}}^{2}\left\|B \Delta_{k}^{\tau_{m}}\right\|^{2} d t d P\right)^{1 / 2}
\end{aligned}
$$

Recall that if $\langle B x, x\rangle \leq m$, then $\|B x\|_{W^{\prime}} \leq m^{1 / 2}\|B\|^{1 / 2}$. Then the integrand is bounded for a.e. $t$ by $\|Z\|_{\mathscr{L}_{2}}^{2} 4 m\|B\|$. Next use the result of Lemma 73.6 .2 and the dominated convergence theorem to conclude that the above converges to 0 as $k \rightarrow \infty$. Then from the assumption that $\tau_{m}=\infty$ for all $m$ large enough,

$$
P\left(A_{k}\right)=\sum_{m=1}^{\infty} P\left(A_{k} \cap\left(\left[\tau_{m}=\infty\right] \backslash\left[\tau_{m-1}<\infty\right]\right)\right)
$$

Now $\sum_{m} P\left(\left[\tau_{m}=\infty\right] \backslash\left[\tau_{m-1}<\infty\right]\right)=1$ and so, one can apply the dominated convergence theorem to conclude that

$$
\lim _{k \rightarrow \infty} P\left(A_{k}\right)=\sum_{m=1}^{\infty} \lim _{k \rightarrow \infty} P\left(A_{k} \cap\left(\left[\tau_{m}=\infty\right] \backslash\left[\tau_{m-1}<\infty\right]\right)\right)=0
$$

Lemma 73.6.4 Let $X$ be as in Situation 73.2.1 and let $X_{k}^{l}$ be as in Lemma 73.1.1 corresponding to $X$ above. Let $X_{k}^{l}$ and $X_{k}^{r}$ both converge to $X$ in $K$ and also

$$
B X_{k}^{l}, B X_{k}^{r} \rightarrow B X \text { in } L^{2}\left([0, T] \times \Omega, W^{\prime}\right)
$$

Say

$$
\begin{align*}
X_{k}^{l}(t) & =\sum_{j=0}^{m_{k}} X\left(t_{j}\right) \mathscr{X}_{\left[t_{j}, t_{j+1)}\right.}(t),  \tag{73.6.22}\\
B X_{k}^{l}(t) & =\sum_{j=0}^{m_{k}} B X\left(t_{j}\right) \mathscr{X}_{\left[t_{j}, t_{j+1)}\right.}(t) \tag{73.6.23}
\end{align*}
$$

Then the sum in 73.6.23 is progressively measurable into $W^{\prime}$. As mentioned earlier, we can take $X(0) \equiv 0$ in the definition of the "left step function".

Proof: This follows right away from the definition of progressively measurable.
One can take a further subsequence such that uniform convergence of the stochastic integral is obtained.

Lemma 73.6.5 Let $X(s)-X_{k}^{l}(s) \equiv \Delta_{k}(s)$. Then the following limit occurs.

$$
\lim _{k \rightarrow \infty} P\left(\left[\sup _{t \in[0, T]}\left|\int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B \Delta_{k} \circ J d W\right| \geq \varepsilon\right]\right)=0
$$

The stochastic integral

$$
\int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B X \circ J d W
$$

makes sense because $B X$ is $W^{\prime}$ progressively measurable and is in $L^{2}\left([0, T] \times \Omega ; W^{\prime}\right)$. Also, there exists a further subsequence, still denoted as $k$ such that

$$
\int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B X_{k}^{l} \circ J d W \rightarrow \int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B X \circ J d W
$$

uniformly on $[0, T]$ for a.e. $\omega$.
Proof: This follows from Lemma 73.6.3. The last conclusion follows from the usual use of the Borel Cantelli lemma. There exists a further subsequence, still denoted with subscript $k$ such that

$$
P\left(\left[\sup _{t \in[0, T]}\left|\int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B \Delta_{k} \circ J d W\right| \geq \frac{1}{k}\right]\right)<2^{-k}
$$

Then by the Borel Cantelli lemma, one can enlarge the set of measure zero such that for $\omega \notin N$,

$$
\sup _{t \in[0, T]}\left|\int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B \Delta_{k} \circ J d W\right|<\frac{1}{k}
$$

for all $k$ large enough. That is, the claimed uniform convergence holds.
From now on, the sequence will either be this subsequence or a further subsequence.

### 73.7 The Ito Formula

Now at long last, here is the first version of the Ito formula valid on the partition points.
Lemma 73.7.1 In Situation 73.2.1, let $D$ be as above, the union of all the positive mesh points for all the $\mathscr{P}_{k}$. Also assume $X_{0} \in L^{2}(\Omega ; W)$. Then for $\omega \notin N$ the exceptional set of measure zero in $\Omega$ and every $t \in D$,

$$
\begin{align*}
\langle B X(t), X(t)\rangle= & \left\langle B X_{0}, X_{0}\right\rangle+\int_{0}^{t}\left(2\langle Y(s), X(s)\rangle+\langle B Z, Z\rangle_{\mathscr{L}_{2}}\right) d s \\
& +2 \int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B X \circ J d W \tag{73.7.24}
\end{align*}
$$

where, in the above formula,

$$
\langle B Z, Z\rangle_{\mathscr{L}_{2}} \equiv\left(R^{-1} B Z, Z\right)_{\mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)}
$$

for $R$ the Riesz map from $W$ to $W^{\prime}$.
Note first that for $\left\{g_{i}\right\}$ an orthonormal basis for $Q^{1 / 2}(U)$,

$$
\left(R^{-1} B Z, Z\right)_{\mathscr{L}_{2}} \equiv \sum_{i}\left(R^{-1} B Z\left(g_{i}\right), Z\left(g_{i}\right)\right)_{W}=\sum_{i}\left\langle B Z\left(g_{i}\right), Z\left(g_{i}\right)\right\rangle_{W^{\prime} W} \geq 0
$$

Proof: Let $t \in D$. Then $t \in \mathscr{P}_{k}$ for all $k$ large enough. Consider 73.5.18,

$$
\begin{gather*}
\langle B X(t), X(t)\rangle-\left\langle B X_{0}, X_{0}\right\rangle=e(k)+2 \int_{0}^{t}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u \\
+2 \int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B X_{k}^{l} \circ J d W+\sum_{j=0}^{q_{k}-1}\left\langle B\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right), M\left(t_{j+1}\right)-M\left(t_{j}\right)\right\rangle \\
-\sum_{j=1}^{q_{k}-1}\left\langle B\left(\Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right), \Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right\rangle \tag{73.7.25}
\end{gather*}
$$

where $t_{q_{k}}=t, \Delta X\left(t_{j}\right)=X\left(t_{j+1}\right)-X\left(t_{j}\right)$ and $e(k) \rightarrow 0$ in probability. By Lemma 73.6.5 the stochastic integral on the right converges uniformly for $t \in[0, T]$ to

$$
2 \int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B X \circ J d W
$$

for $\omega$ off a set of measure zero. The deterministic integral on the right converges uniformly for $t \in[0, T]$ to

$$
2 \int_{0}^{t}\langle Y(u), X(u)\rangle d u
$$

thanks to Lemma 73.6.2.

$$
\begin{aligned}
\left|\int_{0}^{t}\langle Y(u), X(u)\rangle d u-\int_{0}^{t}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u\right| & \leq \int_{0}^{T}\|Y(u)\|_{V^{\prime}}\left\|X(u)-X_{k}^{r}(u)\right\|_{V} \\
& \leq\|Y\|_{L^{p^{\prime}}([0, T])} 2^{-k}
\end{aligned}
$$

for all $k$ large enough. Consider the fourth term. It equals

$$
\begin{equation*}
\sum_{j=0}^{q_{k}-1}\left(R^{-1} B\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right), M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)_{W} \tag{73.7.26}
\end{equation*}
$$

where $R^{-1}$ is the Riesz map from $W$ to $W^{\prime}$. This equals

$$
\begin{aligned}
& \frac{1}{4}\left(\sum_{j=0}^{q_{k}-1}\left\|R^{-1} B M\left(t_{j+1}\right)+M\left(t_{j+1}\right)-\left(R^{-1} B M\left(t_{j}\right)+M\left(t_{j}\right)\right)\right\|^{2}\right. \\
& \left.-\sum_{j=0}^{q_{k}-1}\left\|R^{-1} B M\left(t_{j+1}\right)-M\left(t_{j+1}\right)-\left(R^{-1} B M\left(t_{j}\right)-M\left(t_{j}\right)\right)\right\|^{2}\right)
\end{aligned}
$$

From Theorem 63.6.4, as $k \rightarrow \infty$, the above converges in probability to ( $t_{q_{k}}=t$ )

$$
\frac{1}{4}\left(\left[R^{-1} B M+M\right](t)-\left[R^{-1} B M-M\right](t)\right)
$$

However, from the description of the quadratic variation of $M$, the above equals

$$
\frac{1}{4}\left(\int_{0}^{t}\left\|R^{-1} B Z+Z\right\|_{\mathscr{L}_{2}}^{2} d s-\int_{0}^{t}\left\|R^{-1} B Z-Z\right\|_{\mathscr{L}_{2}}^{2} d s\right)
$$

which equals

$$
\int_{0}^{t}\left(R^{-1} B Z, Z\right)_{\mathscr{L}_{2}} d s \equiv \int_{0}^{t}\langle B Z, Z\rangle_{\mathscr{L}_{2}} d s
$$

This is what was desired.
Note that in the case of a Gelfand triple, when $W=H=H^{\prime}$, the term $\langle B Z, Z\rangle_{\mathscr{L}_{2}}$ will end up reducing to nothing more than $\|Z\|_{\mathscr{L}_{2}}^{2}$.

Thus all the terms in 73.7.25 converge in probability except for the last term which also must converge in probability because it equals the sum of terms which do. It remains to find what this last term converges to. Thus

$$
\begin{aligned}
& \langle B X(t), X(t)\rangle-\left\langle B X_{0}, X_{0}\right\rangle=2 \int_{0}^{t}\langle Y(u), X(u)\rangle d u \\
& +2 \int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B X \circ J d W+\int_{0}^{t}\langle B Z, Z\rangle_{\mathscr{L}_{2}} d s-a
\end{aligned}
$$

where $a$ is the limit in probability of the term

$$
\begin{equation*}
\sum_{j=1}^{q_{k}-1}\left\langle B\left(\Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right), \Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right\rangle \tag{73.7.27}
\end{equation*}
$$

Let $P_{n}$ be the projection onto span $\left(e_{1}, \cdots, e_{n}\right)$ where $\left\{e_{k}\right\}$ is an orthonormal basis for $W$ with each $e_{k} \in V$. Then using

$$
B X\left(t_{j+1}\right)-B X\left(t_{j}\right)-\left(B M\left(t_{j+1}\right)-B M\left(t_{j}\right)\right)=\int_{t_{j}}^{t_{j+1}} Y(s) d s
$$

the troublesome term of 73.7.27 above is of the form

$$
\begin{aligned}
& \sum_{j=1}^{q_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(s), \Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right\rangle d s \\
& =\sum_{j=1}^{q_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(s), \Delta X\left(t_{j}\right)-P_{n} \Delta M\left(t_{j}\right)\right\rangle d s \\
& \quad+\sum_{j=1}^{q_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(s),-\left(I-P_{n}\right) \Delta M\left(t_{j}\right)\right\rangle d s
\end{aligned}
$$

which equals

$$
\begin{align*}
& \sum_{j=1}^{q_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(s), X\left(t_{j+1}\right)-X\left(t_{j}\right)-P_{n}\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right\rangle d s  \tag{73.7.28}\\
& +\sum_{j=1}^{q_{k}-1}\left\langle B\left(\Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right),-\left(I-P_{n}\right)\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right\rangle \tag{73.7.29}
\end{align*}
$$

The reason for the $P_{n}$ is to get $P_{n}\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)$ in $V$. The sum in 73.7.29 is dominated by

$$
\begin{align*}
& \left(\sum_{j=1}^{q_{k}-1}\left\langle B\left(\Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right),\left(\Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right)\right\rangle\right)^{1 / 2} \\
& \left(\sum_{j=1}^{q_{k}-1}\left|\left\langle B\left(I-P_{n}\right) \Delta M\left(t_{j}\right),\left(I-P_{n}\right) \Delta M\left(t_{j}\right)\right\rangle\right|^{2}\right)^{1 / 2} \tag{73.7.30}
\end{align*}
$$

Now it is known from the above that $\sum_{j=1}^{q_{k}-1}\left\langle B\left(\Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right),\left(\Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right)\right\rangle$ converges in probability to $a \geq 0$. If you take the expectation of the square of the other factor, it is no larger than

$$
\begin{gathered}
\|B\| E\left(\sum_{j=1}^{q_{k}-1}\left\|\left(I-P_{n}\right) \Delta M\left(t_{j}\right)\right\|_{W}^{2}\right) \\
=\|B\| E\left(\sum_{j=1}^{q_{k}-1}\left\|\left(I-P_{n}\right) \int_{t_{j}}^{t_{j+1}} Z(s) d W(s)\right\|_{W}^{2}\right) \\
=\|B\| \sum_{j=1}^{q_{k}-1} E\left(\left\|\int_{t_{j}}^{t_{j+1}}\left(I-P_{n}\right) Z(s) d W(s)\right\|^{2}\right) \\
=\|B\| \sum_{j=1}^{q_{k}-1} E\left(\int_{t_{j}}^{t_{j+1}}\left\|\left(I-P_{n}\right) Z(s)\right\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)}^{2} d s\right) \\
\leq\|B\| E\left(\int_{0}^{T}\left\|\left(I-P_{n}\right) Z(s)\right\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)}^{2} d s\right)
\end{gathered}
$$

Now letting $\left\{g_{i}\right\}$ be an orthonormal basis for $Q^{1 / 2} U$,

$$
\begin{equation*}
=\|B\| \int_{\Omega} \int_{0}^{T} \sum_{i=1}^{\infty}\left\|\left(I-P_{n}\right) Z(s)\left(g_{i}\right)\right\|_{W}^{2} d s d P \tag{73.7.31}
\end{equation*}
$$

The integrand $\sum_{i=1}^{\infty}\left\|\left(I-P_{n}\right) Z(s)\left(g_{i}\right)\right\|_{W}^{2}$ converges to 0 . Also, it is dominated by

$$
\sum_{i=1}^{\infty}\left\|Z(s)\left(g_{i}\right)\right\|_{W}^{2} \equiv\|Z\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)}^{2}
$$

which is given to be in $L^{1}([0, T] \times \Omega)$. Therefore, from the dominated convergence theorem, the expression in 73.7 .31 converges to 0 as $n \rightarrow \infty$.

Thus the expression in 73.7 .30 is of the form $f_{k} g_{n k}$ where $f_{k}$ converges in probability to $a^{1 / 2}$ as $k \rightarrow \infty$ and $g_{n k}$ converges in probability to 0 as $n \rightarrow \infty$ independently of $k$. Now this implies $f_{k} g_{n k}$ converges in probability to 0 . Here is why.

$$
\begin{aligned}
P\left(\left[\left|f_{k} g_{n k}\right|>\varepsilon\right]\right) & \leq P\left(2 \delta\left|f_{k}\right|>\varepsilon\right)+P\left(2 C_{\delta}\left|g_{n k}\right|>\varepsilon\right) \\
& \leq P\left(2 \delta\left|f_{k}-a^{1 / 2}\right|+2 \delta\left|a^{1 / 2}\right|>\varepsilon\right)+P\left(2 C_{\delta}\left|g_{n k}\right|>\varepsilon\right)
\end{aligned}
$$

where $\delta\left|f_{k}\right|+C_{\delta}\left|g_{k n}\right|>\left|f_{k} g_{n k}\right|$ and $\lim _{\delta \rightarrow 0} C_{\delta}=\infty$. Pick $\delta$ small enough that $\varepsilon-2 \delta a^{1 / 2}>$ $\varepsilon / 2$. Then this is dominated by

$$
\leq P\left(2 \delta\left|f_{k}-a^{1 / 2}\right|>\varepsilon / 2\right)+P\left(2 C_{\delta}\left|g_{n k}\right|>\varepsilon\right)
$$

Fix $n$ large enough that the second term is less than $\eta$ for all $k$. Now taking $k$ large enough, the above is less than $\eta$. It follows the expression in 73.7.30 and consequently in 73.7.29 converges to 0 in probability.

Now consider the other term 73.7.28 using the $n$ just determined. This term is of the form

$$
\begin{aligned}
& \sum_{j=1}^{q_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(s), X\left(t_{j+1}\right)-X\left(t_{j}\right)-P_{n}\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right\rangle d s= \\
& \quad \sum_{j=1}^{q_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(s), X_{k}^{r}(s)-X_{k}^{l}(s)-P_{n}\left(M_{k}^{r}(s)-M_{k}^{l}(s)\right)\right\rangle d s \\
& =\int_{t_{1}}^{t}\left\langle Y(s), X_{k}^{r}(s)-X_{k}^{l}(s)-P_{n}\left(M_{k}^{r}(s)-M_{k}^{l}(s)\right)\right\rangle d s
\end{aligned}
$$

where $M_{k}^{r}$ denotes the step function

$$
M_{k}^{r}(t)=\sum_{i=0}^{m_{k}-1} M\left(t_{i+1}\right) \mathscr{X}_{\left(t_{i}, t_{i+1}\right]}(t)
$$

and $M_{k}^{l}$ is defined similarly. The term

$$
\int_{t_{1}}^{t}\left\langle Y(s), P_{n}\left(M_{k}^{r}(s)-M_{k}^{l}(s)\right)\right\rangle d s
$$

converges to 0 for a.e. $\omega$ as $k \rightarrow \infty$ thanks to continuity of $t \rightarrow M(t)$. However, more is needed than this. Define the stopping time

$$
\tau_{p}=\inf \left\{t>0:\|M(t)\|_{W}>p\right\}
$$

Then $\tau_{p}=\infty$ for all $p$ large enough, this for a.e. $\omega$. Let

$$
\begin{gather*}
A_{k}=\left[\left|\int_{t_{1}}^{t}\left\langle Y(s), P_{n}\left(M_{k}^{r}(s)-M_{k}^{l}(s)\right)\right\rangle d s\right|>\varepsilon\right] \\
P\left(A_{k}\right)=\sum_{p=0}^{\infty} P\left(A_{k} \cap\left(\left[\tau_{p}=\infty\right] \backslash\left[\tau_{p-1}<\infty\right]\right)\right) \tag{73.7.32}
\end{gather*}
$$

Now

$$
\begin{aligned}
& P\left(A_{k} \cap\left(\left[\tau_{p}=\infty\right] \backslash\left[\tau_{p-1}<\infty\right]\right)\right) \leq P\left(A_{k} \cap\left(\left[\tau_{p}=\infty\right]\right)\right) \\
\leq & P\left(\left[\left|\int_{t_{1}}^{t}\left\langle Y(s), P_{n}\left(\left(M^{\tau_{p}}\right)_{k}^{r}(s)-\left(M^{\tau_{p}}\right)_{k}^{l}(s)\right)\right\rangle d s\right|>\varepsilon\right]\right)
\end{aligned}
$$

This is so because if $\tau_{p}=\infty$, then it has no effect but also it could happen that the defining inequality may hold even if $\tau_{p}<\infty$ hence the inequality. This is no larger than an expression of the form

$$
\begin{equation*}
\frac{C_{n}}{\varepsilon} \int_{\Omega} \int_{0}^{T}\|Y(s)\|_{V^{\prime}}\left\|\left(M^{\tau_{p}}\right)_{k}^{r}(s)-\left(M^{\tau_{p}}\right)_{k}^{l}(s)\right\|_{W} d s d P \tag{73.7.33}
\end{equation*}
$$

The inside integral converges to 0 by continuity of $M$. Also, thanks to the stopping time, the inside integral is dominated by an expression of the form

$$
\int_{0}^{T}\|Y(s)\|_{V^{\prime}} 2 p d s
$$

and this is a function in $L^{1}(\Omega)$ by assumption on $Y$. It follows that the integral in 73.7.33 converges to 0 as $k \rightarrow \infty$ by the dominated convergence theorem. Hence

$$
\lim _{k \rightarrow \infty} P\left(A_{k} \cap\left(\left[\tau_{p}=\infty\right]\right)\right)=0
$$

Since the sets $\left[\tau_{p}=\infty\right] \backslash\left[\tau_{p-1}<\infty\right]$ are disjoint, the sum of their probabilities is finite. Hence there is a dominating function in 73.7 .32 and so, by the dominated convergence theorem applied to the sum,

$$
\lim _{k \rightarrow \infty} P\left(A_{k}\right)=\sum_{p=0}^{\infty} \lim _{k \rightarrow \infty} P\left(A_{k} \cap\left(\left[\tau_{p}=\infty\right] \backslash\left[\tau_{p-1}<\infty\right]\right)\right)=0
$$

Thus $\int_{t_{1}}^{t}\left\langle Y(s), P_{n}\left(M_{k}^{r}(s)-M_{k}^{l}(s)\right)\right\rangle d s$ converges to 0 in probability as $k \rightarrow \infty$.
Now consider

$$
\begin{aligned}
\left|\int_{t_{1}}^{t}\left\langle Y(s), X_{k}^{r}(s)-X_{k}^{l}(s)\right\rangle d s\right| \leq & \int_{0}^{T}\left|\left\langle Y(s), X_{k}^{r}(s)-X(s)\right\rangle\right| d s \\
& +\int_{0}^{T}\left|\left\langle Y(s), X_{k}^{l}(s)-X(s)\right\rangle\right| d s
\end{aligned}
$$

$$
\leq 2\|Y(\cdot, \omega)\|_{L^{p^{\prime}}(0, T)} 2^{-k}
$$

for all $k$ large enough, this by Lemma 73.6.2. Therefore,

$$
\sum_{j=1}^{q_{k}-1}\left\langle B\left(\Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right), \Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right\rangle
$$

converges to 0 in probability. This establishes the desired formula for $t \in D$.
In fact, the formula 73.7.24 is valid for all $t \in N_{\omega}^{C}$.
Theorem 73.7.2 In Situation 73.2.1, for $\omega$ off a set of measure zero, for every $t \notin N_{\omega}$,

$$
\begin{align*}
\langle B X(t), X(t)\rangle= & \left\langle B X_{0}, X_{0}\right\rangle+\int_{0}^{t}\left(2\langle Y(s), X(s)\rangle+\langle B Z, Z\rangle_{\mathscr{L}_{2}}\right) d s \\
& +2 \int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B X \circ J d W \tag{73.7.34}
\end{align*}
$$

Also, there exists a unique continuous, progressively measurable function $\langle B X, X\rangle$ such that it equals $\langle B X(t), X(t)\rangle$ for a.e. $t$ and $\langle B X, X\rangle(t)$ equals the right side of the above for all $t$. In addition to this,

$$
\begin{gather*}
E(\langle B X, X\rangle(t))= \\
E\left(\left\langle B X_{0}, X_{0}\right\rangle\right)+E\left(\int_{0}^{t}\left(2\langle Y(s), X(s)\rangle+\langle B Z, Z\rangle_{\mathscr{L}_{2}}\right) d s\right) \tag{73.7.35}
\end{gather*}
$$

Also the quadratic variation of the stochastic integral in 73.7.34 is dominated by

$$
\begin{equation*}
C \int_{0}^{t}\|Z\|_{\mathscr{L}_{2}}^{2}\|B X\|_{W^{\prime}}^{2} d s \tag{73.7.36}
\end{equation*}
$$

for a suitable constant $C$. Also $t \rightarrow B X(t)$ is continuous with values in $W^{\prime}$ for $t \in N_{\omega}^{C}$.
Proof: Let $t \in N_{\omega}^{C} \backslash D$. For $t>0$, let $t(k)$ denote the largest point of $\mathscr{P}_{k}$ which is less than $t$. Suppose $t(m)<t(k)$. Hence $m \leq k$. Then

$$
B X(t(m))=B X_{0}+\int_{0}^{t(m)} Y(s) d s+B \int_{0}^{t(m)} Z(s) d W(s)
$$

a similar formula holding for $X(t(k))$. Thus for $t>t(m), t \notin N_{\omega}$,

$$
B(X(t)-X(t(m)))=\int_{t(m)}^{t} Y(s) d s+B \int_{t(m)}^{t} Z(s) d W(s)
$$

which is the same sort of thing studied so far except that it starts at $t(m)$ rather than at 0 and $B X_{0}=0$. Therefore, from Lemma 73.7.1 it follows

$$
\begin{gathered}
\langle B(X(t(k))-X(t(m))), X(t(k))-X(t(m))\rangle \\
=\int_{t(m)}^{t(k)}\left(2\langle Y(s), X(s)-X(t(m))\rangle+\langle B Z, Z\rangle_{\mathscr{L}_{2}}\right) d s
\end{gathered}
$$

$$
\begin{equation*}
+2 \int_{t(m)}^{t(k)}\left(Z \circ J^{-1}\right)^{*} B(X(s)-X(t(m))) \circ J d W \tag{73.7.37}
\end{equation*}
$$

Consider that last term. It equals

$$
\begin{equation*}
2 \int_{t(m)}^{t(k)}\left(Z \circ J^{-1}\right)^{*} B\left(X(s)-X_{m}^{l}(s)\right) \circ J d W \tag{73.7.38}
\end{equation*}
$$

This is dominated by

$$
\begin{aligned}
& 2 \mid \int_{0}^{t(k)}\left(Z \circ J^{-1}\right)^{*} B\left(X(s)-X_{m}^{l}(s)\right) \circ J d W \\
& -\int_{0}^{t(m)}\left(Z \circ J^{-1}\right)^{*} B\left(X(s)-X_{m}^{l}(s)\right) \circ J d W \mid \\
\leq & 4 \sup _{t \in[0, T]}\left|\int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B\left(X(s)-X_{m}^{l}(s)\right) \circ J d W\right|
\end{aligned}
$$

In Lemma 73.6 .5 the above expression was shown to converge to 0 in probability. Therefore, by the usual appeal to the Borel Canteli lemma, there is a subsequence still referred to as $\{m\}$, such that it converges to 0 pointwise in $\omega$ for all $\omega$ off some set of measure 0 as $m \rightarrow \infty$. It follows there is a set of measure 0 including the earlier one such that for $\omega$ not in that set, 73.7 .38 converges to 0 in $\mathbb{R}$. Similar reasoning shows the first term on the right in the non stochastic integral of 73.7 .37 is dominated by an expression of the form

$$
4 \int_{0}^{T}\left|\left\langle Y(s), X(s)-X_{m}^{l}(s)\right\rangle\right| d s
$$

which clearly converges to 0 thanks to Lemma 73.6.2. Finally, it is obvious that

$$
\lim _{m \rightarrow \infty} \int_{t(m)}^{t(k)}\langle B Z, Z\rangle_{\mathscr{L}_{2}} d s=0 \text { for a.e. } \omega
$$

due to the assumptions on $Z$. For $\left\{g_{i}\right\}$ an orthonormal basis of $Q^{1 / 2}(U)$,

$$
\begin{aligned}
\langle B Z, Z\rangle_{\mathscr{L}_{2}} & \equiv \sum_{i}\left(R^{-1} B Z\left(g_{i}\right), Z\left(g_{i}\right)\right)=\sum_{i}\left\langle B Z\left(g_{i}\right), Z\left(g_{i}\right)\right\rangle \\
& \leq\|B\| \sum_{i}\left\|Z\left(g_{i}\right)\right\|_{W}^{2} \in L^{1}(0, T) \text { a.e. }
\end{aligned}
$$

This shows that for $\omega$ off a set of measure 0

$$
\lim _{m, k \rightarrow \infty}\langle B(X(t(k))-X(t(m))), X(t(k))-X(t(m))\rangle=0
$$

Then for $x \in W$,

$$
\begin{aligned}
& |\langle B(X(t(k))-X(t(m))), x\rangle| \\
\leq & \langle B(X(t(k))-X(t(m))), X(t(k))-X(t(m))\rangle^{1 / 2}\langle B x, x\rangle^{1 / 2} \\
\leq & \langle B(X(t(k))-X(t(m))), X(t(k))-X(t(m))\rangle^{1 / 2}\|B\|^{1 / 2}\|x\|_{W}
\end{aligned}
$$

and so

$$
\lim _{m, k \rightarrow \infty}\|B X(t(k))-B X(t(m))\|_{W^{\prime}}=0
$$

Recall $t$ was arbitrary in $N_{\omega}^{C}$ and $\{t(k)\}$ is a sequence converging to $t$. Then the above has shown that $\{B X(t(k))\}_{k=1}^{\infty}$ is a convergent sequence in $W^{\prime}$. Does it converge to $B X(t)$ ? Let $\xi(t) \in W^{\prime}$ be what it converges to. Letting $v \in V$ then, since the integral equation shows that $t \rightarrow B X(t)$ is continuous into $V^{\prime}$,

$$
\langle\xi(t), v\rangle=\lim _{k \rightarrow \infty}\langle B X(t(k)), v\rangle=\langle B X(t), v\rangle,
$$

and now, since $V$ is dense in $W$, this implies $\xi(t)=B X(t)=B(X(t))$. Recall also that it was shown earlier that $B X$ is weakly continuous into $W^{\prime}$ hence the strong convergence of $\{B X(t(k))\}_{k=1}^{\infty}$ in $W^{\prime}$ implies that it converges to $B X(t)$, this for any $t \in N_{\omega}^{C}$.

For every $t \in D$ and for $\omega$ off the exceptional set of measure zero described earlier,

$$
\begin{align*}
\langle B(X(t)), X(t)\rangle= & \left\langle B X_{0}, X_{0}\right\rangle+\int_{0}^{t}\left(2\langle Y(s), X(s)\rangle+\langle B Z, Z\rangle_{\mathscr{L}_{2}} d s\right) d s \\
& +2 \int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B X \circ J d W \tag{73.7.39}
\end{align*}
$$

Does this formula hold for all $t \in[0, T]$ ? Maybe not. However, it will hold for $t \notin N_{\omega}$. So let $t \notin N_{\omega}$.

$$
\begin{gathered}
|\langle B X(t(k)), X(t(k))\rangle-\langle B X(t), X(t)\rangle| \\
\leq|\langle B X(t(k)), X(t(k))\rangle-\langle B X(t), X(t(k))\rangle| \\
+|\langle B X(t), X(t(k))\rangle-\langle B X(t), X(t)\rangle| \\
=|\langle B(X(t(k))-X(t)), X(t(k))\rangle|+|\langle B(X(t(k))-X(t)), X(t)\rangle|
\end{gathered}
$$

Then using the Cauchy Schwarz inequality on each term,

$$
\begin{aligned}
\leq & \langle B(X(t(k))-X(t)), X(t(k))-X(t)\rangle^{1 / 2} \\
& \cdot\left(\langle B X(t(k)), X(t(k))\rangle^{1 / 2}+\langle B X(t), X(t)\rangle^{1 / 2}\right)
\end{aligned}
$$

As before, one can use the lower semicontinuity of

$$
t \rightarrow\langle B(X(t(k))-X(t)), X(t(k))-X(t)\rangle
$$

on $N_{\omega}^{C}$ along with the boundedness of $\langle B X(t), X(t)\rangle$ also shown earlier $\mathbf{o f f} N_{\omega}$ to conclude

$$
\begin{gathered}
|\langle B X(t(k)), X(t(k))\rangle-\langle B X(t), X(t)\rangle| \\
\leq C\langle B(X(t(k))-X(t)), X(t(k))-X(t)\rangle^{1 / 2} \\
\leq C \lim \inf _{m \rightarrow \infty}\langle B(X(t(k))-X(t(m))), X(t(k))-X(t(m))\rangle^{1 / 2}<\varepsilon
\end{gathered}
$$

provided $k$ is sufficiently large. Since $\varepsilon$ is arbitrary,

$$
\lim _{k \rightarrow \infty}\langle B X(t(k)), X(t(k))\rangle=\langle B X(t), X(t)\rangle
$$

It follows that the formula 73.7 .39 is valid for all $t \notin N_{\omega}$. Now define the function $\langle B X, X\rangle(t)$ as

$$
\langle B X, X\rangle(t) \equiv\left\{\begin{array}{c}
\langle B(X(t)), X(t)\rangle, t \notin N_{\omega} \\
\text { The right side of 73.7.39 if } t \in N_{\omega}
\end{array}\right.
$$

Then in short, $\langle B X, X\rangle(t)$ equals the right side of 73.7.39 for all $t \in[0, T]$ and is consequently progressively measurable and continuous. Furthermore, for a.e. $t$, this function equals $\langle B(X(t)), X(t)\rangle$. Since it is known on a dense subset, it must be unique.

This implies that $t \rightarrow B X(t)$ is continuous with values in $W^{\prime}$ for $t \notin N_{\omega}$. Here is why. The fact that the formula 73.7.39 holds for all $t \notin N_{\omega}$ implies that $t \rightarrow\langle B X(t), X(t)\rangle$ is continuous on $N_{\omega}^{C}$. Then for $x \in W$,

$$
\begin{equation*}
|\langle B X(t)-B X(s), x\rangle| \leq\langle B(X(t)-X(s)), X(t)-X(s)\rangle^{1 / 2}\|B\|^{1 / 2}\|x\|_{W} \tag{73.7.40}
\end{equation*}
$$

Also

$$
\begin{aligned}
& \langle B(X(t)-X(s)), X(t)-X(s)\rangle \\
= & \langle B X(t), X(t)\rangle+\langle B X(s), X(s)\rangle-2\langle B X(t), X(s)\rangle
\end{aligned}
$$

By weak continuity of $t \rightarrow B X(t)$ shown earlier,

$$
\lim _{t \rightarrow s}\langle B X(t), X(s)\rangle=\langle B X(s), X(s)\rangle
$$

Therefore,

$$
\lim _{t \rightarrow s}\langle B(X(t)-X(s)), X(t)-X(s)\rangle=0
$$

and so the inequality 73.7.40 implies the continuity of $t \rightarrow B X(t)$ into $W^{\prime}$ for $t \notin N_{\omega}$. Note that by assumption this function is continuous into $V^{\prime}$ for all $t$. It was also shown that it is weakly continuous into $W^{\prime}$ on $[0, T]$ and hence it is bounded in $W^{\prime}$.

Now consider the claim about the expectation. Since the stochastic integral

$$
2 \int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B X \circ J d W
$$

is only a local martingale, it is necessary to employ a stopping time. We use the function $\langle B X, X\rangle$ to define this stopping time as

$$
\tau_{p} \equiv \inf \{t>0:\langle B X, X\rangle(t)>p\}
$$

This is the first hitting time of a continuous process and so it is a valid stopping time. Using this, leads to

$$
\langle B X, X\rangle^{\tau_{p}}(t)=\left\langle B X_{0}, X_{0}\right\rangle+\int_{0}^{t} \mathscr{X}_{\left[0, \tau_{p}\right]}(s)\left(2\langle Y(s), X(s)\rangle+\langle B Z, Z\rangle_{\mathscr{L}_{2}} d s\right) d s
$$

$$
\begin{equation*}
+2 \int_{0}^{t} \mathscr{X}_{\left[0, \tau_{p}\right]}(s)\left(Z \circ J^{-1}\right)^{*} B X^{\tau_{p}} \circ J d W \tag{73.7.41}
\end{equation*}
$$

By continuity of $\langle B X, X\rangle, \tau_{p}=\infty$ for all $p$ large enough. Take expectation of both sides of the above. In the integrand of the last term, $B X$ refers to the function $B X(t, \omega) \equiv$ $B(X(t, \omega))$ and so it is progressively measurable because $X$ is assumed to be so. Hence $B X^{\tau_{p}}$ is also progressively measurable and for a.e. Also, for a.e. $s,\left\|B X\left(s \wedge \tau_{p}\right)\right\|_{W^{\prime}} \leq$ $\sqrt{p} \sqrt{\|B\|}$. Therefore, one can take expectations and get

$$
\begin{gathered}
E\left(\langle B X, X\rangle^{\tau_{p}}(t)\right)=E\left(\left\langle B X_{0}, X_{0}\right\rangle\right) \\
+E\left(\int_{0}^{t} \mathscr{X}_{\left[0, \tau_{p}\right]}(s)\left(2\langle Y(s), X(s)\rangle+\langle B Z, Z\rangle_{\mathscr{L}_{2}} d s\right) d s\right)
\end{gathered}
$$

Now let $p \rightarrow \infty$ and use the monotone convergence theorem on the left and the dominated convergence theorem on the right to obtain the desired result 73.7.35. The claim about the quadratic variation follows from Corollary 65.11.1.

## Chapter 74

## A More Attractive Version

The following lemma is convenient.
Lemma 74.0.1 Let $f_{n} \rightarrow f$ in $L^{p}([0, T] \times \Omega, E)$. Then there exists a subsequence $n_{k}$ and a set of measure zero $N$ such that if $\omega \notin N$, then

$$
f_{n_{k}}(\cdot, \omega) \rightarrow f(\cdot, \omega)
$$

in $L^{p}([0, T], E)$ and for a.e. t.
Proof: We have

$$
\begin{aligned}
P\left(\left[\left\|f_{n}-f\right\|_{L^{p}([0, T], E)}>\lambda\right]\right) & \leq \frac{1}{\lambda} \int_{\Omega}\left\|f_{n}-f\right\|_{L^{p}([0, T], E)} d P \\
& \leq \frac{1}{\lambda}\left\|f_{n}-f\right\|_{L^{p}([0, T] \times \Omega, E)}
\end{aligned}
$$

Hence there exists a subsequence $n_{k}$ such that

$$
P\left(\left[\left\|f_{n_{k}}-f\right\|_{L^{p}([0, T], E)}>2^{-k}\right]\right) \leq 2^{-k}
$$

Then by the Borel Cantelli lemma, it follows that there exists a set of measure zero $N$ such that for all $k$ large enough and $\omega \notin N$,

$$
\left\|f_{n_{k}}-f\right\|_{L^{p}([0, T], E)} \leq 2^{-k}
$$

Now by the usual arguments used in proving completeness, $f_{n_{k}}(t) \rightarrow f(t)$ for a.e.t.
Also, we have the approximation lemma proved earlier, Lemma 65.3.1.
Lemma 74.0.2 Let $\Phi:[0, T] \times \Omega \rightarrow V$, be $\mathscr{B}([0, T]) \times \mathscr{F}$ measurable and suppose

$$
\Phi \in K \equiv L^{p}([0, T] \times \Omega ; E), p \geq 1
$$

Then there exists a sequence of nested partitions, $\mathscr{P}_{k} \subseteq \mathscr{P}_{k+1}$,

$$
\mathscr{P}_{k} \equiv\left\{t_{0}^{k}, \cdots, t_{m_{k}}^{k}\right\}
$$

such that the step functions given by

$$
\begin{aligned}
\Phi_{k}^{r}(t) & \equiv \sum_{j=1}^{m_{k}} \Phi\left(t_{j}^{k}\right) \mathscr{X}_{\left(t_{j-1}^{k}, t_{j}^{k}\right]}(t) \\
\Phi_{k}^{l}(t) & \equiv \sum_{j=1}^{m_{k}} \Phi\left(t_{j-1}^{k}\right) \mathscr{X}_{\left[t_{j-1}^{k}, t_{j}^{k}\right)}(t)
\end{aligned}
$$

both converge to $\Phi$ in $K$ as $k \rightarrow \infty$ and

$$
\lim _{k \rightarrow \infty} \max \left\{\left|t_{j}^{k}-t_{j+1}^{k}\right|: j \in\left\{0, \cdots, m_{k}\right\}\right\}=0
$$

Also, each $\Phi\left(t_{j}^{k}\right), \Phi\left(t_{j-1}^{k}\right)$ is in $L^{p}(\Omega ; E)$. One can also assume that $\Phi(0)=0$. The mesh points $\left\{t_{j}^{k}\right\}_{j=0}^{m_{k}}$ can be chosen to miss a given set of measure zero. In addition to this, we can assume that

$$
\left|t_{j}^{k}-t_{j-1}^{k}\right|=2^{-n_{k}}
$$

except for the case where $j=1$ or $j=m_{n_{k}}$ when this might not be so. In the case of the last subinterval defined by the partition, we can assume

$$
\left|t_{m}^{k}-t_{m-1}^{k}\right|=\left|T-t_{m-1}^{k}\right| \geq 2^{-\left(n_{k}+1\right)}
$$

### 74.1 The Situation

Now consider the following situation. There are real separable Banach spaces $V, W$ such that $W$ is a Hilbert space and

$$
V \subseteq W, \quad W^{\prime} \subseteq V^{\prime}
$$

where $V$ is dense in $W$. Also let $B \in \mathscr{L}\left(W, W^{\prime}\right)$ satisfy

$$
\langle B w, w\rangle \geq 0,\langle B u, v\rangle=\langle B v, u\rangle
$$

Note that $B$ does not need to be one to one. Also allowed is the case where $B$ is the Riesz map. It could also happen that $V=W$. Assume that $B=B(\omega)$ where $B$ is $\mathscr{F}_{0}$ measurable into $\mathscr{L}\left(W, W^{\prime}\right)$. This dependence on $\omega$ will be suppressed in the interest of simpler notation. For convenience, assume $\|B(\omega)\|$ is bounded. This is assumed mainly so that an estimate can be made on $\left\langle B X_{0}, X_{0}\right\rangle$ for $X_{0}$ given in $L^{2}(\Omega)$. It probably suffices to simply give an estimate on $\left\|\left\langle B X_{0}, X_{0}\right\rangle\right\|_{L^{1}(\Omega)}$ along with something else on the Ito integral. However, it seems at this time like this is more trouble than it is worth.

Situation 74.1.1 Let $X$ have values in $V$ and satisfy the following

$$
\begin{equation*}
B X(t)=B X_{0}+\int_{0}^{t} Y(s) d s+B M(t) \tag{74.1.1}
\end{equation*}
$$

$X_{0} \in L^{2}(\Omega ; W)$ and is $\mathscr{F}_{0}$ measurable. Here $M(t)$ is a continuous $L^{2}$ martingale having values in $W$. By this is meant that $\lim _{t \rightarrow 0+}\|M(t)\|_{L^{2}(\Omega)}=0$ and for each $\omega, \lim _{t \rightarrow 0+} M(t)=$ $0,\|M\|_{W}^{2} \in L^{2}([0, T] \times \Omega)$. Assume that $d[M]=k d m$ for $k \in L^{1}([0, T] \times \Omega)$, that is, the measure determined by the quadratic variation for the martingale is absolutely continuous with respect to Lebesgue measure as just described.

Assume Y satisfies

$$
Y \in K^{\prime} \equiv L^{p^{\prime}}\left([0, T] \times \Omega ; V^{\prime}\right)
$$

the $\sigma$ algebra of measurable sets defining $K^{\prime}$ will be the progressively measurable sets. Here $1 / p^{\prime}+1 / p=1, p>1$.

Also the sense in which the equation holds is as follows. For a.e. $\omega$, the equation holds in $V^{\prime}$ for all $t \in[0, T]$. Thus we are considering a particular representative $X$ for which this happens. Also it is only assumed that $B X(t)=B(X(t))$ for a.e. $t$. Thus $B X$ is the name
of a function having values in $V^{\prime}$ for which $B X(t)=B(X(t))$ for a.e. $t$. Assume that $X$ is progressively measurable also and $X \in L^{p}([0, T] \times \Omega, V)$.

The goal is to prove the following Ito formula valid for a.e. $t$ for each $\omega$ off a set of measure zero.

$$
\begin{gather*}
\langle B X(t), X(t)\rangle=\left\langle B X_{0}, X_{0}\right\rangle+\int_{0}^{t}(2\langle Y(s), X(s)\rangle) d s \\
+\left[R^{-1} B M, M\right](t)+2 \int_{0}^{t}\langle B X, d M\rangle \tag{74.1.2}
\end{gather*}
$$

where $R$ is the Riesz map from $W$ to $W^{\prime}$. The most significant feature of the last term is that it is a local martingale. The third term on the right is the covariation of the two martingales $R^{-1} B M$ and $M$. It will follow from the argument that this will be nonnegative.

Note that the assumptions on $M$ imply that $[M] \in L^{1}([0, T] \times \Omega)$.

### 74.2 Preliminary Results

Here are discussed some preliminary results which will be needed. From the integral equation, if $\phi \in L^{q}(\Omega ; V)$ and $\psi \in C_{c}^{\infty}(0, T)$ for $q=\max (p, 2)$,

$$
\begin{aligned}
& \int_{\Omega} \int_{0}^{T}\left((B X)(t)-B M(t)-B X_{0}\right) \psi^{\prime} \phi d t d P \\
= & \int_{\Omega} \int_{0}^{T} \int_{0}^{t} Y(s) \psi^{\prime}(t) d s \phi d t d P
\end{aligned}
$$

Then the term on the right equals

$$
\int_{\Omega} \int_{0}^{T} \int_{s}^{T} Y(s) \psi^{\prime}(t) d t d s \phi(\omega) d P=\int_{\Omega}\left(-\int_{0}^{T} Y(s) \psi(s) d s\right) \phi(\omega) d P
$$

It follows that, since $\phi$ is arbitrary,

$$
\int_{0}^{T}\left((B X)(t)-B M(t)-B X_{0}\right) \psi^{\prime}(t) d t=-\int_{0}^{T} Y(s) \psi(s) d s
$$

in $L^{q^{\prime}}\left(\Omega ; V^{\prime}\right)$ and so the weak time derivative of

$$
t \rightarrow(B X)(t)-B M(t)-B X_{0}
$$

equals $Y$ in $L^{q^{\prime}}\left([0, T] ; L^{q^{\prime}}\left(\Omega, V^{\prime}\right)\right)$.Thus, by Theorem 34.2.9, for a.e. $t$,

$$
B(X(t)-M(t))=B X_{0}+\int_{0}^{t} Y(s) d s \text { in } L^{q^{\prime}}\left(\Omega, V^{\prime}\right)
$$

That is,

$$
(B X)(t)=B X_{0}+\int_{0}^{t} Y(s) d s+B M(t), t \notin \hat{N}, m(\hat{N})=0
$$

holds in $L^{q^{\prime}}\left(\Omega, V^{\prime}\right)$ where $(B X)(t)=B(X(t))$ a.e. $t$ in this space, for all $t \notin \hat{N}$, a set of Lebesgue measure zero, in addition to holding for all $t$ for each $\omega$. Now let $\left\{t_{k}^{n}\right\}_{k=1 n=1}^{m_{n} \infty}$ be partitions for which, from Lemma 74.0.2 there are left and right step functions $X_{k}^{l}, X_{k}^{r}$, which converge in $L^{p}([0, T] \times \Omega ; V)$ to $X$ and such that each $\left\{t_{k}^{n}\right\}_{k=1}^{m_{n}}$ has empty intersection with the set of measure zero $\hat{N}$ where, in $L^{q^{\prime}}\left(\Omega ; V^{\prime}\right),(B X)(t) \neq B(X(t))$ in $L^{q^{\prime}}\left(\Omega ; V^{\prime}\right)$. Thus for $t_{k}$ a generic partition point,

$$
B X\left(t_{k}\right)=B\left(X\left(t_{k}\right)\right) \text { in } L^{q^{\prime}}\left(\Omega ; V^{\prime}\right)
$$

Hence there is an exceptional set of measure zero, $N\left(t_{k}\right) \subseteq \Omega$ such that for

$$
\omega \notin N\left(t_{k}\right), B X\left(t_{k}\right)(\omega)=B\left(X\left(t_{k}, \omega\right)\right) .
$$

Define an exceptional set $N \subseteq \Omega$ to be the union of all these $N\left(t_{k}\right)$. There are countably many and so $N$ is also a set of measure zero. Then for $\omega \notin N$, and $t_{k}$ any mesh point at all, $B X\left(t_{k}\right)(\omega)=B\left(X\left(t_{k}, \omega\right)\right)$. This will be important in what follows. In addition to this, from the integral equation, for each of these $\omega \notin N, B X(t)(\omega)=B(X(t, \omega))$ for all $t \notin N_{\omega} \subseteq[0, T]$ where $N_{\omega}$ is a set of Lebesgue measure zero. Thus the $t_{k}$ from the various partitions are always in $N_{\omega}$. By Lemma 69.4.1, there exists a countable set $\left\{e_{i}\right\}$ of vectors in $V$ such that

$$
\left\langle B e_{i}, e_{j}\right\rangle=\delta_{i j}
$$

and for each $x \in W$,

$$
\langle B x, x\rangle=\sum_{i=0}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}, B x=\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i}
$$

By this lemma, if $B=B(\omega)$ where $B$ is $\mathscr{F}_{0}$ measurable into $\mathscr{L}\left(W, W^{\prime}\right)$, then the $e_{i}$ are also $\mathscr{F}_{0}$ measurable into $V$. Thus the conclusion of the above discussion is that at the mesh points, it is valid to write

$$
\begin{aligned}
\left\langle(B X)\left(t_{k}\right), X\left(t_{k}\right)\right\rangle & =\left\langle B\left(X\left(t_{k}\right)\right), X\left(t_{k}\right)\right\rangle \\
& =\sum_{i}\left\langle(B X)\left(t_{k}\right), e_{i}\right\rangle^{2}=\sum_{i}\left\langle B\left(X\left(t_{k}\right)\right), e_{i}\right\rangle^{2}
\end{aligned}
$$

just as would be the case if $(B X)(t)=B(X(t))$ for every $t$. In all which follows, the mesh points will be like this and an appropriate set of measure zero which may be replaced with a larger set of measure zero finitely many times is being neglected. Obviously, one can take a subsequence of the sequence of partitions described above without disturbing the above observations. We will denote these partitions as $\mathscr{P}_{k}$. Thus we obtain the following interesting lemma.

Lemma 74.2.1 In the above situation, there exists a set of measure zero $N \subseteq \Omega$ and $a$ dense subset of $[0, T], D$ such that for $\omega \notin N, B X(t, \omega)=B(X(t, \omega))$ for all $t \in D$. This set $D$ is the union of nested paritions $\left\{\mathscr{P}_{k}\right\}=\left\{t_{j}^{k}\right\}_{j=1, k=1}^{m_{k} \infty}$ such that the left and right step functions $\left\{X_{k}^{l}\right\},\left\{X_{k}^{r}\right\}$ converge to $X$ in $L^{p}([0, T] \times \Omega ; V)$. There is also a set of Lebesgue
measure zero $\hat{N} \subseteq[0, T]$ such that $B X(t)=B(X(t))$ in $L^{q^{\prime}}\left(\Omega ; V^{\prime}\right)$ for all $t \notin \hat{N}$. Thus for such $t, B X(t)(\omega)=B(X(t, \omega))$ for a.e. $\omega$. In particular, for such $t \notin \hat{N}$,

$$
\langle B X(t)(\omega), X(t, \omega)\rangle=\sum_{i}\left\langle B(X(t)), e_{i}\right\rangle^{2} \text { a.e. } \omega
$$

$D$ has empty intersection with $\hat{N}$. There is also a set of Lebesgue measure zero $N_{\omega}$ for each $\omega \notin N$ defined by $B X(t, \omega)=B(X(t, \omega))$ for all $t \notin N_{\omega}$.

Now define a stopping time.

$$
\begin{equation*}
\sigma_{q}^{n} \equiv \inf \left\{t:\left\langle B X_{n}^{l}(t), X_{n}^{l}(t)\right\rangle>q\right\} \tag{74.2.3}
\end{equation*}
$$

Thus this pertains to the $n^{t h}$ partition. Since $X_{n}^{l}$ is right continuous, this will be a well defined stopping time. Thus, for $t$ one of the partition points,

$$
\begin{equation*}
\left\langle B X^{\sigma_{q}^{n}}(t, \omega), X^{\sigma_{q}^{n}}(t, \omega)\right\rangle \leq q \tag{74.2.4}
\end{equation*}
$$

From the definition of $X_{n}^{l}$ and the observation that these partitions are nested,

$$
\lim _{n \rightarrow \infty} \sigma_{q}^{n} \equiv \sigma_{q}
$$

exists because this is a decreasing sequence. There are more available times to consider as $n$ gets larger and so when the inf is taken, it can only get smaller. Thus

$$
\left[\sigma_{q} \leq t\right]=\cap_{m=1}^{\infty} \cup_{k=1}^{\infty} \cap_{n \geq k}\left[\sigma_{q}^{n} \leq t+\frac{1}{m}\right] \in \cap_{m=1}^{\infty} \mathscr{F}_{t+(1 / m)}=\mathscr{F}_{t}
$$

since it is assumed that the filtration is normal. Thus this appears to be a stopping time. However, I don't know how to use this.

Theorem 74.2.2 Let $\left\{t_{j}^{n}\right\}_{j=0}^{m_{n}}$ be the above sequence of partitions of the sort in Lemma 74.0.2 such that if

$$
X_{n}^{l}(t) \equiv \sum_{j=0}^{m_{n}-1} X\left(t_{j}^{n}\right) \mathscr{X}_{\left[t_{j}^{n}, t_{j+1}^{n}\right)}(t)
$$

then $X_{n} \rightarrow X$ in $L^{p}([0, T] \times \Omega, V)$ with the other conditions holding which were discussed above. In particular, $B X(t)=B(X(t))$ for $t$ one of these mesh points. Then the expression

$$
\begin{align*}
& \sum_{j=0}^{m_{n}-1}\left\langle B\left(M\left(t_{j+1}^{n} \wedge t\right)-M\left(t_{j}^{n} \wedge t\right)\right), X\left(t_{j}^{n}\right)\right\rangle \\
= & \sum_{j=0}^{m_{n}-1}\left\langle B X\left(t_{j}^{n}\right),\left(M\left(t_{j+1}^{n} \wedge t\right)-M\left(t_{j}^{n} \wedge t\right)\right)\right\rangle \tag{74.2.5}
\end{align*}
$$

is a local martingale

$$
\int_{0}^{t}\left\langle B X_{k}^{l}, d M\right\rangle
$$

with $\left\{\sigma_{q}^{n}\right\}_{q=1}^{\infty}$ being a localizing sequence.

Proof: This follows from Lemma 66.0.20. This can be seen because, thanks to the fact that $B X_{k}^{l \sigma_{q}^{k}}$ is bounded, the function $B X_{k}^{l}$ is in the set $\mathscr{G}$ described there. This is a place where we use that $d[M]=k d t$.

### 74.3 The Main Estimate

The argument will be based on a formula which follows in the next lemma.
Lemma 74.3.1 In Situation 74.1.1 the following formula holds for a.e. $\omega$ for $0<s<t$. In the following, $\langle\cdot, \cdot\rangle$ denotes the duality pairing between $V, V^{\prime}$.

$$
\begin{gather*}
\langle B X(t), X(t)\rangle=\langle B X(s), X(s)\rangle+ \\
+2 \int_{s}^{t}\langle Y(u), X(t)\rangle d u+\langle B(M(t)-M(s)), M(t)-M(s)\rangle \\
-\langle B X(t)-B X(s)-(M(t)-M(s)), X(t)-X(s)-(M(t)-M(s))\rangle \\
+2\langle B X(s), M(t)-M(s)\rangle \tag{74.3.6}
\end{gather*}
$$

Also for $t>0$

$$
\begin{align*}
& \langle B X(t), X(t)\rangle=\left\langle B X_{0}, X_{0}\right\rangle+2 \int_{0}^{t}\langle Y(u), X(t)\rangle d u+2\left\langle B X_{0}, M(t)\right\rangle+ \\
& \langle B M(t), M(t)\rangle-\left\langle B X(t)-B X_{0}-B M(t), X(t)-X_{0}-M(t)\right\rangle \tag{74.3.7}
\end{align*}
$$

Proof: From the formula which is assumed to hold,

$$
\begin{aligned}
& B X(t)=B X_{0}+\int_{0}^{t} Y(u) d u+B M(t) \\
& B X(s)=B X_{0}+\int_{0}^{s} Y(u) d u+B M(s)
\end{aligned}
$$

Then

$$
B M(t)-B M(s)+\int_{s}^{t} Y(u) d u=B X(t)-B X(s)
$$

It follows that

$$
\begin{gathered}
\langle B(M(t)-M(s)), M(t)-M(s)\rangle- \\
\langle B X(t)-B X(s)-(M(t)-M(s)), X(t)-X(s)-(M(t)-M(s))\rangle \\
+2\langle B X(s), M(t)-M(s)\rangle \\
=\quad\langle B(M(t)-M(s)), M(t)-M(s)\rangle-\langle B X(t)-B X(s), X(t)-X(s)\rangle \\
+2\langle B X(t)-B X(s), M(t)-M(s)\rangle \\
-\langle B(M(t)-M(s)), M(t)-M(s)\rangle+2\langle B X(s), M(t)-M(s)\rangle
\end{gathered}
$$

Some terms cancel and this yields

$$
\begin{aligned}
=-\langle B X(t)- & B X(s), X(t)-X(s)\rangle+2\langle B X(t), M(t)-M(s)\rangle \\
=-\langle B X(t)- & B X(s), X(t)-X(s)\rangle+2\langle B(M(t)-M(s)), X(t)\rangle \\
= & -\langle B(X(t)-X(s)), X(t)-X(s)\rangle \\
& +2\left\langle B X(t)-B X(s)-\int_{s}^{t} Y(u) d u, X(t)\right\rangle \\
= & -\langle B X(t), X(t)\rangle-\langle B X(s), X(s)\rangle \\
& +2\langle B X(t), X(s)\rangle+2\langle B X(t), X(t)\rangle \\
& -2\langle B X(s), X(t)\rangle-2 \int_{s}^{t}\langle Y(u), X(t)\rangle d u \\
= & \langle B X(t), X(t)\rangle-\langle B X(s), X(s)\rangle-2 \int_{s}^{t}\langle Y(u), X(t)\rangle d u
\end{aligned}
$$

Therefore,

$$
\begin{gathered}
\langle B X(t), X(t)\rangle-\langle B X(s), X(s)\rangle \\
=2 \int_{s}^{t}\langle Y(u), X(t)\rangle d u+\langle B(M(t)-M(s)), M(t)-M(s)\rangle \\
-\langle B X(t)-B X(s)-(M(t)-M(s)), X(t)-X(s)-(M(t)-M(s))\rangle \\
+2\langle B X(s), M(t)-M(s)\rangle
\end{gathered}
$$

The following phenomenal estimate holds and it is this estimate which is the main idea in proving the Ito formula. The last assertion about continuity is like the well known result that if $y \in L^{p}(0, T ; V)$ and $y^{\prime} \in L^{p^{\prime}}\left(0, T ; V^{\prime}\right)$, then $y$ is actually continuous a.e. with values in $H$, for $V, H, V^{\prime}$ a Gelfand triple. Later, this continuity result is strengthened further to give strong continuity.
Lemma 74.3.2 In the Situation 74.1.1, the following holds for all $t \notin \hat{N}$,

$$
\begin{align*}
& E(\langle B X(t), X(t)\rangle) \\
< & C\left(\|Y\|_{K^{\prime}},\|X\|_{K}, E([M](T)),\left\|\left\langle B X_{0}, X_{0}\right\rangle\right\|_{L^{1}(\Omega)}\right)<\infty . \tag{74.3.8}
\end{align*}
$$

where $K, K^{\prime}$ were defined earlier. In fact,

$$
E\left(\sup _{t \in[0, T]} \sum_{i}\left\langle B X(t), e_{i}\right\rangle^{2}\right) \leq C\left(\|Y\|_{K^{\prime}},\|X\|_{K}, E([M](T)),\left\|\left\langle B X_{0}, X_{0}\right\rangle\right\|_{L^{1}(\Omega)}\right)
$$

Also, $C$ is a continuous function of its arguments, increasing in each one, and $C(0,0,0,0)=$ 0 . Thus for a.e. $\omega$,

$$
\sup _{t \notin N_{\omega}^{C}}\langle B X(t, \omega), X(t, \omega)\rangle \leq C(\omega)<\infty
$$

Also for $\omega$ off a set of measure zero described earlier, $t \rightarrow B X(t)(\omega)$ is weakly continuous with values in $W^{\prime}$ on $[0, T]$. Also $t \rightarrow\langle B X(t), X(t)\rangle$ is lower semicontinuous on $N_{\omega}^{C}$.

Proof: Consider the formula in Lemma 74.3.1.

$$
\begin{align*}
& \qquad\langle B X(t), X(t)\rangle=\langle B X(s), X(s)\rangle \\
& +2 \int_{s}^{t}\langle Y(u), X(t)\rangle d u+\langle B(M(t)-M(s)), M(t)-M(s)\rangle \\
& -\langle B(X(t)-X(s)-(M(t)-M(s))), X(t)-X(s)-(M(t)-M(s))\rangle \\
& +2\langle B X(s), M(t)-M(s)\rangle \tag{74.3.9}
\end{align*}
$$

Now let $t_{j}$ denote a point of $\mathscr{P}_{k}$ from Lemma 74.0.2. Then for $t_{j}>0, X\left(t_{j}\right)$ is just the value of $X$ at $t_{j}$ but when $t=0$, the definition of $X(0)$ in this step function is $X(0) \equiv 0$. Thus

$$
\begin{aligned}
& \sum_{j=1}^{m-1}\left\langle B X\left(t_{j+1}\right), X\left(t_{j+1}\right)\right\rangle-\left\langle B X\left(t_{j}\right), X\left(t_{j}\right)\right\rangle \\
& +\left\langle B X\left(t_{1}\right), X\left(t_{1}\right)\right\rangle-\left\langle B X_{0}, X_{0}\right\rangle \\
& \quad=\left\langle B X\left(t_{m}\right), X\left(t_{m}\right)\right\rangle-\left\langle B X_{0}, X_{0}\right\rangle
\end{aligned}
$$

Using the formula in Lemma 74.3.1, for $t=t_{m}$ this yields

$$
\begin{align*}
& \left\langle B X\left(t_{m}\right), X\left(t_{m}\right)\right\rangle-\left\langle B X_{0}, X_{0}\right\rangle=2 \sum_{j=1}^{m-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u+ \\
& +2 \sum_{j=1}^{m-1}\left\langle B X\left(t_{j}\right), M\left(t_{j+1}\right)-M\left(t_{j}\right)\right\rangle \\
& \quad+\sum_{j=1}^{m-1}\left\langle B\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right), M\left(t_{j+1}\right)-M\left(t_{j}\right)\right\rangle \\
& \quad-\sum_{j=1}^{m-1}\left\langle B\left(X\left(t_{j+1}\right)-X\left(t_{j}\right)-\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right)\right. \\
& \left.X\left(t_{j+1}\right)-X\left(t_{j}\right)-\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right\rangle \\
& +2 \int_{0}^{t_{1}}\left\langle Y(u), X\left(t_{1}\right)\right\rangle d u+2\left\langle B X_{0}, M\left(t_{1}\right)\right\rangle+\left\langle B M\left(t_{1}\right), M\left(t_{1}\right)\right\rangle \\
& \quad-\left\langle B\left(X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right), X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right\rangle \tag{74.3.10}
\end{align*}
$$

First consider

$$
2 \int_{0}^{t_{1}}\left\langle Y(u), X\left(t_{1}\right)\right\rangle d u+2\left\langle B X_{0}, M\left(t_{1}\right)\right\rangle+\left\langle B M\left(t_{1}\right), M\left(t_{1}\right)\right\rangle .
$$

Each term of the above converges to 0 for a.e. $\omega$ as $k \rightarrow \infty$ and in $L^{1}(\Omega)$. This follows right away for the second two terms from the assumptions on $M$ given in the situation.

Recall it was assumed that $\|B(\omega)\|$ is bounded. This is where it is convenient to make this assumption. Consider the first term. This term is dominated by

$$
\begin{aligned}
& \left(\int_{0}^{t_{1}}\|Y(u)\|^{p^{\prime}} d u\right)^{1 / p^{\prime}}\left(\int_{0}^{T}\left\|X_{k}^{r}(u)\right\|^{p} d u\right)^{1 / p} \\
\leq & C(\omega)\left(\int_{0}^{t_{1}}\|Y(u)\|^{p^{\prime}} d u\right)^{1 / p^{\prime}},\left(\int_{\Omega} C(\omega)^{p} d P\right)^{1 / p}<\infty
\end{aligned}
$$

Hence this converges to 0 for a.e. $\omega$ and also converges to 0 in $L^{1}(\Omega)$.
At this time, not much is known about the last term in 74.3.10, but it is negative and is about to be neglected anyway.

The second term on the right equals

$$
2 \int_{t_{1}}^{t_{m}}\left\langle B X_{k}^{l}, d M\right\rangle=2 \int_{0}^{t_{m}}\left\langle B X_{k}^{l}, d M\right\rangle+e(k)
$$

where $e(k) \rightarrow 0$ for a.e. $\omega$ and in $L^{1}(\Omega)$. Also note that since $\left\langle B M\left(t_{1}\right), M\left(t_{1}\right)\right\rangle$ converges to 0 in $L^{1}(\Omega)$ and for a.e. $\omega$, the sum involving

$$
\left\langle B\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right), M\left(t_{j+1}\right)-M\left(t_{j}\right)\right\rangle
$$

can be started at 0 rather than 1 at the expense of adding in a term which converges to 0 a.e. and in $L^{1}(\Omega)$. Thus 74.3 .10 is of the form

$$
\begin{align*}
& \left\langle B X\left(t_{m}\right), X\left(t_{m}\right)\right\rangle-\left\langle B X_{0}, X_{0}\right\rangle=e(k)+2 \int_{0}^{t_{m}}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u+ \\
& \quad+2 \int_{0}^{t_{m}}\left\langle B X_{k}^{l}, d M\right\rangle \\
& \quad+\sum_{j=0}^{m-1}\left\langle B\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right), M\left(t_{j+1}\right)-M\left(t_{j}\right)\right\rangle \\
& \quad-\sum_{j=1}^{m-1}\left\langle B\left(X\left(t_{j+1}\right)-X\left(t_{j}\right)-\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right)\right. \\
& \left.\quad X\left(t_{j+1}\right)-X\left(t_{j}\right)-\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right\rangle \\
& \quad-\left\langle B\left(X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right), X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right\rangle \tag{74.3.11}
\end{align*}
$$

where $e(k) \rightarrow 0$ for a.e. $\omega$ and also in $L^{1}(\Omega)$.
Now it follows, on discarding the negative terms,

$$
\begin{aligned}
& \left\langle B X\left(t_{m}\right), X\left(t_{m}\right)\right\rangle-\left\langle B X_{0}, X_{0}\right\rangle \leq e(k)+2 \int_{0}^{t_{m}}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u+ \\
+ & 2 \int_{0}^{t_{m}}\left\langle B X_{k}^{l}, d M\right\rangle+\sum_{j=0}^{m-1}\left\langle B\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right), M\left(t_{j+1}\right)-M\left(t_{j}\right)\right\rangle
\end{aligned}
$$

Therefore,

$$
\begin{align*}
\sup _{t_{m} \in \mathscr{P}_{k}}\left\langle B X\left(t_{m}\right)\right. & \left., X\left(t_{m}\right)\right\rangle \leq\left\langle B X_{0}, X_{0}\right\rangle+e(k)+2 \int_{0}^{T}\left|\left\langle Y(u), X_{k}^{r}(u)\right\rangle\right| d u+ \\
& +2 \sup _{t_{m} \in \mathscr{P}_{k}}\left|\int_{0}^{t_{m}}\left\langle B X_{k}^{l}, d M\right\rangle\right| \\
& +\sum_{j=0}^{m-1}\left\langle B\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right), M\left(t_{j+1}\right)-M\left(t_{j}\right)\right\rangle \tag{74.3.12}
\end{align*}
$$

where there are $m_{k}+1$ points in $\mathscr{P}_{k}$. Consider that last term. It is no larger than

$$
\|B\| \sum_{j=0}^{m-1}\left\|M\left(t_{j+1}\right)-M\left(t_{j}\right)\right\|^{2}
$$

Say the last point in the partition is $t_{p}=T$ and consider the sum

$$
\begin{aligned}
& \sum_{j=0}^{p-1}\left\|M\left(t_{j+1} \wedge t\right)-M\left(t_{j} \wedge t\right)\right\|^{2}=\sum_{j=0}^{p-1}\left\|M^{t_{j+1}}-M^{t_{j}}\right\|^{2}(t) \\
= & \sum_{j=0}^{p-1}\left[M^{t_{j+1}}-M^{t_{j}}\right](t)+N_{j}(t)=\sum_{j=0}^{p-1}[M]^{t_{j+1}}(t)-[M]^{t_{j}}(t)+N_{j}(t)
\end{aligned}
$$

for $N_{j}$ a martingale which equals 0 for $t \leq t_{j}$. Now when you put in $t=t_{m}$, this becomes

$$
\sum_{j=0}^{m-1}[M]^{t_{j+1}}\left(t_{m}\right)-[M]^{t_{j}}\left(t_{m}\right)+N_{j}\left(t_{m}\right)
$$

Thus the expectation of that last term in 74.3 .12 is no larger than

$$
\|B\| E\left(\sum_{j=0}^{m-1}[M]^{t_{j+1}}\left(t_{m}\right)-[M]^{t_{j}}\left(t_{m}\right)\right)=\|B\| E\left([M]\left(t_{m}\right)\right)
$$

The next task is to take the expectation of both sides of 74.3.12. Of course there is a small problem with things not being in $L^{1}$. Hence it is appropriate to localize with the stopping time $\sigma_{q}^{k}$ defined in 74.2.3. That is, we obtain all of the above with $X$ replaced with $X^{\sigma_{q}^{k}}$, stopping the original integral equation by introducing $\mathscr{X}_{\left[0, \sigma_{q}^{k}\right]}$ in the integrals. Then carry out the following argument and pass to a limit as $q \rightarrow \infty$. In fact $\sigma_{q}^{k}=\infty$ if $q$ is large enough. Then carry out everything with $X^{\sigma_{q}^{k}}$. We don't write it, but this is what is being done in the following argument.

$$
E\left(\sup _{t_{m} \in \mathscr{P}_{k}}\left\langle B X\left(t_{m}\right), X\left(t_{m}\right)\right\rangle\right) \leq E\left(\left\langle B X_{0}, X_{0}\right\rangle\right)+E(|e(k)|)+2\|Y\|_{K^{\prime}}\left\|X_{k}^{r}\right\|_{K}
$$

$$
+2 E\left(\sup _{t_{m} \in \mathscr{P}_{k}}\left|\int_{0}^{t_{m}}\left\langle B X_{k}^{l}, d M\right\rangle\right|\right)+\|B\| E([M](T))
$$

Now using the Burkholder Davis Gundy inequality and the inequality for the quadratic variation of that funny integral involving $\left\langle B X_{k}^{l}, d M\right\rangle$,

$$
\begin{aligned}
\leq & E\left(\left\langle B X_{0}, X_{0}\right\rangle\right)+E(|e(k)|)+2\|Y\|_{K^{\prime}}\left\|X_{k}^{r}\right\|_{K} \\
& +C E\left(\left(\int_{0}^{T}\left\|B X_{k}^{l}\right\|^{2} d[M]\right)^{1 / 2}\right)+\|B\| E([M](T))
\end{aligned}
$$

Now $\|B v\|_{W^{\prime}}^{2} \leq\|B\|\langle B v, v\rangle$. Hence the above reduces to the following after adjusting the constant $C$,

$$
\begin{aligned}
\leq & E\left(\left\langle B X_{0}, X_{0}\right\rangle\right)+E(|e(k)|)+2\|Y\|_{K^{\prime}}\left\|X_{k}^{r}\right\|_{K} \\
& +C E\left(\left(\int_{0}^{T}\left\langle B X_{k}^{l}, X_{k}^{l}\right\rangle d[M]\right)^{1 / 2}\right)+\|B\| E([M](T)) \\
\leq & \frac{1}{2} \sup _{t_{m} \in \mathscr{P}_{k}}\left\langle B X\left(t_{m}\right), X\left(t_{m}\right)\right\rangle+(C+\|B\|) E([M](T)) \\
& +C\left(E\left(\left\langle B X_{0}, X_{0}\right\rangle\right),\|Y\|_{K^{\prime}},\left\|X_{k}^{r}\right\|_{K}\right)+E(|e(k)|)
\end{aligned}
$$

It follows on subtracting the first term on the right and adjusting constants again,

$$
\begin{gathered}
E\left(\sup _{t_{m} \in \mathscr{P}_{k}}\left\langle B X\left(t_{m}\right), X\left(t_{m}\right)\right\rangle\right) \\
\leq(C+\|B\|) E([M](T))+C\left(E\left(\left\langle B X_{0}, X_{0}\right\rangle\right),\|Y\|_{K^{\prime}},\left\|X_{k}^{r}\right\|_{K}\right)+E(|e(k)|)
\end{gathered}
$$

Now let $q \rightarrow \infty$ and use the monotone convergence theorem which yields the above for un-modified $X$.

Observe that these partitions are nested and that the constant $C(\cdots)$ is continuous and increasing in each argument with $C(\mathbf{0})=0, C(\cdots)$ not depending on $T$. Thus the left side is increasing and for given $\varepsilon>0$, there exists $N$ such that $k \geq N$ implies the right side is no larger than

$$
\begin{equation*}
C\left(E\left(\left\langle B X_{0}, X_{0}\right\rangle\right),\|Y\|_{K^{\prime}},\|X\|_{K}\right)+(C+\|B\|) E([M](T))+\varepsilon \tag{74.3.13}
\end{equation*}
$$

Now let $D$ denote the union of these nested partitions. Then from the monotone convergence theorem,

$$
E\left(\sup _{t \in D}\langle B X(t), X(t)\rangle\right)
$$

is no larger than the right side of 74.3 .13 . Since this is true for all $\varepsilon>0$, it follows

$$
\begin{equation*}
E\left(\sup _{t \in D}\langle B X(t), X(t)\rangle\right) \leq C\left(E\left(\left\langle B X_{0}, X_{0}\right\rangle\right),\|Y\|_{K^{\prime}},\|X\|_{K}, E([M](T))\right) \tag{74.3.14}
\end{equation*}
$$

where $C(\cdots)$ is increasing in each argument, continuous, and $C(\mathbf{0})=0$. Thus, enlarging $N$, for $\omega \notin N$,

$$
\begin{equation*}
\sup _{t \in D}\langle B X(t), X(t)\rangle=C(\omega)<\infty \tag{74.3.15}
\end{equation*}
$$

where $\int_{\Omega} C(\omega) d P<\infty$. By Lemma 69.4.1, there exists a countable set $\left\{e_{i}\right\}$ of vectors in $V$ such that

$$
\left\langle B e_{i}, e_{j}\right\rangle=\delta_{i j}
$$

and for each $x \in W$,

$$
\langle B x, x\rangle=\sum_{i=0}^{\infty}\left\langle B x, e_{i}\right\rangle^{2}, B x=\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i}
$$

Thus for $t$ not in a set of measure zero off which $B X(t)=B(X(t))$,

$$
\langle B X(t), X(t)\rangle=\sum_{i=0}^{\infty}\left\langle B X(t), e_{i}\right\rangle^{2}=\sup _{m} \sum_{k=1}^{m}\left\langle B X(t), e_{i}\right\rangle^{2}
$$

Now from the formula for $B X(t)$, it follows that $B X$ is continuous into $V^{\prime}$. For any $t \notin \hat{N}$ so that $(B X)(t)=B(X(t))$ in $L^{q^{\prime}}\left(\Omega ; V^{\prime}\right)$ and letting $t_{k} \rightarrow t$ where $t_{k} \in D$, Fatou's lemma implies

$$
\begin{aligned}
& E(\langle B X(t), X(t)\rangle)=\sum_{i} E\left(\left\langle B X(t), e_{i}\right\rangle^{2}\right)=\sum_{i} \lim _{k \rightarrow \infty} E\left(\left\langle B X\left(t_{k}\right), e_{i}\right\rangle^{2}\right) \\
& \quad \leq \lim \inf _{k \rightarrow \infty} \sum_{i} E\left(\left\langle B X\left(t_{k}\right), e_{i}\right\rangle^{2}\right)=\lim _{k \rightarrow \infty} E\left(\left\langle B X\left(t_{k}\right), X\left(t_{k}\right)\right\rangle\right) \\
& \quad \leq C\left(\|Y\|_{K^{\prime}},\|X\|_{K},\|Z\|_{J},\left\|\left\langle B X_{0}, X_{0}\right\rangle\right\|_{L^{1}(\Omega)}\right)
\end{aligned}
$$

In addition to this, for arbitrary $t \in[0, T]$, and $t_{k} \rightarrow t$ from $D$,

$$
\sum_{i}\left\langle B X(t), e_{i}\right\rangle^{2} \leq \lim _{k \rightarrow \infty} \inf _{k} \sum_{i}\left\langle B X\left(t_{k}\right), e_{i}\right\rangle^{2} \leq \sup _{s \in D}\langle B X(s), X(s)\rangle
$$

Hence

$$
\begin{aligned}
\sup _{t \in[0, T]} \sum_{i}\left\langle B X(t), e_{i}\right\rangle^{2} & \leq \sup _{s \in D}\langle B X(s), X(s)\rangle \\
& =\sup _{s \in D} \sum_{i}\left\langle B X(s), e_{i}\right\rangle^{2} \leq \sup _{t \in[0, T]} \sum_{i}\left\langle B X(t), e_{i}\right\rangle^{2}
\end{aligned}
$$

It follows that $\sup _{t \in[0, T]} \sum_{i}\left\langle B X(t), e_{i}\right\rangle^{2}$ is measurable and

$$
\begin{aligned}
& E\left(\sup _{t \in[0, T]} \sum_{i}\left\langle B X(t), e_{i}\right\rangle^{2}\right) \leq E\left(\sup _{s \in D}\langle B X(s), X(s)\rangle\right) \\
\leq & C\left(\|Y\|_{K^{\prime}},\|X\|_{K},\|Z\|_{J},\left\|\left\langle B X_{0}, X_{0}\right\rangle\right\|_{L^{1}(\Omega)}\right)
\end{aligned}
$$

And so, for $\omega$ off a set of measure zero, $\sup _{t \in[0, T]} \sum_{i}\left\langle B X(t), e_{i}\right\rangle^{2}$ is bounded above. Include this exceptional set in $N$.

Also for $t \notin N_{\omega}$ and a given $\omega \notin N$, letting $t_{k} \rightarrow t$ for $t_{k} \in D$,

$$
\begin{aligned}
\langle B X(t), X(t)\rangle & =\sum_{i}\left\langle B X(t), e_{i}\right\rangle^{2} \leq \lim _{k \rightarrow \infty} \inf _{k \rightarrow \infty} \sum_{i}\left\langle B X\left(t_{k}\right), e_{i}\right\rangle^{2} \\
& =\lim _{k \rightarrow \infty}\left\langle B X\left(t_{k}\right), X\left(t_{k}\right)\right\rangle \leq \sup _{t \in D}\langle B X(t), X(t)\rangle
\end{aligned}
$$

and so

$$
\sup _{t \notin N_{\omega}}\langle B X(t), X(t)\rangle \leq \sup _{t \in D}\langle B X(t), X(t)\rangle \leq \sup _{t \notin N_{\omega}}\langle B X(t), X(t)\rangle
$$

From 74.3.15,

$$
\sup _{t \notin N_{\omega}}\langle B X(t), X(t)\rangle=C(\omega) \text { a.e. } \omega
$$

where $\int_{\Omega} C(\omega) d P<\infty$. In particular, $\sup _{t \notin N_{\omega}}\langle B X(t), X(t)\rangle$ is bounded for a.e. $\omega$ say for $\omega \notin N$ where $N$ includes the earlier sets of measure zero. This shows that $B X(t)$ is bounded in $W^{\prime}$ for $t \in N_{\omega}^{C}$.

If $v \in V$, then for $\omega \notin N$,

$$
\lim _{t \rightarrow s}\langle B X(t), v\rangle=\langle B X(s), v\rangle, t, s
$$

Therefore, since for such $\omega,\|B X(t)\|_{W^{\prime}}$ is bounded for $t \notin N_{\omega}$, the above holds for all $v \in W$ also. Therefore, for a.e. $\omega, t \rightarrow B X(t, \omega)$ is weakly continuous with values in $W^{\prime}$ for $t \notin N_{\omega}$.

Note also that

$$
\begin{align*}
& \int_{0}^{T} \int_{\Omega}\|B X(t)\|^{2} d P d t \leq \int_{\Omega} \int_{0}^{T}\|B\|^{1 / 2}\langle B X(t), X(t)\rangle d t d P \\
\leq & C\left(\|Y\|_{K^{\prime}},\|X\|_{K},\|Z\|_{J},\left\|\left\langle B X_{0}, X_{0}\right\rangle\right\|_{L^{1}(\Omega)}\right)\|B\|^{1 / 2} T \tag{74.3.16}
\end{align*}
$$

Eventually, it is shown that in fact, the function $t \rightarrow B X(t, \omega)$ is continuous with values in $W^{\prime}$. The above shows that $B X \in L^{2}\left([0, T] \times \Omega, W^{\prime}\right)$.

Finally consider the claim of weak continuity of $B X$ into $W^{\prime}$. From the integral equation, $B X$ is continuous into $V^{\prime}$. Also $t \rightarrow B X(t)$ is bounded in $W^{\prime}$ on $N_{\omega}^{C}$. Let $s \in[0, T]$ be arbitrary. I claim that if $t_{n} \rightarrow s, t_{n} \in D$, it follows that $B X\left(t_{n}\right) \rightarrow B X(s)$ weakly in $W^{\prime}$. If not, then there is a subsequence, still denoted as $t_{n}$ such that $B X\left(t_{n}\right) \rightarrow Y$ weakly in $W^{\prime}$ but $Y \neq B X(s)$. However, the continuity into $V^{\prime}$ means that for all $v \in V$,

$$
\langle Y, v\rangle=\lim _{n \rightarrow \infty}\left\langle B X\left(t_{n}\right), v\right\rangle=\langle B X(s), v\rangle
$$

which is a contradiction since $V$ is dense in $W$. This establishes the claim. Also this shows that $B X(s)$ is bounded in $W^{\prime}$.

$$
|\langle B X(s), w\rangle|=\lim _{n \rightarrow \infty}\left|\left\langle B X\left(t_{n}\right), w\right\rangle\right| \leq \lim \inf _{n \rightarrow \infty}\left\|B X\left(t_{n}\right)\right\|_{W^{\prime}}\|w\|_{W} \leq C(\omega)\|w\|_{W}
$$

Now a repeat of the above argument shows that $s \rightarrow B X(s)$ is weakly continuous into $W^{\prime}$.

### 74.4 A Simplification Of The Formula

This estimate in Lemma 74.3 .2 also provides a way to simplify one of the formulas derived earlier in the case that $X_{0} \in L^{p}(\Omega, V)$ so that $X-X_{0} \in L^{p}([0, T] \times \Omega, V)$. Refer to 74.3.11. One term there is

$$
\left\langle B\left(X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right), X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right\rangle
$$

Also,

$$
\begin{gathered}
\left\langle B\left(X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right), X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right\rangle \\
\leq 2\left\langle B\left(X\left(t_{1}\right)-X_{0}\right), X\left(t_{1}\right)-X_{0}\right\rangle+2\left\langle B M\left(t_{1}\right), M\left(t_{1}\right)\right\rangle
\end{gathered}
$$

It was observed above that $2\left\langle B M\left(t_{1}\right), M\left(t_{1}\right)\right\rangle \rightarrow 0$ a.e. and also in $L^{1}(\Omega)$ as $k \rightarrow \infty$. Apply the above lemma to $\left\langle B\left(X\left(t_{1}\right)-X_{0}\right), X\left(t_{1}\right)-X_{0}\right\rangle$ using $\left[0, t_{1}\right]$ instead of $[0, T]$. The new $X_{0}$ equals 0 . Then from the estimate 74.3.8, it follows that

$$
E\left(\left\langle B\left(X\left(t_{1}\right)-X_{0}\right), X\left(t_{1}\right)-X_{0}\right\rangle\right) \rightarrow 0
$$

as $k \rightarrow \infty$. Taking a subsequence, we could also assume that

$$
\left\langle B\left(X\left(t_{1}\right)-X_{0}\right), X\left(t_{1}\right)-X_{0}\right\rangle \rightarrow 0
$$

a.e. $\omega$ as $k \rightarrow \infty$. Then, using this subsequence, it would follow from 74.3.11,

$$
\begin{gather*}
\left\langle B X\left(t_{m}\right), X\left(t_{m}\right)\right\rangle-\left\langle B X_{0}, X_{0}\right\rangle=e(k)+ \\
2 \int_{0}^{t_{m}}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u++2 \int_{0}^{t_{m}}\left\langle B X_{k}^{l}, d M\right\rangle \\
+\sum_{j=0}^{m-1}\left\langle B\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right), M\left(t_{j+1}\right)-M\left(t_{j}\right)\right\rangle \\
-\sum_{j=1}^{m-1}\left\langle B\left(\Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right), \Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right\rangle \tag{74.4.17}
\end{gather*}
$$

where $e(k) \rightarrow 0$ in $L^{1}(\Omega)$ and a.e. $\omega$ and

$$
\Delta X\left(t_{j}\right) \equiv X\left(t_{j+1}\right)-X\left(t_{j}\right)
$$

$\Delta M\left(t_{j}\right)$ being defined similarly. Note how this eliminated the need to consider the term

$$
\left\langle B\left(X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right), X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right\rangle
$$

in passing to a limit. This is a very desirable thing to be able to conclude.
Can you obtain something similar even in case $X_{0}$ is not assumed to be in $L^{p}(\Omega, V)$ ? Let $X_{0 k} \in L^{p}(\Omega, V) \cap L^{2}(\Omega, W), X_{0 k} \rightarrow X_{0}$ in $L^{2}(\Omega, W)$. Then from the usual arguments involving the Cauchy Schwarz inequality,

$$
\begin{aligned}
\left\langle B\left(X\left(t_{1}\right)-X_{0}\right), X\left(t_{1}\right)-X_{0}\right\rangle^{1 / 2} \leq & \left\langle B\left(X\left(t_{1}\right)-X_{0 k}\right), X\left(t_{1}\right)-X_{0 k}\right\rangle^{1 / 2} \\
& +\left\langle B\left(X_{0 k}-X_{0}\right), X_{0 k}-X_{0}\right\rangle^{1 / 2}
\end{aligned}
$$

Also, restoring the superscript to identify the parition,

$$
B\left(X\left(t_{1}^{k}\right)-X_{0 k}\right)=B\left(X_{0}-X_{0 k}\right)+\int_{0}^{t_{1}^{k}} Y(s) d s+B M\left(t_{1}^{k}\right)
$$

Of course $\left\|X-X_{0 k}\right\|_{K}$ is not bounded, but for each $k$ it is finite. There is a sequence of partitions $\mathscr{P}_{k},\left\|\mathscr{P}_{k}\right\| \rightarrow 0$ such that all the above holds. In the definitions of $K, K^{\prime}, E([M](T))$ replace $[0, T]$ with $[0, t]$ and let the resulting spaces be denoted by $K_{t}, K_{t}^{\prime}$. Let $n_{k}$ denote a subsequence of $\{k\}$ such that

$$
\left\|X-X_{0 k}\right\|_{K_{t_{1}} n_{k}}<1 / k
$$

Then from the above lemma,

$$
\begin{gather*}
E\left(\left\langle B\left(X\left(t_{1}^{n_{k}}\right)-X_{0 k}\right), X\left(t_{1}^{n_{k}}\right)-X_{0 k}\right\rangle\right) \\
\leq C\left(\left\langle B\left(X_{0}-X_{0 k}\right), X_{0}-X_{0 k}\right\rangle_{L^{1}(\Omega)},\|Y\|_{K_{t_{n_{1}}^{\prime}}^{\prime}},\left\|X-X_{0 k}\right\|_{K_{t_{1} n_{k}}}, E\left([M]\left(t_{1}^{n_{k}}\right)\right)\right)  \tag{74.4.18}\\
\leq C\left(\left\langle B\left(X_{0}-X_{0 k}\right), X_{0}-X_{0 k}\right\rangle_{L^{1}(\Omega)},\|Y\|_{\substack{K_{n_{k}}^{\prime} \\
t_{1}}}, \frac{1}{k}, E\left([M]\left(t_{1}^{n_{k}}\right)\right)\right)
\end{gather*}
$$

Hence

$$
\begin{gathered}
E\left(\left\langle B\left(X\left(t_{1}^{n_{k}}\right)-X_{0}\right), X\left(t_{1}^{n_{k}}\right)-X_{0}\right\rangle\right) \\
\leq 2 E\left(\left\langle B\left(X\left(t_{1}^{n_{k}}\right)-X_{0 k}\right), X\left(t_{1}^{n_{k}}\right)-X_{0 k}\right\rangle\right)+2 E\left(\left\langle B\left(X_{0 k}-X_{0}\right), X_{0 k}-X_{0}\right\rangle\right) \\
\leq \quad 2 C\left(\left\langle B\left(X_{0}-X_{0 k}\right), X_{0}-X_{0 k}\right\rangle_{L^{1}(\Omega)},\|Y\|_{\substack{K_{t_{1}}^{\prime},}}, \frac{1}{k}, E\left([M]\left(t_{1}^{n_{k}}\right)\right)\right) \\
\quad+2\|B\|\left\|X_{0 k}-X_{0}\right\|_{L^{2}(\Omega, W)}^{2}
\end{gathered}
$$

which converges to 0 as $k \rightarrow \infty$. It follows that there exists a suitable subsequence such that 74.4.17 holds even in the case that $X_{0}$ is only known to be in $L^{2}(\Omega, W)$. From now on, assume this subsequence for the partitions $\mathscr{P}_{k}$. Thus $k$ will really be $n_{k}$ and it suffices to consider the limit as $k \rightarrow \infty$ of the equation of 74.4.17. To emphasize this point again, the reason for the above observations is to argue that, even when $X_{0}$ is only in $L^{2}(\Omega, W)$, one can neglect

$$
\left\langle B\left(X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right), X\left(t_{1}\right)-X_{0}-M\left(t_{1}\right)\right\rangle
$$

in passing to the limit as $k \rightarrow \infty$ provided a suitable subsequence is used.

### 74.5 Convergence

Convergence will be shown for a subsequence and from now on every sequence will be a subsequence of this one. Since $B X \in L^{2}\left([0, T] \times \Omega ; W^{\prime}\right)$ which was shown above, there exists a sequence of partitions of the sort described above such that also, in addition to the other claims

$$
B X_{k}^{l} \rightarrow B X, B X_{k}^{r} \rightarrow B X
$$

in $L^{2}\left([0, T] \times \Omega, W^{\prime}\right)$. Then the next lemma improves on this.

Lemma 74.5.1 There exists a subsequence still denoted with the subscript $k$ and an enlarged set of measure zero $N$ including the earlier one such that $B X_{k}^{l}(t), B X_{k}^{r}(t)$ also converges pointwise a.e. to to $B X(t)$ in $W^{\prime}$ and $X_{k}^{l}(t), X_{k}^{r}(t)$ converge pointwise a.e. in $V$ to $X(t)$ for $\omega \notin N$ as well as having convergence of $X_{k}^{l}(\cdot, \omega)$ to $X(\cdot, \omega)$ in $L^{p}([0, T] ; V)$ and $B X_{k}^{l}(\cdot, \omega)$ to $B X(\cdot, \omega)$ in $L^{2}\left([0, T] ; W^{\prime}\right)$.

Proof: To see that such a sequence exists, let $n_{k}$ be such that

$$
\begin{aligned}
& \int_{\Omega} \int_{0}^{T}\left\|B X_{n_{k}}^{r}(t)-B X(t)\right\|_{W^{\prime}}^{2} d t d P+\int_{\Omega} \int_{0}^{T}\left\|X_{n_{k}}^{r}(t)-X(t)\right\|_{V}^{p} d t d P+ \\
& \int_{\Omega} \int_{0}^{T}\left\|B X_{n_{k}}^{l}(t)-B X(t)\right\|_{W^{\prime}}^{2} d t d P+\int_{\Omega} \int_{0}^{T}\left\|X_{n_{k}}^{l}(t)-X(t)\right\|_{V}^{p} d t d P<4^{-k} .
\end{aligned}
$$

Then

$$
\begin{gathered}
P\left(\int_{0}^{T}\left\|B X_{n_{k}}^{l}(t)-B X(t)\right\|_{W^{\prime}}^{2} d t+\int_{0}^{T}\left\|X_{n_{k}}^{r}(t)-X(t)\right\|_{V}^{p} d t+\right. \\
\left.\int_{0}^{T}\left\|B X_{n_{k}}^{l}(t)-B X(t)\right\|_{W^{\prime}}^{2} d t+\int_{0}^{T}\left\|X_{n_{k}}^{l}(t)-X(t)\right\|_{V}^{p} d t>2^{-k}\right) \\
\leq 2^{k}\left(4^{-k}\right)=2^{-k}
\end{gathered}
$$

and so by Borel Cantelli lemma, there is a set of measure zero $N$ such that if $\omega \notin N$,

$$
\begin{aligned}
& \int_{0}^{T}\left\|B X_{n_{k}}^{l}(t)-B X(t)\right\|_{W^{\prime}}^{2} d t+\int_{0}^{T}\left\|X_{n_{k}}^{r}(t)-X(t)\right\|_{V}^{p} d t+ \\
& \int_{0}^{T}\left\|B X_{n_{k}}^{l}(t)-B X(t)\right\|_{W^{\prime}}^{2} d t+\int_{0}^{T}\left\|X_{n_{k}}^{l}(t)-X(t)\right\|_{V}^{p} d t \leq 2^{-k}
\end{aligned}
$$

for all $k$ large enough. By the usual proof of completeness of $L^{p}$, it follows that $X_{n_{k}}^{l}(t) \rightarrow$ $X(t)$ for a.e. $t$, this for each $\omega \notin N$, a similar assertion holding for $X_{n_{k}}^{r}$. Also $B X_{n_{k}}^{l}(t) \rightarrow$ $B X(t)$ for a.e. $t$, similar for $B X_{n_{k}}^{r}(t)$. Denote these subsequences as $\left\{X_{k}^{r}\right\}_{k=1}^{\infty},\left\{X_{k}^{l}\right\}_{k=1}^{\infty}$.

Define the following stopping time.

$$
\tau_{p} \equiv \inf \left\{t: \sum_{i}\left\langle B X(t), e_{i}\right\rangle^{2}>p\right\}
$$

By Lemma 74.3.2 $\tau_{p}=\infty$ for all $p$ large enough off some set of measure zero. Also, $B X(t)(\omega)=B(X(t, \omega))$ for a.e. $t$ and so for a.e.t, $\langle B X(t), X(t)\rangle=\sum_{i}\left\langle B X(t), e_{i}\right\rangle^{2}$ and so $\left\|B X^{\tau_{p}}(t)\right\|_{W^{\prime}} \leq\|B\| \sqrt{p}$ for a.e.t. Hence $B X^{\tau_{p}} \in L^{\infty}\left([0, T] \times \Omega, W^{\prime}\right)$.

Lemma 74.5.2 The process $\int_{0}^{t}\left\langle B X_{k}^{l}, d M\right\rangle$ converges in probability as $k \rightarrow \infty$ to $\int_{0}^{t}\langle B X, d M\rangle$ which is a local martingale. Also, there is a subsequence and an enlarged set of measure zero $N$ such that for $\omega$ not in this set, the convergence is uniform on $[0, T]$.

Proof: By assumption, $d[M]=k d t$ for some $k \in L^{1}([0, T] \times \Omega)$ and so $B X^{\tau_{p}} \in \mathscr{G}$ where $\mathscr{G}$ was the class of functions for which one can write $\int_{0}^{t}\langle B X, d M\rangle$. By the Burkholder Davis Gundy inequality,

$$
\begin{align*}
& P\left(\sup _{t}\left|\int_{0}^{t \wedge \tau_{p}}\left\langle B\left(X_{k}^{l}\right)-B X, d M\right\rangle\right|>\varepsilon\right) \\
= & P\left(\sup _{t}\left|\int_{0}^{t} \mathscr{X}_{\left[0, \tau_{p}\right]}\left\langle B\left(X_{k}^{l}\right)-B X, d M\right\rangle\right|>\varepsilon\right) \\
\leq & \frac{C}{\varepsilon} \int_{\Omega}\left(\int_{0}^{T \wedge \tau_{p}}\left\|B\left(X_{k}^{l}\right)-B X\right\|_{W}^{2} k d t\right)^{1 / 2} d P \\
= & \frac{C}{\varepsilon} \int_{\Omega}\left(\int_{0}^{T} \mathscr{X}_{\left[0, \tau_{p}\right]}\left\|B\left(X_{k}^{l}\right)-B X\right\|_{W}^{2} k d t\right)^{1 / 2} d P \tag{74.5.19}
\end{align*}
$$

Let

$$
A_{k}=\left[\sup _{t}\left|\int_{0}^{t}\left\langle B X_{k}^{l}-B X, d M\right\rangle\right|>\varepsilon\right]
$$

Then, since $\tau_{p}=\infty$ for all $p$ large enough,

$$
A_{k}=\cup_{p=0}^{\infty} A_{k} \cap\left(\left[\tau_{p}=\infty\right] \backslash\left[\tau_{p-1} \neq \infty\right]\right)
$$

Consider $B X_{k}^{l \tau_{p}}$. If $t>\tau_{p}$, what of the values of $B X_{k}^{l \tau_{p}}$ ? It equals $B X(s)$ where $s$ is one of the mesh points $s \leq \tau_{p}$ because this is a left step function. Therefore,

$$
\begin{gathered}
\left\langle B X_{k}^{l \tau_{p}}(s), X_{k}^{l \tau_{p}}(s)\right\rangle=\left\langle B\left(X_{k}^{l \tau_{p}}(s)\right), X_{k}^{l \tau_{p}}(s)\right\rangle \\
=\sum_{i}\left\langle B X(s), e_{i}\right\rangle^{2} \leq p
\end{gathered}
$$

As to $\mathscr{X}_{\left[0, \tau_{p}\right]} B X$, it follows that for all $t \leq \tau_{p}$ you have $\sum_{i}\left\langle B X(t), e_{i}\right\rangle^{2} \leq p$ and so, since this equals $\langle B(X(t)), X(t)\rangle$ a.e. $t$, it follows that $\left\|\mathscr{X}_{\left[0, \tau_{p}\right]} B X(t)\right\|_{W^{\prime}}$ is bounded by a constant depending on $p$ for a.e.t. It follows that $B X$ and $B X_{k}^{l}$ are bounded. Now by Lemma 74.5.1, $B X_{k}^{l}(t) \rightarrow B X(t)$ a.e. $t$ and the term $\left\|B\left(X_{k}^{l}\right)-B X\right\|_{W}^{2}$ is essentially bounded. Therefore, in 74.5.19, the integral converges to 0 . From this formula,

$$
\begin{aligned}
& P\left(A_{k} \cap\left(\left[\tau_{p}=\infty\right] \backslash\left[\tau_{p-1} \neq \infty\right]\right)\right) \leq P\left(A_{k} \cap\left(\left[\tau_{p}=\infty\right]\right)\right) \\
& \quad \leq \frac{C}{\varepsilon} \int_{\Omega}\left(\int_{0}^{T} \mathscr{X}_{\left[0, \tau_{p}\right]}\left\|B\left(X_{k}^{l}\right)-B X\right\|_{W}^{2} k d t\right)^{1 / 2} d P
\end{aligned}
$$

Thus

$$
\lim _{k \rightarrow \infty} P\left(A_{k} \cap\left(\left[\tau_{p}=\infty\right] \backslash\left[\tau_{p-1} \neq \infty\right]\right)\right)=0
$$

Then

$$
P\left(A_{k}\right)=\sum_{p=1}^{\infty} P\left(A_{k} \cap\left(\left[\tau_{p}=\infty\right] \backslash\left[\tau_{p-1} \neq \infty\right]\right)\right)
$$

and taking limits using the dominated convergence theorem on the sum on the right,

$$
\lim _{k \rightarrow \infty} P\left(A_{k}\right)=\sum_{p=1}^{\infty} \lim _{k \rightarrow \infty} P\left(A_{k} \cap\left(\left[\tau_{p}=\infty\right] \backslash\left[\tau_{p-1} \neq \infty\right]\right)\right)=0
$$

This proves convergence in probability.

$$
\lim _{k \rightarrow \infty} P\left(\sup _{t}\left|\int_{0}^{t}\left\langle B\left(X_{k}^{l}\right)-B X, d M\right\rangle\right|>\varepsilon\right)=0
$$

Then selecting a subsequence, still denoted with $k$, we can obtain

$$
P\left(\sup _{t}\left|\int_{0}^{t}\left\langle B\left(X_{k}^{l}\right)-B X, d M\right\rangle\right|>\frac{1}{k}\right)<2^{-k}
$$

and so, by the Borel Cantelli lemma, there is a set of measure zero $N$ such that for this subsequence, for all $\omega \notin N$,

$$
\sup _{t}\left|\int_{0}^{t}\left\langle B\left(X_{k}^{l}\right)-B X, d M\right\rangle\right| \leq \frac{1}{k}
$$

for all $k$ large enough. Thus convergence is uniform.
From now on, include $N$ in the exceptional set and every subsequence will be a subsequence of this one.

### 74.6 The Ito Formula

Now at long last, here is the first version of the Ito formula valid on the partition points.
Lemma 74.6.1 In Situation 74.1.1, let $D$ be as above, the union of all the positive mesh points for all the $\mathscr{P}_{k}$. Also assume $X_{0} \in L^{2}(\Omega ; W)$. Then for $\omega \notin N$ the exceptional set of measure zero in $\Omega$ and every $t \in D$,

$$
\begin{gather*}
\langle B X(t), X(t)\rangle=\left\langle B X_{0}, X_{0}\right\rangle+\int_{0}^{t} 2\langle Y(s), X(s)\rangle d s \\
+\left[R^{-1} B M, M\right](t)+2 \int_{0}^{t}\langle B X, d M\rangle \tag{74.6.20}
\end{gather*}
$$

for $R$ the Riesz map from $W$ to $W^{\prime}$. The covariation term $\left[R^{-1} B M, M\right](t)$ is nonnegative.
Proof: Let $t \in D$. Then $t \in \mathscr{P}_{k}$ for all $k$ large enough. Consider 74.4.17,

$$
\begin{gathered}
\langle B X(t), X(t)\rangle-\left\langle B X_{0}, X_{0}\right\rangle=e(k)+2 \int_{0}^{t}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u \\
+2 \int_{0}^{t}\left\langle B X_{k}^{l}, d M\right\rangle+\sum_{j=0}^{q_{k}-1}\left\langle B\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right), M\left(t_{j+1}\right)-M\left(t_{j}\right)\right\rangle
\end{gathered}
$$

$$
\begin{equation*}
-\sum_{j=1}^{q_{k}-1}\left\langle B\left(\Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right), \Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right\rangle \tag{74.6.21}
\end{equation*}
$$

where $t_{q_{k}}=t, \Delta X\left(t_{j}\right)=X\left(t_{j+1}\right)-X\left(t_{j}\right)$ and $e(k) \rightarrow 0$ in probability. By Lemma 74.5.2 the stochastic integral on the right converges uniformly for $t \in[0, T]$ to

$$
2 \int_{0}^{t}\langle B X, d M\rangle
$$

for $\omega$ off a set of measure zero. The deterministic integral on the right converges uniformly for $t \in[0, T]$ to

$$
2 \int_{0}^{t}\langle Y(u), X(u)\rangle d u
$$

Thanks to Lemma 74.5.1.

$$
\begin{aligned}
& \left|\int_{0}^{t}\langle Y(u), X(u)\rangle d u-\int_{0}^{t}\left\langle Y(u), X_{k}^{r}(u)\right\rangle d u\right| \\
\leq & \int_{0}^{T}\|Y(u)\|_{V^{\prime}}\left\|X(u)-X_{k}^{r}(u)\right\|_{V} \\
\leq & \|Y\|_{L^{p^{\prime}}([0, T])}\left(2^{-k}\right)^{1 / p}
\end{aligned}
$$

for all $k$ large enough. Consider the fourth term. It equals

$$
\begin{equation*}
\sum_{j=0}^{q_{k}-1}\left(R^{-1} B\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right), M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)_{W} \tag{74.6.22}
\end{equation*}
$$

where $R^{-1}$ is the Riesz map from $W$ to $W^{\prime}$. This equals

$$
\begin{aligned}
& \frac{1}{4}\left(\sum_{j=0}^{q_{k}-1}\left\|R^{-1} B M\left(t_{j+1}\right)+M\left(t_{j+1}\right)-\left(R^{-1} B M\left(t_{j}\right)+M\left(t_{j}\right)\right)\right\|^{2}\right. \\
& \left.-\sum_{j=0}^{q_{k}-1}\left\|R^{-1} B M\left(t_{j+1}\right)-M\left(t_{j+1}\right)-\left(R^{-1} B M\left(t_{j}\right)-M\left(t_{j}\right)\right)\right\|^{2}\right)
\end{aligned}
$$

From Theorem 63.6.4, as $k \rightarrow \infty$, the above converges in probability to ( $t_{q_{k}}=t$ )

$$
\frac{1}{4}\left(\left[R^{-1} B M+M\right](t)-\left[R^{-1} B M-M\right](t)\right) \equiv\left[R^{-1} B M, M\right](t)
$$

Also note that from 74.6.22, this term must be nonnegative since it is a limit of nonnegative quantities. This is what was desired.

Thus all the terms in 74.6 .21 converge in probability except for the last term which also must converge in probability because it equals the sum of terms which do. It remains to find what this last term converges to. Thus

$$
\langle B X(t), X(t)\rangle-\left\langle B X_{0}, X_{0}\right\rangle=2 \int_{0}^{t}\langle Y(u), X(u)\rangle d u
$$

$$
+2 \int_{0}^{t}\langle B X, d M\rangle+\left[R^{-1} B M, M\right](t)-a
$$

where $a$ is the limit in probability of the term

$$
\begin{equation*}
\sum_{j=1}^{q_{k}-1}\left\langle B\left(\Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right), \Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right\rangle \tag{74.6.23}
\end{equation*}
$$

Let $P_{n}$ be the projection onto span $\left(e_{1}, \cdots, e_{n}\right)$ where $\left\{e_{k}\right\}$ is an orthonormal basis for $W$ with each $e_{k} \in V$. Then using

$$
B X\left(t_{j+1}\right)-B X\left(t_{j}\right)-\left(B M\left(t_{j+1}\right)-B M\left(t_{j}\right)\right)=\int_{t_{j}}^{t_{j+1}} Y(s) d s
$$

the troublesome term of 74.6.23 above is of the form

$$
\begin{aligned}
& \sum_{j=1}^{q_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(s), \Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right\rangle d s \\
& =\sum_{j=1}^{q_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(s), \Delta X\left(t_{j}\right)-P_{n} \Delta M\left(t_{j}\right)\right\rangle d s \\
& \quad+\sum_{j=1}^{q_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(s),-\left(I-P_{n}\right) \Delta M\left(t_{j}\right)\right\rangle d s
\end{aligned}
$$

which equals

$$
\begin{align*}
& \sum_{j=1}^{q_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(s), X\left(t_{j+1}\right)-X\left(t_{j}\right)-P_{n}\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right\rangle d s  \tag{74.6.24}\\
& +\sum_{j=1}^{q_{k}-1}\left\langle B\left(\Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right),-\left(I-P_{n}\right)\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right\rangle \tag{74.6.25}
\end{align*}
$$

The reason for the $P_{n}$ is to get $P_{n}\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)$ in $V$. The sum in 74.6.25 is dominated by

$$
\begin{align*}
& \left(\sum_{j=1}^{q_{k}-1}\left\langle B\left(\Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right),\left(\Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right)\right\rangle\right)^{1 / 2} . \\
& \left(\sum_{j=1}^{q_{k}-1}\left|\left\langle B\left(I-P_{n}\right) \Delta M\left(t_{j}\right),\left(I-P_{n}\right) \Delta M\left(t_{j}\right)\right\rangle\right|^{2}\right)^{1 / 2} \tag{74.6.26}
\end{align*}
$$

Now it is known from the above that

$$
\sum_{j=1}^{q_{k}-1}\left\langle B\left(\Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right),\left(\Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right)\right\rangle
$$

converges in probability to $a \geq 0$. If you take the expectation of the square of the other factor, it is no larger than

$$
\begin{gathered}
\|B\| E\left(\sum_{j=1}^{q_{k}-1}\left\|\left(I-P_{n}\right) \Delta M\left(t_{j}\right)\right\|_{W}^{2}\right) \\
=\|B\| E\left(\sum_{j=1}^{q_{k}-1}\left\|\left(I-P_{n}\right)\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right\|_{W}^{2}\right) \\
=\|B\| \sum_{j=1}^{q_{k}-1} E\left(\left\|\left(I-P_{n}\right)\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right\|_{W}^{2}\right)
\end{gathered}
$$

Then

$$
\begin{gathered}
\left\|\left(I-P_{n}\right)\left(M\left(t_{j+1} \wedge t\right)-M\left(t_{j} \wedge t\right)\right)\right\|_{W}^{2}=\left[\left(1-P_{n}\right) M^{t_{j+1}}-\left(1-P_{n}\right) M^{t_{j}}\right](t)+N(t) \\
=\left[\left(1-P_{n}\right) M\right]^{t_{j+1}}(t)-\left[\left(1-P_{n}\right) M\right]^{t_{j}}(t)+N(t)
\end{gathered}
$$

for $N(t)$ a martingale. In particular, taking $t=t_{q_{k}}$, the above reduces to

$$
\begin{aligned}
& \|B\| \sum_{j=1}^{q_{k}-1} E\left(\left\|\left(I-P_{n}\right)\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right\|_{W}^{2}\right) \\
= & \|B\| \sum_{j=1}^{q_{k}-1} E\left(\left[\left(1-P_{n}\right) M\right]\left(t_{j+1}\right)-\left[\left(1-P_{n}\right) M\right]\left(t_{j}\right)\right) \\
= & \|B\| E\left(\left[\left(1-P_{n}\right) M\right]\left(t_{q_{k}}\right)\right)=\|B\| E\left(\left\|\left(1-P_{n}\right) M\left(t_{q_{k}}\right)\right\|_{W}^{2}\right)
\end{aligned}
$$

From maximal theorems, Theorem 62.9.4,

$$
\|B\| E\left(\sup _{t_{q_{k}}}\left\|\left(1-P_{n}\right) M\left(t_{q_{k}}\right)\right\|_{W}^{2}\right) \leq 2\|B\| E\left(\left\|\left(1-P_{n}\right) M(T)\right\|_{W}^{2}\right)
$$

and this on the right converges to zero as $n \rightarrow \infty$ by assumption that $M(t)$ is in $L^{2}$ and the dominated convergence theorem. In particular, this shows that

$$
\left(\sum_{j=1}^{q_{k}-1}\left|\left\langle B\left(I-P_{n}\right) \Delta M\left(t_{j}\right),\left(I-P_{n}\right) \Delta M\left(t_{j}\right)\right\rangle\right|^{2}\right)^{1 / 2}
$$

converges to 0 in $L^{2}(\Omega)$ independent of $k$ as $n \rightarrow \infty$.
Thus the expression in 74.6 .26 is of the form $f_{k} g_{n k}$ where $f_{k}$ converges in probability to $a^{1 / 2}$ as $k \rightarrow \infty$ and $g_{n k}$ converges in probability to 0 as $n \rightarrow \infty$ independent of $k$. Now this implies $f_{k} g_{n k}$ converges in probability to 0 . Here is why.

$$
\begin{aligned}
P\left(\left[\left|f_{k} g_{n k}\right|>\varepsilon\right]\right) & \leq P\left(2 \delta\left|f_{k}\right|>\varepsilon\right)+P\left(2 C_{\delta}\left|g_{n k}\right|>\varepsilon\right) \\
& \leq P\left(2 \delta\left|f_{k}-a^{1 / 2}\right|+2 \delta\left|a^{1 / 2}\right|>\varepsilon\right)+P\left(2 C_{\delta}\left|g_{n k}\right|>\varepsilon\right)
\end{aligned}
$$

where $\delta\left|f_{k}\right|+C_{\delta}\left|g_{k n}\right|>\left|f_{k} g_{n k}\right|$ and $\lim _{\delta \rightarrow 0} C_{\delta}=\infty$. Pick $\delta$ small enough that $\varepsilon-2 \delta a^{1 / 2}>$ $\varepsilon / 2$. Then this is dominated by

$$
\leq P\left(2 \delta\left|f_{k}-a^{1 / 2}\right|>\varepsilon / 2\right)+P\left(2 C_{\delta}\left|g_{n k}\right|>\varepsilon\right)
$$

Fix $n$ large enough that the second term is less than $\eta$ for all $k$. Now taking $k$ large enough, the above is less than $\eta$. It follows the expression in 74.6 .26 and consequently in 74.6 .25 converges to 0 in probability.

