# Analysis of Functions of Complex and Many Variables 

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March 8, 2024

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CONTENTS

## Preface

This book is on multi-variable real analysis with an introduction to complex analysis. It is for advanced undergraduate students and beginning graduate students. It is NOT SUITABLE as a first course in calculus. I assume the reader has had a course in calculus which does an honest job of presenting the Riemann integral of a function of one variable and knows the usual things about completeness of $\mathbb{R}$ and its algebraic properties although these things are reviewed. This usually implies having had a reasonably good course in analysis for functions of one variable, but my book Calculus of one and many variables would suffice. Also, it is expected that the reader knows what a field is and that $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are fields along with the usual elementary things found in an undergraduate linear algebra course, such as row operations and linear transformations, linear independence and vector spaces. I also assume the reader has knowledge of math induction and well ordering and the topics in a typical pre-calculus course. If not, read the first part of my calculus book. I also assume the reader is familiar with the pre-calculus topics involving $\mathbb{C}$, the complex numbers. If not, see my single variable advanced calculus book or my pre-calculus book http://www.centerofmath.org/textbooks/pre_calc/index.html (2012).

This book has a modern approach to real analysis and an introduction to complex analysis which includes those things which are of most interest to me. It also has an introduction to functional analysis if there is time, but I have tried to write it in a manner which would allow the omission of these topics if it were desired to only include real and complex analysis.

The main direction in the complex analysis part is toward classical nineteenth century analysis although it does include an introduction to methods of complex analysis in spectral theory of operators on a Banach space.

I am presenting some very interesting theorems more than once. I think it is good to see different ways of proving them. Often these theorems appear for the first time in the exercises. Sometimes it seems like complex analysis is unrelated to real analysis because of the lack of pathology. I am trying to merge the two and point out similarities as well as differences. I hope that by doing so, better understanding of both subjects will be acquired. For example, the introduction to the zeta function is heavily dependent on real Lebesgue theory of integration.

I am only assuming that the contours have finite total variation. I realize that one can get all of the main theorems in this subject by considering only piecewise $C^{1}$ curves which lie in an open set on which the function is analytic, but I think the extra effort is justified because it is less fussy and other books on this subject which contain far more complex analysis than the introduction discussed in this book also do it this way. A more elementary introduction which does feature piecewise $C^{1}$ curves is in my single variable advanced calculus book.

I have also tried to include all of the hard topology which is usually omitted but nevertheless used in analysis books. This includes things like the Brouwer fixed point theorem and the Jordan curve theorem. I think that the presentation of introductory mathematics should be opposite to what is encountered in religion. To me, the significance of mathematics is the extent to which one is not required to accept that which is not obvious on faith in the decrees of authority figures. It is hazardous to make such an attempt, especially in topics outside of one's expertise, but I think it is worth the effort to at least give it a try. This book is not a research monograph written for experts. I will use $\equiv$ to mean that something is being defined. Some problems have $\uparrow$ which means to do the above problem first.

CONTENTS

## Part I

## Preliminary Topics

## Chapter 1

## Basic Notions

The reader should be familiar with most of the topics in this chapter. However, it is often the case that set notation is not familiar and so a short discussion of this is included first. Complex numbers are then considered in somewhat more detail. This book is on analysis of functions of real variables or a complex variable. The basic arithmetic of complex numbers needs to be well understood at the outset.

### 1.1 Sets and Set Notation

A set is just a collection of things called elements. Often these are also referred to as points in calculus. For example $\{1,2,3,8\}$ would be a set consisting of the elements $1,2,3$, and 8. To indicate that 3 is an element of $\{1,2,3,8\}$, it is customary to write $3 \in\{1,2,3,8\}$. $9 \notin\{1,2,3,8\}$ means 9 is not an element of $\{1,2,3,8\}$. Sometimes a rule specifies a set. For example you could specify a set as all integers larger than 2 . This would be written as $S=\{x \in \mathbb{Z}: x>2\}$. This notation says: the set of all integers, $x$, such that $x>2$.

If $A$ and $B$ are sets with the property that every element of $A$ is an element of $B$, then $A$ is a subset of $B$. For example, $\{1,2,3,8\}$ is a subset of $\{1,2,3,4,5,8\}$, in symbols, $\{1,2,3,8\} \subseteq\{1,2,3,4,5,8\}$. It is sometimes said that " $A$ is contained in $B$ " or even " $B$ contains $A$ ". The same statement about the two sets may also be written as $\{1,2,3,4,5,8\} \supseteq$ $\{1,2,3,8\}$.

The union of two sets is the set consisting of everything which is an element of at least one of the sets, $A$ or $B$. As an example of the union of two sets $\{1,2,3,8\} \cup\{3,4,7,8\}=$ $\{1,2,3,4,7,8\}$ because these numbers are those which are in at least one of the two sets. In general

$$
A \cup B \equiv\{x: x \in A \text { or } x \in B\}
$$

Be sure you understand that something which is in both $A$ and $B$ is in the union. It is not an exclusive or.

The intersection of two sets, $A$ and $B$ consists of everything which is in both of the sets. Thus $\{1,2,3,8\} \cap\{3,4,7,8\}=\{3,8\}$ because 3 and 8 are those elements the two sets have in common. In general,

$$
A \cap B \equiv\{x: x \in A \text { and } x \in B\} .
$$

The symbol $[a, b]$ where $a$ and $b$ are real numbers, denotes the set of real numbers $x$, such that $a \leq x \leq b$ and $[a, b)$ denotes the set of real numbers such that $a \leq x<b$. $(a, b)$ consists of the set of real numbers $x$ such that $a<x<b$ and ( $a, b]$ indicates the set of numbers $x$ such that $a<x \leq b .[a, \infty)$ means the set of all numbers $x$ such that $x \geq a$ and $(-\infty, a]$ means the set of all real numbers which are less than or equal to $a$. These sorts of sets of real numbers are called intervals. The two points $a$ and $b$ are called endpoints of the interval. Other intervals such as $(-\infty, b)$ are defined by analogy to what was just explained. In general, the curved parenthesis indicates the end point it sits next to is not included while the square parenthesis indicates this end point is included. The reason that there will always be a curved parenthesis next to $\infty$ or $-\infty$ is that these are not real numbers. Therefore, they cannot be included in any set of real numbers.

A special set which needs to be given a name is the empty set also called the null set, denoted by $\emptyset$. Thus $\emptyset$ is defined as the set which has no elements in it. Mathematicians like to say the empty set is a subset of every set. The reason they say this is that if it were not so, there would have to exist a set $A$, such that $\emptyset$ has something in it which is not in $A$.

However, $\emptyset$ has nothing in it and so the least intellectual discomfort is achieved by saying $\emptyset \subseteq A$.

If $A$ and $B$ are two sets, $A \backslash B$ denotes the set of things which are in $A$ but not in $B$. Thus

$$
A \backslash B \equiv\{x \in A: x \notin B\}
$$

Set notation is used whenever convenient.
To illustrate the use of this notation relative to intervals consider three examples of inequalities. Their solutions will be written in the notation just described.

Example 1.1.1 Solve the inequality $2 x+4 \leq x-8$
$x \leq-12$ is the answer. This is written in terms of an interval as $(-\infty,-12]$.
Example 1.1.2 Solve the inequality $(x+1)(2 x-3) \geq 0$.
The solution is $x \leq-1$ or $x \geq \frac{3}{2}$. In terms of set notation this is denoted by $(-\infty,-1] \cup$ $\left[\frac{3}{2}, \infty\right)$.

Example 1.1.3 Solve the inequality $x(x+2) \geq-4$.
This is true for any value of $x$. It is written as $\mathbb{R}$ or $(-\infty, \infty)$.
Something is in the Cartesian product of a set whose elements are sets if it consists of a single thing taken from each set in the family. Thus $(1,2,3) \in\{1,4, .2\} \times\{1,2,7\} \times$ $\{4,3,7,9\}$ because it consists of exactly one element from each of the sets which are separated by $\times$. Also, this is the notation for the Cartesian product of finitely many sets. If $\mathscr{S}$ is a set whose elements are sets, $\prod_{A \in \mathscr{S}} A$ signifies the Cartesian product.

The Cartesian product is the set of choice functions, a choice function being a function which selects exactly one element of each set of $\mathscr{S}$. You may think the axiom of choice, stating that the Cartesian product of a nonempty family of nonempty sets is nonempty, is innocuous but there was a time when many mathematicians were ready to throw it out because it implies things which are very hard to believe, things which never happen without the axiom of choice.

### 1.2 The Schroder Bernstein Theorem

It is very important to be able to compare the size of sets in a rational way. The most useful theorem in this context is the Schroder Bernstein theorem which is the main result to be presented in this section. The Cartesian product is discussed above. The next definition reviews this and defines the concept of a function.

## Definition 1.2.1 Let $X$ and $Y$ be sets.

$$
X \times Y \equiv\{(x, y): x \in X \text { and } y \in Y\}
$$

A relation is defined to be a subset of $X \times Y$. A function $f$, also called a mapping, is a relation which has the property that if $(x, y)$ and $\left(x, y_{1}\right)$ are both elements of the $f$, then $y=y_{1}$. The domain of $f$ is defined as

$$
D(f) \equiv\{x:(x, y) \in f\}
$$

written as $f: D(f) \rightarrow Y$. Another notation which is used is the following

$$
f^{-1}(y) \equiv\{x \in D(f): f(x)=y\}
$$

This is called the inverse image.
It is probably safe to say that most people do not think of functions as a type of relation which is a subset of the Cartesian product of two sets. A function is like a machine which takes inputs, $x$ and makes them into a unique output, $f(x)$. Of course, that is what the above definition says with more precision. An ordered pair, $(x, y)$ which is an element of the function or mapping has an input, $x$ and a unique output $y$,denoted as $f(x)$ while the name of the function is $f$. "mapping" is often a noun meaning function. However, it also is a verb as in " $f$ is mapping $A$ to $B "$. That which a function is thought of as doing is also referred to using the word "maps" as in: $f$ maps $X$ to $Y$. However, a set of functions may be called a set of maps so this word might also be used as the plural of a noun. There is no help for it. You just have to suffer with this nonsense.

The following theorem which is interesting for its own sake will be used to prove the Schroder Bernstein theorem, proved by Dedekind in 1887. The proof given here is like the version in Hewitt and Stromberg [22].
Theorem 1.2.2 Let $f: X \rightarrow Y$ and $g: Y \rightarrow X$ be two functions. Then there exist sets $A, B, C, D$, such that

$$
\begin{gathered}
A \cup B=X, C \cup D=Y, A \cap B=\emptyset, C \cap D=\emptyset \\
f(A)=C, g(D)=B .
\end{gathered}
$$

The following picture illustrates the conclusion of this theorem.


Proof:Consider the empty set, $\emptyset \subseteq X$. If $y \in Y \backslash f(\emptyset)$, then $g(y) \notin \emptyset$ because $\emptyset$ has no elements. Also, if $A, B, C$, and $D$ are as described above, $A$ also would have this same property that the empty set has. However, $A$ is probably larger. Therefore, say $A_{0} \subseteq X$ satisfies $\mathscr{P}$ if whenever $y \in Y \backslash f\left(A_{0}\right), g(y) \notin A_{0}$.

$$
\mathscr{A} \equiv\left\{A_{0} \subseteq X: A_{0} \text { satisfies } \mathscr{P}\right\}
$$

Let $A=\cup \mathscr{A}$. If $y \in Y \backslash f(A)$, then for each $A_{0} \in \mathscr{A}, y \in Y \backslash f\left(A_{0}\right)$ and so $g(y) \notin A_{0}$. Since $g(y) \notin A_{0}$ for all $A_{0} \in \mathscr{A}$, it follows $g(y) \notin A$. Hence $A$ satisfies $\mathscr{P}$ and is the largest subset of $X$ which does so. Now define

$$
C \equiv f(A), D \equiv Y \backslash C, B \equiv X \backslash A
$$

It only remains to verify that $g(D)=B$. It was just shown that $g(D) \subseteq B$.
Suppose $x \in B=X \backslash A$. Then $A \cup\{x\}$ does not satisfy $\mathscr{P}$ and so there exists $y \in$ $Y \backslash f(A \cup\{x\}) \subseteq D$ such that $g(y) \in A \cup\{x\}$. But $y \notin f(A)$ and so since $A$ satisfies $\mathscr{P}$, it follows $g(y) \notin A$. Hence $g(y)=x$ and so $x \in g(D)$. Hence $g(D)=B$.

Theorem 1.2.3 (Schroder Bernstein) If $f: X \rightarrow Y$ and $g: Y \rightarrow X$ are one to one, then there exists $h: X \rightarrow Y$ which is one to one and onto.

Proof:Let $A, B, C, D$ be the sets of Theorem1.2.2 and define

$$
h(x) \equiv\left\{\begin{array}{cc}
f(x) & \text { if } x \in A \\
g^{-1}(x) & \text { if } x \in B
\end{array}\right.
$$

Then $h$ is the desired one to one and onto mapping.
Recall that the Cartesian product may be considered as the collection of choice functions.

Definition 1.2.4 Let $I$ be a set and let $X_{i}$ be a set for each $i \in I . f$ is a choice function written as

$$
f \in \prod_{i \in I} X_{i}
$$

if $f(i) \in X_{i}$ for each $i \in I$.
The axiom of choice says that if $X_{i} \neq \emptyset$ for each $i \in I$, for $I$ a set, then

$$
\prod_{i \in I} X_{i} \neq \emptyset
$$

Sometimes the two functions, $f$ and $g$ are onto but not one to one. It turns out that with the axiom of choice, a similar conclusion to the above may be obtained.

Corollary 1.2.5 If $f: X \rightarrow Y$ is onto and $g: Y \rightarrow X$ is onto, then there exists $h: X \rightarrow Y$ which is one to one and onto.

Proof: For each $y \in Y, f^{-1}(y) \equiv\{x \in X: f(x)=y\} \neq \emptyset$. Therefore, by the axiom of choice, there exists $f_{0}^{-1} \in \prod_{y \in Y} f^{-1}(y)$ which is the same as saying that for each $y \in Y$, $f_{0}^{-1}(y) \in f^{-1}(y)$. Similarly, there exists $g_{0}^{-1}(x) \in g^{-1}(x)$ for all $x \in X$. Then $f_{0}^{-1}$ is one to one because if $f_{0}^{-1}\left(y_{1}\right)=f_{0}^{-1}\left(y_{2}\right)$, then

$$
y_{1}=f\left(f_{0}^{-1}\left(y_{1}\right)\right)=f\left(f_{0}^{-1}\left(y_{2}\right)\right)=y_{2} .
$$

Similarly $g_{0}^{-1}$ is one to one. Therefore, by the Schroder Bernstein theorem, there exists $h: X \rightarrow Y$ which is one to one and onto.

Definition 1.2.6 $A$ set $S$, is finite if there exists a natural number $n$ and a map $\theta$ which maps $\{1, \cdots, n\}$ one to one and onto $S$. $S$ is infinite if it is not finite. A set $S$, is called countable if there exists a map $\theta$ mapping $\mathbb{N}$ one to one and onto $S$.(When $\theta$ maps a set $A$ to a set $B$, this will be written as $\theta: A \rightarrow B$ in the future.) Here $\mathbb{N} \equiv\{1,2, \cdots\}$, the natural numbers. $S$ is at most countable if there exists a map $\theta: \mathbb{N} \rightarrow S$ which is onto.

The property of being at most countable is often referred to as being countable because the question of interest is normally whether one can list all elements of the set, designating a first, second, third etc. in such a way as to give each element of the set a natural number. The possibility that a single element of the set may be counted more than once is often not important.
Theorem 1.2.7 If $X$ and $Y$ are both at most countable, then $X \times Y$ is also at most countable. If either $X$ or $Y$ is countable, then $X \times Y$ is also countable.

Proof:It is given that there exists a mapping $\eta: \mathbb{N} \rightarrow X$ which is onto. Define $\eta(i) \equiv x_{i}$ and consider $X$ as the set $\left\{x_{1}, x_{2}, x_{3}, \cdots\right\}$. Similarly, consider $Y$ as the set $\left\{y_{1}, y_{2}, y_{3}, \cdots\right\}$. It follows the elements of $X \times Y$ are included in the following rectangular array.

$$
\begin{array}{lllll}
\left(x_{1}, y_{1}\right) & \left(x_{1}, y_{2}\right) & \left(x_{1}, y_{3}\right) & \cdots & \leftarrow \text { Those which have } x_{1} \text { in first slot. } \\
\left(x_{2}, y_{1}\right) & \left(x_{2}, y_{2}\right) & \left(x_{2}, y_{3}\right) & \cdots & \leftarrow \text { Those which have } x_{2} \text { in first slot. } \\
\left(x_{3}, y_{1}\right) & \left(x_{3}, y_{2}\right) & \left(x_{3}, y_{3}\right) & \cdots & \leftarrow \text { Those which have } x_{3} \text { in first slot. }
\end{array}
$$

Follow a path through this array as follows.


Thus the first element of $X \times Y$ is $\left(x_{1}, y_{1}\right)$, the second element of $X \times Y$ is $\left(x_{1}, y_{2}\right)$, the third element of $X \times Y$ is $\left(x_{2}, y_{1}\right)$ etc. This assigns a number from $\mathbb{N}$ to each element of $X \times Y$. Thus $X \times Y$ is at most countable.

It remains to show the last claim. Suppose without loss of generality that $X$ is countable. Then there exists $\alpha: \mathbb{N} \rightarrow X$ which is one to one and onto. Let $\beta: X \times Y \rightarrow \mathbb{N}$ be defined by $\beta((x, y)) \equiv \alpha^{-1}(x)$. Thus $\beta$ is onto $\mathbb{N}$. By the first part there exists a function from $\mathbb{N}$ onto $X \times Y$. Therefore, by Corollary 1.2.5, there exists a one to one and onto mapping from $X \times Y$ to $\mathbb{N}$.

Note that by induction, $\prod_{i=1}^{n} X_{i}$ is at most countable if each $X_{i}$ is.
Theorem 1.2.8 If $X$ and $Y$ are at most countable, then $X \cup Y$ is at most countable. If either $X$ or $Y$ are countable, then $X \cup Y$ is countable.

Proof:As in the preceding theorem,

$$
X=\left\{x_{1}, x_{2}, x_{3}, \cdots\right\}
$$

and

$$
Y=\left\{y_{1}, y_{2}, y_{3}, \cdots\right\}
$$

Consider the following array consisting of $X \cup Y$ and path through it.


Thus the first element of $X \cup Y$ is $x_{1}$, the second is $x_{2}$ the third is $y_{1}$ the fourth is $y_{2}$ etc.
Consider the second claim. By the first part, there is a map from $\mathbb{N}$ onto $X \times Y$. Suppose without loss of generality that $X$ is countable and $\alpha: \mathbb{N} \rightarrow X$ is one to one and onto. Then define $\beta(y) \equiv 1$, for all $y \in Y$, and $\beta(x) \equiv \alpha^{-1}(x)$. Thus, $\beta$ maps $X \times Y$ onto $\mathbb{N}$ and this shows there exist two onto maps, one mapping $X \cup Y$ onto $\mathbb{N}$ and the other mapping $\mathbb{N}$ onto $X \cup Y$. Then Corollary 1.2.5 yields the conclusion.

In fact, the countable union of countable sets is also at most countable.
Theorem 1.2.9 Let $A_{i}$ be a countable set. Thus $A_{i}=\left\{r_{j}^{i}\right\}_{j=1}^{\infty}$. Then $\cup_{i=1}^{\infty} A_{i}$ is also at most a countable set. If it is an infinite set, then it is countable.

Proof: This is proved like Theorem 1.2.7 arrange $\cup_{i=1}^{\infty} A_{i}$ as follows.

| $r_{1}^{1}$ | $r_{2}^{1}$ | $r_{3}^{1}$ | $\cdots$ |
| :---: | :---: | :---: | :---: |
| $r_{1}^{2}$ | $r_{2}^{2}$ | $r_{3}^{2}$ | $\cdots$ |
| $r_{1}^{3}$ | $r_{2}^{3}$ | $r_{3}^{3}$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ |  |
|  |  |  |  |

Now take a route through this rectangular array as in Theorem 1.2.7, identifying an enumeration in the order in which the displayed elements are encountered as done in that theorem. Thus there is an onto mapping from $\mathbb{N}$ to $\cup_{i=1}^{\infty} A_{i}$ and so $\cup_{i=1}^{\infty} A_{i}$ is at most countable, meaning its elements can be enumerated. However, if any of the $A_{i}$ is infinite or if the union is, then there is an onto map from $\cup_{i=1}^{\infty} A_{i}$ onto $\mathbb{N}$ and so from Corollary 1.2.5, there would be a one to one and onto map between $\mathbb{N}$ and $\cup_{i=1}^{\infty} A_{i}$.

Note that by induction this shows that if you have any finite set whose elements are countable sets, then the union of these is countable.

### 1.3 Equivalence Relations

There are many ways to compare elements of a set other than to say two elements are equal or the same. For example, in the set of people let two people be equivalent if they have the same weight. This would not be saying they were the same person, just that they weighed the same. Often such relations involve considering one characteristic of the elements of a set and then saying the two elements are equivalent if they are the same as far as the given characteristic is concerned.

Definition 1.3.1 Let $S$ be a set. $\sim$ is an equivalence relation on $S$ if it satisfies the following axioms.

1. $x \sim x$ for all $x \in S$. (Reflexive)
2. If $x \sim y$ then $y \sim x$. (Symmetric)
3. If $x \sim y$ and $y \sim z$, then $x \sim z$. (Transitive)

Definition 1.3.2 $[x]$ denotes the set of all elements of $S$ which are equivalent to $x$ and $[x]$ is called the equivalence class determined by $x$ or just the equivalence class of $x$.

With the above definition one can prove the following simple theorem.

Theorem 1.3.3 Let $\sim$ be an equivalence relation defined on a set, $S$ and let $\mathscr{H}$ denote the set of equivalence classes. Then if $[x]$ and $[y]$ are two of these equivalence classes, either $x \sim y$ and $[x]=[y]$ or it is not true that $x \sim y$ and $[x] \cap[y]=\emptyset$.

Proof: If $x \sim y$, then if $z \in[y]$, you have $x \sim y$ and $y \sim z$ so $x \sim z$ which shows that $[y] \subseteq[x]$. Similarly, $[x] \subseteq[y]$. If it is not the case that $x \sim y$, then there can be no intersection of $[x]$ and $[y]$ because if $z$ were in this intersection, then $x \sim z, z \sim y$ so $x \sim y$.

### 1.4 The Hausdorff Maximal Theorem

The Hausdorff maximal theorem or something like it is often very useful. I will use it whenever convenient because its use typically makes a much longer and involved argument shorter. However, sometimes its use is absolutely essential. First is the definition of what is meant by a partial order.

Definition 1.4.1 A nonempty set $\mathscr{F}$ is called a partially ordered set if it has a partial order denoted by $\prec$. This means it satisfies the following. If $x \prec y$ and $y \prec z$, then $x \prec z$. Also $x \prec x$. It is like $\subseteq$ on the set of all subsets of a given set. It is not the case that given two elements of $\mathscr{F}$ that they are related. In other words, you cannot conclude that either $x \prec y$ or $y \prec x$. A chain, denoted by $\mathscr{C} \subseteq \mathscr{F}$ has the property that it is totally ordered meaning that if $x, y \in \mathscr{C}$, either $x \prec y$ or $y \prec x$. A maximal chain is a chain $\mathscr{C}$ which has the property that there is no strictly larger chain. In other words, if $x \in \mathscr{F} \backslash \cup \mathscr{C}$, then $\mathscr{C} \cup\{x\}$ is no longer a chain.

Here is the Hausdorff maximal theorem. The proof is a proof by contradiction. We assume there is no maximal chain and then show this cannot happen. The axiom of choice is used in choosing the $x_{\mathscr{C}}$ right at the beginning of the argument.

## Theorem 1.4.2 Let $\mathscr{F}$ be a nonempty partially ordered set with order $\prec$. Then there exists a maximal chain.

Proof: Suppose not. Then for $\mathscr{C}$ a chain, let $\theta \mathscr{C}$ denote $\mathscr{C} \cup\left\{x_{\mathscr{C}}\right\}$. Thus for $\mathscr{C}$ a chain, $\boldsymbol{\theta} \mathscr{C}$ is a larger chain which has exactly one more element of $\mathscr{F}$. Since $\mathscr{F} \neq \emptyset$, pick $x_{0} \in$ $\mathscr{F}$. Note that $\left\{x_{0}\right\}$ is a chain. Let $\mathscr{X}$ be the set of all chains $\mathscr{C}$ such that $x_{0} \in \cup \mathscr{C}$. Thus $\mathscr{X}$ contains $\left\{x_{0}\right\}$. Call two chains comparable if one is a subset of the other. Also, if $\mathscr{S}$ is a nonempty subset of $\mathscr{F}$ in which all chains are comparable, then $\cup \mathscr{S}$ is also a chain. From now on $\mathscr{S}$ will always refer to a nonempty set of chains in which any pair are comparable. Then summarizing,

1. $x_{0} \in \cup \mathscr{C}$ for all $\mathscr{C} \in \mathscr{X}$.
2. $\left\{x_{0}\right\} \in \mathscr{X}$
3. If $\mathscr{C} \in \mathscr{X}$ then $\theta \mathscr{C} \in \mathscr{X}$.
4. If $\mathscr{S} \subseteq \mathscr{X}$ then $\cup \mathscr{S} \in \mathscr{X}$.

A subset $\mathscr{Y}$ of $\mathscr{X}$ will be called a "tower" if $\mathscr{Y}$ satisfies 1.) - 4.). Let $\mathscr{Y}_{0}$ be the intersection of all towers. Then $\mathscr{Y}_{0}$ is also a tower, the smallest one. Then the next claim might seem to be so because if not, $\mathscr{Y}_{0}$ would not be the smallest tower.

Claim 1: If $\mathscr{C}_{0} \in \mathscr{Y}_{0}$ is comparable to every chain $\mathscr{C} \in \mathscr{Y}_{0}$, then if $\mathscr{C}_{0} \subsetneq \mathscr{C}$, it must be the case that $\theta \mathscr{C}_{0} \subseteq \mathscr{C}$. In other words, $x_{\mathscr{C}_{0}} \in \cup \mathscr{C}$. The symbol $\subsetneq$ indicates proper subset.

This is done by considering a set $\mathscr{B} \subseteq \mathscr{Y}_{0}$ consisting of $\mathscr{D}$ which acts like $\mathscr{C}$ in the above and showing that it actually equals $\mathscr{Y}_{0}$ because it is a tower.

Proof of Claim 1: Consider $\mathscr{B} \equiv\left\{\mathscr{D} \in \mathscr{Y}_{0}: \mathscr{D} \subseteq \mathscr{C}_{0}\right.$ or $\left.x_{\mathscr{C}_{0}} \in \cup \mathscr{D}\right\}$. Let $\mathscr{Y}_{1} \equiv \mathscr{Y}_{0} \cap \mathscr{B}$. I want to argue that $\mathscr{Y}_{1}$ is a tower. By definition all chains of $\mathscr{Y}_{1}$ contain $x_{0}$ in their unions. If $\mathscr{D} \in \mathscr{Y}_{1}$, is $\theta \mathscr{D} \in \mathscr{Y}_{1}$ ? If $\mathscr{S} \subseteq \mathscr{Y}$, is $\cup \mathscr{S} \in \mathscr{Y}_{1}$ ? Is $\left\{x_{0}\right\} \in \mathscr{B}$ ?
$\left\{x_{0}\right\}$ cannot properly contain $\mathscr{C}_{0}$ since $x_{0} \in \cup \mathscr{C}_{0}$. Therefore, $\mathscr{C}_{0} \supseteq\left\{x_{0}\right\}$ so $\left\{x_{0}\right\} \in \mathscr{B}$.
If $\mathscr{S} \subseteq \mathscr{Y}_{1}$, and $\mathscr{D} \equiv \cup \mathscr{S}$, is $\mathscr{D} \in \mathscr{Y}_{1}$ ? Since $\mathscr{Y}_{0}$ is a tower, $\mathscr{D}$ is comparable to $\mathscr{C}_{0}$. If $\mathscr{D} \subseteq \mathscr{C}_{0}$, then $\mathscr{D}$ is in $\mathscr{B}$. Otherwise $\mathscr{D} \supseteq \mathscr{C}_{0}$ and in this case, why is $\mathscr{D}$ in $\mathscr{B}$ ? Why is $x_{\mathscr{C}_{0}} \in \cup \mathscr{D}$ ? The chains of $\mathscr{S}$ are in $\mathscr{B}$ so one of them, called $\tilde{\mathscr{C}}$ must properly contain $\mathscr{C}_{0}$ and so $x_{\mathscr{C}_{0}} \in \cup \tilde{C} \subseteq \cup \mathscr{D}$. Therefore, $\mathscr{D} \in \mathscr{B} \cap \mathscr{Y}_{0}=\mathscr{Y}_{1}$. 4.) holds. Two cases remain, to show that $\mathscr{Y}_{1}$ satisfies 3.).
case 1: $\mathscr{D} \supseteq \mathscr{C}_{0}$. Then by definition of $\mathscr{B}, x_{\mathscr{C}_{0}} \in \cup \mathscr{D}$ and so $x_{\mathscr{C}_{0}} \in \cup \theta \mathscr{D}$ so $\theta \mathscr{D} \in \mathscr{Y}_{1}$.
case 2: $\mathscr{D} \subseteq \mathscr{C}_{0} . \theta \mathscr{D} \in \mathscr{Y}_{0}$ so $\theta \mathscr{D}$ is comparable to $\mathscr{C}_{0}$. First suppose $\theta \mathscr{D} \supsetneq \mathscr{C}_{0}$. Thus $\mathscr{D} \subseteq \mathscr{C}_{0} \varsubsetneqq \mathscr{D} \cup\left\{x_{\mathscr{D}}\right\}$. If $x \in \mathscr{C}_{0}$ and $x$ is not in $\mathscr{D}$ then $\mathscr{D} \cup\{x\} \subseteq \mathscr{C}_{0} \subsetneq \mathscr{D} \cup\left\{x_{\mathscr{D}}\right\}$. This is impossible. Consider $x$. Thus in this case that $\theta \mathscr{D} \supsetneq \mathscr{C}_{0}, \mathscr{D}=\mathscr{C}_{0}$. It follows that $x_{\mathscr{D}}=x_{\mathscr{C}_{0}} \in \cup \theta \mathscr{C}_{0}=\cup \theta \mathscr{D}$ and so $\theta \mathscr{D} \in \mathscr{Y}_{1}$. The other case is that $\theta \mathscr{D} \subseteq \mathscr{C}_{0}$ so $\theta \mathscr{D} \in \mathscr{B}$ by definition. This shows 3.) so $\mathscr{Y}_{1}$ is a tower and must equal $\mathscr{Y}_{0}$.

Claim 2: Any two chains in $\mathscr{Y}_{0}$ are comparable.
Proof of Claim 2: Let $\mathscr{Y}_{1}$ consist of all chains of $\mathscr{Y}_{0}$ which are comparable to every chain of $\mathscr{Y}_{0} .\left\{x_{0}\right\}$ is in $\mathscr{Y}_{1}$ by definition. All chains of $\mathscr{Y}_{0}$ have $x_{0}$ in their union. If $\mathscr{S} \subseteq \mathscr{Y}_{1}$, is $\cup \mathscr{S} \in \mathscr{Y}_{1}$ ? Given $\mathscr{D} \in \mathscr{Y}_{0}$ either every chain of $\mathscr{S}$ is contained in $\mathscr{D}$ or at least one contains $\mathscr{D}$. Either way $\mathscr{D}$ is comparable to $\cup \mathscr{S}$ so $\cup \mathscr{S} \in \mathscr{Y}_{1}$. It remains to show 3.). Let $\mathscr{C} \in \mathscr{Y}_{1}$ and $\mathscr{D} \in \mathscr{Y}_{0}$. Since $\mathscr{C}$ is comparable to all chains in $\mathscr{Y}_{0}$, it follows from Claim 1 either $\mathscr{C} \subsetneq \mathscr{D}$ when $x_{\mathscr{C}} \in \cup \mathscr{D}$ and $\theta \mathscr{C} \subseteq \mathscr{D}$ or $\mathscr{C} \supseteq \mathscr{D}$ when $\theta \mathscr{C} \supseteq \mathscr{D}$. Hence $\mathscr{Y}_{1}=\mathscr{Y}_{0}$ because $\mathscr{Y}_{0}$ is as small as possible.

Since every pair of chains in $\mathscr{Y}_{0}$ are comparable and $\mathscr{Y}_{0}$ is a tower, it follows that $\cup \mathscr{Y}_{0} \in \mathscr{Y}_{0}$ so $\cup \mathscr{Y}_{0}$ is a chain. However, $\theta \cup \mathscr{Y}_{0}$ is a chain which properly contains $\cup \mathscr{Y}_{0}$ and since $\mathscr{Y}_{0}$ is a tower, $\theta \cup \mathscr{Y}_{0} \in \mathscr{Y}_{0}$. Thus $\cup\left(\theta \cup \mathscr{Y}_{0}\right) \supseteq \cup\left(\cup \mathscr{Y}_{0}\right) \supseteq \cup\left(\theta \cup \mathscr{Y}_{0}\right)$ which is a contradiction. Therefore, for some chain $\mathscr{C}$ it is impossible to obtain the $x_{C}$ described above and so, this $\mathscr{C}$ is a maximal chain.

If $X$ is a nonempty set, $\leq$ is an order on $X$ if

$$
x \leq x
$$

and if $x, y \in X$, then

$$
\text { either } x \leq y \text { or } y \leq x
$$

and

$$
\text { if } x \leq y \text { and } y \leq z \text { then } x \leq z
$$

$\leq$ is a well order and say that $(X, \leq)$ is a well-ordered set if every nonempty subset of $X$ has a smallest element. More precisely, if $S \neq \emptyset$ and $S \subseteq X$ then there exists an $x \in S$ such that $x \leq y$ for all $y \in S$. A familiar example of a well-ordered set is the natural numbers.

Lemma 1.4.3 The Hausdorff maximal principle implies every nonempty set can be wellordered.

Proof: Let $X$ be a nonempty set and let $a \in X$. Then $\{a\}$ is a well-ordered subset of $X$. Let

$$
\mathscr{F}=\{S \subseteq X: \text { there exists a well order for } S\}
$$

Thus $\mathscr{F} \neq \emptyset$. For $S_{1}, S_{2} \in \mathscr{F}$, define $S_{1} \prec S_{2}$ if $S_{1} \subseteq S_{2}$ and there exists a well order for $S_{2}$, $\leq_{2}$ such that

$$
\left(S_{2}, \leq_{2}\right) \text { is well-ordered }
$$

and if

$$
y \in S_{2} \backslash S_{1} \text { then } x \leq_{2} y \text { for all } x \in S_{1},
$$

and if $\leq_{1}$ is the well order of $S_{1}$ then the two orders are consistent on $S_{1}$. Then observe that $\prec$ is a partial order on $\mathscr{F}$. By the Hausdorff maximal principle, let $\mathscr{C}$ be a maximal chain in $\mathscr{F}$ and let

$$
X_{\infty} \equiv \cup \mathscr{C}
$$

Define an order, $\leq$, on $X_{\infty}$ as follows. If $x, y$ are elements of $X_{\infty}$, pick $S \in \mathscr{C}$ such that $x, y$ are both in $S$. Then if $\leq_{S}$ is the order on $S$, let $x \leq y$ if and only if $x \leq_{S} y$. This definition is well defined because of the definition of the order, $\prec$. Now let $U$ be any nonempty subset of $X_{\infty}$. Then $S \cap U \neq \emptyset$ for some $S \in \mathscr{C}$. Because of the definition of $\leq$, if $y \in S_{2} \backslash S_{1}, S_{i} \in \mathscr{C}$, then $x \leq y$ for all $x \in S_{1}$. Thus, if $y \in X_{\infty} \backslash S$ then $x \leq y$ for all $x \in S$ and so the smallest element of $S \cap U$ exists and is the smallest element in $U$. Therefore $X_{\infty}$ is well-ordered. Now suppose there exists $z \in X \backslash X_{\infty}$. Define the following order, $\leq_{1}$, on $X_{\infty} \cup\{z\}$.

$$
\begin{aligned}
& x \leq_{1} y \text { if and only if } x \leq y \text { whenever } x, y \in X_{\infty} \\
& \qquad x \leq_{1} z \text { whenever } x \in X_{\infty} .
\end{aligned}
$$

Then let

$$
\tilde{\mathscr{C}}=\left\{S \in \mathscr{C} \text { or } X_{\infty} \cup\{z\}\right\}
$$

Then $\tilde{\mathscr{C}}$ is a strictly larger chain than $\mathscr{C}$ contradicting maximality of $\mathscr{C}$. Thus $X \backslash X_{\infty}=\emptyset$ and this shows $X$ is well-ordered by $\leq$. This proves the lemma.

With these two lemmas the main result follows.

Theorem 1.4.4 The following are equivalent.
The axiom of choice
The Hausdorff maximal principle
The well-ordering principle.

Proof: It only remains to prove that the well-ordering principle implies the axiom of choice. Let $I$ be a nonempty set and let $X_{i}$ be a nonempty set for each $i \in I$. Let $X=\cup\left\{X_{i}\right.$ : $i \in I\}$ and well order $X$. Let $f(i)$ be the smallest element of $X_{i}$. Then $f \in \prod_{i \in I} X_{i}$.

There are some other equivalences to the axiom of choice proved in the book by Hewitt and Stromberg [22].

### 1.4.1 The Hamel Basis

A Hamel basis is nothing more than the correct generalization of the notion of a basis for a finite dimensional vector space to vector spaces which are possibly not of finite dimension.

Definition 1.4.5 Let $X$ be a vector space. A Hamel basis is a subset of $X, \Lambda$ such that every vector of $X$ can be written as a finite linear combination of vectors of $\Lambda$ and the vectors of $\Lambda$ are linearly independent in the sense that if $\left\{x_{1}, \cdots, x_{n}\right\} \subseteq \Lambda$ and $\sum_{k=1}^{n} c_{k} x_{k}=$ 0 . Then each $c_{k}=0$.

The main result is the following theorem.

## Theorem 1.4.6 Let $X$ be a nonzero vector space. Then it has a Hamel basis.

Proof: Let $x_{1} \in X$ and $x_{1} \neq 0$. Let $\mathscr{F}$ denote the collection of subsets of $X, \Lambda$ containing $x_{1}$ with the property that the vectors of $\Lambda$ are linearly independent as described in Definition 1.4.5 partially ordered by set inclusion. By the Hausdorff maximal theorem, there exists a maximal chain, $\mathscr{C}$ Let $\Lambda=\cup \mathscr{C}$. Since $\mathscr{C}$ is a chain, it follows that if $\left\{x_{1}, \cdots, x_{n}\right\} \subseteq \mathscr{C}$ then there exists a single $\Lambda^{\prime} \in \mathbb{C}$ containing all these vectors. Therefore, if $\sum_{k=1}^{n} c_{k} x_{k}=0$ it follows each $c_{k}=0$. Thus the vectors of $\Lambda$ are linearly independent. Is every vector of $X$ a finite linear combination of vectors of $\Lambda$ ?

Suppose not. Then there exists $z$ which is not equal to a finite linear combination of vectors of $\Lambda$. Consider $\Lambda \cup\{z\}$. If $c z+\sum_{k=1}^{m} c_{k} x_{k}=0$ where the $x_{k}$ are vectors of $\Lambda$, then if $c \neq 0$ this contradicts the condition that $z$ is not a finite linear combination of vectors of $\Lambda$. Therefore, $c=0$ and now all the $c_{k}$ must equal zero because it was just shown $\Lambda$ is linearly independent. It follows $\mathscr{C} \cup\{\Lambda \cup\{z\}\}$ is a strictly larger chain than $\mathscr{C}$ and this is a contradiction. Therefore, $\Lambda$ is a Hamel basis as claimed.

### 1.5 Real and Complex Numbers

I am assuming the reader is familiar with the field of complex numbers which can be considered as points in the plane, the complex number $x+i y$ being the point obtained by graphing the ordered pair $(x, y)$. I assume the reader knows about the complex conjugate $\overline{x+i y} \equiv x-i y$ and all its properties such as, for $z, w \in \mathbb{C}, \overline{(z+w)}=\bar{z}+\bar{w}$ and $\overline{z w}=\bar{z} \bar{w}$. Also recall that for $z \in \mathbb{C},|z| \equiv \sqrt{x^{2}+y^{2}}$ where $z=x+i y$ and that the triangle inequalities hold: $|z+w| \leq|z|+|w|$ and $|z-w| \geq||z|-|w||$ and $|z|=(z \bar{z})^{1 / 2}$. This is the time to review these things. If you have not seen them, read my single variable advanced calculus book or the first part of my calculus book. Any good pre-calculus book has these topics.

Also recall that complex numbers, are often written in the so called polar form which is described next. Suppose $z=x+i y$ is a complex number. Then

$$
x+i y=\sqrt{x^{2}+y^{2}}\left(\frac{x}{\sqrt{x^{2}+y^{2}}}+i \frac{y}{\sqrt{x^{2}+y^{2}}}\right) .
$$

Now note that

$$
\left(\frac{x}{\sqrt{x^{2}+y^{2}}}\right)^{2}+\left(\frac{y}{\sqrt{x^{2}+y^{2}}}\right)^{2}=1
$$

and so

$$
\left(\frac{x}{\sqrt{x^{2}+y^{2}}}, \frac{y}{\sqrt{x^{2}+y^{2}}}\right)
$$

is a point on the unit circle. Therefore, there exists a unique angle $\theta \in[0,2 \pi)$ such that

$$
\cos \theta=\frac{x}{\sqrt{x^{2}+y^{2}}}, \sin \theta=\frac{y}{\sqrt{x^{2}+y^{2}}}
$$

The polar form of the complex number is then $r(\cos \theta+i \sin \theta)$ where $\theta$ is this angle just described and $r=\sqrt{x^{2}+y^{2}} \equiv|z|$.

$$
r=\sqrt{x^{2}+y^{2}} \left\lvert\,\right.
$$

### 1.5.1 Roots Of Complex Numbers

A fundamental identity is the formula of De Moivre which follows.
Theorem 1.5.1 Let $r>0$ be given. Then if $n$ is a positive integer,

$$
[r(\cos t+i \sin t)]^{n}=r^{n}(\cos n t+i \sin n t)
$$

Proof: It is clear the formula holds if $n=1$. Suppose it is true for $n$.

$$
[r(\cos t+i \sin t)]^{n+1}=[r(\cos t+i \sin t)]^{n}[r(\cos t+i \sin t)]
$$

which by induction equals

$$
\begin{gathered}
=r^{n+1}(\cos n t+i \sin n t)(\cos t+i \sin t) \\
=r^{n+1}((\cos n t \cos t-\sin n t \sin t)+i(\sin n t \cos t+\cos n t \sin t)) \\
=r^{n+1}(\cos (n+1) t+i \sin (n+1) t)
\end{gathered}
$$

by the formulas for the cosine and sine of the sum of two angles.
Corollary 1.5.2 Let $z$ be a non zero complex number. Then for $k \in \mathbb{N}$, there are always exactly $k k^{\text {th }}$ roots of $z$ in $\mathbb{C}$.

Proof: Let $z=x+i y$ and let $z=|z|(\cos t+i \sin t)$ be the polar form of the complex number. By De Moivre's theorem, a complex number $r(\cos \alpha+i \sin \alpha)$, is a $k^{t h}$ root of $z$ if and only if

$$
r^{k}(\cos k \alpha+i \sin k \alpha)=|z|(\cos t+i \sin t)
$$

This requires $r^{k}=|z|$ and so $r=|z|^{1 / k}$ and also both $\cos (k \alpha)=\cos t$ and $\sin (k \alpha)=\sin t$. This can only happen if $k \alpha=t+2 l \pi$ for $l$ an integer. Thus

$$
\alpha=\frac{t+2 l \pi}{k}, l \in \mathbb{Z}
$$

and so the $k^{\text {th }}$ roots of $z$ are of the form

$$
|z|^{1 / k}\left(\cos \left(\frac{t+2 l \pi}{k}\right)+i \sin \left(\frac{t+2 l \pi}{k}\right)\right), l \in \mathbb{Z}
$$

Since the cosine and sine are periodic of period $2 \pi$, there are exactly $k$ distinct numbers which result from this formula.

Example 1.5.3 Find the three cube roots of $i$.
First note that $i=1\left(\cos \left(\frac{\pi}{2}\right)+i \sin \left(\frac{\pi}{2}\right)\right)$. Using the formula in the proof of the above corollary, the cube roots of $i$ are

$$
1\left(\cos \left(\frac{(\pi / 2)+2 l \pi}{3}\right)+i \sin \left(\frac{(\pi / 2)+2 l \pi}{3}\right)\right)
$$

where $l=0,1,2$. Therefore, the roots are

$$
\cos \left(\frac{\pi}{6}\right)+i \sin \left(\frac{\pi}{6}\right), \cos \left(\frac{5}{6} \pi\right)+i \sin \left(\frac{5}{6} \pi\right), \cos \left(\frac{3}{2} \pi\right)+i \sin \left(\frac{3}{2} \pi\right)
$$

Thus the cube roots of $i$ are $\frac{\sqrt{3}}{2}+i\left(\frac{1}{2}\right), \frac{-\sqrt{3}}{2}+i\left(\frac{1}{2}\right)$, and $-i$.
The ability to find $k^{t h}$ roots can also be used to factor some polynomials.
Example 1.5.4 Factor the polynomial $x^{3}-27$.
First find the cube roots of 27. By the above procedure using De Moivre's theorem, these cube roots are $3,3\left(\frac{-1}{2}+i \frac{\sqrt{3}}{2}\right)$, and $3\left(\frac{-1}{2}-i \frac{\sqrt{3}}{2}\right)$. Therefore, $x^{3}-27=$

$$
(x-3)\left(x-3\left(\frac{-1}{2}+i \frac{\sqrt{3}}{2}\right)\right)\left(x-3\left(\frac{-1}{2}-i \frac{\sqrt{3}}{2}\right)\right)
$$

Note also $\left(x-3\left(\frac{-1}{2}+i \frac{\sqrt{3}}{2}\right)\right)\left(x-3\left(\frac{-1}{2}-i \frac{\sqrt{3}}{2}\right)\right)=x^{2}+3 x+9$ and so

$$
x^{3}-27=(x-3)\left(x^{2}+3 x+9\right)
$$

where the quadratic polynomial $x^{2}+3 x+9$ cannot be factored without using complex numbers.

Note that even though the polynomial $x^{3}-27$ has all real coefficients, it has some complex zeros, $\frac{-1}{2}+i \frac{\sqrt{3}}{2}$ and $\frac{-1}{2}-i \frac{\sqrt{3}}{2}$. These zeros are complex conjugates of each other. It is always this way. You should show this is the case. To see how to do this, see Problems 17 and 18 below.

Another fact for your information is the fundamental theorem of algebra. This theorem says that any polynomial of degree at least 1 having any complex coefficients always has a root in $\mathbb{C}$. This is sometimes referred to by saying $\mathbb{C}$ is algebraically complete. Gauss is usually credited with giving a proof of this theorem in 1797 but many others worked on it
and the first completely correct proof was due to Argand in 1806. For more on this theorem, you can google fundamental theorem of algebra and look at the interesting Wikipedia article on it. Proofs of this theorem usually involve the use of techniques from calculus even though it is really a result in algebra. A proof and plausibility explanation is given later.

Recall the quadratic formula which gives solutions to $a x^{2}+b x+c=0$ which holds for any $a, b, c \in \mathbb{C}$ with $a \neq 0$. This is also good to review from any good pre-calculus book. My book published with
http://www.centerofmath.org/textbooks/pre_calc/index.html(2012) has all of these elementary considerations. Most are in my on line calculus text or Volume 1 of the one published by World Scientific.

### 1.5.2 The Complex Exponential

Here is a short review of the complex exponential.
It was shown above that every complex number is of the form $r(\cos \theta+i \sin \theta)$ where $r \geq 0$. Laying aside the zero complex number, this shows that every non zero complex number is of the form $e^{\alpha}(\cos \beta+i \sin \beta)$. We write this in the form $e^{\alpha+i \beta}$. Having done so, does it follow that the expression preserves the most important property of the function $t \rightarrow e^{(\alpha+i \beta) t}$ for $t$ real, that

$$
\left(e^{(\alpha+i \beta) t}\right)^{\prime}=(\alpha+i \beta) e^{(\alpha+i \beta) t} ?
$$

By the definition just given which does not contradict the usual definition in case $\beta=0$ and the usual rules of differentiation in calculus,

$$
\begin{aligned}
\left(e^{(\alpha+i \beta) t}\right)^{\prime} & =\left(e^{\alpha t}(\cos (\beta t)+i \sin (\beta t))\right)^{\prime} \\
& =e^{\alpha t}[\alpha(\cos (\beta t)+i \sin (\beta t))+(-\beta \sin (\beta t)+i \beta \cos (\beta t))]
\end{aligned}
$$

Now consider the other side. From the definition it equals

$$
\begin{gathered}
(\alpha+i \beta)\left(e^{\alpha t}(\cos (\beta t)+i \sin (\beta t))\right)=e^{\alpha t}[(\alpha+i \beta)(\cos (\beta t)+i \sin (\beta t))] \\
=e^{\alpha t}[\alpha(\cos (\beta t)+i \sin (\beta t))+(-\beta \sin (\beta t)+i \beta \cos (\beta t))]
\end{gathered}
$$

which is the same thing. This is of fundamental importance in differential equations. It shows that there is no change in going from real to complex numbers for $\omega$ in the consideration of the problem $y^{\prime}=\omega y, y(0)=1$. The solution is always $e^{\omega t}$. The formula just discussed, that

$$
e^{\alpha}(\cos \beta+i \sin \beta)=e^{\alpha+i \beta}
$$

is Euler's formula. He originally conceived of this formula by considering power series of $\cos$ and $\sin$ and re arranging the order of the infinite sums.

### 1.6 A Normed Vector Space $\mathbb{F}^{p}$

In this book $\mathbb{F}$ will denote either the complex numbers $\mathbb{C}$ or the real numbers $\mathbb{R}$. For $p$ a positive integer,

$$
\mathbb{F}^{p} \equiv\left\{\left(a_{1}, \cdots, a_{p}\right): a_{k} \in \mathbb{F}\right\}
$$

That is, it consists of ordered lists of $p$ numbers from $\mathbb{F}$. These will be denoted as $\mathbf{x}$, bold faced. For now, $\|\mathbf{x}\|_{\infty} \equiv \max \left\{\left|x_{k}\right|: k=1,2, \ldots, p\right\}$. Thus, to say that $\mathbf{x}_{k} \rightarrow \mathbf{x}$ will mean that $\lim _{k \rightarrow \infty}\left\|\mathbf{x}_{k}-\mathbf{x}\right\|_{\infty}=0$ which happens if and only if the entries of $\mathbf{x}_{k}$ converge to the corresponding entries of $\mathbf{x}$. This is called a norm and more will be said about these later. The following is important.

Axioms of a Norm

$$
\begin{align*}
& \left.\|\mathbf{x}\| \geq 0 \text { and }\|\mathbf{x}\|=0 \text { if and only if } \mathbf{x}=\mathbf{0}, \text { (each } x_{k}=0\right)  \tag{1.1}\\
& \qquad \begin{array}{c}
\text { If } \alpha \in \mathbb{F}, \text { then }\|\alpha \mathbf{x}\|=|\alpha|\|\mathbf{x}\| \\
\|\mathbf{x}+\mathbf{y}\| \leq\|\mathbf{x}\|+\|\mathbf{y}\|
\end{array} \tag{1.2}
\end{align*}
$$

Only the last property is not obvious. However,

$$
\begin{align*}
\|\mathbf{x}+\mathbf{y}\| & \equiv \max \left\{\left|x_{k}+y_{k}\right|: k \leq p\right\} \leq \max \left\{\left|x_{k}\right|+\left|y_{k}\right|: k \leq p\right\} \\
& \leq \max \left\{\left|x_{k}\right|: k \leq p\right\}+\max \left\{\left|y_{k}\right|: k \leq p\right\} \equiv\|\mathbf{x}\|+\|\mathbf{y}\| \tag{1.4}
\end{align*}
$$

Recall that $\mathbb{F}$ is complete. See my book Analysis of Functions of One Variable, for example. It follows easily that $\mathbb{F}^{p}$ is also complete because any Cauchy Sequence in $\mathbb{F}^{p}$ has each entry a Cauchy sequence in $\mathbb{F}$ and so it converges. This is in the following proposition.

Definition 1.6.1 $\left\{\mathbf{x}^{n}\right\}$ is a Cauchy sequence in $\mathbb{F}^{p}$ means that for all $\varepsilon>0$ there exists $n_{\varepsilon}$ such that if $m, n \geq n_{\mathcal{\varepsilon}}$, then $\left\|\mathbf{x}^{n}-\mathbf{x}^{m}\right\|<\varepsilon$. A sequence $\left\{\mathbf{x}^{n}\right\}$ is said to converge if there exists $\mathbf{x}$ such that $\lim _{n \rightarrow \infty} \mathbf{x}^{n}=\mathbf{x}$.

Proposition 1.6.2 If $\left\{\mathbf{x}^{n}\right\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{F}^{p}$, then there exists $\mathbf{x} \in \mathbb{F}^{p}$ such that $\lim _{n \rightarrow \infty}\left\|\mathbf{x}^{n}-\mathbf{x}\right\|=0$.

Proof: For each $k,\left\{x_{k}^{n}\right\}$ is a Cauchy sequence. Thus, there exists $x_{k} \in \mathbb{F}$ such that $\lim _{n \rightarrow \infty} x_{k}^{n}=x_{k}$. Therefore, letting $\mathbf{x} \equiv\left(x_{1}, \cdots, x_{p}\right), \lim _{n \rightarrow \infty}\left\|\mathbf{x}^{n}-\mathbf{x}\right\|=0$.

Definition 1.6.3 Letting $\left\{\mathbf{x}^{n}\right\}$ be a sequence of vectors in $\mathbb{F}^{p}, \sum_{k=1}^{\infty} \mathbf{x}^{k}$ is said to converge if there exists $\mathbf{s}$ such that $\lim _{n \rightarrow \infty} \sum_{j=1}^{n} \mathbf{x}^{j}=\mathbf{s}$.

The Weierstrass $M$ test is a convenient way to consider convergence of series in $\mathbb{F}^{p}$.
Proposition 1.6.4 If there exists $M_{k}$ such that $M_{k} \geq\left\|\mathbf{x}^{k}\right\|$, and if $\sum_{k} M_{k}$ converges, then so does $\sum_{k} \mathbf{x}^{k}$.

Proof: For $m<n$,

$$
\left\|\sum_{k=1}^{n} \mathbf{x}^{k}-\sum_{k=1}^{m-1} \mathbf{x}^{k}\right\| \leq \sum_{k=m}^{n}\left\|\mathbf{x}^{k}\right\| \leq \sum_{k=m}^{\infty} M_{k}
$$

and if $m$ is large enough, the term on the right is no more than $\varepsilon$ by the standard material in Calculus. Therefore, the partial sums are a Cauchy sequence and must converge thanks to Proposition 1.6.2.

This is not the best norm for $\mathbb{F}^{p}$ however. That will be described next.

### 1.7 Inner Product Spaces

Definition 1.7.1 A vector space $V$ with field of scalars $\mathbb{C}$ or $\mathbb{R}$ is called an inner product space if it has an inner product $(\cdot, \cdot): V \times V \rightarrow \mathbb{C}$ which satisfies the following axioms.

$$
\begin{equation*}
(\mathbf{x}, \mathbf{y})=\overline{(\mathbf{y}, \mathbf{x})} \tag{1.5}
\end{equation*}
$$

For $a, b \in F$,

$$
\begin{gather*}
(a \mathbf{x}+b \mathbf{y}, \mathbf{z})=a(\mathbf{x}, \mathbf{z})+b(\mathbf{y}, \mathbf{z})  \tag{1.6}\\
(\mathbf{x}, \mathbf{x}) \geq 0 \text { and equals } 0 \text { if and only if } \mathbf{x}=\mathbf{0} \tag{1.7}
\end{gather*}
$$

Note that

$$
(\mathbf{z}, a \mathbf{x}+b \mathbf{y})=\overline{(a \mathbf{x}+b \mathbf{y}, \mathbf{z})}=\overline{a(\mathbf{x}, \mathbf{z})+b(\mathbf{y}, \mathbf{z})}=\bar{a}(\mathbf{z}, \mathbf{x})+\bar{b}(\mathbf{z}, \mathbf{y})
$$

The Cauchy Schwarz inequality is a fundamental result which always holds in such a context.

Proposition 1.7.2 Let $V$ be an inner product space. Then for $|\mathbf{x}| \equiv(\mathbf{x}, \mathbf{x})^{1 / 2}$, it follows that $|(\mathbf{x}, \mathbf{y})| \leq|\mathbf{x}||\mathbf{y}|$ and equality holds if and only if one vector is a scalar multiple of the other. $|\cdot|$ satsifies the axioms of a norm 1.1-1.3. Also, $|\mathbf{x}-\mathbf{y}|^{2}+|\mathbf{x}+\mathbf{y}|^{2}=2|\mathbf{x}|^{2}+2|\mathbf{y}|^{2}$.

Proof: There is $\theta \in \mathbb{C}$ such that $|\theta|=1$ and $\bar{\theta}(\mathbf{x}, \mathbf{y})=|(\mathbf{x}, \mathbf{y})|$. Then consider $p(t) \equiv$ $(\mathbf{x}+t \theta \mathbf{y}, \mathbf{x}+t \theta \mathbf{y})$. Thus $p(t) \geq 0$ for all $t \in \mathbb{R}$. From the axioms,

$$
\begin{aligned}
p(t) & =(\mathbf{x}, \mathbf{x})+2 \operatorname{Re} t \bar{\theta}(\mathbf{x}, \mathbf{y})+t^{2}(\mathbf{y}, \mathbf{y}) \\
& =(\mathbf{x}, \mathbf{x})+2 t|(\mathbf{x}, \mathbf{y})|+t^{2}(\mathbf{y}, \mathbf{y}) \geq 0
\end{aligned}
$$

Assume $(\mathbf{y}, \mathbf{y})>0$. Thus, the polynomial $p(t)$ has no real roots or only one. By the quadratic formula, $4|(\mathbf{x}, \mathbf{y})|^{2}-4(\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y}) \leq 0$ which is a restatement of the Cauchy Schwarz inequality. If $(\mathbf{y}, \mathbf{y})=0$, then $p(t)$ cannot be nonnegative for all $t \in \mathbb{R}$ unless $(\mathbf{x}, \mathbf{y})=0$ and so the inequality holds.

For the last claim, note that if one vector is a real multiple of the other equality holds from application of the axioms and definitions of $|\cdot|$. To go the other direction, equality holds if and only if the $4|(\mathbf{x}, \mathbf{y})|^{2}-4(\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y})=0$ if and only if the polynomial has exactly one real root if and only if for some $t$ real, $p(t)=0$ which implies for that $t$, $\mathbf{x}+t \theta \mathbf{y}=\mathbf{0}$ and so one vector is a multiple of the other.

As to $|\cdot|$ satisfying the axioms of a norm, these are all obvious except the triangle inequality which is shown next.

$$
\begin{align*}
|\mathbf{x}+\mathbf{y}|^{2} & =(\mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y})=(\mathbf{x}, \mathbf{x})+(\mathbf{x}, \mathbf{y})+(\mathbf{y}, \mathbf{x})+(\mathbf{y}, \mathbf{y}) \\
& =|\mathbf{x}|^{2}+2 \operatorname{Re}(\mathbf{x}, \mathbf{y})+|\mathbf{y}|^{2} \leq|\mathbf{x}|^{2}+2|(\mathbf{x}, \mathbf{y})|+|\mathbf{y}|^{2} \\
& \leq|\mathbf{x}|^{2}+2|\mathbf{x}| \mathbf{y}\left|+|\mathbf{y}|^{2}=(|\mathbf{x}|+|\mathbf{y}|)^{2}\right. \tag{1.8}
\end{align*}
$$

As to the last assertion, the parallelogram identity, it follows from a computation.

$$
\begin{aligned}
|\mathbf{x}-\mathbf{y}|^{2}+|\mathbf{x}+\mathbf{y}|^{2} & =(\mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y})+(\mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}) \\
& =|\mathbf{x}|^{2}+|\mathbf{y}|^{2}-2 \operatorname{Re}(\mathbf{x}, \mathbf{y})+|\mathbf{x}|^{2}+|\mathbf{y}|^{2}+2 \operatorname{Re}(\mathbf{x}, \mathbf{y}) \\
& =2|\mathbf{x}|^{2}+2|\mathbf{y}|^{2} \mathbf{\square}
\end{aligned}
$$

Note that the axiom which says $(\mathbf{x}, \mathbf{x})=0$ only if $\mathbf{x}=\mathbf{0}$ can be removed in the first part of this proposition and still obtain the Cauchy Schwarz inequality. The only thing used was that $(\mathbf{x}, \mathbf{x}) \geq 0$. Thus the Cauchy Schwarz inequality $|(\mathbf{x}, \mathbf{y})| \leq|\mathbf{x}||\mathbf{y}|$ even if you remove the second part of 1.7.

As a special case, let $\mathbf{x}, \mathbf{y} \in \mathbb{F}^{p}$ and let $(\mathbf{x}, \mathbf{y}) \equiv \sum_{k=1}^{p} x_{k} \overline{y_{k}}$. You can verify easily that 1.5-1.7 are satisfied. Therefore, in this case we have the Cauchy Schwarz inequality of Cauchy.

Corollary 1.7.3 Let $z_{j}, w_{j}$ be complex numbers. Then

$$
\left|\sum_{j=1}^{p} z_{j} \overline{w_{j}}\right| \leq\left(\sum_{j=1}^{p}\left|z_{j}\right|^{2}\right)^{1 / 2}\left(\sum_{j=1}^{p}\left|w_{j}\right|^{2}\right)^{1 / 2}
$$

The norm in an inner product space is $|\mathbf{x}| \equiv(\mathbf{x}, \mathbf{x})^{1 / 2}$.
Proposition 1.7.4 Each of $\|\cdot\|_{\infty}$ and $|\cdot|$ satisfy the axioms of a norm on $\mathbb{F}^{p}$, 1.1-1.3.
Note that the above two norms are equivalent in the sense that

$$
\begin{equation*}
\|\mathbf{x}\|_{\infty} \leq|\mathbf{x}| \leq \sqrt{p}\|\mathbf{x}\|_{\infty} \tag{*}
\end{equation*}
$$

Thus in all analytical considerations, it doesn't matter which norm is used. The two norms have the same Cauchy sequences for example. Actually, any two norms are equivalent, which will be shown later. The significance of the Euclidean norm $|\cdot|$ is geometrical. See the Problem 21 on Page 76 for example.

The triangle inequality holds for $|u| \equiv(u, u)^{1 / 2}$ for any inner product space by the same proof given in 1.8.

The fundamental result pertaining to the inner product just discussed is the Gram Schmidt process presented next.

Definition 1.7.5 $A$ set of vectors $\left\{\mathbf{v}_{1}, \cdots, v_{k}\right\}$ is called orthonormal if

$$
\left(\mathbf{v}_{i}, \mathbf{v}_{j}\right)=\delta_{i j} \equiv\left\{\begin{array}{l}
1 \text { if } i=j \\
0 \text { if } i \neq j
\end{array}\right.
$$

Then there is a very easy proposition which follows this.
Proposition 1.7.6 Suppose $\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{k}\right\}$ is an orthonormal set. Then it is linearly independent.

Proof: Suppose $\sum_{i=1}^{k} c_{i} \mathbf{v}_{i}=\mathbf{0}$. Then taking inner products with $\mathbf{v}_{j}$,

$$
0=\left(\mathbf{0}, \mathbf{v}_{j}\right)=\sum_{i} c_{i}\left(\mathbf{v}_{i}, \mathbf{v}_{j}\right)=\sum_{i} c_{i} \boldsymbol{\delta}_{i j}=c_{j}
$$

Since $j$ is arbitrary, this shows the set is linearly independent as claimed.
It turns out that if $X$ is any subspace of $\mathbb{F}^{m}$, then there exists an orthonormal basis for $X$. This follows from the use of the next lemma applied to a basis for $X$. Recall first that from linear algebra, every subspace of $\mathbb{F}^{m}$ has a basis.

Lemma 1.7.7 Let $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{n}\right\}$ be a linearly independent subset of $\mathbb{F}^{p}, p \geq n$. Then there exist orthonormal vectors $\left\{\mathbf{u}_{1}, \cdots, \mathbf{u}_{n}\right\}$ which have the property that for each $k \leq n$, $\operatorname{span}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{k}\right)=\operatorname{span}\left(\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}\right)$.

Proof: Let $\mathbf{u}_{1} \equiv \mathbf{x}_{1} /\left|\mathbf{x}_{1}\right|$. Thus for $k=1, \operatorname{span}\left(\mathbf{u}_{1}\right)=\operatorname{span}\left(\mathbf{x}_{1}\right)$ and $\left\{\mathbf{u}_{1}\right\}$ is an orthonormal set. Now suppose for some $k<n, \mathbf{u}_{1}, \cdots, \mathbf{u}_{k}$ have been chosen such that $\left(\mathbf{u}_{j}, \mathbf{u}_{l}\right)=\delta_{j l}$ and span $\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{k}\right)=\operatorname{span}\left(\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}\right)$. Then define

$$
\begin{equation*}
\mathbf{u}_{k+1} \equiv \frac{\mathbf{x}_{k+1}-\sum_{j=1}^{k}\left(\mathbf{x}_{k+1}, \mathbf{u}_{j}\right) \mathbf{u}_{j}}{\left|\mathbf{x}_{k+1}-\sum_{j=1}^{k}\left(\mathbf{x}_{k+1}, \mathbf{u}_{j}\right) \mathbf{u}_{j}\right|} \tag{1.9}
\end{equation*}
$$

where the denominator is not equal to zero because the $\mathbf{x}_{j}$ form a basis, and so

$$
\mathbf{x}_{k+1} \notin \operatorname{span}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{k}\right)=\operatorname{span}\left(\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}\right)
$$

Thus by induction,

$$
\mathbf{u}_{k+1} \in \operatorname{span}\left(\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}, \mathbf{x}_{k+1}\right)=\operatorname{span}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{k}, \mathbf{x}_{k+1}\right) .
$$

Also, $\mathbf{x}_{k+1} \in \operatorname{span}\left(\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}, \mathbf{u}_{k+1}\right)$ which is seen easily by solving 1.9 for $\mathbf{x}_{k+1}$ and it follows

$$
\operatorname{span}\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{k}, \mathbf{x}_{k+1}\right)=\operatorname{span}\left(\mathbf{u}_{1}, \cdots, \mathbf{u}_{k}, \mathbf{u}_{k+1}\right)
$$

If $l \leq k$,

$$
\begin{gathered}
\left(\mathbf{u}_{k+1}, \mathbf{u}_{l}\right)=C\left(\left(\mathbf{x}_{k+1}, \mathbf{u}_{l}\right)-\sum_{j=1}^{k}\left(\mathbf{x}_{k+1}, \mathbf{u}_{j}\right)\left(\mathbf{u}_{j}, \mathbf{u}_{l}\right)\right)= \\
C\left(\left(\mathbf{x}_{k+1}, \mathbf{u}_{l}\right)-\sum_{j=1}^{k}\left(\mathbf{x}_{k+1}, \mathbf{u}_{j}\right) \delta_{l j}\right)=C\left(\left(\mathbf{x}_{k+1}, \mathbf{u}_{l}\right)-\left(\mathbf{x}_{k+1}, \mathbf{u}_{l}\right)\right)=0 .
\end{gathered}
$$

The vectors, $\left\{\mathbf{u}_{j}\right\}_{j=1}^{n}$, generated in this way are therefore orthonormal because each vector has unit length.

The following lemma is a fairly simple observation about the Gram Schmidt process which says that if you start with orthonormal vectors, the process will not undo what you already have.

Lemma 1.7.8 Suppose $\left\{\mathbf{w}_{1}, \cdots, \mathbf{w}_{r}, \mathbf{v}_{r+1}, \cdots, \mathbf{v}_{p}\right\}$ is a linearly independent set of vectors such that $\left\{\mathbf{w}_{1}, \cdots, \mathbf{w}_{r}\right\}$ is an orthonormal set of vectors. Then when the Gram Schmidt process is applied to the vectors in the given order, it will not change any of the $\mathbf{w}_{1}, \cdots, \mathbf{w}_{r}$.

Proof: Let $\left\{\mathbf{u}_{1}, \cdots, \mathbf{u}_{p}\right\}$ be the orthonormal set delivered by the Gram Schmidt process. Then $\mathbf{u}_{1}=\mathbf{w}_{1}$ because by definition, $\mathbf{u}_{1} \equiv \mathbf{w}_{1} /\left|\mathbf{w}_{1}\right|=\mathbf{w}_{1}$. Now suppose $\mathbf{u}_{j}=\mathbf{w}_{j}$ for all $j \leq k \leq r$. Then if $k<r$, consider the definition of $\mathbf{u}_{k+1}$.

$$
\mathbf{u}_{k+1} \equiv \frac{\mathbf{w}_{k+1}-\sum_{j=1}^{k+1}\left(\mathbf{w}_{k+1}, \mathbf{u}_{j}\right) \mathbf{u}_{j}}{\left|\mathbf{w}_{k+1}-\sum_{j=1}^{k+1}\left(\mathbf{w}_{k+1}, \mathbf{u}_{j}\right) \mathbf{u}_{j}\right|}
$$

By induction, $\mathbf{u}_{j}=\mathbf{w}_{j}$ and so this reduces to $\mathbf{w}_{k+1} /\left|\mathbf{w}_{k+1}\right|=\mathbf{w}_{k+1}$.

Lemma 1.7.9 Suppose $V, W$ are two inner product spaces which have orthonormal bases,

$$
\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{r}\right\},\left\{\mathbf{w}_{1}, \cdots, \mathbf{w}_{r}\right\}
$$

respectively. Let A map $V$ to $W$ be defined by

$$
A\left(\sum_{k=1}^{r} c_{k} \mathbf{v}_{k}\right) \equiv \sum_{k=1}^{r} c_{k} \mathbf{w}_{k}
$$

Then $|A \mathbf{v}|=|\mathbf{v}|$. That is, A preserves Euclidean norms.
Proof: This follows right away from a computation. If $\left\{\mathbf{u}_{1}, \cdots, \mathbf{u}_{r}\right\}$ is orthonormal, then

$$
\left|\sum_{k=1}^{r} c_{k} \mathbf{u}_{k}\right|^{2}=\left(\sum_{k=1}^{r} c_{k} \mathbf{u}_{k}, \sum_{k=1}^{r} c_{k} \mathbf{u}_{k}\right)=\sum_{j, k} c_{k} \overline{c_{j}}\left(\mathbf{u}_{k}, \mathbf{u}_{j}\right)=\sum_{k} c_{k} \overline{c_{k}}=\sum_{k}\left|c_{k}\right|^{2}
$$

Therefore, $\left|A\left(\sum_{k=1}^{r} c_{k} \mathbf{v}_{k}\right)\right|^{2}=\left|\sum_{k=1}^{r} c_{k} \mathbf{w}_{k}\right|^{2}=\sum_{k}\left|c_{k}\right|^{2}=\left|\sum_{k=1}^{r} c_{k} \mathbf{v}_{k}\right|^{2}$.

### 1.8 Polynomials

Polynomials are a lot like integers. The notion of division is important for polynomials in the same way that it is for integers.
Definition 1.8.1 $A$ polynomial is an expression of the form

$$
a_{n} \lambda^{n}+a_{n-1} \lambda^{n-1}+\cdots+a_{1} \lambda+a_{0}
$$

$a_{n} \neq 0$ where the $a_{i}$ come from a field of scalars. In this book, the field will be $\mathbb{R}$ or $\mathbb{C}$, but the field of scalars could be any field. Two polynomials are equal means that the coefficients match for each power of $\lambda$. The degree of a polynomial is the largest power of $\lambda$. Thus the degree of the above polynomial is $n$. Addition of polynomials is defined in the usual way as is multiplication of two polynomials.The leading term in the above polynomial is $a_{n} \lambda^{n}$. The coefficient of the leading term is called the leading coefficient. It is called a monic polynomial when $a_{n}=1$.

Note that the degree of the zero polynomial is not defined in the above. Multiplication of polynomials has an important property.

Lemma 1.8.2 If $f(\lambda) g(\lambda)=0$, then either $f(\lambda)=0$ or $g(\lambda)=0$. That is, there are no nonzero divisors of 0 .

Proof: Let $f(\boldsymbol{\lambda})$ have degree $n$ and $g(\lambda)$ degree $m$. If $m+n=0$, it is easy to see that the conclusion holds. Suppose the conclusion holds for $m+n \leq M$ and suppose $m+n=M+1$. Then

$$
\begin{aligned}
f(\lambda) g(\lambda)= & \left(a_{0}+a_{1} \lambda+\cdots+a_{n-1} \lambda^{n-1}+a_{n} \lambda^{n}\right) . \\
& \left(b_{0}+b_{1} \lambda+\cdots+b_{m-1} \lambda^{m-1}+b_{m} \lambda^{m}\right) \\
= & \left(a(\lambda)+a_{n} \lambda^{n}\right)\left(b(\lambda)+b_{m} \lambda^{m}\right) \\
= & a(\lambda) b(\lambda)+b_{m} \lambda^{m} a(\lambda)+a_{n} \lambda^{n} b(\lambda)+a_{n} b_{m} \lambda^{n+m}
\end{aligned}
$$

Either $a_{n}=0$ or $b_{m}=0$. Suppose $b_{m}=0$. Then $\left(a(\lambda)+a_{n} \lambda^{n}\right) b(\lambda)=0$. By induction, one of these polynomials in the product is 0 . If $b(\lambda) \neq 0$, then this shows $a_{n}=0$ and $a(\lambda)=0$ so $f(\lambda)=0$. If $b(\lambda)=0$, then, since $b_{m}=0, g(\lambda)=0$. The argument is similar if $a_{n}=0$.

Lemma 1.8.3 Let $f(\lambda)$ and $g(\lambda) \neq 0$ be polynomials. Then there exist polynomials, $q(\lambda)$ and $r(\lambda)$ such that

$$
f(\lambda)=q(\lambda) g(\lambda)+r(\lambda)
$$

where the degree of $r(\lambda)$ is less than the degree of $g(\lambda)$ or $r(\lambda)=0$. These polynomials $q(\lambda)$ and $r(\lambda)$ are unique.

Proof: Suppose that $f(\boldsymbol{\lambda})-q(\boldsymbol{\lambda}) g(\boldsymbol{\lambda})$ is never equal to 0 for any $q(\boldsymbol{\lambda})$. If it is, then the conclusion follows. Now suppose

$$
\begin{equation*}
r(\boldsymbol{\lambda})=f(\boldsymbol{\lambda})-q(\boldsymbol{\lambda}) g(\boldsymbol{\lambda}) \tag{*}
\end{equation*}
$$

where the degree of $r(\lambda)$ is as small as possible. Let it be $m$. Suppose $m \geq n$ where $n$ is the degree of $g(\lambda)$. Say $r(\lambda)=b \lambda^{m}+a(\lambda)$ where $a(\lambda)$ is 0 or has degree less than $m$ while $g(\lambda)=\hat{b} \lambda^{n}+\hat{a}(\lambda)$ where $\hat{a}(\lambda)$ is 0 or has degree less than $n$. Then

$$
r(\lambda)-\frac{b}{\hat{b}} \lambda^{m-n} g(\lambda)=b \lambda^{m}+a(\lambda)-\left(b \lambda^{m}+\frac{b}{\hat{b}} \lambda^{m-n} \hat{a}(\lambda)\right)=a(\lambda)-\tilde{a}(\lambda)
$$

a polynomial having degree less than $m$. Therefore,

$$
a(\lambda)-\tilde{a}(\lambda)=\overbrace{(f(\lambda)-q(\lambda) g(\lambda))}^{=r(\lambda)}-\frac{b}{\hat{b}} \lambda^{m-n} g(\lambda)=f(\lambda)-\hat{q}(\lambda) g(\lambda)
$$

which is of the same form as $*$ having smaller degree. However, $m$ was as small as possible. Hence $m<n$ after all.

As to uniqueness, if you have $r(\lambda), \hat{r}(\lambda), q(\lambda), \hat{q}(\lambda)$ which work, then you would have

$$
(\hat{q}(\lambda)-q(\lambda)) g(\lambda)=r(\lambda)-\hat{r}(\lambda)
$$

Now if the polynomial on the right is not zero, then neither is the one on the left. Hence this would involve two polynomials which are equal although their degrees are different. This is impossible. Hence $r(\boldsymbol{\lambda})=\hat{r}(\boldsymbol{\lambda})$ and so, the above lemma shows $\hat{q}(\boldsymbol{\lambda})=q(\boldsymbol{\lambda})$.

Definition 1.8.4 Let $p(\lambda)=a_{n} \lambda^{n}+\cdots+a_{1} \lambda+a_{0}$ be a polynomial. Then for $\alpha$ a scalar $p(\alpha) \equiv a_{n} \alpha^{n}+\cdots+a_{1} \alpha+a_{0}$. A scalar $\alpha$ is a root of the polynomial means $p(\alpha)=0$.

Proposition 1.8.5 $\alpha$ is a root of $p(\lambda)$ if and only if $p(\lambda)=(\lambda-\alpha) q(\lambda)$ for some polynomial $q(\lambda)$.

Proof: By the division algorithm, $p(\lambda)=(\lambda-\alpha) q(\lambda)+r$ where $r$ has degree 0 so is a scalar. Then $\alpha$ is a root if and only if $r=0$ from this formula.

Definition 1.8.6 A polynomial $f$ is divides a polynomial $g$ if $g(\lambda)=f(\lambda) r(\lambda)$ for some polynomial $r(\lambda)$. Let $\left\{\phi_{i}(\lambda)\right\}$ be a finite set of polynomials. The greatest common divisor will be the monic polynomial $q(\lambda)$ such that $q(\lambda)$ divides each $\phi_{i}(\lambda)$ and if $p(\lambda)$ divides each $\phi_{i}(\lambda)$, then $p(\lambda)$ divides $q(\lambda)$. The finite set of polynomials $\left\{\phi_{i}\right\}$ is said to be relatively prime if their greatest common divisor is 1. A polynomial $f(\lambda)$ is irreducible if there is no polynomial with coefficients in $\mathbb{F}$ which divides it except nonzero scalar multiples of $f(\lambda)$ and constants. In other words, it is not possible to write $f(\lambda)=a(\lambda) b(\lambda)$ where each of $a(\lambda), b(\lambda)$ have degree less than the degree of $f(\lambda)$ unless one of $a(\lambda), b(\lambda)$ is $a$ constant.

Proposition 1.8.7 The greatest common divisor is unique.
Proof: Suppose both $q(\lambda)$ and $q^{\prime}(\lambda)$ work. Then $q(\lambda)$ divides $q^{\prime}(\lambda)$ and the other way around and so $q^{\prime}(\lambda)=q(\lambda) l(\lambda), q(\lambda)=l^{\prime}(\lambda) q^{\prime}(\lambda)$. Therefore, the two must have the same degree. Hence $l^{\prime}(\lambda), l(\lambda)$ are both constants. However, this constant must be 1 because both $q(\lambda)$ and $q^{\prime}(\lambda)$ are monic.

Theorem 1.8.8 Let $\left\{\phi_{i}(\lambda)\right\}$ be polynomials, not all of which are zero polynomials. Then it follows that there exists a greatest common divisor and it equals the monic polynomial $\psi(\lambda)$ of smallest degree such that there exist polynomials $r_{i}(\lambda)$ satisfying $\psi(\lambda)=\sum_{i=1}^{p} r_{i}(\lambda) \phi_{i}(\lambda)$.

Proof: Let $S$ denote the set of monic polynomials of the form $\sum_{i=1}^{p} r_{i}(\lambda) \phi_{i}(\lambda)$. where $r_{i}(\lambda)$ is a polynomial. Then $S \neq \emptyset$ because some $\phi_{i}(\lambda) \neq 0$. Then let the $r_{i}$ be chosen such that the degree of the expression $\sum_{i=1}^{p} r_{i}(\lambda) \phi_{i}(\lambda)$ is as small as possible. Letting $\psi(\lambda)$ equal this sum, it remains to verify it is the greatest common divisor. First, does it divide each $\phi_{i}(\lambda)$ ? Suppose it fails to divide $\phi_{1}(\lambda)$. Then by Lemma 1.8.3, $\phi_{1}(\lambda)=$ $\psi(\lambda) l(\lambda)+r(\lambda)$ where degree of $r(\lambda)$ is less than that of $\psi(\lambda)$. Then dividing $r(\lambda)$ by the leading coefficient if necessary and denoting the result by $\psi_{1}(\lambda)$, it follows the degree of $\psi_{1}(\lambda)$ is less than the degree of $\psi(\lambda)$ and $\psi_{1}(\lambda)$ equals for some $a \in \mathbb{F}$

$$
\begin{aligned}
\psi_{1}(\lambda)= & \left(\phi_{1}(\lambda)-\psi(\lambda) l(\lambda)\right) a=\left(\phi_{1}(\lambda)-\sum_{i=1}^{p} r_{i}(\lambda) \phi_{i}(\lambda) l(\lambda)\right) a \\
& =\left(\left(1-r_{1}(\lambda)\right) \phi_{1}(\lambda)+\sum_{i=2}^{p}\left(-r_{i}(\lambda) l(\lambda)\right) \phi_{i}(\lambda)\right) a
\end{aligned}
$$

This is one of the polynomials in $S$. Therefore, $\psi(\lambda)$ does not have the smallest degree after all because the degree of $\psi_{1}(\lambda)$ is smaller. This is a contradiction. Therefore, $\psi(\lambda)$ divides $\phi_{1}(\lambda)$. Similarly it divides all the other $\phi_{i}(\lambda)$.

If $p(\lambda)$ divides all the $\phi_{i}(\lambda)$, then it divides $\psi(\lambda)$ because of the formula for $\psi(\lambda)$ which equals $\sum_{i=1}^{p} r_{i}(\lambda) \phi_{i}(\lambda)$. Thus $\psi(\lambda)$ satisfies the condition to be the greatest common divisor. This shows the greatest common divisor exists and equals the above description of it.

Lemma 1.8.9 Suppose $\phi(\lambda)$ and $\psi(\lambda)$ are monic polynomials which are irreducible and not equal. Then they are relatively prime.

Proof: Suppose $\eta(\lambda)$ is a nonconstant polynomial. If $\eta(\lambda)$ divides $\phi(\lambda)$, then since $\phi(\lambda)$ is irreducible, $\phi(\lambda)=\eta(\lambda) \tilde{a}$ for some constant $\tilde{a}$. Thus $\eta(\lambda)$ equals $a \phi(\lambda)$ for some $a \in \mathbb{F}$. If $\eta(\lambda)$ divides $\psi(\lambda)$ then it must be of the form $b \psi(\lambda)$ for some $b \in \mathbb{F}$ and so it follows $\eta(\lambda)=a \phi(\lambda)=b \psi(\lambda), \psi(\lambda)=\frac{a}{b} \phi(\lambda)$ but both $\psi(\lambda)$ and $\phi(\lambda)$ are monic polynomials which implies $a=b$ and so $\psi(\lambda)=\phi(\lambda)$. This is assumed not to happen. It follows the only polynomials which divide both $\psi(\lambda)$ and $\phi(\lambda)$ are constants and so the two polynomials are relatively prime. Thus a polynomial which divides them both must be a constant, and if it is monic, then it must be 1 . Thus 1 is the greatest common divisor.

Lemma 1.8.10 Let $\psi(\lambda)$ be an irreducible monic polynomial not equal to 1 which divides

$$
\prod_{i=1}^{p} \phi_{i}(\lambda)^{k_{i}}, k_{i} \text { a positive integer }
$$

where each $\phi_{i}(\lambda)$ is an irreducible monic polynomial not equal to 1 . Then $\psi(\lambda)$ equals some $\phi_{i}(\lambda)$.

Proof : Say $\psi(\lambda) l(\lambda)=\prod_{i=1}^{p} \phi_{i}(\lambda)^{k_{i}}$. Suppose $\psi(\lambda) \neq \phi_{i}(\lambda)$ for all $i$. Then these two $\psi(\lambda)$ and $\phi_{i}(\lambda)$ are relatively prime and by Lemma 1.8.9, there exist polynomials $m_{i}(\lambda), n_{i}(\lambda)$ such that

$$
\begin{aligned}
1 & =\psi(\lambda) m_{i}(\lambda)+\phi_{i}(\lambda) n_{i}(\lambda) \\
\phi_{i}(\lambda) n_{i}(\lambda) & =1-\psi(\lambda) m_{i}(\lambda)
\end{aligned}
$$

It follows that

$$
\prod_{i=1}^{p}\left(\phi_{i}(\lambda) n_{i}(\lambda)\right)^{k_{i}}=\prod_{i=1}^{p}\left(1-\psi(\lambda) m_{i}(\lambda)\right)^{k_{i}}=1+\psi(\lambda) g(\lambda)
$$

where $g(\lambda)$ is the polynomial which multiplies $\psi(\lambda)$ in that product. Hence, for $n(\lambda)=$ $\prod_{i} n_{i}(\lambda)^{k_{i}}$,

$$
\begin{gathered}
n(\lambda) \prod_{i=1}^{p} \phi_{i}(\lambda)^{k_{i}}=n(\lambda) l(\lambda) \psi(\lambda)=1+\psi(\lambda) g(\lambda) \\
\psi(\lambda)(n(\lambda) l(\lambda)-g(\lambda))=1
\end{gathered}
$$

which is impossible because $\psi(\lambda) \neq 1$.
Of course, since coefficients are in a field, you can drop the stipulation that the polynomials are monic and replace the conclusion with: $\psi(\lambda)$ is a multiple of some $\phi_{i}(\lambda)$.

Now here is a simple lemma about canceling monic polynomials. It follows easily from Lemma 1.8.2. That Lemma could also be obtained from a simple modification of the argument given here.

Lemma 1.8.11 Suppose $p(\lambda)$ is a monic polynomial and $q(\lambda)$ is a polynomial such that $p(\lambda) q(\boldsymbol{\lambda})=0$. Then $q(\boldsymbol{\lambda})=0$. Also if $p(\boldsymbol{\lambda}) q_{1}(\boldsymbol{\lambda})=p(\boldsymbol{\lambda}) q_{2}(\boldsymbol{\lambda})$ then $q_{1}(\boldsymbol{\lambda})=$ $q_{2}(\lambda)$.

Proof: Let $p(\lambda)=\sum_{j=1}^{k} p_{j} \lambda^{j}, q(\lambda)=\sum_{i=1}^{n} q_{i} \lambda^{i}, p_{k}=1$.Then the product equals $\sum_{j=1}^{k} \sum_{i=1}^{n} p_{j} q_{i} \lambda^{i+j}$. If not all $q_{i}=0$, let $q_{m}$ be the last coefficient which is nonzero. Then the above is of the form $\sum_{j=1}^{k} \sum_{i=1}^{m} p_{j} q_{i} \lambda^{i+j}=0$. Consider the $\lambda^{m+k}$ term. There is only
one and it is $p_{k} q_{m} \lambda^{m+k}$. Since $p_{k}=1, q_{m}=0$ after all. The second part follows from $p(\boldsymbol{\lambda})\left(q_{1}(\boldsymbol{\lambda})-q_{2}(\boldsymbol{\lambda})\right)=0$.

The following is the analog of the fundamental theorem of arithmetic for polynomials.
Theorem 1.8.12 Let $f(\lambda)$ be a nonconstant polynomial with coefficients in $\mathbb{F}$. Then there is some $a \in \mathbb{F}$ such that $f(\lambda)=a \prod_{i=1}^{n} \phi_{i}(\lambda)$ where $\phi_{i}(\lambda)$ is an irreducible nonconstant monic polynomial and repeats are allowed. Furthermore, this factorization is unique in the sense that any two of these factorizations have the same nonconstant factors in the product, possibly in different order and the same constant a. Every subset of $\left\{\phi_{i}(\lambda), i=1, \ldots, n\right\}$ having at least two elements is relatively prime.

Proof: That such a factorization exists is obvious. If $f(\lambda)$ is irreducible, you are done. Factor out the leading coefficient. If not, then $f(\lambda)=a \phi_{1}(\lambda) \phi_{2}(\lambda)$ where these are monic polynomials. Continue doing this with the $\phi_{i}$ and eventually arrive at a factorization of the desired form.

It remains to argue the factorization is unique except for order of the factors. Suppose

$$
a \prod_{i=1}^{n} \phi_{i}(\lambda)=b \prod_{i=1}^{m} \psi_{i}(\lambda)
$$

where the $\phi_{i}(\lambda)$ and the $\psi_{i}(\lambda)$ are all irreducible monic nonconstant polynomials and $a, b \in \mathbb{F}$. If $n>m$, then by Lemma 1.8.10, each $\psi_{i}(\lambda)$ equals one of the $\phi_{j}(\lambda)$. By the above cancellation lemma, Lemma 1.8.11, you can cancel all these $\psi_{i}(\lambda)$ with appropriate $\phi_{j}(\lambda)$ and obtain a contradiction because the resulting polynomials on either side would have different degrees. Similarly, it cannot happen that $n<m$. It follows $n=m$ and the two products consist of the same polynomials. Then it follows $a=b$. If you have such a subset of the $\phi_{i}(\lambda)$, the monic polynomial of smallest degree which divides them all must be 1 because none of the $\phi_{i}(\lambda)$ divide any other since they are all irreducible.

The following corollary will be well used. This corollary seems rather believable but does require a proof.

Corollary 1.8.13 Let $q(\lambda)=\prod_{i=1}^{p} \phi_{i}(\lambda)^{k_{i}}$ where the $k_{i}$ are positive integers and the $\phi_{i}(\lambda)$ are irreducible distinct monic polynomials. Suppose also that $p(\lambda)$ is a monic polynomial which divides $q(\lambda)$. Then $p(\lambda)=\prod_{i=1}^{p} \phi_{i}(\lambda)^{r_{i}}$ where $r_{i}$ is a nonnegative integer no larger than $k_{i}$.

Proof: Using Theorem 1.8.12, let $p(\lambda)=b \prod_{i=1}^{s} \psi_{i}(\lambda)^{r_{i}}$ where the $\psi_{i}(\lambda)$ are each irreducible and monic and $b \in \mathbb{F}$. Since $p(\lambda)$ is monic, $b=1$. Then there exists a polyno$\operatorname{mial} g(\lambda)$ such that $p(\lambda) g(\lambda)=g(\lambda) \prod_{i=1}^{s} \psi_{i}(\lambda)^{r_{i}}=\prod_{i=1}^{p} \phi_{i}(\lambda)^{k_{i}}$. Hence $g(\lambda)$ must be monic. Therefore,

$$
p(\boldsymbol{\lambda}) g(\boldsymbol{\lambda})=\overbrace{\prod_{i=1}^{s} \psi_{i}(\boldsymbol{\lambda})^{r_{i}}}^{p(\lambda)} \prod_{j=1}^{l} \eta_{j}(\boldsymbol{\lambda})=\prod_{i=1}^{p} \phi_{i}(\boldsymbol{\lambda})^{k_{i}}
$$

for $\eta_{j}$ monic and irreducible. By uniqueness, each $\psi_{i}(\lambda)$ equals one of the $\phi_{j}(\lambda)$ and the same holding true of the $\eta_{i}(\lambda)$. Therefore, $p(\lambda)$ is of the desired form because you can cancel the $\eta_{j}(\lambda)$ from both sides.

### 1.9 The Method of Partial Fractions

A very useful method is the method of partial fractions having to do with rational functions, quotients of polynomials. In applications known to me, these are usually thought of as functions of $\lambda$ and this is what we like to call such quotients, but everything is based only on the usual algebraic manipulations for polynomials. Algebra of quotients of polynomials will involve the usual algebraic processes just as with polynomials. That is $\frac{p(\lambda)}{q(\lambda)}=\frac{\hat{p}(\lambda)}{\hat{q}(\lambda)}$ will mean $p(\lambda) \hat{q}(\boldsymbol{\lambda})=\hat{p}(\boldsymbol{\lambda}) q(\boldsymbol{\lambda})$, and we know what it means for two polynomials to be equal.

Proposition 1.9.1 Suppose $r(\boldsymbol{\lambda})=\frac{a(\lambda)}{p(\lambda)^{m}}$ where $a(\boldsymbol{\lambda})$ is a polynomial and $p(\boldsymbol{\lambda})$ is a polynomial of degree at least 1. Then

$$
r(\lambda)=q(\lambda)+\sum_{k=1}^{m} \frac{b_{k}(\lambda)}{p(\lambda)^{k}}, \text { where degree of } b_{k}(\lambda)<\text { degree of } p(\lambda) \text { or } b_{k}(\lambda)=0
$$

Proof: Suppose first that $m=1$. If the degree of $a(\lambda)$ is larger than the degree of $p(\lambda)$, then do the division algorithm to write $a(\lambda)=p(\lambda) q(\lambda)+\hat{a}(\lambda)$ where the degree of $\hat{a}(\lambda)$ is less than the degree of $p(\lambda)$ or else $\hat{a}(\lambda)=0$. Thus the expression reduces to

$$
\frac{p(\lambda) q(\lambda)+m(\lambda)}{p(\lambda)}=q(\lambda)+\frac{\hat{a}(\lambda)}{p(\lambda)}
$$

and now it is in the desired form. Thus the Proposition is true if $m=1$. Suppose it is true for $m-1 \geq 1$. Then there is nothing to show if the degree of $a(\lambda)$ is less than the degree of $p(\lambda)$, so assume the degree of $a(\lambda)$ is larger than the degree of $p(\lambda)$. Then use the division algorithm as above and write

$$
\frac{a(\lambda)}{p(\lambda)^{m}}=\frac{p(\lambda) q(\lambda)+\hat{a}(\lambda)}{p(\lambda)^{m}}
$$

where the degree of $\hat{a}(\lambda)$ is less than the degree of $p(\lambda)$ or else is 0 . Then the above equals

$$
\frac{a(\lambda)}{p(\lambda)^{m}}=\frac{q(\lambda)}{p(\lambda)^{m-1}}+\frac{\hat{a}(\lambda)}{p(\lambda)^{m}}
$$

and by induction on the first term on the right, this proves the proposition.
With this, the general partial fractions theorem is next. From Theorem 1.8.12, every polynomial $q(\lambda)$ has a factorization of the form $\prod_{i=1}^{M} p_{i}(\lambda)^{m_{i}}$ where the $p_{i}(\lambda)$ are irreducible, meaning they cannot be factored further. Thus the polynomials $p_{i}(\lambda)$ are distinct and relatively prime as is every subset having at least two of these $p_{i}(\lambda)$.

Proposition 1.9.2 Let $\frac{a(\lambda)}{b(\lambda)}=\frac{a(\lambda)}{\prod_{i=1}^{M} p_{i}(\lambda)^{m_{i}}}$ be any rational function where the $p_{i}(\lambda)$ are distinct irreducible polynomials, meaning they can't be factored any further as described in the chapter and each $m_{i}$ is a nonnegative integer.

Then there are polynomials $q(\lambda)$ and $n_{k i}(\lambda)$ with the degree of $n_{k i}(\lambda)$ less than the degree of $p_{i}(\lambda)$ or $n_{k i}(\lambda)=0$, such that

$$
\begin{equation*}
\frac{a(\lambda)}{b(\lambda)}=q(\lambda)+\sum_{i=1}^{M} \sum_{k=1}^{m_{i}} \frac{n_{k i}(\lambda)}{p_{i}(\lambda)^{k}} \tag{1.10}
\end{equation*}
$$

Proof: Suppose first that $\sum_{i=1}^{M} m_{i}=1$. Then the rational function is of the form $\frac{a(\lambda)}{p(\lambda)}$ and this can be placed in the desired form by an application of the division algorithm as above. Suppose now that this proposition is true if $\sum_{i=1}^{M} m_{i} \leq n$ for some $n \geq 1$ and suppose you have

$$
\frac{a(\lambda)}{b(\lambda)}=\frac{a(\lambda)}{\prod_{j=1}^{M} p_{j}(\lambda)^{m_{j}}}, \quad \sum_{j=1}^{M} m_{j}=n+1, \text { each } m_{j} \geq 0
$$

If some $m_{j}=n+1$, then one obtains the situation of Proposition 1.9.1. Therefore, it suffices to assume that no $m_{j}=n+1$ so there are at least two $m_{j}$ which are nonzero.

Every subset of the $\left\{p_{1}(\boldsymbol{\lambda}), p_{2}(\boldsymbol{\lambda}), \ldots, p_{M}(\boldsymbol{\lambda})\right\}$ having at least two $p_{i}(\boldsymbol{\lambda})$ is relatively prime because these polynomials are all irreducible. Therefore, there are polynomials $b_{i}(\lambda)$ such that $b_{i}(\lambda)=0$ if $m_{i}=0$ and $\sum_{i=1}^{M} b_{i}(\lambda) p_{i}(\lambda)=1$. Then multiply by this to obtain

$$
\frac{a(\lambda)}{b(\lambda)}=\frac{a(\lambda)}{\prod_{j=1}^{M} p_{j}(\lambda)^{m_{j}}}=\frac{a(\lambda) \sum_{i=1}^{M} b_{i}(\lambda) p_{i}(\lambda)}{\prod_{j=1}^{M} p_{j}(\lambda)^{m_{j}}}=\sum_{i=1}^{M} \frac{a(\lambda) b_{i}(\lambda) p_{i}(\lambda)}{\prod_{j=1}^{M} p_{j}(\lambda)^{m_{j}}}
$$

Now in the $i^{t h}$ term of the sum, the $p_{i}(\lambda)$ in the top cancels with exactly one of the factors in the bottom or else the term is 0 . It follows that the original $\frac{a(\lambda)}{b(\lambda)}$ is of the form $\sum_{i=1}^{N} \frac{\hat{a}_{i}(\lambda)}{\prod_{j=1}^{M} p_{j}(\lambda)^{m_{i j}}}$ where $\sum_{j=1}^{M} m_{i j} \leq n$. By induction applied to each of the terms in this sum, one obtains $\frac{a(\lambda)}{b(\lambda)}$ equal to an expression of the form in 1.10.

Proposition 1.9.3 The partial fractions expansion is unique.
Proof: Suppose $q(\lambda)+\sum_{i=1}^{M} \sum_{k=1}^{m_{i}} \frac{n_{k i}(\lambda)}{p_{i}(\lambda)^{k}}=0$. Multiply both sides by the following product. $\prod_{j \neq i} p_{j}(\lambda)^{m_{i}} p_{i}(\lambda)^{m_{i}-1}$. Then you get an expression of the form $\hat{q}(\lambda)+\frac{n_{m_{i}}(\lambda)}{p_{i}(\lambda)}=$ 0 where $\hat{q}(\boldsymbol{\lambda})$ is a nonzero polynomial. Thus $\hat{q}(\boldsymbol{\lambda}) p_{i}(\boldsymbol{\lambda})=-n_{m_{i} i}(\boldsymbol{\lambda})$ which is impossible because the two polynomials have different degrees. Next multiply both sides by $\prod_{j \neq i} p_{j}(\lambda)^{m_{i}} p_{i}(\lambda)^{m_{i}-2}$ and by the same reasoning conclude that $n_{m_{i-1} i}(\lambda)=0$. Continuing this way, you see that each $n_{m_{i} i}(\lambda)=0$. Next do the same to show the $n_{k i}(\lambda)=0$ for a different $i$. Thus all of the $n_{k i}(\lambda)=0$ and so $q(\lambda)=0$ also. If you have two partial fractions expansions, subtract one from the other and apply what was just shown.

Once you know the correct form for the partial fractions, it is just a matter of multiplying out and doing linear algebra to find it.

### 1.10 The Fundamental Theorem of Algebra

The fundamental theorem of algebra states that every non constant polynomial having coefficients in $\mathbb{C}$ has a zero in $\mathbb{C}$. If $\mathbb{C}$ is replaced by $\mathbb{R}$, this is not true because of the example, $x^{2}+1=0$. This theorem is a very remarkable result and notwithstanding its title, all the proofs depend on either analysis or topology in some way. It was first mostly proved by Gauss in 1797. The first complete proof was given by Argand in 1806. The proof given later in the book follows Rudin [39]. See also Hardy [20] for a similar proof, more discussion and references. The shortest proofs are in the theory of complex analysis and are also presented later. Here is an informal explanation of this theorem which shows why it is reasonable to believe in the fundamental theorem of algebra.

## Theorem 1.10.1 Let $p(z)=a_{n} z^{n}+a_{n-1} z^{n-1}+\cdots+a_{1} z+a_{0}$ where each $a_{k}$ is $a$

 complex number and $a_{n} \neq 0, n \geq 1$. Then there exists $w \in \mathbb{C}$ such that $p(w)=0$.To begin with, here is the informal explanation. Dividing by the leading coefficient $a_{n}$, there is no loss of generality in assuming that the polynomial is of the form

$$
p(z)=z^{n}+a_{n-1} z^{n-1}+\cdots+a_{1} z+a_{0}
$$

If $a_{0}=0$, there is nothing to prove because $p(0)=0$. Therefore, assume $a_{0} \neq 0$. From the polar form of a complex number $z$, it can be written as $|z|(\cos \theta+i \sin \theta)$. Thus, by DeMoivre's theorem, $z^{n}=|z|^{n}(\cos (n \theta)+i \sin (n \theta))$ It follows that $z^{n}$ is some point on the circle of radius $|z|^{n}$

Denote by $C_{r}$ the circle of radius $r$ in the complex plane which is centered at 0 . Then if $r$ is sufficiently large and $|z|=r$, the term $z^{n}$ is far larger than the rest of the polynomial. It is on the circle of radius $|z|^{n}$ while the other terms are on circles of fixed multiples of $|z|^{k}$ for $k \leq n-1$. Thus, for $r$ large enough, $A_{r}=\left\{p(z): z \in C_{r}\right\}$ describes a closed curve which misses the inside of some circle having 0 as its center. It won't be as simple as suggested in the following picture, but it will be a closed curve thanks to De Moivre's theorem and the observation that the cosine and sine are periodic. Now shrink $r$. Eventually, for $r$ small enough, the non constant terms are negligible and so $A_{r}$ is a curve which is contained in some circle centered at $a_{0}$ which has 0 on the outside.

Thus it is reasonable to believe that for some $r$ during this shrinking process, the set $A_{r}$ must hit 0 . It follows that $p(z)=0$ for some $z$.

For example, consider the polynomial $x^{3}+x+$ $1+i$. It has no real zeros. However, you could let $z=r(\cos t+i \sin t)$ and insert this into the polyno$r$ small
mial. Thus you would want to find a point where

$$
(r(\cos t+i \sin t))^{3}+r(\cos t+i \sin t)+1+i=0+0 i
$$

Expanding this expression on the left to write it in terms of real and imaginary parts, you get on the left

$$
r^{3} \cos ^{3} t-3 r^{3} \cos t \sin ^{2} t+r \cos t+1+i\left(3 r^{3} \cos ^{2} t \sin t-r^{3} \sin ^{3} t+r \sin t+1\right)
$$

Thus you need to have both the real and imaginary parts equal to 0 . In other words, you need to have $(0,0)=$

$$
\left(r^{3} \cos ^{3} t-3 r^{3} \cos t \sin ^{2} t+r \cos t+1,3 r^{3} \cos ^{2} t \sin t-r^{3} \sin ^{3} t+r \sin t+1\right)
$$

for some value of $r$ and $t$. First here is a graph of this parametric function of $t$ for $t \in[0,2 \pi]$ on the left, when $r=4$. Note how the graph misses the origin $0+i 0$. In fact, the closed curve is in the exterior of a circle which has the point $0+i 0$ on its inside.


Next is the graph when $r=.5$. Note how the closed curve is included in a circle which has $0+i 0$ on its outside. As you shrink $r$ you get closed curves. At first, these closed curves enclose $0+i 0$ and later, they exclude $0+i 0$. Thus one of them should pass through this point. In fact, consider the curve which results when $r=1.386$ which is the graph on the right. Note how for this value of $r$ the curve passes through the point $0+i 0$. Thus for some $t, 1.386(\cos t+i \sin t)$ is a solution of the equation $p(z)=0$ or very close to one.

### 1.11 Some Topics from Analysis

Recall from calculus that if $A$ is a nonempty set, $\sup _{a \in A} f(a)$ denotes the least upper bound of $f(A)$ or if this set is not bounded above, it equals $\infty$. Also $\inf _{a \in A} f(a)$ denotes the greatest lower bound of $f(A)$ if this set is bounded below and it equals $-\infty$ if $f(A)$ is not bounded below. Thus to say $\sup _{a \in A} f(a)=\infty$ is just a way to say that $A$ is not bounded above. The existence of these quantities is what we mean when we say that $\mathbb{R}$ is complete.

Definition 1.11.1 Let $f(a, b) \in[-\infty, \infty]$ for $a \in A$ and $b \in B$ where $A, B$ are sets which means that $f(a, b)$ is either a number, $\infty$, or $-\infty$. The symbol, $+\infty$ is interpreted as a point out at the end of the number line which is larger than every real number. Of course there is no such number. That is why it is called $\infty$. The symbol, $-\infty$ is interpreted similarly. Then $\sup _{a \in A} f(a, b)$ means $\sup \left(S_{b}\right)$ where $S_{b} \equiv\{f(a, b): a \in A\}$.

Unlike limits, you can take the sup in different orders.
Lemma 1.11.2 Let $f(a, b) \in[-\infty, \infty]$ for $a \in A$ and $b \in B$ where $A, B$ are sets. Then

$$
\sup _{a \in A} \sup _{b \in B} f(a, b)=\sup _{b \in B} \sup _{a \in A} f(a, b) .
$$

Proof: Note that for all $a, b, f(a, b) \leq \sup _{b \in B} \sup _{a \in A} f(a, b)$ and therefore, for all $a$, $\sup _{b \in B} f(a, b) \leq \sup _{b \in B} \sup _{a \in A} f(a, b)$. Therefore,

$$
\sup _{a \in A} \sup _{b \in B} f(a, b) \leq \sup _{b \in B} \sup _{a \in A} f(a, b) .
$$

Repeat the same argument interchanging $a$ and $b$, to get the conclusion of the lemma.

## Theorem 1.11.3 Let $a_{i j} \geq 0$. Then $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i j}=\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{i j}$.

Proof: First note there is no trouble in defining these sums because the $a_{i j}$ are all nonnegative. If a sum diverges, it only diverges to $\infty$ and so $\infty$ is the value of the sum. Next note that $\sum_{j=r}^{\infty} \sum_{i=r}^{\infty} a_{i j} \geq \sup _{n} \sum_{j=r}^{\infty} \sum_{i=r}^{n} a_{i j}$ because for all $j, \sum_{i=r}^{\infty} a_{i j} \geq \sum_{i=r}^{n} a_{i j}$. Therefore,

$$
\begin{aligned}
& \sum_{j=r}^{\infty} \sum_{i=r}^{\infty} a_{i j} \geq \sup _{n} \sum_{j=r}^{\infty} \sum_{i=r}^{n} a_{i j}=\sup _{n} \lim _{m \rightarrow \infty} \sum_{j=r}^{m} \sum_{i=r}^{n} a_{i j} \\
& =\sup _{n} \lim _{m \rightarrow \infty} \sum_{i=r}^{n} \sum_{j=r}^{m} a_{i j}=\sup _{n} \sum_{i=r}^{n} \lim _{m \rightarrow \infty} \sum_{j=r}^{m} a_{i j} \\
& =\sup _{n} \sum_{i=r}^{n} \sum_{j=r}^{\infty} a_{i j}=\lim _{n \rightarrow \infty} \sum_{i=r}^{n} \sum_{j=r}^{\infty} a_{i j}=\sum_{i=r}^{\infty} \sum_{j=r}^{\infty} a_{i j}
\end{aligned}
$$

Interchanging the $i$ and $j$ in the above argument proves the theorem.

Corollary 1.11.4 If $a_{i j} \geq 0$, then $\sum_{j=r}^{\infty} \sum_{i=r}^{\infty} a_{i j}=\sum_{i, j} a_{i j}$ the last symbol meaning for $\mathbb{N}_{r}$ the integers larger than or equal to $r$,

$$
\sup \left\{\sum_{(i, j) \in S} a_{i j} \text { where } S \text { is a finite subset of } \mathbb{N}_{r} \times \mathbb{N}_{r}\right\}
$$

Proof: $\sum_{(i, j) \in S} a_{i j} \leq \sum_{j=r}^{\infty} \sum_{i=r}^{\infty} a_{i j}$ and so $\sum_{i, j} a_{i j} \leq \sum_{j=r}^{\infty} \sum_{i=r}^{\infty} a_{i j}$. Let $\lambda<\sum_{j=r}^{\infty} \sum_{i=r}^{\infty} a_{i j}$. Then there exists $M$ such that $\sum_{j=r}^{M} \sum_{i=r}^{\infty} a_{i j}=\sum_{i=r}^{\infty} \sum_{j=r}^{M} a_{i j}>\lambda$. Now there is $N$ such that $\lambda<\sum_{i=r}^{N} \sum_{j=r}^{M} a_{i j}<\sum_{i, j} a_{i j}$. Since $\lambda$ is arbitrary, it follows that $\sum_{j=r}^{\infty} \sum_{i=r}^{\infty} a_{i j} \leq \sum_{i, j} a_{i j}$

These theorems are special cases of Fubini's theorem in Lebesgue integration as is shown later.

Corollary 1.11.5 If $\sum_{i, j}\left|a_{i j}\right|<\infty$, then $\sum_{i=r}^{\infty} \sum_{j=r}^{\infty} a_{i j}=\sum_{j=r}^{\infty} \sum_{i=r}^{\infty} a_{i j}$. Here $a_{i j}$ are complex numbers.

Proof: First note that $\sum_{j=r}^{\infty} a_{i j}, \sum_{j=1}^{\infty} \operatorname{Re} a_{i j}, \sum_{j=1}^{\infty} \operatorname{Im} a_{i j}$ exist. This is because, for $b_{i j}=$ $a_{i j}, \operatorname{Re} a_{i j} \operatorname{Im} a_{i j},\left|\sum_{j=p}^{q} b_{i j}\right| \leq \sum_{j=p}^{\infty}\left|a_{i j}\right|$ which is small if $p$ and $q>p$ are large enough because $\Sigma_{j}\left|a_{i j}\right|$ exists. Thus the partial sums form a Cauchy sequence and therefore, these converge. This follows from the assumption that $\mathbb{R}$ and $\mathbb{C}$ are complete which means that Cauchy sequences converge. Now also, for the same $b_{i j}$, if $q>p$, then

$$
\left|\sum_{i=p}^{q} \sum_{j=r}^{\infty} b_{i j}\right| \leq \sum_{i=p}^{\infty}\left|\sum_{j=r}^{\infty} b_{i j}\right| \leq \sum_{i=p}^{\infty} \sum_{j=r}^{\infty}\left|a_{i j}\right|
$$

which is small if $p$ is large enough because $\sum_{i=r}^{\infty} \sum_{j=r}^{\infty}\left|a_{i j}\right|<\infty$. Thus the partial sums form a Cauchy sequence and so $\sum_{i=r}^{\infty} \sum_{j=r}^{\infty} b_{i j}$ exists. Similarly $\sum_{i=r}^{\infty} \sum_{j=r}^{\infty} b_{i j}$ exists. Now $\left|a_{i j}\right|+\operatorname{Re} a_{i j} \geq 0$. Thus

$$
\sum_{i=r}^{\infty} \sum_{j=r}^{\infty}\left|a_{i j}\right|+\operatorname{Re} a_{i j}=\sum_{j=r}^{\infty} \sum_{i=r}^{\infty}\left|a_{i j}\right|+\operatorname{Re} a_{i j}
$$

Thus, from the definition of the infinite sums,

$$
\sum_{i=r}^{\infty} \sum_{j=r}^{\infty}\left|a_{i j}\right|+\sum_{i=r}^{\infty} \sum_{j=r}^{\infty} \operatorname{Re} a_{i j}=\sum_{j=r}^{\infty} \sum_{i=r}^{\infty}\left|a_{i j}\right|+\sum_{j=r}^{\infty} \sum_{i=r}^{\infty} \operatorname{Re} a_{i j}
$$

Subtracting that which is known to be equal from both sides leads to the following equation: $\sum_{i=r}^{\infty} \sum_{j=r}^{\infty} \operatorname{Re} a_{i j}=\sum_{j=r}^{\infty} \sum_{i=r}^{\infty} \operatorname{Re} a_{i j}$. A similar equation holds by the same reasoning for $\operatorname{Im} a_{i j}$ in place of $\operatorname{Re} a_{i j}$. Then this implies that $\sum_{i=r}^{\infty} \sum_{j=r}^{\infty} a_{i j}=\sum_{j=r}^{\infty} \sum_{i=r}^{\infty} a_{i j}$.

### 1.11.1 $\lim$ sup and $\lim$ inf

Sometimes the limit of a sequence does not exist. For example, if $a_{n}=(-1)^{n}$, then $\lim _{n \rightarrow \infty} a_{n}$ does not exist. This is because the terms of the sequence are a distance of 1 apart. Therefore there can't exist a single number such that all the terms of the sequence are ultimately within $1 / 4$ of that number. The nice thing about lim sup and liminf is that they always exist. First here is a simple lemma and definition.

Definition 1.11.6 Denote by $[-\infty, \infty]$ the real line along with symbols $\infty$ and $-\infty$. It is understood that $\infty$ is larger than every real number and $-\infty$ is smaller than every real number. Then if $\left\{A_{n}\right\}$ is an increasing sequence of points of $[-\infty, \infty], \lim _{n \rightarrow \infty} A_{n}$ equals $\infty$ if the only upper bound of the set $\left\{A_{n}\right\}$ is $\infty$. If $\left\{A_{n}\right\}$ is bounded above by a real number, then $\lim _{n \rightarrow \infty} A_{n}$ is defined in the usual way and equals the least upper bound of $\left\{A_{n}\right\}$. If $\left\{A_{n}\right\}$ is a decreasing sequence of points of $[-\infty, \infty], \lim _{n \rightarrow \infty} A_{n}$ equals $-\infty$ if the only lower bound of the sequence $\left\{A_{n}\right\}$ is $-\infty$. If $\left\{A_{n}\right\}$ is bounded below by a real number, then $\lim _{n \rightarrow \infty} A_{n}$ is defined in the usual way and equals the greatest lower bound of $\left\{A_{n}\right\}$. More simply, if $\left\{A_{n}\right\}$ is increasing, $\lim _{n \rightarrow \infty} A_{n} \equiv \sup \left\{A_{n}\right\}$ and if $\left\{A_{n}\right\}$ is decreasing then $\lim _{n \rightarrow \infty} A_{n} \equiv \inf \left\{A_{n}\right\}$.

Lemma 1.11.7 Let $\left\{a_{n}\right\}$ be a sequence of real numbers and let

$$
U_{n} \equiv \sup \left\{a_{k}: k \geq n\right\}
$$

Then $\left\{U_{n}\right\}$ is a decreasing sequence. Also if $L_{n} \equiv \inf \left\{a_{k}: k \geq n\right\}$, then $\left\{L_{n}\right\}$ is an increasing sequence. Therefore, $\lim _{n \rightarrow \infty} L_{n}$ and $\lim _{n \rightarrow \infty} U_{n}$ both exist.

Proof: Let $W_{n}$ be an upper bound for $\left\{a_{k}: k \geq n\right\}$. Then since these sets are getting smaller, it follows that for $m<n, W_{m}$ is an upper bound for $\left\{a_{k}: k \geq n\right\}$. In particular if $W_{m}=U_{m}$, then $U_{m}$ is an upper bound for $\left\{a_{k}: k \geq n\right\}$ and so $U_{m}$ is at least as large as $U_{n}$, the least upper bound for $\left\{a_{k}: k \geq n\right\}$. The claim that $\left\{L_{n}\right\}$ is decreasing is similar.

From the lemma, the following definition makes sense.
Definition 1.11.8 Let $\left\{a_{n}\right\}$ be any sequence of points of $[-\infty, \infty]$

$$
\begin{aligned}
{\lim \sup _{n \rightarrow \infty}}^{a_{n}} & \equiv \lim _{n \rightarrow \infty} \sup \left\{a_{k}: k \geq n\right\} \\
\lim \inf _{n \rightarrow \infty} a_{n} & \equiv \lim _{n \rightarrow \infty} \inf \left\{a_{k}: k \geq n\right\} .
\end{aligned}
$$

Theorem 1.11.9 Suppose $\left\{a_{n}\right\}$ is a sequence of real numbers and that both

$$
\lim _{n \rightarrow \infty} \sup _{n} a_{n} \text { and } \lim _{n \rightarrow \infty} \inf _{n} a_{n}
$$

are real numbers. Then $\lim _{n \rightarrow \infty} a_{n}$ exists if and only if

$$
\lim _{n \rightarrow \infty} \inf _{n}=\lim \sup _{n \rightarrow \infty} a_{n}
$$

and in this case,

$$
\lim _{n \rightarrow \infty} a_{n}=\lim \inf _{n \rightarrow \infty} a_{n}=\lim \sup _{n \rightarrow \infty} a_{n} .
$$

Proof: First note that $\sup \left\{a_{k}: k \geq n\right\} \geq \inf \left\{a_{k}: k \geq n\right\}$ and so,

$$
\lim \sup _{n \rightarrow \infty} a_{n} \equiv \lim _{n \rightarrow \infty} \sup \left\{a_{k}: k \geq n\right\} \geq \lim _{n \rightarrow \infty} \inf \left\{a_{k}: k \geq n\right\} \equiv \lim _{n \rightarrow \infty} \inf _{n} a_{n} .
$$

Suppose first that $\lim _{n \rightarrow \infty} a_{n}$ exists and is a real number $a$. Then from the definition of a limit, there exists $N$ corresponding to $\varepsilon / 6$ in the definition. Hence, if $m, n \geq N$, then

$$
\left|a_{n}-a_{m}\right| \leq\left|a_{n}-a\right|+\left|a-a_{n}\right|<\frac{\varepsilon}{6}+\frac{\varepsilon}{6}=\frac{\varepsilon}{3} .
$$

From the definition of $\sup \left\{a_{k}: k \geq N\right\}$, there exists $n_{1} \geq N$ such that

$$
\sup \left\{a_{k}: k \geq N\right\} \leq a_{n_{1}}+\varepsilon / 3
$$

Similarly, there exists $n_{2} \geq N$ such that

$$
\inf \left\{a_{k}: k \geq N\right\} \geq a_{n_{2}}-\varepsilon / 3
$$

It follows that $\sup \left\{a_{k}: k \geq N\right\}-\inf \left\{a_{k}: k \geq N\right\} \leq\left|a_{n_{1}}-a_{n_{2}}\right|+\frac{2 \varepsilon}{3}<\varepsilon$. Since the sequence, $\left\{\sup \left\{a_{k}: k \geq N\right\}\right\}_{N=1}^{\infty}$ is decreasing and $\left\{\inf \left\{a_{k}: k \geq N\right\}\right\}_{N=1}^{\infty}$ is increasing, it follows that

$$
0 \leq \lim _{N \rightarrow \infty} \sup \left\{a_{k}: k \geq N\right\}-\lim _{N \rightarrow \infty} \inf \left\{a_{k}: k \geq N\right\} \leq \varepsilon
$$

Since $\varepsilon$ is arbitrary, this shows

$$
\begin{equation*}
\lim _{N \rightarrow \infty} \sup \left\{a_{k}: k \geq N\right\}=\lim _{N \rightarrow \infty} \inf \left\{a_{k}: k \geq N\right\} \tag{1.11}
\end{equation*}
$$

Next suppose 1.11 and both equal $a \in \mathbb{R}$. Then

$$
\lim _{N \rightarrow \infty}\left(\sup \left\{a_{k}: k \geq N\right\}-\inf \left\{a_{k}: k \geq N\right\}\right)=0
$$

Since $\sup \left\{a_{k}: k \geq N\right\} \geq \inf \left\{a_{k}: k \geq N\right\}$ it follows that for every $\varepsilon>0$, there exists $N$ such that $\sup \left\{a_{k}: k \geq N\right\}-\inf \left\{a_{k}: k \geq N\right\}<\varepsilon$, and for every $N, \inf \left\{a_{k}: k \geq N\right\} \leq a \leq$ $\sup \left\{a_{k}: k \geq N\right\}$. Thus if $n \geq N,\left|a-a_{n}\right|<\varepsilon$ which implies that $\lim _{n \rightarrow \infty} a_{n}=a$. In case

$$
a=\infty=\lim _{N \rightarrow \infty} \sup \left\{a_{k}: k \geq N\right\}=\lim _{N \rightarrow \infty} \inf \left\{a_{k}: k \geq N\right\}
$$

then if $r \in \mathbb{R}$ is given, there exists $N$ such that $\inf \left\{a_{k}: k \geq N\right\}>r$ which is to say that $\lim _{n \rightarrow \infty} a_{n}=\infty$. The case where $a=-\infty$ is similar except you use $\sup \left\{a_{k}: k \geq N\right\}$.

The significance of limsup and liminf, in addition to what was just discussed, is contained in the following theorem which follows quickly from the definition.

Theorem 1.11.10 Suppose $\left\{a_{n}\right\}$ is a sequence of points of $[-\infty, \infty]$. Define $\lambda$ by $\lambda=\lim \sup _{n \rightarrow \infty} a_{n}$. Then if $b>\lambda$, it follows there exists $N$ such that whenever $n \geq N, a_{n} \leq b$. If $c<\lambda$, then $a_{n}>c$ for infinitely many values of $n$. Let $\gamma=\liminf _{n \rightarrow \infty} a_{n}$. Then if $d<\gamma$, it follows there exists $N$ such that whenever $n \geq N, a_{n} \geq d$. If $e>\gamma$, it follows $a_{n}<e$ for infinitely many values of $n$.

The proof of this theorem is left as an exercise for you. It follows directly from the definition and it is the sort of thing you must do yourself. Here is one other simple proposition.

Proposition 1.11.11 Let $\lim _{n \rightarrow \infty} a_{n}=a>0$. Then $\limsup \operatorname{sim}_{n \rightarrow \infty} a_{n} b_{n}=a \lim \sup _{n \rightarrow \infty} b_{n}$.
Proof: This follows from the definition. Let $\lambda_{n}=\sup \left\{a_{k} b_{k}: k \geq n\right\}$. For all $n$ large enough, $a_{n}>a-\varepsilon$ where $\varepsilon$ is small enough that $a-\varepsilon>0$. Therefore,

$$
\lambda_{n} \geq \sup \left\{b_{k}: k \geq n\right\}(a-\varepsilon)
$$

for all $n$ large enough. Then

$$
\lim _{n \rightarrow \infty} \sup _{n} b_{n}=\lim _{n \rightarrow \infty} \lambda_{n} \geq \lim _{n \rightarrow \infty}\left(\sup \left\{b_{k}: k \geq n\right\}(a-\varepsilon)\right)=(a-\varepsilon) \limsup _{n \rightarrow \infty} b_{n}
$$

Similar reasoning shows $\limsup _{n \rightarrow \infty} a_{n} b_{n} \leq(a+\varepsilon) \limsup _{n \rightarrow \infty} b_{n}$. Now since $\varepsilon>0$ is arbitrary, the conclusion follows.

A fundamental existence theorem is the nested interval lemma.

### 1.11.2 Nested Interval Lemma

A fundamental existence theorem is the nested interval lemma.
Lemma 1.11.12 Let $I_{k}=\left[a_{k}, b_{k}\right]$ and suppose $I_{k} \supseteq I_{k+1}$ for all $k$. Then there is a point $x \in \cap_{k=1}^{\infty} I_{k}$.

Proof: Suppose $k \leq l$. Then $a_{k} \leq a_{l} \leq b_{l}$. On the other hand, suppose $k>l$. Then $a_{k} \leq$ $b_{k} \leq b_{l}$. Let $a \equiv \sup _{k} a_{k}$. Then from what was just observed, $a \leq b_{l}$ for each $l$. Therefore, also $a \leq \inf _{l} b_{l}$, so $a_{l} \leq a \leq b_{l}$ for every $l$ showing that $a$ is a point in all these intervals.

### 1.11.3 Multiplication of Series

Here the main interest is in series of real or complex numbers although this could certainly be generalized. The following is a major result about multiplying series. It is Mertens theorem.

Theorem 1.11.13 Suppose $\sum_{i=r}^{\infty} a_{i}$ and $\sum_{j=r}^{\infty} b_{j}$ are two series which both converge absolutely ${ }^{1}$. Then $\left(\sum_{i=r}^{\infty} a_{i}\right)\left(\sum_{j=r}^{\infty} b_{j}\right)=\sum_{n=r}^{\infty} c_{n}$ where $c_{n}=\sum_{k=r}^{n} a_{k} b_{n-k+r}$.

Proof: Let $p_{n k}=1$ if $r \leq k \leq n$ and $p_{n k}=0$ if $k>n$. Then $c_{n}=\sum_{k=r}^{\infty} p_{n k} a_{k} b_{n-k+r}$. Also,

$$
\begin{gathered}
\sum_{k=r}^{\infty} \sum_{n=r}^{\infty} p_{n k}\left|a_{k}\right|\left|b_{n-k+r}\right|=\sum_{k=r}^{\infty}\left|a_{k}\right| \sum_{n=r}^{\infty} p_{n k}\left|b_{n-k+r}\right| \\
=\sum_{k=r}^{\infty}\left|a_{k}\right| \sum_{n=k}^{\infty}\left|b_{n-k+r}\right|=\sum_{k=r}^{\infty}\left|a_{k}\right| \sum_{n=k}^{\infty}\left|b_{n-(k-r)}\right|=\sum_{k=r}^{\infty}\left|a_{k}\right| \sum_{m=r}^{\infty}\left|b_{m}\right|<\infty .
\end{gathered}
$$

Therefore, from Corollary 1.11.5,

$$
\begin{aligned}
\sum_{n=r}^{\infty} c_{n} & =\sum_{n=r}^{\infty} \sum_{k=r}^{n} a_{k} b_{n-k+r}=\sum_{n=r}^{\infty} \sum_{k=r}^{\infty} p_{n k} a_{k} b_{n-k+r} \\
& =\sum_{k=r}^{\infty} a_{k} \sum_{n=r}^{\infty} p_{n k} b_{n-k+r}=\sum_{k=r}^{\infty} a_{k} \sum_{n=k}^{\infty} b_{n-k+r}=\sum_{k=r}^{\infty} a_{k} \sum_{m=r}^{\infty} b_{m}
\end{aligned}
$$

It follows that $\sum_{n=r}^{\infty} c_{n}$ converges absolutely. Also, you can see by induction that you can multiply any number of absolutely convergent series together and obtain a series which is absolutely convergent. Next, here are some similar results related to Merten's theorem. In this theorem, $z$ is a variable in some set called $K$.

Lemma 1.11.14 Let $\sum_{n=0}^{\infty} a_{n}(z)$ and $\sum_{n=0}^{\infty} b_{n}(z)$ be two convergent series for $z \in K$ which satisfy the conditions of the Weierstrass $M$ test. Thus there exist positive constants, $A_{n}$ and $B_{n}$ such that $\left|a_{n}(z)\right| \leq A_{n},\left|b_{n}(z)\right| \leq B_{n}$ for all $z \in K$ and $\sum_{n=0}^{\infty} A_{n}<\infty, \sum_{n=0}^{\infty} B_{n}<\infty$. Then defining the Cauchy product, $c_{n}(z) \equiv \sum_{k-0}^{n} a_{n-k}(z) b_{k}(z)$, it follows $\sum_{n=0}^{\infty} c_{n}(z)$ also converges absolutely and uniformly on $K$ because $c_{n}(z)$ satisfies the conditions of the Weierstrass M test. Therefore,

$$
\begin{equation*}
\sum_{n=0}^{\infty} c_{n}(z)=\left(\sum_{k=0}^{\infty} a_{k}(z)\right)\left(\sum_{n=0}^{\infty} b_{n}(z)\right) \tag{1.12}
\end{equation*}
$$

[^0]Proof: $\left|c_{n}(z)\right| \leq \sum_{k=0}^{n}\left|a_{n-k}(z)\right|\left|b_{k}(z)\right| \leq \sum_{k=0}^{n} A_{n-k} B_{k}$. Also, from Theorem 1.11.3,

$$
\sum_{n=0}^{\infty} \sum_{k=0}^{n} A_{n-k} B_{k}=\sum_{k=0}^{\infty} \sum_{n=k}^{\infty} A_{n-k} B_{k}=\sum_{k=0}^{\infty} B_{k} \sum_{n=0}^{\infty} A_{n}<\infty .
$$

The claim of 1.12 follows from Merten's theorem, Theorem 1.11.13.
Corollary 1.11.15 Let $P$ be a polynomial and let $\sum_{n=0}^{\infty} a_{n}(z)$ converge uniformly and absolutely on $K$ such that the $\left|a_{n}(z)\right| \leq A_{n}, \sum_{n} A_{n}<\infty$. Then there exists a series for $P\left(\sum_{n=0}^{\infty} a_{n}(z)\right)$ denoted as $\sum_{n=0}^{\infty} c_{n}(z)$, which also converges absolutely and uniformly for $z \in K$ because $c_{n}(z)$ also satisfies the conditions of the Weierstrass $M$ test.

### 1.12 Root Test

The root test has to do with when a series of complex numbers converges. I am assuming the reader has been exposed to infinite series. However, this that I am about to explain is a little more general than what is usually seen in calculus.

Theorem 1.12.1 Let $\mathbf{a}_{k} \in \mathbb{F}^{p}$ and consider $\sum_{k=1}^{\infty} \mathbf{a}_{k}$. Then this series converges absolutely if $\lim \sup _{k \rightarrow \infty}\left|\mathbf{a}_{k}\right|^{1 / k}=r<1$. The series diverges spectacularly if $\lim \sup _{k \rightarrow \infty}\left|\mathbf{a}_{k}\right|^{1 / k}>$ 1 and if $\lim \sup _{k \rightarrow \infty}\left|\mathbf{a}_{k}\right|^{1 / k}=1$, the test fails.

Proof: Suppose first that limsup $\lim _{k \rightarrow \infty}\left|\mathbf{a}_{k}\right|^{1 / k}=r<1$. Then letting $R \in(r, 1)$, it follows from the definition of limsup that for all $k$ large enough, $\left|\mathbf{a}_{k}\right|^{1 / k} \leq R$. Hence there exists $N$ such that if $k \geq N$, then $\left|\mathbf{a}_{k}\right| \leq R^{k}$. Let $M_{k}=\left|\mathbf{a}_{k}\right|$ for $k<N$ and let $M_{k}=R^{k}$ for $k \geq N$. Then

$$
\sum_{k=1}^{\infty} M_{k} \leq \sum_{k=1}^{N-1}\left|\mathbf{a}_{k}\right|+\frac{R^{N}}{1-R}<\infty
$$

and so, by the Weierstrass $M$ test applied to the series of constants, the series converges and also converges absolutely. If limsup $\operatorname{sum}_{k \rightarrow \infty}\left|\mathbf{a}_{k}\right|^{1 / k}=r>1$, then letting $r>R>1$, it follows that for infinitely many $k,\left|\mathbf{a}_{k}\right|>R^{k}$ and so there is a subsequence which is unbounded. In particular, the series cannot converge and in fact diverges spectacularly. In case that the $\lim \sup =1$, you can consider $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges by calculus and $\sum_{n=1}^{\infty} \frac{1}{n^{2}}$ which converges, also from calculus. However, the limsup equals 1 for both of these.

There is no change in the proof if $\mathbf{a}_{k}$ is in a complete normed vector space which will be mentioned later.

This is a major theorem because the limsup always exists. As an important application, here is a corollary which emphasizes one aspect of the above theorem.

Corollary 1.12.2 If $\sum_{k} \mathbf{a}_{k}$ converges, then $\limsup \sin _{k \rightarrow \infty}\left|\mathbf{a}_{k}\right|^{1 / k} \leq 1$.
If the sequence has values in $X$ a complete normed linear space discussed below, there is no change in the conclusion or proof of the above theorem. You just replace $|\cdot|$ with $\|\cdot\|$ the symbol for the norm. Here $\|\cdot\|$ is a norm if it satisfies 1.1-1.3.

### 1.13 Exercises

1. Prove by induction that $\sum_{k=1}^{n} k^{3}=\frac{1}{4} n^{4}+\frac{1}{2} n^{3}+\frac{1}{4} n^{2}$.
2. Prove by induction that whenever $n \geq 2, \sum_{k=1}^{n} \frac{1}{\sqrt{k}}>\sqrt{n}$.
3. Prove by induction that $1+\sum_{i=1}^{n} i(i!)=(n+1)$ !.
4. The binomial theorem states $(x+y)^{n}=\sum_{k=0}^{n}\binom{n}{k} x^{n-k} y^{k}$ where

$$
\binom{n+1}{k}=\binom{n}{k}+\binom{n}{k-1} \text { if } k \in[1, n],\binom{n}{0} \equiv 1 \equiv\binom{n}{n}
$$

Prove the binomial theorem by induction. Next show $\binom{n}{k}=\frac{n!}{(n-k)!k!}, 0!\equiv 1$
5. Let $z=5+i 9$. Find $z^{-1}$.
6. Let $z=2+i 7$ and let $w=3-i 8$. Find $z w, z+w, z^{2}$, and $w / z$.
7. Give the complete solution to $x^{4}+16=0$.
8. Graph the complex cube roots of 8 in the complex plane. Do the same for the four fourth roots of 16 .
9. If $z$ is a complex number, show there exists $\omega$ a complex number with $|\omega|=1$ and $\omega z=|z|$.
10. De Moivre's theorem says $[r(\cos t+i \sin t)]^{n}=r^{n}(\cos n t+i \sin n t)$ for $n$ a positive integer. Does this formula continue to hold for all integers $n$, even negative integers? Explain.
11. You already know formulas for $\cos (x+y)$ and $\sin (x+y)$ and these were used to prove De Moivre's theorem. Now using De Moivre's theorem, derive a formula for $\sin (5 x)$ and one for $\cos (5 x)$.
12. If $z$ and $w$ are two complex numbers and the polar form of $z$ involves the angle $\theta$ while the polar form of $w$ involves the angle $\phi$, show that in the polar form for $z w$ the angle involved is $\theta+\phi$. Also, show that in the polar form of a complex number $z, r=|z|$.
13. Factor $x^{3}+8$ as a product of linear factors.
14. Write $x^{3}+27$ in the form $(x+3)\left(x^{2}+a x+b\right)$ where $x^{2}+a x+b$ cannot be factored any more using only real numbers.
15. Completely factor $x^{4}+16$ as a product of linear factors.
16. Factor $x^{4}+16$ as the product of two quadratic polynomials each of which cannot be factored further without using complex numbers.
17. If $z, w$ are complex numbers prove $\overline{z w}=\overline{z w}$. Then show by induction $\overline{\prod_{j=1}^{n} z_{j}}=$ $\prod_{j=1}^{n} \overline{z_{j}}$. Also verify that $\overline{\sum_{k=1}^{m} z_{k}}=\sum_{k=1}^{m} \overline{z_{k}}$. In words this says the conjugate of a product equals the product of the conjugates and the conjugate of a sum equals the sum of the conjugates.
18. Suppose $p(x)=a_{n} x^{n}+a_{n-1} x^{n-1}+\cdots+a_{1} x+a_{0}$ where all the $a_{k}$ are real numbers. Suppose also that $p(z)=0$ for some $z \in \mathbb{C}$. Show it follows that $p(\bar{z})=0$ also.
19. Show that $1+i, 2+i$ are the only two zeros to $p(x)=x^{2}-(3+2 i) x+(1+3 i)$ so the zeros do not necessarily come in conjugate pairs if the coefficients are not real.
20. I claim that $1=-1$. Here is why. $-1=i^{2}=\sqrt{-1} \sqrt{-1}=\sqrt{(-1)^{2}}=\sqrt{1}=1$. This is clearly a remarkable result but is there something wrong with it? If so, what is wrong?
21. De Moivre's theorem is really a grand thing. I plan to use it now for rational exponents, not just integers.

$$
1=1^{(1 / 4)}=(\cos 2 \pi+i \sin 2 \pi)^{1 / 4}=\cos (\pi / 2)+i \sin (\pi / 2)=i
$$

Therefore, squaring both sides it follows $1=-1$ as in the previous problem. What does this tell you about De Moivre's theorem? Is there a profound difference between raising numbers to integer powers and raising numbers to non integer powers?
22. Review Problem 10 at this point. Now here is another question: If $n$ is an integer, is it always true that $(\cos \theta-i \sin \theta)^{n}=\cos (n \theta)-i \sin (n \theta)$ ? Explain.
23. Suppose you have any polynomial in $\cos \theta$ and $\sin \theta$. By this I mean an expression of the form $\sum_{\alpha=0}^{m} \sum_{\beta=0}^{n} a_{\alpha \beta} \cos ^{\alpha} \theta \sin ^{\beta} \theta$ where $a_{\alpha \beta} \in \mathbb{C}$. Can this always be written in the form $\sum_{\gamma=-(n+m)}^{m+n} b_{\gamma} \cos \gamma \theta+\sum_{\tau=-(n+m)}^{n+m} c_{\tau} \sin \tau \theta$ ? Explain.
24. Show that $\mathbb{C}$ cannot be considered an ordered field. Hint: Consider $i^{2}=-1$.
25. Suppose $p(x)=a_{n} x^{n}+a_{n-1} x^{n-1}+\cdots+a_{1} x+a_{0}$ is a polynomial and it has $n$ zeros, $z_{1}, z_{2}, \cdots, z_{n}$ listed according to multiplicity. ( $z$ is a root of multiplicity $m$ if the polynomial $f(x)=(x-z)^{m}$ divides $p(x)$ but $(x-z) f(x)$ does not.) Show that $p(x)=a_{n}\left(x-z_{1}\right)\left(x-z_{2}\right) \cdots\left(x-z_{n}\right)$.
26. Give the solutions to the following quadratic equations having real coefficients.
(a) $x^{2}-2 x+2=0$
(d) $x^{2}+4 x+9=0$
(b) $3 x^{2}+x+3=0$
(c) $x^{2}-6 x+13=0$
(e) $4 x^{2}+4 x+5=0$
27. Give the solutions to the following quadratic equations having complex coefficients. Note how the solutions do not come in conjugate pairs as they do when the equation has real coefficients.
(a) $x^{2}+2 x+1+i=0$
(d) $x^{2}-4 i x-5=0$
(b) $4 x^{2}+4 i x-5=0$
(c) $4 x^{2}+(4+4 i) x+1+2 i=0$
(e) $3 x^{2}+(1-i) x+3 i=0$
28. Prove the fundamental theorem of algebra for quadratic polynomials having coefficients in $\mathbb{C}$. That is, show that an equation of the form $a x^{2}+b x+c=0$ where $a, b, c$ are complex numbers, $a \neq 0$ has a complex solution. Hint: Consider the fact, noted earlier that the expressions given from the quadratic formula do in fact serve as solutions.
29. Verify DeMorgan's laws,

$$
\begin{aligned}
& (\cup\{A: A \in \mathscr{C}\})^{C}=\cap\left\{A^{C}: A \in \mathscr{C}\right\} \\
& (\cap\{A: A \in \mathscr{C}\})^{C}=\cup\left\{A^{C}: A \in \mathscr{C}\right\}
\end{aligned}
$$

where $\mathscr{C}$ consists of a set whose elements are subsets of a given set $S$. Hint: This says the complement of a union is the intersection of the complements and the complement of an intersection is the union of the complements. You need to show each set on either side of the equation is a subset of the other side.
30. Find the partial fractions expansion of $\frac{x^{6}+3 x^{4}-x^{3}-4 x^{2}-2 x-4}{x\left(x^{4}-4\right)}$ with field of scalars equal to $\mathbb{Q}$ the rational numbers. Then find it for field of scalars equal to $\mathbb{R}$. Note $\left(x^{4}-4\right)=$ $\left(x^{2}+2\right)\left(x^{2}-2\right)$ and both of these are irreducible with field of scalars $\mathbb{Q}$ but the second is not irreducible with field of scalars $\mathbb{R}$ because $x^{2}-2=(x-\sqrt{2})(x+\sqrt{2})$. You will need to first do a division because the degree of the top is larger than the degree of the bottom.
31. If you have any polynomial $p(\lambda)$ with coefficients from a field of scalars, show $p(\lambda)=\prod_{k=1}^{m} q_{k}(\lambda)^{r_{k}}$ where the $r_{k}$ are positive integers and the polynomials $\left\{q_{k}(\lambda)\right\}$ are irreducible, meaning they cannot be factored further. $\left(q_{k}(\lambda)=\phi(\lambda) \psi(\lambda)\right.$ then one of $\phi(\lambda)$ or $\psi(\lambda)$ is a scalar.) Explain why any subset of $\left\{q_{k}(\lambda)\right\}$ having two or more entries is relatively prime.