Now consider the other term 74.6.24 using the $n$ just determined. This term is of the form

$$
\begin{aligned}
& \sum_{j=1}^{q_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(s), X\left(t_{j+1}\right)-X\left(t_{j}\right)-P_{n}\left(M\left(t_{j+1}\right)-M\left(t_{j}\right)\right)\right\rangle d s= \\
& \quad \sum_{j=1}^{q_{k}-1} \int_{t_{j}}^{t_{j+1}}\left\langle Y(s), X_{k}^{r}(s)-X_{k}^{l}(s)-P_{n}\left(M_{k}^{r}(s)-M_{k}^{l}(s)\right)\right\rangle d s \\
& =\int_{t_{1}}^{t}\left\langle Y(s), X_{k}^{r}(s)-X_{k}^{l}(s)-P_{n}\left(M_{k}^{r}(s)-M_{k}^{l}(s)\right)\right\rangle d s
\end{aligned}
$$

where $M_{k}^{r}$ denotes the step function

$$
M_{k}^{r}(t)=\sum_{i=0}^{m_{k}-1} M\left(t_{i+1}\right) \mathscr{X}_{\left(t_{i}, t_{i+1}\right]}(t)
$$

and $M_{k}^{l}$ is defined similarly. The term

$$
\int_{t_{1}}^{t}\left\langle Y(s), P_{n}\left(M_{k}^{r}(s)-M_{k}^{l}(s)\right)\right\rangle d s
$$

converges to 0 for a.e. $\omega$ as $k \rightarrow \infty$ thanks to continuity of $t \rightarrow M(t)$. However, more is needed than this. Define the stopping time

$$
\tau_{p}=\inf \left\{t>0:\|M(t)\|_{W}>p\right\}
$$

Then $\tau_{p}=\infty$ for all $p$ large enough, this for a.e. $\omega$. Let

$$
\begin{gather*}
A_{k}=\left[\left|\int_{t_{1}}^{t}\left\langle Y(s), P_{n}\left(M_{k}^{r}(s)-M_{k}^{l}(s)\right)\right\rangle d s\right|>\varepsilon\right] \\
P\left(A_{k}\right)=\sum_{p=0}^{\infty} P\left(A_{k} \cap\left(\left[\tau_{p}=\infty\right] \backslash\left[\tau_{p-1}<\infty\right]\right)\right) \tag{74.6.27}
\end{gather*}
$$

Now

$$
\begin{aligned}
& P\left(A_{k} \cap\left(\left[\tau_{p}=\infty\right] \backslash\left[\tau_{p-1}<\infty\right]\right)\right) \leq P\left(A_{k} \cap\left(\left[\tau_{p}=\infty\right]\right)\right) \\
\leq & P\left(\left[\left|\int_{t_{1}}^{t}\left\langle Y(s), P_{n}\left(\left(M^{\tau_{p}}\right)_{k}^{r}(s)-\left(M^{\tau_{p}}\right)_{k}^{l}(s)\right)\right\rangle d s\right|>\varepsilon\right]\right)
\end{aligned}
$$

This is so because if $\tau_{p}=\infty$, then it has no effect but also it could happen that the defining inequality may hold even if $\tau_{p}<\infty$ hence the inequality. This is no larger than an expression of the form

$$
\begin{equation*}
\frac{C_{n}}{\varepsilon} \int_{\Omega} \int_{0}^{T}\|Y(s)\|_{V^{\prime}}\left\|\left(M^{\tau_{p}}\right)_{k}^{r}(s)-\left(M^{\tau_{p}}\right)_{k}^{l}(s)\right\|_{W} d s d P \tag{74.6.28}
\end{equation*}
$$

The inside integral converges to 0 by continuity of $M$. Also, thanks to the stopping time, the inside integral is dominated by an expression of the form

$$
\int_{0}^{T}\|Y(s)\|_{V^{\prime}} 2 p d s
$$

and this is a function in $L^{1}(\Omega)$ by assumption on $Y$. It follows that the integral in 74.6.28 converges to 0 as $k \rightarrow \infty$ by the dominated convergence theorem. Hence

$$
\lim _{k \rightarrow \infty} P\left(A_{k} \cap\left(\left[\tau_{p}=\infty\right]\right)\right)=0
$$

Since the sets $\left[\tau_{p}=\infty\right] \backslash\left[\tau_{p-1}<\infty\right]$ are disjoint, the sum of their probabilities is finite. Hence there is a dominating function in 74.6 .27 and so, by the dominated convergence theorem applied to the sum,

$$
\lim _{k \rightarrow \infty} P\left(A_{k}\right)=\sum_{p=0}^{\infty} \lim _{k \rightarrow \infty} P\left(A_{k} \cap\left(\left[\tau_{p}=\infty\right] \backslash\left[\tau_{p-1}<\infty\right]\right)\right)=0
$$

Thus $\int_{t_{1}}^{t}\left\langle Y(s), P_{n}\left(M_{k}^{r}(s)-M_{k}^{l}(s)\right)\right\rangle d s$ converges to 0 in probability as $k \rightarrow \infty$.
Now consider

$$
\begin{aligned}
&\left|\int_{t_{1}}^{t}\left\langle Y(s), X_{k}^{r}(s)-X_{k}^{l}(s)\right\rangle d s\right| \leq \int_{0}^{T}\left|\left\langle Y(s), X_{k}^{r}(s)-X(s)\right\rangle\right| d s \\
&+\int_{0}^{T}\left|\left\langle Y(s), X_{k}^{l}(s)-X(s)\right\rangle\right| d s \\
& \leq 2\|Y(\cdot, \omega)\|_{L^{p^{\prime}}(0, T)}\left(2^{-k}\right)^{1 / p}
\end{aligned}
$$

for all $k$ large enough, this by Lemma 74.5.1. Therefore,

$$
\sum_{j=1}^{q_{k}-1}\left\langle B\left(\Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right), \Delta X\left(t_{j}\right)-\Delta M\left(t_{j}\right)\right\rangle
$$

converges to 0 in probability. This establishes the desired formula for $t \in D$.
In fact, the formula 74.6.20 is valid for all $t \in N_{\omega}^{C}$.
Theorem 74.6.2 In Situation 74.1.1, for $\omega$ off a set of measure zero, it follows that for every $t \in N_{\omega}^{C}$,

$$
\langle B X(t), X(t)\rangle=\left\langle B X_{0}, X_{0}\right\rangle+\int_{0}^{t} 2\langle Y(s), X(s)\rangle d s
$$

$$
\begin{equation*}
\left[R^{-1} B M, M\right](t)+2 \int_{0}^{t}\langle B X, d M\rangle \tag{74.6.29}
\end{equation*}
$$

Also, there exists a unique continuous, progressively measurable function which is denoted here as $\langle B X, X\rangle$ such that it equals $\langle B X(t), X(t)\rangle$ for a.e. $t$ and $\langle B X, X\rangle(t)$ equals the right side of the above for all $t$. In addition to this,

$$
\begin{gather*}
E(\langle B X, X\rangle(t))= \\
E\left(\left\langle B X_{0}, X_{0}\right\rangle\right)+E\left(\int_{0}^{t} 2\langle Y(s), X(s)\rangle d s+\left[R^{-1} B M, M\right](t)\right) \tag{74.6.30}
\end{gather*}
$$

Also the quadratic variation of the stochastic integral in 74.6.29 is dominated by

$$
\begin{equation*}
\int_{0}^{t}\|B X\|_{W^{\prime}}^{2} d[M] \tag{74.6.31}
\end{equation*}
$$

Also $t \rightarrow B X(t)$ is continuous with values in $W^{\prime}$ for $t \in N_{\omega}^{C}$.
Proof: Let $t \in N_{\omega}^{C} \backslash D$. For $t>0$, let $t(k)$ denote the largest point of $\mathscr{P}_{k}$ which is less than $t$. Suppose $t(m)<t(k)$. Hence $m \leq k$. Then

$$
B X(t(m))=B X_{0}+\int_{0}^{t(m)} Y(s) d s+B M(t(m))
$$

a similar formula holding for $X(t(k))$. Thus for $t>t(m), t \in N_{\omega}^{C}$,

$$
B(X(t)-X(t(m)))=\int_{t(m)}^{t} Y(s) d s+B(M(t)-M(t(m)))
$$

which is the same sort of thing studied so far except that it starts at $t(m)$ rather than at 0 and $B X_{0}=0$. Therefore, from Lemma 74.6.1 it follows

$$
\begin{align*}
& \langle B(X(t(k))-X(t(m))), X(t(k))-X(t(m))\rangle \\
= & \int_{t(m)}^{t(k)} 2\langle Y(s), X(s)-X(t(m))\rangle d s \\
+ & {\left[R^{-1} B M, M\right](t(k))-\left[R^{-1} B M, M\right](t(m)) } \\
& +2 \int_{t(m)}^{t(k)}\langle B(X-X(t(m))), d M\rangle \tag{74.6.32}
\end{align*}
$$

Consider that last term. It equals

$$
\begin{equation*}
2 \int_{t(m)}^{t(k)}\left\langle B\left(X-X_{m}^{l}\right), d M\right\rangle \tag{74.6.33}
\end{equation*}
$$

This is dominated by

$$
2\left|\int_{0}^{t(k)}\left\langle B\left(X-X_{m}^{l}\right), d M\right\rangle-\int_{0}^{t(m)}\left\langle B\left(X-X_{m}^{l}\right), d M\right\rangle\right|
$$

$$
\leq 4 \sup _{t \in[0, T]}\left|\int_{0}^{t}\left\langle B\left(X-X_{m}^{l}\right), d M\right\rangle\right|
$$

In Lemma 74.5.2 the above expression converges to 0 . It follows there is a set of measure 0 including the earlier one such that for $\omega$ not in that set, 74.6 .33 converges to 0 in $\mathbb{R}$. Similar reasoning shows the first term on the right in the non stochastic integral of 74.6.32 is dominated by an expression of the form

$$
4 \int_{0}^{T}\left|\left\langle Y(s), X(s)-X_{m}^{l}(s)\right\rangle\right| d s
$$

which clearly converges to 0 thanks to Lemma 74.5.1. Finally, it is obvious that

$$
\lim _{m, k \rightarrow \infty}\left[R^{-1} B M, M\right](t(k))-\left[R^{-1} B M, M\right](t(m))=0 \text { for a.e. } \omega
$$

due to the continuity of the quadratic variation.
This shows that for $\omega$ off a set of measure 0

$$
\lim _{m, k \rightarrow \infty}\langle B(X(t(k))-X(t(m))), X(t(k))-X(t(m))\rangle=0
$$

Then for $x \in W$,

$$
\begin{aligned}
& |\langle B(X(t(k))-X(t(m))), x\rangle| \\
\leq & \langle B(X(t(k))-X(t(m))), X(t(k))-X(t(m))\rangle^{1 / 2}\langle B x, x\rangle^{1 / 2} \\
\leq & \langle B(X(t(k))-X(t(m))), X(t(k))-X(t(m))\rangle^{1 / 2}\|B\|^{1 / 2}\|x\|_{W}
\end{aligned}
$$

and so

$$
\lim _{m, k \rightarrow \infty}\|B X(t(k))-B X(t(m))\|_{W^{\prime}}=0
$$

Recall $t$ was arbitrary and $\{t(k)\}$ is a sequence converging to $t$. Then the above has shown that $\{B X(t(k))\}_{k=1}^{\infty}$ is a convergent sequence in $W^{\prime}$. Does it converge to $B X(t)$ ? Let $\xi(t) \in$ $W^{\prime}$ be what it converges to. Letting $v \in V$ then, since the integral equation shows that $t \rightarrow B X(t)$ is continuous into $V^{\prime}$,

$$
\langle\xi(t), v\rangle=\lim _{k \rightarrow \infty}\langle B X(t(k)), v\rangle=\langle B X(t), v\rangle,
$$

and now, since $V$ is dense in $W$, this implies $\xi(t)=B X(t)=B(X(t))$ since $t \notin N_{\omega}$. Recall also that it was shown earlier that $B X$ is weakly continuous into $W^{\prime}$ on $[0, T]$ hence the strong convergence of $\{B X(t(k))\}_{k=1}^{\infty}$ in $W^{\prime}$ implies that it converges to $B X(t)$, this for any $t \in N_{\omega}^{C}$.

For every $t \in D$ and for $\omega$ off the exceptional set of measure zero described earlier,

$$
\begin{gather*}
\langle B(X(t)), X(t)\rangle=\left\langle B X_{0}, X_{0}\right\rangle+\int_{0}^{t} 2\langle Y(s), X(s)\rangle d s+ \\
{\left[R^{-1} B M, M\right](t)+2 \int_{0}^{t}\langle B X, d M\rangle} \tag{74.6.34}
\end{gather*}
$$

Does this formula hold for all $t \in[0, T]$ ? Maybe not. However, it will hold for $t \notin N_{\omega}$. So let $t \notin N_{\omega}$.

$$
\begin{gathered}
|\langle B X(t(k)), X(t(k))\rangle-\langle B X(t), X(t)\rangle| \\
\leq \begin{array}{l}
|\langle B X(t(k)), X(t(k))\rangle-\langle B X(t), X(t(k))\rangle| \\
+|\langle B X(t), X(t(k))\rangle-\langle B X(t), X(t)\rangle|
\end{array} \\
=|\langle B(X(t(k))-X(t)), X(t(k))\rangle|+|\langle B(X(t(k))-X(t)), X(t)\rangle|
\end{gathered}
$$

Then using the Cauchy Schwarz inequality on each term,

$$
\begin{aligned}
\leq & \langle B(X(t(k))-X(t)), X(t(k))-X(t)\rangle^{1 / 2} \\
& \cdot\left(\langle B X(t(k)), X(t(k))\rangle^{1 / 2}+\langle B X(t), X(t)\rangle^{1 / 2}\right)
\end{aligned}
$$

As before, one can use the lower semicontinuity of

$$
t \rightarrow\langle B(X(t(k))-X(t)), X(t(k))-X(t)\rangle
$$

on $N_{\omega}^{C}$ along with the boundedness of $\langle B X(t), X(t)\rangle$ also shown earlier $\mathbf{o f f} N_{\omega}$ to conclude

$$
\begin{gathered}
|\langle B X(t(k)), X(t(k))\rangle-\langle B X(t), X(t)\rangle| \\
\leq C\langle B(X(t(k))-X(t)), X(t(k))-X(t)\rangle^{1 / 2} \\
\leq C \lim _{m \rightarrow \infty} \inf _{m}\langle B(X(t(k))-X(t(m))), X(t(k))-X(t(m))\rangle^{1 / 2}<\varepsilon
\end{gathered}
$$

provided $k$ is sufficiently large. Since $\varepsilon$ is arbitrary,

$$
\lim _{k \rightarrow \infty}\langle B X(t(k)), X(t(k))\rangle=\langle B X(t), X(t)\rangle
$$

It follows that the formula 74.6 .34 is valid for all $t \in N_{\omega}^{C}$. Now define the function $\langle B X, X\rangle(t)$ as

$$
\langle B X, X\rangle(t) \equiv\left\{\begin{array}{c}
\langle B(X(t)), X(t)\rangle, t \notin N_{\omega} \\
\text { The right side of 74.6.34 if } t \in N_{\omega}
\end{array}\right.
$$

Then in short, $\langle B X, X\rangle(t)$ equals the right side of 74.6 .34 for all $t \in[0, T]$ and is consequently progressively measurable and continuous. Furthermore, for a.e. $t$, this function equals $\langle B(X(t)), X(t)\rangle$. Since it is known on a dense subset, it must be unique.

This implies that $t \rightarrow B X(t)$ is continuous with values in $W^{\prime}$ for $t \in N_{\omega}^{C}$. Here is why. The fact that the formula 74.6 .34 holds for all $t \in N_{\omega}^{C}$ implies that $t \rightarrow\langle B X(t), X(t)\rangle$ is continuous on $N_{\omega}^{C}$. Then for $x \in W, t, s \in N_{\omega}^{C}$

$$
\begin{equation*}
|\langle B X(t)-B X(s), x\rangle| \leq\langle B(X(t)-X(s)), X(t)-X(s)\rangle^{1 / 2}\|B\|^{1 / 2}\|x\|_{W} \tag{74.6.35}
\end{equation*}
$$

Also

$$
\begin{aligned}
& \langle B(X(t)-X(s)), X(t)-X(s)\rangle \\
= & \langle B X(t), X(t)\rangle+\langle B X(s), X(s)\rangle-2\langle B X(t), X(s)\rangle
\end{aligned}
$$

By weak continuity of $t \rightarrow B X(t)$ shown earlier,

$$
\lim _{t \rightarrow s}\langle B X(t), X(s)\rangle=\langle B X(s), X(s)\rangle .
$$

Therefore,

$$
\lim _{t \rightarrow s}\langle B(X(t)-X(s)), X(t)-X(s)\rangle=0
$$

and so the inequality 74.6.35 implies the continuity of $t \rightarrow B X(t)$ into $W^{\prime}$ for $t \notin N_{\omega}$. Note that by assumption this function is continuous into $V^{\prime}$ for all $t$.

Now consider the claim about the expectation. Use the function $\langle B X, X\rangle$ to define a stopping time as

$$
\tau_{p} \equiv \inf \{t>0:\langle B X, X\rangle(t)>p\}
$$

This is the first hitting time of a continuous process and so it is a valid stopping time. Using this, leads to

$$
\begin{gather*}
\langle B X, X\rangle^{\tau_{p}}(t)=\left\langle B X_{0}, X_{0}\right\rangle+\int_{0}^{t} \mathscr{X}_{\left[0, \tau_{p}\right]} 2\langle Y(s), X(s)\rangle d s+ \\
{\left[R^{-1} B M, M\right]^{\tau_{p}}(t)+2 \int_{0}^{t} \mathscr{X}_{\left[0, \tau_{p}\right]}\langle B X, d M\rangle} \tag{74.6.36}
\end{gather*}
$$

The term at the end is now a martingale because $\mathscr{X}_{\left[0, \tau_{p}\right]} B X$ is bounded. Hence the expectation of the martingale at the end equals 0 . Thus you obtain

$$
\begin{gathered}
E\left(\langle B X, X\rangle^{\tau_{p}}(t)\right)=E\left(\left\langle B X_{0}, X_{0}\right\rangle\right) \\
+E\left(\int_{0}^{t} \mathscr{X}_{\left[0, \tau_{p}\right]} 2\langle Y(s), X(s)\rangle d s\right)+E\left(\left[R^{-1} B M, M\right]^{\tau_{p}}(t)\right)
\end{gathered}
$$

Now use the monotone convergence theorem and the dominated convergence theorem to pass to a limit as $p \rightarrow \infty$ and obtain 74.6.30. The claim about the quadratic variation follows from Theorem 66.0.22.

What of the special case where $W=H=H^{\prime}$ and you are in the context of a Gelfand triple

$$
V \subseteq H=H^{\prime} \subseteq V^{\prime}
$$

and $B$ is simply the identity. Then we obtain the following theorem as a special case.
Theorem 74.6.3 In Situation 74.1.1 in which $W=H=H^{\prime}$ and $B=I$, it follows that off a set of measure zero, for every $t \in[0, T]$, there is a set of measure zero $N$ such that for $\omega \notin N$, there is a continuous function $\langle X, X\rangle$ which equals $|X(t)|_{H}^{2}$ for a.e. $t$ such that

$$
\begin{align*}
\langle X, X\rangle(t) & =\left|X_{0}\right|_{H}^{2}+\int_{0}^{t} 2\langle Y(s), X(s)\rangle d s \\
+ & {[M](t)+2 \int_{0}^{t}(X, d M) } \tag{74.6.37}
\end{align*}
$$

Furthermore,off a set of measure zero, $t \rightarrow X(t)$ is continuous as a map into $H$ for a.e. $\omega$. In addition to this,

$$
\begin{gather*}
E(\langle X, X\rangle(t))= \\
E\left(\left|X_{0}\right|^{2}\right)+E\left(\int_{0}^{t} 2\langle Y(s), X(s)\rangle d s\right)+E([M](t)) \tag{74.6.38}
\end{gather*}
$$

The quadratic variation of the stochastic integral satisfies

$$
\left[\int_{0}^{(\cdot)}(X, d M)\right](t) \leq \int_{0}^{t}\|X\|_{H}^{2} d[M]
$$

It is more attractive to write $|X(t)|_{H}^{2}$ in place of $\langle X, X\rangle(t)$. However, I guess this is not strictly right although the discrepancy is only on a set of measure zero so it seems fairly harmless to indulge in this sloppiness. However, for $t \notin N_{\omega}$,

$$
|X(t)|_{H}^{2}=\sum_{i}\left(X(t), e_{i}\right)^{2}
$$

where the orthonormal basis $\left\{e_{i}\right\}$ is in $V$. Then for $s \in N_{\omega}$, you can get the following. Let $t_{n} \rightarrow s$ where $t_{n} \in N_{\omega}$. Then in the above notation,

$$
\sum_{i}\left\langle X(s), e_{i}\right\rangle^{2} \leq \lim \inf _{n \rightarrow \infty} \sum_{i}\left(X\left(t_{n}\right), e_{i}\right)_{H}^{2}=\lim _{n \rightarrow \infty}\left|X\left(t_{n}\right)\right|_{H}^{2} \leq C(\omega)
$$

It follows that in fact $X(s) \in H$ and you can take $X(s)=\sum_{i}\left\langle X(s), e_{i}\right\rangle e_{i} \in H$ because $\sum_{i}\left\langle X(s), e_{i}\right\rangle^{2}<\infty$. Hence

$$
|X(s)|^{2}=\sum_{i}\left\langle X(s), e_{i}\right\rangle^{2} \leq \lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty}\left|X\left(t_{n}\right)\right|_{H}^{2}
$$

so $X$ has values in $H$ and is lower semicontinuous on $[0, T]$.

## Chapter 75

## Some Nonlinear Operators

In this chapter is a description and properties of some standard nonlinear maps.

### 75.1 An Assortment Of Nonlinear Operators

Definition 75.1.1 For $V$ a real Banach space, $A: V \rightarrow V^{\prime}$ is a pseudomonotone map if whenever

$$
\begin{equation*}
u_{n} \rightharpoonup u \tag{75.1.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\lim \sup _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-u\right\rangle \leq 0 \tag{75.1.2}
\end{equation*}
$$

it follows that for all $v \in V$,

$$
\begin{equation*}
\lim \inf _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-v\right\rangle \geq\langle A u, u-v\rangle \tag{75.1.3}
\end{equation*}
$$

The half arrows denote weak convergence.
Definition 75.1.2 $A: V \rightarrow V^{\prime}$ is monotone if for all $v, u \in V$,

$$
\langle A u-A v, u-v\rangle \geq 0
$$

and $A$ is Hemicontinuous iffor all $v, u \in V$,

$$
\lim _{t \rightarrow 0+}\langle A(u+t(v-u)), u-v\rangle=\langle A u, u-v\rangle .
$$

Theorem 75.1.3 Let $V$ be a Banach space and let $A: V \rightarrow V^{\prime}$ be monotone and hemicontinuous. Then $A$ is pseudomonotone.

Proof: Let $A$ be monotone and Hemicontinuous. First here is a claim.
Claim: If 75.1.1 and 75.1.2 hold, then $\lim _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-u\right\rangle=0$.
Proof of the claim: Since $A$ is monotone,

$$
\left\langle A u_{n}-A u, u_{n}-u\right\rangle \geq 0
$$

so

$$
\left\langle A u_{n}, u_{n}-u\right\rangle \geq\left\langle A u, u_{n}-u\right\rangle .
$$

Therefore,

$$
0=\lim \inf _{n \rightarrow \infty}\left\langle A u, u_{n}-u\right\rangle \leq \lim \inf _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-u\right\rangle \leq \lim \sup _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-u\right\rangle \leq 0
$$

Now using that $A$ is monotone again, then letting $t>0$,

$$
\left\langle A u_{n}-A(u+t(v-u)), u_{n}-u+t(u-v)\right\rangle \geq 0
$$

and so

$$
\left\langle A u_{n}, u_{n}-u+t(u-v)\right\rangle \geq\left\langle A(u+t(v-u)), u_{n}-u+t(u-v)\right\rangle .
$$

Taking the liminf on both sides and using the claim and $t>0$,

$$
t \lim \inf _{n \rightarrow \infty}\left\langle A u_{n}, u-v\right\rangle \geq t\langle A(u+t(v-u)),(u-v)\rangle
$$

Next divide by $t$ and use the Hemicontinuity of $A$ to conclude that

$$
\lim \inf _{n \rightarrow \infty}\left\langle A u_{n}, u-v\right\rangle \geq\langle A u, u-v\rangle
$$

From the claim,

$$
\begin{aligned}
\lim \inf _{n \rightarrow \infty}\left\langle A u_{n}, u-v\right\rangle & =\lim \inf _{n \rightarrow \infty}\left(\left\langle A u_{n}, u_{n}-v\right\rangle+\left\langle A u_{n}, u-u_{n}\right\rangle\right) \\
& =\lim \inf _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-v\right\rangle \geq\langle A u, u-v\rangle .
\end{aligned}
$$

Monotonicity is very important in the above proof. The next example shows that even if the operator is linear and bounded, it is not necessarily pseudomonotone.

Example 75.1.4 Let $H$ be any Hilbert space and let $A: H \rightarrow H^{\prime}$ be given by

$$
\langle A x, y\rangle \equiv(-x, y)_{H}
$$

Then A fails to be pseudomonotone.
Proof: Let $\left\{x_{n}\right\}_{n=1}^{\infty}$ be an orthonormal set of vectors in $H$. Then Parsevall's inequality implies

$$
\|x\|^{2} \geq \sum_{n=1}^{\infty}\left|\left(x_{n}, x\right)\right|^{2}
$$

and so for any $x \in H, \lim _{n \rightarrow \infty}\left(x_{n}, x\right)=0$. Thus $x_{n} \rightharpoonup 0 \equiv x$. Also

$$
\begin{gathered}
\lim \sup _{n \rightarrow \infty}\left\langle A x_{n}, x_{n}-x\right\rangle= \\
\lim \sup _{n \rightarrow \infty}\left\langle A x_{n}, x_{n}-0\right\rangle=\lim \sup _{n \rightarrow \infty}\left(-| | x_{n} \|^{2}\right)=-1 \leq 0
\end{gathered}
$$

If $A$ were pseudomonotone, we would need to be able to conclude that for all $y \in H$,

$$
\lim \inf _{n \rightarrow \infty}\left\langle A x_{n}, x_{n}-y\right\rangle \geq\langle A x, x-y\rangle=0
$$

However,

$$
\lim \inf _{n \rightarrow \infty}\left\langle A x_{n}, x_{n}-0\right\rangle=-1<0=\langle A 0,0-0\rangle
$$

Now the following proposition is useful.
Proposition 75.1.5 Suppose $A: V \rightarrow V^{\prime}$ is pseudomonotone and bounded where $V$ is separable. Then it must be demicontinuous. This means that if $u_{n} \rightarrow u$, then $A u_{n} \rightharpoonup A u$.

Proof: Since $u_{n} \rightarrow u$ is strong convergence and since $A u_{n}$ is bounded, it follows

$$
\limsup _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-u\right\rangle=\lim _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-u\right\rangle=0
$$

Suppose this is not so that $A u_{n}$ converges weakly to $A u$. Since $A$ is bounded, there exists a subsequence, still denoted by $n$ such that $A u_{n} \rightharpoonup \xi$ weak $*$. I need to verify $\xi=A u$. From the above, it follows that for all $v \in V$

$$
\begin{aligned}
\langle A u, u-v\rangle & \leq \lim \inf _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-v\right\rangle \\
& =\lim \inf _{n \rightarrow \infty}\left\langle A u_{n}, u-v\right\rangle=\langle\xi, u-v\rangle
\end{aligned}
$$

Hence $\xi=A u$.
There is another type of operator which is more general than pseudomonotone.
Definition 75.1.6 Let $A: V \rightarrow V^{\prime}$ be an operator. Then $A$ is called type $M$ if whenever $u_{n} \rightharpoonup u$ and $A u_{n} \rightharpoonup \xi$, and

$$
\lim \sup _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}\right\rangle \leq\langle\xi, u\rangle
$$

it follows that $A u=\xi$.
Proposition 75.1.7 If $A$ is pseudomonotone, then $A$ is type $M$.
Proof: Suppose $A$ is pseudomonotone and $u_{n} \rightharpoonup u$ and $A u_{n} \rightharpoonup \xi$, and

$$
\lim \sup _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}\right\rangle \leq\langle\xi, u\rangle
$$

Then

$$
\lim \sup _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-u\right\rangle=\lim \sup _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}\right\rangle-\langle\xi, u\rangle \leq 0
$$

Hence

$$
\lim \inf _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-v\right\rangle \geq\langle A u, u-v\rangle
$$

for all $v \in V$. Consequently, for all $v \in V$,

$$
\begin{aligned}
\langle A u, u-v\rangle & \leq \liminf _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-v\right\rangle \\
& =\lim _{n \rightarrow \infty}\left(\left\langle A u_{n}, u-v\right\rangle+\left\langle A u_{n}, u_{n}-u\right\rangle\right) \\
& =\langle\xi, u-v\rangle+\lim _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-u\right\rangle \leq\langle\xi, u-v\rangle
\end{aligned}
$$

and so $A u=\xi$.
An interesting result is the the following which states that a monotone linear function added to a type M is also type M .

Proposition 75.1.8 Suppose $A: V \rightarrow V^{\prime}$ is type $M$ and suppose $L: V \rightarrow V^{\prime}$ is monotone, bounded and linear. Then $L+A$ is type $M$. Let $V$ be separable or reflexive so that the weak convergences in the following argument are valid.

Proof: Suppose $u_{n} \rightharpoonup u$ and $A u_{n}+L u_{n} \rightharpoonup \xi$ and also that

$$
\lim \sup _{n \rightarrow \infty}\left\langle A u_{n}+L u_{n}, u_{n}\right\rangle \leq\langle\xi, u\rangle
$$

Does it follow that $\xi=A u+L u$ ? Suppose not. There exists a further subsequence, still called $n$ such that $L u_{n} \rightharpoonup L u$. This follows because $L$ is linear and bounded. Then from monotonicity,

$$
\left\langle L u_{n}, u_{n}\right\rangle \geq\left\langle L u_{n}, u\right\rangle+\left\langle L(u), u_{n}-u\right\rangle
$$

Hence with this further subsequence, the limsup is no larger and so

$$
\lim \sup _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}\right\rangle+\lim _{n \rightarrow \infty}\left(\left\langle L u_{n}, u\right\rangle+\left\langle L(u), u_{n}-u\right\rangle\right) \leq\langle\xi, u\rangle
$$

and so

$$
\lim \sup _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}\right\rangle \leq\langle\xi-L u, u\rangle
$$

It follows since $A$ is type $M$ that $A u=\xi-L u$, which contradicts the assumption that $\xi \neq$ $A u+L u$.

There is also the following useful generalization of the above proposition.
Corollary 75.1.9 Suppose $A: V \rightarrow V^{\prime}$ is type $M$ and suppose $L: V \rightarrow V^{\prime}$ is monotone, bounded and linear. Then for $u_{0} \in V$ define $M(u) \equiv L\left(u-u_{0}\right)$. Then $M+A$ is type $M$. Let $V$ be separable or reflexive so that the weak convergences in the following argument are valid.

Proof: Suppose $u_{n} \rightharpoonup u$ and $A u_{n}+M u_{n} \rightharpoonup \xi$ and also that

$$
\lim \sup _{n \rightarrow \infty}\left\langle A u_{n}+M u_{n}, u_{n}\right\rangle \leq\langle\xi, u\rangle
$$

Does it follow that $\xi=A u+M u$ ? Suppose not. By assumption, $u_{n}-u_{0} \rightharpoonup u-u_{0}$ and so, since $L$ is bounded, there is a further subsequence, still called $n$ such that

$$
M u_{n}=L\left(u_{n}-u_{0}\right) \rightharpoonup L\left(u-u_{0}\right)=M u .
$$

Since $M$ is monotone,

$$
\left\langle M u_{n}-M u, u_{n}-u\right\rangle \geq 0
$$

Thus

$$
\left\langle M u_{n}, u_{n}\right\rangle-\left\langle M u_{n}, u\right\rangle-\left\langle M u, u_{n}\right\rangle+\langle M u, u\rangle \geq 0
$$

and so

$$
\left\langle M u_{n}, u_{n}\right\rangle \geq\left\langle M u_{n}, u\right\rangle+\left\langle M u, u_{n}-u\right\rangle
$$

Hence with this further subsequence, the lim sup is no larger and so

$$
\lim \sup _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}\right\rangle+\lim _{n \rightarrow \infty}\left(\left\langle M u_{n}, u\right\rangle+\left\langle M(u), u_{n}-u\right\rangle\right) \leq\langle\xi, u\rangle
$$

and so

$$
\lim \sup _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}\right\rangle \leq\langle\xi-M u, u\rangle
$$

It follows since $A$ is type $M$ that $A u=\xi-M u$, which contradicts the assumption that $\xi \neq A u+M u$.

The following is Browder's lemma. It is a very interesting application of the Brouwer fixed point theorem.

Lemma 75.1.10 (Browder) Let $K$ be a convex closed and bounded set in $\mathbb{R}^{n}$ and let $A$ : $K \rightarrow \mathbb{R}^{n}$ be continuous and $\mathbf{f} \in \mathbb{R}^{n}$. Then there exists $\mathbf{x} \in K$ such that for all $\mathbf{y} \in K$,

$$
(\mathbf{f}-A \mathbf{x}, \mathbf{y}-\mathbf{x}) \leq 0
$$

Proof: Let $P_{K}$ denote the projection onto $K$. Thus $P_{K}$ is Lipschitz continuous.

$$
\mathbf{x} \rightarrow P_{K}(\mathbf{f}-A \mathbf{x}+\mathbf{x})
$$

is a continuous map from $K$ to $K$. By the Brouwer fixed point theorem, it has a fixed point $\mathbf{x} \in K$. Therefore, for all $\mathbf{y} \in K$,

$$
(\mathbf{f}-A \mathbf{x}+\mathbf{x}-\mathbf{x}, \mathbf{y}-\mathbf{x})=(\mathbf{f}-A \mathbf{x}, \mathbf{y}-\mathbf{x}) \leq 0
$$

From this lemma, there is an interesting theorem on surjectivity.
Proposition 75.1.11 Let $A: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be continuous and coercive,

$$
\lim _{|\mathbf{x}| \rightarrow \infty} \frac{\left(A\left(\mathbf{x}+\mathbf{x}_{0}\right), \mathbf{x}\right)}{|\mathbf{x}|}=\infty
$$

for some $\mathbf{x}_{0}$. Then for all $\mathbf{f} \in \mathbb{R}^{n}$, there exists $\mathbf{x} \in \mathbb{R}^{n}$ such that $A \mathbf{x}=\mathbf{f}$.
Proof: Define the closed convex sets $B_{n} \equiv \overline{B\left(\mathbf{x}_{0}, n\right)}$. By Browder's lemma, there exists $\mathbf{x}_{n}$ such that

$$
\left(\mathbf{f}-A \mathbf{x}_{n}, \mathbf{y}-\mathbf{x}_{n}\right) \leq 0
$$

for all $\mathbf{y} \in B_{n}$. Then taking $\mathbf{y}=\mathbf{x}_{0}$, it follows from the coercivity condition that the $\mathbf{x}_{n}-\mathbf{x}_{0}$ are bounded. It follows that for large $n, \mathbf{x}_{n}$ is an interior point of $B_{n}$. Therefore,

$$
\left(\mathbf{f}-A \mathbf{x}_{n}, \mathbf{z}\right) \leq 0
$$

for all $\mathbf{z}$ in some open ball centered at $\mathbf{x}_{0}$. Hence $\mathbf{f}=A \mathbf{x}_{n}$.
Lemma 75.1.12 Let $A: V \rightarrow V^{\prime}$ be type $M$ and bounded and suppose $V$ is reflexive or $V$ is separable. Then $A$ is demicontinuous.

Proof: Suppose $u_{n} \rightarrow u$ and $A u_{n}$ fails to converge weakly to $A u$. Then there is a further subsequence, still denoted as $u_{n}$ such that $A u_{n} \rightharpoonup \zeta \neq A u$. Then thanks to the strong convergence, you have

$$
\lim \sup _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}\right\rangle=\left\langle\zeta, u_{n}\right\rangle
$$

which implies $\zeta=A u$ after all.
With these lemmas and the above proposition, there is a very interesting surjectivity result.

Theorem 75.1.13 Let $A: V \rightarrow V^{\prime}$ be type $M$, bounded, and coercive

$$
\begin{equation*}
\lim _{\|u\| \rightarrow \infty} \frac{\left\langle A\left(u+u_{0}\right), u\right\rangle}{\|u\|}=\infty \tag{75.1.4}
\end{equation*}
$$

for some $u_{0}$, where $V$ is a separable reflexive Banach space. Then $A$ is surjective.
Proof: Since $V$ is separable, there exists an increasing sequence of finite dimensional subspaces $\left\{V_{n}\right\}$ such that $\overline{\cup_{n} V_{n}}=V$. Say span $\left(v_{1}, \cdots, v_{n}\right)=V_{n}$. Then consider the following diagram.

$$
\begin{array}{lllll}
\mathbb{R}^{n} & \stackrel{\theta^{*}}{\leftarrow} & V_{n}^{\prime} & \stackrel{i^{*}}{\leftarrow} & V^{\prime} \\
\mathbb{R}^{n} & \stackrel{\theta}{\rightarrow} & V_{n} & \xrightarrow{i} & V
\end{array}
$$

Here the map $\theta$ is the one which does the following.

$$
\theta(\mathbf{x})=\sum_{i=1}^{n} x_{i} v_{i}
$$

The map $i$ is the inclusion map. Consider the map $\theta^{*} i^{*} A i \theta$. By Lemma 75.1.12 this map is continuous. The map $\theta$ is continuous, one to one, and onto. Thus its inverse is also continuous. Let $\mathbf{x}_{0}$ correspond to $u_{0}$. Then for some constant $C$,

$$
\frac{\left(\theta^{*} i^{*} A i \theta\left(\mathbf{x}+\mathbf{x}_{0}\right), \mathbf{x}\right)}{|\mathbf{x}|} \geq \frac{\left\langle A i \theta\left(\mathbf{x}+\mathbf{x}_{0}\right), i \theta \mathbf{x}\right\rangle}{C\|i \theta \mathbf{x}\|_{V}}
$$

and to say $|\mathbf{x}| \rightarrow \infty$ is the same as saying that $\|i \theta \mathbf{x}\|_{V} \rightarrow \infty$. Hence $\theta^{*} i^{*} A i \theta$ is coercive. Let $f \in V^{\prime}$. Then from 75.1.11, there exists $\mathbf{x}_{n}$ such that

$$
\theta^{*} i^{*} A i \theta \mathbf{x}_{n}=\theta^{*} i^{*} \mathbf{f}
$$

Thus, $i^{*} A i \theta \mathbf{x}_{n}=i^{*} f$ and this implies that for $v_{n} \equiv \theta \mathbf{x}_{n}$,

$$
i^{*} A i v_{n}=i^{*} f
$$

In other words,

$$
\begin{equation*}
\left\langle A v_{n}, y\right\rangle=\langle f, y\rangle \tag{75.1.5}
\end{equation*}
$$

for all $y \in V_{n}$. Then from the coercivity condition 75.1.4, the $v_{n}$ are bounded independent of $n$. Since $V$ is reflexive, there is a subsequence, still called $\left\{v_{n}\right\}$ which converges weakly to $v \in V$. Since $A$ is bounded, it can also be assumed that $A v_{n} \rightharpoonup \zeta \in V^{\prime}$. Then

$$
\lim \sup _{n \rightarrow \infty}\left\langle A v_{n}, v_{n}\right\rangle=\lim \sup _{n \rightarrow \infty}\left\langle f, v_{n}\right\rangle=\langle f, v\rangle
$$

Also, passing to the limit in 75.1.5,

$$
\langle\zeta, y\rangle=\langle f, y\rangle
$$

for any $y \in V_{n}$, this for any $n$. Since the union of these $V_{n}$ is dense, it follows that the above equation holds for all $y \in V$. Therefore, $f=\zeta$ and so

$$
\lim \sup _{n \rightarrow \infty}\left\langle A v_{n}, v_{n}\right\rangle=\lim \sup _{n \rightarrow \infty}\left\langle f, v_{n}\right\rangle=\langle f, v\rangle=\langle\zeta, v\rangle
$$

Since $A$ is type $M$,

$$
A v=\zeta=f
$$

### 75.2 Duality Maps

The duality map is an attempt to duplicate some of the features of the Riesz map in Hilbert space which is discussed in the chapter on Hilbert space.

Definition 75.2.1 A Banach space is said to be strictly convex if whenever $\|x\|=\|y\|$ and $x \neq y$, then

$$
\left\|\frac{x+y}{2}\right\|<\|x\| .
$$

$F: X \rightarrow X^{\prime}$ is said to be a duality map if it satisfies the following: a.) $\|F(x)\|=\|x\|^{p-1} . b$.) $F(x)(x)=\|x\|^{p}$, where $p>1$.

Duality maps exist. Here is why. Let

$$
F(x) \equiv\left\{x^{*}:\left\|x^{*}\right\| \leq\|x\|^{p-1} \text { and } x^{*}(x)=\|x\|^{p}\right\}
$$

Then $F(x)$ is not empty because you can let $f(\alpha x)=\alpha\|x\|^{p}$. Then $f$ is linear and defined on a subspace of $X$. Also

$$
\sup _{\|\alpha x\| \leq 1}|f(\alpha x)|=\sup _{\|\alpha x\| \leq 1}|\alpha|\|x\|^{p} \leq\|x\|^{p-1}
$$

Also from the definition,

$$
f(x)=\|x\|^{p}
$$

and so, letting $x^{*}$ be a Hahn Banach extension, it follows $x^{*} \in F(x)$. Also, $F(x)$ is closed and convex. It is clearly closed because if $x_{n}^{*} \rightarrow x^{*}$, the condition on the norm clearly holds and also the other one does too. It is convex because

$$
\left\|x^{*} \lambda+(1-\lambda) y^{*}\right\| \leq \lambda\left\|x^{*}\right\|+(1-\lambda)\left\|y^{*}\right\| \leq \lambda\|x\|^{p-1}+(1-\lambda)\|x\|^{p-1}
$$

If the conditions hold for $x^{*}$, then we can show that in fact $\left\|x^{*}\right\|=\|x\|^{p-1}$. This is because

$$
\left\|x^{*}\right\| \geq\left|x^{*}\left(\frac{x}{\|x\|}\right)\right|=\frac{1}{\|x\|}\left|x^{*}(x)\right|=\|x\|^{p-1}
$$

Now how many things are in $F(x)$ assuming the norm on $X^{\prime}$ is strictly convex? Suppose $x_{1}^{*}$, and $x_{2}^{*}$ are two things in $F(x)$. Then by convexity, so is $\left(x_{1}^{*}+x_{2}^{*}\right) / 2$. Hence by strict convexity, if the two are different, then

$$
\left\|\frac{x_{1}^{*}+x_{2}^{*}}{2}\right\|=\|x\|^{p-1}<\frac{1}{2}\left\|x_{1}^{*}\right\|+\frac{1}{2}\left\|x_{2}^{*}\right\|=\|x\|^{p-1}
$$

which is a contradiction. Therefore, $F$ is an actual mapping.
What are some of its properties? First is one which is similar to the Cauchy Schwarz inequality. Since $p-1=p / p^{\prime}$,

$$
\sup _{\|y\| \leq 1}|\langle F x, y\rangle|=\|x\|^{p / p^{\prime}}
$$

and so for arbitrary $y \neq 0$,

$$
\begin{aligned}
|\langle F x, y\rangle| & =\|y\|\left|\left\langle F x, \frac{y}{\|y\| \|}\right\rangle\right| \leq\|y\|\|x\|^{p / p^{\prime}} \\
& =|\langle F y, y\rangle|^{1 / p}|\langle F x, x\rangle|^{1 / p^{\prime}}
\end{aligned}
$$

Next we can show that $F$ is monotone.

$$
\begin{aligned}
\langle F x-F y, x-y\rangle & =\langle F x, x\rangle-\langle F x, y\rangle-\langle F y, x\rangle+\langle F y, y\rangle \\
& \geq\|x\|^{p}+\|y\|^{p}-\|y\|\|x\|^{p / p^{\prime}}-\|y\|^{p / p^{\prime}}\|x\| \\
\geq\|x\|^{p}+\|y\|^{p} & -\left(\frac{\|y\|^{p}}{p}+\frac{\|x\|^{p}}{p^{\prime}}\right)-\left(\frac{\|y\|^{p}}{p^{\prime}}+\frac{\|x\|^{p}}{p}\right)=0
\end{aligned}
$$

Next it can be shown that $F$ is hemicontinuous. By the construction, $F(x+t y)$ is bounded as $t \rightarrow 0$. Let $t \rightarrow 0$ be a subsequence such that

$$
F(x+t y) \rightarrow \xi \text { weak } *
$$

Then we ask: Does $\xi$ do what it needs to do in order to be $F(x)$ ? The answer is yes. First of all $\|F(x+t y)\|=\|x+t y\|^{p-1} \rightarrow\|x\|^{p-1}$. The set

$$
\left\{x^{*}:\left\|x^{*}\right\| \leq\|x\|^{p-1}+\varepsilon\right\}
$$

is closed and convex and so it is weak $*$ closed as well. For all small enough $t$, it follows $F(x+t y)$ is in this set. Therefore, the weak limit is also in this set and it follows $\|\xi\| \leq$ $\|x\|^{p-1}+\varepsilon$. Since $\varepsilon$ is arbitrary, it follows $\|\xi\| \leq\|x\|^{p-1}$. Is $\xi(x)=\|x\|^{p}$ ? We have

$$
\begin{aligned}
\|x\|^{p} & =\lim _{t \rightarrow 0}\|x+t y\|^{p}=\lim _{t \rightarrow 0}\langle F(x+t y), x+t y\rangle \\
& =\lim _{t \rightarrow 0}\langle F(x+t y), x\rangle=\langle\xi, x\rangle
\end{aligned}
$$

and so, $\xi$ does what it needs to do to be $F(x)$. This would be clear if $\|\xi\|=\|x\|^{p-1}$. However, $|\langle\xi, x\rangle|=\|x\|^{p}$ and so $\|\xi\| \geq\left|\left\langle\xi, \frac{x}{\|x\|}\right\rangle\right|=\|x\|^{p-1}$. Thus $\|\xi\|=\|x\|^{p-1}$ which shows $\xi$ does everyting it needs to do to equal $F(x)$ and so it is $F(x)$. Since this conclusion follows for any convergent sequence, it follows that $F(x+t y)$ converges to $F(x)$ weakly as $t \rightarrow 0$. This is what it means to be hemicontinuous. This proves the following theorem. One can show also that $F$ is demicontinuous which means strongly convergent sequences go to weakly convergent sequences. Here is a proof for the case where $p=2$. You can clearly do the same thing for arbitrary $p$.

Lemma 75.2.2 Let $F$ be a duality map for $p=2$ where $X, X^{\prime}$ are reflexive and have strictly convex norms. (If $X$ is reflexive, there is always an equivalent strictly convex norm [8].) Then $F$ is demicontinuous.

Proof: Say $x_{n} \rightarrow x$. Then does it follow that $F x_{n} \rightharpoonup F x$ ? Suppose not. Then there is a subsequence, still denoted as $x_{n}$ such that $x_{n} \rightarrow x$ but $F x_{n} \rightharpoonup y \neq F x$ where here $\rightharpoonup$ denotes weak convergence. This follows from the Eberlein Smulian theorem. Then

$$
\langle y, x\rangle=\lim _{n \rightarrow \infty}\left\langle F x_{n}, x_{n}\right\rangle=\lim _{n \rightarrow \infty}\left\|x_{n}\right\|^{2}=\|x\|^{2}
$$

Also, there exists $z,\|z\|=1$ and $\langle y, z\rangle \geq\|y\|-\varepsilon$. Then

$$
\|y\|-\varepsilon \leq\langle y, z\rangle=\lim _{n \rightarrow \infty}\left\langle F x_{n}, z\right\rangle \leq \lim \inf _{n \rightarrow \infty}\left\|F x_{n}\right\|=\lim \inf _{n \rightarrow \infty}\left\|x_{n}\right\|=\|x\|
$$

and since $\varepsilon$ is arbitrary, $\|y\| \leq\|x\|$. It follows from the above construction of $F x$, that $y=F x$ after all, a contradiction.

Theorem 75.2.3 Let $X$ be a reflexive Banach space with $X^{\prime}$ having strictly convex norm ${ }^{1}$. Then for $p>1$, there exists a mapping $F: X \rightarrow X^{\prime}$ which is bounded, monotone, hemicontinuous, coercive in the sense that $\lim _{|x| \rightarrow \infty}\langle F x, x\rangle /|x|=\infty$, which also satisfies the inequalities

$$
|\langle F x, y\rangle| \leq|\langle F x, x\rangle|^{1 / p^{\prime}}|\langle F y, y\rangle|^{1 / p}
$$

Note that these conclusions about duality maps show that they map onto the dual space.
The duality map was onto and it was monotone. This was shown above. Consider the form of a duality map for the $L^{p}$ spaces. Let $F: L^{p} \rightarrow\left(L^{p}\right)^{\prime}$ be the one which satisfies

$$
\|F f\|=\|f\|^{p-1},\langle F f, f\rangle=\|f\|^{p}
$$

Then in this case,

$$
F f=|f|^{p-2} \bar{f}
$$

This is because it does what it needs to do.

$$
\begin{aligned}
\|F f\|_{L^{p^{\prime}}} & =\left(\int_{\Omega}\left(|f|^{p-1}\right)^{p^{\prime}} d \mu\right)^{1 / p^{\prime}}=\left(\int_{\Omega}\left(|f|^{p / p^{\prime}}\right)^{p^{\prime}} d \mu\right)^{1 / p^{\prime}} \\
& =\left(\int_{\Omega}|f|^{p} d \mu\right)^{1-(1 / p)}=\left(\left(\int_{\Omega}|f|^{p} d \mu\right)^{1 / p}\right)^{p-1}=\|f\|_{L^{p}}^{p-1}
\end{aligned}
$$

while it is obvious that

$$
\langle F f, f\rangle=\int_{\Omega}|f|^{p} d \mu=\|f\|_{L^{p}(\Omega)}^{p}
$$

Now here is an interesting inequality which I will only consider in the case where the quantities are real valued.

Lemma 75.2.4 Let $p>2$. Then for $a, b$ real numbers,

$$
\left(|a|^{p-2} a-|b|^{p-2} b\right)(a-b) \geq C|a-b|^{p}
$$

for some constant $C$ independent of $a, b$.

[^44]Proof: There is nothing to show if $a=b$. Without loss of generality, assume $a>b$. Also assume $p \geq 2$. There is nothing to show if $p=2$. I want to show that there exists a constant $C$ such that for $a>b$,

$$
\begin{equation*}
\frac{|a|^{p-2} a-|b|^{p-2} b}{|a-b|^{p-1}} \geq C \tag{75.2.6}
\end{equation*}
$$

First assume also that $b \geq 0$. Now it is clear that as $a \rightarrow \infty$, the quotient above converges to 1. Take the derivative of this quotient. This yields

$$
(p-1)|a-b|^{p-2} \frac{|a|^{p-2}|a-b|-\left(|a|^{p-2} a-|b|^{p-2} b\right)}{|a-b|^{2 p-2}}
$$

Now remember $a>b$. Then the above reduces to

$$
(p-1)|a-b|^{p-2} b \frac{|b|^{p-2}-|a|^{p-2}}{|a-b|^{2 p-2}}
$$

Since $b \geq 0$, this is negative and so 1 would be a lower bound. Now suppose $b<0$. Then the above derivative is negative for $b<a \leq-b$ and then it is positive for $a>-b$. It equals 0 when $a=-b$. Therefore the quotient in 75.2.6 achieves its minimum value when $a=-b$. This value is

$$
\frac{|-b|^{p-2}(-b)-|b|^{p-2} b}{|-b-b|^{p-1}}=|b|^{p-2} \frac{-2 b}{|2 b|^{p-1}}=|b|^{p-2} \frac{1}{|2 b|^{p-2}}=\frac{1}{2^{p-2}}
$$

Therefore, the conclusion holds whenever $p \geq 2$. That is

$$
\left(|a|^{p-2} a-|b|^{p-2} b\right)(a-b) \geq \frac{1}{2^{p-2}}|a-b|^{p}
$$

This proves the lemma.
This holds for $p>1$ also, but I don't remember how to show this at this time.
However, in the context of strictly convex norms on the reflexive Banach space $X$, the following important result holds. I will give it for the case where $p=2$ since this is the case of most interest.

Theorem 75.2.5 Let $X$ be a reflexive Banach space and $X, X^{\prime}$ have strictly convex norms as discussed above. Let $F$ be the duality map with $p=2$. Then $F$ is strictly monotone. This means

$$
\langle F u-F v, u-v\rangle \geq 0
$$

and it equals 0 if and only if $u-v$.
Proof: First why is it monotone? By definition of $F,\langle F(u), u\rangle=\|u\|^{2}$ and $\|F(u)\|=$ $\|u\|$. Then

$$
|\langle F u, v\rangle|=\left|\left\langle F u, \frac{v}{\|v\|}\right\rangle\right|\|v\| \leq\|F u\|\|v\|=\|u\|\|v\|
$$

Hence

$$
\begin{aligned}
\langle F u-F v, u-v\rangle & =\|u\|^{2}+\|v\|^{2}-\langle F u, v\rangle-\langle F v, u\rangle \\
& \geq\|u\|^{2}+\|v\|^{2}-2\|u\|\|v\| \geq 0
\end{aligned}
$$

Now suppose $\|x\|=\|y\|=1$ but $x \neq y$. Then

$$
\left\langle F x, \frac{x+y}{2}\right\rangle \leq\left\|\frac{x+y}{2}\right\|<\frac{\|x\|+\|y\|}{2}=1
$$

It follows that

$$
\frac{1}{2}\langle F x, x\rangle+\frac{1}{2}\langle F x, y\rangle=\frac{1}{2}+\frac{1}{2}\langle F x, y\rangle<1
$$

and so

$$
\langle F x, y\rangle<1
$$

For arbitrary $x, y, x /\|x\| \neq y /\|y\|$

$$
\langle F x, y\rangle=\|x\|\|y\|\left\langle F\left(\frac{x}{\|x\|}\right),\left(\frac{y}{\|y\|}\right)\right\rangle
$$

It is easy to check that $F(\alpha x)=\alpha F(x)$. Therefore,

$$
|\langle F x, y\rangle|=\|x\|\|y\|\left\langle F\left(\frac{x}{\|x\|}\right),\left(\frac{y}{\|y\|}\right)\right\rangle<\|x\|\|y\|
$$

Now say that $x \neq y$ and consider

$$
\langle F x-F y, x-y\rangle
$$

First suppose $x=\alpha y$. Then the above is

$$
\begin{aligned}
\langle F(\alpha y)-F y,(\alpha-1) y\rangle & =(\alpha-1)\left(\langle F(\alpha y), y\rangle-\|y\|^{2}\right) \\
& =(\alpha-1)\left(\langle\alpha F(y), y\rangle-\|y\|^{2}\right) \\
& =(\alpha-1)^{2}\|y\|^{2}>0
\end{aligned}
$$

The other case is that $x /\|x\| \neq y /\|y\|$ and in this case,

$$
\begin{gathered}
\langle F x-F y, x-y\rangle=\|x\|^{2}+\|y\|^{2}-\langle F x, y\rangle-\langle F y, x\rangle \\
>\|x\|^{2}+\|y\|^{2}-2\|x\|\|y\| \geq 0
\end{gathered}
$$

Thus $F$ is strictly monotone as claimed.
Another useful observation about duality maps for $p=2$ is that $\left\|F^{-1} y^{*}\right\|_{V}=\left\|y^{*}\right\|_{V^{\prime}}$. This is because

$$
\left\|y^{*}\right\|_{V^{\prime}}=\left\|F F^{-1} y^{*}\right\|_{V^{\prime}}=\left\|F^{-1} y^{*}\right\|_{V}
$$

also from similar reasoning,

$$
\left\langle y^{*}, F^{-1} y^{*}\right\rangle=\left\langle F F^{-1} y^{*}, F^{-1} y^{*}\right\rangle=\left\|F^{-1} y^{*}\right\|_{V}^{2}=\left\|y^{*}\right\|_{V^{\prime}}^{2}
$$

## Chapter 76

## Implicit Stochastic Equations

### 76.1 Introduction

In this chapter, implicit evolution equations are considered. These are of the form

$$
B u(t, \omega)-B u_{0}(\omega)+\int_{0}^{t} A(s, u(t, \omega), \omega) d s=\int_{0}^{t} f(s) d s+B \int_{0}^{t} \Phi d W
$$

the term on the end being a stochastic integral. The novelty is in allowing $B$ to be an operator which could vanish or have other interesting features. Thus the integral equation could degenerate to a non stochastic elliptic equation. This generalization of evolution equations has proven useful in the study of deterministic evolution equations and we give some interesting examples which indicate that this may be true in the case of stochastic equations also. In any case, it is an interesting generalization and equations of the usual form are recovered by using a Gelfand triple in which $B=I$.

Like deterministic equations, there are many ways to consider stochastic equations. Here it is based on an approach due to Bardos and Brezis [14] which avoids the consideration of finite dimensional problems. A generalized Ito formula is summarized in the next section. It is Theorem 76.2.3.

### 76.2 Preliminary Results

Let $X$ have values in $W$ and satisfy the following

$$
\begin{equation*}
B X(t)=B X_{0}+\int_{0}^{t} Y(s) d s+B \int_{0}^{t} Z(s) d W(s) \tag{76.2.1}
\end{equation*}
$$

$X_{0} \in L^{2}(\Omega ; W)$ and is $\mathscr{F}_{0}$ measurable, where $Z$ is $\mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)$ progressively measurable and

$$
\|Z\|_{L^{2}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right)}<\infty .
$$

This is what is needed to define the stochastic integral in the above formula. Here $Q$ is a nonnegative self adjoint operator defined on a separable real Hilbert space $U$. In what follows, $J$ will denote a one to one Hilbert Schmidt operator mapping $Q^{1 / 2} U$ into another separable Hilbert space $U_{1}$. For more explanation on this situation see [108].

Assume $X, Y$ satisfy

$$
X \in K \equiv L^{p}([0, T] \times \Omega ; V), Y \in K^{\prime}=L^{p^{\prime}}\left([0, T] \times \Omega ; V^{\prime}\right)
$$

where $1 / p^{\prime}+1 / p=1, p>1$, and $X, Y$ are progressively measurable into $V$ and $V^{\prime}$ respectively.

The sense in which the equation 76.2 .1 holds is as follows. For a.e. $\omega$, the equation holds in $V^{\prime}$ for all $t \in[0, T]$. Assume that

$$
\begin{gathered}
X \in L^{2}([0, T] \times \Omega, W), \\
B X \in L^{2}\left([0, T] \times \Omega, \mathscr{B}([0, T]) \times \mathscr{F}, W^{\prime}\right), X \in L^{p}([0, T] \times \Omega, \mathscr{B}([0, T]) \times \mathscr{F}, V)
\end{gathered}
$$

Note that, since $X$ is progressively measurable into $V$, this implies that $B X$ is progressively measurable into $W^{\prime}$. Also $W(t)$ is a $J J^{*}$ Wiener process on $U_{1}$ in the following diagram. ( $W$ is a cylindrical Wiener process.)

$$
\begin{aligned}
& U \\
& \downarrow \quad Q^{1 / 2} \\
& Q^{1 / 2} U \\
& \downarrow \quad \Phi \\
& W
\end{aligned}
$$

We will also make use of the following generalization of familiar concepts from Hilbert space.

Lemma 76.2.1 Suppose $V, W$ are separable Banach spaces, $W$ also a Hilbert space such that $V$ is dense in $W$ and $B \in \mathscr{L}\left(W, W^{\prime}\right)$ satisfies

$$
\langle B x, x\rangle \geq 0,\langle B x, y\rangle=\langle B y, x\rangle, B \neq 0 .
$$

Then there exists a countable set $\left\{e_{i}\right\}$ of vectors in $V$ such that

$$
\left\langle B e_{i}, e_{j}\right\rangle=\delta_{i j}
$$

and for each $x \in W$,

$$
\langle B x, x\rangle=\sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}
$$

and also

$$
B x=\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i},
$$

the series converging in $W^{\prime}$.
Then in the above situation, we have the following fundamental estimate.
Lemma 76.2.2 In the above situation where, off a set of measure zero, 76.2.1 holds for all $t \in[0, T]$, and $X$ is progressively measurable into $V$,

$$
\begin{aligned}
& E\left(\sup _{t \in[0, T]}\langle B X, X\rangle(t)\right) \\
< & C\left(\|Y\|_{K^{\prime}},\|X\|_{K},\|Z\|_{J},\left\|\left\langle B X_{0}, X_{0}\right\rangle\right\|_{L^{1}(\Omega)}\right)<\infty .
\end{aligned}
$$

where $\langle B X, X\rangle(t)=\langle B(X(t)), X(t)\rangle$ a.e. and $\langle B X, X\rangle$ is progressively measurable and continuous in $t$.

$$
\begin{aligned}
J & =L^{2}\left([0, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U ; W\right)\right), K \equiv L^{p}([0, T] \times \Omega ; V) \\
K^{\prime} & \equiv L^{p^{\prime}}\left([0, T] \times \Omega ; V^{\prime}\right)
\end{aligned}
$$

Also, $C$ is a continuous function of its arguments and $C(0,0,0,0)=0$. Thus for a.e. $\omega$,

$$
\sup _{t \in[0, T]}\langle B X, X\rangle(t) \leq C(\omega)<\infty .
$$

For a.e. $\omega, t \rightarrow B X(t, \omega)$ is weakly continuous with values in $W^{\prime}$ for $t$ off a set of measure zero. Also $t \rightarrow\langle B X(t), X(t)\rangle$ is lower semicontinuous off a set of measure zero.

Then from this fundamental lemma, the following Ito formula is valid. The proof of this theorem follows the same methods used for a similar result in [108].

Theorem 76.2.3 Off a set of measure zero, for every $t \in[0, T]$,

$$
\begin{gather*}
\langle B X, X\rangle(t)=\left\langle B X_{0}, X_{0}\right\rangle+\int_{0}^{t}\left(2\langle Y(s), X(s)\rangle+\langle B Z, Z\rangle_{\mathscr{L}_{2}}\right) d s \\
+2 \int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B X \circ J d W \tag{76.2.2}
\end{gather*}
$$

Also

$$
\begin{gather*}
E(\langle B X, X\rangle(t))= \\
E\left(\left\langle B X_{0}, X_{0}\right\rangle\right)+E\left(\int_{0}^{t}\left(2\langle Y(s), X(s)\rangle+\langle B Z, Z\rangle_{\mathscr{L}_{2}}\right) d s\right) \tag{76.2.3}
\end{gather*}
$$

The quadratic variation of the stochastic integral is dominated by

$$
\begin{equation*}
C \int_{0}^{t}\|Z\|_{\mathscr{L}_{2}}^{2}\|B X\|_{W^{\prime}}^{2} d s \tag{76.2.4}
\end{equation*}
$$

for a suitable constant $C$. Also $t \rightarrow B X(t)$ is continuous with values in $W^{\prime}$ for $t \in N_{\omega}^{C}$.
We will often abuse the notation and write $\langle B X(t), X(t)\rangle$ instead of the more precise $\langle B X, X\rangle(t)$. No harm is done because these two are equal a.e.

In addition to the above, we will use the following basic theorems about nonlinear operators. This is Proposition 75.1.8 above.

Proposition 76.2.4 Suppose $A: V \rightarrow V^{\prime}$ is type $M$, see [91], and suppose $L: V \rightarrow V^{\prime}$ is monotone, bounded and linear. Here $V$ is a separable reflexive Banach space. Then $L+A$ is type $M$.

As an important example, we give the following definition.
Definition 76.2.5 Let $f:[0, T] \times \Omega \rightarrow V$

$$
\tau_{h} f(t, \omega) \equiv\left\{\begin{array}{l}
f(t-h, \omega) \text { if } t \geq h \\
0 \text { if } t<h
\end{array}\right.
$$

Then letting $B$ be a monotone nonnegative, self adjoint operator, $B: W \rightarrow W^{\prime}$ for $W$ a separable Hilbert space, consider the linear operator $L: L^{2}(0, T, W) \equiv \mathscr{W} \rightarrow L^{2}\left(0, T, W^{\prime}\right) \equiv$ $\mathscr{W}^{\prime}$ given as

$$
L u \equiv\left(\frac{I-\tau_{h}}{h}\right) B u .
$$

Is it the case that $L$ is monotone? Clearly it is linear and so it suffices to consider $\langle L u, u\rangle_{\mathscr{W}^{\prime}, \mathscr{W}}$ which equals

$$
\begin{align*}
& \frac{1}{h} \int_{0}^{T}\langle B u(t), u(t)\rangle d t-\frac{1}{h} \int_{h}^{T}\langle B u(t-h), u(t)\rangle d t \\
& =\frac{1}{h} \int_{0}^{T}\langle B u(t), u(t)\rangle d t-\frac{1}{h} \int_{0}^{T-h}\langle B u(t), u(t+h)\rangle d t \\
& \geq \frac{1}{h} \int_{0}^{T}\langle B u(t), u(t)\rangle d t \\
& -\frac{1}{h} \int_{0}^{T-h}\left(\frac{1}{2}\langle B u(t), u(t)\rangle+\frac{1}{2}\langle B u(t+h), u(t+h)\rangle\right) d t \\
& =\frac{1}{2 h} \int_{0}^{T-h}\langle B u(t), u(t)\rangle d t+\frac{1}{h} \int_{T-h}^{T}\langle B u(t), u(t)\rangle d t \\
& -\frac{1}{2 h} \int_{0}^{T-h}\langle B u(t+h), u(t+h)\rangle d t \\
& =\frac{1}{2 h} \int_{0}^{T-h}\langle B u(t), u(t)\rangle d t+\frac{1}{h} \int_{T-h}^{T}\langle B u(t), u(t)\rangle d t \\
& -\frac{1}{2 h} \int_{h}^{T}\langle B u(t), u(t)\rangle d t \\
& =\frac{1}{2 h} \int_{h}^{T-h}\langle B u(t), u(t)\rangle d t+\frac{1}{2 h} \int_{0}^{h}\langle B u(t), u(t)\rangle d t \\
& +\frac{1}{h} \int_{T-h}^{T}\langle B u(t), u(t)\rangle d t-\frac{1}{2 h} \int_{h}^{T}\langle B u(t), u(t)\rangle d t \\
& =\frac{1}{2 h} \int_{h}^{T-h}\langle B u(t), u(t)\rangle d t+\frac{1}{h} \int_{T-h}^{T}\langle B u(t), u(t)\rangle d t \\
& -\frac{1}{2 h} \int_{h}^{T-h}\langle B u(t), u(t)\rangle d t \\
& +\frac{1}{2 h} \int_{0}^{h}\langle B u(t), u(t)\rangle d t-\frac{1}{2 h} \int_{T-h}^{T}\langle B u(t), u(t)\rangle d t \\
& =\frac{1}{2 h} \int_{T-h}^{T}\langle B u(t), u(t)\rangle d t+\frac{1}{2 h} \int_{0}^{h}\langle B u(t), u(t)\rangle d t \geq 0 \tag{76.2.5}
\end{align*}
$$

The following is a restatement of Theorem 75.1.13

Theorem 76.2.6 Let $A: V \rightarrow V^{\prime}$ be type $M$, bounded, and coercive

$$
\begin{equation*}
\lim _{\|u\| \rightarrow \infty} \frac{\left\langle A\left(u+u_{0}\right), u\right\rangle}{\|u\|}=\infty \tag{76.2.6}
\end{equation*}
$$

for some $u_{0} \in V$, where $V$ is a separable reflexive Banach space. Then $A$ is surjective.
In addition, there is a fundamental definition and theorem about weak derivatives which will be used.

Definition 76.2.7 Let $f \in L^{1}\left(a, b, V^{\prime}\right)$ where $V^{\prime}$ is the dual of a Banach space $V$. Let $\mathscr{D}^{*}(a, b)$ linear mappings from $C_{c}^{\infty}(a, b)$ to $V^{\prime}$. Then we can consider $f \in \mathscr{D}^{*}(a, b)$, the linear transformations defined on $C_{c}^{\infty}(a, b)$ as follows.

$$
f(\phi) \equiv \int_{a}^{b} f \phi d s
$$

This is well defined due to regularity considerations for Lebesgue measure. Then define $D f \in \mathscr{D}^{*}(a, b)$ by

$$
D f(\phi) \equiv-\int_{a}^{b} f \phi^{\prime} d s
$$

To say that $D f \in L^{1}\left(a, b, V^{\prime}\right)$ is to say that there exists $g \in L^{1}\left(a, b, V^{\prime}\right)$ such that

$$
D f(\phi) \equiv-\int_{a}^{b} f \phi^{\prime} d s=\int_{a}^{b} g \phi d s
$$

for all $\phi \in C_{c}^{\infty}(a, b)$. Note that regularity considerations imply that $g$ is unique if it exists.
The following is Theorem 69.2.9.
Theorem 76.2.8 Suppose that $f$ and $D f$ are both in $L^{1}\left(a, b, V^{\prime}\right)$. Then $f$ is equal to $a$ continuous function a.e., still denoted by $f$ and

$$
f(x)=f(a)+\int_{a}^{x} D f(t) d t
$$

In the next section are theorems about how shifts in time relate to progressive measurability.

### 76.3 The Existence Of Approximate Solutions

The situation is as follows. There are spaces $V \subseteq W$ where $V$ is a reflexive separable Banach space and $W$ is a separable Hilbert space. It is assumed that $V$ is dense in $W$. Define the spaces

$$
\mathscr{V} \equiv L^{p}([0, T] \times \Omega, V), \mathscr{W} \equiv L^{2}([0, T] \times \Omega, W)
$$

where in each case, the $\sigma$ algebra of measurable sets will be the progressively measurable sets. Thus, from the Riesz representation theorem,

$$
\mathscr{V}^{\prime}=L^{p^{\prime}}\left([0, T] \times \Omega, V^{\prime}\right), \mathscr{W}^{\prime}=L^{2}\left([0, T] \times \Omega, W^{\prime}\right)
$$

It will be assumed for the sake of convenience that $p \geq 2$. It follows that

$$
\mathscr{V} \subseteq \mathscr{W}, \mathscr{W}^{\prime} \subseteq \mathscr{V}^{\prime}
$$

The entire presentation will be based on the following lemma.
Lemma 76.3.1 Let $\mathscr{V} \equiv L^{p}([0, T] \times \Omega, V)$ where $V$ is a separable Banach space and the $\sigma$ algebra of measurable sets consists of those which are progressively measurable. Then for $h \in(0, T), \tau_{h}: \mathscr{V} \rightarrow \mathscr{V}$.

Proof: First consider $Q$ which is a progressively measurable set. Is it the case that $\tau_{h} \mathscr{X}_{Q}$ is also progressively measurable? Define $Q+h$ as

$$
Q+h \equiv\{(t+h, \omega):(t, \omega) \in Q\}
$$

Then

$$
\tau_{h} \mathscr{X}_{Q}(t, \omega)=\left\{\begin{array}{l}
\mathscr{X}_{Q+h}(t, \omega) \text { if } t \geq h \\
0 \text { if } t<h
\end{array}\right.
$$

Is this function progressively measurable? For $(s, \omega) \in[0, t] \times \Omega$, we have the following

$$
\begin{gathered}
0<\alpha \leq 1,\left[(s, \omega): \tau_{h} \mathscr{X}_{Q}(s, \omega) \geq \alpha\right]=[h, t] \times \Omega \cap(Q+h) \\
\alpha>1,\left[(s, \omega): \tau_{h} \mathscr{X}_{Q}(s, \omega) \geq \alpha\right]=\emptyset \in \mathscr{B}([0, t]) \times \mathscr{F}_{t} \\
\alpha \leq 0,\left[(s, \omega): \tau_{h} \mathscr{X}_{Q}(s, \omega) \geq \alpha\right]=[0, t] \times \Omega \in \mathscr{B}([0, t]) \times \mathscr{F}_{t}
\end{gathered}
$$

It suffices to show that for $t \geq h,[h, t] \times \Omega \cap(Q+h)$ is $\mathscr{B}([0, t]) \times \mathscr{F}_{t}$ measurable. It is known that $[0, t] \times \Omega \cap Q$ is $\mathscr{B}([0, t]) \times \mathscr{F}_{t}$ measurable and also that $[0, t-h] \times \Omega \cap Q$ is $\mathscr{B}([0, t-h]) \times \mathscr{F}_{t-h}$ measurable. Let

$$
\mathscr{G} \equiv\left\{Q \in \mathscr{B}([0, t-h]) \times \mathscr{F}_{t-h}:[h, t] \times \Omega \cap Q+h \in \mathscr{B}([0, t]) \times \mathscr{F}_{t}\right\}
$$

First consider $I \times B$ where $I$ is an interval in $\mathscr{B}([0, t-h])$ and $B \in \mathscr{F}_{t-h}$. Then

$$
\begin{gathered}
=h+I \times B \\
{[h, t] \times \Omega \cap(I+h) \times B=I^{\prime} \times B}
\end{gathered}
$$

where $I^{\prime}$ is in $\mathscr{B}([0, t])$ and of course $B \in \mathscr{F}_{t-h} \subseteq \mathscr{F}_{t}$. Thus the sets of this form, are in $\mathscr{G}$. Next suppose $Q \in \mathscr{G}$. Is $Q^{C} \in \mathscr{G}$ ?

$$
\left([h, t] \times \Omega \cap\left(Q^{C}+h\right)\right) \cup[h, t] \times \Omega \cap(Q+h) \cup[0, h) \times \Omega=[0, t] \times \Omega
$$

Then all of these disjoint sets but the first are in $\mathscr{B}([0, t]) \times \mathscr{F}_{t}$. It follows that the first is also in $\mathscr{B}([0, t]) \times \mathscr{F}_{t}$. It is clear that $\mathscr{G}$ is also closed with respect to countable disjoint unions. Therefore, $\mathscr{G}$ contains the $\pi$ system of sets of the form $I \times B$ just described. It follows that $\mathscr{G}=\mathscr{B}([0, t-h]) \times \mathscr{F}_{t-h}$.

Now if $Q$ is progressively measurable, then $[0, t-h] \times \Omega \cap Q$ is $\mathscr{B}([0, t-h]) \times \mathscr{F}_{t-h}$ measurable and so from what was just shown, $[h, t] \times \Omega \cap Q+h \in \mathscr{B}([0, t]) \times \mathscr{F}_{t}$. Thus $\tau_{h} \mathscr{X}_{Q}$ is progressively measurable. It follows that if $f \in \mathscr{V}$, you could consider $\phi(f)$ for
$\phi \in V^{\prime}$ and the positive and negative parts of this function. Each of these is the limit of a sequence of simple functions involving combinations of indicator functions of the form $\mathscr{X}_{Q}$. Thus $\tau_{h} \phi(f)=\phi\left(\tau_{h} f\right)$ is the limit of simple functions involving combinations of functions $\tau_{h} \mathscr{X}_{Q}$ and, as just shown, these simple functions are progressively measurable. Thus $\tau_{h} f$ is also progressively measurable by the Pettis theorem.

This Lemma states that you can do $\tau_{h}$ to progressively measurable functions and end up with one which is progressively measurable. Let

$$
B \in \mathscr{L}\left(W, W^{\prime}\right)
$$

satisfy

$$
\begin{equation*}
\langle B x, y\rangle=\langle B y, x\rangle,\langle B x, x\rangle \geq 0 \tag{76.3.7}
\end{equation*}
$$

Also suppose that

$$
\begin{equation*}
A \text { is monotone and hemicontinuous from } \mathscr{V} \text { to } \mathscr{V}^{\prime} \tag{76.3.8}
\end{equation*}
$$

This means the operator is monotone:

$$
\langle A u-A u, u-v\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \geq 0
$$

and hemicontinuous:

$$
\lim _{t \rightarrow 0}\langle A(u+t v), w\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}=\langle A u, w\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}
$$

Also we assume that $A$ is bounded and takes the form

$$
A u(t, \omega)=A(t, u(t, \omega), \omega)
$$

for $u \in \mathscr{V}$. Such an operator is type $M$ and this is what we use. Such an operator is defined by:

If $u_{n} \rightarrow u$ weakly in $\mathscr{V}$ and $A u_{n} \rightarrow \xi$ weakly in $\mathscr{V}^{\prime}$ and $\lim \sup _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}\right\rangle \leq\langle\xi, u\rangle$
Then the above implies

$$
A u=\xi
$$

We define $\mathscr{V}_{\omega}$ as $L^{p}(0, T, V)$ with the definition of $\mathscr{V}_{\omega}^{\prime}$ similar, the subscript denoting that $\omega$ is fixed, the $\sigma$ algebra of measurable sets being the Borel sets, $\mathscr{B}([0, T])$. Also,

$$
\begin{equation*}
(t, u, \omega) \rightarrow A(t, u, \omega) \tag{76.3.9}
\end{equation*}
$$

is progressively measurable.
Suppose $A(\omega)$ is monotone and hemicontinuous and bounded from $\mathscr{V}_{\omega}$ to $\mathscr{V}_{\omega}^{\prime}$. Thus

$$
\begin{equation*}
A(\omega) \text { is type } M \text { from } \mathscr{V}_{\omega} \text { to } \mathscr{V}_{\omega}^{\prime} \tag{76.3.10}
\end{equation*}
$$

where

$$
A(\omega) u \equiv A(t, u, \omega)
$$

We assume the estimates found in the next lemma.

Lemma 76.3.2 If $p \geq 2$ and

$$
\begin{gather*}
\langle A(t, u, \omega), u\rangle_{V} \geq \delta\|u\|_{V}^{p}-c(t, \omega)  \tag{76.3.11}\\
\|A(t, u, \omega)\|_{V^{\prime}} \leq k\|u\|_{V}^{p-1}+c^{1 / p^{\prime}}(t, \omega) \tag{76.3.12}
\end{gather*}
$$

where $c \geq 0, c \in L^{1}([0, T] \times \Omega)$, then if $(t, \omega) \rightarrow q(t, \omega)$ is in $\mathscr{V}$,, it follows that for a.e. $\omega$, similar inequalities hold for $\bar{A}$ given by

$$
\begin{equation*}
\bar{A}(t, u, \omega) \equiv A(t, u+q(t, \omega), \omega) \tag{76.3.13}
\end{equation*}
$$

Proof: Letting $q$ be progressively measurable, $q(t, \omega) \in V$ only consider $\omega$ such that $t \rightarrow q(t, \omega)$ is in $L^{p}(0, T, V)$.

$$
\langle\bar{A}(t, u, \omega), u\rangle=
$$

$$
\langle A(t, u+q(t, \omega), \omega), u\rangle=\langle A(t, u+q(t, \omega), \omega), u+q(t, \omega)\rangle
$$

$$
-\langle A(t, u+q(t, \omega), \omega), q(t, \omega)\rangle
$$

$$
\geq \delta\|u+q(t, \omega)\|_{V}^{p}-k\|u+q(t, \omega)\|_{V}^{p-1}\|q(t, \omega)\|_{V}-c^{1 / p^{\prime}}(t, \omega)\|q(t, \omega)\|_{V}-c(t, \omega)
$$

$$
\geq \delta\|u+q(t, \omega)\|_{V}^{p}-k\|u+q(t, \omega)\|_{V}^{p-1}\|q(t, \omega)\|_{V}-\|q(t, \omega)\|_{V}^{p}-2 c(t, \omega)
$$

$$
\geq \frac{\delta}{2}\|u+q(t, \omega)\|_{V}^{p}-C(k, \delta, T)\|q(t, \omega)\|_{V}^{p}-2 c(t, \omega)
$$

Now

$$
\|u+q(t, \omega)\| \geq\|u\|-\|q(t, \omega)\|
$$

and so by convexity,

$$
\frac{\|u+q(t, \omega)\|^{p}+\|q(t, \omega)\|^{p}}{2} \geq\left(\frac{\|u+q(t, \omega)\|+\|q(t, \omega)\|}{2}\right)^{p} \geq\left(\frac{\|u\|}{2}\right)^{p}
$$

This implies

$$
\|u+q(t, \omega)\|^{p} \geq 2\left(\frac{\|u\|^{p}}{2^{p}}-\frac{\|q(t, \omega)\|^{p}}{2}\right)
$$

Therefore,

$$
\begin{gathered}
\langle\bar{A}(t, u, \omega), u\rangle= \\
\langle A(t, u+q(t, \omega), \omega), u\rangle \geq \frac{\delta}{2}\left(2\left(\frac{\|u\|^{p}}{2^{p}}-\frac{\|q(t, \omega)\|^{p}}{2}\right)\right) \\
-C(k, \delta, T)\|q(t, \omega)\|_{V}^{p}-2 c(t, \omega) \\
\geq \frac{\delta}{2^{p}}\|u\|^{p}-c^{\prime}(t, \omega)
\end{gathered}
$$

where $c^{\prime} \in L^{1}([0, T] \times \Omega)$.

Consider the other inequality. Let $\|z\|_{V} \leq 1$.Then

$$
|\langle A(t, u+q(t, \omega), \omega), z\rangle| \leq k\|u+q(t, \omega)\|^{p-1}+c^{1 / p^{\prime}}(t, \omega)
$$

Since $p \geq 2$, a convexity argument shows that

$$
\begin{aligned}
\langle A(t, u+q(t, \omega), \omega), z\rangle & \leq k\left(2^{p-2}\|u\|^{p-1}+2^{p-2}\|q(t, \omega)\|^{p-1}\right)+c^{1 / p^{\prime}}(t, \omega) \\
& =2^{p-2} k\|u\|^{p-1}+(\bar{c}(t, \omega))^{1 / p^{\prime}}
\end{aligned}
$$

where $\bar{c} \in L^{1}([0, T] \times \Omega)$. Thus the same two inequalities continue to hold.
In what follows, $c \geq 0$ and is in $L^{1}([0, T] \times \Omega)$, the $\sigma$ algebra being $\mathscr{B}([0, T]) \times \mathscr{F}_{T}$.

$$
\begin{gather*}
\langle A(t, u, \omega), u\rangle_{V} \geq \delta\|u\|_{V}^{p}-c(t, \omega)  \tag{76.3.14}\\
\|A(t, u, \omega)\|_{V^{\prime}} \leq k\|u\|_{V}^{p-1}+c^{1 / p^{\prime}}(t, \omega) \tag{76.3.15}
\end{gather*}
$$

Letting $\bar{A}$ be defined above in 76.3.13,

$$
\bar{A}(t, u, \omega) \equiv A(t, u+q, \omega) \equiv \bar{A}(\omega)(t, u)
$$

Assume the following pathwise uniqueness condition which is the hypothesis of the following lemma.

Lemma 76.3.3 Suppose it is true that whenever $u, v \in \mathscr{V}_{\omega}$ and

$$
\begin{equation*}
B u(t)-B v(t)+\int_{0}^{t} A(u)-A(v)=0 \tag{76.3.16}
\end{equation*}
$$

it follows that $u=v$. Then if

$$
\begin{align*}
(B u)^{\prime}+\bar{A}(\omega) u & =f \text { in } \mathscr{V}_{\omega}^{\prime}, B u(0)=B u_{0} \\
(B v)^{\prime}+\bar{A}(\omega) v & =f \text { in } \mathscr{V}_{\omega}^{\prime}, B v(0)=B u_{0} \tag{76.3.17}
\end{align*}
$$

it follows that $u=v$ in $\mathscr{V}_{\omega}$. Here $u_{0} \in W$.
Proof: If $(B u)^{\prime}+\bar{A}(\omega) u=f$ and $(B v)^{\prime}+\bar{A}(\omega) v=f$, then

$$
B u(t)-B v(t)+\int_{0}^{t} A(u+q)-A(v+q) d s=0
$$

Hence

$$
B(u(t)+q(t))-B(v(t)+q(t))+\int_{0}^{t} A(u+q)-A(v+q) d s=0
$$

and so $u+q=v+q$ showing that $u=v$.
We give the following measurability lemma.

Lemma 76.3.4 Suppose $f_{n}$ is progressively measurable and converges weakly to $\bar{f}$ in

$$
L^{\alpha}\left([0, T] \times \Omega, X, \mathscr{B}([0, T]) \times \mathscr{F}_{T}\right), \alpha>1
$$

where $X$ is a reflexive separable Banach space. Also suppose that for each $\omega \notin N$ a set of measure zero,

$$
f_{n}(\cdot, \omega) \rightarrow f(\cdot, \omega) \text { weakly in } L^{\alpha}(0, T, X)
$$

Then there is an enlarged set of measure zero, still denoted as $N$ such that for $\omega \notin N$,

$$
\bar{f}(\cdot, \omega)=f(\cdot, \omega) \text { in } L^{\alpha}(0, T, X)
$$

Also $\bar{f}$ is progressively measurable.
Proof: By the Pettis theorem, $\bar{f}$ is progressively measurable. Letting

$$
\phi \in L^{\alpha^{\prime}}\left([0, T] \times \Omega, X^{\prime}, \mathscr{B}([0, T]) \times \mathscr{F}_{T}\right),
$$

it is known that for a.e. $\omega$,

$$
\int_{0}^{T}\left\langle\phi(t, \omega), f_{n}(t, \omega)\right\rangle d t \rightarrow \int_{0}^{T}\langle\phi(t, \omega), f(t, \omega)\rangle d t
$$

Therefore, the function of $\omega$ on the right is at least $\mathscr{F}_{T}$ measurable. Now let

$$
g \in L^{\infty}\left(\Omega, X^{\prime}, \mathscr{F}_{T}\right)
$$

and let $\psi \in C([0, T])$. Then for $1<p \leq \alpha$,

$$
\begin{aligned}
& \int_{\Omega}\left|\int_{0}^{T}\left\langle g(\omega) \psi(t), f_{n}(t, \omega)\right\rangle d t\right|^{p} d P \\
\leq & C(T) \int_{\Omega}\|g\|_{L^{\infty}\left(\Omega, X^{\prime}\right)}^{p} \int_{0}^{T}|\psi(t)|^{p}\left\|f_{n}(t, \omega)\right\|_{X}^{p} d t d P \\
\leq & C(T, g, \psi) \int_{\Omega} \int_{0}^{T}\left\|f_{n}(t, \omega)\right\|_{X}^{p} d t d P \leq C<\infty
\end{aligned}
$$

for some $C$. Since $\int_{0}^{T}\left\langle g(\omega) \psi(t), f_{n}(t, \omega)\right\rangle d t$ is bounded in $L^{p}(\Omega)$ independent of $n$ because $\int_{\Omega} \int_{0}^{T}\left\|f_{n}(t, \omega)\right\|_{X}^{p} d t d P$ is given to be bounded, it follows that the functions

$$
\omega \rightarrow \int_{0}^{T}\left\langle g(\omega) \psi(t), f_{n}(t, \omega)\right\rangle d t
$$

are uniformly integrable and so it follows from the Vitali convergence theorem that

$$
\int_{\Omega} \int_{0}^{T}\left\langle g(\omega) \psi(t), f_{n}(t, \omega)\right\rangle d t d P \rightarrow \int_{\Omega} \int_{0}^{T}\langle g(\omega) \psi(t), f(t, \omega)\rangle d t d P
$$

But also from the assumed weak convergence to $\bar{f}$

$$
\int_{\Omega} \int_{0}^{T}\left\langle g(\omega) \psi(t), f_{n}(t, \omega)\right\rangle d t d P \rightarrow \int_{\Omega} \int_{0}^{T}\langle g(\omega) \psi(t), \bar{f}(t, \omega)\rangle d t d P
$$

It follows that

$$
\int_{\Omega}\left\langle g(\omega), \int_{0}^{T}(f-\bar{f}) \psi(t) d t\right\rangle d P=0
$$

This is true for every such $g \in L^{\infty}\left(\Omega, X^{\prime}\right)$, and so for a fixed $\psi \in C([0, T])$ and the Riesz representation theorem,

$$
\int_{\Omega}\left\|\int_{0}^{T}(f-\bar{f}) \psi(t) d t\right\|_{X} d P=0
$$

Therefore, there exists $N_{\psi}$ such that if $\omega \notin N_{\psi}$, then

$$
\int_{0}^{T}(f-\bar{f}) \psi(t) d t=0
$$

Enlarge $N$, the exceptional set to also include $\cup_{\psi \in \mathscr{D}} N_{\psi}$ where $\mathscr{D}$ is a countable dense subset of $C([0, T])$. Therefore, if $\omega \notin N$, then the above holds for all $\psi \in C([0, T])$. It follows that for such $\omega, f(t, \omega)=\bar{f}(t, \omega)$ for a.e. $t$. Therefore, $f(\cdot, \omega)=\bar{f}(\cdot, \omega)$ in $L^{\alpha}(0, T, X)$ for all $\omega \notin N$.

Then one can obtain the following existence theorem using a technique of Bardos and Brezis [14].

Lemma 76.3.5 Let $q \in \mathscr{V}$ and let the conditions 76.3.14-76.3.17 be valid. Let $f \in \mathscr{V}^{\prime}$ be given. Then for each $\omega$ off a set of measure zero, there exists $u(\cdot, \omega) \in \mathscr{V}_{\omega}$ such that $(B u)^{\prime}(\cdot, \omega) \in \mathscr{V}_{\omega}^{\prime}$ and

$$
B u(0, \omega)=0
$$

and also the following equation holds in $\mathscr{V}_{\omega}^{\prime}$ for a.e. $\omega$

$$
(B u)^{\prime}(\cdot, \omega)+\bar{A}(\omega)(\cdot, u(\cdot, \omega))=f(\cdot, \omega)
$$

In addition to this, it can be assumed that $(t, \omega) \rightarrow u(t, \omega)$ is progressively measurable into $V$. That is, for each $\omega$ off a set of measure zero, $t \rightarrow u(t, \omega)$ can be modified on a set of measure zero in $[0, T]$ such that the resulting $u$ is progressively measurable.

Proof: Consider the equation

$$
\begin{equation*}
L_{h} B u+\bar{A} u=\frac{1}{h}\left(I-\tau_{h}\right)(B u)+\bar{A} u=f \text { in } \mathscr{V}^{\prime} \tag{76.3.18}
\end{equation*}
$$

By Proposition 76.2.4 and Theorem 76.2.6, there exists a solution to the above equation if the left side is coercive. However, it was shown above in the computations leading to 76.2.5 that $L_{h} \circ B$ is monotone. Hence the coercivity follows right away from Lemma 76.3.2.

Thus 76.3.18 holds in $\mathscr{V}^{\prime}$. It follows that, indexing the solution by $h$,

$$
\int_{\Omega} \int_{0}^{T}\left\|\frac{1}{h}\left(I-\tau_{h}\right)\left(B u_{h}\right)+\bar{A} u_{h}-f\right\|_{V^{\prime}}^{p^{\prime}} d t d P=0
$$

and so there exists a set of measure zero $N_{h}$ such that for $\omega \notin N_{h}$, the following equation holds in $\mathscr{V}_{\omega}^{\prime}$

$$
\frac{1}{h}\left(I-\tau_{h}\right)\left(B u_{h}(\cdot, \omega)\right)+\bar{A}(\omega)\left(u_{h}(\cdot, \omega)\right)=f(\cdot, \omega)
$$

Let $h$ denote a sequence converging to 0 and let $N$ be a set of measure zero which includes $\cup_{h} N_{h}$.

Letting $u_{h} \in \mathscr{V}$ be the above solution to 76.3 .18 , it also follows from the above estimates 76.3.14-76.3.15 that for $\omega$ off $N,\left\|u_{h}(\cdot, \omega)\right\|_{\mathscr{V}_{\omega}}$ is bounded independent of $h$. Thus, for such $\omega$ off this set, there exists a subsequence still called $u_{h}$ such that the following convergences hold.

$$
\begin{gathered}
u_{h} \rightharpoonup u \text { in } \mathscr{V}_{\omega} \\
\bar{A}(\omega) u_{h} \rightharpoonup \xi \text { in } \mathscr{V}_{\omega}^{\prime} \\
\frac{1}{h}\left(I-\tau_{h}\right)\left(B u_{h}\right) \rightharpoonup \zeta \text { in } \mathscr{V}_{\omega}^{\prime}
\end{gathered}
$$

First we need to identify $\zeta$. Let $\phi \in C^{\infty}([0, T])$ where $\phi=0$ near $T$ and let $w \in V$. Then

$$
\begin{gathered}
\left\langle\int_{0}^{T} \zeta \phi, w\right\rangle=\lim _{h \rightarrow 0}\left\langle\int_{0}^{T} \frac{1}{h}\left(I-\tau_{h}\right)\left(B u_{h}\right), w \phi\right\rangle \\
=\lim _{h \rightarrow 0}\left\langle\int_{0}^{T} \frac{B u_{h}(t)}{h} \phi(t)-\int_{h}^{T} \frac{B u_{h}(t-h)}{h} \phi(t), w\right\rangle \\
=\lim _{h \rightarrow 0}\left\langle\int_{0}^{T} \frac{B u_{h}(t)}{h} \phi(t)-\int_{0}^{T-h} \frac{B u_{h}(t)}{h} \phi(t+h), w\right\rangle \\
=\lim _{h \rightarrow 0}\left(\left\langle\int_{0}^{T-h} B u_{h} \frac{\phi(t)-\phi(t+h)}{h}, w\right\rangle+\int_{T-h}^{T} \frac{B u_{h}(t)}{h} \phi(t)\right) \\
=\left\langle-\int_{0}^{T} B u(t) \phi^{\prime}(t), w\right\rangle
\end{gathered}
$$

Since this holds for all $\phi \in C_{c}^{\infty}(0, T)$, it follows that $\zeta=(B u)^{\prime}$. Hence letting $\phi$ be an arbitrary function in $C^{\infty}([0, T])$ which equals zero near $T$, this implies from the above that

$$
\begin{aligned}
&\left\langle-\int_{0}^{T} \zeta \phi, w\right\rangle=\left\langle\int_{0}^{T} B u(t) \phi^{\prime}(t), w\right\rangle \\
&=\left\langle\int_{0}^{T}\left(B u(0)+\int_{0}^{t}(B u)^{\prime}(s) d s\right) \phi^{\prime}(t), w\right\rangle \\
&=\int_{0}^{T}\langle B u(0), w\rangle \phi^{\prime}(t) d t+\left\langle\int_{0}^{T} \int_{0}^{t}(B u)^{\prime}(s) d s \phi^{\prime}(t), w\right\rangle \\
&=-\langle B u(0), w\rangle \phi(0)+\left\langle\int_{0}^{T}(B u)^{\prime}(s) \int_{s}^{T} \phi^{\prime}(t) d t d s, w\right\rangle \\
&=-\langle B u(0), w\rangle \phi(0)-\left\langle\int_{0}^{T}(B u)^{\prime}(s) \phi(s) d s, w\right\rangle
\end{aligned}
$$

Hence, since $\zeta=(B u)^{\prime}$,

$$
0=-\langle B u(0), w\rangle \phi(0)
$$

Then it follows that, $\langle B u(0), w\rangle=0$. Since $w$ was arbitrary, $B u(0)=0$ and $\zeta=(B u)^{\prime}$.
Thus, passing to a limit in 76.3.18,

$$
(B u)^{\prime}+\xi=f \text { in } \mathscr{V}_{\omega}^{\prime}, B u(0)=0
$$

It is desired to identify $\xi$ with $\bar{A}(\omega) u$. First let

$$
L_{h} \equiv \frac{I-\tau_{h}}{h}
$$

Then

$$
L_{h}(B u)(t)=\left\{\begin{array}{l}
\frac{1}{h} \int_{t-h}^{t}(B u)^{\prime} d s \text { if } t \geq h \\
\frac{1}{h} \int_{0}^{t}(B u)^{\prime} d s \text { if } t<h
\end{array}\right.
$$

Then from standard considerations involving approximate identities,

$$
\begin{equation*}
\lim _{h \rightarrow 0} L_{h}(B u)=(B u)^{\prime} \text { strongly in } \mathscr{V}_{\omega}^{\prime} \tag{76.3.19}
\end{equation*}
$$

Thus

$$
\begin{gathered}
\left\langle L_{h}\left(B u_{h}\right)-(B u)^{\prime}, u_{h}-u\right\rangle= \\
\geq\left\langle L_{h}\left(B u_{h}\right)-L_{h}(B u), u_{h}-u\right\rangle+\left\langle L_{h}(B u)-(B u)^{\prime}, u_{h}-u\right\rangle \\
\geq\left\langle L_{h}(B u)-(B u)^{\prime}, u_{h}-u\right\rangle
\end{gathered}
$$

and the above strong convergence implies that this converges to 0 . Therefore, from 76.3.18,

$$
L_{h} B u_{h}+\bar{A} u_{h}=f \text { in } \mathscr{V}_{\omega}^{\prime}
$$

and so

$$
\left\langle L_{h} B u_{h}, u_{h}-u\right\rangle+\left\langle\bar{A} u_{h}, u_{h}-u\right\rangle=\left\langle f, u_{h}-u\right\rangle
$$

From the above,

$$
\left\langle(B u)^{\prime}, u_{h}-u\right\rangle+\left\langle L_{h}(B u)-(B u)^{\prime}, u_{h}-u\right\rangle+\left\langle\bar{A} u_{h}, u_{h}-u\right\rangle \leq\left\langle f, u_{h}-u\right\rangle
$$

and so, taking limsup ${ }_{h \rightarrow 0}$ of both sides, it follows from 76.3.19 that

$$
\lim \sup _{h \rightarrow 0}\left\langle\bar{A} u_{h}, u_{h}-u\right\rangle \leq 0, \quad \lim \sup _{h \rightarrow 0}\left\langle\bar{A} u_{h}, u_{h}\right\rangle \leq\langle\xi, u\rangle
$$

Since $A$ is monotone and hemicontinuous, the same is true of $\bar{A}$ and so

$$
\bar{A} u=\xi
$$

Thus

$$
\begin{equation*}
\left((B u)^{\prime}(\cdot, \omega)\right)+\bar{A}(\omega) u(\cdot, \omega)=f(\cdot, \omega) \text { in } \mathscr{V}_{\omega}^{\prime}, B u(0, \omega)=0 \tag{76.3.20}
\end{equation*}
$$

It follows from the uniqueness assumption 76.3 .17 that for each $\omega$ off a set of measure zero, there exists a unique solution to

$$
\begin{aligned}
(B u)^{\prime}(\cdot, \omega)+\bar{A}(\cdot, u(\cdot, \omega), \omega) & =f(\cdot, \omega) \text { in } \mathscr{V}_{\omega}^{\prime}, \\
B(u(\cdot, \omega))(0) & =0
\end{aligned}
$$

You can consider the function of two variables $u(t, \omega)$. Is this function progressively measurable? Right now, this is not clear because we have done nothing more than solve a problem for each $\omega$.

However, we can at least say that $u_{h}$ is progressively measurable because $u_{h} \in \mathscr{V}$. Recall also that

$$
\frac{1}{h}\left(I-\tau_{h}\right) B u_{h}+\bar{A}(\omega) u_{h}=f, u_{h} \in \mathscr{V}
$$

Next we show that because of uniqueness, one can assume that $u$ is progressively measurable. To do this, we show that the sequence for which convergence holds in the above can be chosen independent of $\omega$.

Claim: A single sequence $h \rightarrow 0$ works for all $\omega$ off a set of measure zero.
Proof: Since there is only one solution to the above initial value problem for $\omega \notin N$, then letting $h \rightarrow 0$ be a single sequence, one can conclude that $u_{h}(\cdot, \omega) \rightharpoonup u(\cdot, \omega)$ in $\mathscr{V}_{\omega}=$ $L^{p}(0, T, V)$. Otherwise, from the above argument, one could obtain another subsequence which converges to a solution different than $u(\cdot, \omega)$ which would violate uniqueness.

From the coercivity condition, it follows that there exists a constant $C(f)$ depending on $f$ such that for all $h$,

$$
\left\|u_{h}\right\|_{\mathscr{V}} \leq C(f)
$$

Therefore, there is a further subsequence still denoted by $h$ such that

$$
\begin{equation*}
u_{h} \rightharpoonup \bar{u} \text { in } L^{p}([0, T] \times \Omega ; V) \tag{76.3.21}
\end{equation*}
$$

where the measurable sets are just the product measurable sets $\mathscr{B}([0, T]) \times \mathscr{F}_{T}$. Then it follows from Lemma 76.3.4 that $u(\cdot, \omega)=\bar{u}(\cdot, \omega)$ in $\mathscr{V}_{\omega}$ for all $\omega$ off a set of measure zero. It follows that in all of the above, we could substitute $\bar{u}$ for $u$ at least for $\omega$ off a single set of measure zero. Thus $u$ can be assumed progressively measurable.

Note the importance of path uniqueness in obtaining the result on progressive measurability of the solutions.

We will write $u$ rather than $\bar{u}$ to save notation. Now with this lemma, it is easy to obtain the following proposition.

Proposition 76.3.6 Let $q \in \mathscr{V}$ such that $t \rightarrow q(t, \omega)$ is continuous and $q(0, \omega)=0$, and let the conditions 76.3.14-76.3.17 be valid. Also let $u_{0} \in L^{2}(\Omega, V)$ such that $u_{0}$ is $\mathscr{F}_{0}$ measurable. Let $f \in \mathscr{V}^{\prime}$ be given. Then for each $\omega$ off a set of measure zero, there exists $u(\cdot, \omega) \in \mathscr{V}_{\omega}$ such that $(B u)^{\prime}(\cdot, \omega) \in \mathscr{V}_{\omega}^{\prime}$ and

$$
B u(0, \omega)=B u_{0}
$$

and also the following equation holds in $\mathscr{V}_{\omega}^{\prime}$ for a.e. $\omega$

$$
(B u-B q)^{\prime}(\cdot, \omega)+A(\cdot, u(\cdot, \omega), \omega)=f(\cdot, \omega)
$$

In addition to this, it can be assumed that $(t, \omega) \rightarrow u(t, \omega)$ is progressively measurable into $V$. That is, for each $\omega$ off a set of measure zero, $t \rightarrow u(t, \omega)$ can be modified on a set of measure zero in $[0, T]$ such that the resulting $u$ is progressively measurable. Then one also obtains that $u$ is the unique solution to the integral equation which holds for a.e. $\omega$

$$
\begin{equation*}
B u(t, \omega)-B u_{0}+\int_{0}^{t} A(s, u(s, \omega), \omega) d s=\int_{0}^{t} f(s, \omega) d s+B q(t, \omega) \tag{76.3.22}
\end{equation*}
$$

Proof: Recall

$$
\bar{A}(\omega)(t, u) \equiv A(t, u+q(t, \omega), \omega)
$$

where $q$ was in $\mathscr{V}$. Therefore, replace this definition of $\bar{A}$ with

$$
\bar{A}(\omega)(t, u) \equiv A\left(t, u+q(t, \omega)+u_{0}, \omega\right)
$$

Then from Lemma 76.3.5, there exists $w \in \mathscr{V}$ such that

$$
(B w)^{\prime}(\cdot, \omega)+A\left(\cdot, w(\cdot, \omega)+q(\cdot, \omega)+u_{0}(\omega), \omega\right)=f(\cdot, \omega), B w(0)=0
$$

Let $u(t, \omega)=w(t, \omega)+q(t, \omega)+u_{0}(\omega)$. Then for fixed $\omega, B u(0)=B w(0)+B u_{0}=B u_{0}$. Also

$$
(B(u-q))^{\prime}+A(\cdot, u, \omega)=f(\cdot, \omega), B u(0)=B u_{0}
$$

Then an integration yields 76.3.22. Uniqueness follows from the above uniqueness assumption 76.3.17.

One can easily generalize this using an exponential shift argument.
Corollary 76.3.7 Suppose the situation of the above proposition but that all that is known is that $\lambda B+A$ is monotone and hemicontinuous on $\mathscr{V}_{\omega}$ and $\mathscr{V}$ for all $\lambda$ sufficiently large. Then defining

$$
\left\langle A_{\lambda}(t, w, \omega), v\right\rangle_{V^{\prime}, V} \equiv\left\langle e^{-\lambda t} A\left(t, e^{\lambda t} w, \omega\right), v\right\rangle_{V^{\prime}, V}
$$

for such $\lambda$, it follows that $\lambda B+A_{\lambda}$ is also monotone and hemicontinuous. Then replace the coercivity, and boundedness conditions above with the following weaker conditions

$$
\begin{equation*}
\lambda\langle B u, u\rangle+\langle A(t, u, \omega), u\rangle_{V} \geq \delta\|u\|_{V}^{p}-c(t, \omega) \tag{76.3.23}
\end{equation*}
$$

for all $\lambda$ large enough.

$$
\begin{equation*}
\|A(t, u, \omega)\|_{V^{\prime}} \leq k\|u\|_{V}^{p-1}+c^{1 / p^{\prime}}(t, \omega) \tag{76.3.24}
\end{equation*}
$$

where $c \in L^{1}([0, T] \times \Omega), c \geq 0$. Then the conclusion of Proposition 76.3 .6 is still valid. There exists a unique $u \in \mathscr{V}$ such that for a.e. $\omega$,

$$
\begin{equation*}
(B u-B q)^{\prime}(\cdot, \omega)+A(\omega)(u(\cdot, \omega))=f(\cdot, \omega), B(u-q)(0)=B u_{0} \tag{76.3.25}
\end{equation*}
$$

Proof: That $\lambda B+A_{\lambda}$ is monotone and hemicontinuous follows from the definition. Also, from the above estimates,

$$
\lambda\langle B u, u\rangle+\left\langle A_{\lambda}(t, u, \omega), u\right\rangle_{V} \geq e^{-2 \lambda t}\left(\lambda\left\langle B\left(e^{\lambda t} u\right), e^{\lambda t} u\right\rangle+\left\langle A\left(t, e^{\lambda t} u, \omega\right), e^{\lambda t} u\right\rangle\right)
$$

$$
\begin{gathered}
\geq e^{-2 \lambda t}\left(\delta\left\|e^{\lambda t} u\right\|_{V}^{p}-c(t, \omega)\right) \geq e^{-2 \lambda t}\left(\delta\left\|e^{\lambda t} u\right\|_{V}^{p}-e^{\lambda p t} e^{-\lambda p t} c(t, \omega)\right) \\
\geq e^{-2 \lambda t} e^{p \lambda t}\left(\delta\|u\|_{V}^{p}-e^{-\lambda p t} c(t, \omega)\right) \geq \delta\|u\|_{V}^{p}-e^{-\lambda p t} c(t, \omega)
\end{gathered}
$$

which is of the right form.
Similarly

$$
\begin{gathered}
\left\|\lambda B w+A_{\lambda}(t, w, \omega)\right\|_{V^{\prime}} \leq\|\lambda B w\|_{V^{\prime}}+\left\|e^{-\lambda t} A\left(t, e^{\lambda t} w, \omega\right)\right\|_{V^{\prime}} \\
\leq \lambda\|B\|\|w\|_{V}+e^{-\lambda t}\left\|A\left(t, e^{\lambda t} w, \omega\right)\right\|_{V^{\prime}} \\
\leq \lambda\|B\|\|w\|_{V}+e^{-\lambda t} k\left\|e^{\lambda t} w\right\|^{p-1}+e^{-\lambda t} c^{1 / p^{\prime}}(t, \omega)
\end{gathered}
$$

Since $p \geq 2$, this is no larger than

$$
\begin{aligned}
\leq & (\lambda\|B\|)^{p /\left(p-p^{\prime}\right)}+\|w\|_{V}^{p-1}+e^{(p-1) \lambda t} e^{-\lambda t} k\|w\|_{V}^{p-1}+e^{-\lambda t} c^{1 / p^{\prime}}(t, \omega) \\
& \leq\left(e^{(p-2) \lambda T} k+1\right)\|w\|_{V}^{p-1}+e^{-\lambda t} c^{1 / p^{\prime}}(t, \omega)+(\lambda\|B\|)^{p /\left(p-p^{\prime}\right)} \\
& \equiv \bar{k}\|w\|_{V}^{p-1}+\bar{c}(t, \omega)^{1 / p^{\prime}}
\end{aligned}
$$

Now note that $w$ is a solution to

$$
\begin{gathered}
\quad B\left(w-e^{-\lambda(\cdot)} q\right)^{\prime}+\lambda B w+e^{-\lambda(\cdot)} A\left(t, e^{\lambda(\cdot)} w, \omega\right) \\
=e^{-\lambda(\cdot)} f(\cdot, \omega)+\lambda B e^{-\lambda(\cdot)} q(\cdot, \omega) \text { in } \mathscr{V}_{\omega} \\
B\left(w-e^{-\lambda(\cdot)} q\right)(0)=B u_{0}
\end{gathered}
$$

if and only if $u(t) \equiv e^{\lambda t} w(t)$ is a solution to

$$
(B(u-q))^{\prime}+A(t, u, \omega)=f(\cdot, \omega), B(u-q)(0)=B u_{0}
$$

Thus the necessary uniqueness condition holds for the initial value problem for $w$ and hence it follows from Proposition 76.3.6 that there exists a unique progressively measurable solution to the initial value problem for $w$ and hence a unique progressively measurable solution to the above one for $u$.

Now suppose the situation of the above corollary and let $E$ be a separable Hilbert space which is dense in $V$ and let

$$
\Phi \in L^{2}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, E\right)\right), \Phi \text { being progressively measurable }
$$

so that one can consider the stochastic integral $\int_{0}^{t} \Phi d W$. Let

$$
\tau_{n} \equiv \inf \left\{t:\left\|\int_{0}^{t} \Phi d W\right\|_{E}>2^{n}\right\}
$$

Thus

$$
\left\|\int_{0}^{t \wedge \tau_{n}} \Phi d W\right\|_{E} \leq 2^{n}
$$

Then you could pick $u_{0} \in L^{p}(\Omega, V), u_{0}$ being $\mathscr{F}_{0}$ measurable, and let

$$
q(t, \omega)=\int_{0}^{t \wedge \tau_{n}} \Phi d W
$$

The result is clearly in $\mathscr{V}$ and is continuous in $t$. Therefore, from Corollary 76.3.7, there exists a unique solution $u \in \mathscr{V}$ to the initial value problem

$$
\left(B u-B \int_{0}^{t \wedge \tau_{n}} \Phi d W\right)^{\prime}(\cdot, \omega)+A(\omega)(u(\cdot, \omega))=f(\cdot, \omega), B u(0)=B u_{0}
$$

Integrating, one obtains a unique solution $u_{n} \in \mathscr{V}$ to the integral equation

$$
B u_{n}(t, \omega)-B u_{0}(\omega)+\int_{0}^{t} A\left(s, u_{n}, \omega\right) d s=\int_{0}^{t} f(s, \omega) d s+B \int_{0}^{t \wedge \tau_{n}} \Phi d W
$$

This holds in $\mathscr{V}_{\omega}^{\prime}$ and is so for all $\omega$ off a set of measure zero $N_{n}$. Let $N=\cup_{n} N_{n}$. For $\omega \notin N, t \rightarrow \int_{0}^{t} \Phi d W$ is continuous and so for all $n$ large enough, $\tau_{n}=\infty$. Thus for a fixed $\omega$, it follows that for all $n$ large enough $\tau_{n}=\infty$ and so one obtains

$$
B u_{n}(t, \omega)-B u_{0}(\omega)+\int_{0}^{t} A\left(s, u_{n}, \omega\right) d s=\int_{0}^{t} f(s, \omega) d s+B \int_{0}^{t} \Phi d W
$$

Then for $k$ some other index sufficiently large, the same holds for $u_{k}$. By the uniqueness assumption 76.3.17, $u_{k}(t, \omega)=u_{n}(t, \omega)$ and so it follows that $\lim _{n \rightarrow \infty} u_{n}(t, \omega)$ exists because for each $\omega$ off a set of measure zero, there is eventually no change in $u_{n}$. Defining $u(t, \omega) \equiv \lim _{n \rightarrow \infty} u_{n}(t, \omega) \equiv u_{n}(t, \omega)$ for all $n$ large enough, it follows that $u$ is progressively measurable since it is the pointwise limit of progressively measurable functions and

$$
B u(t, \omega)-B u_{0}(\omega)+\int_{0}^{t} A(s, u, \omega) d s=\int_{0}^{t} f(s, \omega) d s+B \int_{0}^{t} \Phi d W
$$

This has shown the following lemma.
Lemma 76.3.8 Let $(t, u, \omega) \rightarrow A(t, u, \omega)$ be progressively measurable into $V^{\prime}$ and suppose for some $\lambda$,

$$
\begin{aligned}
\lambda B+A(\omega) & : \\
\lambda B+A & : \mathscr{V}_{\omega} \rightarrow \mathscr{V}_{\omega}^{\prime} \\
\lambda & \mathscr{V} \rightarrow \mathscr{V}^{\prime}
\end{aligned}
$$

are both monotone bounded and hemicontinuous. Also suppose the two estimates giving boundedness and coercivity 76.3.23-76.3.24 of Corollary 76.3.7 above. Here V,W are as described above $V \subseteq W, W^{\prime} \subseteq V^{\prime}, W$ is a separable Hilbert space and $V$ is a separable reflexive Banach space. $B: W \rightarrow W^{\prime}$ is nonnegative and self adjoint. Let $f \in \mathscr{V}^{\prime}$ and let $u_{0} \in L^{p}(\Omega, V)$ where $u_{0}$ is $\mathscr{F}_{0}$ measurable. Then if $\Phi \in L^{2}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, E\right)\right), \Phi$
being progressively measurable, into $E$, where $E$ is a Hilbert space dense in $V$ with $\|u\|_{E} \geq$ $\|u\|_{V}$, then there exists a unique solution to the integral equation

$$
B u(t)-B u_{0}+\int_{0}^{t} A(s, u, \omega) d s=\int_{0}^{t} f(s, \omega) d s+B \int_{0}^{t} \Phi d W
$$

in the sense that $u$ is in $\mathscr{V}$ and there exists a set of measure zero $N$ such that if $\omega \notin N$, then the above integral equation holds for all $t$.

### 76.4 The General Case

Suppose $\lambda B+A(\omega), \lambda B+A$ are both monotone bounded and hemicontinuous on $\mathscr{V}_{\omega}$ and $\mathscr{V}$ respectively for $\lambda$ sufficiently large. Also suppose the two estimates giving boundedness and coercivity 76.3.23-76.3.24 of Corollary 76.3.7 above. We strengthen the assumption that $\lambda B+A(\omega)$ is monotone as follows. In the usual case where $B$ is the identity, this conclusion is obvious, but here we need to assume it.

$$
\begin{equation*}
\langle(\lambda B+A(\omega))(u)-(\lambda B+A(\omega))(v), u-v\rangle \geq \delta\|u-v\|_{U}^{\alpha}, \alpha \geq 1 \tag{76.4.26}
\end{equation*}
$$

where here $U$ is a reflexive Banach space such that $V \subseteq U$ and the inclusion map is continuous, $V$ being dense in $U$. In regards to this monotonicity condition, here is a simple lemma which will be used later.

Lemma 76.4.1 Suppose $u_{n} \rightarrow w$ weakly in $\mathscr{V}_{\omega}$ and that for a.e.t, $u_{n}(t) \rightarrow u(t)$ in $U$. Then $w(t)=u(t)$ a.e.

Proof: You know that $\left\|u_{n}\right\|_{L^{p}([0, T], V)}$ is bounded. Now consider $\phi \in U^{\prime}$ and $\psi \in$ $C([0, T])$. Then the weak convergence implies

$$
\lim _{n \rightarrow \infty} \int_{0}^{T}\left\langle\phi, u_{n}\right\rangle_{U^{\prime}, U} \psi d t=\int_{0}^{T}\langle\phi, w\rangle_{U^{\prime}, U} \psi d t
$$

because it is also the case that $u_{n} \rightarrow w$ weakly in $L^{p}([0, T], U)$. However, the fact that $\left\|u_{n}\right\|_{L^{p}([0, T], V)}$ is bounded means that, by the assumed pointwise convergence,

$$
\lim _{n \rightarrow \infty} \int_{0}^{T}\left\langle\phi, u_{n}\right\rangle_{U^{\prime}, U} \psi d t=\int_{0}^{T}\langle\phi, u\rangle_{U^{\prime}, U} \psi d t
$$

It follows that

$$
\int_{0}^{T}\langle\phi, u-w\rangle \psi d t=0
$$

Since this is true for all $\psi \in C([0, T])$, there exists a set of measure zero $Q_{\phi}$ such that for $t \notin Q_{\phi}$,

$$
\langle\phi, u(t)-w(t)\rangle=0
$$

Letting $Q=\cup_{\phi \in D} Q_{\phi}$, where $D$ is a countable dense subset of $U^{\prime}$, it follows that for $t \notin Q$, the above holds for all $\phi \in U^{\prime}$. Hence $u(t)=w(t)$ for $t \notin Q$ and $m(Q)=0$.

Typically $\alpha=2$ and $U=W$.

Recall that

$$
V \subseteq W, \quad W^{\prime} \subseteq V^{\prime}
$$

each space dense in the one to its right and the inclusion maps are continuous.
Assume only

$$
\Phi \in L^{2}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right)
$$

By density of $E$ into $W$, there exists a sequence

$$
\Phi_{n} \in L^{2}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, E\right)\right)
$$

such that

$$
\begin{gathered}
\left\|\Phi_{n}-\Phi\right\|_{L^{2}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right)} \rightarrow 0 \\
\left\|\Phi_{n}\right\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)} \leq\|\Phi\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)}
\end{gathered}
$$

Also let $u_{0 n} \in L^{p}(\Omega, V)$ where $u_{0 n}$ is $\mathscr{F}_{0}$ measurable and such that $u_{0 n} \in L^{p}(\Omega, V)$ and

$$
\left\|u_{0 n}(\omega)-u_{0}(\omega)\right\|_{W} \rightarrow 0,\left\langle B u_{0 n}, u_{0 n}\right\rangle \leq 2\left\langle B u_{0}, u_{0}\right\rangle
$$

for each $\omega$. The existence of such an approximating sequence follows from density considerations of $E$ into $V$ and of $V$ into $W$.

By Lemma 76.3.8 there is a solution $u_{n}$ to the integral equation

$$
\begin{equation*}
B u_{n}(t)-B u_{0 n}+\int_{0}^{t} A\left(s, u_{n}, \omega\right) d s=\int_{0}^{t} f(s, \omega) d s+B \int_{0}^{t} \Phi_{n} d W \tag{76.4.27}
\end{equation*}
$$

Then by the Implicit Ito formula there is a set of measure zero such that for all $n, m$

$$
\begin{aligned}
& \frac{1}{2}\left\langle B\left(u_{n}-u_{m}\right), u_{n}-u_{m}\right\rangle(t)-\frac{1}{2}\left\langle B u_{0 n}-B u_{0 m}, u_{0 n}-u_{0 m}\right\rangle \\
& +\delta \int_{0}^{t}\left\|u_{n}-u_{m}\right\|_{U}^{\alpha} d s \\
& \leq \lambda \int_{0}^{t}\left\langle B\left(u_{n}-u_{m}\right), u_{n}-u_{m}\right\rangle(s) d s \\
& \quad+\frac{1}{2} \int_{0}^{t}\left\langle B\left(\Phi_{n}-\Phi_{m}\right), \Phi_{n}-\Phi_{m}\right\rangle_{\mathscr{L}_{2}} d s+M_{m n}(t)
\end{aligned}
$$

Also the last term is a martingale whose quadratic variation satisfies

$$
\begin{gathered}
{\left[M_{m n}\right](t) \leq C \int_{0}^{t}\left\|\Phi_{n}-\Phi_{m}\right\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)}^{2}\left\|B\left(u_{n}-u_{m}\right)\right\|_{W^{\prime}}^{2} d s} \\
\leq C \int_{0}^{t}\left\|\Phi_{n}-\Phi_{m}\right\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)}^{2}\left\langle B u_{n}-B u_{m}, u_{n}-u_{m}\right\rangle d s
\end{gathered}
$$

Then from Gronwall's inequality, and adjusting the constants,

$$
\begin{aligned}
& \left\langle B\left(u_{n}-u_{m}\right), u_{n}-u_{m}\right\rangle(t)+\int_{0}^{t}\left\|u_{n}-u_{m}\right\|_{U}^{\alpha} d s \\
\leq & C\left(u_{0 n}-u_{0 m}, \Phi_{n}-\Phi_{m}\right)+C(T) M_{m n}^{*}(t)
\end{aligned}
$$

where the expectation of the first constant on the right converges to 0 as $m, n \rightarrow \infty$. Here

$$
M_{n m}^{*}(t)=\sup _{s \in[0, t]}\left|M_{n m}(t)\right|
$$

Since $M^{*}$ is increasing, this implies that after adjusting constants,

$$
\begin{aligned}
& \sup _{s \in[0, t]}\left(\left\langle B\left(u_{n}-u_{m}\right), u_{n}-u_{m}\right\rangle(s)+\int_{0}^{t}\left\|u_{n}-u_{m}\right\|_{U}^{\alpha} d s\right) \\
\leq & C\left(u_{0 n}-u_{0 m}, \Phi_{n}-\Phi_{m}\right)+C(T) M_{m n}^{*}(t)
\end{aligned}
$$

Then taking expectations and using the Burkholder Davis Gundy inequality,

$$
\begin{gathered}
E\left(\sup _{s \in[0, t]}\left(\left\langle B\left(u_{n}-u_{m}\right), u_{n}-u_{m}\right\rangle(s)+\int_{0}^{t}\left\|u_{n}-u_{m}\right\|_{U}^{\alpha} d s\right)\right) \\
\leq C\left(u_{0 n}-u_{0 m}, \Phi_{n}-\Phi_{m}\right)+ \\
C(T) \int_{\Omega}\left(\int_{0}^{t}\left\|\Phi_{n}-\Phi_{m}\right\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)}^{2}\left\langle B\left(u_{n}-u_{m}\right), u_{n}-u_{m}\right\rangle d s\right)^{1 / 2} d P \\
\leq C_{n, m}+2 C \int_{\Omega_{s \in[0, t]} \sup \left\langle B\left(u_{n}-u_{m}\right), u_{n}-u_{m}\right\rangle^{1 / 2}(s)} \begin{array}{c}
\left(\int_{0}^{t}\left\|\Phi_{n}-\Phi_{m}\right\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)}^{2}\right)^{1 / 2} d P
\end{array} .
\end{gathered}
$$

Then adjusting the constants,

$$
\begin{gathered}
E\left(\sup _{s \in[0, t]}\left(\left\langle B\left(u_{n}-u_{m}\right), u_{n}-u_{m}\right\rangle(s)+\int_{0}^{t}\left\|u_{n}-u_{m}\right\|_{U}^{\alpha} d s\right)\right) \\
\quad \leq C_{n, m}+C \int_{\Omega} \int_{0}^{T}\left\|\Phi_{n}-\Phi_{m}\right\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)}^{2} d t d P \equiv C_{n, m}
\end{gathered}
$$

where $C_{n, m} \rightarrow 0$ as $n, m \rightarrow \infty$. In particular, it is true for $t=T$

$$
E\left(\sup _{s \in[0, T]}\left\langle B\left(u_{n}-u_{m}\right), u_{n}-u_{m}\right\rangle(s)+\int_{0}^{T}\left\|u_{n}-u_{m}\right\|_{U}^{\alpha} d s\right) \leq C_{n, m}
$$

Then

$$
P\left(\sup _{s \in[0, T]}\left\langle B\left(u_{n}-u_{m}\right), u_{n}-u_{m}\right\rangle(s)+\int_{0}^{T}\left\|u_{n}-u_{m}\right\|_{U}^{\alpha} d s \geq \lambda\right) \leq \frac{C_{n, m}}{\lambda}
$$

Now take a subsequence such that if $m>n_{k}, C_{n_{k}, m}<4^{-k}$. Then the above inequality implies that

$$
P\binom{\sup _{s \in[0, T]}\left\langle B\left(u_{n}-u_{m}\right), u_{n}-u_{m}\right\rangle(s)}{+\int_{0}^{T}\left\|u_{n_{k}}-u_{n_{k+1}}\right\|_{U}^{\alpha} d s \geq 2^{-k}} \leq \frac{4^{-k}}{2^{-k}}=2^{-k}
$$

and so, by the Borel Cantelli lemma, there is a set of measure zero $N$ including all earlier exceptional sets of measure zero such that for $\omega \notin N$,

$$
\sup _{s \in[0, T]}\left\langle B\left(u_{n}-u_{m}\right), u_{n}-u_{m}\right\rangle(s)+\int_{0}^{T}\left\|u_{n_{k}}-u_{n_{k+1}}\right\|_{U}^{\alpha} d s<2^{-k}
$$

for all $k$ large enough. We will denote this new subsequence by $\left\{u_{n}\right\}$. Thus for such $\omega$, it follows that $\left\{B u_{n}\right\}$ is a Cauchy sequence in $C\left(N_{\omega}^{C}, W^{\prime}\right)$ for $N_{\omega}$ an exceptional set of measure zero where $B\left(u_{n}-u_{m}\right)(t) \neq B\left(u_{n}(t)-u_{m}(t)\right)$ and also $\left\{u_{n}\right\}$ is a Cauchy sequence in $L^{\alpha}(0, T, U)$. It follows

$$
\begin{align*}
& B u_{n} \rightarrow z \text { strongly in } C\left(N_{\omega}^{C}, W^{\prime}\right) \text { with uniform norm }  \tag{76.4.28}\\
& \lim _{m, n \rightarrow \infty} \sup _{s \in[0, T]}\left\langle B\left(u_{n}-u_{m}\right), u_{n}-u_{m}\right\rangle(s)=0 \tag{76.4.29}
\end{align*}
$$

There exists $u \in L^{\alpha}(0, T, U)$ such that for $\omega \notin N$,

$$
\begin{equation*}
\left\|u_{n}-u\right\|_{L^{\alpha}(0, T, U)} \rightarrow 0, u_{n}(t, \omega) \rightarrow u(t, \omega) \text { for a.e.t in } U \tag{76.4.30}
\end{equation*}
$$

Of course a technical issue is the fact that $B$ is a degenerate operator which might not be invertible. In the above limit, we do not know that $z=B u$ for some $u$. We resolve this issue by obtaining pointwise estimates for a given $\omega$ and then pass to a limit. After this, a time integration will give the desired result. There are easier ways to do this if $B$ is not degenerate.

From now on, this or a subsequence of this one will be the sequence of interest. Return to 76.4.27 and use the Ito formula again. Thus using the estimates,

$$
\begin{aligned}
& \frac{1}{2}\left\langle B u_{n}, u_{n}\right\rangle(t)-\frac{1}{2}\left\langle B u_{0 n}, u_{0 n}\right\rangle+\delta \int_{0}^{t}\left\|u_{n}\right\|_{V}^{p} d s-\lambda \int_{0}^{t}\left\langle B u_{n}, u_{n}\right\rangle d s \\
= & \frac{1}{2} \int_{0}^{t}\left\langle B \Phi_{n}, \Phi_{n}\right\rangle d s+\int_{0}^{t} c(s, \omega) d s+\int_{0}^{t}\left\langle f, u_{n}\right\rangle d s+M_{n}(t)
\end{aligned}
$$

where $M_{n}(t)$ is a local martingale whose quadratic variation satisfies

$$
\left[M_{n}\right](t) \leq C \int_{0}^{t}\left\|\Phi_{n}\right\|_{\mathscr{L}_{2}}^{2}\left\|B u_{n}\right\|_{W}^{2} d s
$$

Then adjusting the constants,

$$
\left\langle B u_{n}, u_{n}\right\rangle(t)+\int_{0}^{t}\left\|u_{n}\right\|_{V}^{p} d s \leq C\left(u_{0 n}, \Phi_{n}, f, c\right)+C M_{n}^{*}(t)
$$

where the expectation of the first constant on the right is no larger than a constant $C$ which is independent of $n$. Since the right term is increasing in $t$,

$$
\begin{equation*}
\sup _{s \in[0, t]}\left\langle B u_{n}, u_{n}\right\rangle(s)+\int_{0}^{t}\left\|u_{n}\right\|_{V}^{p} d s \leq C\left(u_{0 n}, \Phi_{n}, f, c\right)+C M_{n}^{*}(t) \tag{76.4.31}
\end{equation*}
$$

Now using the Burkholder Davis Gundy inequality as before and taking the expectation,

$$
\begin{gathered}
E\left(\sup _{s \in[0, t]}\left\langle B u_{n}, u_{n}\right\rangle(s)\right)+E \int_{0}^{t}\left\|u_{n}\right\|_{V}^{p} d s \\
\leq C+C \int_{\Omega}\left(\int_{0}^{t}\left\|\Phi_{n}\right\|_{\mathscr{L}_{2}}^{2}\left\|B u_{n}\right\|_{W}^{2} d s\right)^{1 / 2} d P \\
\leq C+C \int_{\Omega}\left(\int_{0}^{t}\left\|\Phi_{n}\right\|_{\mathscr{L}_{2}}^{2}\left\langle B u_{n}, u_{n}\right\rangle d s\right)^{1 / 2} d P \\
\leq C+C \int_{\Omega_{s \in[0, t]}} \sup \left\langle B u_{n}, u_{n}\right\rangle^{1 / 2}(s)\left(\int_{0}^{t}\left\|\Phi_{n}\right\|_{\mathscr{L}_{2}}^{2} d s\right)^{1 / 2} d P
\end{gathered}
$$

Then adjusting the constants and using the approximation properties of $\Phi_{n}$ given above, there is a constant $C$ independent of $n, t \leq T$ such that

$$
E\left(\sup _{s \in[0, t]}\left\langle B u_{n}, u_{n}\right\rangle(s)\right)+E \int_{0}^{t}\left\|u_{n}\right\|_{V}^{p} d s \leq C
$$

In particular

$$
\begin{equation*}
E\left(\sup _{s \in[0, T]}\left\langle B u_{n}, u_{n}\right\rangle(s)\right)+E \int_{0}^{T}\left\|u_{n}\right\|_{V}^{p} d s \leq C \tag{76.4.32}
\end{equation*}
$$

Next use monotonicity to obtain

$$
\begin{aligned}
\frac{1}{2}\left\langle B u_{r}-B u_{q}, u_{r}-u_{q}\right\rangle(t) \leq & \frac{1}{2} \int_{0}^{t}\left(\left(\Phi_{r}-\Phi_{q}\right) \circ J^{-1}\right)^{*} B\left(u_{r}-u_{q}\right) \circ J d W \\
& +C_{\lambda} \int_{0}^{t}\left\langle B u_{r}-B u_{q}, u_{r}-u_{q}\right\rangle d s+\int_{0}^{t}\left\|\Phi_{r}-\Phi_{q}\right\|^{2} d s
\end{aligned}
$$

and so, from Gronwall's inequality, there is a constant $C$ which is independent of $r, q$ such that

$$
\left\langle B u_{r}-B u_{q}, u_{r}-u_{q}\right\rangle(t) \leq C M_{r q}(t) \leq C M_{r q}^{*}(T)+C \int_{0}^{t}\left\|\Phi_{r}-\Phi_{q}\right\|^{2} d s
$$

where $M_{r q}$ refers to that local martingale on the right. Thus also

$$
\begin{equation*}
\sup _{t \in[0, T]}\left\langle B u_{r}-B u_{q}, u_{r}-u_{q}\right\rangle(t) \leq C M_{r q}(t) \leq C M_{r q}^{*}(T)+C \int_{0}^{T}\left\|\Phi_{r}-\Phi_{q}\right\|^{2} d s \tag{76.4.33}
\end{equation*}
$$

Taking the expectation and using the Burkholder Davis Gundy inequality again, and similar estimates to the above, using appropriate stopping times as needed, we obtain

$$
E\left(\sup _{t \in[0, T]}\left\langle B u_{r}-B u_{q}, u_{r}-u_{q}\right\rangle(t)\right) \leq C \int_{\Omega} \int_{0}^{T}\left\|\Phi_{r}-\Phi_{q}\right\|^{2} d t d P
$$

Now the right side converges to 0 as $r, q \rightarrow \infty$ and so there is a subsequence, denoted with the index $k$ such that if $p>k$,

$$
\begin{equation*}
E\left(\sup _{t \in[0, T]}\left\langle B u_{k}-B u_{p}, u_{k}-u_{p}\right\rangle(t)\right) \leq \frac{1}{2^{k}} \tag{76.4.34}
\end{equation*}
$$

Then consider the earlier local martingales. One of these is of the form

$$
M_{k}=\int_{0}^{t}\left(\Phi_{k} \circ J^{-1}\right)^{*} B u_{k} \circ J d W
$$

Then by the Burkholder Davis Gundy inequality and modifying constants as appropriate,

$$
\begin{gathered}
E\left(\left(M_{k}-M_{k+1}\right)^{*}\right) \\
\leq \quad C \int_{\Omega}\left(\int_{0}^{T}\left\|\left(\Phi_{k} \circ J^{-1}\right)^{*} B u_{k}-\left(\Phi_{k+1} \circ J^{-1}\right)^{*} B u_{k+1}\right\|^{2} d t\right)^{1 / 2} d P \\
\leq C \int_{\Omega}\binom{\int_{0}^{T}\left\|\Phi_{k}-\Phi_{k+1}\right\|^{2}\left\langle B u_{k}, u_{k}\right\rangle}{+\left\|\Phi_{k+1}\right\|^{2}\left\langle B u_{k}-B u_{k+1}, u_{k}-u_{k+1}\right\rangle d t}^{1 / 2} d P \\
\leq \quad C \int_{\Omega}\left(\int_{0}^{T}\left\|\Phi_{k}-\Phi_{k+1}\right\|^{2}\left\langle B u_{k}, u_{k}\right\rangle d t\right)^{1 / 2} \\
\quad+C \int_{\Omega}\left(\int_{0}^{T}\left\|\Phi_{k+1}\right\|^{2}\left\langle B u_{k}-B u_{k+1}, u_{k}-u_{k+1}\right\rangle d t\right)^{1 / 2} d P \\
\leq C \int_{\Omega} \sup _{t}\left\langle B u_{k}, u_{k}\right\rangle^{1 / 2}\left(\int_{0}^{T}\left\|\Phi_{k}-\Phi_{k+1}\right\|^{2} d t\right)^{1 / 2} d P \\
+C \int_{\Omega} \sup _{t}\left\langle B u_{k}-B u_{k+1}, u_{k}-u_{k+1}\right\rangle^{1 / 2}\left(\int_{0}^{T}\left\|\Phi_{k+1}\right\|^{2} d t\right)^{1 / 2} d P \\
\leq C\left(\int_{\Omega} \sup _{t}\left\langle B u_{k}, u_{k}\right\rangle d P\right)^{1 / 2}\left(\int_{\Omega} \int_{0}^{T}\left\|\Phi_{k}-\Phi_{k+1}\right\|^{2} d t d P\right)^{1 / 2} \\
+C\left(\int_{\Omega} \sup _{t}\left\langle B u_{k}-B u_{k+1}, u_{k}-u_{k+1}\right\rangle d P\right)^{1 / 2}\left(\int_{\Omega} \int_{0}^{T}\left\|\Phi_{k+1}\right\|^{2} d t d P\right)^{1 / 2}
\end{gathered}
$$

From the above inequalities, after adjusting the constants, the above is no larger than an expression of the form $C\left(\frac{1}{2}\right)^{k / 2}$ which is a summable sequence. Then

$$
\sum_{k} \int_{\Omega} \sup _{t \in[0, T]}\left|M_{k}(t)-M_{k+1}(t)\right| d P<\infty
$$

Then $\left\{M_{k}\right\}$ is a Cauchy sequence in $M_{T}^{1}$ and so there is a continuous martingale $M$ such that

$$
\begin{equation*}
\lim _{k \rightarrow \infty} E\left(\sup _{t}\left|M_{k}(t)-M(t)\right|\right)=0 \tag{76.4.35}
\end{equation*}
$$

Taking a further subsequence if needed, one can also have

$$
P\left(\sup _{t}\left|M_{k}(t)-M(t)\right|>\frac{1}{k}\right) \leq \frac{1}{2^{k}}
$$

and so by the Borel Cantelli lemma, there is a set of measure zero such that off this set, $\sup _{t}\left|M_{k}(t)-M(t)\right|$ converges to 0 . Hence for such $\omega, M_{k}^{*}(T)$ is bounded independent of $k$. Thus for $\omega$ off a set of measure zero, 76.4.31 implies that for such $\omega$,

$$
\sup _{s \in[0, T]}\left\langle B u_{r}, u_{r}\right\rangle(s)+\int_{0}^{T}\left\|u_{r}(s)\right\|_{V}^{p} d s \leq C(\omega)
$$

where $C(\omega)$ does not depend on the index $r$, this for the subsequence just described which will be the sequence of interest in what follows. Using the boundedness assumption for $A$, one also obtains an estimate of the form

$$
\begin{equation*}
\sup _{s \in[0, T]}\left\langle B u_{r}, u_{r}\right\rangle(s)+\int_{0}^{T}\left\|u_{r}(s)\right\|_{V}^{p} d s+\int_{0}^{T}\left\|z_{r}\right\|_{V^{\prime}}^{p^{\prime}} \leq C(\omega) \tag{76.4.36}
\end{equation*}
$$

Lemma 76.4.2 There is a subsequence, still indexed by $n$ and a set of measure zero $N$, containing all the preceding sets of measure zero such that for $\omega \notin N$,

$$
\sup _{s \in[0, T]}\left\langle B u_{n}, u_{n}\right\rangle(s)+\int_{0}^{T}\left\|u_{n}\right\|_{V}^{p} d s \leq C(\omega)<\infty
$$

From the theory of the stochastic integral, there is a further subsequence of the above such that

$$
\int_{0}^{t} \Phi_{n} d W \rightarrow \int_{0}^{t} \Phi d W \text { strongly in } C([0, T], W)
$$

for all $\omega$ off a set of measure zero. Enlarge the exceptional set $N$ and only use subsequences of this one so that both the above estimate in the lemma and the above convergence hold for $\omega \notin N$. Recall the integral equation solved.

$$
\begin{equation*}
B u_{n}(t)-B u_{0 n}+\int_{0}^{t} A\left(s, u_{n}, \omega\right) d s=\int_{0}^{t} f(s, \omega) d s+B \int_{0}^{t} \Phi_{n} d W \tag{76.4.37}
\end{equation*}
$$

Thus

$$
\left(B u_{n}-B \int_{0}^{(\cdot)} \Phi_{n} d W-B u_{0 n}\right)^{\prime}+A u_{n}=f
$$

Then for $\omega \notin N$, a subsequence of the one for which the above lemma holds, still denoted as $\left\{u_{n}\right\}$ yields the following convergences,

$$
\begin{gather*}
u_{n} \rightarrow u \text { weakly in } \mathscr{V}_{\omega}  \tag{76.4.38}\\
A u_{n} \rightharpoonup \xi \text { weakly in } \mathscr{V}_{\omega}^{\prime}  \tag{76.4.39}\\
\left(B u_{n}-B \int_{0}^{(\cdot)} \Phi_{n} d W-B u_{0 n}\right)^{\prime} \rightharpoonup \zeta \text { weakly in } \mathscr{V}_{\omega}^{\prime} \tag{76.4.40}
\end{gather*}
$$

By the earlier convergence 76.4.30, this $u$ is the same as the one in 76.4.30.
Consider $\zeta$. Let $\psi$ be infinitely differentiable and equal to 0 near $T$ and let $g \in V$. Then since $B u_{n}(0)=B u_{0 n}$,

$$
\begin{gathered}
\int_{0}^{T}\langle\zeta, \psi g\rangle d t=\lim _{n \rightarrow \infty} \int_{0}^{T}\left\langle\left(B u_{n}-B \int_{0}^{(\cdot)} \Phi_{n} d W-B u_{0 n}\right)^{\prime}, \psi g\right\rangle d t \\
=-\lim _{n \rightarrow \infty} \int_{0}^{T}\left\langle\left(B u_{n}-B \int_{0}^{(\cdot)} \Phi_{n} d W-B u_{0 n}\right), \psi^{\prime} g\right\rangle d t \\
\quad=-\int_{0}^{T}\left\langle\psi^{\prime} B g,\left(u-\int_{0}^{(\cdot)} \Phi d W-u_{0}\right)\right\rangle d t \\
\quad=-\int_{0}^{T}\left\langle B\left(u-\int_{0}^{(\cdot)} \Phi d W-u_{0}\right), \psi^{\prime} g\right\rangle d t
\end{gathered}
$$

which shows that

$$
\zeta=\left(B\left(u-\int_{0}^{(\cdot)} \Phi d W-u_{0}\right)\right)^{\prime}
$$

in the sense of $V^{\prime}$ valued distributions. Also from the above,

$$
\begin{aligned}
\int_{0}^{T}\langle\zeta, \psi g\rangle d t= & \left\langle B u(0)-B u_{0}, \psi(0) g\right\rangle \\
& +\int_{0}^{T}\left\langle\left(B u-B \int_{0}^{(\cdot)} \Phi d W-B u_{0}\right), \psi^{\prime} g\right\rangle d t \\
= & \left\langle B u(0)-B u_{0}, \psi(0) g\right\rangle+\int_{0}^{T}\langle\zeta, \psi g\rangle d t
\end{aligned}
$$

Hence $B(u(0, \omega))=B u_{0}$. Thus this has shown that

$$
\left(B\left(u-\int_{0}^{(\cdot)} \Phi d W-u_{0}\right)\right)^{\prime}+\xi(\cdot, \omega)=f(\cdot, \omega) \text { in } \mathscr{V}_{\omega}^{\prime}, B u(0)=B u_{0}
$$

Thus integrating this, we get

$$
\begin{equation*}
B u(t, \omega)-B u_{0}(\omega)+\int_{0}^{t} \xi(s, \omega) d s=\int_{0}^{t} f(s, \omega) d s+B \int_{0}^{t} \Phi d W \tag{76.4.41}
\end{equation*}
$$

Lemma 76.4.3 The above sequence does not depend on $\omega \notin N$. In fact, it is not necessary to take a further subsequence.

Proof: In fact, it is not necessary to take a subsequence to get the convergences 76.4.38 - 76.4.40. This is because of the pointwise convergence of 76.4.30 and Lemma 76.4.1. If the original sequence did not converge, then there would be two subsequences converging weakly to two different functions in $\mathscr{V}_{\omega} v, w$ which is impossible because of 76.4.30 and this lemma since it would require $v(t)=w(t)$ a.e.

The question at this point is whether $u$ is progressively measurable. From the assumed estimates, the Ito formula, and 76.4.27, the same kind of estimates used earlier show that there exists an estimate of the form

$$
\left\|u_{n}\right\|_{\mathscr{V}} \leq C
$$

Therefore, there exists a further subsequence such that

$$
u_{n} \rightarrow \bar{u} \text { weakly in } \mathscr{V}
$$

It follows from Lemma 76.3.4 that off an enlarged exceptional set of measure zero, still denoted as $N$,

$$
\bar{u}(\cdot, \omega)=u(\cdot, \omega) \text { in } \mathscr{V}_{\omega}
$$

Hence we can assume that $u$ is progressively measurable into $V$. It follows that $B u$ is progressively measurable into $W^{\prime}$.

Thus also

$$
(t, \omega) \rightarrow \int_{0}^{t} \xi(s, \omega) d s
$$

is progressively measurable into $V^{\prime}$.
Of course the next task is to identify $\xi$. This is always a problem even in the non stochastic case. Here it is especially difficult because in order to identify $\xi$ we need to use the implicit Ito formula which only holds if $\xi$ is sufficiently measurable. However, we have obtained $\xi$ as a weak limit for fixed $\omega$. Therefore, this is a significant issue. In stochastic evolution problems where $B=I$ this is not as difficult because one gets $\xi$ as a weak limit in $\mathscr{V}$ and then $\xi$ is progressively measurable. We cannot do it this way and still get the best results in which there is a solution to the integral equation which holds for all $t$ off a set of measure zero because of the degenerate nature of the operator $B$. However, $\xi$ is only an equivalence class of functions. We show in the next lemma that there exists a representative of this equivalence class for each $\omega$ off an exceptional set of measure zero such that the resulting $\xi$ is progressively measurable. This will enable us to use the implicit Ito formula and indentify $\xi$.

The following lemma will allow the use of the Ito formula and eventually identify $\xi$.
Lemma 76.4.4 Enlarging the exceptional set, one can assume that $\xi$ is also progressively measurable. In fact, if

$$
\xi_{n} \equiv \int_{t-(1 / n)}^{t} \xi d s
$$

is known to be progressively measurable, $\xi(t, \omega) \equiv 0$ for $t<0$, then there exists a set of measure zero $N$ such that for $\omega \notin N, \xi(t, \omega)=\bar{\xi}(t, \omega)$ for all $t$ off a set of measure zero and $\bar{\xi}$ is progressively measurable.

Proof: Define

$$
\xi_{n} \equiv n \int_{t-(1 / n)}^{t} \xi d s
$$

where $\xi$ is defined to be zero for $t \leq 0$. Then by what was just shown, this is progressively measurable. Also, standard approximate identity arguments verify that for each $\omega, \xi_{n} \rightarrow$
$\xi$ in $\mathscr{V}_{\omega}$. Next note that the set where $\xi_{n}$ is not a Cauchy sequence is a progressively measurable set. It equals

$$
\cup_{n} \cap_{m} \cup_{k, l \geq m}\left[(t, \omega):\left\|\xi_{l}(t, \omega)-\xi_{k}(t, \omega)\right\|>\frac{1}{n}\right] \equiv S
$$

Now for $p>0$

$$
\lim _{m \rightarrow \infty} P\left(\sup _{p>0}\left\|\xi_{m+p}-\xi_{m}\right\|_{\mathscr{V}_{\omega}}>\varepsilon\right)=0
$$

This is because of the convergence of $\xi_{n}$ to $\xi$ in $\mathscr{V}_{\omega}$. Therefore, there is a subsequence still called $\xi_{n}$ such that

$$
P\left(\sup _{p>0}\left\|\xi_{n+p}-\xi_{n}\right\|_{\mathscr{V}_{\omega}}>2^{-n}\right)<2^{-n}
$$

and so there is an enlarged set of measure zero, still denoted as $N$ such that all of the above considerations hold for $\omega \notin N$ and also for $\omega \notin N$,

$$
\sup _{p>0}\left\|\xi_{n+p}-\xi_{n}\right\|_{\mathscr{V}_{\omega}} \leq 2^{-n}
$$

for all $n$ large enough. Now let $S$ defined above, correspond to this particular subsequence. Let $S(\omega)$ be those $t$ such that $(t, \omega) \in S$. Then $S(\omega)$ is a set of measure zero for each $\omega \notin N$ because the above inequality implies that $t \rightarrow \xi_{n}(t, \omega)$ is a Cauchy sequence off a set of measure zero which by definition is $S(\omega)$. Then consider $\left\{\xi_{n}(t, \omega) \mathscr{X}_{S^{C}}(t, \omega)\right\}$. For each $\omega$ off $N$, this converges for all $t$. Thus it converges pointwise to a function $\bar{\xi}$ which must be progressively measurable. However, $t \rightarrow \bar{\xi}(t, \omega)$ must also equal $t \rightarrow \xi(t, \omega)$ in $\mathscr{V}_{\omega}$ by the above construction. Therefore, we can assume without loss of generality that $\xi$ is itself progressively measurable.

From the weak convergence of $u_{n}$ to $u$ in $\mathscr{V}_{\omega}$,

$$
B u_{n} \rightarrow B u \text { weakly in } \mathscr{V}_{\omega}^{\prime}
$$

and so

$$
(\lambda B+A(\omega)) u_{n} \rightarrow \lambda B u+\xi \text { weakly in } \mathscr{V}_{\omega}^{\prime}
$$

Now the above convergences and the integral equation imply that off the exceptional set $N$, for each $t$

$$
B u_{n}(t) \rightarrow B u(t) \text { weakly in } V^{\prime}
$$

From a generalization of standard theorems in Hilbert space, stated in Lemma 76.2.1 there exist vectors $\left\{e_{i}\right\} \subseteq V$ such that

$$
\left\langle B u_{n}(t), u_{n}(t)\right\rangle=\sum_{i=1}^{\infty}\left|\left\langle B u_{n}(t), e_{i}\right\rangle\right|^{2}
$$

Hence

$$
\lim \inf _{n \rightarrow \infty}\left\langle B u_{n}(t), u_{n}(t)\right\rangle \geq \sum_{i=1}^{\infty} \lim \inf _{n \rightarrow \infty}\left|\left\langle B u_{n}(t), e_{i}\right\rangle\right|^{2}
$$

$$
\begin{equation*}
=\sum_{i=1}^{\infty}\left|\left\langle B u(t), e_{i}\right\rangle\right|^{2}=\langle B u(t), u(t)\rangle \tag{76.4.42}
\end{equation*}
$$

Thus the above inequalities and formulas hold for a.e. $t$.
Return to the equation 76.4.41. Define the stopping time

$$
\tau_{p} \equiv \inf \left\{t \in[0, T]:\langle B u, u\rangle(t)+\int_{0}^{t}\|\xi\|_{V^{\prime}}^{p^{\prime}} d s>p\right\}
$$

From 76.4.28 and the fact that $\xi \in \mathscr{V}_{\omega}^{\prime}$, it follows that $\tau_{p}=\infty$ for all $p$ large enough. Then stop the equation using this stopping time.

$$
\begin{aligned}
& B u^{\tau_{p}}(t, \omega)-B u_{0}(\omega)+\int_{0}^{t} \mathscr{X}_{\left[0, \tau_{p}\right]} \xi^{\tau_{p}}(s, \omega) d s \\
= & \int_{0}^{t} \mathscr{X}_{\left[0, \tau_{p}\right]} f(s, \omega) d s+B \int_{0}^{t} \mathscr{X}_{\left[0, \tau_{p}\right]} \Phi d W
\end{aligned}
$$

From the implicit Ito formula Theorem 76.2.3, for a.e. $t$,

$$
\begin{aligned}
& \frac{1}{2}\left\langle B u^{\tau_{p}}(t), u^{\tau_{p}}(t)\right\rangle-\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle+\int_{0}^{t} \mathscr{X}_{\left[0, \tau_{p}\right]}\left\langle\xi^{\tau_{p}}, u^{\tau_{p}}\right\rangle d s \\
= & \frac{1}{2} \int_{0}^{t} \mathscr{X}_{\left[0, \tau_{p}\right]}\langle B \Phi, \Phi\rangle d s \\
+ & \int_{0}^{t} \mathscr{X}_{\left[0, \tau_{p}\right]}\left\langle f, u^{\tau_{p}}\right\rangle d s+\int_{0}^{t} \mathscr{X}_{\left[0, \tau_{p}\right]}\left(\Phi \circ J^{-1}\right)^{*} B u^{\tau_{p}} \circ J d W
\end{aligned}
$$

Then letting $p \rightarrow \infty$ this yields the following formula for a.e. $t$

$$
\begin{align*}
& \frac{1}{2}\langle B u(t), u(t)\rangle-\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle+\int_{0}^{t}\langle\lambda B u+\xi, u\rangle d s=\frac{1}{2} \int_{0}^{t}\langle B \Phi, \Phi\rangle d s \\
& \quad+\int_{0}^{t}\langle f, u\rangle d s+\int_{0}^{t}\left(\Phi \circ J^{-1}\right)^{*} B u \circ J d W+\int_{0}^{t}\langle\lambda B u, u\rangle d s \tag{76.4.43}
\end{align*}
$$

Lemma 76.4.5 It is true that

$$
\lim _{n \rightarrow \infty} \int_{0}^{T}\left\langle B u_{n}, u_{n}\right\rangle d t=\int_{0}^{T}\langle B u, u\rangle d t
$$

Proof: From 76.4.28 $B u_{n} \rightarrow z$ strongly in $C\left(N_{\omega}^{C}, W^{\prime}\right)$. But also, for each $t, B u_{n}(t) \rightarrow$ $B u(t)$ weakly in $V^{\prime}$ and so $z(t)=B u(t)$. This strong convergence in $C\left(N_{\omega}^{C}, W^{\prime}\right)$ along with the uniform norm with the weak convergence of $u_{n}$ to $u$ in $\mathscr{V}_{\omega}$ is sufficient to obtain the above limit.

You might think that

$$
\int_{0}^{T}\left(\Phi_{n} \circ J^{-1}\right)^{*} B u_{n} \circ J d W \rightarrow \int_{0}^{T}\left(\Phi \circ J^{-1}\right)^{*} B u \circ J d W
$$

but this is not entirely clear. It will be true in the case that in 76.4.26, $\alpha=2$ and $U=W$ and this is shown later. However, it is not clearly true here unless it is also the case that $\Phi \in L^{2}\left(\Omega, L^{\infty}\left([0, T], \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right)\right)$.

Lemma 76.4.6 If $\Phi \in L^{2}\left(\Omega, L^{\infty}\left([0, T], \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right)\right)$ then

$$
\int_{0}^{T}\left(\Phi_{n} \circ J^{-1}\right)^{*} B u_{n} \circ J d W \rightarrow \int_{0}^{T}\left(\Phi \circ J^{-1}\right)^{*} B u \circ J d W
$$

Proof:

$$
\begin{gather*}
E\left(\left|\int_{0}^{T}\left(\Phi_{n} \circ J^{-1}\right)^{*} B u_{n} \circ J d W-\int_{0}^{T}\left(\Phi \circ J^{-1}\right)^{*} B u \circ J d W\right|\right) \\
\leq E\left(\left|\int_{0}^{T}\left(\Phi_{n} \circ J^{-1}\right)^{*} B u_{n} \circ J d W-\int_{0}^{T}\left(\Phi \circ J^{-1}\right)^{*} B u_{n} \circ J d W\right|\right) \\
+E\left(\left|\int_{0}^{T}\left(\Phi \circ J^{-1}\right)^{*} B u_{n} \circ J d W-\int_{0}^{T}\left(\Phi \circ J^{-1}\right)^{*} B u \circ J d W\right|\right) \\
\leq \\
\quad \int_{\Omega}\left(\left(\int_{0}^{T}\left\|\Phi_{n}-\Phi\right\|_{\mathscr{L}_{2}}^{2}\left\langle B u_{n}, u_{n}\right\rangle\right)^{1 / 2}\right) d P  \tag{76.4.44}\\
\\
\quad+\int_{\Omega}\left(\int_{0}^{T}\|\Phi\|_{\mathscr{L}_{2}}^{2}\left\langle B u_{n}-B u, u_{n}-u\right\rangle d t\right)^{1 / 2} d P
\end{gather*}
$$

Consider that second term. It is no larger than

$$
\begin{aligned}
& \int_{\Omega}\|\Phi\|_{L^{\infty}\left([0, T], \mathscr{L}_{2}\right)}\left(\int_{0}^{T}\left\langle B u_{n}-B u, u_{n}-u\right\rangle d t\right)^{1 / 2} d P \\
\leq & \left(\int_{\Omega}\|\Phi\|_{L^{\infty}\left([0, T], \mathscr{L}_{2}\right)}^{2}\right)^{1 / 2}\left(\int_{\Omega} \int_{0}^{T}\left\langle B u_{n}-B u, u_{n}-u\right\rangle d t d P\right)^{1 / 2}
\end{aligned}
$$

Now consider the following. Letting the $e_{i}$ be the special vectors of Lemma 34.4.2, it follows,

$$
\begin{aligned}
\int_{\Omega} \int_{0}^{T} & \left\langle B u_{n}-B u, u_{n}-u\right\rangle d t d P=\int_{\Omega} \int_{0}^{T} \sum_{i=1}^{\infty}\left\langle B u_{n}-B u, e_{i}\right\rangle^{2} d t d P \\
& =\int_{\Omega} \int_{0}^{T} \sum_{i=1}^{\infty} \lim _{p \rightarrow \infty} \inf _{p \rightarrow \infty}\left\langle B u_{n}-B u_{p}, e_{i}\right\rangle^{2} d t d P \\
& \leq \lim _{p \rightarrow \infty} \inf _{\Omega} \int_{\Omega} \int_{0}^{T} \sum_{i=1}^{\infty}\left\langle B u_{n}-B u_{p}, e_{i}\right\rangle^{2} d t d P \\
& =\lim _{p \rightarrow \infty} \inf _{\Omega} \int_{0}^{T}\left\langle B u_{n}-B u_{p}, u_{n}-u_{p}\right\rangle d t d P \leq \frac{T}{2^{n}}
\end{aligned}
$$

The last inequality follows from 76.4.34. Therefore, the second term in 76.4.44 is no larger than $\left(C(T, \Phi) / 2^{n}\right)^{1 / 2}$ which converges to 0 as $n \rightarrow \infty$. Now consider the first term in 76.4.44.

$$
\int_{\Omega}\left(\left(\int_{0}^{T}\left\|\Phi_{n}-\Phi\right\|_{\mathscr{L}_{2}}^{2}\left\langle B u_{n}, u_{n}\right\rangle\right)^{1 / 2}\right) d P
$$

$$
\begin{aligned}
& \leq \int_{\Omega_{t \in[0, T]} \sup _{t}\left\langle B u_{n}, u_{n}\right\rangle^{1 / 2}(t)\left(\left(\int_{0}^{T}\left\|\Phi_{n}-\Phi\right\|_{\mathscr{L}_{2}}^{2} d t\right)^{1 / 2}\right) d P}^{\leq\left(\int_{\Omega_{t \in[0, T]}} \sup _{t]}\left\langle B u_{n}, u_{n}\right\rangle(t)\right)^{1 / 2}\left(\int_{\Omega} \int_{0}^{T}\left\|\Phi_{n}-\Phi\right\|_{\mathscr{L}_{2}}^{2} d t\right)^{1 / 2}}
\end{aligned}
$$

From 76.4.32

$$
\leq C\left(\int_{\Omega} \int_{0}^{T}\left\|\Phi_{n}-\Phi\right\|_{\mathscr{L}_{2}}^{2} d t\right)^{1 / 2}
$$

which converges to 0 .
Return now to the equation solved by $u_{n}$ in 76.4.37. Apply the Ito formula to this one. This yields for a.e. $t$,

$$
\begin{align*}
\frac{1}{2}\left\langle B u_{n}(t), u_{n}(t)\right\rangle & -\frac{1}{2}\left\langle B u_{0 n}, u_{0 n}\right\rangle+\int_{0}^{t}\left\langle A(\omega) u_{n}, u_{n}\right\rangle d s=\frac{1}{2} \int_{0}^{t}\left\langle B \Phi_{n}, \Phi_{n}\right\rangle d s \\
& +\int_{0}^{t}\left\langle f, u_{n}\right\rangle d s+\int_{0}^{t}\left(\Phi_{n} \circ J^{-1}\right)^{*} B u_{n} \circ J d W \tag{76.4.45}
\end{align*}
$$

Assume without loss of generality that $T$ is not in the exceptional set. If not, consider all $T^{\prime}$ close to $T$ such that $T^{\prime}$ is not in the exceptional set.

$$
\begin{gathered}
\int_{0}^{T}\left\langle(\lambda B+A(\omega)) u_{n}, u_{n}\right\rangle d s \\
=\frac{1}{2}\left\langle B u_{0 n}, u_{0 n}\right\rangle-\frac{1}{2}\left\langle B u_{n}(T), u_{n}(T)\right\rangle+\int_{0}^{T}\left\langle f, u_{n}\right\rangle d s \\
+\int_{0}^{T}\left(\Phi_{n} \circ J^{-1}\right)^{*} B u_{n} \circ J d W+\frac{1}{2} \int_{0}^{T}\left\langle B \Phi_{n}, \Phi_{n}\right\rangle d s+\int_{0}^{T}\left\langle\lambda B u_{n}, u_{n}\right\rangle d s
\end{gathered}
$$

Now it follows from 76.4.42 applied to $t=T$ and the above lemma that

$$
\begin{gathered}
\lim \sup _{n \rightarrow \infty} \int_{0}^{T}\left\langle(\lambda B+A(\omega)) u_{n}, u_{n}\right\rangle d s \\
\leq \frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle-\frac{1}{2}\langle B u(T), u(T)\rangle+\int_{0}^{T}\langle f, u\rangle d s \\
+\int_{0}^{T}\left(\Phi \circ J^{-1}\right)^{*} B u \circ J d W+\frac{1}{2} \int_{0}^{T}\langle B \Phi, \Phi\rangle d s+\int_{0}^{T}\langle\lambda B u, u\rangle d s
\end{gathered}
$$

and from 76.4.43, the expression on the right equals $\int_{0}^{T}\langle\lambda B u+\xi, u\rangle d s$. Hence

$$
\lim \sup _{n \rightarrow \infty} \int_{0}^{T}\left\langle(\lambda B+A(\omega)) u_{n}, u_{n}\right\rangle d s \leq \int_{0}^{T}\langle\lambda B u+\xi, u\rangle d s
$$

Then since $\lambda B+A(\omega)$ is monotone and hemicontinuous, it is type $M$ and so this requires $A(\omega) u=\xi$.

Hence we obtain

$$
B u(t)-B u_{0}(\omega)+\int_{0}^{t} A(\omega)(u) d s=\int_{0}^{t} f(s, \omega) d s+B \int_{0}^{t} \Phi d W
$$

This is a solution for a given $\omega \notin N$. Also, a stopping time argument like the above and the coercivity estimates for $A$ along with the implicit Ito formula show that $u \in \mathscr{V}$. This yields the existence part of the following existence and uniqueness theorem.

Theorem 76.4.7 Suppose $\mathscr{V} \equiv L^{p}([0, T] \times \Omega, V)$ where $p \geq 2$,with the $\sigma$ algebra of progressively measurable sets and $\mathscr{V}_{\omega}=L^{p}([0, T], V)$.

$$
\begin{aligned}
\Phi & \in L^{2}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right) \cap L^{2}\left(\Omega, L^{\infty}\left([0, T], \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right)\right), \\
f & \in \quad \mathscr{V}^{\prime} \equiv L^{p^{\prime}}\left([0, T] \times \Omega, V^{\prime}\right)
\end{aligned}
$$

and both are progressively measurable. Suppose that

$$
\lambda B+A(\omega): \mathscr{V}_{\omega} \rightarrow \mathscr{V}_{\omega}^{\prime}, \lambda B+A: \mathscr{V} \rightarrow \mathscr{V}^{\prime}
$$

are monotone hemicontinuous and bounded where

$$
A(\omega) u(t) \equiv A(t, u(t), \omega)
$$

and $(t, u, \omega) \rightarrow A(t, u, \omega)$ is progressively measurable. Also suppose for $p \geq 2$, the coercivity, and the boundedness conditions

$$
\begin{equation*}
\lambda\langle B u, u\rangle+\langle A(t, u, \omega), u\rangle_{V} \geq \delta\|u\|_{V}^{p}-c(t, \omega) \tag{76.4.46}
\end{equation*}
$$

where $c \in L^{1}([0, T] \times \Omega)$ for all $\lambda$ large enough. Also,

$$
\begin{equation*}
\|A(t, u, \omega)\|_{V^{\prime}} \leq k\|u\|_{V}^{p-1}+c^{1 / p^{\prime}}(t, \omega) \tag{76.4.47}
\end{equation*}
$$

also suppose the monotonicity condition for all $\lambda$ large enough.

$$
\begin{equation*}
\langle(\lambda B+A(\omega))(u)-(\lambda B+A(\omega))(v), u-v\rangle \geq \delta\|u-v\|_{U}^{\alpha} \tag{76.4.48}
\end{equation*}
$$

Then if $u_{0} \in L^{2}(\Omega, W)$ with $u_{0} \mathscr{F}_{0}$ measurable, there exists a unique solution $u(\cdot, \omega) \in \mathscr{V}_{\omega}$ with $u \in \mathscr{V}\left(L^{p}([0, T] \times \Omega, V)\right.$ and progressively measurable) such that for $\omega$ off a set of measure zero,

$$
B u(t, \omega)-B u_{0}(\omega)+\int_{0}^{t} A(s, u(s, \omega), \omega) d s=\int_{0}^{t} f d s+B \int_{0}^{t} \Phi d W
$$

It is also assumed that $V$ is a reflexive separable real Banach space.
Proof: The uniqueness assertion follows easily from the monotonicity condition.
Now we remove the assumption that $\Phi \in L^{2}\left(\Omega, L^{\infty}\left([0, T], \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right)\right)$. Everything is the same except for the need for a different argument to show that

$$
\int_{0}^{T}\left(\Phi_{n} \circ J^{-1}\right)^{*} B u_{n} \circ J d W \rightarrow \int_{0}^{T}\left(\Phi \circ J^{-1}\right)^{*} B u \circ J d W
$$

In this case we assume

$$
\langle(\lambda B+A(\omega))(u)-(\lambda B+A(\omega))(v), u-v\rangle \geq \delta\|u-v\|_{W}^{2}
$$

Then repeating the above argument with this change yields set of measure zero, still denoted as $N$ such that for $\omega \notin N$

$$
\begin{equation*}
\int_{0}^{T}\left\|u_{n}-u_{n+1}\right\|_{W}^{2} d s \leq 2^{-n} \tag{76.4.49}
\end{equation*}
$$

for all $n$ large enough. Hence for such $\omega, u_{n}(\cdot, \omega)$ is Cauchy in $L^{2}([0, T], W)$ and in fact $u_{n}(t, \omega)$ is a Cauchy sequence in $W$. Thus $\left\{u_{n}(\cdot, \omega)\right\}$ converges in $L^{2}([0, T], W)$ to $u(\cdot, \omega) \in L^{2}([0, T], W)$ and by the above considerations involving continuous dependence of $V$ into $W$, it follows that $u(\cdot, \omega)$ will be the same as the $u$ from the above convergences. Now this convergence implies that in addition, for a.e. $t$,

$$
\begin{gather*}
\lim _{n \rightarrow \infty}\left\langle B u_{n}(t, \omega)-B u(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle=0  \tag{76.4.50}\\
\lim _{n \rightarrow \infty} \int_{0}^{T}\left\langle B u_{n}(t, \omega)-B u(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle d t=0
\end{gather*}
$$

What is known from 76.4.35 is that for

$$
M_{n}(t)=\int_{0}^{t}\left(\Phi_{n} \circ J^{-1}\right)^{*} B u_{n} \circ J d W
$$

there is a continuous martingale $M \in M_{T}^{1}$ such that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} E\left(\sup _{t \in[0, T]}\left|M_{n}(t)-M(t)\right|\right)=0 \tag{76.4.51}
\end{equation*}
$$

Define a stopping time

$$
\tau_{p} \equiv \inf \left\{t:\langle B u, u\rangle(t)+\sup _{n}\left\langle B u_{n}, u_{n}\right\rangle(t)>p\right\}
$$

This is a good enough stopping time because the function used to define it as a hitting time is lower semicontinuous.

Lemma 76.4.8 $\int_{0}^{T}\left(\Phi_{n} \circ J^{-1}\right)^{*} B u_{n} \circ J d W \rightarrow \int_{0}^{T}\left(\Phi \circ J^{-1}\right)^{*} B u \circ J d W$ in probability. Also there is a futher subsequence and set of measure zero such that off this set,

$$
\lim _{n \rightarrow \infty}\left(\sup _{t \in[0, T]}\left|\int_{0}^{t}\left(\Phi \circ J^{-1}\right)^{*} B u \circ J d W-\int_{0}^{t}\left(\Phi_{n} \circ J^{-1}\right)^{*} B u_{n} \circ J d W\right|\right)=0 .
$$

In particular, what is needed here is valid,

$$
\lim _{n \rightarrow \infty} \int_{0}^{T}\left(\Phi_{n} \circ J^{-1}\right)^{*} B u_{n} \circ J d W=\int_{0}^{T}\left(\Phi \circ J^{-1}\right)^{*} B u \circ J d W
$$

Proof: Let $\varepsilon>0$. Then define

$$
A_{n}=\left\{\omega:\left|\int_{0}^{T}\left(\Phi_{n} \circ J^{-1}\right)^{*} B u_{n} \circ J d W-\int_{0}^{T}\left(\Phi \circ J^{-1}\right)^{*} B u \circ J d W\right|>\varepsilon\right\}
$$

Then

$$
A_{n}=\cup_{p=1}^{\infty} A_{n} \cap\left(\left[\tau_{p}=\infty\right] \backslash\left[\tau_{p-1}<\infty\right]\right),\left[\tau_{0}<\infty\right] \equiv \emptyset
$$

the sets in the union being disjoint. Then $A \cap\left(\left[\tau_{p}=\infty\right]\right) \subseteq$

$$
\left\{\omega:\left|\begin{array}{c}
\int_{0}^{T} \mathscr{X}_{\left[0, \tau_{p}\right]}\left(\Phi_{n} \circ J^{-1}\right)^{*} B u_{n} \circ J d W \\
-\int_{0}^{T} \mathscr{X}_{\left[0, \tau_{p}\right]}\left(\Phi \circ J^{-1}\right)^{*} B u \circ J d W
\end{array}\right|>\varepsilon\right\}
$$

Then as before,

$$
\begin{align*}
& E\left(\left|\int_{0}^{T} \mathscr{X}_{\left[0, \tau_{p}\right]}\left(\Phi_{n} \circ J^{-1}\right)^{*} B u_{n} \circ J d W-\int_{0}^{T} \mathscr{X}_{\left[0, \tau_{p}\right]}\left(\Phi \circ J^{-1}\right)^{*} B u \circ J d W\right|\right) \\
& \leq \int_{\Omega}\left(\left(\int_{0}^{T}\left\|\Phi_{n}-\Phi\right\|_{\mathscr{L}_{2}}^{2} \mathscr{X}_{\left[0, \tau_{p}\right]}\left\langle B u_{n}, u_{n}\right\rangle\right)^{1 / 2}\right) d P \\
&+\int_{\Omega}\left(\int_{0}^{T} \mathscr{X}_{\left[0, \tau_{p}\right]}\|\Phi\|_{\mathscr{L}_{2}}^{2}\left\langle B u_{n}-B u, u_{n}-u\right\rangle d t\right)^{1 / 2} d P \tag{76.4.52}
\end{align*}
$$

Consider the second term. It is no larger than

$$
\left(\int_{\Omega} \int_{0}^{T} \mathscr{X}_{\left[0, \tau_{p}\right]}\|\Phi\|_{\mathscr{L}_{2}}^{2}\left\langle B u_{n}-B u, u_{n}-u\right\rangle d t d P\right)^{1 / 2}
$$

Now $t \rightarrow\left\langle B u_{n}, u_{n}\right\rangle$ is continuous and so $\mathscr{X}_{\left[0, \tau_{p}\right]}\left\langle B u_{n}, u_{n}\right\rangle(t) \leq p$. If not, then you would have $\left\langle B u_{n}, u_{n}\right\rangle(t)>p$ for some $t \leq \tau_{p}$ and so, by continuity, there would be $s<t \leq \tau_{p}$ for which $\left\langle B u_{n}, u_{n}\right\rangle(s)>p$ contrary to the definition of $\tau_{p}$. Then $\mathscr{X}_{\left[0, \tau_{p}\right]}\left\langle B u_{n}-B u, u_{n}-u\right\rangle$ is bounded a.e. and also converges to 0 for a.e. $t \leq \tau_{p}$ as $n \rightarrow \infty$. Therefore, off a set of measure zero, including the set where $t \rightarrow\|\Phi\|_{\mathscr{L}_{2}}^{2}$ is not in $L^{1}$, the double integral converges to 0 by the dominated convergence theorem. As to the first integral in 76.4.52, it is dominated by

$$
\begin{aligned}
& \int_{\Omega} \mathscr{X}_{\left[0, \tau_{p}\right]} \sup _{t \in\left[0, \tau_{p}\right]}\left\langle B u_{n}, u_{n}\right\rangle^{1 / 2}(t)\left(\int_{0}^{T}\left\|\Phi_{n}-\Phi\right\|_{\mathscr{L}_{2}}^{2}\right)^{1 / 2} d P \\
\leq & \left(\int_{\Omega_{t \in[0, T]}} \sup \left\langle B u_{n}, u_{n}\right\rangle d P\right)^{1 / 2}\left(\int_{\Omega} \int_{0}^{T}\left\|\Phi_{n}-\Phi\right\|_{\mathscr{L}_{2}}^{2} d t d P\right)^{1 / 2}
\end{aligned}
$$

From the estimate 76.4.32,

$$
\leq C\left(\int_{\Omega} \int_{0}^{T}\left\|\Phi_{n}-\Phi\right\|_{\mathscr{L}_{2}}^{2} d t d P\right)^{1 / 2}
$$

for a constant $C$ independent of $n$. Therefore,

$$
\lim _{n \rightarrow \infty} E\left(\left|\begin{array}{l}
\int_{0}^{T} \mathscr{X}_{\left[0, \tau_{p}\right]}\left(\Phi_{n} \circ J^{-1}\right)^{*} B u_{n} \circ J d W \\
-\int_{0}^{T} \mathscr{X}_{\left[0, \tau_{p}\right]}\left(\Phi \circ J^{-1}\right)^{*} B u \circ J d W
\end{array}\right|\right)=0
$$

Hence

$$
P\left(A_{n} \cap\left[\tau_{p}=\infty\right]\right) \leq \frac{1}{\varepsilon} E\left(\left|\begin{array}{l}
\int_{0}^{T} \mathscr{X}_{\left[0, \tau_{p}\right]}\left(\Phi_{n} \circ J^{-1}\right)^{*} B u_{n} \circ J d W \\
-\int_{0}^{T} \mathscr{X}_{\left[0, \tau_{p}\right]}\left(\Phi \circ J^{-1}\right)^{*} B u \circ J d W
\end{array}\right|\right)
$$

and so

$$
\lim _{n \rightarrow \infty} P\left(A_{n} \cap\left[\tau_{p}=\infty\right]\right)=0
$$

Then

$$
P\left(A_{n}\right)=\sum_{p=1}^{\infty} P\left(A_{n} \cap\left(\left[\tau_{p}=\infty\right] \backslash\left[\tau_{p-1}<\infty\right]\right)\right)
$$

and so from the dominated convergence theorem,

$$
\lim _{n \rightarrow \infty} P\left(A_{n}\right)=\sum_{p=1}^{\infty} \lim _{n \rightarrow \infty} P\left(A_{n} \cap\left(\left[\tau_{p}=\infty\right] \backslash\left[\tau_{p-1}<\infty\right]\right)\right)=\sum_{p} 0=0 .
$$

There was nothing special about $T$. The same argument holds for all $t$ and so $M(t)$ mentioned above has been identified as $\int_{0}^{t}\left(\Phi \circ J^{-1}\right)^{*} B u \circ J d W$. Then from 76.4.51

$$
\lim _{n \rightarrow \infty} E\left(\sup _{t \in[0, T]}\left|\int_{0}^{t}\left(\Phi \circ J^{-1}\right)^{*} B u \circ J d W-\int_{0}^{t}\left(\Phi_{n} \circ J^{-1}\right)^{*} B u_{n} \circ J d W\right|\right)=0
$$

It follows from the usual Borel Cantelli argument that there is a set of measure zero and a further subsequence such that off this set, all the above convergences happen and also

$$
\int_{0}^{t}\left(\Phi_{n} \circ J^{-1}\right)^{*} B u_{n} \circ J d W \rightarrow \int_{0}^{t}\left(\Phi \circ J^{-1}\right)^{*} B u \circ J d W
$$

uniformly on $[0, T]$.
The rest of the argument is identical. This yields the following theorem.
Theorem 76.4.9 Suppose $\mathscr{V} \equiv L^{p}([0, T] \times \Omega, V)$ where $p \geq 2$, with the $\sigma$ algebra of progressively measurable sets and $\mathscr{V}_{\omega}=L^{p}([0, T], V)$.

$$
\begin{aligned}
\Phi & \in L^{2}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right) \\
f & \in \quad \mathscr{V}^{\prime} \equiv L^{p^{\prime}}\left([0, T] \times \Omega, V^{\prime}\right)
\end{aligned}
$$

and both are progressively measurable. Suppose that

$$
\lambda B+A(\omega): \mathscr{V}_{\omega} \rightarrow \mathscr{V}_{\omega}^{\prime}, \lambda B+A: \mathscr{V} \rightarrow \mathscr{V}^{\prime}
$$

are monotone hemicontinuous and bounded where

$$
A(\omega) u(t) \equiv A(t, u(t), \omega)
$$

and $(t, u, \omega) \rightarrow A(t, u, \omega)$ is progressively measurable. Also suppose for $p \geq 2$, the coercivity, and the boundedness conditions

$$
\begin{equation*}
\lambda\langle B u, u\rangle+\langle A(t, u, \omega), u\rangle_{V} \geq \delta\|u\|_{V}^{p}-c(t, \omega) \tag{76.4.53}
\end{equation*}
$$

where $c \in L^{1}([0, T] \times \Omega)$ for all $\lambda$ large enough. Also,

$$
\begin{equation*}
\|A(t, u, \omega)\|_{V^{\prime}} \leq k\|u\|_{V}^{p-1}+c^{1 / p^{\prime}}(t, \omega) \tag{76.4.54}
\end{equation*}
$$

also suppose the monotonicity condition for all $\lambda$ large enough.

$$
\begin{equation*}
\langle(\lambda B+A(\omega))(u)-(\lambda B+A(\omega))(v), u-v\rangle \geq \delta\|u-v\|_{W}^{2} \tag{76.4.55}
\end{equation*}
$$

Then if $u_{0} \in L^{2}(\Omega, W)$ with $u_{0} \mathscr{F}_{0}$ measurable, there exists a unique solution $u(\cdot, \omega) \in \mathscr{V}_{\omega}$ with $u \in \mathscr{V}\left(L^{p}([0, T] \times \Omega, V)\right.$ and progressively measurable) such that for $\omega$ off a set of measure zero,

$$
B u(t, \omega)-B u_{0}(\omega)+\int_{0}^{t} A(s, u(s, \omega), \omega) d s=\int_{0}^{t} f d s+B \int_{0}^{t} \Phi d W
$$

It is also assumed that $V$ is a reflexive separable real Banach space.

### 76.5 Replacing $\Phi$ With $\sigma(u)$

It is not hard to include the case where $\Phi$ is replaced with a function $\sigma(u)$. We make the following assumptions. For each $r>0$ there exists $\lambda$ large enough that

$$
\langle\lambda B(u)+A(u)-(\lambda B(\hat{u})+A(\hat{u})), u-\hat{u}\rangle \geq r\|u-\hat{u}\|_{W}^{2}
$$

Note that in the case where $B=I$ and there is a conventional Gelfand triple, $V, H, V^{\prime}$, this kind of condition is obvious if $\lambda I+A$ is monotone for some $\lambda$. Thus this is not an unreasonable assumption to make although it is stronger than some of the assumptions used above with the integral given by $\int_{0}^{t} \Phi d W$.

As to $\sigma$ we make the following assumptions.

$$
\begin{gathered}
(t, u, \omega) \in[0, T] \times W \times \Omega \rightarrow \sigma(t, u, \omega) \text { is progressively measurable into } W \\
\|\sigma(t, u, \omega)\|_{W} \leq C+C\|u\|_{W} \\
\|\sigma(t, u, \omega)-\sigma(t, \hat{u}, \omega)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)} \leq K\|u-\hat{u}\|_{W}
\end{gathered}
$$

That is, it has linear growth and is Lipschitz.
Let $\lambda$ correspond to $r$ where $r-\|B\| K^{2}>4$. Also let $T$ be such that

$$
\hat{C} e^{\lambda T} K^{2}<3
$$

where $\hat{C}$ is a constant used in the Burkholder Davis Gundy inequality. This is a restriction on the size of $K$. Thus we only give a solution if $K$ is small enough. Later, this will be removed in the most interesting case. This will give a local solution valid for a fixed $T>0$ and then the global solution can be obtained by applying this result on the succession of intervals $[0, T],[T, 2 T],[3 T, 4 T]$, and so forth.

From Theorem 76.4.9, if $w \in \mathscr{W}$, there exists a unique solution $u$ to

$$
B u(t, \omega)-B u_{0}(\omega)+\int_{0}^{t} A(s, u(s, \omega), \omega) d s=\int_{0}^{t} f d s+B \int_{0}^{t} \sigma(w) d W
$$

holding in the sense described there. Let $u_{i}$ result from $w_{i}$. Then from the implicit Ito formula and the above monotonicity estimate,

$$
\begin{aligned}
& \frac{1}{2}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle(t)+r \int_{0}^{t}\left\|u_{1}-u_{2}\right\|_{W}^{2} d s \\
&-\lambda \int_{0}^{t}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle d s \\
&-\int_{0}^{t}\left\langle B \sigma\left(u_{1}\right)-B \sigma\left(u_{2}\right), \sigma\left(u_{1}\right)-\sigma\left(u_{2}\right)\right\rangle_{\mathscr{L}_{2}} d s \leq M^{*}(t)
\end{aligned}
$$

where the right side is of the form $\sup _{s \in[0, t]}|M(s)|$ where $M(t)$ is a local martingale having quadratic variation dominated by

$$
\begin{equation*}
C \int_{0}^{t}\left\|\sigma\left(w_{1}\right)-\sigma\left(w_{2}\right)\right\|^{2}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle d s \tag{76.5.56}
\end{equation*}
$$

Therefore, since $M^{*}$ is increasing in $t$, it follows from the Lipschitz condition on $\sigma$ that

$$
\begin{aligned}
& \frac{1}{2}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle(t)+r \int_{0}^{t}\left\|u_{1}-u_{2}\right\|_{W}^{2} d s \\
& -\lambda \int_{0}^{t}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle d s-\|B\| K^{2} \int_{0}^{t}\left\|u_{1}-u_{2}\right\|_{W}^{2} d s \leq M^{*}(t)
\end{aligned}
$$

Thus, from the assumption about $r$,

$$
\begin{aligned}
& \sup _{s \in[0, t]}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle(s)+4 \int_{0}^{t}\left\|u_{1}-u_{2}\right\|_{W}^{2} d s \\
\leq & \lambda \int_{0}^{t}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle d s+2 M^{*}(t)
\end{aligned}
$$

Then applying Gronwall's inequality,

$$
\sup _{s \in[0, t]}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle(s)+4 \int_{0}^{t}\left\|u_{1}-u_{2}\right\|_{W}^{2} d s \leq 2 e^{\lambda T} M^{*}(t)
$$

Now take expectations and use the Burkholder Davis Gundy inequality. The expectation of the right side is then dominated by

$$
2 \hat{C} e^{\lambda T} \int_{\Omega}\left(\int_{0}^{t}\left\|\sigma\left(w_{1}\right)-\sigma\left(w_{2}\right)\right\|_{\mathscr{L}_{2}}^{2}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle d s\right)^{1 / 2} d P
$$

$$
\begin{gathered}
\leq\left[\begin{array}{c}
\int_{\Omega} \sup _{s \in[0, t]}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle^{1 / 2} . \\
2 \hat{C} e^{\lambda T}\left(\int_{0}^{t} K^{2}\left\|w_{1}-w_{2}\right\|_{W}^{2} d t\right)^{1 / 2} d P
\end{array}\right] \\
\leq E\left(\frac{1}{2} \sup _{s \in[0, t]}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle(s)\right) \\
+\hat{C} e^{\lambda T} E\left(\int_{0}^{t} K^{2}\left\|w_{1}-w_{2}\right\|_{W}^{2} d t\right)
\end{gathered}
$$

It follows that, after adjusting constants as needed, one gets an inequality of the following form.

$$
\begin{aligned}
& \frac{1}{2} E\left(\sup _{s \in[0, t]}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle(s)\right)+4 \int_{\Omega} \int_{0}^{t}\left\|u_{1}-u_{2}\right\|_{W}^{2} d s d P \\
\leq & \hat{C} e^{\lambda T} E\left(\int_{0}^{t} K^{2}\left\|w_{1}-w_{2}\right\|_{W}^{2} d t\right)
\end{aligned}
$$

This holds for every $t \leq T$ and so, from the estimate on the size of $T$, it follows that

$$
\int_{0}^{T} \int_{\Omega}\left\|u_{1}-u_{2}\right\|_{W}^{2} d s d P \leq \frac{3}{4} \int_{0}^{T} \int_{\Omega}\left\|w_{1}-w_{2}\right\|_{W}^{2} d t d P
$$

Therefore, there is a unique fixed point to this mapping which takes $w \in \mathscr{W}$ to $u$ the solution to the integral equation. We denote it as $u$. Thus $u$ is progressively measurable and for $\omega$ off a set of measure zero, we have a solution to the integral equation

$$
\begin{aligned}
& B u(t, \omega)-B u_{0}(\omega)+\int_{0}^{t} A(s, u(s, \omega), \omega) d s \\
= & \int_{0}^{t} f d s+B \int_{0}^{t} \sigma(u) d W, t \in[0, T]
\end{aligned}
$$

Now the same argument can be repeated for the succession of intervals mentioned above. However, you need to be careful that at $T$, you have $B u(T, \omega)=B(u(T, \omega))$ for $\omega$ off a set of measure zero. If this is not so, you locate $T^{\prime}$ close to $T$ for which it is so as in Lemma 73.3.1 and use this $T^{\prime}$ instead, but these are mainly technical issues. This proves the following existence and uniqueness theorem.

Theorem 76.5.1 Suppose $f \in \mathscr{V}^{\prime}$ is progressively measurable and that

$$
(t, \omega) \rightarrow \sigma(t, \omega, u(t, \omega))
$$

is progressively measurable whenever u is. Suppose that

$$
\lambda B+A(\omega): \mathscr{V}_{\omega} \rightarrow \mathscr{V}_{\omega}^{\prime}, \lambda B+A: \mathscr{V} \rightarrow \mathscr{V}^{\prime}
$$

are monotone hemicontinuous and bounded where

$$
A(\omega) u(t) \equiv A(t, u(t), \omega)
$$

and $(t, u, \omega) \rightarrow A(t, u, \omega)$ is progressively measurable. Also suppose for $p \geq 2$, the coercivity, and the boundedness conditions

$$
\begin{equation*}
\lambda\langle B u, u\rangle+\langle A(t, u, \omega), u\rangle_{V} \geq \delta\|u\|_{V}^{p}-c(t, \omega) \tag{76.5.57}
\end{equation*}
$$

for all $\lambda$ large enough.

$$
\begin{equation*}
\|A(t, u, \omega)\|_{V^{\prime}} \leq k\|u\|_{V}^{p-1}+c^{1 / p^{\prime}}(t, \omega) \tag{76.5.58}
\end{equation*}
$$

where $c \in L^{1}([0, T] \times \Omega)$. Also suppose the monotonicity condition that for all $r>0$ there exists $\lambda$ such that

$$
\langle(\lambda B+A(\omega))(u)-(\lambda B+A(\omega))(v), u-v\rangle \geq r\|u-v\|_{W}^{2}
$$

Also suppose that

$$
\begin{gathered}
(t, u, \omega) \in[0, T] \times W \times \Omega \rightarrow \sigma(t, u, \omega) \text { is progressively measurable into } W \\
\|\sigma(t, u, \omega)\|_{W} \leq C+C\|u\|_{W} \\
\|\sigma(t, u, \omega)-\sigma(t, \hat{u}, \omega)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)} \leq K\|u-\hat{u}\|_{W}
\end{gathered}
$$

Then if $u_{0} \in L^{2}(\Omega, W)$ with $u_{0} \mathscr{F}_{0}$ measurable, there exists a unique solution $u(\cdot, \omega) \in \mathscr{V}_{\omega}$ with $u \in \mathscr{V}\left(L^{p}([0, T] \times \Omega, V)\right.$ and progressively measurable) such that for $\omega$ off a set of measure zero,

$$
B u(t, \omega)-B u_{0}(\omega)+\int_{0}^{t} A(s, u(s, \omega), \omega) d s=\int_{0}^{t} f d s+B \int_{0}^{t} \sigma(u) d W
$$

In case $B$ is the Riesz map, you do not have to make any assumption on the size of $K$. Thus

$$
\langle B u, u\rangle=\|u\|_{W}^{2}
$$

The case of most interest is the usual one where $V \subseteq W=W^{\prime} \subseteq V^{\prime}$, the case of a Gelfand triple in which $B$ is the identity. As to $\sigma$, the assumption is made that

$$
\begin{gathered}
\|\sigma(t, \omega, u)\|_{W} \leq C+C\|u\|_{W} \\
\left\|\sigma\left(t, \omega, u_{1}\right)-\sigma\left(t, \omega, u_{2}\right)\right\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)} \leq K\left\|u_{1}-u_{2}\right\|_{W}
\end{gathered}
$$

Of course it is also assumed that whenever $u$ has values in $W$ and is progressively measurable, $(t, \omega) \rightarrow \sigma(t, \omega, u(t, \omega))$ is also progressively measurable into $\mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)$.

Letting $w_{i} \in L^{2}([0, T] \times \Omega, W)$ each $w_{i}$ being progressively measurable, the above assumptions imply that there exists a solution $u_{i}$ to the integral equation

$$
B u_{i}(t, \omega)-B u_{0}(\omega)+\int_{0}^{t} A\left(u_{i}\right) d s=\int_{0}^{t} f(s, \omega) d s+B \int_{0}^{t} \sigma\left(w_{i}\right) d W
$$

here we write $\sigma\left(w_{i}\right)$ for short instead of $\sigma\left(t, \omega, w_{i}\right)$. First, consider

$$
w \in L^{2}([0, T] \times \Omega, W) \cap L^{\infty}\left([0, T], L^{2}(\Omega, W)\right)
$$

and let $u$ be the solution which results from placing $w$ in $\sigma$. Then from the estimates,

$$
\begin{gathered}
\langle B u, u\rangle(t)-\left\langle B u_{0}, u_{0}\right\rangle+\delta \int_{0}^{t}\|u\|_{V}^{p} d s=2 \int_{0}^{t}\langle f, u\rangle d s+C\left(b_{3}, b_{4}, b_{5}\right) \\
+\lambda \int_{0}^{t}\langle B u, u\rangle d s+\int_{0}^{t}\langle B \sigma(w), \sigma(w)\rangle_{\mathscr{L}_{2}} d s+2 M^{*}(t) \\
\leq 2 \int_{0}^{t}\langle f, u\rangle d s+C\left(b_{3}, b_{4}, b_{5}\right)+\lambda \int_{0}^{t}\langle B u, u\rangle d s+\int_{0}^{t}\left(C+C\|w\|_{W}^{2}\right) d s+2 M^{*}(t)
\end{gathered}
$$

where $M^{*}(t)=\sup _{s \in[0, t]}|M(s)|$ and the quadratic variation of $M$ is no larger than

$$
\int_{0}^{t}\|\sigma(w)\|^{2}\langle B(u), u\rangle d s
$$

Then using Gronwall's inequality, one obtains an inequality of the form

$$
\sup _{s \in[0, T]}\langle B u, u\rangle(s) \leq C+C\left(M^{*}(t)+\int_{0}^{t}\|w\|_{W}^{2} d s\right)
$$

where $C=C\left(u_{0}, f, \delta, \lambda, b_{3}, b_{4}, b_{5}, T\right)$ and is integrable. Then take expectation. By Burkholder Davis Gundy inequality and adjusting constants as needed,

$$
\begin{aligned}
& E\left(\sup _{s \in[0, T]}\langle B u, u\rangle(s)\right) \\
\leq & C+C \int_{\Omega} \int_{0}^{T}\|w\|_{W}^{2} d s d P+C \int_{\Omega}\left(\int_{0}^{T}\|\sigma(w)\|^{2}\langle B(u), u\rangle d s\right)^{1 / 2} d P \\
\leq & C+C \int_{\Omega} \int_{0}^{T}\|w\|_{W}^{2} d s d P+C \int_{\Omega} \sup _{s \in[0, T]}\langle B u, u\rangle^{1 / 2}(s)\left(\int_{0}^{T}\|\sigma(w)\|^{2} d s\right)^{1 / 2} d P \\
\leq & C+C \int_{\Omega} \int_{0}^{T}\|w\|_{W}^{2} d s d P+\frac{1}{2} E\left(\sup _{s \in[0, T]}\langle B u, u\rangle(s)\right)+C \int_{\Omega} \int_{0}^{T}\left(C+C\|w\|_{W}^{2}\right)
\end{aligned}
$$

Thus

$$
E(\langle B u, u\rangle(t)) \leq E\left(\sup _{s \in[0, T]}\langle B u, u\rangle(s)\right) \leq C+C \int_{\Omega} \int_{0}^{T}\|w\|_{W}^{2} d s d P
$$

and so

$$
\|u\|_{L^{\infty}\left([0, T], L^{2}(\Omega, W)\right)}^{2} \leq C+C \int_{\Omega} \int_{0}^{T}\|w\|_{W}^{2} d s d P
$$

which implies $u \in L^{\infty}\left([0, T], L^{2}(\Omega, W)\right)$ and is progressively measurable.
Using the monotonicity assumption, there is a suitable $\lambda$ such that

$$
\begin{aligned}
& \frac{1}{2}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle(t)+r \int_{0}^{t}\left\|u_{1}-u_{2}\right\|_{W}^{2} d s \\
& -\lambda \int_{0}^{t}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle d s
\end{aligned}
$$

$$
-\int_{0}^{t}\left\langle B \sigma\left(u_{1}\right)-B \sigma\left(u_{2}\right), \sigma\left(u_{1}\right)-\sigma\left(u_{2}\right)\right\rangle_{\mathscr{L}_{2}} d s \leq M^{*}(t)
$$

where the right side is of the form $\sup _{s \in[0, t]}|M(s)|$ where $M(t)$ is a local martingale having quadratic variation dominated by

$$
\begin{equation*}
C \int_{0}^{t}\left\|\sigma\left(w_{1}\right)-\sigma\left(w_{2}\right)\right\|^{2}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle d s \tag{76.5.59}
\end{equation*}
$$

Then by assumption and using Gronwall's inequality, there is a constant $C=C(\lambda, K, T)$ such that

$$
\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle(t) \leq C M^{*}(t)
$$

Then also, since $M^{*}$ is increasing,

$$
\sup _{s \in[0, t]}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle(s) \leq C M^{*}(t)
$$

Taking expectations and from the Burkholder Davis Gundy inequality,

$$
\begin{gathered}
E\left(\sup _{s \in[0, t]}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle(s)\right) \\
\leq C \int_{\Omega}\left(\int_{0}^{t}\left\|\sigma\left(w_{1}\right)-\sigma\left(w_{2}\right)\right\|^{2}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle\right)^{1 / 2} d P \\
\leq C \int_{\Omega} \sup _{s \in[0, t]}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle^{1 / 2}(s)\left(\int_{0}^{t}\left\|\sigma\left(w_{1}\right)-\sigma\left(w_{2}\right)\right\|^{2}\right)^{1 / 2} d P
\end{gathered}
$$

Then it follows after adjusting constants that there exists an inequality of the form

$$
E\left(\sup _{s \in[0, t]}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle(s)\right) \leq C E\left(\int_{0}^{t}\left\|\sigma\left(w_{1}\right)-\sigma\left(w_{2}\right)\right\|_{\mathscr{L}_{2}}^{2} d s\right)
$$

Hence

$$
E\left(\sup _{s \in[0, t]}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle(t)\right) \leq C K^{2} E\left(\int_{0}^{t}\left\|w_{1}-w_{2}\right\|_{W}^{2} d s\right)
$$

Thus, for each $t \leq T$

$$
\int_{\Omega}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle(t) d P \leq C K^{2} E\left(\int_{0}^{t}\left\|w_{1}-w_{2}\right\|_{W}^{2} d s\right)
$$

one can consider the map $\psi(w) \equiv u$ as described above. Then the above inequality implies

$$
\begin{aligned}
& E\left(\left\langle B\left(\psi^{n} w_{1}-\psi^{n} w_{2}\right), \psi^{n} w_{1}-\psi^{n} w_{2}\right\rangle(t)\right) \\
\leq & C K^{2} E\left(\int_{0}^{t}\left\|\psi^{n-1} w_{1}-\psi^{n-1} w_{2}\right\|_{W}^{2} d t_{1}\right)
\end{aligned}
$$

$$
\begin{gathered}
=C K^{2} E\left(\int_{0}^{t}\left\langle B\left(\psi^{n-1} w_{1}-\psi^{n-1} w_{2}\right), \psi^{n-1} w_{1}-\psi^{n-1} w_{2}\right\rangle\left(t_{1}\right) d t_{1}\right) \\
\leq\left(C K^{2}\right)^{2} E\left(\int_{0}^{t} \int_{0}^{t_{1}}\left\langle B\left(\psi^{n-2} w_{1}-\psi^{n-2} w_{2}\right), \psi^{n-2} w_{1}-\psi^{n-2} w_{2}\right\rangle\left(t_{2}\right) d t_{2} d t_{1}\right) \\
\cdots \leq\left(C K^{2}\right)^{n} E\left(\int_{0}^{t} \int_{0}^{t_{1}} \cdots \int_{0}^{t_{n-1}}\left\langle B\left(w_{1}-w_{2}\right), w_{1}-w_{2}\right\rangle\left(t_{n}\right) d t_{n} \cdots d t_{2} d t_{1}\right) \\
=\left(C K^{2}\right)^{n} \int_{0}^{t} \int_{0}^{t_{1}} \cdots \int_{0}^{t_{n-1}} E\left(\left\langle B\left(w_{1}-w_{2}\right), w_{1}-w_{2}\right\rangle\left(t_{n}\right)\right) d t_{n} \cdots d t_{2} d t_{1} \\
\leq\left(C K^{2}\right)^{n} \sup _{t} E\left(\left\langle B\left(w_{1}-w_{2}\right), w_{1}-w_{2}\right\rangle(t)\right) \frac{T^{n}}{n!}<\frac{1}{2}\left\|w_{1}-w_{2}\right\|_{L^{\infty}\left([0, T], L^{2}(\Omega, W)\right)}^{2}
\end{gathered}
$$

provided $n$ is sufficiently large. It follows that

$$
\left\|\psi^{n} w_{1}-\psi^{n} w_{2}\right\|_{L^{\infty}\left([0, T], L^{2}(\Omega, W)\right)}^{2} \leq \frac{1}{2}\left\|w_{1}-w_{2}\right\|_{L^{\infty}\left([0, T], L^{2}(\Omega, W)\right)}^{2}
$$

for all $n$ sufficiently large. Hence, if one begins with

$$
w \in L^{\infty}\left([0, T], L^{2}(\Omega, W)\right) \cap L^{2}([0, T] \times \Omega, W),
$$

the sequence of iterates $\left\{\psi^{n} w\right\}_{n=1}^{\infty}$ converges to a fixed point $u$ in $L^{\infty}\left([0, T], L^{2}(\Omega, W)\right)$. This $u$ is automatically in $L^{2}([0, T] \times \Omega, W)$ and is progressively measurable since each of the iterates is progressively measurable. This proves the following theorem.
Theorem 76.5.2 Suppose $f \in \mathscr{V}^{\prime}$ is progressively measurable and that $(t, \omega) \rightarrow \sigma(t, u, \omega)$ is progressively measurable whenever u is. Suppose that $B: W \rightarrow W^{\prime}$ is a Riesz map.

$$
\lambda B+A(\omega): \mathscr{V}_{\omega} \rightarrow \mathscr{V}_{\omega}^{\prime}, \lambda B+A: \mathscr{V} \rightarrow \mathscr{V}^{\prime}
$$

are monotone hemicontinuous and bounded where

$$
A(\omega) u(t) \equiv A(t, u(t), \omega)
$$

and $(t, u, \omega) \rightarrow A(t, u, \omega)$ is progressively measurable. Also suppose for $p \geq 2$, the coercivity, and the boundedness conditions

$$
\begin{equation*}
\lambda\langle B u, u\rangle+\langle A(t, u, \omega), u\rangle_{V} \geq \delta\|u\|_{V}^{p}-c(t, \omega) \tag{76.5.60}
\end{equation*}
$$

for all $\lambda$ large enough.

$$
\begin{equation*}
\|A(t, u, \omega)\|_{V^{\prime}} \leq k\|u\|_{V}^{p-1}+c^{1 / p^{\prime}}(t, \omega) \tag{76.5.61}
\end{equation*}
$$

where $c \in L^{1}([0, T] \times \Omega)$. Also suppose that

$$
\begin{gathered}
\|\sigma(t, u, \omega)\|_{W} \leq C+C\|u\|_{W} \\
\|\sigma(t, u, \omega)-\sigma(t, \hat{u}, \omega)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)} \leq K\|u-\hat{u}\|_{W}
\end{gathered}
$$

Then if $u_{0} \in L^{2}(\Omega, W)$ with $u_{0} \mathscr{F}_{0}$ measurable, there exists a unique solution $u(\cdot, \omega) \in \mathscr{V}_{\omega}$ with $u \in \mathscr{V}\left(L^{p}([0, T] \times \Omega, V)\right.$ and progressively measurable) such that for $\omega$ off a set of measure zero,

$$
B u(t, \omega)-B u_{0}(\omega)+\int_{0}^{t} A(s, u(s, \omega), \omega) d s=\int_{0}^{t} f d s+B \int_{0}^{t} \sigma(u) d W
$$

### 76.6 Examples

Here we give some examples. The first is a standard example, the porous media equation, which is discussed well in [116]. For stochastic versions of this example, see [108]. The generalization to stochastic equations does not require the theory developed here. We will show, however, that it can be considered in terms of the theory of this paper without much difficulty using an approach proposed in [23]. These examples involve operators which are not monotone, in the usual way but they can be transformed into equations which do fit the above theory.

Example 76.6.1 The stochastic porous media equation is

$$
u_{t}-\Delta\left(u|u|^{p-2}\right)=f, u(0)=u_{0}, u=0 \text { on } \partial U
$$

where here $U$ is a bounded open set in $\mathbb{R}^{n}, n \leq 3$ having Lipschitz boundary. One can consider a stochastic version of this as a solution to the following integral equation

$$
\begin{equation*}
u(t)-u_{0}+\int_{0}^{t}(-\Delta)\left(u|u|^{p-2}\right) d s=\int_{0}^{t} \Phi d W+\int_{0}^{t} f d s \tag{76.6.62}
\end{equation*}
$$

where here

$$
\Phi \in L^{2}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right) \cap L^{2}\left(\Omega, L^{\infty}\left([0, T], \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)\right)
$$

$H=L^{2}(U)$ and the equation holds in the manner described above in $H^{-1}(U)$. Assume $p \geq 2$ and $f \in L^{2}((0, T) \times \Omega, H)$.

One can consider this as an implicit integral equation of the form

$$
\begin{equation*}
(-\Delta)^{-1} u(t)-(-\Delta)^{-1} u_{0}+\int_{0}^{t} u|u|^{p-2} d s=(-\Delta)^{-1} \int_{0}^{t} \Phi d W+(-\Delta)^{-1} \int_{0}^{t} f d s \tag{76.6.63}
\end{equation*}
$$

where $-\Delta$ is the Riesz map of $H_{0}^{1}(U)$ to $H^{-1}(U)$. Then we can also consider $(-\Delta)^{-1}$ as a map from $L^{2}(U)$ to $L^{2}(U)$ as follows.

$$
(-\Delta)^{-1} f=u \text { where }-\Delta u=f, u=0 \text { on } \partial U
$$

Thus we let $W=L^{2}(U)=H$ and $V=L^{p}(U)$. Let $B \equiv(-\Delta)^{-1}$ on $L^{2}(U)$ as just described. Let $A(u)=u|u|^{p-2}$. It is obvious that the necessary coercivity condition holds. In addition, there is a strong monotonicity condition which holds. Therefore, if $u_{0} \in L^{2}\left(\Omega, L^{2}(U)\right)$ and $\mathscr{F}_{0}$ measurable, Theorem 76.4.7 applies and we can conclude that there exists a unique solution to the integral equation 76.6 .63 in the sense described in this theorem. Here $u \in$ $L^{p}\left([0, T] \times \Omega, L^{p}(U)\right)$ and is progressively measurable, the integral equation holding for all $t$ for $\omega$ off a set of measure zero. Since $A$ satisfies for some $\delta>0$ an inequality of the form

$$
\langle A u-A v, u-v\rangle \geq \delta\|u-v\|_{L^{p}(U)}^{p}
$$

it follows easily from the above methods that the solution is also unique. In fact, this follows right away from Theorem 76.4.7 because $\left\langle-\Delta^{-1} u, u\right\rangle=\|u\|_{H^{-1}}^{2}$.

Also note that from the integral equation,

$$
(-\Delta)^{-1}\left(u(t)-u_{0}-\int_{0}^{t} \Phi d W\right)+\int_{0}^{t} u|u|^{p-2} d s=(-\Delta)^{-1} \int_{0}^{t} f d s
$$

and so, since $(-\Delta)$ is the Riesz map on $H_{0}^{1}(U)$, the integral equation above shows that off a set of measure zero,
$\int_{0}^{t} u|u|^{p-2} d s=(-\Delta)^{-1}\left(\int_{0}^{t} f d s-\left(u(t)-u_{0}-\int_{0}^{t} \Phi d W\right)\right) \in L^{2}\left(0, T, H_{0}^{1}(U) \cap H^{2}(U)\right)$
by elliptic regularity results. Without that stochastic integral, one could assert that $|u|^{\frac{p-2}{2}} u \in$ $L^{2}\left(0, T, H_{0}^{1}(U)\right)$. This is shown in [23]. However, it appears that no such condition can be obtained here because of the nowhere differentiability of the stochastic integral, even if more is assumed on $u_{0}$ and $\Phi$.

Also in this reference is a treatment of the Stefan problem. The Stefan problem involves a partial differential equation

$$
u_{t}-\sum_{i} \frac{\partial}{\partial x_{i}}\left(k(u) \frac{\partial u}{\partial x_{i}}\right)=f, \text { on } U \times[0, T] \equiv Q
$$

for $(x, t) \notin S$ where $u$ is the temperature and $k(u)$ has a jump at $\sigma$ and $S$ is given by $u(x, t)=$ $\sigma$. It is assumed that $0<k_{1} \leq k(r) \leq k_{2}<\infty$ for all $r \in \mathbb{R}$. For example, its graph could be of the form


On $S$ there is a jump condition which is assumed to hold. Namely

$$
b n_{t}-\left(k(u+) u_{, i}(+)-k(u-) u_{, i}(-)\right) n_{i}=0
$$

where the sum is taken over repeated indices and $b>0 . u(+)$ is the "limit" as $\left(x^{\prime}, t^{\prime}\right) \rightarrow$ $(x, t) \in S$ where $\left(x^{\prime}, t^{\prime}\right) \in S_{+}, u(-)$ defined similarly. Also $\mathbf{n}$ will denote the unit normal which goes from $S_{+} \equiv\{(x, t): u(x, t)>\sigma\}$ toward $S_{-} \equiv\{(x, t): u(x, t)<\sigma\}$.

$$
\mathbf{n}=\left(n_{t}, n_{x_{1}}, \cdots, n_{x_{n}}\right)
$$

In addition, there is an initial condition and boundary conditions

$$
u(x, 0)=u_{0}(x) \notin S, u(x, t)=0 \text { on } \partial U
$$

The idea is to obtain a variational formulation of this thing. To do this, let $K(r) \equiv \int_{0}^{r} k(s) d s$. Thus in the case of the above picture, the graph of $K(r)$ would look like


Now let $\beta(t)$ be a function which satisfies

$$
\beta^{\prime}(t)=\frac{1}{k\left(K^{-1}(t)\right)} \text { for } t \neq \tau \equiv K(\sigma)
$$

and it has a jump equal to $b$ at $\tau$.


Let $v=K(u)$. Then for $u \neq \sigma$, equivalently $v \neq \tau$,

$$
v_{t}=K^{\prime}(u) u_{t}=k\left(K^{-1}(v)\right) u_{t}
$$

and so

$$
u_{t}=\frac{1}{k\left(K^{-1}(v)\right)} v_{t}=\frac{d}{d t}(\beta(v))
$$

Also,

$$
v_{, i}=K^{\prime}(u) u_{, i}=k(u) u_{, i}
$$

and so

$$
u_{, i}=\frac{1}{k(u)} v_{, i}
$$

Hence

$$
\left(k(u) u_{, i}\right)_{, i}=\left(k(u) \frac{1}{k(u)} v_{, i}\right)_{, i}=\Delta v
$$

Thus, off the set $S$,

$$
\beta(v)_{t}-\Delta v=f
$$

Now let $\phi \in L^{2}\left(0, T, H_{0}^{1}(U)\right)$ with $\phi(x, T)=0$. Then assume $S$ is sufficiently smooth that things like divergence theorem apply. Also note that $u=\sigma$ is the same as $v=\tau$.

$$
\begin{aligned}
& \int_{Q}\left(\beta(v)_{t}-\Delta v\right) \phi=\int_{S_{+}}\left(\beta(v)_{t}-\Delta v\right) \phi+\int_{S_{-}}\left(\beta(v)_{t}-\Delta v\right) \phi \\
&= \int_{S_{+}}(\beta(v) \phi)_{t}-\beta(v) \phi_{t}-\left(v_{, i} \phi\right)_{, i}+v_{, i} \phi_{, i} \\
&+\int_{S_{-}}(\beta(v) \phi)_{t}-\beta(v) \phi_{t}-\left(v_{, i} \phi\right)_{, i}+v_{, i} \phi_{, i}
\end{aligned}
$$

Now using the divergence theorem, and continuing these formal manipulations, the above reduces to

$$
\begin{aligned}
& \int_{S} \beta(v(+)) \phi n_{t}-\left(v_{, i}(+) \phi\right) n_{i}+\int_{S_{+}}-\beta(v) \phi_{t}+v_{, i} \phi_{, i}-\int_{U \cap S_{+}} \beta(v(x, 0)) \phi(x, 0) \\
+ & \int_{S}-\beta(v(-)) \phi n_{t}+\left(v_{, i}(-) \phi\right) n_{i}+\int_{S_{-}}-\beta(v) \phi_{t}+v_{, i} \phi_{, i}-\int_{U \cap S_{-}} \beta(v(x, 0)) \phi(x, 0)
\end{aligned}
$$

Combining the two integrals over $S$ yields

$$
\int_{S}\left(b n_{t}-\left(v_{, i}(+)-v_{, i}(-)\right) n_{i}\right) \phi=\int_{S}\left(b n_{t}-\left(k(u) u_{, i}(+)-k(u) u_{, i}(-)\right) n_{i}\right) \phi=0
$$

by assumption. Therefore, including $f$, we obtain

$$
\int_{Q}-\beta(v) \phi_{t}+v_{, i} \phi_{, i}-\int_{U} \beta(v(x, 0)) \phi(x, 0)=\int_{Q} f \phi
$$

which implies, using the initial condition

$$
\int_{U} \beta\left(v_{0}\right) \phi(x, 0)+\int_{Q}\left((\beta(v))^{\prime} \phi+v_{, i} \phi_{, i}\right)-\int_{U} \beta(v(x, 0)) \phi(x, 0)=\int_{Q} f \phi
$$

Regard $\beta$ as a maximal monotone graph and let $\alpha(t) \equiv \beta^{-1}(t)$. Thus $\alpha$ is single valued. It just has a horizontal place corresponding to the jump in $\beta$. Then let $w=\beta(v)$ so that $v=\alpha(w)$.


Then in terms of $w$, the above equals

$$
\int_{U} w_{0} \phi(x, 0)+\int_{Q}\left(w^{\prime} \phi+\alpha(w)_{, i} \phi_{, i}\right)-\int_{U} w(x, 0) \phi(x, 0)=\int_{Q} f \phi
$$

and so this simplifies to

$$
w^{\prime}-\Delta(\alpha(w))=f, w(0)=w_{0}
$$

where $\alpha$ maps onto $\mathbb{R}$ and is monotone and satisfies

$$
\begin{aligned}
\left(\alpha\left(r_{1}\right)-\alpha\left(r_{2}\right)\right)\left(r_{1}-r_{2}\right) & \geq 0,|\alpha(r)| \leq m|r| \\
\left|\alpha\left(r_{1}\right)-\alpha\left(r_{2}\right)\right| & \leq m\left|r_{1}-r_{2}\right|, \alpha(r) r \geq \delta|r|^{2}
\end{aligned}
$$

for some $\delta, m>0$. Then $K^{-1}(\alpha(w))=u$ where $u$ is the original dependent variable. Obviously, the original function $k$ could have had more than one jump and you would handle it the same way by defining $\beta$ to be like $K^{-1}$ except for having appropriate jumps at the values of $K(u)$ corresponding to the jumps in $k$. This explains the following example.

Example 76.6.2 It can be shown that the Stefan problem can be reduced to the consideration of an equation of the form

$$
w_{t}-\Delta(\alpha(w))=f, w(0)=w_{0}
$$

where $\alpha: L^{2}\left(0, T, L^{2}(U)\right) \rightarrow L^{2}\left(0, T, L^{2}(U)\right)$ is monotone hemicontinuous and coercive, $\alpha$ being a single valued function. Here $f$ is the same which occurred in the original partial differential equation

$$
u_{t}-\sum_{i} \frac{\partial}{\partial x_{i}}\left(k(u) \frac{\partial u}{\partial x_{i}}\right)=f
$$

Thus a stochastic Stefan problem could be considered in the form

$$
\left(-\Delta^{-1}\right) w(t)-(-\Delta)^{-1} w_{0}+\int_{0}^{t} \alpha(w) d s=\left(-\Delta^{-1}\right) \int_{0}^{t} f d s+\left(-\Delta^{-1}\right) \int_{0}^{t} \Phi d W
$$

This example can be included in the above general theory because

$$
\left(\left(-\Delta^{-1}\right) u-\left(-\Delta^{-1}\right) v, u-v\right) \geq\|u-v\|_{V^{\prime}}^{2}, V^{\prime} \equiv H^{-1}, V=H_{0}^{1}(U)
$$

This is seen as follows. $V, L^{2}(U), V^{\prime}$ is a Gelfand triple. Then $-\Delta$ is the Riesz map $R$ from $H_{0}^{1}(U)$ to $H^{-1}(U)$. then you have

$$
\left(y, R^{-1} y\right)_{L^{2}(U)}=\left\langle R R^{-1} y, R^{-1} y\right\rangle_{V^{\prime}, V}=\left\|R^{-1} y\right\|_{V}^{2}=\|y\|_{V^{\prime}}^{2}
$$

Next we give a simple example which is a singular and degenerate equation. This is a model problem which illustrates how the theory can be used. This problem is mixed parabolic and stochastic and nonlinear elliptic. It is a singular equation because the coefficient $b$ can be unbounded. The existence of a solution is easy to obtain from the above theory but it does not follow readily from other methods. If $p=2$ it is an abstract version of stochastic heat equation which could model a material in which the density becomes vanishingly small in some regions and very large in other regions.

Example 76.6.3 Suppose $U$ is a bounded open set in $\mathbb{R}^{3}$ and $b(\mathbf{x}) \geq 0, b \in L^{p}(U), p \geq 4$ for simplicity. Consider the degenerate stochastic initial boundary value problem

$$
\begin{aligned}
b(\cdot) u(t, \cdot)-b(\cdot) u_{0}(\cdot)-\int_{0}^{t} \nabla \cdot\left(|\nabla u|^{p-2} \nabla u\right) & =b \int_{0}^{t} \Phi d W \\
u & =0 \text { on } \partial U
\end{aligned}
$$

where $\Phi \in L^{2}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right)$ for $W=H_{0}^{1}(U)$.
To consider this equation and initial condition, it suffices to let $W=H_{0}^{1}(U), V=$ $W_{0}^{1, p}(U)$,

$$
\begin{aligned}
A & : \quad V \rightarrow V^{\prime},\langle A u, v\rangle=\int_{U}|\nabla u|^{p-2} \nabla u \cdot \nabla v d x \\
B & : \quad W \rightarrow W^{\prime},\langle B u, v\rangle=\int_{U} b(x) u(x) v(x) d x
\end{aligned}
$$

Then by the Sobolev embedding theorem, $B$ is obviously self adjoint, bounded and nonnegative. This follows from a short computation:

$$
\begin{aligned}
& \left|\int_{U} b(x) u(x) v(x) d x\right| \leq\|v\|_{L^{4}(U)}\left(\int_{U}|b(x)|^{4 / 3}|u(x)|^{4 / 3} d x\right)^{3 / 4} \\
& \leq\|v\|_{H_{0}^{1}(U)}\left(\left(\int|b(x)|^{4} d x\right)^{1 / 3}\left(\int\left(|u(x)|^{4 / 3}\right)^{3 / 2}\right)^{2 / 3}\right)^{3 / 4} \\
& =\|v\|_{H_{0}^{1}(U)}\|b\|_{L^{4}(U)}\|u\|_{L^{2}(U)} \leq\|b\|_{L^{4}}\|u\|_{H_{0}^{1}}\|v\|_{H_{0}^{1}}
\end{aligned}
$$

Also for some $\delta>0$

$$
\langle A u-A v, u-v\rangle \geq \delta\|u-v\|_{V}^{p}
$$

The nonlinear operator is obviously monotone and hemicontinuous. As for $u_{0}$, it is only necessary to assume $u_{0} \in L^{2}(\Omega, W)$ and $\mathscr{F}_{0}$ measurable. Then Theorem 76.4.7 gives the existence of a solution in the sense that for a.e. $\omega$ the integral equation holds for all $t$. Note that $b$ can be unbounded and may also vanish. Thus the equation can degenerate to the case of a non stochastic nonlinear elliptic equation.

The existence theorems can easily be extended to include the situation where $\Phi$ is replaced with a function of the unknown function $u$. This is done by splitting the time interval into small sub intervals of length $h$ and retarding the function in the stochastic integral, like a standard proof of the Peano existence theorem. Then the Ito formula is applied to obtain estimates and a limit is taken.

Other examples of the usefulness of this theory will result when one considers stochastic versions of systems of partial differential equations in which there is a nonlinear coupling between a parabolic equation and a nonlinear elliptic equation. These kinds of problems occur, for example as quasistatic damage problems in which the damage parameter satisfies a parabolic equation and the balance of momentum is a nonlinear elliptic equation and the two equations are coupled in a nonlinear way.

### 76.7 Other Examples, Inclusions

The above general result can also be used as a starting point for evolution inclusions or other situations where one does not have hemicontinuous operators. Assume here that

$$
\Phi \in L^{\infty}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)
$$

We will use the following simple observation. Let $\alpha>2$. Let $\|\Phi\|_{\infty}$ denote the norm in $L^{\infty}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$. By the Burkholder Davis Gundy inequality,

$$
\begin{gathered}
\int_{\Omega}\left(\left|\int_{s}^{t} \Phi d W\right|\right)^{\alpha} d P \leq \\
\int_{\Omega}\left(\sup _{r \in[s, t]}\left|\int_{s}^{r} \Phi d W\right|\right)^{\alpha} d P \leq C \int_{\Omega}\left(\int_{s}^{t}\|\Phi\|^{2} d \tau\right)^{\alpha / 2} d P \\
\leq C\|\Phi\|_{\infty}^{\alpha} \int_{\Omega}\left(\int_{s}^{t} d \tau\right)^{\alpha / 2} d P=C\|\Phi\|_{\infty}^{\alpha}|t-s|^{\alpha / 2}
\end{gathered}
$$

By the Kolmogorov Čentsov theorem, this shows that $t \rightarrow \int_{0}^{t} \Phi d W$ is Holder continuous with exponent

$$
\gamma<\frac{(\alpha / 2)-1}{\alpha}=\frac{1}{2}-\frac{1}{\alpha}
$$

Since $\alpha>2$ is arbitrary, this shows that for any $\gamma<1 / 2$, the stochastic integral is Holder continuous with exponent $\gamma$. This is exactly the same kind of continuity possessed by the Wiener process. We state this as the following lemma.

Lemma 76.7.1 Let $\Phi \in L^{\infty}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$ then for any $\gamma<1 / 2$, the stochastic integral $\int_{0}^{t} \Phi d W$ is Holder continuous with exponent $\gamma$.

To begin with, we consider a stochastic inclusion. Suppose, in the context of Theorem 76.4.7, that $V$ is a closed subspace of $W^{\sigma, p}(U), \sigma>1$ which contains $C_{c}^{\infty}(U)$ where $U$ is an open bounded set in $\mathbb{R}^{n}$, different than the Hilbert space $U$. (In case the matrix $A$ which follows equals 0 , it suffices to take $\sigma \geq 1$.) Let

$$
\sum_{i, j} a_{i, j}(\mathbf{x}) \xi_{i} \xi_{j} \geq 0, a_{i j}=a_{j i}
$$

where the $a_{i, j} \in C^{1}(\bar{U})$. Denoting by $A$ the matrix whose $i j^{t h}$ entry is $a_{i j}$, let

$$
W \equiv\left\{u \in L^{2}(U):\left(u, \quad A^{1 / 2} \nabla u\right) \in L^{2}(U)^{n+1}\right\}
$$

with a norm given by

$$
\|u\|_{W} \equiv\left(\int_{U}\left(u v+\sum_{i, j} a_{i j}(\mathbf{x}) \partial_{i} u \partial_{j} v\right) d x\right)^{1 / 2}
$$

$B: W \rightarrow W^{\prime}$ be given by

$$
\langle B u, v\rangle \equiv \int_{U}\left(u v+\sum_{i, j} a_{i j}(\mathbf{x}) \partial_{i} u \partial_{j} v\right) d x
$$

so that $B$ is the Riesz map for this space. The case where the $a_{i j}$ could vanish is allowed. Thus $B$ is a positive self adjoint operator and is therefore, included in the above discussion. In this example, it will be significant that $B$ is one to one and does not vanish.

This operator maps onto $L^{2}(U)$ because of basic considerations concerning maximal monotone operators. This is because

$$
\langle D u, v\rangle \equiv \int_{U} \sum_{i, j} a_{i j}(\mathbf{x}) \partial_{i} u \partial_{j} v d x
$$

can be obtained as a subgradient of a convex lower semicontinuous and proper functional defined on $L^{2}(U)$. Therefore, the operator is maximal monotone on $L^{2}(U)$ which means that $I+D$ is onto. The domain of $D$ consists of all $u \in L^{2}(U)$ such that

$$
D u=-\sum_{i, j} \partial_{j}\left(a_{i j}(\mathbf{x}) \partial_{i} u\right) \in L^{2}(U)
$$

along with suitable boundary conditions determined by the choice of $V$. It follows that if $u+D u=B u=f \in H=L^{2}(U)$, then

$$
u-\sum_{i, j} \partial_{j}\left(a_{i j} \partial_{i} u\right)=f
$$

Therefore,

$$
\begin{gathered}
\|u\|_{L^{2}(U)}^{2}+\int_{U} \sum_{i, j} a_{i j}(\mathbf{x}) \partial_{i} u \partial_{j} u=\|u\|_{W}^{2}= \\
=(f, u) \leq\|f\|_{L^{2}(U)}\|u\|_{L^{2}(U)} \leq\|f\|_{L^{2}(U)}\|u\|_{W}
\end{gathered}
$$

which shows that the map $B^{-1}: H=L^{2}(U) \rightarrow W$ is continuous.
Next suppose that $\Phi \in L^{\infty}\left([0, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$. Then by continuity of the mapping $B^{-1}$, it follows that $\Psi \equiv B^{-1} \Phi$ satisfies $\Psi \in L^{\infty}\left([0, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right)$. Thus $\Phi=B \Psi$. In addition to this, to simplify the presentation, assume in addition that

$$
\begin{gathered}
\langle A(t, u, \omega)-A(t, v, \omega), u-v\rangle \geq \delta^{2}\|u-v\|_{V}^{p} \\
\langle A(t, u, \omega), u\rangle \geq \delta^{2}\|u\|_{V}^{p}
\end{gathered}
$$

Also assume the uniqueness condition of Lemma 76.3.16 is satisfied. Consider the following graph.


There is a monotone Lipschitz function $J_{n}$ which is approximating a function with the indicated jump. For a convex function $\phi$, we denote by $\partial \phi$ its subgradient. Thus for $y \in \partial \phi(x)$

$$
(y, u) \leq \phi(x+u)-\phi(x) .
$$

Denote the Lipschitz function as $J_{n}$ and the maximal monotone graph which it is approximating as $J$. Thus $J$ denotes the ordered pairs $(x, y)$ which are of the form $(0, y)$ for $|y| \leq 1$ along with ordered pairs $(x, 1), x>0$ and ordered pairs $(x,-1)$ for $x<0$. The graph of $J$ is illustrated in the above picture and is a maximal monotone graph. Thus $J=\partial \phi$ where $\phi(r)=|r|$. As illustrated in the graph, $J_{n}$ is piecewise linear.

Let $\phi_{n}(r) \equiv \int_{0}^{r} J_{n}(s) d s$. It follows easily that $\phi_{n}(r) \rightarrow \phi(r)$ uniformly on $\mathbb{R}$. Also let $h \geq 0$ be progressively measurable and uniformly bounded by $M$ and let $u_{0} \in L^{2}(\Omega, W), u_{0}$ $\mathscr{F}_{0}$ measurable. Then from the above theorems, there exists a unique solution to the integral equation

$$
B u_{n}(t)-B u_{0}+\int_{0}^{t} A\left(s, u_{n}, \omega\right) d s+\int_{0}^{t} h(s, \omega) J_{n}\left(u_{n}\right) d s=B \int_{0}^{t} \Psi d W
$$

the last term equaling $\int_{0}^{t} \Phi d W$. The integral equation holds off a set of measure zero and is progressively measurable.