## Chapter 2

## Basic Topology and Algebra

Next are metric spaces which have no algebra involved.

### 2.1 Metric Spaces

It was shown above that $\|\mathbf{x}+\mathbf{y}\| \leq\|\mathbf{x}\|+\|\mathbf{y}\|$ where $\|\cdot\|=|\cdot|$ or $\|\cdot\|_{\infty}$. This was called the triangle inequality. Thus, in particular,

$$
\|\mathbf{x}-\mathbf{y}\|+\|\mathbf{y}-\mathbf{z}\| \geq\|\mathbf{x}-\mathbf{z}\|
$$

A metric space is a nonempty set $X$ along with a distance function $d: X \times X \rightarrow[0, \infty)$ which satisfies the following axioms.

1. $d(x, y)=d(y, x)$
2. $d(x, y)+d(y, z) \geq d(x, z)$
3. $d(x, x)=0$ and $d(x, y)=0$ if and only if $x=y$

Definition 2.1.1 In a metric space we say $\lim _{n \rightarrow \infty} x_{n}=x, x_{n} \rightarrow x$, if and only if $\lim _{n \rightarrow \infty} d\left(x_{n}, x\right)=0$.
Proposition 2.1.2 The limit is well defined. That is, if $x, x^{\prime}$ are both limits of a sequence, then $x=x^{\prime}$.

Proof: From the definition, there exist $N, N^{\prime}$ such that if $n \geq N$, then $d\left(x, x_{n}\right)<\varepsilon / 2$ and if $n \geq N^{\prime}$, then $d\left(x, x_{n}\right)<\varepsilon / 2$. Then let $M \geq \max \left(N, N^{\prime}\right)$. Let $n>M$. Then $d\left(x, x^{\prime}\right) \leq$ $d\left(x, x_{n}\right)+d\left(x_{n}, x^{\prime}\right)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$. Since $\varepsilon$ is arbitrary, this shows that $x=x^{\prime}$ because $d\left(x, x^{\prime}\right)=0$.

Thus $\mathbb{F}^{p}$ with either of the norms discussed is an example of a metric space if we define $d(\mathbf{x}, \mathbf{y}) \equiv\|\mathbf{x}-\mathbf{y}\|$. A metric space is significantly more general than a normed vector space like $\mathbb{R}^{n}$ because it does not have any algebraic vector space properties associated with it. The only thing of importance is the distance function. There are many things which are metric spaces which are of interest and are not vector spaces. For example, you could consider the surface of the earth. It is not a subspace of $\mathbb{R}^{3}$ but it is very meaningful to ask for the distance between points on the earth. Because of this, I am going to use the language of metric spaces when referring to things which only involve topological considerations. It is customary to not bother to make the symbol for something in the space bold face so I will follow this simplified notation when referring to metric space which does not necessarily have any vector space attributes.

A useful result is in the following lemma.
Lemma 2.1.3 Suppose $x_{n} \rightarrow x$ and $y_{n} \rightarrow y$. Then $d\left(x_{n}, y_{n}\right) \rightarrow d(x, y)$.
Proof: Consider the following.

$$
d(x, y) \leq d\left(x, x_{n}\right)+d\left(x_{n}, y\right) \leq d\left(x, x_{n}\right)+d\left(x_{n}, y_{n}\right)+d\left(y_{n}, y\right)
$$

so $d(x, y)-d\left(x_{n}, y_{n}\right) \leq d\left(x, x_{n}\right)+d\left(y_{n}, y\right)$. Similarly

$$
d\left(x_{n}, y_{n}\right)-d(x, y) \leq d\left(x, x_{n}\right)+d\left(y_{n}, y\right)
$$

and so

$$
\left|d\left(x_{n}, y_{n}\right)-d(x, y)\right| \leq d\left(x, x_{n}\right)+d\left(y_{n}, y\right)
$$

and the right side converges to 0 as $n \rightarrow \infty$.

### 2.2 Closed and Open Sets

The definition of open and closed sets is next. This is in an arbitrary metric space.
Definition 2.2.1 An open ball, denoted as $B(x, r)$ is defined as follows.

$$
B(x, r) \equiv\{y: d(x, y)<r\}
$$

A set $U$ is said to be open if whenever $x \in U$, it follows that there is $r>0$ such that $B(x, r) \subseteq U$. More generally, a point $x$ is said to be an interior point of $U$ if there exists such a ball. In words, an open set is one for which every point is an interior point.

For example, you could have $X$ be a subset of $\mathbb{R}$ and $d(x, y)=|x-y|$.
Then the first thing to show is the following.
Proposition 2.2.2 An open ball is an open set.
Proof: Suppose $y \in B(x, r)$. We need to verify that $y$ is an interior point of $B(x, r)$. Let $\delta=r-d(x, y)$. Then if $z \in B(y, \delta)$, it follows that

$$
d(z, x) \leq d(z, y)+d(y, x)<\delta+d(y, x)=r-d(x, y)+d(y, x)=r
$$

Thus $y \in B(y, \delta) \subseteq B(x, r)$.
Definition 2.2.3 Let $S$ be a nonempty subset of a metric space. Then $p$ is a limit point (accumulation point) of $S$ if for every $r>0$ there exists a point different than $p$ in $B(p, r) \cap S$. Sometimes people denote the set of limit points as $S^{\prime}$.


The following proposition is fairly obvious from the above definition and will be used whenever convenient. It is equivalent to the above definition and so it can take the place of the above definition if desired.

Proposition 2.2.4 A point $x$ is a limit point of the nonempty set $A$ if and only if every $B(x, r)$ contains infinitely many points of $A$.

Proof: $\Leftarrow$ is obvious. Consider $\Rightarrow$. Let $x$ be a limit point. Let $r_{1}=1$. Then $B\left(x, r_{1}\right)$ contains $a_{1} \neq x$. If $\left\{a_{1}, \cdots, a_{n}\right\}$ have been chosen none equal to $x$ and with no repeats in the list, let $0<r_{n}<\min \left(\frac{1}{n}, \min \left\{d\left(a_{i}, x\right), i=1,2, \cdots n\right\}\right)$. Then let $a_{n+1} \in B\left(x, r_{n}\right)$. Thus every $B(x, r)$ contains $B\left(x, r_{n}\right)$ for all $n$ large enough and hence it contains $a_{k}$ for $k \geq n$ where the $a_{k}$ are distinct, none equal to $x$.

Next there is an important theorem about limit points and convergent sequences.
Theorem 2.2.5 Let $S \neq \emptyset$. Then $p$ is a limit point of $S$ if and only if there exists $a$ sequence of distinct points of $S,\left\{x_{n}\right\}$ none of which equal $p$ such that $\lim _{n \rightarrow \infty} x_{n}=p$.

Proof: $\Longrightarrow$ Suppose $p$ is a limit point. Why does there exist the promised convergent sequence? Let $x_{1} \in B(p, 1) \cap S$ such that $x_{1} \neq p$. If $x_{1}, \cdots, x_{n}$ have been chosen, let $x_{n+1} \neq p$ be in $B\left(p, \delta_{n+1}\right) \cap S$ where

$$
\delta_{n+1}=\min \left\{\frac{1}{n+1}, d\left(x_{i}, p\right), i=1,2, \cdots, n\right\} .
$$

Then this constructs the necessary convergent sequence.
$\Longleftarrow$ Conversely, if such a sequence $\left\{x_{n}\right\}$ exists, then for every $r>0, B(p, r)$ contains $x_{n} \in S$ for all $n$ large enough. Hence, $p$ is a limit point because none of these $x_{n}$ are equal to $p$.

Definition 2.2.6 $A$ set $H$ is closed means $H^{C}$ is open.
Note that this says that the complement of an open set is closed. If $V$ is open, then the complement of its complement is itself. Thus $\left(V^{C}\right)^{C}=V$ an open set. Hence $V^{C}$ is closed. Thus, open sets are complements of closed sets and closed sets are complements of open sets.

Then the following theorem gives the relationship between closed sets and limit points.
Theorem 2.2.7 A set $H$ is closed if and only if it contains all of its limit points.
Proof: $\Longrightarrow$ Let $H$ be closed and let $p$ be a limit point. We need to verify that $p \in H$. If it is not, then since $H$ is closed, its complement is open and so there exists $\delta>0$ such that $B(p, \delta) \cap H=\emptyset$. However, this prevents $p$ from being a limit point.
$\Longleftarrow$ Next suppose $H$ has all of its limit points. Why is $H^{C}$ open? If $p \in H^{C}$ then it is not a limit point and so there exists $\delta>0$ such that $B(p, \delta)$ has no points of $H$. In other words, $H^{C}$ is open. Hence $H$ is closed.

Corollary 2.2.8 $A$ set $H$ is closed if and only if whenever $\left\{h_{n}\right\}$ is a sequence of points of $H$ which converges to a point $x$, it follows that $x \in H$.

Proof: $\Longrightarrow$ Suppose $H$ is closed and $h_{n} \rightarrow x$. If $x \in H$ there is nothing left to show. If $x \notin H$, then from the definition of limit, it is a limit point of $H$ because none of the $h_{n}$ are equal to $x$. Hence $x \in H$ after all.
$\Longleftarrow$ Suppose the limit condition holds, why is $H$ closed? Let $x \in H^{\prime}$ the set of limit points of $H$. By Theorem 2.2.5 there exists a sequence of points of $H,\left\{h_{n}\right\}$ such that $h_{n} \rightarrow x$. Then by assumption, $x \in H$. Thus $H$ contains all of its limit points and so it is closed by Theorem 2.2.7.

Next is the important concept of a subsequence.
Definition 2.2.9 Let $\left\{x_{n}\right\}_{n=1}^{\infty}$ be a sequence. Then if $n_{1}<n_{2}<\cdots$ is a strictly increasing sequence of indices, we say $\left\{x_{n_{k}}\right\}_{k=1}^{\infty}$ is a subsequence of $\left\{x_{n}\right\}_{n=1}^{\infty}$.

The really important thing about subsequences is that they preserve convergence.
Theorem 2.2.10 Let $\left\{x_{n_{k}}\right\}$ be a subsequence of a convergent sequence $\left\{x_{n}\right\}$ where $x_{n} \rightarrow x$. Then $\lim _{k \rightarrow \infty} x_{n_{k}}=x$ also.

Proof: Let $\varepsilon>0$ be given. Then there exists $N$ such that $d\left(x_{n}, x\right)<\varepsilon$ if $n \geq N$. It follows that if $k \geq N$, then $n_{k} \geq N$ and so $d\left(x_{n_{k}}, x\right)<\varepsilon$ if $k \geq N$. This is what it means to say $\lim _{k \rightarrow \infty} x_{n_{k}}=x$.

In the case of $\mathbb{F}^{p}$, if you have two norms which are equivalent in the sense that

$$
\delta\|\mathbf{x}\| \leq\|\mathbf{x}\|_{1} \leq \Delta\|\mathbf{x}\|
$$

then a set $U$ is open with respect to one norm if and only if it is open with respect to the other. Indeed, if $\mathbf{x} \in U$ which is open with respect to $\|\cdot\|$, then there is $B_{\|\cdot\|}(\mathbf{x}, \boldsymbol{\delta}) \subseteq U$ but from the above inequality,

$$
B_{\|\cdot\|_{1}}\left(\mathbf{x}, \frac{1}{\Delta} \delta\right) \subseteq B_{\|\cdot\|}(\mathbf{x}, \delta) \subseteq U
$$

because if $\|\mathbf{x}-\mathbf{y}\|_{1}<\frac{\delta}{\Delta}$, then $\|\mathbf{x}-\mathbf{y}\| \leq \Delta\|\mathbf{x}-\mathbf{y}\|_{1}<\Delta \frac{\delta}{\Delta}=\delta$. Now observe that

$$
\frac{1}{\Delta}\|\mathbf{x}\|_{1} \leq\|\mathbf{x}\| \leq \frac{1}{\delta}\|\mathbf{x}\|_{1}
$$

You should write down the reasoning to this carefully.
Notice that $B_{\infty}(\mathbf{p}, r)=\prod_{i=1}^{n}\left(p_{i}-r, p_{i}+r\right)$ a product of open intervals. This is especially convenient. You should carefully write down the reasoning for this from the definition of $\|\cdot\|_{\infty}$.

Now go back to the notion of a general metric space.
Theorem 2.2.11 The intersection of any finite collection of open sets is open. The union of any collection of open sets is open. The intersection of any collection of closed sets is closed and the union of any finite collection of closed sets is closed.

Proof: To see that any union of open sets is open, note that every point $p$ of the union is in at least one of the open sets $U$. Therefore, it is an interior point of $U$ and hence an interior point of the entire union.

Now let $\left\{U_{1}, \cdots, U_{m}\right\}$ be some open sets and suppose $p \in \cap_{k=1}^{m} U_{k}$. Then there exists $r_{k}>0$ such that $B\left(p, r_{k}\right) \subseteq U_{k}$. Let $0<r \leq \min \left(r_{1}, r_{2}, \cdots, r_{m}\right)$. Then $B(p, r) \subseteq \cap_{k=1}^{m} U_{k}$ and so the finite intersection is open. Note that if the finite intersection is empty, there is nothing to prove because it is certainly true in this case that every point in the intersection is an interior point because there aren't any such points.

Suppose $\left\{H_{1}, \cdots, H_{m}\right\}$ is a finite set of closed sets. Then $\cup_{k=1}^{m} H_{k}$ is closed if its complement is open. However, from DeMorgan's laws, Problem 29 on Page 36,

$$
\left(\cup_{k=1}^{m} H_{k}\right)^{C}=\cap_{k=1}^{m} H_{k}^{C}
$$

a finite intersection of open sets which is open by what was just shown.
Next let $\mathscr{C}$ be a set consisting of closed sets. Then

$$
(\cap \mathscr{C})^{C}=\cup\left\{H^{C}: H \in \mathscr{C}\right\}
$$

a union of open sets which is therefore open by the first part of the proof. Thus $\cap \mathscr{C}$ is closed. This proves the theorem.

Now back to $\mathbb{F}^{p}$ we can conclude that every point in an open set is a limit point of the open set.

Example 2.2.12 Consider $A=B(\mathbf{x}, \boldsymbol{\delta})$, an open ball in $\mathbb{F}^{p}$. Then every point of $B(\mathbf{x}, \boldsymbol{\delta})$ is a limit point of $A$.

If $\mathbf{z} \in B(\mathbf{x}, \boldsymbol{\delta})$, consider $\mathbf{z}+\frac{1}{k}(\mathbf{x}-\mathbf{z}) \equiv \mathbf{w}_{k}$ for $k \in \mathbb{N}$. Then

$$
\begin{gathered}
\left\|\mathbf{w}_{k}-\mathbf{x}\right\|=\left\|\mathbf{z}+\frac{1}{k}(\mathbf{x}-\mathbf{z})-\mathbf{x}\right\| \\
=\left\|\left(1-\frac{1}{k}\right) \mathbf{z}-\left(1-\frac{1}{k}\right) \mathbf{x}\right\|=\frac{k-1}{k}\|\mathbf{z}-\mathbf{x}\|<\delta
\end{gathered}
$$

and also $\left\|\mathbf{w}_{k}-\mathbf{z}\right\| \leq \frac{1}{k}\|\mathbf{x}-\mathbf{z}\|<\delta / k$ so $\mathbf{w}_{k} \rightarrow \mathbf{z}$. Furthermore, the $\mathbf{w}_{k}$ are distinct. Thus $\mathbf{z}$ is a limit point of $A$ as claimed. This is because every ball containing $\mathbf{z}$ contains infinitely many of the $\mathbf{w}_{k}$ and since they are all distinct, they can't all be equal to $\mathbf{z}$.

In a general metric space, peculiar things can occur. In particular, you can have a nonempty open set which has no limit points at all.

Example 2.2.13 Let $\Omega \neq \emptyset$ and define for $x, y \in \Omega, d(x, y)=0$ if $x=y$ and $d(x, y)=1$ if $x \neq y$. Then you can show that this is a perfectly good metric space on $\Omega$. However, every set is both open and closed. There are also no limit points for any nonempty set since $B(x, 1 / 2)=\{x\}$. You should consider why every set is both open and closed.

Next is the definition of what is meant by the closure of a set.
Definition 2.2.14 Let $A$ be a nonempty subset of $X$ for $X$ a metric space. Then $\bar{A}$ is defined to be the intersection of all closed sets which contain $A$. This is called the closure of $A$. Note $X$ is one such closed set which contains $A$.

Lemma 2.2.15 Let $A$ be a nonempty set in $X$. Then $\bar{A}$ is a closed set and

$$
\bar{A}=A \cup A^{\prime}
$$

where $A^{\prime}$ denotes the set of limit points of $A$.
Proof: First of all, denote by $\mathscr{C}$ the set of closed sets which contain $A$. Then define $\bar{A} \equiv \cap \mathscr{C}$. This is a closed set from Theorem 2.2.11.

The interesting part is the next claim. First note that from the definition, $A \subseteq \bar{A}$ so if $x \in A$, then $x \in \bar{A}$. Now consider $y \in A^{\prime}$ but $y \notin A$. If $y \notin \bar{A}$, a closed set, then there exists $B(y, r) \subseteq \bar{A}^{C}$. Thus $y$ cannot be a limit point of $A$, a contradiction. Therefore, $A \cup A^{\prime} \subseteq \bar{A}$

Next suppose $x \in \bar{A}$ and suppose $x \notin A$. Is $x \in A^{\prime}$ ? If not, then there is $r>0$ such that $B(x, r) \cap A=\emptyset$. But then $B(x, r)^{C}$ is a closed set containing $A$ so from the definition, it also contains $\bar{A}$ which is contrary to the assertion that $x \in \bar{A}$. Hence if $x \notin A$, then $x \in A^{\prime}$ and so $A \cup A^{\prime} \supseteq \bar{A}$

### 2.3 Sequences and Cauchy Sequences

It was discussed above what is meant by convergence of a sequence in a metric space. It was shown that if a sequence converges, then so does every subsequence. The context in this section will be a metric space $(X, d)$.

Of course the converse does not hold. Consider $a_{k}=(-1)^{k}$ it has a subsequence converging to 1 but the sequence does not converge. However, if you have a Cauchy sequence, defined next, then convergence of a subsequence does imply convergence of the Cauchy sequence. This is a very important observation.

Definition 2.3.1 $\left\{x_{k}\right\}$ is a Cauchy sequence if and only if the following holds. For every $\varepsilon>0$, there exists $n_{\varepsilon}$ such that if $k, l \geq n_{\varepsilon}$, then $d\left(x_{k}, x_{l}\right)<\varepsilon$

All Cauchy sequences are bounded.
Theorem 2.3.2 The set of terms in a Cauchy sequence $\left\{x_{n}\right\}$ in $X$ is bounded in the sense that there exists $M$ such that for all $n, d\left(x_{n}, x_{1}\right)<M$. Also, if any sequence converges, then it is Cauchy.

Proof: Let $\varepsilon=1$ in the definition of a Cauchy sequence and let $n>n_{1}$. Then from the definition, $d\left(x_{n}, x_{n_{1}}\right)<1$. It follows that if $n \geq n_{1}$,

$$
d\left(x_{n}, x_{1}\right)<d\left(x_{n}, x_{n_{1}}\right)+d\left(x_{n_{1}}, x_{1}\right)<1+d\left(x_{n_{1}}, x_{1}\right)
$$

and if $n<n_{1}, d\left(x_{n}, x_{1}\right) \leq \max \left\{d\left(x_{k}, x_{1}\right): k \leq n_{1}\right\}$ and so for any $n$,

$$
d\left(x_{n}, x_{1}\right)<1+d\left(x_{n_{1}}, x_{1}\right)+\max \left\{d\left(x_{k}, x_{1}\right): k \leq n_{1}\right\}
$$

If $\lim _{n \rightarrow \infty} x_{n}=x$, then if $\varepsilon>0$ is given, there is $N$ such that if $n>N$, then $d\left(x_{n}, x\right)<\varepsilon / 2$. Thus, if $n, m>N$, then $d\left(x_{n}, x_{m}\right) \leq d\left(x_{n}, x\right)+d\left(x, x_{m}\right)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$ and so the sequence is Cauchy whenever it converges.

Here is the theorem which says that if a subsequence of a Cauchy sequence converges, then so does the Cauchy sequence.
Theorem 2.3.3 Let $\left\{x_{n}\right\}$ be a Cauchy sequence. Then it converges to $x$ if and only if some subsequence converges to $x$.

Proof: $\Longrightarrow$ This was just done above. Indeed, if the sequence converges, then every subsequence converges to the same thing.
$\Longleftarrow$ Suppose now that $\left\{x_{n}\right\}$ is a Cauchy sequence and $\lim _{k \rightarrow \infty} x_{n_{k}}=x$. Then there exists $N_{1}$ such that if $k>N_{1}$, then $d\left(x_{n_{k}}, x\right)<\varepsilon / 2$. From the definition of what it means to be Cauchy, there exists $N_{2}$ such that if $m, n \geq N_{2}$, then $d\left(x_{m}, x_{n}\right)<\varepsilon / 2$. Let $N \geq \max \left(N_{1}, N_{2}\right)$. Then if $k \geq N$, then $n_{k} \geq N$ and so $d\left(x, x_{k}\right) \leq d\left(x, x_{n_{k}}\right)+d\left(x_{n_{k}}, x_{k}\right)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$. It follows from the definition that $\lim _{k \rightarrow \infty} x_{k}=x$.
Definition 2.3.4 A metric space is said to be complete if every Cauchy sequence converges.

Note that if you have equivalent norms on $\mathbb{F}^{p}$, then the Cauchy sequences are the same. The following lemma says that $\mathbb{R}$ is complete.
Lemma 2.3.5 Let $\left\{x_{k}\right\}$ be a Cauchy sequence in $\mathbb{R}$. Then it converges.
Proof: From Theorem 2.3.2, the entire sequence is contained in some closed interval $I \equiv I_{0}=[a, b]$. Divide this interval into two equal intervals by splitting it at its midpoint. Then one of these contains $x_{k}$ for infinitely many $k$. Call this interval $I_{1}$. Now split it in half and let $I_{2}$ be a half which contains $x_{k}$ for infinitely many $k$. Continue this way. Then $\left\{I_{k}\right\}$ is sequence of nested closed intervals each of which contains $x_{k}$ for infinitely many $k$, the length of $I_{k}$ being $2^{-k}$ times the length of $I_{0}$. Pick $n_{1}<n_{2}<\cdots$, where $x_{n_{k}} \in I_{k}$. Thus this is a Cauchy sequence. By the nested interval lemma, there is $x$ a point of $I_{0}$ which is in each of these nested intervals. Then $\left|x-x_{n_{k}}\right| \leq 2^{-k}$ (length of $I_{0}$ ) and so this subsequence converges to $x \in I_{0}$. Now by Theorem 2.3.3, the original Cauchy sequence converges to $x$.

Next is a corollary which says that $\mathbb{C}$ is also complete.

Corollary 2.3.6 Let $\left\{z_{k}\right\}$ be a Cauchy sequence in $\mathbb{C}$. Then it converges.
Proof: Say $z_{k}=x_{k}+i y_{k}$. Then from the way we define distance in $\mathbb{C},\left\{z_{k}\right\}$ is Cauchy if and only if $\left\{x_{k}\right\}$ and $\left\{y_{k}\right\}$ are both Cauchy. Indeed, $\left|z_{k}-z_{m}\right|^{2} \equiv\left|x_{k}-x_{m}\right|^{2}+\left|y_{k}-y_{m}\right|^{2}$. Therefore, there exists $x, y$ such that $x_{k} \rightarrow x, y_{k} \rightarrow y$. It follows $z_{k} \rightarrow x+i y$.

Now consider the case that the metric space is $\mathbb{F}^{p}$.
Theorem 2.3.7 A sequence $\left\{\mathbf{x}^{k}\right\}$ converges in $\mathbb{F}^{p}$ if and only if it is a Cauchy sequence.

Proof: $\Leftarrow$ Let $\mathbf{x}^{k}=\left(x_{1}^{k}, \cdots, x_{p}^{k}\right)$. Then since this is a Cauchy sequence, the components are Cauchy, and so, from what was just shown, $\lim _{k \rightarrow \infty} x_{i}^{k}=x_{i}$ which, from the way we define the norm implies $\mathbf{x}^{k} \rightarrow \mathbf{x} \equiv\left(\begin{array}{lll}x_{1} & \cdots & x_{p}\end{array}\right)$
$\Rightarrow$ If $\lim _{k \rightarrow \infty} \mathbf{x}^{k}=\mathbf{x}$, then for $k$ large enough, $\left|\mathbf{x}^{k}-\mathbf{x}\right|<\varepsilon / 2$. Hence if $k, m$ are large enough,

$$
\left|\mathbf{x}^{k}-\mathbf{x}^{m}\right| \leq\left|\mathbf{x}^{k}-\mathbf{x}\right|+\left|\mathbf{x}-\mathbf{x}^{m}\right|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon
$$

### 2.4 Separability and Complete Separability

Definition 2.4.1 A metric space is called separable if there exists a countable dense subset $D$. This means two things. First, $D$ is countable, and second, that if $x$ is any point and $r>0$, then $B(x, r) \cap D \neq \emptyset$. A metric space is called completely separable if there exists a countable collection of nonempty open sets $\mathscr{B}$ such that every open set is the union of some subset of $\mathscr{B}$. This collection of open sets is called a countable basis.

For those who like to fuss about empty sets, the empty set is open and it is indeed the union of a subset of $\mathscr{B}$ namely the empty subset.

Theorem 2.4.2 A metric space is separable if and only if it is completely separable. In fact a separable metric space has a countable basis of balls. Also $\mathbb{F}^{p}$ is separable.

Proof: $\Longleftarrow$ Let $\mathscr{B}$ be the special countable collection of open sets and for each $B \in \mathscr{B}$, let $p_{B}$ be a point of $B$. Then let $\mathscr{P} \equiv\left\{p_{B}: B \in \mathscr{B}\right\}$. To be specific, let $p_{B}$ be the center of $B$. If $B(x, r)$ is any ball, then it is the union of sets of $\mathscr{B}$ and so there is a point of $\mathscr{P}$ in it. Since $\mathscr{B}$ is countable, so is $\mathscr{P}$.
$\Longrightarrow$ Let $D$ be the countable dense set and let

$$
\mathscr{B} \equiv\{B(d, r): d \in D, r \in \mathbb{Q} \cap[0, \infty)\}
$$

Then $\mathscr{B}$ is countable because the Cartesian product of countable sets is countable. It suffices to show that every ball is the union of these sets. Let $B(x, R)$ be a ball. Let $y \in B(y, \delta) \subseteq B(x, R)$. Then there exists $d \in B\left(y, \frac{\delta}{10}\right)$. Let $\varepsilon \in \mathbb{Q}$ and $\frac{\delta}{10}<\varepsilon<\frac{\delta}{5}$. Then $y \in B(d, \varepsilon) \in \mathscr{B}$. Is $B(d, \varepsilon) \subseteq B(x, R)$ ? If so, then the desired result follows because this would show that every $y \in B(x, R)$ is contained in one of these sets of $\mathscr{B}$ which is contained in $B(x, R)$ showing that $B(x, R)$ is the union of sets of $\mathscr{B}$. Let $z \in B(d, \varepsilon) \subseteq B\left(d, \frac{\delta}{5}\right)$. Then $d(y, z) \leq d(y, d)+d(d, z)<\frac{\delta}{10}+\varepsilon<\frac{\delta}{10}+\frac{\delta}{5}<\delta$. Hence $B(d, \varepsilon) \subseteq B(y, \delta) \subseteq B(x, r)$. Therefore, every ball is the union of sets of $\mathscr{B}$ and, since every open set is the union of balls, it follows that every open set is the union of sets of $\mathscr{B}$.

As for the last claim, let $\mathbb{Q}$ be the rational numbers. Then obviously $\mathbb{Q}^{p}$ is dense in $\mathbb{R}^{p}$ and $(\mathbb{Q}+i \mathbb{Q})^{p}$ is dense in $\mathbb{C}^{p}$. There are countably many points in $(\mathbb{Q}+i \mathbb{Q})^{p}$ by induction applied to Theorem 1.2.7.

Definition 2.4.3 Let $S$ be a nonempty set. Then a set of open sets $\mathscr{C}$ is called an open cover of $S$ if $\cup \mathscr{C} \supseteq \mathscr{S}$. (It covers up the set $S$. Think lilly pads covering the surface of a pond.)

One of the important properties possessed by separable metric spaces is the Lindeloff property.
Definition 2.4.4 A metric space has the Lindeloff property if whenever $\mathscr{C}$ is an open cover of a set $S$, there exists a countable subset of $\mathscr{C}$ denoted here by $\mathscr{B}$ such that $\mathscr{B}$ is also an open cover of $S$.

## Theorem 2.4.5 Every separable metric space has the Lindeloff property.

Proof: Let $\mathscr{C}$ be an open cover of a set $S$. Let $\mathscr{B}$ be a countable basis. Such exists by Theorem 2.4.2. Let $\hat{\mathscr{B}}$ denote those sets of $\mathscr{B}$ which are contained in some set of $\mathscr{C}$. Thus $\hat{\mathscr{B}}$ is a countable open cover of $S$. Now for $B \in \mathscr{\mathscr { B }}$, let $U_{B}$ be a set of $\mathscr{C}$ which contains $B$. Letting $\widehat{\mathscr{C}}$ denote these sets $U_{B}$ it follows that $\widehat{\mathscr{C}}$ is countable and is an open cover of $S$.

Note how the axiom of choice was used in the above where we let $U_{B}$ be a set of $\mathscr{C}$ which contains $B$.

Definition 2.4.6 A Polish space is a complete separable metric space. These things turn out to be very useful in probability theory and in other areas.

Now it is convenient to consider the distance function in a metric space $(X, d)$.
Definition 2.4.7 Let $S$ be a nonempty set in $X$ and let $x \in X$. Then the distance of $x$ to the set $S$ is defined as

$$
\operatorname{dist}(x, S) \equiv \inf \{d(x, y): y \in S\}
$$

The main result concerning this function is that it is Lipschitz continuous as described in the following theorem.

Theorem 2.4.8 Let $S \neq \emptyset$ and consider $f(x) \equiv \operatorname{dist}(x, S)$, then

$$
|f(x)-f(\hat{x})| \leq d(x, \hat{x})
$$

Proof: Say $\operatorname{dist}(x, S) \leq \operatorname{dist}(\hat{x}, S)$. Otherwise, reverse the argument which follows. Then for a suitable choice of $y \in S$,

$$
|\operatorname{dist}(x, S)-\operatorname{dist}(\hat{x}, S)|=\operatorname{dist}(\hat{x}, S)-\operatorname{dist}(x, S) \leq \operatorname{dist}(\hat{x}, S)-(d(x, y)-\varepsilon)
$$

Then

$$
\begin{aligned}
& |\operatorname{dist}(x, S)-\operatorname{dist}(\hat{x}, S)| \leq d(\hat{x}, y)-(d(x, y)-\varepsilon) \\
& \quad \leq d(\hat{x}, x)+d(x, y)-d(x, y)+\varepsilon=d(\hat{x}, x)+\varepsilon
\end{aligned}
$$

Since $\varepsilon$ is arbitrary, this shows the claimed result.

### 2.5 Compactness and Continuous Functions

As usual, we are not worrying about empty sets. Fussing over these is usually a waste of time. Thus if a set is mentioned, the default is that it is nonempty.

Definition 2.5.1 A metric space $K$ is compact if whenever $\mathscr{C}$ is an open cover of $K$, there exists a finite subset of $\mathscr{C}\left\{U_{1}, \cdots, U_{n}\right\}$ such that $K \subseteq \cup_{k=1}^{n} U_{k}$. In words, every open cover admits a finite sub-cover.

Directly from this definition is the following proposition.
Proposition 2.5.2 If $K$ is a closed, nonempty subset of a nonempty compact set $H$, then $K$ is compact.

Proof: Let $\mathscr{C}$ be an open cover for $K$. Then $\mathscr{C} \cup\left\{K^{C}\right\}$ is an open cover for $H$. Thus there are finitely many sets from this last collection of open sets, $U_{1}, \cdots, U_{m}$ which covers $H$. Include only those which are in $\mathscr{C}$. These cover $K$ because $K^{C}$ covers no points of $K$.

This is the real definition given above. However, in metric spaces, it is equivalent to another definition called sequentially compact.

Definition 2.5.3 A metric space $K$ is sequentially compact means that whenever $\left\{x_{n}\right\} \subseteq K$, there exists a subsequence $\left\{x_{n_{k}}\right\}$ such that $\lim _{k \rightarrow \infty} x_{n_{k}}=x \in K$ for some point $x$. In words, every sequence has a subsequence which converges to a point in the set.

There is a fundamental property possessed by a sequentially compact set in a metric space which is described in the following proposition. The special number described is called a Lebesgue number.

Proposition 2.5.4 Let $K$ be a sequentially compact set in a metric space and let $\mathscr{C}$ be an open cover of $K$. Then there exists a number $\delta>0$ such that whenever $x \in K$, it follows that $B(x, \delta)$ is contained in some set of $\mathscr{C}$.

Proof: If $\mathscr{C}$ is an open cover of $K$ and has no Lebesgue number, then for each $n \in \mathbb{N}, \frac{1}{n}$ is not a Lebesgue number. Hence there exists $x_{n} \in K$ such that $B\left(x_{n}, \frac{1}{n}\right)$ is not contained in any set of $\mathscr{C}$. By sequential compactness, there is a subsequence $\left\{x_{n_{k}}\right\}$ such that $x_{n_{k}} \rightarrow x \in K$. Now there is $r>0$ such that $B(x, r) \subseteq U \in \mathscr{C}$. Let $k$ be large enough that $\frac{1}{n_{k}}<\frac{r}{2}$ and also large enough that $x_{n_{k}} \in B\left(x, \frac{r}{2}\right)$. Then $B\left(x_{n_{k}}, \frac{1}{n_{k}}\right) \subseteq B\left(x_{n_{k}}, \frac{r}{2}\right) \subseteq B(x, r)$ contrary to the requirement that $B\left(x_{n_{k}}, \frac{1}{n_{k}}\right)$ is not contained in any set of $\mathscr{C}$.

In any metric space, these two definitions of compactness are equivalent.
Theorem 2.5.5 Let $K$ be a nonempty subset of a metric space $(X, d)$. Then it is compact if and only if it is sequentially compact.

Proof: $\Leftarrow$ Suppose $K$ is sequentially compact. Let $\mathscr{C}$ be an open cover of $K$. By Proposition 2.5.4 there is a Lebesgue number $\boldsymbol{\delta}>0$. Let $x_{1} \in K$. If $B\left(x_{1}, \boldsymbol{\delta}\right)$ covers $K$, then pick a set of $\mathscr{C}$ containing this ball and this set will be a finite subset of $\mathscr{C}$ which covers $K$. If $B\left(x_{1}, \delta\right)$ does not cover $K$, let $x_{2} \notin B\left(x_{1}, \delta\right)$. Continue this way obtaining $x_{k}$ such that $d\left(x_{k}, x_{j}\right) \geq \delta$ whenever $k \neq j$. Thus eventually $\left\{B\left(x_{i}, \delta\right)\right\}_{i=1}^{n}$ must cover $K$ because if not, you could get a sequence $\left\{x_{k}\right\}$ which has every pair of points further apart than $\delta$ and hence
it has no Cauchy subsequence. Therefore, by Lemma 2.3.2, it would have no convergent subsequence. This would contradict $K$ is sequentially compact. Now for $B\left(x_{i}, \delta\right)$, pick $U_{i} \in \mathscr{C}$ and $\left\{U_{1}, \ldots, U_{m}\right\}$ covers $K$.
$\Rightarrow$ Now suppose $K$ is compact. If it is not sequentially compact, then there exists a sequence $\left\{x_{n}\right\}$ which has no convergent subsequence to a point of $K$. In particular, no point of this sequence is repeated infinitely often. The set of points $\cup_{n}\left\{x_{n}\right\}$ has no limit point in $K$. If it did, you would have a subsequence converging to this point since every ball containing this point would contain infinitely many points of $\cup_{n}\left\{x_{n}\right\}$. Now consider the sets $H_{n} \equiv \cup_{k \geq n}\left\{x_{k}\right\} \cup H^{\prime}$ where $H^{\prime}$ denotes all limit points of $\cup_{n}\left\{x_{n}\right\}$ in $X$ which is the same as the limit points of $\cup_{k \geq n}\left\{x_{k}\right\}$. Therefore, each $H_{n}$ is closed thanks to Theorem 2.2.7. Now let $U_{n} \equiv H_{n}^{C}$. This is an increasing sequence of open sets whose union contains $K$ thanks to the fact that there is no constant subsequence. However, none of these open sets covers $K$ because $U_{n}$ is missing $x_{n}$, violating the definition of compactness.
$\Rightarrow$ Another proof of the second part of the above is as follows. Suppose $K$ is not sequentially compact. Then there is $\left\{x_{n}\right\}$ such that no $x \in K$ is the limit of a convergent subsequence. Hence if $x \in K$, there is $r_{x}>0$ such that $B\left(x, r_{x}\right)$ contains $x_{n}$ for only finitely many $n$. Otherwise, $B\left(x, \frac{1}{k}\right)$ would contain $x_{n}$ for infinitely many $n$ and there would exist $\left\{x_{n_{k}}\right\}_{k=1}^{\infty}$ with $n_{k}<n_{k+1}$ for all $k$ and $x_{n_{k}} \in B\left(x, \frac{1}{k}\right)$ so this subsequence would converge to $x$. By compactness, there are finitely many of these balls $B\left(x, r_{x}\right)$ which cover $K$. Now this is a contradiction because one of these balls must now contain $x_{n}$ for infinitely many $n$.
Definition 2.5.6 Let $X$ be a metric space. Then a finite set of points $\left\{x_{1}, \cdots, x_{n}\right\}$ is called an $\varepsilon$ net if $X \subseteq \cup_{k=1}^{n} B\left(x_{k}, \varepsilon\right)$. If, for every $\varepsilon>0$ a metric space has an $\varepsilon$ net, then we say that the metric space is totally bounded.

Lemma 2.5.7 If a metric space $(K, d)$ is sequentially compact, then it is separable and totally bounded.

Proof: Pick $x_{1} \in K$. If $B\left(x_{1}, \varepsilon\right) \supseteq K$, then stop. Otherwise, pick $x_{2} \notin B\left(x_{1}, \varepsilon\right)$. Continue this way. If $\left\{x_{1}, \cdots, x_{n}\right\}$ have been chosen, either

$$
K \subseteq \cup_{k=1}^{n} B\left(x_{k}, \varepsilon\right)
$$

in which case, you have found an $\varepsilon$ net or this does not happen in which case, you can pick $x_{n+1} \notin \cup_{k=1}^{n} B\left(x_{k}, \varepsilon\right)$. The process must terminate since otherwise, the sequence would need to have a convergent subsequence which is not possible because every pair of terms is farther apart than $\varepsilon$. See Lemma 2.3.2. Thus for every $\varepsilon>0$, there is an $\varepsilon$ net. Thus the metric space is totally bounded. Let $N_{\varepsilon}$ denote an $\varepsilon$ net. Let $D=\cup_{k=1}^{\infty} N_{1 / 2^{k}}$. Then this is a countable dense set. It is countable because it is the countable union of finite sets and it is dense because given a point, there is a point of $D$ within $1 / 2^{k}$ of it.

Also recall that a complete metric space is one for which every Cauchy sequence converges to a point in the metric space.

The following is the main theorem which relates these concepts. Note that if $(X, d)$ is a metric space, then so is $(S, d)$ whenever $S \subseteq X$. You simply use the metric on $S$.
Theorem 2.5.8 For $(X, d)$ a metric space, the following are equivalent.

1. $(X, d)$ is compact.
2. $(X, d)$ is sequentially compact.

3．$(X, d)$ is complete and totally bounded．
Proof：By Theorem 2．5．5，the first two conditions are equivalent．
$2 . \Rightarrow 3$ ．If $(X, d)$ is sequentially compact，then by Lemma 2．5．7，it is totally bounded． If $\left\{x_{n}\right\}$ is a Cauchy sequence，then there is a subsequence which converges to $x \in X$ by assumption．However，from Theorem 2．3．3 this requires the original Cauchy sequence to converge．Thus $(X, d)$ is complete and totally bounded．
$3 . \Rightarrow 2$ ．Suppose $\left\{x_{k}\right\}$ is a sequence in $X$ ．It suffices to show it has a Cauchy subse－ quence．By assumption there are finitely many open balls of radius $1 / n$ covering $X$ ．This for each $n \in \mathbb{N}$ ．Therefore，for $n=1$ ，there is one of the balls，having radius 1 which con－ tains $x_{k}$ for infinitely many $k$ ．Therefore，there is a subsequence with every term contained in this ball of radius 1 ．Now do for this subsequence what was just done for $\left\{x_{k}\right\}$ ．There is a further subsequence contained in a ball of radius $1 / 2$ ．Continue this way．Denote the $i^{t h}$ subsequence as $\left\{x_{k i}\right\}_{k=1}^{\infty}$ ．Arrange them as shown

```
x 11, 和, 利利,
x}12,\mp@subsup{x}{22}{},\mp@subsup{x}{32}{},\mp@subsup{x}{42}{}
x 13},\mp@subsup{x}{23}{},\mp@subsup{x}{33}{},\mp@subsup{x}{43}{}
    \vdots
```

Thus all terms of $\left\{x_{k i}\right\}_{k=1}^{\infty}$ are contained in a ball of radius $1 / i$ ．Consider now the diagonal sequence defined as $y_{k} \equiv x_{k k}$ ．Given $n$ ，each $y_{k}$ is contained in a ball of radius $1 / n$ whenever $k \geq n$ ．Thus $\left\{y_{k}\right\}$ is a subsequence of the original sequence and $\left\{y_{k}\right\}$ is a Cauchy sequence． By completeness of $X$ ，this converges to some $x \in X$ which shows that every sequence in $X$ has a convergent subsequence．This shows 3 ．）$\Rightarrow 2$ ．）．

Lemma 2．5．9 The closed interval $[a, b]$ in $\mathbb{R}$ is compact and every Cauchy sequence in $\mathbb{R}$ converges．

Proof：To show this，suppose it is not．Then there is an open cover $\mathscr{C}$ which admits no finite subcover for $[a, b] \equiv I_{0}$ ．Consider the two intervals $\left[a, \frac{a+b}{2}\right],\left[\frac{a+b}{2}, b\right]$ ．One of these，maybe both cannot be covered with finitely many sets of $\mathscr{C}$ since otherwise，there would be a finite collection of sets from $\mathscr{C}$ covering $[a, b]$ ．Let $I_{1}$ be the interval which has no finite subcover．Now do for it what was done for $I_{0}$ ．Split it in half and pick the half which has no finite covering of sets of $\mathscr{C}$ ．Thus there is a＂nested＂sequence of closed intervals $I_{0} \supseteq I_{1} \supseteq I_{2} \cdots$ ，each being half of the preceding interval．Say $I_{n}=\left[a_{n}, b_{n}\right]$ ．By the nested interval Lemma，Lemma 1．11．12，there is a point $x$ in all these intervals．The point is unique because the lengths of the intervals converge to 0 ．This point is in some $O \in \mathscr{C}$ ．Thus for some $\delta>0,[x-\delta, x+\delta]$ ，having length $2 \delta$ ，is contained in $O$ ．For $k$ large enough，the interval $\left[a_{k}, b_{k}\right]$ has length less than $\delta$ but contains $x$ ．Therefore，it is contained in $[x-\delta, x+\delta]$ and so must be contained in a single set of $\mathscr{C}$ contrary to the construction．This contradiction shows that in fact $[a, b]$ is compact．

The second claim was proved earlier，but here it is again．If $\left\{x_{n}\right\}$ is a Cauchy sequence， then it is contained in some interval $[a, b]$ which is compact．Hence there is a subsequence which converges to some $x \in[a, b]$ ．By Theorem 2．3．3 the original Cauchy sequence con－ verges to $x$ ．

Now the next corollary pertains more specifically to $\mathbb{R}^{p}$ ．
Corollary 2．5．10 For each $r>0, Q \equiv[-r, r]^{p} \equiv \prod_{i=1}^{p}[-r, r]$ is compact in $\mathbb{R}^{p}$ ．

Proof: Let $\left\{\mathbf{x}^{k}\right\}_{k=1}^{\infty}$ be a sequence in $Q$. Then for each $i,\left\{x_{i}^{k}\right\}_{k=1}^{\infty}$ is contained in $[-r, r]$. Therefore, taking a succession of $p$ subsequences, one obtains a subsequence of $\left\{\mathbf{x}^{k}\right\}$ denoted as $\left\{\mathbf{x}^{n_{k}}\right\}$ such that for each $i \leq p,\left\{x_{i}^{n_{k}}\right\}$ converges to some $x_{i} \in[-r, r]$. It follows that $\lim _{k \rightarrow \infty} \mathbf{x}^{n_{k}}=\mathbf{x}$ where the $i^{t h}$ component of $\mathbf{x}$ is $x_{i}$.

Since $\mathbb{R}^{p}$ is a metric space, Theorem 2.5.8 implies the following theorem.
Theorem 2.5.11 A nonempty set $K$ contained in $\mathbb{R}^{p}$ is compact if and only if it is sequentially compact.

Now the general result called the Heine Borel theorem comes right away.
Theorem 2.5.12 For $K \subseteq \mathbb{R}^{p}$ a nonempty set, the following are equivalent.

1. $K$ is compact.
2. $K$ is sequentially compact.
3. $K$ is closed and bounded.

Proof: The first two are equivalent from Theorem 2.5.5. It remains to show that these are equivalent to closed and bounded.
$\Rightarrow$ Suppose the first two hold. Why is $K$ bounded? If not, there is $\mathbf{k}_{n} \in K \backslash B(\mathbf{0}, n)$. Then $\left\{\mathbf{k}_{n}\right\}$ cannot have a Cauchy subsequence and so no subsequence can converge thanks to Theorem 2.3.3. Why is $K$ closed? Using Corollary 2.2.8, it suffices to show that if $\mathbf{k}_{n} \rightarrow \mathbf{k}$, then $\mathbf{k} \in K$. We know that $\left\{\mathbf{k}_{n}\right\}$ is a Cauchy sequence by Theorem 2.3.2. Since $K$ is sequentially compact, a subsequence converges to some $\mathbf{l} \in K$. However, from Theorem 2.3.3, the original sequence also converges to $\mathbf{l}$ and $\operatorname{so} \mathbf{l}=\mathbf{k}$. Thus $\mathbf{k} \in K$. The following is another proof that $K$ is closed given $K$ is compact.
$\Leftarrow$ Suppose now that $K$ is closed and bounded. Then it is a closed subset of $[-r, r]^{p}$ for large $r$. Thus, it is a closed subset of a compact set by Corollary 2.5.10. Therefore, it is compact by Proposition 2.5.2.

As shown above, every closed interval $[a, b]$ is compact and sequentially compact. Next is an easy observation about the product of compact sets. The proof was essentially used above.

Corollary 2.5.13 Suppose $K_{i}$ is a compact subset of $\mathbb{R}$. Then $K \equiv \prod_{i=1}^{p} K_{i}$ is a compact subset of $\mathbb{R}^{p}$.

Proof: This is easiest to see in terms of sequential compactness. Let $\left\{\mathbf{x}_{n}\right\}_{n=1}^{\infty}$ be a sequence in $K$. Say $\mathbf{x}_{n}=\left(\begin{array}{cccc}x_{n}^{1} & x_{n}^{2} & \cdots & x_{n}^{p}\end{array}\right)$. By sequential compactness of each $K_{i}$, it follows that taking $p$ subsequences, one can obtain a subsequence, still denoted by $\left\{\mathbf{x}_{n}\right\}$ such that for each $i \leq p, \lim _{n \rightarrow \infty} x_{n}^{i}=x^{i} \in K_{i}$. Then $\mathbf{x}_{n} \rightarrow \mathbf{x} \in K$.

Since $\mathbb{C}^{p}$ is just $\mathbb{R}^{2 p}$, closed and bounded sets are compact in $\mathbb{C}^{p}$ also as a special case of the above.

A useful corollary of this theorem is the following, sometimes called the Weierstrass Bolzano theorem.

Corollary 2.5.14 Let $\left\{\mathbf{x}_{k}\right\}_{k=1}^{\infty}$ be a bounded sequence in $\mathbb{R}^{p}$ or $\mathbb{C}^{p}$. Then it has a convergent subsequence.

Proof: The given sequence is contained in some set of the form $\prod_{i=1}^{p}[-r, r]$ which is a compact set as shown in Corollary 2.5.13. Hence the given sequence has a convergent subsequence. Here we regard $\mathbb{C}^{p}$ as $\mathbb{R}^{2 p}$.

It is always the case that a compact set in a metric space is a closed set. In fact, this is true for any Hausdorff space. What is done in general is to axiomatize the idea of a metric space to define a general topological space as follows. Here $X$ is a nonempty set.

1. Let $\tau$ be the collection of open sets called the topology, $\tau \subseteq \mathscr{P}(X)$. Then if $\mathscr{C} \subseteq \tau$, $\cap \mathscr{C} \in \tau$.
2. If $U_{i} \in \tau$ for $i=1,2, \cdots, n$, then $\cap_{i=1}^{n} U_{i} \in \tau$

Definition 2.5.15 Hausdorff space is a general topological space which has the property that if $x \neq y$, then there exist open sets $U_{x}$ and $U_{y}$ containing $x, y$ respectively such that $U_{x} \cap U_{y}=\emptyset$.

Proposition 2.5.16 If $K$ is a compact subset of a Hausdorff space, then it is closed. In particular, this holds for any metric space.

Proof: Let $K$ be a nonempty compact set and suppose $p \notin K$. Then for each $x \in K$, there are open sets $U_{x}, V_{x}$ such that $x \in V_{x}$ and $p \in U_{x}$ and $U_{x} \cap V_{x}=\emptyset$. Then since $V$ is compact, there are finitely many $V_{x}$ which cover $K$ say $V_{x_{1}}, \cdots, V_{x_{n}}$. Then let $U=\cap_{i=1}^{n} U_{x_{i}}$. It follows $p \in U$ and $U$ has empty intersection with $K$. In fact $U$ has empty intersection with $\cup_{i=1}^{n} V_{x_{i}}$ because it is contained in each $U_{x_{i}}$. Since $U$ is an open set and $p \in K^{C}$ is arbitrary, it follows $K^{C}$ is an open set.

The following is a very important property pertaining to compact sets. It is a surprising result. However, it follows from the definition of compactness.

Proposition 2.5.17 Suppose $\mathscr{F}$ is a nonempty collection of nonempty compact sets with the finite intersection property. This means that the intersection of any finite subset of $\mathscr{F}$ is nonempty. Then $\cap \mathscr{F} \neq \emptyset$.

Proof: If the conclusion were not so, $\cup\left\{F^{C}: F \in \mathscr{F}\right\}=X$ and so, in particular, picking some $F_{0} \in \mathscr{F},\left\{F^{C}: F \in \mathscr{F}\right\}$ would be an open cover of $F_{0}$. A point in $F_{0}$ is not in $F_{0}^{C}$ so it must be in one of the above sets $F \neq F_{0}$. Since $F_{0}$ is compact, some finite subcover, $F_{1}^{C}, \cdots, F_{m}^{C}$ exists, $F_{0} \subseteq \cup_{k=1}^{m} F_{k}^{C}$. Therefore, the finite intersection property is violated because

$$
F_{0} \cap\left(\cap_{k=1}^{m} F_{k}\right) \subseteq\left(\cup_{k=1}^{m} F_{k}^{C}\right) \cap\left(\cap_{k=1}^{m} F_{k}\right)=\left(\cap_{k=1}^{m} F_{k}\right)^{C} \cap\left(\cap_{k=1}^{m} F_{k}\right)=\emptyset
$$

Note that absolutely no mention was made of context. This is because this finite intersection property is always true whenever you have a set of compact sets. Of course, in this book, we typically have in mind a metric space. I am just pointing out that all of it generalizes.

### 2.5.1 Continuous Functions

The following is a fairly general definition of what it means for a function to be continuous. It includes everything seen in typical calculus classes as a special case.

Definition 2.5.18 Let $f: X \rightarrow Y$ be a function where $(X, d)$ and $(Y, \rho)$ are metric spaces. Then $f$ is continuous at $x \in X$ if and only if the following condition holds. For every $\varepsilon>0$, there exists $\delta>0$ such that if $d(\hat{x}, x)<\delta$, then $\rho(f(\hat{x}), f(x))<\varepsilon$. If $f$ is continuous at every $x \in X$ we say that $f$ is continuous on $X$. The notation $f^{-1}(S)$ means $\{x \in X: f(x) \in S\}$. It is called the inverse image of $S$.

For example, you could have a real valued function $f(x)$ defined on an interval $[0,1]$. In this case you would have $X=[0,1]$ and $Y=\mathbb{R}$ with the distance given by $d(x, y)=|x-y|$. Then the following theorem is the main result. Recall that if $(X, d)$ is a metric space and $S \subseteq X$ is a nonempty subset, then $(S, d)$ is also a metric space so this latter case is included in what follows.

Theorem 2.5.19 Let $f: X \rightarrow Y$ where $(X, d)$ and $(Y, \rho)$ are metric spaces. Then the following two are equivalent.
a $f$ is continuous at $x \in D(f)$.
$b$ Whenever $x_{n} \rightarrow x \in D(f)$, each $x_{n} \in D(f)$, it follows that $f\left(x_{n}\right) \rightarrow f(x)$.
Also, the following are equivalent.
c $f$ is continuous on $X$.
$d$ Whenever $V$ is open in $Y$, it follows that $f^{-1}(V) \equiv\{x \in X: f(x) \in V\}$ is open in $X$.
$e$ Whenever $H$ is closed in $Y$, it follows that $f^{-1}(H)$ is closed in $X$.
Proof: $\mathrm{a} \Rightarrow \mathrm{b}$ : Let $f$ be continuous at $x$ and suppose $x_{n} \rightarrow x$. Then let $\varepsilon>0$ be given. By continuity, there exists $\delta>0$ such that if $d(\hat{x}, x)<\delta$, then $\rho(f(\hat{x}), f(x))<\varepsilon$. Since $x_{n} \rightarrow x$, it follows that there exists $N$ such that if $n \geq N$, then $d\left(x_{n}, x\right)<\delta$ and so, if $n \geq N$, it follows that $\rho\left(f\left(x_{n}\right), f(x)\right)<\varepsilon$. Since $\varepsilon>0$ is arbitrary, it follows that $f\left(x_{n}\right) \rightarrow f(x)$.
$\mathrm{b} \Rightarrow \mathrm{a}$ : Suppose b holds but $f$ fails to be continuous at $x$. Then there exists $\varepsilon>0$ such that for all $\delta>0$, there exists $\hat{x}$ such that $d(\hat{x}, x)<\delta$ but $\rho(f(\hat{x}), f(x)) \geq \varepsilon$. Letting $\delta=1 / n$, there exists $x_{n}$ such that $d\left(x_{n}, x\right)<1 / n$ but $\rho\left(f\left(x_{n}\right), f(x)\right) \geq \varepsilon$. Now this is a contradiction because by assumption, the fact that $x_{n} \rightarrow x$ implies that $f\left(x_{n}\right) \rightarrow f(x)$. In particular, for large enough $n, \rho\left(f\left(x_{n}\right), f(x)\right)<\varepsilon$ contrary to the construction.
$\mathrm{c} \Rightarrow \mathrm{d}$ : Let $V$ be open in $Y$. Let $x \in f^{-1}(V)$ so that $f(x) \in V$. Since $V$ is open, there exists $\varepsilon>0$ such that $B(f(x), \varepsilon) \subseteq V$. Since $f$ is continuous at $x$, it follows that there exists $\delta>0$ such that if $\hat{x} \in B(x, \boldsymbol{\delta})$, then $f(\hat{x}) \in B(f(x), \boldsymbol{\varepsilon}) \subseteq V .(f(B(x, \boldsymbol{\delta})) \subseteq B(f(x), \varepsilon))$ In other words, $B(x, \delta) \subseteq f^{-1}(B(f(x), \varepsilon)) \subseteq f^{-1}(V)$ which shows that, since $x$ was an arbitrary point of $f^{-1}(V)$, every point of $f^{-1}(V)$ is an interior point which implies $f^{-1}(V)$ is open.
$\mathrm{d} \Rightarrow \mathrm{e}$ : Let $H$ be closed in $Y$. Then $f^{-1}(H)^{C}=f^{-1}\left(H^{C}\right)$ which is open by assumption. Hence $f^{-1}(H)$ is closed because its complement is open.
$\mathrm{e} \Rightarrow \mathrm{d}$ : Let $V$ be open in $Y$. Then $f^{-1}(V)^{C}=f^{-1}\left(V^{C}\right)$ which is assumed to be closed. This is because the complement of an open set is a closed set. Thus $f^{-1}(V)$ is open because its complement is closed.
$\mathrm{d} \Rightarrow \mathrm{c}$ : Let $x \in X$ be arbitrary. Is it the case that $f$ is continuous at $x$ ? Let $\varepsilon>0$ be given. Then $B(f(x), \varepsilon)$ is an open set in $V$ (Recall that open balls are open.) and so $x \in f^{-1}(B(f(x), \varepsilon))$ which is given to be open. Hence there exists $\delta>0$ such that $x \in$ $B(x, \delta) \subseteq f^{-1}(B(f(x), \varepsilon))$. Thus, $f(B(x, \delta)) \subseteq B(f(x), \varepsilon)$ so if $d(x, \hat{x})<\delta$ meaning that
$\hat{x} \in B(x, \boldsymbol{\delta})$, then $\rho(f(\hat{x}), f(x))<\varepsilon$ meaning that $f(\hat{x}) \in B(f(x), \varepsilon)$. Thus $f$ is continuous at $x$ for every $x$.

In the case where $\mathbf{f}: D(\mathbf{f}) \subseteq \mathbb{R}^{p} \rightarrow \mathbb{R}^{q}$, the above definition takes the following more familiar form.

Definition 2.5.20 $A$ function $\mathbf{f}: D(\mathbf{f}) \subseteq \mathbb{R}^{p} \rightarrow \mathbb{R}^{q}$ is continuous at $\mathbf{x} \in D(\mathbf{f})$ if for each $\varepsilon>0$ there exists $\delta>0$ such that whenever $\mathbf{y} \in D(\mathbf{f})$ and

$$
|\mathbf{y}-\mathbf{x}|<\delta
$$

it follows that

$$
|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})|<\varepsilon .
$$

$\mathbf{f}$ is continuous if it is continuous at every point of $D(\mathbf{f})$.
This is equivalent to the same statement with $\|\cdot\|_{\infty}$ in place of $|\cdot|$ because

$$
\|\mathbf{x}\|_{\infty} \leq|\mathbf{x}| \equiv\left(\sum_{k=1}^{p}\left|x_{k}\right|^{2}\right)^{1 / 2} \leq \sqrt{p}\|\mathbf{x}\|_{\infty}
$$

and it will be shown a little later that any two norms satisfy an inequality of the above sort so the choice of norm does not affect whether a function is continuous in the sense that if it is continuous with respect to one norm, then it is continuous for the other.

Corollary 2.5.21 $\mathbf{f}: \mathbb{R}^{p} \rightarrow \mathbb{R}^{q}$ is continuous if and only if $\mathbf{f}^{-1}(V)$ is open in $\mathbb{R}^{p}$ whenever $V$ is open in $\mathbb{R}^{q}$ and $\mathbf{f}^{-1}(C)$ is closed in $\mathbb{R}^{p}$ whenever $C$ is closed in $\mathbb{R}^{q}$.

Recall how the function $x \rightarrow \operatorname{dist}(x, S)$ was continuous. Theorem 2.5.19 implies

$$
\left\{x: \operatorname{dist}(x, S)>\frac{1}{k}\right\} \text { is open, }\left\{x: \operatorname{dist}(x, S) \geq \frac{1}{k}\right\} \text { is closed }
$$

and so forth.
Now here are some basic properties of continuous functions which have values in $\mathbb{R}^{p}$ or $\mathbb{R}$ so that it makes sense to add and multiply by scalars. However, no context is specified for property 3. which holds for $\mathbf{f}, \mathbf{g}$ having values and domains in metric space.

Theorem 2.5.22 The following assertions are valid.

1. The function $a \mathbf{f}+b \mathbf{g}$ is continuous at $\mathbf{x}$ when $\mathbf{f}, \mathbf{g}$ are continuous at $\mathbf{x} \in D(\mathbf{f}) \cap D(\mathbf{g})$ and $a, b \in \mathbb{R}$.
2. If and $f$ and $g$ are each real valued functions continuous at $\mathbf{x}$, then $f g$ is continuous at $\mathbf{x}$. If, in addition to this, $g(\mathbf{x}) \neq 0$, then $f / g$ is continuous at $\mathbf{x}$.
3. If $\mathbf{f}$ is continuous at $\mathbf{x}, \mathbf{f}(\mathbf{x}) \in D(\mathbf{g})$, and $\mathbf{g}$ is continuous at $\mathbf{f}(\mathbf{x})$, then $\mathbf{g} \circ \mathbf{f}$ is continuous at $\mathbf{x}$.
4. If $\mathbf{f}=\left(f_{1}, \cdots, f_{q}\right): D(\mathbf{f}) \rightarrow \mathbb{R}^{q}$, then $\mathbf{f}$ is continuous if and only if each $f_{k}$ is a continuous real valued function.
5. The function $f: \mathbb{R}^{p} \rightarrow \mathbb{R}$, given by $f(\mathbf{x})=|\mathbf{x}|$ is continuous.

Proof: Begin with (1). Let $\varepsilon>0$ be given. Let $\mathbf{x}_{n} \rightarrow \mathbf{x}$. Then by assumption, $\mathbf{f}\left(\mathbf{x}_{n}\right) \rightarrow$ $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}\left(\mathbf{x}_{n}\right) \rightarrow \mathbf{g}(\mathbf{x})$ whenever $\mathbf{x}_{n} \rightarrow \mathbf{x}$ with $\mathbf{x}_{n} \in D(\mathbf{f}) \cap D(\mathbf{g})$. Thus

$$
\left|(a \mathbf{f}+b \mathbf{g})\left(\mathbf{x}_{n}\right)-(a \mathbf{f}+b \mathbf{g})(\mathbf{x})\right| \rightarrow 0 .
$$

Now begin on (2). This also follows from properties of convergence of sequences of real numbers from beginning calculus. For example, letting $\mathbf{x}_{n} \rightarrow \mathbf{x}$ as above where $g(\mathbf{x}) \neq 0,\left|g\left(\mathbf{x}_{n}\right)-g(\mathbf{x})\right|<|g(\mathbf{x})| / 2$ for all $n$ large enough and so, for all large enough $n$, from the triangle inequality, $\frac{3|g(\mathbf{x})|}{2} \geq\left|g\left(\mathbf{x}_{n}\right)\right| \geq \frac{1}{2}|g(\mathbf{x})|$

$$
\begin{aligned}
\left|\frac{f\left(\mathbf{x}_{n}\right)}{g\left(\mathbf{x}_{n}\right)}-\frac{f(\mathbf{x})}{g(\mathbf{x})}\right| & =\left|\frac{f\left(\mathbf{x}_{n}\right) g(\mathbf{x})-f(\mathbf{x}) g\left(\mathbf{x}_{n}\right)}{g(\mathbf{x}) g\left(\mathbf{x}_{n}\right)}\right| \\
& \leq 2 \frac{1}{|g(\mathbf{x})|^{2}}\left|f\left(\mathbf{x}_{n}\right) g(\mathbf{x})-f(\mathbf{x}) g\left(\mathbf{x}_{n}\right)\right| \\
\leq & 2 \frac{1}{|g(\mathbf{x})|^{2}}\binom{\left|f\left(\mathbf{x}_{n}\right) g(\mathbf{x})-f(\mathbf{x}) g(\mathbf{x})\right|}{+\left|f(\mathbf{x}) g(\mathbf{x})-f(\mathbf{x}) g\left(\mathbf{x}_{n}\right)\right|} \\
& =2 \frac{1}{|g(\mathbf{x})|^{2}}\binom{|g(\mathbf{x})|\left|f\left(\mathbf{x}_{n}\right)-f(\mathbf{x})\right|}{+|f(\mathbf{x})|\left|g(\mathbf{x})-g\left(\mathbf{x}_{n}\right)\right|}
\end{aligned}
$$

which converges to 0 by assumption.
Now begin on (3). In terms of sequences, if $\mathbf{x}_{n} \rightarrow \mathbf{x}$, then $\mathbf{f}\left(\mathbf{x}_{n}\right) \rightarrow \mathbf{f}(\mathbf{x})$ and so

$$
\lim _{n \rightarrow \infty} \mathbf{g}\left(\mathbf{f}\left(\mathbf{x}_{n}\right)\right)=\mathbf{g}(\mathbf{f}(\mathbf{x}))
$$

Thus $\mathbf{g} \circ \mathbf{f}$ is continuous at $\mathbf{x}$.
Part (4) says: If $\mathbf{f}=\left(f_{1}, \cdots, f_{q}\right): D(\mathbf{f}) \rightarrow \mathbb{R}^{q}$, then $\mathbf{f}$ is continuous if and only if each $f_{k}$ is a continuous real valued function at $\mathbf{x}$. Then letting $\mathbf{x}_{n} \rightarrow \mathbf{x}$

$$
\begin{aligned}
& \max \left(\left|f_{i}\left(\mathbf{x}_{n}\right)-f_{i}(\mathbf{x})\right|, i=1, \ldots, q\right) \leq\left|\mathbf{f}\left(\mathbf{x}_{n}\right)-\mathbf{f}(\mathbf{x})\right| \equiv\left(\sum_{i=1}^{q}\left|f_{i}\left(\mathbf{x}_{n}\right)-f_{i}(\mathbf{x})\right|^{2}\right)^{1 / 2} \\
\leq & \sqrt{q} \max \left(\left|f_{i}\left(\mathbf{x}_{n}\right)-f_{i}(\mathbf{x})\right|, i=1, \ldots, q\right)
\end{aligned}
$$

Thus $\mathbf{f}\left(\mathbf{x}_{n}\right) \rightarrow \mathbf{f}(\mathbf{x})$ if and only if each $f_{k}\left(\mathbf{x}_{n}\right) \rightarrow f(\mathbf{x})$ and this shows (4).
To verify part (5), the triangle inequality implies $\left\|\mathbf{x}_{n}\left|-\left|\mathbf{x} \| \leq\left|\mathbf{x}_{n}-\mathbf{x}\right|\right.\right.\right.$ so if $\mathbf{x}_{n} \rightarrow \mathbf{x}$, then $\left|\mathbf{x}_{n}\right| \rightarrow|\mathbf{x}|$.

### 2.5.2 Limits of Functions

I will feature limits of functions which have values in some $\mathbb{R}^{p}$. First of all, you can only consider limits at limit points of the domain as explained below. It isn't any harder to formulate this in terms of metric spaces, so this is what I will do. You can let the metric space be $\mathbb{R}^{p}$ if you like.