Then from the Ito formula, one obtains, using the monotonicity of $J_{n}$ an estimate in which $C$ does not depend on $n$

$$
\frac{1}{2} E\left\langle B u_{n}(t), u_{n}(t)\right\rangle-\frac{1}{2} E\left\langle B u_{0}, u_{0}\right\rangle+E \int_{0}^{t}\left\langle A u_{n}, u_{n}\right\rangle_{V} d s \leq C
$$

In particular, this holds for $n=1$. Therefore, adjusting the constant, it follows that

$$
\int_{\Omega}\left\langle B u_{1}(t), u_{1}(t)\right\rangle+\int_{\Omega} \int_{0}^{T}\left\|u_{1}\right\|_{V}^{p} d t d s \leq C
$$

Consequently, there exists a set of measure zero $N$ such that for $\omega \notin N$,

$$
\begin{equation*}
\left\langle B u_{1}(t), u_{1}(t)\right\rangle+\int_{0}^{T}\left\|u_{1}\right\|_{V}^{p} d t \leq C(\omega) \tag{76.7.64}
\end{equation*}
$$

From the integral equation, it follows that, enlarging $N$ by including countably many sets of measure zero, for $\omega \notin N$

$$
\begin{gathered}
B u_{n}(t)-B u_{1}(t)+\int_{0}^{t} A\left(s, u_{n}, \omega\right)-A\left(s, u_{1}, \omega\right) d s \\
\quad+\int_{0}^{t} h(s, \omega) J_{n}\left(u_{n}\right)-h(s, \omega) J_{1}\left(u_{1}\right) d s=0
\end{gathered}
$$

Now it is certainly true that $\left|J_{n}\left(u_{n}\right)-J_{1}\left(u_{n}\right)\right| \leq 2$. Thus

$$
\begin{gathered}
\int_{0}^{t}\left\langle h(s, \omega) J_{n}\left(u_{n}\right)-h(s, \omega) J_{1}\left(u_{1}\right), u_{n}-u_{1}\right\rangle d s \\
=\int_{0}^{t}\left\langle h(s, \omega) J_{n}\left(u_{n}\right)-h(s, \omega) J_{1}\left(u_{n}\right), u_{n}-u_{1}\right\rangle d s \\
+\int_{0}^{t}\left\langle h(s, \omega)\left(J_{1}\left(u_{n}\right)-J_{1}\left(u_{1}\right)\right), u_{n}-u_{1}\right\rangle d s \\
\geq-2 M \int_{0}^{t}\left|u_{n}-u_{1}\right| d s
\end{gathered}
$$

Therefore, from the Ito formula and for $\omega \notin N$,

$$
\begin{aligned}
& \frac{1}{2}\left\langle B u_{n}(t)-B u_{1}(t), u_{n}(t)-u_{1}(t)\right\rangle+\delta^{2} \int_{0}^{t}\left\|u_{n}-u_{1}\right\|_{V}^{p} d s \\
\leq & \int_{0}^{t} 2 M\left|u_{n}-u_{1}\right| d s \leq\left(2+\frac{1}{2} \int_{0}^{t}\left|u_{n}-u_{1}\right|^{2} d s\right) M \\
\leq & \left(2+\frac{1}{2} \int_{0}^{t}\left\langle B u_{n}-B u_{1}, u_{n}-u_{1}\right\rangle d s\right) M
\end{aligned}
$$

where $M$ is an upper bound to $h$. Then by Gronwall's inequality

$$
\frac{1}{2}\left\langle B u_{n}(t)-B u_{1}(t), u_{n}(t)-u_{1}(t)\right\rangle \leq 2 M e^{M T}
$$

Hence

$$
\frac{1}{2}\left\langle B u_{n}(t)-B u_{1}(t), u_{n}(t)-u_{1}(t)\right\rangle+\delta^{2} \int_{0}^{t}\left\|u_{n}-u_{1}\right\|_{V}^{p} d s \leq 2 M+T M 2 e^{T M}
$$

It follows from 76.7.64 that for all $\omega \notin N$ and adjusting the constant,

$$
\begin{equation*}
\left\langle B u_{n}(t), u_{n}(t)\right\rangle+\int_{0}^{T}\left\|u_{n}\right\|_{V}^{p} d s \leq C(\omega) \tag{76.7.65}
\end{equation*}
$$

for all $n$, where $C(\omega)$ depends only on $\omega$.
For $\omega \notin N$, the above estimate implies there exists a further subsequence, still called $n$ such that

$$
\begin{gathered}
B u_{n} \rightarrow B u \text { weak } * \text { in } L^{\infty}\left(0, T, W^{\prime}\right) \\
u_{n} \rightarrow u \text { weak } * \operatorname{in} L^{\infty}(0, T, H) \\
u_{n} \rightarrow u \text { weakly in } \mathscr{V}_{\omega}
\end{gathered}
$$

From the integral equation solved and the assumption that $A$ is bounded, it can also be assumed that

$$
\begin{gather*}
\left(B\left(u_{n}-\int_{0}^{(\cdot)} \Psi d W\right)\right)^{\prime} \rightarrow\left(B\left(u-\int_{0}^{(\cdot)} \Psi d W\right)\right)^{\prime} \text { weakly in } \mathscr{V}_{\omega}^{\prime}  \tag{76.7.66}\\
B u(0)=B u_{0} \\
A u_{n} \rightarrow \xi \text { weakly in } \mathscr{V}_{\omega}^{\prime}
\end{gather*}
$$

It is known that $u_{n}$ is bounded in $\mathscr{V}_{\omega}$. Also it is known that $\left(B\left(u_{n}-\int_{0}^{(\cdot)} \Psi d W\right)\right)^{\prime}$ is bounded in $\mathscr{V}_{\omega}^{\prime}$. Therefore, $B\left(u_{n}-\int_{0}^{(\cdot)} \Psi d W\right)$ satisfies a Holder condition into $V^{\prime}$. Since $\Psi$ is in $L^{\infty}, \int_{0}^{(\cdot)} \Psi d W$ satisfies a Holder condition, and so $B u_{n}$ satisfies a Holder condition into $V^{\prime}$ while $B u_{n}$ is bounded in $\mathscr{W}_{\omega}^{\prime}$. By compactness of the embedding of $V$ into $W$, it follows that $W^{\prime}$ embeds compactly into $V^{\prime}$. This is sufficient to conclude that $\left\{B u_{n}\right\}$ is precompact in $\mathscr{W}_{\omega}^{\prime}$. The proof is similar to one given by Lions. [91] page 57. See Theorem 69.5.6. Since $B$ is the Riesz map, this implies that $\left\{u_{n}\right\}$ is precompact in $\mathscr{W}_{\omega}$ and hence in $\mathscr{H}_{\omega}$.

Therefore, one can take a further subsequence and conclude that

$$
u_{n} \rightarrow u \text { strongly in } \mathscr{H}_{\omega} \equiv L^{2}\left([0, T], L^{2}(U)\right)
$$

Therefore, a further subsequence, still denoted by $n$ satisfies

$$
u_{n}(t) \rightarrow u(t) \text { in } L^{2}(U) \text { for a.e. } t
$$

We can also assume that

$$
J_{n}\left(u_{n}\right) \rightarrow \zeta \text { weak } * \operatorname{in} L^{\infty}\left(0, T, L^{\infty}(U)\right)
$$

From the integral equation solved,

$$
\begin{align*}
& \left\langle\left(B\left(u_{n}-\int_{0}^{(\cdot)} \Psi d W\right)\right)^{\prime}, u_{n}-u\right\rangle_{V_{\omega}} \\
& +\left\langle A\left(t, u_{n}\right)+h(t, \omega) J_{n}\left(u_{n}\right), u_{n}-u\right\rangle=0 \tag{76.7.67}
\end{align*}
$$

We claim that

$$
\int_{0}^{t}\left\langle\left(B\left(u_{n}-\int_{0}^{(\cdot)} \Psi d W\right)\right)^{\prime}-\left(B\left(u-\int_{0}^{(\cdot)} \Psi d W\right)\right)^{\prime}, u_{n}-u\right\rangle d s \geq 0
$$

The difficulty is that $\int_{0}^{(\cdot)} \Psi d W$ is only in $W$. To see that the conclusion is so, note that it is clear from a computation that

$$
\int_{0}^{t}\left\langle\begin{array}{c}
\frac{1-\tau(h)}{h}\left(B u_{n}-B \int_{0}^{(\cdot)} \Psi d W\right)  \tag{76.7.68}\\
-\frac{1-\tau(h)}{h}\left(B u-B \int_{0}^{(\cdot)} \Psi d W\right), u_{n}-u
\end{array}\right\rangle d s \geq 0
$$

Claim: The above is indeed nonnegative.
Proof: Denote by $q(t)$ the stochastic integral, $u_{n}$ as $u$ and $u$ as $v$ to save notation. Then the left side of the above equals

$$
\begin{aligned}
& \frac{1}{h} \int_{0}^{t}\langle B(u-q)-B(v-q), u-v\rangle d s \\
& -\frac{1}{h} \int_{h}^{t}\langle B(u(s-h)-q(s-h))-B(v(s-h)-q(s-h)), u(s)-v(s)\rangle d s \\
& \geq \frac{1}{h} \int_{0}^{t}\langle B(u-q)-B(v-q), u-v\rangle d s \\
& -\frac{1}{2 h} \int_{h}^{t}\left\langle\begin{array}{c}
B(u(s-h)-q(s-h))-B(v(s-h)-q(s-h)), \\
(u(s-h)-v(s-h))
\end{array}\right\rangle d s \\
& -\frac{1}{2 h} \int_{h}^{t}\langle B(u-q)-B(v-q), u-v\rangle d s \\
& \geq \frac{1}{h} \int_{0}^{t}\langle B(u-q)-B(v-q), u-v\rangle d s \\
& -\frac{1}{2 h} \int_{0}^{t-h}\langle B(u(s)-q(s))-B(v(s)-q(s)),(u(s)-v(s))\rangle d s \\
& -\frac{1}{2 h} \int_{h}^{t}\langle B(u-q)-B(v-q), u-v\rangle d s \\
& =\frac{1}{h} \int_{t-h}^{t}\langle B(u-q)-B(v-q), u-v\rangle d s \\
& +\frac{1}{h} \int_{0}^{t-h}\langle B(u-q)-B(v-q), u-v\rangle d s \\
& -\frac{1}{2 h} \int_{0}^{t-h}\langle B(u-q)-B(v-q),(u-v)\rangle d s \\
& -\frac{1}{2 h} \int_{h}^{t}\langle B(u-q)-B(v-q), u-v\rangle d s
\end{aligned}
$$

$$
\begin{aligned}
= & \frac{1}{h} \int_{t-h}^{t}\langle B(u-q)-B(v-q), u-v\rangle d s \\
& +\frac{1}{2 h} \int_{0}^{t-h}\langle B(u-q)-B(v-q),(u-v)\rangle d s \\
& -\frac{1}{2 h} \int_{h}^{t-h}\langle B(u-q)-B(v-q), u-v\rangle d s \\
& -\frac{1}{2 h} \int_{t-h}^{t}\langle B(u-q)-B(v-q), u-v\rangle d s \\
= & \frac{1}{2 h} \int_{t-h}^{t}\langle B(u-q)-B(v-q), u-v\rangle d s \\
& +\frac{1}{2 h} \int_{0}^{h}\langle B(u-q)-B(v-q),(u-v)\rangle d s
\end{aligned}
$$

which is nonnegative as can be seen by replacing $u-v$ with $(u-q)-(v-q)$ and using monotonicity of $B$.

Now pass to a limit in 76.7.68 as $h \rightarrow 0$ to get the desired inequality. Therefore, from 76.7.67,

$$
\lim \sup _{n \rightarrow \infty} \int_{0}^{T}\left\langle A\left(t, u_{n}\right)+h(t, \omega) J_{n}\left(u_{n}\right), u_{n}-u\right\rangle d t \leq 0
$$

From the above strong convergence, the left side equals

$$
\lim \sup _{n \rightarrow \infty} \int_{0}^{T}\left\langle A\left(t, u_{n}\right), u_{n}-u\right\rangle d t \leq 0
$$

It follows that for all $v \in \mathscr{V}_{\omega}$,

$$
\begin{aligned}
& \int_{0}^{T}\langle A(t, u), u-v\rangle d t \\
\leq & \lim _{n \rightarrow \infty} \inf _{n} \int_{0}^{T}\left\langle A\left(t, u_{n}\right), u_{n}-v\right\rangle d t \\
= & \lim \sup _{n \rightarrow \infty}\left[\int_{0}^{T}\left\langle A\left(t, u_{n}\right), u_{n}-u\right\rangle d t+\int_{0}^{T}\left\langle A\left(t, u_{n}\right), u-v\right\rangle d t\right] \\
\leq & \int_{0}^{T}\langle\xi, u-v\rangle d t
\end{aligned}
$$

Since $v$ is arbitrary, $A(\cdot, u)=\xi \in \mathscr{V}_{\omega}^{\prime}$. Passing to the limit in the integral equation yields

$$
B u(t)-B u_{0}+\int_{0}^{t} A(s, u) d s+\int_{0}^{t} h(s, \omega) \zeta(s, \omega) d s=\int_{0}^{t} \Phi d W
$$

What is $h(s, \omega) \zeta(s, \omega)$ ?

$$
\int_{0}^{T}\left\langle h(s, \omega) J_{n}\left(u_{n}(s)\right), v(s)-u_{n}(s)\right\rangle d s \leq \int_{0}^{T} h(s, \omega)\left(\phi_{n}(v)-\phi_{n}\left(u_{n}\right)\right) d s
$$

Passing to the limit and using the strong convergence described above along with the uniform convergence of $\phi_{n}$ to $\phi$,

$$
\int_{0}^{T}(h(s, \omega) \zeta(s), v(s)-u(s))_{H} d s \leq \int_{0}^{T} h(s, \omega)(\phi(v(s))-\phi(u(s))) d s
$$

Hence,

$$
\int_{0}^{T}(h(s, \omega) \phi(v(s))-h(s, \omega) \phi(u(s)))-(h(s, \omega) \zeta(s), v(s)-u(s))_{H} d s \geq 0
$$

for any choice of $v \in \mathscr{H}_{\omega}$. It follows that for a.e. $s, h(s, \omega) \zeta(s) \in \partial_{v}(h(s, \omega) \phi(u(s)))$.
This has shown that for each $\omega \notin N$, there exists a solution to the integral equation

$$
\begin{equation*}
B u(t)-B u_{0}+\int_{0}^{t} A(s, u) d s+\int_{0}^{t} h(s, \omega) \zeta(s, \omega) d s=\int_{0}^{t} \Phi d W \tag{76.7.69}
\end{equation*}
$$

where for a.e. $s, h(s, \omega) \zeta(s, \omega) \in \partial_{v}(h(s, \omega) \phi(u(s)))$. Suppose you have two such solutions $\left(u_{1}, \zeta_{1}\right)$ and $\left(u_{2}, \zeta_{2}\right)$. Then

$$
B u_{1}(t)-B u_{2}(t)+\int_{0}^{t} A\left(s, u_{1}\right)-A\left(s, u_{2}\right) d s+\int_{0}^{t} h(s, \omega)\left(\zeta_{1}(s, \omega)-\zeta_{2}(s, \omega)\right) d s=0
$$

Then from monotonicity of the subgradient it follows that $u_{1}=u_{2}$. Then the two integral equations yield that for a.e. $t$

$$
\begin{aligned}
& \left(B\left(u_{1}-\int_{0}^{(\cdot)} \Psi d W\right)\right)^{\prime}(t)+A\left(s, u_{1}(t)\right)+h(t, \omega) \zeta_{1}(t, \omega)= \\
= & \left(B\left(u_{2}-\int_{0}^{(\cdot)} \Psi d W\right)\right)^{\prime}(t)+A\left(s, u_{2}(t)\right)+h(t, \omega) \zeta_{2}(t, \omega)=0
\end{aligned}
$$

Therefore, for a.e. $t, h(t, \omega) \zeta_{1}(t, \omega)=h(t, \omega) \zeta_{2}(t, \omega)$. Thus the solution to the integral equation for each $\omega$ off a set of measure zero is unique.

At this point it is not clear that $(t, \omega) \rightarrow u(t, \omega)$ is progressively measurable. We claim that for $\omega \notin N$ it is not necessary to take a subsequence in the above. This is because the above argument shows that if $u_{n}$ fails to converge weakly, then there would exist two subsequences converging weakly to two different solutions to the integral equation which would contradict uniqueness.

Therefore, for $\omega \notin N, u_{n}(\cdot, \omega) \rightarrow u(\cdot, \omega)$ weakly in $\mathscr{V}_{\omega}$ for a single sequence. Using the estimate 76.3.24 it also follows that for a further subsequence still denoted as $u_{n}$,

$$
u_{n} \rightharpoonup \bar{u} \text { in } L^{p}([0, T] \times \Omega ; V)
$$

where the measurable sets are just the product measurable sets $\mathscr{B}([0, T]) \times \mathscr{F}_{T}$. By Lemma 76.3.4 for $\omega$ off a set of measure zero, $u(\cdot, \omega)=\bar{u}(\cdot, \omega)$ in $\mathscr{V}_{\omega}$ where $\bar{u}$ is progressively measurable. It follows that in all of the above, we could substitute $\bar{u}$ for $u$ at least for $\omega$ off a single set of measure zero. Thus $u$ can be assumed progressively measurable. The above argument along with technical details related to exponential shift considerations proves the following theorem.

Theorem 76.7.2 In the situation of Corollary 76.4.9 where $V$ is a closed subspace of $W^{\sigma, p}(U), \sigma>1$ and $W$ is as described above for $U$ a bounded open set, $u_{0} \in L^{2}(\Omega, W), u_{0}$ $\mathscr{F}_{0}$ measurable. Suppose $\lambda I+A(t, u, \omega)$ satisfies

$$
\langle\lambda I+A(t, u, \omega)-(\lambda I+A(t, v, \omega)), u-v\rangle \geq \delta^{2}\|u-v\|_{V}^{p}
$$

for all $\lambda$ large enough. Also assume $\Phi \in L^{\infty}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$ with $\Phi=B \Psi$ where

$$
\Psi \in L^{\infty}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right)
$$

and progressively measurable. Then there exists a unique solution to the integral equation

$$
\begin{equation*}
B u(t)-B u_{0}+\int_{0}^{t} A(s, u) d s+\int_{0}^{t} h(s, \omega) \zeta(s, \omega) d s=\int_{0}^{t} \Phi d W \tag{76.7.70}
\end{equation*}
$$

where for a.e. $s, h(s, \omega) \zeta(s, \omega) \in \partial_{u}(h(s, \omega) \phi(u(s)))$ where $\phi(r) \equiv|r|$. The symbol $\partial_{u}$ is the subgradient of $\phi(u)$. Written in terms of inclusions, there exists a set of measure zero such that off this set,

$$
\begin{aligned}
\left(B\left(u-\int_{0}^{(\cdot)} \Phi d W\right)\right)^{\prime}+A(t, u) & \in \partial_{u}(h(t, \omega) \phi(u(s))) \text { a.e. } t \\
u(0) & =u_{0}
\end{aligned}
$$

Note that one can replace

$$
\Phi \in L^{\infty}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)
$$

with $\Phi \in L^{2}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$ along with an assumption that $t \rightarrow \Phi(t, \omega)$ is continuous. This can be done by defining a stopping time

$$
\tau_{n} \equiv \inf \{t:\|\Phi(t)\|>n\}
$$

Then from the above example, there exists a solution to the integral equation off a set of measure zero

$$
B u_{n}(t)-B u_{0}+\int_{0}^{t} A\left(s, u_{n}\right) d s+\int_{0}^{t} h(s, \omega) \zeta_{n}(s, \omega) d s=\int_{0}^{t \wedge \tau_{n}} \Phi d W
$$

Since $\Phi$ is a continuous process, $\tau_{n}=\infty$ for all $n$ large enough. Hence, one can replace the above with the desired integral equation. Of course the size of $n$ depends on $\omega$, but we can define

$$
u(t, \omega)=\lim _{n \rightarrow \infty} u_{n}(t, \omega)
$$

because by uniqueness which comes from monotonicity, if for a particular $\omega$, both $n, k$ are sufficiently large, then $u_{n}=u_{k}$. Thus $u$ is progressively measurable and is the desired solution.

Next we show that the above theory can also be used as a starting point for some second order in time problems. Consider a beam which has a point mass of mass $m$ attached to
one end. Suppose for sake of illustration that the left end is clamped, $u(0, t)=u_{x}(0, t)=0$, while the right end which has the attached mass is free to move, $u_{x}(1, t)=0$, and the beam occupies the interval $[0,1]$ in material coordinates. Then the stress is $\sigma=-u_{x x x}$ and balance of momentum is

$$
u_{t t}=\sigma_{x}+f
$$

where $f$ is a body force. Thus, letting

$$
w \in V \equiv\left\{w \in H^{2}(0,1): w(0)=w_{x}(0)=0, w_{x}(1)=0\right\}
$$

be a test function,

$$
\begin{aligned}
\int_{0}^{1} u_{t t} w d x & =\left.\sigma w\right|_{0} ^{1}+\int_{0}^{1}(-\sigma) w_{x} d x+\int_{0}^{1} f w d x \\
& =-m u_{t t}(1, t) w(1, t)+\int_{0}^{1} u_{x x x} w_{x} d x+\int_{0}^{1} f w d x
\end{aligned}
$$

Doing another integration by parts and using the boundary conditions, it follows that an appropriate variational formulation for this problem is

$$
\int_{0}^{1} u_{t t} w d x+m \gamma_{1} u_{t t} \gamma_{1} w+\int_{0}^{1} u_{x x} w_{x x} d x=\int_{0}^{1} f w d x
$$

where here $\gamma_{1}$ is the trace map on the right end.
Letting

$$
u(t)=u_{0}+\int_{0}^{t} v(s) d s
$$

where $u(0, t)=u_{0}$, we can write the above variational equation in the form

$$
(B v)^{\prime}+A u=f, B v(0)=B v_{0}
$$

where we assume that $v_{0} \in W$ where $W$ is the closure of $V$ in $H^{1}(0,1)$ and the operators are given by

$$
\begin{aligned}
& B: \quad W \rightarrow W^{\prime},\langle B u, w\rangle \equiv \int_{0}^{1} u w d x+m \gamma_{1} u \gamma_{1} w \\
& A: \quad V \rightarrow V^{\prime},\langle A u, w\rangle \equiv \int_{0}^{1} u_{x x} w_{x x} d x
\end{aligned}
$$

Thus in terms of an integral equation, this would be of the form

$$
B v(t)-B v_{0}+\int_{0}^{t} A(u) d s=\int_{0}^{t} f d s
$$

This suggests a stochastic version of the form

$$
B v(t)-B v_{0}+\int_{0}^{t} A(u) d s=\int_{0}^{t} f d s+\int_{0}^{t} \Phi d W
$$

where $\Phi \in L^{\infty}\left((0, T) \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$ for $H=L^{2}(0,1)$. As in the previous example, simple considerations involving maximal monotone operators imply that there exists $\Psi \in$ $L^{\infty}\left((0, T) \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right)$ such that $\Phi=B \Psi$. We also assume that $f \in \mathscr{V}^{\prime}$ and $u_{0}, v_{0}$ are in $L^{2}(\Omega, V)$ and $L^{2}(\Omega, W)$ respectively, both being $\mathscr{F}_{0}$ measurable. The above equation does not fit the general theory developed earlier because it is second order in time and is a stochastic version of a hyperbolic equation rather than a parabolic one. We consider it using a parabolic regularization which can be studied with the above general theory along with a simple fixed point argument.

Consider the approximate problem which is to find a solution to

$$
\begin{equation*}
B v(t)-B v_{0}+\varepsilon \int_{0}^{t} A v d s+\int_{0}^{t} A(u) d s=\int_{0}^{t} f d s+\int_{0}^{t} \Phi d W \tag{76.7.71}
\end{equation*}
$$

where $u$ is given above as an integral of $v$. First we argue that there exists a unique solution to the above integral equation and then we pass to a limit as $\varepsilon \rightarrow 0$. Let $u \in \mathscr{V}$ be given.

From Corollary 76.4.9 there exists a unique solution $v$ to 76.7.71. Now suppose $u_{1}, u_{2}$ are two given in $\mathscr{V}$ and denote by $v_{i}$ the corresponding $v$ which solves the above. Then from the Ito formula or standard considerations,

$$
\begin{aligned}
& \frac{1}{2} E\left\langle B\left(v_{1}(t)-v_{2}(t)\right), v_{1}(t)-v_{2}(t)\right\rangle+\varepsilon E \int_{0}^{t}\left\|v_{1}-v_{2}\right\|_{V}^{2} d s \\
\leq & \frac{\varepsilon}{2} E \int_{0}^{t}\left\|v_{1}-v_{2}\right\|_{V}^{2} d s+C_{\varepsilon} E \int_{0}^{t}\left\|u_{1}-u_{2}\right\|_{V}^{2} d s
\end{aligned}
$$

Now define a mapping $\theta$ from $\mathscr{V}$ to $\mathscr{V}$ as follows. Begin with $v$ then obtain

$$
\begin{equation*}
u(t) \equiv u_{0}+\int_{0}^{t} v(s) d s \tag{76.7.72}
\end{equation*}
$$

Use this $u$ in 76.7.71. Then $\theta v$ is the solution to 76.7 .71 which corresponds to $u$. Then the above inequality shows that

$$
\begin{gathered}
\int_{0}^{t} \int_{\Omega}\left\|\theta v_{1}(s)-\theta v_{2}(s)\right\|^{2} d P d s \leq \frac{C_{\varepsilon}}{\varepsilon} \int_{0}^{t} \int_{\Omega}\left\|u_{1}-u_{2}\right\|_{V}^{2} d P d s \\
\leq \frac{C_{\varepsilon}}{\varepsilon} C_{T} \int_{0}^{t} \int_{0}^{s} \int_{\Omega}\left\|v_{1}(r)-v_{2}(r)\right\|_{V}^{2} d P d r d s
\end{gathered}
$$

It follows that a high enough power of $\theta$ is a contraction map on $L^{2}\left(0, T, L^{2}(\Omega, V)\right)$ and so there exists a unique fixed point. This yields a unique solution to the above approximate problem 76.7.71 in which $u, v$ are related by 76.7.72.

Next we let $\varepsilon \rightarrow 0$. Index the above solution with $\varepsilon$. By the Ito formula again,

$$
\begin{gathered}
\frac{1}{2} E\left\langle B v_{\varepsilon}(t), v_{\varepsilon}(t)\right\rangle-\frac{1}{2} E\left\langle B v_{0}, v_{0}\right\rangle+\varepsilon \int_{0}^{t} E\left\|v_{\varepsilon}\right\|_{V}^{2} d s \\
\quad+\frac{1}{2} E\left\|u_{\varepsilon}(t)\right\|_{V}^{2}-\frac{1}{2} E\left\|u_{0}\right\|_{V}^{2}=\int_{0}^{t} E\left\langle f, v_{\varepsilon}\right\rangle d s
\end{gathered}
$$

Then one can obtain an estimate and pass to a limit as $\varepsilon \rightarrow 0$ obtaining the following convergences.

$$
\begin{gathered}
\varepsilon v_{\varepsilon} \rightarrow 0 \text { strongly in } \mathscr{V} \\
B v_{\varepsilon} \rightarrow B v \text { weak } * \operatorname{in} L^{\infty}\left(0, T, L^{2}\left(\Omega, W^{\prime}\right)\right) \\
u_{\varepsilon}(t) \rightarrow u(t) \text { weak } * \text { in } L^{\infty}\left(0, T, L^{2}(\Omega, V)\right)
\end{gathered}
$$

Then one can simply pass to a limit in the approximate integral equation and obtain, thanks to linearity considerations, that

$$
\begin{equation*}
B v(t)-B v_{0}+\int_{0}^{t} A(u) d s=\int_{0}^{t} f d s+\int_{0}^{t} \Phi d W, u(t)=u_{0}+\int_{0}^{t} v(s) d s \tag{76.7.73}
\end{equation*}
$$

the equation holding in $\mathscr{V}^{\prime}$. Thus for a.e. $\omega$, the above holds for a.e. $t$. It is possible to work harder and have the equation holding for all $t$. This involves using the other form of the Ito formula, estimating for a fixed $\omega$ as done above and then arguing that by uniqueness one can use a single subsequence which works independent of $\omega$.

Example 76.7.3 Let $u_{0} \in L^{2}(\Omega, V)$ where $V$ is described above and let $v_{0} \in L^{2}(\Omega, W)$ for $W$ described above. Let both of these initial conditions be progressively measurable. Also let $f \in \mathscr{V}^{\prime}$ and $\Phi \in L^{\infty}\left((0, T) \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$. Then there exists a unique solution to the the integral equation 76.7 .73 which can be written in the form

$$
B u_{t}(t)-B v_{0}+\int_{0}^{t} A\left(u_{0}+\int_{0}^{t} u_{t}(r) d r\right) d s=\int_{0}^{t} f d s+\int_{0}^{t} \Phi d W
$$

Note that a more standard model involves no point mass at the tip of the beam. This would be done the same way but it would not require the generalized Ito formula presented earlier. A more standard version would work.

One can find many other examples where this generalized Ito formula is a useful tool to study various kinds of stochastic partial differential equations. We have presented five examples above in which it was helpful to have the extra generality.

## Chapter 77

## Stochastic Inclusions

### 77.1 The General Context

The situation is as follows. There are spaces $V \subseteq W$ where $V, W$ are reflexive separable Banach spaces. It is assumed that $V$ is dense in $W$. Define the space for $p>1$

$$
\mathscr{V} \equiv L^{p}([0, T] ; V)
$$

where in each case, the $\sigma$ algebra of measurable sets will be $\mathscr{B}([0, T])$ the Borel measurable sets. Thus, from the Riesz representation theorem,

$$
\mathscr{V}^{\prime}=L^{p^{\prime}}\left([0, T] ; V^{\prime}\right),
$$

We also assume $(\Omega, \mathscr{F}, P)$ is a complete probability space. That is, if $P(E)=0$ and $F \subseteq E$, then $F \in \mathscr{F}$. Also

$$
V \subseteq W, \quad W^{\prime} \subseteq V^{\prime}
$$

$B(\omega)$ will be a linear operator, $B(\omega): W \rightarrow W^{\prime}$ which satisfies

1. $\langle B(\omega) x, y\rangle=\langle B(\omega) y, x\rangle$
2. $\langle B(\omega) x, x\rangle \geq 0$ and equals 0 if and only if $x=0$.
3. $\omega \rightarrow B(\omega)$ is a measurable $\mathscr{L}\left(W, W^{\prime}\right)$ valued function.

In the above formulae, $\langle\cdot, \cdot\rangle$ denotes the duality pairing of the Banach space $W$, with its dual space. We will use this notation in the present paper, the exact specification of which Banach space being determined by the context in which this notation occurs.

For example, you could simply take $W=H=H^{\prime}$ and $B$ the identity and consider a standard Gelfand triple where $H$ is a Hilbert space and $B$ equal to the identity. An interesting feature is the requirement that $B(\omega)$ be one to one. It would be interesting to include the case of degenerate $B$, but $B$ one to one includes the case of most interest just mentioned. Also a more general set of assumptions will allow the inclusion of this case of degenerate $B(\omega)$ also.

We assume always that the norm on the various reflexive Banach spaces is strictly convex.

### 77.2 Some Fundamental Theorems

The following fundamental result will be very useful. It says essentially that if $(B u)^{\prime} \in$ $L^{p^{\prime}}\left(0, T ; V^{\prime}\right)$ and $u \in L^{p}(0, T ; V)$ then the map $u \rightarrow B u(t)$ is continuous as a map from

$$
X \equiv\left\{u \in L^{p}([0, T] ; V):(B u)^{\prime} \in L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)\right\}
$$

having norm equal to

$$
\|u\|_{X} \equiv\|u\|_{L^{p}(0, T, V)}+\left\|(B u)^{\prime}\right\|_{L^{p^{\prime}}\left(0, T ; V^{\prime}\right)}
$$

to $W^{\prime}$. There is also a convenient integration by parts formula, Theorem 34.4.3. For convenience, the dependence of $B$ on $\omega$ is often suppressed. This is not a problem because the entire approach will be to consider the situation for fixed $\omega$.

Theorem 77.2.1 Let $V \subseteq W, W^{\prime} \subseteq V^{\prime}$ be separable Banach spaces, and let $Y \in L^{p^{\prime}}\left(0, T ; V^{\prime}\right)$ and

$$
\begin{equation*}
B u(t)=B u_{0}+\int_{0}^{t} Y(s) d s \text { in } V^{\prime}, u_{0} \in W, B u(t)=B(u(t)) \text { for a.e. } t \tag{77.2.1}
\end{equation*}
$$

Thus $Y=(B u)^{\prime}$ as a weak derivative in the sense of $V^{\prime}$ valued distributions. It is known that $u \in L^{p}(0, T, V)$ for $p>1$. Then $t \rightarrow B u(t)$ is continuous into $W^{\prime}$ for $t$ off a set of measure zero $N$ and also there exists a continuous function $t \rightarrow\langle B u, u\rangle(t)$ such that for all $t \notin N,\langle B u, u\rangle(t)=\langle B(u(t)), u(t)\rangle, B u(t)=B(u(t))$, and for all $t$,

$$
\frac{1}{2}\langle B u, u\rangle(t)=\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle+\int_{0}^{t}\langle Y(s), u(s)\rangle d s
$$

Note that the formula 77.2 .1 shows that $B u_{0}=B u(0)$. Also it shows that $t \rightarrow\langle B u, u\rangle(t)$ is continuous. To emphasize this a little more, $B u$ is the name of a function. $B u(t)=$ $B(u(t))$ for a.e. $t$ and $t \rightarrow B u(t)$ is continuous into $V^{\prime}$ on $[0, T]$ because of the integral equation.

Theorem 77.2.2 In the above corollary, the map $u \rightarrow B u(t)$ is continuous as a map from $X$ to $V^{\prime}$. Also if $Y$ denotes those $f \in L^{p}([0, T] ; V)$ for which $f^{\prime} \in L^{p}([0, T] ; V)$, so that $f$ has a representative such that $f(t)=f(0)+\int_{0}^{t} f^{\prime}(s) d s$, then if $\|f\|_{Y} \equiv\|f\|_{L^{p}([0, T] ; V)}+$ $\left\|f^{\prime}\right\|_{L^{p}([0, T] ; V)}$ the map $f \rightarrow f(t)$ is continuous.

Proof: First, why is $u \rightarrow B u(0)$ continuous? Say $u, v \in X$ and say $p \geq 2$ first.

$$
B u(t)-B v(t)=B u(0)-B v(0)+\int_{0}^{t}(B u)^{\prime}(s)-(B v)^{\prime}(s) d s
$$

and so,

$$
\begin{gathered}
\left(\int_{0}^{T}\|B u(0)-B v(0)\|_{V^{\prime}}^{p^{\prime}} d t\right)^{1 / p^{\prime}} \leq\left(\int_{0}^{T}\|B u(t)-B v(t)\|_{V^{\prime}}^{p^{\prime}} d t\right)^{1 / p^{\prime}} \\
+\left(\int_{0}^{T}\left\|\int_{0}^{t}(B u)^{\prime}(s)-(B v)^{\prime}(s) d s\right\|^{p^{\prime}} d t\right)^{1 / p^{\prime}}
\end{gathered}
$$

and so

$$
\begin{gathered}
\|B u(0)-B v(0)\|_{V^{\prime}} T^{1 / p^{\prime}} \leq \\
\left(\|B\|\|u-v\|_{L^{p^{\prime}}([0, T] ; V)}+T^{1 / p^{\prime}}\left\|(B u)^{\prime}-(B v)^{\prime}\right\|_{L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)}\right) \\
\leq C(\|B\|, T)\|u-v\|_{X}
\end{gathered}
$$

Thus $u \rightarrow B u(0)$ is continuous into $V^{\prime}$. If $p<2$, then you do something similar.

$$
\begin{gathered}
\left(\int_{0}^{T}\|B u(0)-B v(0)\|_{V^{\prime}}^{p} d t\right)^{1 / p} \leq\left(\int_{0}^{T}\|B u(t)-B v(t)\|_{V^{\prime}}^{p} d t\right)^{1 / p} \\
+\left(\int_{0}^{T}\left\|\int_{0}^{t}(B u)^{\prime}(s)-(B v)^{\prime}(s) d s\right\|^{p} d t\right)^{1 / p} \\
\|B u(0)-B v(0)\|_{V^{\prime}} T^{1 / p} \leq\|B\|\|u-v\|_{L^{p}}+C(T)\left\|(B u)^{\prime}-(B v)^{\prime}\right\|_{L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)} \\
\leq C(\|B\|, T)\|u-v\|_{X}
\end{gathered}
$$

However, one could just as easily have done this for an arbitrary $s<T$ by repeating the argument for

$$
B u(t)=B u(s)+\int_{s}^{t}(B u)^{\prime}(r) d r
$$

Thus this mapping is certainly continuous into $V^{\prime}$. The last assertion is similar.
Also of use will be the following generalization of the Ascoli Arzela theorem. [117], Theorem 69.5.4.

Theorem 77.2.3 Let $q>1$ and let $E \subseteq W \subseteq X$ where the injection map is continuous from $W$ to $X$ and compact from $E$ to $W$. Let $S$ be defined by

$$
\left\{u \text { such that }\|u(t)\|_{E} \leq R \text { for all } t \in[a, b], \text { and }\|u(s)-u(t)\|_{X} \leq R|t-s|^{1 / q}\right\}
$$

Thus $S$ is bounded in $L^{\infty}(0, T, E)$ and in addition, the functions are uniformly Holder continuous into $X$. Then $S \subseteq C([a, b] ; W)$ and if $\left\{u_{n}\right\} \subseteq S$, there exists a subsequence, $\left\{u_{n_{k}}\right\}$ which converges to a function $u \in C([a, b] ; W)$ in the following way.

$$
\lim _{k \rightarrow \infty}\left\|u_{n_{k}}-u\right\|_{\infty, W}=0
$$

Next is a major measurable selection theorem which forms an essential part of showing the existence of measurable solutions. See Theorem 70.2.1. The following is not dependent on there being a measure but in the applications there is typically a probability measure and often a set of measure zero which occurs in a natural way so an exceptional set of measure zero is included in the statement of the theorem but it has absolutely nothing to do with a set of measure zero as will be seen by just letting the exceptional set be $\emptyset$.

Theorem 77.2.4 Let $V$ be a reflexive separable Banach space with dual $V^{\prime}$, and let $p, p^{\prime}$ be such that $p>1$ and $\frac{1}{p}+\frac{1}{p^{\prime}}=1$. Let the functions $t \rightarrow u_{n}(t, \omega)$, for $n \in \mathbb{N}$, be in $L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)$ and $(t, \omega) \rightarrow u_{n}(t, \omega)$ be $\mathscr{B}([0, T]) \times \mathscr{F} \equiv \mathscr{P}$ measurable into $V^{\prime}$. Suppose there is a set of measure zero $N \subseteq \Omega$ such that if $\omega \notin N$, then

$$
\sup _{t \in[0, T]}\left\|u_{n}(t, \omega)\right\|_{V^{\prime}} \leq C(\omega),
$$

for all $n$. Also, suppose for each $\omega \notin N$, each subsequence of $\left\{u_{n}\right\}$ has a further subsequence that converges weakly in $L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)$ to $v(\cdot, \omega) \in L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)$ such that the function $t \rightarrow v(t, \omega)$ is weakly continuous into $V^{\prime}$.

Then, there exists a product measurable function $u$ such that $t \rightarrow u(t, \omega)$ is weakly continuous into $V^{\prime}$ and for each $\omega \notin N$, a subsequence $u_{n(\omega)}$ such that $u_{n(\omega)}(\cdot, \omega) \rightarrow u(\cdot, \omega)$ weakly in $L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)$.

We prove the theorem in steps given below. Let $X=\prod_{k=1}^{\infty} C([0, T])$ and note that when it is equipped with the product topology, then one can consider $X$ as a metric space using the metric

$$
d(\mathbf{f}, \mathbf{g}) \equiv \sum_{k=1}^{\infty} 2^{-k} \frac{\left\|f_{k}-g_{k}\right\|}{1+\left\|f_{k}-g_{k}\right\|}
$$

where $\mathbf{f}=\left(f_{1}, f_{2}, \ldots\right), \mathbf{g}=\left(g_{1}, g_{2}, \ldots\right) \in X$, and the norm is the maximum norm in $C([0, T])$. With this metric, $X$ is complete and separable.

Lemma 77.2.5 Let $\left\{\mathbf{f}_{n}\right\}$ be a sequence in $X$ and suppose that each one of the components $f_{n k}$ is bounded by $C=C(k)$ in $C^{0,1}([0, T])$. Then, there exists a subsequence $\left\{\mathbf{f}_{n_{j}}\right\}$ that converges to some $\mathbf{f} \in X$ as $n_{j} \rightarrow \infty$. Thus, $\left\{\mathbf{f}_{n}\right\}$ is pre-compact in $X$.

Proof: By the Ascoli-Arzelà theorem, there exists a subsequence $\left\{\mathbf{f}_{n_{1}}\right\}$ such that the sequence of the first components $f_{n_{1} 1}$ converges in $C([0, T])$. Then, taking a subsequence, one can obtain $\left\{n_{2}\right\}$ a subsequence of $\left\{n_{1}\right\}$ such that both the first and second components of $\mathbf{f}_{n_{2}}$ converge. Continuing in this way one obtains a sequence of subsequences, each a subsequence of the previous one such that $\mathbf{f}_{n_{j}}$ has the first $j$ components converging to functions in $C([0, T])$. Therefore, the diagonal subsequence has the property that it has every component converging to a function in $C([0, T])$. The resulting function is $\mathbf{f} \in$ $\prod_{k} C([0, T])$.

Now, for $m \in \mathbb{N}$ and $\phi \in V$, define $l_{m}(t) \equiv \max (0, t-(1 / m))$ and

$$
\psi_{m, \phi}: L^{p^{\prime}}\left([0, T] ; V^{\prime}\right) \rightarrow C([0, T])
$$

by

$$
\psi_{m, \phi} u(t) \equiv \int_{0}^{T}\left\langle m \phi \mathscr{X}_{\left[l_{m}(t), t\right]}(s), u(s)\right\rangle_{V} d s=m \int_{l_{m}(t)}^{t}\langle\phi, u(s)\rangle_{V} d s
$$

Here, $\mathscr{X}_{\left[l_{m}(t), t\right]}(\cdot)$ is the indicator function of the interval $\left[l_{m}(t), t\right]$ and $\langle\cdot, \cdot\rangle_{V}=\langle\cdot, \cdot\rangle_{V}$ is the duality pairing between $V$ and $V^{\prime}$.

Let $\mathscr{D}=\left\{\phi_{r}\right\}_{r=1}^{\infty}$ denote a countable dense subset of $V$. Then the pairs $(\phi, m)$ for $\phi \in \mathscr{D}$ and $m \in \mathbb{N}$ form a countable set. Let $\left(m_{k}, \phi_{r_{k}}\right)$ denote an enumeration of the pairs $(m, \phi) \in \mathbb{N} \times \mathscr{D}$. To simplify the notation, we set

$$
f_{k}(u)(t) \equiv \psi_{m_{k}, \phi_{r_{k}}}(u)(t)=m_{k} \int_{l_{m_{k}}(t)}^{t}\left\langle\phi_{r_{k}}, u(s)\right\rangle_{V} d s
$$

For fixed $\omega \notin N$ and $k$, the functions $\left\{t \rightarrow f_{k}\left(u_{j}(\cdot, \omega)\right)(t)\right\}_{j}$ are uniformly bounded and equicontinuous because they are in $C^{0,1}([0, T])$. Indeed, we have for $\omega \notin N$,

$$
\left|f_{k}\left(u_{j}(\cdot, \omega)\right)(t)\right|=\left|m_{k} \int_{l_{m_{k}}(t)}^{t}\left\langle\phi_{r_{k}}, u_{j}(s, \omega)\right\rangle_{V} d s\right| \leq C(\omega)\left\|\phi_{r_{k}}\right\|_{V}
$$

and for $t \leq t^{\prime}$

$$
\begin{aligned}
& \left|f_{k}\left(u_{j}(\cdot, \omega)\right)(t)-f_{k}\left(u_{j}(\cdot, \omega)\right)\left(t^{\prime}\right)\right| \\
\leq & \left|m_{k} \int_{l_{m_{k}}(t)}^{t}\left\langle\phi_{r_{k}}, u_{j}(s, \omega)\right\rangle_{V} d s-m_{k} \int_{l_{m_{k}}\left(t^{\prime}\right)}^{t^{\prime}}\left\langle\phi_{r_{k}}, u_{j}(s, \omega)\right\rangle_{V} d s\right| \\
\leq & 2 m_{k}\left|t^{\prime}-t\right| C(\omega)\left\|\phi_{r_{k}}\right\|_{V^{\prime}}
\end{aligned}
$$

By Lemma 77.2.5, the set of functions $\left\{\mathscr{X}_{N^{C}}(\omega) \mathbf{f}\left(u_{j}(\cdot, \omega)\right)\right\}_{j=n}^{\infty}$ is pre-compact in $X=$ $\prod_{k} C([0, T])$. We now define a set valued map $\Gamma^{n}: \Omega \rightarrow X$ by

$$
\Gamma^{n}(\omega) \equiv \overline{\cup_{j \geq n}\left\{\mathscr{X}_{N^{C}}(\omega) \mathbf{f}\left(u_{j}(\cdot, \omega)\right)\right\}}
$$

where the closure is taken in $X$. Then $\Gamma^{n}(\omega)$ is the closure of a pre-compact set in $X$ and so $\Gamma^{n}(\omega)$ is compact in $X$. From the definition, a function $\mathbf{f}$ is in $\Gamma^{n}(\omega)$ if and only if $d\left(\mathbf{f}, \mathscr{X}_{N^{C}}(\omega) \mathbf{f}\left(w_{l}\right)\right) \rightarrow 0$ as $l \rightarrow \infty$, where each $w_{l}$ is one of the $u_{j}(\cdot, \omega)$ for $j \geq n$. In the topology on $X$, this happens iff for every $k$,

$$
f_{k}(t)=\lim _{l \rightarrow \infty} m_{k} \int_{l_{m_{k}}(t)}^{t}\left\langle\phi_{r_{k}}, \mathscr{X}_{N^{C}}(\omega) w_{l}(s, \omega)\right\rangle_{V} d s
$$

where the limit is the uniform limit in $t$.
Lemma 77.2.6 The mapping $\omega \rightarrow \Gamma^{n}(\omega)$ is an $\mathscr{F}$ measurable set-valued map with values in $X$. If $\sigma$ is a measurable selection, then for each $t, \omega \rightarrow \sigma(t, \omega)$ is $\mathscr{F}$ measurable and $(t, \omega) \rightarrow \sigma(t, \omega)$ is $\mathscr{B}([0, T]) \times \mathscr{F}$ measurable.

We note that if $\sigma$ is a measurable selection then $\sigma(\omega) \in \Gamma^{n}(\omega)$, so $\sigma=\sigma(\cdot, \omega)$ is a continuous function. To have $\sigma$ measurable would mean that $\sigma_{k}^{-1}($ open $) \in \mathscr{F}$, where the open set is in $C([0, T])$.

Proof: Let $O$ be a basic open set in $X$. Then $O=\prod_{k=1}^{\infty} O_{k}$, where $O_{k}$ is a proper open set of $C([0, T])$ only for $k \in\left\{k_{1}, \cdots, k_{r}\right\}$. Thus there is a proper open set in these positions and in every other position the open set is the whole space $C([0, T])$. We need to show that

$$
\Gamma^{n-}(O) \equiv\left\{\omega: \Gamma^{n}(\omega) \cap O \neq \emptyset\right\} \in \mathscr{F}
$$

Now, $\Gamma^{n-}(O)=\cap_{i=1}^{r}\left\{\omega: \Gamma^{n}(\omega)_{k_{i}} \cap O_{k_{i}} \neq \emptyset\right\}$, so we consider whether

$$
\begin{equation*}
\left\{\omega: \Gamma^{n}(\omega)_{k_{i}} \cap O_{k_{i}} \neq \emptyset\right\} \in \mathscr{F} . \tag{77.2.2}
\end{equation*}
$$

From the definition of $\Gamma^{n}(\omega)$, this is equivalent to the condition that

$$
f_{k_{i}}\left(\mathscr{X}_{N^{C}}(\omega) u_{j}(\cdot, \omega)\right)=\left(\mathbf{f}\left(\mathscr{X}_{N^{C}}(\omega) u_{j}(\cdot, \omega)\right)\right)_{k_{i}} \in O_{k_{i}}
$$

for some $j \geq n$, and so the set in 77.2.2 is of the form

$$
\cup_{j=n}^{\infty}\left\{\omega:\left(\mathbf{f}\left(\mathscr{X}_{N^{C}}(\omega) u_{j}(\cdot, \omega)\right)\right)_{k_{i}} \in O_{k_{i}}\right\} .
$$

Now $\omega \rightarrow\left(\mathbf{f}\left(\mathscr{X}_{N^{C}}(\omega) u_{j}(\cdot, \omega)\right)\right)_{k_{i}}$ is $\mathscr{F}$ measurable into $C([0, T])$ and so the above set is in $\mathscr{F}$. To see this, let $g \in C([0, T])$ and consider the inverse image of the ball with radius $r$ and center $g$,

$$
B(g, r)=\left\{\omega:\left\|\left(\mathscr{X}_{N^{C}}(\omega) \mathbf{f}\left(u_{j}(\cdot, \omega)\right)\right)_{k_{i}}-g\right\|_{C([0, T])}<r\right\} .
$$

By continuity considerations,

$$
\begin{aligned}
& \left\|\left(\mathscr{X}_{N^{C}}(\omega) \mathbf{f}\left(u_{j}(\cdot, \omega)\right)\right)_{k_{i}}-g\right\|_{C([0, T])} \\
= & \sup _{t \in \mathbb{Q} \cap[0, T]}\left|\left(\mathscr{X}_{N^{C}}(\omega) \mathbf{f}\left(u_{j}(t, \omega)\right)\right)_{k_{i}}-g(t)\right|,
\end{aligned}
$$

which is the sup over countably many $\mathscr{F}$ measurable functions. Thus, it is $\mathscr{F}$ measurable. Since every open set is the countable union of such balls, it follows that the claim about $\mathscr{F}$ measurability is valid. Hence, $\Gamma^{n-}(O)$ is $\mathscr{F}$ measurable whenever $O$ is a basic open set.

Now, $X$ is a separable metric space and so every open set is a countable union of these basic sets. Let $U \subseteq X$ be open with $U=\cup_{l=1}^{\infty} O_{l}$ where $O_{l}$ is a basic open set as above. Then,

$$
\Gamma^{n-}(U)=\cup_{l=1}^{\infty} \Gamma^{n-}\left(O_{l}\right) \in \mathscr{F} .
$$

The existence of a measurable selection follows from the standard theory of measurable multi-functions [10, 70] see [70] starting on Page 141 for all the necessay stuff on measurable multifunctions or Section 48. If $\sigma$ is one of these measurable selections, the evaluation at $t$ is $\mathscr{F}$ measurable. Thus, $\omega \rightarrow \sigma(t, \omega)$ is $\mathscr{F}$ measurable with values in $\mathbb{R}^{\infty}$. Also, $t \rightarrow \sigma(t, \omega)$ is continuous, and so it follows that in fact $\sigma$ is product measurable as claimed.

Definition 77.2.7 Let $\Gamma(\omega) \equiv \cap_{n=1}^{\infty} \Gamma^{n}(\omega)$.
Lemma 77.2.8 $\Gamma$ is a nonempty $\mathscr{F}$ measurable set-valued function with values in compact subsets of $X$. There exists a measurable selection $\gamma$ such that $(t, \omega) \rightarrow \gamma(t, \omega)$ is $\mathscr{P}$ measurable. Also, for each $\omega$, there exists a subsequence, $u_{n(\omega)}(\cdot, \omega)$ such that for each $k$,

$$
\begin{aligned}
\gamma_{k}(t, \omega) & =\lim _{n(\omega) \rightarrow \infty} \mathbf{f}\left(\mathscr{X}_{N^{C}}(\omega) u_{n(\omega)}(t, \omega)\right)_{k} \\
& =\lim _{n(\omega) \rightarrow \infty} m_{k} \int_{l_{m_{k}}(t)}^{t}\left\langle\phi_{r_{k}}, \mathscr{X}_{N^{C}}(\omega) u_{n(\omega)}(s, \omega)\right\rangle_{V} d s .
\end{aligned}
$$

Proof: From the definition of $\Gamma(\omega)=\cap_{n=1}^{\infty} \Gamma^{n}(\omega)$ it follows that $\omega \rightarrow \Gamma(\omega)$ is a compact set-valued map in $X$ and is nonempty because each $\Gamma^{n}(\omega)$ is nonempty and compact, and the $\Gamma^{n}(\omega)$ are nested. We next show that $\omega \rightarrow \Gamma(\omega)$ is $\mathscr{F}$ measurable. Indeed, each $\Gamma^{n}$ is compact valued and $\mathscr{F}$ measurable so, if $F$ is closed,

$$
\Gamma(\omega) \cap F=\cap_{n=1}^{\infty} \Gamma^{n}(\omega) \cap F
$$

and the left-hand side is not empty if and only if each $\Gamma^{n}(\omega) \cap F \neq \emptyset$. Thus, for $F$ closed,

$$
\{\omega: \Gamma(\omega) \cap F \neq \emptyset\}=\cap_{n}\left\{\omega: \Gamma^{n}(\omega) \cap F \neq \emptyset\right\}
$$

and so

$$
\Gamma^{-}(F)=\cap_{n} \Gamma^{n-}(F) \in \mathscr{F}
$$

The last claim follows from the theory of multi-functions, see, e.g., [10, 70] or Section 48. See Proposition 48.1.4. The fact that $\Gamma^{n}(\omega)$ is compact, $\Gamma^{n}$ is measurable and $\Gamma^{n-}(U) \in$ $\mathscr{F}$, for $U$ open, imply the strong measurability of $\Gamma^{n}[10,70]$ see also Section 48 , and also that $\Gamma^{n-}(F) \in \mathscr{F}$. Thus, $\omega \rightarrow \Gamma(\omega)$ is a nonempty compact valued in $X$ and $\mathscr{F}$ measurable. We are using the theorem which says that when $\Gamma$ has compact values, then one can conclude that strong measurability and measurability coincide. This is why we can say that $\Gamma^{n-}(F) \in \mathscr{F}$.

The standard theory [10, 70], Section 48, also guarantees the existence of an $\mathscr{F}$ measurable selection $\omega \rightarrow \gamma(\omega)$ with $\gamma(\omega) \in \Gamma(\omega)$, for each $\omega$, and also that $t \rightarrow \gamma_{k}(t, \omega)$ (the $k^{t h}$ component of $\gamma$ ) is continuous. Next, we consider the product measurability of $\gamma_{k}$. We know that $\omega \rightarrow \gamma_{k}(\omega)$ is $\mathscr{F}$ measurable into $C([0, T])$ and since pointwise evaluation is continuous, $\omega \rightarrow \gamma_{k}(t, \omega)$ is $\mathscr{F}$ measurable. (This is nothing more than a case of the general result that a continuous function of a measurable function is measurable.) Then, since $t \rightarrow \gamma_{k}(t, \omega)$ is continuous, it follows that $\gamma_{k}$ is a $\mathscr{P}$ measurable real valued function and that $\gamma$ is a $\mathscr{P}$ measurable $\mathbb{R}^{\infty}$ valued function. Since $\gamma(\omega) \in \Gamma(\omega)$, it follows that for each $n, \gamma(\omega) \in \Gamma^{n}(\omega)$. Therefore, there exists $j_{n} \geq n$ such that for each $\omega$,

$$
d\left(\mathbf{f}\left(\mathscr{X}_{N^{C}}(\omega) u_{j_{n}}(\cdot, \omega)\right), \gamma(\omega)\right)<2^{-n}
$$

Therefore, for a suitable subsequence $\left\{u_{n(\omega)}(\cdot, \omega)\right\}$, we have

$$
\gamma(\omega)=\lim _{n(\omega) \rightarrow \infty} \mathbf{f}\left(\mathscr{X}_{N^{C}}(\omega) u_{n(\omega)}(\cdot, \omega)\right),
$$

for each $\omega$. In particular, for each $k$

$$
\begin{gather*}
\gamma_{k}(t, \omega)=\lim _{n(\omega) \rightarrow \infty} \mathbf{f}\left(\mathscr{X}_{N^{C}}(\omega) u_{n(\omega)}(t, \omega)\right)_{k} \\
=\lim _{n(\omega) \rightarrow \infty} m_{k} \int_{l_{m_{k}}(t)}^{t}\left\langle\phi_{r_{k}}, \mathscr{X}_{N^{C}}(\omega) u_{n(\omega)}(s, \omega)\right\rangle_{V} d s \tag{77.2.3}
\end{gather*}
$$

for each $t$.
Note that it is not clear that $(t, \omega) \rightarrow \mathbf{f}\left(\mathscr{X}_{N^{C}}(\omega) u_{n(\omega)}(t, \omega)\right)$ is $\mathscr{P}$ measurable, although $(t, \omega) \rightarrow \gamma(t, \omega)$ is $\mathscr{P}$ measurable.

Now here is the proof of the theorem.
Proof of Theorem 77.2.4 By assumption, there exists a further subsequence, still denoted by $n(\omega)$, such that, in addition to 77.2.3, the weak limit

$$
\lim _{n(\omega) \rightarrow \infty} \mathscr{X}_{N^{C}}(\omega) u_{n(\omega)}(\cdot, \omega)=u(\cdot, \omega)
$$

exists in $L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)$ such that $t \rightarrow u(t, \omega)$ is weakly continuous into $V^{\prime}$. Then, 77.2.3 also holds for this further subsequence and in addition,

$$
\begin{aligned}
& m_{k} \int_{l_{m_{k}}(t)}^{t}\left\langle\phi_{r_{k}}, u(s, \omega)\right\rangle_{V} d s \\
= & \lim _{n(\omega) \rightarrow \infty} m_{k} \int_{l_{m_{k}}(t)}^{t}\left\langle\phi_{r_{k}}, \mathscr{X}_{N^{C}}(\omega) u_{n(\omega)}(s, \omega)\right\rangle_{V} d s \\
= & \gamma_{k}(t, \omega) .
\end{aligned}
$$

Letting $\phi \in \mathscr{D}$ be given, there exists a subsequence, denoted by $k$, such that $m_{k} \rightarrow \infty$ and $\phi_{r_{k}}=\phi$. Recall $\left(m_{k}, \phi_{r_{k}}\right)$ denoted an enumeration of the pairs $(m, \phi) \in \mathbb{N} \times \mathscr{D}$. Then, passing to the limit and using the assumed continuity of $s \rightarrow u(s, \omega)$, the left-hand side of this equality converges to $\langle\phi, u(s, \omega)\rangle_{V}$ and so the right-hand side, $\gamma_{k}(t, \omega)$, must also converge and for each $\omega$. Since the right-hand side is a product measurable function of $(t, \omega)$, it follows that the pointwise limit is also product measurable. Hence, $(t, \omega) \rightarrow$ $\langle\phi, u(t, \omega)\rangle_{V}$ is product measurable for each $\phi \in \mathscr{D}$. Since $\mathscr{D}$ is a dense set, it follows that $(t, \omega) \rightarrow\langle\phi, u(t, \omega)\rangle_{V}$ is $\mathscr{P}$ measurable for all $\phi \in V$ and so by the Pettis theorem, [127], $(t, \omega) \rightarrow u(t, \omega)$ is $\mathscr{P}$ measurable into $V^{\prime}$.

Actually, one can say more about the measurability of the approximating sequence and in fact, we can obtain one for which $\omega \rightarrow u_{n(\omega)}(t, \omega)$ is also $\mathscr{F}$ measurable.

Lemma 77.2.9 Suppose that $u_{n(\omega)} \rightarrow u$ weakly in $L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)$, where $u$ is product measurable, and $\left\{u_{n(\omega)}\right\}$ is a subsequence of $\left\{u_{n}\right\}$, such that there exists a set of measure zero $N \subseteq \Omega$ and

$$
\sup _{t \in[0, T]}\left\|u_{n}(t, \omega)\right\|_{V^{\prime}}<C(\omega), \text { for } \omega \notin N
$$

Then, there exists a subsequence of $\left\{u_{n}\right\}$, denoted as $\left\{u_{k(\omega)}\right\}$, such that $u_{k(\omega)} \rightarrow u$ weakly in $L^{p^{\prime}}\left([0, T] ; V^{\prime}\right), \omega \rightarrow k(\omega)$ is $\mathscr{F}$ measurable, and $\omega \rightarrow u_{k(\omega)}(t, \omega)$ is also $\mathscr{F}$ measurable, for each $\omega \notin N$.

Proof: Assume that $f, g \in L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)$ and let $\left\{\phi_{k}\right\}$ be a countable dense subset of $L^{p}([0, T] ; V)$. Then, a bounded set in $L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)$ with the weak topology can be considered a complete metric space using the metric

$$
d(f, g) \equiv \sum_{j=1}^{\infty} 2^{-j} \frac{\left|\left\langle\phi_{k}, f-g\right\rangle\right|}{1+\left|\left\langle\phi_{k}, f-g\right\rangle\right|}
$$

Now, let $k(\omega)$ be the first index of $\left\{u_{n}\right\}$ that is at least as large as $k$ and such that

$$
d\left(\mathscr{X}_{N^{C}}(\omega) u_{k(\omega)}, u\right) \leq 2^{-k}
$$

Such an index exists because there exists a convergent sequence $\mathscr{X}_{N^{C}}(\omega) u_{n(\omega)}$ that converge weakly to $u$. In fact,

$$
\{\omega: k(\omega)=l\}=\left\{\omega: d\left(u_{l}, u\right) \leq 2^{-k}\right\} \cap \cap_{j=1}^{l-1}\left\{\omega: d\left(u_{j}, u\right)>2^{-k}\right\}
$$

Since $u$ is product measurable and each $u_{l}$ is also product measurable, these are all measurable sets with respect to $\mathscr{F}$ and so $\omega \rightarrow k(\omega)$ is $\mathscr{F}$ measurable. Now, we have that $\mathscr{X}_{N^{C}}(\omega) u_{k(\omega)} \rightarrow u$ weakly in $L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)$, for each $\omega$, and each function is $\mathscr{F}$ measurable because

$$
u_{k(\omega)}(t, \omega)=\sum_{j=1}^{\infty} \mathscr{X}_{[k(\omega)=j]} u_{j}(t, \omega),
$$

and every term in the sum is $\mathscr{F}$ measurable.
Theorem 77.2.4 can be generalized in a very nice way. It is a better result because you don't need to assume anything so strong as to have the functions bounded. One does not need any assumption that the limit is weakly continuous into $V^{\prime}$. You can also have the functions take values in either $V$ or $V^{\prime}$. The following is not dependent on there being a measure but in the applications there is typically a probability measure and often a set of measure zero which occurs in a natural way so an exceptional set of measure zero is included in the statement of the theorem.

Theorem 77.2.10 Let $V$ be a reflexive separable Banach space with dual $V^{\prime}$, and let $p, p^{\prime}$ be such that $p>1$ and $\frac{1}{p}+\frac{1}{p^{\prime}}=1$. Let the functions $t \rightarrow u_{n}(t, \omega)$, for $n \in \mathbb{N}$, be in $L^{p}([0, T] ; V) \equiv \mathscr{V}$ and $(t, \omega) \rightarrow u_{n}(t, \omega)$ be $\mathscr{B}([0, T]) \times \mathscr{F} \equiv \mathscr{P}$ measurable into V. Suppose there is a set of measure zero $N \subseteq \Omega$ such that if $\omega \notin N$, then

$$
\left\|u_{n}(\cdot, \omega)\right\|_{\mathscr{V}} \leq C(\omega)
$$

for all $n$. (Thus, by weak compactness, for each $\omega$, each subsequence of $\left\{u_{n}\right\}$ has a further subsequence that converges weakly in $\mathscr{V}$ to $v(\cdot, \omega) \in \mathscr{V}$. (v not known to be $\mathscr{P}$ measurable))

Then, there exists a product measurable function $u$ such that $t \rightarrow u(t, \omega)$ is in $\mathscr{V}$ and for each $\omega \notin N$, a subsequence $u_{n(\omega)}$ such that $u_{n(\omega)}(\cdot, \omega) \rightarrow u(\cdot, \omega)$ weakly in $\mathscr{V}$.

We prove the theorem in steps given below. Let $X=\prod_{k=1}^{\infty} C([0, T])$ and note that when it is equipped with the product topology, then one can consider $X$ as a metric space using the metric

$$
d(\mathbf{f}, \mathbf{g}) \equiv \sum_{k=1}^{\infty} 2^{-k} \frac{\left\|f_{k}-g_{k}\right\|}{1+\left\|f_{k}-g_{k}\right\|}
$$

where $\mathbf{f}=\left(f_{1}, f_{2}, \ldots\right), \mathbf{g}=\left(g_{1}, g_{2}, \ldots\right) \in X$, and the norm is the maximum norm in $C([0, T])$. With this metric, $X$ is complete and separable.

Lemma 77.2.11 Let $\left\{\mathbf{f}_{n}\right\}$ be a sequence in $X$ and suppose that each one of the components $f_{n k}$ is bounded by $C=C(k)$ in $C^{0,\left(1 / p^{\prime}\right)}([0, T])$. Then, there exists a subsequence $\left\{\mathbf{f}_{n_{j}}\right\}$ that converges to some $\mathbf{f} \in X$ as $n_{j} \rightarrow \infty$. Thus, $\left\{\mathbf{f}_{n}\right\}$ is pre-compact in $X$.

Proof: This follows right away from Tychonoff's theorem and the compactness of the embedding of the Holder space into $C([0,1])$.

Now, for $m \in \mathbb{N}$ and $\phi \in V^{\prime}$, define $l_{m}(t) \equiv \max (0, t-(1 / m))$ and $\psi_{m, \phi}: \mathscr{V} \rightarrow C([0, T])$ by

$$
\psi_{m, \phi} u(t) \equiv \int_{0}^{T}\left\langle m \phi \mathscr{X}_{\left[l_{m}(t), t\right]}(s), u(s)\right\rangle_{V} d s=m \int_{l_{m}(t)}^{t}\langle\phi, u(s)\rangle_{V} d s
$$

Here, $\mathscr{X}_{\left[l_{m}(t), t\right]}(\cdot)$ is the characteristic function of the interval $\left[l_{m}(t), t\right]$ and $\langle\cdot, \cdot\rangle_{V}=\langle\cdot, \cdot\rangle_{V}$ is the duality pairing between $V$ and $V^{\prime}$.

Let $\mathscr{D}=\left\{\phi_{r}\right\}_{r=1}^{\infty}$ denote a countable subset of $V^{\prime}$. Then the pairs $(\phi, m)$ for $\phi \in \mathscr{D}$ and $m \in \mathbb{N}$ form a countable set. Let $\left(m_{k}, \phi_{r_{k}}\right)$ denote an enumeration of the pairs $(m, \phi) \in$ $\mathbb{N} \times \mathscr{D}$. To simplify the notation, we set

$$
f_{k}(u)(t) \equiv \psi_{m_{k}, \phi_{r_{k}}}(u)(t)=m_{k} \int_{l_{m_{k}}(t)}^{t}\left\langle\phi_{r_{k}}, u(s)\right\rangle_{V} d s
$$

For fixed $\omega \notin N$ and $k$, the functions $\left\{t \rightarrow f_{k}\left(u_{j}(\cdot, \omega)\right)(t)\right\}_{j}$ are uniformly bounded and equicontinuous because they are in $C^{0,1 / p^{\prime}}([0, T])$. Indeed, we have for $\omega \notin N$,

$$
\begin{aligned}
\left|f_{k}\left(u_{j}(\cdot, \omega)\right)(t)\right| & =\left|m_{k} \int_{l_{m_{k}}(t)}^{t}\left\langle\phi_{r_{k}}, u_{j}(s, \omega)\right\rangle_{V} d s\right| \\
& \leq m\left\|\phi_{r_{k}}\right\| T^{1 / p^{\prime}}\left|\int_{0}^{T}\left\|u_{j}(s, \omega)\right\|_{V}^{p} d s\right|^{1 / p} \leq C(\omega) m\left\|\phi_{r_{k}}\right\|_{V^{\prime}} T^{1 / p^{\prime}}
\end{aligned}
$$

and for $t \leq t^{\prime}$

$$
\begin{aligned}
& \left|f_{k}\left(u_{j}(\cdot, \omega)\right)(t)-f_{k}\left(u_{j}(\cdot, \omega)\right)\left(t^{\prime}\right)\right| \\
\leq & \left|m_{k} \int_{l_{m_{k}}(t)}^{t}\left\langle\phi_{r_{k}}, u_{j}(s, \omega)\right\rangle_{V} d s-m_{k} \int_{l_{m_{k}}\left(t^{\prime}\right)}^{t^{\prime}}\left\langle\phi_{r_{k}}, u_{j}(s, \omega)\right\rangle_{V} d s\right| \\
\leq & 2 m_{k}\left|t^{\prime}-t\right|^{1 / p^{\prime}} C(\omega)\left\|\phi_{r_{k}}\right\|_{V^{\prime}} .
\end{aligned}
$$

By Lemma 77.2.11, the set of functions $\left\{\mathscr{X}_{N^{C}}(\omega) \mathbf{f}\left(u_{j}(\cdot, \omega)\right)\right\}_{j=n}^{\infty}$ is pre-compact in $X=$ $\prod_{k} C([0, T])$. We now define a set valued map $\Gamma^{n}: \Omega \rightarrow X$ by

$$
\Gamma^{n}(\omega) \equiv \overline{\cup_{j \geq n}\left\{\mathscr{X}_{N^{C}}(\omega) \mathbf{f}\left(u_{j}(\cdot, \omega)\right)\right\}}
$$

where the closure is taken in $X$. Then $\Gamma^{n}(\omega)$ is the closure of a pre-compact set in $X$ and so $\Gamma^{n}(\omega)$ is compact in $X$. From the definition, a function $\mathbf{f}$ is in $\Gamma^{n}(\omega)$ if and only if $d\left(\mathbf{f}, \mathscr{X}_{N^{C}}(\omega) \mathbf{f}\left(w_{l}\right)\right) \rightarrow 0$ as $l \rightarrow \infty$, where each $w_{l}$ is one of the $u_{j}(\cdot, \omega)$ for $j \geq n$. In the topology on $X$, this happens iff for every $k$,

$$
f_{k}(t)=\lim _{l \rightarrow \infty} m_{k} \int_{l_{m_{k}}(t)}^{t}\left\langle\phi_{r_{k}}, \mathscr{X}_{N^{C}}(\omega) w_{l}(s, \omega)\right\rangle_{V} d s
$$

where the limit is the uniform limit in $t$.

Lemma 77.2.12 The mapping $\omega \rightarrow \Gamma^{n}(\omega)$ is an $\mathscr{F}$ measurable set-valued map with values in $X$. If $\sigma$ is a measurable selection, then for each $t, \omega \rightarrow \sigma(t, \omega)$ is $\mathscr{F}$ measurable and $(t, \omega) \rightarrow \sigma(t, \omega)$ is $\mathscr{B}([0, T]) \times \mathscr{F}$ measurable.

We note that if $\sigma$ is a measurable selection then $\sigma(\omega) \in \Gamma^{n}(\omega)$, so $\sigma=\sigma(\cdot, \omega)$ is a continuous function. To have $\sigma$ measurable would mean that $\sigma_{k}^{-1}($ open $) \in \mathscr{F}$, where the open set is in $C([0, T])$.

Proof: Let $O$ be a basic open set in $X$. Then $O=\prod_{k=1}^{\infty} O_{k}$, where $O_{k}$ is a proper open set of $C([0, T])$ only for $k \in\left\{k_{1}, \cdots, k_{r}\right\}$. Thus there is a proper open set in these positions and in every other position the open set is the whole space $C([0, T])$. We need to show that

$$
\Gamma^{n-}(O) \equiv\left\{\omega: \Gamma^{n}(\omega) \cap O \neq \emptyset\right\} \in \mathscr{F}
$$

Now, $\Gamma^{n-}(O)=\cap_{i=1}^{r}\left\{\omega: \Gamma^{n}(\omega)_{k_{i}} \cap O_{k_{i}} \neq \emptyset\right\}$, so we consider whether

$$
\begin{equation*}
\left\{\omega: \Gamma^{n}(\omega)_{k_{i}} \cap O_{k_{i}} \neq \emptyset\right\} \in \mathscr{F} . \tag{77.2.4}
\end{equation*}
$$

From the definition of $\Gamma^{n}(\omega)$, this is equivalent to the condition that

$$
f_{k_{i}}\left(\mathscr{X}_{N^{C}}(\omega) u_{j}(\cdot, \omega)\right)=\left(\mathbf{f}\left(\mathscr{X}_{N^{C}}(\omega) u_{j}(\cdot, \omega)\right)\right)_{k_{i}} \in O_{k_{i}}
$$

for some $j \geq n$, and so the set in 77.2.4 is of the form

$$
\cup_{j=n}^{\infty}\left\{\omega:\left(\mathbf{f}\left(\mathscr{X}_{N^{C}}(\omega) u_{j}(\cdot, \omega)\right)\right)_{k_{i}} \in O_{k_{i}}\right\} .
$$

Now $\omega \rightarrow\left(\mathbf{f}\left(\mathscr{X}_{N^{C}}(\omega) u_{j}(\cdot, \omega)\right)\right)_{k_{i}}$ is $\mathscr{F}$ measurable into $C([0, T])$ and so the above set is in $\mathscr{F}$. To see this, let $g \in C([0, T])$ and consider the inverse image of the ball with radius $r$ and center $g$,

$$
B(g, r)=\left\{\omega:\left\|\left(\mathscr{X}_{N^{C}}(\omega) \mathbf{f}\left(u_{j}(\cdot, \omega)\right)\right)_{k_{i}}-g\right\|_{C([0, T])}<r\right\} .
$$

By continuity considerations,

$$
\begin{aligned}
& \left\|\left(\mathscr{X}_{N^{C}}(\omega) \mathbf{f}\left(u_{j}(\cdot, \omega)\right)\right)_{k_{i}}-g\right\|_{C([0, T])} \\
= & \sup _{t \in \mathbb{Q} \cap[0, T]}\left|\left(\mathscr{X}_{N^{C}}(\omega) \mathbf{f}\left(u_{j}(t, \omega)\right)\right)_{k_{i}}-g(t)\right|,
\end{aligned}
$$

which is the sup over countably many $\mathscr{F}$ measurable functions. Thus, it is $\mathscr{F}$ measurable. Since every open set is the countable union of such balls, it follows that the claim about $\mathscr{F}$ measurability is valid. Hence, $\Gamma^{n-}(O)$ is $\mathscr{F}$ measurable whenever $O$ is a basic open set.

Now, $X$ is a separable metric space and so every open set is a countable union of these basic sets. Let $U \subseteq X$ be open with $U=\cup_{l=1}^{\infty} O_{l}$ where $O_{l}$ is a basic open set as above. Then,

$$
\Gamma^{n-}(U)=\cup_{l=1}^{\infty} \Gamma^{n-}\left(O_{l}\right) \in \mathscr{F} .
$$

The existence of a measurable selection follows from the standard theory of measurable multi-functions [10, 70] see [70] starting on Page 141 for all the necessary stuff on measurable multifunctions or Section 48. If $\sigma$ is one of these measurable selections, the evaluation at $t$ is $\mathscr{F}$ measurable. Thus, $\omega \rightarrow \sigma(t, \omega)$ is $\mathscr{F}$ measurable with values in $\mathbb{R}^{\infty}$. Also, $t \rightarrow \sigma(t, \omega)$ is continuous, and so it follows that in fact $\sigma$ is product measurable as claimed.

Definition 77.2.13 Let $\Gamma(\omega) \equiv \cap_{n=1}^{\infty} \Gamma^{n}(\omega)$.
Lemma 77.2.14 $\Gamma$ is a nonempty $\mathscr{F}$ measurable set-valued function with values in compact subsets of $X$. There exists a measurable selection $\gamma$ such that $(t, \omega) \rightarrow \gamma(t, \omega)$ is $\mathscr{P}$ measurable. Also, for each $\omega$, there exists a subsequence, $u_{n(\omega)}(\cdot, \omega)$ such that for each $k$,

$$
\begin{aligned}
\gamma_{k}(t, \omega) & =\lim _{n(\omega) \rightarrow \infty} \mathbf{f}\left(\mathscr{X}_{N^{C}}(\omega) u_{n(\omega)}(t, \omega)\right)_{k} \\
& =\lim _{n(\omega) \rightarrow \infty} m_{k} \int_{l_{m_{k}}(t)}^{t}\left\langle\phi_{r_{k}}, \mathscr{X}_{N^{C}}(\omega) u_{n(\omega)}(s, \omega)\right\rangle_{V} d s .
\end{aligned}
$$

Proof: From the definition of $\Gamma(\omega)=\cap_{n=1}^{\infty} \Gamma^{n}(\omega)$ it follows that $\omega \rightarrow \Gamma(\omega)$ is a compact set-valued map in $X$ and is nonempty because each $\Gamma^{n}(\omega)$ is nonempty and compact, and the $\Gamma^{n}(\omega)$ are nested. We next show that $\omega \rightarrow \Gamma(\omega)$ is $\mathscr{F}$ measurable. Indeed, each $\Gamma^{n}$ is compact valued and $\mathscr{F}$ measurable so, if $F$ is closed,

$$
\Gamma(\omega) \cap F=\cap_{n=1}^{\infty} \Gamma^{n}(\omega) \cap F
$$

and the left-hand side is not empty if and only if each $\Gamma^{n}(\omega) \cap F \neq \emptyset$. Thus, for $F$ closed,

$$
\{\omega: \Gamma(\omega) \cap F \neq \emptyset\}=\cap_{n}\left\{\omega: \Gamma^{n}(\omega) \cap F \neq \emptyset\right\}
$$

and so

$$
\Gamma^{-}(F)=\cap_{n} \Gamma^{n-}(F) \in \mathscr{F} .
$$

The last claim follows from the theory of multi-functions, see, e.g., [10, 70] or Section 48. The fact that $\Gamma^{n}(\omega)$ is compact, $\Gamma^{n}$ is measurable and $\Gamma^{n-}(U) \in \mathscr{F}$, for $U$ open, imply the strong measurability of $\Gamma^{n}[10,70]$ see also Section 48, and also that $\Gamma^{n-}(F) \in \mathscr{F}$. Thus, $\omega \rightarrow \Gamma(\omega)$ is nonempty compact valued in $X$ and $\mathscr{F}$ measurable. We are using the theorem which says that when $\Gamma$ has compact values, then one can conclude that strong measurability and measurability coincide. See Proposition 48.1.4. This is why we can say that $\Gamma^{n-}(F) \in \mathscr{F}$.

The standard theory [10, 70], Section 48, also guarantees the existence of an $\mathscr{F}$ measurable selection $\omega \rightarrow \gamma(\omega)$ with $\gamma(\omega) \in \Gamma(\omega)$, for each $\omega$, and also that $t \rightarrow \gamma_{k}(t, \omega)$ (the $k^{\text {th }}$ component of $\gamma$ ) is continuous. Next, we consider the product measurability of $\gamma_{k}$. We know that $\omega \rightarrow \gamma_{k}(\omega)$ is $\mathscr{F}$ measurable into $C([0, T])$ and since pointwise evaluation is continuous, $\omega \rightarrow \gamma_{k}(t, \omega)$ is $\mathscr{F}$ measurable. (This is nothing more than a case of the general result that a continuous function of a measurable function is measurable.) Then, since $t \rightarrow \gamma_{k}(t, \omega)$ is continuous, it follows that $\gamma_{k}$ is a $\mathscr{P}$ measurable real valued function
and that $\gamma$ is a $\mathscr{P}$ measurable $\mathbb{R}^{\infty}$ valued function. Since $\gamma(\omega) \in \Gamma(\omega)$, it follows that for each $n, \gamma(\omega) \in \Gamma^{n}(\omega)$. Therefore, there exists $j_{n} \geq n$ such that for each $\omega$,

$$
d\left(\mathbf{f}\left(\mathscr{X}_{N^{C}}(\omega) u_{j_{n}}(\cdot, \omega)\right), \gamma(\omega)\right)<2^{-n}
$$

Therefore, for a suitable subsequence $\left\{u_{n(\omega)}(\cdot, \omega)\right\}$, we have

$$
\gamma(\omega)=\lim _{n(\omega) \rightarrow \infty} \mathbf{f}\left(\mathscr{X}_{N^{C}}(\omega) u_{n(\omega)}(\cdot, \omega)\right),
$$

for each $\omega$. In particular, for each $k$

$$
\begin{gather*}
\gamma_{k}(t, \omega)=\lim _{n(\omega) \rightarrow \infty} \mathbf{f}\left(\mathscr{X}_{N^{C}}(\omega) u_{n(\omega)}(t, \omega)\right)_{k} \\
=\lim _{n(\omega) \rightarrow \infty} m_{k} \int_{l_{m_{k}}(t)}^{t}\left\langle\phi_{r_{k}}, \mathscr{X}_{N^{C}}(\omega) u_{n(\omega)}(s, \omega)\right\rangle_{V} d s \tag{77.2.5}
\end{gather*}
$$

for each $t$.
Note that it is not clear that $(t, \omega) \rightarrow \mathbf{f}\left(\mathscr{X}_{N^{C}}(\omega) u_{n(\omega)}(t, \omega)\right)$ is $\mathscr{P}$ measurable, although $(t, \omega) \rightarrow \gamma(t, \omega)$ is $\mathscr{P}$ measurable.

Now here is the proof of the theorem.
Proof of Theorem 77.2.10 By assumption, there exists a further subsequence, still denoted by $n(\omega)$, such that, the weak limit

$$
\lim _{n(\omega) \rightarrow \infty} \mathscr{X}_{N^{C}}(\omega) u_{n(\omega)}(\cdot, \omega)=v(\cdot, \omega)
$$

exists in $\mathscr{V}$. Then,

$$
\begin{aligned}
& m_{k} \int_{l_{m_{k}}(t)}^{t}\left\langle\phi_{r_{k}}, v(s, \omega)\right\rangle_{V} d s \\
= & \lim _{n(\omega) \rightarrow \infty} m_{k} \int_{l_{m_{k}}(t)}^{t}\left\langle\phi_{r_{k}}, \mathscr{X}_{N^{C}}(\omega) u_{n(\omega)}(s, \omega)\right\rangle_{V} d s \\
= & \gamma_{k}(t, \omega), \text { product measurable } .
\end{aligned}
$$

Letting $\phi \in \mathscr{D}$ be given, there exists a subsequence, denoted by $k$, such that $m_{k} \rightarrow \infty$ and $\phi_{r_{k}}=\phi$. Recall $\left(m_{k}, \phi_{r_{k}}\right)$ denoted an enumeration of the pairs $(m, \phi) \in \mathbb{N} \times \mathscr{D}$. For a given $\phi \in \mathscr{D}$ denote this sequence by $m_{\phi}$. Thus we have measurability of

$$
(t, \omega) \rightarrow m_{\phi} \int_{l_{m_{\phi}}(t)}^{t}\langle\phi, v(s, \omega)\rangle_{V} d s
$$

for each $\phi \in \mathscr{D}$.
Now we will be a little more careful about the countable set $\mathscr{D}$. Iterate the following. Let $\phi_{1} \neq 0$. Let $\mathscr{F}$ denote linearly independent subsets of $V^{\prime}$ which contain $\phi_{1}$ such that the elements are further apart than $1 / 5$. Let $\mathscr{C}$ denote a maximal chain. Thus $\cup \mathscr{C}$ is also in $\mathscr{F}$. If $W:=\overline{\operatorname{span} \cup \mathscr{C}}$ fails to be all of $V^{\prime}$, then there would exist $\psi \notin W$ such that the distance of
$\psi$ to the closed subspace $W$ is at least $1 / 5$. Now $\mathscr{C}, \cup\{\mathscr{C} \cup\{\psi\}\}$ would violate maximality of $\mathscr{C}$. Hence $W=V^{\prime}$. Now it follows that $\mathscr{C}$ must be countable since otherwise, $V^{\prime}$ would fail to be separable. Let $M$ be the rational linear combinations of $\mathscr{D}$. It must be dense in $V^{\prime}$. Note that linear combinations of the $\phi_{i}$ are uniquely determined because none is a linear combination of the others. Now define a linear mapping on $M$ which makes sense for $(t, \omega)$ on a certain set.

Definition 77.2.15 Let $E$ be those points $(t, \omega)$ such that the following limit exists for each $\phi \in \mathscr{D}$

$$
\Lambda(t, \omega) \phi \equiv \lim _{m_{\phi} \rightarrow \infty} m_{\phi} \int_{l_{m_{\phi}}(t)}^{t}\langle\phi, v(s, \omega)\rangle d s
$$

The set of points where the limit of measurable functions exists is always measurable so $E$ is a measurable set. Extend this mapping linearly. That is, for $\psi \in M, \psi \equiv \sum_{i} a_{i} \phi_{i}$,

$$
\Lambda(t, \omega) \psi \equiv \sum_{i} a_{i} \Lambda(t, \omega) \phi_{i}=\sum_{i} a_{i}\left(\lim _{m_{\phi_{i}} \rightarrow \infty} m_{\phi_{i}} \int_{l_{m_{\phi_{i}}}(t)}^{t}\left\langle\phi_{i}, v(s, \omega)\right\rangle d s\right)
$$

Thus $(t, \omega) \rightarrow \Lambda(t, \omega) \psi$ is product measurable, being the sum of limits of product measurable functions. Let $G$ denote those $(t, \omega)$ in $E$ such that there exists a constant $C(t, \omega)$ such that for all $\psi \in M$,

$$
|\Lambda(t, \omega) \psi| \leq C(t, \omega)\|\psi\|
$$

Lemma 77.2.16 G is product measurable.
Proof: This follows from the formula

$$
E \cap G^{C}=\cap_{n} \cup_{\psi \in M}\{(t, \omega) \in E:|\Lambda(t, \omega) \psi|>n\|\psi\|\}
$$

which is clearly product measurable because $(t, \omega) \rightarrow \Lambda(t, \omega) \psi$ is. Thus, since $E$ is measurable, it follows that $E \cap G=G$ is also.

For $(t, \omega) \in G, \Lambda(t, \omega)$ has a unique extension to all of $V$, the dual space of $V^{\prime}$, still denoted as $\Lambda(t, \omega)$. By the Riesz representation theorem, for $(t, \omega) \in G$, there exists $u(t, \omega) \in V$,

$$
\Lambda(t, \omega) \psi=\langle\psi, u(t, \omega)\rangle_{V^{\prime}, V}
$$

Thus $(t, \omega) \rightarrow \mathscr{X}_{G}(t, \omega) u(t, \omega)$ is product measurable by the Pettis theorem. Let $u=0$ off $G$. We know $G$ is product measurable. For each $\omega,\{t:(t, \omega) \in G\}$ has full measure. This involves the fundamental theorem of calclulus.

Fix $\omega$. By the fundamental theorem of calculus,

$$
\lim _{m \rightarrow \infty} m \int_{l_{m}(t)}^{t} v(s, \omega) d s=v(t, \omega) \text { in } V
$$

for a.e. $t$ say for all $t \notin N(\omega) \subseteq[0, T]$. Of course we do not know that $\omega \rightarrow v(t, \omega)$ is measurable. However, the existence of this limit for $t \notin N(\omega)$ implies that for every $\phi \in V^{\prime}$,

$$
\lim _{m \rightarrow \infty}\left|m \int_{l_{m}(t)}^{t}\langle\phi, v(s, \omega)\rangle d s\right| \leq C(t, \omega)\|\phi\|
$$

for some $C(t, \omega)$. Here $m$ does not depend on $\phi$. Thus, in particular, this holds for a subsequence and so for each $t \notin N(\omega),(t, \omega) \in G$ because for each $\phi \in \mathscr{D}$,

$$
\lim _{m_{\phi} \rightarrow \infty} m_{\phi} \int_{l_{m_{\phi}}(t)}^{t}\langle\phi, v(s, \omega)\rangle d s \text { exists and satisfies the above inequality. }
$$

Hence, for all $\psi \in M$,

$$
\Lambda(t, \omega) \psi=\langle\psi, u(t, \omega)\rangle_{V^{\prime}, V}
$$

where $u$ is product measurable.
Also, for $t \notin N(\omega)$ and $\phi \in \mathscr{D}$,

$$
\langle\phi, u(t, \omega)\rangle_{V^{\prime}, V}=\Lambda(t, \omega) \phi \equiv \lim _{m_{\phi} \rightarrow \infty} m_{\phi} \int_{l_{m_{\phi}}(t)}^{t}\langle\phi, v(s, \omega)\rangle d s=\langle\phi, v(t, \omega)\rangle_{V^{\prime}, V}
$$

therefore, for all $\phi \in M$

$$
\langle\phi, u(t, \omega)\rangle_{V^{\prime}, V}=\langle\phi, v(t, \omega)\rangle_{V^{\prime}, V}
$$

and hence $u(t, \omega)=v(t, \omega)$. Thus, for each $\omega$, the product measurable function $u$ satisfies $u(t, \omega)=v(t, \omega)$ for a.e. $t$. Hence $u(\cdot, \omega)=v(\cdot, \omega)$ in $\mathscr{V}$.

Of course a similar theorem will hold with essentially the identical proof if the functions take values in $V^{\prime}$.

One can also combine the two theorems to obtain a useful result for limits of functions in $\mathscr{V}^{\prime}$ and $\mathscr{V}$. You just let

$$
X=\prod_{k=1}^{\infty} C([0, T]) \times C([0, T])
$$

and let $\left\{\phi_{k}\right\}$ be a dense subset of $V^{\prime}$ while $\left\{\eta_{k}\right\}$ is a dense subset of $V$. Then the mappings are given by

$$
\psi_{m, \phi} u(t)=m \int_{l_{m}(t)}^{t}\langle\phi, u(s)\rangle_{V^{\prime}, V} d s, \psi_{m, \eta} y(t)=m \int_{l_{m}(t)}^{t}\langle\eta, y(s)\rangle_{V, V^{\prime}} d s
$$

and one considers for each $\phi_{k}, \eta_{k}$,

$$
\left(m_{k} \int_{l_{m_{k}}(t)}^{t}\left\langle\phi_{k}, u(s)\right\rangle_{V^{\prime}, V} d s, m_{k} \int_{l_{m_{k}}(t)}^{t}\left\langle\eta_{k}, y(s)\right\rangle_{V, V^{\prime}} d s\right)
$$

This time you need to use an enumeration of $\mathbb{N} \times V^{\prime} \times V$ and in the last step, you must use a subsequence still denoted with $k$ such that $m_{k} \rightarrow \infty$ but $\phi_{k}=\phi$ and $\eta_{k}=\eta$ for $\phi, \eta$ two given elements of $V^{\prime}$ and $V$ respectively. Then repeating the above argument, one obtains the following generalization.

Theorem 77.2.17 Let $V$ be a reflexive separable Banach space with dual $V^{\prime}$, and let $p, p^{\prime}$ be such that $p>1$ and $\frac{1}{p}+\frac{1}{p^{\prime}}=1$. Let the functions $t \rightarrow u_{n}(t, \omega)$, for $n \in \mathbb{N}$, be in $L^{p}([0, T] ; V) \equiv \mathscr{V}$ and $(t, \omega) \rightarrow u_{n}(t, \omega)$ be $\mathscr{B}([0, T]) \times \mathscr{F} \equiv \mathscr{P}$ measurable into $V$. Also
let the functions $t \rightarrow y_{n}(t, \omega)$ be in $\mathscr{V}^{\prime}$ and $(t, \omega) \rightarrow y_{n}(t, \omega)$ is $\mathscr{P}$ measurable into $V^{\prime}$. Suppose there is a set of measure zero $N \subseteq \Omega$ such that if $\omega \notin N$, then

$$
\sup _{t \in[0, T]}\left\|y_{n}(t, \omega)\right\|_{V^{\prime}}+\left\|u_{n}(\cdot, \omega)\right\|_{\mathscr{V}} \leq C(\omega)
$$

for all $n$. (Thus, by weak compactness, for each $\omega$, each subsequence of $\left\{u_{n}\right\}$ has a further subsequence that converges weakly in $\mathscr{V}$ to $v(\cdot, \omega) \in \mathscr{V}$. (v not known to be $\mathscr{P}$ measurable)) Suppose that each subsequence of $\left\{y_{n}(\cdot, \omega)\right\}$ has a subsequence which converges weakly in $\mathscr{V}^{\prime}$ to $z(\cdot, \omega) \in \mathscr{V}^{\prime}$ such that the function $t \rightarrow z(t, \omega)$ is weakly continuous into $V^{\prime}$.

Then, there exist product measurable functions $u, y$ such that $t \rightarrow u(t, \omega)$ is in $\mathscr{V}, t \rightarrow$ $y(t, \omega)$ is weakly continuous into $V^{\prime}$ and for each $\omega \notin N$, a subsequence of $\mathbb{N}$ denoted by $\{n(\omega)\}$ such that $u_{n(\omega)}(\cdot, \omega) \rightarrow u(\cdot, \omega)$ weakly in $\mathscr{V}$ and $y_{n(\omega)}(\cdot, \omega)$ converges weakly to $y(\cdot, \omega)$ in $\mathscr{V}^{\prime}$.

Note that the conclusion of the proposition holds if $p=1$ and $\mathscr{V}=L^{1}([0, T], V)$.
Here is something else about being measurable into $\mathscr{V}$ or $\mathscr{V}^{\prime}$. Such functions have representatives which are product measurable.

Lemma 77.2.18 Let $f(\cdot, \omega) \in \mathscr{V}^{\prime}$. Then if $\omega \rightarrow f(\cdot, \omega)$ is measurable into $\mathscr{V}^{\prime}$, it follows that for each $\omega$, there exists a representative $\hat{f}(\cdot, \omega) \in \mathscr{V}^{\prime}, \hat{f}(\cdot, \omega)=f(\cdot, \omega)$ in $\mathscr{V}^{\prime}$ such that $(t, \omega) \rightarrow \hat{f}(t, \omega)$ is product measurable. If $f(\cdot, \omega) \in \mathscr{V}^{\prime}$ and $(t, \omega) \rightarrow f(t, \omega)$ is product measurable, then $\omega \rightarrow f(\cdot, \omega)$ is measurable into $\mathscr{V}^{\prime}$. The same holds replacing $\mathscr{V}^{\prime}$ with $\mathscr{V}$.

Proof: If a function $f$ is measurable into $\mathscr{V}^{\prime}$, then there exist simple functions $f_{n}$

$$
\lim _{n \rightarrow \infty}\left\|f_{n}(\omega)-f(\omega)\right\|_{\mathscr{V}^{\prime}}=0,\left\|f_{n}(\omega)\right\| \leq 2\|f(\omega)\|_{\mathscr{V}^{\prime}} \equiv C(\omega)
$$

Now one of these simple functions is of the form $\sum_{i=1}^{M} c_{i} \mathscr{X}_{E_{i}}(\omega)$ where $c_{i} \in \mathscr{V}^{\prime}$. Therefore, there is no loss of generality in assuming that $c_{i}(t)=\sum_{j=1}^{N} d_{j}^{i} \mathscr{X}_{F_{j}}(t)$ where $d_{j}^{i} \in V^{\prime}$. Hence we can assume each $f_{n}$ is product measurable into $\mathscr{B}\left(V^{\prime}\right) \times \mathscr{F}$. Then by Theorem 77.2.10, there exists $\hat{f}(\cdot, \omega) \in \mathscr{V}^{\prime}$ such that $\hat{f}$ is product measurable and a subsequence $f_{n(\omega)}$ converging weakly in $\mathscr{V}^{\prime}$ to $\hat{f}(\cdot, \omega)$ for each $\omega$. Thus $f_{n(\omega)}(\omega) \rightarrow f(\omega)$ strongly in $\mathscr{V}^{\prime}$ and $f_{n(\omega)}(\omega) \rightarrow \hat{f}(\omega)$ weakly in $\mathscr{V}^{\prime}$. Therefore, $\hat{f}(\omega)=f(\omega)$ in $\mathscr{V}^{\prime}$ and so it can be assumed that if $f$ is measurable into $\mathscr{V}^{\prime}$ then for each $\omega$, it has a representative $\hat{f}(\omega)$ such that $(t, \omega) \rightarrow \hat{f}(t, \omega)$ is product measurable.

If $f$ is product measurable into $V^{\prime}$ and each $f(\cdot, \omega) \in \mathscr{V}^{\prime}$, does it follow that $f$ is measurable into $\mathscr{V}^{\prime}$ ? By measurability, $f(t, \omega)=\lim _{n \rightarrow \infty} \sum_{i=1}^{m_{n}} c_{i}^{n} \mathscr{X}_{E_{i}^{n}}(t, \omega)=\lim _{n \rightarrow \infty} f_{n}(t, \omega)$ where $E_{i}^{n}$ is product measurable and we can assume $\left\|f_{n}(t, \omega)\right\|_{V^{\prime}} \leq 2\|f(t, \omega)\|$. Then by product measurability, $\omega \rightarrow f_{n}(\cdot, \omega)$ is measurable into $\mathscr{V}^{\prime}$ because if $g \in \mathscr{V}$ then

$$
\omega \rightarrow\left\langle f_{n}(\cdot, \omega), g\right\rangle
$$

is of the form

$$
\omega \rightarrow \sum_{i=1}^{m_{n}} \int_{0}^{T}\left\langle c_{i}^{n} \mathscr{X}_{E_{i}^{n}}(t, \omega), g(t)\right\rangle d t \text { which is } \omega \rightarrow \sum_{i=1}^{m_{n}} \int_{0}^{T}\left\langle c_{i}^{n}, g(t)\right\rangle \mathscr{X}_{E_{i}^{n}}(t, \omega) d t
$$

and this is $\mathscr{F}$ measurable since $E_{i}^{n}$ is product measurable. Thus, it is measurable into $\mathscr{V}^{\prime}$ as desired and

$$
\langle f(\cdot, \omega), g\rangle=\lim _{n \rightarrow \infty}\left\langle f_{n}(\cdot, \omega), g\right\rangle, \omega \rightarrow\left\langle f_{n}(\cdot, \omega), g\right\rangle \text { is } \mathscr{F} \text { measurable. }
$$

By the Pettis theorem, $\omega \rightarrow\langle f(\cdot, \omega), g\rangle$ is measurable into $\mathscr{V}^{\prime}$. Obviously, the conclusion is the same for these two conditions if $\mathscr{V}^{\prime}$ is replaced with $\mathscr{V}$.

The following theorem is also useful. It is really a generalization of the familiar Gram Schmidt process. It is Lemma 34.4.2.

Theorem 77.2.19 Suppose $V, W$ are separable Banach spaces, such that $V$ is dense in $W$ and $B \in \mathscr{L}\left(W, W^{\prime}\right)$ satisfies

$$
\langle B x, x\rangle \geq 0,\langle B x, y\rangle=\langle B y, x\rangle, B \neq 0
$$

Then there exists a countable set $\left\{e_{i}\right\}$ of vectors in $V$ such that

$$
\left\langle B e_{i}, e_{j}\right\rangle=\delta_{i j}
$$

and for each $x \in W$,

$$
\langle B x, x\rangle=\sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}
$$

and also

$$
B x=\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i},
$$

the series converging in $W^{\prime}$. In case $B=B(\omega)$ where $\omega \rightarrow B(\omega)$ is measurable into $\mathscr{L}\left(W, W^{\prime}\right)$, these vectors $e_{i}$ will also depend on $\omega$ and will be measurable functions of $\omega$.

### 77.3 Preliminary Results

We use the following well known theorem [91]. It is Theorem 34.7.6.
Theorem 77.3.1 Let $E \subseteq F \subseteq G$ where the injection map is continuous from $F$ to $G$ and compact from $E$ to $F$. Let $p \geq 1$, let $q>1$, and define

$$
\begin{gathered}
S \equiv\left\{u \in L^{p}([a, b], E): \text { for some } C,\|u(t)-u(s)\|_{G} \leq C|t-s|^{1 / q}\right. \\
\text { and } \left.\|u\|_{L^{p}([a, b], E)} \leq R\right\} .
\end{gathered}
$$

Thus $S$ is bounded in $L^{p}([a, b], E)$ and Holder continuous into $G$. Then $S$ is precompact in $L^{p}([a, b], F)$. This means that if $\left\{u_{n}\right\}_{n=1}^{\infty} \subseteq S$, it has a subsequence $\left\{u_{n_{k}}\right\}$ which converges in $L^{p}([a, b], F)$.

We recall the following theorem which is proved in [99] and earlier, Theorem 25.5.2 for what will suffice here.

Theorem 77.3.2 If $A$ and $B$ are pseudo monotone and bounded then $A+B$ is also pseudo monotone and bounded.

Also the following result, found in [91] is well known.
Theorem 77.3.3 If a single valued map, $A: X \rightarrow X^{\prime}$ is monotone, hemicontinuous, and bounded, then $A$ is pseudo monotone. Furthermore, the duality map, $J^{-1}: X \rightarrow X^{\prime}$ which satisfies $\left\langle J^{-1} f, f\right\rangle=\|f\|^{2},\left\|J^{-1} f\right\|_{X}=\|f\|_{X}$ is strictly monotone hemicontinuous and bounded. So is the duality map $F: X \rightarrow X^{\prime}$ which satisfies $\|F f\|_{X^{\prime}}=\|f\|_{X}^{p-1},\langle F f, f\rangle=$ $\|f\|_{X}^{p}$ for $p>1$.

The following fundamental result will be of use in what follows. There is somewhat more in this than will be needed. In this paper, $B$ is a possibly degenerate operator satisfying only the following:

$$
\begin{equation*}
B \in \mathscr{L}\left(W, W^{\prime}\right),\langle B u, u\rangle \geq 0,\langle B u, v\rangle=\langle B v, u\rangle \tag{77.3.6}
\end{equation*}
$$

where here $V \subseteq W$ and $V$ is dense in $W$. In the case where $B=B(\omega)$, we will assume for the sake of simplicity that

$$
B(\omega)=k(\omega) B, k(\omega) \geq 0, k \text { being } \mathscr{F} \text { measurable }
$$

Allowing $B$ to depend on $\omega$ introduces some technical considerations so if there is no interest in this, simply assume $B$ is independent of $\omega$. This includes all cases of most interest.

Lemma 77.3.4 Suppose $V, W$ are separable Banach spaces such that $V$ is dense in $W$ and $B \in \mathscr{L}\left(W, W^{\prime}\right)$ satisfies

$$
\langle B x, x\rangle \geq 0,\langle B x, y\rangle=\langle B y, x\rangle, B \neq 0 .
$$

Then there exists a countable set $\left\{e_{i}\right\}$ of vectors in $V$ such that

$$
\left\langle B e_{i}, e_{j}\right\rangle=\delta_{i j}
$$

and for each $x \in W$,

$$
\langle B x, x\rangle=\sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}
$$

and also

$$
B x=\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i}
$$

the series converging in $W^{\prime}$. If $B=B(\omega)$ and $B$ is $\mathscr{F}$ measurable into $\mathscr{L}\left(W, W^{\prime}\right)$ and if the $e_{i}=e_{i}(\omega)$ are as described above, then these $e_{i}$ are measurable into $V$. If $t \rightarrow B(t, \omega)$ is $C^{1}\left([0, T], \mathscr{L}\left(W, W^{\prime}\right)\right)$ and iffor each $w \in W$,

$$
\left\langle B^{\prime}(t, \omega) w, w\right\rangle \leq k_{w, \omega}(t)\langle B(t, \omega) w, w\rangle
$$

Where $k_{w, \omega} \in L^{1}([0, T])$, then the vectors $e_{i}(t)$ can be chosen to also be right continuous functions of $t$.

The following has to do with the values of $B u$ and gives an integration by parts formula.
Corollary 77.3.5 Let $V \subseteq W, W^{\prime} \subseteq V^{\prime}$ be separable Banach spaces, and $B \in \mathscr{L}\left(W, W^{\prime}\right)$ is nonnegative and self adjoint. Also suppose $t \rightarrow B(u(t))$ has a weak derivative $(B u)^{\prime} \in$ $L^{p^{\prime}}\left(0, T, V^{\prime}\right)$ for $u \in L^{p}(0, T, V)$. Then there is a continuous function denoted as $t \rightarrow B u(t)$ which equals $B(u(t))$ a.e. $t$. Say for $t \notin N$. Suppose $B u(0)=B u_{0}, u_{0} \in W$. Then

$$
\begin{equation*}
B u(t)=B u_{0}+\int_{0}^{t}(B u)^{\prime}(s) d s \text { in } V^{\prime} \tag{77.3.7}
\end{equation*}
$$

Then $t \rightarrow B u(t)$ is in $C\left(N^{C}, W^{\prime}\right)$ and also for such $t$,

$$
\frac{1}{2}\langle B u(t), u(t)\rangle=\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle+\int_{0}^{t}\left\langle(B u)^{\prime}(s), u(s)\right\rangle d s
$$

There exists a continuous function $t \rightarrow\langle B u, u\rangle(t)$ which equals the right side of the above for all $t$ and equals $\langle B(u(t)), u(t)\rangle$ off $N$. This also satisfies

$$
\sup _{t \in[0, T]}\langle B u, u\rangle(t) \leq C\left(\left\|(B u)^{\prime}\right\|_{L^{p^{\prime}}\left(0, T, V^{\prime}\right)},\|u\|_{L^{p}(0, T, V)}\right)
$$

This also makes it easy to verify continuity of pointwise evaluation of $B u$. Let $L u=$ $(B u)^{\prime}$.

$$
\begin{gather*}
u \in D(L) \equiv X \equiv\left\{u \in L^{p}(0, T, V): L u \equiv(B u)^{\prime} \in L^{p^{\prime}}\left(0, T, V^{\prime}\right)\right\} \\
\|u\|_{X} \equiv \max \left(\|u\|_{L^{p}(0, T, V)},\|L u\|_{L^{p^{\prime}}\left(0, T, V^{\prime}\right)}\right) \tag{77.3.8}
\end{gather*}
$$

Since $L$ is closed, this $X$ is a Banach space.
Then the following theorem is obtained.
Theorem 77.3.6 $\operatorname{Say}(B u)^{\prime} \in L^{p^{\prime}}\left(0, T, V^{\prime}\right)$ so

$$
B u(t)=B u(0)+\int_{0}^{t}(B u)^{\prime}(s) d s \text { in } V^{\prime}
$$

the map $u \rightarrow B u(t)$ is continuous as a map from $X$ to $V^{\prime}$. Also, if $Y$ denotes those $f \in$ $L^{p}([0, T], V)$ for which $f^{\prime} \in L^{p}([0, T], V)$, so that $f$ has a representative such that $f(t)=$ $f(0)+\int_{0}^{t} f^{\prime}(s) d s$, then if $\|f\|_{Y} \equiv\|f\|_{L^{p}([0, T], V)}+\left\|f^{\prime}\right\|_{L^{p}([0, T], V)}$, the map $f \rightarrow f(t)$ is continuous.

Also one can obtain the following for $p>1$.
Proposition 77.3.7 Let

$$
X=\left\{u \in L^{p}(0, T, V) \equiv \mathscr{V}: L u \equiv(B u)^{\prime} \in L^{p^{\prime}}\left(0, T, V^{\prime}\right)\right\}
$$

where $V$ is a reflexive Banach space. Let a norm on $X$ be given by

$$
\|u\|_{X} \equiv \max \left(\|u\|_{\mathscr{V}},\|L u\|_{\mathscr{V}^{\prime}}\right)
$$

Then there is a continuous function $t \rightarrow\langle B u, v\rangle(t)$ such that

$$
\langle B u, v\rangle(t)=\langle B(u(t)), v(t)\rangle
$$

a.e. $t$ such that

$$
\sup _{t \in[0, T]}|\langle B u, v\rangle(t)| \leq C\|u\|_{X}\|v\|_{X}
$$

and if $K: X \rightarrow X^{\prime}$

$$
\langle K u, v\rangle \equiv \int_{0}^{T}\langle L u, v\rangle d s+\langle B u, v\rangle(0)
$$

Then $K$ is continuous and linear and

$$
\langle K u, u\rangle=\frac{1}{2}[\langle B u, u\rangle(T)+\langle B u, u\rangle(0)]
$$

If $u \in X$ and $B u(0)=0$ then there exists a sequence $\left\{u_{n}\right\}$ such that $\left\|u_{n}-u\right\|_{X} \rightarrow 0$ but $u_{n}(t)=0$ for all $t$ close to 0 .

### 77.4 Measurable Approximate Solutions

The main result in this section is the following theorem. Its proof follows a method due to Brezis and Lions [91] adapted to the case considered here where the operator is set valued. In this theorem, we let $F: V \rightarrow V^{\prime}$ be the duality map $\langle F u, u\rangle=\|u\|^{p},\|F u\|=\|u\|^{p-1}$ for $p>1$.

As above, $L u=(B u)^{\prime}$. In addition to this, define $\Lambda$ to be the restriction of $L$ to those $u \in X$ which have $B u(0)=0$. Thus

$$
D(\Lambda)=\{u \in X: B u(0)=0\}
$$

Then one can show that $\Lambda^{*}$ is monotone. It is not hard to see that this should be the case. Let $v \in D\left(\Lambda^{*}\right)$ and suppose it is smooth. Then

$$
\int_{0}^{T}\langle\Lambda u, v\rangle d t=\langle B u(T), v(T)\rangle-\int_{0}^{T}\left\langle B u, v^{\prime}\right\rangle d t
$$

and so, if $\left|\int_{0}^{T}\langle\Lambda u, v\rangle d t\right| \leq C\|u\|_{\mathscr{V}}$, then we need to have $v(T)=0$ and $\Lambda^{*} v=-B v^{\prime}$. Now it is just a matter of doing the computations to verify that

$$
\left\langle\Lambda^{*} v, v\right\rangle \geq 0
$$

Lemma 77.4.1 Let $K$ and L be as in Proposition 77.3.7. Then for each $f \in \mathscr{V}^{\prime}$ and $u_{0} \in W$, there exists a unique $u \in X$ such that

$$
\begin{equation*}
\langle K u, v\rangle+F u=\langle f, v\rangle+\left\langle B v(0), u_{0}\right\rangle \tag{77.4.9}
\end{equation*}
$$

for all $v \in X$. Also, the mapping which takes $\left(f, u_{0}\right)$ to this solution is demicontinuous in the sense that if $f_{n} \rightarrow f$ strongly in $\mathscr{V}^{\prime}$ and $u_{0 n} \rightarrow u_{0}$ in $W$, then $u_{n} \rightarrow u$ weakly in $\mathscr{V}$.

Proof: Let $J^{-1}$ be the duality map mentioned above and define $H_{\varepsilon}: X \rightarrow X^{\prime}$ by

$$
\left\langle H_{\varepsilon}(u), v\right\rangle=\varepsilon\left\langle L v, J^{-1} L u\right\rangle+\langle F u, v\rangle+\langle K u, v\rangle
$$

for all $v \in X$. Then $H_{\varepsilon}$ is pseudo monotone because it is monotone, bounded, and hemicontinuous. This follows from Theorem 77.3.2, and 77.3.3. It is also easy to see that $H_{\varepsilon}$ is coercive.

$$
\frac{\left\langle H_{\varepsilon}(u), u\right\rangle}{\|u\|_{X}}=\varepsilon \frac{\|L u\|_{\mathscr{V}^{\prime}}^{2}}{\|u\|_{X}}+\frac{\|u\|_{\mathscr{V}}^{p}}{\|u\|_{X}}+\frac{1}{2}[\langle B u(T), u(T)\rangle+\langle B u, u\rangle(0)] \frac{1}{\|u\|_{X}}
$$

If not, then there is $\left\|u_{n}\right\|_{X} \rightarrow \infty$ but for some $M$,

$$
\varepsilon \frac{\left\|L u_{n}\right\|_{\mathscr{V}^{\prime}}^{2}}{\left\|u_{n}\right\|_{X}}+\frac{\left\|u_{n}\right\|_{\mathscr{V}}^{p}}{\left\|u_{n}\right\|_{X}}+\frac{1}{2}\left[\left\langle B u_{n}(T), u_{n}(T)\right\rangle+\left\langle B u_{n}, u_{n}\right\rangle(0)\right] \frac{1}{\left\|u_{n}\right\|_{X}} \leq M
$$

Then one of $\left\|u_{n}\right\|_{\mathscr{V}}$ or $\left\|L u_{n}\right\|_{\mathscr{V}^{\prime}}$ is unbounded. Either way, a contradiction is obtained. Thus $H_{\varepsilon}$ is coercive bounded, and pseudomonotone. It follows that it maps onto $X^{\prime}$.

There exists $u_{\varepsilon} \in X$ such that for all $v \in X$,

$$
\begin{equation*}
\varepsilon\left\langle L v, J^{-1} L u_{\varepsilon}\right\rangle+\left\langle F u_{\varepsilon}, v\right\rangle+\left\langle K u_{\varepsilon}, v\right\rangle=\langle f, v\rangle+\left\langle B v(0), u_{0}\right\rangle . \tag{77.4.10}
\end{equation*}
$$

In 77.4.10, let $v=u_{\varepsilon}$. Using the inequality,

$$
\begin{aligned}
\left|\left\langle B v(0), u_{0}\right\rangle\right| & \leq\langle B v, v\rangle^{1 / 2}(0)\left\langle B u_{0}, u_{0}\right\rangle^{1 / 2} \\
& \leq \frac{1}{2}\langle B v, v\rangle(0)+\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle
\end{aligned}
$$

it follows that

$$
\begin{aligned}
& \left\langle F u_{\mathcal{E}}, u_{\varepsilon}\right\rangle+\frac{1}{2}\left[\left\langle B u_{\mathcal{E}}, u_{\varepsilon}\right\rangle(T)+\left\langle B u_{\varepsilon}, u_{\varepsilon}\right\rangle(0)\right] \\
\leq & \|f\|_{\mathscr{V}^{\prime}}\left\|u_{\varepsilon}\right\|_{\mathscr{V}}+\frac{1}{2}\left\langle B u_{\varepsilon}, u_{\varepsilon}\right\rangle(0)+\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle
\end{aligned}
$$

Thus

$$
\left\|u_{\mathcal{E}}\right\|_{\mathscr{V}}^{p}+\frac{1}{2}\left\langle B u_{\mathcal{E}}, u_{\varepsilon}\right\rangle(T) \leq \frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle+\|f\|_{\mathscr{V}}\left\|u_{\mathcal{E}}\right\|_{\mathscr{V}}
$$

which implies that there exists a constant $C$ independent of $\varepsilon$ such that

$$
\begin{equation*}
\left\|u_{\mathcal{E}}\right\|_{\mathscr{V}} \leq C . \tag{77.4.11}
\end{equation*}
$$

Now let $v \in D(\Lambda)$. Thus $v \in X$ and $B v(0)=0$ so the last term of 77.4.10 equals 0 . The term, $\left\langle B u_{\varepsilon}, v\right\rangle(0)$ found in the definition of $\left\langle K u_{\varepsilon}, v\right\rangle$ also equals 0 . This follows from

$$
\left\langle B u_{\varepsilon}, v\right\rangle(0)=\lim _{n \rightarrow \infty}\left\langle B u_{\varepsilon}, v_{n}\right\rangle(0)=0 .
$$

where $v_{n}=0$ near 0 and converges to $v$ in $X$ by Proposition 77.3.7. Therefore, for $v \in D(\Lambda)$, a dense subset of $\mathscr{V}$,

$$
\varepsilon\left\langle\Lambda v, J^{-1} L u_{\varepsilon}\right\rangle+\left\langle F u_{\varepsilon}, v\right\rangle+\left\langle L u_{\mathcal{\varepsilon}}, v\right\rangle=\langle f, v\rangle .
$$

It follows that $J^{-1} L u_{\mathcal{\varepsilon}} \in D\left(\Lambda^{*}\right)$ and so for all $v \in D(\Lambda)$,

$$
\begin{equation*}
\varepsilon\left\langle\Lambda^{*} J^{-1} L u_{\varepsilon}, v\right\rangle+\left\langle F u_{\varepsilon}, v\right\rangle+\left\langle L u_{\varepsilon}, v\right\rangle=\langle f, v\rangle . \tag{77.4.12}
\end{equation*}
$$

Since $D(\Lambda)$ is dense in $\mathscr{V}$, this equation holds for all $v \in \mathscr{V}$ and so in particular, it holds for $v=J^{-1} L u_{\varepsilon}$. Therefore,

$$
\begin{equation*}
-\left\|F u_{\mathcal{E}}\right\|_{\mathscr{V}^{\prime}}\left\|L u_{\mathcal{E}}\right\|_{\mathscr{V}^{\prime}}+\left\|L u_{\mathcal{E}}\right\|_{\mathscr{V}^{\prime}}^{2} \leq\|f\|_{\mathscr{V}^{\prime}}\left\|L u_{\mathcal{E}}\right\|_{\mathscr{V}^{\prime}} \tag{77.4.13}
\end{equation*}
$$

It follows from 77.4.13, 77.4.11 that $\left\|L u_{\mathcal{E}}\right\|_{\mathscr{V}^{\prime}}$ is bounded independent of $\varepsilon$. Therefore, there exists a sequence $\varepsilon \rightarrow 0$ such that

$$
\begin{gather*}
u_{\varepsilon} \rightharpoonup u \text { in } \mathscr{V},  \tag{77.4.14}\\
K u_{\varepsilon} \rightharpoonup K u \text { in } X^{\prime},  \tag{77.4.15}\\
F u_{\varepsilon} \rightharpoonup u^{*} \text { in } \mathscr{V}^{\prime},  \tag{77.4.16}\\
B u_{\varepsilon}(0) \rightharpoonup B u(0) \text { in } W^{\prime} . \tag{77.4.17}
\end{gather*}
$$

In 77.4.10 replace $v$ with $u_{\varepsilon}-u$. Using $J^{-1}$ is monotone,

$$
\begin{gather*}
\varepsilon\left\langle L u_{\varepsilon}-L u, J^{-1} L u\right\rangle+\left\langle F u_{\varepsilon}+K u_{\varepsilon}, u_{\varepsilon}-u\right\rangle \\
\leq\left\langle f, u_{\varepsilon}-u\right\rangle+\left\langle B\left(u_{\varepsilon}-u\right)(0), u_{0}\right\rangle \tag{77.4.18}
\end{gather*}
$$

Formula 77.4.17 applied to the last term of 77.4.18 implies

$$
\begin{equation*}
\lim \sup _{\varepsilon \rightarrow 0}\left\langle F u_{\varepsilon}+K u_{\varepsilon}, u_{\varepsilon}-u\right\rangle \leq 0 \tag{77.4.19}
\end{equation*}
$$

By pseudomonotonicity,

$$
\lim _{\varepsilon \rightarrow 0} \inf _{\varepsilon \rightarrow 0}\left\langle F u_{\varepsilon}+K u_{\varepsilon}, u_{\varepsilon}-u\right\rangle \geq\langle F u+K u, u-u\rangle=0
$$

so $\lim _{\varepsilon \rightarrow 0}\left\langle F u_{\varepsilon}+K u_{\varepsilon}, u_{\varepsilon}-u\right\rangle=0$ and so

$$
\begin{gathered}
\left\langle u^{*}+K u, u-v\right\rangle \\
\lim _{\varepsilon \rightarrow 0} \inf _{\varepsilon \rightarrow 0}\left(\left\langle F u_{\varepsilon}+K u_{\varepsilon}, u_{\varepsilon}-u\right\rangle+\left\langle F u_{\varepsilon}+K u_{\varepsilon}, u-v\right\rangle\right)= \\
\lim _{\varepsilon \rightarrow 0} \inf _{\varepsilon \rightarrow 0}\left\langle F u_{\varepsilon}+K u_{\varepsilon}, u_{\varepsilon}-v\right\rangle \geq\langle F u+K u, u-v\rangle
\end{gathered}
$$

and so $u^{*}=F u$ and from 77.4.10,

$$
\begin{equation*}
\langle K u, v\rangle+\langle F u, v\rangle=\langle f, v\rangle+\left\langle B v(0), u_{0}\right\rangle \tag{77.4.20}
\end{equation*}
$$

Thus for every $v \in X$,

$$
\int_{0}^{T}\left\langle(B u)^{\prime}, v\right\rangle d s+\langle B u, v\rangle(0)+\int_{0}^{T}\langle F u, v\rangle d s=\int_{0}^{T}\langle f, v\rangle d s+\left\langle B v(0), u_{0}\right\rangle
$$

So let $v$ be smooth and equal to 0 except for $t \in[0, \delta]$ and equals $v_{0}$ at 0 . Then as $\delta \rightarrow 0$, the integrals become increasingly small and so

$$
\left\langle B u(0), v_{0}\right\rangle=\left\langle B v_{0}, u_{0}\right\rangle=\left\langle B u_{0}, v_{0}\right\rangle
$$

and since $v_{0}$ is arbitrary in $V$, then it follows that $B u(0)=B u_{0}$. Thus this has provided a solution $u$ to the system

$$
(B u)^{\prime}+F u=f, B u(0)=B u_{0}, u \in X
$$

It remains to consider the assertion about continuity. First note that the solution to the above initial value problem is unique due to the strict monotonicity of $F$. In fact, if there are two solutions, $u, w$, then

$$
\frac{1}{2}\|B u(t)-B w(t)\|_{W}^{2}+\int_{0}^{t}\langle F u-F w, u-w\rangle d s=0
$$

and so, in particular, $\langle F u-F w, u-w\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}=0$ which implies $u=w$ in $\mathscr{V}$.
Let $u$ be the solution which goes with $\left(f, u_{0}\right)$ and let $u_{n}$ denote the solution which goes with $\left(f_{n}, u_{0 n}\right)$ where it is assumed that $f_{n} \rightarrow f$ in $\mathscr{V}^{\prime}$ and $u_{0 n} \rightarrow u_{0}$ in $W$. It is desired to show that $u_{n} \rightarrow u$ weakly in $\mathscr{V}$. First note that the $u_{n}$ are bounded in $\mathscr{V}$ because

$$
\frac{1}{2}\left\langle B u_{n}, u_{n}\right\rangle(T)-\frac{1}{2}\left\langle B u_{0 n}, u_{0 n}\right\rangle+\int_{0}^{T}\left\|u_{n}\right\|_{V}^{p} d s=\int_{0}^{T}\left\langle f_{n}, u_{n}\right\rangle d s \leq\left\|f_{n}\right\|_{\mathscr{V}^{\prime}}\left\|u_{n}\right\|_{\mathscr{V}}
$$

and this clearly implies that $\left\|u_{n}\right\|_{\mathscr{V}}$ is indeed bounded. Thus if this sequence fails to converge weakly to $u$, it must be the case that there is a subsequence, still denoted as $u_{n}$ which converges weakly to $w \neq u$ in $\mathscr{V}$. Then by the fact that $F$ is bounded, there is an estimate of the form

$$
\left\|u_{n}\right\|_{\mathscr{V}}+\left\|L u_{n}\right\|_{\mathscr{V}^{\prime}} \leq C
$$

Thus, a further subsequence satisfies

$$
\begin{aligned}
u_{n} & \rightarrow w \text { weakly in } \mathscr{V} \\
L u_{n} & \rightarrow L w \text { weakly in } \mathscr{V}^{\prime} \\
F u_{n} & \rightarrow \xi \text { weakly in } \mathscr{V}^{\prime}
\end{aligned}
$$

then

$$
\begin{aligned}
& \int_{0}^{T}\left\langle\left(B\left(u_{n}-w\right)\right)^{\prime}, u_{n}-w\right\rangle d t \\
= & \frac{1}{2}\left\langle B\left(u_{n}-w\right),\left(u_{n}-w\right)\right\rangle(T)-\frac{1}{2}\left\langle B\left(u_{n}-w\right),\left(u_{n}-w\right)\right\rangle(0) \\
\geq & -\frac{1}{2}\left\langle B\left(u_{n}-w\right),\left(u_{n}-w\right)\right\rangle(0)=-\frac{1}{2}\left\langle B\left(u_{n 0}-u_{0}\right), u_{n 0}-u_{0}\right\rangle
\end{aligned}
$$

It follows

$$
\left\langle L w, u_{n}-w\right\rangle_{\mathscr{V}}+\left\langle F u_{n}, u_{n}-w\right\rangle_{\mathscr{V}}-\frac{1}{2}\left\langle B\left(u_{n 0}-u_{0}\right), u_{n 0}-u_{0}\right\rangle \leq\left\langle f_{n}, u_{n}-w\right\rangle
$$

and so $\lim \sup _{n \rightarrow \infty}\left\langle F u_{n}, u_{n}-w\right\rangle \leq 0$. Then as before, $\xi=F w$ and one obtains

$$
(B w)^{\prime}+F w=f, \quad B w(0)=B u_{0}, w \in X
$$

contradicting uniqueness. Hence $u_{n} \rightarrow u$ weakly as claimed.
Now suppose $(\Omega, \mathscr{F})$ is a measurable space and $B=B(\omega)$ and is measurable into $\mathscr{L}\left(W, W^{\prime}\right)$ and $f:[0, T] \times \Omega \rightarrow V^{\prime}$ is product measurable, $\mathscr{B}([0, T]) \times \mathscr{F}$ measurable where $\mathscr{B}([0, T])$ denotes the Borel sets. Also, it is assumed that for each $\omega, f(\cdot, \omega) \in \mathscr{V}^{\prime}$. The following lemma ties together these ideas. It is Lemma 77.2.18 proved above. It is stated here for convenience.

Lemma 77.4.2 Let $f(\cdot, \omega) \in \mathscr{V}^{\prime}$. Then if $\omega \rightarrow f(\cdot, \omega)$ is measurable into $\mathscr{V}^{\prime}$, it follows that for each $\omega$, there exists a representative $\hat{f}(\cdot, \omega) \in \mathscr{V}^{\prime}, \hat{f}(\cdot, \omega)=f(\cdot, \omega)$ in $\mathscr{V}^{\prime}$ such that $(t, \omega) \rightarrow \hat{f}(t, \omega)$ is product measurable. If $f(\cdot, \omega) \in \mathscr{V}^{\prime}$ and $(t, \omega) \rightarrow f(t, \omega)$ is product measurable, then $\omega \rightarrow f(\cdot, \omega)$ is measurable into $\mathscr{V}^{\prime}$. The same holds replacing $\mathscr{V}^{\prime}$ with $\mathscr{V}$.

Now consider the initial value problem

$$
\begin{align*}
(B(\omega) u(\cdot, \omega))^{\prime}+F u(\cdot, \omega) & =f(\cdot, \omega), \\
B(\omega) u(0, \omega) & =B(\omega) u_{0}(\omega), u(\cdot, \omega) \in X \tag{77.4.21}
\end{align*}
$$

where we also assume $u_{0}$ is $\mathscr{F}$ measurable into $W$. From Lemma 77.4.2,

$$
\omega \rightarrow\left(f(\cdot, \omega), u_{0}(\omega)\right)
$$

is measurable into $\mathscr{V}^{\prime} \times W$. That is,

$$
\left(f, u_{0}\right)^{-1}(U)=\left\{\omega:\left(f(\cdot, \omega), u_{0}(\omega)\right) \in U\right\} \in \mathscr{F}
$$

for $U$ an open set in $\mathscr{V}^{\prime} \times W$. From Lemma 77.4.1, the map $\Phi_{\omega}$ which takes $\left(f, u_{0}\right)$ to the solution $u$ is demicontinuous. We desire to argue that $u$ is measurable into $\mathscr{V}$. In doing so, it is easiest to assume that $B$ does not depend on $\omega$. However, the dependence on $\omega$ can be included using the approximation assumption for $B(\omega)$ mentioned earlier.

Letting $f_{n}(\cdot, \omega) \rightarrow f(\cdot, \omega)$ where $f_{n}$ is a simple function and $u_{0 n}(\omega) \rightarrow u_{0}(\omega)$ where $u_{0 n}$ is also a simple function, it follows that

$$
\Phi_{\omega}\left(f_{n}(\cdot, \omega), u_{0 n}(\omega)\right) \rightarrow \Phi_{\omega}\left(f(\cdot, \omega), u_{0}(\omega)\right)=u
$$

weakly. Here $\Phi_{\omega}$ would be a continuous function of $\mathscr{V}^{\prime} \times W$.
Lemma 77.4.3 Suppose $f(\cdot, \omega) \in \mathscr{V}^{\prime}$ for each $\omega$ and that $(t, \omega) \rightarrow f(t, \omega)$ is product measurable into $V^{\prime}$. Also $u_{0}$ is $\mathscr{F}$ measurable into $W$ and

$$
B(\omega)=k(\omega) B, k(\omega) \geq 0, k \text { measurable }
$$

Then for each $\omega \in \Omega$, there exists a unique solution $u(\cdot, \omega)$ in $\mathscr{V}$ satisfying

$$
\begin{aligned}
(B(\omega) u(\cdot, \omega))^{\prime}+F u(\cdot, \omega) & =f(\cdot, \omega), \\
B(\omega) u(0, \omega) & =B(\omega) u_{0}(\omega), u(\cdot, \omega) \in X
\end{aligned}
$$

This solution has a representative which satisfies $(t, \omega) \rightarrow u(t, \omega)$ is product measurable into $V$.

Proof: Let $B_{n}(\omega) \equiv k_{n}(\omega) B$ where $\left\{k_{n}(\omega)\right\}$ is an increasing sequence of simple functions converging pointwise to $k(\omega)$. Replace $B(\omega)$ with $B_{n}(\omega)$. Then define

$$
\left\langle K_{n} u, v\right\rangle \equiv \int_{0}^{T}\left\langle L_{n} u, v\right\rangle d s+\langle B u, v\rangle(0)
$$

where $L_{n}$ is defined as

$$
L_{n} u=\left(B_{n}(\omega) u\right)^{\prime}
$$

for $B_{n}$ having values in $\mathscr{L}\left(W, W^{\prime}\right)$ such that $B_{n}(\omega) \rightarrow B(\omega)$ and each of these is self adjoint and nonnegative. Let $u_{n}$ be the solution to the above initial value problem

$$
\left\langle K_{n} u_{n}, v\right\rangle+F u_{n}=\left\langle f_{n}, v\right\rangle+\left\langle B v(0), u_{0 n}\right\rangle
$$

in which $u_{0 n}$ and $f_{n}$ are simple functions converging to $u_{0}$ and $f$ in $W$ and $\mathscr{V}^{\prime}$ respectively for each $\omega$. Thus these have constant values in $\mathscr{V}^{\prime}$ or $W$ on finitely many measurable subsets of $\Omega$. Since $B_{n}$ is constant on measurable sets, it follows that $u_{n}(\cdot, \omega)$ is also a constant element of $\mathscr{V}$ on each of finitely many measurable sets. Hence $u_{n}(\cdot, \omega)$ is measurable into $\mathscr{V}$. Then fixing $\omega$, and letting $v=u_{n}$,

$$
\frac{1}{2}\left[\left\langle B u_{n}, u_{n}\right\rangle(T)+\left\langle B u_{n}, u_{n}\right\rangle(0)\right]+\int_{0}^{T}\left\|u_{n}\right\|_{V}^{p} d s=\int_{0}^{T}\left\langle f_{n}, u_{n}\right\rangle d s+\left\langle B v(0), u_{0 n}\right\rangle
$$

Thus, since $F$ is bounded, one obtains an inequality of the form

$$
\left\|u_{n}\right\|_{\mathscr{V}}+\left\|\left(B_{n}(\omega) u_{n}\right)^{\prime}\right\|_{\mathscr{V}^{\prime}} \leq C
$$

Then there is a subsequence such that

$$
\begin{aligned}
& u_{n} \rightarrow u \text { weakly in } \mathscr{V} \\
& B_{n}(\omega) u_{n} \rightarrow B(\omega) u \text { weak } * \text { in } L^{\infty}\left([0, T] ; V^{\prime}\right) \\
& B_{n}(\omega) u_{n} \rightarrow B(\omega) u \text { weakly in } \mathscr{V}^{\prime} \\
& F u_{n} \rightarrow \xi \text { weakly in } \mathscr{V}^{\prime} \\
&\left(B_{n}(\omega) u_{n}\right)^{\prime} \rightarrow(B(\omega) u)^{\prime} \text { weakly in } \mathscr{V}^{\prime}
\end{aligned}
$$

Also, suppressing the dependence on $\omega$,

$$
\left(B_{n} u_{n}\right)(t)=B_{n} u_{0 n}+\int_{0}^{t}\left(B_{n} u\right)^{\prime}(s) d s
$$

and so in fact,

$$
\left(B_{n} u_{n}\right)(t) \rightarrow(B u)(t) \text { in } V^{\prime} \text { for each } t
$$

Also,

$$
\begin{aligned}
\left\langle\left(B_{n} u_{n}\right)^{\prime}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}+\left\langle F u_{n}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} & =\left\langle f_{n}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \\
\left\langle k_{n}(\omega)\left(B u_{n}\right)^{\prime}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}+\left\langle F u_{n}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} & =\left\langle f_{n}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}
\end{aligned}
$$

Thus, by monotonicity,

$$
\begin{gathered}
\left\langle k_{n}(\omega)\left(B u_{n}\right)^{\prime}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}=\left\langle k(\omega)\left(B u_{n}\right)^{\prime}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \\
+\left\langle\left(k_{n}(\omega)-k(\omega)\right)\left(B u_{n}\right)^{\prime}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \\
\geq\left\langle k(\omega)(B u)^{\prime}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}+\left\langle\left(k_{n}(\omega)-k(\omega)\right)\left(B u_{n}\right)^{\prime}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}
\end{gathered}
$$

The last term in the above expression converges to 0 due to the convergence of $k_{n}(\omega)$ to $k(\omega)$. Thus

$$
\left\langle(B(\omega) u)^{\prime}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}+\left\langle F u_{n}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \leq\left\langle f_{n}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}
$$

and so

$$
\limsup _{n \rightarrow \infty}\left\langle F u_{n}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \leq 0
$$

Then as before, one can conclude that $F u=\xi$. Then passing to the limit gives the desired solution to the equation, this for each $\omega$. However, by uniqueness, it follows that if $\bar{u}$ is the solution to the evolution equation of Lemma 77.4.1, then for each $\omega, u=\bar{u}$ in $\mathscr{V}$. Also this $u$ just obtained is measurable into $\mathscr{V}$ thanks to the Pettis theorem. Therefore, $\bar{u}$ can be modified on a set of measure zero for each fixed $\omega$ to equal $u$ a function measurable into $\mathscr{V}$. Hence there exists a solution to the evolution equation of this lemma $u$ which is measurable into $\mathscr{V}$. By the Lemma 77.4.2, it follows that there is a representative for $u$ which is product measurable into $V$.

### 77.5 The Main Result

The main result is an existence theorem for product measurable solutions to the system

$$
\begin{align*}
(B(\omega) u(\cdot, \omega))^{\prime}+u^{*}(\cdot, \omega) & =f(\cdot, \omega) \text { in } \mathscr{V}^{\prime} \\
B(\omega) u(0, \omega) & =B(\omega) u_{0}(\omega) \tag{77.5.22}
\end{align*}
$$

where $u^{*}(\cdot, \omega) \in A(u(\cdot, \omega), \omega)$. It is Theorem 77.5.6 below. First are some assumptions.
Here $I$ will denote a subinterval of $[0, T]$, of the form $I=[0, \hat{T}], \hat{T} \leq T$, and $\mathscr{V}_{I} \equiv$ $L^{p}(I, V)$ with similar things defined analogously. We assume only that $p>1$.

Definition 77.5.1 For $X$ a reflexive Banach space, we say $A: X \rightarrow \mathscr{P}\left(X^{\prime}\right)$ is pseudomonotone and bounded if the following hold.

1. The set $A u$ is nonempty, closed and convex for all $u \in X$. A takes bounded sets to bounded sets.
2. If $u_{i} \rightarrow u$ weakly in $X$ and $u_{i}^{*} \in A u_{i}$ is such that

$$
\begin{equation*}
\lim \sup _{i \rightarrow \infty}\left\langle u_{i}^{*}, u_{i}-u\right\rangle \leq 0 \tag{77.5.23}
\end{equation*}
$$

then, for each $v \in X$, there exists $u^{*}(v) \in A u$ such that

$$
\begin{equation*}
\lim _{i \rightarrow \infty} \inf _{i \rightarrow}\left\langle u_{i}^{*}, u_{i}-v\right\rangle \geq\left\langle u^{*}(v), u-v\right\rangle \tag{77.5.24}
\end{equation*}
$$

Now suppose the following for the operator $A(\cdot, \omega) . A(\cdot, \omega): \mathscr{V}_{I} \rightarrow \mathscr{V}_{I}^{\prime}$ for each $I$ a subinterval of $[0, T]$ and

$$
\begin{equation*}
A(\cdot, \omega): \mathscr{V}_{I} \rightarrow \mathscr{P}\left(\mathscr{V}_{I}^{\prime}\right) \text { is bounded, } \tag{77.5.25}
\end{equation*}
$$

If, for $u \in \mathscr{V}$,

$$
u^{*} \mathscr{X}_{[0, \hat{T}]} \in A\left(u \mathscr{X}_{[0, \hat{T}]}, \omega\right)
$$

for each $\hat{T}$ in an increasing sequence converging to $T$, then

$$
\begin{equation*}
u^{*} \in A(u, \omega) \tag{77.5.26}
\end{equation*}
$$

For some $\hat{p} \geq p$, assume the specific estimate

$$
\begin{equation*}
\sup \left\{\left\|u^{*}\right\|_{\mathscr{V}_{I}^{\prime}}: u^{*} \in A(u, \omega)\right\} \leq a(\omega)+b(\omega)\|u\|_{\mathscr{V}_{I}}^{\hat{p}-1} \tag{77.5.27}
\end{equation*}
$$

where $a(\omega), b(\omega)$ are nonnegative. Note that the growth could be quadratic in case $p=2$. Also assume the coercivity condition:

$$
\begin{equation*}
\left.\left.\lim _{\substack{\|u\|_{\mathscr{V}} \rightarrow \infty \\ u \in X_{r}}} \frac{\inf \left\{2\left\langle u^{*}, u\right\rangle_{\mathscr{V}}, \mathscr{V}\right.}{}+\langle B u, u\rangle(T): u^{*} \in A(u, \omega)\right\}\right) \tag{77.5.28}
\end{equation*}
$$

or alternatively the following specific estimate valid for each $t \leq T$ and for some $\lambda(\omega) \geq 0$,

$$
\begin{equation*}
\inf \left(\int_{0}^{t}\left\langle u^{*}, u\right\rangle+\lambda(\omega)\langle B u, u\rangle d t: u^{*} \in A(u, \omega)\right) \geq \delta(\omega) \int_{0}^{t}\|u\|_{V}^{p} d s-m(\omega) \tag{77.5.29}
\end{equation*}
$$

where $m(\omega)$ is some nonnegative constant, $\delta(\omega)>0$. Note that the estimate is a coercivity condition on $\lambda B+A$ rather than on $A$ but is more specific than 77.5.28.

Let $U$ be a Banach space dense in $V$ and that if $u_{i} \rightharpoonup u$ in $\mathscr{V}_{I}$ and $u_{i}^{*} \in A\left(u_{i}\right)$ with $u_{i}^{*} \rightharpoonup u^{*}$ in $\mathscr{V}_{I}^{\prime}$ and $\left(B u_{i}\right)^{\prime} \rightharpoonup(B u)^{\prime}$ in $\mathscr{U}_{r I}^{\prime}$, $\rightharpoonup$ denoting weak convergence, then if

$$
\limsup _{i \rightarrow \infty}\left\langle u_{i}^{*}, u_{i}-u\right\rangle_{\mathscr{V}_{I}^{\prime}, \mathscr{V}_{I}} \leq 0
$$

it follows that for all $v \in \mathscr{V}_{I}$, there exists $u^{*}(v) \in A u$ such that

$$
\begin{equation*}
\lim _{i \rightarrow \infty}\left\langle u_{i}^{*}, u_{i}-v\right\rangle_{\mathscr{V}_{I}^{\prime}, \mathscr{V}_{I}} \geq\left\langle u^{*}(v), u-v\right\rangle_{\mathscr{V}_{I}^{\prime}, \mathscr{V}_{I}} \tag{77.5.30}
\end{equation*}
$$

where $r>\max (\hat{p}, 2)$, and we replace $p$ with $r$ and $I$ an arbitrary subinterval of the form $[0, \hat{T}], \hat{T}<T$, for $[0, T]$, and $U$ for $V$ where indicated. Here

$$
\mathscr{U}_{r I} \equiv L^{r}(I ; U)
$$

Note that we are not assuming $A$ is pseudomonotone, just that it satisfies a similar limit condition. Typically, this limit condition holds because of a use of the compact embedding of theorem 77.3.1 or similar result and it does not matter whether $U$ is a small subset of $V$ as long as it is dense in $V$.

Here is an alternate limit condition. Let $U$ be a Banach space dense in $V$ and that if $u_{i} \rightharpoonup u$ in $\mathscr{V}_{I}$ and $u_{i}^{*} \in A\left(u_{i}\right)$ with $u_{i}^{*} \rightharpoonup u^{*}$ in $\mathscr{V}_{I}^{\prime}$ and $t \rightarrow B u_{i}(t)$ is continuous and

$$
\begin{equation*}
\sup _{i} \sup _{t \neq s} \frac{\left\|B u_{i}(t)-B u_{i}(s)\right\|_{U^{\prime}}}{|t-s|^{\alpha}} \leq C \tag{77.5.31}
\end{equation*}
$$

then if

$$
\begin{equation*}
\lim \sup _{i \rightarrow \infty}\left\langle u_{i}^{*}, u_{i}-u\right\rangle_{\mathscr{V}_{I}^{\prime}, \mathscr{V}_{I}} \leq 0 \tag{77.5.32}
\end{equation*}
$$

it follows that for all $v \in \mathscr{V}_{I}$, there exists $u^{*}(v) \in A u$ such that

$$
\begin{equation*}
\lim _{i \rightarrow \infty}\left\langle u_{i}^{*}, u_{i}-v\right\rangle_{\mathscr{Y}_{I}^{\prime}, \mathscr{V}_{I}} \geq\left\langle u^{*}(v), u-v\right\rangle_{\mathscr{V}_{I}^{\prime}, \mathscr{V}_{I}} \tag{77.5.33}
\end{equation*}
$$

This alternate condition is implied by 77.5 .30 but the conditions under which either condition holds are likely to depend on some sort of compactness which will be useable for either limit condition. Technically if you assume this alternate condition, you are assuming more, but I don't have any examples to show that it would be actually assuming more.

For $\omega \rightarrow u(\cdot, \omega)$ measurable into $\mathscr{V}$,

$$
\begin{equation*}
\omega \rightarrow A(u(\cdot, \omega), \omega) \text { has a measurable selection into } \mathscr{V}^{\prime} \tag{77.5.34}
\end{equation*}
$$

This last condition means there is a function $\omega \rightarrow u^{*}(\omega)$ which is measurable into $\mathscr{V}^{\prime}$ such that $u^{*}(\omega) \in A(u(\cdot, \omega), \omega)$. This is assured to take place if the following standard measurability condition is satisfied for all $O$ open in $\mathscr{V}^{\prime}$ :

$$
\begin{equation*}
\{\omega: A(u(\cdot, \omega), \omega) \cap O \neq \emptyset\} \in \mathscr{F} \tag{77.5.35}
\end{equation*}
$$

See for example [70], [10]. Our assumption is implied by this one but they are not equivalent. Thus what is considered here generalizes an assumption that $\omega \rightarrow A(u(\cdot, \omega), \omega)$ is set valued measurable.

Note that this condition would hold if $u \rightarrow A(t, u, \omega)$ is bounded and pseudomonotone as a single valued map from $V$ to $V^{\prime}$ and $(t, \omega) \rightarrow A(t, u, \omega)$ is product measurable into $V^{\prime}$. One would use the demicontinuity of $u \rightarrow A(\cdot, u, \omega)$ which comes from the pseudo monotone and bounded assumption and consider a sequence of simple functions $u_{n}(t, \omega) \rightarrow u(t, \omega)$ in $V$ for $u$ measurable, each $u(\cdot, \omega)$ being in $\mathscr{V}$, Then the measurability of $A\left(t, u_{n}, \omega\right)$ would attach to $A(t, u, \omega)$ in the limit. More generally, here is a useful lemma. It is about preserving the existence of a measurable representative under the assumption that the values are closed and convex.

Lemma 77.5.2 Suppose $\omega \rightarrow A(u, \omega)$ has a measurable selection in $\mathscr{V}^{\prime}$ for u a given element of $\mathscr{V}$ not dependent on $\omega$ and for each $\omega, A(u, \omega)$ is a closed bounded, convex set in $\mathscr{V}^{\prime}$. Also suppose $u \rightarrow A(u, \omega)$ is upper semicontinuous from the strong topology of $\mathscr{V}$ to the weak topology of $\mathscr{V}^{\prime}$. That is, if $u_{n} \rightarrow u$ in $\mathscr{V}$ strongly, then if $O$ is a weakly open set containing $A(u, \omega)$, it follows that $A\left(u_{n}, \omega\right) \in O$ for all n large enough. Then whenever $u$ is measurable into $\mathscr{V}$, it follows that there is a measurable selection for $\omega \rightarrow A(u(\omega), \omega)$ into $\mathscr{V}^{\prime}$.

Proof: Let $\omega \rightarrow u(\omega)$ be measurable into $\mathscr{V}$ and let $u_{n}(\omega) \rightarrow u(\omega)$ in $\mathscr{V}$ where $u_{n}$ is a simple function

$$
u_{n}(\omega)=\sum_{k=1}^{m_{n}} c_{k}^{n} \mathscr{X}_{E_{k}^{n}}(\omega), \text { the } E_{k}^{n} \text { disjoint, } \Omega=\cup_{k} E_{k}^{n}
$$

each $c_{k}^{n}$ being in $\mathscr{V}$. Then by assumption, there is a measurable selection for $\omega \rightarrow A\left(c_{k}^{n}, \omega\right)$ denoted as $\omega \rightarrow y_{k}^{n}(\omega)$. Thus $\omega \rightarrow y_{k}^{n}(\omega)$ is measurable into $\mathscr{V}^{\prime}$ and $y_{k}^{n}(\omega) \in A\left(c_{k}^{n}, \omega\right)$ for all $\omega \in \Omega$. Then consider

$$
y^{n}(\omega)=\sum_{k=1}^{m_{n}} y_{k}^{n}(\omega) \mathscr{X}_{E_{k}^{n}}(\omega)
$$

It is measurable and for $\omega \in E_{k}^{n}$ it equals $y_{k}^{n}(\omega) \in A\left(c_{k}^{n}, \omega\right)=A\left(u_{n}(\omega), \omega\right)$. Thus $y^{n}$ is a measurable selection of $\omega \rightarrow A\left(u_{n}(\omega), \omega\right)$. By the estimates, for each $\omega$ these $y^{n}(\omega)$ lie in a bounded subset of $\mathscr{V}^{\prime}$. The bound might depend on $\omega$ of course. By Theorem 77.2.10 and Lemma 77.4.2 there is a measurable into $\mathscr{V}^{\prime}$ function $\omega \rightarrow y(\omega)$ and a subsequence for each $\omega, y^{n(\omega)}(\omega)$ such that $y^{n(\omega)}(\omega) \rightarrow y(\omega)$ weakly in $\mathscr{V}^{\prime}$. By the Pettis theorem, $y$ is measurable into $\mathscr{V}^{\prime}$. Where is $y(\omega)$ ? If $y(\omega) \notin A(u(\omega), \omega)$, then there would exist $z(\omega) \in \mathscr{V}$ such that $\langle y(\omega), z\rangle>r>\langle w, z\rangle$ for all $w \in A(u(\omega), \omega)$. Let $O=$ $\left\{w \in \mathscr{V}^{\prime}\right.$ such that $\left.r>\langle w, z\rangle\right\}$. Then $O$ contains $A(u(\omega), \omega)$ and is a weakly open set. It follows from the upper semicontinuity assumption that $y^{n(\omega)}(\omega) \in O$ for all $n(\omega)$ large enough. Thus $r>\left\langle y^{n(\omega)}(\omega), z\right\rangle$. But by weak convergence,

$$
\langle y(\omega), z\rangle>r \geq \lim _{n(\omega) \rightarrow \infty}\left\langle y^{n(\omega)}(\omega), z\right\rangle=\langle y(\omega), z\rangle
$$

contradicting $y(\omega) \notin A(u(\omega), \omega)$. Hence $y(\omega) \in A(u(\omega), \omega)$ and $\omega \rightarrow y(\omega)$ is a measurable selection.

In fact, this is just a special case of a general result in the next theorem. It says essentially that having a measurable selection is preserved when going from constant to measurable functions. In this theorem, $V$ is a reflexive separable Banach space. This is difficult to show for measurable multifunctions. It is Theorem 48.3.1 and was proved earlier. A proof is given here also.

Theorem 77.5.3 Suppose $\omega \rightarrow A(u, \omega)$ has a measurable selection in $V^{\prime}$ for u a given element of $V$ not dependent on $\omega$ and for each $\omega, A(u, \omega)$ is a closed, convex set in $V^{\prime}$ and $A(\cdot, \omega)$ is bounded. Also suppose $u \rightarrow A(u, \omega)$ is upper semicontinuous from the strong topology of $V$ to the weak topology of $V^{\prime}$. That is, if $u_{n} \rightarrow u$ in $V$ strongly, then if $O$ is a weakly open set containing $A(u, \omega)$, it follows that $A\left(u_{n}, \omega\right) \in O$ for all $n$ large enough. Conclusion: whenever $\omega \rightarrow u(\omega)$ is measurable into $V$, it follows that there is a measurable selection for $\omega \rightarrow A(u(\omega), \omega)$ into $V^{\prime}$.

Proof: Let $\omega \rightarrow u(\omega)$ be measurable into $V$ and let $u_{n}(\omega) \rightarrow u(\omega)$ in $V$ where $u_{n}$ is a simple function

$$
u_{n}(\omega)=\sum_{k=1}^{m_{n}} c_{k}^{n} \mathscr{X}_{E_{k}^{n}}(\omega), \text { the } E_{k}^{n} \text { disjoint, } \Omega=\cup_{k} E_{k}^{n}
$$

each $c_{k}^{n}$ being in $V$. We can assume that $\left\|u_{n}(\omega)\right\| \leq 2\|u(\omega)\|$ for all $\omega$. Then by assumption, there is a measurable selection for $\omega \rightarrow A\left(c_{k}^{n}, \omega\right)$ denoted as $\omega \rightarrow y_{k}^{n}(\omega)$. Thus $\omega \rightarrow y_{k}^{n}(\omega)$ is measurable into $V^{\prime}$ and $y_{k}^{n}(\omega) \in A\left(c_{k}^{n}, \omega\right)$ for all $\omega \in \Omega$. Then consider

$$
y^{n}(\omega)=\sum_{k=1}^{m_{n}} y_{k}^{n}(\omega) \mathscr{X}_{E_{k}^{n}}(\omega)
$$

It is measurable and for $\omega \in E_{k}^{n}$ it equals $y_{k}^{n}(\omega) \in A\left(c_{k}^{n}, \omega\right)=A\left(u_{n}(\omega), \omega\right)$. Thus $y^{n}$ is a measurable selection of $\omega \rightarrow A\left(u_{n}(\omega), \omega\right)$. By the estimates, for each $\omega$ these $y^{n}(\omega)$ lie in a bounded subset of $V^{\prime}$. The bound might depend on $\omega$ of course.

Now let $\left\{z_{i}\right\}$ be a countable dense subset of $V$. Then let $X \equiv \prod_{i=1}^{\infty} \mathbb{R}$. It is a Polish space. Let

$$
\begin{gathered}
\mathbf{f}\left(y^{j}\right)(\omega) \equiv \prod_{i=1}^{\infty}\left\langle y^{j}(\omega), z_{i}\right\rangle \\
\Gamma_{n}(\omega) \equiv \overline{\cup_{k \geq n} \mathbf{f}\left(y^{k}\right)(\omega)},
\end{gathered}
$$

the closure taken in $X$. Now $y^{k}(\omega) \in A\left(u_{k}(\omega), \omega\right)$ and so by assumption, since $\left\|u_{k}(\omega)\right\| \leq$ $2\|u(\omega)\|$ it is bounded in $V^{\prime}$, this for each $\omega$.

Thus the components of $\mathbf{f}\left(y^{j}\right)(\omega)$ lie in a compact subset of $\mathbb{R}$, this for each $\omega$. It follows from Tychanoff's theorem that $\Gamma_{n}(\omega)$ is a compact subset of the Polish space $X$.

Claim: $\Gamma_{n}$ is measurable into $X$.
Proof of claim: It is necessary to show that $\Gamma_{n}^{-}(U) \equiv\left\{\omega: \Gamma_{n}(U) \cap U \neq \emptyset\right\}$ is measurable whenever $U$ is open. It suffices to verify this for $U$ a basic open set in the topology of $X$. Thus let $U=\prod_{i=1}^{\infty} O_{i}$ where $O_{i}$ is a proper open subset of $\mathbb{R}$ only for $i \in\left\{j_{1}, \cdots, j_{n}\right\}$. Then

$$
\Gamma_{n}^{-}(U)=\cup_{j \geq n} \cap_{r=1}^{n}\left\{\omega:\left\langle y^{j}(\omega), z_{j_{r}}\right\rangle \in O_{j_{r}}\right\}
$$

which is a measurable set thanks to $y^{j}$ being measurable.
In addition to this, $\Gamma_{n}(\omega)$ is compact, as explained above. Therefore, $\Gamma_{n}$ is also strongly measurable meaning $\Gamma_{n}^{-}(F)$ is measurable for all $F$ closed. Now let $\Gamma(\omega) \equiv \cap_{n=1}^{\infty} \Gamma_{n}(\omega)$. It is a nonempty closed subset of $X$ and if $F$ is closed in $X$,

$$
\Gamma^{-}(F)=\cap_{n=1}^{\infty} \Gamma_{n}^{-}(F)
$$

a measurable set since each $\Gamma_{n}^{-}(F)$ is measurable. Thus $\Gamma$ is a measurable multifunction and so it has a measurable selection $\omega \rightarrow \mathbf{z}(\omega)$. Thus by definition, for each $i, z_{i}(\omega)=$ $\lim _{n(\omega) \rightarrow \infty}\left\langle y^{n(\omega)}(\omega), z_{i}\right\rangle$ for some subsequence indexed by $n(\omega)$. The sequence given as $\left\{y^{n(\omega)}(\omega)\right\}$ is bounded in $V^{\prime}$ and so there is a subsequence still denoted as $\left\{y^{n(\omega)}\right\}$ which converges weakly to $y(\omega)$. Thus $z_{i}(\omega)=\left\langle y(\omega), z_{i}\right\rangle$ for each $i$. Since $\omega \rightarrow z_{i}(\omega)$ is measurable, it follows from density of the $\left\{z_{i}\right\}$ that $y$ is weakly, hence strongly measurable, this by the Pettis theorem. Now $y(\omega)=\lim _{n(\omega) \rightarrow \infty} y^{n(\omega)}(\omega)$. But

$$
y^{n(\omega)}(\omega) \in A\left(u_{n(\omega)}(\omega), \omega\right)
$$

which is a convex closed set for which $u \rightarrow A(u, \omega)$ is upper semicontinuous and $u_{n(\omega)} \rightarrow u$ so in fact, $y(\omega) \in A(u(\omega), \omega)$. This is the claimed measurable selection.

Note that we are not assuming that $\omega \rightarrow A(u, \omega)$ is measurable, only that it has a measurable selection and of course the upper semicontinuity and that the values are closed and convex. Also note that $\omega \rightarrow \Gamma(\omega)$ is measurable so there is a dense subset of measurable functions $\left\{\mathbf{z}_{k}(\omega)\right\}$ each being measurable into $X$. However, we don't know much about $\Gamma(\omega)$ other than it is measurable into $X$.

Also $\mathscr{V}$ could be replaced with $L^{p}(I, V)$ where $I$ is any interval and nothing changes.
The condition leading to 77.5 .26 will typically be satisfied. For example, suppose $u^{*} \in$ $A(u, \omega)$ means that

$$
u^{*}(t) \in A\left(t, u(t), \int_{0}^{t} u(s) d s, \omega\right)
$$

for a.e. $t$. where $A$ has values in $\mathscr{P}\left(V^{\prime}\right)$. Then to say that $u^{*} \mathscr{X}_{[0, \hat{Y}]} \in A\left(u \mathscr{X}_{[0, \hat{T}]}, \omega\right)$ for each $\hat{T}$ in an increasing sequence converging to $T$ would imply the above holding for a.e. $t$. While the above is the typical situation one would expect to see, the following proposition is also interesting.

Proposition 77.5.4 Suppose $A(\cdot, \omega): \mathscr{V} \rightarrow \mathscr{P}\left(\mathscr{V}^{\prime}\right)$ is upper semicontinuous as a map from the strong topology of $\mathscr{V}$ to the weak topology of $\mathscr{V}^{\prime}$ and has closed convex values. Then if

$$
u^{*} \mathscr{X}_{[0, \hat{T}]} \in A\left(u \mathscr{X}_{[0, \hat{T}]}, \omega\right)
$$

for each $\hat{T}$ in an increasing sequence converging to $T$, then

$$
\begin{equation*}
u^{*} \in A(u, \omega) \tag{77.5.36}
\end{equation*}
$$

Proof: Let $\hat{T}_{n} \uparrow T$ such that $u^{*} \mathscr{X}_{\left[0, \hat{T}_{n}\right]} \in A\left(u \mathscr{X}_{\left[0, \hat{T}_{n}\right]}, \omega\right)$. Then if $u^{*} \notin A(u, \omega)$, there exists $z \in \mathscr{V}$ such that

$$
\left\langle u^{*}, z\right\rangle>r>\left\langle w^{*}, z\right\rangle
$$

for all $w^{*} \in A(u, \omega)$. Now $u^{*} \mathscr{X}_{\left[0, \hat{T}_{n}\right]} \rightarrow u^{*}$ in $\mathscr{V}^{\prime}$ and $u \mathscr{X}_{\left[0, \hat{T}_{n}\right]} \rightarrow u$ in $\mathscr{V}$. Letting $O$ be the weakly open set, $\left\{z^{*}:\left\langle z^{*}, z\right\rangle<r\right\}$, it follows that this $O$ is a weakly open set which contains $A(u, \omega)$. Hence, by upper semicontinuity, $\left\langle u^{*} \mathscr{X}_{\left[0, \hat{T}_{n}\right]}, z\right\rangle<r$ for all $n$ large enough. Hence, passing to a limit, one obtains $\left\langle u^{*}, z\right\rangle>r \geq\left\langle u^{*}, z\right\rangle$, a contradiction. Thus $u^{*} \in A(u, \omega)$.

Let $r>\max (\hat{p}, 2)$. Recall that $\hat{p} \geq p$ and the growth had to do with $\hat{p}$. Let $\mathscr{U}$ and $\mathscr{U}_{I}$ be defined by analogy with $\mathscr{V}$ and $\mathscr{V}_{I}$ where $\mathscr{U} \equiv L^{r}([0, T], U)$. Here $U$ is a Hilbert space which is dense in $V$ and embedds compactly into $V,\|u\|_{U} \geq\|u\|_{V}$. Also let $F: U \rightarrow U^{\prime}$ be the duality map for $r$. Thus

$$
\|F u\|_{U^{\prime}}=\|u\|_{U}^{r-1},\langle F u, u\rangle=\|u\|_{U}^{r}
$$

Also define the following notation for small positive $h$.

$$
\tau_{h} g(t) \equiv\left\{\begin{array}{l}
g(t-h) \text { if } t>h \\
0 \text { if } t \leq h
\end{array}\right.
$$

Let $\omega \rightarrow u_{0}(\omega)$ be $\mathscr{F}$ measurable into $W$. Also let $\omega \rightarrow f(\cdot, \omega)$ be $\mathscr{F}$ measurable into $\mathscr{V}^{\prime}, \omega \rightarrow B(\omega)$ measurable into $\mathscr{L}\left(W, W^{\prime}\right)$. Now let $u_{h}$ for $h>0$ and small, be the unique solution to the initial value problem

$$
\begin{align*}
\left(B(\omega) u_{h}(\cdot, \omega)\right)^{\prime}+\varepsilon F u_{h}(\cdot, \omega) & =f(\cdot, \omega)-u_{h}^{*}(\cdot, \omega) \text { in } \mathscr{U}^{\prime},  \tag{77.5.37}\\
B u_{h}(0, \omega) & =B u_{0}(\omega)
\end{align*}
$$

where $u_{h}^{*} \in A\left(\tau_{h} u, \omega\right)$ is a $\mathscr{F}$ measurable selection into $\mathscr{V}^{\prime}$. Since $F$ is monotone bounded and hemicontinuous, there is no problem with it being pseudomonotone from $X_{r I}$ to $X_{r I}^{\prime}$. Such a solution exists on $[0, h]$ by the above reasoning. Let this solution be denoted by $u_{1}$. Then use it to define a solution to the evolution equation on $[0,2 h]$ called $u_{2}$. By uniqueness, these coincide on $[0, h]$. Then use $u_{2}$ to extend to a solution on $[0,3 h]$ called $u_{3}$. Then $u_{3}=u_{2}$ on $[0,2 h]$. Continue this way to obtain a solution valid on $[0, T]$. By Lemma 77.4.3, this solution may be assumed to be measurable into $\mathscr{U}^{\prime}$. One gets this by using the lemma on a succession of intervals $[0, h],[0,2 h]$, and so forth.

Now acting on $u_{h}$ and suppressing the dependence on $\omega$ in most places, it follows from the assumed estimates (Note how the assumption on growth was used here.) that

$$
\begin{align*}
& \frac{1}{2}\left\langle B u_{h}, u_{h}\right\rangle(T)-\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle+\varepsilon \int_{0}^{T}\left\|u_{h}\right\|_{U}^{r} d s \\
& \leq\left(\int_{0}^{T}\|f\|_{V^{\prime}}^{p^{\prime}} d s\right)^{1 / p^{\prime}}\left(\int_{0}^{T}\left\|u_{h}\right\|_{V}^{p}\right)^{1 / p} \\
&+\int_{0}^{T}\left(a+b\left\|\tau_{h} u_{h}\right\|_{V}^{\hat{p}-1}\right)\left\|u_{h}\right\|_{V} d s \\
& \leq\|f\|_{V^{\prime}}^{\hat{p}^{\prime}}+\left\|u_{h}\right\|_{\mathscr{V}}^{\hat{p}}+\left\|u_{h}\right\|_{\mathscr{V}}^{\hat{p}}+a T^{1 / \hat{p}^{\prime}}+b\left\|u_{h}\right\|_{\mathscr{V}}^{\hat{p}} \tag{77.5.38}
\end{align*}
$$

which is of the form

$$
\leq C\left(\|f\|_{\mathscr{V}^{\prime}}^{\hat{p}^{\prime}}, a(\omega) T\right)+(2+b)\left\|u_{h}\right\|_{\mathscr{V}}^{\hat{p}}
$$

Now here is where it is good that $\hat{p}<r$.

$$
\begin{aligned}
\left\|u_{h}\right\|_{\mathscr{V}}^{\hat{p}} \leq & \int_{0}^{T} \frac{1}{\delta} \delta\left\|u_{h}\right\|_{U}^{\hat{p}} d s \\
& \leq\left(\int_{0}^{T} \delta^{r / \hat{p}}\left\|u_{h}\right\|_{U}^{r} d s\right)^{\hat{p} / r}\left(\int_{0}^{T} \frac{1}{\delta^{r /(r-\hat{p})}} 1^{r / r-\hat{p}}\right)^{(r-\hat{p}) / r} \\
& \leq \frac{1}{\delta^{r /(r-\hat{p})}} \frac{T^{(r-\hat{p}) / r}}{r}(r-\hat{p})+\frac{\hat{p} \delta^{r / \hat{p}}\left\|u_{h}\right\|_{\mathscr{U}}^{r}}{r}
\end{aligned}
$$

Thus this has shown

$$
\begin{aligned}
& \frac{1}{2}\left\langle B u_{h}, u_{h}\right\rangle(T)-\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle+\varepsilon\left\|u_{h}\right\|_{\mathscr{U}}^{r} \\
\leq & C\left(\|f\|_{\mathscr{V}^{\prime}}^{\hat{p}^{\prime}}, a(\omega) T\right)+\frac{1}{\delta^{r /(r-\hat{p})}} \frac{T^{(r-\hat{p}) / r}}{r}(r-\hat{p})+\frac{\hat{p} \delta^{r / \hat{p}}\left\|u_{h}\right\|_{\mathscr{U}}^{r}}{r}
\end{aligned}
$$

Then for $\delta$ small enough, depending on $\varepsilon$,

$$
\frac{\hat{p}}{r} \delta^{r / \hat{p}}<\frac{\varepsilon}{2}
$$

And so the inequality ending at 77.5 .38 yields

$$
\left\langle B u_{h}, u_{h}\right\rangle(T)+\varepsilon\left\|u_{h}\right\|_{\mathscr{U}}^{r} \leq C\left(\|f\|_{\mathscr{V}^{\prime}}^{\hat{p}^{\prime}},(a(\omega)+1) T, \varepsilon\right)+\left\langle B u_{0}, u_{0}\right\rangle
$$

From 77.5.37 and the boundedness of the various operators, $\left(B(\omega) u_{h}(\cdot, \omega)\right)^{\prime}$ is bounded in $\mathscr{U}^{\prime}$. Thus, summarizing these estimates yields the following for fixed $\varepsilon$

$$
\begin{equation*}
\left\|\left(B(\omega) u_{h}(\cdot, \omega)\right)^{\prime}\right\|_{\mathscr{U}^{\prime}}+\left\|u_{h}\right\|_{\mathscr{U}}+\left\|u_{h}^{*}\right\|_{\mathscr{V}^{\prime}} \leq C \tag{77.5.39}
\end{equation*}
$$

where $C$ does not depend on $h$ although it does depend on $\varepsilon$ and of course on $\omega$. Then one can get a subsequence, still denoted with $h$ such that as $h \rightarrow 0$,

$$
\begin{gather*}
u_{h} \rightarrow u \text { weakly in } \mathscr{U}  \tag{77.5.40}\\
\tau_{h} u_{h} \rightarrow u \text { weakly in } \mathscr{U}  \tag{77.5.41}\\
\left(B(\omega) u_{h}(\cdot, \omega)\right)^{\prime} \rightarrow(B(\omega) u)^{\prime} \text { weakly in } \mathscr{U}^{\prime}  \tag{77.5.42}\\
u_{h} \rightarrow u \text { strongly in } \mathscr{V}  \tag{77.5.43}\\
u_{h}^{*} \rightarrow u^{*} \text { weakly in } \mathscr{V}^{\prime}  \tag{77.5.44}\\
F u_{h} \rightarrow \xi \in \mathscr{U}^{\prime}  \tag{77.5.45}\\
B u(0, \omega)=B u_{0}(\omega) \tag{77.5.46}
\end{gather*}
$$

The fourth of these comes from a use of Theorem 77.3.1. We need to argue that $u^{*} \in$ $A(u, \omega)$. From the equation and initial conditions of 77.5.37, it follows from monotonicity conditions and the observation that $\mathscr{V}^{\prime}$ is contained in $\mathscr{U}^{\prime}$ that

$$
\begin{gathered}
\left\langle\left(B(\omega) u_{h}(\cdot, \omega)\right)^{\prime}, u_{h}-u\right\rangle+\left\langle\varepsilon F u_{h}(\cdot, \omega), u_{h}-u\right\rangle \\
+\left\langle u_{h}^{*}(\cdot, \omega), u_{h}-u\right\rangle=\left\langle f(\cdot, \omega), u_{h}-u\right\rangle
\end{gathered}
$$

and so

$$
\begin{gathered}
\left\langle(B(\omega) u(\cdot, \omega))^{\prime}, u_{h}-u\right\rangle+\left\langle\varepsilon F u_{h}(\cdot, \omega), u_{h}-u\right\rangle_{\mathscr{U}^{\prime}, \mathscr{U}} \\
+\left\langle u_{h}^{*}(\cdot, \omega), u_{h}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \leq\left\langle f(\cdot, \omega), u_{h}-u\right\rangle
\end{gathered}
$$

by the strong convergence of 77.5 .43 , it follows that the third term converges to 0 as $h \rightarrow 0$. This is because the estimate 77.5 .27 implies that the $u_{h}^{*}$ are bounded, and then the strong convergence gives the desired result. Hence

$$
\limsup _{h \rightarrow 0}\left\langle\varepsilon F u_{h}(\cdot, \omega), u_{h}-u\right\rangle_{\mathscr{U}^{\prime}, \mathscr{U}} \leq 0
$$

and since $F$ is monotone and hemicontinuous, it follows that in fact,

$$
\lim _{h \rightarrow 0}\left\langle\varepsilon F u_{h}(\cdot, \omega), u_{h}-u\right\rangle_{\mathscr{U}^{\prime}, \mathscr{U}}=0
$$

so for $v \in \mathscr{U}$ arbitrary,

$$
\begin{aligned}
\langle\varepsilon \xi, u-v\rangle= & \lim \inf _{h \rightarrow 0}\left(\left\langle\varepsilon F u_{h}(\cdot, \omega), u-u_{h}\right\rangle_{\mathscr{U}^{\prime}, \mathscr{U}}+\left\langle\varepsilon F u_{h}(\cdot, \omega), u_{h}-v\right\rangle_{\mathscr{U}^{\prime}, \mathscr{U}}\right) \\
& =\lim _{h \rightarrow 0}\left\langle\varepsilon F u_{h}(\cdot, \omega), u_{h}-v\right\rangle_{\mathscr{U}^{\prime}, \mathscr{U}} \geq\langle\varepsilon F u, u-v\rangle
\end{aligned}
$$

and so, since $v$ is an arbitrary element of $\mathscr{U}$, it follows that $\xi=F(u)$.
Now consider the other term involving $u_{h}^{*}$. Recall that $u_{h}^{*} \in A\left(\tau_{h} u_{h}, \omega\right)$.

$$
\begin{aligned}
\left\|\tau_{h} u_{h}-u_{h}\right\|_{\mathscr{V}} & \leq\left\|\tau_{h} u_{h}-\tau_{h} u\right\|_{\mathscr{V}}+\left\|\tau_{h} u-u\right\|_{\mathscr{V}} \\
& \leq\left\|u_{h}-u\right\|_{\mathscr{V}}+\left\|\tau_{h} u-u\right\|_{\mathscr{V}}
\end{aligned}
$$

and both of these on the right converge to 0 thanks to continuity of translation and 77.5.43. Therefore,

$$
\lim _{h \rightarrow 0}\left\langle u_{h}^{*}(\cdot, \omega), \tau_{h} u_{h}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}=0
$$

It follows that

$$
\begin{aligned}
\left\langle u^{*}, u-v\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} & =\lim \inf _{h \rightarrow 0}\left(\left\langle u_{h}^{*}(\cdot, \omega), u-\tau_{h} u_{h}\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}+\left\langle u_{h}^{*}(\cdot, \omega), \tau_{h} u_{h}-v\right\rangle\right) \\
& \geq \lim \inf _{h \rightarrow 0}\left\langle u_{h}^{*}(\cdot, \omega), \tau_{h} u_{h}-v\right\rangle \geq\left\langle u^{*}(v), u-v\right\rangle
\end{aligned}
$$

where $u^{*}(v) \in A(u, \omega)$. Then it follows that $u^{*} \in A(u, \omega)$ because if not, then by separation theorems, there would exist $v$ such that

$$
\left\langle u^{*}, u-v\right\rangle_{V^{\prime}, \mathscr{V}}<\left\langle w^{*}, u-v\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}
$$

for all $w^{*} \in A(u, \omega)$ which contradicts the above inequality. Thus, passing to the limit in 77.5.37,

$$
\begin{align*}
(B(\omega) u(\cdot, \omega))^{\prime}+\varepsilon F u(\cdot, \omega)+u^{*} & =f(\cdot, \omega) \text { in } \mathscr{U}^{\prime},  \tag{77.5.47}\\
B u(0, \omega) & =B u_{0}(\omega)
\end{align*}
$$

Here $u^{*} \in A(u, \omega)$. Of course nothing is known about the measurability of $u^{*}, u$. All that has been obtained in the above is a solution for each fixed $\omega$. However, each of the functions $u_{h}, u_{h}^{*}$ is measurable. Also we have the estimate 77.5.39. By Theorem 77.2.10, there are functions $\hat{u}(\cdot, \omega), \hat{u}^{*}(\cdot, \omega)$ and a subsequence with subscript $h(\omega)$ such that the following weak convergences in $\mathscr{V}$ and $\mathscr{V}^{\prime}$ take place

$$
u_{h(\omega)}(\cdot, \omega) \rightharpoonup \hat{u}(\cdot, \omega), u_{h(\omega)}^{*}(\cdot, \omega) \rightharpoonup \hat{u}^{*}(\cdot, \omega)
$$

such that the functions $(t, \omega) \rightarrow \hat{u}(t, \omega),(t, \omega) \rightarrow \hat{u}^{*}(t, \omega)$ are product measurable into $V$ and $V^{\prime}$ respectively. The above argument shows that for each $\omega$, there is a further subsequence, still denoted with subscript $h(\omega)$ such that $u_{h(\omega)}(\cdot, \omega) \rightarrow u(\cdot, \omega)$ weakly in $\mathscr{V}$ and $u_{h(\omega)}^{*}(\cdot, \omega) \rightarrow u^{*}(\cdot, \omega)$ weakly in $\mathscr{V}^{\prime}$ such that $\left(u(\cdot, \omega), u^{*}(\cdot, \omega)\right)$ is a solution to the evolution equation for each $\omega$. By uniqueness of limits, $u(\cdot, \omega)=\hat{u}(\cdot, \omega)$, similar for $\hat{u}^{*}$. Thus this solution which is defined for each $\omega$ has a representative for each $\omega$ such that the resulting functions of $t, \omega$ are product measurable into $V, V^{\prime}$ respectively. This proves the following lemma.

Lemma 77.5.5 For each $\varepsilon>0$ there exists a solution to

$$
\begin{align*}
(B(\omega) u(\cdot, \omega))^{\prime}+\varepsilon F u(\cdot, \omega)+u^{*}(\cdot, \omega) & =f(\cdot, \omega) \text { in } \mathscr{U}^{\prime},  \tag{77.5.48}\\
B u(0, \omega) & =B u_{0}(\omega) \\
u^{*}(\cdot, \omega) & \in A(u(\cdot, \omega), \omega) \tag{77.5.49}
\end{align*}
$$

this solution satisfies $(t, \omega) \rightarrow u(t, \omega)$ is product measurable into $V$. Also $(t, \omega) \rightarrow u^{*}(t, \omega)$, and $(t, \omega) \rightarrow B(\omega) u(t, \omega)$ are product measurable into $V^{\prime}$ and $W^{\prime}$ respectively.

Next it is desired to remove the regularizing term $\varepsilon F u$. This will involve another use of Theorem 77.2.10. Denote by $u_{\varepsilon}$ the solution to the above lemma. Then act on $u_{\varepsilon}$ on both sides. This yields

$$
\begin{gather*}
\frac{1}{2}\left\langle B u_{\varepsilon}, u_{\varepsilon}\right\rangle(T)-\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle+\varepsilon \int_{0}^{T}\left\langle F u_{\varepsilon}, u_{\varepsilon}\right\rangle d s \\
\quad+\int_{0}^{T}\left\langle u_{\varepsilon}^{*}, u_{\varepsilon}\right\rangle d s=\int_{0}^{T}\left\langle f, u_{\varepsilon}\right\rangle d s \tag{77.5.50}
\end{gather*}
$$

Then by the coercivity assumption,

$$
\lim _{\substack{\|u\|_{\mathscr{V}} \rightarrow \infty \\ u \in X_{r}}} \frac{\inf \left\{2\left\langle u^{*}, u\right\rangle+\langle B u, u\rangle(T): u^{*} \in A(u, \omega)\right\}}{\|u\|_{\mathscr{V}}}=\infty
$$

it follows that

$$
\begin{equation*}
\varepsilon\left\langle F u_{\varepsilon}, u_{\varepsilon}\right\rangle_{\mathscr{U}^{\prime}, \mathscr{U}}+\left\|u_{\varepsilon}\right\|_{\mathscr{V}} \leq C\left(u_{0}, f\right) \tag{77.5.51}
\end{equation*}
$$

where the constant on the right does not depend on $\varepsilon$. Then

$$
\varepsilon F u_{\varepsilon} \rightarrow 0 \text { strongly in } \mathscr{U}^{\prime}
$$

this follows because from properties of the duality map,

$$
\begin{aligned}
\left\langle\varepsilon F u_{\varepsilon}, v\right\rangle & \leq \varepsilon\left\langle F u_{\varepsilon}, u_{\varepsilon}\right\rangle^{1 / r^{\prime}}\langle F v, v\rangle^{1 / r} \\
& =\varepsilon^{1 / r^{\prime}}\left\langle F u_{\mathcal{E}}, u_{\mathcal{E}}\right\rangle^{1 / r^{\prime}} \varepsilon^{1 / r}\|v\|_{\mathscr{U}} \leq C \varepsilon^{1 / r}\|v\|_{\mathscr{U}}
\end{aligned}
$$

Then since $A$ is bounded, there is a constant $C$ independent of $\varepsilon$ such that

$$
\begin{equation*}
\left\|u_{\varepsilon}^{*}\right\|_{\mathscr{V}^{\prime}}+\left\|\left(B u_{\varepsilon}\right)^{\prime}\right\|_{\mathscr{U}^{\prime}}+\left\|u_{\varepsilon}\right\|_{\mathscr{V}} \leq C \tag{77.5.52}
\end{equation*}
$$

It follows there is a subsequence, still denoted with $\varepsilon$ such that

$$
\begin{align*}
u_{\varepsilon}^{*} & \rightarrow u^{*} \text { weakly in } \mathscr{V}^{\prime}  \tag{77.5.53}\\
\left(B u_{\varepsilon}\right)^{\prime} & \rightarrow(B u)^{\prime} \text { weakly in } \mathscr{U}^{\prime},  \tag{77.5.54}\\
u_{\varepsilon} & \rightarrow u \text { weakly in } \mathscr{V} . \tag{77.5.55}
\end{align*}
$$

Also

$$
\frac{1}{2}\left\langle B u_{\varepsilon}, u_{\varepsilon}\right\rangle(T)-\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle+\left\langle u_{\varepsilon}^{*}, u_{\varepsilon}\right\rangle+\varepsilon\left\langle F u_{\varepsilon}, u_{\varepsilon}\right\rangle=\left\langle f, u_{\varepsilon}\right\rangle
$$

Assume $T$ is such that

$$
\left\langle B u_{\varepsilon}, u_{\varepsilon}\right\rangle(T)=\left\langle B\left(u_{\varepsilon}(T)\right), u_{\varepsilon}(T)\right\rangle, B u_{\varepsilon}(T)=B\left(u_{\varepsilon}(T)\right)
$$

for all $\varepsilon$ in the sequence converging to 0 and also

$$
B u(T)=B(u(T)),\langle B u, u\rangle(T)=\langle B(u(T)), u(T)\rangle
$$

If not, carry out the argument for $\hat{T}$ close to $T$ for which this condition does hold. We have the integral equation

$$
B u_{\varepsilon}(t)-B u_{0}+\int_{0}^{t} u_{\varepsilon}^{*} d s+\int_{0}^{t} \varepsilon F u_{\varepsilon} d s=\int_{0}^{t} f d s
$$

and so $B u_{\varepsilon}(t)$ converges to $B u(t)$ in $U^{\prime}$ weakly. This follows right away from the convergence of $\left(B u_{\varepsilon}\right)^{\prime}$ in the above. Also from the above equation,

$$
B u(t)-B u_{0}+\int_{0}^{t} u^{*} d s=\int_{0}^{t} f d s
$$

Thus

$$
\begin{aligned}
B u(0) & =B u_{0} \\
(B u)^{\prime}+u^{*} & =f \text { in } \mathscr{U}^{\prime}
\end{aligned}
$$

Since $\mathscr{V}^{\prime} \subseteq \mathscr{U}^{\prime}$,

$$
\frac{1}{2}\langle B u, u\rangle(t)-\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle+\int_{0}^{t}\left\langle u^{*}, u\right\rangle_{V^{\prime}, V} d s=\int_{0}^{t}\langle f, u\rangle d s
$$

Also

$$
\begin{gather*}
\frac{1}{2}\left\langle B u_{\varepsilon}, u_{\varepsilon}\right\rangle(t)-\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle+\int_{0}^{t}\left\langle u_{\mathcal{\varepsilon}}^{*}, u_{\varepsilon}\right\rangle_{V^{\prime}, V} d s \\
\quad+\int_{0}^{t}\left\langle\varepsilon F u_{\varepsilon}, u_{\varepsilon}\right\rangle d s=\int_{0}^{t}\left\langle f, u_{\varepsilon}\right\rangle d s \tag{77.5.56}
\end{gather*}
$$

Now let $\left\{e_{i}\right\}$ be the vectors of Lemma 77.3.4 where these are in $U$. Thus

$$
\left\langle B u_{\varepsilon}, u_{\varepsilon}\right\rangle(T)=\sum_{i=1}^{\infty}\left\langle B\left(u_{\varepsilon}(T)\right), e_{i}\right\rangle^{2}
$$

Hence, by Fatou's lemma,

$$
\begin{aligned}
\liminf _{\varepsilon \rightarrow 0}\left\langle B u_{\varepsilon}, u_{\varepsilon}\right\rangle(T) & =\lim \inf _{\varepsilon \rightarrow 0} \sum_{i=1}^{\infty}\left\langle B\left(u_{\varepsilon}(T)\right), e_{i}\right\rangle^{2} \\
& \geq \sum_{i=1}^{\infty} \lim \inf _{\varepsilon \rightarrow 0}\left\langle B\left(u_{\varepsilon}(T)\right), e_{i}\right\rangle^{2} \\
& =\sum_{i=1}^{\infty} \liminf _{\varepsilon \rightarrow 0}\left\langle B u_{\varepsilon}(T), e_{i}\right\rangle^{2} \\
& =\sum_{i=1}^{\infty}\left\langle B u(T), e_{i}\right\rangle^{2} \\
& =\langle B(u(T)), u(T)\rangle=\langle B u, u\rangle(T)
\end{aligned}
$$

From 77.5.56, letting $t=T$,

$$
\begin{aligned}
\limsup _{\varepsilon \rightarrow 0}\left\langle u_{\varepsilon}^{*}, u_{\varepsilon}\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} & \leq \limsup _{\varepsilon \rightarrow 0}\left(\left\langle f, u_{\varepsilon}\right\rangle+\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle-\frac{1}{2}\left\langle B u_{\varepsilon}, u_{\varepsilon}\right\rangle(T)\right) \\
& \leq\langle f, u\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}+\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle-\frac{1}{2}\langle B u, u\rangle(T)=\left\langle u^{*}, u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}
\end{aligned}
$$

It follows that

$$
\limsup _{\varepsilon \rightarrow 0}\left\langle u_{\varepsilon}^{*}, u_{\varepsilon}-u\right\rangle \leq\left\langle u^{*}, u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}-\left\langle u^{*}, u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}=0
$$

and so

$$
\liminf _{\varepsilon \rightarrow 0}\left\langle u_{\varepsilon}^{*}, u_{\varepsilon}-v\right\rangle \geq\left\langle u^{*}(v), u-v\right\rangle
$$

for any $v \in \mathscr{V}$ where $u^{*}(v) \in A(u, \omega)$. In particular for $v=u$. Hence

$$
\liminf _{\varepsilon \rightarrow 0}\left\langle u_{\varepsilon}^{*}, u_{\varepsilon}-u\right\rangle \geq\left\langle u^{*}(v), u-u\right\rangle=0 \geq \lim \sup _{\varepsilon \rightarrow 0}\left\langle u_{\varepsilon}^{*}, u_{\varepsilon}-u\right\rangle
$$

showing that $\lim _{\varepsilon \rightarrow 0}\left\langle u_{\varepsilon}^{*}, u_{\mathcal{E}}-u\right\rangle=0$. Thus

$$
\begin{aligned}
\left\langle u^{*}, u-v\right\rangle & \geq \lim \inf _{\varepsilon \rightarrow 0}\left(\left\langle u_{\varepsilon}^{*}, u-u_{\varepsilon}\right\rangle+\left\langle u_{\varepsilon}^{*}, u_{\varepsilon}-v\right\rangle\right) \\
& =\lim \inf _{\varepsilon \rightarrow 0}\left\langle u_{\varepsilon}^{*}, u_{\varepsilon}-v\right\rangle \geq\left\langle u^{*}(v), u-v\right\rangle
\end{aligned}
$$

This implies $u^{*} \in A(u, \omega)$ because if not, then by separation theorems, there exists $v \in \mathscr{V}$ such that for all $w^{*} \in A(u, \omega)$,

$$
\left\langle u^{*}, u-v\right\rangle<\left\langle w^{*}, u-v\right\rangle
$$

contrary to what was shown above. Thus this obtains

$$
B u(t)-B u_{0}+\int_{0}^{t} u^{*} d s=\int_{0}^{t} f d s
$$

where $u^{*} \in A(u, \omega)$. In case $B u_{\varepsilon}(T) \neq B\left(u_{\varepsilon}(T)\right)$, you do the same argument for $\hat{T}<T$ where $B u_{\varepsilon}(\hat{T})=B\left(u_{\varepsilon}(\hat{T})\right)$ for all $\varepsilon$ and for $u$. Then the above argument shows that $u^{*} \mathscr{X}_{[0, \hat{T}]} \in A\left(\mathscr{X}_{[0, \hat{T}]} u, \omega\right)$. This being true for every such $\hat{T}<T$ implies that it holds on $[0, T]$ and shows part of the following theorem which is the main result.

Theorem 77.5.6 Let the conditions on A hold 77.5.25-77.5.28, 77.5.30-77.5.34. Also let $B$ satisfy 77.3.6 and assume, if it depends on $\omega$, it is of the form

$$
B(\omega)=k(\omega) B, k(\omega) \geq 0, k \text { measurable }
$$

Let $u_{0}$ be $\mathscr{F}$ measurable into $W$, and let $f$ be product measurable into $V^{\prime}, f(\cdot, \omega) \in \mathscr{V}^{\prime}$. Then there exists a solution to the following evolution inclusion

$$
\begin{align*}
(B(\omega) u(\cdot, \omega))^{\prime}+u^{*}(\cdot, \omega) & =f(\cdot, \omega) \text { in } \mathscr{V}^{\prime} \\
B(\omega) u(0, \omega) & =B(\omega) u_{0}(\omega) \tag{77.5.57}
\end{align*}
$$

where $u^{*}(\cdot, \omega) \in A(u(\cdot, \omega), \omega)$. In addition to this, $(t, \omega) \rightarrow u(t, \omega)$ is product measurable into $V$ and $(t, \omega) \rightarrow u^{*}(t, \omega)$ is product measurable into $V^{\prime}$.

In place of the coercivity condition 77.5 .28 assume the coercivity condition involving both $B$ and $A$ given in 77.5.29. Then

$$
\begin{align*}
(B(\omega) u(\cdot, \omega))^{\prime}+u^{*}(\cdot, \omega) & =f(\cdot, \omega) \text { in } \mathscr{U}^{\prime}  \tag{77.5.58}\\
B(\omega) u(0, \omega) & =B(\omega) u_{0}(\omega) \tag{77.5.59}
\end{align*}
$$

Thus the following holds in $V^{\prime}$

$$
\begin{aligned}
(B(\omega) u(\cdot, \omega))(t)-B(\omega) u_{0}(\omega)+\int_{0}^{t} u^{*}(\cdot, \omega) d s & =\int_{0}^{t} f(s, \omega) d s \\
(B u)^{\prime} & \in \mathscr{V}^{\prime}
\end{aligned}
$$

Proof of Theorem 77.5.6: First consider the claim about replacing the coercivity condition. Returning to 77.5.50, one obtains by integrating up to $t$ and adding $\lambda \int_{0}^{t}\left\langle B u_{\varepsilon}, u_{\varepsilon}\right\rangle d s$ to both sides,

$$
\begin{gather*}
\frac{1}{2}\left\langle B u_{\varepsilon}, u_{\varepsilon}\right\rangle(t)-\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle+\varepsilon \int_{0}^{t}\left\langle F u_{\varepsilon}, u_{\varepsilon}\right\rangle d s \\
+\int_{0}^{t}\left\langle u_{\varepsilon}^{*}, u_{\varepsilon}\right\rangle d s+\lambda \int_{0}^{t}\left\langle B u_{\varepsilon}, u_{\varepsilon}\right\rangle d s=\int_{0}^{t}\left\langle f, u_{\varepsilon}\right\rangle d s+\lambda \int_{0}^{t}\left\langle B u_{\varepsilon}, u_{\varepsilon}\right\rangle d s \tag{77.5.60}
\end{gather*}
$$

Then from the estimate 77.5.29,

$$
\begin{array}{r}
\frac{1}{2}\left\langle B u_{\varepsilon}, u_{\varepsilon}\right\rangle(t)-\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle+\varepsilon \int_{0}^{t}\left\langle F u_{\varepsilon}, u_{\varepsilon}\right\rangle d s \\
+\delta(\omega) \int_{0}^{t}\left\|u_{\varepsilon}\right\|_{V}^{p} d s-m(\omega)=\int_{0}^{t}\left\langle f, u_{\varepsilon}\right\rangle d s+\lambda \int_{0}^{t}\left\langle B u_{\varepsilon}, u_{\varepsilon}\right\rangle d s \tag{77.5.61}
\end{array}
$$

From this, it is a routine use of Gronwall's inequality to obtain the estimate

$$
\begin{equation*}
\varepsilon\left\langle F u_{\mathcal{E}}, u_{\varepsilon}\right\rangle_{\mathscr{U}^{\prime}, \mathscr{U}}+\left\|u_{\varepsilon}\right\|_{\mathscr{V}} \leq C\left(u_{0}, f, \lambda, \omega\right) \tag{77.5.62}
\end{equation*}
$$

Then the rest of the argument is the same. You obtain the following in $U^{\prime}$.

$$
B(\omega) u(t, \omega)-B(\omega) u_{0}(\omega)+\int_{0}^{t} u^{*}(\cdot, \omega) d s=\int_{0}^{t} f(s, \omega) d s
$$

Since all terms but the first are in $V^{\prime}$, the equation holds in $V^{\prime}$. Also, the equation in 77.5.58 shows that $(B(\omega) u(\cdot, \omega))^{\prime} \in \mathscr{V}^{\prime}$.

It only remains to show that there is a product measurable solution. The above argument has shown that there exists a solution for each $\omega$. This is another application of Theorem 77.2.10. For the sequence defined in the convergences 77.5.53-77.5.55, there is an estimate 77.5.52. Therefore, the conditions of this theorem hold and there exists a subsequence denoted with $\varepsilon(\omega)$ such that

$$
\begin{aligned}
& u_{\mathcal{E}(\omega)}(\cdot, \omega) \quad \rightarrow \hat{u}(\cdot, \omega) \text { weakly in } \mathscr{V} \\
& u_{\varepsilon(\omega)}^{*}(\cdot, \omega) \quad \rightarrow \quad \hat{u}^{*}(\cdot, \omega) \text { weakly in } \mathscr{V}^{\prime}
\end{aligned}
$$

where the $\hat{u}$ and $\hat{u}^{*}$ are product measurable. Now the above argument shows that for each $\omega$ there exists a further subsequence, still denoted with $\varepsilon(\omega)$ such that convergence to a solution to the evolution inclusion is obtained $\left(u(\cdot, \omega), u^{*}(\cdot, \omega)\right)$. Then by uniqueness of limits, $\hat{u}(\cdot, \omega)=u(\cdot, \omega)$ in $\mathscr{V}$, similar for $u^{*}$ and $\hat{u}^{*}$. Hence there is a solution to the above evolution problem which satisfies the claimed product measurability.

One can give a very interesting generalization of the above theorem.
Theorem 77.5.7 In the context of Theorem 77.5.6, let $q(t, \omega)$ be a product measurable function into $V$ such that $t \rightarrow q(t, \omega)$ is continuous, $q(0, \omega)=0$.

Then, there exists a solution $u$ of the integral equation

$$
B u(t, \omega)+\int_{0}^{t} z(s, \omega) d s=\int_{0}^{t} f(s, \omega) d s+B u_{0}(\omega)+B q(t, \omega)
$$

where $(t, \omega) \rightarrow u(t, \omega)$ is product measurable. Moreover, for each $\omega, B u(t, \omega)=B(u(t, \omega))$ for a.e. $t$ and $z(\cdot, \omega) \in A(u(\cdot, \omega), \omega)$ for a.e. $t, z$ is product measurable into $V^{\prime}$. Also, for each $a \in[0, T]$,

$$
B u(t, \omega)+\int_{a}^{t} z(s, \omega) d s=\int_{a}^{t} f(s, \omega) d s+B u(a, \omega)+B q(t, \omega)-B q(a, \omega)
$$

Proof: Define a stopping time

$$
\tau_{r} \equiv \inf \{t:|q(t, \omega)|>r\}
$$

Then this is the first hitting time of an open set by a continuous random variable and so it is a valid stopping time. Then for each $r$, let

$$
A_{r}(\omega, w) \equiv A\left(\omega, w+q^{\tau_{r}}(\cdot, \omega)\right)
$$

where the notation means $q^{\tau_{r}}(t) \equiv q\left(t \wedge \tau_{r}\right)$. Then, since $q^{\tau_{r}}$ is uniformly bounded, all of the necessary estimates and measurability for the solution to the above corollary hold for $A_{r}$ replacing $A$. Therefore, there exists a solution $w_{r}$ to the inclusion

$$
\left(B w_{r}\right)^{\prime}(\cdot, \omega)+A_{r}\left(w_{r}(\cdot, \omega), \omega\right) \ni f(\cdot, \omega), B w_{r}(0, \omega)=B u_{0}(\omega)
$$

Now for fixed $\omega, q^{\tau_{r}}(t, \omega)$ does not change for all $r$ large enough. This is because it is a continuous function of $t$ and so is bounded on the interval $[0, T]$. Thus, for $r$ large enough and fixed $\omega, q^{\tau_{r}}(t, \omega)=q(t, \omega)$. Thus, we obtain

$$
\begin{equation*}
\left\langle B w_{r}(t, \omega), w_{r}(t, \omega)\right\rangle+\int_{0}^{t}\left\|w_{r}(s, \omega)\right\|_{V}^{p} d s \leq C(\omega) \tag{77.5.63}
\end{equation*}
$$

Now, as before one can pass to a limit involving a subsequence, as $r \rightarrow \infty$ and obtain a solution to the integral equation

$$
B w(t, \omega)-B u_{0}(\omega)+\int_{0}^{t} z(s, \omega) d s=\int_{0}^{t} f(s, \omega) d s
$$

where $z(s, \omega) \in A(s, \omega, w(s, \omega)+q(s, \omega))$ for a.e. $s$ and $z$ is product measurable. Then an application of Theorem 77.2.10 shows that there exists a solution $w$ to this integral equation for each $\omega$ which also has $(t, \omega) \rightarrow w(t, \omega)$ product measurable and $(t, \omega) \rightarrow$ $z(t, \omega)$ product measurable. Now just let $u(t, \omega)=w(t, \omega)+q(t, \omega)$.

The last claim follows from letting $t=a$ in the top equation and then subtracting this from the top equation with $t>a$.

### 77.6 Variational Inequalities

We have some good theorems above in the context of 77.5.25-77.5.28, 77.5.30-77.5.34 and $B$ satisfies 77.3.6 and assume, if it depends on $\omega$, it is of the form

$$
B(\omega)=k(\omega) B, k(\omega) \geq 0, k \text { measurable }
$$

Now this will be used to consider variational inequalities.
Let $\mathscr{K}$ be a closed convex subset of $\mathscr{V}$ containing 0 . Let $P: \mathscr{V} \rightarrow \mathscr{V}^{\prime}$ be an operator of penalization. Thus $P=0$ on $\mathscr{K}$ and is monotone and demicontinuous and nonzero off $\mathscr{K}$.

$$
P u=F\left(u-\operatorname{proj}_{\mathscr{K}}(u)\right)
$$

where $F$ is the duality map such that $\langle F u, u\rangle=\|u\|^{2},\|F u\|=\|u\|$. Then $A(\cdot, \omega)+n P$ satisfies the conditions for Theorem 77.5.6 assuming $A(\cdot, \omega)$ satisfies the conditions of this theorem. Then by Theorem 77.5.6, there exists a solution $u_{n}$ such that $(t, \omega) \rightarrow$ $u_{n}(t, \omega),(t, \omega) \rightarrow u_{n}^{*}(t, \omega)$ are product measurable, and for each $\omega$,

$$
\begin{align*}
\left(B u_{n}\right)^{\prime}+u_{n}^{*}(\cdot, \omega)+n P\left(u_{n}(\cdot, \omega)\right) & =f(\cdot, \omega) \text { in } \mathscr{V}^{\prime} \\
B u_{n}(0, \omega) & =0 \tag{77.6.64}
\end{align*}
$$

Here $B$ is as described in that theorem. Using $0 \in \mathscr{K}$ and monotonicity of $P$, the estimates for $A$ lead to an estimate of the form

$$
\left\|u_{n}(\cdot, \omega)\right\|_{\mathscr{V}}+\left\|u_{n}^{*}(\cdot, \omega)\right\|_{\mathscr{V}^{\prime}} \leq C(\omega)
$$

Then there is a subsequence

$$
u_{n} \rightarrow u \text { weakly in } \mathscr{V}
$$

$$
\begin{gathered}
u_{n}^{*} \rightarrow u^{*} \text { weakly in } \mathscr{V}^{\prime} \\
P u_{n} \rightarrow \xi \text { weakly in } \mathscr{V}^{\prime}
\end{gathered}
$$

Let $\Lambda$ denote those $v \in \mathscr{V}$ such that $(B v)^{\prime} \in \mathscr{V}^{\prime}$ and $B v(0)=0$. Then for $v \in \Lambda$,

$$
\left\langle\left(B u_{n}\right)^{\prime}, u_{n}-v\right\rangle+\left\langle u_{n}^{*}(\cdot, \omega), u_{n}-v\right\rangle+n\left\langle P\left(u_{n}(\cdot, \omega)\right), u_{n}-v\right\rangle=\left\langle f(\cdot, \omega), u_{n}-v\right\rangle
$$

Thus by monotonicity considerations,

$$
\begin{equation*}
\left\langle(B v)^{\prime}, u_{n}-v\right\rangle+\left\langle u_{n}^{*}(\cdot, \omega), u_{n}-v\right\rangle+n\left\langle P\left(u_{n}(\cdot, \omega)\right), u_{n}-v\right\rangle \leq\left\langle f(\cdot, \omega), u_{n}-v\right\rangle \tag{*}
\end{equation*}
$$

It follows that

$$
\begin{aligned}
\lim \sup _{n \rightarrow \infty}\left\langle P\left(u_{n}(\cdot, \omega)\right), u_{n}-v\right\rangle & \leq 0 \\
\lim \sup _{n \rightarrow \infty}\left\langle P\left(u_{n}(\cdot, \omega)\right), u_{n}-u\right\rangle & \leq\langle-\xi, u-v\rangle
\end{aligned}
$$

Now, since $\Lambda$ is dense, $v$ can be chosen as close as desired to $u$ and hence

$$
\lim \sup _{n \rightarrow \infty}\left\langle P\left(u_{n}(\cdot, \omega)\right), u_{n}-u\right\rangle \leq 0
$$

Since $P$ is monotone, in fact the limit exists in the above. Therefore, for any $v \in \Lambda$ and $*$,

$$
\lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty}\left(\left\langle P\left(u_{n}(\cdot, \omega)\right), u_{n}-v\right\rangle\right) \geq\langle P u, u-v\rangle
$$

and so

$$
\langle P u, u-v\rangle \leq 0
$$

for all $v \in \Lambda$. It follows that $P u=0$ and so $u \in \mathscr{K}$.
Now for $v \in \Lambda \cap \mathscr{K}$, monotonicity considerations imply

$$
\left\langle(B v)^{\prime}, u_{n}-v\right\rangle+\left\langle u_{n}^{*}(\cdot, \omega), u_{n}-u\right\rangle+\left\langle u_{n}^{*}(\cdot, \omega), u-v\right\rangle \leq\left\langle f(\cdot, \omega), u_{n}-v\right\rangle
$$

Then

$$
\begin{equation*}
\left\langle u_{n}^{*}(\cdot, \omega), u_{n}-u\right\rangle \leq\left\langle f(\cdot, \omega), u_{n}-v\right\rangle-\left\langle(B v)^{\prime}, u_{n}-v\right\rangle-\left\langle u_{n}^{*}(\cdot, \omega), u-v\right\rangle \tag{77.6.65}
\end{equation*}
$$

Then

$$
\lim _{n \rightarrow \infty}\left\langle u_{n}^{*}(\cdot, \omega), u_{n}-u\right\rangle \leq\langle f(\cdot, \omega), u-v\rangle+\left\langle(B v)^{\prime}, v-u\right\rangle+\left\langle u^{*}(\cdot, \omega), v-u\right\rangle
$$

We assume the existence of a regularizing sequence. If $u \in \mathscr{K}$ there exists $u_{i} \rightarrow u$ weakly in $\mathscr{V}$ such that

$$
\limsup _{i \rightarrow \infty}\left\langle\left(B u_{i}\right)^{\prime}, u_{i}-u\right\rangle_{\mathscr{V}} \leq 0
$$

In the above inequality, let $v=u_{i}$

$$
\lim \sup _{n \rightarrow \infty}\left\langle u_{n}^{*}(\cdot, \omega), u_{n}-u\right\rangle \leq\left\langle f(\cdot, \omega), u-u_{i}\right\rangle+\left\langle\left(B u_{i}\right)^{\prime}, u_{i}-u\right\rangle+\left\langle u^{*}(\cdot, \omega), u_{i}-u\right\rangle
$$

Then take limsup $\sin _{i \rightarrow 0}$ of both sides to obtain

$$
\lim \sup _{n \rightarrow \infty}\left\langle u_{n}^{*}(\cdot, \omega), u_{n}-u\right\rangle \leq 0
$$

Now assume the usual limit condition holds for $A(\cdot, \omega)$. In practice, this typically means $A(\cdot, \omega)$ will be single valued, monotone and hemicontinuous because there is no control on the time derivative. However, we will go ahead and assume just that the limit condition holds. This would also take place if $A(\cdot, \omega)$ were defined on $\mathscr{V}$ and maximal monotone, for example. Then for every $v \in \mathscr{V}$,

$$
\lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty}\left\langle u_{n}^{*}(\cdot, \omega), u_{n}-v\right\rangle \geq\left\langle u^{*}(v), u-v\right\rangle
$$

where $u^{*}(v) \in A(u, \omega)$. In particular, this holds for $v=u$ and so

$$
\lim \inf _{n \rightarrow \infty}\left\langle u_{n}^{*}(\cdot, \omega), u_{n}-u\right\rangle \geq 0 \geq \lim \sup _{n \rightarrow \infty}\left\langle u_{n}^{*}(\cdot, \omega), u_{n}-u\right\rangle
$$

showing the the limit exists. Then

$$
\begin{aligned}
\left\langle u^{*}(v), u-v\right\rangle & \leq \lim _{n \rightarrow \infty}\left(\left\langle u_{n}^{*}(\cdot, \omega), u_{n}-v\right\rangle\right) \\
& =\lim _{n \rightarrow \infty}\left(\left\langle u_{n}^{*}(\cdot, \omega), u_{n}-u\right\rangle+\left\langle u_{n}^{*}(\cdot, \omega), u-v\right\rangle\right) \\
& =\left\langle u^{*}, u-v\right\rangle
\end{aligned}
$$

and since this is true for all $v \in \mathscr{V}$ it follows that $u^{*} \in A(u(\cdot, \omega), \omega)$ since otherwise, separation theorems would give a contradiction. If $u^{*}$ were not in $A(u(\cdot, \omega), \omega)$ there would exist $v$ such that for all $z^{*} \in A(u, \omega)$,

$$
\left\langle z^{*}, u-v\right\rangle>\left\langle u^{*}, u-v\right\rangle
$$

contrary to the above. Therefore, in 77.6 .65 we can take the limit of both sides and conclude that for every $v \in \mathscr{K}$ such that $(B v)^{\prime} \in \mathscr{V}^{\prime}, B v(0)=0$,

$$
\left\langle(B v)^{\prime}, u-v\right\rangle+\left\langle u^{*}, u-v\right\rangle \leq\langle f(\cdot, \omega), u-v\rangle
$$

where $u^{*} \in A(u, \omega)$
This has proved the first part of the following theorem which gives measurable solutions to a variational inequality.

Theorem 77.6.1 Suppose $A(\cdot, \omega)$ is monotone hemicontinuous bounded and single valued and coercive as a map from $\mathscr{V}$ to $\mathscr{V}^{\prime}$. Suppose also that for $\omega \rightarrow u(\omega)$ measurable into $\mathscr{V}$, it follows that $\omega \rightarrow A(u(\omega), \omega)$ is measurable into $\mathscr{V}^{\prime}$. Let $\mathscr{K}$ be a closed convex subset of $\mathscr{V}$ containing 0 and let $B \in \mathscr{L}\left(W, W^{\prime}\right)$ be self adjoint and nonnegative as above. Let there be a regularizing sequence $\left\{u_{i}\right\}$ for each $u \in \mathscr{K}$ satisfying $B u_{i}(0)=0,\left(B u_{i}\right)^{\prime} \in \mathscr{V}^{\prime}, u_{i} \in \mathscr{K}$,

$$
\lim \sup _{i \rightarrow \infty}\left\langle\left(B u_{i}\right)^{\prime}, u_{i}-u\right\rangle \leq 0
$$

Then for each $\omega$, there exists a solution to

$$
\left\langle(B v)^{\prime}, u-v\right\rangle+\langle A(u(\cdot, \omega), \omega), u(\cdot, \omega)-v\rangle \leq\langle f(\cdot, \omega), u-v\rangle
$$

valid for all $v \in \mathscr{K}$ such that $(B v)^{\prime} \in \mathscr{V}^{\prime}, B v(0)=0$, and $(t, \omega) \rightarrow u(t, \omega)$, is $\mathscr{B}([0, T]) \times \mathscr{F}$ measurable.

Proof: This follows from Theorem 77.2.10. This is because there is an estimate of the right sort for the measurable functions $u_{n}(\cdot, \omega)$ and $u_{n}^{*}(\cdot, \omega)$ and the above argument which shows that a subsequence has a convergent subsequence which converges appropriately to a solution.

You can have $\mathscr{K}=\mathscr{K}(\omega)$. There would be absolutely no change in the above theorem. You just need to have the operator of penalization satisfy $\omega \rightarrow P(u(\omega), \omega)=$ $F\left(u(\omega)-\operatorname{proj}_{\mathscr{K}(\omega)} u(\omega)\right)$ is measurable into $\mathscr{V}^{\prime}$ provided $\omega \rightarrow u(\omega)$ is measurable into $\mathscr{V}$. What are the conditions on the set valued $\omega \rightarrow \mathscr{K}(\omega)$ which will cause this to take place?

Lemma 77.6.2 Let $\omega \rightarrow \mathscr{K}(\omega)$ be measurable into $\mathscr{V}$. Then $\omega \rightarrow \operatorname{proj}_{\mathscr{K}(\omega)} u(\omega)$ is also measurable into $\mathscr{V}$ if $\omega \rightarrow u(\omega)$ is measurable.

Proof: It follows from standard results on measurable multi-functions [70] that there is a countable collection $\left\{w_{n}(\omega)\right\}, \omega \rightarrow w_{n}(\omega)$ being measurable and $w_{n}(\omega) \in \mathscr{K}(\omega)$ for each $\omega$ such that for each $\omega, \mathscr{K}(\omega)=\overline{\cup_{n} w_{n}(\omega)}$. Let

$$
d_{n}(\omega) \equiv \min \left\{\left\|u(\omega)-w_{k}(\omega)\right\|, k \leq n\right\}
$$

Let $u_{1}(\omega) \equiv w_{1}(\omega)$. Let

$$
u_{2}(\omega)=w_{1}(\omega)
$$

on the set

$$
\left\{\omega:\left\|u(\omega)-w_{1}(\omega)\right\|<\left\{\left\|u(\omega)-w_{2}(\omega)\right\|\right\}\right\}
$$

and

$$
u_{2}(\omega) \equiv w_{2}(\omega) \text { off the above set. }
$$

Thus $\left\|u_{2}(\omega)-u(\omega)\right\|=d_{2}$. Let

$$
\begin{aligned}
& u_{3}(\omega)=w_{1}(\omega) \text { on }\left\{\begin{array}{c}
\omega:\left\|u(\omega)-w_{1}(\omega)\right\| \\
<\left\|u(\omega)-w_{j}(\omega)\right\|, j=2,3
\end{array}\right\} \equiv S_{1} \\
& u_{3}(\omega)=w_{2}(\omega) \text { on } S_{1} \cap\left\{\begin{array}{c}
\omega:\left\|u(\omega)-w_{1}(\omega)\right\| \\
<\left\|u(\omega)-w_{j}(\omega)\right\|, j=3
\end{array}\right\} \\
& u_{3}(\omega)=w_{3}(\omega) \text { on the remainder of } \Omega
\end{aligned}
$$

Thus $\left\|u_{3}(\omega)-u(\omega)\right\|=d_{3}$. Continue this way, obtaining $u_{n}(\omega)$ such that

$$
\left\|u_{n}(\omega)-u(\omega)\right\|=d_{n}(\omega)
$$

and $u_{n}(\omega) \in \mathscr{K}(\omega)$ with $u_{n}$ measurable. Thus, in effect one picks the closest of all the $w_{k}(\omega)$ for $k \leq n$ as the value of $u_{n}(\omega)$ and $u_{n}$ is measurable and by density in $\mathscr{K}(\omega)$ of $\left\{w_{n}(\omega)\right\}$ for each $\omega,\left\{u_{n}(\omega)\right\}$ must be a minimizing sequence for

$$
\lambda(\omega) \equiv \inf \{\|u(\omega)-z\|: z \in \mathscr{K}(\omega)\}
$$

Then it follows that $u_{n}(\omega) \rightarrow \operatorname{proj}_{\mathscr{K}(\omega)} u(\omega)$ weakly in $\mathscr{V}$. Here is why: Suppose it fails to converge to $\operatorname{proj}_{\mathscr{K}(\omega)} u(\omega)$. Since it is minimizing, it is a bounded sequence. Thus
there would be a subsequence, still denoted as $u_{n}(\omega)$ which converges to some $q(\omega) \neq$ $\operatorname{proj}_{\mathscr{K}(\omega)} u(\omega)$. Then

$$
\lambda(\omega)=\lim _{n \rightarrow \infty}\left\|u(\omega)-u_{n}(\omega)\right\| \geq\|u(\omega)-q(\omega)\|
$$

because convex and lower semicontinuous is weakly lower semicontinuous. But this implies $q(\omega)=\operatorname{proj}_{\mathscr{K}(\omega)}(u(\omega))$ because the projection map is well defined thanks to strict convexity of the norm used. This is a contradiction. Hence $\operatorname{proj}_{\mathscr{K}(\omega)} u(\omega)=\lim _{n \rightarrow \infty} u_{n}(\omega)$ and so is a measurable function. It follows that $\omega \rightarrow P(u(\omega), \omega)$ is measurable into $\mathscr{V}$.

The following corollary is now immediate.
Corollary 77.6.3 Suppose $A(\cdot, \omega)$ is monotone hemicontinuous bounded, single valued, and coercive as a map from $\mathscr{V}$ to $\mathscr{V}^{\prime}$. Suppose also that for $\omega \rightarrow u(\omega)$ measurable into $\mathscr{V}$, it follows that $\omega \rightarrow A(u(\omega), \omega)$ is measurable into $\mathscr{V}^{\prime}$. Let $\mathscr{K}(\omega)$ be a closed convex subset of $\mathscr{V}$ containing 0 and $\omega \rightarrow \mathscr{K}(\omega)$ is a set valued measurable multifunction. Let $B \in$ $\mathscr{L}\left(W, W^{\prime}\right)$ be self adjoint and nonnegative as above. Let there be a regularizing sequence $\left\{u_{i}\right\}$ for each $u \in \mathscr{K}$ satisfying $B u_{i}(0)=0,\left(B u_{i}\right)^{\prime} \in \mathscr{V}^{\prime}, u_{i} \in \mathscr{K}$,

$$
\lim \sup _{i \rightarrow \infty}\left\langle\left(B u_{i}\right)^{\prime}, u_{i}-u\right\rangle \leq 0
$$

Then for each $\omega$, there exists a solution to

$$
\left\langle(B v)^{\prime}, u-v\right\rangle+\langle A(u(\cdot, \omega), \omega), u(\cdot, \omega)-v\rangle \leq\langle f(\cdot, \omega), u(\cdot, \omega)-v\rangle
$$

valid for all $v \in \mathscr{K}(\omega)$ with $(B v)^{\prime} \in \mathscr{V}^{\prime}, B v(0)=0$, and $(t, \omega) \rightarrow u(t, \omega)$, is $\mathscr{B}([0, T]) \times \mathscr{F}$ measurable.

Proof: The proof is identical to the above. One obtains a measurable solution to 77.6.64 in which $P$ is replaced with $P(\cdot, \omega)$. Then one proceeds in exactly the same steps as before and finally uses Theorem 77.2.10 to obtain the measurability of a solution to the variational inequality.

What does it mean for $u(\omega) \in \mathscr{K}(\omega)$ for each $\omega$ ? It means that there is a sequence of the $w_{n}\left\{w_{n(\omega)}\right\}$ such that each $w_{n}$ is measurable into $\mathscr{V}$ implying that for each $\omega$ there is a representative $t \rightarrow w_{n}(t, \omega)$ such that the resulting $(t, \omega) \rightarrow w_{n}(t, \omega)$ is product measurable and $\left\|u(\cdot, \omega)-w_{n(\omega)}(\cdot, \omega)\right\|_{\mathscr{V}} \rightarrow 0$. Thus there is no reason to think that $(t, \omega) \rightarrow u(t, \omega)$ is product measurable. The message of the above corollary says that nevertheless, there is a measurable solution to the variational inequality.

### 77.7 Progressively Measurable Solutions

In the context of uniqueness of the evolution initial value problem for fixed $\omega$, one can prove theorems about progressively measurable solutions fairly easily. First is a definition of the term progressively measurable.
Definition 77.7.1 Let $\mathscr{F}_{t}$ be an increasing in $t$ set of $\sigma$ algebras of sets of $\mathscr{F}$. Thus each $\mathscr{F}_{t}$ is a $\sigma$ algebra and if $s \leq t$, then $\mathscr{F}_{s} \leq \mathscr{F}_{t}$. This set of $\sigma$ algebras is called a filtration. A set $S \subseteq[0, T] \times \Omega$ is called progressively measurable iffor every $t \in[0, T]$,

$$
S \cap[0, t] \times \Omega \in \mathscr{B}([0, t]) \times \mathscr{F}_{t}
$$

Denote by $\mathscr{P}$ the progressively measurable sets. This is a $\sigma$ algebra of subsets of $[0, T] \times \Omega$. A function $g$ is progressively measurable if $\mathscr{X}_{[0, t]} g$ is $\mathscr{B}([0, t]) \times \mathscr{F}_{t}$ measurable for each $t$.

Let $A$ satisfy the bounded condition 77.5.25, the condition on subintervals 77.5.26, the specific boundedness estimate 77.5 .27 , the specific coercivity estimate involving $B$ and $A$ in 77.5.29, and the limit condition 77.5.30. In place of the condition on the existence of a measurable selection 77.5.34, we will assume the following condition.

Condition 77.7.2 For each $t \leq T$, if $\omega \rightarrow u(\cdot, \omega)$ is $\mathscr{F}_{t}$ measurable into $\mathscr{V}_{[0, t]}$, then there exists a $\mathscr{F}_{t}$ measurable selection of $A(u(\cdot, \omega), \omega)$ into $\mathscr{V}_{[0, t]}^{\prime}$.

Note that $u(\cdot, \omega)$ is in $\mathscr{V}_{[0, t]}$ so $u(t, \omega) \in V$.
In this section, we assume that $\omega \rightarrow B(\omega)$ is $\mathscr{F}_{0}$ measurable into $\mathscr{L}\left(W, W^{\prime}\right)$. For convenience, here are the conditions used on $A$.

For the operator $A(\cdot, \omega) \cdot A(\cdot, \omega): \mathscr{V}_{I} \rightarrow \mathscr{V}_{I}^{\prime}$ for each $I$ a subinterval of $[0, T], I=[0, \hat{T}]$ and

$$
\begin{equation*}
A(\cdot, \omega): \mathscr{V}_{I} \rightarrow \mathscr{P}\left(\mathscr{V}_{I}^{\prime}\right) \text { is bounded, } \tag{77.7.66}
\end{equation*}
$$

If, for $u \in \mathscr{V}$,

$$
u^{*} \mathscr{X}_{[0, \hat{T}]} \in A\left(u \mathscr{X}_{[0, \hat{T}]}, \omega\right)
$$

for each $\hat{T}$ in an increasing sequence converging to $T$, then

$$
\begin{equation*}
u^{*} \in A(u, \omega) \tag{77.7.67}
\end{equation*}
$$

For some $\hat{p} \geq p$, assume the specific estimate

$$
\begin{equation*}
\sup \left\{\left\|u^{*}\right\|_{\mathscr{V}_{I}^{\prime}}: u^{*} \in A(u, \omega)\right\} \leq a(\omega)+b(\omega)\|u\|_{\mathscr{V}_{I}}^{\hat{p}-1} \tag{77.7.68}
\end{equation*}
$$

where $a(\omega), b(\omega)$ are nonnegative. Note that the growth could be quadratic in case $p=2$. This really just says there is polynomial growth. Also assume the coercivity condition:

$$
\begin{equation*}
\lim _{\substack{\|u\|_{\mathscr{V}} \rightarrow \infty \\ u \in X_{r}}} \frac{\inf \left\{2\left\langle u^{*}, u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}+\langle B u, u\rangle(T): u^{*} \in A(u, \omega)\right\}}{\|u\|_{\mathscr{V}}}=\infty, \tag{77.7.69}
\end{equation*}
$$

or alternatively the following specific estimate valid for each $t \leq T$ and for some $\lambda(\omega) \geq 0$,

$$
\begin{equation*}
\inf \left(\int_{0}^{t}\left\langle u^{*}, u\right\rangle+\lambda(\omega)\langle B u, u\rangle d t: u^{*} \in A(u, \omega)\right) \geq \delta(\omega) \int_{0}^{t}\|u\|_{V}^{p} d s-m(\omega) \tag{77.7.70}
\end{equation*}
$$

where $m(\omega)$ is some nonnegative constant, $\delta(\omega)>0$. Note that the estimate is a coercivity condition on $\lambda B+A$ rather than on $A$ but is more specific than 77.5.28.

Let $U$ be a Banach space dense in $V$ and that if $u_{i} \rightharpoonup u$ in $\mathscr{V}_{I}$ and $u_{i}^{*} \in A\left(u_{i}\right)$ with $u_{i}^{*} \rightharpoonup u^{*}$ in $\mathscr{V}_{I}^{\prime}$ and $\left(B u_{i}\right)^{\prime} \rightharpoonup(B u)^{\prime}$ in $\mathscr{U}_{r I}^{\prime}$, $\rightharpoonup$ denoting weak convergence, then if

$$
\lim \sup _{i \rightarrow \infty}\left\langle u_{i}^{*}, u_{i}-u\right\rangle_{\mathscr{V}_{I}^{\prime}, \mathscr{y}_{I}} \leq 0
$$

it follows that for all $v \in \mathscr{V}_{I}$, there exists $u^{*}(v) \in A u$ such that

$$
\begin{equation*}
\lim _{i \rightarrow \infty}\left\langle u_{i}^{*}, u_{i}-v\right\rangle_{\mathscr{V}_{I}^{\prime}, \mathscr{V}_{I}} \geq\left\langle u^{*}(v), u-v\right\rangle_{\mathscr{V}_{I}^{\prime}, \mathscr{V}_{I}} \tag{77.7.71}
\end{equation*}
$$

where $r>\max (\hat{p}, 2)$, and we replace $p$ with $r$ and $I$ an arbitrary subinterval of the form $[0, \hat{T}], \hat{T}<T$, for $[0, T]$, and $U$ for $V$ where indicated. Here

$$
\mathscr{U}_{r I} \equiv L^{r}(I ; U)
$$

The theorem to be shown is the following.
Theorem 77.7.3 Assume the above conditions, 77.7.66, 77.7.67, 77.7.68, 77.7.70, 77.7.71, and the Condition 77.7.2. Let $u_{0}$ be $\mathscr{F}_{0}$ measurable and $\omega \rightarrow B(\omega)$ also $\mathscr{F}_{0}$ measurable and $(t, \omega) \rightarrow \mathscr{X}_{[0, t]}(t) f(t, \omega)$ is $\mathscr{B}([0, t]) \times \mathscr{F}_{t}$ product measurable into $V^{\prime}$ for each $t$. Also assume that for each $\omega$, there is at most one solution to the evolution equation

$$
\begin{aligned}
(B(\omega) u(\cdot, \omega))(t)-B(\omega) u_{0}(\omega)+\int_{0}^{t} u^{*}(\cdot, \omega) d s & =\int_{0}^{t} f(s, \omega) d s \\
u^{*}(\cdot, \omega) & \in A(u(\cdot, \omega), \omega)
\end{aligned}
$$

for $t \in[0, \hat{T}]$ for each $\hat{T} \leq T$. Then there exists a unique solution $\left(u(\cdot, \omega), u^{*}(\cdot, \omega)\right)$ in $\mathscr{V} \times \mathscr{V}^{\prime}$ to the above integral equation for each $\omega$. This solution satisfies $(t, \omega) \rightarrow$ $\left(u(t, \omega), u^{*}(t, \omega)\right)$ is progressively measurable into $V \times V^{\prime}$.

Proof: Let $\mathscr{T}$ denote subsets of $(0, T]$ which contain $T$ such that for $S \in \mathscr{T}$, there exists a solution $u_{S}$ for each $\omega$ to the above integral equation on $[0, T]$ such that $(t, \omega) \rightarrow$ $\mathscr{X}_{[0, s]}(t) u_{S}(t, \omega)$ is $\mathscr{B}([0, s]) \times \mathscr{F}_{s}$ measurable for each $s \in S$. Then $\{T\} \in \mathscr{T}$. If $S, S^{\prime}$ are in $\mathscr{T}$, then $S \leq S^{\prime}$ will mean that $S \subseteq S^{\prime}$ and also $u_{S}(t, \omega)=u_{S^{\prime}}(t, \omega)$ in $V$ for all $t \in S$, similar for $u_{S}^{*}$ and $u_{S^{\prime}}^{*}$. Note that equality must hold in $\mathscr{V}$ by uniqueness. Now let $\mathscr{C}$ denote a maximal chain. Is $\cup \mathscr{C} \equiv S_{\infty}$ all of $(0, T]$ ? What is $u_{S_{\infty}}$ ? Define $u_{S_{\infty}}(t, \omega)$ the common value of $u_{S}(t, \omega)$ for all $S$ in $\mathscr{C}$, which contain $t \in S_{\infty}$. If $s \in S_{\infty}$, then it is in some $S \in \mathscr{C}$ and so the product measurability condition holds for this $s$. Thus $S_{\infty}$ is a maximal element of the partially ordered set. Is $S_{\infty}$ all of $(0, T]$ ? Suppose $\hat{s} \notin S_{\infty}, T>\hat{s}>0$.

From Theorem 77.5.6 there exists a solution to the integral equation on $[0, \hat{s}]$ called $u_{1}$ such that $(t, \omega) \rightarrow u_{1}(t, \omega)$ is $\mathscr{B}([0, \hat{s}]) \times \mathscr{F}_{\hat{s}}$ measurable, similar for $u_{1}^{*}$. By the same theorem, there is a solution on $[0, T], u_{2}$ which is $\mathscr{B}([0, T]) \times \mathscr{F}_{T}$ measurable. Now by uniqueness, $u_{2}(\cdot, \omega)=u_{1}(\cdot, \omega)$ in $\mathscr{V}_{[0, \hat{s}]}$, similar for $u_{i}^{*}$. Therefore, no harm is done in redefining $u_{2}$ on $[0, \hat{s}]$ so that $u_{2}(t, \omega)=u_{1}(t, \omega)$ for all $t \in[0, \hat{s}]$, similar for $u^{*}$. Denote these functions as $\hat{u}, \hat{u}^{*}$. By uniqueness, $u_{S_{\infty}}(\cdot, \omega)=\hat{u}(\cdot, \omega)$ in $L^{p}([0, \hat{s}], V)$. Thus no harm is done by re-defining $\hat{u}(s, \omega)$ to equal $u_{S_{\infty}}(s, \omega)$ for $s<\hat{s}$ and $\hat{u}(\hat{s}, \omega)$ at $\hat{s}$. As to $s>\hat{s}$ also re define $\hat{u}(s, \omega) \equiv u_{S_{\infty}}(s, \omega)$ for such $s$. By uniqueness, the two are equal in $\mathscr{V}_{[\hat{s}, T]}$ and so no change occurs in the solution of the integral equation. Now $S_{\infty}$ was not maximal after all. $S_{\infty} \cup\{\hat{s}\}$ is larger. This contradiction shows that in fact, $S_{\infty}=(0, T]$.