Definition 2.5.23 Let $f: D(f) \subseteq X \rightarrow Y$ where $(X, d)$ and $(Y, \rho)$ are metric spaces. For $x$ a limit point of $D(f)$, meaning that $B(x, r)$ contains points of $D(f)$ other than $x$ for each $r>0, \lim _{y \rightarrow x} f(y)=z \in Y$ means the following.

For every $\varepsilon>0$, there exists $\delta>0$ such that if $0<d(x, y)<\delta$ and $y \in D(f)$, then $\rho(f(y), z)<\varepsilon$.

Note that $x$ must be a limit point of $D(f)$ in order to take the limit at $x$. This will be clear from the next proposition which says that the limit, if it exists, is well defined.

Proposition 2.5.24 Let $x$ be a limit point of $D(f)$ where $f: D(f) \subseteq X \rightarrow Y$ as in the above definition. If $\lim _{y \rightarrow x} f(x)=z$ and $\lim _{y \rightarrow x} f(y)=\hat{z}$, then $z=\hat{z}$.

Proof: Let $\delta$ be small enough to go with $\varepsilon / 3$ in the case of both $z, \hat{z}$. Then, since $x$ is a limit point, there exists $y \in B(x, \delta) \cap D(f), y \neq x$. Then

$$
\rho(z, \hat{z}) \leq \rho(z, f(y))+\rho(f(y), \hat{z})<\frac{2 \varepsilon}{3}<\varepsilon
$$

Since $\varepsilon$ is arbitrary, this shows that $z=\hat{z}$.

### 2.5.3 The Extreme Value Theorem and Uniform Continuity

These topics work in any metric space or even more general settings. First is a theorem which says that the continuous image of a compact set is compact.

Theorem 2.5.25 Let $f: X \rightarrow Y$ where $(X, d)$ and $(Y, \rho)$ are metric spaces and $f$ is continuous on $X$. Then if $K \subseteq X$ is compact, it follows that $f(K)$ is compact in $(Y, \rho)$.

Proof: Let $\mathscr{C}$ be an open cover of $f(K)$. Denote by $f^{-1}(\mathscr{C})$ the sets

$$
\left\{f^{-1}(U): U \in \mathscr{C}\right\}
$$

Each of these is an open set by Theorem 2.5.19. Then $f^{-1}(\mathscr{C})$ is an open cover of $K$. It follows there are finitely many,

$$
\left\{f^{-1}\left(U_{1}\right), \cdots, f^{-1}\left(U_{n}\right)\right\}
$$

which covers $K$. It follows that $\left\{U_{1}, \cdots, U_{n}\right\}$ is an open cover for $f(K)$.
The following is the important extreme values theorem for a real valued function defined on a compact set.

Theorem 2.5.26 Let $K$ be a compact metric space and suppose $f: K \rightarrow \mathbb{R}$ is a continuous function. That is, $\mathbb{R}$ is the metric space where the metric is given by $d(x, y)=$ $|x-y|$. Then $f$ achieves its maximum and minimum values on $K$.

Proof: Let $\lambda=\sup \{f(x): x \in K\}$. Then from the definition of sup, you have the existence of a sequence $\left\{x_{n}\right\} \subseteq K$ such that $\lim _{n \rightarrow \infty} f\left(x_{n}\right)=\lambda$. There is a subsequence still called $\left\{x_{n}\right\}$ which converges to some $x \in K$. From continuity, $\lambda=\lim _{n \rightarrow \infty} f\left(x_{n}\right)=f(x)$ and so $f$ achieves its maximum value at $x$. Similar reasoning shows that it achieves its minimum value on $K$.

Definition 2.5.27 Let $f:(X, d) \rightarrow(Y, \rho)$ be a function. Then it is said to be uniformly continuous on $X$ if for every $\varepsilon>0$ there exists a $\delta>0$ such that whenever $x, \hat{x}$ are two points of $X$ with $d(x, \hat{x})<\delta$, it follows that $\rho(f(x), f(\hat{x}))<\varepsilon$.

Note the difference between this and continuity. With continuity, the $\delta$ could depend on $x$ but here it works for any pair of points in $X$.

There is a remarkable result concerning compactness and uniform continuity.

Theorem 2.5.28 Let $f:(K, d) \rightarrow(Y, \rho)$ be a continuous function where $K$ is a compact metric space. Then $f$ is uniformly continuous.

Proof: Suppose $f$ fails to be uniformly continuous. Then there exists $\varepsilon>0$ and pairs of points $x_{n}, \hat{x}_{n}$ such that $d\left(x_{n}, \hat{x}_{n}\right)<1 / n$ but $\rho\left(f\left(x_{n}\right), f\left(\hat{x}_{n}\right)\right) \geq \varepsilon$. Since $K$ is compact, it is sequentially compact and so there exists a subsequence, still denoted as $\left\{x_{n}\right\}$ such that $x_{n} \rightarrow x \in K$. Then also $\hat{x}_{n} \rightarrow x$ also and so by Lemma 2.1.3, $0=\rho(f(x), f(x))=$ $\lim _{n \rightarrow \infty} \rho\left(f\left(x_{n}\right), f\left(\hat{x}_{n}\right)\right) \geq \varepsilon$ which is a contradiction.

Later in the book, I will consider the fundamental theorem of algebra. However, here is a fairly short proof based on the extreme value theorem. You may have to fill in a few details however. In particular, note that $(\mathbb{C},|\cdot|)$ is the same as $\left(\mathbb{R}^{2},|\cdot|\right)$ where $|\cdot|$ is the standard norm on $\mathbb{R}^{2}$. Thus closed and bounded sets are compact in $(\mathbb{C},|\cdot|)$. Also, the above theorems apply for $\mathbb{R}^{p}$ and so they also apply for $\mathbb{C}^{p}$ because it is the same as $\mathbb{R}^{2 p}$.

Proposition 2.5.29 Let $p(z)=a_{0}+a_{1} z+\cdots+a_{n-1} z^{n-1}+z^{n}$ be a nonconstant polynomial where each $a_{i} \in \mathbb{C}$. Then there is a root to this polynomial.

Proof: Suppose the nonconstant polynomial $p(z)=a_{0}+a_{1} z+\cdots+z^{n}$, has no zero in C. Since $\lim _{|z| \rightarrow \infty}|p(z)|=\infty$, there is a $z_{0}$ with $\left|p\left(z_{0}\right)\right|=\min _{z \in \mathbb{C}}|p(z)|>0$ Why? (The growth condition shows that you can restrict attention to a closed and bounded set and then apply the extreme value theorem.) Then let $q(z)=\frac{p\left(z+z_{0}\right)}{p\left(z_{0}\right)}$. This is also a polynomial which has no zeros and the minimum of $|q(z)|$ is 1 and occurs at $z=0$. Since $q(0)=1$, it follows $q(z)=1+a_{k} z^{k}+r(z)$ where $r(z)$ consists of higher order terms. Here $a_{k}$ is the first coefficient which is nonzero. Choose a sequence, $z_{n} \rightarrow 0$, such that $a_{k} z_{n}^{k}<0$. For example, let $-a_{k} z_{n}^{k}=(1 / n)$. Then

$$
\left|q\left(z_{n}\right)\right|=\left|1+a_{k} z^{k}+r(z)\right| \leq 1-1 / n+\left|r\left(z_{n}\right)\right|=1+a_{k} z_{n}^{k}+\left|r\left(z_{n}\right)\right|<1
$$

for all $n$ large enough because $\left|r\left(z_{n}\right)\right|$ is small compared with $\left|a_{k} z_{n}^{k}\right|$ since it involves higher order terms. This is a contradiction.

The idea is that if $k<m$, then for $z$ sufficiently small, $z^{m}$ is very small relative to $z^{k}$.
Here is another very interesting theorem about continuity and compactness. It says that if you have a continuous function defined on a compact set $K$ then $f(K)$ is also compact. If $f$ is one to one, then its inverse is also continuous.

Theorem 2.5.30 Let $K$ be a compact set in some metric space and let $f: K \rightarrow f(K)$ be continuous. Then $f(K)$ is compact. If $f$ is one to one, then $f^{-1}$ is also continuous.

Proof: The first part is in Theorem 2.5.25. However, I will give a different proof. As explained above, compactness and sequential compactness are the same in the setting of metric space. Suppose then that $\left\{f\left(x_{k}\right)\right\}_{k=1}^{\infty}$ is a sequence in $f(K)$. Since $K$ is compact, there is a subsequence, still denoted as $\left\{x_{k}\right\}_{k=1}^{\infty}$ such that $x_{k} \rightarrow x \in K$. Then by continuity, $f\left(x_{k}\right) \rightarrow f(x)$ and so $f(K)$ is compact as claimed.

Next suppose $f$ is one to one. If you have $f\left(x_{k}\right) \rightarrow f(x)$, does it follow that $x_{k} \rightarrow x$ ? If not, then by compactness, there is a subsequence, still denoted as $\left\{x_{k}\right\}_{k=1}^{\infty}$ such that $x_{k} \rightarrow$ $\hat{x} \in K, x \neq \hat{x}$. Then by continuity, it also happens that $f\left(x_{k}\right) \rightarrow f(\hat{x})$ and so $f(x)=f(\hat{x})$ which is a contradiction. Therefore, $x_{k} \rightarrow x$ as desired, showing that $f^{-1}$ is continuous.

### 2.5.4 Convergence of Functions

First is the definition of a Banach space. It is really just a generalization of the familiar $\mathbb{R}^{p}$. For many things of interest in this book, the Banach space will be $\mathbb{R}^{p}$ or $\mathbb{C}^{p}$.
Definition 2.5.31 A Banach space is a complete normed linear space. It can be either real or complex, depending on the field of scalars. That is, it is a normed vector space in which every Cauchy sequence converges. In particular, it is a metric space in which $d(\mathbf{x}, \mathbf{y}) \equiv\|\mathbf{x}-\mathbf{y}\|$ so all the theory of metric space applies. In particular open balls really are open.

Here the discussion is specialized to vector valued functions having values in some Banach space $X$. Most if not all of it will work for general metric spaces.

There are two kinds of convergence for a sequence of functions described in the next definition, pointwise convergence and uniform convergence. Of the two, uniform convergence is far better and tends to be the kind of convergence most encountered in complex analysis. Pointwise convergence is more often encounted in real analysis and necessitates much more difficult theorems.
Definition 2.5.32 Let $X, Y$ be Banach spaces where $\|\cdot\|$ will denote the norm in either one. $S \subseteq X$ and let $\mathbf{f}_{n}: S \rightarrow Y$ for $n=1,2, \cdots$. Then $\left\{\mathbf{f}_{n}\right\}$ is said to converge pointwise to $\mathbf{f}$ on $S$ if for all $\mathbf{x} \in S$,

$$
\mathbf{f}_{n}(\mathbf{x}) \rightarrow \mathbf{f}(\mathbf{x}) \text {, that is } \lim _{n \rightarrow \infty}\left\|\mathbf{f}_{n}(\mathbf{x})-\mathbf{f}(\mathbf{x})\right\|=0
$$

for each $\mathbf{x}$. The sequence is said to converge uniformly to $\mathbf{f}$ on $S$ if

$$
\lim _{n \rightarrow \infty}\left(\sup _{\mathbf{x} \in S}\left\|\mathbf{f}_{n}(\mathbf{x})-\mathbf{f}(\mathbf{x})\right\|\right)=0
$$

$\sup _{\mathbf{x} \in S}\left\|\mathbf{f}_{n}(\mathbf{x})-\mathbf{f}(\mathbf{x})\right\|$ is denoted as $\left\|\mathbf{f}_{n}-\mathbf{f}\right\|_{\infty}$ or just $\left\|\mathbf{f}_{n}-\mathbf{f}\right\|$ for short. $\|\cdot\|$ is called the uniform norm. More generally, it suffices in the above to let $S$ just be a metric space.

The following picture illustrates the above definition.


The wriggly function is uniformly close to the not so wriggly one.
To illustrate the difference in the two types of convergence, here is a standard example.
Example 2.5.33 Let

$$
f(x) \equiv\left\{\begin{array}{l}
0 \text { if } x \in[0,1) \\
1 \text { if } x=1
\end{array}\right.
$$

Also let $f_{n}(x) \equiv x^{n}$ for $x \in[0,1]$. Then $f_{n}$ converges pointwise to $f$ on $[0,1]$ but does not converge uniformly to $f$ on $[0,1]$.


Note how the target function is not continuous although each function in the sequence is. The next theorem shows that this kind of loss of continuity never occurs when you have uniform convergence. The theorem holds generally when $S \subseteq X$ a metric space and $\mathbf{f}, \mathbf{f}_{n}$ have values in $Y$ another metric space. Note that you could simply refer to $S$ as the metric space if you want.

Theorem 2.5.34 Let $S$ be a subset of a Banach space or just a metric space and let $\mathbf{f}_{n}: S \rightarrow Y$ be continuous where $Y$ is a Banach space and let $\mathbf{f}_{n}$ converge uniformly to $\mathbf{f}$ on $S$. Then if each $\mathbf{f}_{n}$ is continuous at $\mathbf{x} \in S$, it follows that $\mathbf{f}$ is also continuous at $\mathbf{x}$.

Proof: Let $\varepsilon>0$ be given. Let $N$ be such that if $n \geq N$, then

$$
\sup _{\mathbf{y} \in S}\left\|\mathbf{f}_{n}(\mathbf{y})-\mathbf{f}(\mathbf{y})\right\| \equiv\left\|\mathbf{f}_{n}-\mathbf{f}\right\|_{\infty}<\frac{\varepsilon}{3}
$$

Pick such an $n$. Then by continuity of $\mathbf{f}_{n}$ at $\mathbf{x}$, there exists $\delta>0$ such that if $\|\mathbf{y}-\mathbf{x}\|<\delta$ or $d(\mathbf{x}, \mathbf{y})<\boldsymbol{\delta}$, then $\left\|\mathbf{f}_{n}(\mathbf{y})-\mathbf{f}_{n}(\mathbf{x})\right\|<\frac{\varepsilon}{3}$. Then if $\|\mathbf{y}-\mathbf{x}\|<\delta$ or $d(\mathbf{x}, \mathbf{y})<\delta, \mathbf{y} \in S$, then

$$
\begin{aligned}
\|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})\| & \leq\left\|\mathbf{f}(\mathbf{x})-\mathbf{f}_{n}(\mathbf{x})\right\|+\left\|\mathbf{f}_{n}(\mathbf{x})-\mathbf{f}_{n}(\mathbf{y})\right\|+\left\|\mathbf{f}_{n}(\mathbf{y})-\mathbf{f}(\mathbf{y})\right\| \\
& <\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon
\end{aligned}
$$

Thus $\mathbf{f}$ is continuous at $\mathbf{x}$ as claimed.
Definition 2.5.35 Let $\mathbf{f}_{n}: S \rightarrow Y$. Then $\mathbf{f}_{n}$ is said to be uniformly Cauchy if for every $\varepsilon>0$, there exists $N$ such that if $m, n>\varepsilon$, then if $m, n \geq N$, then $\left\|\mathbf{f}_{n}-\mathbf{f}_{m}\right\| \equiv$ $\sup _{\mathbf{x} \in S}\left|\mathbf{f}_{n}(\mathbf{x})-\mathbf{f}_{m}(\mathbf{x})\right|<\boldsymbol{\varepsilon}$

## Observation 2.5.36 $\|\cdot\|$ satisfies the axioms of a norm.

Consider the triangle inequality.

$$
\|\mathbf{f}(\mathbf{x})+\mathbf{g}(\mathbf{x})\| \leq\|\mathbf{f}(\mathbf{x})\|+\|\mathbf{g}(\mathbf{x})\| \leq\|\mathbf{f}\|+\|\mathbf{g}\|
$$

and so

$$
\|\mathbf{f}+\mathbf{g}\| \equiv \sup _{\mathbf{x} \in S}\|\mathbf{f}(\mathbf{x})+\mathbf{g}(\mathbf{x})\| \leq \sup _{\mathbf{x} \in S}\|\mathbf{f}(\mathbf{x})\|+\sup _{\mathbf{x} \in S}\|\mathbf{g}(\mathbf{x})\|=\|\mathbf{f}\|+\|\mathbf{g}\|
$$

As to the axiom about scalars,

$$
\|\alpha \mathbf{f}\| \equiv \sup _{\mathbf{x} \in S}\|\alpha \mathbf{f}(\mathbf{x})\|=\sup _{\mathbf{x} \in S}|\alpha|\|\mathbf{f}(\mathbf{x})\|=|\alpha| \sup _{\mathbf{x} \in S}\|\mathbf{f}(\mathbf{x})\| \equiv|\alpha|\|\mathbf{f}\|
$$

It is clear that $\|\mathbf{f}\| \geq 0$. If $\|\mathbf{f}\|=0$, then clearly $\mathbf{f}(\mathbf{x})=\mathbf{0}$ for each $\mathbf{x}$ and so $\mathbf{f}=\mathbf{0}$.
Theorem 2.5.37 Let $Y$ be a Banach space and $S$ a metric space. $\mathbf{f}_{n}: S \rightarrow Y$ be bounded functions: $\sup _{\mathbf{x} \in S}\left\|\mathbf{f}_{n}(\mathbf{x})\right\|=C_{n}<\infty$. Then there exists bounded $\mathbf{f}: S \rightarrow Y$ such that $\lim _{n \rightarrow \infty}\left\|\mathbf{f}-\mathbf{f}_{n}\right\|=0$ if and only if $\left\{\mathbf{f}_{n}\right\}$ is uniformly Cauchy.

Proof: $\Leftarrow$ First suppose $\left\{\mathbf{f}_{n}\right\}$ is uniformly Cauchy. Then for each $\mathbf{x}$,

$$
\begin{equation*}
\left\|\mathbf{f}_{n}(\mathbf{x})-\mathbf{f}_{m}(\mathbf{x})\right\| \leq\left\|\mathbf{f}_{n}-\mathbf{f}_{m}\right\| \tag{2.1}
\end{equation*}
$$

and it follows that for each $\mathbf{x},\left\{\mathbf{f}_{n}(\mathbf{x})\right\}$ is a Cauchy sequence. By completeness of $Y$, this converges. Let $\mathbf{f}(\mathbf{x})$ be that to which it converges. Now pick $N$ such that for $m, n \geq$ $N,\left\|\mathbf{f}_{n}-\mathbf{f}_{m}\right\|<\varepsilon / 2$. Then in 2.1,

$$
\left\|\mathbf{f}_{n}(\mathbf{x})\right\| \leq\left\|\mathbf{f}_{N}(\mathbf{x})\right\|+\varepsilon / 2 \leq C_{N}+1
$$

so, since $\mathbf{x}$ is arbitrary, it follows each $\mathbf{f}_{n}$ is bounded since $\left\|\mathbf{f}_{n}(\mathbf{x})\right\|$ is bounded above by

$$
\max \left\{C_{N}+\varepsilon / 2, C_{k}, k \leq N\right\}
$$

Also, for $n \geq N$,

$$
\left\|\mathbf{f}_{n}(\mathbf{x})-\mathbf{f}(\mathbf{x})\right\| \leq\left\|\mathbf{f}_{n}(\mathbf{x})-\mathbf{f}_{m}(\mathbf{x})\right\|+\left\|\mathbf{f}_{m}(\mathbf{x})-\mathbf{f}(\mathbf{x})\right\|<\varepsilon / 2+\left\|\mathbf{f}_{m}(\mathbf{x})-\mathbf{f}(\mathbf{x})\right\|
$$

Now, take the limit as $m \rightarrow \infty$ on the right to obtain that for all $\mathbf{x},\left\|\mathbf{f}_{n}(\mathbf{x})-\mathbf{f}(\mathbf{x})\right\| \leq \varepsilon / 2$. Therefore, $\left\|\mathbf{f}_{n}-\mathbf{f}\right\|<\varepsilon$ if $n \geq N$ so, since $\varepsilon$ is arbitrary, this shows that $\lim _{n \rightarrow \infty}\left\|\mathbf{f}-\mathbf{f}_{n}\right\|=0$.
$\Rightarrow$ Conversely, if there exists bounded $\mathbf{f}: S \rightarrow Y$ to which $\left\{\mathbf{f}_{n}\right\}$ converges uniformly, why is $\left\{\mathbf{f}_{n}\right\}$ uniformly Cauchy?

$$
\begin{aligned}
\left\|\mathbf{f}_{n}(\mathbf{x})-\mathbf{f}_{m}(\mathbf{x})\right\| & \leq\left\|\mathbf{f}_{n}(\mathbf{x})-\mathbf{f}(\mathbf{x})\right\|+\left\|\mathbf{f}(\mathbf{x})-\mathbf{f}_{m}(\mathbf{x})\right\| \\
& \leq\left\|\mathbf{f}_{n}-\mathbf{f}\right\|+\left\|\mathbf{f}-\mathbf{f}_{m}\right\|
\end{aligned}
$$

By assumption, there is $N$ such that if $n \geq N$, then $\left\|\mathbf{f}_{n}-\mathbf{f}\right\|<\varepsilon / 3$. Then if $m, n \geq N$, it follows that for any $\mathbf{x} \in S$,

$$
\left\|\mathbf{f}_{n}(\mathbf{x})-\mathbf{f}_{m}(\mathbf{x})\right\|<\varepsilon / 3+\varepsilon / 3=2 \varepsilon / 3
$$

Then $\sup _{\mathbf{x} \in S}\left\|\mathbf{f}_{n}(\mathbf{x})-\mathbf{f}_{m}(\mathbf{x})\right\| \equiv\left\|\mathbf{f}_{n}-\mathbf{f}_{m}\right\| \leq 2 \varepsilon / 3<\varepsilon$. Hence $\left\{\mathbf{f}_{n}\right\}$ is uniformly Cauchy.
Now here is an example of an infinite dimensional space which is also complete.
Example 2.5.38 Denote by $B C(S ; Y)$ the bounded continuous functions defined on $S$ with values in $Y$ for $Y$ a real or complex Banach space. Then this is a complex vector space with norm $\|\cdot\|$ defined above. It is also complete. Here $S$ is some metric space.

It is obvious that this is a vector space. Indeed, it is a subspace of the set of functions having values in $Y$ and it is clear that the given set of functions is closed with respect to the vector space operations. It was explained above in Observation 2.5.36 that this uniform norm really is a norm. It remains to verify completeness. Suppose then that $\left\{\mathbf{f}_{n}\right\}$ is a Cauchy sequence. By Theorem 2.5.37, there exists a bounded function $\mathbf{f}$ such that $\mathbf{f}_{n}$ converges to $\mathbf{f}$ uniformly. Then by Theorem 2.5 .34 it follows that $\mathbf{f}$ is also continuous. Since every Cauchy sequence converges, this says that $B C(S ; Y)$ is complete.

I will be a little vague about $K$ other than to say it is compact. You can assume it is a compact metric space if desired.

Proposition 2.5.39 Let $K$ be a compact set and consider the continuous functions defined on $K$ with values in $X, C(K ; X)$ with the uniform norm

$$
\|\mathbf{f}\| \equiv \max \left\{\|\mathbf{f}(\mathbf{x})\|_{X}: \mathbf{x} \in K\right\}
$$

Then this is a complete normed linear space, a Banach space.

Proof: It is a repeat of the proof of Theorem 2.5.37.
One can consider convergence of infinite series the same way as done in calculus.
Definition 2.5.40 The symbol $\sum_{k=1}^{\infty} \mathbf{f}_{k}(\mathbf{x})$ is defined as the limit of the sequence of partial sums $\lim _{n \rightarrow \infty} \sum_{k=1}^{n} \mathbf{f}_{k}(\mathbf{x})$ provided this limit exists. This is called pointwise convergence of the infinite sum. The infinite sum is said to converge uniformly on a set $S$ if the sequence of paritial sums converges uniformly, that is

$$
\lim _{n \rightarrow \infty}\left\|\sum_{k=1}^{\infty} \mathbf{f}_{k}-\sum_{k=1}^{n} \mathbf{f}_{k}\right\| \equiv \lim _{n \rightarrow \infty}\left(\sup _{\mathbf{x} \in S}\left\|\sum_{k=1}^{\infty} \mathbf{f}_{k}(\mathbf{x})-\sum_{k=1}^{n} \mathbf{f}_{k}(\mathbf{x})\right\|\right)=0
$$

Note how this theorem includes the case of $\sum_{k=1}^{\infty} \mathbf{a}_{k}$ as a special case. Here the $\mathbf{a}_{k}$ don't depend on $\mathbf{x}$.

The following theorem is very useful. It tells how to recognize that an infinite sum is converging or converging uniformly. First is a little lemma which reviews standard calculus.

Lemma 2.5.41 Suppose $M_{k} \geq 0$ and $\sum_{k=1}^{\infty} M_{k}$ converges. It follows that

$$
\lim _{m \rightarrow \infty} \sum_{k=m}^{\infty} M_{k}=0
$$

Proof: By assumption, there is $N$ such that if $m \geq N$, then if $n>m$,

$$
\left|\sum_{k=1}^{n} M_{k}-\sum_{k=1}^{m} M_{k}\right|=\sum_{k=m+1}^{n} M_{k}<\varepsilon / 2
$$

Then letting $n \rightarrow \infty$, one can pass to a limit and conclude that $\sum_{k=m+1}^{\infty} M_{k}<\varepsilon$. It follows that for $m>N, \sum_{k=m}^{\infty} M_{k}<\varepsilon$. The part about passing to a limit follows from the fact that $n \rightarrow \sum_{k=m+1}^{n} M_{k}$ is an increasing sequence which is bounded above by $\sum_{k=1}^{\infty} M_{k}$. Therefore, it converges by completeness of $\mathbb{R}$.

Theorem 2.5.42 Let $Y$ be a Banach space, $\mathbb{C}^{p}$ for example, $\mathbf{f}_{k}: S \rightarrow Y$. For $\mathbf{x} \in S$, if $\sum_{k=1}^{\infty}\left\|\mathbf{f}_{k}(\mathbf{x})\right\|<\infty$, then $\sum_{k=1}^{\infty} \mathbf{f}_{k}(\mathbf{x})$ converges pointwise. If there exists $M_{k}$ such that $M_{k} \geq\left\|\mathbf{f}_{k}(\mathbf{x})\right\|$ for all $\mathbf{x} \in S$, then $\sum_{k=1}^{\infty} \mathbf{f}_{k}(\mathbf{x})$ converges uniformly.

Proof: $\|\cdot\|$ will denote either the uniform norm or the norm in $X$ depending on context. Let $m<n$. Then $\left\|\sum_{k=1}^{n} \mathbf{f}_{k}(\mathbf{x})-\sum_{k=1}^{m} \mathbf{f}_{k}(\mathbf{x})\right\| \leq \sum_{k=m}^{\infty}\left\|\mathbf{f}_{k}(\mathbf{x})\right\|<\varepsilon / 2$ whenever $m$ is large enough due to the assumption that $\sum_{k=1}^{\infty}\left\|\mathbf{f}_{k}(\mathbf{x})\right\|<\infty$. Thus the partial sums are a Cauchy sequence and so the series converges pointwise.

If $M_{k} \geq\left\|\mathbf{f}_{k}(\mathbf{x})\right\|$ for all $\mathbf{x} \in S$, then for $M$ large enough,

$$
\left\|\sum_{k=1}^{n} \mathbf{f}_{k}(\mathbf{x})-\sum_{k=1}^{m} \mathbf{f}_{k}(\mathbf{x})\right\| \leq \sum_{k=m}^{\infty}\left\|\mathbf{f}_{k}(\mathbf{x})\right\| \leq \sum_{k=m}^{\infty} M_{k}<\varepsilon / 2
$$

Thus, taking sup, $\left\|\sum_{k=1}^{n} \mathbf{f}_{k}(\cdot)-\sum_{k=1}^{m} \mathbf{f}_{k}(\cdot)\right\| \leq \varepsilon / 2<\varepsilon$ and so the partial sums are uniformly Cauchy sequence. Hence they converge uniformly to what is defined as $\sum_{k=1}^{\infty} \mathbf{f}_{k}(\mathbf{x})$ for $\mathbf{x} \in S$.

The latter part of this theorem is called the Weierstrass $M$ test. As a very interesting application, consider the question of nowhere differentiable functions. This considers the simple case of functions having values in the Banach space $\mathbb{R}$.

Consider the following description of a function. The following is the graph of the function on $[0,1]$.


The height of the function is $1 / 2$ and the slope of the rising line is 1 while the slope of the falling line is -1 . Now extend this function to the whole real line to make it periodic of period 1. This means $f(x+n)=f(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$, the integers. In other words to find the graph of $f$ on $[1,2]$ you simply slide the graph of $f$ on $[0,1]$ a distance of 1 to get the same tent shaped thing on $[1,2]$. Continue this way. The following picture illustrates what a piece of the graph of this function looks like. Some might call it an infinite sawtooth.


Now define $g(x) \equiv \sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^{k} f\left(4^{k} x\right)$. Letting $M_{k}=(3 / 4)^{k}$, an application of the Weierstrass $M$ test, Theorem 2.5.42 shows $g$ is everywhere continuous. This is because each function in the sum is continuous and the series converges uniformly on $\mathbb{R}$. However, this function is nowhere differentiable. This is shown next.

Let $\delta_{m}= \pm \frac{1}{4}\left(4^{-m}\right)$ where we assume $m>2$. That of interest will be $m \rightarrow \infty$.

$$
\frac{g\left(x+\delta_{m}\right)-g(x)}{\boldsymbol{\delta}_{m}}=\frac{\sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^{k}\left(f\left(4^{k}\left(x+\delta_{m}\right)\right)-f\left(4^{k} x\right)\right)}{\delta_{m}}
$$

If you take $k>m$,

$$
\begin{aligned}
f\left(4^{k}\left(x+\delta_{m}\right)\right)-f\left(4^{k} x\right) & =f\left(4^{k}\left(x \pm \frac{1}{4}\left(4^{-m}\right)\right)\right)-f\left(4^{k} x\right) \\
& =f(4^{k} x \pm \overbrace{\frac{1}{4} 4^{k-m}}^{\text {integer }})-f\left(4^{k} x\right)=0
\end{aligned}
$$

Therefore,

$$
\frac{g\left(x+\delta_{m}\right)-g(x)}{\delta_{m}}=\frac{1}{\delta_{m}} \sum_{k=0}^{m}\left(\frac{3}{4}\right)^{k}\left(f\left(4^{k}\left(x+\delta_{m}\right)\right)-f\left(4^{k} x\right)\right)
$$

The absolute value of the last term in the sum is $\left|\left(\frac{3}{4}\right)^{m}\left(f\left(4^{m}\left(x+\delta_{m}\right)\right)-f\left(4^{m} x\right)\right)\right|$ and we choose the sign of $\boldsymbol{\delta}_{m}$ such that both $4^{m}\left(x+\boldsymbol{\delta}_{m}\right)$ and $4^{m} x$ are in some interval which is of the form $[k / 2,(k+1) / 2)$ which is certainly possible because the distance between these two points is $1 / 4$ and such half open intervals include all of $\mathbb{R}$. Thus, since $f$ has slope $\pm 1$ on the interval just mentioned,

$$
\left|\left(\frac{3}{4}\right)^{m}\left(f\left(4^{m}\left(x+\delta_{m}\right)\right)-f\left(4^{m} x\right)\right)\right|=\left(\frac{3}{4}\right)^{m} 4^{m}\left|\delta_{m}\right|=3^{m}\left|\delta_{m}\right|
$$

As to the other terms, $0 \leq f(x) \leq 1 / 2$ and so

$$
\left|\sum_{k=0}^{m-1}\left(\frac{3}{4}\right)^{k}\left(f\left(4^{k}\left(x+\delta_{m}\right)\right)-f\left(4^{k} x\right)\right)\right| \leq \sum_{k=0}^{m-1}\left(\frac{3}{4}\right)^{k}=\frac{1-(3 / 4)^{m}}{1 / 4}=4-4\left(\frac{3}{4}\right)^{m}
$$

Thus $\left|\frac{g\left(x+\delta_{m}\right)-g(x)}{\delta_{m}}\right| \geq 3^{m}-\left(4-4\left(\frac{3}{4}\right)^{m}\right) \geq 3^{m}-4$. Since $\delta_{m} \rightarrow 0$ as $m \rightarrow \infty, g^{\prime}(x)$ does not exist because the difference quotients are not bounded.

This proves the following theorem.
Theorem 2.5.43 There exists a function defined on $\mathbb{R}$ which is continuous and bounded but fails to have a derivative at any point.

Proof: It only remains to verify that the function just constructed is bounded. However, $0 \leq g(x) \leq \frac{1}{2} \sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^{k}=2$.

Note that you could consider $(\varepsilon / 2) g(x)$ to get a function which is continuous, has values between 0 and $\varepsilon$ which has no derivative.

### 2.6 Tietze Extension Theorem

This is an interesting theorem which holds in arbitrary normal topological spaces. However, I am specializing to a metric space $X$ to keep the emphasis on that which is most familiar. The presentation depends on Lemma 2.4.8.

Lemma 2.6.1 Let $H, K$ be two nonempty disjoint closed subsets of a metric space $X$. Then there exists a continuous function, $g: X \rightarrow[-1 / 3,1 / 3]$ such that $g(H)=-1 / 3$, $g(K)=1 / 3, g(X) \subseteq[-1 / 3,1 / 3]$.

Proof: Let $f(\mathbf{x}) \equiv \frac{\operatorname{dist}(\mathbf{x}, H)}{\operatorname{dist}(\mathbf{x}, H)+\operatorname{dist}(\mathbf{x}, K)}$. The denominator is never equal to zero because if $\operatorname{dist}(\mathbf{x}, H)=0$, then $\mathbf{x} \in H$ because $H$ is closed. (To see this, pick $\mathbf{h}_{k} \in B(\mathbf{x}, 1 / k) \cap H$. Then $\mathbf{h}_{k} \rightarrow \mathbf{x}$ and since $H$ is closed, $\mathbf{x} \in H$.) Similarly, if $\operatorname{dist}(\mathbf{x}, K)=0$, then $\mathbf{x} \in K$ and so the denominator is never zero as claimed because it is not possible for a point to be in both $H$ and $K$. Hence $f$ is continuous and from its definition, $f=0$ on $H$ and $f=1$ on $K$. Now let $g(\mathbf{x}) \equiv \frac{2}{3}\left(f(\mathbf{x})-\frac{1}{2}\right)$. Then $g$ has the desired properties.

Definition 2.6.2 For $f: M \subseteq X \rightarrow \mathbb{R}$, define $\|f\|_{M}$ as

$$
\sup \{|f(x)|: x \in M\}
$$

Lemma 2.6.3 Suppose $M$ is a closed set in $X$ and suppose $f: M \rightarrow[-1,1]$ is continuous at every point of $M$. Then there exists a function $g$ which is defined and continuous on all of $\mathbb{R}^{p}$ such that $\|f-g\|_{M}<\frac{2}{3}, g(X) \subseteq[-1 / 3,1 / 3]$.

Proof: Let $H=f^{-1}([-1,-1 / 3]), K=f^{-1}([1 / 3,1])$. Thus $H$ and $K$ are disjoint closed subsets of $M$. Suppose first $H, K$ are both nonempty. Then by Lemma 2.6.1 there exists $g$ such that $g$ is a continuous function defined on all of $X$ and $g(H)=-1 / 3, g(K)=1 / 3$, and $g\left(\mathbb{R}^{p}\right) \subseteq[-1 / 3,1 / 3]$. It follows $\|f-g\|_{M}<2 / 3$. If $H=\emptyset$, then $f$ has all its values in $[-1 / 3,1]$ and so letting $g \equiv 1 / 3$, the desired condition is obtained. If $K=\emptyset$, let $g \equiv-1 / 3$. If both $H, K=\emptyset$, there isn't much to show. Just let $g(x)=0$ for all $x$.

Lemma 2.6.4 Suppose $M$ is a closed set in $X$ and suppose $f: M \rightarrow[-1,1]$ is continuous at every point of $M$. Then there exists a function $g$ which is defined and continuous on all of $X$ such that $g=f$ on $M$ and $g$ has its values in $[-1,1]$.

Proof: Using Lemma 2.6.3, let $g_{1}$ be such that $g_{1}(X) \subseteq[-1 / 3,1 / 3]$ and

$$
\left\|f-g_{1}\right\|_{M} \leq \frac{2}{3}
$$

Suppose $g_{1}, \cdots, g_{m}$ have been chosen such that $g_{j}(X) \subseteq[-1 / 3,1 / 3]$ and

$$
\begin{equation*}
\left\|f-\sum_{i=1}^{m}\left(\frac{2}{3}\right)^{i-1} g_{i}\right\|_{M}<\left(\frac{2}{3}\right)^{m} \tag{2.2}
\end{equation*}
$$

This has been done for $m=1$. Then $\left\|\left(\frac{3}{2}\right)^{m}\left(f-\sum_{i=1}^{m}\left(\frac{2}{3}\right)^{i-1} g_{i}\right)\right\|_{M} \leq 1$ and so

$$
\left(\frac{3}{2}\right)^{m}\left(f-\sum_{i=1}^{m}\left(\frac{2}{3}\right)^{i-1} g_{i}\right)
$$

can play the role of $f$ in the first step of the proof. Therefore, there exists $g_{m+1}$ defined and continuous on all of $X$ such that its values are in $[-1 / 3,1 / 3]$ and

$$
\left\|\left(\frac{3}{2}\right)^{m}\left(f-\sum_{i=1}^{m}\left(\frac{2}{3}\right)^{i-1} g_{i}\right)-g_{m+1}\right\|_{M} \leq \frac{2}{3}
$$

Hence $\left\|\left(f-\sum_{i=1}^{m}\left(\frac{2}{3}\right)^{i-1} g_{i}\right)-\left(\frac{2}{3}\right)^{m} g_{m+1}\right\|_{M} \leq\left(\frac{2}{3}\right)^{m+1}$. It follows that there exists a sequence, $\left\{g_{i}\right\}$ such that each has its values in $[-1 / 3,1 / 3]$ and for every $m 2.2$ holds. Then let

$$
g(\mathbf{x}) \equiv \sum_{i=1}^{\infty}\left(\frac{2}{3}\right)^{i-1} g_{i}(\mathbf{x})
$$

It follows $|g(\mathbf{x})| \leq\left|\sum_{i=1}^{\infty}\left(\frac{2}{3}\right)^{i-1} g_{i}(\mathbf{x})\right| \leq \sum_{i=1}^{m}\left(\frac{2}{3}\right)^{i-1} \frac{1}{3} \leq 1$ and $\left|\left(\frac{2}{3}\right)^{i-1} g_{i}(\mathbf{x})\right| \leq\left(\frac{2}{3}\right)^{i-1} \frac{1}{3}$ so the Weierstrass $M$ test applies and shows convergence is uniform. Therefore $g$ must be continuous by Theorem 2.5.34. The estimate 2.2 implies $f=g$ on $M$.

The following is the Tietze extension theorem.
Theorem 2.6.5 Let $M$ be a closed nonempty subset of $X$ a metric space and let $f: M \rightarrow[a, b]$ be continuous at every point of $M$. Then there exists a function, $g$ continuous on all of $X$ which coincides with $f$ on $M$ such that $g(X) \subseteq[a, b]$.

Proof: Let $f_{1}(\mathbf{x})=1+\frac{2}{b-a}(f(\mathbf{x})-b)$. Then $f_{1}$ satisfies the conditions of Lemma 2.6.4 and so there exists $g_{1}: X \rightarrow[-1,1]$ such that $g_{1}$ is continuous on $X$ and equals $f_{1}$ on $M$. Let $g(\mathbf{x})=\left(g_{1}(\mathbf{x})-1\right)\left(\frac{b-a}{2}\right)+b$. This works.

For $\mathbf{x} \in M$,

$$
\begin{gathered}
g(\mathbf{x})=\left(\left(1+\frac{2}{b-a}(f(\mathbf{x})-b)\right)-1\right)\left(\frac{b-a}{2}\right)+b \\
=\left(\left(\frac{2}{b-a}(f(\mathbf{x})-b)\right)\right)\left(\frac{b-a}{2}\right)+b=(f(\mathbf{x})-b)+b=f(\mathbf{x})
\end{gathered}
$$

Also $1+\frac{2}{b-a}(f(\mathbf{x})-b) \in[-1,1]$ so $\frac{2}{b-a}(f(\mathbf{x})-b) \in[-2,0]$ and

$$
(f(\mathbf{x})-b) \in[-b+a, 0], f(\mathbf{x}) \in[a, b]
$$

### 2.7 Equivalence of Norms

As mentioned above, it makes absolutely no difference which norm you decide to use on $\mathbb{R}^{p}$. This holds in general finite dimensional normed spaces and is shown here. Of course the main interest here is where the normed linear space is $\left(\mathbb{R}^{p},\|\cdot\|\right)$ but it is no harder to present the more general result where you have a finite dimensional vector space $V$ which has a norm. If you have not seen such things, just let $V$ be $\mathbb{R}^{p}$ in what follows or consider the problems at end of the chapter.
Definition 2.7.1 Let $(V,\|\cdot\|)$ be a normed linear space and let a basis for $V$ consist of the vectors $\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{n}\right\}$. For $\mathbf{x} \in V$, let its component vector in $\mathbb{F}^{p}$ be $\left(\alpha_{1}, \cdots, \alpha_{n}\right)$ so that $\mathbf{x}=\sum_{i} \alpha_{i} \mathbf{v}_{i}$. Then define

$$
\theta \mathbf{x} \equiv \alpha=\left(\begin{array}{lll}
\alpha_{1} & \cdots & \alpha_{n}
\end{array}\right)^{T}
$$

Thus $\theta$ is well defined, one to one and onto from $V$ to $\mathbb{F}^{p}$. It is also linear and its inverse $\theta^{-1}$ satisfies all the same algebraic properties as $\theta$. In particular, $(V,\|\cdot\|)$ could be $\left(\mathbb{R}^{p},\|\cdot\|\right)$ where $\|\cdot\|$ is some norm on $\mathbb{R}^{p}$.

The following fundamental lemma comes from the extreme value theorem for continuous functions defined on a compact set. Let

$$
f(\alpha) \equiv\left\|\sum_{i} \alpha_{i} \mathbf{v}_{i}\right\| \equiv\left\|\theta^{-1} \alpha\right\|
$$

Then it is clear that $f$ is a continuous function. This is because $\alpha \rightarrow \sum_{i} \alpha_{i} \mathbf{v}_{i}$ is a continuous map into $V$ and from the triangle inequality $\mathbf{x} \rightarrow\|\mathbf{x}\|$ is continuous as a map from $V$ to $\mathbb{R}$.

Lemma 2.7.2 There exists $\delta>0$ and $\Delta \geq \delta$ such that

$$
\delta=\min \{f(\alpha):|\alpha|=1\}, \Delta=\max \{f(\alpha):|\alpha|=1\}
$$

Also,

$$
\begin{align*}
\delta|\alpha| & \leq\left\|\theta^{-1} \alpha\right\| \leq \Delta|\alpha|  \tag{2.3}\\
\delta|\theta \mathbf{v}| & \leq\|\mathbf{v}\| \leq \Delta|\theta \mathbf{v}| \tag{2.4}
\end{align*}
$$

Proof: These numbers exist thanks to the extreme value theorem, Theorem 2.5.26. It cannot be that $\delta=0$ because if it were, you would have $|\alpha|=1$ but $\sum_{j=1}^{n} \alpha_{k} \mathbf{v}_{j}=\mathbf{0}$ which is impossible since $\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{n}\right\}$ is linearly independent. The first of the above inequalities follows from

$$
\delta \leq\left\|\theta^{-1} \frac{\alpha}{|\alpha|}\right\|=f\left(\frac{\alpha}{|\alpha|}\right) \leq \Delta
$$

the second follows from observing that $\theta^{-1} \alpha$ is a generic vector $\mathbf{v}$ in $V$.
Now we can draw several conclusions about $(V,\|\cdot\|)$ for $V$ finite dimensional.
Theorem 2.7.3 Let $(V,\|\cdot\|)$ be a finite dimensional normed linear space. Then the compact sets are exactly those which are closed and bounded. Also $(V,\|\cdot\|)$ is complete. If $K$ is a closed and bounded set in $(V,\|\cdot\|)$ and $f: K \rightarrow \mathbb{R}$, then $f$ achieves its maximum and minimum on $K$.

Proof: First note that the inequalities 2.3 and 2.4 show that both $\theta^{-1}$ and $\theta$ are continuous. Thus these take convergent sequences to convergent sequences.

Let $\left\{\mathbf{w}_{k}\right\}_{k=1}^{\infty}$ be a Cauchy sequence. Then from 2.4, $\left\{\theta \mathbf{w}_{k}\right\}_{k=1}^{\infty}$ is a Cauchy sequence. Thanks to Theorem 2.5.26, it converges to some $\beta \in \mathbb{F}^{p}$. It follows that $\lim _{k \rightarrow \infty} \theta^{-1} \theta \mathbf{w}_{k}=$ $\lim _{k \rightarrow \infty} \mathbf{w}_{k}=\theta^{-1} \beta \in V$. This shows completeness.

Next let $K$ be a closed and bounded set. Let $\left\{\mathbf{w}_{k}\right\} \subseteq K$. Then $\left\{\theta \mathbf{w}_{k}\right\} \subseteq \theta K$ which is also a closed and bounded set thanks to the inequalities 2.3 and 2.4. Thus there is a subsequence still denoted with $k$ such that $\theta \mathbf{w}_{k} \rightarrow \beta \in \mathbb{F}^{p}$. Then as just done, $\mathbf{w}_{k} \rightarrow \theta^{-1} \beta$. Since $K$ is closed, it follows that $\theta^{-1} \beta \in K$.

Finally, why are the only compact sets those which are closed and bounded? Let $K$ be compact. If it is not bounded, then there is a sequence of points of $K,\left\{\mathbf{k}^{m}\right\}_{m=1}^{\infty}$ such that $\left\|\mathbf{k}^{m}\right\| \geq m$. It follows that it cannot have a convergent subsequence because the points are further apart from each other than $1 / 2$. Hence $K$ is not sequentially compact and consequently it is not compact. It follows that $K$ is bounded. It follows from Proposition 2.5.16 that $K$ is closed.

The last part is identical to the proof in Theorem 2.5.26. You just take a convergent subsequence of a minimizing (maximizing) sequence and exploit continuity.

Next is the theorem which states that any two norms on a finite dimensional vector space are equivalent. In particular, any two norms on $\mathbb{R}^{p}$ are equivalent.

Theorem 2.7.4 Let $\|\cdot\|,\||\cdot \||$ be two norms on $V$ a finite dimensional vector space. Then they are equivalent, which means there are constants $0<a<b$ such that for all $\mathbf{v}$,

$$
a\|\mathbf{v}\| \leq\||\mathbf{v}\|\mid \leq b\| \mathbf{v} \|
$$

Proof: In Lemma 2.7.2, let $\delta, \Delta$ go with $\|\cdot\|$ and $\hat{\delta}, \hat{\Delta}$ go with $\|\mid \cdot\| \|$. Then using the inequalities of this lemma,

$$
\|\mathbf{v}\| \leq \Delta|\theta \mathbf{v}| \leq \frac{\Delta}{\hat{\delta}}\left\|\left|\mathbf{v}\left\|\left|\leq \frac{\Delta \hat{\Delta}}{\hat{\delta}}\right| \theta \mathbf{v} \left\lvert\, \leq \frac{\Delta}{\delta} \hat{\Delta} \frac{\hat{\delta}}{\hat{\delta}}\right.\right\| \mathbf{v} \|\right.\right.
$$

and so $\frac{\hat{\delta}}{\Delta}\|\mathbf{v}\| \leq\left\|\left|\mathbf{v}\left\|\left\lvert\, \leq \frac{\hat{\Delta}}{\delta}\right.\right\| \mathbf{v} \|\right.\right.$. Thus the norms are equivalent.

### 2.8 Norms on Linear Maps

To begin with, the notion of a linear map is just a function which is linear. Such a function, denoted by $L$, and mapping $\mathbb{R}^{n}$ to $\mathbb{R}^{m}$ is linear means

$$
L\left(\sum_{i=1}^{m} x_{i} \mathbf{v}_{i}\right)=\sum_{i=1}^{m} x_{i} L \mathbf{v}_{i}
$$

In other words, it distributes across additions and allows one to factor out scalars. Hopefully this is familiar from linear algebra. If not, have a look at a Linear Algebra book. Any of my on line books has this material.
Definition 2.8.1 We use the symbol $\mathscr{L}\left(\mathbb{R}^{n}, \mathbb{R}^{m}\right)$ to denote the space of linear transformations, also called linear operators, which map $\mathbb{R}^{n}$ to $\mathbb{R}^{m}$. For $L \in \mathscr{L}\left(\mathbb{R}^{n}, \mathbb{R}^{m}\right)$ one can always consider it as an $m \times n$ matrix $A$ as follows. Let

$$
A=\left(\begin{array}{llll}
L \mathbf{e}_{1} & L \mathbf{e}_{2} & \cdots & L \mathbf{e}_{n}
\end{array}\right)
$$

where in the above $\mathbf{L}_{\mathbf{i}}$ is the $i^{\text {th }}$ column. Define the sum and scalar multiple of linear transformations in the natural manner. That is, for L,M linear transformations and $\alpha, \beta$ scalars,

$$
(\alpha L+\beta M)(\mathbf{x}) \equiv \alpha L(\mathbf{x})+\beta M(\mathbf{x})
$$

Observation 2.8.2 With the above definition of sums and scalar multiples of linear transformations, the result of such a linear combination of linear transformations is itself linear. Indeed, for $\mathbf{x}, \mathbf{y}$ vectors and $a, b$ scalars,

$$
\begin{aligned}
(\alpha L & +\beta M)(a \mathbf{x}+b \mathbf{y}) \equiv \alpha L(a \mathbf{x}+b \mathbf{y})+\beta M(a \mathbf{x}+b \mathbf{y}) \\
& =\alpha a L(\mathbf{x})+\alpha b L(\mathbf{y})+\beta a M(\mathbf{x})+\beta b M(\mathbf{y}) \\
& =a(\alpha L(\mathbf{x})+\beta M(\mathbf{x}))+b(\alpha L(\mathbf{y})+\beta M(\mathbf{y})) \\
& =a(\alpha L+\beta M)(\mathbf{x})+b(\alpha L+\beta M)(\mathbf{y})
\end{aligned}
$$

Also, a linear combination of linear transformations corresponds to the linear combination of the corresponding matrices in which addition is defined in the usual manner as addition of corresponding entries. To see this, note that if A is the matrix of $L$ and $B$ the matrix of M,

$$
(\alpha L+\beta M) \mathbf{e}_{i} \equiv(\alpha A+\beta B) \mathbf{e}_{i}=\alpha A \mathbf{e}_{i}+\beta B \mathbf{e}_{i}
$$

by the usual rules of matrix multiplication. Thus the $i^{\text {th }}$ column of $(\alpha A+\beta B)$ is the linear combination of the $i^{\text {th }}$ columns of $A$ and $B$ according to usual rules of matrix multiplication.

Proposition 2.8.3 For $L \in \mathscr{L}\left(\mathbb{R}^{n}, \mathbb{R}^{m}\right)$, the matrix defined above satisfies

$$
A \mathbf{x}=L \mathbf{x}, \mathbf{x} \in \mathbb{R}^{n}
$$

and if any $m \times n$ matrix $A$ does satisfy $A \mathbf{x}=L \mathbf{x}$, then $A$ is given in the above definition.
Proof: $A \mathbf{x}=L \mathbf{x}$ for all $\mathbf{x}$ if and only if for $\mathbf{x}=\sum_{i=1}^{n} x_{i} \mathbf{e}_{i}$

$$
A \mathbf{x}=L\left(\sum_{i=1}^{n} x_{i} \mathbf{e}_{i}\right)=\sum_{i=1}^{n} x_{i} L\left(\mathbf{e}_{i}\right) \equiv\left(\begin{array}{llll}
L \mathbf{e}_{1} & L \mathbf{e}_{2} & \cdots & L \mathbf{e}_{n}
\end{array}\right)\left(\begin{array}{c}
x_{1} \\
x_{2} \\
\vdots \\
x_{n}
\end{array}\right)
$$

if and only if for every $\mathbf{x} \in \mathbb{R}^{n}$,

$$
A \mathbf{x}=\left(\begin{array}{llll}
L \mathbf{e}_{1} & L \mathbf{e}_{2} & \cdots & L \mathbf{e}_{n}
\end{array}\right) \mathbf{x}
$$

which happens if and only if $A=\left(\begin{array}{llll}L \mathbf{e}_{1} & L \mathbf{e}_{2} & \cdots & L \mathbf{e}_{n}\end{array}\right)$.
Definition 2.8.4 The norm of a linear transformation of $A \in \mathscr{L}\left(\mathbb{R}^{n}, \mathbb{R}^{m}\right)$ is defined as

$$
\|A\| \equiv \sup \left\{\|A \mathbf{x}\|_{\mathbb{R}^{m}}:\|\mathbf{x}\|_{\mathbb{R}^{n}} \leq 1\right\}<\infty
$$

Then $\|A\|$ is referred to as the operator norm of the linear transformation $A$.

It is an easy exercise to verify that $\|\cdot\|$ is a norm on $\mathscr{L}\left(\mathbb{R}^{n}, \mathbb{R}^{m}\right)$ and it is always the case that $\|A \mathbf{x}\|_{\mathbb{R}^{m}} \leq\|A\|\|\mathbf{x}\|_{\mathbb{R}^{n}}$. This is shown next. Furthermore, you should verify that you can replace $\leq 1$ with $=1$ in the definition. Thus $\|A\| \equiv \sup \left\{\|A \mathbf{x}\|_{\mathbb{R}^{m}}:\|\mathbf{x}\|_{\mathbb{R}^{n}}=1\right\}$.

It is necessary to verify that this norm is actually well defined.
Lemma 2.8.5 The operator norm is well defined. Let $A \in \mathscr{L}\left(\mathbb{R}^{n}, \mathbb{R}^{m}\right)$.
Proof: We can use the matrix of the linear transformation with matrix multiplication interchangeably with the linear transformation. This follows from the above considerations. Suppose $\lim _{k \rightarrow \infty} \mathbf{v}^{k}=\mathbf{v}$ in $\mathbb{R}^{n}$. Does it follow that $A \mathbf{v}^{k} \rightarrow A \mathbf{v}$ ? This is indeed the case with the usual Euclidean norm and therefore, it is also true with respect to any other norm by the equivalence of norms (Theorem 2.7.4). To see this,

$$
\begin{aligned}
& \left|A \mathbf{v}^{k}-A \mathbf{v}\right| \equiv\left(\sum_{i=1}^{m}\left|\left(A \mathbf{v}^{k}\right)_{i}-(A \mathbf{v})_{i}\right|^{2}\right)^{1 / 2} \leq\left(\sum_{i=1}^{m}\left|\sum_{j=1}^{n} A_{i j}\left(v_{j}^{k}-v_{j}\right)\right|^{2}\right)^{1 / 2} \\
& \leq\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{n}\left|A_{i j}\right|\left|v_{j}^{k}-v_{j}\right|\right)^{2}\right)^{1 / 2} \leq\left|\mathbf{v}^{k}-\mathbf{v}\right|\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{n}\left|A_{i j}\right|\right)^{2}\right)^{1 / 2}
\end{aligned}
$$

Thus $A$ is continuous. Then also $\mathbf{v} \rightarrow\|A \mathbf{v}\|_{\mathbb{R}^{m}}$ is a continuous function by the triangle inequality. Indeed,

$$
|\|A \mathbf{v}\|-\|A \mathbf{u}\|| \leq\|A \mathbf{v}-A \mathbf{u}\|_{\mathbb{R}^{m}}
$$

Now let $D$ be the closed ball of radius 1 in $V$. By Theorem 2.7.3, this set $D$ is compact and so

$$
\max \left\{\|A \mathbf{v}\|_{\mathbb{R}^{m}}:\|\mathbf{v}\|_{\mathbb{R}^{n}} \leq 1\right\} \equiv\|A\|<\infty
$$

Then we have the following theorem.
Theorem 2.8.6 Let $\mathbb{R}^{n}$ and $\mathbb{R}^{m}$ be finite dimensional normed linear spaces of dimension $n$ and $m$ respectively and denote by $\|\cdot\|$ the norm on either $\mathbb{R}^{n}$ or $\mathbb{R}^{m}$. Then if $A$ is any linear function mapping $\mathbb{R}^{n}$ to $\mathbb{R}^{m}$, then $A \in \mathscr{L}\left(\mathbb{R}^{n}, \mathbb{R}^{m}\right)$ and $\left(\mathscr{L}\left(\mathbb{R}^{n}, \mathbb{R}^{m}\right),\|\cdot\|\right)$ is a complete normed linear space of dimension nm with

$$
\|A \mathbf{x}\| \leq\|A\|\|\mathbf{x}\|
$$

Also if $A \in \mathscr{L}\left(\mathbb{R}^{n}, \mathbb{R}^{m}\right)$ and $B \in \mathscr{L}\left(\mathbb{R}^{m}, \mathbb{R}^{p}\right)$ where $\mathbb{R}^{n}, \mathbb{R}^{m}, \mathbb{R}^{p}$ are normed linear spaces,

$$
\|B A\| \leq\|B\|\|A\|
$$

Proof: It is necessary to show the norm defined on linear transformations really is a norm. Again the triangle inequality is the only property which is not obvious. It remains to show this and verify $\|A\|<\infty$. This last follows from the above Lemma 2.8.5. Thus the norm is at least well defined. It remains to verify its properties.

$$
\begin{gathered}
\|A+B\| \equiv \sup \{\|(A+B)(\mathbf{x})\|:\|\mathbf{x}\| \leq 1\} \\
\leq \sup \{\|A \mathbf{x}\|:\|\mathbf{x}\| \leq 1\}+\sup \{\|B \mathbf{x}\|:\|\mathbf{x}\| \leq 1\} \equiv\|A\|+\|B\|
\end{gathered}
$$

Next consider the assertion about the dimension of $\mathscr{L}\left(\mathbb{R}^{n}, \mathbb{R}^{m}\right)$. This is fairly obvious because a basis for the space of $m \times n$ matrices is clearly the matrices $E_{i j}$ which has a 1 in the $i j^{\text {th }}$ position and a 0 everywhere else. By Theorem 2.7.4 $\left(\mathscr{L}\left(\mathbb{R}^{n}, \mathbb{R}^{m}\right),\|\cdot\|\right)$ is complete. If $x \neq \mathbf{0}$,

$$
\|A \mathbf{x}\| \frac{1}{\|\mathbf{x}\|}=\left\|A \frac{\mathbf{x}}{\|\mathbf{x}\|}\right\| \leq\|A\|
$$

Thus $\|A \mathbf{x}\| \leq\|A\|\|\mathbf{x}\|$.
Consider the last claim.

$$
\|B A\| \equiv \sup _{\|\mathbf{x}\| \leq 1}\|B(A(\mathbf{x}))\| \leq\|B\| \sup _{\|\mathbf{x}\| \leq 1}\|A \mathbf{x}\|=\|B\|\|A\|
$$

What does it mean to say that $A^{k} \rightarrow A$ in terms of this operator norm? In words, this happens if and only if the $i j^{t h}$ entry of $A^{k}$ converges to the $i j^{t h}$ entry of $A$ for each $i j$.

Proposition 2.8.7 $\lim _{k \rightarrow \infty}\left\|A^{k}-A\right\|=0$ if and only if for every $i, j$

$$
\lim _{k \rightarrow \infty}\left|A_{i j}^{k}-A_{i j}\right|=0
$$

Proof: If $A$ is an $m \times n$ matrix, then $A_{i j}=\mathbf{e}_{i}^{T} A \mathbf{e}_{j}$. Suppose now that

$$
\left\|A^{k}-A\right\| \rightarrow 0
$$

Then in terms of the usual Euclidean norm and using the Cauchy Schwarz inequality,

$$
\begin{gather*}
\left|A_{i j}^{k}-A_{i j}\right|=\left|\mathbf{e}_{i}^{T}\left(A^{k}-A\right) \mathbf{e}_{j}\right|= \\
\left|\left(\mathbf{e}_{i},\left(A^{k}-A\right) \mathbf{e}_{j}\right)\right| \leq\left|\mathbf{e}_{i}\right|\left|\left(A^{k}-A\right) \mathbf{e}_{j}\right| \leq\left\|A^{k}-A\right\| \tag{2.5}
\end{gather*}
$$

If the operator norm is taken with respect to $\|\cdot\|$, some other norm than the Euclidean norm, then the right side of the above after $\leq$

$$
\left|\left(A^{k}-A\right) \mathbf{e}_{j}\right| \leq \Delta\left\|\left(A^{k}-A\right) \mathbf{e}_{j}\right\| \leq \Delta\left\|A^{k}-A\right\|\left\|\mathbf{e}_{j}\right\|
$$

Thus convergence in operator norm implies pointwise convergence of the entries of $A^{k}$ to the corresponding entries of $A$.

Next suppose the entries of $A^{k}$ converge to the corresponding entries of $A$. If $\|\mathbf{v}\| \leq 1$, and to save notation, let $B^{k}=A^{k}-A$. Then

$$
\begin{aligned}
& \left|\left(A^{k}-A\right) \mathbf{v}\right|=\left|\left(\begin{array}{llll}
\left(\sum_{j} B_{1 j}^{k} v_{j}\right. & \sum_{j} B_{2 j}^{k} v_{j} & \cdots & \sum_{j} B_{m j} v_{j}
\end{array}\right)^{T}\right| \\
& =\left(\sum_{i}\left(\sum_{j}\left|B_{i j}^{k}\right|\left|v_{j}\right|\right)^{2}\right)^{1 / 2} \leq|\mathbf{v}|\left(\sum_{i}\left(\sum_{j}\left|B_{i j}^{k}\right|\right)^{2}\right)^{1 / 2}
\end{aligned}
$$

By equivalence of norms,

$$
\delta\left\|B^{k} \mathbf{v}\right\| \leq \Delta\|\mathbf{v}\|\left(\sum_{i}\left(\sum_{j}\left|B_{i j}^{k}\right|\right)^{2}\right)^{1 / 2}
$$

and so if $\|\mathbf{v}\| \leq 1$,

$$
\delta\left\|B^{k} \mathbf{v}\right\| \leq \Delta\left(\sum_{i}\left(\sum_{j}\left|B_{i j}^{k}\right|\right)^{2}\right)^{1 / 2}
$$

and so

$$
\left\|B^{k}\right\| \leq \frac{\Delta}{\delta}\left(\sum_{i}\left(\sum_{j}\left|B_{i j}^{k}\right|\right)^{2}\right)^{1 / 2}
$$

the quantity on the right converging to 0 .
More generally, you might want to consider linear transformations $\mathscr{L}(V, W)$ where $V, W$ are finite dimensional normed linear spaces. In this case, the operator norm defined above is the same and it is well defined.

Proposition 2.8.8 Let $L \in \mathscr{L}(V, W)$ where $V, W$ are normed linear spaces with $V$ finite dimensional. Then the operator norm defined above as

$$
\|L\| \equiv \sup _{\|v\| \leq 1}\|L v\|_{W}
$$

is finite and satisfies all the axioms of a norm. Also $\|L v\| \leq\|L\|\|v\|$.
Proof: I won't bother with the subscript on the norms and allow this to be determined by context in what follows. Let $\left(v_{1}, \ldots, v_{n}\right)$ be a basis for $V$. For $v \in V$, let $v=\sum_{j=1}^{n} a_{j} v_{j}$ so the $a_{j}$ are the coordinates of $v$ in $\mathbb{F}$. Let $\mathbf{h}(v) \equiv \mathbf{a}$ where $\mathbf{a}=\left(a_{1}, \ldots, a_{n}\right)$ with $v=\sum_{j=1}^{n} a_{j} v_{j}$. Thus $\mathbf{h}(v)_{i}=a_{i}$. Obviously $\mathbf{h}$ is linear, one to one, and onto. Define $\||v\|\| \equiv|\mathbf{h}(v)|$ where the last is the usual Euclidean norm on $\mathbb{F}^{n}$. Then this is obviously a norm because $\|\|v\| \mid=0$ if and only if $\mathbf{h}(v)=0$ if and only if $v=0$. It is also clear that $\||\alpha v \||\equiv| \mathbf{h}(\alpha v)|=|\alpha||\mathbf{h}(v)|$ whenever $\alpha$ is a scalar. It only remains to show the triangle inequality. However, this is also easy because we know it for $|\cdot|$.

$$
\||v+w\||\equiv| \mathbf{h}(v+w)|\leq|\mathbf{h}(v)|+|\mathbf{h}(w)| \equiv\||v\| \|+\||w \|| .
$$

It follows the two norms are equivalent. Thus $\||v\|\mid \leq \Delta\| v \|$ for some $\Delta$. Hence

$$
\begin{aligned}
\sup _{\|v\| \leq 1}\|L v\|_{W} & \leq \sup _{\| \| v \| \leq \Delta}\|L v\|_{W}=\sup _{|\mathbf{h}(v)| \leq \Delta}\left\|L \sum_{i} \mathbf{h}(v)_{i} v_{i}\right\| \\
& \leq \sup _{|\mathbf{h}(v)| \leq \Delta} \sum_{i}\left|\mathbf{h}\left(v_{i}\right)\right|\left\|L v_{i}\right\|_{W} \\
& \leq \sup _{|\mathbf{h}(v)| \leq \Delta}\left(\sum_{i}\left|\mathbf{h}(v)_{i}\right|^{2}\right)^{1 / 2}\left(\sum_{i}\left\|L v_{i}\right\|^{2}\right)^{1 / 2} \\
& \leq \Delta\left(\sum_{i}\left\|L v_{i}\right\|^{2}\right)^{1 / 2}<\infty
\end{aligned}
$$

Thus the operator norm is well defined. That it satisfies the axioms of a norm on $\mathscr{L}(V, W)$ follows in the same way as above for the case where $V=\mathbb{R}^{n}$ and $W=\mathbb{R}^{m}$.

In particular, consider the triangle inequality.

$$
\begin{aligned}
\|L+\hat{L}\| & \equiv \sup _{\|v\| \leq 1}\|L(v)+\hat{L}(v)\| \leq \sup _{\|v\| \leq 1}(\|L(v)\|+\|\hat{L}(v)\|) \\
& \leq \sup _{\|v\| \leq 1}\|L v\|+\sup _{\|v\| \leq 1}\|\hat{L} v\| \equiv\|L\|+\|\hat{L}\|
\end{aligned}
$$

As to the last claim, if $v \neq 0,\left\|L\left(\frac{v}{\|v\|}\right)\right\| \leq\|L\|$ and so $\|L(v)\| \leq\|L\|\|v\|$.

### 2.9 General Banach Spaces

The above is about linear maps defined on finite dimensional spaces. What if instead, you have Banach spaces which are just complete normed linear spaces? What then? I will quit writing functions and vectors in bold face here. It turns out that in this case, you assume the linear maps are continuous, not just linear.

Theorem 2.9.1 Let $X$ and $Y$ be two normed linear spaces and let $L: X \rightarrow Y$ be linear $(L(a x+b y)=a L(x)+b L(y)$ for $a, b$ scalars and $x, y \in X)$. The following are equivalent
a.) $L$ is continuous at 0
b.) $L$ is continuous
c.) There exists $K>0$ such that $\|L x\|_{Y} \leq K\|x\|_{X}$ for all $x \in X$ ( $L$ is bounded).

Proof: a.) $\Rightarrow \mathrm{b}$.) Let $x_{n} \rightarrow x$. It is necessary to show that $L x_{n} \rightarrow L x$. But $\left(x_{n}-x\right) \rightarrow 0$ and so from continuity at 0 , it follows

$$
L\left(x_{n}-x\right)=L x_{n}-L x \rightarrow 0
$$

so $L x_{n} \rightarrow L x$. This shows a.) implies b.).
b.) $\Rightarrow$ c.) Since $L$ is continuous, $L$ is continuous at 0 . Hence $\|L x\|_{Y}<1$ whenever $\|x\|_{X} \leq$ $\delta$ for some $\delta$. Therefore, suppressing the subscript on the $\|\cdot\|$, it follows that $\left\|L\left(\frac{\delta x}{\|x\|}\right)\right\| \leq$ 1. Hence $\|L x\| \leq \frac{1}{\delta}\|x\|$.
c.) $\Rightarrow$ a.) follows from the inequality given in $c$. .).

Definition 2.9.2 Let $L: X \rightarrow Y$ be linear and continuous where $X$ and $Y$ are normed linear spaces. Denote the set of all such continuous linear maps by $\mathscr{L}(X, Y)$ and define

$$
\begin{equation*}
\|L\|=\sup \{\|L x\|:\|x\| \leq 1\} \tag{2.6}
\end{equation*}
$$

This is called the operator norm.
Note that from Theorem 2.9.1, $\|L\|$ is well defined because of part c .) of that Theorem.
The next lemma follows immediately from the definition of the norm and the assumption that $L$ is linear.

Lemma 2.9.3 With $\|L\|$ defined in 2.6, $\mathscr{L}(X, Y)$ is a normed linear space. Also $\|L x\| \leq$ $\|L\|\|x\|$.