Theorem 77.7.4 Assume the above conditions, 77.7.66, 77.7.67, 77.7.68, 77.7.70, 77.7.71, and the Condition 77.7.2. Let $u_{0}$ be $\mathscr{F}_{0}$ measurable and $\omega \rightarrow B(\omega)$ also $\mathscr{F}_{0}$ measurable and $(t, \omega) \rightarrow \mathscr{X}_{[0, t]}(t) f(t, \omega)$ is $\mathscr{B}([0, t]) \times \mathscr{F}_{t}$ product measurable into $V^{\prime}$ for each $t$.

$$
B(\omega)=k(\omega) B, k(\omega) \geq 0, k \text { measurable }
$$

Also let $t \rightarrow q(t, \omega)$ be continuous and $q$ is progressively measurable into $V$. Suppose there is at most one solution to

$$
\begin{equation*}
B u(t, \omega)+\int_{0}^{t} z(s, \omega) d s=\int_{0}^{t} f(s, \omega) d s+B u_{0}(\omega)+B q(t, \omega) \tag{77.7.72}
\end{equation*}
$$

for each $\omega$. Then the solution $u$ to the above integral equation is progressively measurable and so is $z$. Moreover, for each $\omega$, both $B u(t, \omega)=B(u(t, \omega))$ a.e. $t$ and $z(\cdot, \omega) \in$ $A(u(\cdot, \omega), \omega)$. Also, for each $a \in[0, T]$,

$$
B u(t, \omega)+\int_{a}^{t} z(s, \omega) d s=\int_{a}^{t} f(s, \omega) d s+B u(a, \omega)+B q(t, \omega)-B q(a, \omega)
$$

Proof: By Theorem 77.5 .7 there exists a solution to 77.7 .72 which is $\mathscr{B}([0, T]) \times \mathscr{F}_{T}$ measurable. Now, as in the proof of Theorem 77.5 .7 one can define a new operator

$$
A_{r}(w, \omega) \equiv A\left(\omega, w+q^{\tau_{r}}(\cdot, \omega)\right)
$$

where $\tau_{r}$ is the stopping time defined there. Then, since $q$ is progressively measurable, the progressively measurable condition is satisfied for this new operator. Hence by Theorem 77.7.3 there exists a unique solution $w_{r}$ which is progressively measurable to the integral equation

$$
B w_{r}(t, \omega)+\int_{0}^{t} z_{r}(s, \omega) d s=\int_{0}^{t} f(s, \omega) d s+B u_{0}(\omega)
$$

where $z_{r}(\cdot, \omega) \in A_{r}(w(\cdot, \omega), \omega)$. Then as in Theorem 77.7.3 you can let $r \rightarrow \infty$ and eventually $q^{\tau_{r}}(\cdot, \omega)=q(\cdot, \omega)$. Then, passing to a limit, it follows that for a given $\omega$, there is a solution to

$$
\begin{aligned}
B w(t, \omega)+\int_{0}^{t} z(s, \omega) d s & =\int_{0}^{t} f(s, \omega) d s+B u_{0}(\omega) \\
z(\cdot, \omega) & \in A(w(\cdot, \omega)+q(\cdot, \omega), \omega)
\end{aligned}
$$

which is progressively measurable because $w(\cdot, \omega)=\lim _{r \rightarrow \infty} w_{r}(\cdot, \omega)$ in $\mathscr{V}$ each $w_{r}$ being progressively measurable. Note how uniqueness for fixed $\omega$ is important in this argument. Recall that

$$
\tau_{r} \equiv \inf \{t:|q(t, \omega)|>r\}
$$

By continuity, eventually, for a given $\omega, \tau_{r}=\infty$ and so no further change takes place in $q^{\tau_{r}}(\cdot, \omega)$ for that $\omega$. By uniqueness, the same is true of the solution $w_{r}(\cdot, \omega)$ and so pointwise convergence takes place for the $w_{r}$. Without uniqueness holding, this becomes very unclear. Thus for each $\hat{T}<T, \omega \rightarrow w(\cdot, \omega)$ is measurable into $\mathscr{V}[0, \hat{T}]$. Then by Lemma 77.4.2, $w$ has a representative in $\mathscr{V}$ for each $\omega$ such that the resulting function satisfies $(t, \omega) \rightarrow \mathscr{X}_{[0, \hat{T}]}(t) w(t, \omega)$ is $\mathscr{B}([0, \hat{T}]) \times \mathscr{F}_{\hat{T}}$ measurable into $V$. Thus one can assume that $w$ is progressively measurable. Now as in Theorem 77.5.7, Define $u=w+q$.

The last claim follows from letting $t=a$ in the top equation and then subtracting this from the top equation with $t>a$.

### 77.8 Adding A Quasi-bounded Operator

Recall the following conditions for the various operators.

## Bounded and coercive conditions

$A(\cdot, \omega) \cdot A(\cdot, \omega): \mathscr{V}_{I} \rightarrow \mathscr{V}_{I}^{\prime}$ for each $I$ a subinterval of $[0, T] I=[0, \hat{T}], \hat{T} \leq T$

$$
\begin{equation*}
A(\cdot, \omega): \mathscr{V}_{I} \rightarrow \mathscr{P}\left(\mathscr{V}_{I}^{\prime}\right) \text { is bounded, } \tag{77.8.73}
\end{equation*}
$$

If, for $u \in \mathscr{V}$,

$$
u^{*} \mathscr{X}_{[0, \hat{T}]} \in A\left(u \mathscr{X}_{[0, \hat{T}]}, \omega\right)
$$

for each $\hat{T}$ in an increasing sequence converging to $T$, then

$$
\begin{equation*}
u^{*} \in A(u, \omega) \tag{77.8.74}
\end{equation*}
$$

Assume the specific estimate

$$
\begin{equation*}
\sup \left\{\left\|u^{*}\right\|_{\mathscr{V}_{I}^{\prime}}: u^{*} \in A(u, \omega)\right\} \leq a(\omega)+b(\omega)\|u\|_{\mathscr{V}_{I}}^{p-1} \tag{77.8.75}
\end{equation*}
$$

where $a(\omega), b(\omega)$ are nonnegative. Also assume the following coercivity estimate valid for each $t \leq T$ and for some $\lambda(\omega) \geq 0$,

$$
\begin{equation*}
\inf \left(\int_{0}^{t}\left\langle u^{*}, u\right\rangle+\lambda(\omega)\langle B u, u\rangle d t: u^{*} \in A(u, \omega)\right) \geq \delta(\omega) \int_{0}^{t}\|u\|_{V}^{p} d s-m(\omega) \tag{77.8.76}
\end{equation*}
$$

where $m(\omega)$ is some nonnegative constant, $\delta(\omega)>0$.

## Limit condition

Let $U$ be a Banach space dense in $V$ and that if $u_{i} \rightharpoonup u$ in $\mathscr{V}_{I}$ and $u_{i}^{*} \in A\left(u_{i}\right)$ with $u_{i}^{*} \rightharpoonup u^{*}$ in $\mathscr{V}_{I}^{\prime}$ and $\left(B u_{i}\right)^{\prime} \rightharpoonup(B u)^{\prime}$ in $\mathscr{U}_{r I}^{\prime}$, $\rightharpoonup$ denoting weak convergence, then if

$$
\limsup _{i \rightarrow \infty}\left\langle u_{i}^{*}, u_{i}-u\right\rangle_{\mathscr{Y}_{I}^{\prime}, \mathscr{y}_{I}} \leq 0
$$

it follows that for all $v \in \mathscr{V}_{I}$, there exists $u^{*}(v) \in A u$ such that

$$
\begin{equation*}
\lim _{i \rightarrow \infty}\left\langle u_{i}^{*}, u_{i}-v\right\rangle_{\mathscr{V}_{I}^{\prime}, \mathscr{V}_{I}} \geq\left\langle u^{*}(v), u-v\right\rangle_{\mathscr{V}_{I}^{\prime}, \mathscr{V}_{I}} \tag{77.8.77}
\end{equation*}
$$

where $r>\max (p, 2)$, and we replace $p$ with $r$ and $I$ an arbitrary subinterval of the form $[0, \hat{T}], \hat{T}<T$, for $[0, T]$, and $U$ for $V$ where indicated. Here

$$
\mathscr{U}_{r I} \equiv L^{r}(I ; U)
$$

Typically, $U$ is compactly embedded in $V$.

## Measurability condition

For $\omega \rightarrow u(\cdot, \omega)$ measurable into $\mathscr{V}$,

$$
\begin{equation*}
\omega \rightarrow A(u(\cdot, \omega), \omega) \text { has a measurable selection into } \mathscr{V}^{\prime} \tag{77.8.78}
\end{equation*}
$$

This last condition means there is a function $\omega \rightarrow u^{*}(\omega)$ which is measurable into $\mathscr{V}^{\prime}$ such that $u^{*}(\omega) \in A(u(\cdot, \omega), \omega)$.

As for the operator $B$ it is either independent of $\omega$ and is a nonnegative self adjoint operator mapping $W$ to $W^{\prime}$ or else it is of the form $k(\omega) B$ where $k \geq 0$ and is measurable.

We will assume here that $p>1$. Then the following main result was obtained above. It is Theorem 77.5.7.

Theorem 77.8.1 If 77.8.73-77.8.78 and $B$ as described above,let $q(t, \omega)$ be a product measurable function into $V$ such that $t \rightarrow q(t, \omega)$ is continuous, $q(0, \omega)=0$.

Then, there exists a solution $u$ of the integral equation

$$
B u(t, \omega)+\int_{0}^{t} z(s, \omega) d s=\int_{0}^{t} f(s, \omega) d s+B u_{0}(\omega)+B q(t, \omega)
$$

where $(t, \omega) \rightarrow u(t, \omega)$ is measurable. Moreover, for each $\omega, B u(t, \omega)=B(u(t, \omega))$ for a.e. $t$ and $z(\cdot, \omega) \in A(\omega, u(\cdot, \omega))$ for a.e. $t$. Also, for each $a \in[0, T]$,

$$
B u(t, \omega)+\int_{a}^{t} z(s, \omega) d s=\int_{a}^{t} f(s, \omega) d s+B u(a, \omega)+B q(t, \omega)-B q(a, \omega)
$$

The idea here is to show that everything works as well if a suitable unbounded maximal monotone operator is added in. The result is interesting but not as interesting as it might be. This is because the maximal monotone operator must be quasi bounded. Still it is interesting to note that the above holds for some unbounded operators. This has been pointed out in the case where there are no stochastic effects in a recent paper [54]. This generalizes this result by considering the measurability of solutions and allowing for possibly degenerate leading operator $B$.

To begin with assume $q=0$.
Now let $G: D(G) \subseteq \mathscr{V} \rightarrow \mathscr{P}\left(\mathscr{V}^{\prime}\right)$ be maximal monotone. Also assume that $0 \in D(G)$. Then you have

$$
\left\langle u^{*}, u\right\rangle \geq\left\langle g^{*}, u\right\rangle \text { if } u^{*} \in G u
$$

for every $g^{*} \in G(0)$. Hence

$$
\begin{equation*}
\left\langle u^{*}, u\right\rangle \geq-|G(0)|\|u\|_{\mathscr{V}} \text { if } u^{*} \in G u \tag{77.8.79}
\end{equation*}
$$

where $|G(0)| \equiv \inf \left\{\left\|y^{*}\right\|_{\mathscr{V}^{\prime}}: y^{*} \in G(0)\right\}$.
There is a standard way of approximating $G$ with bounded demicontinuous operators which is reviewed next. It is all in Barbu [13]. See Section 25.7.4. Since $G$ is maximal monotone,

$$
0 \in F\left(x_{\mu}-x\right)+\mu^{p-1} G\left(x_{\mu}\right)
$$

where $F$ is a duality map for $p$, the one used in the above theorem. Barbu uses only $p=2$ but it works just as well for arbitrary $p>1$ with the minor modifications used here. To see
this, you consider $\hat{G}(y) \equiv G(x+y)$. Then $\hat{G}$ is also maximal monotone and so there exists a solution to

$$
0 \in F(\hat{x})+\mu^{p-1} \hat{G}(\hat{x})=F(\hat{x})+\mu^{p-1} G(x+\hat{x})
$$

Now let $x_{\mu}=x+\hat{x}$ so $\hat{x}=x_{\mu}-x$. Hence

$$
0 \in F\left(x_{\mu}-x\right)+\mu^{p-1} G x_{\mu}
$$

The symbol $\lim \sup _{n, n \rightarrow \infty} a_{m n}$ means $\lim _{N \rightarrow \infty}\left(\sup _{m \geq N, n \geq N} a_{m n}\right)$.
Lemma 77.8.2 Suppose $\limsup \lim _{n \rightarrow \infty} a_{m n} \leq 0$. Then $\limsup \lim _{m \rightarrow \infty}\left(\limsup _{n \rightarrow \infty} a_{m n}\right) \leq 0$.
Proof: Suppose the first inequality. Then for $\varepsilon>0$, there exists $N$ such that if $n, m$ are both as large as $N$, then $a_{m n} \leq \varepsilon$. Thus $\sup _{n \geq N} a_{m n} \leq \varepsilon$ provided $m \geq N$ also. Hence for such $m$,

$$
\lim _{n \rightarrow \infty}\left(\sup _{n \geq N} a_{m n}\right) \leq \varepsilon
$$

for each $m \geq N$. It follows $\limsup _{m \rightarrow \infty}\left(\limsup _{n \rightarrow \infty} a_{m n}\right) \leq \varepsilon$. Since $\varepsilon$ is arbitrary, this proves the lemma.

Then here is a simple observation.
Lemma 77.8.3 Let $G: D(G) \subseteq X \rightarrow \mathscr{P}\left(X^{\prime}\right)$ where $X$ is a Banach space be maximal monotone and let $v_{n} \in G u_{n}$ and

$$
u_{n} \rightarrow u, v_{n} \rightarrow v \text { weakly. }
$$

Also suppose that

$$
\lim \sup _{m, n \rightarrow \infty}\left\langle v_{n}-v_{m}, u_{n}-u_{m}\right\rangle \leq 0
$$

or

$$
\limsup _{n \rightarrow \infty}\left\langle v_{n}-v, u_{n}-u\right\rangle \leq 0
$$

Then $[u, v] \in \mathscr{G}(G)$ and $\left\langle v_{n}, u_{n}\right\rangle \rightarrow\langle v, u\rangle$.
Proof: By monotonicity,

$$
\begin{aligned}
0 & \geq \lim \sup _{m, n \rightarrow \infty}\left\langle v_{n}-v_{m}, u_{n}-u_{m}\right\rangle \\
& \geq \lim _{m, n \rightarrow \infty}\left\langle v_{n}-v_{m}, u_{n}-u_{m}\right\rangle \geq 0
\end{aligned}
$$

and so

$$
\lim _{m, n \rightarrow \infty}\left\langle v_{n}-v_{m}, u_{n}-u_{m}\right\rangle=0
$$

Suppose then that $\left\langle v_{n}, u_{n}\right\rangle$ fails to converge to $\langle v, u\rangle$. Then there is a subsequence, still denoted with subscript $n$ such that $\left\langle v_{n}, u_{n}\right\rangle \rightarrow \mu \neq\langle v, u\rangle$. Let $\varepsilon>0$. Then there exists $M$ such that if $n, m>M$, then

$$
\left|\left\langle v_{n}, u_{n}\right\rangle-\mu\right|<\varepsilon,\left|\left\langle v_{n}-v_{m}, u_{n}-u_{m}\right\rangle\right|<\varepsilon
$$

Then if $m, n>M$,

$$
\left|\left\langle v_{n}-v_{m}, u_{n}-u_{m}\right\rangle\right|=\left|\left\langle v_{n}, u_{n}\right\rangle+\left\langle v_{m}, u_{m}\right\rangle-\left\langle v_{n}, u_{m}\right\rangle-\left\langle v_{m}, u_{n}\right\rangle\right|<\varepsilon
$$

Hence it is also true that

$$
\left|\left\langle v_{n}, u_{n}\right\rangle+\left\langle v_{m}, u_{m}\right\rangle-\left\langle v_{n}, u_{m}\right\rangle-\left\langle v_{m}, u_{n}\right\rangle\right| \leq\left|2 \mu-\left(\left\langle v_{n}, u_{m}\right\rangle+\left\langle v_{m}, u_{n}\right\rangle\right)\right|<3 \varepsilon
$$

Now take a limit first with respect to $n$ and then with respect to $m$ to obtain

$$
|2 \mu-(\langle v, u\rangle+\langle v, u\rangle)|<3 \varepsilon
$$

Since $\varepsilon$ is arbitrary, $\mu=\langle v, u\rangle$ after all. Hence the claim that $\left\langle v_{n}, u_{m}\right\rangle \rightarrow\langle v, u\rangle$ is verified. Next suppose $[x, y] \in \mathscr{G}(G)$ and consider

$$
\begin{aligned}
& \langle v-y, u-x\rangle=\langle v, u\rangle-\langle v, x\rangle-\langle y, u\rangle+\langle y, x\rangle \\
& =\lim _{n \rightarrow \infty}\left(\left\langle v_{n}, u_{n}\right\rangle-\left\langle v_{n}, x\right\rangle-\left\langle y, u_{n}\right\rangle+\langle y, x\rangle\right) \\
& =\lim _{n \rightarrow \infty}\left\langle v_{n}-y, u_{n}-x\right\rangle \geq 0
\end{aligned}
$$

and since $[x, y]$ is arbitrary, it follows that $v \in G u$.
Next suppose $\lim \sup _{n \rightarrow \infty}\left\langle v_{n}-v, u_{n}-u\right\rangle \leq 0$. It is not known that $[u, v] \in \mathscr{G}(G)$.

$$
\begin{aligned}
\lim \sup _{n \rightarrow \infty}\left[\left\langle v_{n}, u_{n}\right\rangle-\left\langle v, u_{n}\right\rangle-\left\langle v_{n}, u\right\rangle+\langle v, u\rangle\right] & \leq 0 \\
\lim \sup _{n \rightarrow \infty}\left\langle v_{n}, u_{n}\right\rangle-\langle v, u\rangle & \leq 0
\end{aligned}
$$

Thus $\lim \sup _{n \rightarrow \infty}\left\langle v_{n}, u_{n}\right\rangle \leq\langle v, u\rangle$. Now let $[x, y] \in \mathscr{G}(G)$

$$
\begin{aligned}
& \langle v-y, u-x\rangle=\langle v, u\rangle-\langle v, x\rangle-\langle y, u\rangle+\langle y, x\rangle \\
& \geq \lim \sup _{n \rightarrow \infty}\left[\left\langle v_{n}, u_{n}\right\rangle-\left\langle v_{n}, x\right\rangle-\left\langle y, u_{n}\right\rangle+\langle y, x\rangle\right] \\
& \geq \lim _{n \rightarrow \infty}\left[\left\langle v_{n}-y, u_{n}-x\right\rangle\right] \geq 0
\end{aligned}
$$

Hence $[u, v] \in \mathscr{G}(G)$. Now

$$
\lim \sup _{n \rightarrow \infty}\left\langle v_{n}-v, u_{n}-u\right\rangle \leq 0 \leq \lim _{n \rightarrow \infty} \inf _{n}\left\langle v_{n}-v, u_{n}-u\right\rangle
$$

the second coming from monotonicity and the fact that $v \in G u$. Therefore,

$$
\lim _{n \rightarrow \infty}\left\langle v_{n}-v, u_{n}-u\right\rangle=0
$$

which shows that $\lim _{n \rightarrow \infty}\left\langle v_{n}, u_{n}\right\rangle=\langle v, u\rangle$.
Similar reasoning implies

Lemma 77.8.4 Suppose $A$ is a set valued operator, $A: X \rightarrow \mathscr{P}(X)$ and $u_{n}^{*} \in A u_{n}$. Suppose also that $u_{n} \rightarrow u$ weakly and $u_{n}^{*} \rightarrow u^{*}$ weakly. Suppose also that

$$
\lim \sup _{m, n \rightarrow \infty}\left\langle u_{n}^{*}-u_{m}^{*}, u_{n}-u_{m}\right\rangle \leq 0
$$

Then one can conclude that

$$
\lim \sup _{n \rightarrow \infty}\left\langle u_{n}^{*}, u_{n}-u\right\rangle \leq 0
$$

Proof: It is assumed that

$$
\lim \sup _{m, n \rightarrow \infty}\left(\left\langle u_{n}^{*}, u_{n}\right\rangle+\left\langle u_{m}^{*}, u_{m}\right\rangle-\left(\left\langle u_{n}^{*}, u_{m}\right\rangle+\left\langle u_{m}^{*}, u_{n}\right\rangle\right)\right) \leq 0
$$

Then is it the case that $\limsup _{n \rightarrow \infty}\left\langle u_{n}^{*}, u_{n}\right\rangle \leq\left\langle u^{*}, u\right\rangle$ ? Let $\mu$ equal $\lim \sup _{n \rightarrow \infty}\left\langle u_{n}^{*}, u_{n}\right\rangle$. Then in the above, it implies

$$
\left(2 \mu-\left(\left\langle u_{n}^{*}, u_{m}\right\rangle+\left\langle u_{m}^{*}, u_{n}\right\rangle\right)\right)<\varepsilon
$$

whenever $m, n$ large enough. Thus taking $\limsup _{n \rightarrow \infty} \limsup _{m \rightarrow \infty}$ of the above, you get

$$
\left(2 \mu-\left(\left\langle u^{*}, u\right\rangle+\left\langle u^{*}, u\right\rangle\right)\right)<\varepsilon
$$

Thus you at least need $\mu \leq\left\langle u^{*}, u\right\rangle$. That is, $\limsup _{n \rightarrow \infty}\left\langle u_{n}^{*}, u_{n}\right\rangle \leq\left\langle u^{*}, u\right\rangle$. Hence

$$
\lim \sup _{n \rightarrow \infty}\left\langle u_{n}^{*}, u_{n}-u\right\rangle=\lim \sup _{n \rightarrow \infty}\left\langle u_{n}^{*}, u_{n}\right\rangle-\left\langle u^{*}, u\right\rangle \leq\left\langle u^{*}, u\right\rangle-\left\langle u^{*}, u\right\rangle=0
$$

Definition 77.8.5 Let $x_{\mu}$ just defined be denoted by $J_{\mu} x$ and define also

$$
G_{\mu}(x) \equiv-\mu^{-(p-1)} F\left(x_{\mu}-x\right)
$$

This $x_{\mu}$ is defined as follows.

$$
0 \in F\left(x_{\mu}-x\right)+\mu^{p-1} G x_{\mu}
$$

Later, we will write $J_{\mu}$ u for $u_{\mu}$. Thus

$$
0=F\left(J_{\mu} u-u\right)+\mu^{p-1} z_{\mu}, z_{\mu} \in G\left(J_{\mu} u\right)
$$

Also from this definition,

$$
G_{\mu}(u)=-\mu^{-(p-1)} F\left(J_{\mu} u-u\right)=z_{\mu} \in G\left(J_{\mu} u\right)
$$

Then there are some things which can be said about these operators.
Theorem 77.8.6 The following hold. Here $V$ is a reflexive Banach space with strictly convex norm. $G: D(G) \rightarrow \mathscr{P}\left(V^{\prime}\right)$ is maximal monotone. Then

1. $J_{\mu}$ and $G_{\mu}$ are bounded single valued operators defined on $V$. Bounded means they take bounded sets to bounded sets. Also $G_{\mu}$ is a monotone operator.
2. $G_{\mu}, J_{\mu}$ are demicontinuous. That is, strongly convergent sequences are mapped to weakly convergent sequences.
3. For every

$$
x \in D(G),\left\|G_{\mu}(x)\right\| \leq|G x| \equiv \inf \left\{\left\|y^{*}\right\|: y^{*} \in G x\right\}
$$

For every $x \in \overline{\operatorname{conv}(D(G))}$, it follows that $\lim _{\mu \rightarrow 0} J_{\mu}(x)=x$. The new symbol means the closure of the convex hull. It is the closure of the set of all convex combinations of points of $D(G)$.

Then $A(\cdot, \omega)+G_{\mu}$ will be bounded and have the same limit properties as $A(\cdot, \omega)$. As to measurability, $G$ and hence $G_{\mu}$ do not depend on $\omega$ and so the measurability condition will hold.

What about the estimates? We need to consider the estimates. Recall what these were:

$$
\begin{equation*}
\sup \left\{\left\|u^{*}\right\|_{\mathscr{V}_{I}^{\prime}}: u^{*} \in A(u, \omega)\right\} \leq a(\omega)+b(\omega)\|u\|_{\mathscr{V}_{I}}^{p-1} \tag{77.8.80}
\end{equation*}
$$

where $a(\omega), b(\omega)$ are nonnegative. Also assume the following coercivity estimate valid for each $t \leq T$ and for some $\lambda(\omega) \geq 0$,

$$
\begin{equation*}
\inf \left(\int_{0}^{t}\left\langle u^{*}, u\right\rangle+\lambda(\omega)\langle B u, u\rangle d t: u^{*} \in A(u, \omega)\right) \geq \delta(\omega) \int_{0}^{t}\|u\|_{V}^{p} d s-m(\omega) \tag{77.8.81}
\end{equation*}
$$

where $m(\omega)$ is some nonnegative constant, $\delta(\omega)>0$.
The coercivity is not too bad. This is because $G_{\mu}$ is monotone and $0 \in D(G)$. Therefore,

$$
\left\langle G_{\mu} u, u\right\rangle=\left\langle G_{\mu} u-G_{\mu}(0), u\right\rangle \geq 0
$$

so

$$
\left\langle G_{\mu} u, u\right\rangle \geq-|G(0)|\|u\| \geq \frac{\delta(\omega)}{2}\|u\|^{p}-\hat{m}(\omega)
$$

so the coercivity condition 77.8 .81 will end up holding for $A+G_{\mu}$. However, more needs to be considered for the growth condition.

From the definition of $u_{\mu}$, there exists $z_{\mu} \in G u_{\mu}$

$$
0=F\left(u_{\mu}-u\right)+\mu^{p-1} z_{\mu}
$$

Then from the choice of $F$, it is also the duality map from $\mathscr{V}$ to $\mathscr{V}^{\prime}$ corresponding to $p>2$.

$$
\begin{aligned}
& 0=\left\langle F\left(u_{\mu}-u\right), u_{\mu}\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}+\mu^{p-1}\left\langle z_{\mu}, u_{\mu}\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \\
& \geq\left\langle F\left(u_{\mu}-u\right), u_{\mu}\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}-\mu^{p-1}|G(0)|\left\|u_{\mu}\right\|_{\mathscr{V}} \\
&=\left\|u_{\mu}-u\right\|_{\mathscr{V}}^{p}+\left\langle F\left(u_{\mu}-u\right), u\right\rangle-\mu|G(0)|\left\|u_{\mu}\right\| \\
& \geq\left\|u_{\mu}-u\right\|^{p}-\left\|u_{\mu}-u\right\|^{p-1}\|u\|-\mu|G(0)|\left\|u_{\mu}\right\| \\
& \geq \frac{1}{p}\left\|u_{\mu}-u\right\|^{p}-\frac{1}{p}\|u\|^{p}-\mu|G(0)|\left\|u_{\mu}\right\|
\end{aligned}
$$

Thus

$$
0 \geq\left|\left\|u_{\mu}\right\|-\|u\|\right|^{p}-\|u\|^{p}-p \mu|G(0)|\left\|u_{\mu}\right\|
$$

This requires that there is some constant $C$ such that $\left\|u_{\mu}\right\| \leq C\|u\|+C$. The details follow.
Let $a=\left\|u_{\mu}\right\|, b=\|u\|$. Then they are both positive and

$$
0 \geq|a-b|^{p}-b^{p}-\alpha a
$$

where $\alpha=p \mu|G(0)|$. Want to say $a \leq C b+C$ for some $C$. This is the conclusion of the following lemma.

Lemma 77.8.7 Suppose $0 \geq|a-b|^{p}-b^{p}-\alpha a$ for $a, b \geq 0$ and $\alpha>0$. Then there exists a constant $C$ such that

$$
a \leq C b+C
$$

Proof: If $b \geq a$, then there is nothing to show. Therefore, it suffices to show that the desired inequality holds for $a>b$. Thus from now on, $a>b$.

$$
0 \geq(a-b)^{p}-b^{p}-\alpha a
$$

Suppose $a>n b+n$. Let $x=b / a$. Then for $x \in[0,1]$,

$$
\begin{aligned}
0 & \geq(1-x)^{p}-x^{p}-\alpha \frac{1}{a^{p-1}} \\
& \geq(1-x)^{p}-x^{p}-\alpha \frac{1}{(n b+n)^{p-1}} \\
& \geq(1-x)^{p}-x^{p}-\alpha \frac{1}{(n)^{p-1}}
\end{aligned}
$$

Now for all $n$ large enough, the right side is a decreasing function of $x$ which is positive at $x=0$ and negative at $x=1$. Thus $x$ corresponds to the place where this function is negative. Taking a limit as $n \rightarrow \infty$, it follows that we must have

$$
x \geq \delta, \delta \in(0,1)
$$

It is where $(1-x)^{p}-x^{p}=0$. Thus $x=\frac{b}{a} \geq \delta$. Then, since $a>n b+n$,

$$
\frac{1}{\delta} b \geq a>n b+n
$$

Now this is a contradiction when $n$ is taken increasingly large. Hence, for large enough $n, a \leq n b+n$.

It follows that $\left\|u_{\mu}\right\| \leq C\|u\|+C$ for some $C$. Hence,

$$
\begin{aligned}
\left\|G_{\mu} u\right\| & \leq \frac{1}{\mu^{p-1}}\left\|u_{\mu}-u\right\|^{p-1} \leq \frac{1}{\mu^{p-1}}\left(\left\|u_{\mu}\right\|+\|u\|\right)^{p-1} \\
& \leq \frac{2^{p-2}}{\mu^{p-1}}\left(\left\|u_{\mu}\right\|^{p-1}+\|u\|^{p-1}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \leq \frac{2^{p-2}}{\mu^{p-1}}\left((C\|u\|+C)^{p-1}+\|u\|^{p-1}\right) \\
& \leq \frac{2^{p-2}}{\mu^{p-1}}\left(2^{p-2}\left(C^{p-1}\|u\|^{p-1}+C^{p-1}\right)+\|u\|^{p-1}\right) \\
& \leq C_{\mu}\|u\|^{p-1}+C_{\mu}
\end{aligned}
$$

This is the case that $p \geq 2$. The case that $p>1$ but $p<2$ is easier. In this case,

$$
\frac{1}{\mu^{p-1}}\left(\left\|u_{\mu}\right\|+\|u\|\right)^{p-1} \leq \frac{1}{\mu^{p-1}}\left(\left\|u_{\mu}\right\|^{p-1}+\|u\|^{p-1}\right)
$$

A similar inequality holds. Thus the necessary growth condition is obtained for $G_{\mu}$ and consequently, the necessary growth condition remains valid for $G_{\mu}+A$. It was noted earlier that the coercivity estimate continues to hold.

It follows that there exists a solution to the integral equation

$$
B u(t, \omega)+\int_{0}^{t} z(s, \omega) d s+\int_{0}^{t} G_{\mu}(u(s, \omega)) d s=\int_{0}^{t} f(s, \omega) d s+B u_{0}(\omega)
$$

where $z(\cdot, \omega) \in A(u(\cdot, \omega), \omega)$ which has the measurability described above. That is, both $u$ and $z$ are product measurable. Then acting on $u \mathscr{X}_{[0, t]}$ and using the estimates valid for $\lambda$ large enough, one can get an estimate of the form

$$
\begin{equation*}
\frac{1}{2}\langle B u, u\rangle(t)-\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle+\int_{0}^{t}\|u(s)\|_{V}^{p} d s+\int_{0}^{t}\left\langle G_{\mu} u, u\right\rangle d s \leq \lambda \int_{0}^{t}\langle B u, u\rangle d s+C(f) \tag{77.8.82}
\end{equation*}
$$

Now $G_{\mu}$ is monotone and so,

$$
\left\langle G_{\mu} u, u\right\rangle=\left\langle G_{\mu} u-G_{\mu} 0, u\right\rangle+\left\langle G_{\mu}(0), u\right\rangle \geq\left\langle G_{\mu}(0), u\right\rangle \geq-|G(0)|\|u\|
$$

It follows easily from standard manipulations and 77.8 .82 that $\|u\|_{\mathscr{V}}$ is bounded independent of $\mu$. That is, there is a constant $C$ independent of $\mu$ such that

$$
\begin{equation*}
\|u\|_{\mathscr{V}} \leq C \tag{77.8.83}
\end{equation*}
$$

The details follow. The above inequality 77.8.82 implies that by acting on $u \mathscr{X}_{[0, t]}$,

$$
\frac{1}{2}\langle B u, u\rangle(t)-\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle+\int_{0}^{t}\|u(s)\|_{V}^{p} d s-\int_{0}^{t}|G(0)|\|u\|_{V} d s \leq \lambda \int_{0}^{t}\langle B u, u\rangle d s+C(f)
$$

Then by Gronwall's inequality and adjusting constants,

$$
\begin{equation*}
\langle B u, u\rangle(t)+\int_{0}^{t}\|u(s)\|_{V}^{p} d s \leq C\left(u_{0}, f, \lambda\right)+C(\lambda) \int_{0}^{t}|G(0)|\|u\|_{V} d s \tag{77.8.84}
\end{equation*}
$$

so it is clear that there is an inequality of the form

$$
\sup _{t \in[0, T]}\langle B u, u\rangle(t)+\int_{0}^{T}\|u(s)\|_{V}^{p} d s \leq C\left(u_{0}, f, \lambda\right)
$$

Then returning to 77.8.82, all terms are bounded except $\int_{0}^{t}\left\langle G_{\mu} u, u\right\rangle d s$, so this term must also be bounded for $t=T$ also. Thus

$$
\left|\int_{0}^{T}\left\langle G_{\mu} u, u\right\rangle d t\right| \leq C
$$

where $C$ is independent of $\mu$. We denote by $u_{\mu}$ the solution to the above equation.
Here is the definition of quasi-bounded.
Definition 77.8.8 A set valued operator $G$ is quasi-bounded if whenever $x \in D(G)$ and $x^{*} \in G x$ are such that

$$
\left|\left\langle x^{*}, x\right\rangle\right|,\|x\| \leq M
$$

it follows that $\left\|x^{*}\right\| \leq K_{M}$. Bounded would mean that if $\|x\| \leq M$, then $\left\|x^{*}\right\| \leq K_{M}$. Here you only know this if there is another condition.

Assumption 77.8.9 $G: D(G) \rightarrow \mathscr{P}\left(\mathscr{V}^{\prime}\right)$ is quasi-bounded and maximal monotone.
By Proposition 25.7.23 an example of a quasi-bounded operator is a maximal monotone operator $G$ for which $0 \in \operatorname{int}(D(G))$.

Now $G_{\mu} u_{\mu} \in G J_{\mu} u_{\mu}$ as noted above. Therefore, there exists $g_{\mu} \in G\left(J_{\mu} u_{\mu}\right)$ such that

$$
\begin{gather*}
C \geq\left\langle G_{\mu} u_{\mu}, u_{\mu}\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}=\left\langle g_{\mu}, u_{\mu}\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}=\left\langle g_{\mu}, J_{\mu} u_{\mu}\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}+\left\langle g_{\mu}, u_{\mu}-J_{\mu} u_{\mu}\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}  \tag{77.8.85}\\
\geq-|G(0)|\left\|J_{\mu} u_{\mu}\right\|_{\mathscr{V}}+\left\langle-\frac{1}{\mu^{p-1}} F\left(J_{\mu} u_{\mu}-u_{\mu}\right), u_{\mu}-J_{\mu} u_{\mu}\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \\
\quad \geq-|G(0)|\left\|J_{\mu} u_{\mu}\right\|_{\mathscr{V}}+\frac{1}{\mu^{p-1}}\left\|J_{\mu} u_{\mu}-u_{\mu}\right\|_{\mathscr{V}}^{p} \tag{77.8.86}
\end{gather*}
$$

Thus the fact that $\left\|u_{\mu}\right\|$ is bounded independent of $\mu$ implies that $\left\|J_{\mu} u_{\mu}\right\|$ is also bounded and that in fact $\left\|u_{\mu}-J_{\mu} u_{\mu}\right\|_{\mathscr{V}} \rightarrow 0$ as $\mu \rightarrow 0$. This follows from consideration of the last line of the above formula. Note also that

$$
\begin{equation*}
\left\langle g_{\mu}, u_{\mu}-J_{\mu} u_{\mu}\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}=\frac{1}{\mu^{p-1}}\left\|J_{\mu} u_{\mu}-u_{\mu}\right\|_{\mathscr{V}}^{p} \text { is bounded. } \tag{77.8.87}
\end{equation*}
$$

Then from 77.8.85, it follows that $\left\langle g_{\mu}, J_{\mu} u_{\mu}\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}$ is bounded. By the assumption that $G$ is quasi-bounded, $g_{\mu}$ must also be bounded.

Then we have shown

$$
\begin{equation*}
B u_{\mu}(t, \omega)+\int_{0}^{t} z_{\mu}(s, \omega) d s+\int_{0}^{t} g_{\mu}(s, \omega) d s=\int_{0}^{t} f(s, \omega) d s+B u_{0}(\omega) \tag{77.8.88}
\end{equation*}
$$

where

$$
\begin{equation*}
\left\|g_{\mu}\right\|_{\mathscr{V}^{\prime}}+\left\|z_{\mu}\right\|_{\mathscr{V}^{\prime}}+\sup _{t \in[0, T]}\left\langle B u_{\mu}, u_{\mu}\right\rangle(t)+\left\|J_{\mu} u_{\mu}\right\|_{\mathscr{V}}+\left\|u_{\mu}\right\|_{\mathscr{V}}+\left\|\left(B u_{\mu}\right)^{\prime}\right\|_{\mathscr{V}^{\prime}} \leq C \tag{77.8.89}
\end{equation*}
$$

The last term in the sum being bounded follows from the integral equation and the fundamental theorem of calculus along with the boundedness $f, g_{\mu}, z_{\mu}$. In addition to this, the estimate 77.8.86 implies

$$
\begin{equation*}
\lim _{\mu \rightarrow 0}\left\|J_{\mu} u_{\mu}-u_{\mu}\right\|_{\mathscr{V}}=0 \tag{77.8.90}
\end{equation*}
$$

There is a subsequence, $\mu \rightarrow 0$ still denoted as $\mu$ such that

$$
\begin{gather*}
g_{\mu} \rightarrow g \text { weakly in } \mathscr{V}^{\prime}  \tag{77.8.91}\\
z_{\mu} \rightarrow z \text { weakly in } \mathscr{V}^{\prime}  \tag{77.8.92}\\
u_{\mu} \rightarrow u \text { weakly in } \mathscr{V}  \tag{77.8.93}\\
J_{\mu} u_{\mu} \rightarrow u \text { weakly in } \mathscr{V}  \tag{77.8.94}\\
\left(B u_{\mu}\right)^{\prime} \rightarrow(B u)^{\prime} \text { weakly in } \mathscr{V}^{\prime}  \tag{77.8.95}\\
B u_{\mu}(t) \rightarrow B u(t) \text { weakly in } V^{\prime} \tag{77.8.96}
\end{gather*}
$$

Now consider two of these for $\mu$ and $\nu$. Subtract and act on $u_{\mu}-u_{\nu}$. Then one obtains

$$
\left\langle B u_{\mu}-B u_{v}, u_{\mu}-u_{v}\right\rangle(t)+\int_{0}^{t}\left\langle z_{\mu}-z_{v}, u_{\mu}-u_{v}\right\rangle+\int_{0}^{t}\left\langle g_{\mu}-g_{v}, u_{\mu}-v_{v}\right\rangle=0
$$

Consider that last term for $t=T$. It equals

$$
\begin{aligned}
\int_{0}^{T}\left\langle G_{\mu} u_{\mu}-\right. & \left.G_{v} u_{v}, u_{\mu}-u_{v}\right\rangle=\overbrace{\int_{0}^{T}\left\langle G_{\mu} u_{\mu}-G_{v} u_{v}, J_{\mu} u_{\mu}-J_{v} u_{v}\right\rangle}^{\geq 0} \\
& +\int_{0}^{T}\left\langle g_{\mu}-g_{v}, u_{\mu}-J_{\mu} u_{\mu}-\left(u_{v}-J_{v} u_{v}\right)\right\rangle \\
& =\int_{0}^{T}\left\langle g_{\mu}-g_{v}, J_{\mu} u_{\mu}-J_{v} u_{v}\right\rangle+\varepsilon(\mu, v)
\end{aligned}
$$

where

$$
\begin{gathered}
|\varepsilon(\mu, v)| \leq\left(\int_{0}^{T}\left(\left\|g_{\mu}\right\|+\left\|g_{v}\right\|\right)^{p^{\prime}}\right)^{1 / p^{\prime}}\left(\int_{0}^{T}\left(\left\|u_{\mu}-J_{\mu} u_{\mu}\right\|+\left\|u_{v}-J_{v} u_{V}\right\|\right)^{p}\right)^{1 / p} \\
\leq 2 C\left(\left\|u_{\mu}-J_{\mu} u_{\mu}\right\|_{\mathscr{V}}+\left\|u_{v}-J_{v} u_{v}\right\|_{\mathscr{V}}\right)
\end{gathered}
$$

Adjusting constants and using 77.8.87,

$$
\leq C\left(\mu^{(1-(1 / p))}+v^{(1-(1 / p))}\right)
$$

Thus

$$
\int_{0}^{T}\left\langle G_{\mu} u_{\mu}-G_{v} u_{v}, u_{\mu}-u_{v}\right\rangle=\int_{0}^{T}\left\langle g_{\mu}-g_{v}, J_{\mu} u_{\mu}-J_{v} u_{v}\right\rangle+\varepsilon(\mu, v)
$$

where $\lim _{\mu, v \rightarrow 0} \varepsilon(\mu, v)=0$. It follows from 77.8.97

$$
\begin{aligned}
& \lim \sup _{\mu, v \rightarrow 0}\left(\int_{0}^{T}\left\langle z_{\mu}-z_{v}, u_{\mu}-u_{v}\right\rangle d s+\varepsilon(\mu, v)\right) \\
= & \lim \sup _{\mu, v \rightarrow 0}\left(\int_{0}^{T}\left\langle z_{\mu}-z_{v}, u_{\mu}-u_{v}\right\rangle d s\right) \leq 0
\end{aligned}
$$

From Lemma 77.8.4,

$$
\limsup _{\mu \rightarrow 0}\left\langle z_{\mu}, u_{\mu}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \leq 0
$$

By the limit condition for $A(\cdot, \omega)$, for each $v \in \mathscr{V}$, there exists $z(v) \in A u$ such that

$$
\begin{aligned}
\lim _{\mu \rightarrow 0} \inf _{\mu}\left\langle z_{\mu}, u_{\mu}-v\right\rangle & =\lim _{\mu \rightarrow 0}\left(\left\langle z_{\mu}, u_{\mu}-u\right\rangle+\left\langle z_{\mu}, u-v\right\rangle\right) \\
& =\langle z, u-v\rangle \geq\langle z(v), u-v\rangle
\end{aligned}
$$

Since $A(u, \omega)$ is convex and closed, separation theorems imply that $z \in A u$. Return to the equation solved.

$$
\left(B u_{\mu}\right)^{\prime}+z_{\mu}+g_{\mu}=f
$$

Then act on $u_{\mu}-u$ and use monotonicity arguments to write

$$
\begin{equation*}
\left\langle(B u)^{\prime}, u_{\mu}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}+\left\langle z_{\mu}, u_{\mu}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}+\left\langle g_{\mu}, u_{\mu}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \leq\left\langle f, u_{\mu}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \tag{77.8.98}
\end{equation*}
$$

Then it was shown above that

$$
0 \geq \lim \sup _{\mu \rightarrow 0}\left\langle z_{\mu}, u_{\mu}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \geq \lim \inf _{\mu \rightarrow 0}\left\langle z_{\mu}, u_{\mu}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \geq\langle z(u), u-u\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}=0
$$

and so, from 77.8.98,

$$
\lim _{\mu \rightarrow 0}\left\langle g_{\mu}, u_{\mu}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}=\lim _{\mu \rightarrow 0}\left\langle g_{\mu}, J_{\mu} u_{\mu}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}=0
$$

and so

$$
\lim _{\mu \rightarrow 0}\left\langle g_{\mu}, J_{\mu} u_{\mu}\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}=\langle g, u\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}
$$

Now let $[a, b] \in \mathscr{G}(G)$. Then

$$
\langle b-g, a-u\rangle=\lim _{\mu \rightarrow 0}\left\langle b-g_{\mu}, a-J_{\mu} u_{\mu}\right\rangle \geq 0
$$

because $g_{\mu} \in G\left(J_{\mu} u_{\mu}\right)$. Since $G$ is maximal monotone, it follows that $[u, g] \in \mathscr{G}(G)$.
This has shown that for each $\omega$ fixed, and every sequence of solutions to the integral equation $\left\{u_{\mu}\right\}$, each function $\left\{B u_{\mu}\right\}$ being product measurable by Theorem 77.5.7, there exists a subsequence which converges to a solution $u$ to the integral equation. In particular, $t \rightarrow B u(t)$ is weakly continuous into $V^{\prime}$. Then by the fundamental measurable selection theorem, Theorem 77.2.10, there exists a product measurable function $\bar{u}(t, \omega)$ with values in $V$ weakly continuous in $t$ and a sequence depending on $\omega,\left\{u_{\mu(\omega)}\right\}$ such that for each
$\omega, \lim _{\mu(\omega) \rightarrow 0} u_{\mu(\omega)}(\cdot, \omega)=\bar{u}(\cdot, \omega)$ weakly in $\mathscr{V}$. However, from the above argument, for each $\omega$, there is a further subsequence, still denoted with subscript $\mu(\omega)$ such that in $\mathscr{V}^{\prime}$,

$$
\lim _{\mu(\omega) \rightarrow 0} u_{\mu(\omega)}(\cdot, \omega)=u(\cdot, \omega)
$$

where $u$ is a solution to the integral equation. Since $u(\cdot, \omega)=\bar{u}(\cdot, \omega)$ in $\mathscr{V}$ it follows that these must be equal a.e. and hence $(t, \omega) \rightarrow u(t, \omega)$ is product measurable. This proves the following theorem.

Theorem 77.8.10 Suppose 77.8.73-77.8.78 and B as described above and $u_{0}$ is $\mathscr{F}$ measurable. Also let $G: D(G) \subseteq V \rightarrow \mathscr{P}\left(V^{\prime}\right)$ be maximal monotone and quasi-bounded.

Then, there exists a solution $u$ of the integral equation

$$
B u(t, \omega)+\int_{0}^{t} z(s, \omega) d s+\int_{0}^{t} g(s, \omega) d s=\int_{0}^{t} f(s, \omega) d s+B u_{0}(\omega),
$$

where $(t, \omega) \rightarrow u(t, \omega)$ is product measurable $(t, \omega) \rightarrow z(t, \omega)$ also. Moreover, for each $\omega, B u(t, \omega)=B(u(t, \omega))$ a.e. $t$ and $z(\cdot, \omega) \in A(u(\cdot, \omega), \omega)$, and $g(\cdot, \omega) \in G(u(\cdot, \omega))$ for each $\omega$.

Note that in the case of most interest where you have a Gelfand triple and $B$ is the identity, the fundamental theorem of calculus implies easily that $\omega \rightarrow z(s, \omega)+g(s, \omega)$ is measurable for a.e. $s$. One can also generalize to the following in which a measurable $q(t, \omega)$ is added.

Corollary 77.8.11 Suppose 77.8.73-77.8.78 and $B$ as described above and $u_{0}$ is $\mathscr{F}$ measurable. Also let $G: D(G) \subseteq V \rightarrow \mathscr{P}\left(V^{\prime}\right)$ be maximal monotone and quasi-bounded. Let $(t, \omega) \rightarrow q(t, \omega)$ be product measurable into $V$ and let $t \rightarrow q(t, \omega)$ be continuous, $q(0, \omega)=0$. Then, there exists a solution $u$ of the integral equation

$$
B u(t, \omega)+\int_{0}^{t} z(s, \omega) d s+\int_{0}^{t} g(s, \omega) d s=\int_{0}^{t} f(s, \omega) d s+B u_{0}(\omega)+B q(t, \omega)
$$

where $(t, \omega) \rightarrow u(t, \omega),(t, \omega) \rightarrow z(t, \omega), g(t, \omega)$ are product measurable. Moreover, for each $\omega, B u(t, \omega)=B(u(t, \omega))$ a.e. $t$ and $z(\cdot, \omega) \in A(u(\cdot, \omega), \omega)$ for a.e. $t$, and $g(\cdot, \omega) \in$ $G(u(\cdot, \omega))$ for each $\omega$.

Proof: Define a stopping time

$$
\tau_{n}(\omega) \equiv \inf \{t: q(t, \omega)>n\}
$$

Then let $\tilde{A}(\cdot, \omega) \equiv A\left(q^{\tau_{n}}(\cdot, \omega)+w, \omega\right)$. Then $\tilde{A}$ satisfies the same properties as $A$ and so there exists a solution to the integral equation

$$
B w_{n}(t, \omega)+\int_{0}^{t} z_{n}(s, \omega) d s+\int_{0}^{t} g_{n}(s, \omega) d s=\int_{0}^{t} f(s, \omega) d s+B u_{0}(\omega)
$$

where $w_{n}, z_{n}, g_{n}$ are product measurable, $z_{n}(\cdot, \omega) \in \tilde{A}\left(w_{n}(\cdot, \omega), \omega\right)$ a.e. $t$. By continuity of $t \rightarrow q(t, \omega), \tau_{n}=\infty$ for all $n$ sufficiently large and so $q(t, \omega)=q^{\tau_{n}}(t, \omega)$. As before,
for each $\omega$, one obtains the convergences 77.8.91-77.8.96 as $n \rightarrow \infty$. As before, for each $\omega, z(\cdot, \omega) \in \tilde{A}(w(\cdot, \omega), \omega)$ a.e. where $t \rightarrow w(t, \omega)$ is the function to which $w_{n}(\cdot, \omega)$ converges weakly. Note that the estimates allowing this to happen are dependent on $\omega$. However, one can apply Theorem 77.2.10 as before and obtain a solution to

$$
B w(t, \omega)+\int_{0}^{t} z(s, \omega) d s+\int_{0}^{t} g(s, \omega) d s=\int_{0}^{t} f(s, \omega) d s+B u_{0}(\omega)
$$

such that $w, z, g$ are product measurable into $V$ or $V^{\prime}$ and,$z(\cdot, \omega) \in \tilde{A}(w(\cdot, \omega), \omega)$. Now let $u(t, \omega)=w(t, \omega)+q(t, \omega)$ to obtain the existence of the desired solution in the corollary.

## Chapter 78

## A Different Approach

### 78.1 Summary Of The Problem

The situation is as follows. There are spaces $V \subseteq W$ where $V, W$ are reflexive separable Banach spaces. It is assumed that $V$ is dense in $W$. Define the space for $p>1$

$$
\mathscr{V} \equiv L^{p}([0, T] ; V)
$$

where in each case, the $\sigma$ algebra of measurable sets will be $\mathscr{B}([0, T])$ the Borel measurable sets. Thus, from the Riesz representation theorem,

$$
\mathscr{V}^{\prime}=L^{p^{\prime}}\left([0, T] ; V^{\prime}\right),
$$

We also assume $(\Omega, \mathscr{F})$ is a measurable space. No measure is needed. Also

$$
V \subseteq W, \quad W^{\prime} \subseteq V^{\prime}, V \text { dense in } W
$$

$B(t)$ will be a linear operator, $B(t): W \rightarrow W^{\prime}$ which satisfies

1. $\langle B(t) x, y\rangle=\langle B(t) y, x\rangle$
2. $\langle B(t) x, x\rangle \geq 0$
3. $B \in C^{1}\left([0, T] ; \mathscr{L}\left(W, W^{\prime}\right)\right)$ so in particular, the time derivative is bounded.

In the above formulae, $\langle\cdot, \cdot\rangle$ denotes the duality pairing of the Banach space $W$, with its dual space. We will use this notation in the present paper, the exact specification of which Banach space being determined, by the context in which this notation occurs.

For example, you could simply take $W=H=H^{\prime}$ and $B$ the identity and consider a standard Gelfand triple where $H$ is a Hilbert space and $B$ equal to the identity.

The product measurable sets are those in the smallest $\sigma$ algebra which contains the measurable rectangles $B \times A$ where $B \in \mathscr{B}([0, T]), A \in \mathscr{F}$. The paper is about the existence of product measurable solutions to the system

$$
\begin{align*}
(B u(\cdot, \omega))^{\prime}+u^{*}(\cdot, \omega) & =f(\cdot, \omega) \text { in } \mathscr{V}^{\prime} \\
B u(0, \omega) & =B u_{0}(\omega)  \tag{78.1.1}\\
u^{*}(\cdot, \omega) & \in A(u(\cdot, \omega), \omega) . \tag{78.1.2}
\end{align*}
$$

The evolution inclusion is well understood for fixed $\omega$. However, we will show the existence of a solution $u, u^{*}$ such that $(t, \omega) \rightarrow u(t, \omega)$ and $(t, \omega) \rightarrow u^{*}(t, \omega)$ are product measurable in this solution. Essentially, we show the existence of a measurable selection in the set of solutions. There are no assumptions made on the measurable space. It is just a set with a $\sigma$ algebra of subsets. Essentially this involves showing that the usual limit processes preserve measurability in some sense. The main theorems in this paper are essentially measurable selection theorems for the set of solutions to these implicit inclusions.

To begin with, we will assume $p \geq 2$. The reason for this is that we want to consider

$$
\int_{0}^{T}\left\langle B^{\prime} u, u\right\rangle d t
$$

and this won't make sense unless $p \geq 2$. However, this restriction is not necessary if $B$ is a constant operator, as we show in the succeeding section.

### 78.1.1 General Assumptions On $A$

The case $A(u, \omega)$ for $u \in \mathscr{V}$ given by $A(u, \omega)(t)=A(u(t), \omega)$ is included as a special case. In addition to this commonly used situation, we are including the case where $A(u, \omega)(t)$ depends on past values of $u(s)$ for $s \leq t$. This makes our theory useful in situations where the problem is second order in $t$. The following definition is the standard one [99].

Definition 78.1.1 For $X$ a reflexive Banach space, we say $A: X \rightarrow \mathscr{P}\left(X^{\prime}\right)$ is pseudomonotone and bounded if the following hold.

1. The set $A u$ is nonempty, closed and convex for all $u \in X$. A takes bounded sets to bounded sets.
2. If $u_{i} \rightarrow u$ weakly in $X$ and $u_{i}^{*} \in A u_{i}$ is such that

$$
\begin{equation*}
\limsup _{i \rightarrow \infty}\left\langle u_{i}^{*}, u_{i}-u\right\rangle \leq 0 \tag{78.1.3}
\end{equation*}
$$

then, for each $v \in X$, there exists $u^{*}(v) \in A u$ such that

$$
\begin{equation*}
\lim _{i \rightarrow \infty}\left\langle u_{i}^{*}, u_{i}-v\right\rangle \geq\left\langle u^{*}(v), u-v\right\rangle \tag{78.1.4}
\end{equation*}
$$

We will assume in this section in which $B$ is time dependent that $p \geq 2$. The specific assumptions on $A(\cdot, \omega)$ are described next.

- growth estimate

Assume the specific estimate

$$
\begin{equation*}
\sup \left\{\left\|u^{*}\right\|_{\mathscr{V}^{\prime}}: u^{*} \in A(u, \omega)\right\} \leq a(\omega)+b(\omega)\|u\|_{\mathscr{V}}^{\hat{p}-1} \tag{78.1.5}
\end{equation*}
$$

where $a(\omega), b(\omega)$ are nonnegative and $\hat{p} \geq p$.

## - coercivity estimate

Also assume the coercivity condition: valid for each $t \leq T$,

$$
\begin{gather*}
\inf \left(\int_{0}^{t}\left\langle u^{*}, u\right\rangle d s: u^{*} \in A(u, \omega)\right) \\
+\frac{1}{2} \int_{0}^{t}\left\langle B^{\prime} u, u\right\rangle d s \geq \delta(\omega) \int_{0}^{t}\|u\|_{V}^{p} d s-m(\omega) \tag{78.1.6}
\end{gather*}
$$

where $m(\omega)$ is some nonnegative constant, $\delta(\omega)>0$. In fact, it is often enough to assume the left side is given by

$$
\inf \left(\int_{0}^{t}\left\langle u^{*}, u\right\rangle+\lambda(\omega)\langle B u, u\rangle d s: u^{*} \in A(u, \omega)\right)
$$

for some $\lambda(\omega)$ by using a suitable exponential shift argument and changing the dependent variable. We will sometimes denote weak convergence by $\rightharpoonup$.

## - limit condition

If $u_{i} \rightharpoonup u$ in $\mathscr{V}$ and $\left(B u_{i}\right)^{\prime} \rightharpoonup(B u)^{\prime}$ in $\mathscr{V}^{\prime}, u_{i}^{*} \in A\left(u_{i}\right), \rightharpoonup$ denoting weak convergence, then if

$$
\limsup _{i \rightarrow \infty}\left\langle u_{i}^{*}, u_{i}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \leq 0
$$

it follows that for all $v \in \mathscr{V}$, there exists $u^{*}(v) \in A u$ such that

$$
\begin{equation*}
\lim _{i \rightarrow \infty} \inf _{i \rightarrow \infty}\left\langle u_{i}^{*}, u_{i}-v\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \geq\left\langle u^{*}(v), u-v\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \tag{78.1.7}
\end{equation*}
$$

## - measurability condition

For $\omega \rightarrow u(\cdot, \omega)$ measurable into $\mathscr{V}$,

$$
\begin{equation*}
\omega \rightarrow A(u(\cdot, \omega), \omega) \text { has a measurable selection into } \mathscr{V}^{\prime} \tag{78.1.8}
\end{equation*}
$$

This last condition means there is a function $\omega \rightarrow u^{*}(\omega)$ which is measurable into $\mathscr{V}^{\prime}$ such that $u^{*}(\omega) \in A(u(\cdot, \omega), \omega)$. This is assured to take place if the following standard measurability condition is satisfied for all $O$ open in $\mathscr{V}^{\prime}$ :

$$
\begin{equation*}
\{\omega: A(u(\cdot, \omega), \omega) \cap O \neq \emptyset\} \in \mathscr{F} \tag{78.1.9}
\end{equation*}
$$

See for example [70], [10] or the chapter on measurable multifunctions Chapter 48. Our assumption is implied by this one but they are not equivalent. Thus what is considered here is more general than an assumption that $\omega \rightarrow A(u(\cdot, \omega), \omega)$ is set valued measurable.

Note that this condition would hold if $u \rightarrow A(t, u, \omega)$ is bounded and pseudomonotone as a single valued map from $V$ to $V^{\prime}$ and $(t, \omega) \rightarrow A(t, u, \omega)$ is product measurable into $V^{\prime}$ for each $u$. One would use the demicontinuity of $u \rightarrow A(t, u, \omega)$ which comes from a pseudo monotone and bounded assumption and consider a sequence of simple functions $u_{n}(t, \omega) \rightarrow u(t, \omega)$ in $V$ for $u$ measurable, each $u_{n}(\cdot, \omega)$ being in $\mathscr{V}$, Then the measurability of $A\left(t, u_{n}, \omega\right)$ would attach to $A(t, u, \omega)$ in the limit. In the situation where $A(\cdot, \omega)$ satisfies a suitable upper semicontinuity condition, it is enough to assume only that $\omega \rightarrow A(u, \omega)$ has a measurable selection for each $u \in V$. This is a straightforward exercise in approximating with simple functions and then using upper semicontinuity instead of continuity.

We assume always that the norm on the various reflexive Banach spaces is strictly convex.

### 78.1.2 Preliminary Results

We use the following well known theorem [91]. It is stated here for the situation in which a Holder condition is given rather than a bound on weak derivatives. See Theorem 34.7.6 on Page 1219.

Theorem 78.1.2 Let $E \subseteq F \subseteq G$ where the injection map is continuous from $F$ to $G$ and compact from $E$ to $F$. Let $p \geq 1$, let $q>1$, and define

$$
\begin{gathered}
S \equiv\left\{u \in L^{p}([a, b], E): \text { for some } C,\|u(t)-u(s)\|_{G} \leq C|t-s|^{1 / q}\right. \\
\text { and } \left.\|u\|_{L^{p}([a, b], E)} \leq R\right\}
\end{gathered}
$$

Thus $S$ is bounded in $L^{p}([a, b], E)$ and Holder continuous into $G$. Then $S$ is precompact in $L^{p}([a, b], F)$. This means that if $\left\{u_{n}\right\}_{n=1}^{\infty} \subseteq S$, it has a subsequence $\left\{u_{n_{k}}\right\}$ which converges in $L^{p}([a, b], F)$. The same conclusion can be drawn if it is known instead of the Holder condition that $\left\|u^{\prime}\right\|_{L^{1}([a, b] ; X)}$ is bounded.

Next are some measurable selection theorems which form an essential part of showing the existence of measurable solutions. They are not dependent on there being a measure but in the applications of most interest to us, there is typically a probability measure. First is a basic selection theorem for a set of limits. See Lemma 48.2.2 on Page 1541.

Theorem 78.1.3 Let $U$ be a separable reflexive Banach space. Suppose there is a sequence $\left\{u_{j}(\omega)\right\}_{j=1}^{\infty}$ in $U$, where $\omega \rightarrow u_{j}(\omega)$ is measurable and for each $\omega$,

$$
\sup _{j}\left\|u_{j}(\omega)\right\|_{U}<\infty
$$

Then there exists a function $\omega \rightarrow u(\omega)$ with values in $U$ such that $\omega \rightarrow u(\omega)$ is measurable, and a subsequence $n(\omega)$, depending on $\omega$, such that

$$
\lim _{n(\omega) \rightarrow \infty} u_{n(\omega)}(\omega)=u(\omega) \text { weakly in } U
$$

Next is a specialization to the situation where the Banach space is a function space. The proof is in [88]. This gives a result on product measurability. It is Theorem 77.2.10 on Page 2601.

Theorem 78.1.4 Let $V$ be a reflexive separable Banach space with dual $V^{\prime}$, and let $p, p^{\prime}$ be such that $p>1$ and $\frac{1}{p}+\frac{1}{p^{\prime}}=1$. Let the functions $t \rightarrow u_{n}(t, \omega)$, for $n \in \mathbb{N}$, be in $L^{p}([0, T] ; V) \equiv \mathscr{V}$ and $(t, \omega) \rightarrow u_{n}(t, \omega)$ be $\mathscr{B}([0, T]) \times \mathscr{F} \equiv \mathscr{P}$ measurable into $V$. Suppose

$$
\left\|u_{n}(\cdot, \omega)\right\|_{\mathscr{V}} \leq C(\omega)
$$

for all $n$. (Thus, by weak compactness, for each $\omega$, each subsequence of $\left\{u_{n}\right\}$ has a further subsequence that converges weakly in $\mathscr{V}$ to $v(\cdot, \omega) \in \mathscr{V} .(v$ not known to be $\mathscr{P}$ measurable))

Then, there exists a product measurable function $u$ such that $t \rightarrow u(t, \omega)$ is in $\mathscr{V}$ and for each $\omega$ a subsequence $u_{n(\omega)}$ such that $u_{n(\omega)}(\cdot, \omega) \rightarrow u(\cdot, \omega)$ weakly in $\mathscr{V}$.

Next is what it means to be measurable into $\mathscr{V}$ or $\mathscr{V}^{\prime}$. Such functions have representatives which are product measurable.

Lemma 78.1.5 Let $f(\cdot, \omega) \in \mathscr{V}^{\prime}$. Then if $\omega \rightarrow f(\cdot, \omega)$ is measurable into $\mathscr{V}^{\prime}$, it follows that for each $\omega$, there exists a representative $\hat{f}(\cdot, \omega) \in \mathscr{V}^{\prime}, \hat{f}(\cdot, \omega)=f(\cdot, \omega)$ in $\mathscr{V}^{\prime}$ such that $(t, \omega) \rightarrow \hat{f}(t, \omega)$ is product measurable. If $f(\cdot, \omega) \in \mathscr{V}^{\prime}$ and $(t, \omega) \rightarrow f(t, \omega)$ is product measurable, then $\omega \rightarrow f(\cdot, \omega)$ is measurable into $\mathscr{V}^{\prime}$. The same holds replacing $\mathscr{V}^{\prime}$ with $\mathscr{V}$.

Proof: If a function $f$ is measurable into $\mathscr{V}^{\prime}$, then there exist simple functions $f_{n}$

$$
\lim _{n \rightarrow \infty}\left\|f_{n}(\omega)-f(\omega)\right\|_{\mathscr{V}^{\prime}}=0,\left\|f_{n}(\omega)\right\| \leq 2\|f(\omega)\|_{\mathscr{V}^{\prime}} \equiv C(\omega)
$$

Now one of these simple functions is of the form $\sum_{i=1}^{M} c_{i} \mathscr{X}_{E_{i}}(\omega)$ where $c_{i} \in \mathscr{V}^{\prime}$. Therefore, there is no loss of generality in assuming that

$$
c_{i}(t)=\sum_{j=1}^{N} d_{j}^{i} \mathscr{X}_{F_{j}}(t)
$$

where $d_{j}^{i} \in V^{\prime}$. Hence we can assume each $f_{n}$ is product measurable into $\mathscr{B}\left(V^{\prime}\right) \times \mathscr{F}$. Then by Theorem 78.1.4, there exists $\hat{f}(\cdot, \omega) \in \mathscr{V}^{\prime}$ such that $\hat{f}$ is product measurable and a subsequence $f_{n(\omega)}$ converging weakly in $\mathscr{V}^{\prime}$ to $\hat{f}(\cdot, \omega)$ for each $\omega$. Thus $f_{n(\omega)}(\omega) \rightarrow f(\omega)$ strongly in $\mathscr{V}^{\prime}$ and $f_{n(\omega)}(\omega) \rightarrow \hat{f}(\omega)$ weakly in $\mathscr{V}^{\prime}$. Therefore, $\hat{f}(\omega)=f(\omega)$ in $\mathscr{V}^{\prime}$ and so it can be assumed that if $f$ is measurable into $\mathscr{V}^{\prime}$ then for each $\omega$, it has a representative $\hat{f}(\omega)$ such that $(t, \omega) \rightarrow \hat{f}(t, \omega)$ is product measurable.

If $f$ is product measurable into $V^{\prime}$ and each $f(\cdot, \omega) \in \mathscr{V}^{\prime}$, does it follow that $f$ is measurable into $\mathscr{V}^{\prime}$ ? By measurability, $f(t, \omega)=\lim _{n \rightarrow \infty} \sum_{i=1}^{m_{n}} c_{i}^{n} \mathscr{X}_{E_{i}^{n}}(t, \omega)=\lim _{n \rightarrow \infty} f_{n}(t, \omega)$ where $E_{i}^{n}$ is product measurable and we can assume $\left\|f_{n}(t, \omega)\right\|_{V^{\prime}} \leq 2\|f(t, \omega)\|$. Then by product measurability, $\omega \rightarrow f_{n}(\cdot, \omega)$ is measurable into $\mathscr{V}^{\prime}$ because if $g \in \mathscr{V}$ then

$$
\omega \rightarrow\left\langle f_{n}(\cdot, \omega), g\right\rangle
$$

is of the form

$$
\omega \rightarrow \sum_{i=1}^{m_{n}} \int_{0}^{T}\left\langle c_{i}^{n} \mathscr{X}_{E_{i}^{n}}(t, \omega), g(t)\right\rangle d t \text { which is } \omega \rightarrow \sum_{i=1}^{m_{n}} \int_{0}^{T}\left\langle c_{i}^{n}, g(t)\right\rangle \mathscr{X}_{E_{i}^{n}}(t, \omega) d t
$$

and this is $\mathscr{F}$ measurable since $E_{i}^{n}$ is product measurable. Thus, it is measurable into $\mathscr{V}^{\prime}$ as desired and

$$
\langle f(\cdot, \omega), g\rangle=\lim _{n \rightarrow \infty}\left\langle f_{n}(\cdot, \omega), g\right\rangle, \omega \rightarrow\left\langle f_{n}(\cdot, \omega), g\right\rangle \text { is } \mathscr{F} \text { measurable. }
$$

By the Pettis theorem, $\omega \rightarrow\langle f(\cdot, \omega), g\rangle$ is measurable into $\mathscr{V}^{\prime}$. Obviously, the conclusion is the same for these two conditions if $\mathscr{V}^{\prime}$ is replaced with $\mathscr{V}$.

The following theorem is also useful. It is really a generalization of the familiar Gram Schmidt process. See Lemma 34.4.2 on Page 1179.

Theorem 78.1.6 Suppose $V, W$ are separable Banach spaces, such that $V$ is dense in $W$ and $B \in \mathscr{L}\left(W, W^{\prime}\right)$ satisfies

$$
\langle B x, x\rangle \geq 0,\langle B x, y\rangle=\langle B y, x\rangle, B \neq 0
$$

Then there exists a countable set $\left\{e_{i}\right\}$ of vectors in $V$ such that

$$
\left\langle B e_{i}, e_{j}\right\rangle=\delta_{i j}
$$

and for each $x \in W$,

$$
\langle B x, x\rangle=\sum_{i=1}^{\infty}\left|\left\langle B x, e_{i}\right\rangle\right|^{2}
$$

and also

$$
B x=\sum_{i=1}^{\infty}\left\langle B x, e_{i}\right\rangle B e_{i}
$$

the series converging in $W^{\prime}$. In case $B=B(\omega)$ where $\omega \rightarrow B(\omega)$ is measurable into $\mathscr{L}\left(W, W^{\prime}\right)$, these vectors $e_{i}$ will also depend on $\omega$ and will be measurable functions of $\omega$. In particular, we could let $\omega=t$ with the Lebesgue measurable sets.

The following result, found in [91] is well known. See Section 25.2 on Page 832.
Theorem 78.1.7 If a single valued map, $A: X \rightarrow X^{\prime}$ is monotone, hemicontinuous, and bounded, then $A$ is pseudo monotone. Furthermore, the duality map, $J^{-1}: X \rightarrow X^{\prime}$ which satisfies $\left\langle J^{-1} f, f\right\rangle=\|f\|^{2},\left\|J^{-1} f\right\|_{X}=\|f\|_{X}$ is strictly monotone hemicontinuous and bounded. So is the duality map $F: X \rightarrow X^{\prime}$ which satisfies $\|F f\|_{X^{\prime}}=\|f\|_{X}^{p-1},\langle F f, f\rangle=$ $\|f\|_{X}^{p}$ for $p>1$.

The following fundamental result will be of use in what follows. There is somewhat more in this than will be needed. $B$ is a possibly degenerate operator satisfying only the following:

$$
\begin{equation*}
B \in \mathscr{L}\left(W, W^{\prime}\right),\langle B u, u\rangle \geq 0,\langle B u, v\rangle=\langle B v, u\rangle \tag{78.1.10}
\end{equation*}
$$

where here $V \subseteq W$ and $V$ is dense in $W$.
Also one can obtain the following for $p \geq 2$. It is an integration by parts formula. See Theorem 34.6.4 on Page 1211.

Proposition 78.1.8 Let $p \geq 2$ in what follows. For $u, v \in X$, the following hold. If $B$ is time independent, then it is not necessary to assume $p \geq 2$. It is enough to assume $p>1$.

1. $t \rightarrow\langle B(t) u(t), v(t)\rangle_{W^{\prime}, W}$ equals an absolutely continuous function a.e., denoted by $\langle B u, v\rangle(\cdot)$.
2. $\langle L u(t), u(t)\rangle=\frac{1}{2}\left[\langle B u, u\rangle^{\prime}(t)+\left\langle B^{\prime}(t) u(t), u(t)\right\rangle\right]$ a.e. $t$
3. $|\langle B u, v\rangle(t)| \leq C\|u\|_{X}\|v\|_{X}$ for some $C>0$ and for all $t \in[0, T]$.
4. $t \rightarrow B(t) u(t)$ equals a function in $C\left(0, T ; W^{\prime}\right)$ a.e., denoted by $B u(\cdot)$.
5. $\sup \left\{\|B u(t)\|_{W^{\prime}}, t \in[0, T]\right\} \leq C\|u\|_{X}$ for some $C>0$.

If $K: X \rightarrow X^{\prime}$ is given by

$$
\langle K u, v\rangle_{X^{\prime}, X} \equiv \int_{0}^{T}\langle L u(t), v(t)\rangle d t+\langle B u, v\rangle(0),
$$

then
6. $K$ is linear, continuous and weakly continuous.
7. $\langle K u, u\rangle=\frac{1}{2}[\langle B u, u\rangle(T)+\langle B u, u\rangle(0)]+\frac{1}{2} \int_{0}^{T}\left\langle B^{\prime}(t) u(t), u(t)\right\rangle d t$.
8. If $B u(0)=0$, for $u \in X$, there exists $u_{n} \rightarrow u$ in $X$ such that $u_{n}(t)$ is 0 near 0 . A similar conclusion could be deduced at $T$ if $B u(T)=0$.

Fussing with $p \geq 2$ is necessary only because of the consideration of

$$
\int_{0}^{T}\left\langle B^{\prime}(t) u(t), u(t)\right\rangle d t
$$

If $p<2$, this term might not make sense. The last assertion about approximation makes possible the following corollary.

Corollary 78.1.9 If $B u(0)=0$ for $u \in X$, then $\langle B u, u\rangle(0)=0$. The converse is also true. An analogous result will hold with 0 replaced with $T$.

Proof: Let $u_{n} \rightarrow u$ in $X$ with $u_{n}(t)=0$ for all $t$ close enough to 0 . For $t$ off a set of measure zero consisting of the union of sets of measure zero corresponding to $u_{n}$ and $u$,

$$
\begin{gathered}
\left\langle B u_{n}, u_{n}\right\rangle(t)=\left\langle B(t) u_{n}(t), u_{n}(t)\right\rangle,\langle B u, u\rangle(t)=\langle B(t) u(t), u(t)\rangle, \\
\left\langle B\left(u-u_{n}\right), u\right\rangle(t)=\left\langle B(t)\left(u(t)-u_{n}(t)\right), u(t)\right\rangle \\
\left\langle B u_{n}, u-u_{n}\right\rangle(t)=\left\langle B(t) u_{n}(t), u(t)-u_{n}(t)\right\rangle
\end{gathered}
$$

Then, considering such $t$,

$$
\begin{aligned}
\langle B(t) u(t), u(t)\rangle-\left\langle B(t) u_{n}(t), u_{n}(t)\right\rangle= & \left\langle B(t)\left(u(t)-u_{n}(t)\right), u(t)\right\rangle \\
& +\left\langle B(t) u_{n}(t), u(t)-u_{n}(t)\right\rangle
\end{aligned}
$$

Hence from Theorem 78.1.8,

$$
\left|\langle B(t) u(t), u(t)\rangle-\left\langle B(t) u_{n}(t), u_{n}(t)\right\rangle\right| \leq C\left\|u-u_{n}\right\|_{X}\left(\|u\|_{X}+\left\|u_{n}\right\|_{X}\right)
$$

Thus if $n$ is sufficiently large,

$$
\left|\langle B(t) u(t), u(t)\rangle-\left\langle B(t) u_{n}(t), u_{n}(t)\right\rangle\right|<\varepsilon
$$

So let $n$ be fixed and this large and now let $t_{k} \rightarrow 0$ to obtain

$$
\left\langle B\left(t_{k}\right) u_{n}\left(t_{k}\right), u_{n}\left(t_{k}\right)\right\rangle=0
$$

for $k$ large enough. Hence

$$
\langle B u, u\rangle(0)=\lim _{k \rightarrow \infty}\left\langle B\left(t_{k}\right) u\left(t_{k}\right), u\left(t_{k}\right)\right\rangle<\varepsilon
$$

Since $\varepsilon$ is arbitrary, $\langle B u, u\rangle(0)=0$.
Next suppose $\langle B u, u\rangle(0)=0$. Then letting $v \in X$, with $v$ smooth,

$$
\langle B u(0), v(0)\rangle=\langle B u, v\rangle(0)=\langle B u, u\rangle^{1 / 2}(0)\langle B v, v\rangle^{1 / 2}(0)=0
$$

and it follows that $B u(0)=0$.
Note also that this shows that if $(B v)^{\prime} \in L^{p^{\prime}}\left(0, T ; V^{\prime}\right)$ as well as $(B u)^{\prime}$, then there is a continuous function

$$
t \rightarrow\langle B(u+v), u+v\rangle(t)
$$

which equals $\langle B(u(t)+v(t)), u(t)+v(t)\rangle$ for $a . e . t$ and so, defining

$$
\langle B u, v\rangle(t) \equiv(\langle B u, u\rangle(t)+\langle B v, v\rangle(t)-\langle B(u+v), u+v\rangle(t)) \frac{1}{2}
$$

It follows that $t \rightarrow\langle B u, v\rangle(t)$ is continuous and equals $\langle B(u(t)), v(t)\rangle$ a.e. $t$.
This also makes it easy to verify continuity of pointwise evaluation of $B u$.
Let $L u=(B u)^{\prime}$.

$$
\begin{gather*}
u \in D(L) \equiv X \equiv\left\{u \in L^{p}(0, T, V): L u \equiv(B u)^{\prime} \in L^{p^{\prime}}\left(0, T, V^{\prime}\right)\right\} \\
\|u\|_{X} \equiv \max \left(\|u\|_{L^{p}(0, T, V)},\|L u\|_{L^{p^{\prime}}\left(0, T, V^{\prime}\right)}\right) \tag{78.1.11}
\end{gather*}
$$

Since $L$ is closed, this $X$ is a Banach space. To see that $L$ is closed, suppose $u_{n} \rightarrow u$ in $\mathscr{V}$ and $\left(B u_{n}\right)^{\prime} \rightarrow \xi$ in $\mathscr{V}^{\prime}$. Is $\xi=(B u)^{\prime}$ ? Letting $\phi \in C_{c}^{\infty}([0, T])$ and $v \in V$,

$$
\begin{equation*}
\int_{0}^{T}\langle\xi, \phi v\rangle_{V^{\prime}, V}=\lim _{n \rightarrow \infty} \int_{0}^{T}\left\langle\left(B u_{n}\right)^{\prime}, \phi v\right\rangle=\lim _{n \rightarrow \infty}-\int_{0}^{T}\left\langle B u_{n}, \phi^{\prime} v\right\rangle \tag{78.1.12}
\end{equation*}
$$

We can take a subsequence, still denoted with $n$ such that $u_{n}(t) \rightarrow u(t)$ pointwise a.e. Also

$$
\int_{0}^{T}\left|\left\langle B u_{n}, \phi^{\prime} v\right\rangle\right|^{p} \leq \int_{0}^{T}\left\|u_{n}\right\|_{V}^{p} d t C\left(\phi^{\prime}, v\right)
$$

and these terms on the right are uniformly bounded by the assumption that $u_{n}$ is bounded in $\mathscr{V}$. Therefore, by the Vitali convergence theorem, and using the subsequence just described, we can pass to the limit in 78.1.12.

$$
\left\langle\int_{0}^{T} \xi \phi d t, v\right\rangle=\left\langle-\int_{0}^{T}(B u) \phi^{\prime} d t, v\right\rangle
$$

Since $v$ is arbitrary, this shows that

$$
\int_{0}^{T} \xi \phi d t=-\int_{0}^{T}(B u) \phi^{\prime} d t \text { in } V^{\prime}
$$

and so $\xi=(B u)^{\prime}$ because this is what is meant by $(B u)^{\prime}$. Hence $L$ is indeed closed and $X$ is a Banach space. It is also a reflexive Banach space because it is isometric to a closed subspace of the reflexive Banach space $\mathscr{V} \times \mathscr{V}^{\prime}$. Also, the following is useful. See Theorem 34.4.7 on Page 1193.

Theorem 78.1.10 If $Y$ denotes those $f \in L^{p}([0, T] ; V)$ for which $f^{\prime} \in L^{p}([0, T] ; V)$, so that $f$ has a representative such that $f(t)=f(0)+\int_{0}^{t} f^{\prime}(s) d$ s a.e. $t$, then if $\|f\|_{Y} \equiv$ $\|f\|_{L^{p}([0, T] ; V)}+\left\|f^{\prime}\right\|_{L^{p}([0, T] ; V)}$ the map $f \rightarrow f(t)$ is continuous in the sense that $\|f(t)\| \leq$ $C\left(\|f\|_{Y}\right)$.

We also have the following general theory about existence of measurable solutions to elliptic problems. First are conditions which a nonlinear set valued map should satisfy. In what follows, $X$ denotes a reflexive separable Banach space with dual $X^{\prime},(\Omega, \mathscr{F})$ is a measurable space, and $A(\cdot, \omega): X \rightarrow \mathscr{P}\left(X^{\prime}\right)$, for $\omega \in \Omega$, denotes a set valued operator. We make the following assumptions on such an operator:

- $H_{1}$ Measurability condition. For each $u \in X$, there is a measurable selection $z(\omega)$ such that

$$
z(\omega) \in \mathscr{A}(u, \omega)
$$

- $H_{2}$ Values of $\mathscr{A}$. $\mathscr{A}(\cdot, \omega): X \rightarrow \mathscr{P}\left(X^{\prime}\right)$ has bounded, closed, nonempty, and convex values. $A(\cdot, \omega)$ maps bounded sets to bounded sets.
- $H_{3}$ Limit conditions, $\mathscr{A}(\cdot, \omega)$ is pseudomonotone:

$$
\text { If } u_{n} \rightharpoonup u \text { and } \lim \sup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle \leq 0, \text { for } z_{n} \in \mathscr{A}\left(u_{n}, \omega\right),
$$

then for each $v$, there exists $z(v) \in \mathscr{A}(u, \omega)$ such that

$$
\lim _{k \rightarrow \infty} \inf _{k}\left\langle z_{n}, u_{n}-v\right\rangle \geq\langle z(v), u-v\rangle
$$

In our use of the above, the space $X$ will be a space of functions defined on $[0, T]$ to be described more later.

We note that for a fixed $\omega$, the operator $A(\cdot, \omega)$ described earlier is set-valued, bounded and pseudomonotone as a map from $X$ to $\mathscr{P}(X)$. Moreover, the sum of two of such operators is set-valued, bounded and pseudomonotone, Theorem 48.5.2 below. The limit condition $H_{3}$ implies that $A(\cdot, \omega)$ is upper-semicontinuous from the strong topology to the weak topology. This can be used to show that when $\omega \rightarrow u(\omega)$ is measurable, then $A(u(\omega), \omega)$ has a measurable selection assuming only that $\omega \rightarrow A(u, \omega)$ has a measurable selection for fixed $u \in X$. Here is a well known result on the sum of pseudomonotone operators. See Theorem 25.5.1 on Page 855.

Theorem 78.1.11 Assume that A and B are set-valued, bounded and pseudomonotone operators. Then, their sum is also a set-valued, bounded and pseudomonotone operator. Moreover, if $u_{n} \rightarrow u$ weakly, $z_{n} \rightarrow z, z_{n} \in A\left(u_{n}\right), w_{n} \rightarrow w$ weakly with $w_{n} \in A\left(u_{n}\right)$, and

$$
\limsup _{n \rightarrow \infty}\left\langle z_{n}+w_{n}, u_{n}-u\right\rangle \leq 0
$$

then,

$$
\lim \inf _{n \rightarrow \infty}\left\langle z_{n}+w_{n}, u_{n}-v\right\rangle \geq\langle z(v)+w(v), u-v\rangle
$$

for $z(v) \in A(u), w(v) \in B(u)$ and in fact, $z \in A(u)$ and $w \in B(u)$.
We now state our result on measurable solutions to general elliptic variational inequalities that may contain sums of set-valued, bounded and pseudomonotone operators. Then the following is proved in [2]. See Theorem 48.5.3 on Page 1566.

Theorem 78.1.12 Let $\omega \rightarrow K(\omega)$ be a measurable set-valued function, where $K(\omega) \subset V$ is convex, closed and bounded. Let the operators $A(\cdot, \cdot)$ and $B(\cdot, \cdot)$ satisfy assumptions $H_{1}-H_{3}$. Finally, let $\omega \rightarrow f(\omega)$ be measurable with values in $V^{\prime}$.

Then, there exists a measurable function $\omega \rightarrow u(\omega) \in K(\omega)$ such that $\omega \rightarrow w^{A}(\omega)$, and $\omega \rightarrow w^{B}(\omega)$ with $w^{A}(\omega) \in A(u(\omega), \omega)$ and $w^{B}(\omega) \in B(u(\omega), \omega)$, and

$$
\left\langle f(\omega)-\left(w^{A}(\omega)+w^{B}(\omega)\right), z-u(\omega)\right\rangle \leq 0
$$

for all $z \in K(\omega)$.
If it is only known that $K(\omega)$ is closed and convex, the same conclusion holds true if it is also known that for some $z(\omega) \in K(\omega), A(\cdot, \omega)+B(\cdot, \omega)$ is coercive, that is

$$
\lim _{\|v\| \rightarrow \infty} \inf \left\{\frac{\left\langle z^{*}, v-z\right\rangle}{\|v\|}: z^{*} \in(A(v, \omega)+B(v, \omega))\right\}=\infty .
$$

Instead of two operators, one could have the sum of finitely many with the same conclusions.

### 78.2 Measurable Solutions To Evolution Inclusions

The main result in this section is Theorem 78.2.2 below. It gives an existence theorem for many evolution inclusions. We are assuming that $A(\cdot, \omega): \mathscr{V} \rightarrow \mathscr{P}\left(\mathscr{V}^{\prime}\right)$ satisfies some conditions presented earlier: These are 78.1.1-78.1.1. Then we can regard $A(\cdot, \omega)$ as a set valued pseudomonotone map from $X$ to $\mathscr{P}\left(X^{\prime}\right)$. It is clear that $A(u, \omega)$ is a closed convex set in $X^{\prime}$ because if $z_{n}^{*} \rightarrow z^{*}$ in $X^{\prime}, z_{n}^{*} \in A(u, \omega)$, then $z^{*} \in A(u, \omega)$ because a subsequence, still denoted as $z_{n}^{*}$ converges weakly to some $w^{*} \in A(u, \omega)$ in $V^{\prime}$. Since $X$ is dense in $\mathscr{V}$, this requires $w^{*}=z^{*}$. The necessary limit conditions for pseudomonotone are nothing more than the assumed conditions in 78.1.1. Also, we will assume in this section that $p \geq 2$. This restriction is necessary because of the desire to consider time dependent $B$ and the assumption 78.1 .6 which involves a term $\int_{0}^{T}\left\langle B^{\prime} u, u\right\rangle$ which might not make sense if $p$ were only larger than 1 . If $B$ were not time dependent, this assumption would not be necessary and the argument given here would continue to be valid. We essentially show this is the case in the following section in which we also consider a more general coercivity condition than 78.1.1, but one can see that there is no change in the argument and it is in fact simpler if we assume $B$ is constant.

As above, $5, K: X \rightarrow X^{\prime}$ can be defined as

$$
\langle K u, v\rangle \equiv \int_{0}^{T}\langle L u, v\rangle d s+\langle B u, v\rangle(0)
$$

Note that from Proposition 78.1.8, if $v \in X$ and $B v(0)=0$, then

$$
\begin{equation*}
\langle K u, v\rangle=\int_{0}^{T}\langle L u, v\rangle d s \tag{78.2.13}
\end{equation*}
$$

This is because, from the Cauchy Schwarz inequality and continuity of $\langle B u, u\rangle(\cdot)$,

$$
\langle B u, v\rangle(0) \leq\langle B u, u\rangle^{1 / 2}(0)\langle B v, v\rangle^{1 / 2}(0)
$$

and if $B v(0)=0$, then from Corollary 78.1.9, $\langle B v, v\rangle^{1 / 2}(0)=0$. From the above Proposition 78.1.8, this operator $K$ is hemicontinuous and bounded and monotone as a map from $X$ to $X^{\prime}$. Thus $K+A(\cdot, \omega)$ is a set valued pseudomonotone map for which we can apply Theorem 78.1.12 and obtain existence theorems for measurable solutions to variational inequalities right away, but we want to obtain solutions to an evolution equation in which $K(\omega)=V$ and the above theorem does not apply because the sum of these two operators is not coercive. Therefore, we consider another operator which, when added, will result in coercivity. Let $J: \mathscr{V} \rightarrow \mathscr{V}^{\prime}$ be the duality map for 2 . Thus $\|J u\|_{\mathscr{V}^{\prime}}=\|u\|_{\mathscr{V}}$ and $\langle J u, u\rangle=\|u\|_{\mathscr{V}}^{2}$. Then $J^{-1}: \mathscr{V}^{\prime} \rightarrow \mathscr{V}$ also satisfies $\left\langle f, J^{-1} f\right\rangle=\|f\|_{V^{\prime}}^{2}$.

The main result in this section is based on methods due to Brezis [22] and Lions [91] adapted to the case considered here where the operator is set valued, and we consider measurability. We define the operator $M: X \rightarrow X^{\prime}$ by

$$
\langle M u, v\rangle \equiv\left\langle L v, J^{-1} L u\right\rangle_{V^{\prime}, \mathscr{V}} \text { where as above, } L u=(B u)^{\prime} .
$$

Then let $f$ be measurable into $\mathscr{V}^{\prime}$. Thus, in particular, $f(\omega) \in \mathscr{V}^{\prime}$ for each $\omega$. Consider the approximate problem and a solution $u_{\varepsilon}$ to

$$
\begin{equation*}
\varepsilon M u_{\varepsilon}(\omega)+K u_{\varepsilon}(\omega)+w_{\varepsilon}^{*}(\omega)=f(\omega)+g(\omega), w_{\varepsilon}^{*}(\omega) \in A\left(u_{\varepsilon}(\omega), \omega\right) \tag{78.2.14}
\end{equation*}
$$

Where $g(\omega) \in X^{\prime}$ is given by

$$
\langle g(\omega), v\rangle \equiv\left\langle B v(0), u_{0}(\omega)\right\rangle
$$

where $u_{0}(\omega)$ is a given function measurable into $W$. Now for $u \in X$, we let

$$
\mathscr{A}(u, \omega)=\varepsilon M u+K u+A(u, \omega)
$$

Then by the assumptions on $A(\cdot, \omega)$, there is $u^{*}(\omega)$ for which $\omega \rightarrow u^{*}(\omega)$ is measurable into $\mathscr{V}^{\prime}$, hence measurable into $X^{\prime}$. Therefore, $\omega \rightarrow \mathscr{A}(u, \omega)$ has a measurable selection, namely $\varepsilon M u+K u+u^{*}(\omega)$ and so condition 78.1.2 is verified.

By Theorem 78.1.12, a solution to 78.2.14 will exist with both $u_{\varepsilon}$ and $w_{\varepsilon}^{*}$ measurable if we can argue that the sum of the operators $\varepsilon M+K+A(\cdot, \omega)$ is coercive, since this is the sum of pseudomonotone operators. From 78.1.1

$$
\inf \left(\int_{0}^{T}\left\langle u^{*}, u\right\rangle d s: u^{*} \in A(u, \omega)\right)+\frac{1}{2} \int_{0}^{T}\left\langle B^{\prime} u, u\right\rangle \geq \delta(\omega) \int_{0}^{T}\|u\|_{V}^{p} d s-m(\omega)
$$

and so routine considerations show that $\varepsilon M+K+A(\cdot, \omega)$ does indeed satisfy a suitable coercivity estimate for each positive $\varepsilon$. Thus we have the following existence theorem for approximate solutions.

Lemma 78.2.1 Let $f$ be measurable into $\mathscr{V}^{\prime}$ and let A satisfy the conditions 78.1.1-78.1.1. Then for $K$ and $M$ defined as above, it follows there exist measurable $u_{\varepsilon}$ and $w_{\varepsilon}^{*}$ satisfying 78.2.14.

Note this implies that, suppressing dependence on $\omega$,

$$
\left\langle B u_{\varepsilon}, v\right\rangle(0)=\left\langle B v(0), u_{0}\right\rangle
$$

for all $v \in X$. Thus, letting $v$ be a smooth function with values in $V$

$$
\left\langle B u_{\varepsilon}(0), v(0)\right\rangle=\left\langle B u_{0}, v(0)\right\rangle
$$

Since $V$ is dense in $W$, this requires $B u_{\varepsilon}(0)=B u_{0}$.
Now define $\Lambda$ to be the restriction of $L$ to those $u \in X$ which have $B u(0)=0$. Thus by Corollary 78.1.9,

$$
D(\Lambda)=\{u \in X: B u(0)=0\}=\{u \in X:\langle B u, u\rangle(0)=0\}
$$

and if $v \in D(\Lambda), u \in X$, then as noted earlier,

$$
\langle K u, v\rangle=\int_{0}^{T}\langle L u, v\rangle d s
$$

Also, one can show an estimate for $\Lambda^{*}$.
You can define $D(T) \equiv\left\{u \in \mathscr{V}: u^{\prime} \in \mathscr{V}, u(T)=0\right\}$ and let $T u=-B u^{\prime}$. Then

$$
\begin{aligned}
\langle T u, u\rangle & =-\int_{0}^{T}\left\langle B u^{\prime}, u\right\rangle=-\left.\langle B u, u\rangle\right|_{0} ^{T}+\int_{0}^{T}\left\langle(B u)^{\prime}, u\right\rangle \\
& =\langle B u, u\rangle(0)+\int_{0}^{T}\left\langle B^{\prime} u, u\right\rangle+\int_{0}^{T}\left\langle B u^{\prime}, u\right\rangle
\end{aligned}
$$

and so we obtain

$$
\begin{equation*}
2\langle T u, u\rangle \geq \int_{0}^{T}\left\langle B^{\prime} u, u\right\rangle \tag{78.2.15}
\end{equation*}
$$

Then one shows that $T^{*}=\Lambda$ and that the graph of $\Lambda^{*}$ is the closure of the graph of $T$ thus showing that $\Lambda^{*}$ also satisfies an inequality like 78.2 .15 for $u \in D\left(\Lambda^{*}\right)$.

From 78.2.14,

$$
\begin{aligned}
& \varepsilon\left\langle L v, J^{-1} L u_{\varepsilon}\right\rangle_{V^{\prime}, \mathscr{V}}+\left\langle K u_{\varepsilon}(\omega), v\right\rangle_{X^{\prime}, X}+\left\langle w_{\varepsilon}^{*}(\omega), v\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \\
= & \langle f(\omega), v\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}+\langle g(\omega), v\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}
\end{aligned}
$$

If we restrict to $v \in D(\Lambda)$ so $B v(0)=0$, then it reduces to

$$
\varepsilon\left\langle\Lambda v, J^{-1} L u_{\varepsilon}\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}+\left\langle L u_{\varepsilon}(\omega), v\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}+\left\langle w_{\varepsilon}^{*}(\omega), v\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}=\langle f(\omega), v\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}
$$

and so $J^{-1} L u_{\varepsilon} \in D\left(\Lambda^{*}\right)$. Thus, since $D(\Lambda)$ is dense in $\mathscr{V}$, it follows that

$$
\varepsilon \Lambda^{*} J^{-1} L u_{\varepsilon}+L u_{\varepsilon}+w_{\varepsilon}^{*}=f \text { in } \mathscr{V}^{\prime}
$$

Then act on $J^{-1} L u_{\varepsilon}$ on both sides in the above. This yields for some $C$ dependent on $B^{\prime}$ an inequality of the following form.

$$
\begin{equation*}
-\varepsilon C\left\|L u_{\varepsilon}\right\|^{2}+\left\|L u_{\varepsilon}\right\|^{2}+\left\langle w_{\varepsilon}^{*}, J^{-1} L u_{\varepsilon}\right\rangle \leq\left\langle f, J^{-1} L u_{\varepsilon}\right\rangle \tag{78.2.16}
\end{equation*}
$$

Also, acting on both sides of 78.2 .14 with $u_{\varepsilon}$ and using the formula for $\langle K u, u\rangle$,

$$
\begin{aligned}
& \varepsilon\left\langle L u_{\varepsilon}, J^{-1} L u_{\varepsilon}\right\rangle+\frac{1}{2}\left[\left\langle B u_{\varepsilon}, u_{\varepsilon}\right\rangle(T)+\left\langle B u_{\varepsilon}, u_{\varepsilon}\right\rangle(0)\right] \\
&+\frac{1}{2} \int_{0}^{T}\left\langle B^{\prime}(t) u_{\varepsilon}(t), u_{\varepsilon}(t)\right\rangle d t+\left\langle w_{\varepsilon}^{*}, u_{\varepsilon}\right\rangle_{V^{\prime}, \mathscr{V}}=\left\langle f, u_{\varepsilon}\right\rangle+\left\langle B u_{\varepsilon}(0), u_{0}(\omega)\right\rangle \\
&=\left\langle f, u_{\varepsilon}\right\rangle+\left\langle B u_{0}(\omega), u_{0}(\omega)\right\rangle
\end{aligned}
$$

It follows easily from the coercivity condition 78.1 .6 that $u_{\varepsilon}$ is bounded in $\mathscr{V}$ and consequently $w_{\varepsilon}^{*}$ is bounded in $\mathscr{V}^{\prime}$, this from the growth estimate 78.1.1. Now from 78.2.16, it also follows that $\left\|L u_{\mathcal{\varepsilon}}\right\|_{\mathscr{V}^{\prime}}$ is bounded for small $\varepsilon$. Thus

$$
\left\|L u_{\varepsilon}(\omega)\right\|_{\mathscr{V}^{\prime}}+\left\|u_{\varepsilon}(\omega)\right\|_{\mathscr{V}}+\left\|w_{\varepsilon}^{*}(\omega)\right\|_{\mathscr{V}^{\prime}} \leq C(\omega)<\infty
$$

$C(\omega)$ independent of small $\varepsilon$. By Theorem 78.1.3, there is a subsequence $\varepsilon(\omega) \rightarrow 0$ such that

$$
\begin{gather*}
\left(L u_{\varepsilon(\omega)}(\omega), u_{\varepsilon(\omega)}(\omega), w_{\varepsilon(\omega)}^{*}(\omega), B u_{\mathcal{E}(\omega)}(\omega)(0)\right) \rightarrow \\
(L u(\omega), u(\omega), \xi(\omega), B u(\omega)(0)) \tag{78.2.17}
\end{gather*}
$$

in $\mathscr{V}^{\prime} \times \mathscr{V} \times \mathscr{V}^{\prime} \times V^{\prime}$ weakly and $\omega \rightarrow(L u(\omega), u(\omega), \xi(\omega))$ is measurable into $\mathscr{V}^{\prime} \times \mathscr{V} \times$ $\mathscr{V}^{\prime}$. It follows that $B u(\omega)(0)=B u_{0}(\omega)$ because each $B u_{\varepsilon(\omega)}(\omega)(0)=B u_{0}(\omega)$. Note that this also shows that $K u_{\varepsilon} \rightharpoonup K u$ in $X^{\prime}$. Thus, suppressing the dependence on $\omega$, use 78.2.14 to act on $u_{\varepsilon}-u$ and obtain

$$
\varepsilon\left\langle L u_{\varepsilon}-L u, J^{-1} L u_{\varepsilon}\right\rangle+\left\langle K u_{\varepsilon}, u_{\varepsilon}-u\right\rangle+\left\langle w_{\varepsilon}^{*}, u_{\varepsilon}-u\right\rangle=\left\langle f, u_{\varepsilon}-u\right\rangle+\left\langle g, u_{\varepsilon}-u\right\rangle
$$

Using monotonicity of $J^{-1}$,

$$
\varepsilon\left\langle L u_{\varepsilon}-L u, J^{-1} L u\right\rangle+\left\langle K u_{\varepsilon}, u_{\varepsilon}-u\right\rangle+\left\langle w_{\varepsilon}^{*}, u_{\varepsilon}-u\right\rangle \leq\left\langle f, u_{\varepsilon}-u\right\rangle+\left\langle g, u_{\varepsilon}-u\right\rangle
$$

Now $\left(B u_{\varepsilon}-B u\right)(0)=0$. Therefore, $u_{\varepsilon}-u \in D(\Lambda)$ and so

$$
\varepsilon\left\langle\Lambda^{*} J^{-1} L u, u_{\varepsilon}-u\right\rangle+\left\langle K u_{\varepsilon}, u_{\varepsilon}-u\right\rangle+\left\langle w_{\varepsilon}^{*}, u_{\varepsilon}-u\right\rangle \leq\left\langle f, u_{\varepsilon}-u\right\rangle+\left\langle g, u_{\varepsilon}-u\right\rangle
$$

Recall that $K$ is monotone, bounded and hemicontinuous. In fact, it is monotone and linear. Hence, $K+A$ is pseudomonotone. Then from the above,

$$
\limsup _{\varepsilon \rightarrow 0}\left\langle K u_{\varepsilon}+w_{\varepsilon}^{*}, u_{\varepsilon}-u\right\rangle \leq 0
$$

Now these weak convergences in 78.2.17 include the weak convergence of $u_{\varepsilon}$ to $u$ in $X$. Thus, since $K+A(\cdot, \omega)$ is pseudomonotone as a map from $X$ to $\mathscr{P}\left(X^{\prime}\right)$, for every $v \in X$, there exists $w^{*}(v) \in K(u)+A(u, \omega)$ such that

$$
\lim _{\varepsilon \rightarrow 0}\left\langle K u_{\varepsilon}+w_{\varepsilon}^{*}, u_{\varepsilon}-v\right\rangle \geq\left\langle w^{*}(v), u-v\right\rangle
$$

In particular, this holds if $v=u$ which shows that

$$
\lim _{\varepsilon \rightarrow 0}\left\langle K\left(u_{\varepsilon}\right)+w_{\varepsilon}^{*}, u_{\varepsilon}-u\right\rangle=0
$$

It follows then that for $v \in X$,

$$
\begin{aligned}
\langle\xi+K u, u-v\rangle & =\lim _{\varepsilon \rightarrow 0}\left\langle w_{\varepsilon}^{*}+K u_{\varepsilon}, u-v\right\rangle \\
& =\lim _{\varepsilon \rightarrow 0}\left[\left\langle w_{\varepsilon}^{*}+K u_{\varepsilon}, u-u_{\varepsilon}\right\rangle+\left\langle w_{\varepsilon}^{*}+K u_{\varepsilon}, u_{\varepsilon}-v\right\rangle\right] \\
& \geq \lim _{\varepsilon \rightarrow 0}\left\langle w_{\varepsilon}^{*}+K u_{\varepsilon}, u_{\varepsilon}-v\right\rangle \geq\left\langle w^{*}(v), u-v\right\rangle
\end{aligned}
$$

Since $v$ is arbitrary, separation theorems imply that

$$
\xi(\omega)+K u(\omega) \equiv w^{*}(\omega)+K u(\omega) \in A(u(\omega), \omega)+K u(\omega)
$$

Then passing to the limit in 78.2 .14 , we have

$$
\begin{gather*}
K u(\omega)+w^{*}(\omega)=f(\omega)+g(\omega) \text { in } X^{\prime}, \\
w^{*}(\omega) \in A(u(\omega), \omega), B u(\omega)(0)=B u_{0}(\omega) \tag{78.2.18}
\end{gather*}
$$

and $L u, w^{*}, u$ are all measurable into the appropriate spaces. This implies for each $v \in X$,

$$
\int_{0}^{T}\langle L u, v\rangle+\langle B u, v\rangle(0)+\int_{0}^{T}\left\langle w^{*}, v\right\rangle=\int_{0}^{T}\langle f, v\rangle+\left\langle B v(0), u_{0}\right\rangle
$$

In particular, letting $v=u$,

$$
\begin{aligned}
\int_{0}^{T}\langle L u, u\rangle+\langle B u, u\rangle(0)+\int_{0}^{T}\left\langle w^{*}, u\right\rangle & =\int_{0}^{T}\langle f, v\rangle+\left\langle B u(0), u_{0}\right\rangle \\
& =\int_{0}^{T}\langle f, v\rangle+\left\langle B u_{0}, u_{0}\right\rangle
\end{aligned}
$$

Thanks to 78.2.18, this shows that $\langle B u, u\rangle(0)=\left\langle B u_{0}, u_{0}\right\rangle$. Also it follows from Theorem 78.1.8.

This has proved the following theorem.
Theorem 78.2.2 Let $p \geq 2$ and let A satisfy 78.1.1-78.1.1 and let $f$ be measurable into $\mathscr{V}^{\prime}$ and let $u_{0}$ be measurable into $W$. Then there exists a solution to 78.2 .18 such that $L u, w^{*}, u$ are all measurable. We also have for $u$ this solution that for fixed $\omega,\left\langle B u_{0}, u_{0}\right\rangle=\langle B u, u\rangle(0)$.

We also have the following corollary which gives measurable solutions to periodic problems. Of course there is no uniqueness for such periodic problems so this is another place where our theory is applicable. In this corollary, we assume for the sake of simplicity that $B(t)=B$ a constant. Thus, it is not necessary to assume $p \geq 2$ in the following corollary.

Corollary 78.2.3 Let A satisfy 78.1.1-78.1.1, $p>1$, and let $f$ be measurable into $\mathscr{V}^{\prime}$. Then there exists a solution to

$$
\begin{aligned}
L u(\omega)+w^{*}(\omega) & =f(\omega), B u(0, \omega)=B u(T, \omega), \\
w^{*}(\omega) & \in A(u(\omega), \omega)
\end{aligned}
$$

such that $L u, w^{*}, u$ are all measurable.
Proof: Define $\Lambda$ as the restriction of $L$ to the space $\{u \in D(L): B u(0)=B u(T)\}$. This enables periodic conditions. Let

$$
D(T) \equiv\left\{v \in \mathscr{V}: v^{\prime} \in \mathscr{V} \text { and } v(T)=v(0)\right\}, T v=-B v^{\prime}
$$

Then consider $T^{*}$. If $u \in D(\Lambda), v \in D(T)$,

$$
-\int_{0}^{T}\left\langle B v^{\prime}, u\right\rangle=-\left.\langle B u, v\rangle\right|_{0} ^{T}+\int_{0}^{T}\left\langle(B u)^{\prime}, v\right\rangle
$$

and so, since the boundary term vanishes, this shows that $D(\Lambda) \subseteq D\left(T^{*}\right)$ and that $T^{*}=\Lambda$ on $D(\Lambda)$.

Next let $u \in D\left(T^{*}\right)$. By definition, this means that

$$
\begin{equation*}
|\langle T v, u\rangle| \leq C_{u}\|v\|_{\mathscr{V}} \tag{*}
\end{equation*}
$$

So let $v \in C_{c}^{\infty}([0, T] ; V)$.

$$
\langle T v, u\rangle=-\int_{0}^{T}\left\langle B v^{\prime}, u\right\rangle=-\int_{0}^{T}\left\langle B u, v^{\prime}\right\rangle
$$

From the Riesz representation theorem, there exists a unique $(B u)^{\prime}$ such that the above equals $\int_{0}^{T}\left\langle(B u)^{\prime}, v\right\rangle$ and by density of $C_{c}^{\infty}([0, T] ; V)$ this shows $T^{*} u=(B u)^{\prime}=L u$. Thus $T^{*}=L$ on $D\left(T^{*}\right)$ and in particular $(B u)^{\prime} \in \mathscr{V}^{\prime}$. It remains to consider the boundary conditions. For $u \in D\left(T^{*}\right)$ and $v \in D(T)$,

$$
\langle T v, u\rangle=-\int_{0}^{T}\left\langle B v^{\prime}, u\right\rangle=-\left.\langle B u, v\rangle\right|_{0} ^{T}+\int_{0}^{T}\left\langle(B u)^{\prime}, v\right\rangle
$$

The boundary term is of the form

$$
\langle B u(0)-B u(T), v(0)\rangle
$$

If $*$ is to hold for all $v \in D(T)$ we must have $B u(0)=B u(T)$. If the difference is $\xi \neq 0$, you would need to have

$$
|\langle\xi, v(0)\rangle| \leq C_{u}\|v\|_{\mathscr{V}}
$$

for all $v \in D(T)$. So pick $v \in D(T)$ such that $|\langle\xi, v(0)\rangle|=\delta>0$ and consider a piecewise linear function $\psi_{n}$ which is one at 0 and $T$ but zero on $[1 / n, T-(1 / n)]$. Then if $v_{n}=\psi_{n} v$, the left side is $\delta$ for all $n$ but the right converges to 0 .

This shows that $D\left(T^{*}\right)=D(\Lambda)$ and $T^{*}=\Lambda$. Now it follows that $T^{* *}=\Lambda^{*}$ and so $\Lambda^{*}$ is monotone because $T$ is and the graph of $\Lambda^{*}$ is the closure of the graph of $T$. Indeed,

$$
\int_{0}^{T}\left\langle-B v^{\prime}, v\right\rangle=\int_{0}^{T}\left\langle B v^{\prime}, v\right\rangle
$$

so $\langle T v, v\rangle=0$. The same is true of $\Lambda^{*}$.
Now in this case we let $X$ be the same as before. Then we consider the approximate problem for $u_{\varepsilon}$ given by

$$
\begin{aligned}
& \varepsilon\left\langle L v, J^{-1}\left(L u_{\varepsilon}\right)\right\rangle+\left\langle L u_{\varepsilon}(\omega), v\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}+ \\
& \frac{1}{2}\langle B u, v\rangle(0)-\frac{1}{2}\langle B u, v\rangle(T)+\left\langle w_{\varepsilon}^{*}(\omega), v\right\rangle \\
&=\langle f(\omega), v\rangle, w_{\varepsilon}^{*}(\omega) \in A\left(u_{\varepsilon}(\omega), \omega\right)
\end{aligned}
$$

Then using monotonicity of $\Lambda^{*}$ and $\Lambda$ as before, one obtains the existence of a measurable solution. To see that the necessary monotonicity holds, note that

$$
\langle L u, u\rangle_{V^{\prime}, V}+\frac{1}{2}\langle B u, u\rangle(0)-\frac{1}{2}\langle B u, u\rangle(T)=0
$$

This follows from 5 and 7. Indeed, these imply that

$$
\langle L u, u\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}+\langle B u, u\rangle(0)=\frac{1}{2}[\langle B u, u\rangle(T)+\langle B u, u\rangle(0)]
$$

and so the above follows. The rest of the argument is similar to that used to prove Theorem 78.2.2. At the end you will obtain that

$$
\frac{1}{2}\langle B u, v\rangle(0)-\frac{1}{2}\langle B u, v\rangle(T)=0
$$

which will require that $B u(T)=B u(0)$ since $v$ is arbitrary.

### 78.3 Relaxed Coercivity Condition

This section is devoted to proving Theorem 78.3.2 below. It includes a more general coercivity condition and uses a slightly modified limit condition. Also, it removes the restriction that $p \geq 2$, which was made because of the terms involving $B^{\prime}$. However, we will specialize to the case where $B$ does not depend on $t$. It seems that this will be necessary because if one is required to consider $\left\langle B^{\prime} u, u\right\rangle$ then this won't make sense unless $p \geq 2$. In what follows $p>1$.

Let $U$ be dense in $V$ with the embedding compact, $U$ being a separable reflexive Banach space. It is always possible to get such a space, (In fact, it can be assumed a Hilbert space.) but in applications of most interest to us, it can be obtained by Sobolev embedding theorems. We will let $r>\max (2, p)$ and $\mathscr{U}_{r}=L^{r}([0, T] ; U)$. Also, for $I=[0, \hat{T}], \hat{T}<$ $T$, we will denote as $\mathscr{V}_{I}$ the space $L^{p}(I ; V)$ with a similar usage of this notation in other
situations. If $u \in \mathscr{V}$, then we will always consider $u \in \mathscr{V}_{I}$ also by simply considering its restriction to $I$. With this convention, it is clear that if $u$ is measurable into $\mathscr{V}$ then it is also measurable into $\mathscr{V}_{I}$.