Proof: Let $x \neq 0$ then $x /\|x\|$ has norm equal to 1 and so $\left\|L\left(\frac{x}{\|x\|}\right)\right\| \leq\|L\|$. Therefore, multiplying both sides by $\|x\|,\|L x\| \leq\|L\|\|x\|$. This is obviously a linear space. It remains
to verify the operator norm really is a norm. First of all, if $\|L\|=0$, then $L x=0$ for all $\|x\| \leq 1$. It follows that for any $x \neq 0,0=L\left(\frac{x}{\|x\|}\right)$ and so $L x=0$. Therefore, $L=0$. Also, if $c$ is a scalar,

$$
\|c L\|=\sup _{\|x\| \leq 1}\|c L(x)\|=|c| \sup _{\|x\| \leq 1}\|L x\|=|c|\|L\|
$$

It remains to verify the triangle inequality. Let $L, M \in \mathscr{L}(X, Y)$.

$$
\begin{aligned}
\|L+M\| & \equiv \sup _{\|x\| \leq 1}\|(L+M)(x)\| \leq \sup _{\|x\| \leq 1}(\|L x\|+\|M x\|) \\
& \leq \sup _{\|x\| \leq 1}\|L x\|+\sup _{\|x\| \leq 1}\|M x\|=\|L\|+\|M\| .
\end{aligned}
$$

This shows the operator norm is really a norm as hoped.
As a review, consider the space of linear transformations defined on $\mathbb{R}^{n}$ having values in $\mathbb{R}^{m}$. The fact the transformation is linear automatically imparts continuity to it. To show this, you might recall that every such linear transformation can be realized in terms of matrix multiplication.

Thus, in finite dimensions the algebraic condition that an operator is linear is sufficient to imply the topological condition that the operator is continuous. The situation is not so simple in infinite dimensional spaces such as $C\left(X ; \mathbb{R}^{n}\right)$. This explains the imposition of the topological condition of continuity as a criterion for membership in $\mathscr{L}(X, Y)$ in addition to the algebraic condition of linearity.

Theorem 2.9.4 If $Y$ is a Banach space, then $\mathscr{L}(X, Y)$ is also a Banach space.
Proof: Let $\left\{L_{n}\right\}$ be a Cauchy sequence in $\mathscr{L}(X, Y)$ and let $x \in X$.

$$
\left\|L_{n} x-L_{m} x\right\| \leq\|x\|\left\|L_{n}-L_{m}\right\| .
$$

Thus $\left\{L_{n} x\right\}$ is a Cauchy sequence. Let

$$
L x=\lim _{n \rightarrow \infty} L_{n} x
$$

Then, clearly, $L$ is linear because if $x_{1}, x_{2}$ are in $X$, and $a, b$ are scalars, then

$$
\begin{aligned}
L\left(a x_{1}+b x_{2}\right) & =\lim _{n \rightarrow \infty} L_{n}\left(a x_{1}+b x_{2}\right) \\
& =\lim _{n \rightarrow \infty}\left(a L_{n} x_{1}+b L_{n} x_{2}\right) \\
& =a L x_{1}+b L x_{2} .
\end{aligned}
$$

Also $L$ is continuous. To see this, note that $\left\{\left\|L_{n}\right\|\right\}$ is a Cauchy sequence of real numbers because $\mid\left\|L_{n}\right\|-\left\|L_{m}\right\|\|\leq\| L_{n}-L_{m} \|$. Hence there exists $K>\sup \left\{\left\|L_{n}\right\|: n \in \mathbb{N}\right\}$. Thus, if $x \in X$,

$$
\|L x\|=\lim _{n \rightarrow \infty}\left\|L_{n} x\right\| \leq K\|x\|
$$

Definition 2.9.5 More generally, given Banach spaces $X, Y, \mathscr{L}(X, Y)$ is the space of continuous linear maps from which map $X$ to $Y$.

### 2.10 Connected Sets

Stated informally, connected sets are those which are in one piece. In order to define what is meant by this, I will first consider what it means for a set to not be in one piece. This is called separated. Connected sets are defined in terms of not being separated. This is why theorems about connected sets sometimes seem a little tricky. It is defined in terms of what it is not, rather than what it is. Much of this works fine in more general settings, but I will only consider the context of $\mathbb{R}^{p}$ because this is what is of interest in this book and I don't want to keep changing the context in order to get the most general versions. Now is a definition about what it means to not be connected. This is called separated.

## Definition 2.10.1 $A$ set, $S$ in $\mathbb{R}^{p}$, is separated if there exist sets $A, B$ such that

$$
S=A \cup B, A, B \neq \emptyset, \text { and } \bar{A} \cap B=\bar{B} \cap A=\emptyset
$$

In this case, the sets A and B are said to separate S. A set is connected if it is not separated. Remember $\bar{A}$ denotes the closure of the set $A$.

One of the most important theorems about connected sets is the following.
Theorem 2.10.2 Suppose $\mathscr{U}$ is a set of connected sets and that there exists a point $p$ which is in all of these connected sets. Then $K \equiv \cup \mathscr{U}$ is connected.

Proof: The argument is dependent on Lemma 2.2.15. Suppose

$$
K=A \cup B
$$

where $\bar{A} \cap B=\bar{B} \cap A=\emptyset, A \neq \emptyset, B \neq \emptyset$. Then $p$ is in one of these sets. Say $p \in A$. Then if $U \in \mathscr{U}$, it must be the case that $U \subseteq A$ since if not, you would have

$$
U=(A \cap U) \cup(B \cap U)
$$

and the limit points of $A \cap U$ cannot be in $B$ hence not in $B \cap U$ while the limit points of $B \cap U$ cannot be in $A$ hence not in $A \cap U$. Thus $B=\emptyset$. It follows that $K$ cannot be separated and so it is connected.

The intersection of connected sets is not necessarily connected as is shown by the following picture.


Theorem 2.10.3 Let $\mathbf{f}: X \rightarrow \mathbb{R}^{m}$ be continuous where $X$ is connected. Then $\mathbf{f}(X)$ is also connected.

Proof: To do this you show $\mathbf{f}(X)$ is not separated. Suppose to the contrary that $\mathbf{f}(X)=$ $A \cup B$ where $A$ and $B$ separate $\mathbf{f}(X)$. Then consider the sets $\mathbf{f}^{-1}(A)$ and $\mathbf{f}^{-1}(B)$. If $\mathbf{z} \in$ $\mathbf{f}^{-1}(B)$, then $\mathbf{f}(\mathbf{z}) \in B$ and so $\mathbf{f}(\mathbf{z})$ is not a limit point of $A$. Therefore, there exists an open set, $U$ containing $\mathbf{f}(\mathbf{z})$ such that $U \cap A=\emptyset$. But then, the continuity of $\mathbf{f}$ implies that $\mathbf{f}^{-1}(U)$ is an open set containing $\mathbf{z}$ such that $\mathbf{f}^{-1}(U) \cap \mathbf{f}^{-1}(A)=\emptyset$. Therefore, $\mathbf{f}^{-1}(B)$ contains no limit points of $\mathbf{f}^{-1}(A)$. Similar reasoning implies $\mathbf{f}^{-1}(A)$ contains no limit points of $\mathbf{f}^{-1}(B)$. It follows that $X$ is separated by $\mathbf{f}^{-1}(A)$ and $\mathbf{f}^{-1}(B)$, contradicting the assumption that $X$ was connected.

An arbitrary set can be written as a union of maximal connected sets called connected components. This is the concept of the next definition.

Definition 2.10.4 Let $S$ be a set and let $\mathbf{p} \in S$. Denote by $C_{\mathbf{p}}$ the union of all connected subsets of $S$ which contain $\mathbf{p}$. This is called the connected component determined by p.

Theorem 2.10.5 Let $C_{\mathbf{p}}$ be a connected component of a set $S$. Then $C_{\mathbf{p}}$ is a connected set and if $C_{\mathbf{p}} \cap C_{\mathbf{q}} \neq \emptyset$, then $C_{\mathbf{p}}=C_{\mathbf{q}}$.

Proof: Let $\mathscr{C}$ denote the connected subsets of $S$ which contain $\mathbf{p}$. By Theorem 2.10.2, $\cup \mathscr{C}=C_{\mathbf{p}}$ is connected. If $\mathbf{x} \in C_{\mathbf{p}} \cap C_{\mathbf{q}}$, then from Theorem 2.10.2, $C_{\mathbf{p}} \supseteq C_{\mathbf{p}} \cup C_{\mathbf{q}}$ and so $C_{\mathbf{p}} \supseteq C_{\mathbf{q}}$. The inclusion goes the other way by the same reason.

This shows the connected components of a set are equivalence classes and partition the set.

A set $I$ is an interval in $\mathbb{R}$ if and only if whenever $x, y \in I$ then $[x, y] \subseteq I$. The following theorem is about the connected sets in $\mathbb{R}$.

Theorem 2.10.6 $A$ set $C$ in $\mathbb{R}$ is connected if and only if $C$ is an interval.
Proof: Let $C$ be connected. If $C$ consists of a single point $p$, there is nothing to prove. The interval is just $[p, p]$. Suppose $p<q$ and $p, q \in C$. You need to show $(p, q) \subseteq C$. If

$$
x \in(p, q) \backslash C
$$

let $C \cap(-\infty, x) \equiv A$, and $C \cap(x, \infty) \equiv B$. Then $C=A \cup B$ and the sets $A$ and $B$ separate $C$ contrary to the assumption that $C$ is connected.

Conversely, let $I$ be an interval. Suppose $I$ is separated by $A$ and $B$. Pick $x \in A$ and $y \in B$. Suppose without loss of generality that $x<y$. Now define the set,

$$
S \equiv\{t \in[x, y]:[x, t] \subseteq A\}
$$

and let $l$ be the least upper bound of $S$. Then $l \in \bar{A}$ so $l \notin B$ which implies $l \in A$. But if $l \notin \bar{B}$, then for some $\delta>0,(l, l+\delta) \cap B=\emptyset$. contradicting the definition of $l$ as an upper bound for $S$. Therefore, $l \in \bar{B}$ which implies $l \notin A$ after all, a contradiction. It follows $I$ must be connected.

This yields a generalization of the intermediate value theorem from one variable calculus.

Corollary 2.10.7 Let $E$ be a connected set in $\mathbb{R}^{p}$ and suppose $f: E \rightarrow \mathbb{R}$ and that $y \in\left(f\left(e_{1}\right), f\left(e_{2}\right)\right)$ where $e_{i} \in E$. Then there exists $e \in E$ such that $f(e)=y$.

Proof: From Theorem 2.10.3, $f(E)$ is a connected subset of $\mathbb{R}$. By Theorem 2.10.6 $f(E)$ must be an interval. In particular, it must contain $y$. This proves the corollary.

The following theorem is a very useful description of the open sets in $\mathbb{R}$.
Theorem 2.10.8 Let $U$ be an open set in $\mathbb{R}$. Then there exist countably many disjoint open sets $\left\{\left(a_{i}, b_{i}\right)\right\}_{i=1}^{\infty}$ such that $U=\cup_{i=1}^{\infty}\left(a_{i}, b_{i}\right)$.

Proof: Let $p \in U$ and let $z \in C_{p}$, the connected component determined by $p$. Since $U$ is open, there exists, $\delta>0$ such that $(z-\delta, z+\delta) \subseteq U$. It follows from Theorem 2.10.2 that $(z-\delta, z+\delta) \subseteq C_{p}$. This shows $C_{p}$ is open. By Theorem 2.10.6, this shows $C_{p}$ is an open interval, $(a, b)$ where $a, b \in[-\infty, \infty]$. There are therefore at most countably many of these connected components because each must contain a rational number and the rational numbers are countable. Denote by $\left\{\left(a_{i}, b_{i}\right)\right\}_{i=1}^{\infty}$ the set of these connected components.

Definition 2.10.9 $A$ set $E$ in $\mathbb{R}^{p}$ is arcwise connected if for any two points, $\mathbf{p}, \mathbf{q} \in$ $E$, there exists a closed interval, $[a, b]$ and a continuous function, $\gamma:[a, b] \rightarrow E$ such that $\gamma(a)=p$ and $\gamma(b)=q$. The set of points $\gamma([a, b])$ is called an arc, Jordan arc, or a simple curve.

An example of an arcwise connected space would be any subset of $\mathbb{R}^{p}$ which is the continuous image of an interval. Arcwise connected is not the same as connected. A well known example is the following.

$$
\begin{equation*}
\left\{\left(x, \sin \frac{1}{x}\right): x \in(0,1]\right\} \cup\{(0, y): y \in[-1,1]\} \tag{2.7}
\end{equation*}
$$

You can verify that this set of points in $\mathbb{R}^{2}$ is not arcwise connected but is connected.
Lemma 2.10.10 In $\mathbb{R}^{p}, B(\mathbf{z}, r)$ is arcwise connected.
Proof: This is easy from the convexity of the set. If $\mathbf{x}, \mathbf{y} \in B(\mathbf{z}, r)$, then let $\gamma(t)=$ $\mathbf{x}+t(\mathbf{y}-\mathbf{x})$ for $t \in[0,1]$.

$$
\begin{aligned}
\|\mathbf{x}+t(\mathbf{y}-\mathbf{x})-\mathbf{z}\| & =\|(1-t)(\mathbf{x}-\mathbf{z})+t(\mathbf{y}-\mathbf{z})\| \\
& \leq(1-t)\|\mathbf{x}-\mathbf{z}\|+t\|\mathbf{y}-\mathbf{z}\| \\
& <(1-t) r+t r=r
\end{aligned}
$$

showing $\gamma(t)$ stays in $B(\mathbf{z}, r)$
Proposition 2.10.11 If $X \neq \emptyset$ is arcwise connected, then it is connected.
Proof: Let $p \in X$. Then by assumption, for any $x \in X$, there is an arc joining $p$ and $x$. This arc is connected because it is the continuous image of an interval which is connected. Since $x$ is arbitrary, every $x$ is in a connected subset of $X$ which contains $p$. Hence $C_{p}=X$ and so $X$ is connected.

Theorem 2.10.12 Let $U$ be an open subset of $\mathbb{R}^{p}$. Then $U$ is arcwise connected if and only if $U$ is connected. Also the connected components of an open set are open sets.

Proof: By Proposition 2.10 .11 it is only necessary to verify that if $U$ is connected and open, then $U$ is arcwise connected. Pick $\mathbf{p} \in U$. Say $\mathbf{x} \in U$ satisfies $\mathscr{P}$ if there exists a continuous function, $\gamma:[a, b] \rightarrow U$ such that $\gamma(a)=\mathbf{p}$ and $\gamma(\mathbf{b})=\mathbf{x}$.

$$
A \equiv\{\mathbf{x} \in U \text { such that } \mathbf{x} \text { satisfies } \mathscr{P} .\}
$$

If $\mathbf{x} \in A$, then Lemma 2.10 .10 implies $B(\mathbf{x}, r) \subseteq U$ is arcwise connected for small enough $r$. Thus letting $\mathbf{y} \in B(\mathbf{x}, r)$, there exist intervals, $[a, b]$ and $[c, d]$ and continuous functions having values in $U, \gamma, \eta$ such that $\gamma(a)=\mathbf{p}, \gamma(b)=\mathbf{x}, \eta(c)=\mathbf{x}$, and $\eta(d)=\mathbf{y}$. Then let $\gamma_{1}:[a, b+d-c] \rightarrow U$ be defined as

$$
\gamma_{1}(t) \equiv\left\{\begin{array}{l}
\gamma(t) \text { if } t \in[a, b] \\
\eta(t+c-b) \text { if } t \in[b, b+d-c]
\end{array}\right.
$$

Then it is clear that $\gamma_{1}$ is a continuous function mapping $\mathbf{p}$ to $\mathbf{y}$ and showing that $B(\mathbf{x}, r) \subseteq$ $A$. Therefore, $A$ is open. $A \neq \emptyset$ because since $U$ is open there is an open set, $B(\mathbf{p}, \delta)$ containing $\mathbf{p}$ which is contained in $U$ and is arcwise connected.

Now consider $B \equiv U \backslash A$. I claim this is also open. If $B$ is not open, there exists a point $\mathbf{z} \in B$ such that every open set containing $\mathbf{z}$ is not contained in $B$. Therefore, letting $B(\mathbf{z}, \boldsymbol{\delta})$ be such that $\mathbf{z} \in B(\mathbf{z}, \boldsymbol{\delta}) \subseteq U$, there exist points of $A$ contained in $B(\mathbf{z}, \boldsymbol{\delta})$. But then, a repeat of the above argument shows $z \in A$ also. Hence $B$ is open and so if $B \neq \emptyset$, then $U=B \cup A$ and so $U$ is separated by the two sets $B$ and $A$ contradicting the assumption that $U$ is connected. Note that, since $B$ is open, it contains no limit points of $A$ and since $A$ is open, it contains no limit points of $B$.

It remains to verify the connected components are open. Let $\mathbf{z} \in C_{\mathbf{p}}$ where $C_{\mathbf{p}}$ is the connected component determined by $\mathbf{p}$. Then picking $B(\mathbf{z}, \delta) \subseteq U, C_{\mathbf{p}} \cup B(\mathbf{z}, \delta)$ is connected and contained in $U$ and so it must also be contained in $C_{\mathbf{p}}$. Thus $\mathbf{z}$ is an interior point of $C_{\mathbf{p}}$.

As an application, consider the following corollary.
Corollary 2.10.13 Let $f: \Omega \rightarrow \mathbb{Z}$ be continuous where $\Omega$ is a connected nonempty open set in $\mathbb{R}^{p}$. Then $f$ must be a constant.

Proof: Suppose not. Then it achieves two different values, $k$ and $l \neq k$. Then $\Omega=$ $f^{-1}(l) \cup f^{-1}(\{m \in \mathbb{Z}: m \neq l\})$ and these are disjoint nonempty open sets which separate $\Omega$. To see they are open, note

$$
f^{-1}(\{m \in \mathbb{Z}: m \neq l\})=f^{-1}\left(\cup_{m \neq l}\left(m-\frac{1}{6}, m+\frac{1}{6}\right)\right)
$$

which is the inverse image of an open set while $f^{-1}(l)=f^{-1}\left(\left(l-\frac{1}{6}, l+\frac{1}{6}\right)\right)$ also an open set.

### 2.11 Completion of Metric Spaces

Let $(X, d)$ be a metric space $X \neq \emptyset$. Perhaps this is not a complete metric space. In other words, it may be that Cauchy Sequences do not converge. Of course if $x \in X$ and if $x_{n}=x$ for all $n$ then $\left\{x_{n}\right\}$ is a Cauchy sequence and it converges to $x$.

Lemma 2.11.1 Denote by $\mathbf{x}$ a Cauchy sequence $\mathbf{x}$ being short for $\left\{x_{n}\right\}_{n=1}^{\infty}$. Then if $\mathbf{x}, \mathbf{y}$ are two Cauchy sequences, $\lim _{n \rightarrow \infty} d\left(x_{n}, y_{n}\right)$ exists.

Proof: Let $\varepsilon>0$ be given and let $N$ be so large that whenever $n, m \geq N$, it follows that $d\left(x_{n}, x_{m}\right), d\left(y_{n}, y_{m}\right)<\varepsilon / 2$. Then for such $n, m$

$$
\begin{aligned}
\left|d\left(x_{n}, y_{n}\right)-d\left(x_{m}, y_{m}\right)\right| & \leq\left|d\left(x_{n}, y_{n}\right)-d\left(x_{n}, y_{m}\right)\right|+\left|d\left(x_{n}, y_{m}\right)-d\left(x_{m}, y_{m}\right)\right| \\
& \leq d\left(y_{n}, y_{m}\right)+d\left(x_{n}, x_{m}\right)<\varepsilon
\end{aligned}
$$

by Theorem 2.4.8. Therefore, $\left\{d\left(x_{n}, y_{n}\right)\right\}_{n}$ is a Cauchy sequence in $\mathbb{R}$ and so it converges.
Definition 2.11.2 Let $\mathbf{x} \sim \mathbf{y}$ when $\lim _{n \rightarrow \infty} d\left(x_{n}, y_{n}\right)=0$.
Lemma 2.11.3 ~ is an equivalence relation.
Proof: Clearly $\mathbf{x} \sim \mathbf{x}$ and if $\mathbf{x} \sim \mathbf{y}$ then $\mathbf{y} \sim \mathbf{x}$. Suppose then that $\mathbf{x} \sim \mathbf{y}$ and $\mathbf{y} \sim \mathbf{z}$. Is $\mathbf{x} \sim \mathbf{z}$ ?

$$
d\left(x_{n}, z_{n}\right) \leq d\left(x_{n}, y_{n}\right)+d\left(y_{n}, z_{n}\right)
$$

and both of those terms on the right converge to 0 .
Definition 2.11.4 Denote by $[\mathbf{x}]$ the equivalence class determined by the Cauchy sequence $\mathbf{x}$. Let $d([\mathbf{x}],[\mathbf{y}]) \equiv \lim _{n \rightarrow \infty} d\left(x_{n}, y_{n}\right)$.

Theorem 2.11.5 Denote by $\hat{X}$ the set of equivalence classes. Then d defined above is a metric, $\hat{X}$ with this is a complete metric space, and $X$ can be considered a dense subset of $\hat{X}$.

Proof: That $d$ just defined is a metric is obvious from the fact that the original metric $d$ satisfies the triangle inequality. It is also clear that $d([\mathbf{x}],[\mathbf{y}]) \geq 0$ and that if $[\mathbf{x}]=[\mathbf{y}]$ if and only if $d([\mathbf{x}],[\mathbf{y}])=0$.

It remains to show that $(\hat{X}, d)$ is complete. Let $\left\{[\mathbf{x}]_{n}\right\}_{n}$ be a Cauchy sequence. From Theorem 2.3.3 it suffices to show the convergence of a subsequence. There is a subsequence, denoted as $\left\{\left[\mathbf{x}^{n}\right]\right\}$ where $\mathbf{x}^{n}$ is a representative of $[\mathbf{x}]_{n}$ such that $d\left(\left[\mathbf{x}^{n}\right],\left[\mathbf{x}^{n+1}\right]\right)<$ $4^{-n}$. Thus there is an increasing sequence $\left\{k_{n}\right\}$ such that $d\left(x_{k}^{n}, x_{l}^{n+1}\right)<2^{-n}$ if $k, l \geq k_{n}$ where $k_{n}$ is increasing in $n$. Let $\mathbf{y}=\left\{x_{k_{n}}^{n}\right\}_{n=1}^{\infty}$. For $m \geq k_{n}$ and the triangle inequality,

$$
\begin{aligned}
d\left(x_{m}^{n}, y_{m}\right) & =d\left(x_{m}^{n}, x_{k_{m}}^{m}\right) \leq d\left(x_{m}^{n}, x_{k_{n}}^{n}\right)+d\left(x_{k_{n}}^{n}, x_{k_{m}}^{m}\right) \leq 2^{-n}+\sum_{j=n}^{m-1} d\left(x_{k_{j}}^{j}, x_{k_{m}}^{j+1}\right) \\
& <2^{-n}+\sum_{j=n}^{m-1} 2^{-j}<2^{-n}+2^{-(n-1)}<2^{-(n-2)}
\end{aligned}
$$

Then $\mathbf{y}$ is a Cauchy sequence since it is a subsequence of one and also $d\left(\left[\mathbf{x}^{n}\right],[\mathbf{y}]\right) \rightarrow 0$.
To show that $X$ is dense in $\hat{X}$, let $[\mathbf{x}]$ be given. Then for $m$ large enough, $d\left(x_{k}, x_{m}\right)<\varepsilon$ whenever $k \geq m$. It suffices to let $\mathbf{y}$ be the constant Cauchy sequence always equal to $x_{m}$.

### 2.12 Exercises

1. Explain carefully why in $\mathbb{R}^{n}, B_{\infty}(\mathbf{p}, r)=\prod_{i=1}^{n}\left(p_{i}-r, p_{i}+r\right)$
2. Say $\|\cdot\|,\|\cdot\|_{1}$ are two equivalent norms. Explain carefully why if $\mathbf{x}$ is a limit point of a set $A$ with respect to $\|\cdot\|$ then it is also a limit point with respect to $\|\cdot\|_{1}$. Also show that if $\mathbf{x}_{n} \rightarrow \mathbf{x}$ with respect to $\|\cdot\|$, then the same is true with respect to $\|\cdot\|_{1}$.
3. If you have $X$ a normed linear space and $Y$ is a Banach space,(complete normed linear space) show that $\mathscr{L}(X, Y)$ is a Banach space with respect to the operator norm.
4. If $X, Y$ are normed linear spaces, verify that $A: X \rightarrow Y$ is in $\mathscr{L}(X, Y)$ if and only if $A$ is continuous at each $x \in X$ if and only if $A$ is continuous at 0 .
5. Generalize the root test, Theorem 1.12.1 to the situation where the $\mathbf{a}_{k}$ are in a complete normed linear space.
6. Suppose $X$ is a Banach space and $\left\{B_{n}\right\}$ is a sequence of closed sets in $X$ such that $B_{n} \supseteq B_{n+1}$ for all $n$ and no $B_{n}$ is empty. Also suppose that the diameter of $B_{n}$ converges to 0 . Recall the diameter is given by $\operatorname{diam}(B) \equiv \sup \{\|x-y\|: x, y \in B\}$. Thus these sets $B_{n}$ are nested and $\operatorname{diam}\left(B_{n}\right) \rightarrow 0$. Verify that there is a unique point in the intersection of all these sets.
7. If $X$ is a Banach space, and $Y$ is the span of finitely many vectors in $X$, show that $Y$ is closed.
8. If $X$ is an infinite dimensional Banach space, show that there exists a sequence $\left\{x_{n}\right\}_{n=1}^{\infty}$ such that $\left\|x_{n}\right\| \leq 1$ but for any $m \neq n,\left\|x_{n}-x_{m}\right\| \geq 1 / 4$. Thus in infinite dimensional Banach spaces, closed and bounded sets are no longer compact as they are in $\mathbb{F}^{n}$.
9. In the proof of the fundamental theorem of algebra, explain why there exists $z_{0}$ such that for $p(z)$ a polynomial with complex coefficients, $\left|p\left(z_{0}\right)\right|=\min _{z \in \mathbb{C}}|p(z)|>0$
10. Explain why a compact set in $\mathbb{R}$ has a largest point and a smallest point. Now if $f: K \rightarrow \mathbb{R}$ for $K$ compact and $f$ continuous, give another proof of the extreme value theorem from using that $f(K)$ is compact.
11. Generalize Theorem 2.5 .34 to the case where $f_{n}: S \rightarrow T$ where $S, T$ are metric spaces. Give an appropriate definition for uniform convergence which will imply uniform convergence transfers continuity from $f_{n}$ to the target function $f$.
12. A function $f: X \rightarrow \mathbb{R}$ for $X$ a normed linear space is lower semicontinuous if, whenever $x_{n} \rightarrow x, f(x) \leq \liminf _{n \rightarrow \infty} f\left(x_{n}\right)$ It is upper semicontinuous if, whenever $x_{n} \rightarrow x, f(x) \geq \limsup _{n \rightarrow \infty} f\left(x_{n}\right)$ Explain why, if $K$ is compact and $f$ is upper semicontinuous then $f$ achieves its maximum and if $K$ is compact and $f$ is lower semicontinuous, then $f$ achieves its minimum on $K$.
13. Suppose $f_{n}: S \rightarrow Y$ where $S$ is a nonempty subset of $X$ a normed linear space and suppose that $Y$ is a Banach space (complete normed linear space). Generalize the theorem in the chapter to this case: Let $f_{n}: S \rightarrow Y$ be bounded functions: $\sup _{x \in S}\left|f_{n}(x)\right|=$ $C_{n}<\infty$. Then there exists bounded $f: S \rightarrow Y$ such that $\lim _{n \rightarrow \infty}\left\|f-f_{n}\right\|=0$ if and only if $\left\{f_{n}\right\}$ is uniformly Cauchy. Also show that $B C(S ; Y)$ is a Banach space.
14. Show that no interval $[a, b] \subseteq \mathbb{R}$ can be countable. Hint: First show $[0,1]$ is not countable. You might do this by noting that every point in this interval can be written
as $\sum_{k=1}^{\infty} 2^{-k} a_{k}$ where $a_{k}$ is either 0 or 1 . Let $\mathscr{F}$ be $\cup_{n} \mathscr{P}(\{1,2, \cdots, n\})$. Explain why $\mathscr{F}$ is countable. Then let $\mathscr{S} \equiv \mathscr{P}(\mathbb{N}) \backslash \mathscr{F}$. Explain why $\mathscr{S}$ is uncountable. Let $C$ be all points of the form $\sum_{k=1}^{m} 2^{-k} a_{k}$ where $a_{k}$ is 0 or 1 . Explain why $C$ is countable. Let $J=[0,1] \backslash C$. Now let $\theta: \mathscr{S} \rightarrow J$ be given by $\theta(S)=\sum_{k \in S} 2^{-k}$. Explain why $\theta$ is one to one onto $J$. If $[0,1]$ is countable, show there are onto mappings as indicated $\mathbb{N} \rightarrow[0,1] \rightarrow J \rightarrow \mathscr{S}$ showing that $\mathscr{S}$ is countable after all.
15. Using the above problem as needed, let $B$ be a countable set of real numbers. Say $B=\left\{b_{n}\right\}_{n=1}^{\infty}$. Let

$$
f_{n}(t) \equiv\left\{\begin{array}{l}
1 \text { if } t \in\left\{b_{1}, \cdots, b_{n}\right\} \\
0 \text { otherwise }
\end{array}\right.
$$

Let $g(t) \equiv \sum_{k=1}^{\infty} 2^{-k} f_{k}(t)$. Explain why $g$ is continuous on $\mathbb{R} \backslash B$ and discontinuous on $B$. Note that $B$ could be the rational numbers.
16. Consider $\mathbb{R} \backslash\{0\}$. Show this is not connected.
17. Show $S \equiv\left\{\left(x, \sin \left(\frac{1}{x}\right)\right)\right.$ if $\left.x>0\right\} \cup\{(0, y):|y| \leq 1\}$ is connected but not arcwise connected.
18. Let $A$ be an $m \times n$ matrix. Then $A^{*}$, called the adjoint matrix, is obtained from $A$ by taking the transpose and then the conjugate. For example,

$$
\left(\begin{array}{cc}
i & 1 \\
1+i & 2 \\
3 & 1-i
\end{array}\right)^{*}=\left(\begin{array}{ccc}
-i & 1-i & 3 \\
1 & 2 & 1+i
\end{array}\right)
$$

Formally, $\left(A^{*}\right)_{i j}=\overline{A_{j i}}$. Show $(A \mathbf{x}, \mathbf{y})=\left(\mathbf{x}, A^{*} \mathbf{y}\right)$ and $(\mathbf{x}, B \mathbf{y})=\left(B^{*} \mathbf{x}, \mathbf{y}\right)$. The inner product is described in the chapter. Recall $(\mathbf{x}, \mathbf{y}) \equiv \sum_{j} x_{j} \overline{y_{j}}$.
19. Let $X$ be a subspace of $\mathbb{F}^{m}$ having dimension $d$ and let $\mathbf{y} \in \mathbb{F}^{m}$. Show that $\mathbf{x} \in$ $X$ is closest to $\mathbf{y}$ in the Euclidean norm $|\cdot|$ out of all vectors in $X$ if and only if $(\mathbf{y}-\mathbf{x}, \mathbf{u})=0$ for all $\mathbf{u} \in X$. Next show there exists such a closest point and it equals $\sum_{j=1}^{d}\left(\mathbf{y}, \mathbf{u}_{j}\right) \mathbf{u}_{j}$ for $\left\{\mathbf{u}_{j}\right\}_{j=1}^{d}$ an orthonormal basis for $X$.
20. Let $A: \mathbb{F}^{n} \rightarrow \mathbb{F}^{m}$ be an $m \times n$ matrix. (Note how it is being considered as a linear transformation.) Show $\operatorname{Im}(A) \equiv\left\{A \mathbf{x}: \mathbf{x} \in \mathbb{F}^{n}\right\}$ is a subspace of $\mathbb{F}^{m}$. If $\mathbf{y} \in \mathbb{F}^{m}$ is given, show that there exists $\mathbf{x}$ such that $\mathbf{y}-A \mathbf{x}$ is as small as possible ( $A \mathbf{x}$ is the point of $\operatorname{Im}(A)$ closest to $\mathbf{y})$ and it is a solution to the least squares equation $A^{*} A \mathbf{x}=A^{*} \mathbf{y}$. Hint: You might want to use Problem 18.
21. Show that the usual norm in $\mathbb{F}^{n}$ given by $|\mathbf{x}|=(\mathbf{x}, \mathbf{x})^{1 / 2}$ satisfies the following identities, the first of them being the parallelogram identity and the second being the polarization identity.

$$
\begin{aligned}
|\mathbf{x}+\mathbf{y}|^{2}+|\mathbf{x}-\mathbf{y}|^{2} & =2|\mathbf{x}|^{2}+2|\mathbf{y}|^{2} \\
\operatorname{Re}(\mathbf{x}, \mathbf{y}) & =\frac{1}{4}\left(|\mathbf{x}+\mathbf{y}|^{2}-|\mathbf{x}-\mathbf{y}|^{2}\right)
\end{aligned}
$$

Show that these identities hold in any inner product space, not just $\mathbb{F}^{n}$. By definition, an inner product space is just a vector space which has an inner product.
22. Let $K$ be a nonempty closed and convex set in an inner product space $(X,|\cdot|)$ which is complete. For example, $\mathbb{F}^{n}$ or any other finite dimensional inner product space. Let $y \notin K$ and let $\lambda=\inf \{|y-x|: x \in K\}$. Let $\left\{x_{n}\right\}$ be a minimizing sequence. That is $\lambda=\lim _{n \rightarrow \infty}\left|y-x_{n}\right|$ Explain why such a minimizing sequence exists. Next explain the following using the parallelogram identity in the above problem as follows.

$$
\begin{gathered}
\left|y-\frac{x_{n}+x_{m}}{2}\right|^{2}=\left|\frac{y}{2}-\frac{x_{n}}{2}+\frac{y}{2}-\frac{x_{m}}{2}\right|^{2} \\
=-\left|\frac{y}{2}-\frac{x_{n}}{2}-\left(\frac{y}{2}-\frac{x_{m}}{2}\right)\right|^{2}+\frac{1}{2}\left|y-x_{n}\right|^{2}+\frac{1}{2}\left|y-x_{m}\right|^{2}
\end{gathered}
$$

Hence

$$
\begin{aligned}
\left|\frac{x_{m}-x_{n}}{2}\right|^{2} & =-\left|y-\frac{x_{n}+x_{m}}{2}\right|^{2}+\frac{1}{2}\left|y-x_{n}\right|^{2}+\frac{1}{2}\left|y-x_{m}\right|^{2} \\
& \leq-\lambda^{2}+\frac{1}{2}\left|y-x_{n}\right|^{2}+\frac{1}{2}\left|y-x_{m}\right|^{2}
\end{aligned}
$$

Next explain why the right hand side converges to 0 as $m, n \rightarrow \infty$. Thus $\left\{x_{n}\right\}$ is a Cauchy sequence and converges to some $x \in X$. Explain why $x \in K$ and $|x-y|=\lambda$. Thus there exists a closest point in $K$ to $y$. Next show that there is only one closest point. Hint: To do this, suppose there are two $x_{1}, x_{2}$ and consider $\frac{x_{1}+x_{2}}{2}$ using the parallelogram law to show that this average works better than either of the two points which is a contradiction unless they are really the same point. This theorem is of enormous significance.
23. Let $K$ be a closed convex nonempty set in a complete inner product space $(H,|\cdot|)$ (Hilbert space) and let $y \in H$. Denote the closest point to $y$ by $P x$. Show that $P x$ is characterized as being the solution to the following variational inequality

$$
\operatorname{Re}(z-P x, y-P x) \leq 0
$$

for all $z \in K$. Hint: Let $x \in K$. Then, due to convexity, a generic thing in $K$ is of the form $x+t(z-x), t \in[0,1]$ for every $z \in K$. Then

$$
|x+t(z-x)-y|^{2}=|x-y|^{2}+t^{2}|z-x|^{2}-t 2 \operatorname{Re}(z-x, y-x)
$$

If $x=P y$, then the minimum value of this on the left occurs when $t=0$. Function defined on $[0,1]$ has its minimum at $t=0$. What does it say about the derivative of this function at $t=0$ ? Next consider the case that for some $x$ the inequality $\operatorname{Re}(z-x, y-x) \leq 0$. Explain why this shows $x=P y$.
24. Using Problem 23 and Problem 22 show the projection map, $P$ onto a closed convex subset of a complete inner product space is Lipschitz continuous with Lipschitz constant 1. That is $|P x-P y| \leq|x-y|$.
25. Suppose $S$ is an uncountable set and suppose $f(s)$ is a positive number for each $s \in S$. Also let $\hat{S}$ denote a finite subset of $S$. Show that

$$
\sup \left\{\sum_{s \in \hat{S}} f(s): \hat{S} \subseteq S\right\}=\infty
$$

26. Let $Y$ be a normed vector space and suppose $h:[a, b] \rightarrow Y$ is differentiable on $(a, b)$ meaning

$$
\lim _{h \rightarrow 0} \frac{f(t+h)-f(t)}{h}=f^{\prime}(t)
$$

continuous on $[a, b]$ and $h^{\prime}(t)=0$. Then $\|h(b)-h(a)\|=0$. Show this. Hint: Let

$$
S \equiv\{t \in[a, b]: \text { for all } s \in[a, t],\|h(s)-h(a)\| \leq \varepsilon(s-a)\}
$$

Then let $t \equiv \sup S$. By continuity, $\|h(t)-h(a)\| \leq \varepsilon(t-a)$. Suppose $t<b$. If strict inequality holds, then this will persist for $s$ near $t$ and violate the definition of $t$. Therefore, $\|h(t)-h(a)\|=\varepsilon(t-a)$. Then, still assuming $t<b$, there exists $h_{k} \downarrow 0$ and

$$
\varepsilon\left(t-a+h_{k}\right)<\left\|h\left(t+h_{k}\right)-h(a)\right\| \leq\left\|h\left(t+h_{k}\right)-h(t)\right\|+\|h(t)-h(a)\|
$$

Now we have $\varepsilon<\frac{1}{h_{k}}\left\|h\left(t+h_{k}\right)-h(t)\right\|$ and passing to a limit, $\varepsilon<\left\|h^{\prime}(t)\right\|$ a contradiction.
27. Let $Y$ be a normed vector space and suppose $h:[a, b] \rightarrow Y$ is differentiable on $(a, b)$ meaning

$$
\lim _{h \rightarrow 0} \frac{f(t+h)-f(t)}{h}=f^{\prime}(t)
$$

continuous on $[a, b]$ and $\left\|h^{\prime}(t)\right\| \leq M$ for all $t \in(a, b)$. Then

$$
\|h(b)-h(a)\| \leq M|b-a| .
$$

This is called the mean value inequality. Show this. Hint: Let

$$
S \equiv\{t \in[a, b]: \text { for all } s \in[a, t],\|h(s)-h(a)\| \leq(M+\varepsilon)(s-a)\}
$$

## Chapter 3

## Stone Weierstrass Approximation Theorem

### 3.1 The Bernstein Polynomials

These polynomials give an explicit description of a sequence of polynomials which converge uniformly to a continuous function. Recall that if you have a bounded function defined on some set $S$ with values in $Y$.

$$
\|f\|_{\infty} \equiv \sup \{\|f(x)\|: x \in S\}
$$

This is one way to measure distance between functions.
Lemma 3.1.1 The following estimate holds for $x \in[0,1]$ and $m \geq 2$.

$$
\sum_{k=0}^{m}\binom{m}{k}(k-m x)^{2} x^{k}(1-x)^{m-k} \leq \frac{1}{4} m
$$

Proof: First of all, from the binomial theorem

$$
\begin{aligned}
& \sum_{k=0}^{m}\binom{m}{k}\left(e^{t(k-m x)}\right) x^{k}(1-x)^{m-k}=e^{-t m x} \sum_{k=0}^{m}\binom{m}{k}\left(e^{t k}\right) x^{k}(1-x)^{m-k} \\
& \quad=e^{-t m x}\left(1-x+x e^{t}\right)^{m} \equiv e^{-t m x} g(t)^{m}, g(0)=1, g^{\prime}(0)=g^{\prime \prime}(0)=x
\end{aligned}
$$

Take a derivative with respect to $t$ twice.

$$
\begin{gathered}
\sum_{k=0}^{m}\binom{m}{k}(k-m x)^{2} e^{t(k-m x)} x^{k}(1-x)^{m-k} \\
=\quad(m x)^{2} e^{-t m x} g(t)^{m}+2(-m x) e^{-t m x} m g(t)^{m-1} g^{\prime}(t) \\
+e^{-t m x}\left[m(m-1) g(t)^{m-2} g^{\prime}(t)^{2}+m g(t)^{m-1} g^{\prime \prime}(t)\right]
\end{gathered}
$$

Now let $t=0$ and note that the right side is $m\left(x-x^{2}\right) \leq m / 4$ for $x \in[0,1]$. Thus

$$
\sum_{k=0}^{m}\binom{m}{k}(k-m x)^{2} x^{k}(1-x)^{m-k}=m x-m x^{2} \leq m / 4
$$

With this preparation, here is the first version of the Weierstrass approximation theorem. I will allow $f$ to have values in a complete normed linear space. Thus, $f \in C([0,1] ; X)$ where $X$ is a Banach space, Definition 2.5.31. Thus this is a function which is continuous with values in $X$ as discussed earlier with metric spaces.

Theorem 3.1.2 Let $f \in C([0,1] ; X)$ and let the norm be denoted by $\|\cdot\|$.

$$
p_{m}(x) \equiv \sum_{k=0}^{m}\binom{m}{k} x^{k}(1-x)^{m-k} f\left(\frac{k}{m}\right) .
$$

Then these polynomials converge uniformly to $f$ on $[0,1]$.

Proof: Let $\|f\|_{\infty}$ denote the largest value of $\|f(x)\|$. By uniform continuity of $f$, there exists a $\delta>0$ such that if $\left|x-x^{\prime}\right|<\delta$, then $\left\|f(x)-f\left(x^{\prime}\right)\right\|<\varepsilon / 2$. By the binomial theorem,

$$
\begin{gathered}
\left\|p_{m}(x)-f(x)\right\| \leq \sum_{k=0}^{m}\binom{m}{k} x^{k}(1-x)^{m-k}\left\|f\left(\frac{k}{m}\right)-f(x)\right\| \\
\leq \sum_{\left|\frac{k}{m}-x\right|<\delta}\binom{m}{k} x^{k}(1-x)^{m-k}\left\|f\left(\frac{k}{m}\right)-f(x)\right\|+ \\
2\|f\|_{\infty} \sum_{\left|\frac{k}{m}-x\right| \geq \delta}\binom{m}{k} x^{k}(1-x)^{m-k}
\end{gathered}
$$

Therefore,

$$
\begin{aligned}
& \leq \sum_{k=0}^{m}\binom{m}{k} x^{k}(1-x)^{m-k} \frac{\varepsilon}{2}+2\|f\|_{\infty} \sum_{(k-m x)^{2} \geq m^{2} \delta^{2}}\binom{m}{k} x^{k}(1-x)^{m-k} \\
& \quad \leq \frac{\varepsilon}{2}+2\|f\|_{\infty} \frac{1}{m^{2} \delta^{2}} \sum_{k=0}^{m}\binom{m}{k}(k-m x)^{2} x^{k}(1-x)^{m-k} \\
& \quad \leq \frac{\varepsilon}{2}+2\|f\|_{\infty} \frac{1}{4} m \frac{1}{\delta^{2} m^{2}}<\varepsilon
\end{aligned}
$$

provided $m$ is large enough. Thus $\left\|p_{m}-f\right\|_{\infty}<\varepsilon$ when $m$ is large enough.
Note that we do not need to have $X$ be complete in order for this to hold. It would have sufficed to have simply let $X$ be a normed linear space.

Corollary 3.1.3 If $f \in C([a, b] ; X)$ where $X$ is a normed linear space, then there exists a sequence of polynomials which converge uniformly to $f$ on $[a, b]$. The coefficients of these polynomials are in $X$.

Proof: Let $l:[0,1] \rightarrow[a, b]$ be one to one, linear and onto. Then $f \circ l$ is continuous on $[0,1]$ and so if $\varepsilon>0$ is given, there exists a polynomial $p$ such that for all $x \in[0,1],\|p(x)-f \circ l(x)\|<\varepsilon$. Therefore, letting $y=l(x)$, it follows that for all $y \in[a, b]$,

$$
\left\|p\left(l^{-1}(y)\right)-f(y)\right\|<\varepsilon .
$$

The exact form of the polynomial is as follows.

$$
\begin{gather*}
p(x)=\sum_{k=0}^{m}\binom{m}{k} x^{k}(1-x)^{m-k} f\left(l\left(\frac{k}{m}\right)\right) \\
p\left(l^{-1}(y)\right)=\sum_{k=0}^{m}\binom{m}{k}\left(l^{-1}(y)\right)^{k}\left(1-l^{-1}(y)\right)^{m-k} f\left(l\left(\frac{k}{m}\right)\right) \tag{3.1}
\end{gather*}
$$

### 3.2 The Case of Compact Sets

There is a profound generalization of the Weierstrass approximation theorem due to Stone. It has to be one of the most elegant and insightful theorems in mathematics.

Definition 3.2.1 $\mathscr{A}$ is an algebra of real valued functions if $\mathscr{A}$ is a real vector space and if whenever $f, g \in \mathscr{A}$ then $f g \in \mathscr{A}$.

There is a generalization of the Weierstrass theorem due to Stone in which an interval will be replaced by a compact or locally compact set and polynomials will be replaced with elements of an algebra satisfying certain axioms.

Corollary 3.2.2 On the interval $[-M, M]$, there exist polynomials $p_{n}$ such that

$$
p_{n}(0)=0
$$

and

$$
\lim _{n \rightarrow \infty}\left\|p_{n}-\mid \cdot\right\| \|_{\infty}=0
$$

recall that $\|f\|_{\infty} \equiv \sup _{t \in[-M, M]}|f(t)|$.
Proof: By Corollary 3.1.3 there exists a sequence of polynomials, $\left\{\tilde{p}_{n}\right\}$ such that $\tilde{p}_{n} \rightarrow$ $|\cdot|$ uniformly. Then let $p_{n}(t) \equiv \tilde{p}_{n}(t)-\tilde{p}_{n}(0)$.

In what follows, $x$ will be a point in $\mathbb{R}^{p}$. However, this could be generalized. Note that $\mathbb{C}^{p}$ can be considered as $\mathbb{R}^{2 p}$.

Definition 3.2.3 An algebra of functions $\mathscr{A}$ defined on $A$, annihilates no point of $A$ if for all $x \in A$, there exists $g \in \mathscr{A}$ such that $g(x) \neq 0$. The algebra separates points if whenever $x_{1} \neq x_{2}$, then there exists $g \in \mathscr{A}$ such that $g\left(x_{1}\right) \neq g\left(x_{2}\right)$.

The following generalization is known as the Stone Weierstrass approximation theorem.
Theorem 3.2.4 Let $A$ be a compact set in $\mathbb{R}^{p}$ and let $\mathscr{A} \subseteq C(A ; \mathbb{R})$ be an algebra of functions which separates points and annihilates no point. Then $\mathscr{A}$ is dense in $C(A ; \mathbb{R})$.

Proof: First here is a lemma.
Lemma 3.2.5 Let $c_{1}$ and $c_{2}$ be two real numbers and let $x_{1} \neq x_{2}$ be two points of $A$. Then there exists a function $f_{x_{1} x_{2}}$ such that

$$
f_{x_{1} x_{2}}\left(x_{1}\right)=c_{1}, f_{x_{1} x_{2}}\left(x_{2}\right)=c_{2} .
$$

Proof of the lemma: Let $g \in \mathscr{A}$ satisfy $g\left(x_{1}\right) \neq g\left(x_{2}\right)$. Such a $g$ exists because the algebra separates points. Since the algebra annihilates no point, there exist functions $h$ and $k$ such that $h\left(x_{1}\right) \neq 0, k\left(x_{2}\right) \neq 0$. Then let

$$
u \equiv g h-g\left(x_{2}\right) h, v \equiv g k-g\left(x_{1}\right) k
$$

It follows that $u\left(x_{1}\right) \neq 0$ and $u\left(x_{2}\right)=0$ while $v\left(x_{2}\right) \neq 0$ and $v\left(x_{1}\right)=0$. Let

$$
f_{x_{1} x_{2}} \equiv \frac{c_{1} u}{u\left(x_{1}\right)}+\frac{c_{2} v}{v\left(x_{2}\right)}
$$

This proves the lemma. Now continue the proof of Theorem 3.2.4.
First note that $\overline{\mathscr{A}}$ satisfies the same axioms as $\mathscr{A}$ but in addition to these axioms, $\overline{\mathscr{A}}$ is closed. The closure of $\mathscr{A}$ is taken with respect to the usual norm on $C(A),\|f\|_{\infty} \equiv$ $\max \{|f(x)|: x \in A\}$. Thus $\overline{\mathscr{A}}$ consists, by definition, of all functions in $\mathscr{A}$ along with all uniform limits of these functions. Suppose $f \in \overline{\mathscr{A}}$ and suppose $M$ is large enough that $\|f\|_{\infty}<M$. Using Corollary 3.2.2, let $p_{n}$ be a sequence of polynomials such that $\left\|p_{n}-\mid \cdot\right\| \|_{\infty} \rightarrow 0, p_{n}(0)=0$. It follows that $p_{n} \circ f \in \overline{\mathscr{A}}$ and so $|f| \in \overline{\mathscr{A}}$ whenever $f \in \overline{\mathscr{A}}$. Also note that

$$
\max (f, g)=\frac{|f-g|+(f+g)}{2}, \min (f, g)=\frac{(f+g)-|f-g|}{2} .
$$

Therefore, this shows that if $f, g \in \overline{\mathscr{A}}$ then $\max (f, g), \min (f, g) \in \overline{\mathscr{A}}$. By induction, if $f_{i}, i=1,2, \cdots, m$ are in $\overline{\mathscr{A}}$ then

$$
\max \left(f_{i}, i=1,2, \cdots, m\right), \min \left(f_{i}, i=1,2, \cdots, m\right) \in \overline{\mathscr{A}}
$$

Now let $h \in C(A ; \mathbb{R})$ and let $x \in A$. Use Lemma 3.2.5 to obtain $f_{x y}$, a function of $\overline{\mathscr{A}}$ which agrees with $h$ at $x$ and $y$. Letting $\varepsilon>0$, there exists an open set $U(y)$ containing $y$ such that

$$
f_{x y}(z)>h(z)-\varepsilon \text { if } z \in U(y) .
$$

Since $A$ is compact, let $U\left(y_{1}\right), \cdots, U\left(y_{l}\right)$ cover $A$. Let

$$
f_{x} \equiv \max \left(f_{x y_{1}}, f_{x y_{2}}, \cdots, f_{x y_{l}}\right)
$$

Then $f_{x} \in \overline{\mathscr{A}}$ and $f_{x}(z)>h(z)-\varepsilon$ for all $z \in A$ and $f_{x}(x)=h(x)$. This implies that for each $x \in A$ there exists an open set $V(x)$ containing $x$ such that for $z \in V(x), f_{x}(z)<h(z)+\varepsilon$. Let $V\left(x_{1}\right), \cdots, V\left(x_{m}\right)$ cover $A$ and let $f \equiv \min \left(f_{x_{1}}, \cdots, f_{x_{m}}\right)$.Therefore, $f(z)<h(z)+\varepsilon$ for all $z \in A$ and since $f_{x}(z)>h(z)-\varepsilon$ for all $z \in A$, it follows $f(z)>h(z)-\varepsilon$ also and so $|f(z)-h(z)|<\varepsilon$ for all $z$. Since $\varepsilon$ is arbitrary, this shows $h \in \overline{\mathscr{A}}$ and proves $\overline{\mathscr{A}}=C(A ; \mathbb{R})$.

### 3.3 The Case of a Closed Set in $\mathbb{R}^{p}$

You can extend this theory to the case where $A=X$ a closed set. More generally, this is done with a locally compact Hausdorff space but this kind of space has not been considered here.

Definition 3.3.1 Let $X$ be a closed set in $\mathbb{R}^{p} . C_{0}(X)$ denotes the space of real or complex valued continuous functions defined on $X$ with the property that if $f \in C_{0}(X)$, then for each $\varepsilon>0$ there exists a compact set $K$ such that $|f(x)|<\varepsilon$ for all $x \in X \backslash K$. Define

$$
\|f\|_{\infty}=\sup \{|f(x)|: x \in X\} .
$$

These functions are said to vanish at infinity.
Lemma 3.3.2 For $X$ a closed set in $\mathbb{R}^{p}$ with the above norm, $C_{0}(X)$ is a complete space, meaning that every Cauchy sequence converges.

Proof: Let $\left\{f_{n}\right\}$ be a Cauchy sequence of functions in $C_{0}(X)$. Then in particular, $\left\{f_{n}(x)\right\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{F}$. Let $f(x) \equiv \lim _{n \rightarrow \infty} f_{n}(x)$.Let $\varepsilon>0$ be given. Then there exists $N$ such that for any $x \in X$,

$$
\left|f_{m}(x)-f_{n}(x)\right| \leq\left\|f_{m}-f_{n}\right\|_{\infty}<\varepsilon / 3
$$

for $m, n \geq N$. Thus, picking $n \geq N$ and taking a limit as $m \rightarrow \infty,\left|f(x)-f_{n}(x)\right| \leq \varepsilon / 3$ since $x$ was arbitrary,

$$
\begin{equation*}
\sup _{x \in X}\left|f(x)-f_{n}(x)\right| \leq \varepsilon / 3 \tag{3.2}
\end{equation*}
$$

By assumption, there exists a compact set $K$ such that if $x \notin K$ then $\left\|f_{N}\right\|_{\infty}<\varepsilon / 3$. Thus, from 3.2,

$$
\sup _{x \notin K}|f(x)| \leq 2 \varepsilon / 3<\varepsilon
$$

It remains to verify that $f$ is continuous. Letting $N$ be as the above, let $x, y \in X$. Then

$$
|f(x)-f(y)| \leq\left|f(x)-f_{N}(x)\right|+\left|f_{N}(x)-f_{N}(y)\right|+\left|f_{N}(y)-f(y)\right|
$$

By continuity of $f_{N}$ at $x$, there exists $\delta>0$ such that if $|x-y|<\delta$ for $y \in X$, it follows that $\left|f_{N}(x)-f_{N}(y)\right|<\varepsilon / 3$. Then for $|y-x|<\delta$,

$$
\begin{aligned}
|f(x)-f(y)| & \leq \sup _{x \in X}\left|f(x)-f_{N}(x)\right|+\frac{\varepsilon}{3}+\sup _{y \in X}\left|f(y)-f_{N}(y)\right| \\
& <\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon
\end{aligned}
$$

showing that $f$ is continuous. Thus the sequence of functions converges uniformly to a function $f \in C_{0}(X)$ which is what it means to be complete. Every Cauchy sequence converges. Indeed 3.2 says that $\left\|f-f_{n}\right\|_{\infty}<\varepsilon$ for $f \in C_{0}(X)$.

The above refers to functions which have values in $\mathbb{C}$ but the same proof works for functions which have values in any complete normed linear space.

In the case where the functions in $C_{0}(X)$ all have real values, I will denote the resulting space by $C_{0}(X ; \mathbb{R})$ with similar meanings in other cases.

The following has to do with a trick which will enable a result valid on sets which are only closed rather than compact. In general, you consider a locally compact Hausdorff space instead of a closed subset of $\mathbb{R}^{p}$. Consider the unit sphere in $\mathbb{R}^{p+1}$, centered at the point $(0, \cdots, 0,1) \equiv(\overrightarrow{0}, 1)$.

$$
S^{p} \equiv\left\{\vec{x} \in \mathbb{R}^{p+1}:\left(x_{n+1}-1\right)^{2}+\sum_{k=1}^{n} x_{k}^{2}=1\right\}
$$

Define a map from $\mathbb{R}^{p}$ which is identified with $\mathbb{R}^{p} \times\{0\}$ to the surface of this sphere as follows. Extend a line from the point, $\vec{p}$ in $\mathbb{R}^{p}$ to the point $(\overrightarrow{0}, 2)$ on the top of this sphere and let $\theta(p)$ denote the point of this sphere which the line intersects.


This map $\theta$ is one to one onto $S^{p} \backslash\{(\overrightarrow{0}, 2)\}$. More precisely, if you have $\left(\vec{a}, a_{n+1}\right)$ on $S^{p} \backslash(\overrightarrow{0}, 2)$ to get $\theta^{-1}\left(\vec{a}, a_{n+1}\right)$, you form the line from $(\overrightarrow{0}, 2)$ through this point and see where it hits $\mathbb{R}^{p}$. The line is $(\overrightarrow{0}, 2)+t\left(\left(\vec{a}, a_{n+1}\right)-(\overrightarrow{0}, 2)\right)$ and it hits $\mathbb{R}^{p}$ when $2+$ $t\left(a_{n+1}-2\right)=0$ which is when $t=\frac{2}{2-a_{n+1}}$. Thus $\theta^{-1}\left(\vec{a}, a_{n+1}\right)=\left(\frac{2 \vec{a}}{2-a_{n+1}}, 0\right)$. From this formula, it is clear that $\theta^{-1}$ is continuous and one to one. It is also onto because if $\vec{x} \in \mathbb{R}^{p}$ you can take the line from $(\overrightarrow{0}, 2)$ to $(\vec{x}, 0)$ and where it intersects $S^{p}$ is the point which is wanted. It is also easy to see from this that $\theta$ is continuous. Indeed, suppose $\vec{x}_{k} \rightarrow \vec{x}$ in $\mathbb{R}^{p}$. Does it follow that $\theta\left(\vec{x}_{k}\right) \rightarrow \theta(\vec{x})$ ? We know that $\left\{\vec{x}_{k}\right\}$ is bounded since it converges. Therefore, there is an open ball, $B((\overrightarrow{0}, 2), r)$ such that $\theta\left(\vec{x}_{k}\right) \in S^{p} \backslash B((\overrightarrow{0}, 2), r) \equiv K$ a compact set. If $\theta\left(\vec{x}_{k}\right)$ fails to converge to $\theta(\vec{x})$, then there is a subsequence, still denoted as $\theta\left(\vec{x}_{k}\right)$ such that $\theta\left(\vec{x}_{k}\right) \rightarrow y \in K$ where $y \neq \theta(\vec{x})$. But then, the continuity of $\theta^{-1}$ implies $x_{k} \rightarrow \theta^{-1}(y)$ and so $\theta^{-1}(y)=x$ which implies $y=\theta(x)$, a contradiction. Thus both $\theta$ and $\theta^{-1}$ are continuous, one to one and onto mappings between $\mathbb{R}^{p}$ and $S^{p} \backslash\{(\overrightarrow{0}, 2)\}$.

Theorem 3.3.3 Let $\mathscr{A}$ be an algebra of functions of $C_{0}(X, \mathbb{R})$ which separates the points of the closed set $X \subseteq \mathbb{R}^{p}$ and annihilates no point of $X$. Then $\mathscr{A}$ is dense in $C_{0}(X ; \mathbb{R})$.

Proof: $\widetilde{\mathscr{A}}$ denote all finite linear combinations of the form

$$
\left\{\sum_{i=1}^{n} c_{i} \widetilde{f}_{i}+c_{0}: f \in \mathscr{A}, c_{i} \in \mathbb{R}\right\}
$$

where for $f \in C_{0}(X ; \mathbb{R})$,

$$
\widetilde{f}(x) \equiv\left\{\begin{array}{l}
f\left(\theta^{-1}(x)\right) \text { if } x \in \theta(X) \\
0 \text { if } x=(\overrightarrow{0}, 2)
\end{array} .\right.
$$

Then $\widetilde{\mathscr{A}}$ is obviously an algebra of functions in $C\left(S^{p} ; \mathbb{R}\right)$. It separates points because this is true of $\mathscr{A}$. Similarly, it annihilates no point because of the inclusion of $c_{0}$ an arbitrary element of $\mathbb{R}$ in the definition of $\widetilde{\mathscr{A}}$ above. Therefore from Theorem 3.2.4, $\widetilde{\mathscr{A}}$ is dense in $C\left(S^{p} ; \mathbb{R}\right)$. Letting $f \in C_{0}(X ; \mathbb{R})$, it follows $\widetilde{f} \in C\left(S^{p} ; \mathbb{R}\right)$. It is clearly continuous on $\theta(X)$. What about at $(\overrightarrow{0}, 2)$ ? If you have $x_{n} \rightarrow(\overrightarrow{0}, 2)$, then $\left|\theta^{-1}\left(x_{n}\right)\right| \rightarrow \infty$ and therefore, since $f \in C_{0}, f\left(\theta^{-1}\left(x_{n}\right)\right) \equiv \widetilde{f}\left(x_{n}\right) \rightarrow 0 \equiv \widetilde{f}((\overrightarrow{0}, 2))$ and so indeed $\tilde{f}$ is in $C\left(S^{p} ; \mathbb{R}\right)$ as claimed. Thus there exists a sequence $\left\{h_{n}\right\} \subseteq \widetilde{\mathscr{A}}$ such that $h_{n}$ converges uniformly to $\widetilde{f}$. Now $h_{n}$ is
of the form $\sum_{i=1}^{m_{n}} c_{i}^{n} \widetilde{f_{i}^{n}}+c_{0}^{n}$ and since $\widetilde{f}((\overrightarrow{0}, 2))=0$, you can take each $c_{0}^{n}=0$ and so this has shown that in particular, specializing to $S^{p} \backslash\{(\overrightarrow{0}, 2)\}$,

$$
\lim _{n \rightarrow \infty} \sup _{z \in X}\left|\sum_{i=1}^{m_{n}} c_{i}^{n} f_{i}^{n}\left(\theta^{-1}(\theta(z))\right)-f\left(\theta^{-1}(\theta(z))\right)\right|=0
$$

where $f_{i}^{n} \in \mathscr{A}$. Thus

$$
\lim _{n \rightarrow \infty} \sup _{z \in X}\left|\sum_{i=1}^{m_{n}} c_{i}^{n} f_{i}^{n}(z)-f(z)\right|=0
$$

However, the sum gives a sequence of elements of $\mathscr{A}$ which are converging uniformly to $f$ on $X$.

### 3.4 The Case of Complex Valued Functions

What about the general case where $C_{0}(X)$ consists of complex valued functions and the field of scalars is $\mathbb{C}$ rather than $\mathbb{R}$ ? The following is the version of the Stone Weierstrass theorem which applies to this case. You have to assume that for $f \in \mathscr{A}$ it follows $\bar{f} \in \mathscr{A}$.

Lemma 3.4.1 Let $z$ be a complex number. Then

$$
\operatorname{Re}(z)=\operatorname{Im}(i \bar{z}), \operatorname{Im}(z)=\operatorname{Re}(i \bar{z})
$$

Proof: The following computation comes from the definition of real and imaginary parts.

$$
\begin{aligned}
& \operatorname{Re}(z)=\frac{z+\bar{z}}{2}=\frac{i z+i \bar{z}}{2 i}=\frac{i \bar{z}-\overline{(i \bar{z})}}{2 i}=\operatorname{Im}(i \bar{z}) \\
& \operatorname{Im}(z)=\frac{z-\bar{z}}{2 i}=\frac{i \bar{z}-i z}{2}=\frac{i \bar{z}+\overline{(i \bar{z})}}{2}=\operatorname{Re}(i \bar{z})
\end{aligned}
$$

Theorem 3.4.2 Suppose $\mathscr{A}$ is an algebra of functions in $C_{0}(X)$, which separates the points of $X$ and annihilates no point of $X$, a closed subset of $\mathbb{R}^{p}$ and has the property that if $f \in \mathscr{A}$, then $\bar{f} \in \mathscr{A}$. Then $\mathscr{A}$ is dense in $C_{0}(X)$.

Proof: Let $\operatorname{Re} \mathscr{A} \equiv\{\operatorname{Re} f: f \in \mathscr{A}\}, \operatorname{Im} \mathscr{A} \equiv\{\operatorname{Im} f: f \in \mathscr{A}\}$.
Claim 1: $\operatorname{Re} \mathscr{A}=\operatorname{Im} \mathscr{A}$
Proof of claim: A typical element of $\operatorname{Re} \mathscr{A}$ is $\operatorname{Re} f$ where $f \in \mathscr{A}$, then from Lemma 3.4.1, $\operatorname{Re}(f)=\operatorname{Im}(i \bar{f}) \in \operatorname{Im} \mathscr{A}$. Thus $\operatorname{Re} \mathscr{A} \subseteq \operatorname{Im} \mathscr{A}$. By assumption, $i \bar{f} \in \mathscr{A}$. The other direction works the same. Just use the other formula in Lemma 3.4.1.

Claim 2: Both $\operatorname{Re} \mathscr{A}$ and $\operatorname{Im} \mathscr{A}$ are real algebras.
Proof of claim: It is obvious these are both real vector spaces. Since these are equal, it suffices to consider $\operatorname{Re} \mathscr{A}$. It remains to show that $\operatorname{Re} \mathscr{A}$ is closed with respect to products.

$$
\frac{f+\bar{f}}{2} \frac{g+\bar{g}}{2}=\frac{1}{4}[f g+f \bar{g}+\bar{f} g+\overline{f g}]=\frac{1}{4}[2 \operatorname{Re}(f g)+2 \operatorname{Re}(\bar{f} g)]
$$

Now by assumption, $f g \in \mathscr{A}$ and so $\operatorname{Re}(f g) \in \operatorname{Re} \mathscr{A}$. Also $\operatorname{Re}(\bar{f} g) \in \operatorname{Re} \mathscr{A}$ because both $\bar{f}, g$ are in $\mathscr{A}$ and it is an algebra. Thus, the above is in $\operatorname{Re} \mathscr{A}$ because, as noted, this is a real vector space.

Claim 3: $\mathscr{A}=\operatorname{Re} \mathscr{A}+i \operatorname{Im} \mathscr{A}$
Proof of claim: If $f \in \mathscr{A}$, then

$$
f=\frac{f+\bar{f}}{2}+i \frac{f-\bar{f}}{2 i} \in \operatorname{Re} \mathscr{A}+i \operatorname{Im} \mathscr{A}
$$

so $\mathscr{A} \subseteq \operatorname{Re} \mathscr{A}+i \operatorname{Im} \mathscr{A}$. Now for $f, g \in \mathscr{A}$

$$
\operatorname{Re}(f)+i \operatorname{Im}(g) \equiv \frac{f+\bar{f}}{2}+i\left(\frac{g-\bar{g}}{2 i}\right)=\frac{f+g}{2}+\frac{\bar{f}-\bar{g}}{2} \in \mathscr{A}
$$

because $\mathscr{A}$ is closed with respect to conjugates. Thus $\operatorname{Re} \mathscr{A}+i \operatorname{Im} \mathscr{A} \subseteq \mathscr{A}$.
Both $\operatorname{Re} \mathscr{A}$ and $\operatorname{Im} \mathscr{A}$ must separate the points. Here is why: If $x_{1} \neq x_{2}$, then there exists $f \in \mathscr{A}$ such that $f\left(x_{1}\right) \neq f\left(x_{2}\right) . \operatorname{If} \operatorname{Im} f\left(x_{1}\right) \neq \operatorname{Im} f\left(x_{2}\right)$, this shows there is a function in $\operatorname{Im} \mathscr{A}, \operatorname{Im} f$ which separates these two points. If $\operatorname{Im} f$ fails to separate the two points, then $\operatorname{Re} f$ must separate the points and so, by Lemma 3.4.1,

$$
\operatorname{Re} f\left(x_{1}\right)=\operatorname{Im}\left(i \bar{f}\left(x_{1}\right)\right) \neq \operatorname{Re} f\left(x_{2}\right)=\operatorname{Im}\left(i \bar{f}\left(x_{2}\right)\right)
$$

Thus $\operatorname{Im} \mathscr{A}$ separages the points. Similarly $\operatorname{Re} \mathscr{A}$ separates the points using a similar argument or because it is equal to $\operatorname{Im} \mathscr{A}$.

Neither $\operatorname{Re} \mathscr{A}$ nor $\operatorname{Im} \mathscr{A}$ annihilate any point. This is easy to see because if $x$ is a point, there exists $f \in \mathscr{A}$ such that $f(x) \neq 0$. Thus either $\operatorname{Re} f(x) \neq 0$ or $\operatorname{Im} f(x) \neq 0$. If $\operatorname{Im} f(x) \neq 0$, this shows this point is not annihilated by $\operatorname{Im} \mathscr{A}$. Since they are equal, $\operatorname{Re} \mathscr{A}$ does not annihilate this point either.

It follows from Theorem 3.3.3 that $\operatorname{Re} \mathscr{A}$ and $\operatorname{Im} \mathscr{A}$ are dense in the real valued functions of $C_{0}(X)$. Let $f \in C_{0}(X)$. Then there exists $\left\{h_{n}\right\} \subseteq \operatorname{Re} \mathscr{A}$ and $\left\{g_{n}\right\} \subseteq \operatorname{Im} \mathscr{A}$ such that $h_{n} \rightarrow \operatorname{Re} f$ uniformly and $g_{n} \rightarrow \operatorname{Im} f$ uniformly. Therefore, $h_{n}+i g_{n} \in \mathscr{A}$ and it converges to $f$ uniformly.