Then the modified conditions on $A: \mathscr{V}_{I} \rightarrow \mathscr{P}\left(\mathscr{V}_{I}^{\prime}\right)$ are as follows for $A(u, \omega)$ a convex closed set in $\mathscr{V}_{I}^{\prime}$ whenever $u \in \mathscr{V}_{I}$.

## - growth estimate

Assume the specific estimate for $u \in \mathscr{V}_{I}$.

$$
\begin{equation*}
\sup \left\{\left\|u^{*}\right\|_{\mathscr{V}_{I}^{\prime}}: u^{*} \in A(u, \omega)\right\} \leq a(\omega)+b(\omega)\|u\|_{\mathscr{Y}_{I}}^{\hat{p}-1} \tag{78.3.19}
\end{equation*}
$$

where $a(\omega), b(\omega)$ are nonnegative, $\hat{p} \geq p$.

## - coercivity estimate

Also assume the coercivity condition: valid for each $t \leq T$ and for some $\lambda(\omega) \geq 0$,

$$
\begin{gather*}
\inf \left(\int_{0}^{t}\left\langle u^{*}, u\right\rangle+\lambda(\omega)\langle B u, u\rangle d s: u^{*} \in A(u, \omega)\right) \\
\geq \delta(\omega) \int_{0}^{t}\|u\|_{V}^{p} d s-m(\omega) \tag{78.3.20}
\end{gather*}
$$

where $m(\omega)$ is some nonnegative constant for fixed $\omega$, and $\delta(\omega)>0$. No uniformity in $\omega$ is necessary.

## - Limit conditions

Let $U$ be a Banach space dense and compact in $V$ and that if $u_{i} \rightharpoonup u$ in $\mathscr{V}_{I}$ and $u_{i}^{*} \in A\left(u_{i}, \omega\right)$ with $\left(B u_{n}\right)^{\prime} \rightarrow(B u)^{\prime}$ weakly in $\mathscr{U}_{r I}^{\prime}$, then if

$$
\begin{equation*}
\limsup _{i \rightarrow \infty}\left\langle u_{i}^{*}, u_{i}-u\right\rangle_{\mathscr{V}_{I}^{\prime}, \mathscr{V}_{I}} \leq 0 \tag{78.3.21}
\end{equation*}
$$

it follows that for all $v \in \mathscr{V}_{I}$, there exists $u^{*}(v) \in A u$ such that

$$
\begin{equation*}
\lim _{i \rightarrow \infty}\left\langle u_{i}^{*}, u_{i}-v\right\rangle_{\mathscr{V}_{I}^{\prime}, \mathscr{V}_{I}} \geq\left\langle u^{*}(v), u-v\right\rangle_{\mathscr{V}_{I}^{\prime}, \mathscr{V}_{I}} \tag{78.3.22}
\end{equation*}
$$

You typically obtain this kind of thing from Theorem 78.1.2 applied to lower order terms along with some sort of compactness of the embedding of $V$ into $W$.

## - measurability condition

For $\omega \rightarrow u(\cdot, \omega)$ measurable into $\mathscr{V}$,

$$
\begin{equation*}
\omega \rightarrow A\left(\mathscr{X}_{I} u(\cdot, \omega), \omega\right) \text { has a measurable selection into } \mathscr{V}_{I}^{\prime} \tag{78.3.23}
\end{equation*}
$$

This condition means there is a function $\omega \rightarrow u^{*}(\omega)$ which is measurable into $\mathscr{V}_{I}^{\prime}$ such that $u^{*}(\omega) \in A\left(\mathscr{X}_{I} u(\cdot, \omega), \omega\right)$. This is assured to take place if the following standard measurability condition is satisfied for all $O$ open in $\mathscr{V}_{I}^{\prime}$ :

$$
\begin{equation*}
\left\{\omega: A\left(\mathscr{X}_{I} u(\cdot, \omega), \omega\right) \cap O \neq \emptyset\right\} \in \mathscr{F} \tag{78.3.24}
\end{equation*}
$$

A sufficient condition for this condition is that $\omega \rightarrow A(u(\cdot, \omega), \omega)$ has a measurable selection into $\mathscr{V}^{\prime}$ for any $\omega \rightarrow u(\cdot, \omega)$ measurable into $\mathscr{V}$ and if $u^{*} \in A(u(\cdot, \omega), \omega)$, then $\mathscr{X}_{I} u^{*} \in A\left(\mathscr{X}_{I} u(\cdot, \omega), \omega\right)$, and this is typical of what we will always consider, in which the values of $u^{*}$ are dependent on the earlier values of $u$ only.

Let $F$ be the duality map for $r>\max (\hat{p}, 2)$. Thus

$$
\langle F u, u\rangle=\|u\|^{r}, \quad\|F u\|=\|u\|^{r-1}
$$

and is a demicontinuous map. Let $X$ be those $u \in \mathscr{U}_{r}$ such that $(B u)^{\prime} \in \mathscr{U}_{r}^{\prime}$ with a convenient norm given by $\max \left(\|u\|_{\mathscr{U}_{r}},\left\|(B u)^{\prime}\right\|_{\mathscr{U}_{r}^{\prime}}\right)$. Then if we let $\mathscr{U}_{r I}$ play the role of $\mathscr{V}_{I}$ in Theorem 78.2.2, we obtain the following lemma as a corollary of this theorem.

Lemma 78.3.1 Let A satisfy 78.3-78.3 and let $f$ be measurable into $\mathscr{V}^{\prime}$ and let $u_{0}$ be measurable into $W$. Then for $\varepsilon>0$, there exists a solution to

$$
\begin{equation*}
L u+\varepsilon F u+u^{*}=f, B u(0, \omega)=B u_{0}(\omega) \tag{78.3.25}
\end{equation*}
$$

such that $L u, u^{*}, u$ are all measurable into $\mathscr{U}_{r}^{\prime}, \mathscr{U}_{r}^{\prime}$, and $\mathscr{U}_{r}$ respectively, $u^{*}(\omega) \in A(u, \omega)$. In other terms, for $v \in X=\left\{u \in \mathscr{U}_{r}: L u \in \mathscr{U}_{r}^{\prime}\right\}$

$$
\begin{gather*}
\int_{0}^{T}\langle L u, v\rangle+\varepsilon \int_{0}^{T}\langle F u, v\rangle+\int_{0}^{T}\left\langle u^{*}, v\right\rangle+ \\
\langle B u, v\rangle(0)=\int_{0}^{T}\langle f, v\rangle+\left\langle B v(0), u_{0}\right\rangle \tag{78.3.26}
\end{gather*}
$$

Proof: Using easy estimates and the definition that $r>\max (\hat{p}, 2), \hat{p} \geq p$, (Recall that $\hat{p}$ determined the polynomial growth of $\left\|u^{*}\right\|_{\mathscr{V}^{\prime}}$ where $\left.u^{*} \in A(u, \omega)\right)$ it is routine to show that the earlier coercivity condition holds for $\varepsilon F+A(\cdot, \omega)$. Indeed, we have the following from the above assumptions.

$$
\begin{gathered}
\inf \left(\int_{0}^{t}\left\langle u^{*}, u\right\rangle+\lambda(\omega)\langle B u, u\rangle d s: u^{*} \in A(u, \omega)\right) \\
\geq \delta(\omega) \int_{0}^{t}\|u\|_{V}^{p} d s-m(\omega)
\end{gathered}
$$

Thus,

$$
\begin{gathered}
\inf \left(\int_{0}^{t}\left\langle u^{*}, u\right\rangle: u^{*} \in A(u, \omega)\right) \geq \delta(\omega) \int_{0}^{t}\|u\|_{V}^{p} d s \\
-C_{B} \lambda(\omega) \int_{0}^{t} \eta\left(\frac{1}{\eta}\right)\|u\|_{V}^{2}-m(\omega)
\end{gathered}
$$

Then the right side is no smaller than

$$
\begin{aligned}
& -C_{B} \lambda(\omega) \int_{0}^{t}\left(\frac{1}{\eta}\right) \eta\|u\|_{V}^{2}-m(\omega) \\
\geq & -C_{B,} \lambda(\omega) \eta^{r / 2} \int_{0}^{t}\|u\|_{U}^{r}-C_{B} \lambda(\omega) T\left(\frac{1}{\eta}\right)^{r /(r-2)}-m(\omega)
\end{aligned}
$$

Then picking $\eta$ small enough, we obtain $C_{B} \lambda(\omega) \eta^{r / 2}<\varepsilon / 2$.
Both operators $\varepsilon F$ and $A(\cdot, \omega)$ are pseudomonotone as maps from $X$ to $\mathscr{P}\left(X^{\prime}\right)$ where $X$ is defined in terms of $\mathscr{U}_{r I}$ as before. Therefore, the existence of the measurable solution is obtained.

Denoting with $u_{\varepsilon}$ the above solution, suppose $L u_{\varepsilon} \rightarrow L u$ weakly in $\mathscr{U}_{r}^{\prime}$ with $L u=$ $(B u)^{\prime} \in \mathscr{V}^{\prime}$ and $u_{\varepsilon} \rightarrow u$ weakly in $\mathscr{V}$ and $u_{\varepsilon}^{*} \rightarrow u^{*}$ in $\mathscr{V}^{\prime}, \varepsilon F u_{\varepsilon} \rightarrow 0$ strongly in $\mathscr{U}_{r}^{\prime}$. Thus, passing to the limit in 78.3 .25 we obtain $L u \in \mathscr{V}^{\prime}$ because it equals something in $\mathscr{V}^{\prime}$. We will show this below by an argument that $\varepsilon F u_{\varepsilon} \rightarrow 0$ strongly in $\mathscr{U}_{r}^{\prime}$.

Written differently, the $u_{\varepsilon}$ satisfy the following for all $v \in X$.

$$
\begin{gather*}
\int_{0}^{T}\left\langle L u_{\mathcal{\varepsilon}}, v\right\rangle+\left\langle B u_{\varepsilon}, v\right\rangle(0)+\int_{0}^{T}\left\langle u_{\varepsilon}^{*}, v\right\rangle \\
+\varepsilon \int_{0}^{T}\left\langle F u_{\mathcal{\varepsilon}}, u_{\varepsilon}\right\rangle=\int_{0}^{T}\langle f, v\rangle+\left\langle B v(0), u_{0}\right\rangle \tag{78.3.27}
\end{gather*}
$$

and

$$
B u_{\varepsilon}(t)=B u_{0}+\int_{0}^{t} L u_{\varepsilon}(s) d s
$$

The weak convergence of $L u_{\varepsilon}$ implies that $B u_{\varepsilon}(t) \rightarrow B u(t)$ in $U^{\prime}$. Thus

$$
B u(t)=B u_{0}+\int_{0}^{t} L u(s) d s
$$

and so $B u(0)=B u_{0}$. We will show that there exist suitable subsequences such that the kind of convergence just described will hold.

Using the equation to act on $u$ in 78.3 .25 or in 78.3 .27 , we obtain from the assumed coercivity condition the following for fixed $\omega$,

$$
\begin{align*}
& \frac{1}{2}\langle B u, u\rangle(t)-\frac{1}{2}\langle B u, u\rangle(0)+\varepsilon \int_{0}^{t}\|u\|_{U}^{r} d s+\delta(\omega) \int_{0}^{t}\|u\|_{V}^{p} d s-m(\omega) \\
\leq & \lambda(\omega) \int_{0}^{t}\langle B u, u\rangle(s) d s+\int_{0}^{t}\langle f, u\rangle(s) d s \tag{78.3.28}
\end{align*}
$$

From Gronwall's inequality, one obtains an estimate of the form

$$
\langle B u, u\rangle(t)+\varepsilon \int_{0}^{T}\|u\|_{U}^{r} d s+\int_{0}^{T}\|u\|_{V}^{p} d s \leq C(f, \omega)
$$

where the constant depends only on the indicated quantities. It follows from this and the definition of the duality map $F$ that if $u_{\varepsilon}$ is the solution to Lemma 78.3.1, then $\varepsilon F u_{\varepsilon} \rightarrow 0$ strongly in $\mathscr{U}_{r}^{\prime}$. Also, the estimates for $A$ and the above estimate implies that $L u_{\varepsilon}$ is bounded in $\mathscr{U}_{r}^{\prime}$. Thus we have an inequality of the form

$$
\left\langle B u_{\varepsilon}, u_{\varepsilon}\right\rangle(t)+\varepsilon \int_{0}^{T}\left\|u_{\mathcal{E}}\right\|_{U}^{r} d s+\left\|u_{\mathcal{E}}\right\|_{\mathscr{V}}^{p}+\left\|L u_{\mathcal{E}}\right\|_{\mathscr{U}_{r}^{\prime}}+\left\|u_{\mathcal{\varepsilon}}^{*}\right\|_{\mathscr{V}^{\prime}} \leq C(f, \omega)
$$

Of course each of these $u_{\mathcal{\varepsilon}}, u_{\varepsilon}^{*}$ are measurable into $\mathscr{V}$ and $\mathscr{U}^{\prime}$ respectively. By density considerations, $u_{\varepsilon}^{*}$ is also measurable into $\mathscr{V}^{\prime}$. It follows from Theorem 78.1.3 that there
exists $\left(u, u^{*}\right)$ which is measurable into $\mathscr{V} \times \mathscr{V}^{\prime}$ and a sequence with $\varepsilon(\omega)$ such that as $\varepsilon(\omega) \rightarrow 0,\left(u_{\varepsilon(\omega)}(\omega), u_{\varepsilon(\omega)}^{*}(\omega)\right) \rightarrow\left(u(\omega), u^{*}(\omega)\right)$ in $\mathscr{V} \times \mathscr{V}^{\prime}$. Then, taking a further subsequence, we can obtain the following convergences for fixed $\omega$.

$$
\begin{aligned}
u_{\varepsilon(\omega)}(\omega) & \rightarrow u(\omega) \text { weakly in } \mathscr{V} \\
u_{\varepsilon(\omega)}^{*}(\omega) & \rightarrow u^{*}(\omega) \text { weakly in } \mathscr{V}^{\prime} \\
L u_{\varepsilon(\omega)} & \rightarrow L u \text { weakly in } \mathscr{U}_{r}^{\prime}
\end{aligned}
$$

These convergences continue to hold for $\mathscr{V}$ and $\mathscr{U}_{r}^{\prime}$ replaced with $\mathscr{V}_{I}$ and $\mathscr{U}_{r I}^{\prime}$ and we simply consider the restrictions of the functions to $I$. The problem here is that we do not know that $u$ is in $\mathscr{U}_{r}$. This is why it is necessary to take a little different approach.

Letting $\sigma>0$, there exists $\hat{T}(\omega)>T-\sigma$ such that for each $\varepsilon(\omega)$ in that sequence,

$$
\left\langle B u_{\varepsilon(\omega)}, u_{\varepsilon(\omega)}\right\rangle(\hat{T})=\left\langle B\left(u_{\varepsilon(\omega)}(\hat{T})\right), u_{\varepsilon(\omega)}(\hat{T})\right\rangle, B u_{\varepsilon}(\hat{T})=B\left(u_{\varepsilon}(\hat{T})\right)
$$

for all $\varepsilon(\omega)$ in the sequence converging to 0 and also

$$
B u(\hat{T})=B(u(\hat{T})),\langle B u, u\rangle(\hat{T})=\langle B(u(\hat{T})), u(\hat{T})\rangle
$$

Now let $\left\{e_{i}\right\}$ be the vectors of Theorem 78.1.6 where these are in $U$. Thus for $\hat{T}$,

$$
\left\langle B u_{\varepsilon}, u_{\varepsilon}\right\rangle(\hat{T})=\left\langle B u_{\varepsilon}(\hat{T}), u_{\varepsilon}(\hat{T})\right\rangle=\sum_{i=1}^{\infty}\left\langle B\left(u_{\varepsilon}(\hat{T})\right), e_{i}\right\rangle^{2}
$$

Hence, by Fatou's lemma,

$$
\begin{align*}
\liminf _{\varepsilon \rightarrow 0}\left\langle B u_{\varepsilon}, u_{\varepsilon}\right\rangle(\hat{T}) & =\lim \inf _{\varepsilon \rightarrow 0} \sum_{i=1}^{\infty}\left\langle B\left(u_{\varepsilon}(\hat{T})\right), e_{i}\right\rangle^{2} \\
& \geq \sum_{i=1}^{\infty} \liminf _{\varepsilon \rightarrow 0}\left\langle B\left(u_{\varepsilon}(\hat{T})\right), e_{i}\right\rangle^{2} \\
& =\sum_{i=1}^{\infty} \liminf _{\varepsilon \rightarrow 0}\left\langle B u_{\varepsilon}(\hat{T}), e_{i}\right\rangle^{2} \\
& =\sum_{i=1}^{\infty}\left\langle B u(\hat{T}), e_{i}\right\rangle^{2} \\
& =\langle B(u(\hat{T})), u(\hat{T})\rangle=\langle B u, u\rangle(\hat{T}) \tag{78.3.29}
\end{align*}
$$

Then by 78.3.25, we can obtain

$$
\begin{gather*}
\frac{1}{2}\left\langle B u_{\varepsilon}, u_{\varepsilon}\right\rangle(\hat{T})-\frac{1}{2}\left\langle B u_{\varepsilon}, u_{\varepsilon}\right\rangle(0)+ \\
\int_{0}^{\hat{T}} \varepsilon\left\langle F u_{\varepsilon}, u_{\varepsilon}\right\rangle d t+\int_{0}^{\hat{T}}\left\langle u_{\varepsilon}^{*}, u_{\varepsilon}\right\rangle=\int_{0}^{\hat{T}}\left\langle f, u_{\varepsilon}\right\rangle \tag{78.3.30}
\end{gather*}
$$

From what was shown above, $\left\langle B u_{\varepsilon}, u_{\mathcal{E}}\right\rangle(0)=\left\langle B u_{0}, u_{0}\right\rangle$. Now passing to the limit as $\varepsilon \rightarrow 0$,

$$
L u+u^{*}=f
$$

in $\mathscr{U}_{r}^{\prime}$. But every term is in $\mathscr{V}^{\prime}$ except the first and so it is also in $\mathscr{V}^{\prime}$. Also, we know that $\left\langle B u_{\mathcal{E}}, u_{\varepsilon}\right\rangle(0)=\left\langle B u_{0}, u_{0}\right\rangle$ and by Theorem 78.1.8, $\langle B u, u\rangle(0)=\left\langle B u_{0}, u_{0}\right\rangle$ also. Then the integration by parts formula yields

$$
\frac{1}{2}\langle B u, u\rangle(\hat{T})-\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle+\int_{0}^{\hat{T}}\left\langle u^{*}, u\right\rangle d t=\int_{0}^{\hat{T}}\langle f, u\rangle d t
$$

which shows

$$
\int_{0}^{\hat{T}}\left\langle u^{*}, u\right\rangle d t=\int_{0}^{\hat{T}}\langle f, u\rangle d t-\frac{1}{2}\langle B u, u\rangle(\hat{T})+\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle
$$

Then from 78.3.30 and the lower semicontinuity shown in 78.3.29, it follows that

$$
\begin{aligned}
\limsup _{\varepsilon \rightarrow 0} \int_{0}^{\hat{T}}\left\langle u_{\varepsilon}^{*}, u_{\varepsilon}\right\rangle & \leq \int_{0}^{\hat{T}}\langle f, u\rangle d t+\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle-\lim _{\varepsilon \rightarrow 0} \inf _{2} \frac{1}{2}\left\langle B u_{\varepsilon}, u_{\varepsilon}\right\rangle(\hat{T}) \\
& \leq \int_{0}^{\hat{T}}\langle f, u\rangle d t+\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle-\frac{1}{2}\langle B u, u\rangle(\hat{T})=\int_{0}^{\hat{T}}\left\langle u^{*}, u\right\rangle d t
\end{aligned}
$$

Thus we have $u_{\varepsilon} \rightarrow u$ weakly in $\mathscr{V}_{I}$ and $\left(B u_{\varepsilon}\right)^{\prime} \rightarrow(B u)^{\prime}$ weakly in $\mathscr{U}_{r I}^{\prime}$,

$$
\lim \sup _{\varepsilon \rightarrow 0} \int_{0}^{\hat{T}}\left\langle u_{\varepsilon}^{*}, u_{\varepsilon}-u\right\rangle \leq \int_{0}^{\hat{T}}\left\langle u^{*}, u\right\rangle-\int_{0}^{\hat{T}}\left\langle u^{*}, u\right\rangle=0
$$

Therefore, by the limit condition 78.3, for any $v \in \mathscr{V}$

$$
\lim _{\varepsilon \rightarrow 0} \inf _{0} \int_{0}^{\hat{T}}\left\langle u_{\varepsilon}^{*}, u_{\varepsilon}-v\right\rangle \geq \int_{0}^{\hat{T}}\left\langle u^{*}(v), u-v\right\rangle, \text { some } u^{*}(v) \in A(u, \omega)
$$

In particular, this holds for $u$ and so, in fact, $\int_{0}^{\hat{T}}\left\langle u_{\varepsilon}^{*}, u_{\varepsilon}-u\right\rangle$ converges to 0 . Therefore,

$$
\begin{aligned}
\int_{0}^{\hat{T}}\left\langle u^{*}, u-v\right\rangle & =\lim _{\varepsilon \rightarrow 0} \int_{0}^{\hat{T}}\left\langle u_{\varepsilon}^{*}, u-v\right\rangle \\
& \geq \lim _{\varepsilon \rightarrow 0} \inf _{\varepsilon \rightarrow 0}\left(\int_{0}^{\hat{T}}\left\langle u_{\varepsilon}^{*}, u-u_{\varepsilon}\right\rangle+\int_{0}^{\hat{T}}\left\langle u_{\varepsilon}^{*}, u_{\varepsilon}-v\right\rangle\right) \\
& \geq \int_{0}^{\hat{T}}\left\langle u^{*}(v), u-v\right\rangle, \text { some } u^{*}(v) \in A(u, \omega)
\end{aligned}
$$

since $v$ is arbitrary, this shows from separation theorems that $u^{*}(\omega) \in A(u(\omega), \omega)$ in $\mathscr{V}_{[0, \hat{T}]}^{\prime}$.
This has proved the following theorem in which a more general coercivity condition is used.

Theorem 78.3.2 Suppose the conditions on A,78.3-78.3. Also let $u_{0}$ be measurable into $W$ and $f$ measurable into $\mathscr{V}^{\prime}$. Let $B \in \mathscr{L}\left(W, W^{\prime}\right)$ be nonnegative and self adjoint as described above. Let $\sigma>0$ be small. Then there exist functions $u, u^{*}$ measurable into $\mathscr{V}_{[0, T-\sigma]} \times \mathscr{V}_{[0, T-\sigma]}^{\prime}$ such that $u^{*}(\omega) \in A\left(\mathscr{X}_{[0, T-\sigma]} u(\omega), \omega\right)$ for each $\omega$ and for $t \leq T-\sigma$, for each $\omega$,

$$
B u(t)-B u_{0}+\int_{0}^{t} u^{*}(s) d s=\int_{0}^{t} f(s) d s
$$

Note that if for a given $\omega$ there is a unique solution to the evolution equation, then we can obtain the solution on $(0, T)$. However, $\sigma$ was totally arbitrary so it seems like there is not much difference between the above and the obtimum solution. However, one could also index the above solutions relative to $\sigma$, take an appropriate extension of each on $(T-\sigma, T)$ and get similar estimates and pass to a limit as above as $\sigma \rightarrow 0$ and thereby obtain a measurable solution valid on $(0, T)$. This time, it will be clear that $L u, L u_{\sigma}$ are both in $\mathscr{V}^{\prime}$ so a monotonicity condition will hold for $L$ without the delicate argument given above which caused a smaller interval to be considered. Thus the following corollary will hold if enough additional details are considered. The issue does not seem sufficiently significant to justify the consideration of these details.

Corollary 78.3.3 In the situation of Theorem 78.3.2 there exists the same kind of measurable solution valid on $(0, T)$. This time, $u^{*}, u$ are measurable into $\mathscr{V}$ and $\mathscr{V}^{\prime}$ respectively.

One can give a very interesting generalization of Theorem 78.3.2.
Theorem 78.3.4 In the context of Theorem 78.3.2, let $q(t, \omega)$ be a product measurable function into $V$ such that $t \rightarrow q(t, \omega)$ is continuous, $q(0, \omega)=0$.

Then for each small $\sigma$, there exists a solution $u$ of the integral equation

$$
B u(t, \omega)+\int_{0}^{t} u^{*}(s, \omega) d s=\int_{0}^{t} f(s, \omega) d s+B u_{0}(\omega)+B q(t, \omega), t \leq T-\sigma
$$

where $(t, \omega) \rightarrow u(t, \omega)$ is product measurable. Moreover, for each $\omega, B u(t, \omega)=B(u(t, \omega))$ for a.e. $t$ and $u^{*}(\cdot, \omega) \in A(u(\cdot, \omega), \omega)$ for a.e. $t, u^{*}$ is product measurable into $V^{\prime}$. Also, for each $a \in[0, T-\sigma]$,

$$
B u(t, \omega)+\int_{a}^{t} u^{*}(s, \omega) d s=\int_{a}^{t} f(s, \omega) d s+B u(a, \omega)+B q(t, \omega)-B q(a, \omega)
$$

Proof: Define a stopping time

$$
\tau_{r}(\omega) \equiv \inf \{t:|q(t, \omega)|>r\}
$$

Then this is the first hitting time of an open set by a continuous random variable and so it is a valid stopping time. Then for each $r$, let

$$
A_{r}(\omega, w) \equiv A\left(\omega, w+q^{\tau_{r}}(\cdot, \omega)\right)
$$

where the notation means $q^{\tau_{r}}(t) \equiv q\left(t \wedge \tau_{r}\right)$. Then, since $q^{\tau_{r}}$ is uniformly bounded, all of the necessary estimates and measurability for the solution to the above corollary hold for $A_{r}$ replacing $A$. Therefore, there exists a solution $w_{r}$ to the inclusion

$$
\left(B w_{r}\right)^{\prime}(\cdot, \omega)+A_{r}\left(w_{r}(\cdot, \omega), \omega\right) \ni f(\cdot, \omega), B w_{r}(0, \omega)=B u_{0}(\omega), t \in[0, T-\sigma / 2]
$$

Now for fixed $\omega, q^{\tau_{r}}(t, \omega)$ does not change for all $r$ large enough. This is because it is a continuous function of $t$ and so is bounded on the interval [ $0, T-\sigma / 2]$. Thus, for $r$ large enough and fixed $\omega, q^{\tau_{r}}(t, \omega)=q(t, \omega)$. Thus, we obtain

$$
\begin{equation*}
\left\langle B w_{r}(t, \omega), w_{r}(t, \omega)\right\rangle+\int_{0}^{t}\left\|w_{r}(s, \omega)\right\|_{V}^{p} d s \leq C(\omega) \tag{78.3.31}
\end{equation*}
$$

Now, as before in the proof of Theorem 78.3.2 one can pass to a limit involving a subsequence, as $r(\omega) \rightarrow \infty$ and obtain a solution to the integral equation

$$
B w(t, \omega)-B u_{0}(\omega)+\int_{0}^{t} u^{*}(s, \omega) d s=\int_{0}^{t} f(s, \omega) d s, t \in[0, T-\sigma]
$$

where $u^{*}(\omega) \in A(w(s, \omega)+q(s, \omega), \omega)$ and $u^{*}, w$ are measurable into $\mathscr{V}_{[0, T-\sigma]}^{\prime}$. Now let $u(t, \omega)=w(t, \omega)+q(t, \omega)$.

The last claim follows from letting $t=a$ in the top equation and then subtracting this from the top equation with $t>a$.

### 78.4 Progressively Measurable Solutions

In the context of uniqueness of the evolution initial value problem for fixed $\omega$, one can prove theorems about progressively measurable solutions fairly easily.

First is a definition of the term progressively measurable.
Definition 78.4.1 Let $\mathscr{F}_{t}$ be an increasing in $t$ set of $\sigma$ algebras of sets of $\Omega$ where $(\Omega, \mathscr{F})$ is a measurable space. Thus each $\mathscr{F}_{t}$ is a $\sigma$ algebra and if $s \leq t$, then $\mathscr{F}_{s} \leq \mathscr{F}_{t}$. This set of $\sigma$ algebras is called a filtration. A set $S \subseteq[0, T] \times \Omega$ is called progressively measurable if for every $t \in[0, T]$,

$$
S \cap[0, t] \times \Omega \in \mathscr{B}([0, t]) \times \mathscr{F}_{t}
$$

Denote by $\mathscr{P}$ the progressively measurable sets. This is a $\sigma$ algebra of subsets of $[0, T] \times \Omega$. A function $g$ is progressively measurable if $\mathscr{X}_{[0, t]} g$ is $\mathscr{B}([0, t]) \times \mathscr{F}_{t}$ measurable for each $t$.

Let $A$ satisfy the conditions 78.3-78.3 but the last condition will be modified as follows.

Condition 78.4.2 For each $t \leq T$, if $\omega \rightarrow u(\cdot, \omega)$ is $\mathscr{F}_{t}$ measurable into $\mathscr{V}_{[0, t]}$, then there exists a $\mathscr{F}_{t}$ measurable selection of $A\left(\mathscr{X}_{[0, t]} u(\cdot, \omega), \omega\right)$ into $\mathscr{V}_{[0, t]}^{\prime}$.

Note that $u(\cdot, \omega)$ is in $\mathscr{V}_{[0, t]}$ so $u(t, \omega) \in V$.
The theorem to be shown is the following.

Theorem 78.4.3 Assume the above conditions, 78.3-78.3, and 78.4.2. Let $u_{0}$ be $\mathscr{F}_{0}$ measurable and $(t, \omega) \rightarrow \mathscr{X}_{[0, t]}(t) f(t, \omega)$ is $\mathscr{B}([0, t]) \times \mathscr{F}_{t}$ product measurable into $V^{\prime}$ for each $t$. Also assume that for each $\omega$, there is at most one solution $\left(u, u^{*}\right)$ to the evolution equation

$$
\begin{align*}
B u(\omega)(t)-B u_{0}(\omega)+\int_{0}^{t} u^{*}(\cdot, \omega) d s & =\int_{0}^{t} f(s, \omega) d s  \tag{78.4.32}\\
u^{*}(\cdot, \omega) & \in A(u(\cdot, \omega), \omega)
\end{align*}
$$

for $t \in[0, T]$. Then there exists a unique solution $\left(u(\cdot, \omega), u^{*}(\cdot, \omega)\right)$ in $\mathscr{V}_{[0, T]} \times \mathscr{V}_{[0, T]}^{\prime}$ to the above integral equation for each $\omega, t \in(0, T)$. This solution satisfies

$$
(t, \omega) \rightarrow\left(u(t, \omega), u^{*}(t, \omega)\right)
$$

is progressively measurable into $V \times V^{\prime}$.
Proof: First note that Theorem 78.3.2 there exists a solution on $[0, T-\sigma]$ for each small $\sigma>0$. Then by uniqueness, there exists a solution on $(0, T)$. Let $\mathscr{T}$ denote subsets of $(0, T-\sigma]$ which contain $T-\sigma$ such that for $S \in \mathscr{T}$, there exists a solution $u_{S}$ for each $\omega$ to the above integral equation on $[0, T-\sigma]$ such that $(t, \omega) \rightarrow \mathscr{X}_{[0, s]}(t) u_{S}(t, \omega)$ is $\mathscr{B}([0, s]) \times$ $\mathscr{F}_{s}$ measurable for each $s \in S$. Then $\{T-\sigma\} \in \mathscr{T}$. If $S, S^{\prime}$ are in $\mathscr{T}$, then $S \leq S^{\prime}$ will mean that $S \subseteq S^{\prime}$ and also $u_{S}(t, \omega)=u_{S^{\prime}}(t, \omega)$ in $V$ for all $t \in S$, similar for $u_{S}^{*}$ and $u_{S^{\prime}}^{*}$. Note how we are considering a particular representative of a function in $\mathscr{V}_{[0, T-\sigma]}$ and $\mathscr{V}_{[0, T-\sigma]}^{\prime}$ because of the pointwise condition. Now let $\mathscr{C}$ denote a maximal chain. Is $\cup \mathscr{C} \equiv S_{\infty}$ all of $(0, T-\sigma]$ ? What is $u_{S_{\infty}}$ ? Define $u_{S_{\infty}}(t, \omega)$ the common value of $u_{S}(t, \omega)$ for all $S$ in $\mathscr{C}$, which contain $t \in S_{\infty}$. If $s \in S_{\infty}$, then it is in some $S \in \mathscr{C}$ and so the product measurability condition holds for this $s$. Thus $S_{\infty}$ is a maximal element of the partially ordered set. Is $S_{\infty}$ all of $(0, T-\sigma]$ ? Suppose $\hat{s} \notin S_{\infty}, T-\sigma>\hat{s}>0$.

From Theorem 78.3.2 there exists a solution to the integral equation 78.4.32 on $[0, \hat{s}]$ called $u_{1}$ such that $(t, \omega) \rightarrow u_{1}(t, \omega)$ is $\mathscr{B}([0, \hat{s}]) \times \mathscr{F}_{\hat{s}}$ measurable, similar for $u_{1}^{*}$. By the same theorem, there is a solution on $[0, T-\sigma], u_{2}$ which is $\mathscr{B}([0, T-\sigma]) \times \mathscr{F}_{[0, T-\sigma]}$ measurable. Now by uniqueness, $u_{2}(\cdot, \omega)=u_{1}(\cdot, \omega)$ in $\mathscr{V}_{[0, \hat{s}]}$, similar for $u_{i}^{*}$. Therefore, no harm is done in re-defining $u_{2}, u_{2}^{*}$ on $[0, \hat{s}]$ so that $u_{2}(t, \omega)=u_{1}(t, \omega)$, for all $t \in[0, \hat{s}]$, similar for $u^{*}$. Denote these functions as $\hat{u}, \hat{u}^{*}$. By uniqueness, $u_{S_{\infty}}(\cdot, \omega)=\hat{u}(\cdot, \omega)$ in $L^{p}([0, \hat{s}], V)$. Thus no harm is done by re-defining $\hat{u}(s, \omega)$ to equal $u_{S_{\infty}}(s, \omega)$ for $s<\hat{s}$ and $u_{1}(\hat{s}, \omega)$ at $\hat{s}$. As to $s>\hat{s}$ also re define $\hat{u}(s, \omega) \equiv u_{S_{\infty}}(s, \omega)$ for such $s$. By uniqueness, the two are equal in $\mathscr{V}_{[\hat{s}, T-\sigma]}$ and so no change occurs in the solution of the integral equation. Now $S_{\infty}$ was not maximal after all. $S_{\infty} \cup\{\hat{s}\}$ is larger. This contradiction shows that in fact, $S_{\infty}=(0, T-\sigma]$. Thus there exists a unique progressively measurable solution to 78.4.32 on $[0, T-\sigma]$ for each small $\sigma$. Thus we can simply use uniqueness to conclude the existence of a unique progressively measurable solution on $[0, T)$.
Theorem 78.4.4 Assume the above conditions, 78.3-78.3, and 78.4.2. Let $u_{0}$ be $\mathscr{F}_{0}$ measurable and $(t, \omega) \rightarrow \mathscr{X}_{[0, t]}(t) f(t, \omega)$ is $\mathscr{B}([0, t]) \times \mathscr{F}_{t}$ product measurable into $V^{\prime}$ for each $t \in[0, T-\sigma]$. Also let $t \rightarrow q(t, \omega)$ be continuous and $q$ is progressively measurable into $V$. Suppose there is at most one solution to

$$
\begin{equation*}
B u(t, \omega)+\int_{0}^{t} u^{*}(s, \omega) d s=\int_{0}^{t} f(s, \omega) d s+B u_{0}(\omega)+B q(t, \omega) \tag{78.4.33}
\end{equation*}
$$

for each $\omega$. Then the solution $u$ to the above integral equation is progressively measurable and so is $u^{*}$. Moreover, for each $\omega, u^{*}(\cdot, \omega) \in A(u(\cdot, \omega), \omega)$. Also, for each $a \in[0, T]$,

$$
B u(\omega)(t)+\int_{a}^{t} u^{*}(s, \omega) d s=\int_{a}^{t} f(s, \omega) d s+B u(\omega)(a)+B q(t, \omega)-B q(a, \omega)
$$

Proof: By Theorem 78.3.4 there exists a solution to 78.4 .33 which is $\mathscr{B}([0, T-\sigma]) \times$ $\mathscr{F}_{T-\sigma}$ measurable. Since this is true for all $\sigma>0$, there exists a unique $\mathscr{B}([0, \hat{T}]) \times \mathscr{F}_{\hat{T}}$ measurable solution for each $\hat{T}<T$. Now, as in the proof of Theorem 78.3.4 one can define a new operator

$$
A_{r}(w, \omega) \equiv A\left(\omega, w+q^{\tau_{r}}(\cdot, \omega)\right)
$$

where $\tau_{r}$ is the stopping time defined there. Then, since $q$ is progressively measurable, the progressively measurable condition is satisfied for this new operator. Hence by Theorem 78.4.3 there exists a unique solution $w$ which is progressively measurable to the integral equation

$$
B w_{r}(t, \omega)+\int_{0}^{t} u_{r}^{*}(s, \omega) d s=\int_{0}^{t} f(s, \omega) d s+B u_{0}(\omega)
$$

where $u_{r}^{*}(\cdot, \omega) \in A_{r}(w(\cdot, \omega), \omega)$. Then you can let $r \rightarrow \infty$ and eventually $q^{\tau_{r}}(\cdot, \omega)=$ $q(\cdot, \omega)$. Thus there is a solution to

$$
\begin{aligned}
B w(t, \omega)+\int_{0}^{t} u^{*}(s, \omega) d s & =\int_{0}^{t} f(s, \omega) d s+B u_{0}(\omega) \\
u^{*}(\cdot, \omega) & \in A(w(\cdot, \omega)+B q(\cdot, \omega), \omega)
\end{aligned}
$$

which is progressively measurable because $w(\cdot, \omega)=\lim _{r \rightarrow \infty} w_{r}(\cdot, \omega)$ in $\mathscr{V}$ each $w_{r}$ being progressively measurable. Uniqueness is needed in passing to the limit. Thus for each $\hat{T}<T, \omega \rightarrow w(\cdot, \omega)$ is measurable into $\mathscr{V}_{[0, \hat{T}]}$. Then by Lemma 78.1.5, $w$ has a representative in $\mathscr{V}_{[0, \hat{T}]}$ for each $\omega$ such that the resulting function satisfies $(t, \omega) \rightarrow \mathscr{X}_{[0, \hat{T}]}(t) w(t, \omega)$ is $\mathscr{B}([0, \hat{T}]) \times \mathscr{F}_{\hat{T}}$ measurable into $V$. Thus one can assume that $w$ is progressively measurable. Now as in Theorem 78.3.4, Define $u=w+q$. It follows by uniqueness that there exists a unique progressively measurable solution to 78.4 .33 on $(0, T)$.

The last claim follows from letting $t=a$ in the top equation and then subtracting this from the top equation with $t>a$.

## Chapter 79

## Including Stochastic Integrals

### 79.1 The Case of Uniqueness

You can include stochastic integrals in the above formulation. In this section and from now on, we will assume that $W$ is a Hilbert space because the stochastic integrals featured here will have values in $W$ and the version of the stochastic integral to be considered here will be the Ito integral. Here is a brief review of this integral.

Let $U$ be a separable real Hilbert space and let $Q: U \rightarrow U$ be self adjoint and nonnegative. Also $H$ will be a separable real Hilbert space. $\mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)$ will denote the Hilbert Schmidt operators which map $Q^{1 / 2} U$ to $H$. Here $Q^{1 / 2} U$ is the Hilbert space which has an inner product given by

$$
(y, z) \equiv\left(Q^{-1 / 2} y, Q^{-1 / 2} z\right)
$$

where $Q^{-1 / 2} y$ denotes $x$ such that $Q^{1 / 2} x=y$ and out of all such $x$, this is the one which has the smallest norm. It is like the Moore Penrose inverse in linear algebra. Then one can define a stochastic integral

$$
\int_{0}^{t} \Phi d W
$$

where $\Phi \in L^{2}\left([0, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$ where here $\Phi$ is progressively measurable with respect to the filtration $\mathscr{F}_{t}$. This filtration will be

$$
\mathscr{F}_{t}=\overline{\cap_{p>t} \sigma(W(r)-W(s): 0 \leq s \leq r \leq p)}
$$

The horizontal line indicates completion. The symbol

$$
\sigma(W(r)-W(s): 0 \leq s \leq r \leq p)
$$

indicates the smallest $\sigma$ algebra for which all those increments are measurable. Here $W(t)$ is a Wiener process which has values in $U_{1}$, some other Hilbert space, maybe $H$. There is a Hilbert Schmidt operator $J \in \mathscr{L}_{2}\left(Q^{1 / 2} U, U_{1}\right)$ such that $W(t)=\sum_{i=1}^{\infty} \psi_{i}(t) J e_{i}$ where here the $\psi_{i}$ are independent real Wiener processes. You could take $U, U_{1}$ to both be $H$. This is following [108]. Then the stochastic integral has the following properties.

1. $\int_{0}^{t} \Phi d W$ is a martingale with respect to $\mathscr{F}_{t}$ with values in $H$, equal to 0 when $t=0$.
2. One has the Ito isometry

$$
E\left(\left\|\int_{0}^{t} \Phi d W\right\|_{H}^{2}\right)=\int_{0}^{t}\|\Phi\|_{\mathscr{L}_{2}}^{2} d s
$$

3. One can localize as follows. For $\tau$ a stopping time,

$$
\int_{0}^{t \wedge \tau} \Phi d W=\int_{0}^{t} \mathscr{X}_{[0, \tau]} \Phi d W
$$

4. One can also generalize to the case where $\Phi$ is only progressively measurable and instead of being in $L^{2}\left([0, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)$, you have only that

$$
P\left(\int_{0}^{T}\|\Phi(t)\|_{\mathscr{L}_{2}}^{2} d t<\infty\right)=1
$$

This is done by using an appropriate sequence of stopping times called a localizing sequence. More generally a local martingale is a stochastic process $M(t)$ adapted to the filtration for which there is a locallizing sequence of stopping times $\left\{\tau_{n}\right\}$ such that $\lim _{n \rightarrow \infty} \tau_{n}=\infty$ and $M^{\tau_{n}}$ is a martingale. Local martingales will occur in the estimates which are encountered in what follows.
5. Denoting by $M(t)$ the stochastic integral, $M(t)=\int_{0}^{t} \Phi d W$, the quadratic variation is given by

$$
[M](t)=\int_{0}^{t}\|\Phi\|_{\mathscr{L}_{2}}^{2} d s
$$

6. We will also need a part of the Burkholder Davis Gundy inequality [77], Theorem 63.4.4 which in terms of this stochastic integral is of the form

$$
\int_{\Omega} M^{*} d P \leq C E\left(\left(\int_{0}^{T}\|\Phi\|_{\mathscr{L}_{2}}^{2} d s\right)^{1 / 2}\right), C \text { some constant }
$$

where $M(t)$ is the above stochastic integral and

$$
M^{*} \equiv \sup \left\{\|M(t)\|_{H}: t \in[0, T]\right\}
$$

Now let $\Phi \in L^{2}\left([0, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right)$. Let an orthonormal basis for $Q^{1 / 2} U$ be $\left\{g_{i}\right\}$ and an orthonormal basis for $W$ be $\left\{f_{i}\right\}$. Then $\left\{f_{i} \otimes g_{i}\right\}$ is an orthonormal basis for $\mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)$. Hence,

$$
\Phi=\sum_{i} \sum_{j} \Phi_{i j} f_{i} \otimes g_{j}
$$

where $f_{i} \otimes g_{j}(y) \equiv\left(g_{j}, y\right)_{Q^{1 / 2} U} f_{i}$. Let $E$ be a separable real Hilbert space which is dense in $V$. Then without loss of generality, one can assume that the orthonormal basis for $W$ are all vectors in $E$. Thus the orthogonal projection of $\Phi$ onto the closed subspace

$$
\operatorname{span}\left(\left\{f_{i} \otimes g_{i}\right\}, i, j \leq n\right)
$$

given by

$$
\Phi_{n} \equiv \sum_{i=1}^{n} \sum_{j=1}^{n} \Phi_{i j} f_{i} \otimes g_{j}
$$

Then $\Phi_{n} \in L^{2}\left([0, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, E\right)\right)$ and also

$$
\lim _{n \rightarrow \infty}\left\|\Phi_{n}-\Phi\right\|_{L^{2}\left([0, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right)}=0
$$

and $\int_{0}^{t} \Phi_{n} d W$ is continuous and progressively measurable into $E$ hence into $V$. We can take a subsequence such that $\left\|\Phi_{n}-\Phi\right\|_{L^{2}\left([0, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right)}<2^{-n}$ and this will be assumed whenever convenient.

Note that if $P_{n}$ is the orthogonal projection onto $\operatorname{span}\left(f_{1}, \cdots, f_{n}\right)$, then

$$
\begin{aligned}
\left|P_{n} \Phi(y)\right|_{W} & =\left|P_{n} \sum_{i} \sum_{j} \Phi_{i j} f_{i} \otimes g_{j}(y)\right|_{W} \\
& =\left|P_{n} \sum_{i} \sum_{j} \Phi_{i j} f_{i}\left(y, g_{j}\right)\right|_{W} \\
& =\left|\sum_{i=1}^{n} \sum_{j} \Phi_{i j} f_{i}\left(y, g_{j}\right)\right|_{W} \\
& \geq\left|\sum_{i=1}^{n} \sum_{j=1}^{n} \Phi_{i j} f_{i}\left(y, g_{j}\right)\right|_{W}=\left|\Phi_{n}(y)\right|_{W}
\end{aligned}
$$

Thus

$$
\left|\int_{s}^{t} \Phi_{n} d W\right|_{W} \leq\left|\int_{s}^{t} P_{n} \Phi d W\right|_{W}=\left|P_{n} \int_{s}^{t} \Phi d W\right|_{W} \leq\left|\int_{S}^{t} \Phi d W\right|_{W}
$$

The following corollary will be useful.
Corollary 79.1.1 Let $\Phi_{n}$ be as described above. Then

$$
\left\|\Phi_{n}(t, \omega)\right\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)} \leq\|\Phi(t, \omega)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)}
$$

where $\left\|\Phi_{n}(t, \omega)\right\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)} \uparrow\|\Phi(t, \omega)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)}$

$$
\Phi \in L^{\alpha}\left(\Omega ; L^{\infty}\left([0, T], \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right)\right) \cap L^{2}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right)
$$

where $\alpha>2$. Then off a set of measure zero, the stochastic integrals $\int_{0}^{t} \Phi_{n} d W$ satisfy

$$
\sup _{n} \sup _{t \neq s} \frac{\left\|\int_{s}^{t} \Phi_{n} d W\right\|}{|t-s|^{\gamma}}<C(\omega), \gamma<1 / 2, \gamma=\frac{(\alpha / 2)-1}{\alpha}
$$

Proof: Let, $\alpha>2$. As explained above, $\left|\int_{s}^{r} \Phi_{n} d W\right| \leq\left|\int_{s}^{r} \Phi d W\right|$. Thus by the Burkholder Davis Gundy inequality,

$$
\begin{aligned}
& \sup _{n}\left|\int_{s}^{r} \Phi_{n} d W\right| \leq\left|\int_{s}^{r} \Phi d W\right| \\
& \int_{\Omega}\left(\left|\int_{s}^{t} \Phi d W\right|\right)^{\alpha} d P \leq C \int_{\Omega}\left(\int_{S}^{t}\|\Phi\|^{2} d \tau\right)^{\alpha / 2} d P \\
& \leq C \int_{\Omega}\|\Phi\|_{L^{\infty}\left([0, T], \mathscr{L}_{2}\left(Q^{1 / 2} U, H\right)\right)}^{\alpha}|t-s|^{\alpha / 2} \\
& \leq C\|\Phi\|_{L^{\alpha}\left(\Omega ; L^{\infty}\left([0, T], \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right)\right)|t-s|^{\alpha / 2}}^{\alpha} \\
& \equiv C|t-s|^{\alpha / 2}
\end{aligned}
$$

Then by the Kolmogorov Čentsov theorem, for $\gamma$ as given,

$$
E\left(\sup _{0 \leq s<t \leq T} \sup _{n} \frac{\left|\int_{s}^{t} \Phi_{n} d W\right|}{(t-s)^{\gamma}}\right) \leq E\left(\sup _{0 \leq s<t \leq T} \frac{\left|\int_{s}^{t} \Phi d W\right|}{(t-s)^{\gamma}}\right) \leq C
$$

where $\gamma<\beta / \alpha$ where, $\beta+1=\alpha / 2$. Thus for $\gamma<\frac{(\alpha / 2)-1}{\alpha}$,

$$
\sup _{n} \sup _{0 \leq s<t \leq T} \frac{\left|\int_{s}^{t} \Phi_{n} d W\right|}{(t-s)^{\gamma}} \leq C(\omega)
$$

for all $\omega$ off a set of measure zero.
Recall the following conditions for the various operators.

## Bounded and coercive conditions

$A(\cdot, \omega) \cdot A(\cdot, \omega): \mathscr{V}_{I} \rightarrow \mathscr{V}_{I}^{\prime}$ for each $I$ a subinterval of $[0, T] I=[0, \hat{T}], \hat{T} \leq T$

$$
\begin{equation*}
A(\cdot, \omega): \mathscr{V}_{I} \rightarrow \mathscr{P}\left(\mathscr{V}_{I}^{\prime}\right) \text { is bounded, } \tag{79.1.1}
\end{equation*}
$$

If, for $u \in \mathscr{V}$,

$$
u^{*} \mathscr{X}_{[0, \hat{T}]} \in A\left(u \mathscr{X}_{[0, \hat{T}]}, \omega\right)
$$

for each $\hat{T}$ in an increasing sequence converging to $T$, then

$$
\begin{equation*}
u^{*} \in A(u, \omega) \tag{79.1.2}
\end{equation*}
$$

Assume the specific estimate

$$
\begin{equation*}
\sup \left\{\left\|u^{*}\right\|_{\mathscr{V}_{I}^{\prime}}: u^{*} \in A(u, \omega)\right\} \leq a(\omega)+b(\omega)\|u\|_{\mathscr{V}_{I}}^{p-1} \tag{79.1.3}
\end{equation*}
$$

where $a(\omega), b(\omega)$ are nonnegative. Note that here we use $p$ and not $\hat{p} \geq p$ as done earlier. It is likely that this could be generalized by introduction of a suitable regularizing duality map multiplied by $\varepsilon$ and letting $\varepsilon \rightarrow 0$. You would do everything here adding in $\varepsilon F$ where $F$ is the duality map $F: U \rightarrow U^{\prime}$ for $r$ where $r>\hat{p} \geq p$ and keep it in the definition of $A$. Here $U$ is a Hilbert space embedded compactly into $V$ and dense in $V$. Then you would let $\varepsilon \rightarrow 0$ and observe that $\varepsilon F u_{\varepsilon} \rightarrow 0$ in $\mathscr{U}_{r}^{\prime}$. Also assume the following coercivity estimate valid for each $t \leq T$ and for some $\lambda(\omega) \geq 0$,

$$
\begin{equation*}
\inf \left(\int_{0}^{t}\left\langle u^{*}, u\right\rangle+\lambda(\omega)\langle B u, u\rangle d t: u^{*} \in A(u, \omega)\right) \geq \delta(\omega) \int_{0}^{t}\|u\|_{V}^{p} d s-m(\omega) \tag{79.1.4}
\end{equation*}
$$

where $m(\omega)$ is some nonnegative constant, $\delta(\omega)>0$.

## Monotonicity

It will also be assumed that $\lambda(\omega) B+A$ is monotone in the sense that

$$
\int_{0}^{t}\left\langle\lambda(\omega) B u+u^{*}-\lambda(\omega) B v+v^{*}, u-v\right\rangle d s \geq 0
$$

for a suitable choice of $\lambda(\omega)$ whenever $u^{*} \in A(u, \omega), v^{*} \in A(v, \omega)$.

## Limit condition

Let $U$ be a Banach space dense in $V$ and that if $u_{i} \rightharpoonup u$ in $\mathscr{V}_{I}$ and $u_{i}^{*} \in A\left(u_{i}\right)$ with $u_{i}^{*} \rightharpoonup u^{*}$ in $\mathscr{V}_{I}^{\prime}$ and $t \rightarrow B u_{i}(t)$ is continuous and

$$
\begin{equation*}
\sup _{i} \sup _{t \neq s} \frac{\left\|B u_{i}(t)-B u_{i}(s)\right\|_{U^{\prime}}}{|t-s|^{\alpha}} \leq C \tag{79.1.5}
\end{equation*}
$$

then if

$$
\begin{equation*}
\limsup _{i \rightarrow \infty}\left\langle u_{i}^{*}, u_{i}-u\right\rangle_{\mathscr{V}_{I}^{\prime}, \mathscr{V}_{I}} \leq 0 \tag{79.1.6}
\end{equation*}
$$

it follows that for all $v \in \mathscr{V}_{I}$, there exists $u^{*}(v) \in A u$ such that

$$
\begin{equation*}
\lim _{i \rightarrow \infty}\left\langle u_{i}^{*}, u_{i}-v\right\rangle_{\mathscr{V}_{I}^{\prime}, \mathscr{V}_{I}} \geq\left\langle u^{*}(v), u-v\right\rangle_{\mathscr{V}_{I}^{\prime}, \mathscr{V}_{I}} \tag{79.1.7}
\end{equation*}
$$

As to $B(\omega)$, it is $k(\omega) B$ where $B \in \mathscr{L}\left(W, W^{\prime}\right)$ and is self adjoint and nonnegative where $k$ is $\mathscr{F}_{0}$ measurable.

## Progressively measurable condition

Condition 79.1.2 For each $t \leq T$, if $\omega \rightarrow u(\cdot, \omega)$ is $\mathscr{F}_{t}$ measurable into $\mathscr{V}_{[0, t]}$, then there exists a $\mathscr{F}_{t}$ measurable selection of $A(u(\cdot, \omega), \omega)$ into $\mathscr{V}_{[0, t]}^{\prime}$.

Then there is a theorem. It was Theorem 77.7 .4 which gave existence and uniqueness of progressively measurable solutions $u$ to the integral equation.

Theorem 79.1.3 Assume the above conditions, 79.1.1-, 79.1.7 along with the progressive measurability condition 79.1.2. Let $u_{0}$ be $\mathscr{F}_{0}$ measurable and $\omega \rightarrow B(\omega)$ also $\mathscr{F}_{0}$ measurable and $(t, \omega) \rightarrow \mathscr{X}_{[0, t]}(t) f(t, \omega)$ is $\mathscr{B}([0, t]) \times \mathscr{F}_{t}$ product measurable into $V^{\prime}$ for each $t$.

$$
B(\omega)=k(\omega) B, k(\omega) \geq 0, k \text { measurable }
$$

Also let $t \rightarrow q(t, \omega)$ be continuous and $q$ is progressively measurable into $V$. Suppose there is at most one solution to

$$
\begin{equation*}
B u(t, \omega)+\int_{0}^{t} z(s, \omega) d s=\int_{0}^{t} f(s, \omega) d s+B u_{0}(\omega)+B q(t, \omega), \tag{79.1.8}
\end{equation*}
$$

for each $\omega$. Then the solution to the above integral equation u is progressively measurable. Moreover, for each $\omega$, both $B u(t, \omega)=B(u(t, \omega))$ for a.e. $t$ and $z(t, \omega) \in A(u(t, \omega), \omega)$ for a.e. t. Also, for each $a \in[0, T]$,

$$
B u(t, \omega)+\int_{a}^{t} z(s, \omega) d s=\int_{a}^{t} f(s, \omega) d s+B u(a, \omega)+B q(t, \omega)-B q(a, \omega)
$$

Letting $q(t)=\int_{0}^{t} \Phi_{n} d W$ defined above with the filtration also being the one obtained from the Wiener process, this implies the following theorem. The $\sigma$ algebra of progressively measurable sets will be denoted by $\mathscr{P}$.

Theorem 79.1.4 Assume the above conditions, 79.1.1-, 79.1.7 along with the progressive measurability condition 79.1.2. Also assume there is at most one solution to 79.1.8 where

$$
q(t, \cdot) \equiv \int_{0}^{t} \Phi_{n} d W
$$

Then there exists a $\mathscr{P}$ measurable $u_{n}$ such that also $z_{n}$ is progressively measurable

$$
B u_{n}(t, \omega)-B u_{0}(\omega)+\int_{0}^{t} z_{n}(s, \omega) d s=\int_{0}^{t} f(s, \omega) d s+B \int_{0}^{t} \Phi_{n} d W
$$

where for each $\omega, z_{n}(\cdot, \omega) \in A\left(u_{n}(\cdot, \omega), \omega\right)$. The function $B u_{n}(t, \omega)=B\left(u_{n}(t, \omega)\right)$ for a.e. $t$.

This gives an existence theorem for the inclusion of a stochastic integral. However, it is desired to get a similar result for $\Phi$ rather than $\Phi_{n}$. Next is the Ito formula which is useable because of the progressive measurability of $u_{n}, z_{n}$. This formula applies to the following situation.

Situation 79.1.5 Let $X$ have values in $V$ and satisfy the following

$$
\begin{equation*}
B X(t)=B X_{0}+\int_{0}^{t} Y(s) d s+B \int_{0}^{t} Z(s) d W(s) \tag{79.1.9}
\end{equation*}
$$

$X_{0} \in L^{2}(\Omega ; W)$ and is $\mathscr{F}_{0}$ measurable, where $Z$ is $\mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)$ progressively measurable and

$$
\|Z\|_{L^{2}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right)}<\infty .
$$

This is what is needed to define the stochastic integral in the above formula.
Assume $X, Y$ satisfy

$$
B X, Y \in K^{\prime} \equiv L^{p^{\prime}}\left([0, T] \times \Omega ; V^{\prime}\right)
$$

the $\sigma$ algebra of measurable sets defining $K^{\prime}$ will be the progressively measurable sets. Here $1 / p^{\prime}+1 / p=1, p>1$.

Also the sense in which the equation holds is as follows. For a.e. $\omega$, the equation holds in $V^{\prime}$ for all $t \in[0, T]$. Thus we are considering a particular representative $X$ of $K$ for which this happens. Also it is only assumed that $B X(t)=B(X(t))$ for a.e. $t$. Thus $B X$ is the name of a function having values in $V^{\prime}$ for which $B X(t)=B(X(t))$ for a.e. $t$, all $t \notin N_{\omega}$ a set of measure zero. Assume that $X$ is progressively measurable also and $X \in L^{p}([0, T] \times \Omega, V)$.

Then in the above situation, we obtain the following integration by parts formula which is called the Ito formula. This particular version is presented in Theorem 73.7.2 and is a generalization of work of Krylov. A proof of the case of a Gelfand triple in which $B=I$ is in [108].

Theorem 79.1.6 In Situation 79.1.5, for $\omega$ off a set of measure zero, for every $t \in N_{\omega}^{C}$, the measure of $N_{\omega}$ equalling 0 ,

$$
\begin{gather*}
\langle B X(t), X(t)\rangle=\left\langle B X_{0}, X_{0}\right\rangle+\int_{0}^{t} 2\langle Y(s), X(s)\rangle d s+ \\
\int_{0}^{t}\langle B Z, Z\rangle_{\mathscr{L}_{2}} d s+2 M(t) \tag{79.1.10}
\end{gather*}
$$

where $M(t)$ is a stochastic integral and a local martingale equal to 0 when $t=0$. Also, there exists a unique continuous, progressively measurable function denoted as $\langle B X, X\rangle$ such that it equals $\langle B X(t), X(t)\rangle$ for a.e. $t$ and $\langle B X, X\rangle(t)$ equals the right side of the above for all $t$. In addition to this,

$$
\begin{gather*}
E(\langle B X, X\rangle(t))= \\
E\left(\left\langle B X_{0}, X_{0}\right\rangle\right)+E\left(\int_{0}^{t}\left(2\langle Y(s), X(s)\rangle+\langle B Z, Z\rangle_{\mathscr{L}_{2}}\right) d s\right) \tag{79.1.11}
\end{gather*}
$$

Also the quadratic variation of $M(t)$ in 79.1.10 is dominated by

$$
\begin{equation*}
C \int_{0}^{t}\|Z\|_{\mathscr{L}_{2}}^{2}\|B X\|_{W^{\prime}}^{2} d s \tag{79.1.12}
\end{equation*}
$$

for a suitable constant $C$. Also $t \rightarrow B X(t)$ is continuous with values in $W^{\prime}$ for $t \in N_{\omega}^{C}$. In fact, this martingale can be written as

$$
\int_{0}^{t}\left(Z \circ J^{-1}\right)^{*} B X \circ J d W
$$

That ugly integral displayed above can be written in the form

$$
\int_{0}^{t}\langle B X, d N\rangle
$$

where $N(t)=\int_{0}^{t} Z(s) d W$.
Now we consider the meaning of the symbol $\langle B Z, Z\rangle_{\mathscr{L}_{2}}$. You begin with a complete orthonormal set $\left\{g_{k}\right\}$ in $Q^{1 / 2} U$. Then to say that $Z$ has values in $\mathscr{L}_{2}\left(Q^{1 / 2} U ; W\right)$ is to say that $\sum_{j} \sum_{i}\left(Z\left(g_{i}\right), e_{j}\right)^{2}=\sum_{i}\left\|Z\left(g_{i}\right)\right\|_{W}^{2}<\infty$ where $\left\{e_{j}\right\}$ is an orthonormal basis in $W$. You can let it be the one used earlier where each is actually in $V$ or even in $E$. Then the symbol means

$$
\left(R^{-1} B Z, Z\right)_{\mathscr{L}_{2}}
$$

where $R$ is the Riesz map from the Hilbert space $W$ to its dual space. Thus it equals

$$
\sum_{i}\left(R^{-1} B Z\left(g_{i}\right), Z\left(g_{i}\right)\right)_{W}=\sum_{i}\left\langle B Z\left(g_{i}\right), Z\left(g_{i}\right)\right\rangle
$$

so it is seen to be nonnegative.

Now apply this Ito formula to Theorem 79.1.4 in which we make the assumptions there on $\left\|u_{0}\right\| \in L^{2}(\Omega)$ and that $f \in L^{p^{\prime}}\left([0, T] \times \Omega ; V^{\prime}\right)$ where the $\sigma$ algebra is $\mathscr{P}$ the progressively measurable $\sigma$ algebra, and

$$
\Phi \in L^{2}\left(\Omega, L^{2}\left([0, T], \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right)\right)
$$

which implies the same is true of $\Phi_{n}$. This yields, from the assumed estimates, an expression of the form where $\delta>0$ is a suitable constant.

$$
\begin{align*}
& \frac{1}{2}\left\langle B u_{n}, u_{n}\right\rangle(t)-\frac{1}{2}\left\langle B u_{0}, u_{0}\right\rangle+\delta \int_{0}^{t}\left\|u_{n}(s)\right\|_{V}^{p} d s \\
\leq & \lambda \int_{0}^{t}\left\langle B u_{n}, u_{n}\right\rangle(s) d s+\int_{0}^{t}\left\langle f, u_{n}\right\rangle_{V^{\prime}, V} d s+\int_{0}^{t} c(s, \omega) d s \\
& +\int_{0}^{t}\left\langle B \Phi_{n}, \Phi_{n}\right\rangle_{\mathscr{L}_{2}} d s+M_{n}(t) \tag{79.1.13}
\end{align*}
$$

where $c \in L^{1}([0, T] \times \Omega)$. Then taking expectations or using that part of the Ito formula,

$$
\begin{aligned}
& \frac{1}{2} E\left(\left\langle B u_{n}, u_{n}\right\rangle(t)\right)+\delta E\left(\int_{0}^{T}\left\|u_{n}(s)\right\|_{V}^{p} d s\right) \\
\leq & \lambda \int_{0}^{t} E\left(\left\langle B u_{n}, u_{n}\right\rangle(s)\right) d s+\int_{0}^{t} E\left(\left\langle f, u_{n}\right\rangle_{V^{\prime}, V}\right) d s+C\left(\Phi, u_{0}\right)
\end{aligned}
$$

Then by Gronwall's inequality and some simple manipulations,

$$
E\left(\left\langle B u_{n}, u_{n}\right\rangle(t)\right)+E\left(\int_{0}^{T}\left\|u_{n}(s)\right\|_{V}^{p} d s\right) \leq C\left(T, f, u_{0}, \Phi\right)
$$

Then using obvious estimates and Gronwall's inequality in 79.1.13, this yields an inequality of the form

$$
\left\langle B u_{n}, u_{n}\right\rangle(t)-\left\langle B u_{0}, u_{0}\right\rangle+\int_{0}^{t}\left\|u_{n}(s)\right\|_{V}^{p} d s \leq C(f, \lambda, c)+\|B\| \int_{0}^{t}\left\|\Phi_{n}\right\|_{\mathscr{L}_{2}}^{2} d s+M_{n}^{*}(t)
$$

where the random variable $C(f, \lambda, c)$ is nonnegative and is integrable. Now $t \rightarrow M_{n}^{*}(t)$ is increasing as is the integral on the right. Hence it follows that, modifying the constants,

$$
\begin{gather*}
\sup _{s \in[0, t]}\left\langle B u_{n}, u_{n}\right\rangle(s)+\int_{0}^{t}\left\|u_{n}(s)\right\|_{V}^{p} d s \\
\leq C\left(f, \lambda, c, u_{0}\right)+2\|B\| \int_{0}^{t}\left\|\Phi_{n}\right\|_{\mathscr{L}_{2}}^{2} d s+2 M_{n}^{*}(t) \tag{79.1.14}
\end{gather*}
$$

Next take the expectation of both sides and use the Burkholder Davis Gundy inequality along with the description of the quadratic variation of the martingale $M_{n}(t)$. This yields

$$
\begin{aligned}
& E\left(\sup _{s \in[0, t]}\left\langle B u_{n}, u_{n}\right\rangle(s)\right)+E\left(\int_{0}^{t}\left\|u_{n}(s)\right\|_{V}^{p} d s\right) \\
\leq & C+2\|B\| E\left(\int_{0}^{t}\left\|\Phi_{n}\right\|_{\mathscr{L}_{2}}^{2} d s\right)
\end{aligned}
$$

$$
+C \int_{\Omega}\left(\int_{0}^{t}\left\|B u_{n}\right\|_{W}^{2}\left\|\Phi_{n}\right\|_{\mathscr{L}_{2}}^{2} d s\right)^{1 / 2} d P
$$

Now $\|B w\|=\sup _{\|v\| \leq 1}\langle B w, v\rangle \leq\langle B w, w\rangle^{1 / 2}$. Also $\int_{0}^{t}\left\|\Phi_{n}\right\|_{\mathscr{L}_{2}}^{2} d s \leq \int_{0}^{T}\|\Phi\|_{\mathscr{L}_{2}}^{2} d s$ and so the above inequality implies

$$
\begin{aligned}
& E\left(\sup _{s \in[0, t]}\left\langle B u_{n}, u_{n}\right\rangle(s)\right)+E\left(\int_{0}^{t}\left\|u_{n}(s)\right\|_{V}^{p} d s\right) \\
\leq & C(f, \lambda, c, \Phi)+C \int_{\Omega} \sup _{s \in[0, t]}\left\langle B u_{n}, u_{n}\right\rangle^{1 / 2}(s)\left(\int_{0}^{t}\|\Phi\|_{\mathscr{L}_{2}}^{2}\right)^{1 / 2} d P
\end{aligned}
$$

Then adjusting the constants yields

$$
\begin{gather*}
\frac{1}{2} E\left(\sup _{s \in[0, T]}\left\langle B u_{n}, u_{n}\right\rangle(s)\right)+E\left(\int_{0}^{T}\left\|u_{n}(s)\right\|_{V}^{p} d s\right) \\
\leq C+C \int_{\Omega} \int_{0}^{T}\|\Phi\|_{\mathscr{L}_{2}}^{2} d t d P \equiv C \tag{79.1.15}
\end{gather*}
$$

If needed, you could use a stopping time to be sure that $E\left(\sup _{s \in[0, T]}\left\langle B u_{n}, u_{n}\right\rangle(s)\right)<\infty$ and then let it converge to $\infty$.

From the integral equation,

$$
B u_{n}(t)-B u_{m}(t)+\int_{0}^{t} z_{n}-z_{m} d s=B \int_{0}^{t}\left(\Phi_{n}-\Phi_{m}\right) d W
$$

Then using the monotonicity assumption and the Ito formula,

$$
\begin{gathered}
\frac{1}{2}\left\langle B u_{n}-B u_{m}, u_{n}-u_{m}\right\rangle(t) \leq \lambda \int_{0}^{t}\left\langle B u_{n}-B u_{m}, u_{n}-u_{m}\right\rangle d s s \\
+\int_{0}^{t}\left\langle B\left(\Phi_{n}-\Phi_{m}\right), \Phi_{n}-\Phi_{m}\right\rangle d+\int_{0}^{t}\left(\left(\Phi_{n}-\Phi_{m}\right) \circ J^{-1}\right)^{*} B\left(u_{n}-u_{m}\right) \circ J d W
\end{gathered}
$$

and so, from Gronwall's inequality, there is a constant $C$ which is independent of $m, n$ such that

$$
\left\langle B u_{n}-B u_{m}, u_{n}-u_{m}\right\rangle(t) \leq C M_{n m}(t) \leq C M_{n m}^{*}(T)+C \int_{0}^{t}\left\|\Phi_{n}-\Phi_{m}\right\|_{\mathscr{L}_{2}}^{2} d s
$$

where $M_{n m}$ refers to that local martingale on the right. Thus also

$$
\begin{equation*}
\sup _{t \in[0, T]}\left\langle B u_{n}-B u_{m}, u_{n}-u_{m}\right\rangle(t) \leq C M_{n m}(t) \leq C M_{n m}^{*}(T)+C \int_{0}^{T}\left\|\Phi_{n}-\Phi_{m}\right\|_{\mathscr{L}_{2}}^{2} d s \tag{79.1.16}
\end{equation*}
$$

Taking the expectation and using the Burkholder Davis Gundy inequality again in a similar manner to the above,

$$
E\left(\sup _{t \in[0, T]}\left\langle B u_{n}-B u_{m}, u_{n}-u_{m}\right\rangle(t)\right) \leq C \int_{\Omega} \int_{0}^{T}\left\|\Phi_{n}-\Phi_{m}\right\|_{\mathscr{L}_{2}}^{2} d t d P
$$

Now the right side converges to 0 as $m, n \rightarrow \infty$ and so there is a subsequence, denoted with the index $k$ such that whenever $m>k$,

$$
E\left(\sup _{t \in[0, T]}\left\langle B u_{k}-B u_{m}, u_{k}-u_{m}\right\rangle(t)\right) \leq \frac{1}{2^{k}}
$$

Note how this implies

$$
\begin{equation*}
\int_{\Omega} \int_{0}^{T}\left\langle B u_{k}-B u_{m}, u_{k}-u_{m}\right\rangle d t d P \leq \frac{T}{2^{k}} \tag{79.1.17}
\end{equation*}
$$

Then consider the martingales $M_{k}(t)$ considered earlier. One of these is of the form

$$
M_{k}=\int_{0}^{t}\left(\Phi_{k} \circ J^{-1}\right)^{*} B u_{k} \circ J d W
$$

Then by the Burkholder Davis Gundy inequality and modifying constants as appropriate,

$$
\begin{gathered}
E\left(\left(M_{k}-M_{k+1}\right)^{*}\right) \\
\leq C \int_{\Omega}\left(\int_{0}^{T}\left\|\left(\Phi_{k} \circ J^{-1}\right)^{*} B u_{k}-\left(\Phi_{k+1} \circ J^{-1}\right)^{*} B u_{k+1}\right\|^{2} d t\right)^{1 / 2} d P \\
\leq C \int_{\Omega}\binom{\int_{0}^{T}\left\|\Phi_{k}-\Phi_{k+1}\right\|^{2}\left\langle B u_{k}, u_{k}\right\rangle}{+\left\|\Phi_{k+1}\right\|^{2}\left\langle B u_{k}-B u_{k+1}, u_{k}-u_{k+1}\right\rangle d t}^{1 / 2} d P \\
\leq \quad C \int_{\Omega}\left(\int_{0}^{T}\left\|\Phi_{k}-\Phi_{k+1}\right\|^{2}\left\langle B u_{k}, u_{k}\right\rangle d t\right)^{1 / 2} \\
\quad+C \int_{\Omega}\left(\int_{0}^{T}\left\|\Phi_{k+1}\right\|^{2}\left\langle B u_{k}-B u_{k+1}, u_{k}-u_{k+1}\right\rangle d t\right)^{1 / 2} d P \\
\leq C \int_{\Omega} \sup _{t}\left\langle B u_{k}, u_{k}\right\rangle^{1 / 2}\left(\int_{0}^{T}\left\|\Phi_{k}-\Phi_{k+1}\right\|^{2} d t\right)^{1 / 2} d P \\
+C \int_{\Omega} \sup _{t}\left\langle B u_{k}-B u_{k+1}, u_{k}-u_{k+1}\right\rangle^{1 / 2}\left(\int_{0}^{T}\left\|\Phi_{k+1}\right\|^{2} d t\right)^{1 / 2} d P \\
\leq C\left(\int_{\Omega} \sup _{t}\left\langle B u_{k}, u_{k}\right\rangle d P\right)^{1 / 2}\left(\int_{\Omega} \int_{0}^{T}\left\|\Phi_{k}-\Phi_{k+1}\right\|^{2} d t d P\right)^{1 / 2} \\
+C\left(\int_{\Omega} \sup _{t}\left\langle B u_{k}-B u_{k+1}, u_{k}-u_{k+1}\right\rangle d P\right)^{1 / 2}\left(\int_{\Omega} \int_{0}^{T}\left\|\Phi_{k+1}\right\|^{2} d t d P\right)^{1 / 2}
\end{gathered}
$$

From the above inequality, 79.1.15 and after adjusting the constants, the above is no larger than an expression of the form $C\left(\frac{1}{2}\right)^{k / 2}$ which is a summable sequence. Then

$$
\sum_{k} \int_{\Omega_{t \in[0, T]}} \sup _{t}\left|M_{k}(t)-M_{k+1}(t)\right| d P<\infty
$$

Thus $\left\{M_{k}\right\}$ is a Cauchy sequence in $M_{T}^{1}$ and so there is a continuous martingale $M$ such that

$$
\lim _{k \rightarrow \infty} E\left(\sup _{t}\left|M_{k}(t)-M(t)\right|\right)=0
$$

Taking a further subsequence if needed, one can also have

$$
P\left(\sup _{t}\left|M_{k}(t)-M(t)\right|>\frac{1}{k}\right) \leq \frac{1}{2^{k}}
$$

and so by the Borel Cantelli lemma, there is a set of measure zero such that off this set, $\sup _{t}\left|M_{k}(t)-M(t)\right|$ converges to 0 . Hence for such $\omega, M_{k}^{*}(T)$ is bounded independent of $k$. Thus for $\omega$ off a set of measure zero, 79.1.14 implies that for such $\omega$,

$$
\sup _{s \in[0, T]}\left\langle B u_{r}, u_{r}\right\rangle(s)+\int_{0}^{T}\left\|u_{r}(s)\right\|_{V}^{p} d s \leq C(\omega)
$$

where $C(\omega)$ does not depend on the index $r$, this for the subsequence just described which will be the sequence of interest in what follows. Using the boundedness assumption for $A$, one also obtains an estimate of the form

$$
\begin{equation*}
\sup _{s \in[0, T]}\left\langle B u_{r}, u_{r}\right\rangle(s)+\int_{0}^{T}\left\|u_{r}(s)\right\|_{V}^{p} d s+\int_{0}^{T}\left\|z_{r}\right\|_{V^{\prime}}^{p^{\prime}} \leq C(\omega) \tag{79.1.18}
\end{equation*}
$$

The idea here is to take weak limits converging to a function $u$ and then identify $z(\cdot, \omega)$ as being in $A(u, \omega)$ but this will involve a difficulty. It will require a use of the above Ito formula and this will need $u$ to be progressively measurable. By uniqueness, it would seem that this could be concluded by arguing that one does not need to take a subsequence due to uniqueness but the problem is that we won't know the limit of the sequence is a solution unless we use the Ito formula. This is why we make the extra assumption that for $z_{i}(\cdot, \omega) \in A\left(u_{i}, \omega\right)$ and for all $\lambda$ large enough,

$$
\begin{equation*}
\left\langle\lambda B u_{1}(t)+z_{1}(t)-\left(\lambda B u_{2}(t)+z_{2}(t)\right), u_{1}(t)-u_{2}(t)\right\rangle \geq \delta\left\|u_{1}(t)-u_{2}(t)\right\|_{\hat{V}}^{\alpha}, \alpha \geq 1 \tag{79.1.19}
\end{equation*}
$$

where here $\hat{V}$ will be a Banach space such that $V$ is dense in $\hat{V}$ and the embedding is continuous. As mentioned, this is not surprising in the case of most interest where there is a Gelfand triple and $B=I$ and $A$ is defined pointwise with no memory terms involving time integrals. Then using the integral equation for $r=p, q, p<q$ along with the conclusion of the Ito formula above,

$$
\begin{aligned}
& E\left(\left\langle B\left(u_{n}-u_{m}\right), u_{n}-u_{m}\right\rangle(t)\right)+E\left(\int_{0}^{t}\left\|u_{n}-u_{m}\right\|_{\hat{V}}^{\alpha} d s\right) \\
\leq & E\left(\int_{0}^{t}\|B\|\left\|\Phi_{n}-\Phi_{m}\right\|_{\mathscr{L}_{2}}^{2} d s\right) \equiv e(m, n)
\end{aligned}
$$

Hence, the right side converges to 0 as $m, n \rightarrow \infty$ from the dominated convergence theorem. In particular,

$$
\begin{equation*}
E\left(\int_{0}^{T}\left\|u_{n}-u_{m}\right\|_{\hat{V}}^{\alpha} d s\right) \leq E\left(\int_{0}^{T}\|B\|\left\|\Phi_{n}-\Phi_{m}\right\|_{\mathscr{L}_{2}}^{2} d s\right) \equiv e(m, n) \tag{79.1.20}
\end{equation*}
$$

Then also

$$
P\left(\int_{0}^{T}\left\|u_{n}-u_{m}\right\|_{\hat{V}}^{\alpha} d s>\lambda\right) \leq \frac{e(m, n)}{\lambda}
$$

and so there exists a subsequence, denoted by $r$ such that

$$
P\left(\int_{0}^{T}\left\|u_{r}-u_{r+1}\right\|_{\hat{V}}^{\alpha} d s \leq 2^{-r}\right)<2^{-r}
$$

Thus, by the Borel Cantelli lemma, there is a further enlarged set of measure zero, still denoted as $N$ such that for $\omega \notin N$

$$
\int_{0}^{T}\left\|u_{r}-u_{r+1}\right\|_{\hat{V}}^{\alpha} d s \leq 2^{-r}
$$

for all $r$ large enough. Hence, by the usual proof of completeness, for these $\omega$,

$$
\left\{u_{r}(\cdot, \omega)\right\}
$$

is Cauchy in $L^{\alpha}([0, T], \hat{V})$ and also $u_{r}(t, \omega)$ converges to some $u(t, \omega)$ pointwise in $\hat{V}$ for a.e. $t$. In addition, from 79.1.20 these functions are a Cauchy sequence in $L^{\alpha}([0, T] \times \Omega ; \hat{V})$ with respect to the $\sigma$ algebra of progressively measurable sets. Thus from Lemma 76.3.4, it can be assumed that for $\omega$ off the set of measure zero, $(t, \omega) \rightarrow u(t, \omega)$ is progressively measurable. From now on, this will be the sequence or a further subsequence. For $\omega \notin N$, a set of measure zero and 79.1.18, there is a further subsequence for which the following convergences occur as $r \rightarrow \infty$.

$$
\begin{align*}
& u_{r} \rightarrow u \text { weakly in } \mathscr{V}  \tag{79.1.21}\\
& B\left(u_{r}\right) \rightarrow B(u) \text { weakly in } \mathscr{V}^{\prime}  \tag{79.1.22}\\
& z_{r} \rightarrow z \text { weakly in } \mathscr{V}^{\prime}  \tag{79.1.23}\\
&\left(B\left(u_{r}-\int_{0}^{(\cdot)} \Phi_{r} d W\right)\right)^{\prime} \rightarrow\left(B\left(u-\int_{0}^{(\cdot)} \Phi d W\right)\right)^{\prime} \text { weakly in } \mathscr{V}^{\prime}  \tag{79.1.24}\\
& \int_{0}^{(\cdot)} \Phi_{r} d W \rightarrow \int_{0}^{(\cdot)} \Phi d W \text { uniformly in } C([0, T] ; W)  \tag{79.1.25}\\
& B u_{r}(t) \rightarrow B u(t) \text { weakly in } V^{\prime}  \tag{79.1.26}\\
& B u(0)=B u_{0}  \tag{79.1.27}\\
& B u(t)=B(u(t)) \text { a.e. } t \tag{79.1.28}
\end{align*}
$$

In addition to this, we can choose the subsequence such that

$$
\begin{equation*}
\sup _{r} \sup _{t \neq s} \frac{\left\|\int_{s}^{t} \Phi_{r} d W\right\|}{|t-s|^{\gamma}}<C(\omega)<\infty \tag{79.1.29}
\end{equation*}
$$

This is thanks to Corollary 79.1.1. The boundedness of the operator $A$, in particular the given estimates, imply that $z_{r}$ is bounded in $L^{p^{\prime}}\left([0, T] \times \Omega, V^{\prime}\right)$. Thus a subsequence can be obtained which yields weak convergence of $z_{r}$ in $L^{p^{\prime}}\left([0, T] \times \Omega, V^{\prime}\right)$ and then Lemma
76.3.4 may be applied to conclude that off a set of measure zero, $z$ is progressively measurable.

The claim 79.1.26 and 79.1.27 follow from the continuity of the evaluation map defined on $X$, Theorem 77.2.2. The claim in 79.1.28 follows from 79.1.22 and the convergence 79.1.26. To see this, let $\psi \in C_{c}^{\infty}(0, T)$.

$$
\begin{aligned}
\int_{0}^{T} B u(t) \psi(t) d t & =\lim _{r \rightarrow \infty} \int_{0}^{T} B u_{r}(t) \psi(t) d t \\
& =\lim _{r \rightarrow \infty} \int_{0}^{T} B\left(u_{r}(t)\right) \psi(t) d t=\int_{0}^{T} B(u(t)) \psi(t) d t
\end{aligned}
$$

Since this is true for all such $\psi$, it follows that $B u(t)=B(u(t))$ for a.e. $t$. Passing to a limit in the integral equation yields the following for $\omega$ off a set of measure zero,

$$
B u(t, \omega)-B u_{0}(\omega)+\int_{0}^{t} z(s, \omega) d s=\int_{0}^{t} f(s, \omega) d s+B \int_{0}^{t} \Phi_{n} d W
$$

In the following claim, assume $\Phi \in L^{\alpha}\left(\Omega, L^{\infty}\left([0, T], \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right)\right), \alpha>2$
Claim: $\lim _{r \rightarrow \infty} \int_{0}^{T}\left(\Phi_{r} \circ J^{-1}\right)^{*} B u_{r} \circ J d W=\int_{0}^{T}\left(\Phi \circ J^{-1}\right)^{*} B u \circ J d W$ off a set of measure zero.

Proof of claim:

$$
\begin{aligned}
& E\left(\left|\int_{0}^{T}\left(\Phi_{r} \circ J^{-1}\right)^{*} B u_{r} \circ J d W-\int_{0}^{T}\left(\Phi \circ J^{-1}\right)^{*} B u \circ J d W\right|\right) \\
\leq & E\left(\left|\int_{0}^{T}\left(\Phi_{r} \circ J^{-1}\right)^{*} B u_{r} \circ J d W-\int_{0}^{T}\left(\Phi \circ J^{-1}\right)^{*} B u_{r} \circ J d W\right|\right) \\
& +E\left(\left|\int_{0}^{T}\left(\Phi \circ J^{-1}\right)^{*} B u_{r} \circ J d W-\int_{0}^{T}\left(\Phi \circ J^{-1}\right)^{*} B u \circ J d W\right|\right)
\end{aligned}
$$

Then, by the Burkholder Davis Gundy inequality,

$$
\begin{gathered}
\leq \int_{\Omega}\left(\int_{0}^{T}\left\|\Phi_{r}-\Phi\right\|^{2}\left\langle B u_{r}, u_{r}\right\rangle\right)^{1 / 2} d P \\
+\int_{\Omega}\left(\int_{0}^{T}\|\Phi\|^{2}\left\langle B u_{r}-B u, u_{r}-u\right\rangle\right)^{1 / 2} d P \\
\leq \int_{\Omega} \sup _{t}\left\langle B u_{r}(t), u_{r}(t)\right\rangle^{1 / 2}\left(\int_{0}^{T}\left\|\Phi_{r}-\Phi\right\|^{2} d t\right)^{1 / 2} d P \\
+\int_{\Omega}\left\|\Phi_{n}\right\|_{L^{\infty}\left([0, T], \mathscr{L}_{2}\right)}\left(\int_{0}^{T}\left\langle B u_{r}-B u, u_{r}-u\right\rangle\right)^{1 / 2} d P \\
\leq\left(\int_{\Omega} \sup _{t}\left\langle B u_{r}(t), u_{r}(t)\right\rangle d P\right)^{1 / 2}\left(\int_{\Omega} \int_{0}^{T}\left\|\Phi_{r}-\Phi\right\|^{2} d t\right)^{1 / 2}
\end{gathered}
$$

$$
\begin{equation*}
+\left(\int_{\Omega}\left\|\Phi_{n}\right\|_{L^{\infty}\left([0, T], \mathscr{L}_{2}\right)}^{2}\right)^{1 / 2}\left(\int_{\Omega} \int_{0}^{T}\left\langle B u_{r}-B u, u_{r}-u\right\rangle d t d P\right)^{1 / 2} \tag{79.1.30}
\end{equation*}
$$

Letting the $e_{i}$ be the special vectors of Theorem 77.2.19,

$$
\begin{aligned}
& \int_{\Omega} \int_{0}^{T}\left\langle B u_{r}-B u, u_{r}-u\right\rangle d t d P=\int_{\Omega} \int_{0}^{T} \sum_{i}\left\langle B u_{r}-B u, e_{i}\right\rangle^{2} d t d P \\
&=\int_{\Omega} \int_{0}^{T} \sum_{i} \lim _{p \rightarrow \infty} \inf _{p}\left\langle B u_{r}-B u_{p}, e_{i}\right\rangle^{2} d t d P \\
& \quad \leq \lim \inf _{p \rightarrow \infty} \int_{\Omega} \int_{0}^{T} \sum_{i}\left\langle B u_{r}-B u_{p}, e_{i}\right\rangle^{2} d t d P \\
&= \liminf _{p \rightarrow \infty} \int_{\Omega} \int_{0}^{T} \sum_{i}\left\langle B u_{r}-B u_{p}, e_{i}\right\rangle^{2} d t d P \\
&=\lim _{p \rightarrow \infty} \int_{\Omega} \int_{0}^{T}\left\langle B u_{r}-B u_{p}, u_{r}-u_{p}\right\rangle d t d P
\end{aligned}
$$

Now by 79.1.17, the last expression is no larger than $T / 2^{r}$ and so

$$
\int_{\Omega} \int_{0}^{T}\left\langle B u_{r}-B u, u_{r}-u\right\rangle d t d P \leq \frac{T}{2^{r}}
$$

Then, from 79.1.30,

$$
\begin{gathered}
E\left(\left|\int_{0}^{T}\left(\Phi_{r} \circ J^{-1}\right)^{*} B u_{r} \circ J d W-\int_{0}^{T}\left(\Phi \circ J^{-1}\right)^{*} B u \circ J d W\right|\right) \\
\leq\left(\int_{\Omega} \sup _{t}\left\langle B u_{r}(t), u_{r}(t)\right\rangle d P\right)^{1 / 2}\left(\int_{\Omega} \int_{0}^{T}\left\|\Phi_{r}-\Phi\right\|^{2} d t\right)^{1 / 2}+C\left(\frac{T}{2^{r}}\right)^{1 / 2} \\
\leq C\left(\int_{\Omega} \int_{0}^{T}\left\|\Phi_{r}-\Phi\right\|^{2} d t\right)^{1 / 2}+C\left(\frac{T}{2^{r}}\right)^{1 / 2}<C 2^{-r}+C\left(\frac{T}{2^{r}}\right)^{1 / 2}
\end{gathered}
$$

which clearly converges to 0 as $r \rightarrow \infty$. Since the right side is summable, one obtains also pointwise convergence. This proves the claim.

From the above considerations using the space $\hat{V}$, it follows that this $u$ is the same as the one just obtained in the sense that for $\omega$ off $N$, the two are equal for a.e. $t$. Thus we take $u$ to be this common function. Hence there is a set of measure zero such that $(t, \omega) \rightarrow \mathscr{X}_{N^{C}} u(t, \omega)$ is progressively measurable in the above convergences. Also, this shows that we are taking $u \in L^{p}([0, T] \times \Omega ; V)$. From the measurability of $u_{r}, u$, we can obtain a dense countable subset $\left\{t_{k}\right\}$ and an enlarged set of measure zero $N$ such that for $\omega \notin N, B u\left(t_{k}, \omega\right)=B\left(u\left(t_{k}, \omega\right)\right)$ and $B u_{r}\left(t_{k}, \omega\right)=B\left(u_{r}\left(t_{k}, \omega\right)\right)$ for all $t_{k}$ and $r$. This uses the same argument as in Lemma 73.3.1.

It remains to verify that $z(\cdot, \omega) \in A(u(\cdot, \omega), \omega)$. It follows from the above considerations that the Ito formula above can be used at will. Assume that for a given $\omega \notin$
$N, B u(T, \omega)=B(u(T, \omega))$, similar for $B u_{r}$. If not, just do the following argument for all $T^{\prime}$ close to $T$, letting $T^{\prime}$ be in the dense subset just described. Then from the integral equation solved, and letting $\left\{e_{i}\right\}$ be the special set described in Theorem 77.2.19 and suppressing the dependence on $\omega$,

$$
\begin{aligned}
& \sum_{i=1}^{\infty}\left\langle B u_{r}(T), e_{i}\right\rangle^{2}-\sum_{i=1}^{\infty}\left\langle B u_{0}, e_{i}\right\rangle+2 \int_{0}^{T}\left\langle z_{r}, u_{r}\right\rangle d s \\
= & 2 \int_{0}^{T}\left\langle f, u_{r}\right\rangle d s+2 \int_{0}^{T}\left(\Phi_{r} \circ J^{-1}\right)^{*} B u_{r} \circ J d W
\end{aligned}
$$

Thus also

$$
\begin{align*}
& 2 \int_{0}^{T}\left\langle z_{r}, u_{r}\right\rangle d s=-\sum_{i=1}^{\infty}\left\langle B u_{r}(T), e_{i}\right\rangle^{2}+\sum_{i=1}^{\infty}\left\langle B u_{0}, e_{i}\right\rangle \\
& +2 \int_{0}^{T}\left\langle f, u_{r}\right\rangle d s+2 \int_{0}^{T}\left(\Phi_{r} \circ J^{-1}\right)^{*} B u_{r} \circ J d W \tag{79.1.31}
\end{align*}
$$

A similar formula to 79.1 .31 holds for $u$. Thus

$$
\begin{aligned}
& 2 \int_{0}^{T}\langle z, u\rangle d s=-\sum_{i=1}^{\infty}\left\langle B u(T), e_{i}\right\rangle^{2}+\sum_{i=1}^{\infty}\left\langle B u_{0}, e_{i}\right\rangle \\
& +2 \int_{0}^{T}\langle f, u\rangle d s+2 \int_{0}^{T}\left(\Phi \circ J^{-1}\right)^{*} B u \circ J d W
\end{aligned}
$$

It follows from 79.1.25 and the other convergences that

$$
\lim \sup _{r \rightarrow \infty} \int_{0}^{T}\left\langle z_{r}, u_{r}\right\rangle d s \leq \int_{0}^{T}\langle z, u\rangle d s
$$

Hence

$$
\lim \sup _{r \rightarrow \infty}\left\langle z_{r}, u_{r}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \leq 0
$$

Now from the limit condition, for any $v \in \mathscr{V}$, there exists a $z(v) \in A(u(\cdot, \omega), \omega)$ such that

$$
\begin{aligned}
\langle z, u-v\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} & \geq \lim _{r \rightarrow \infty}\left(\left\langle z_{r}, u_{r}-u\right\rangle+\left\langle z_{r}, u-v\right\rangle\right) \\
& \geq \lim _{r \rightarrow \infty} \inf _{r \rightarrow}\left\langle z_{r}, u_{r}-v\right\rangle \geq\langle z(v), u-v\rangle
\end{aligned}
$$

The reason the limit condition applies is the estimate 79.1.29 and the convergence 79.1.24 which shows that

$$
B\left(u_{r}-\int_{0}^{(\cdot)} \Phi_{r} d W\right)
$$

satisfy a Holder condition into $V^{\prime}$. Then the estimate 79.1.29 implies that the $B \int_{0}^{(\cdot)} \Phi_{r} d W$ are bounded in a Holder norm and so the same is true of the $B u_{r}$. Thus the situation of the limit condition 79.1.7 is obtained. Then it follows from separation theorems and the fact that $A(u(\cdot, \omega), \omega)$ is closed and convex that $z(\cdot, \omega) \in A(u(\cdot, \omega), \omega)$. This has proved the following Theorem.

Theorem 79.1.7 Assume the above conditions, 79.1.1-, 79.1.7 along with the progressive measurability condition 79.1.2. Also assume there is at most one solution to 79.1.8 where

$$
q(t, \cdot) \equiv \int_{0}^{t} \Phi d W
$$

Then there exists a $\mathscr{P}$ measurable u such that also z is progressively measurable

$$
B u(t, \omega)-B u_{0}(\omega)+\int_{0}^{t} z(s, \omega) d s=\int_{0}^{t} f(s, \omega) d s+B \int_{0}^{t} \Phi d W
$$

where for each $\omega, z(\cdot, \omega) \in A(u(\cdot, \omega), \omega)$. The function $B u(t, \omega)=B(u(t, \omega))$ for a.e. $t$. Here

$$
\Phi \in L^{\alpha}\left(\Omega ; L^{\infty}\left([0, T], \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right)\right) \cap L^{2}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right), \alpha>2
$$

and $u_{0} \in L^{2}(\Omega, W), f \in L^{p^{\prime}}\left([0, T] \times \Omega ; V^{\prime}\right)$.
The following corollary comes right away from the above and uniqueness for fixed $\omega$.
Corollary 79.1.8 In the situation of Theorem 79.1.7, change the conditions on $\Phi$. Instead of letting

$$
\Phi \in L^{\alpha}\left(\Omega ; L^{\infty}\left([0, T], \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right)\right) \cap L^{2}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right)
$$

assume that $\Phi \in L^{2}\left([0, T] \times \Omega ; \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)\right)$ and that $t \rightarrow \Phi(t, \omega)$ is continuous into $\mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)$. Then there exists a unique solution to the integral equation of Theorem 79.1.7.

Proof: Let $\tau_{m}=\inf \left\{t:\|\Phi(t, \omega)\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)}>m\right\}$. Then $\Phi^{\tau_{m}}$ is uniformly bounded above by $m$ and $\lim _{m \rightarrow \infty} \tau_{m}=\infty$. Hence $\Phi^{\tau_{m}}$ is in the necessary space for the conclusion of Theorem 79.1.7 to hold. Letting $m \rightarrow \infty$ and using uniqueness, one finds a solution to the integral equation.

### 79.2 Replacing $\Phi$ With $\sigma(u)$

One can replace $\Phi$ with $\sigma(u)$ provided $B$ maps $W$ one to one onto $W^{\prime}$. This includes the most common case of a Gelfand triple in which $B=I$ and $V \subseteq H=H^{\prime} \subseteq V^{\prime}$. We also need to assume that $A$ is defined pointwise as described below rather than possibly having memory terms involved.

Theorem 79.2.1 In the situation of Theorem 79.1.7, suppose 1-79.4.2 and progressive measurability condition 79.1.2 but A defined pointwise,

$$
A(u, \omega)(t)=A(u(t, \omega), t, \omega)
$$

and suppose $f$ is progressively measurable and is in $L^{p^{\prime}}\left(\Omega ; L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)\right)$. Assume

$$
\langle B u, u\rangle \geq \delta\|u\|_{W}^{2}
$$

so that an equivalent norm on $W$ is $\langle B u, u\rangle^{1 / 2}$. Assume the monotonicity assumption: for $z_{i} \in A\left(u_{i}, \omega\right)$,

$$
\begin{equation*}
\left\langle\lambda B u_{1}+z_{1}-\left(\lambda B u_{2}+z_{2}\right), u_{1}-u_{2}\right\rangle \geq \delta\left\|u_{1}-u_{2}\right\|_{W}^{2} \tag{79.2.32}
\end{equation*}
$$

for all $\lambda$ large enough. Suppose $\sigma(t, \omega, u) \in \mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)$ and has the growth properties

$$
\begin{gather*}
\|\sigma(t, \omega, u)\|_{W} \leq C+C\|u\|_{W} \\
\left\|\sigma\left(t, \omega, u_{1}\right)-\sigma\left(t, \omega, u_{2}\right)\right\|_{\mathscr{L}_{2}\left(Q^{1 / 2} U, W\right)} \leq K\left\|u_{1}-u_{2}\right\|_{W} \tag{79.2.33}
\end{gather*}
$$

and $(t, \omega) \rightarrow \sigma(t, \omega, u(t, \omega))$ is progressively measurable whenever $(t, \omega) \rightarrow u(t, \omega)$ is. Then there exists a unique solution $u$ to the integral equation

$$
B u(t)-B u_{0}+\int_{0}^{t} z d s=\int_{0}^{t} f d s+B \int_{0}^{t} \sigma(s, \cdot, u) d W
$$

The case of most interest is the usual one where $V \subseteq W=W^{\prime} \subseteq V^{\prime}$, the case of a Gelfand triple in which $B$ is the identity. We also are assuming that $A$ does not have memory terms. We obtain $(t, \omega) \rightarrow \sigma(t, \omega, u(t, \omega))$ progressively measurable if $(t, \omega) \rightarrow \sigma(t, \omega, u)$ is progressively measurable for each $u \in W$ thanks to continuity in $u$ which comes from 79.2.33.

Proof: For given $w \in L^{2}(\Omega, C([0, T], W))$ each $w$ being progressively measurable, define $u=\psi(w)$ as the solution to the integral equation

$$
B u(t, \omega)-B u_{0}(\omega)+\int_{0}^{t} z(s, \omega) d s=\int_{0}^{t} f(s, \omega) d s+B \int_{0}^{t} \sigma(w) d W
$$

which exists by above assumptions and Corollary 79.1.8. Here we write $\sigma(w)$ for short instead of $\sigma(t, \omega, w)$. From Theorem 73.7.2, $\langle B u, u\rangle$ is continuous hence bounded and so $B u$ is in $L^{\infty}\left([0, T], W^{\prime}\right)$ which implies $u \in L^{\infty}([0, T], W)$. Since $B u$ is essentially bounded in $W^{\prime}$ and equals a continuous function in $V^{\prime}$, it follows from density considerations that $B u_{i}$ can be re defined on a set of meausure zero to be weakly continuous into $W^{\prime}$ hence weakly continuous into $V^{\prime}$. This re definition must yield the integral equation because all other terms than $B u$ are continuous. However, this implies that if we let $u(t)=B^{-1}(B u)(t)$, then $u$ is weakly continuous into $W$. By continuity of $\|u\|^{2} \equiv\langle B u, u\rangle$, this shows that in fact, $u$ is continuous into $W$ thanks to the uniform continuity of the Hilbert space norm. Thus $u(\cdot, \omega) \in C([0, T], W)$.

Then from the estimates,

$$
\begin{array}{r}
\langle B u, u\rangle(t)-\left\langle B u_{0}, u_{0}\right\rangle+\delta \int_{0}^{t}\|u\|_{V}^{p} d s=2 \int_{0}^{t}\langle f, u\rangle d s+C\left(b_{3}, b_{4}, b_{5}\right) \\
+\lambda \int_{0}^{t}\langle B u, u\rangle d s+\int_{0}^{t}\langle B \sigma(w), \sigma(w)\rangle_{\mathscr{L}_{2}} d s+2 M^{*}(t) \\
\leq 2 \int_{0}^{t}\langle f, u\rangle d s+C\left(b_{3}, b_{4}, b_{5}\right)+\lambda \int_{0}^{t}\langle B u, u\rangle d s+\int_{0}^{t}\left(C+C\|w\|_{W}^{2}\right) d s+2 M^{*}(t)
\end{array}
$$

where $M^{*}(t)=\sup _{s \in[0, t]}|M(s)|$ and the quadratic variation of $M$ is no larger than

$$
\int_{0}^{t}\|\sigma(w)\|^{2}\langle B u, u\rangle d s
$$

Then using Gronwall's inequality, one obtains an inequality of the form

$$
\sup _{s \in[0, t]}\langle B u, u\rangle(s) \leq C+C\left(M^{*}(t)+\int_{0}^{t}\|w\|_{W}^{2} d s\right)
$$

where $C=C\left(u_{0}, f, \delta, \lambda, b_{3}, b_{4}, b_{5}, T\right)$ and is integrable. Then take expectation. By Burkholder Davis Gundy inequality and adjusting constants as needed,

$$
\begin{aligned}
& E\left(\sup _{s \in[0, T]}\langle B u, u\rangle(s)\right) \\
\leq & C+C \int_{\Omega} \int_{0}^{T}\|w\|_{W}^{2} d s d P+C \int_{\Omega}\left(\int_{0}^{T}\|\sigma(w)\|^{2}\langle B(u), u\rangle d s\right)^{1 / 2} d P \\
\leq & C+C \int_{\Omega} \int_{0}^{T}\|w\|_{W}^{2} d s d P+C \int_{\Omega} \sup _{s \in[0, T]}\langle B u, u\rangle^{1 / 2}(s)\left(\int_{0}^{T}\|\sigma(w)\|^{2} d s\right)^{1 / 2} d P \\
\leq & C+C \int_{\Omega} \int_{0}^{T}\|w\|_{W}^{2} d s d P+\frac{1}{2} E\left(\sup _{s \in[0, T]}\langle B u, u\rangle(s)\right)+C \int_{\Omega} \int_{0}^{T}\left(C+C\|w\|_{W}^{2}\right)
\end{aligned}
$$

Thus

$$
E(\langle B u, u\rangle(t)) \leq E\left(\sup _{s \in[0, T]}\langle B u, u\rangle(s)\right) \leq C+C \int_{\Omega} \int_{0}^{T}\|w\|_{W}^{2} d s d P
$$

and so

$$
\|u\|_{L^{2}(\Omega, C([0, T] ; W))}^{2} \leq C+C \int_{\Omega} \int_{0}^{T}\|w\|_{W}^{2} d s d P
$$

which implies $u \in L^{2}(\Omega, C([0, T] ; W))$ and is progressively measurable.
Using the monotonicity assumption, there is a suitable $\lambda$ such that

$$
\begin{aligned}
& \frac{1}{2}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle(t)+r \int_{0}^{t}\left\|u_{1}-u_{2}\right\|_{W}^{2} d s \\
&-\lambda \int_{0}^{t}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle d s \\
&-\int_{0}^{t}\left\langle B \sigma\left(u_{1}\right)-B \sigma\left(u_{2}\right), \sigma\left(u_{1}\right)-\sigma\left(u_{2}\right)\right\rangle_{\mathscr{L}_{2}} d s \leq M^{*}(t)
\end{aligned}
$$

where the right side is of the form $\sup _{s \in[0, t]}|M(s)|$ where $M(t)$ is a local martingale having quadratic variation dominated by

$$
\begin{equation*}
C \int_{0}^{t}\left\|\sigma\left(w_{1}\right)-\sigma\left(w_{2}\right)\right\|^{2}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle d s \tag{79.2.34}
\end{equation*}
$$

Then by assumption and using Gronwall's inequality, there is a constant $C=C(\lambda, K, T)$ such that

$$
\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle(t) \leq C M^{*}(t)
$$

Then also, since $M^{*}$ is increasing,

$$
\sup _{s \in[0, t]}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle(s) \leq C M^{*}(t)
$$

Taking expectations and from the Burkholder Davis Gundy inequality,

$$
\begin{gathered}
E\left(\sup _{s \in[0, t]}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle(s)\right) \\
\leq C \int_{\Omega}\left(\int_{0}^{t}\left\|\sigma\left(w_{1}\right)-\sigma\left(w_{2}\right)\right\|^{2}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle\right)^{1 / 2} d P \\
\leq C \int_{\Omega} \sup _{s \in[0, t]}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle^{1 / 2}(s)\left(\int_{0}^{t}\left\|\sigma\left(w_{1}\right)-\sigma\left(w_{2}\right)\right\|^{2}\right)^{1 / 2} d P
\end{gathered}
$$

Then it follows after adjusting constants that there exists an inequality of the form

$$
E\left(\sup _{s \in[0, t]}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle(s)\right) \leq C E\left(\int_{0}^{t}\left\|\sigma\left(w_{1}\right)-\sigma\left(w_{2}\right)\right\|_{\mathscr{L}_{2}}^{2} d s\right)
$$

Hence

$$
\begin{gathered}
E\left(\sup _{s \in[0, t]}\left\langle B\left(u_{1}-u_{2}\right), u_{1}-u_{2}\right\rangle(s)\right) \leq C K^{2} E\left(\int_{0}^{t}\left\|w_{1}-w_{2}\right\|_{W}^{2} d s\right) \\
E\left(\sup _{s \in[0, t]}\left\|u_{1}(s)-u_{2}(s)\right\|_{W}^{2}\right) \leq C K^{2} E\left(\int_{0}^{t} \sup _{\tau \in[0, s]}\left\|w_{1}(\tau)-w_{2}(\tau)\right\|_{W}^{2} d \tau\right)
\end{gathered}
$$

You can iterate this inequality and obtain for $\psi\left(w_{i}\right)$ defined as $u_{i}$ in the above, the following inequality.

$$
\begin{aligned}
& E\left(\sup _{s \in[0, t]}\left\|\psi^{n} w_{1}-\psi^{n} w_{2}\right\|_{W}^{2}(s)\right) \\
\leq & \left(C K^{2}\right)^{n} E\left(\int_{0}^{t} \int_{0}^{t_{1}} \cdots \int_{0}^{t_{n-1}} \sup _{t_{n} \in\left[0, t_{n-1}\right]}\left\|w_{1}\left(t_{n}\right)-w_{2}\left(t_{n}\right)\right\|_{W}^{2} d t_{n} \cdots d t_{2} d t_{1}\right)
\end{aligned}
$$

Then, letting $t=T$,

$$
\begin{aligned}
E\left(\sup _{s \in[0, T]}\left\|\psi^{n} w_{1}-\psi^{n} w_{2}\right\|_{W}^{2}(s)\right) & \leq E\left(\sup _{t \in[0, T]}\left\|w_{1}(t)-w_{2}(t)\right\|^{2}\right) \frac{T^{n}}{n!} \\
& \leq \frac{1}{2} E\left(\sup _{t \in[0, T]}\left\|w_{1}-w_{2}\right\|_{W}^{2}(t)\right)
\end{aligned}
$$

provided $n$ is large enough and so $\psi$ has a high enough power a contraction map. Hence, if one begins with $w \in L^{2}(\Omega, C([0, T] ; W))$, the sequence of iterates $\left\{\psi^{n} w\right\}_{n=1}^{\infty}$ must converge to some fixed point $u$ in $L^{2}(\Omega, C([0, T], W))$. This $u$ is progressively measurable since each of the iterates is progressively measurable. This fixed point is the solution to the integral equation.