### 3.5 Exercises

1. Let $\phi_{n}(x)=\left(1-x^{2}\right)^{n}$ for $|x| \leq 1$. For $f$ a continuous function defined on $[-1,1]$, extend it to have $f(x)=f(1)$ for $x>1$ and $f(x)=f(-1)$ for $x<-1$. Consider $p_{n}(x) \equiv \int_{x-1}^{x+1} \phi_{n}(x-y) f(y) d y=\int_{-1}^{1} \phi_{n}(y) f(x-y) d y$. This involves elementary calculus and change of variables. Show that $p_{n}(x)$ is a polynomial and that $p_{n}$ converges uniformly to $f$ on $[-1,1]$. This is the way Weierstrass originally proved the famous approximation theorem.
2. In fact the Bernstein polynomials apply for $f$ having values in a normed linear space and a similar result will hold. Give such a generalization.
3. Consider a continuous function $f$ defined on the box $\prod_{k=1}^{p}[0,1] \equiv[0,1]^{p}$. Then consider $f\left(t_{1}\right) \equiv f\left(t_{1}, \cdots\right)$ as a continuous function having values in $C\left([0,1]^{p-1}\right)$. Then the Bernstein polynomials are of the form $\sum_{k=0}^{m}\binom{m}{k} f\left(\frac{k}{m}, \cdots\right) t_{1}^{k}\left(1-t_{1}\right)^{m-k}$. Now repeat the process on these coefficients $f\left(\frac{k}{m}, \cdots\right)$ which can be considered functions in $C\left([0,1]^{p-2}\right), t_{2} \rightarrow f\left(\frac{k}{m}, t_{2}, \cdots\right)$. Continuing this way, show there is a polynomial

$$
\sum_{k_{1}, \cdots, k_{m}} a_{k_{1}, \cdots, k_{n}} t_{1}^{k_{1}} \cdots t^{k_{m}}
$$

which is uniformly close to $f$ on $[0,1]$. Extend to the case of a box $\prod_{k=1}^{p}\left[a_{k}, b_{k}\right]$. The continuous function $f$ is only required to have values in some normed linear space.
4. Consider the following $p_{\mathbf{m}}(\mathbf{x}) \equiv$

$$
\begin{gather*}
\sum_{k_{1}=0}^{m_{1}} \cdots \sum_{k_{n}=0}^{m_{n}}\binom{m_{1}}{k_{1}}\binom{m_{2}}{k_{2}} \cdots\binom{m_{n}}{k_{n}} x_{1}^{k_{1}}\left(1-x_{1}\right)^{m_{1}-k_{1}} x_{2}^{k_{2}}\left(1-x_{2}\right)^{m_{2}-k_{2}} \\
\cdots x_{n}^{k_{n}}\left(1-x_{n}\right)^{m_{n}-k_{n}} f\left(\frac{k_{1}}{m_{1}}, \cdots, \frac{k_{n}}{m_{n}}\right) \tag{3.3}
\end{gather*}
$$

where $f:[0,1]^{n} \rightarrow X$ a normed linear space. Show that for all $\varepsilon$, there exists $n$ such that $\left\|p_{\mathbf{m}}-f\right\|<\varepsilon$ if $\min \left(m_{1}, \cdots, m_{n}\right)>0$. Hint: Consider Lemma 3.1.1 first and you may see how to do this.
5. Theorem 2.5.43 gave an example of a function which is everywhere continuous and nowhere differentiable. The first examples of this sort were given by Weierstrass in 1872 who gave an example involving an infinite series in which each term had all derivatives everywere and yet the uniform limit had no derivative anywhere. Using the example of Theorem 2.5.43, give an example of an infinite series of functions, each term being a polynomial defined on $[0,1], \sum_{k=1}^{\infty} p_{k}(x)=f(x)$ for which it makes absolutely no sense to write $f^{\prime}(x)=\sum_{k=1}^{\infty} p_{k}^{\prime}(x)$ because $f^{\prime}$ fails to exist at any point. In other words, you cannot differentiate an infinite series term by term. The derivative of a sum is not the sum of the derivatives when dealing with an infinite "sum". Also show that if you have any differentiable function $g$ and $\varepsilon>0$, there exists a nowhere differentiable function $h$ such that $\|g-h\|<\varepsilon$. This is in stark contrast with what will be presented in complex analysis in which, thanks to the Cauchy integral formula, uniform convergence of differentiable functions does lead to a differentiable function. Hint: Use Weierstrass approximation theorem and telescoping series to get the example of a series which can't be differentiated term by term.
6. If $f, f^{\prime}$ are both continuous, suppose $p_{n} \rightarrow f$ uniformly where the $p_{n}$ are the Bernstein polynomials. Show that then $p_{n}^{\prime} \rightarrow f^{\prime}$ uniformly also.
7. Use the above problem to show that if $f$ is continuous and defined on $\prod_{k=1}^{p}[0,1]$ and if also all the partial derivatives of $f$ are continuous, then if $p_{n} \rightarrow f$ uniformly with the $p_{n}$ being the Bernstein polynomials discussed in Problem 3, then the partial derivatives of these $p_{n}$ converge uniformly to the corresponding partial derivatives of $f$. Extend to the case where $f$ is defined on $\prod_{k=1}^{p}\left[a_{k}, b_{k}\right]$.
8. In contrast to Problem 6, consider the sequence of functions

$$
\left\{f_{n}(x)\right\}_{n=1}^{\infty}=\left\{\frac{x}{1+n x^{2}}\right\}_{n=1}^{\infty}
$$

Show it converges uniformly to $f(x) \equiv 0$. However, $f_{n}^{\prime}(0)$ converges to 1 , not $f^{\prime}(0)$. Hint: To show the first part, find the value of $x$ which maximizes the function $\left|\frac{x}{1+n x^{2}}\right|$. You know how to do this. Then plug it in and you will have an estimate sufficient to verify uniform convergence. This shows how special the Bernstein polynomials are.
9. Show using the Weierstrass approximation theorem that if $f$ is a continuous, real valued function on $[a, b]$, then it has an antiderivative. Hint: Let $p_{n} \rightarrow f$ uniformly and let $P_{n}^{\prime}=p_{n}, P_{n}(a)=0$. It is obvious that a polynomial has an antiderivative. Now use the uniform convergence of the $p_{n}$ and the mean value theorem from single variable calculus to show that $\left\{P_{n}\right\}$ also converges uniformly to some function $F$ and that $F$ is the desired antiderivative.

$$
\begin{aligned}
& \frac{F(x+h)-F(x)}{h} \\
= & \frac{F(x+h)-P_{n}(x+h)}{h}+\frac{P_{n}(x+h)-P_{n}(x)}{h}+\frac{P_{n}(x)-F(x)}{h} \\
= & \varepsilon_{n(h)}(h)+\frac{P_{n}(x+h)-P_{n}(x)}{h}=\varepsilon_{n(h)}(h)+p_{n}\left(x+\theta_{h} h\right) \\
= & \varepsilon_{n(h)}(h)+\left(p_{n}\left(x+\theta_{h} h\right)-f\left(x+\theta_{h} h\right)\right)+\left(f\left(x+\theta_{h} h\right)-f(x)\right)
\end{aligned}
$$

Here $n(h)$ is so large that $\lim _{h \rightarrow 0} \varepsilon_{n(h)}(h)=0$. Now pass to a limit.
10. In the above problem, explain, using the mean value theorem from calculus, how you could define $\int_{a}^{b} f(x) d x \equiv F(b)-F(a)$ where $F$ is an antiderivative, and thereby obtain the integral of elementary calculus.

## Part II

## Real Analysis

## Chapter 4

## The Derivative

The derivative is a linear transformation. In this chapter are the principal results about the derivative.

### 4.1 Basic Definitions

The derivative is a linear transformation. This may not be entirely clear from a beginning calculus course because they like to say it is a slope which is a number. As observed by Deudonne,
"...In the classical teaching of Calculus, this idea (that the derivative is a linear transformation) is immediately obscured by the accidental fact that, on a onedimensional vector space, there is a one-to-one correspondence between linear forms and numbers, and therefore the derivative at a point is defined as a number instead of a linear form. This slavish subservience to the shibboleth ${ }^{1}$ of numerical interpretation at any cost becomes much worse when dealing with functions of several variables..."

The concept of derivative generalizes right away to functions of many variables but only if you regard a number which is identified as the derivative in single variable calculus as a linear transformation on $\mathbb{R}$. However for functions of many variables, no attempt will be made to consider derivatives from one side or another. This is because when you consider functions of many variables, there isn't a well defined side. However, it is certainly the case that there are more general notions which include such things. I will present a fairly general notion of the derivative of a function which is defined on an open subset of a normed vector space which has values in a normed vector space. The case of most interest is that of a function which maps an open set in $\mathbb{F}^{n}$ to $\mathbb{F}^{m}$ but it is no more trouble to consider the extra generality and it is sometimes useful to have this extra generality because sometimes you want to consider functions defined, for example on subspaces of $\mathbb{F}^{n}$ and it is nice to not have to trouble with ad hoc considerations. Also, you might want to consider $\mathbb{F}^{n}$ with some norm other than the usual one.

For most of what follows, it is not important for the vector spaces to be finite dimensional provided you make the following definition of what is meant by $\mathscr{L}(X, Y)$ which is automatic if $X$ is finite dimensional. See Proposition 2.8.8.

## Definition 4.1.1 Let $\left(X,\|\cdot\|_{X}\right)$ and $\left(Y,\|\cdot\|_{Y}\right)$ be two normed linear spaces. Then

 $\mathscr{L}(X, Y)$ denotes the set of linear maps from $X$ to $Y$ which also satisfy the following condition. For $L \in \mathscr{L}(X, Y)$,$$
\lim _{\|x\|_{X} \leq 1}\|L x\|_{Y} \equiv\|L\|<\infty
$$

To save notation, I will use $\|\cdot\|$ as a norm on either $X, Y$ or $\mathscr{L}(X, Y)$ and allow the context to determine which it is.

Let $U$ be an open set in $X$, and let $\mathbf{f}: U \rightarrow Y$ be a function.

[^1]Definition 4.1.2 $A$ function $\mathbf{g}$ is $\mathbf{o}(\mathbf{v})$ if

$$
\begin{equation*}
\lim _{\|\mathbf{v}\| \rightarrow 0} \frac{\mathbf{g}(\mathbf{v})}{\|\mathbf{v}\|}=\mathbf{0} \tag{4.1}
\end{equation*}
$$

A function $\mathbf{f}: U \rightarrow Y$ is differentiable at $\mathbf{x} \in U$ if there exists a linear transformation $L \in$ $\mathscr{L}(X, Y)$ such that

$$
\mathbf{f}(\mathbf{x}+\mathbf{v})=\mathbf{f}(\mathbf{x})+L \mathbf{v}+\mathbf{o}(\mathbf{v})
$$

This linear transformation $L$ is the definition of $D \mathbf{f}(\mathbf{x})$. This derivative is often called the Frechet derivative.

Note that from Theorem 2.7.4 the question whether a given function is differentiable is independent of the norm used on the finite dimensional vector space. That is, a function is differentiable with one norm if and only if it is differentiable with another norm. In infinite dimensions, this is not clearly so and in this case, simply regard the norm as part of the definition of the normed linear space which incidentally will also typically be assumed to be a complete normed linear space.

The definition 4.1 means the error, $\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})-L \mathbf{v}$ converges to $\mathbf{0}$ faster than $\|\mathbf{v}\|$. Thus the above definition is equivalent to saying

$$
\begin{equation*}
\lim _{\|\mathbf{v}\| \rightarrow 0} \frac{\|\mathbf{f}(\mathbf{x}+\mathbf{v})-(\mathbf{f}(\mathbf{x})+L \mathbf{v})\|}{\|\mathbf{v}\|}=0 \tag{4.2}
\end{equation*}
$$

or equivalently,

$$
\begin{equation*}
\lim _{\mathbf{y} \rightarrow \mathbf{x}} \frac{\|\mathbf{f}(\mathbf{y})-(\mathbf{f}(\mathbf{x})+D \mathbf{f}(\mathbf{x})(\mathbf{y}-\mathbf{x}))\|}{\|\mathbf{y}-\mathbf{x}\|}=0 . \tag{4.3}
\end{equation*}
$$

The symbol, $\mathbf{o}(\mathbf{v})$ should be thought of as an adjective. Thus, if $t$ and $k$ are constants,

$$
\mathbf{o}(\mathbf{v})=\mathbf{o}(\mathbf{v})+\mathbf{o}(\mathbf{v}), \mathbf{o}(t \mathbf{v})=\mathbf{o}(\mathbf{v}), k \mathbf{o}(\mathbf{v})=\mathbf{o}(\mathbf{v})
$$

and other similar observations hold.

## Theorem 4.1.3 The derivative is well defined.

Proof: First note that for a fixed nonzero vector $\mathbf{v}, \mathbf{o}(t \mathbf{v})=\mathbf{o}(t)$. This is because

$$
\lim _{t \rightarrow 0} \frac{\mathbf{o}(t \mathbf{v})}{|t|}=\lim _{t \rightarrow 0}\|\mathbf{v}\| \frac{\mathbf{o}(t \mathbf{v})}{\|t \mathbf{v}\|}=\mathbf{0}
$$

Now suppose both $L_{1}$ and $L_{2}$ work in the above definition. Then let $\mathbf{v}$ be any vector and let $t$ be a real scalar which is chosen small enough that $t \mathbf{v}+\mathbf{x} \in U$. Then

$$
\mathbf{f}(\mathbf{x}+t \mathbf{v})=\mathbf{f}(\mathbf{x})+L_{1} t \mathbf{v}+\mathbf{o}(t \mathbf{v}), \mathbf{f}(\mathbf{x}+t \mathbf{v})=\mathbf{f}(\mathbf{x})+L_{2} t \mathbf{v}+\mathbf{o}(t \mathbf{v}) .
$$

Therefore, subtracting these two yields $\left(L_{2}-L_{1}\right)(t \mathbf{v})=\mathbf{o}(t \mathbf{v})=\mathbf{o}(t)$. Therefore, dividing by $t$ yields $\left(L_{2}-L_{1}\right)(\mathbf{v})=\frac{\mathbf{o}(t)}{t}$. Now let $t \rightarrow 0$ to conclude that $\left(L_{2}-L_{1}\right)(\mathbf{v})=0$. Since this is true for all $\mathbf{v}$, it follows $L_{2}=L_{1}$.

Lemma 4.1.4 Let $\mathbf{f}$ be differentiable at $\mathbf{x}$. Then $\mathbf{f}$ is continuous at $\mathbf{x}$ and in fact, there exists $K>0$ such that whenever $\|\mathbf{v}\|$ is small enough, $\|\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})\| \leq K\|\mathbf{v}\|$. Also if $\mathbf{f}$ is differentiable at $\mathbf{x}$, then

$$
\mathbf{o}(\|\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})\|)=\mathbf{o}(\mathbf{v})
$$

Proof: From the definition of the derivative, $\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})=D \mathbf{f}(\mathbf{x}) \mathbf{v}+\mathbf{o}(\mathbf{v})$. Let $\|\mathbf{v}\|$ be small enough that $\frac{\mathbf{o}(\|\mathbf{v}\|)}{\|\mathbf{v}\|}<1$ so that $\|\mathbf{o}(\mathbf{v})\| \leq\|\mathbf{v}\|$. Then for such $\mathbf{v}$,

$$
\|\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})\| \leq\|D \mathbf{f}(\mathbf{x}) \mathbf{v}\|+\|\mathbf{v}\| \leq(\|D \mathbf{f}(\mathbf{x})\|+1)\|\mathbf{v}\|
$$

This proves the lemma with $K=\|D \mathbf{f}(\mathbf{x})\|+1$. Recall the operator norm discussed in Definitions 2.8.4, 4.1.1.

The last assertion is implied by the first as follows. Define

$$
\mathbf{h}(\mathbf{v}) \equiv\left\{\begin{array}{l}
\frac{\mathbf{o}(\|\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})\|)}{\|\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})\|} \text { if }\|\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})\| \neq 0 \\
\mathbf{0} \text { if }\|\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})\|=0
\end{array}\right.
$$

Then $\lim _{\|\mathbf{v}\| \rightarrow 0} \mathbf{h}(\mathbf{v})=\mathbf{0}$ from continuity of $\mathbf{f}$ at $\mathbf{x}$ which is implied by the first part. Also from the above estimate,

$$
\left\|\frac{\mathbf{o}(\|\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})\|)}{\|\mathbf{v}\|}\right\|=\|\mathbf{h}(\mathbf{v})\| \frac{\|\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})\|}{\|\mathbf{v}\|} \leq\|\mathbf{h}(\mathbf{v})\|(\|D \mathbf{f}(\mathbf{x})\|+1)
$$

This establishes the second claim.
Here $\|D \mathbf{f}(\mathbf{x})\|$ is the operator norm of the linear transformation, $D \mathbf{f}(\mathbf{x})$. This will always be the case unless specified to be otherwise.

### 4.2 The Chain Rule

With the above lemma, it is easy to prove the chain rule.

Theorem 4.2.1 (The chain rule) Let $U$ and $V$ be open sets $U \subseteq X$ and $V \subseteq Y$. Suppose $\mathbf{f}: U \rightarrow V$ is differentiable at $\mathbf{x} \in U$ and suppose $\mathbf{g}: V \rightarrow Z$ is differentiable at $\mathbf{f}(\mathbf{x}) \in V$ where $Z$ is a normed linear space. Then $\mathbf{g} \circ \mathbf{f}$ is differentiable at $\mathbf{x}$ and

$$
D(\mathbf{g} \circ \mathbf{f})(\mathbf{x})=D \mathbf{g}(\mathbf{f}(\mathbf{x})) D \mathbf{f}(\mathbf{x})
$$

Proof: This follows from a computation. Let $B(\mathbf{x}, r) \subseteq U$ and let $r$ also be small enough that for $\|\mathbf{v}\| \leq r$, it follows that $\mathbf{f}(\mathbf{x}+\mathbf{v}) \in V$. Such an $r$ exists because $\mathbf{f}$ is continuous at $\mathbf{x}$. For $\|\mathbf{v}\|<r$, the definition of differentiability of $\mathbf{g}$ and $\mathbf{f}$ implies

$$
\begin{gather*}
\mathbf{g}(\mathbf{f}(\mathbf{x}+\mathbf{v}))-\mathbf{g}(\mathbf{f}(\mathbf{x}))= \\
= \\
=D \mathbf{g}(\mathbf{f}(\mathbf{x}))(\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x}))+\mathbf{o}(\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})) \\
=  \tag{4.4}\\
= \\
D(\mathbf{f}(\mathbf{x}))[D(\mathbf{f}(\mathbf{x}))) D(\mathbf{x}) \mathbf{v}+\mathbf{o}(\mathbf{x}))]+\mathbf{v}(\mathbf{f}(\mathbf{x}+\mathbf{o}(\mathbf{v})-\mathbf{v})+\mathbf{f}(\mathbf{f}(\mathbf{f}(\mathbf{x}+\mathbf{v}))-\mathbf{f}(\mathbf{x})) \\
\\
=(\mathbf{f}(\mathbf{x}))) D(\mathbf{f}(\mathbf{x})) \mathbf{v}+\mathbf{o}(\mathbf{v})
\end{gather*}
$$

By Lemma 4.1.4. From the definition of the derivative, $D(\mathbf{g} \circ \mathbf{f})(\mathbf{x})$ exists and equals $D(\mathbf{g}(\mathbf{f}(\mathbf{x}))) D(\mathbf{f}(\mathbf{x}))$.

### 4.3 The Matrix of the Derivative

The case of interest here is where $X=\mathbb{R}^{n}$ and $Y=\mathbb{R}^{m}$, the function being defined on an open subset of $\mathbb{R}^{n}$. Of course this all generalizes to arbitrary vector spaces and one considers the matrix taken with respect to various bases. As above, $\mathbf{f}$ will be defined and differentiable on an open set $U \subseteq \mathbb{R}^{n}$.

The matrix of $D \mathbf{f}(\mathbf{x})$ is the matrix having the $i^{t h}$ column equal to $D \mathbf{f}(\mathbf{x}) \mathbf{e}_{i}$ and so it is only necessary to compute this. Recall that for $J f(\mathbf{x})$ this matrix,

$$
J \mathbf{f}(\mathbf{x}) \mathbf{v}=D \mathbf{f}(\mathbf{x}) \mathbf{v}
$$

for any $\mathbf{v}$ where on the left the meaning is matrix multiplication. $J \mathbf{f}(\mathbf{x})$ is an $m \times n$ matrix and it is multiplying $\mathbf{v}$, a vector of $\mathbb{R}^{n}$ on the left. This is the matrix taken with respect to the standard basis vectors. Let $t$ be a small real number. Then

$$
\frac{\mathbf{f}\left(\mathbf{x}+t \mathbf{e}_{i}\right)-\mathbf{f}(\mathbf{x})-D \mathbf{f}(\mathbf{x})\left(t \mathbf{e}_{i}\right)}{t}=\frac{\mathbf{o}(t)}{t}
$$

Therefore,

$$
\frac{\mathbf{f}\left(\mathbf{x}+t \mathbf{e}_{i}\right)-\mathbf{f}(\mathbf{x})}{t}=D \mathbf{f}(\mathbf{x})\left(\mathbf{e}_{i}\right)+\frac{\mathbf{o}(t)}{t}
$$

The limit exists on the right and so it exists on the left also. Thus

$$
\frac{\partial \mathbf{f}(\mathbf{x})}{\partial x_{i}} \equiv \lim _{t \rightarrow 0} \frac{\mathbf{f}\left(\mathbf{x}+t \mathbf{e}_{i}\right)-\mathbf{f}(\mathbf{x})}{t}=D \mathbf{f}(\mathbf{x})\left(\mathbf{e}_{i}\right)
$$

and so the matrix of the derivative is just the matrix which has the $i^{t h}$ column equal to the $i^{\text {th }}$ partial derivative of $\mathbf{f}$. Note that this shows that whenever $\mathbf{f}$ is differentiable, it follows that the partial derivatives all exist. It does not go the other way however as discussed later.

Theorem 4.3.1 Let $\mathrm{f}: U \subseteq \mathbb{F}^{n} \rightarrow \mathbb{F}^{m}$ and suppose $\mathbf{f}$ is differentiable at $\mathbf{x}$. Then all the partial derivatives $\frac{\partial f_{i}(\mathbf{x})}{\partial x_{j}}$ exist and if $J \mathbf{f}(\mathbf{x})$ is the matrix of the linear transformation, $D \mathbf{f}(\mathbf{x})$ with respect to the standard basis vectors, then the $i j^{\text {th }}$ entry is given by $\frac{\partial f_{i}}{\partial x_{j}}(\mathbf{x})$ also denoted as $f_{i, j}$ or $f_{i, x_{j}}$. It is the matrix whose $i^{\text {th }}$ column is

$$
\frac{\partial \mathbf{f}(\mathbf{x})}{\partial x_{i}} \equiv \lim _{t \rightarrow 0} \frac{\mathbf{f}\left(\mathbf{x}+t \mathbf{e}_{i}\right)-\mathbf{f}(\mathbf{x})}{t} .
$$

In particular, this says the same as saying that the $i j^{\text {th }}$ entry of this matrix is $\frac{\partial f_{i}(\mathbf{x})}{\partial x_{j}}$.
I will generally not distinguish between the linear transformation $D \mathbf{f}(\mathbf{x})$ and its matrix with respect to the standard basis vectors $J \mathbf{f}(\mathbf{x})$ when the setting is $\mathbb{R}^{n}$ and $\mathbb{R}^{m}$.

If you take another partial derivative, it can be written as $f_{x_{i} x_{j}} \equiv \frac{\partial}{\partial x_{j}} \frac{\partial f}{\partial x_{i}}$. This might be written as $f_{, i j}$ also. I assume the reader has seen partial derivatives in calculus.

What if all the partial derivatives of $\mathbf{f}$ exist? Does it follow that $\mathbf{f}$ is differentiable? Consider the following function, $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$,

$$
f(x, y)=\left\{\begin{array}{l}
\frac{x y}{x^{2}+y^{2}} \text { if }(x, y) \neq(0,0) \\
0 \text { if }(x, y)=(0,0)
\end{array} .\right.
$$

Then from the definition of partial derivatives,

$$
\lim _{h \rightarrow 0} \frac{f(h, 0)-f(0,0)}{h}=\lim _{h \rightarrow 0} \frac{0-0}{h}=0
$$

and

$$
\lim _{h \rightarrow 0} \frac{f(0, h)-f(0,0)}{h}=\lim _{h \rightarrow 0} \frac{0-0}{h}=0
$$

However $f$ is not even continuous at $(0,0)$ which may be seen by considering the behavior of the function along the line $y=x$ and along the line $x=0$. By Lemma 4.1.4 this implies $f$ is not differentiable. Therefore, it is necessary to consider the correct definition of the derivative given above if you want to get a notion which generalizes the concept of the derivative of a function of one variable in such a way as to preserve continuity whenever the function is differentiable.

### 4.4 The Usual Form of the Chain Rule

Let $\mathbf{z} \equiv \mathbf{g}(\mathbf{y})$ and let $\mathbf{y}=\mathbf{f}(\mathbf{x})$. Assuming $D \mathbf{g}(\mathbf{f}(\mathbf{x}))$ exists and $D \mathbf{f}(\mathbf{x})$ both exist and then we have $\mathbf{x} \in U \subseteq \mathbb{R}^{n}$ and $\mathbf{y} \in V \subseteq \mathbb{R}^{m}$ where $U, V$ are open sets and $\mathbf{f}(V) \subseteq V$ with $\mathbf{g}: V \rightarrow \mathbb{R}^{p}$. What is the matrix of $\mathbf{g} \circ \mathbf{f}(\mathbf{x})=\mathbf{g}(\mathbf{y})$ ? Say $\mathbf{g}$ has values in $\mathbb{R}^{p}$. From the chain rule above, and the description of the matrix of the derivative in Theorem 4.3.1,

$$
\begin{gather*}
\left(\begin{array}{cccc}
\frac{\partial \mathbf{z}}{\partial x_{1}} & \frac{\partial \mathbf{z}}{\partial x_{2}} & \cdots & \frac{\partial \mathbf{z}}{\partial x_{n}}
\end{array}\right)=  \tag{4.5}\\
\left(\begin{array}{llll}
\frac{\partial \mathbf{z}}{\partial y_{1}} & \frac{\partial \mathbf{z}}{\partial y_{2}} & \cdots & \frac{\partial \mathbf{z}}{\partial y_{m}}
\end{array}\right)\left(\begin{array}{llll}
\frac{\partial \mathbf{y}}{\partial x_{1}} & \frac{\partial \mathbf{y}}{\partial x_{2}} & \cdots & \frac{\partial \mathbf{y}}{\partial x_{n}}
\end{array}\right)
\end{gather*}
$$

Now from the way we multiply matrices, to find the $i j^{t h}$ entry of the matrix on the right, one multiplies the $i^{\text {th }}$ row of the left matrix with the $j^{\text {th }}$ column of the matrix on the right. Thus the $i j^{t h}$ entry of the matrix on the right is

$$
\sum_{k} \frac{\partial z_{i}}{\partial y_{k}} \frac{\partial y_{k}}{\partial x_{j}}
$$

and by the chain rule, Theorem 4.2.1, this equals the $i j^{t h}$ entry of the matrix of 4.5. That is,

$$
\frac{\partial z_{i}}{\partial x_{j}}=\sum_{k} \frac{\partial z_{i}}{\partial y_{k}} \frac{\partial y_{k}}{\partial x_{j}}
$$

This is stated as the following proposition.
Proposition 4.4.1 Let $\mathbf{z} \equiv \mathbf{g} \circ \mathbf{f}(\mathbf{x})$ and let $\mathbf{y} \equiv \mathbf{f}(\mathbf{x})$, then assuming Df( $\mathbf{x})$ exists and $D \mathbf{g}(\mathbf{f}(\mathbf{x}))$ exists, then $\frac{\partial z_{i}}{\partial x_{j}}=\sum_{k} \frac{\partial z_{i}(\mathbf{y})}{\partial y_{k}} \frac{\partial y_{k}}{\partial x_{j}}$ assuming that all functions make sense. That is $\mathbf{f}: U \rightarrow \mathbf{f}(U) \subseteq V$ and $\mathbf{g}: V \rightarrow \mathbb{R}^{p}$ for $U, V$ open sets in $\mathbb{R}^{n}$ and $\mathbb{R}^{m}$ respectively. Also, since this holds for each $i, \frac{\partial \mathbf{z}}{\partial x_{j}}=\sum_{k} \frac{\partial \mathbf{z}}{\partial y_{k}} \frac{\partial y_{k}}{\partial x_{j}}$.

Some people like to dispense with the summation sign and write instead $\frac{\partial \mathbf{z}}{\partial x_{j}}=\frac{\partial \mathbf{z}}{\partial y_{k}} \frac{\partial y_{k}}{\partial x_{j}}$ where it is understood that summation takes place on the repeated index.

### 4.5 Differentiability and $C^{1}$ Functions

There is a way to get the differentiability of a function from the existence and continuity of the partial derivatives. This is very convenient because these partial derivatives are taken with respect to a one dimensional variable. Of course, the determination of continuity is again a multivariable consideration. The following theorem is the main result.
Definition 4.5.1 When $\mathbb{f}: U \rightarrow \mathbb{R}^{p}$ for $U$ an open subset of $\mathbb{R}^{n}$ and the vector valued functions, $\frac{\partial \mathbf{f}}{\partial x_{i}}$ are all continuous, (equivalently each $\frac{\partial f_{i}}{\partial x_{j}}$ is continuous), the function is said to be $C^{1}(U)$. If all the partial derivatives up to order $k$ exist and are continuous, then the function is said to be $C^{k}$.

It turns out that for a $C^{1}$ function, all you have to do is write the matrix described in Theorem 4.3.1 and this will be the derivative. There is no question of existence for the derivative for such functions. This is the importance of the next theorem.
Theorem 4.5.2 Suppose $\mathbf{f}: U \rightarrow \mathbb{R}^{p}$ where $U$ is an open set in $\mathbb{R}^{n}$. Suppose also that all partial derivatives of $\mathbf{f}$ exist on $U$ and are continuous. Then $\mathbf{f}$ is differentiable at every point of $U$.

Proof: If you fix all the variables but one, you can apply the fundamental theorem of calculus as follows.

$$
\begin{equation*}
\mathbf{f}\left(\mathbf{x}+v_{k} \mathbf{e}_{k}\right)-\mathbf{f}(\mathbf{x})=\int_{0}^{1} \frac{\partial \mathbf{f}}{\partial x_{k}}\left(\mathbf{x}+t v_{k} \mathbf{e}_{k}\right) v_{k} d t \tag{4.6}
\end{equation*}
$$

Here is why. Let $\mathbf{h}(t)=\mathbf{f}\left(\mathbf{x}+t v_{k} \mathbf{e}_{k}\right)=\mathbf{f}\left(x_{1}, \cdots, x_{k}+t v_{k}, x_{k+1}, \cdots, x_{n}\right)$. Then from the chain rule, $\mathbf{h}^{\prime}(t)=\frac{\partial \mathbf{f}}{\partial x_{k}}\left(\mathbf{x}+t v_{k} \mathbf{e}_{k}\right) v_{k}$. Therefore, since $\mathbf{h}^{\prime}$ is continuous, one can apply the fundamental theorem of calculus to each component and write

$$
\mathbf{f}\left(\mathbf{x}+v_{k} \mathbf{e}_{k}\right)-\mathbf{f}(\mathbf{x})=\mathbf{h}(1)-\mathbf{h}(0)=\int_{0}^{1} \mathbf{h}^{\prime}(t) d t=\int_{0}^{1} \frac{\partial \mathbf{f}}{\partial x_{k}}\left(\mathbf{x}+t v_{k} \mathbf{e}_{k}\right) v_{k} d t
$$

Now I will use this observation to prove the theorem. Let $\mathbf{v}=\left(v_{1}, \cdots, v_{n}\right)$ with $|\mathbf{v}|$ sufficiently small. Thus $\mathbf{v}=\sum_{k=1}^{n} v_{k} \mathbf{e}_{k}$. For the purposes of this argument, define $\sum_{k=n+1}^{n} v_{k} \mathbf{e}_{k} \equiv$ $\mathbf{0}$. Then with this convention, $\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})=$

$$
\begin{gather*}
\sum_{i=1}^{n}\left(\mathbf{f}\left(\mathbf{x}+\sum_{k=i}^{n} v_{k} \mathbf{e}_{k}\right)-\mathbf{f}\left(\mathbf{x}+\sum_{k=i+1}^{n} v_{k} \mathbf{e}_{k}\right)\right)=\sum_{i=1}^{n} \int_{0}^{1} \frac{\partial \mathbf{f}}{\partial x_{i}}\left(\mathbf{x}+\sum_{k=i+1}^{n} v_{k} \mathbf{e}_{k}+t v_{i} \mathbf{e}_{i}\right) v_{i} d t \\
=\sum_{i=1}^{n} \int_{0}^{1}\left(\frac{\partial \mathbf{f}}{\partial x_{i}}\left(\mathbf{x}+\sum_{k=i+1}^{n} v_{k} \mathbf{e}_{k}+t v_{i} \mathbf{e}_{i}\right) v_{i}-\frac{\partial \mathbf{f}}{\partial x_{i}}(\mathbf{x}) v_{i}\right) d t  \tag{4.7}\\
\quad+\sum_{i=1}^{n} \int_{0}^{1} \frac{\partial \mathbf{f}}{\partial x_{i}}(\mathbf{x}) v_{i} d t=\sum_{i=1}^{n} \frac{\partial \mathbf{f}}{\partial x_{i}}(\mathbf{x}) v_{i}+\mathbf{o}(\mathbf{v})
\end{gather*}
$$

and this shows $\mathbf{f}$ is differentiable at $\mathbf{x}$. The reason for this is that each term in the sum in 4.7 is $\mathbf{0}(\mathbf{v})$. Indeed, letting $|\cdot|$ be the usual Euclidean norm,

$$
\left|\int_{0}^{1}\left(\frac{\partial \mathbf{f}}{\partial x_{i}}\left(\mathbf{x}+\sum_{k=i+1}^{n} v_{k} \mathbf{e}_{k}+t v_{i} \mathbf{e}_{i}\right)-\frac{\partial \mathbf{f}}{\partial x_{i}}(\mathbf{x})\right) d t v_{i}\right|
$$

$$
\leq\left|\int_{0}^{1}\left(\frac{\partial \mathbf{f}}{\partial x_{i}}\left(\mathbf{x}+\sum_{k=i+1}^{n} v_{k} \mathbf{e}_{k}+t v_{i} \mathbf{e}_{i}\right)-\frac{\partial \mathbf{f}}{\partial x_{i}}(\mathbf{x})\right)\right||\mathbf{v}|
$$

and by continuity, the integral converges to 0 as $|\mathbf{v}| \rightarrow 0$. This follows from uniform continuity of $\frac{\partial \mathbf{f}}{\partial x_{i}}$ on a sufficiently small closed ball containing $\mathbf{x}$.

### 4.6 Mixed Partial Derivatives

Under certain conditions the mixed partial derivatives will always be equal. The simple condition is that if they exist and are continuous, then they are equal. This astonishing fact is due to Euler in 1734 and was proved by Clairaut although not very well. The first satisfactory proof was by Hermann Schwarz in 1873. For reasons I cannot understand, calculus books seldom include a proof of this important result. It is not all that hard. It is based on the mean value theorem for derivatives. Here it is.

Theorem 4.6.1 Suppose $f: U \subseteq \mathbb{R}^{2} \rightarrow \mathbb{R}$ where $U$ is an open set on which $f_{x}, f_{y}$, $f_{x y}$ and $f_{y x}$ exist. Then if $f_{x y}$ and $f_{y x}$ are continuous at the point $(x, y) \in U$, it follows $f_{x y}(x, y)=f_{y x}(x, y)$.

Proof: Since $U$ is open, there exists $r>0$ such that $B((x, y), r) \subseteq U$. Now let $|t|,|s|<$ $r / 2$ and consider

$$
\begin{equation*}
\Delta(s, t) \equiv \frac{1}{s t}\{\overbrace{f(x+t, y+s)-f(x+t, y)}^{h(t)}-\overbrace{(f(x, y+s)-f(x, y))}^{h(0)}\} \tag{4.8}
\end{equation*}
$$

Note that $(x+t, y+s) \in U$ because

$$
|(x+t, y+s)-(x, y)|=|(t, s)|=\left(t^{2}+s^{2}\right)^{1 / 2} \leq\left(\frac{r^{2}}{4}+\frac{r^{2}}{4}\right)^{1 / 2}=\frac{r}{\sqrt{2}}<r
$$

As implied above, $h(t) \equiv f(x+t, y+s)-f(x+t, y)$. Then, by the mean value theorem from calculus and the (one variable) chain rule,

$$
\Delta(s, t)=\frac{1}{s t}(h(t)-h(0))=\frac{1}{s t} h^{\prime}(\alpha t) t=\frac{1}{s}\left(f_{x}(x+\alpha t, y+s)-f_{x}(x+\alpha t, y)\right)
$$

for some $\alpha \in(0,1)$. Applying the mean value theorem again,

$$
\Delta(s, t)=f_{x y}(x+\alpha t, y+\beta s)
$$

where $\alpha, \beta \in(0,1)$.
If the terms $f(x+t, y)$ and $f(x, y+s)$ are interchanged in $4.8, \Delta(s, t)$ is also unchanged and the above argument shows there exist $\gamma, \delta \in(0,1)$ such that

$$
\Delta(s, t)=f_{y x}(x+\gamma t, y+\delta s)
$$

Letting $(s, t) \rightarrow(0,0)$ and using the continuity of $f_{x y}$ and $f_{y x}$ at $(x, y)$,

$$
\lim _{(s, t) \rightarrow(0,0)} \Delta(s, t)=f_{x y}(x, y)=f_{y x}(x, y) .
$$

The following is obtained from the above by simply fixing all the variables except for the two of interest.

Corollary 4.6.2 Suppose $U$ is an open subset of $\mathbb{R}^{n}$ and $f: U \rightarrow \mathbb{R}$ has the property that for two indices, $k, l, f_{x_{k}}, f_{x_{l}}, f_{x_{l} x_{k}}$, and $f_{x_{k} x_{l}}$ exist on $U$ and $f_{x_{k} x_{l}}$ and $f_{x_{l} x_{k}}$ are both continuous at $\mathbf{x} \in U$. Then $f_{x_{k} x_{l}}(\mathbf{x})=f_{x_{l} x_{k}}(\mathbf{x})$.

It is necessary to assume the mixed partial derivatives are continuous in order to assert they are equal. The following is a well known example [4].

Example 4.6.3 Let

$$
f(x, y)=\left\{\begin{array}{l}
\frac{x y\left(x^{2}-y^{2}\right)}{x^{2}+y^{2}} \text { if }(x, y) \neq(0,0) \\
0 \text { if }(x, y)=(0,0)
\end{array}\right.
$$

From the definition of partial derivatives it follows immediately from the definition that $f_{x}(0,0)=f_{y}(0,0)=0$. Using the standard rules of differentiation, for $(x, y) \neq(0,0)$,

$$
f_{x}=y \frac{x^{4}-y^{4}+4 x^{2} y^{2}}{\left(x^{2}+y^{2}\right)^{2}}, f_{y}=x \frac{x^{4}-y^{4}-4 x^{2} y^{2}}{\left(x^{2}+y^{2}\right)^{2}}
$$

Now

$$
f_{x y}(0,0) \equiv \lim _{y \rightarrow 0} \frac{f_{x}(0, y)-f_{x}(0,0)}{y}=\lim _{y \rightarrow 0} \frac{-y^{4}}{\left(y^{2}\right)^{2}}=-1
$$

while

$$
f_{y x}(0,0) \equiv \lim _{x \rightarrow 0} \frac{f_{y}(x, 0)-f_{y}(0,0)}{x}=\lim _{x \rightarrow 0} \frac{x^{4}}{\left(x^{2}\right)^{2}}=1
$$

showing that although the mixed partial derivatives do exist at $(0,0)$, they are not equal there.

Here is a picture of the graph of this function. It looks innocuous but isn't.


### 4.7 A Cofactor Identity

Lemma 4.7.1 Suppose $\operatorname{det}(A)=0$. Then for all sufficiently small nonzero $\varepsilon$, it follows that $\operatorname{det}(A+\varepsilon I) \neq 0$.

Proof: Let $\operatorname{det}(\lambda I-A)=\lambda^{p}+a_{1} \lambda^{p-1}+\cdots+a_{p-1} \lambda+a_{p}$. First suppose $A$ is a $p \times p$ matrix. If $\operatorname{det}(A) \neq 0$, this will still be true for all $\varepsilon$ small enough. Now suppose that $\operatorname{det}(A)=0$. Thus, the constant term of $\operatorname{det}(\lambda I-A)$ is 0 . Consider $\varepsilon I+A \equiv A_{\varepsilon}$ for small real $\varepsilon$. The characteristic polynomial of $A_{\varepsilon}$ is

$$
\operatorname{det}\left(\lambda I-A_{\varepsilon}\right)=\operatorname{det}((\lambda-\varepsilon) I-A)
$$

This is of the form

$$
(\lambda-\varepsilon)^{p}+a_{1}(\lambda-\varepsilon)^{p-1}+\cdots+(\lambda-\varepsilon)^{m} a_{m}
$$

where the $a_{j}$ are the coefficients in the characteristic polynomial for $A$ and $a_{k}=0$ for $k>m, a_{m} \neq 0$. The constant term of this polynomial in $\lambda$ must be nonzero for all $\varepsilon$ small enough because it is of the form

$$
(-1)^{m} \varepsilon^{m} a_{m}+(\text { higher order terms in } \varepsilon)=\varepsilon^{m}\left[a_{m}(-1)^{m}+\varepsilon C(\varepsilon)\right]
$$

which is nonzero for all positive but very small $\varepsilon$. Thus $\varepsilon I+A$ is invertible for all $\varepsilon$ small enough but nonzero.

Recall that for $A$ an $p \times p$ matrix, $\operatorname{cof}(A)_{i j}$ is the determinant of the matrix which results from deleting the $i^{t h}$ row and the $j^{t h}$ column and multiplying by $(-1)^{i+j}$. In the proof and in what follows, I am using $D \mathbf{g}$ to equal the matrix of the linear transformation $D \mathbf{g}$ taken with respect to the usual basis on $\mathbb{R}^{p}$. Thus $(D \mathbf{g})_{i j}=\partial g_{i} / \partial x_{j}$ where $\mathbf{g}=\sum_{i} g_{i} \mathbf{e}_{i}$ for the $\mathbf{e}_{i}$ the standard basis vectors.

Lemma 4.7.2 Let $\mathbf{g}: U \rightarrow \mathbb{R}^{p}$ be $C^{2}$ where $U$ is an open subset of $\mathbb{R}^{p}$. Then

$$
\sum_{j=1}^{p} \operatorname{cof}(D \mathbf{g})_{i j, j}=0
$$

where here $(D \mathbf{g})_{i j} \equiv g_{i, j} \equiv \frac{\partial g_{i}}{\partial x_{j}}$. Also, $\operatorname{cof}(D \mathbf{g})_{i j}=\frac{\partial \operatorname{det}(D \mathbf{g})}{\partial g_{i, j}}$.
Proof: From the cofactor expansion theorem,

$$
\begin{equation*}
\delta_{k j} \operatorname{det}(D \mathbf{g})=\sum_{i=1}^{p} g_{i, k} \operatorname{cof}(D \mathbf{g})_{i j} \tag{4.9}
\end{equation*}
$$

This is because if $k \neq j$, that on the right is the cofactor expansion of a determinant with two equal columns while if $k=j$, it is just the cofactor expansion of the determinant. In particular,

$$
\begin{equation*}
\frac{\partial \operatorname{det}(D \mathbf{g})}{\partial g_{i, j}}=\operatorname{cof}(D \mathbf{g})_{i j} \tag{4.10}
\end{equation*}
$$

which shows the last claim of the lemma. Assume that $D \mathbf{g}(\mathbf{x})$ is invertible to begin with. Differentiate 4.9 with respect to $x_{j}$ and sum on $j$ using the chain rule in Proposition 4.4.1. Note $\operatorname{det} D \mathbf{g}$ is a function of the $g_{r, s}$ which are functions of the $x_{k}$. This yields

$$
\sum_{r, s, j} \delta_{k j} \frac{\partial(\operatorname{det} D \mathbf{g})}{\partial g_{r, s}} g_{r, s j}=\sum_{i j} g_{i, k j}(\operatorname{cof}(D \mathbf{g}))_{i j}+\sum_{i j} g_{i, k} \operatorname{cof}(D \mathbf{g})_{i j, j}
$$

Hence, using $\boldsymbol{\delta}_{k j}=0$ if $j \neq k$ and 4.10,

$$
\sum_{r s}(\operatorname{cof}(D \mathbf{g}))_{r s} g_{r, s k}=\sum_{r s} g_{r, k s}(\operatorname{cof}(D \mathbf{g}))_{r s}+\sum_{i j} g_{i, k} \operatorname{cof}(D \mathbf{g})_{i j, j} .
$$

Subtracting the first sum on the right from both sides and using the equality of mixed partials,

$$
\sum_{i} g_{i, k}\left(\sum_{j}(\operatorname{cof}(D \mathbf{g}))_{i j, j}\right)=0 .
$$

Since it is assumed $D \mathbf{g}$ is invertible, this shows $\sum_{j}(\operatorname{cof}(D \mathbf{g}))_{i j, j}=0$. If $\operatorname{det}(D \mathbf{g})=0$, use Lemma 4.7.1 to let

$$
\mathbf{g}_{k}(\mathbf{x})=\mathbf{g}(\mathbf{x})+\varepsilon_{k} \mathbf{x}
$$

where $\varepsilon_{k} \rightarrow 0$ and $\operatorname{det}\left(D \mathbf{g}+\varepsilon_{k} I\right) \equiv \operatorname{det}\left(D \mathbf{g}_{k}\right) \neq 0$. Then

$$
\sum_{j}(\operatorname{cof}(D \mathbf{g}))_{i j, j}=\lim _{k \rightarrow \infty} \sum_{j}\left(\operatorname{cof}\left(D \mathbf{g}_{k}\right)\right)_{i j, j}=0
$$

### 4.8 Implicit Function Theorem

The implicit function theorem is one of the greatest theorems in mathematics. There are many versions of this theorem which are of far greater generality than the one given here. The proof given here is like one found in one of Caratheodory's books on the calculus of variations. It is not as elegant as some of the others which are based on a contraction mapping principle but it may be shorter and is based on more elementary ideas. For a more elegant proof which generalizes better see my book Real and Abstract Analysis. The proof given here is based on a mean value theorem in the following lemma.
Lemma 4.8.1 Let $U$ be an open set in $\mathbb{R}^{p}$ which contains the line segment $t \rightarrow \mathbf{y}+$ $t(\mathbf{z}-\mathbf{y})$ for $t \in[0,1]$ and let $f: U \rightarrow \mathbb{R}$ be differentiable at $\mathbf{y}+t(\mathbf{z}-\mathbf{y})$ for $t \in(0,1)$ and continuous for $t \in[0,1]$. Then there exists $\mathbf{x}$ on this line segment such that $f(\mathbf{z})-f(\mathbf{y})=$ $D f(\mathbf{x})(\mathbf{z}-\mathbf{y})$.

Proof: Let $h(t) \equiv f(\mathbf{y}+t(\mathbf{z}-\mathbf{y}))$ for $t \in[0,1]$. Then $h$ is continuous on $[0,1]$ and has a derivative, $h^{\prime}(t)=D f(\mathbf{y}+t(\mathbf{z}-\mathbf{y}))(\mathbf{z}-\mathbf{y})$, this by the chain rule. Then by the mean value theorem of one variable calculus, there exists $t \in(0,1)$ such that

$$
f(\mathbf{z})-f(\mathbf{y})=h(1)-h(0)=h^{\prime}(t)=D f(\mathbf{y}+t(\mathbf{z}-\mathbf{y}))(\mathbf{z}-\mathbf{y})
$$

and we let $\mathbf{x}=\mathbf{y}+t(\mathbf{z}-\mathbf{y})$ for this $t$.
Also of use is the following lemma.
Lemma 4.8.2 Let $A$ be an $m \times n$ matrix and suppose that for all $i, j,\left|A_{i j}\right| \leq C$. Then the operator norm satisfies $\|A\| \leq C m n$.

Proof: Note that if $\mathbf{z}$ is a vector, $|\mathbf{z}|=\sup _{|\mathbf{y}| \leq 1}(\mathbf{z}, \mathbf{y})$. Indeed, for $|\mathbf{y}| \leq 1$, the right side is no more than $|\mathbf{z}|$ thanks to the Cauchy Schwarz inequality and this can be achieved by letting $\mathbf{y}=\mathbf{z} /|\mathbf{z}|$.

$$
\begin{aligned}
\|A\| & \equiv \sup _{|\mathbf{x}| \leq 1}|A \mathbf{x}|=\sup _{|\mathbf{x}| \leq 1|\mathbf{y}| \leq 1}|(A \mathbf{x}, \mathbf{y})|=\sup _{|\mathbf{x}| \leq 1|\mathbf{y}| \leq 1} \sup _{i}\left|\sum_{i} \sum_{j} A_{i j} x_{j} y_{i}\right| \\
& \leq \sup _{|\mathbf{x}| \leq 1|\mathbf{y}| \leq 1} \sup _{i} \sum_{j} C\left|x_{j}\right|\left|y_{i}\right| \leq C \sum_{i} \sum_{j}|\mathbf{x}||\mathbf{y}|=\text { Cmn. }
\end{aligned}
$$

Definition 4.8.3 Suppose $U$ is an open set in $\mathbb{R}^{n} \times \mathbb{R}^{m}$ and $(\mathbf{x}, \mathbf{y})$ will denote $a$ typical point of $\mathbb{R}^{n} \times \mathbb{R}^{m}$ with $\mathbf{x} \in \mathbb{R}^{n}$ and $\mathbf{y} \in \mathbb{R}^{m}$. Let $\mathbf{f}: U \rightarrow \mathbb{R}^{p}$ be in $C^{1}(U)$ meaning that all partial derivatives exist and are continuous. Then define

$$
\begin{aligned}
D_{1} \mathbf{f}(\mathbf{x}, \mathbf{y}) & \equiv\left(\begin{array}{ccc}
f_{1, x_{1}}(\mathbf{x}, \mathbf{y}) & \cdots & f_{1, x_{n}}(\mathbf{x}, \mathbf{y}) \\
\vdots & & \vdots \\
f_{p, x_{1}}(\mathbf{x}, \mathbf{y}) & \cdots & f_{p, x_{n}}(\mathbf{x}, \mathbf{y})
\end{array}\right), \\
D_{2} \mathbf{f}(\mathbf{x}, \mathbf{y}) & \equiv\left(\begin{array}{ccc}
f_{1, y_{1}}(\mathbf{x}, \mathbf{y}) & \cdots & f_{1, y_{m}}(\mathbf{x}, \mathbf{y}) \\
\vdots & & \vdots \\
f_{p, y_{1}}(\mathbf{x}, \mathbf{y}) & \cdots & f_{p, y_{m}}(\mathbf{x}, \mathbf{y})
\end{array}\right) .
\end{aligned}
$$

Definition 4.8.4 Let $\delta, \eta>0$ satisfy: $\overline{B\left(\mathbf{x}_{0}, \boldsymbol{\delta}\right)} \times \overline{B\left(\mathbf{y}_{0}, \eta\right)} \subseteq U$ where $\mathbf{f}: U \subseteq \mathbb{R}^{n} \times$ $\mathbb{R}^{m} \rightarrow \mathbb{R}^{p}$ is given as

$$
\mathbf{f}(\mathbf{x}, \mathbf{y})=\left(\begin{array}{c}
f_{1}(\mathbf{x}, \mathbf{y}) \\
f_{2}(\mathbf{x}, \mathbf{y}) \\
\vdots \\
f_{p}(\mathbf{x}, \mathbf{y})
\end{array}\right)
$$

and for $\left(\begin{array}{lll}\mathbf{x}^{1} & \cdots & \mathbf{x}^{n}\end{array}\right) \in \overline{B\left(\mathbf{x}_{0}, \boldsymbol{\delta}\right)}{ }^{p}$ and $\mathbf{y} \in B\left(\mathbf{y}_{0}, \hat{\eta}\right)$ define

$$
J\left(\mathbf{x}^{1}, \cdots, \mathbf{x}^{p}, \mathbf{y}\right) \equiv\left(\begin{array}{ccc}
f_{1, x_{1}}\left(\mathbf{x}^{1}, \mathbf{y}\right) & \cdots & f_{1, x_{n}}\left(\mathbf{x}^{1}, \mathbf{y}\right)  \tag{*}\\
\vdots & & \vdots \\
f_{p, x_{1}}\left(\mathbf{x}^{p}, \mathbf{y}\right) & \cdots & f_{p, x_{n}}\left(\mathbf{x}^{p}, \mathbf{y}\right)
\end{array}\right) .
$$

Thus, its $t^{\text {th }}$ row is $D_{1} f_{i}\left(\mathbf{x}^{i}, \mathbf{y}\right)$. Let $K, r$ be constants.
By Lemma 4.8.1, and $(\mathbf{x}, \mathbf{y}) \in B\left(\mathbf{x}_{0}, \boldsymbol{\delta}\right) \times B\left(\mathbf{y}_{0}, \eta\right) \subseteq U$, and $\mathbf{h}, \mathbf{k}$ sufficiently small, there are $\mathbf{x}^{i}$ on the line segment between $\mathbf{x}$ and $\mathbf{x}+\mathbf{h}$ such that

$$
\begin{gather*}
\mathbf{f ( \mathbf { x } + \mathbf { h } , \mathbf { y } + \mathbf { k } ) - \mathbf { f } ( \mathbf { x } , \mathbf { y } ) = \mathbf { f } ( \mathbf { x } + \mathbf { h } , \mathbf { y } + \mathbf { k } ) - \mathbf { f } ( \mathbf { x } , \mathbf { y } + \mathbf { k } ) + \mathbf { f } ( \mathbf { x } , \mathbf { y } + \mathbf { k } ) - \mathbf { f } ( \mathbf { x } , \mathbf { y } )} \\
=J\left(\mathbf{x}^{1}, \cdots, \mathbf{x}^{p}, \mathbf{y}+\mathbf{k}\right) \mathbf{h}+D_{2} \mathbf{f}(\mathbf{x}, \mathbf{y}) \mathbf{k}+\mathbf{o}(\mathbf{k})  \tag{4.11}\\
=D_{1} \mathbf{f}(\mathbf{x}, \mathbf{y}) \mathbf{h}+D_{2} \mathbf{f}(\mathbf{x}, \mathbf{y}) \mathbf{k}+\mathbf{o}(\mathbf{k})+\left(J\left(\mathbf{x}^{1}, \cdots, \mathbf{x}^{p}, \mathbf{y}+\mathbf{k}\right)-D_{\mathbf{1}} \mathbf{f}(\mathbf{x}, \mathbf{y})\right) \mathbf{h} \tag{4.12}
\end{gather*}
$$

Now by continuity of the partial derivatives, if $\sqrt{|\mathbf{h}|^{2}+|\mathbf{k}|^{2}}$ is sufficiently small,

$$
\left\|J\left(\mathbf{x}^{1}, \cdots, \mathbf{x}^{p}, \mathbf{y}+\mathbf{k}\right)-D_{1} \mathbf{f}(\mathbf{x}, \mathbf{y})\right\|<\varepsilon
$$

and so

$$
\frac{\left|\left(J\left(\mathbf{x}^{1}, \cdots, \mathbf{x}^{p}, \mathbf{y}+\mathbf{k}\right)-D_{1} \mathbf{f}(\mathbf{x}, \mathbf{y})\right) \mathbf{h}\right|}{\sqrt{|\mathbf{h}|^{2}+|\mathbf{k}|^{2}}} \leq \frac{\varepsilon|\mathbf{h}|}{\sqrt{|\mathbf{h}|^{2}+|\mathbf{k}|^{2}}} \leq \varepsilon
$$

and so the last term in 4.12 is $\mathbf{o}((\mathbf{h}, \mathbf{k}))$. Thus $\mathbf{f}(\mathbf{x}+\mathbf{h}, \mathbf{y}+\mathbf{k})-\mathbf{f}(\mathbf{x}, \mathbf{y})$ is of the form

$$
D_{1} \mathbf{f}(\mathbf{x}, \mathbf{y}) \mathbf{h}+D_{2} \mathbf{f}(\mathbf{x}, \mathbf{y}) \mathbf{k}+\mathbf{o}(\mathbf{h}, \mathbf{k})
$$

which shows that $\mathbf{f}$ is differentiable and its derivative is the $p \times(n+m)$ matrix,

$$
\left(D_{1} \mathbf{f}(\mathbf{x}, \mathbf{y}) \quad D_{2} \mathbf{f}(\mathbf{x}, \mathbf{y})\right) .
$$

Proposition 4.8.5 Suppose $g: \overline{B\left(\mathbf{x}_{0}, \delta\right)} \times \overline{B\left(\mathbf{y}_{0}, \eta_{0}\right)} \rightarrow[0, \infty)$ is continuous and

$$
g\left(\mathbf{x}_{0}, \mathbf{y}_{0}\right)=0
$$

and if $\mathbf{x} \neq \mathbf{x}_{0}, g\left(\mathbf{x}, \mathbf{y}_{0}\right)>0$. Then there exists $\eta<\eta_{0}$ such that if $\mathbf{y} \in B\left(\mathbf{y}_{0}, \eta\right)$, then the function $\mathbf{x} \rightarrow g(\mathbf{x}, \mathbf{y})$ achieves its minimum on the open set $B\left(\mathbf{x}_{0}, \boldsymbol{\delta}\right)$.

Proof: If not, then there is a sequence $\mathbf{y}_{k} \rightarrow \mathbf{y}_{0}$ but the minimum of $\mathbf{x} \rightarrow g\left(\mathbf{x}, \mathbf{y}_{k}\right)$ for $\mathbf{x} \in \overline{B\left(\mathbf{x}_{0}, \delta\right)}$ happens on $\partial B\left(\mathbf{x}_{0}, \boldsymbol{\delta}\right) \equiv \partial B \equiv\left\{\mathbf{x}:\left|\mathbf{x}-\mathbf{x}_{0}\right|=\delta\right\}$ at $\mathbf{x}_{k}$. Now $\partial B$ is closed and bounded and so compact. Hence there is a subsequence, still denoted with subscript $k$ such that $\mathbf{x}_{k} \rightarrow \mathbf{x} \in \partial B$ and $\mathbf{y}_{k} \rightarrow \mathbf{y}_{0}$. Let

$$
0<2 \varepsilon<\min \left\{g\left(\hat{\mathbf{x}}, \mathbf{y}_{0}\right): \hat{\mathbf{x}} \in \partial B\right\}
$$

Then for $k$ large,

$$
\left|g\left(\mathbf{x}_{k}, \mathbf{y}_{k}\right)-g\left(\mathbf{x}, \mathbf{y}_{0}\right)\right|<\varepsilon,\left|g\left(\mathbf{x}_{k}, \mathbf{y}_{k}\right)-g\left(\mathbf{x}_{k}, \mathbf{y}_{0}\right)\right|<\varepsilon
$$

the second inequality from uniform continuity. Then from these inequalities, for $k$ large,

$$
\begin{aligned}
g\left(\mathbf{x}_{0}, \mathbf{y}_{k}\right) & \geq g\left(\mathbf{x}_{k}, \mathbf{y}_{k}\right)>g\left(\mathbf{x}_{k}, \mathbf{y}_{0}\right)-\varepsilon \\
& >\min \left\{g\left(\hat{\mathbf{x}}, \mathbf{y}_{0}\right): \hat{\mathbf{x}} \in \partial B\right\}-\varepsilon>2 \varepsilon-\varepsilon=\varepsilon
\end{aligned}
$$

Now let $k \rightarrow \infty$ to conclude that $g\left(\mathbf{x}_{0}, \mathbf{y}_{0}\right) \geq \varepsilon$, a contradiction.
Here is the implicit function theorem. It is based on the mean value theorem from one variable calculus, the extreme value theorem from calculus, and the formula for the inverse of a matrix in terms of the transpose of the cofactor matrix divided by the determinant.

Theorem 4.8.6 (implicit function theorem) Suppose $U$ is an open set in $\mathbb{R}^{n} \times \mathbb{R}^{m}$. Let $\mathbf{f}: U \rightarrow \mathbb{R}^{n}$ be in $C^{1}(U)$ and suppose

$$
\begin{equation*}
\mathbf{f}\left(\mathbf{x}_{0}, \mathbf{y}_{0}\right)=\mathbf{0}, D_{1} \mathbf{f}\left(\mathbf{x}_{0}, \mathbf{y}_{0}\right)^{-1} \text { exists. } \tag{4.13}
\end{equation*}
$$

Then there exist positive constants $\delta, \eta$, such that for every $\mathbf{y} \in B\left(\mathbf{y}_{0}, \eta\right)$ there exists a unique $\mathbf{x}(\mathbf{y}) \in B\left(\mathbf{x}_{0}, \delta\right)$ such that

$$
\begin{equation*}
\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})=\mathbf{0} . \tag{4.14}
\end{equation*}
$$

Furthermore, the mapping, $\mathbf{y} \rightarrow \mathbf{x}(\mathbf{y})$ is in $C^{1}\left(B\left(\mathbf{y}_{0}, \eta\right)\right)$.
Proof: Let

$$
\mathbf{f}(\mathbf{x}, \mathbf{y})=\left(\begin{array}{llll}
f_{1}(\mathbf{x}, \mathbf{y}) & f_{2}(\mathbf{x}, \mathbf{y}) & \cdots & f_{n}(\mathbf{x}, \mathbf{y})
\end{array}\right)^{T} .
$$

Define for $\left(\mathbf{x}^{1}, \cdots, \mathbf{x}^{n}\right) \in{\overline{B\left(\mathbf{x}_{0}, \delta\right)}}^{n}$ and $\mathbf{y} \in B\left(\mathbf{y}_{0}, \eta\right)$ the following matrix.

$$
J\left(\mathbf{x}^{1}, \cdots, \mathbf{x}^{n}, \mathbf{y}\right) \equiv\left(\begin{array}{ccc}
f_{1, x_{1}}\left(\mathbf{x}^{1}, \mathbf{y}\right) & \cdots & f_{1, x_{n}}\left(\mathbf{x}^{1}, \mathbf{y}\right)  \tag{*}\\
\vdots & & \vdots \\
f_{n, x_{1}}\left(\mathbf{x}^{n}, \mathbf{y}\right) & \cdots & f_{n, x_{n}}\left(\mathbf{x}^{n}, \mathbf{y}\right)
\end{array}\right)
$$

Then by the assumption of continuity of all the partial derivatives, there exists $r>0$ and $\delta_{0}, \eta_{0}>0$ such that if $\delta \leq \delta_{0}$ and $\eta \leq \eta_{0}$, it follows that for all $\left(\mathbf{x}^{1}, \cdots, \mathbf{x}^{n}\right) \in{\overline{B\left(\mathbf{x}_{0}, \delta\right)}}^{n} \equiv$ $\overline{B\left(\mathbf{x}_{0}, \delta\right)} \times \overline{B\left(\mathbf{x}_{0}, \delta\right)} \times \cdots \times \overline{B\left(\mathbf{x}_{0}, \delta\right)}$, and $\mathbf{y} \in \overline{B\left(\mathbf{y}_{0}, \eta\right)}$,

$$
\begin{equation*}
\operatorname{det} J\left(\mathbf{x}^{1}, \cdots, \mathbf{x}^{n}, \mathbf{y}\right) \notin(-r, r) \tag{4.15}
\end{equation*}
$$

and $\overline{B\left(\mathbf{x}_{0}, \delta_{0}\right)} \times \overline{B\left(\mathbf{y}_{0}, \eta_{0}\right)} \subseteq U$. Therefore, from the formula for the inverse of a matrix and continuity of all entries of the various matrices, there exists a constant $K$ such that all
entries of $J\left(\mathbf{x}^{1}, \cdots, \mathbf{x}^{n}, \mathbf{y}\right), J\left(\mathbf{x}^{1}, \cdots, \mathbf{x}^{n}, \mathbf{y}\right)^{-1}$, and $D_{2} \mathbf{f}(\mathbf{x}, \mathbf{y})$ have absolute value smaller than $K$ on the convex set $\overline{B\left(\mathbf{x}_{0}, \delta\right)}{ }^{n} \times \overline{B\left(\mathbf{y}_{0}, \eta\right)}$ whenever $\delta, \eta$ are sufficiently small. It is always tacitly assumed that these radii are this small.

Next it is shown that for a given $\mathbf{y} \in B\left(\mathbf{y}_{0}, \eta\right), \eta \leq \eta_{0}$, there is at most one $\mathbf{x} \in B\left(\mathbf{x}_{0}, \delta_{0}\right)$ such that $\mathbf{f}(\mathbf{x}, \mathbf{y})=\mathbf{0}$.

Pick $\mathbf{y} \in B\left(\mathbf{y}_{0}, \eta\right)$ and suppose there exist $\mathbf{x}, \mathbf{z} \in \overline{B\left(\mathbf{x}_{0}, \delta\right)}$ such that

$$
\mathbf{f}(\mathbf{x}, \mathbf{y})=\mathbf{f}(\mathbf{z}, \mathbf{y})=\mathbf{0}
$$

Consider $f_{i}$ and let

$$
h(t) \equiv f_{i}(\mathbf{x}+t(\mathbf{z}-\mathbf{x}), \mathbf{y})
$$

Then $h(1)=h(0)$ and so by the mean value theorem, $h^{\prime}\left(t_{i}\right)=0$ for some $t_{i} \in(0,1)$. Therefore, from the chain rule and for this value of $t_{i}$,

$$
\begin{equation*}
h^{\prime}\left(t_{i}\right)=\sum_{j=1}^{n} \frac{\partial}{\partial x_{j}} f_{i}\left(\mathbf{x}+t_{i}(\mathbf{z}-\mathbf{x}), \mathbf{y}\right)\left(z_{j}-x_{j}\right)=0 \tag{4.16}
\end{equation*}
$$

Then denote by $\mathbf{x}^{i}$ the vector, $\mathbf{x}+t_{i}(\mathbf{z}-\mathbf{x})$. It follows from 4.16 that

$$
J\left(\mathbf{x}^{1}, \cdots, \mathbf{x}^{n}, \mathbf{y}\right)(\mathbf{z}-\mathbf{x})=\mathbf{0}
$$

and so from $4.15 \mathbf{z}-\mathbf{x}=\mathbf{0}$. (The matrix, in the above is invertible since its determinant is nonzero.) Now it will be shown that if $\eta$ is chosen sufficiently small, then for all $\mathbf{y} \in$ $B\left(\mathbf{y}_{0}, \eta\right)$, there exists a unique $\mathbf{x}(\mathbf{y}) \in B\left(\mathbf{x}_{0}, \boldsymbol{\delta}\right)$ such that $\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})=\mathbf{0}$.

Claim: If $\eta$ is small enough, then the function, $\mathbf{x} \rightarrow h_{\mathbf{y}}(\mathbf{x}) \equiv|\mathbf{f}(\mathbf{x}, \mathbf{y})|^{2}$ achieves its minimum value on $\overline{B\left(\mathbf{x}_{0}, \delta\right)}$ at a point of $B\left(\mathbf{x}_{0}, \boldsymbol{\delta}\right)$. This is Proposition 4.8.5.

Choose $\eta<\eta_{0}$ and also small enough that the above claim holds and let $\mathbf{x}(\mathbf{y})$ denote a point of $B\left(\mathbf{x}_{0}, \boldsymbol{\delta}\right)$ at which the minimum of $h_{\mathbf{y}}$ on $\overline{B\left(\mathbf{x}_{0}, \delta\right)}$ is achieved. Since $\mathbf{x}(\mathbf{y})$ is an interior point, you can consider $h_{\mathbf{y}}(\mathbf{x}(\mathbf{y})+t \mathbf{v})$ for $|t|$ small and conclude this function of $t$ has a zero derivative at $t=0$. Now

$$
h_{\mathbf{y}}(\mathbf{x}(\mathbf{y})+t \mathbf{v})=\sum_{i=1}^{n} f_{i}^{2}(\mathbf{x}(\mathbf{y})+t \mathbf{v}, \mathbf{y})
$$

and so from the chain rule,

$$
\frac{d}{d t} h_{\mathbf{y}}(\mathbf{x}(\mathbf{y})+t \mathbf{v})=\sum_{i=1}^{n} \sum_{j=1}^{n} 2 f_{i}(\mathbf{x}(\mathbf{y})+t \mathbf{v}, \mathbf{y}) \frac{\partial f_{i}(\mathbf{x}(\mathbf{y})+t \mathbf{v}, \mathbf{y})}{\partial x_{j}} v_{j}
$$

Therefore, letting $t=0$, it is required that for every $\mathbf{v}$,

$$
\sum_{i=1}^{n} \sum_{j=1}^{n} 2 f_{i}(\mathbf{x}(\mathbf{y}), \mathbf{y}) \frac{\partial f_{i}(\mathbf{x}(\mathbf{y}), \mathbf{y})}{\partial x_{j}} v_{j}=0
$$

In terms of matrices this reduces to $0=2 \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})^{T} D_{1} \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) \mathbf{v}$ for every vector $\mathbf{v}$. Therefore, $\mathbf{0}=\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})^{T} D_{1} \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})$. From 4.15, it follows $\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})=\mathbf{0}$. Multiply by $D_{1} \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})^{-1}$ on the right. This proves the existence of the function $\mathbf{y} \rightarrow \mathbf{x}(\mathbf{y})$ such that $\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})=\mathbf{0}$ for all $\mathbf{y} \in B\left(\mathbf{y}_{0}, \eta\right)$.

It remains to verify this function is a $C^{1}$ function. To do this, let $\mathbf{y}_{1}$ and $\mathbf{y}_{2}$ be points of $B\left(\mathbf{y}_{0}, \eta\right)$. Then as before, consider the $i^{\text {th }}$ component of $\mathbf{f}$ and consider the same argument using the mean value theorem to write

$$
\begin{gather*}
0=f_{i}\left(\mathbf{x}\left(\mathbf{y}_{1}\right), \mathbf{y}_{1}\right)-f_{i}\left(\mathbf{x}\left(\mathbf{y}_{2}\right), \mathbf{y}_{2}\right) \\
=f_{i}\left(\mathbf{x}\left(\mathbf{y}_{1}\right), \mathbf{y}_{1}\right)-f_{i}\left(\mathbf{x}\left(\mathbf{y}_{2}\right), \mathbf{y}_{1}\right)+f_{i}\left(\mathbf{x}\left(\mathbf{y}_{2}\right), \mathbf{y}_{1}\right)-f_{i}\left(\mathbf{x}\left(\mathbf{y}_{2}\right), \mathbf{y}_{2}\right) \\
=D_{1} f_{i}\left(\mathbf{x}^{i}, \mathbf{y}_{1}\right)\left(\mathbf{x}\left(\mathbf{y}_{1}\right)-\mathbf{x}\left(\mathbf{y}_{2}\right)\right)+D_{2} f_{i}\left(\mathbf{x}\left(\mathbf{y}_{2}\right), \mathbf{y}^{i}\right)\left(\mathbf{y}_{1}-\mathbf{y}_{2}\right) \tag{4.17}
\end{gather*}
$$

where $\mathbf{y}^{i}$ is a point on the line segment joining $\mathbf{y}_{1}$ and $\mathbf{y}_{2}$ and $\mathbf{x}^{i}$ is a point on the line segment joining $\mathbf{x}\left(\mathbf{y}_{1}\right)$ and $\mathbf{x}\left(\mathbf{y}_{2}\right)$. Thus

$$
\left(\mathbf{x}\left(\mathbf{y}_{1}\right)-\mathbf{x}\left(\mathbf{y}_{2}\right)\right)=-J\left(\mathbf{x}^{1}, \cdots, \mathbf{x}^{n}, \mathbf{y}_{1}\right)^{-1} M\left(\mathbf{y}_{1}-\mathbf{y}_{2}\right)
$$

where $M$ denotes the matrix having the $i^{\text {th }}$ row equal to $D_{2} f_{i}\left(\mathbf{x}\left(\mathbf{y}_{2}\right), \mathbf{y}^{i}\right)$ all entries being bounded by $K$. It follows that

$$
\left|\mathbf{x}\left(\mathbf{y}_{1}\right)-\mathbf{x}\left(\mathbf{y}_{2}\right)\right| \leq K n\left|M\left(\mathbf{y}_{1}-\mathbf{y}_{2}\right)\right| \leq K^{2} n m\left|\mathbf{y}_{1}-\mathbf{y}_{2}\right|
$$

Thus $\mathbf{y} \rightarrow \mathbf{x}(\mathbf{y})$ is continuous near $\mathbf{y}_{0}$.
Now let $\mathbf{y}_{2}=\mathbf{y}, \mathbf{y}_{1}=\mathbf{y}+h \mathbf{e}_{k}$ for small $h$. Then $M$ described above depends on $h$ and

$$
\lim _{h \rightarrow 0} M(h)=D_{2} \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})
$$

thanks to the continuity of $\mathbf{y} \rightarrow \mathbf{x}(\mathbf{y})$ just shown. Also,

$$
\frac{\mathbf{x}\left(\mathbf{y}+h \mathbf{e}_{k}\right)-\mathbf{x}(\mathbf{y})}{h}=-J\left(\mathbf{x}^{1}(h), \cdots, \mathbf{x}^{n}(h), \mathbf{y}+h \mathbf{e}_{k}\right)^{-1} M(h) \mathbf{e}_{k}
$$

Passing to a limit and using the formula for the inverse of a matrix in terms of the cofactor matrix, and the continuity of $\mathbf{y} \rightarrow \mathbf{x}(\mathbf{y})$ shown above, this yields

$$
\frac{\partial \mathbf{x}}{\partial y_{k}}=-D_{1} \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})^{-1} D_{2} f_{i}(\mathbf{x}(\mathbf{y}), \mathbf{y}) \mathbf{e}_{k}
$$

Then continuity of $\mathbf{y} \rightarrow \mathbf{x}(\mathbf{y})$ and the assumed continuity of the partial derivatives of $\mathbf{f}$ shows that each partial derivative of $\mathbf{y} \rightarrow \mathbf{x}(\mathbf{y})$ exists and is continuous.

This theorem implies the inverse function theorem stated next.
Theorem 4.8 .7 (inverse function theorem) Let $\mathbf{x}_{0} \in U$, an open set in $\mathbb{R}^{n}$, and let $\mathbf{f}: U \rightarrow \mathbb{R}^{n}$. Suppose

$$
\begin{equation*}
\mathbf{f} \text { is } C^{1}(U), \text { and } D \mathbf{f}\left(\mathbf{x}_{0}\right)^{-1} \text { exists. } \tag{4.18}
\end{equation*}
$$

Then there exist open sets $W$, and $V$ such that $\mathbf{x}_{0} \in W \subseteq U, \mathbf{f}: W \rightarrow V$ is one to one and onto, $\mathbf{f}^{-1}$ is $C^{1}$.

Proof: Apply the implicit function theorem to the function $\mathbf{F}(\mathbf{x}, \mathbf{y}) \equiv \mathbf{f}(\mathbf{x})-\mathbf{y}$ where $\mathbf{y}_{0} \equiv \mathbf{f}\left(\mathbf{x}_{0}\right)$. Thus the function $\mathbf{y} \rightarrow \mathbf{x}(\mathbf{y})$ defined in that theorem is $\mathbf{f}^{-1}$ and there is $B\left(\mathbf{y}_{0}, \eta\right)$ where this function is defined. Now let $W \equiv \mathbf{f}^{-1}\left(B\left(\mathbf{y}_{0}, \eta\right)\right)$ and $V \equiv B\left(\mathbf{y}_{0}, \eta\right)$.

### 4.9 More Continuous Partial Derivatives

The implicit function theorem will now be improved slightly. If $\mathbf{f}$ is $C^{k}$, it follows that the function which is implicitly defined is also $C^{k}$, not just $C^{1}$, meaning all mixed partial derivatives of $\mathbf{f}$ up to order $k$ are continuous. Since the inverse function theorem comes as a case of the implicit function theorem, this shows that the inverse function also inherits the property of being $C^{k}$. First some notation is convenient. Let $\alpha=\left(\alpha_{1}, \cdots, \alpha_{n}\right)$ where each $\alpha_{i}$ is a nonnegative integer. Then letting $|\alpha|=\sum_{i} \alpha_{i}$,

$$
D^{\alpha} \mathbf{f}(\mathbf{x}) \equiv \frac{\partial^{|\alpha|} \mathbf{f}}{\partial^{\alpha_{1}} \partial^{\alpha_{2} \ldots \partial^{\alpha_{n}}}}(\mathbf{x}), D^{0} \mathbf{f}(\mathbf{x}) \equiv \mathbf{f}(\mathbf{x})
$$

The symbol on the right means to take the $\alpha_{n}$ partial derivative with respect to $x_{n}$, then the $\alpha_{n-1}$ partial derivative with respect to $x_{n-1}$ of what you just got and so on till you take the $\alpha_{1}$ partial derivative with respect to $x_{1}$. The idea is to show that all mixed partial derivatives such that $|\alpha| \leq k$ exist and are continuous.

Theorem 4.9.1 (implicit function theorem) Suppose $U$ is an open set in $\mathbb{F}^{n} \times \mathbb{F}^{m}$. Let $\mathbf{f}: U \rightarrow \mathbb{F}^{n}$ be in $C^{k}(U)$ and suppose

$$
\begin{equation*}
\mathbf{f}\left(\mathbf{x}_{0}, \mathbf{y}_{0}\right)=\mathbf{0}, D_{1} \mathbf{f}\left(\mathbf{x}_{0}, \mathbf{y}_{0}\right)^{-1} \in \mathscr{L}\left(\mathbb{F}^{n}, \mathbb{F}^{n}\right) \tag{4.19}
\end{equation*}
$$

Then there exist positive constants $\delta, \eta$, such that for every $\mathbf{y} \in B\left(\mathbf{y}_{0}, \eta\right)$ there exists a unique $\mathbf{x}(\mathbf{y}) \in B\left(\mathbf{x}_{0}, \delta\right)$ such that

$$
\begin{equation*}
\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})=\mathbf{0} . \tag{4.20}
\end{equation*}
$$

Furthermore, the mapping $\mathbf{y} \rightarrow \mathbf{x}(\mathbf{y})$ is in $C^{k}\left(B\left(\mathbf{y}_{0}, \eta\right)\right)$.
Proof: From the implicit function theorem $\mathbf{y} \rightarrow \mathbf{x}(\mathbf{y})$ is $C^{1}$. It remains to show that it is $C^{k}$ for $k>1$ assuming that $\mathbf{f}$ is $C^{k}$. From 4.20

$$
\frac{\partial \mathbf{x}}{\partial y^{l}}=-D_{1} \mathbf{f}(\mathbf{x}, \mathbf{y})^{-1} \frac{\partial \mathbf{f}}{\partial y^{l}}
$$

By the formula for the inverse in terms of cofactors, if $\mathbf{f}$ is $C^{2}$, one can use the chain rule to take another continuous derivative. Thus, the following formula holds for $q=1$ and $|\alpha|=q$.

$$
\begin{equation*}
D^{\alpha} \mathbf{x}(\mathbf{y})=\sum_{|\beta| \leq q} M_{\beta}(\mathbf{x}, \mathbf{y}) D^{\beta} \mathbf{f}(\mathbf{x}, \mathbf{y}) \tag{4.21}
\end{equation*}
$$

where $M_{\beta}$ is a matrix whose entries are differentiable functions of $D^{\gamma} \mathbf{x}$ for $|\gamma|<q$ and $D^{\tau} \mathbf{f}(\mathbf{x}, \mathbf{y})$ for $|\tau| \leq q$. This follows easily from the description of $D_{1} \mathbf{f}(\mathbf{x}, \mathbf{y})^{-1}$ in terms of the cofactor matrix and the determinant of $D_{1} \mathbf{f}(\mathbf{x}, \mathbf{y})$. Suppose 4.21 holds for $|\alpha|=q<k$. Then by induction, this yields $\mathbf{x}$ is $C^{q}$. Then

$$
\frac{\partial D^{\alpha} \mathbf{x}(\mathbf{y})}{\partial y^{p}}=\sum_{|\beta| \leq|\alpha|} \frac{\partial M_{\beta}(\mathbf{x}, \mathbf{y})}{\partial y^{p}} D^{\beta} \mathbf{f}(\mathbf{x}, \mathbf{y})+M_{\beta}(\mathbf{x}, \mathbf{y}) \frac{\partial D^{\beta} \mathbf{f}(\mathbf{x}, \mathbf{y})}{\partial y^{p}}
$$

By the chain rule $\frac{\partial M_{\beta}(\mathbf{x}, \mathbf{y})}{\partial y^{p}}$ is a matrix whose entries are differentiable functions of $D^{\tau} \mathbf{f}(\mathbf{x}, \mathbf{y})$ for $|\tau| \leq q+1$ and $D^{\gamma} \mathbf{x}$ for $|\gamma|<q+1$. It follows, since $y^{p}$ was arbitrary, that for any
$|\alpha|=q+1$, a formula like 4.21 holds with $q$ being replaced by $q+1$. Continuing this way, $\mathbf{x}$ is $C^{k}$.

As a simple corollary, this yields the inverse function theorem. You just let $\mathbf{F}(\mathbf{x}, \mathbf{y})=$ $\mathbf{y}-\mathbf{f}(\mathbf{x})$ and apply the implicit function theorem.

Theorem 4.9.2 (inverse function theorem) Let $\mathbf{x}_{0} \in U \subseteq \mathbb{F}^{n}$ and let $\mathbf{f}: U \rightarrow \mathbb{F}^{n}$. Suppose for $k$ a positive integer, $\mathbf{f}$ is $C^{k}(U)$, and $D \mathbf{f}\left(\mathbf{x}_{0}\right)^{-1} \in \mathscr{L}\left(\mathbb{F}^{n}, \mathbb{F}^{n}\right)$. Then there exist open sets $W$, and $V$ such that $\mathbf{x}_{0} \in W \subseteq U, \mathbf{f}: W \rightarrow V$ is one to one and onto, $\mathbf{f}^{-1}$ is $C^{k}$.

### 4.10 Normed Linear Space

The implicit function theorem and inverse function theorem continue to hold if $\mathbb{R}^{n}$ and $\mathbb{R}^{m}$ are replaced by finite dimensional normed linear spaces $X, Y$ respectively of dimension $n$ and $m$.

Theorem 4.10.1 (implicit function theorem) Suppose $U$ is an open set in $X \times Y$ where $X, Y$ are normed linear space of dimension $n, m$ and suppose $\mathbf{f}: U \rightarrow Z$ be in $C^{k}(U)$ where $Z$ is an $n$ dimensional normed linear space. Suppose also

$$
\begin{equation*}
f\left(x_{0}, y_{0}\right)=0, D_{1} f\left(x_{0}, y_{0}\right)^{-1} \text { exists. } \tag{4.22}
\end{equation*}
$$

Then there exist positive constants $\delta, \eta$, such that for every $y \in B\left(y_{0}, \eta\right)$ there exists a unique $x(y) \in B\left(x_{0}, \boldsymbol{\delta}\right)$ such that

$$
\begin{equation*}
f(x(y), y)=0 . \tag{4.23}
\end{equation*}
$$

Furthermore, the mapping, $y \rightarrow x(y)$ is in $C^{k}\left(B\left(y_{0}, \eta\right)\right)$.
Proof: Denote the coordinate maps for $X, Y, Z$ in terms of bases for these spaces by $\theta_{X}, \theta_{Y}, \theta_{Z}$. These are all linear maps and so, since we are in finite dimensions, they are each $C^{k}$ for every positive integer $k$ with respect to any norm on $\mathbb{R}^{n}, \mathbb{R}^{m}$ thanks to Theorem 2.7.4 on equivalence of norms and the same is true of their inverses. Denote by $\mathbf{x}, \mathbf{y}, \mathbf{z}$ the coordinate vectors for $x, y, z \in X, Y, Z$ respectively. Let $\mathbf{f}=\theta_{Z} f$ and note that the conditions for the implicit function theorem, Theorem 4.9.1 for $\mathbf{f}\left(\mathbf{x}_{0}, \mathbf{y}_{0}\right)=\mathbf{0}$ all hold and so this proves the theorem. Since we are in finite dimensions, $D_{1} f\left(x_{0}, y_{0}\right)^{-1}$ exists if $D_{1} f\left(x_{0}, y_{0}\right)$ is one to one which implies $D_{1} \mathbf{f}\left(\mathbf{x}_{0}, \mathbf{y}_{0}\right)^{-1}$ exists.

Of course the inverse function theorem follows from this in the case of normed linear spaces. This also illustrates how you can always reduce to $\mathbb{R}^{p}$ by doing everything in terms of coordinates.

### 4.11 Taylor Approximations

First recall the following one variable calculus theorem. It is in my on line book "Calculus of One and Many Variables" or in any elementary Calculus book. See Problem 16 below on Page 110.

Theorem 4.11.1 Let $h:(-\delta, 1+\delta) \rightarrow \mathbb{R}$ have $m+1$ derivatives. Then there exists $t \in(0,1)$ such that

$$
h(1)=h(0)+\sum_{k=1}^{m} \frac{h^{(k)}(0)}{k!}+\frac{h^{(m+1)}(t)}{(m+1)!} .
$$

Now suppose $U$ is an open set in $\mathbb{R}^{p}$ and $f: U \rightarrow \mathbb{R}$ is $C^{m+1}$ with $\mathbf{x}_{0} \in U$. For $\mathbf{x} \in$ $B\left(\mathbf{x}_{0}, r\right) \subseteq U$, let $h(t)=f\left(\mathbf{x}_{0}+t\left(\mathbf{x}-\mathbf{x}_{0}\right)\right), t \in(0,1)$. Then

$$
\begin{aligned}
h^{\prime}(t) & =\sum_{i} \frac{\partial f\left(\mathbf{x}_{0}+t\left(\mathbf{x}-\mathbf{x}_{0}\right)\right)}{\partial x_{i}}\left(x_{i}-x_{0 i}\right) \\
h^{\prime \prime}(t) & =\sum_{i_{1}, i_{2}} \frac{\partial^{2} f}{\partial x_{i_{1}} \partial x_{i_{2}}}\left(x_{i_{1}}-x_{0 i_{1}}\right)\left(x_{i_{2}}-x_{0 i_{2}}\right)
\end{aligned}
$$

and continuing this way,

$$
\begin{equation*}
h^{(k)}(t)=\sum_{i_{1}, \cdots, i_{k}} \frac{\partial^{k} f}{\partial x_{i_{1}} \partial x_{i_{2}} \cdots \partial x_{i_{k}}} \prod_{j=1}^{k}\left(x_{i_{j}}-x_{0 i_{j}}\right) \tag{4.24}
\end{equation*}
$$

Then the Taylor approximation is of the form $h(1)=f(\mathbf{x})=$

$$
\begin{align*}
& f\left(\mathbf{x}_{0}\right)+\sum_{k=1}^{m} \frac{1}{k!} \sum_{i_{1}, \cdots, i_{k}} \frac{\partial^{k} f\left(\mathbf{x}_{0}\right)}{\partial x_{i_{1}} \partial x_{i_{2}} \cdots \partial x_{i_{k}}} \prod_{j=1}^{k}\left(x_{i_{j}}-x_{0 i_{j}}\right) \\
& +\frac{1}{(m+1)!} \sum_{i_{1}, \cdots, i_{m+1}} \frac{\partial^{m+1} f\left(\mathbf{x}_{0}+t\left(\mathbf{x}-\mathbf{x}_{0}\right)\right)}{\partial x_{i_{1}} \partial x_{i_{2}} \cdots \partial x_{i_{m+1}}} \prod_{j=1}^{m+1}\left(x_{i_{j}}-x_{0 i_{j}}\right) \tag{4.25}
\end{align*}
$$

The last term being the remainder with $t \in(0,1)$. Thus, if the $(m+1)^{s t}$ partial derivatives are all bounded, this shows that if $\left\|\mathbf{x}-\mathbf{x}_{0}\right\|$ is sufficiently small, then the difference between $f(\mathbf{x})$ and that series on the right in 4.25 other than the remainder term will also be very small.

### 4.12 Exercises

1. For $(x, y) \neq(0,0)$, let $f(x, y)=\frac{x y^{4}}{x^{2}+y^{8}}$. Show that this function has a limit as $(x, y) \rightarrow$ $(0,0)$ for $(x, y)$ on an arbitrary straight line through $(0,0)$. Next show that this function fails to have a limit at $(0,0)$.
2. Here are some scalar valued functions of several variables. Determine which of these functions are $o(\mathbf{v})$. Here $\mathbf{v}$ is a vector in $\mathbb{R}^{n}, \mathbf{v}=\left(v_{1}, \cdots, v_{n}\right)$.
(a) $v_{1} v_{2}$
(e) $v_{1}\left(v_{1}+v_{2}+x v_{3}\right)$
(b) $v_{2} \sin \left(v_{1}\right)$
(f) $\left(e^{v_{1}}-1-v_{1}\right)$
(c) $v_{1}^{2}+v_{2}$
(g) $(\mathbf{x} \cdot \mathbf{v})|\mathbf{v}|$
3. Here is a function of two variables. $f(x, y)=x^{2} y+x^{2}$. Find $D f(x, y)$ directly from the definition. Recall this should be a linear transformation which results from multiplication by a $1 \times 2$ matrix. Find this matrix.
4. Let $\mathbf{f}(x, y)=\binom{x^{2}+y}{y^{2}}$. Compute the derivative directly from the definition. This should be the linear transformation which results from multiplying by a $2 \times 2$ matrix. Find this matrix.
5. Find $f_{x}, f_{y}, f_{z}, f_{x y}, f_{y x}, f_{z y}$ for the following. Verify the mixed partial derivatives are equal.
(a) $x^{2} y^{3} z^{4}+\sin (x y z)$
(b) $\sin (x y z)+x^{2} y z$
6. Suppose $f$ is a continuous function and $f: U \rightarrow \mathbb{R}$ where $U$ is an open set and suppose that $\mathbf{x} \in U$ has the property that for all $\mathbf{y}$ near $\mathbf{x}, f(\mathbf{x}) \leq f(\mathbf{y})$. Prove that if $f$ has all of its partial derivatives at $\mathbf{x}$, then $f_{x_{i}}(\mathbf{x})=0$ for each $x_{i}$. Hint: Consider $f(\mathbf{x}+t \mathbf{v})=h(t)$. Argue that $h^{\prime}(0)=0$ and then see what this implies about $D f(\mathbf{x})$.
7. As an important application of Problem 6 consider the following. Experiments are done at $n$ times, $t_{1}, t_{2}, \cdots, t_{n}$ and at each time there results a collection of numerical outcomes. Denote by $\left\{\left(t_{i}, x_{i}\right)\right\}_{i=1}^{p}$ the set of all such pairs and try to find numbers $a$ and $b$ such that the line $x=a t+b$ approximates these ordered pairs as well as possible in the sense that out of all choices of $a$ and $b, \sum_{i=1}^{p}\left(a t_{i}+b-x_{i}\right)^{2}$ is as small as possible. In other words, you want to minimize the function of two variables $f(a, b) \equiv \sum_{i=1}^{p}\left(a t_{i}+b-x_{i}\right)^{2}$. Find a formula for $a$ and $b$ in terms of the given ordered pairs. You will be finding the formula for the least squares regression line.
8. Let $f$ be a function which has continuous derivatives. Show that $u(t, x)=f(x-c t)$ solves the wave equation $u_{t t}-c^{2} \Delta u=0$. What about $u(x, t)=f(x+c t)$ ? Here $\Delta u=$ $u_{x x}$.
9. Show that if $\Delta u=\lambda u$ where $u$ is a function of only $x$, then $e^{\lambda t} u$ solves the heat equation $u_{t}-\Delta u=0$. Here $\Delta u=u_{x x}$.
10. Show that if $f(x)=o(x)$, then $f^{\prime}(0)=0$.
11. Let $f(x, y)$ be defined on $\mathbb{R}^{2}$ as follows. $f\left(x, x^{2}\right)=1$ if $x \neq 0$. Define $f(0,0)=0$, and $f(x, y)=0$ if $y \neq x^{2}$. Show that $f$ is not continuous at $(0,0)$ but that

$$
\lim _{h \rightarrow 0} \frac{f(h a, h b)-f(0,0)}{h}=0
$$

for $(a, b)$ an arbitrary vector. This is called a Gateaux derivative. Thus the Gateaux derivative exists at $(0,0)$ in every direction but $f$ is not even continuous there.
12. Let

$$
f(x, y) \equiv\left\{\begin{array}{l}
\frac{x y^{4}}{x^{2}+y^{8}} \text { if }(x, y) \neq(0,0) \\
0 \text { if }(x, y)=(0,0)
\end{array}\right.
$$

Show that this function is not continuous at $(0,0)$ but that the Gateaux derivative

$$
\lim _{h \rightarrow 0} \frac{f(h a, h b)-f(0,0)}{h}
$$

exists and equals 0 for every vector $(a, b)$.
13. One of the big applications of the implicit function theorem is to the method of Lagrange multipliers. The heuristic explanations usually given in beginning calculus courses are specious. At least this is certainly true of the explanation I use all the time based on pictures and geometric reasoning. They break down as soon as you ask the obvious question whether there is a smooth curve through a point in the level surface. In other words, why does the level surface even look the way we draw it in these courses? To do the method of Lagrange multipliers correctly, you need to use some sort of big theorem and the version involving the implicit function theorem is likely the easiest. Using the implicit function theorem, prove the following theorem which is the general method of Lagrange multipliers.

Theorem 4.12.1 Let $U$ be an open subset of $\mathbb{R}^{n}$ and let $f: U \rightarrow \mathbb{R}$ be a $C^{1}$ function. Then if $\mathbf{x}_{0} \in U$, has the property that
$g_{i}\left(\mathbf{x}_{0}\right)=0, i=1, \cdots, m, g_{i} a C^{1}$ function, and $\mathbf{x}_{0}$ is either a local maximum or local minimum of $f$ on the intersection of the level sets $\left\{\mathbf{x}: g_{i}(\mathbf{x})=0\right\} i=1, \cdots, m$, and if some $m \times m$ submatrix of

$$
D \mathbf{g}\left(\mathbf{x}_{0}\right) \equiv\left(\begin{array}{cccc}
g_{1 x_{1}}\left(\mathbf{x}_{0}\right) & g_{1 x_{2}}\left(\mathbf{x}_{0}\right) & \cdots & g_{1 x_{n}}\left(\mathbf{x}_{0}\right) \\
\vdots & \vdots & & \vdots \\
g_{m x_{1}}\left(\mathbf{x}_{0}\right) & g_{m x_{2}}\left(\mathbf{x}_{0}\right) & \cdots & g_{m x_{n}}\left(\mathbf{x}_{0}\right)
\end{array}\right)
$$

has nonzero determinant, then there exist scalars, $\lambda_{1}, \cdots, \lambda_{m}$ such that

$$
\left(\begin{array}{c}
f_{x_{1}}\left(\mathbf{x}_{0}\right)  \tag{4.26}\\
\vdots \\
f_{x_{n}}\left(\mathbf{x}_{0}\right)
\end{array}\right)=\lambda_{1}\left(\begin{array}{c}
g_{1 x_{1}}\left(\mathbf{x}_{0}\right) \\
\vdots \\
g_{1 x_{n}}\left(\mathbf{x}_{0}\right)
\end{array}\right)+\cdots+\lambda_{m}\left(\begin{array}{c}
g_{m x_{1}}\left(\mathbf{x}_{0}\right) \\
\vdots \\
g_{m x_{n}}\left(\mathbf{x}_{0}\right)
\end{array}\right)
$$

holds.

Hint: Let $\mathbf{F}: U \times \mathbb{R} \rightarrow \mathbb{R}^{m+1}$ be defined by

$$
\mathbf{F}(\mathbf{x}, a) \equiv\left(\begin{array}{c}
f(\mathbf{x})-a  \tag{4.27}\\
g_{1}(\mathbf{x}) \\
\vdots \\
g_{m}(\mathbf{x})
\end{array}\right)
$$

and if the condition holds on rank, and 4.26 fails to hold, then from linear algebra you can use the implicit function theorem to solve for $m+1$ of the $\mathbf{x}$ variables in terms of the others, $a$ being one of them, these other variables being in an open set. In particular $a$ cannot be a local extremum unless 4.26 holds.
14. Now consider the queston about level surfaces. Suppose you have

$$
S=\left\{\mathbf{x} \in \mathbb{R}^{n+1}: f(\mathbf{x})=c\right\} .
$$

We usually refer to this as a level surface in $\mathbb{R}^{n+1}$ and we give examples of things like ellipsoids and spheres. Then everyone is deceived into thinking they know what is going on because of the examples. After this deception, and this is indeed what it
is, we give specious arguments to justify the method of Lagrange multipliers (I have spent my career giving such specious arguments.) by showing that the gradient of the objective function is perpendicular to the direction vector of every smooth curve lying in $S$ at a point where the maximum or minimum exists using the chain rule. One thing which is missing in this kind of stupidity is a consideration whether there even exist such smooth curves. Use the implicit function theorem to give conditions which imply the existence of such smooth curves near a point on $S$.
15. State and give a short proof of the inverse function theorem for normed linear spaces using Theorem 4.10.1.
16. Prove Theorem 4.11.1. Hint: Let $K$ be such that

$$
h(1)=h(0)+\sum_{k=1}^{m} \frac{1}{k!} h^{(k)}(0)+K .
$$

Now define

$$
g(u) \equiv h(1)-\left(h(u)+\sum_{k=1}^{m} \frac{1}{k!} h^{(k)}(u)(1-u)^{k}+K(1-u)^{m+1}\right)
$$

Then $g(0)=0$ and $g(1)=0$ so by the mean value theorem, there is $t \in(0,1)$ where $g^{\prime}(t)=0$. Compute $g^{\prime}(u)$ and simplify then choose the $t$ just mentioned and solve for $K$.
17. Let $\mathbf{f}: \mathbb{R}^{2} \times \mathbb{R} \rightarrow \mathbb{R}^{2}$

$$
\mathbf{f}(x, y, \lambda)=\binom{x+x y+y^{2}+\sin (\lambda)}{x+y^{2}-x^{2}+\lambda}
$$

Then $\mathbf{f}(0,0, \lambda)=\mathbf{0}, D_{1} \mathbf{f}(x, y, \lambda)=\left(\begin{array}{cc}1+y & x+2 y \\ 1-2 x & 2 y\end{array}\right)$ so

$$
D_{1} \mathbf{f}((0,0), 0)=\left(\begin{array}{ll}
1 & 0 \\
1 & 0
\end{array}\right)
$$

Thus you can't say $\mathbf{f}(x, y, \boldsymbol{\lambda})=\mathbf{0}$ defines $(x, y)$ as a function of $\boldsymbol{\lambda}$ near $(0,0,0)$. However, let

$$
\begin{aligned}
Q\binom{\alpha}{\beta} & \equiv\left(\begin{array}{cc}
1 / 2 & 1 / 2 \\
1 / 2 & 1 / 2
\end{array}\right)\binom{\alpha}{\beta}=\binom{\frac{\alpha+\beta}{2}}{\frac{\alpha+\beta}{2}} \\
(I-Q)\binom{\alpha}{\beta} & =\binom{\alpha}{\beta}-\binom{\frac{\alpha+\beta}{2}}{\frac{\alpha+\beta}{2}}=\binom{\frac{1}{2} \alpha-\frac{1}{2} \beta}{\frac{1}{2} \beta-\frac{1}{2} \alpha}
\end{aligned}
$$

The equation $\mathbf{f}(x, y, \lambda)=\mathbf{0}$ can be written in the form

$$
\begin{equation*}
Q \mathbf{f}(x, y, \lambda)=\binom{-\frac{1}{2} x^{2}+\frac{1}{2} x y+x+y^{2}+\frac{1}{2} \lambda+\frac{1}{2} \sin \lambda}{-\frac{1}{2} x^{2}+\frac{1}{2} x y+x+y^{2}+\frac{1}{2} \lambda+\frac{1}{2} \sin \lambda}=\mathbf{0} \tag{4.28}
\end{equation*}
$$

$$
(I-Q) \mathbf{f}(x, y, \lambda)=\binom{\frac{1}{2} x^{2}+\frac{1}{2} y x-\frac{1}{2} \lambda+\frac{1}{2} \sin \lambda}{-\frac{1}{2} x^{2}-\frac{1}{2} y x+\frac{1}{2} \lambda-\frac{1}{2} \sin \lambda}=\mathbf{0}
$$

$D_{x} Q \mathbf{f}(0,0,0)=\binom{1}{1}$ which is one to one on $\mathbb{R}$. Indeed, if $\binom{1}{1} u=\binom{0}{0}$, then $u=0$. By Theorem 4.10.1, the first equation in 4.28 defines $x=x(y, \lambda)$ for small $y, \lambda$. Also, you know it is a $C^{k}$ function for every $k$ so you can use Taylor approximation for functions of many variables to approximate $x(y, \lambda)$. In the top equation, $x_{y}=0$. Also $x_{\lambda}=-1$ so $x(y, \lambda) \approx-\lambda$ other than higher order terms for small $y, \lambda$. Now plug in to the bottom equation

$$
\begin{aligned}
& \frac{1}{2} x^{2}(y, \lambda)+\frac{1}{2} y x(y, \lambda)-\frac{1}{2} \lambda+\frac{1}{2} \sin \lambda \\
= & \frac{1}{2}(-\lambda)^{2}+\frac{1}{2} y(-\lambda)-\frac{1}{2} \lambda+\frac{1}{2} \sin \lambda=0
\end{aligned}
$$

Solve this for $y$ to find $y(\lambda)=-1+\frac{\sin (\lambda)}{\lambda}+\lambda$ at least approximately. This kind of procedure is called the Lyapunov Schmidt procedure. It deals with the case where the partial derivative used in the statement of the implicit function theorem is not invertible. Note how it was possible to solve for a solution $\mathbf{f}(x, y, \lambda)=\mathbf{0}$ in this example.
18. Let $\mathbf{f}((x, y), \lambda)=\binom{x+x y+y^{2}+x \sin (\lambda)}{x+y^{2}-x^{2}+x \lambda}$. One solution to $\mathbf{f}((x, y), \lambda)=\mathbf{0}$ is $x(\lambda)=y(\lambda)=0$. Use the above procedure to show there is a nonzero solution to this non-linear system of equations for small $\lambda$.
19. Let $X, Y$ be finite dimensional vector spaces and let $L \in \mathscr{L}(X, Y)$. Let $\left\{L x_{1}, \ldots, L x_{m}\right\}$ be a basis for $L(X)$. Show that if $\left\{z_{1}, \ldots, z_{r}\right\}$ is a basis for $\operatorname{ker}(L)$, then a basis for $X$ is $\left\{x_{1}, \ldots, x_{m}, z_{1}, \ldots, z_{r}\right\}$ is a basis for $X$. Show that $L$ is one to one on $X_{1} \equiv$ $\operatorname{span}\left(x_{1}, \ldots, x_{m}\right)$.
20. Go through the details of the following argument. Let $\mathbf{f}: U \subseteq \mathbb{R}^{n} \times \mathbb{R}^{m} \rightarrow \mathbb{R}^{n}$ where $U$ is open in $\mathbb{R}^{n} \times \mathbb{R}^{m},(\mathbf{0}, \mathbf{0}) \in U$. Let $\mathbf{f}$ be $C^{k}$ for $k \geq 1$. Also suppose $\mathbf{f}(\mathbf{0}, \mathbf{0})=$ $\mathbf{0}$. If $L=D_{1} \mathbf{f}(0,0)$ and if $L^{-1}$ exists, then by the implicit function theorem, the equation $\mathbf{f}(\mathbf{x}, \boldsymbol{\lambda})=\mathbf{0}$ defines $\mathbf{x}=\mathbf{x}(\lambda)$ for small $\lambda$ and $\mathbf{x}$ is $C^{k}$. Let $\left\{\mathbf{y}_{1}, \ldots, \mathbf{y}_{m}\right\}$ be a basis for $L\left(\mathbb{R}^{n}\right)$ and enlarge to get $\left\{\mathbf{y}_{1}, \ldots, \mathbf{y}_{m}, \mathbf{w}_{m+1}, \ldots, \mathbf{w}_{n}\right\}$ as a basis for $\mathbb{R}^{n}$. Letting $L \mathbf{x}_{k}=\mathbf{y}_{k}$ use the above problem to have a basis for $X$ which is of the form $\left\{\mathbf{x}_{1}, \ldots, \mathbf{x}_{m}, \mathbf{z}_{m+1}, \ldots, \mathbf{z}_{n}\right\}$ with $\left\{\mathbf{z}_{m+1}, \ldots, \mathbf{z}_{n}\right\}$ a basis for $\operatorname{ker}(L)$. Thus, from the above problem $L$ is one to one on $X_{1} \equiv \operatorname{span}\left(\mathbf{x}_{1}, \ldots, \mathbf{x}_{m}\right)$. For $\hat{\mathbf{x}} \in X_{1}$, show $D_{\hat{\mathbf{x}}} \mathbf{f}(\mathbf{0}, \mathbf{0})$ is the restriction of $L$ to $X_{1}$ and so $D_{\hat{\mathbf{x}}} \mathbf{f}(\mathbf{0}, \mathbf{0})$ is one to one on $X_{1}$. Now define the linear map $Q: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ by $Q\left(\sum_{k=1}^{m} a_{k} \mathbf{y}_{k}+\sum_{k=m+1}^{n} b_{k} \mathbf{w}_{k}\right) \equiv \sum_{k=1}^{m} a_{k} \mathbf{y}_{k}$. Thus $Q^{2}=Q$. We can write the original equations $\mathbf{f}(\mathbf{x}, \boldsymbol{\lambda})=\mathbf{0}$ as

$$
\begin{gathered}
Q \mathbf{f}(\hat{\mathbf{x}}, \tilde{\mathbf{x}}, \lambda)=Q \mathbf{f}(\mathbf{x}, \lambda)=\mathbf{0}, \tilde{\mathbf{x}} \in \operatorname{ker}(L) \\
(I-Q) \mathbf{f}(\hat{\mathbf{x}}, \tilde{\mathbf{x}}, \lambda)=\mathbf{0}
\end{gathered}
$$

Thus $Q \mathbf{f}(\mathbf{x}, \boldsymbol{\lambda}) \in \operatorname{span}\left(\mathbf{y}_{1}, \ldots, \mathbf{y}_{m}\right) \equiv Y_{1}$. Now show that for $\hat{\mathbf{x}}$ the variable in $X_{1}$, and if $\mathbf{v} \in X_{1}$, and $D_{\hat{\mathbf{x}}} Q \mathbf{f}(\mathbf{0}, \mathbf{0}, \mathbf{0}) \mathbf{v}=\mathbf{0}$, then $\mathbf{v}=\mathbf{0}$ and so we can apply the implicit function theorem to obtain $\hat{\mathbf{x}}=\hat{\mathbf{x}}(\tilde{\mathbf{x}}, \lambda)$ as the solution to $Q \mathbf{f}(\mathbf{x}, \lambda)=\mathbf{0}$ for $\tilde{\mathbf{x}}, \lambda$ small where here $\tilde{\mathbf{x}}$ is in $\operatorname{ker}(L)$. Since everything in sight is $C^{k}$, one can use Taylor series for
functions of many variables to approximate the solution in these two equations. See the Taylor formula 4.25. This is the general idea in the above two problems.

## Chapter 5

## Line Integrals and Curves

This chapter is on integrals involving one real variable of integration. I will present this in terms of line integrals and point out that the usual Riemann integral which I assume has been seen by the reader comes as a special case. As above, the usual Euclidean norm is indicated by $|\cdot|$.

### 5.1 Existence and Definition

Definition 5.1.1 Let $\gamma:[a, b] \rightarrow \mathbb{R}^{p}$ be a function. Then $\gamma$ is of bounded variation if

$$
\sup \left\{\sum_{i=1}^{n}\left|\gamma\left(t_{i}\right)-\gamma\left(t_{i-1}\right)\right|: a=t_{0}<\cdots<t_{n}=b\right\} \equiv V(\gamma,[a, b])<\infty
$$

where the sums are taken over all possible lists, $\left\{a=t_{0}<\cdots<t_{n}=b\right\}$. The set of points $\gamma([a, b])$ will also be denoted by $\gamma^{*}$. When $\gamma$ is one to one on $[a, b)$ and continuous on $[a, b]$ we call $\gamma^{*}$ a simple curve. You might have $\gamma(a)=\gamma(b)$ when it is called a simple closed curve.

The idea is that it makes sense to talk of the length of the curve $\gamma([a, b])$, defined as $V(\gamma,[a, b])$. For this reason, in the case that $\gamma$ is continuous, such an image of a bounded variation function is called a rectifiable curve.

Definition 5.1.2 Let $\gamma:[a, b] \rightarrow \mathbb{R}^{p}$ be of bounded variation and let $\mathbf{f}: \gamma^{*} \rightarrow \mathbb{R}^{p}$. Letting $P \equiv\left\{t_{0}, \cdots, t_{n}\right\}$ where $a=t_{0}<t_{1}<\cdots<t_{n}=b$, define

$$
\|P\| \equiv \max \left\{\left|t_{j}-t_{j-1}\right|: j=1, \cdots, n\right\}
$$

and the Riemann sum by

$$
S(P) \equiv \sum_{j=1}^{n} \mathbf{f}\left(\gamma\left(\tau_{j}\right)\right) \cdot\left(\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right)
$$

where $\tau_{j} \in\left[t_{j-1}, t_{j}\right]$. (Note this notation is a little sloppy because it does not identify the specific point $\tau_{j} \in\left[t_{j-1}, t_{j}\right]$ used. It is understood that this point is arbitrary.) Define $\int_{\gamma} \mathbf{f} \cdot d \gamma$ as the unique number which satisfies the following condition. For all $\varepsilon>0$ there exists a $\delta>0$ such that if $\|P\| \leq \delta$, then

$$
\left|\int_{\gamma} \mathbf{f} \cdot d \gamma-S(P)\right|<\varepsilon
$$

Sometimes this is written as $\int_{\gamma} \mathbf{f} \cdot d \gamma \equiv \lim _{\|P\| \rightarrow 0} S(P)$. The set of points in the curve, $\gamma([a, b])$ will be denoted sometimes by $\gamma^{*}$. Also, when convenient, I will write $\sum_{P}$ to denote a Riemann sum.

Then $\gamma^{*}$ is a set of points in $\mathbb{R}^{p}$ and as $t$ moves from $a$ to $b, \gamma(t)$ moves from $\gamma(a)$ to $\gamma(b)$. Thus $\gamma^{*}$ has a first point and a last point.

Note that from the above definition, it is obvious that the line integral is linear. Simply let $P_{n}$ refer to a uniform parition of $[a, b]$ and let $\tau_{j}^{n}$ be the midpoint of $\left[t_{j-1}^{n}, t_{j}^{n}\right]$. Then for $a, b$ scalars and $\mathbf{f}, \mathbf{g}$ vector valued functions which have integrals,

$$
\begin{aligned}
& a \int_{\gamma} \mathbf{f} \cdot d \gamma+b \int_{\gamma} \mathbf{g} \cdot d \gamma \\
= & \binom{\lim _{n \rightarrow \infty} a \sum_{P_{n}} \mathbf{f}\left(\gamma\left(\tau_{j}^{n}\right)\right) \cdot\left(\gamma\left(t_{j}^{n}\right)-\gamma\left(t_{j-1}^{n}\right)\right)}{+\lim _{n \rightarrow \infty} b \sum_{P_{n}} \mathbf{g}\left(\gamma\left(\tau_{j}^{n}\right)\right) \cdot\left(\gamma\left(t_{j}^{n}\right)-\gamma\left(t_{j-1}^{n}\right)\right)} \\
= & \lim _{n \rightarrow \infty} \sum_{P_{n}}\left(a \mathbf{f}\left(\gamma\left(\tau_{j}^{n}\right)\right)+b \mathbf{g}\left(\gamma\left(\tau_{j}^{n}\right)\right)\right) \cdot\left(\gamma\left(t_{j}^{n}\right)-\gamma\left(t_{j-1}^{n}\right)\right) \\
\equiv & \int_{\gamma}(a \mathbf{f}+b \mathbf{g}) \cdot d \gamma
\end{aligned}
$$

Another issue is whether the integral depends on the parametrization associated with $\gamma^{*}$ or only on $\gamma^{*}$ and the direction of motion over $\gamma^{*}$. If $\phi:[c, d] \rightarrow[a, b]$ is a continuous nondecreasing function, then $\gamma \circ \phi:[c, d] \rightarrow \mathbb{R}^{p}$ is also of bounded variation and yields the same set of points in $\mathbb{R}^{p}$ with the same first and last points. The next theorem explains that one can use either $\gamma$ or $\gamma \circ \phi$ and get the same integral.

### 5.1.1 Change of Parameter

Theorem 5.1.3 Let $\phi$ be continuous and non-decreasing and $\gamma$ is continuous and bounded variation. Then assuming that $\int_{\gamma} \mathbf{f} \cdot d \gamma$ exists, so does $\int_{\gamma \circ \phi} \mathbf{f} d(\gamma \circ \phi)$ and

$$
\begin{equation*}
\int_{\gamma} \mathbf{f} \cdot d \gamma=\int_{\gamma \circ \phi} \mathbf{f} d(\gamma \circ \phi) . \tag{5.1}
\end{equation*}
$$

Proof: There exists $\delta>0$ such that if $P$ is a partition of $[a, b]$ such that $\|P\|<\delta$, then $\left|\int_{\gamma} \mathbf{f} \cdot d \gamma-S(P)\right|<\varepsilon$. By Theorem 2.5.28, $\phi$ is uniformly continuous so there exists $\sigma>0$ such that if $Q$ is a partition of $[c, d]$ with $\|Q\|<\sigma, Q=\left\{s_{0}, \cdots, s_{n}\right\}$, then $\left|\phi\left(s_{j}\right)-\phi\left(s_{j-1}\right)\right|<\delta$. Thus letting $P$ denote the points in $[a, b]$ given by $\phi\left(s_{j}\right)$ for $s_{j} \in Q$, it follows that $\|P\|<\delta$ and so

$$
\left|\int_{\gamma} \mathbf{f} \cdot d \gamma-\sum_{j=1}^{n} \mathbf{f}\left(\gamma\left(\phi\left(\tau_{j}\right)\right)\right) \cdot\left(\gamma\left(\phi\left(s_{j}\right)\right)-\gamma\left(\phi\left(s_{j-1}\right)\right)\right)\right|<\varepsilon
$$

where $\tau_{j} \in\left[s_{j-1}, s_{j}\right]$. Therefore, from the definition 5.1 holds and $\int_{\gamma \circ \phi} \mathbf{f} \cdot d(\gamma \circ \phi)$ exists and equals $\int_{\gamma} \mathbf{f} \cdot d \gamma$.

This theorem shows that $\int_{\gamma} \mathbf{f} \cdot d \gamma$ is independent of the particular parametrization $\gamma$ used in its computation in the sense that if $\phi$ is any nondecreasing continuous function from another interval, $[c, d]$, mapping onto $[a, b]$, then the same value is obtained by replacing $\gamma$ with $\gamma \circ \phi$.

Lemma 5.1.4 Let $\phi: I \rightarrow \mathbb{R}$ be a function and $I$ is an interval and suppose $\phi$ is $1-$ 1 and continuous on $I$. Then $\phi$ is either strictly increasing or strictly decreasing on $I$. Furthermore, if $\phi$ is one to one and continuous on $[a, b]$ then $\phi^{-1}$ is continuous.

Proof: First it is shown that $\phi$ is either strictly increasing or strictly decreasing on if $I$ is an open interval.

If $\phi$ is not strictly decreasing on $I$, then there exists $x_{1}<y_{1}, x_{1}, y_{1} \in(a, b)$ such that $\left(\phi\left(y_{1}\right)-\phi\left(x_{1}\right)\right)\left(y_{1}-x_{1}\right)>0$. If for some other pair of points, $x_{2}<y_{2}$ with $x_{2}, y_{2} \in(a, b)$, the above inequality does not hold, then since $\phi$ is $1-1$, $\left(\phi\left(y_{2}\right)-\phi\left(x_{2}\right)\right)\left(y_{2}-x_{2}\right)<0$. Let $x_{t} \equiv t x_{1}+(1-t) x_{2}$ and $y_{t} \equiv t y_{1}+(1-t) y_{2}$. Then $x_{t}<y_{t}$ for all $t \in[0,1]$ because

$$
t x_{1} \leq t y_{1} \text { and }(1-t) x_{2} \leq(1-t) y_{2}
$$

with strict inequality holding for at least one of these inequalities since not both $t$ and $(1-t)$ can equal zero. Now define

$$
h(t) \equiv\left(\phi\left(y_{t}\right)-\phi\left(x_{t}\right)\right)\left(y_{t}-x_{t}\right) .
$$

Since $h$ is a continuous function of $t$ and $h(0)<0$, while $h(1)>0$, there exists $t \in(0,1)$ such that $h(t)=0$. Therefore, both $x_{t}$ and $y_{t}$ are points of $I$ and $\phi\left(y_{t}\right)-\phi\left(x_{t}\right)=0$ contradicting the assumption that $\phi$ is one to one. It follows $\phi$ is either strictly increasing or strictly decreasing on $I$.

This property of being either strictly increasing or strictly decreasing on the interior $(a, b)$ of an interval carries over to $[a, b]$ by the continuity of $\phi$ in the case that $\phi$ is defined and continuous on $[a, b]$. Suppose $\phi$ is strictly increasing on $(a, b)$. If $y \in(a, b)$, is it true that $\phi(b)>\phi(y)$ ? If not, you would have $\phi(b) \leq \phi(y)$. Since $\phi$ is one to one, these can't be equal and so $\phi(b)<\phi(y)$. But now, by the intermediate value theorem, there would be $z \in(y, b)$ with $\phi(z)=\frac{\phi(b)+\phi(y)}{2}<\phi(y)$ violating the fact that $\phi$ is increasing on $(a, b)$. It is similar with the other end point.

It only remains to verify $\phi^{-1}$ is continuous if $\phi$ is one to one on $[a, b]$. Suppose then that $s_{n} \rightarrow s$ where $s_{n}$ and $s$ are points of $\phi([a, b])$. It is desired to verify that $\phi^{-1}\left(s_{n}\right) \rightarrow \phi^{-1}(s)$. If this does not happen, there exists $\varepsilon>0$ and a subsequence, still denoted by $s_{n}$ such that $\left|\phi^{-1}\left(s_{n}\right)-\phi^{-1}(s)\right| \geq \varepsilon$. Using the sequential compactness of $[a, b]$ there exists a further subsequence, still denoted by $n$, such that $\phi^{-1}\left(s_{n}\right) \rightarrow t_{1} \in[a, b], t_{1} \neq \phi^{-1}(s)$. Then by continuity of $\phi$, it follows $s_{n} \rightarrow \phi\left(t_{1}\right)$ and so $s=\phi\left(t_{1}\right)$. Therefore, $t_{1}=\phi^{-1}(s)$ after all.

If $\gamma, \eta$ are two continuous one to one parametrizations of a curve $C$, then it follows that $\eta=\gamma \circ\left(\gamma^{-1} \circ \eta\right)$ where $\gamma^{-1} \circ \eta$ is either increasing or decreasing by Lemma 5.1.4.

Given a parametrization $\gamma$ of a curve and an interval $[a, b]$ on which $\gamma$ is defined, there is a natural orientation corresponding to increasing $t \in[a, b]$. Sometimes people write $-\gamma$ to denote the opposite orientation. To obtain this, you could write a parametrization for it as $-\gamma(t) \equiv \gamma(b-t)$ for $t \in[0, b-a]$. Of course this encounters the points of $\gamma^{*}$ in the opposite order. Therefore, in the above definition, $-\int_{\gamma} \mathbf{f} \cdot d \gamma=\int_{\gamma} \mathbf{f} \cdot d(-\gamma)$.The Riemann sums for $\int_{\gamma} \mathbf{f} \cdot d(-\gamma)$ are -1 times the Riemann sums for $\int_{\gamma} \mathbf{f} \cdot d \gamma$.

### 5.1.2 Existence

The fundamental result in this subject is the following theorem.
Theorem 5.1.5 Let $\mathbf{f}: \gamma^{*} \rightarrow \mathbb{R}^{p}$ be continuous and let $\gamma:[a, b] \rightarrow \mathbb{R}^{p}$ be continuous and of bounded variation. Then $\int_{\gamma} \mathbf{f} \cdot d \gamma$ exists. Also letting $\delta_{m}>0$ be such that $|t-s|<\delta_{m}$ implies $\|\mathbf{f}(\gamma(t))-\mathbf{f}(\gamma(s))\|<\frac{1}{m}$,

$$
\begin{equation*}
\left|\int_{\gamma} \mathbf{f} \cdot d \gamma-S(P)\right| \leq \frac{2 V(\gamma,[a, b])}{m} \tag{5.2}
\end{equation*}
$$

whenever $\|P\|<\delta_{m}$, the $\boldsymbol{\delta}_{m}$ decreasing in $m$.
Proof: The function, $\mathbf{f} \circ \gamma$, is uniformly continuous because it is defined on a compact set. Therefore, there exists a decreasing sequence of positive numbers $\left\{\delta_{m}\right\}$ such that if $|s-t|<\delta_{m}$, then $|\mathbf{f}(\gamma(t))-\mathbf{f}(\gamma(s))|<\frac{1}{m}$. Let $F_{m} \equiv \overline{\left\{S(P):\|P\|<\delta_{m}\right\}}$. Thus $F_{m}$ is a closed set. (The symbol, $S(P)$ in the above definition, means to include all sums corresponding to $P$ for any choice of $\tau_{j}$.) It is shown that

$$
\begin{equation*}
\operatorname{diam}\left(F_{m}\right) \leq \frac{2 V(\gamma,[a, b])}{m} \tag{5.3}
\end{equation*}
$$

and then it will follow there exists a unique point, $I \in \cap_{m=1}^{\infty} F_{m}$. This is because $\mathbb{R}$ is complete. It will then follow $I=\int_{\gamma} \mathbf{f}(t) \cdot d \gamma(t)$. To verify 5.3, it suffices to verify that whenever $P$ and $Q$ are partitions satisfying $\|P\|<\delta_{m}$ and $\|Q\|<\delta_{m}$,

$$
\begin{equation*}
|S(P)-S(Q)| \leq \frac{2}{m} V(\gamma,[a, b]) \tag{5.4}
\end{equation*}
$$

Suppose $\|P\|<\delta_{m}$ and $Q \supseteq P$. Then also $\|Q\|<\delta_{m}$. To begin with, suppose that $P \equiv\left\{t_{0}, \cdots, t_{p}, \cdots, t_{n}\right\}$ and $Q \equiv\left\{t_{0}, \cdots, t_{p-1}, t^{*}, t_{p}, \cdots, t_{n}\right\}$. Thus $Q$ contains only one more point than $P$. Letting $S(Q)$ and $S(P)$ be Riemann Steiltjes sums,

$$
\begin{gathered}
S(Q) \equiv \sum_{j=1}^{p-1} \mathbf{f}\left(\gamma\left(\sigma_{j}\right)\right) \cdot\left(\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right)+\mathbf{f}\left(\gamma\left(\sigma_{*}\right)\right) \cdot\left(\gamma\left(t^{*}\right)-\gamma\left(t_{p-1}\right)\right) \\
+\mathbf{f}\left(\gamma\left(\sigma^{*}\right)\right) \cdot\left(\gamma\left(t_{p}\right)-\gamma\left(t^{*}\right)\right)+\sum_{j=p+1}^{n} \mathbf{f}\left(\gamma\left(\sigma_{j}\right)\right) \cdot\left(\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right) \\
S(P) \equiv \sum_{j=1}^{p-1} \mathbf{f}\left(\gamma\left(\tau_{j}\right)\right) \cdot\left(\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right)+ \\
\overbrace{\mathbf{f}\left(\gamma\left(\tau_{p}\right)\right) \cdot\left(\gamma\left(t^{*}\right)-\gamma\left(t_{p-1}\right)\right)+\mathbf{f}\left(\gamma\left(\tau_{p}\right)\right) \cdot\left(\gamma\left(t_{p}\right)-\gamma\left(t^{*}\right)\right)}^{=\mathbf{f}\left(\gamma\left(\tau_{p}\right)\right) \cdot\left(\gamma\left(t_{p}\right)-\gamma\left(t_{p-1}\right)\right)} \\
\quad+\sum_{j=p+1}^{n} \mathbf{f}\left(\gamma\left(\tau_{j}\right)\right) \cdot\left(\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right)
\end{gathered}
$$

Therefore, since $\left|\left(\mathbf{f}\left(\gamma\left(\sigma_{j}\right)\right)-\mathbf{f}\left(\gamma\left(\tau_{j}\right)\right)\right) \cdot\left(\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right)\right| \leq \frac{1}{m}\left|\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right|$,

$$
\begin{align*}
& |S(P)-S(Q)| \leq \sum_{j=1}^{p-1} \frac{1}{m}\left|\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right|+\frac{1}{m}\left|\gamma\left(t^{*}\right)-\gamma\left(t_{p-1}\right)\right|+ \\
& \frac{1}{m}\left|\gamma\left(t_{p}\right)-\gamma\left(t^{*}\right)\right|+\sum_{j=p+1}^{n} \frac{1}{m}\left|\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right| \leq \frac{1}{m} V(\gamma,[a, b]) . \tag{5.5}
\end{align*}
$$

Clearly the extreme inequalities would be valid in 5.5 if $Q$ had more than one extra point. You simply do the above trick more than one time. Let $S(P)$ and $S(Q)$ be Riemann Steiltjes sums for which $\|P\|$ and $\|Q\|$ are less than $\delta_{m}$ and let $R \equiv P \cup Q$. Then from what was just observed,

$$
|S(P)-S(Q)| \leq|S(P)-S(R)|+|S(R)-S(Q)| \leq \frac{2}{m} V(\gamma,[a, b])
$$

and this shows 5.4 which proves 5.3. Therefore, there exists a unique number, $I \in \cap_{m=1}^{\infty} F_{m}$ which satisfies the definition of $\int_{\gamma} \mathbf{f} \cdot d \gamma$.

In case the existence and uniqueness of $I$ is not clear, note that each $F_{m}$ is closed and if you pick a point from each, you get a Cauchy sequence. Thus it converges to a point of $F_{m}$ for each $m$. Hence there is a point in all these $F_{m}$ and since their diameters converge to 0 , there can be no more than one point. This argument would work just as well if $\gamma$ had values in some Banach space.

### 5.1.3 The Riemann Integral

The reader is assumed to be familiar with the Riemann integral but if not, the above is more general and gives the principal results for the Riemann integral of continuous functions very easily. Therefore, here is a slight digression to show this. It is sometimes useful to consider Riemann integrals for functions which have values in a Banach space $X$. The following includes this case also. First is the definition.

Definition 5.1.6 Let $f:[a, b] \rightarrow X$ where $X$ is a Banach space and define a Riemann sum by

$$
S(P) \equiv \sum_{j=1}^{n} f\left(\tau_{j}\right)\left(t_{j}-t_{j-1}\right) \in X
$$

where $\tau_{j} \in\left[t_{j-1}, t_{j}\right]$. (Note this notation is a little sloppy because it does not identify the specific point $\tau_{j} \in\left[t_{j-1}, t_{j}\right]$ used. It is understood that this point is arbitrary.) Define $\int_{a}^{b} f(t) d t$ as the unique element of $X$ which satisfies the following condition. For all $\varepsilon>0$ there exists a $\delta>0$ such that if $\|P\| \leq \delta$, then

$$
\left|\int_{a}^{b} f(t) d t-S(P)\right|<\varepsilon
$$

Sometimes this is written as $\int_{a}^{b} f(t) d t \equiv \lim _{\|P\| \rightarrow 0} S(P)$.
The following is the corresponding theorem for continuous functions being Riemann integrable. I am mainly featuring continuous functions in what follows to avoid technical considerations. However, everything holds for Riemann integrable functions also.

Theorem 5.1.7 Let $f:[a, b] \rightarrow X$ be continuous where $X$ is a Banach space. Then $\int_{a}^{b} f(t) d t$ exists. Also letting $\delta_{m}>0$ be such that $|t-s|<\delta_{m}$ implies $\|f(t)-f(s)\|<\frac{1}{m}$,

$$
\begin{equation*}
\left|\int_{a}^{b} f(t) d t-S(P)\right| \leq \frac{2(b-a)}{m} \tag{5.6}
\end{equation*}
$$

whenever $\|P\|<\delta_{m}$. Also one obtains the triangle inequality for $a<b$

$$
\left\|\int_{a}^{b} f(t) d t\right\| \leq \int_{a}^{b}\|f(t)\| d t
$$

The latter holds whenever $f,\|f\|$ are each Riemann integrable. (Actually, if $f$ is Riemann integrable then $\|f\|$ will be also.) Also the integral is linear. If $f, g$ are Riemann integrable, then for $\alpha, \beta$ scalars,

$$
\int_{a}^{b} \alpha f(t)+\beta g(t) d t=\alpha \int_{a}^{b} f(t) d t+\beta \int_{a}^{b} g(t) d t
$$

Also if $a<b<c$ and $f$ is Riemann integrable, on $[a, c]$, then $\int_{a}^{c} f(t) d t=\int_{a}^{b} f(t) d t+$ $\int_{b}^{c} f(t) d t$.

Proof: The first part is just a slightly simpler version of the proof of Theorem 5.1.5. It remains to consider the triangle inequality and other claims. Let $S(P, f)$ be a Riemann sum for $f$. Then if $f$ is continuous with values in $X$, it follows from the triangle inequality that $t \rightarrow\|f(t)\|$ is also continuous, into $\mathbb{R}$. Then there is a sequence of partitions $P_{m}$ with $\left\|P_{m}\right\| \rightarrow 0$ such that

$$
\left\|\int_{a}^{b} f(t) d t\right\|=\lim _{m \rightarrow \infty}\left\|S\left(P_{m}, f\right)\right\| \leq \lim _{m \rightarrow \infty} S\left(P_{m},\|f\|\right)=\int_{a}^{b}\|f(t)\| d t
$$

The claim about linearity follows right away from taking a limit of sums which are linear. Letting $S\left(P_{m}, f\right) \rightarrow \int_{a}^{b} f(t) d t, S\left(P_{m}, g\right) \rightarrow \int_{a}^{b} g(t) d t$ where these Riemann sums involve the same intermediate points,

$$
\begin{aligned}
\alpha \int_{a}^{b} f(t) d t+\beta \int_{a}^{b} g(t) d t & =\lim _{m \rightarrow \infty}\left(\alpha S\left(P_{m}, f\right)+\beta S\left(P_{m}, g\right)\right) \\
& =\lim _{m \rightarrow \infty} S\left(P_{m}, \alpha f+\beta g\right)=\int_{a}^{b} \alpha f(t)+\beta g(t) d t
\end{aligned}
$$

As for the last part, let $S\left(P_{m}, f\right) \rightarrow \int_{a}^{b} f(t) d t$ and $S\left(\hat{P}_{m}, f\right) \rightarrow \int_{b}^{c} f(t) d t$. Then, letting $Q_{m}=P_{m} \cup \hat{P}_{m}$, we can assume $b \in Q_{m}$ and $S\left(Q_{m}, f\right) \rightarrow \int_{a}^{c} f(t) d t$. Then

$$
\begin{aligned}
\int_{a}^{c} f(t) d t & =\lim _{m \rightarrow \infty} S\left(Q_{m}, f\right)=\lim _{m \rightarrow \infty}\left(S\left(P_{m}, f\right)+S\left(\hat{P}_{m}, f\right)\right) \\
& =\int_{a}^{b} f(t) d t+\int_{b}^{c} f(t) d t
\end{aligned}
$$

Working a little harder, one can show it suffices to have $f, g$ both be only Riemann integrable, not necessarily continuous in all of the above. It is convenient to be able to say that the function is integrable because it is continuous.

Also one obtains easily the fundamental theorem of calculus.
Theorem 5.1.8 Let $f:[a, b] \rightarrow X$ be continuous and let $F(t) \equiv \int_{a}^{t} f(s) d s$. Then $F^{\prime}(t)=f(t)$ for all $t \in[a, b]$, where at the endpoints, the derivative means the appropriate one sided derivative.

Proof: Let $t \in[a, b)$ and let $h$ be small and positive in $[a, b)$.

$$
\begin{gathered}
\left\|\frac{F(t+h)-F(t)}{h}-f(t)\right\|=\left\|\frac{1}{h} \int_{t}^{t+h} f(s) d s-\frac{1}{h} \int_{t}^{t+h} f(t) d s\right\| \\
\leq \frac{1}{h} \int_{t}^{t+h}\|f(s)-f(t)\| d s \leq \frac{1}{h} \int_{t}^{t+h} \varepsilon d s=\varepsilon
\end{gathered}
$$

provided that $h$ is small enough due to continuity of $f$ at $t$. Next suppose $t \in(a, b]$. Then consider $\left\|\frac{F(t+h)-F(t)}{h}-f(t)\right\|$ where $h<0$. Letting $k=-h$,

$$
\frac{F(t+h)-F(t)}{h}=\frac{F(t)-F(t-k)}{k}
$$

and if $k$ is sufficiently small, the same argument holds and you obtain for small negative $h$

$$
\left|\frac{F(t+h)-F(t)}{h}-f(t)\right|=\left|\frac{F(t)-F(t-k)}{k}-f(t)\right| \leq \frac{1}{k} \int_{t-k}^{t}|f(s)-f(t)| d s \leq \varepsilon
$$

provided $|h|=|k|$ is small enough. Hence $F^{\prime}(t)=f(t)$ for all $t \in(a, b)$ and this is also true for one sided derivatives at the end points.

The next lemma is called the mean value inequality.
Lemma 5.1.9 Let $Y$ be a normed vector space and suppose $\mathbf{h}:[a, b] \rightarrow Y$ is differentiable and satisfies $\left\|\mathbf{h}^{\prime}(t)\right\| \leq M, M \geq 0$. Then $\|\mathbf{h}(b)-\mathbf{h}(a)\| \leq M(b-a)$.

Proof: Let $\varepsilon>0$ be given and let

$$
S \equiv\{t \in[a, b]: \text { for all } s \in[a, t],\|\mathbf{h}(s)-\mathbf{h}(a)\| \leq(M+\varepsilon)(s-a)\}
$$

Then $a \in S$. Let $t=\sup S$. Then by continuity of $\mathbf{h}$ it follows

$$
\begin{equation*}
\|\mathbf{h}(t)-\mathbf{h}(a)\|=(M+\varepsilon)(t-a) \tag{5.7}
\end{equation*}
$$

Suppose $t<b$. Then there exist positive numbers, $h_{k}$ decreasing to 0 such that

$$
\left\|\mathbf{h}\left(t+h_{k}\right)-\mathbf{h}(a)\right\|>(M+\varepsilon)\left(t+h_{k}-a\right)
$$

and now it follows from 5.7 and the triangle inequality that

$$
\begin{aligned}
& \left\|\mathbf{h}\left(t+h_{k}\right)-\mathbf{h}(t)\right\|+\|\mathbf{h}(t)-\mathbf{h}(a)\| \\
= & \left\|\mathbf{h}\left(t+h_{k}\right)-\mathbf{h}(t)\right\|+(M+\varepsilon)(t-a)>(M+\varepsilon)\left(t+h_{k}-a\right)
\end{aligned}
$$

and so

$$
\left\|\mathbf{h}\left(t+h_{k}\right)-\mathbf{h}(t)\right\|>(M+\varepsilon) h_{k}
$$

Now dividing by $h_{k}$ and letting $k \rightarrow \infty,\left\|\mathbf{h}^{\prime}(t)\right\| \geq M+\varepsilon$, a contradiction. Thus $t=1$. Since $\varepsilon$ is arbitrary, the conclusion of the lemma follows.

Corollary 5.1.10 Let $f:[a, b] \rightarrow X$ be continuous. Suppose $F^{\prime}(x)=f(x)$ for all $x \in$ $(a, b)$ where $F$ is a continuous function on $[a, b]$. Then $\int_{a}^{b} f(x) d x=F(b)-F(a)$. Also if $f$ is real valued, there exists $y \in(a, b)$ with $\int_{a}^{b} f(x) d x=f(y)(b-a)$.

Proof: Let $G(t) \equiv \int_{a}^{t} f(s) d s$. Then

$$
G(b)-G(a)-(F(b)-F(a))=(G(b)-F(b))-(G(a)-F(a)) .
$$

$(G-F)^{\prime}(x)=0$ so by Lemma 5.1.9,

$$
\|(G(b)-F(b))-(G(a)-F(a))\|=0
$$

Thus, since $G(a)=0, G(b)=\int_{a}^{b} f(s) d s=F(b)-F(a)$. The last assertion is from the mean value theorem.

Although the main interest is in continuous functions, here is an important case.
Theorem 5.1.11 Let $f$ be decreasing and real valued on $[a, b]$ then $f$ is Riemann integrable.

Proof: Let $S_{n}$ be the closure of all Riemann sums for $f$ corresponding to a partition $P$ which has $\|P\| \leq 1 / n$. Let $P=\left\{t_{0}, \ldots, t_{m}\right\}$ be a partition and let $\sum_{i=1}^{m} f\left(\tau_{i}\right)\left(t_{i}-t_{i-1}\right)$ and $\sum_{i=1}^{m} f\left(\hat{\tau}_{i}\right)\left(t_{i}-t_{i-1}\right)$ be two Riemann sums for such a partition $P$. Then, since $f$ is decreasing,

$$
\begin{aligned}
\left|\sum_{i=1}^{m} f\left(\tau_{i}\right)\left(t_{i}-t_{i-1}\right)-\sum_{i=1}^{m} f\left(\hat{\tau}_{i}\right)\left(t_{i}-t_{i-1}\right)\right| & \leq \sum_{i=1}^{m}\left(f\left(t_{i-1}\right)-f\left(t_{i}\right)\right)\left(t_{i}-t_{i-1}\right) \\
& \leq\|P\|(f(a)-f(b))
\end{aligned}
$$

It follows that the diameter of $S_{n}$ is no more than $\frac{1}{n}(f(a)-f(b))$. Therefore, there is a unique point in $\cap_{n=1}^{\infty} S_{n}$ and from the definition, $\lim _{\|P\| \rightarrow 0} S(P)$ exists and is the integral.

The same proof shows that increasing functions are Riemann integrable, and then this generalizes to any function which is either increasing or decreasing on each of finitely many non-overlapping intervals whose union is $[a, b]$ will also be Riemann integrable. Thus all reasonable real valued functions are Riemann integrable.