### 79.3 An Example

A model for a nonlinear beam is the Gao beam in which the transverse vibrations satisfy an equation of the form

$$
w_{t t}+w_{x x x x}+\alpha w_{t x x x x}+\left(1-w_{x}^{2}\right) w_{x x}=f
$$

To this, we can add boundary conditions and initial conditions

$$
\begin{aligned}
w(t, 0) & =w_{x}(t, 0)=w(t, 1)=w_{x}(t, 1)=0 \\
w(0, x) & =w_{0}(x), w_{t}(0, x)=v_{0}(x) \\
w_{0} & \in H_{0}^{2}((0,1))=V, v_{0} \in L^{2}((0,1))=H
\end{aligned}
$$

Also let $W$ be the closure of $V$ in $H^{1}((0,1))$ so $W=H_{0}^{1}((0,1))$. Let $H$ denote $L^{2}((0,1))$. These conditions correspond to a clamped beam. An equivalent norm on $V$ is $\|u\|_{V}=$ $\left|u_{x x}\right|_{H}$. Our theory allows us to include coefficient functions which are progressively measurable but in the interest of simplicity, this technical complication will be omitted. Also, it will follow that there is a pointwise estimate for $w_{x}(t, x)$ thanks to Sobolev embedding theorems and routine arguments involving the term $w_{x x x x}$. Therefore, in the case of a deterministic Gao beam, there is no loss of generality in replacing $w_{x}^{2}$ with $q\left(w_{x}\right)$ where $q$ is a bounded and Lipschitz continuous truncation of $r \rightarrow r^{2}$. To begin with, we make this approximation. Let $Q^{\prime}(r)=q(r)$ and $Q(0)=0$. Thus $Q$ will have linear growth for $|r|$ large and is an odd function. Define operators $L, N$ mapping $V$ to $V^{\prime}$.

$$
\langle L w, u\rangle=\int_{0}^{1} w_{x x} u_{x x} d x,\langle N w, u\rangle=\int_{0}^{1}\left(Q\left(w_{x}\right)-w_{x}\right) u_{x} d x
$$

Then in terms of these operators, and writing in terms of the velocity $v$, we obtain the following abstract formulation.

$$
v^{\prime}+\alpha L v+L w+N w=f \in \mathscr{V}^{\prime}, w(t)=w_{0}+\int_{0}^{t} v d s, v(0)=v_{0}
$$

In terms of an integral equation, this would be

$$
\begin{align*}
v(t)-v_{0}+\alpha \int_{0}^{t} L v(s) d s+\int_{0}^{t} L w d s+\int_{0}^{t} N w d s & =\int_{0}^{t} f d s \\
w(t) & =w_{0}+\int_{0}^{t} v d s \tag{79.3.35}
\end{align*}
$$

Then the abstract version of a stochastic equation is

$$
\begin{align*}
v(t)-v_{0}+\alpha \int_{0}^{t} L v(s) d s+\int_{0}^{t} L w d s+\int_{0}^{t} N w d s & =\int_{0}^{t} f d s+\int_{0}^{t} \sigma(v) d(W 9.3 .36) \\
w(t) & =w_{0}+\int_{0}^{t} v d s \tag{79.3.37}
\end{align*}
$$

where $\sigma$ will be of the form given in Theorem 79.2.1. To show existence of a solution to the above, note that by Theorem 79.2.1, if $w \in L^{2}(\Omega, C([0, T] ; V))$ with $w$ progressively measurable, then there exists a unique solution $v$ to the above integral equation 79.3.36. Let $\psi(w)(t)=w_{0}+\int_{0}^{t} v d s$. Suppose now you have $w_{i} \in L^{2}(\Omega, C([0, T] ; V))$ with $v_{i}$ being the solution corresponding to the fixed $w_{i}$. Then

$$
\begin{gathered}
v_{1}(t)-v_{2}(t)+\alpha \int_{0}^{t} L\left(v_{1}-v_{2}\right)(s) d s+\int_{0}^{t} L\left(w_{1}-w_{2}\right) d s \\
\quad+\int_{0}^{t}\left(N w_{1}-N w_{2}\right) d s=\int_{0}^{t}\left(\sigma\left(v_{1}\right)-\sigma_{2}\left(v_{2}\right)\right) d W
\end{gathered}
$$

From the Ito formula,

$$
\begin{gathered}
\left|v_{1}-v_{2}\right|_{H}^{2}+2 \alpha \int_{0}^{t}\left\|v_{1}-v_{2}\right\|_{V}^{2} d s+\int_{0}^{t}\left\langle\int_{0}^{s} L\left(w_{1}-w_{2}\right) d \tau, v_{1}(s)-v_{2}(s)\right\rangle d s+ \\
\int_{0}^{t}\left\langle\int_{0}^{s} N\left(w_{1}\right)-N\left(w_{2}\right) d \tau, v_{1}(s)-v_{2}(s)\right\rangle d s=2 \int_{0}^{t}\left\|\sigma\left(v_{1}\right)-\sigma_{2}\left(v_{2}\right)\right\|_{\mathscr{L}_{2}}^{2} d s+M(t)
\end{gathered}
$$

where the quadratic variation of $M(t)$ is

$$
\int_{0}^{t}\left\|\sigma\left(v_{1}\right)-\sigma_{2}\left(v_{2}\right)\right\|_{\mathscr{L}_{2}}^{2}\left|v_{1}-v_{2}\right|_{H}^{2} d s
$$

Using the estimates and the Lipschitz condition on $Q$, and letting $C$ be a generic constant depending on the subscripts, routine manipulations yield an inequality of the form

$$
\begin{aligned}
& \left|v_{1}-v_{2}\right|_{H}^{2}+2 \alpha \int_{0}^{t}\left\|v_{1}-v_{2}\right\|_{V}^{2} d s \leq \varepsilon \int_{0}^{t}\left\|v_{1}(s)-v_{2}(s)\right\|_{V}^{2} d s \\
& \quad+C_{\varepsilon} \int_{0}^{t} \int_{0}^{s}\left\|w_{1}-w_{2}\right\|_{V}^{2} d \tau d s+C \int_{0}^{t}\left|v_{1}-v_{2}\right|_{H}^{2} d s+M^{*}(t)
\end{aligned}
$$

Letting $\varepsilon \leq \alpha$, and modifying the constants as needed, Gronwall's inequality yields an inequality of the form

$$
\left|v_{1}-v_{2}\right|_{H}^{2}+\alpha \int_{0}^{t}\left\|v_{1}-v_{2}\right\|_{V}^{2} d s \leq C_{\alpha, T} \int_{0}^{t} \int_{0}^{s}\left\|w_{1}-w_{2}\right\|_{V}^{2} d \tau d s+C_{\alpha, T} M^{*}(t)
$$

Modifying the constants again if necessary, one obtains

$$
\sup _{s \in[0, t]}\left|v_{1}-v_{2}\right|_{H}^{2}(s)+\alpha \int_{0}^{t}\left\|v_{1}-v_{2}\right\|_{V}^{2} d s \leq C_{\alpha, T} \int_{0}^{t} \int_{0}^{s}\left\|w_{1}-w_{2}\right\|_{V}^{2} d \tau d s+C_{\alpha, T} M^{*}(t)
$$

and now, by the Burkholder Davis Gundy inequality,

$$
E\left(\sup _{s \in[0, t]}\left|v_{1}-v_{2}\right|_{H}^{2}(s)\right)+\alpha \int_{0}^{t} E\left(\left\|v_{1}-v_{2}\right\|_{V}^{2}\right) d s \leq
$$

$$
\begin{aligned}
& C_{\alpha, T} \int_{0}^{t} \int_{0}^{s} E\left(\left\|w_{1}-w_{2}\right\|_{V}^{2}\right) d \tau d s \\
& +C_{\alpha, T} \int_{\Omega}\left(\int_{0}^{t}\left\|\sigma\left(v_{1}\right)-\sigma_{2}\left(v_{2}\right)\right\|_{\mathscr{L}_{2}}^{2}\left|v_{1}-v_{2}\right|_{W}^{2} d s\right)^{1 / 2} d P
\end{aligned}
$$

Then the usual manipulations and the Lipschitz condition on $\sigma$ yields an inequality of the form

$$
\begin{gathered}
\frac{1}{2} E\left(\sup _{s \in[0, t]}\left|v_{1}-v_{2}\right|_{H}^{2}(s)\right)+\alpha \int_{0}^{t} E\left(\left\|v_{1}-v_{2}\right\|_{V}^{2}\right) d s \leq \\
\quad C_{\alpha, T} \int_{0}^{t} \int_{0}^{s} E\left(\left\|w_{1}-w_{2}\right\|_{V}^{2}\right) d \tau d s \\
\quad+C_{\alpha, T} \int_{0}^{t} E\left(\sup _{\tau \in[0, s]}\left|v_{1}-v_{2}\right|_{H}^{2}(\tau)\right) d s
\end{gathered}
$$

Thus, Gronwall's inequality yields

$$
\begin{gather*}
E\left(\sup _{s \in[0, t]}\left|v_{1}-v_{2}\right|_{H}^{2}(s)\right)+\int_{0}^{t} E\left(\left\|v_{1}-v_{2}\right\|_{V}^{2}\right) d s \\
\quad \leq C_{\alpha, T} \int_{0}^{t} \int_{0}^{s} E\left(\left\|w_{1}-w_{2}\right\|_{V}^{2}\right) d \tau d s \tag{79.3.38}
\end{gather*}
$$

Now $\psi\left(w_{i}\right)$ is defined as $w_{0}+\int_{0}^{t} v_{i} d s$ and so $\psi\left(w_{i}\right)$ is in $C([0, T] ; V)$ and from the above,

$$
\left\|\psi\left(w_{1}\right)(s)-\psi\left(w_{2}\right)(s)\right\|_{V}^{2} \leq C_{T} \int_{0}^{s}\left\|v_{1}(\tau)-v_{2}(\tau)\right\|_{V}^{2} d \tau
$$

and so

$$
\begin{aligned}
\sup _{s \in[0, t]}\left\|\psi\left(w_{1}\right)(s)-\psi\left(w_{2}\right)(s)\right\|_{V}^{2} & =\left\|\psi\left(w_{1}\right)-\psi\left(w_{2}\right)\right\|_{C([0, t] ; V)}^{2} \\
& \leq C_{T} \int_{0}^{t}\left\|v_{1}(s)-v_{2}(s)\right\|_{V}^{2} d s
\end{aligned}
$$

Using 79.3.38,

$$
\begin{gathered}
E\left(\left\|\psi\left(w_{1}\right)-\psi\left(w_{2}\right)\right\|_{C([0, t] ; V)}^{2}\right) \leq C_{T} \int_{0}^{t} E\left(\left\|v_{1}(s)-v_{2}(s)\right\|_{V}^{2}\right) d s \\
\leq C_{\alpha, T} \int_{0}^{t} \int_{0}^{s} E\left(\left\|w_{1}-w_{2}\right\|_{V}^{2}\right) d \tau d s \leq C_{\alpha, T} \int_{0}^{t} E\left(\left\|w_{1}-w_{2}\right\|_{C([0, s] ; V)}^{2}\right) d s
\end{gathered}
$$

Iterating this inequality shows that for all $n$ large enough,

$$
\left\|\psi^{n}\left(w_{1}\right)-\psi^{n}\left(w_{2}\right)\right\|_{C([0, t] ; V)}^{2} \leq \frac{1}{2}\left\|w_{1}-w_{2}\right\|_{C([0, t] ; V)}^{2}
$$

Letting $t=T$, it follows that $\psi$ has a unique progressively measurable fixed point in $L^{2}(\Omega ; C([0, T] ; V))$. The fixed point is the limit of the sequence $\left\{\psi^{n} w\right\}$ and each function in the sequence is progressively measurable. This is the unique solution to the integral equation.

This is interesting because it is an example of a stochastic equation which is second order in time. If one were to include a point mass on the beam at $x_{0}$, then this would lead to an evolution equation of the form

$$
B v(t)-v_{0}+\alpha \int_{0}^{t} L v(s) d s+\int_{0}^{t} L w d s+\int_{0}^{t} N w d s=\int_{0}^{t} f d s+B \int_{0}^{t} \sigma(v) d W
$$

where $B=I+P$ with $\langle P u, v\rangle=u\left(x_{0}\right) v\left(x_{0}\right)$. If this were done, you would need to adjust $W$ so that $B$ is the Riesz map from $W$ to $W^{\prime}$. You would use the same kind of fixed point argument that was just given. If you wished to consider quasistatic motion of the nonlinear beam in which the acceleration term $w_{t t}$ were neglected, this would involve letting $W=$ $V$ and your operator $B$ in Theorem 79.2 .1 would be given by $\langle B w, u\rangle=\int_{0}^{1} w_{x x} u_{x x}$. Thus Theorem 79.2.1 gives a way to study second order in time equations, and implicit equations in which the leading coefficient involves a self adjoint operator $B$. Without the assumption that $B$ is one to one, we can give an even more general theorem in case $\sigma$ does not depend on the solution, Corollary 79.1.8. This one allows the time differentiated term to even vanish so the differential inclusion could be degenerate. Many other examples are available. One has only to consider set valued problems, for example.

Next we consider the elimination of the truncation function. As mentioned, this is easy for a deterministic equation but not so much for the situation here. Begin with 79.3.3679.3.37 and use the Ito formula. Then

$$
\begin{gathered}
|v(t)|_{H}^{2}+2 \alpha \int_{0}^{t}\|v\|_{V}^{2} d s+2 \int_{0}^{t}\langle L w, v\rangle d s \\
+2 \int_{0}^{t}\langle N w, v\rangle d s-\int_{0}^{t}\|\sigma(v)\|^{2} d s=2 \int_{0}^{t}\langle f, v\rangle d s+M(t)
\end{gathered}
$$

where $M(t)$ is a martingale whose total variation satisfies

$$
[M](t)=\int_{0}^{t}\|\sigma(v)\|^{2}|v|_{H}^{2} d s
$$

Then, using estimates and simple manipulations, for $M^{*}(t) \equiv \sup _{s \leq t}|M(s)|$,

$$
\begin{gathered}
\frac{1}{2}|v(t)|_{H}^{2}+2 \alpha \int_{0}^{t}\|v\|_{V}^{2} d s+\|w(t)\|_{V}^{2}+2 \int_{0}^{t} \int_{0}^{1}\left(Q\left(w_{x}\right)-w_{x}\right) v_{x} d x d s \\
\leq \int_{0}^{t}\left(C+C|v|_{H}^{2}\right) d s+C\left(f, w_{0}, \omega\right)+M^{*}(t)
\end{gathered}
$$

Now let $\Psi^{\prime}=Q$ so that $\Psi(r) \geq 0$ and is quadratic for large $|r|$. Then the above implies

$$
\frac{1}{2}|v(t)|_{H}^{2}+2 \alpha \int_{0}^{t}\|v\|_{V}^{2} d s+\|w(t)\|_{V}^{2}+\int_{0}^{1} \Psi\left(w_{x}\right)-\left|w_{x}(t)\right|^{2} d x
$$

$$
\leq \int_{0}^{t}\left(C+C|v|_{H}^{2}\right) d s+C\left(f, w_{0}, \omega\right)+M^{*}(t)
$$

Using compactness of the embedding of $V$ into $W$, we can simplify this to an inequality of the following form where this involves modifying the constants $C$.

$$
\begin{aligned}
& \frac{1}{2}|v(t)|_{H}^{2}+2 \alpha \int_{0}^{t}\|v\|_{V}^{2} d s+\frac{1}{2}\|w(t)\|_{V}^{2} \\
\leq & \int_{0}^{t}\left(C+C|v|_{H}^{2}\right) d s+C\left(f, w_{0}, \omega\right)+M^{*}(t)
\end{aligned}
$$

Then, since the functions on the right are increasing in $t$, we can modify the constants again and obtain an inequality of the form

$$
\begin{aligned}
& \sup _{s \leq t}|v(s)|_{H}^{2}+\sup _{s \leq t}\|w(t)\|_{V}^{2}+\alpha \int_{0}^{t}\|v\|_{V}^{2} d s \\
\leq & \int_{0}^{t}\left(C+C|v|_{H}^{2}\right) d s+C\left(f, w_{0}, \omega\right)+C M^{*}(t)
\end{aligned}
$$

Take expectations using the Burkholder Davis Gundy inequality and estimates for $\sigma$ and obtain

$$
\begin{gathered}
E\left(\sup _{s \leq t}|v(s)|_{H}^{2}\right)+E\left(\sup _{s \leq t}\|w(t)\|_{V}^{2}\right)+E\left(\alpha \int_{0}^{t}\|v\|_{V}^{2} d s\right) \\
\leq \int_{0}^{t} E\left(C+C\left(\sup _{\tau \leq s}|v(\tau)|_{H}^{2}\right)\right) d s \\
+C\left(f, w_{0}\right)+C \int_{\Omega}\left(\int_{0}^{t}\left(\left(C+C|v|_{H}\right)^{2} \sup _{\tau \leq s}|v(\tau)|_{H}^{2}\right)\right)^{1 / 2} d P \\
\leq \int_{0}^{t} E\left(C+C\left(\sup _{\tau \leq s}|v(\tau)|_{H}^{2}\right)\right) d s+C\left(f, w_{0}\right) \\
\quad+C \int_{\Omega} \sup _{\tau \leq t}|v(\tau)|_{H}\left(\int_{0}^{t}\left(C+C|v|_{H}^{2}\right)\right)^{1 / 2} d P
\end{gathered}
$$

Using Cauchy Schwarz inequality one can simplify the above to

$$
\begin{aligned}
& E\left(\sup _{s \leq t}|v(s)|_{H}^{2}\right)+E\left(\sup _{s \leq t}\|w(t)\|_{V}^{2}\right)+E\left(\alpha \int_{0}^{t}\|v\|_{V}^{2} d s\right) \\
\leq & C\left(f, w_{0}, T\right)+C \int_{0}^{t} E\left(\sup _{\tau \leq s}|v(\tau)|_{H}^{2}\right) d s
\end{aligned}
$$

Then Gronwall's inequality implies

$$
\begin{equation*}
E\left(\sup _{s \leq t}|v(s)|_{H}^{2}\right)+E\left(\sup _{s \leq t}\|w(t)\|_{V}^{2}\right)+E\left(\alpha \int_{0}^{t}\|v\|_{V}^{2} d s\right) \leq C\left(f, w_{0}, T\right) \tag{79.3.39}
\end{equation*}
$$

Thus, letting $t=T$,

$$
\begin{equation*}
E\left(\|v\|_{C([0, T] ; H)}^{2}\right)+E\left(\|w\|_{C([0, T] ; V)}^{2}\right)+E\left(\alpha \int_{0}^{T}\|v\|_{V}^{2} d s\right) \leq C\left(f, w_{0}, T\right) \tag{79.3.40}
\end{equation*}
$$

Now let $w_{n}, v_{n}$ correspond in the above to $q_{n}$ where $q_{n}(r)=r^{2}$ for $|r| \leq 2^{n}$ and $q_{n}(r)=$ $4^{n}$ for $|r| \geq 2^{n}$. Let $N_{n}$ be the operator resulting from $q_{n}$ and let $N$ be the operator defined by

$$
\langle N w, u\rangle \equiv \int_{0}^{1}\left(\frac{w_{x}^{3}}{3}-w_{x}\right) u_{x} d x
$$

By embedding theorems, $\left\|w_{n x}\right\|_{C([0,1])} \leq C\left\|w_{x x}\right\|_{V}$. We define stopping times.

$$
\tau_{n}(\omega) \equiv \inf \left\{t \in[0, T]: C\left\|w_{n x x}(\cdot, \omega)\right\|_{C([0, t])}>2^{n}\right\}
$$

$\inf (\emptyset)$ defined as $\infty$. Then

$$
\begin{aligned}
& v_{n}^{\tau_{n}}(t)-v_{0}+\alpha \int_{0}^{t} \mathscr{X}_{\left[0, \tau_{n}\right]}(s) L v_{n}(s) d s+\int_{0}^{t} \mathscr{X}_{\left[0, \tau_{n}\right]}(s) L w_{n} d s \\
+ & \int_{0}^{t} \mathscr{X}_{\left[0, \tau_{n}\right]}(s) N_{n} w_{n} d s=\int_{0}^{t} \mathscr{X}_{\left[0, \tau_{n}\right]}(s) f d s+\int_{0}^{t} \mathscr{X}_{\left[0, \tau_{n}\right]}(s) \sigma(v) d W
\end{aligned}
$$

Does $\lim _{n \rightarrow \infty} \tau_{n}=\infty$ ? If not so for some $\omega$, then there is a subsequence still denoted with $n$ such that for all $n, \tau_{n}(\omega) \leq T$. This implies

$$
C^{2}\left\|w_{n x x}(\cdot, \omega)\right\|_{C([0, T])}^{2}>4^{n}
$$

but the set of $\omega$ for which the above holds has measure no more than $C^{2} C\left(f, w_{0}, T\right) / 4^{n}$ thanks to 79.3 .40 and so there is a further subsequence and a set of measure zero such that for $\omega$ not in this set, eventually, for all $n$ large enough,

$$
C\left\|w_{n x x}(\cdot, \omega)\right\|_{C([0, T])} \leq 2^{n}
$$

which requires $\tau_{n}=\infty$, contrary to the construction of this subsequence. Thus $\tau_{n}$ converges to $\infty$ off a set of measure zero. Using uniqueness, define $v=v_{n}$ whenever $\tau_{n}=\infty$ and $w=$ $w_{n}$ whenever $\tau_{n}=\infty$. Then from the embedding theorem mentioned above, $w_{x}(t, \omega)^{2}=$ $q_{n}\left(w_{x}(t, \omega)\right)$ and so

$$
\begin{align*}
& v(t)-v_{0}+\alpha \int_{0}^{t} L v(s) d s+\int_{0}^{t} L w d s \\
& +\int_{0}^{t} N w d s=\int_{0}^{t} f d s+\int_{0}^{t} \sigma(v) d W \tag{79.3.41}
\end{align*}
$$

where

$$
\langle N w, u\rangle=\int_{0}^{1}\left(\left(\frac{w_{x}^{3}}{3}\right)-w_{x}\right) u_{x} d x
$$

Of course it would be very interesting in this example to see if you can pass to a limit as $\alpha \rightarrow 0$. We do this in the non probabilistic version of this problem quite easily, but whether it can be done in the stochastic equations considered here is not clear.

### 79.4 Stochastic Inclusions Without Uniqueness ??

We will consider strong solutions to the integral equation

$$
u(t)-u_{0}+\int_{0}^{t} z d s=\int_{0}^{t} f+\int_{0}^{t} \Phi d W
$$

where $z(t, \omega) \in A(u(t, \omega), t, \omega)$ in a situation where there is not necessarily uniqueness for the integral equation for fixed $\omega$. Uniqueness usually comes from monotonicity considerations. I am trying to eliminate these and replace with a monotonicity condition which comes from including $\varepsilon F$ for $F$ a duality map.

### 79.4.1 Estimates

In what follows $[0, T]$ will be a finite interval with no restriction on the size of $T$.
Definition 79.4.1 Recall a filtration is $\left\{\mathscr{F}_{t}\right\}, t \in[0, T]$ where each $\mathscr{F}_{t}$ is a $\sigma$ algebra of sets in $\Omega$ a probability space and these are increasing in $t$. Then the progressively measurable sets $\mathscr{P}$ are $S \subseteq \Omega$ such that

$$
S \cap[0, t] \times \Omega \text { is } \mathscr{B}([0, t]) \times \mathscr{F}_{t} \text { measurable }
$$

You can verify that this is indeed a $\sigma$ algebra of sets in $[0, T] \times \Omega$. Here $\mathscr{B}([0, t])$ is the $\sigma$ algebra of Borel measurable sets on $[0, t]$. We could have used $\mathscr{B}([0, T]) \times \mathscr{F}_{t}$ instead of $\mathscr{B}([0, t]) \times \mathscr{F}_{t}$ in the above because a set is in $\mathscr{B}([0, t])$ if and only if it is the intersection of a Borel set of $[0, T]$ with $[0, t]$. We will always assume that each $\mathscr{F}_{t}$ contains all the sets of measure zero of $\mathscr{F}_{T}$.

We will assume all Banach spaces are separable in what follows.
We will assume $U \subseteq V \subseteq H=H^{\prime} \subseteq V^{\prime} \subseteq U^{\prime}$ where the inclusion map of $V$ into the Hilbert space $H$ is compact and $V$ is dense in $H$ and $U$ is a Banach space which is dense and compact in $V$. One can always obtain such a space. In fact one can always have $U$ be a Hilbert space. In practice this is most easily seen from Sobolev embedding theorems but the existence of this space follows from general abstract considerations.

Then for $p>1$, define

$$
\mathscr{V} \equiv L^{p}([0, T] \times \Omega ; V), \mathscr{U} \equiv L^{r}([0, T] \times \Omega ; V), r \geq \max (2, p)
$$

It follows that the dual space $\mathscr{V}^{\prime}$ can be identified in the usual way as $L^{p^{\prime}}([0, T] \times \Omega ; V)$. Similarly $\mathscr{H}$ will be defined as $L^{2}([0, T] \times \Omega ; H)$. In each instance the relevant $\sigma$ algebra will be the progressively measurable sets. A set $A \subseteq \Omega \times[0, T]$ is progressively measurable if

$$
A \cap[0, t] \times \Omega \in \mathscr{F}_{t} \times \mathscr{B}([0, t])
$$

where $\mathscr{B}([0, t])$ denotes the Borel sets of $[0, t]$, equivalently the intersections of a Borel set of $[0, T]$ with $[0, t]$.

Also define $\mathscr{V}_{\omega}$ as $L^{p}([0, T] ; V)$ where the subscript $\omega$ indicates that $\omega$ is fixed. Let $\mathscr{H}_{\omega}$ and $\mathscr{U}_{\omega}$ be defined similarly. We will assume the following on $A: V \times[0, T] \times \Omega \rightarrow \mathscr{P}\left(V^{\prime}\right)$.

1. $A(\cdot, t, \omega): V \rightarrow \mathscr{P}\left(V^{\prime}\right)$ is pseudomonotone and bounded: $A(u, t, \omega)$ is a closed convex set for each $(t, \omega), u \rightarrow A(u, t, \omega)$ is bounded, and if $u_{n} \rightarrow u$ weakly and

$$
\lim \sup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle \leq 0, z_{n} \in A\left(u_{n}, t, \omega\right)
$$

then for any $v \in V$,

$$
\lim \inf _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-v\right\rangle \geq\langle z(v), u-v\rangle \text { some } z(v) \in A(u, t, \omega)
$$

2. $A(\cdot, t, \omega)$ satisfies the estimates: There exists $b_{1} \geq 0$ and $b_{2} \geq 0$, such that

$$
\begin{equation*}
\|z\|_{V^{\prime}} \leq b_{1}\|u\|_{V}^{p-1}+b_{2}(t, \omega) \tag{79.4.42}
\end{equation*}
$$

for all $z \in A(u, t, \omega), b_{2}(\cdot, \cdot) \in L^{p^{\prime}}([0, T] \times \Omega)$.
3. There exists a positive constant $b_{3}$ and a nonnegative function $b_{4}$ that is $\mathscr{B}([0, T]) \times$ $\mathscr{F}_{T}$ measurable and also $b_{4}(\cdot, \cdot) \in L^{1}([0, T] \times \Omega)$, such that for some $\lambda \geq 0$,

$$
\begin{equation*}
\inf _{z \in A(u, t, \omega)}\langle z, u\rangle \geq b_{3}\|u\|_{V}^{p}-b_{4}(t, \omega)-\lambda|u|_{H}^{2} \tag{79.4.43}
\end{equation*}
$$

One can often reduce to the case that $\lambda=0$ by using an exponential shift argument.
4. The mapping $(t, \omega) \rightarrow A(u(t, \omega), t, \omega)$ is measurable in the sense that

$$
(t, \omega) \rightarrow A(u(t, \omega), t, \omega)
$$

is a progressively measurable multifunction with respect to $\mathscr{P}$ whenever $(t, \omega) \rightarrow$ $u(t, \omega)$ is in $\mathscr{V} \equiv \mathscr{V}_{p} \equiv L^{p}([0, T] \times \Omega ; V, \mathscr{P})$.

As mentioned one can often reduce to the case where $\lambda=0$ in 3 . Indeed, let $A(u, t, \omega)$ be single valued for the sake of simplicity. Let

$$
w=e^{-\lambda t} u
$$

where $u$ satisfies

$$
u(t)-u_{0}+\frac{1}{n} \int_{0}^{t} F u d s+\int_{0}^{t} A(u, t, \omega) d s=\int_{0}^{t} f d s+\int_{0}^{t} \Phi d W
$$

Then this amounts to

$$
\left(u-\int_{0}^{(\cdot)} \Phi d W\right)^{\prime}+\frac{1}{n} F u+A(u, t, \omega)=f, u(0)=u_{0}
$$

In terms of $w$, this is

$$
\left(e^{\lambda(\cdot)} w-e^{\lambda(\cdot)} \int_{0}^{(\cdot)} e^{-\lambda(\cdot)} \Phi d W\right)^{\prime}+\frac{1}{n} F\left(e^{\lambda(\cdot)} w\right)+A\left(e^{\lambda(\cdot)} w, t, \omega\right)=f, u(0)=u_{0}
$$

and this reduces to

$$
\begin{gathered}
\lambda e^{\lambda(\cdot)}\left(w-\int_{0}^{(\cdot)} e^{-\lambda(\cdot)} \Phi d W\right)+e^{\lambda(\cdot)}\left(w-\int_{0}^{(\cdot)} e^{-\lambda(\cdot)} \Phi d W\right)^{\prime} \\
+\frac{1}{n} F\left(e^{\lambda(\cdot)} w\right)+A\left(e^{\lambda(\cdot)} w, t, \omega\right)=f
\end{gathered}
$$

and this reduces to

$$
\begin{gathered}
\lambda\left(w-\int_{0}^{(\cdot)} e^{-\lambda(\cdot)} \Phi d W\right)+\left(w-\int_{0}^{(\cdot)} e^{-\lambda(\cdot)} \Phi d W\right)^{\prime} \\
+e^{-\lambda(\cdot)} \frac{1}{n} F\left(e^{\lambda(\cdot)} w\right)+e^{-\lambda(\cdot)} A\left(e^{\lambda(\cdot)} w, t, \omega\right)=e^{-\lambda(\cdot)} f, w(0)=u_{0}
\end{gathered}
$$

which implies

$$
\begin{aligned}
& \left(w-\int_{0}^{(\cdot)} e^{-\lambda(\cdot)} \Phi d W\right)^{\prime}+\lambda w+e^{-\lambda(\cdot)} \frac{1}{n} F\left(e^{\lambda(\cdot)} w\right) \\
+e^{-\lambda(\cdot)} A\left(e^{\lambda(\cdot)} w, t, \omega\right)= & \lambda \int_{0}^{(\cdot)} e^{-\lambda(\cdot)} \Phi d W+e^{-\lambda(\cdot)} f, w(0)=u_{0}
\end{aligned}
$$

which is equivalent to

$$
w(t)+\int_{0}^{t} e^{-\lambda(\cdot)} \frac{1}{n} F\left(e^{\lambda(\cdot)} w\right) d s+\int_{0}^{t} \lambda w+e^{-\lambda(\cdot)} A\left(e^{\lambda s} w(s), s, \omega\right) d s=\int_{0}^{t} \hat{f}(s) d s
$$

where $\hat{f}(\cdot)=e^{-\lambda(\cdot)} f+\lambda \int_{0}^{(\cdot)} e^{-\lambda(\cdot)} \Phi d W$. You can consider $\tilde{A}(w, t, \omega) \equiv e^{-\lambda t} A\left(e^{\lambda t} w, t, \omega\right)$. This satisfies similar conditions to $A$. If $F$ were a linear Riesz map, then you would get the same type of problem but with $\lambda=0$ for $\lambda$ large enough. It may work in other cases also.

Definition 79.4.2 Let A be given above. Then $z \in \hat{A}(u)$ means that for $u \in \mathscr{V}, z(t, \omega) \in$ $A(u(t, \omega), t, \omega)$ for a.e. $(t, \omega), z \in V^{\prime}$

Thus $\hat{A}: \mathscr{V} \rightarrow \mathscr{P}\left(\mathscr{V}^{\prime}\right)$. Now let $F$ be the duality map from $U$ to $U^{\prime}$ which satisfies

$$
\langle F u, u\rangle=\|u\|_{U}^{r},\|F u\|=\|u\|^{r-1}, r \geq \max (2, p)
$$

Thus $r$ is at least 2 . We assume that $u_{0} \in L^{2}(\Omega ; H)$ and is $\mathscr{F}_{0}$ measurable and for each $n$ there exist $\left(u_{n}, z_{n}\right)$ a progressively measurable solution to the integral equation

$$
\begin{equation*}
u_{n}(t)-u_{0}+\frac{1}{n} \int_{0}^{t} F u_{n} d s+\int_{0}^{t} z_{n} d s=\int_{0}^{t} f d s+\int_{0}^{t} \Phi d W, z_{n} \in \hat{A}\left(u_{n}\right), f \in \mathscr{V} p^{\prime} \tag{79.4.44}
\end{equation*}
$$

in $\mathscr{U}_{\omega}{ }^{\prime}$. This would be the case if $\lambda I+\frac{1}{n} F+A$ were monotone for large enough $\lambda$. Such theorems are now well known and versions of them are in [108]. The message here is about going from the solution to the regularized problem to one which is missing the $F$ term for which $A$ might not be monotone. Therefore, we assume the existence of such a
progressively measurable solution $(u, z)$ to 79.4 .44. Is it possible that without the $\frac{1}{n} F$ it might never happen that $\lambda I+A$ is monotone and yet still have appropriate estimates for $A$ ? I don't have any examples right now. Thus it is not clear that this gives anything new.

The other consideration is a usable limit condition on Nemytskii operators, $\hat{A}$ described in the following lemma. Probably the earliest solution to this problem was given in [17]. These ideas were extended to set valued maps in [18] and to implicit evolution inclusions in [85]. Here the nonlinear operators will depend not just on $t$ but also on $\omega$ where $\omega \in \Omega$ for $(\Omega, \mathscr{F}, P)$ a probability space.

Lemma 79.4.3 Let A be as described above in 1-4. Then for $u \in \mathscr{V}, \hat{A}(u)$ is a closed and convex and bounded subset of $\mathscr{V}^{\prime}$.

Proof: It is clear that $\hat{A}(u)$ is convex. Indeed, if $z, w$ are in this set, and $\lambda \in[0,1]$, then $(\lambda z+(1-\lambda) w)(t, \omega) \in A(u(t, \omega), t, \omega)$ for $(t, \omega)$ off the union of the two exceptional sets corresponding to $z, w$. Now let $\left\{z_{n}\right\}$ be a sequence in $\hat{A}(u)$ which converges to $z$ in $\mathscr{V}^{\prime}$. Then pass to a subsequence which converges pointwise a.e. Let an exceptional set be the union of the exceptional sets for each $z_{n}$. Thus, off some set of measure zero $\Sigma$, $z_{n}(t, \omega) \in A(u(t, \omega), t, \omega)$ and we have $z_{n}(t, \omega) \rightarrow z(t, \omega)$ for $(t, \omega)$ off this exceptional set of measure zero. Now the fact that $A(u, t, \omega)$ is closed for $u \in V$ shows that $z(t, \omega) \in$ $A(u(t, \omega), t, \omega)$.

Definition 79.4.4 For $S$ a set in $[0, T] \times \Omega, S_{\omega}$ will denote $\{t:(t, \omega) \in S\}$.

### 79.4.2 A Limit Theorem

We will make use of the following fundamental measurable selection lemma. It is proved in [87]. We will use this lemma in the context where the measurable space is $(\Omega \times[0, T], \mathscr{P})$ where $\mathscr{P}$ is the $\sigma$ algebra of progressively measurable sets.

Lemma 79.4.5 Let $\Omega$ be a set and let $\mathscr{F}$ be a $\sigma$ algebra of subsets of $\Omega$. Let $U$ be a separable reflexive Banach space. Suppose there is a sequence $\left\{u_{j}(\omega)\right\}_{j=1}^{\infty}$ in $U$, where each $\omega \rightarrow u_{j}(\omega)$ is measurable and for each $\omega, \sup _{i}\left\|u_{i}(\omega)\right\|<\infty$. Then, there exists $u(\omega) \in U$ such that $\omega \rightarrow u(\omega)$ is measurable into $U$, and a subsequence $n(\omega)$, that depends on $\omega$, such that the weak limit

$$
\lim _{n(\omega) \rightarrow \infty} u_{n(\omega)}(\omega)=u(\omega)
$$

holds.
Next we derive some considerations of solutions to 79.4.44. From the Ito formula in 79.4.44,

$$
\begin{gather*}
\frac{1}{2}\left|u_{n}(t)\right|_{H}^{2}-\frac{1}{2}\left|u_{0}\right|_{H}^{2}+\frac{1}{n} \int_{0}^{t}\left\langle F u_{n}, u_{n}\right\rangle d s+\int_{0}^{t}\left\langle z_{n}, u_{n}\right\rangle d s \\
\quad-\frac{1}{2} \int_{0}^{t}\|\Phi\|_{\mathscr{L}_{2}}^{2} d s=\int_{0}^{t}\left\langle f, u_{n}\right\rangle d s+M_{n}(t) \tag{79.4.45}
\end{gather*}
$$

where $M_{n}(t)$ is a local martingale whose quadratic variation is

$$
\left[M_{n}\right](t) \leq C \int_{0}^{t}\|\Phi\|_{\mathscr{L}_{2}}^{2}\left|u_{n}\right|_{H}^{2} d s
$$

Then estimates give $\left\|\frac{1}{n}\left\langle F u_{n}, u_{n}\right\rangle\right\|_{\mathscr{U}^{\prime}}$ bounded as well as

$$
\begin{equation*}
\left\|u_{n}\right\|_{\mathscr{V}},\left\|z_{n}\right\|_{\mathscr{V}^{\prime}} \tag{79.4.46}
\end{equation*}
$$

One takes expectation of 79.4.45 using an appropriate localizing sequence of stopping times if necessary. It follows that there is a subsequence, still denoted with $n$ such that

$$
\begin{align*}
u_{n} & \rightarrow u \text { weakly in } \mathscr{V}  \tag{79.4.47}\\
z_{n} & \rightarrow z \text { weakly in } \mathscr{V}^{\prime} \\
\frac{1}{n} F u_{n} & \rightarrow 0 \text { strongly in } \mathscr{U}^{\prime}
\end{align*}
$$

The last convergence follows from the following argument.

$$
\begin{aligned}
& \int_{0}^{T} \int_{\Omega} \frac{1}{n}\left\langle F u_{n}, w\right\rangle d P d t \\
\leq & \int_{0}^{T} \int_{\Omega} \frac{1}{n^{1 / r^{\prime}}}\left\langle F u_{n}, u_{n}\right\rangle^{1 / r^{\prime}} \frac{1}{n^{1 / r}}\langle F w, w\rangle^{1 / r} d P d t \\
\leq & \left(\int_{0}^{T} \int_{\Omega} \frac{1}{n}\left\langle F u_{n}, u_{n}\right\rangle d P d t\right)^{1 / r^{\prime}}\left(\int_{0}^{T} \int_{\Omega} \frac{1}{n}\|w\|^{r} d P d t\right)^{1 / r} \\
\leq & C \frac{1}{n^{1 / r}}\|w\|_{\mathscr{U}}
\end{aligned}
$$

and so

$$
\left\|\frac{1}{n} F u_{n}\right\|_{\mathscr{U}^{\prime}} \leq \frac{C}{n^{1 / r}}
$$

Recall

$$
\begin{aligned}
& u_{n}(t)-u_{0}+\frac{1}{n} \int_{0}^{t} F u_{n} d s+\int_{0}^{t} z_{n} d s \\
= & \int_{0}^{t} f d s+\int_{0}^{t} \Phi d W, z_{n} \in \hat{A}\left(u_{n}\right)
\end{aligned}
$$

Then if $\phi \in \mathscr{U}$,

$$
\begin{gathered}
\left\langle\frac{1}{n} \int_{0}^{(\cdot)} F u_{n} d s+\int_{0}^{(\cdot)} z_{n} d s, \phi\right\rangle_{\mathscr{U}^{\prime}, \mathscr{U}} \\
=\int_{\Omega} \int_{0}^{T}\left\langle\left(\frac{1}{n} \int_{0}^{r} F u_{n} d s+\int_{0}^{r} z_{n} d s\right), \phi(r, \omega)\right\rangle_{U^{\prime}, U} d r d P
\end{gathered}
$$

Then the above equals

$$
\int_{\Omega} \int_{0}^{T} \frac{1}{n} \int_{0}^{r}\left\langle F u_{n}(s, \omega), \phi(r, \omega)\right\rangle d s d r d P+\int_{0}^{r}\left\langle z_{n}(s, \omega), \phi(r, \omega)\right\rangle d s d r d P
$$

$$
\begin{aligned}
= & \int_{\Omega} \int_{0}^{T} \int_{s}^{T}\left\langle\frac{1}{n} F u_{n}(s, \omega), \phi(r, \omega)\right\rangle d r d s d P \\
& +\int_{\Omega} \int_{0}^{T} \int_{s}^{T}\left\langle z_{n}(s, \omega), \phi(r, \omega)\right\rangle d r d s d P \\
= & \int_{\Omega} \int_{0}^{T}\left\langle\frac{1}{n} F u_{n}(s, \omega), \int_{s}^{T} \phi(r, \omega) d r\right\rangle d s d P \\
& +\int_{\Omega} \int_{0}^{T}\left\langle z_{n}(s, \omega), \int_{s}^{T} \phi(r, \omega) d r\right\rangle d s d P
\end{aligned}
$$

Now $(s, \omega) \rightarrow \int_{s}^{T} \phi(r, \omega) d r$ is also in $\mathscr{U}$ and so the above weak convergences and estimates yield that in the limit, this becomes

$$
\begin{aligned}
& \int_{\Omega} \int_{0}^{T}\left\langle z(s, \omega), \int_{s}^{T} \phi(r, \omega) d r\right\rangle_{U^{\prime}, U} d s d P \\
= & \int_{\Omega} \int_{0}^{T}\left\langle\int_{0}^{r} z d s, \phi(r, \omega)\right\rangle_{U^{\prime}, U} d r d P
\end{aligned}
$$

Thus $u_{n}$ must converge in $\mathscr{U}^{\prime}$ to

$$
u_{0}-\int_{0}^{(\cdot)} z d s+\int_{0}^{(\cdot)} f d s+\int_{0}^{(\cdot)} \Phi d W
$$

Therefore, when passing to a limit, one obtains from 79.4.44

$$
\begin{equation*}
u(\cdot)-u_{0}+\int_{0}^{(\cdot)} z d s=\int_{0}^{(\cdot)} f d s+\int_{0}^{(\cdot)} \Phi d W \text { in } \mathscr{U}_{\omega}^{\prime} \text { for a.e. } \omega \tag{79.4.48}
\end{equation*}
$$

All functions are continuous except the first so we define it so that the above holds pointwise in $t$. Thus, with 79.4.44, off a set of measure zero,

$$
\begin{align*}
u_{n}(t)-u_{0}+\frac{1}{n} \int_{0}^{t} F u_{n} d s+\int_{0}^{t} z_{n} d s & =\int_{0}^{t} f d s+\int_{0}^{t} \Phi d W \\
u(t)-u_{0}+\int_{0}^{t} z d s & =\int_{0}^{t} f d s+\int_{0}^{t} \Phi d W \tag{79.4.49}
\end{align*}
$$

Note that these are now equations which hold for each $t$ for $\omega$ off a set of measure zero. Then it follows that for a.e. $\omega$

$$
\begin{gather*}
\left\|u_{n}(t)+\int_{0}^{t} z_{n} d s-\left(u(t)+\int_{0}^{t} z d s\right)\right\|_{U^{\prime}} \\
=\left\|\frac{1}{n} \int_{0}^{t} F u_{n} d s\right\|_{U^{\prime}} \leq \int_{0}^{T} \frac{1}{n}\left\|F u_{n}\right\| d s \tag{79.4.50}
\end{gather*}
$$

Let $n_{k}>k$ be the first index such that if $l \geq n_{k}$, then

$$
\int_{\Omega} \int_{0}^{T} \frac{1}{l}\left\|F u_{l}\right\| d s d P<4^{-k}
$$

Then

$$
\begin{gathered}
P\left(\omega: \sup _{t \in[0, T]}\left\|u_{n_{k}}(t)+\int_{0}^{t} z_{n_{k}} d s-\left(u(t)+\int_{0}^{t} z d s\right)\right\|_{U^{\prime}}>2^{-k}\right) \\
\leq 2^{k} \int_{\Omega} \int_{0}^{T} \frac{1}{n_{k}}\left\|F u_{n_{k}}\right\| d s d P<2^{-k}
\end{gathered}
$$

Therefore there is a subsequence still denoted with $n$ such that

$$
P\left(\omega: \sup _{t \in[0, T]}\left\|u_{n}(t)+\int_{0}^{t} z_{n} d s-\left(u(t)+\int_{0}^{t} z d s\right)\right\|_{U^{\prime}}>2^{-n}\right) \leq 2^{-n}
$$

It follows that there is a set of measure zero $N$ such that if $\omega \notin N$, then $\omega$ is in only finitely many of the above sets. That is, for $\omega \notin N$,

$$
\sup _{t \in[0, T]}\left\|u_{n}(t, \omega)+\int_{0}^{t} z_{n}(s, \omega) d s-\left(u(t, \omega)+\int_{0}^{t} z(s, \omega) d s\right)\right\|_{U^{\prime}}<2^{-n}
$$

for all $n$ large enough.
Lemma 79.4.6 There is a set of measure zero $N$, an enlargement of the earlier set such that for $\omega \notin N$, and a suitable subsequence, still denoted with n, such that

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left(u_{n}(t, \omega)+\int_{0}^{t} z_{n}(s, \omega) d s\right)=u(t, \omega)+\int_{0}^{t} z(s, \omega) d s \text { in } U^{\prime} \tag{79.4.51}
\end{equation*}
$$

for each $t \in[0, T]$. In addition to this, for $\omega \notin N$,

$$
\begin{equation*}
\left\|u_{n}(t, \omega)-u_{0}(\omega)+\int_{0}^{t}\left(z_{n}(s, \omega)-f(s, \omega)\right) d s-\int_{0}^{t} \Phi d W\right\|_{U^{\prime}} \rightarrow 0 \tag{79.4.52}
\end{equation*}
$$

Proof: The formula 79.4 .51 follows from the above discussion. The second claim follows from the first and the equation satisfied by $u$ 79.4.49.

In the situation of 79.4.44, can we conclude that for the subsequence of Lemma 79.4.6,

$$
\begin{equation*}
\lim \sup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle_{\mathscr{U}^{\prime}, \mathscr{U}} \leq 0 ? \tag{79.4.53}
\end{equation*}
$$

Let $\left\{e_{k}\right\}$ be a complete orthonormal set in $H$, dense in $H$, with each vector in $U$. Thus

$$
\begin{equation*}
u_{n}(t, \omega)=\sum_{k}\left(u_{n}(t, \omega), e_{k}\right)_{H} e_{k},\left|u_{n}(t, \omega)\right|^{2}=\sum_{k}\left(u_{n}(t, \omega), e_{k}\right)^{2} \tag{79.4.54}
\end{equation*}
$$

The following claim is the key idea which will yield 79.4.53.
Claim: $\liminf _{n \rightarrow \infty}\left(u_{n}(t, \omega), e_{k}\right)^{2} \geq\left(u(t, \omega), e_{k}\right)^{2}$ for a.e. $(t, \omega)$.
Proof of claim: Let $B_{\varepsilon}$ be those $(t, \omega)$ such that

$$
\lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty}\left(u_{n}(t, \omega), e_{k}\right)^{2} \leq\left(u(t, \omega), e_{k}\right)^{2}-\varepsilon
$$

and consider $I(w) \equiv \int_{B_{\varepsilon}}\left(w(t, \omega), e_{k}\right)^{2} d(P \times m)$, the measure being product measure. Then $I$ is clearly convex and strongly lower semicontinuous on

$$
\mathscr{H} \equiv L^{2}(\Omega \times[0, T] ; H)
$$

To see that this is strongly lower semicontinuous, suppose $w_{n} \rightarrow w$ in $\mathscr{H}$ but

$$
\lim _{n \rightarrow \infty} \inf _{n} I\left(w_{n}\right)<I(w)
$$

Then take a subsequence, still denoted with $n$ such that the liminf equals lim. Then take a further subsequence, still denoted with $n$ such that $w_{n}(t, \omega) \rightarrow w(t, \omega)$ for a.e. $(t, \omega)$. Then by Fatou's lemma,

$$
I(w) \leq \lim \inf _{n \rightarrow \infty} I\left(w_{n}\right)<I(\omega)
$$

a contradiction. Thus $I$ is strongly lower semicontinuous as claimed. By convexity, it is also weakly lower semicontinuous. Hence by the weak convergence of $u_{n}$ to $u$ 79.4.47,

$$
\begin{gathered}
\lim \inf _{n \rightarrow \infty} \int_{B_{\varepsilon}}\left(u_{n}(t, \omega), e_{k}\right)^{2} d(P \times m) \geq \int_{B_{\varepsilon}}\left(u(t, \omega), e_{k}\right)^{2} d(P \times m) \\
\quad \geq \int_{B_{\varepsilon}} \lim _{n \rightarrow \infty}\left(u_{n}(t, \omega), e_{k}\right)^{2} d(P \times m)+\varepsilon(P \times m)\left(B_{\varepsilon}\right)
\end{gathered}
$$

Thus $(P \times m)\left(B_{\varepsilon}\right)=0$. Since this is so for each $\varepsilon>0$, it must be the case that the claimed inequality is satisfied off a set of measure zero. Let $\Sigma$ denote this progressively measurable set of product measure zero.

Let

$$
\begin{aligned}
& M_{\varepsilon} \equiv\left\{t:(t, \omega) \in \Sigma \text { for } \omega \text { in a set of measure larger than } \varepsilon, N_{\varepsilon}\right\} \\
& \qquad M_{\varepsilon} \equiv\left\{t: \int_{\Omega} \mathscr{X}_{\Sigma}(t, \omega) d P>\varepsilon\right\}
\end{aligned}
$$

If this set has positive measure, then

$$
m\left(M_{\varepsilon}\right) \varepsilon \leq \int_{M_{\varepsilon}} \int_{\Omega} \mathscr{X}_{\Sigma}(t, \omega) d P d t \leq(P \times m)(\Sigma)=0 .
$$

Thus each $M_{\varepsilon}$ has measure zero and so, taking the union of $M_{\varepsilon}$ for $\varepsilon$ a sequence converging to 0 , it follows that for $t \notin M$, defined as $\cup_{\varepsilon} M_{\varepsilon},(t, \omega)$ is in $\Sigma$ only for $\omega$ in a set of measure $\leq \varepsilon$ for each $\varepsilon$. Thus for $t \notin M,(t, \omega)$ is in $\Sigma$ only for $\omega$ in a set of measure zero. Letting $t \notin M$, it follows from 79.4.54 that for a.e. $\omega$

$$
\begin{aligned}
\lim _{n \rightarrow \infty}\left|u_{n}(t, \omega)\right|^{2} & =\lim _{n \rightarrow \infty} \inf _{k}\left(u_{n}(t, \omega), e_{k}\right)^{2} \\
& \geq \sum_{k} \lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty}\left(u_{n}(t, \omega), e_{k}\right)^{2} \\
& \geq \sum_{k}\left(u(t, \omega), e_{k}\right)^{2}=|u(t, \omega)|_{H}^{2}
\end{aligned}
$$

Therefore, from Fatou's lemma, for such $t$,

$$
\lim \inf _{n \rightarrow \infty} \int_{\Omega}\left|u_{n}(t, \omega)\right|^{2} d P \geq \int_{\Omega} \lim \inf _{n \rightarrow \infty}\left|u_{n}(t, \omega)\right|^{2} d P \geq \int_{\Omega}|u(t, \omega)|_{H}^{2} d P
$$

This has proved the following fundamental result.

Lemma 79.4.7 There exists a set $M \subseteq[0, T]$ having measure zero such that for $t \notin M$,

$$
\lim \inf _{n \rightarrow \infty} \int_{\Omega}\left|u_{n}(t, \omega)\right|^{2} d P \geq \int_{\Omega}|u(t, \omega)|_{H}^{2} d P
$$

We will always assume $T \notin M$ because otherwise, we could complete the argument with $\hat{T} \notin M$ arbitrarily close to $T$ and then draw the desired conclusions or settle for drawing the desired conclusions with $[0, \hat{T}]$ replacing $[0, T]$ where $\hat{T}$ is a fixed number as close to $T$ as desired. Note that the inequality is only strengthened by going to a subsequence.

Then from the Ito formula and 79.4.44,

$$
\begin{gather*}
\frac{1}{2}\left|u_{n}(T)\right|_{H}^{2}-\frac{1}{2}\left|u_{0}\right|_{H}^{2}+\frac{1}{n} \int_{0}^{T}\left\langle F u_{n}, u_{n}\right\rangle d s+\int_{0}^{T}\left\langle z_{n}, u_{n}\right\rangle d s \\
\quad-\frac{1}{2} \int_{0}^{T}\|\Phi\|_{\mathscr{L}_{2}}^{2} d s=\int_{0}^{T}\left\langle f, u_{n}\right\rangle d s+M_{n}(T) \tag{79.4.55}
\end{gather*}
$$

where $M_{n}(t)$ is a local martingale, $M(0)=0$. Therefore,

$$
\begin{gathered}
\int_{\Omega} \int_{0}^{T}\left\langle z_{n}, u_{n}\right\rangle d s d P \leq \\
\int_{\Omega} \int_{0}^{T}\left\langle f, u_{n}\right\rangle d s d P-\frac{1}{2} \int_{\Omega}\left(\left|u_{n}(T)\right|_{H}^{2}\right) d P+\frac{1}{2} \int_{\Omega}\left(\left|u_{0}\right|^{2}\right) d P \\
+\frac{1}{2} \int_{\Omega} \int_{0}^{T}\|\Phi\|_{\mathscr{L}_{2}}^{2} d s d P+\overbrace{\int_{\Omega} M_{n}(T) d P}^{=0}
\end{gathered}
$$

To make more precise, one would use a localizing sequence of stopping times for the local martingale, take expectations and then pass to a limit, but the end result will be as above. Then taking limsup $\sin _{n \rightarrow \infty}$ of both sides and using Lemma 79.4.7,

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \sup _{\Omega} \int_{0}^{T}\left\langle z_{n}, u_{n}\right\rangle d s d P \\
\leq & \int_{\Omega} \int_{0}^{T}\langle f, u\rangle d s d P-\frac{1}{2} \lim _{n \rightarrow \infty} \inf _{\Omega}\left|u_{n}(T)\right|_{H}^{2} d P \\
& +\frac{1}{2} \int_{\Omega}\left(\left|u_{0}\right|^{2}\right) d P+\frac{1}{2} \int_{\Omega} \int_{0}^{T}\|\Phi\|_{\mathscr{L}_{2}}^{2} d s d P \\
\leq & \int_{\Omega} \int_{0}^{T}\langle f, u\rangle d s d P+\frac{1}{2} \int_{\Omega}\left(\left|u_{0}\right|^{2}\right) d P \\
& +\frac{1}{2} \int_{\Omega} \int_{0}^{T}| | \Phi \|_{\mathscr{L}_{2}}^{2} d s d P-\frac{1}{2} \int_{\Omega}|u(T)|_{H}^{2} d P
\end{aligned}
$$

On the other hand, from 79.4.48 and the Ito formula,

$$
\begin{aligned}
\langle z, u\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}= & \int_{\Omega} \int_{0}^{T}\langle f, u\rangle d s d P+\frac{1}{2} \int_{\Omega}\left(\left|u_{0}\right|^{2}\right) d P \\
& +\frac{1}{2} \int_{\Omega} \int_{0}^{T}\|\Phi\|_{\mathscr{L}_{2}}^{2} d s d P-\frac{1}{2} \int_{\Omega}|u(T)|_{H}^{2} d P
\end{aligned}
$$

and so

$$
\lim \sup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \leq\langle z, u\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}
$$

Now it follows that

$$
\begin{equation*}
\limsup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \leq\langle z, u\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}-\langle z, u\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}=0 \tag{79.4.56}
\end{equation*}
$$

Thus in the situation of 79.4.44, we get 79.4.56. We state this as the following lemma.
Lemma 79.4.8 Suppose $u_{n}, z_{n}$ are progressively measurable, $z_{n} \in \hat{A}\left(u_{n}\right)$, and

$$
u_{n}(t)-u_{0}+\frac{1}{n} \int_{0}^{t} F u_{n} d s+\int_{0}^{t} z_{n} d s=\int_{0}^{t} f d s+\int_{0}^{t} \Phi d W
$$

Then there is a subsequence, still denoted with $n$ such that 79.4 .56 holds.
Now the following is the main limit lemma which is a statement about this subsequence satisfying 79.4.56. This lemma gives a useable limit condition for the Nemytskii operators.
Lemma 79.4.9 Suppose conditions 1-79.4.2 hold. Also, suppose $U$ is a separable Banach space dense in $V$, a reflexive separable Banach space and $V$ is dense in a Hilbert space $H$ identified with its dual space. Thus

$$
U \subseteq V \subseteq H=H^{\prime} \subseteq V^{\prime} \subseteq U^{\prime}
$$

Hypotheses: For $f \in \mathscr{V}^{\prime}$

$$
\begin{gather*}
u_{n}(t)-u_{0}+\frac{1}{n} \int_{0}^{t} F u_{n} d s+\int_{0}^{t} z_{n} d s=\int_{0}^{t} f d s+\int_{0}^{t} \Phi d W \text { in } U^{\prime}  \tag{**}\\
u_{n} \text { and } z_{n} \text { are } \mathscr{P} \text { measurable } \\
u_{n} \rightarrow u \text { weakly in } \mathscr{V}, z_{n} \rightarrow z \text { weakly in } \mathscr{U}^{\prime} \\
\lim \sup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \leq 0, \tag{*}
\end{gather*}
$$

Note that from the above discussion, * follows ** from taking a suitable subsequence. Assume also that for some set of measure zero $N$, if $\omega \notin N$,

$$
\begin{equation*}
\sup _{t \in[0, T]} \lambda\left|u_{n}(t, \omega)\right|_{H}^{2} \leq C(\omega), C(\cdot) \in L^{1}(\Omega) \tag{79.4.57}
\end{equation*}
$$

(Since we are assuming that $\lambda=0$ in 3, this condition is automatic, but if you did have 79.4.57 for $\lambda>0$, then the following argument will show how to use it.)

Conclusion: If the above conditions hold, then there exists a further subsequence, still denoted with $n$ such that for any $v \in \mathscr{V}$, there exists $z(v) \in \hat{A}(u)$ with

$$
\lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-v\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \geq\langle z(v), u-v\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}
$$

Also $z \in \hat{A}(u)$ and $u, z$ are progressively measurable and

$$
u(t)-u_{0}+\int_{0}^{t} z d s=\int_{0}^{t} f d s+\int_{0}^{t} \Phi d W \text { in } V^{\prime}
$$

for all $\omega$ off a set of measure zero.

Proof: In the following argument, $N$ will be a set of measure zero containing the one of Lemma 79.4.6 and the sequence will always be a subsequence of the subsequence of that lemma. Recall that from Lemma 79.4.6, that for $\omega \notin N$,

$$
\left\|\begin{array}{c}
u_{n}(t, \omega)-u_{0}(\omega)  \tag{79.4.58}\\
+\int_{0}^{t}\left(z_{n}(s, \omega)-f(s, \omega)\right) d s-\int_{0}^{t} \Phi d W
\end{array}\right\|_{U^{\prime}} \rightarrow 0
$$

for this sequence which does not depend on $\omega$ or $t$. From now on, this or a further subsequence will be meant.

From the hypothesis,

$$
\begin{equation*}
u_{n} \rightarrow u \text { weakly in } \mathscr{V}, z_{n} \rightarrow z \text { weakly in } \mathscr{V}^{\prime} \tag{79.4.59}
\end{equation*}
$$

Thus

$$
\begin{equation*}
u(t)-u_{0}+\int_{0}^{t} z d s=\int_{0}^{t} f d s+\int_{0}^{t} \Phi d W \text { for a.e. } \omega \tag{79.4.60}
\end{equation*}
$$

and so

$$
\begin{equation*}
\left\|u(t)-u_{0}+\int_{0}^{t} z d s-\left(\int_{0}^{t} f d s+\int_{0}^{t} \Phi d W\right)\right\|_{U^{\prime}}=0 \tag{79.4.61}
\end{equation*}
$$

Note that in these weak convergences 79.4.59, we can assume the $\sigma$ algebra is just $\mathscr{B}([0, T]) \times \mathscr{F}_{T}$ because the progressive measurability will be preserved in the limit due to the Pettis theorem and the progressive measurability of each $u_{n}, z_{n}$. However, we could also let the $\sigma$ algebra be $\mathscr{P}$ the progressively measurable sets just as well.

Claim: There is a set of measure zero $N$, including the one obtained so far such that for $\omega \notin N$

$$
\lim \inf _{n \rightarrow \infty}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle \geq 0
$$

for a.e. $t$. The exceptional set, denoted as $M_{\omega}$ includes those $t$ for which some $z_{n}(t, \omega) \notin$ $A\left(u_{n}(t, \omega), t, \omega\right)$.

Let $\omega \notin N$ and $t \notin M_{\omega}$. First take a subsequence such that liminf = lim. Then suppose that

$$
\lim \inf _{n \rightarrow \infty}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle<0
$$

Then from the estimates, one can obtain that for a suitable subsequence, $u_{n}(t, \omega) \rightarrow$ $\psi(t, \omega)$ weakly in $V$. Note, $n=n(t, \omega)$. Here is why: From the above inequality, there exists a subsequence $\left\{n_{k}\right\}$, which may depend on $t, \omega$, such that

$$
\begin{align*}
& \lim _{k \rightarrow \infty}\left\langle z_{n_{k}}(t, \omega), u_{n_{k}}(t, \omega)-u(t, \omega)\right\rangle  \tag{79.4.62}\\
= & \lim _{n \rightarrow \infty} \inf _{n}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle<0 . \tag{79.4.63}
\end{align*}
$$

Now, condition 3 implies that for all $k$ large enough,

$$
\begin{align*}
& b_{3}\left\|u_{n_{k}}(t, \omega)\right\|_{V}^{p}-b_{4}(t, \omega)-\lambda\left|u_{n_{k}}(t, \omega)\right|_{H}^{2} \\
< & \left\|z_{n_{k}}(t, \omega)\right\|_{V^{\prime}}\|u(t, \omega)\|_{V} \\
\leq & \left(b_{1}\left\|u_{n_{k}}(t, \omega)\right\|_{V}^{p-1}+b_{2}(t, \omega)\right)\|u(t, \omega)\|_{V} \tag{79.4.64}
\end{align*}
$$

it follows from 79.4.57 that $\left\|u_{n_{k}}(t, \omega)\right\|_{V}$ and consequently $\left\|z_{n_{k}}(t, \omega)\right\|_{V^{\prime}}$ are bounded. Thus, denoting this subsequence with $n$, there is a further subsequence for which $u_{n}(t, \omega) \rightarrow$ $\psi(t, \omega)$ weakly in $V$ which was what was claimed.

But also, it follows from Lemma 79.4.6 that for $\omega \notin N$,

$$
\left\|u_{n}(t, \omega)-u_{0}(\omega)+\int_{0}^{t}\left(z_{n}(s, \omega)-f(s, \omega)\right) d s-\int_{0}^{t} \Phi d W\right\|_{U^{\prime}} \rightarrow 0
$$

where $n$ doesn't depend on $(t, \omega)$.
By convexity, Lemma 79.4.6, and weak semicontinuity considerations, it must be the case that

$$
\begin{aligned}
& \left\|\psi(t, \omega)-u_{0}(\omega)+\int_{0}^{t}(z(s, \omega)-f(s, \omega)) d s-\int_{0}^{t} \Phi d W\right\|_{U^{\prime}} \\
\leq & \lim _{n \rightarrow \infty}\left\|+\int_{0}^{t}\left(z_{n}(s, \omega)-f(s, \omega)\right) d s-\int_{0}^{t} \Phi d W\right\|_{U^{\prime}}=0
\end{aligned}
$$

Here $n=n(t, \omega)$ is a subsequence. But of course, this requires $\psi(t, \omega)=u(t, \omega)$ in $U^{\prime}$ thanks to 79.4 .60 and so in fact, $u_{n}(t, \omega) \rightarrow u(t, \omega)$ weakly.

Now, 79.4.62 and the limit conditions for pseudomonotone operators imply that the liminf condition holds. There exists $z_{\infty} \in A(u(t, \omega), t, \omega)$ such that

$$
\begin{aligned}
& \lim \inf _{n \rightarrow \infty}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle \\
\geq & \left\langle z_{\infty}, u(t, \omega)-u(t, \omega)\right\rangle=0 \\
> & \lim _{k \rightarrow \infty}\left\langle z_{n_{k}}(t, \omega), u_{n_{k}}(t, \omega)-u(t, \omega)\right\rangle \\
= & \lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle
\end{aligned}
$$

which is a contradiction. This completes the proof of the claim.
It follows from this claim that for given $\omega$ off a set of measure zero and $t \notin M_{\omega}$,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \inf _{n \rightarrow}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle \geq 0 \tag{79.4.65}
\end{equation*}
$$

Also, it is assumed that

$$
\limsup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \leq 0
$$

This continues holding for subsequences. From the estimates,

$$
\begin{aligned}
& \int_{\Omega} \int_{0}^{T}\left(b_{3}\left\|u_{n}(t, \omega)\right\|_{V}^{p}-b_{4}(t, \omega)-\lambda\left|u_{n}(t, \omega)\right|_{H}^{2}\right) d t d P \\
\leq & \int_{\Omega} \int_{0}^{T}\|u(t, \omega)\|_{V}\left(\left\|u_{n}(t, \omega)\right\|_{V}^{p-1} b_{1}+b_{2}\right) d t d P
\end{aligned}
$$

so it is routine to get $\left\|u_{n}\right\|_{\mathscr{V}}$ is bounded. This follows from the assumptions, in particular 79.4.57.

Now, the coercivity condition 3 shows that if $y \in \mathscr{V}$, then

$$
\begin{aligned}
\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-y(t, \omega)\right\rangle \geq & b_{3}\left\|u_{n}(t, \omega)\right\|_{V}^{p}-b_{4}(t, \omega)-\lambda\left|u_{n}(t, \omega)\right|_{H}^{2} \\
& -\left(b_{1}\left\|u_{n}(t, \omega)\right\|^{p-1}+b_{2}(t, \omega)\right)\|y(t, \omega)\|_{V}
\end{aligned}
$$

Using $p-1=\frac{p}{p^{\prime}}$, where $\frac{1}{p}+\frac{1}{p^{\prime}}=1$, the right-hand side of this inequality equals

$$
\begin{aligned}
& b_{3}\left\|u_{n}(t, \omega)\right\|_{V}^{p}-b_{4}(t, \omega)-b_{1}\left\|u_{n}(t, \omega)\right\|^{p / p^{\prime}}\|y(t, \omega)\|_{V} \\
& -b_{2}(t, \omega)\|y(t, \omega)\|_{V}-\lambda\left|u_{n}(t, \omega)\right|_{H}^{2}
\end{aligned}
$$

the last term being bounded by a function in $L^{1}([0, T] \times \Omega)$ by assumption. Thus there exists $c_{y}(\cdot, \cdot) \in L^{1}([0, T] \times \Omega)$ and a positive constant $C$ such that

$$
\begin{equation*}
\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-y(t, \omega)\right\rangle \geq-c_{y}(t, \omega)-C\|y(t, \omega)\|_{V}^{p} . \tag{79.4.66}
\end{equation*}
$$

Letting $y=u$, we use Fatou's lemma to write

$$
\begin{gathered}
{\lim \inf _{n \rightarrow \infty}}^{\int_{\Omega}} \int_{0}^{T}\left(\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle+c_{u}(t, \omega)+C\|u(t, \omega)\|_{V}^{p}\right) d t d P \geq \\
\int_{\Omega} \int_{0}^{T} \lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle+\left(c_{u}(t, \omega)+C\|u(t, \omega)\|_{V}^{p}\right) d t d P \\
\geq \int_{\Omega} \int_{0}^{T}\left(c_{u}(t, \omega)+C\|u(t, \omega)\|_{V}^{p}\right) d t d P
\end{gathered}
$$

Here, we added the term $c_{u}(t, \omega)+C\|u(t, \omega)\|_{V}^{p}$ to make the integrand nonnegative in order to apply Fatou's lemma. Thus,

$$
\lim \inf _{n \rightarrow \infty} \int_{\Omega} \int_{0}^{T}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle d t d P \geq 0
$$

Consequently, using the claim in the last inequality,

$$
\begin{aligned}
0 & \geq \lim \sup _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \\
& \geq \lim _{n \rightarrow \infty} \inf _{\Omega} \int_{0}^{T}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle d t d P \\
& \geq \int_{\Omega} \int_{0}^{T} \lim \inf _{n \rightarrow \infty}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle d t d P \geq 0
\end{aligned}
$$

hence, we find that

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}=0 \tag{79.4.67}
\end{equation*}
$$

We need to show that if $y$ is given in $\mathscr{V}$ then

$$
\lim \inf _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-y\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \geq\langle z(y), u-y\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}, \quad z(y) \in \hat{A} u
$$

Suppose to the contrary that there exists $y$ such that

$$
\begin{equation*}
\eta=\lim \inf _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-y\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}<\langle\tilde{z}, u-y\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \tag{79.4.68}
\end{equation*}
$$

for all $\tilde{z} \in \hat{A} u$. Take a subsequence, denoted still with subscript $n$ such that

$$
\eta=\lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-y\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}
$$

Note that this subsequence does not depend on $(t, \omega)$. Thus

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-y\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}<\langle\tilde{z}, u-y\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \text { for all } \tilde{z} \in \hat{A} u \tag{79.4.69}
\end{equation*}
$$

We will obtain a contradiction to this. In what follows, we continue to use the subsequence just described which satisfies the above inequality 79.4.69.

The estimate 79.4.66 implies,

$$
\begin{equation*}
0 \leq\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle^{-} \leq c(t, \omega)+C\|u(t, \omega)\|_{V}^{p} \tag{79.4.70}
\end{equation*}
$$

where $c$ is a function in $L^{1}([0, T] \times \Omega)$. Thanks to 79.4.65,

$$
\lim _{n \rightarrow \infty}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle \geq 0, \text { a.e. }
$$

and, therefore, the following pointwise limit exists,

$$
\lim _{n \rightarrow \infty}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle^{-}=0, \text { a.e. }
$$

and so we may apply the dominated convergence theorem using 79.4.70 and conclude

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \int_{\Omega} \int_{0}^{T}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle^{-} d t d P \\
= & \int_{\Omega} \int_{0}^{T} \lim _{n \rightarrow \infty}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle^{-} d t d P=0
\end{aligned}
$$

Now, it follows from 79.4.67 and the above equation, that

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \int_{\Omega} \int_{0}^{T}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle^{+} d t d P \\
= & \lim _{n \rightarrow \infty} \int_{\Omega} \int_{0}^{T}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle \\
& +\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle^{-} d t d P \\
= & \lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-u\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}=0 .
\end{aligned}
$$

Therefore, both

$$
\int_{\Omega} \int_{0}^{T}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle^{+} d t d P
$$

and

$$
\int_{\Omega} \int_{0}^{T}\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle^{-} d t d P
$$

converge to 0 , thus,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \int_{\Omega} \int_{0}^{T}\left|\left\langle z_{n}(t, \omega), u_{n}(t, \omega)-u(t, \omega)\right\rangle\right| d t d P=0 \tag{79.4.71}
\end{equation*}
$$

From the above, it follows that there exists a further subsequence $\left\{n_{k}\right\}$ not depending on $t, \omega$ such that

$$
\begin{equation*}
\left|\left\langle z_{n_{k}}(t, \omega), u_{n_{k}}(t, \omega)-u(t, \omega)\right\rangle\right| \rightarrow 0 \quad \text { a.e. }(t, \omega) . \tag{79.4.72}
\end{equation*}
$$

By the pseudomonotone limit condition for $A$ there exists $w_{t, \omega} \in A(u(t, \omega), t, \omega)$ such that for a.e. $(t, \omega)$

$$
\begin{aligned}
\alpha(t, \omega) & \equiv \lim \inf _{k \rightarrow \infty}\left\langle z_{n_{k}}(t, \omega), u_{n_{k}}(t, \omega)-y(t, \omega)\right\rangle \\
& =\lim \inf _{k \rightarrow \infty}\left\langle z_{n_{k}}(t, \omega), u(t, \omega)-y(t, \omega)\right\rangle \geq\left\langle w_{t, \omega}, u(t, \omega)-y(t, \omega)\right\rangle
\end{aligned}
$$

Note that $u$ is progressively measurable and if $A(\cdot, t, \omega)$ were single valued, this would give a contradiction at this point. We continue with the case where $A$ is set valued. This case will make use of the measurable selection in Lemma 79.4.5.

On the exceptional set, let $\alpha(t, \omega) \equiv \infty$, and consider the set

$$
F(t, \omega) \equiv\{w \in A(u(t, \omega), t, \omega):\langle w, u(t, \omega)-y(t, \omega)\rangle \leq \alpha(t, \omega)\}
$$

which then satisfies $F(t, \omega) \neq \emptyset$. Now $F(t, \omega)$ is closed and convex in $V^{\prime}$.
We will let $\Sigma$ be a progressively measurable set of measure zero which includes

$$
N \times[0, T] \cup\left\{(t, \omega): \omega \notin N^{C}, t \in M_{\omega}\right\}
$$

Claim $*:(t, x) \rightarrow F(t, \omega)$ has a $\mathscr{P}$ measurable selection off a set of measure zero.
Proof of claim: Letting $B(0, C(t, \omega))$ contain $A(u(t, \omega), t, \omega)$, we can assume $(t, \omega) \rightarrow$ $C(t, \omega)$ is $\mathscr{P}$ measurable by using the estimates and the measurability of $u$. For $\gamma \in \mathbb{N}$, let $S_{\gamma} \equiv\{(t, \omega): C(t, \omega)<\gamma\}$. If it is shown that $F$ has a measurable selection on $S_{\gamma}$, then it follows that it has a measurable selection. Thus in what follows, assume that $(t, \omega) \in S_{\gamma}$.

Define for $(t, \omega) \in S_{\gamma}$

$$
G(t, \omega) \equiv\left\{w:\langle w, u(t, \omega)-y(t, \omega)\rangle<\alpha(t, \omega)+\frac{1}{n},(t, \omega) \in \Sigma^{C} \cap S_{\gamma}\right\} \cap B(0, \gamma)
$$

Thus, it was shown above that this $G(t, \omega) \neq \emptyset$ at least for large enough $\gamma$ that $S_{\gamma} \neq \emptyset$. For $U$ open, $G^{-}(U)$ is defined by

$$
G^{-}(U) \equiv\left\{\begin{array}{c}
(t, \omega) \in S_{\gamma}: \text { for some } w \in U \cap B(0, \gamma)  \tag{*}\\
\langle w, u(t, \omega)-y(t, \omega)\rangle<\alpha(t, \omega)+\frac{1}{n}
\end{array}\right\}
$$

Let $\left\{w_{j}\right\}$ be a dense subset of $U \cap B(0, \gamma)$. This is possible because $V^{\prime}$ is separable. The expression in $*$ equals

$$
\cup_{k=1}^{\infty}\left\{(t, \omega) \in S_{\gamma}:\left\langle w_{k}, u(t, \omega)-y(t, \omega)\right\rangle<\alpha(t, \omega)+\frac{1}{n}\right\}
$$

which is $\mathscr{P}$ measurable. Thus $G$ is a measurable multifunction.
Since $(t, \omega) \rightarrow G(t, \omega)$ is measurable, there is a sequence $\left\{w_{n}(t, \omega)\right\}$ of measurable functions such that $\overline{\cup_{n=1}^{\infty} w_{n}(t, \omega)}$ equals

$$
\overline{G(t, \omega)}=\left\{w:\langle w, u(t, \omega)-y(t, \omega)\rangle \leq \alpha(t, \omega)+\frac{1}{n},(t, \omega) \notin \Sigma\right\} \cap \overline{B(0, \gamma)}
$$

As shown above, there exists $w_{t, \omega}$ in $A(u(t, \omega), t, \omega)$ as well as $G(t, \omega)$. Thus there is a sequence of $w_{r}(t, \omega)$ converging to $w_{t, \omega}$. Of course $r$ will need to depend on $t, \omega$. Since $(t, \omega) \rightarrow A(u(t, \omega), t, \omega)$ is a $\mathscr{P}$ measurable multifunction, it has a countable subset of $\mathscr{P}$ measurable functions $\left\{z_{k}(t, \omega)\right\}$ which is dense in $A(u(t, \omega), t, \omega)$. Let $U_{k}$ be defined as

$$
U_{k}(t, \omega) \equiv \cup_{m} B\left(z_{m}(t, \omega), \frac{1}{k}\right) \subseteq A(u(t, \omega), t, \omega)+B\left(0, \frac{2}{k}\right)
$$

Now define $A_{1 k}=\left\{(t, \omega): w_{1}(t, \omega) \in U_{k}(t, \omega)\right\}$. Then let

$$
A_{2 k}=\left\{(t, \omega) \notin A_{1 k}: w_{2}(t, \omega) \in U_{k}(t, \omega)\right\}
$$

and

$$
A_{3 k}=\left\{(t, \omega) \notin \cup_{i=1}^{2} A_{i k}: w_{3}(t, \omega) \in U_{k}(t, \omega)\right\}
$$

and so forth. Any $(t, \omega) \in S_{\gamma}$ must be contained in one of these $A_{r k}$ for some $r$ since if not so, there would not be a sequence $w_{r}(t, \omega)$ converging to $w_{t, \omega} \in A(u(t, \omega), t, \omega)$. These $A_{r k}$ partition $S_{\gamma} \backslash \Sigma$ and each is measurable since the $\left\{z_{k}(t, \omega)\right\}$ are measurable. Let

$$
\hat{w}_{k}(t, \omega) \equiv \sum_{r=1}^{\infty} \mathscr{X}_{A_{r k}}(t, \omega) w_{r}(t, \omega)
$$

Thus $\hat{w}_{k}(t, \omega)$ is in $U_{k}(t, \omega)$ for all $(t, \omega) \in S_{\gamma}$ and equals exactly one of the $w_{m}(t, \omega) \in$ $\overline{G(t, \omega)}$.

Also, by construction, the $\hat{w}_{k}(\cdot, \cdot)$ are bounded in $L^{\infty}\left(S_{\gamma} ; V^{\prime}\right)$. Therefore, there is a subsequence of these, still called $\hat{w}_{k}$ which converges weakly to a function $w$ in $L^{2}\left(S_{\gamma} ; V^{\prime}\right)$. Thus $w$ is a weak limit point of $\operatorname{co}\left(\cup_{j=k}^{\infty} \hat{w}_{j}\right)$ for each $k$. Therefore, in the open ball $B\left(w, \frac{1}{k}\right) \subseteq L^{2}\left(S_{\gamma} ; V^{\prime}\right)$ with respect to the strong topology, there is a convex combination $\sum_{j=k}^{\infty} c_{j k} \hat{w}_{j}$ (the $c_{j k}$ add to 1 and only finitely many are nonzero) which converges strongly in $L^{2}\left(S_{\gamma} ; V^{\prime}\right)$. Since $\overline{G(t, \omega)}$ is convex and closed, this convex combination is in $\overline{G(t, \omega)}$. Off a set of $\mathscr{P}$ measure zero, we can assume this convergence of $\sum_{j=k}^{\infty} c_{j k} \hat{w}_{j}$ as $k \rightarrow \infty$ happens pointwise a.e. for a suitable subsequence. However,

$$
\sum_{j=k}^{\infty} c_{j k} \hat{w}_{j}(t, \omega) \in U_{k}(t, \omega) \subseteq A(u(t, \omega), t, \omega)+B\left(0, \frac{2}{k}\right)
$$

Thus $w(t, \omega) \in A(u(t, \omega), t, \omega)$ a.e. $(t, \omega)$ because $A(u(t, \omega), t, \omega)$ is a closed set. Since $w$ is the pointwise limit of measurable functions off a set of measure zero, it can be assumed measurable and for a.e. $(t, \omega), w(t, \omega) \in A(u(t, \omega), t, \omega) \cap \overline{G(t, \omega)}$. Now denote this measurable function $w_{n}$. Then

$$
w_{n}(t, \omega) \in A(u(t, \omega), t, \omega),\left\langle w_{n}(t, \omega), u(t, \omega)-y(t, \omega)\right\rangle \leq \alpha(t, \omega)+\frac{1}{n} \text { a.e. }(t, \omega)
$$

These $w_{n}(t, \omega)$ are bounded for each $(t, \omega)$ off a set of measure zero and so by Lemma 79.4.5, there is a $\mathscr{P}$ measurable function $(t, \omega) \rightarrow \hat{z}(t, \omega)$ and a subsequence $w_{n(t, \omega)}(t, \omega) \rightarrow$ $\hat{z}(t, \omega)$ weakly as $n(t, \omega) \rightarrow \infty$. Now $A(u(t, \omega), t, \omega)$ is closed and convex, and $w_{n(t, \omega)}(t, \omega)$ is in $A(u(t, \omega), t, \omega)$, and so $\hat{z}(t, \omega) \in A(u(t, \omega), t, \omega)$ and

$$
\begin{equation*}
\langle\hat{z}(t, \omega), u(t, \omega)-y(t, \omega)\rangle \leq \alpha(t, \omega)=\lim \inf _{k \rightarrow \infty}\left\langle z_{n_{k}}(t, \omega), u_{n_{k}}(t, \omega)-y(t, \omega)\right\rangle \tag{**}
\end{equation*}
$$

Therefore, $t \rightarrow F(t, \omega)$ has a measurable selection on $S_{\gamma}$ excluding a set of measure zero, namely $\hat{z}(t, \omega)$ which will be called $\hat{z}_{\gamma}(t, \omega)$ in what follows.

Then $F(t, \omega)$ has a measurable selection on $[0, T] \times \Omega$ other than a set of measure zero. To see this, enlarge $\Sigma$ to include the exceptional sets of measure zero in the above argument for each $\gamma$. Then partition $[0, T] \times \Omega \backslash \Sigma$ as follows. For $\gamma=1,2, \cdots$, consider $S_{\gamma} \backslash S_{\gamma-1}, \gamma=1,2, \cdots$ for $S_{0}$ defined as $\emptyset$. Then letting $\hat{z}_{\gamma}$ be the selection for $(t, \omega) \in S_{\gamma}$, let $\tilde{z}(t, \omega)=\sum_{\gamma=1}^{\infty} \hat{z}_{\gamma}(t, \omega) \mathscr{X}_{S_{\gamma} \backslash S_{\gamma-1}}(t, \omega)$. The estimates imply $\tilde{z} \in \mathscr{V}^{\prime}$ and so $\tilde{z} \in \hat{A}(u)$.

From the estimates, there exists $h \in L^{1}([0, T] \times \Omega)$ such that

$$
\langle\tilde{z}(t, \omega), u(t, \omega)-y(t, \omega)\rangle \geq-|h(t, \omega)|
$$

Thus, from the above inequality,

$$
\begin{aligned}
& \|h\|_{L^{1}}+\langle\tilde{z}, u-y\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \\
& \leq \int_{\Omega} \int_{0}^{T} \lim _{k \rightarrow \infty} \inf _{k \rightarrow \infty}\left\langle z_{n_{k}}(t, \omega), u_{n_{k}}(t, \omega)-y(t, \omega)\right\rangle+|h(t, \omega)| d t d P \\
& \leq \lim _{k \rightarrow \infty} \inf \left\langle z_{n_{k}}, u_{n_{k}}-y\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}+\|h\|_{L^{1}} \\
& \quad=\lim _{n \rightarrow \infty}\left\langle z_{n}, u_{n}-y\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}+\|h\|_{L^{1}}
\end{aligned}
$$

which contradicts 79.4.69.
Finally, why is $z \in \hat{A}(u)$ ? We have shown that $(t, \omega) \rightarrow F(t, \omega)$ has a measurable selection. Thus, there exists for any $y \in \mathscr{V}, \tilde{z}(y) \in \hat{A}(u)$ such that for a.e. $(t, \omega)$,

$$
\langle\tilde{z}(y)(t, \omega), u(t, \omega)-y(t, \omega)\rangle \leq \lim \inf _{k \rightarrow \infty}\left\langle z_{n_{k}}(t, \omega), u(t, \omega)-y(t, \omega)\right\rangle
$$

Using the same trick involving the estimates and Fatou's lemma, there exists $\tilde{z}$ depending on $y$ such that $\tilde{z}(y) \in \hat{A}(u)$ and

$$
\langle\tilde{z}(y), u-y\rangle_{\mathscr{V}^{\prime}, \mathscr{V}} \leq \lim _{k \rightarrow \infty} \inf _{k \rightarrow \infty}\left\langle z_{n_{k}}, u-y\right\rangle_{\mathscr{V}^{\prime}, \mathscr{V}}=\langle z, u-y\rangle
$$

It follows that since $y$ is arbitrary, it must be the case that $z \in \hat{A}(u)$ thanks to separation theorems and the fact that $\hat{A}(u)$ is convex and closed.

An examination of the above proof yields the following corollary.
Corollary 79.4.10 Replace $\frac{1}{n} F u_{n}$ with $e_{n}$ where $e_{n}$ is progressively measurable and

$$
\left\|e_{n}\right\|_{\mathscr{U}^{\prime}} \rightarrow 0,\left\|e_{n}(\cdot, \omega)\right\|_{\mathscr{U}_{\omega}^{\prime}} \rightarrow 0
$$

where the second convergence happens for each $\omega$ off a set of measure zero and keep the other assumptions the same. Then the same conclusion is obtained.

### 79.4.3 The Main Theorem

We first consider an easier problem and then obtain the result by taking limits. Let $U$ be a Banach space that is compactly embedded and dense in $V$. Also, let $F: U \rightarrow U^{\prime}$ be the duality map

$$
\langle F u, u\rangle=\|u\|_{U}^{r}, \quad\|F u\|_{U^{\prime}}=\|u\|_{U}^{r-1}
$$

where $r>\max (2, p)$. Such a Banach space always exists and is important in probability theory where $(i, U, V)$ is an abstract Wiener space but you can also get it in most applications to partial differential equations from Sobolev embedding theorems. Then the above has proved the following main theorem.

Theorem 79.4.11 Suppose 1-79.4.2 with $\lambda=0$, and suppose $f$ is progressively measurable and is in $L^{p^{\prime}}\left(\Omega ; L^{p^{\prime}}\left([0, T] ; V^{\prime}\right)\right)$. Also let $F$ be the duality map for $r \geq \max (2, p)$ which maps $U$ to $U^{\prime}$ for $U$ a separable Banach space contained and dense in $V$. Let

$$
\Phi \in L^{2}\left([0, T] \times \Omega, \mathscr{L}_{2}\left(Q^{1 / 2} H, H\right)\right) \cap L^{2}\left(\Omega ; L^{\infty}\left([0, T] ; \mathscr{L}_{2}\left(Q^{1 / 2} H, H\right)\right)\right)
$$

Also suppose that for each $n \in \mathbb{N}$, there is a progressively measurable $u_{n}, z_{n}$ such that

$$
u_{n}(t)-u_{0}+\frac{1}{n} \int_{0}^{t} F u_{n} d s+\int_{0}^{t} z_{n} d s=\int_{0}^{t} f d s+\int_{0}^{t} \Phi d W, z_{n} \in \hat{A}\left(u_{n}\right)
$$

Then there exists a solution $u, z$ to the inclusion

$$
u(t)-u_{0}+\int_{0}^{t} z d s=\int_{0}^{t} f d s+\int_{0}^{t} \Phi d W, z \in \hat{A}(u) \text { in } \mathscr{U}_{\omega}^{\prime}
$$

where both $u, z$ are progressively measurable. If $U$ is compact in $V$ and if for every $\varepsilon>0$ there exists $\mu_{\varepsilon} \geq 0$ such that $u \rightarrow \mu_{\varepsilon} u+\varepsilon F u+A u$ is strictly monotone as a map from $U$ to $U^{\prime}$, then the above hypothesis is satisfied.

The reason for assuming $\lambda=0$ is that it is hard to show 79.4.57 otherwise. As noted, it may be possible to reduce to this case using an exponential shift argument. Other than that, it appears you almost have to have $A+\mu I$ monotone for suitable $\mu$ which would end up yielding uniqueness. This is because in order to get the necessary estimates, you would need to have a Cauchy sequence for certain martingales in $M_{T}^{2}$ an appropriate space of continuous martingales. This will end up requiring an assumption of monotonicity. If $\Phi=0$ then of course there would be no problem. Right now, I don't have good examples.

The significant thing about this is that there may be no uniqueness of solutions to the evolution inclusion for fixed $\omega$ but there exists a progressively measurable solution. The case where $\Phi$ is replaced by $\sigma(u)$ is currently unsolved as far as I know unless one has uniqueness of the evolution inclusion for fixed $\omega$. It may well be possible to do something on this in case $\sigma$ is Lipschitz continuous. I don't know yet. If $\sigma$ is only continuous, I am pretty sure this will not be possible because there are examples where strong solutions do not exist. It should be possible to consider

$$
B u(t)-B u_{0}+\int_{0}^{t} z d s=\int_{0}^{t} f d s+B \int_{0}^{t} \Phi d W
$$

where $V \subseteq W, W^{\prime} \subseteq V^{\prime}$ and $B \in \mathscr{L}\left(W, W^{\prime}\right)$ being self adjoint and one to one by using a more technical Ito formula. However, this has not been done. A non probabilitic version is in [85]. I don't know how to resolve the problem in which $B$ is only nonnegative. Good results are certainly available in the non probabilistic setting without the stochastic integral. Some were presented earlier.

## Appendix A

## The Hausdorff Maximal Theorem

First is a review of the definition of a partially ordered set.
Definition A.0.1 A nonempty set $\mathscr{F}$ is called a partially ordered set if it has a partial order denoted by $\prec$. This means it satisfies the following. If $x \prec y$ and $y \prec z$, then $x \prec z$. Also $x \prec x$. It is like $\subseteq$ on the set of all subsets of a given set. It is not the case that given two elements of $\mathscr{F}$ that they are related. In other words, you cannot conclude that either $x \prec y$ or $y \prec x$. A chain, denoted by $\mathscr{C} \subseteq \mathscr{F}$ has the property that it is totally ordered meaning that if $x, y \in \mathscr{C}$, either $x \prec y$ or $y \prec x$. A maximal chain is a chain $\mathscr{C}$ which has the property that there is no strictly larger chain. In other words, if $x \in \mathscr{F} \backslash \cup \mathscr{C}$, then $\mathscr{C} \cup\{x\}$ is no longer a chain so $x$ fails to be related to something in $\mathscr{C}$.

Here is the Hausdorff maximal theorem. The proof is a proof by contradiction. We assume there is no maximal chain and then show this cannot happen. The axiom of choice is used in choosing the $x_{\mathscr{C}}$ right at the beginning of the argument.

Theorem A.0.2 Let $\mathscr{F}$ be a nonempty partially ordered set with order $\prec$. Then there exists a maximal chain.

Proof: Suppose no chain is maximal. Then, from the axiom of choice, for each chain $\mathscr{C}$ there exists $x_{\mathscr{C}} \in \mathscr{F} \backslash \cup \mathscr{C}$ such that $\mathscr{C} \cup\left\{x_{\mathscr{C}}\right\}$ is a chain. For $\mathscr{C}$ a chain, let $\theta \mathscr{C}$ denote $\mathscr{C} \cup\left\{x_{\mathscr{C}}\right\}$. Thus for $\mathscr{C}$ a chain, $\theta \mathscr{C}$ is a larger chain which has exactly one more element of $\mathscr{F}$. Since $\mathscr{F} \neq \emptyset$, pick $x_{0} \in \mathscr{F}$. Note that $\left\{x_{0}\right\}$ is a chain. Let $\mathscr{X}$ be the set of all chains $\mathscr{C}$ such that $x_{0} \in \cup \mathscr{C}$. Thus $\mathscr{X}$ contains $\left\{x_{0}\right\}$. Call two chains comparable if one is a subset of the other. Then summarizing,

1. $x_{0} \in \cup \mathscr{C}$ for all $\mathscr{C} \in \mathscr{X}$.
2. $\left\{x_{0}\right\} \in \mathscr{X}$
3. If $\mathscr{C} \in \mathscr{X}$ then $\theta \mathscr{C} \in \mathscr{X}$.
4. If $\mathscr{S} \subseteq \mathscr{X}$ is nonempty and every pair of chains in $\mathscr{S}$ is comparable, then $\cup \mathscr{S}$ is also a chain in $\mathscr{X}$.

A subset $\mathscr{Y}$ of $\mathscr{X}$ will be called a "tower" if $\mathscr{Y}$ satisfies 1.) - 4.). Let $\mathscr{Y}_{0}$ be the intersection of all towers. Then $\mathscr{Y}_{0}$ is also a tower, the smallest one. Then the next claim might seem to be so because if not, $\mathscr{Y}_{0}$ would not be the smallest tower.

Claim 1: If $\mathscr{C}_{0} \in \mathscr{Y}_{0}$ is comparable to every chain $\mathscr{C} \in \mathscr{Y}_{0}$, then if $\mathscr{C}_{0} \subsetneq \mathscr{C}$, it must be the case that $\theta \mathscr{C}_{0} \subseteq \mathscr{C}$. In other words, $x_{\mathscr{C}_{0}} \in \mathscr{C}$. The symbol $\subsetneq$ indicates proper subset.

This is done by considering a set $\mathscr{B} \subseteq \mathscr{Y}_{0}$ consisting of $\mathscr{D}$ which acts like $\mathscr{C}$ in the above and showing that it actually equals $\mathscr{\mathscr { Y }}_{0}$ because it is a tower.

Proof of Claim 1: Consider $\mathscr{B} \equiv\left\{\mathscr{D} \in \mathscr{Y}_{0}: \mathscr{D} \subseteq \mathscr{C}_{0}\right.$ or $\left.x_{\mathscr{C}_{0}} \in \cup \mathscr{D}\right\}$. Let $\mathscr{Y}_{1} \equiv \mathscr{Y}_{0} \cap \mathscr{B}$. I want to argue that $\mathscr{Y}_{1}$ is a tower. By definition all chains of $\mathscr{Y}_{1}$ contain $x_{0}$ in their unions. If $\mathscr{D} \in \mathscr{Y}_{1}$, is $\boldsymbol{\theta} \mathscr{D} \in \mathscr{Y}_{1}$ ?

If $\mathscr{S}$ is a nonempty subset of $\mathscr{Y}_{1}$ is $\mathscr{D} \equiv \cup \mathscr{S} \in \mathscr{Y}_{1} ? \mathscr{D}$ is in $\mathscr{Y}_{0}$ and so it is comparable to $\mathscr{C}_{0}$. If $\mathscr{D} \subseteq \mathscr{C}_{0}$, then it is in $\mathscr{B}$. Otherwise $\mathscr{D} \supseteq \mathscr{C}_{0}$ and in this case, why is $\mathscr{D}$ in $\mathscr{B}$ ? Why is $x_{\mathscr{C}_{0}} \in \cup \mathscr{D}$ ? The chains of $\mathscr{S}$ are in $\mathscr{B}$ so one of them, called $\tilde{C}$ must properly contain $\mathscr{C}_{0}$ and so $x_{\mathscr{C}_{0}} \in \cup \tilde{\mathscr{C}} \subseteq \cup \mathscr{D}$. Therefore, $\mathscr{D} \in \mathscr{B} \cap \mathscr{Y}_{0}=\mathscr{Y}_{1}$. Two cases remain to show that $\mathscr{Y}_{1}$ satisfies 3.).
case 1: $\mathscr{D} \supseteq \mathscr{C}_{0}$. Then $x_{\mathscr{C}_{0}} \in \cup \mathscr{D}$ and so $x_{\mathscr{C}_{0}} \in \cup \theta \mathscr{D}$ so $\theta \mathscr{D} \in \mathscr{Y}_{1}$.
case 2: $\mathscr{D} \subseteq \mathscr{C}_{0}$. Then $\theta \mathscr{D} \in \mathscr{T}_{0}$ by definition of $\mathscr{Y}_{0}$ and if $\theta \mathscr{D} \supseteq \mathscr{C}_{0}$, it follows that $\mathscr{D} \subseteq \mathscr{C}_{0} \varsubsetneqq \mathscr{D} \cup\left\{x_{\mathscr{D}}\right\}$. If $x \in \mathscr{C}_{0} \backslash \mathscr{D}$ then $x=x_{\mathscr{D}}$. This is where having $\theta \mathscr{C}$ contain exactly one more element of $\mathscr{F}$ is used. But then

$$
\mathscr{C}_{0} \varsubsetneqq \mathscr{D} \cup\left\{x_{\mathscr{D}}\right\} \subseteq \mathscr{C}_{0} \cup\left\{x_{\mathscr{D}}\right\}=\mathscr{C}_{0}
$$

the last equality coming because $x_{\mathscr{D}} \in \mathscr{C}_{0}$. The above is nonsense, and so $\mathscr{C}_{0}=\mathscr{D}$ so $x_{\mathscr{D}}=x_{\mathscr{C}_{0}} \in \cup \theta \mathscr{C}_{0}=\cup \theta \mathscr{D}$ and so $\theta \mathscr{D} \in \mathscr{Y}_{1}$. If $\theta \mathscr{D} \subseteq \mathscr{C}_{0}$ then right away $\theta \mathscr{D} \in \mathscr{B}$. Thus $\mathscr{B}=\mathscr{Y}_{0}$ because $\mathscr{Y}_{1}$ cannot be smaller than $\mathscr{Y}_{0}$. In particular, if $\mathscr{D} \supseteq \mathscr{C}_{0}$, then $x_{\mathscr{C}_{0}} \in \cup \mathscr{D}$ or in other words, $\theta \mathscr{C}_{0} \subseteq \mathscr{D}$.

Claim 2: Any two chains in $\mathscr{Y}_{0}$ are comparable so if $\mathscr{C} \subsetneq \mathscr{D}$, then $\theta \mathscr{C} \subseteq \mathscr{D}$.
Proof of Claim 2: Let $\mathscr{Y}_{1}$ consist of all chains of $\mathscr{Y}_{0}$ which are comparable to every chain of $\mathscr{Y}_{0}$. (Like $\mathscr{C}_{0}$ above.) I want to show that $\mathscr{Y}_{1}$ is a tower. Let $\mathscr{C} \in \mathscr{Y}_{1}$ and $\mathscr{D} \in \mathscr{Y}_{0}$. Since $\mathscr{C}$ is comparable to all chains in $\mathscr{Y}_{0}$, either $\mathscr{C} \subsetneq \mathscr{D}$ or $\mathscr{C} \supseteq \mathscr{D}$. I need to show that $\theta \mathscr{C}$ is comparable with $\mathscr{D}$. The second case is obvious so consider the first that $\mathscr{C} \subsetneq \mathscr{D}$. By Claim $1, \boldsymbol{\theta} \mathscr{C} \subseteq \mathscr{D}$. Since $\mathscr{D}$ is arbitrary, this shows that $\mathscr{Y}_{1}$ is a tower. Hence $\mathscr{Y}_{1}=\mathscr{Y}_{0}$ because $\mathscr{Y}_{0}$ is as small as possible. It follows that every two chains in $\mathscr{Y}_{0}$ are comparable and so if $\mathscr{C} \subsetneq \mathscr{D}$, then $\theta \mathscr{C} \subseteq \mathscr{D}$.

Since every pair of chains in $\mathscr{Y}_{0}$ are comparable and $\mathscr{Y}_{0}$ is a tower, it follows that $\cup \mathscr{Y}_{0} \in \mathscr{Y}_{0}$ so $\cup \mathscr{Y}_{0}$ is a chain. However, $\theta \cup \mathscr{Y}_{0}$ is a chain which properly contains $\cup \mathscr{Y}_{0}$ and since $\mathscr{Y}_{0}$ is a tower, $\theta \cup \mathscr{Y}_{0} \in \mathscr{Y}_{0}$. Thus $\cup\left(\theta \cup \mathscr{Y}_{0}\right) \supseteq \cup\left(\cup \mathscr{Y}_{0}\right) \supseteq \cup\left(\theta \cup \mathscr{Y}_{0}\right)$ which is a contradiction. Therefore, for some chain $\mathscr{C}$ it is impossible to obtain the $x_{C}$ described above and so, this $\mathscr{C}$ is a maximal chain.

If $X$ is a nonempty set,$\leq$ is an order on $X$ if

$$
x \leq x
$$

and if $x, y \in X$, then

$$
\text { either } x \leq y \text { or } y \leq x
$$

and

$$
\text { if } x \leq y \text { and } y \leq z \text { then } x \leq z
$$

$\leq$ is a well order and say that $(X, \leq)$ is a well-ordered set if every nonempty subset of $X$ has a smallest element. More precisely, if $S \neq \emptyset$ and $S \subseteq X$ then there exists an $x \in S$ such that $x \leq y$ for all $y \in S$. A familiar example of a well-ordered set is the natural numbers.

Lemma A.0.3 The Hausdorff maximal principle implies every nonempty set can be wellordered.

Proof: Let $X$ be a nonempty set and let $a \in X$. Then $\{a\}$ is a well-ordered subset of $X$. Let

$$
\mathscr{F}=\{S \subseteq X: \text { there exists a well order for } S\}
$$

Thus $\mathscr{F} \neq \emptyset$. For $S_{1}, S_{2} \in \mathscr{F}$, define $S_{1} \prec S_{2}$ if $S_{1} \subseteq S_{2}$ and there exists a well order for $S_{2}$, $\leq_{2}$ such that

$$
\left(S_{2}, \leq_{2}\right) \text { is well-ordered }
$$

and if

$$
y \in S_{2} \backslash S_{1} \text { then } x \leq_{2} y \text { for all } x \in S_{1},
$$

and if $\leq_{1}$ is the well order of $S_{1}$ then the two orders are consistent on $S_{1}$. Then observe that $\prec$ is a partial order on $\mathscr{F}$. By the Hausdorff maximal principle, let $\mathscr{C}$ be a maximal chain in $\mathscr{F}$ and let

$$
X_{\infty} \equiv \cup \mathscr{C}
$$

Define an order, $\leq$, on $X_{\infty}$ as follows. If $x, y$ are elements of $X_{\infty}$, pick $S \in \mathscr{C}$ such that $x, y$ are both in $S$. Then if $\leq_{S}$ is the order on $S$, let $x \leq y$ if and only if $x \leq_{S} y$. This definition is well defined because of the definition of the order, $\prec$. Now let $U$ be any nonempty subset of $X_{\infty}$. Then $S \cap U \neq \emptyset$ for some $S \in \mathscr{C}$. Because of the definition of $\leq$, if $y \in S_{2} \backslash S_{1}, S_{i} \in \mathscr{C}$, then $x \leq y$ for all $x \in S_{1}$. Thus, if $y \in X_{\infty} \backslash S$ then $x \leq y$ for all $x \in S$ and so the smallest element of $S \cap U$ exists and is the smallest element in $U$. Therefore $X_{\infty}$ is well-ordered. Now suppose there exists $z \in X \backslash X_{\infty}$. Define the following order, $\leq_{1}$, on $X_{\infty} \cup\{z\}$.

$$
\begin{gathered}
x \leq_{1} y \text { if and only if } x \leq y \text { whenever } x, y \in X_{\infty} \\
\qquad x \leq_{1} z \text { whenever } x \in X_{\infty} .
\end{gathered}
$$

Then let

$$
\widetilde{\mathscr{C}}=\left\{S \in \mathscr{C} \text { or } X_{\infty} \cup\{z\}\right\}
$$

Then $\tilde{\mathscr{C}}$ is a strictly larger chain than $\mathscr{C}$ contradicting maximality of $\mathscr{C}$. Thus $X \backslash X_{\infty}=\emptyset$ and this shows $X$ is well-ordered by $\leq$. This proves the lemma.

With these two lemmas the main result follows.
Theorem A.0. 4 The following are equivalent.

## The axiom of choice

## The Hausdorff maximal principle

The well-ordering principle.
Proof: It only remains to prove that the well-ordering principle implies the axiom of choice. Let $I$ be a nonempty set and let $X_{i}$ be a nonempty set for each $i \in I$. Let $X=\cup\left\{X_{i}\right.$ : $i \in I\}$ and well order $X$. Let $f(i)$ be the smallest element of $X_{i}$. Then

$$
f \in \prod_{i \in I} X_{i} .
$$

## A. 1 The Hamel Basis

A Hamel basis is nothing more than the correct generalization of the notion of a basis for a finite dimensional vector space to vector spaces which are possibly not of finite dimension.

Definition A.1.1 Let $X$ be a vector space. A Hamel basis is a subset of $X, \Lambda$ such that every vector of $X$ can be written as a finite linear combination of vectors of $\Lambda$ and the vectors of $\Lambda$ are linearly independent in the sense that if $\left\{x_{1}, \cdots, x_{n}\right\} \subseteq \Lambda$ and

$$
\sum_{k=1}^{n} c_{k} x_{k}=0
$$

then each $c_{k}=0$.
The main result is the following theorem.
Theorem A.1.2 Let $X$ be a nonzero vector space. Then it has a Hamel basis.
Proof: Let $x_{1} \in X$ and $x_{1} \neq 0$. Let $\mathscr{F}$ denote the collection of subsets of $X, \Lambda$ containing $x_{1}$ with the property that the vectors of $\Lambda$ are linearly independent as described in Definition A.1.1 partially ordered by set inclusion. By the Hausdorff maximal theorem, there exists a maximal chain, $\mathscr{C}$ Let $\Lambda=\cup \mathscr{C}$. Since $\mathscr{C}$ is a chain, it follows that if $\left\{x_{1}, \cdots, x_{n}\right\} \subseteq \mathscr{C}$ then there exists a single $\Lambda^{\prime} \in \mathbb{C}$ containing all these vectors. Therefore, if

$$
\sum_{k=1}^{n} c_{k} x_{k}=0
$$

it follows each $c_{k}=0$. Thus the vectors of $\Lambda$ are linearly independent. Is every vector of $X$ a finite linear combination of vectors of $\Lambda$ ?

Suppose not. Then there exists $z$ which is not equal to a finite linear combination of vectors of $\Lambda$. Consider $\Lambda \cup\{z\}$. If

$$
c z+\sum_{k=1}^{m} c_{k} x_{k}=0
$$

where the $x_{k}$ are vectors of $\Lambda$, then if $c \neq 0$ this contradicts the condition that $z$ is not a finite linear combination of vectors of $\Lambda$. Therefore, $c=0$ and now all the $c_{k}$ must equal zero because it was just shown $\Lambda$ is linearly independent. It follows $\mathscr{C} \cup\{\Lambda \cup\{z\}\}$ is a strictly larger chain than $\mathscr{C}$ and this is a contradiction. Therefore, $\Lambda$ is a Hamel basis as claimed. This proves the theorem.

## A. 2 Exercises

1. Zorn's lemma states that in a nonempty partially ordered set, if every chain has an upper bound, there exists a maximal element, $x$ in the partially ordered set. $x$ is maximal, means that if $x \prec y$, it follows $y=x$. Show Zorn's lemma is equivalent to the Hausdorff maximal theorem.
2. Show that if $Y, Y_{1}$ are two Hamel bases of $X$, then there exists a one to one and onto map from $Y$ to $Y_{1}$. Thus any two Hamel bases are of the same size.
3. $\uparrow$ Using the Baire category theorem of the chapter on Banach spaces show that any Hamel basis of a Banach space is either finite or uncountable.
4. $\uparrow$ Consider the vector space of all polynomials defined on $[0,1]$. Does there exist a norm, $\|\cdot\|$ defined on these polynomials such that with this norm, the vector space of polynomials becomes a Banach space (complete normed vector space)?

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[^0]:    ${ }^{1}$ Archimedes 287-212 B.C. found areas of curved regions by stuffing them with simple shapes which he knew the area of and taking a limit. He also made fundamental contributions to physics. The story is told about how he determined that a gold smith had cheated the king by giving him a crown which was not solid gold as had been claimed. He did this by finding the amount of water displaced by the crown and comparing with the amount of water it should have displaced if it had been solid gold.

[^1]:    ${ }^{2}$ This theorem is why Newton and Liebnitz are credited with inventing calculus. The integral had been around for thousands of years and the derivative was by their time well known. However the connection between these two ideas had not been fully made although Newton's predecessor, Isaac Barrow had made some progress in this direction.

[^2]:    ${ }^{3}$ Of course it was proved that if $f$ is continuous on a closed interval, $[a, b]$, then $f \in R([a, b])$ but this is a hard theorem using the difficult result about uniform continuity.

[^3]:    ${ }^{1}$ This is the plural form of basis. We could say basiss but it would involve an inordinate amount of hissing as in "The sixth shiek's sixth sheep is sick". This is the reason that bases is used instead of basiss.

[^4]:    ${ }^{3}$ A special case was first proved by Hamilton in 1853. The general case was announced by Cayley some time later and a proof was given by Frobenius in 1878.

[^5]:    ${ }^{1}$ To a mathematician, the statment: Whenever a pig is born with wings it can fly must be taken as true. We do not consider biological or aerodynamic considerations in such statements. There is no such thing as a winged pig and therefore, all winged pigs must be superb flyers since there can be no example of one which is not. On the other hand we would also consider the statement: Whenever a pig is born with wings it can't possibly fly, as equally true. The point is, you can say anything you want about the elements of the empty set and no one can gainsay your statement. Therefore, such statements are considered as true by default. You may say this is a very strange way of thinking about truth and ultimately this is because mathematics is not about truth. It is more about consistency and logic.

[^6]:    ${ }^{1}$ This is the plural form of basis. We could say basiss but it would involve an inordinate amount of hissing as in "The sixth shiek's sixth sheep is sick". This is the reason that bases is used instead of basiss.

[^7]:    ${ }^{1}$ Note the difference between this picture and the one usually drawn in calculus courses where the little rectangles are upright rather than on their sides. This illustrates a fundamental philosophical difference between the Riemann and the Lebesgue integrals. With the Riemann integral intervals are measured. With the Lebesgue integral, it is inverse images of intervals which are measured.

[^8]:    ${ }^{2}$ The negative part of the real number $x$ is defined to be $x^{-} \equiv \frac{1}{2}(|x|-x)$. Thus $|x|=x^{+}+x^{-}$and $x=x^{+}-x^{-}$.

[^9]:    ${ }^{3}$ Note that, since $g$ is allowed to have the value $\infty$, it is not known that $g \in L^{1}(\Omega)$.

[^10]:    ${ }^{1} 1$ Kings 17, 2 Kings 4, Mathew 14, and Mathew 15 all contain such descriptions. The stuff involved was either oil, bread, flour or fish. In mathematics such things have also been done with sets. In the book by Bruckner Bruckner and Thompson there is an interesting discussion of the Banach Tarski paradox which says it is possible to divide a ball in $\mathbb{R}^{3}$ into five disjoint pieces and assemble the pieces to form two disjoint balls of the same size as the first. The details can be found in: The Banach Tarski Paradox by Wagon, Cambridge University press. 1985. It is known that all such examples must involve the axiom of choice.

[^11]:    ${ }^{3}$ Set of all subsets of $Z$

[^12]:    ${ }^{4}$ In 1940 it was shown by Godel that the continuum hypothesis cannot be disproved. In 1963 it was shown by Cohen that the continuum hypothesis cannot be proved. These assertions are based on the axiom of choice and the Zermelo Frankel axioms of set theory. This topic is far outside the scope of this book and this is only a hopefully interesting historical observation.

[^13]:    ${ }^{1}$ Actually it is only a function of the first but this is not important in what follows.

[^14]:    ${ }^{1}$ Set of all subsets of $Z$

[^15]:    ${ }^{1}$ These spaces are named after Stefan Banach, 1892-1945. Banach spaces are the basic item of study in the subject of functional analysis and will be considered later in this book.

    There is a recent biography of Banach, R. Katuża, The Life of Stefan Banach, (A. Kostant and W. Woyczyński, translators and editors) Birkhauser, Boston (1996). More information on Banach can also be found in a recent short article written by Douglas Henderson who is in the department of chemistry and biochemistry at BYU.

    Banach was born in Austria, worked in Poland and died in the Ukraine but never moved. This is because borders kept changing. There is a rumor that he died in a German concentration camp which is apparently not true. It seems he died after the war of lung cancer.

    He was an interesting character. He hated taking examinations so much that he did not receive his undergraduate university degree. Nevertheless, he did become a professor of mathematics due to his important research. He and some friends would meet in a cafe called the Scottish cafe where they wrote on the marble table tops until Banach's wife supplied them with a notebook which became the "Scotish notebook" and was eventually published.

[^16]:    ${ }^{1}$ Actually, all this works in much more general settings than this.

[^17]:    ${ }^{1}$ As proved above, the assumption that $|\lambda|(\Omega)<\infty$ is redundant.

[^18]:    ${ }^{2}$ Recall this is automatic for a complex measure.

[^19]:    ${ }^{1}$ Note that this follows from the assumed separability of $X, Y$ because the graph is a subset of the separable space $X \times Y$

[^20]:    ${ }^{1}$ René Gateaux was one of the many young French men killed in world war I. This derivative is named after him, but it developed naturally from ideas used in the calculus of variations which were due to Euler and Lagrange back in the 1700 's.

[^21]:    ${ }^{2}$ In Hilbert space, the existence of this projection map is obvious and it is assumed that it exists here.

[^22]:    ${ }^{1}$ It is known that if the space is reflexive, then there is an equivalent norm which is strictly convex. However, in most examples, this strict convexity is obvious.

[^23]:    ${ }^{1}$ In this example, you only know that $f^{\prime}$ exists a.e.

[^24]:    ${ }^{1}$ The definition applies with no change to a general topological space in place of $\mathbb{R}^{n}$.

[^25]:    ${ }^{1}$ The definition applies with no change to a general topological space in place of $\mathbb{R}^{n}$.

[^26]:    ${ }^{2}$ If each $\psi_{U}$ were only continuous, one could conclude $f$ is continuous. Here the main interest is differentiable.

[^27]:    ${ }^{3}$ This means $V$ is the intersection of an open set with $\Gamma$. Equivalently, it means that $V$ is an open set in the traditional way regarding $\Gamma$ as a metric space with the metric it inherits from $\mathbb{R}^{m}$.

[^28]:    ${ }^{1}$ You could also let the norm be given by $\|u\|_{m, p} \equiv \sum_{|\alpha| \leq m}\left\|D^{\alpha} u\right\|_{p}$ or $\|u\|_{m, p} \equiv \max \left\{\left\|D^{\alpha} u\right\|_{p}:|\alpha| \leq m\right\}$ because all norms are equivalent on $\mathbb{R}^{p}$ where $p$ is the number of multi indices no larger than $m$. This is used whenever convenient.

[^29]:    ${ }^{2}$ This is never a problem in $\mathbb{R}^{n}$. In fact, every open covering has a locally finite subcovering in $\mathbb{R}^{n}$ or more generally in any metric space due to Stone's theorem. These are issues best left to you in case you are interested. I am usually interested in bounded sets, $U$, and for these, there is a finite covering.

[^30]:    ${ }^{3}$ This means that $\operatorname{spt}(u) \subseteq B \times(a, b)$ and $u \in C^{1}\left(\overline{V^{-}}\right)$.

[^31]:    ${ }^{1}$ Vector addition is continuous is the property which is used here.

[^32]:    ${ }^{1}$ Giancinto Morera 1856-1909. This theorem or one like it dates from around 1886

[^33]:    ${ }^{1}$ This implies $f$ has no zero on $\Omega$.

[^34]:    ${ }^{2}$ This is the terminology used in Rudin's book Real and Complex Analysis.

[^35]:    ${ }^{1}$ Actually, it is only necessary to assume one of the series converges and the other converges absolutely. This is known as Merten's theorem and may be read in the 1974 book by Apostol listed in the bibliography.

[^36]:    ${ }^{1}$ This says you can specify the first $m_{k}$ derivatives of the function at the point $z_{k}$.

[^37]:    ${ }^{2}$ It is not a field because you can't divide two analytic functions and get another one.

[^38]:    ${ }^{3}$ Wilhelm Blaschke, 1915

[^39]:    ${ }^{4}$ This is a fun thing to say: nonzero zeros.

[^40]:    ${ }^{1}$ A module is like a vector space except instead of a field of scalars, you have a ring of scalars.

[^41]:    ${ }^{2}$ I don't know why it is traditional to refer to this antiderivative as $-\zeta$ rather than $\zeta$ but I am following the convention. I think it is to minimize the number of minus signs in the next expression.

[^42]:    ${ }^{1}$ If it is only a seminorm, it satisfies the same conditions.

[^43]:    ${ }^{1}$ Note how the $\sigma$ algebra $\mathscr{F}_{s}$ are defined, as the intersection of completions of $\sigma$ algebras corresponding to $t$ strictly larger than $s$.

[^44]:    ${ }^{1}$ It is known that if the space is reflexive, then there is an equivalent norm which is strictly convex. However, in most examples, this strict convexity is obvious.