### 5.2 Estimates and Approximations

The following theorem follows easily from the above definitions and theorem.
Theorem 5.2.1 Let $\mathbf{f} \in C\left(\gamma^{*}\right)$ and let $\gamma:[a, b] \rightarrow \mathbb{R}^{p}$ be of bounded variation and continuous. Let $M$ be at least as large as the maximum of $|\mathbf{f}|$ on $\gamma^{*}$. That is,

$$
\begin{equation*}
M \geq \max \{|\mathbf{f} \circ \gamma(t)|: t \in[a, b]\} \tag{5.8}
\end{equation*}
$$

Then

$$
\begin{equation*}
\left|\int_{\gamma} \mathbf{f} \cdot d \gamma\right| \leq M V(\gamma,[a, b]) \tag{5.9}
\end{equation*}
$$

Also if $\left\{\mathbf{f}_{n}\right\}$ is a sequence of functions continuous on $\gamma^{*}$ which is converging uniformly to the function $\mathbf{f}$ on $\gamma^{*}$, then

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \int_{\gamma} \mathbf{f}_{n} \cdot d \gamma=\int_{\gamma} \mathbf{f} \cdot d \gamma \tag{5.10}
\end{equation*}
$$

Proof: Let 5.8 hold. From the proof of Theorem 5.1.5 on existence, when $\|P\|<$ $\delta_{m},\left|\int_{\gamma} \mathbf{f} \cdot d \gamma-S(P)\right| \leq \frac{2}{m} V(\gamma,[a, b])$ and so $\left|\int_{\gamma} \mathbf{f} \cdot d \gamma\right| \leq|S(P)|+\frac{2}{m} V(\gamma,[a, b])$ Then by the triangle inequality and Cauchy Schwarz inequality,

$$
\begin{aligned}
& \leq \sum_{j=1}^{n} M\left|\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right|+\frac{2}{m} V(\gamma,[a, b]) \\
& \leq M V(\gamma,[a, b])+\frac{2}{m} V(\gamma,[a, b])
\end{aligned}
$$

This proves 5.9 since $m$ is arbitrary. To verify 5.10 use the above inequality to write

$$
\begin{aligned}
& \quad\left|\int_{\gamma} \mathbf{f} \cdot d \gamma-\int_{\gamma} \mathbf{f}_{n} \cdot d \gamma\right|=\left|\int_{\gamma}\left(\mathbf{f}-\mathbf{f}_{n}\right) \cdot d \gamma(t)\right| \\
& \leq \max \left\{\left|\mathbf{f} \circ \gamma(t)-\mathbf{f}_{n} \circ \gamma(t)\right|: t \in[a, b]\right\} V(\gamma,[a, b]) .
\end{aligned}
$$

Since the convergence is assumed to be uniform, this proves 5.10.
As an easy example of a curve of bounded variation, here is an easy lemma.

Lemma 5.2.2 Let $\gamma:[a, b] \rightarrow \mathbb{R}^{p}$ be in $C^{1}([a, b])$. Then $V(\gamma,[a, b])<\infty$ so $\gamma$ is of bounded variation.

Proof: You can use Proposition 5.1.7 in the first inequality.

$$
\begin{aligned}
\sum_{j=1}^{n}\left|\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right| & =\sum_{j=1}^{n}\left|\int_{t_{j-1}}^{t_{j}} \gamma^{\prime}(s) d s\right| \leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_{j}}\left|\gamma^{\prime}(s)\right| d s \\
& \leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_{j}}\left\|\gamma^{\prime}\right\|_{\infty} d s=\left\|\gamma^{\prime}\right\|_{\infty}(b-a)
\end{aligned}
$$

Therefore it follows $V(\gamma,[a, b]) \leq\left\|\gamma^{\prime}\right\|_{\infty}(b-a)$. Here

$$
\|\gamma\|_{\infty}=\max \{|\gamma(t)|: t \in[a, b]\}
$$

which exists by Theorem 2.5.26.
Theorem 5.2.3 Let $\gamma:[a, b] \rightarrow \mathbb{R}^{p}$ be continuous and of bounded variation. Let $\Omega$ be an open set containing $\gamma^{*}$ and let $\mathbf{f}: \Omega \rightarrow \mathbb{R}^{p}$ be continuous, and let $\varepsilon>0$ be given. Then there exists $\eta:[a, b] \rightarrow \mathbb{R}^{p}$ such that $\eta(a)=\gamma(a), \gamma(b)=\eta(b), \eta \in C^{1}([a, b])$, and

$$
\begin{gather*}
\|\gamma-\eta\| \leq \varepsilon,\|\gamma-\eta\| \equiv \max \{|\gamma(t)-\eta(t)|: t \in[a, b]\}  \tag{5.11}\\
\left|\int_{\gamma} \mathbf{f} \cdot d \gamma-\int_{\eta} \mathbf{f} d \eta\right|<\varepsilon, \text { all } z \in K  \tag{5.12}\\
V(\eta,[a, b]) \leq V(\gamma,[a, b]) \tag{5.13}
\end{gather*}
$$

Proof: Extend $\gamma$ to be defined on all $\mathbb{R}$ according to $\gamma(t)=\gamma(a)$ if $t<a$ and $\gamma(t)=\gamma(b)$ if $t>b$. Now define for $0 \leq h \leq 1$.

$$
\gamma_{h}(t) \equiv \frac{1}{2 h} \int_{-2 h+t+\frac{2 h}{(b-a)}(t-a)}^{t+\frac{2 h}{(b-a)}(t-a)} \gamma(s) d s, \gamma_{0}(t) \equiv \gamma(t)
$$

where the integral is defined in the obvious way. That is, the $j^{t h}$ component of $\gamma_{h}(t)$ will be $\frac{1}{2 h} \int_{-2 h+t+\frac{2 h}{(b-a)}(t-a)}^{t+\frac{2 h}{(b-a)}(t-a)} \gamma_{j}(s) d s$. Note that $(t, h) \rightarrow \gamma_{h}(t)$ is clearly continuous on the compact set $[a, b] \times[0,1]$ if $\gamma_{0}(0) \equiv \gamma(t)$. This is from the fundamental theorem of calculus, Theorem 5.1.8, and the observation that

$$
t+\frac{2 h}{(b-a)}(t-a)-\left(-2 h+t+\frac{2 h}{(b-a)}(t-a)\right)=2 h
$$

Thus $(t, h) \rightarrow \gamma_{h}(t)$ is uniformly continuous on this set by Theorem 2.5.28. Also, the definition implies

$$
\gamma_{h}(b)=\frac{1}{2 h} \int_{b}^{b+2 h} \gamma(s) d s=\gamma(b), \gamma_{h}(a)=\frac{1}{2 h} \int_{a-2 h}^{a} \gamma(s) d s=\gamma(a) .
$$

By continuity of $\gamma$, the chain rule from beginning calculus, and the fundamental theorem of calculus, Theorem 5.1.8, $\gamma_{h}^{\prime}(t)=$

$$
\begin{aligned}
& \frac{1}{2 h}\left\{\gamma\left(t+\frac{2 h}{b-a}(t-a)\right)\left(1+\frac{2 h}{b-a}\right)-\right. \\
& \left.\gamma\left(-2 h+t+\frac{2 h}{b-a}(t-a)\right)\left(1+\frac{2 h}{b-a}\right)\right\}
\end{aligned}
$$

and so $\gamma_{h} \in C^{1}([a, b])$. Next is a fundamental estimate.
Lemma 5.2.4 $V\left(\gamma_{h},[a, b]\right) \leq V(\gamma,[a, b])$.
Proof: Let $a=t_{0}<t_{1}<\cdots<t_{n}=b$. Then using the definition of $\gamma_{h}$ and changing the variables to make all integrals over $[0,2 h]$,

$$
\begin{gathered}
\sum_{j=1}^{n}\left|\gamma_{h}\left(t_{j}\right)-\gamma_{h}\left(t_{j-1}\right)\right|=\sum_{j=1}^{n}\left|\left[\begin{array}{c}
\frac{1}{2 h} \int_{0}^{2 h} \gamma\left(s-2 h+t_{j}+\frac{2 h}{b-a}\left(t_{j}-a\right)\right) \\
-\gamma\left(s-2 h+t_{j-1}+\frac{2 h}{b-a}\left(t_{j-1}-a\right)\right)
\end{array}\right]\right| \\
\leq \frac{1}{2 h} \int_{0}^{2 h} \sum_{j=1}^{n}\left|\begin{array}{c}
\gamma\left(s-2 h+t_{j}+\frac{2 h}{b-a}\left(t_{j}-a\right)\right) \\
-\gamma\left(s-2 h+t_{j-1}+\frac{2 h}{b-a}\left(t_{j-1}-a\right)\right)
\end{array}\right| d s
\end{gathered}
$$

For a given $s \in[0,2 h]$, the points, $s-2 h+t_{j}+\frac{2 h}{b-a}\left(t_{j}-a\right)$ for $j=1, \cdots, n$ form an increasing list of points in the interval $[a-2 h, b+2 h]$ and so the integrand is bounded above by $V(\gamma,[a-2 h, b+2 h])=V(\gamma,[a, b])$. It follows

$$
\sum_{j=1}^{n}\left|\gamma_{h}\left(t_{j}\right)-\gamma_{h}\left(t_{j-1}\right)\right| \leq V(\gamma,[a, b]) \text { so } V\left(\gamma_{h},[a, b]\right) \leq V(\gamma,[a, b])
$$

With this lemma the proof of Theorem 5.2.3 can be completed without too much trouble. By uniform continuity of $\gamma$, if $h$ is small enough, say $h<\delta_{1}$, then for all $t \in[a, b]$,

$$
\begin{align*}
\left|\gamma(t)-\gamma_{h}(t)\right| & \leq \frac{1}{2 h} \int_{-2 h+t+\frac{2 h}{(b-a)}(t-a)}^{t+\frac{2 h}{(b-a)}(t-a)}|\gamma(s)-\gamma(t)| d s \\
& <\frac{1}{2 h} \int_{-2 h+t+\frac{2 h}{(b-a)}(t-a)}^{t+\frac{2 h}{(b-a)}(t-a)} \varepsilon d s=\varepsilon \tag{5.14}
\end{align*}
$$

This proves 5.11. It remains to verify the approximation of the integrals.
Let $P=\left\{t_{0}, \ldots, t_{n}\right\}$

$$
\begin{align*}
S_{h}(P) & \equiv \sum_{k=1}^{n} \mathbf{f}\left(\gamma_{h}\left(\tau_{k}\right)\right) \cdot\left(\gamma_{h}\left(t_{k}\right)-\gamma_{h}\left(t_{k-1}\right)\right)  \tag{5.15}\\
S(P) & \equiv \sum_{k=1}^{n} \mathbf{f}\left(\gamma\left(\tau_{k}\right)\right) \cdot\left(\gamma\left(t_{k}\right)-\gamma\left(t_{k-1}\right)\right) \tag{5.16}
\end{align*}
$$

From estimates of Theorem 5.1.5 and the fact that the total variation of $\gamma_{h}$ is no more than that of $\gamma$, there exists $\delta_{2}$ such that if $\|P\|<\delta_{2}$, then

$$
\begin{equation*}
\left|\int_{\gamma}^{\mathbf{f}} \cdot d \gamma(t)-S(P)\right|<\frac{\varepsilon}{3},\left|\int_{\gamma_{h}} \mathbf{f} \cdot d \gamma_{h}(t)-S_{h}(P)\right|<\frac{\varepsilon}{3} \tag{5.17}
\end{equation*}
$$

Then consider $\left|S(P)-S_{h}(P)\right|$ where $0<h<\min \left(\delta_{1}, \delta_{2}\right)$. For such a fixed $P$, choose $h$ small enough in 5.15, 5.16 that $\left|S(P)-S_{h}(P)\right|<\frac{\varepsilon}{3}$. Then for this $h$ and $P$,

$$
\begin{aligned}
& \left|\int_{\gamma} \mathbf{f} \cdot d \gamma(t)-\int_{\gamma_{h}} \mathbf{f} \cdot d \gamma_{h}(t)\right| \\
\leq & \left|\int_{\gamma} \mathbf{f} \cdot d \gamma(t)-S(P)\right|+\left|S(P)-S_{h}(P)\right|+\left|\int_{\gamma_{h}} \mathbf{f} \cdot d \gamma_{h}(t)-S_{h}(P)\right| \\
< & \varepsilon / 3+\varepsilon / 3+\varepsilon / 3=\varepsilon
\end{aligned}
$$

Let $\eta \equiv \gamma_{h}$.
This is a very useful theorem because if $\gamma$ is $C^{1}([a, b])$, it is easy to calculate $\int_{\gamma} \mathbf{f} \cdot d \gamma$ and the above theorem allows a reduction to the case where $\gamma$ is $C^{1}$. The next theorem shows how easy it is to compute these integrals in the case where $\gamma$ is $C^{1}$.

Theorem 5.2.5 If $\mathbf{f}: \gamma^{*} \rightarrow \mathbb{R}^{p}$ is continuous and $\gamma:[a, b] \rightarrow \mathbb{R}^{p}$ is in $C^{1}([a, b])$, then

$$
\begin{equation*}
\int_{\gamma} \mathbf{f} \cdot d \gamma=\int_{a}^{b} \mathbf{f}(\gamma(t)) \cdot \gamma^{\prime}(t) d t \tag{5.18}
\end{equation*}
$$

Proof: Let $P=\left\{t_{0}, \cdots, t_{n}\right\}$. Then

$$
S(P)=\sum_{k=1}^{n} \mathbf{f}\left(\gamma\left(\sigma_{k}\right)\right) \cdot\left(\gamma\left(t_{k}\right)-\gamma\left(t_{k-1}\right)\right)=\sum_{k=1}^{n} \sum_{i=1}^{p} f_{i}\left(\gamma\left(\sigma_{k}\right)\right)\left(\gamma_{i}\left(t_{k}\right)-\gamma_{i}\left(t_{k-1}\right)\right)
$$

By the mean value theorem, this is $\sum_{k=1}^{n} \sum_{i=1}^{p} f_{i}\left(\gamma\left(\sigma_{k}\right)\right) \gamma_{i}^{\prime}\left(\tau_{k}^{i}\right)\left(t_{k}-t_{k-1}\right), \tau_{k}^{i} \in\left(t_{k-1}, t_{k}\right)$. This is

$$
\sum_{k=1}^{n} \sum_{i=1}^{p} f_{i}\left(\gamma\left(\sigma_{k}\right)\right) \gamma_{i}^{\prime}\left(\sigma_{k}\right)\left(t_{k}-t_{k-1}\right)+e(\|P\|)
$$

where $\lim _{\|P\| \rightarrow 0} e(\|P\|)=0$. This follows from the uniform continuity of $\gamma_{i}^{\prime}$. Then

$$
\begin{aligned}
\int_{\gamma} \mathbf{f} \cdot d \gamma & =\lim _{\|P\| \rightarrow 0} S(P)=\lim _{\|P\| \rightarrow 0}\left(\sum_{k=1}^{n} \sum_{i=1}^{p} f_{i}\left(\gamma\left(\sigma_{k}\right)\right) \gamma_{i}^{\prime}\left(\sigma_{k}\right)\left(t_{k}-t_{k-1}\right)+e(\|P\|)\right) \\
& =\int_{a}^{b} \mathbf{f}(\gamma(t)) \cdot \gamma^{\prime}(t) d t
\end{aligned}
$$

### 5.2.1 Finding the Length of a $C^{1}$ Curve

It is very easy to find the length of a $C^{1}$ curve.
Proposition 5.2.6 Let $\gamma:[a, b] \rightarrow \mathbb{R}^{p}$ be $C^{1}$. Then $V(\gamma)=\int_{a}^{b}\left|\gamma^{\prime}(t)\right| d t$
Proof: Let $\hat{P}=\left\{t_{0}, \cdots, t_{m}\right\}$ be such that

$$
V(\gamma)-\varepsilon \leq \sum_{j=1}^{m}\left|\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right| \leq V(\gamma)
$$

Now using the same notation, let $P=\left\{t_{0}, \cdots, t_{n}\right\}$ be a partition containing $\hat{P}$ so that the above inequality will hold for all such $P$.

$$
\begin{aligned}
& \sum_{j=1}^{n}\left|\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right|=\sum_{j=1}^{n}\left|\int_{t_{j-1}}^{t_{j}} \gamma^{\prime}(s) d s\right|=\sum_{j=1}^{n}\left(\sum_{k=1}^{p}\left(\int_{t_{j-1}}^{t_{j}} \gamma_{k}^{\prime}(s) d s\right)^{2}\right)^{1 / 2} \\
& =\sum_{j=1}^{n}\left(\sum_{k=1}^{p}\left(\gamma_{k}^{\prime}\left(s_{k j}\right)\left(t_{j}-t_{j-1}\right)\right)^{2}\right)^{1 / 2}=\sum_{j=1}^{n}\left(\sum_{k=1}^{p} \gamma_{k}^{\prime}\left(s_{k j}\right)^{2}\right)^{1 / 2}\left(t_{j}-t_{j-1}\right)
\end{aligned}
$$

where $s_{k j} \in\left(t_{j-1}, t_{j}\right)$. By uniform continuity of $\gamma^{\prime}$, if $\|P\|$ is small enough, the expression on the right equals

$$
\begin{aligned}
& \sum_{j=1}^{n}\left(\left(\sum_{k=1}^{p} \gamma_{k}^{\prime}\left(t_{j-1}\right)^{2}\right)^{1 / 2}+\hat{e}(\|P\|)\right)\left(t_{j}-t_{j-1}\right) \\
= & \sum_{j=1}^{n}\left(\sum_{k=1}^{p} \gamma_{k}^{\prime}\left(t_{j-1}\right)^{2}\right)^{1 / 2}\left(t_{j}-t_{j-1}\right)+e(\|P\|) \\
= & \sum_{j=1}^{n}\left|\gamma^{\prime}\left(t_{j-1}\right)\right|\left(t_{j}-t_{j-1}\right)+e(\|P\|)
\end{aligned}
$$

where $\lim _{\|P\| \rightarrow 0} e(\|P\|)=0$. Then letting $\|P\| \rightarrow 0$, the above converges to $\int_{a}^{b}\left|\gamma^{\prime}(t)\right| d t$ and so

$$
V(\gamma)-\varepsilon \leq \int_{a}^{b}\left|\gamma^{\prime}(t)\right| d t \leq V(\gamma)
$$

since $\varepsilon$ is arbitrary, this shows $V(\gamma)=\int_{a}^{b}\left|\gamma^{\prime}(t)\right| d t$.
Example 5.2.7 Let $\gamma(t) \equiv\left(t, t^{2}, t\right)$ for $t \in[0,1]$. Find the length of the curve determined by this parametrization.
$\left|\gamma^{\prime}(t)\right|=\sqrt{2+4 t^{2}}$ and so the length of the curve is $\int_{0}^{1} \sqrt{2+4 t^{2}} d t$. You can use the standard calculus gimmicks to find this integral or you could find it numerically. It equals $\frac{1}{2} \ln (\sqrt{2}+\sqrt{3})+\frac{1}{2} \sqrt{2} \sqrt{3}$.

### 5.2.2 Curves Defined in Pieces

Definition 5.2.8 If $\gamma_{k}:\left[a_{k}, b_{k}\right] \rightarrow \mathbb{R}^{p}$ is continuous, one to one on $\left[a_{k}, b_{k}\right)$, and of bounded variation, for $k=1, \cdots, m$ and $\gamma_{k}\left(b_{k}\right)=\gamma_{k+1}\left(a_{k}\right)$, define

$$
\begin{equation*}
\int_{\sum_{k=1}^{m} \gamma_{k}} \mathbf{f} \cdot d \gamma \equiv \sum_{k=1}^{m} \int_{\gamma_{k}} \mathbf{f} \cdot d \gamma_{k} \tag{5.19}
\end{equation*}
$$

In addition to this, for $\gamma:[a, b] \rightarrow \mathbb{R}^{p}$, define $-\gamma:[a, b] \rightarrow \mathbb{R}^{p}$ by $-\gamma(t) \equiv \gamma(b+a-t)$. Thus $\gamma$ simply traces out the points of $\gamma^{*}$ in the opposite order.

The following lemma is useful and follows quickly from Theorem 5.1.3. It shows that when you string these finite variation curves together end to end, you could just as well save trouble on the details and consider a single finite variation vector valued function.

Lemma 5.2.9 In the above definition where $\gamma_{k}\left(b_{k}\right)=\gamma_{k+1}\left(a_{k}\right)$, there exists a continuous bounded variation function, $\gamma$ one to one on $\gamma^{-1}\left(\gamma_{k}\left[a_{k}, b_{k}\right)\right)$ which is defined on some closed interval, $[c, d]$, such that $\gamma([c, d])=\cup_{k=1}^{m} \gamma_{k}\left(\left[a_{k}, b_{k}\right]\right)$ and $\gamma(c)=\gamma_{1}\left(a_{1}\right)$ while $\gamma(d)=\gamma_{m}\left(b_{m}\right)$. Furthermore,

$$
\begin{equation*}
\int_{\gamma} \mathbf{f} \cdot d \gamma=\sum_{k=1}^{m} \int_{\gamma_{k}} \mathbf{f} \cdot d \gamma_{k} . \tag{5.20}
\end{equation*}
$$

If $\gamma:[a, b] \rightarrow \mathbb{R}^{p}$ is of bounded variation and continuous, then

$$
\begin{equation*}
\int_{\gamma} \mathbf{f} \cdot d \gamma=-\int_{-\gamma} \mathbf{f} \cdot d \gamma \tag{5.21}
\end{equation*}
$$

Proof: Consider the first claim about the intervals. It is obvious if $m=1$. Suppose then that it holds for $m-1$ and you have $m$ intervals and curves. By induction, there exists $[\hat{c}, \hat{d}]$ and $\hat{\gamma}$ such that $\hat{\gamma}([\hat{c}, \hat{d}])=\cup_{k=1}^{m-1} \gamma_{k}\left(\left[a_{k}, b_{k}\right]\right)$ with $\hat{\gamma}(\hat{c})=\gamma_{1}\left(a_{1}\right), \hat{\gamma}(\hat{d})=\gamma\left(b_{m-1}\right)$, and by assumption, $\gamma_{k}\left(b_{k}\right)=\gamma_{k+1}\left(a_{k}\right)$ for $k \leq m-1$. I will describe an extension of $\hat{\gamma}$ to $\left[\hat{c}, \hat{d}+b_{m}-a_{m}\right]$ which will be defined as $[c, d]$. Thus the new interval coming after $[\hat{c}, \hat{d}]$ will be $\left[\hat{d}, \hat{d}+b_{m}-a_{m}\right]$

$$
\gamma(t) \equiv\left\{\begin{array}{l}
\hat{\gamma}(t) \text { if } t \in[\hat{c}, \hat{d}] \\
\gamma_{m}\left(t-\hat{d}+a_{m}\right) \text { if } t \in\left[\hat{d}, \hat{d}+b_{m}-a_{m}\right]
\end{array}\right.
$$

Now consider the claim in 5.20. In writing the Riemann sums, it can always be assumed that the end points of the intervals $\gamma^{-1}\left(\gamma_{k}\left(\left[a_{k}, b_{k}\right]\right)\right)$ are in the partition since including these points makes $\|P\|$ no larger and the integral is defined in terms of smallness of $\|P\|$. Therefore, 5.20 follows from Theorem 5.1.3 applied to two different parametrizations of $\gamma_{k}^{*}$, the one coming from $\gamma$ and the one coming from $\gamma_{k}$.

Finally consider the last claim. Say $\gamma:[a, b] \rightarrow \mathbb{R}^{n}$. Then $-\gamma(t) \equiv \gamma(a+b-t)$ and so a typical Riemann sum for $-\gamma$ would be

$$
\begin{gathered}
\sum_{i=1}^{n} \mathbf{f}\left(\gamma\left(a+b-\tau_{i}\right)\right) \cdot\left(\gamma\left(a+b-t_{i}\right)-\gamma\left(a+b-t_{i-1}\right)\right), \tau_{i} \in\left[t_{i-1}, t_{i}\right] \\
=-\sum_{i=1}^{n} \mathbf{f}\left(\gamma\left(a+b-\tau_{i}\right)\right) \cdot\left(\gamma\left(a+b-t_{i-1}\right)-\gamma\left(a+b-t_{i}\right)\right)
\end{gathered}
$$

Now $a+b-\tau_{i} \in\left[a+b-t_{i}, a+b-t_{i-1}\right]$ and so the above is just a Riemann sum for $-\int_{\gamma} \mathbf{f} \cdot d \gamma$. Thus, in the limit as $\|P\| \rightarrow 0$, one obtains $-\int_{\gamma} \mathbf{f} \cdot d \gamma$.

### 5.3 Conservative Vector Fields

Recall the gradient of a scalar function $\mathbf{x} \rightarrow F(\mathbf{x}),\left(\begin{array}{lll}F_{x_{1}} & \cdots & F_{x_{p}}\end{array}\right)^{T} \equiv D F^{T}$ the transpose of the matrix of the derivative of $F$. (It is best not to worry too much about this distinction between the gradient and the derivative at this point.)

Theorem 5.3.1 Let $\gamma:[a, b] \rightarrow \mathbb{R}^{p}$ be continuous and of bounded variation. Also suppose $\nabla F=\mathbf{f}$ on $\Omega$, an open set containing $\gamma^{*}$ and $\mathbf{f}$ is continuous on $\Omega$. Then $\int_{\gamma} \mathbf{f} \cdot d \gamma=$ $F(\gamma(b))-F(\gamma(a))$.

Proof: By Theorem 5.2.3 there exists $\eta \in C^{1}([a, b])$ such that $\gamma(a)=\eta(a)$, and $\gamma(b)=$ $\eta(b)$ such that $\left|\int_{\gamma} \mathbf{f} \cdot d \gamma-\int_{\eta} \mathbf{f} \cdot d \eta\right|<\varepsilon$. Then from Theorem 5.2.5, since $\eta$ is in $C^{1}([a, b])$, it follows from the chain rule and the fundamental theorem of calculus that

$$
\begin{aligned}
\int_{\eta} \mathbf{f} \cdot d \eta & =\int_{a}^{b} \mathbf{f}(\eta(t)) \eta^{\prime}(t) d t=\int_{a}^{b} \frac{d}{d t} F(\eta(t)) d t \\
& =F(\eta(b))-F(\eta(a))=F(\gamma(b))-F(\gamma(a))
\end{aligned}
$$

Therefore, $\left|(F(\gamma(b))-F(\gamma(a)))-\int_{\gamma} \mathbf{f} \cdot d \gamma\right|<\varepsilon$ and since $\varepsilon>0$ is arbitrary, this proves the theorem.

Corollary 5.3.2 If $\gamma:[a, b] \rightarrow \mathbb{R}^{p}$ is continuous, has bounded variation, is a closed curve, $\gamma(a)=\gamma(b)$, and $\gamma^{*} \subseteq \Omega$ where $\Omega$ is an open set on which $\nabla F=\mathbf{f}$, then $\int_{\gamma} \mathbf{f} \cdot d \gamma=0$.

Theorem 5.3.3 Let $\Omega$ be a connected open set and let $\mathbf{f}: \Omega \rightarrow \mathbb{R}^{p}$ be continuous. Then $\mathbf{f}$ has a potential $F$ if and only if $\int_{\gamma} \mathbf{f} \cdot d \gamma$ is path independent for all $\gamma$ a bounded variation curve such that $\gamma^{*}$ is contained in $\Omega$. This means the above line integral depends only on $\gamma(a)$ and $\gamma(b)$.

Proof: The first part was proved in Theorem 5.3.1. It remains to verify the existence of a potential in the situation of path independence.

Let $\mathbf{x}_{0} \in \Omega$ be fixed. Let $S$ be the points $\mathbf{x}$ of $\Omega$ which have the property there is a bounded variation curve joining $\mathbf{x}_{0}$ to $\mathbf{x}$. Let $\gamma_{\mathbf{x}_{0} \mathbf{x}}$ denote such a curve. Note first that $S$ is nonempty. To see this, $B\left(\mathbf{x}_{0}, r\right) \subseteq \Omega$ for $r$ small enough. Every $\mathbf{x} \in B\left(\mathbf{x}_{0}, r\right)$ is in $S$. Then $S$ is open because if $\mathbf{x} \in S$, then $B(\mathbf{x}, r) \subseteq \Omega$ for small enough $r$ and if $\mathbf{y} \in B(\mathbf{x}, r)$, you could go take $\gamma_{\mathbf{x}_{0} \mathbf{x}}$ and from $\mathbf{x}$ follow the straight line segment joining $\mathbf{x}$ to $\mathbf{y}$. In addition to this, $\Omega \backslash S$ must also be open because if $\mathbf{x} \in \Omega \backslash S$, then choosing $B(\mathbf{x}, r) \subseteq \Omega$, no point of $B(\mathbf{x}, r)$ can be in $S$ because then you could take the straight line segment from that point to $\mathbf{x}$ and conclude that $\mathbf{x} \in S$ after all. Therefore, since $\Omega$ is connected, it follows $\Omega \backslash S=\emptyset$. Thus for every $\mathbf{x} \in S$, there exists $\gamma_{\mathbf{x}_{0} \mathbf{x}}$, a bounded variation curve from $\mathbf{x}_{0}$ to $\mathbf{x}$.

Define $F(\mathbf{x}) \equiv \int_{\gamma_{\mathbf{x}_{0}} \mathbf{x}} \mathbf{f} \cdot d \gamma_{\mathbf{x}_{0} \mathbf{x}}$. Then $F$ is well defined by assumption. Now let $l_{\mathbf{x}\left(\mathbf{x}+\boldsymbol{e}_{k}\right)}$ denote the linear segment from $\mathbf{x}$ to $\mathbf{x}+t \mathbf{e}_{k}$. Thus to get to $\mathbf{x}+t \mathbf{e}_{k}$ you could first follow $\gamma_{\mathbf{x}_{0} \mathbf{x}}$ to $\mathbf{x}$ and from there follow $l_{\mathbf{x}\left(\mathbf{x}+t_{k}\right)}$ to $\mathbf{x}+t \mathbf{e}_{k}$. Hence $\frac{F\left(\mathbf{x}+\mathrm{t}_{k}\right)-F(\mathbf{x})}{t}=\frac{1}{t} \int_{l_{\mathbf{x}\left(\mathbf{x}+e_{k}\right)}} \mathbf{f}$. $d l_{\mathbf{x}\left(\mathbf{x}+t \mathbf{t}_{k}\right)}=\frac{1}{t} \int_{0}^{t} \mathbf{f}\left(\mathbf{x}+s \mathbf{e}_{k}\right) \cdot \mathbf{e}_{k} d s \rightarrow f_{k}(\mathbf{x})$ by continuity of $\mathbf{f}$. Thus $\nabla F=\mathbf{f}$.

Corollary 5.3.4 Let $\Omega$ be a connected open set and $\mathbf{f}: \Omega \rightarrow \mathbb{R}^{p}$. Then $\mathbf{f}$ has a potential if and only if every closed, $\gamma(a)=\gamma(b)$, bounded variation curve contained in $\Omega$ has the property that $\int_{\gamma} \mathbf{f} \cdot d \gamma=0$.

Proof: Using Lemma 5.2.9, this condition about closed curves is equivalent to the condition that the line integrals of the above theorem are path independent. This proves the corollary.

Such a vector valued function is called conservative. Summarizing the above we have the following major theorem which is called the fundamental theorem of line integrals.

Theorem 5.3.5 Let $\mathrm{f}: \Omega \rightarrow \mathbb{R}^{p}$ be a $C^{1}$ vector field, meaning the partial derivatives exist and are continuous. Also let $\Omega$ be open and connected. Then for $\gamma$ a continuous bounded variation curve, the following are equivalent.

1. $\mathbf{f}$ is conservative meaning $\mathbf{f}=\nabla F$.
2. $\int_{\gamma} \mathbf{f} \cdot d \gamma$ is path independent whenever $\gamma^{*} \subseteq \Omega$.
3. $\int_{\gamma} \mathbf{f} \cdot d \gamma=\mathbf{0}$ whenever $\gamma$ is a closed curve, meaning that $\gamma:[a, b] \rightarrow \mathbb{R}^{p}, \gamma(a)=\gamma(b)$.

### 5.4 Orientation Of Curves

A curve $C$ is a set of points of the form $\gamma([a, b])$ where $\gamma$ is one to one on $[a, b)$. The curve is a simple curve if $\gamma$ is one to one on $[a, b]$ and it is a simple closed curve if $\gamma(a)=\gamma(b)$.

Definition 5.4.1 $A$ set of points $C \subseteq \mathbb{R}^{p}$ is a simple curve if $C=\gamma([a, b])$ for some interval $[a, b]$ where $\gamma$ is continuous on $[a, b]$ and one to one on $[a, b)$. It is a simple curve if $\gamma$ is one to one on $[a, b]$ and it is a simple closed curve if $\gamma(a)=\gamma(b)$.

Lemma 5.4.2 $C$ is a simple closed curve if and only if it is a one to one image of $S^{1}$. Here $S^{1}$ is the unit circle in the plane $x^{2}+y^{2}=1$.

Proof: Say $C=\gamma\left(S^{1}\right)$. Then let $\eta:[0,2 \pi) \rightarrow S^{1}$ be defined by $\eta(t) \equiv(\cos t, \sin t)$. Then it is clear from geometrical reasoning that $\eta$ is one to one and continuous and also that its inverse is continuous. To see the latter fairly easily, note that $\eta^{-1}$ is continuous on $S^{1} \backslash A_{\varepsilon}$ where $A_{\varepsilon}$ is the circular arc on $S^{1}$ corresponding to the angle being in $(2 \pi-\varepsilon, 2 \pi)$ for each $\varepsilon>0$. This will account for all points of $S^{1}$ for $\varepsilon$ small enough. Then $C=\gamma \circ \eta([0,2 \pi))$ and if we use the same formula for $\eta$ when $t=2 \pi$ this shows $C$ is a simple closed curve.

Conversely, if $C$ is a simple closed curve, then changing the parameter domain to be $[0,2 \pi)$, we can get $\gamma:[0,2 \pi) \rightarrow C$ where $\gamma(0)=\gamma(2 \pi), \gamma$ one to one on $[0,2 \pi)$ and continuous on $[0,2 \pi]$. Then consider $\gamma \circ \eta^{-1}: S^{1} \rightarrow C$ is one to one and continuous. Note that $\left(x_{1 n}, x_{2 n}\right) \rightarrow(1,0)$ if and only if $\left(x_{1 n},-x_{2 n}\right) \rightarrow(1,0)$ and $\eta^{-1}\left(\mathbf{x}_{n}\right) \rightarrow 2 \pi$ then $\gamma \circ \eta^{-1}\left(\mathbf{x}_{n}\right) \rightarrow \gamma(2 \pi)$ by continuity of $\gamma$.

Definition 5.4.3 Let $\eta, \gamma$ be continuous one to one parametrizations for a simple curve. That is, $\gamma([a, b])=\eta([c, d])$ and $\gamma, \eta$ are one to one on $[a, b)$ and $[c, d)$ respectively. If $\eta^{-1} \circ \gamma$ is increasing, then $\gamma$ and $\eta$ are said to be equivalent parametrizations and this is written as $\gamma \sim \eta$. It is also said that the two parametrizations give the same orientation for the curve when $\gamma \sim \eta$.

First is a discussion of orientation of simple curves and after that, orientation of a simple closed curve is considered. In simple language, the message is that there are exactly two directions of motion along a simple curve.


Lemma 5.4.4 The following hold for $\sim$.

$$
\begin{gather*}
\gamma \sim \gamma,  \tag{5.22}\\
\text { If } \gamma \sim \eta \text { then } \eta \sim \gamma,  \tag{5.23}\\
\text { If } \gamma \sim \eta \text { and } \eta \sim \theta, \text { then } \gamma \sim \theta . \tag{5.24}
\end{gather*}
$$

Proof: Formula 5.22 is obvious because $\gamma^{-1} \circ \gamma(t)=t$ so it is clearly an increasing function. If $\gamma \sim \eta$ then $\gamma^{-1} \circ \eta$ is increasing. Now $\eta^{-1} \circ \gamma$ must also be increasing because it is the inverse of $\gamma^{-1} \circ \eta$. This verifies 5.23. To see 5.24, $\gamma^{-1} \circ \theta=\left(\gamma^{-1} \circ \eta\right) \circ\left(\eta^{-1} \circ \theta\right)$ and so since both of these functions are increasing, it follows $\gamma^{-1} \circ \theta$ is also increasing.

Definition 5.4.5 Let $\Gamma$ be a simple curve and let $\gamma$ be a parametrization for $\Gamma$. Denoting by $[\gamma]$ the equivalence class of parameterizations determined by the above equivalence relation, the pair $(\Gamma,[\gamma])$ will be called an oriented curve.

In simple language, an oriented curve is one which has a direction of motion specified. If $\gamma \sim \eta$ then $\gamma=\eta \circ\left(\eta^{-1} \circ \gamma\right)$ and $\eta^{-1} \circ \gamma$ is increasing so $\gamma, \eta$ both trace out the curve in the same direction. This shows the following simple lemma which makes clear the meaning of orientation.

Lemma 5.4.6 Suppose $\gamma^{*}=\eta^{*} . \gamma \sim \eta$ if and only if $\gamma=\eta \circ \phi$ where $\phi$ is an increasing function mapping one half open interval to the other. More generally it is always the case that $\gamma=\eta \circ \phi$ for $\phi$ a one to one mapping from one parameter domain to the other. The two parametrizations are equivalent if and only if $\phi$ is increasing. $\phi$ must be either increasing or decreasing.

Note that this shows that when $\gamma^{*}, \eta^{*}$ are simple curves having endpoints, then $\gamma \sim \eta$ if and only if the first endpoints coincide and the last endpoints coincide.

When the parametrizations are equivalent, this says they preserve the direction of motion along the curve. Recall Theorem 5.1.3. Thus Lemma 5.4.6 shows that the line integral depends only on the set of points in the curve and the orientation or direction of motion along the curve.

The orientation of a simple curve is determined by two points on the curve. This is the idea of the following proposition.

Proposition 5.4.7 Let $(\Gamma,[\gamma])$ be an oriented simple curve, $\gamma([a, b])=\gamma^{*}, \gamma$ one to one on $[a, b]$ and let $\mathbf{p}, \mathbf{q}$ be any two distinct points of $\Gamma$. Then $[\gamma]$ is determined by the order of $\gamma^{-1}(\mathbf{p})$ and $\gamma^{-1}(\mathbf{q})$. This means that $\eta \in[\gamma]$ if and only if $\eta^{-1}(\mathbf{p})$ and $\eta^{-1}(\mathbf{q})$ occur in the same order as $\gamma^{-1}(\mathbf{p})$ and $\gamma^{-1}(\mathbf{q})$. In other words, if and only if $\mathbf{p}, \mathbf{q}$ are encountered in the same order with both parametrizations as the parameter increases.

Proof: This follows from Lemma 5.4.6. $\gamma=\eta \circ \phi$ and if the orders of two points are the same, then $\phi$ can't be decreasing and so it is increasing and $\gamma \sim \eta$.

This shows that the direction of motion on the simple curve is determined by any two points and the determination of which is encountered first by any parametrization in the equivalence class of parameterizations which determines the orientation. Sometimes people indicate this direction of motion by drawing an arrow.

Definition 5.4.8 Let $\Gamma$ be a simple closed curve. This means there is $\gamma: S^{1} \rightarrow \Gamma$ which is one to one and onto. As shown in Lemma 5.4.2 this is equivalent to a parametrization $\gamma:[a, b] \rightarrow \Gamma$ such that $\gamma$ is continuous and one to one on $[a, b)$ but $\gamma(a)=\gamma(b)$.

Simple closed curves are the continuous one to one image of the unit circle. However, one can take any point on the simple closed curve and regard that point as the beginning and ending point. This differs from a simple curve in which you have only two points which can be considered the beginning or the end.

Proposition 5.4.9 Let $\Gamma$ be a simple closed curve. Then for any point $\mathbf{p}$ on $\Gamma$, there is a closed interval $[a, b]$ and a continuous map $\gamma:[a, b] \rightarrow \Gamma$ and $\gamma$ one to one on $[a, b)$ such that $\gamma(a)=\gamma(b)=\mathbf{p}$. That is, $\mathbf{p}$ will be the beginning and ending point.

Proof: Let $\xi: S^{1} \rightarrow \Gamma$ be one to one, onto and continuous. Then $\xi^{-1}(\mathbf{p}) \equiv(\cos \alpha, \sin \alpha)$ for some $\alpha \in[0,2 \pi)$. Then let $\theta:[\alpha, \alpha+2 \pi] \rightarrow S^{1}$ be defined by $\theta(t) \equiv(\cos t, \sin t)$. Then $\theta$ is one to one on $[\alpha, \alpha+2 \pi)$ and continuous on $[\alpha, \alpha+2 \pi]$ and $\theta:[\alpha, \alpha+2 \pi] \rightarrow S^{1}$
is onto and $\theta(\alpha)=\theta(\alpha+2 \pi)$. Consider $\xi \circ \theta \equiv \gamma$. Also, $\gamma(\alpha) \equiv \xi\left(\xi^{-1}(\mathbf{p})\right)=\mathbf{p}$, $\gamma(\alpha+2 \pi)=\xi(\theta(\alpha+2 \pi))=\mathbf{p}$. Clearly $\gamma$ is also continuous on $[\alpha, \alpha+2 \pi]$ and is one to one on $[\alpha, \alpha+2 \pi)$ and maps onto $\Gamma$.

Note that another parametrization could have been $\theta(t) \equiv(\cos (-t), \sin (-t))$ where one would define $\xi^{-1}(\mathbf{p}) \equiv(\cos (-\alpha), \sin (-\alpha))$ which would result in a different direction over $\Gamma$.

If $\Gamma=\gamma([a, b])$ where $\gamma$ is one to one on $[a, b)$, continuous on $[a, b]$ and $\gamma(a)=\gamma(b)$, then if $c \in(a, b)$, you cannot have $\gamma([a, c])=\Gamma$ or $\gamma([c, b])=\Gamma$ because this would violate $\gamma$ being one to one on $[a, b)$. Thus $\gamma(c)$ is neither a beginning nor an ending point for this particular parametrization.

Recall that for simple curves two parametrizations $\gamma, \eta$ are equivalent if $\gamma^{-1} \circ \eta$ preserves the direction of motion over an interval. Here two parametrizations will be equivalent if such a composition preserves motion over a circle according to whether it is clockwise or counterclockwise. There is no mystery about clockwise or counter clockwise motion around the unit circle and we can use this much as there being no mystery about the positive or negative motion on an interval to describe orientation of a simple closed curve.

Definition 5.4.10 Let $\Gamma$ be a simple closed curve. Two parametrizations $\gamma: S^{1} \rightarrow \Gamma$ and $\eta: S^{1} \rightarrow \Gamma$ are equivalent if and only if $\eta^{-1} \circ \gamma$ preserves the direction of motion around $S^{1}$. That is, if for increasing $t \in \mathbb{R}, \mathbf{x}(t)$ is a point on $S^{1}$, then if $\mathbf{x}(t)$ moves clockwise for increasing $t$, so does $\eta^{-1} \circ \gamma(\mathbf{x}(t))$ and if $\mathbf{x}(t)$ moves counter clockwise for increasing $t$, then so does $\eta^{-1} \circ \gamma(\mathbf{x}(t))$.
Lemma 5.4.11 The above definition of orientation of a simple closed curve yields an equivalence relation.

Proof: The proof is just like it was earlier in case of simple curves. It is obvious that $\gamma \sim \gamma$. If $\gamma \sim \eta$ so $\eta^{-1} \circ \gamma$ preserves direction of motion. How about $\gamma^{-1} \circ \eta$ ? If $\mathbf{x}(t)$ is moving counter clockwise then $\mathbf{x}(t)=\left(\eta^{-1} \circ \gamma\right)\left(\gamma^{-1} \circ \eta(\mathbf{x}(t))\right)$. This cannot be true unless $t \rightarrow \gamma^{-1} \circ \eta(\mathbf{x}(t))$ moves counter clockwise as well since you cannot have equality in $t$ of two points on $S^{1}$ which move in opposite directions. Similarly, clockwise motion must also be preserved. The transitive law is fairly obvious also. Say $\gamma \sim \eta$ and $\eta \sim \zeta$. Then $\zeta^{-1} \circ \gamma=\left(\zeta^{-1} \circ \eta\right)\left(\eta^{-1} \circ \gamma\right)$ and the two mappings on the right preserve motion on $S^{1}$.

Note that if you pick any three distinct points on a circle, you can list them in any order and determine a unique direction of motion along the circle by moving over it to encounter the points in the order you chose. This is stated in the following proposition. Note how the order of two points will orient a simple curve and three will orient a simple closed curve.

Proposition 5.4.12 If $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are three distinct points on $S^{1}$, then a single direction of motion around $S^{1}$ is determined by listing these in any particular order. For example, if they are listed in order $\mathbf{p q r}$, meaning one goes from $\mathbf{p}$ then to $\mathbf{q}$ then to $\mathbf{r}$, then this describes a simple curve on $S^{1}$ having $\mathbf{p}$ at one end and $\mathbf{r}$ at the other such that $\mathbf{q}$ is a point of this simple curve. The remainder of $S^{1}$ is a simple curve having end points $\mathbf{r}$ and $\mathbf{p}$ which is oriented so that $\mathbf{r}$ is the first and $\mathbf{p}$ is the last.

With the above obvious proposition, we have the following simple way of orienting a simple closed curve and showing that every simple closed curve is the union of two oriented simple curves joined at their ends.

Proposition 5.4.13 Let $\Gamma$ be a simple closed curve. $\gamma\left(S^{1}\right)=\Gamma$ where $\gamma$ is one to one and onto and continuous. Then the orientation of $\Gamma$ determined by $\gamma$ can be determined by either moving clockwise or counter clockwise over $S^{1}$. This orientation is uniquely defined by the order of any three distinct points of $\Gamma$. This also presents $\Gamma$ as two oriented simple curves joined at their ends. If you have two simple curves joined at their end points $\mathbf{p}, \mathbf{q}$ and at no other points where the first simple curve goes from $\mathbf{p}$ to $\mathbf{q}$ and the second from $\mathbf{q}$ to $\mathbf{p}$, this results in a simple closed curve.

Proof: If the points are $\mathbf{a}, \mathbf{b}, \mathbf{c}$ listed in that order, one would apply Proposition 5.4.12 to $\mathbf{p}=\gamma^{-1}(\mathbf{a}), \mathbf{q}=\gamma^{-1}(\mathbf{b}), \mathbf{r}=\gamma^{-1}(\mathbf{c})$.

Now consider the last claim about joining two simple curves. By assumption there are $\gamma_{i}:\left[a_{i}, b_{i}\right] \rightarrow \Gamma_{i}$ with end points $\gamma_{i}(a), \gamma_{i}(b)$ consisting of the points $\mathbf{p}, \mathbf{q}$. Without loss of generality, one can change the parameter if necessary to have

$$
\gamma_{1}\left(a_{1}\right)=\mathbf{p}, \gamma_{1}\left(b_{1}\right)=\mathbf{q}, \gamma_{2}\left(a_{2}\right)=\mathbf{q}, \gamma_{2}\left(b_{2}\right)=\mathbf{p}
$$

Now change the parameter again to have $\left[a_{1}, b_{1}\right]=[0,1]$ and $\left[a_{2}, b_{2}\right]=[1,2]$. Let

$$
\gamma(t) \equiv\left\{\begin{array}{l}
\gamma_{1}(t) \text { if } t \in[0,1], \gamma_{1}(1)=\mathbf{q} \\
\gamma_{2}(t) \text { if } t \in[1,2], \gamma_{2}(2)=\mathbf{p}
\end{array}\right.
$$

Then $\gamma$ is one to one on $[0,2)$ and has $\gamma(0)=\gamma(2)$. Thus if $\Gamma \equiv \gamma([0,2])$, then $\Gamma$ is a simple closed curve.

We will mainly use orientations on simple curves to specify the direction of motion on a simple closed curve as in the following Proposition. The situation is illustrated by the following picture in which there are two simple closed curves which share a simple curve denoted by $l$. The case we have in mind is in the plane but it would work as well if it were only the case that the two simple closed curves have only the two points $\mathbf{p}, \mathbf{q}$ in common.


Proposition 5.4.14 Let $\Gamma_{1}$ and $\Gamma_{2}$ be two simple closed, oriented curves and let their intersection be $l$. Suppose also that $l$ is itself a simple curve not a point. Also suppose the orientation of $l$ when considered a part of $\Gamma_{1}$ is opposite its orientation when considered a part of $\Gamma_{2}$. Then if the open segment ( $l$ except for its endpoints) of $l$ is removed, the result is a simple closed oriented curve $\Gamma$. This $\Gamma$ has the same orientation as each of $\Gamma_{j}$.


Proof: Say the orientation for $\Gamma_{1}$ comes from $\mathbf{p}$ then $\mathbf{q}$ written as pq. Then if $\mathbf{r} \in \Gamma_{1}$ not in $l$, the ordered list pqr must deliver the orientation of $\Gamma_{1}$. Then the orientation of $\Gamma_{2}$ must come from qps for $\mathbf{s} \in \Gamma_{2}$ not in $l$ by assumption the orientation of $\Gamma_{2}$ involves first $\mathbf{q}$ then $\mathbf{p}$. Thus the part of $\Gamma_{1}$ other than the open segment of $l$ would have the orientation qp. This determines an orientation for $\Gamma$ and does not contradict what is given exactly because the orientation of $l$ is opposite when considered as part of $\Gamma_{1}$ and $\Gamma_{2}$. Thus, if $\gamma$ is a parametrization for $\Gamma$ and $\gamma_{j}, j=1,2$ are corresponding parametrizations for $\Gamma_{j}$, these
have the same orientation if and only if the orientations of $l$ are opposite as part of $\Gamma_{1}$ and $\Gamma_{2}$. Note how this would not work if the segment $l$ were oriented the same way as part of each $\Gamma_{i}$ because you could not get a single orientation for $\Gamma$. You would be describing two different orientations on $S^{1}$.


I will restate the above proposition to emphasize the role of the segment $l$ in determining orientation.

Corollary 5.4.15 Let the intersection of simple closed curves, $\Gamma_{1}$ and $\Gamma_{2}$ consist of the simple curve $l$. Then place opposite orientations on $l$, and use these two different orientations to specify orientations of $\Gamma_{1}$ and $\Gamma_{2}$. Then letting $\Gamma$ denote the simple closed curve which is obtained from deleting the open segment of $l$, there exists an orientation for $\Gamma$ which is consistent with the orientations of $\Gamma_{1}$ and $\Gamma_{2}$ obtained from the given specification of opposite orientations on $l$.

### 5.5 Exercises

1. Let $\mathbf{r}(t)=\left(\ln (t), \frac{t^{2}}{2}, \sqrt{2} t\right)$ for $t \in[1,2]$. Find the length of this curve.
2. Let $\mathbf{r}(t)=\left(\frac{2}{3} t^{3 / 2}, t, t\right)$ for $t \in[0,1]$. Find the length of this curve.
3. Let $\mathbf{r}(t)=(t, \cos (3 t), \sin (3 t))$ for $t \in[0,1]$. Find the length of this curve.
4. Suppose for $t \in[0, \pi]$ the position of an object is given by

$$
\mathbf{r}(t)=\left(\begin{array}{lll}
t & \cos 2 t & \sin 2 t
\end{array}\right)
$$

Let $\mathbf{F}(x, y, z)=\left(\begin{array}{ccc}2 x y & x^{2}+2 z y & y^{2}\end{array}\right)$ Find $\int_{\mathbf{r}} \mathbf{F} \cdot d \mathbf{r}$.
5. Show the mean value theorem for integrals. Suppose $f \in C([a, b])$. Then there exists $x \in[a, b]$, in fact $x$ can be taken in $(a, b)$, such that $f(x)(b-a)=\int_{a}^{b} f(t) d t$. Hint: Use fundamental theorem of calculus.
6. In this problem is a short argument showing a version of what has become known as Fubini's theorem. Suppose $f \in C([a, b] \times[c, d])$. Then

$$
\int_{a}^{b} \int_{c}^{d} f(x, y) d y d x=\int_{c}^{d} \int_{a}^{b} f(x, y) d x d y
$$

First explain why the two iterated integrals make sense. Hint: To prove the two iterated integrals are equal, let $a=x_{0}<x_{1}<\cdots<x_{n}=b$ and $c=y_{0}<y_{1}<\cdots<$ $y_{m}=d$ be two partitions of $[a, b]$ and $[c, d]$ respectively. Then explain why

$$
\begin{aligned}
\int_{a}^{b} \int_{c}^{d} f(x, y) d y d x & =\sum_{i=1}^{n} \sum_{j=1}^{m} \int_{x_{i-1}}^{x_{i}} \int_{y_{j-1}}^{y_{j}} f(s, t) d t d s \\
\int_{c}^{d} \int_{a}^{b} f(x, y) d x d y & =\sum_{j=1}^{m} \sum_{i=1}^{n} \int_{y_{j-1}}^{y_{j}} \int_{x_{i-1}}^{x_{i}} f(s, t) d s d t
\end{aligned}
$$

Now use the mean value theorem for integrals to write

$$
\int_{x_{i-1}}^{x_{i}} \int_{y_{j-1}}^{y_{j}} f(s, t) d t d s=f\left(\hat{s}_{i}, \hat{t}_{j}\right)\left(x_{i}-x_{i-1}\right)\left(y_{i}-y_{i-1}\right)
$$

do something similar for $\int_{y_{j-1}}^{y_{j}} \int_{x_{i-1}}^{x_{i}} f(s, t) d s d t$ and then observe that the difference between the sums can be made as small as desired by simply taking suitable partitions. A complete treatment of Fubini's theorem is later.
7. This chapter is on line integrals. It was almost exclusively oriented toward having $\gamma$ continuous. There is a similar thing called a Riemann Stieltjes integral, written as $\int_{a}^{b} f(t) d g(t)$. A function $f$ (assume here it is scalar valued for simplicity although this is not necessary) is said to be Riemann Stieltjes integrable if there is a number, $I$ such that for all $\varepsilon>0$ there exists $\delta$ such that if $\|P\|<\delta$, then

$$
\left|\sum_{i=1}^{n} f\left(\tau_{i}\right)\left(g\left(t_{i}\right)-g\left(t_{i-1}\right)\right)-I\right|<\varepsilon
$$

for any Riemann Stieltjes sum defined as the above in which $\tau_{i} \in\left[t_{i-1}, t_{i}\right]$. This $I$ is denoted as $\int_{a}^{b} f(t) d g(t)$ and we will say that $f \in R([a, b], g)$. Show that if $g$ is of bounded variation and $f$ is continuous, then $\int_{a}^{b} f(t) d g(t)$ exists. Note the difference between this and $\int_{a}^{b} f(g(t)) d g(t)$ which is a case of line integrals considered in this chapter and how either includes the ordinary Riemann integral $\int_{a}^{b} f(t) d t$.
8. Suppose $\int_{a}^{b} f d g$ exists. Explain the following: Let $P \equiv\left\{x_{0}, \cdots, x_{n}\right\}$ and let $t_{i} \in$ $\left[x_{i-1}, x_{i}\right]$.

$$
\begin{gathered}
f g(b)-f g(a)-\sum_{i=1}^{n} g\left(t_{i}\right)\left(f\left(x_{i}\right)-f\left(x_{i-1}\right)\right) \\
=\sum_{i=1}^{n} f\left(x_{i}\right) g\left(x_{i}\right)-f\left(x_{i-1}\right) g\left(x_{i-1}\right)-\sum_{i=1}^{n} g\left(t_{i}\right)\left(f\left(x_{i}\right)-f\left(x_{i-1}\right)\right) \\
=\sum_{i=1}^{n} f\left(x_{i}\right)\left(g\left(x_{i}\right)-g\left(t_{i}\right)\right)+\sum_{i=1}^{n} f\left(x_{i-1}\right)\left(g\left(t_{i}\right)-g\left(x_{i-1}\right)\right)
\end{gathered}
$$

and if $\|P\|$ is small enough, this is a Riemann sum for $\int_{a}^{b} f d g$ which is closer to $\int_{a}^{b} f d g$ than $\varepsilon$. Use to explain why if $\int_{a}^{b} f d g$ exists, then so does $\int_{a}^{b} g d f$ and $\int_{a}^{b} f d g+$ $\int_{a}^{b} g d f=f g(b)-f g(a)$. Note how this says roughly that $d(f g)=f d g+g d f$. As an example, suppose $g(t)=t$ and $t \rightarrow f(t)$ is decreasing. In particular, it is of bounded variation. Thus $\int_{a}^{b} g d f$ exists. It follows then that $\int_{a}^{b} f d g=\int_{a}^{b} f(t) d t$ exists.
9. Let $f$ be increasing and $g$ continuous on $[a, b]$. Then there exists $c \in[a, b]$ such that

$$
\int_{a}^{b} g d f=g(c)(f(b)-f(c))
$$

Hint: First note $g$ Riemann Stieltjes integrable because it is continuous. Since $g$ is continuous, you can let $m=\min \{g(x): x \in[a, b]\}$ and $M=\max \{g(x): x \in[a, b]\}$ Then $m \int_{a}^{b} d f \leq \int_{a}^{b} g d f \leq M \int_{a}^{b} d f$ Now if $f(b)-f(a) \neq 0$, you could divide by it
and conclude $m \leq \frac{\int_{a}^{b} g d f}{f(b)-f(a)} \leq M$. You need to explain why $\int_{a}^{b} d f=f(b)-f(a)$. Next use the intermediate value theorem to get the term in the middle equal to $g(c)$ for some $c$. What happens if $f(b)-f(a)=0$ ? Modify the argument and fill in the details to show the conclusion still follows.
10. Suppose $g$ is increasing and $f$ is continuous and of bounded variation. Then it follows that $g \in R([a, b], f)$. Show there exists $c \in[a, b]$ such that

$$
\int_{a}^{b} g d f=g(a) \int_{a}^{c} d f+g(b) \int_{c}^{b} d f
$$

This is called the second mean value theorem for integrals. Hint: Use integration by parts.

$$
\int_{a}^{b} g d f=-\int_{a}^{b} f d g+f(b) g(b)-f(a) g(a)
$$

Now use the first mean value theorem, the result of Problem 9 to substitute something for $\int_{a}^{b} f d g$ and then simplify.
11. Let $U$ be an open subset of $\mathbb{R}^{n}$ and suppose that $f:[a, b] \times U \rightarrow \mathbb{R}$ satisfies

$$
(x, y) \rightarrow \frac{\partial f}{\partial y_{i}}(x, y),(x, y) \rightarrow f(x, y)
$$

are all continuous. Show that $\int_{a}^{b} f(x, y) d x, \int_{a}^{b} \frac{\partial f}{\partial y_{i}}(x, y) d x$ all make sense and that in fact $\frac{\partial}{\partial y_{i}}\left(\int_{a}^{b} f(x, y) d x\right)=\int_{a}^{b} \frac{\partial f}{\partial y_{i}}(x, y) d x$ Also explain why $y \rightarrow \int_{a}^{b} \frac{\partial f}{\partial y_{i}}(x, y) d x$ is continuous. Hint: You will need to use the theorems from one variable calculus about the existence of the integral for a continuous function. You may also want to use theorems about uniform continuity of continuous functions defined on compact sets.
12. Show $\int_{0}^{\infty} e^{-t^{2}} d t=\frac{\sqrt{\pi}}{2}$. Hint: For $x \geq 0, f(x)=\left(\int_{0}^{x} e^{-t^{2}} d t\right)^{2}$. Then

$$
f^{\prime}(x)=2 e^{-x^{2}}\left(\int_{0}^{x} e^{-t^{2}} d t\right)=2 e^{-x^{2}}\left(\int_{0}^{1} e^{-x^{2} u^{2}} d u\right)
$$

Now integrate by parts to get

$$
f(x)=-\left.e^{-t^{2}}\left(\int_{0}^{1} e^{-u^{2} t^{2}} d u\right)\right|_{0} ^{x}-\int_{0}^{x} e^{-t^{2}} \int_{0}^{1} 2 u^{2} t e^{-u^{2} t^{2}} d u d t
$$

Now interchange the order of integration using Problem 6. Then do the integrations and let $x \rightarrow \infty$.
13. For $x>0, \Gamma(x) \equiv \int_{0}^{\infty} e^{-t} t^{x-1} d t \equiv \lim _{R \rightarrow \infty} \int_{1 / R}^{R} e^{-t} t^{x-1} d t$. Show this limit exists. $x \rightarrow$ $\Gamma(x)$ is the gamma function. Also show that $\Gamma(x+1)=x \Gamma(x), \Gamma(1)=1$. How does $\Gamma(n)$ for $n$ a positive integer compare with $(n-1)$ !?
14. Suppose $\Gamma$ is a simple curve and $\hat{\Gamma}$ is a simple closed curve. Does there exist a one to one continuous function $\mathbf{g}$ which maps $\Gamma$ onto $\hat{\Gamma}$ ? Explain why or why not.
15. Suppose $\Gamma$ is a simple closed curve. Show there exists a continuous function $\mathbf{f}: \Gamma \rightarrow$ $\Gamma$ such that for all $\mathbf{x} \in \Gamma, \mathbf{f}(\mathbf{x}) \neq \mathbf{x}$. However, if $\Gamma$ is a simple curve, show that if $\mathbf{f}: \Gamma \rightarrow \Gamma$ is continuous, then there is some $\mathbf{x} \in \Gamma$ such that $\mathbf{f}(\mathbf{x})=\mathbf{x}$. Hint: For this last part, show first that if $\mathbf{h}:[0,1] \rightarrow[0,1]$ is continuous, then $\mathbf{h}(x)=x$ for some $x \in[0,1]$.
16. These two problems are on elementary calculus. Recall $\ln (n) \equiv \int_{1}^{n}(1 / t) d t$. Show that for $n \in \mathbb{N}$,

$$
\frac{1}{2}(\ln (n+1)+\ln (n)) \leq \int_{n}^{n+1} \ln (t) d t \leq \ln (n+1 / 2)
$$

This follows easily from consideration of the following graph. The larger trapezoid is obtained from the tangent line through $\left(n+\frac{1}{2}, \ln \left(n+\frac{1}{2}\right)\right)$.

(a) Now $\sum_{k=1}^{n-1}\left(\int_{k}^{k+1} \ln (t) d t-\frac{1}{2}(\ln (k+1)+\ln (k))\right)$ is an increasing sequence of partial sums.
(b) Next consider the following computations coming from the above inequalities

$$
\begin{gathered}
\int_{1}^{n} \ln (t) d t-\sum_{k=1}^{n-1} \frac{1}{2}(\ln (k+1)+\ln (k)) \\
=\sum_{k=1}^{n-1}\left(\int_{k}^{k+1} \ln (t) d t-\frac{1}{2}(\ln (k+1)+\ln (k))\right) \\
\leq \sum_{k=1}^{n-1} \ln (k+1 / 2)-\frac{1}{2}(\ln (k+1)+\ln (k)) \\
=\sum_{k=1}^{n-1} \frac{1}{2}(\ln (k+1 / 2)-\ln (k))-\sum_{k=2}^{n} \frac{1}{2}(\ln (k)-\ln (k-1 / 2)) \\
=\frac{1}{2} \ln \left(\frac{3}{2}\right)-\frac{1}{2}(\ln (n)-\ln (n-1 / 2)) \leq \frac{1}{2} \ln \left(\frac{3}{2}\right)
\end{gathered}
$$

Therefore, the series in part a.) converges to some $c$.
(c) Note that the series in a.) equals

$$
S_{n} \equiv \int_{1}^{n} \ln (t) d t-\sum_{k=1}^{n-1} \frac{1}{2}(\ln (k+1)+\ln (k))
$$

Hence $\lim _{n \rightarrow \infty} \exp \left(S_{n}\right)=e^{c}$. Thus

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \frac{\exp (n \ln n-n+1)}{\prod_{k=1}^{n-1} \exp (\sqrt{(k+1) k})} \\
= & \frac{n^{n} e^{-n} e}{\prod_{k=1}^{n-1} \sqrt{k+1} \prod_{k=1}^{n-1} \sqrt{k}} \\
= & \frac{n^{n} e^{-n} e n^{1 / 2}}{\prod_{k=1}^{n-1} \sqrt{k+1} \prod_{k=1}^{n} \sqrt{k}}=\frac{n^{n+(1 / 2)} e}{n!e^{n}}=e^{c}
\end{aligned}
$$

Thus there is a constant $k=1 / e^{1-c}$ such that $\lim _{n \rightarrow \infty} \frac{n!e^{n}}{n^{(n+(1 / 2))}}=k$. It is possible to show that $k=\sqrt{2 \pi}$ but in most applications, it suffices to know the existence of the limit. This is Stirling's formula.
17. Show $k=\sqrt{2 \pi}$ as follows. Verify the following:

$$
\int_{0}^{\pi / 2} \sin ^{2 m}(x)=\frac{(2 m-1)(2 m-3) \cdots 1}{2^{m} m!} \frac{\pi}{2}
$$

$\int_{0}^{\pi / 2} \sin ^{2 m+1}(x)=\frac{2^{m} m!}{(2 m+1)(2 m-1) \cdots 3}, \int_{0}^{\pi / 2} \sin ^{n}(x)=\frac{n-1}{n} \int_{0}^{\pi / 2} \sin ^{n-2}(x)$. Show the following using the above.

$$
\begin{aligned}
& \left(\frac{2 m+1}{2 m}\right) 1 \geq \frac{\int_{0}^{\pi / 2} \sin ^{2 m}(x)}{\frac{2 m}{2 m+1} \int_{0}^{\pi / 2} \sin ^{2 m-1}(x)}=\frac{\int_{0}^{\pi / 2} \sin ^{2 m}(x)}{\int_{0}^{\pi / 2} \sin ^{2 m+1}(x)} \\
= & \frac{\frac{(2 m-1)(2 m-3) \cdots 1}{2^{m} m!} \frac{\pi}{2}}{\frac{2^{m} m!}{(2 m+1)(2 m-1) \cdots 3}}=\frac{(2 m+1)(2 m-1)^{2}(2 m-3)^{2} \cdots 1^{2}}{4^{m}(m!)^{2}} \frac{\pi}{2} \geq 1
\end{aligned}
$$

Then multiply on the top and bottom by $(2 m)^{2}(2(m-1))^{2} \cdots 2^{2}$. Obtain

$$
\lim _{m \rightarrow \infty} \frac{2^{m} m!2^{m}(m!)}{\sqrt{(2 m+1)}(2 m)!} \sqrt{\frac{2}{\pi}}=1
$$

Now $\lim _{m \rightarrow \infty} \frac{(m!)^{2} e^{2 m}}{k^{2} m^{2 m+1}}=1=\lim _{m \rightarrow \infty} \frac{(2 m)!e^{2 m}}{k(2 m)^{2 m+(1 / 2)}}$ where $k$ is the constant of the above problem. Thus

$$
\lim _{m \rightarrow \infty} \frac{k^{2} m^{2 m+1}(2 m)!e^{2 m}}{(m!)^{2} e^{2 m} k(2 m)^{2 m+(1 / 2)}}=1
$$

Now

$$
\begin{aligned}
& \lim _{m \rightarrow \infty} \frac{2^{m} m!2^{m}(m!)}{\sqrt{(2 m+1)}(2 m)!} \sqrt{\frac{2}{\pi}} \frac{k^{2} m^{2 m+1}(2 m)!e^{2 m}}{(m!)^{2} e^{2 m} k(2 m)^{2 m+(1 / 2)}} \\
= & \lim _{m \rightarrow \infty} \frac{1}{\sqrt{(m+(1 / 2))}} \sqrt{\frac{1}{2 \pi}} k m^{1 / 2}=\sqrt{\frac{1}{2 \pi}} k=1
\end{aligned}
$$

## Chapter 6

## Measures and Measurable Functions

The Lebesgue integral is much better than the Rieman integral. This has been known for over 100 years. It is much easier to generalize to many dimensions and it is much easier to use in applications. It is also this integral which is most important in probability. However, this integral is more abstract. This chapter will develop the notion of measures which are used for this integral. Complex analysis does not usually require the use of this superior integral however, but the approach we take here will involve integration of a function of more than one variable and when you do this, the Riemann integral becomes totally insufferable, forcing one to consider things like the Jordan content of the boundary and so forth.
Definition 6.0.1 Let $\Omega$ be a nonempty set. $\mathscr{F} \subseteq P(\Omega)$, the set of all subsets of $\Omega$, is called a $\sigma$ algebra if it contains $\emptyset, \Omega$, and is closed with respect to countable unions and complements. That is, if $\left\{A_{n}\right\}_{n=1}^{\infty}$ is countable and each $A_{n} \in \mathscr{F}$, then $\cup_{n=1}^{\infty} A_{m} \in \mathscr{F}$ also and if $A \in \mathscr{F}$, then $\Omega \backslash A \equiv A^{C} \in \mathscr{F}$. It is clear that any intersection of $\sigma$ algebras is a $\sigma$ algebra. If $\mathscr{K} \subseteq P(\Omega), \sigma(\mathscr{K})$ is the smallest $\sigma$ algebra which contains $\mathscr{K}$.

Observation 6.0.2 For $A_{i} \in \mathscr{F}$ a $\sigma$ algebra, then $\cap_{i=1}^{\infty} A_{i}=\left(\cup_{i=1}^{\infty} A_{i}^{C}\right)^{C} \in \mathscr{F}$.
Thus countable unions, countable intersections, and complements of sets of $\mathscr{F}$ stay in $\mathscr{F}$.

### 6.1 Measurable Functions

Then for functions which have values in $(-\infty, \infty]$ we have the following Lemma.
Notation 6.1.1 In whatever context, $f^{-1}(S) \equiv\{\omega \in \Omega: f(\omega) \in S\}$. It is called the inverse image of $S$ and everything in the theory of the Lebesgue integral is formulated in terms of this. Sometimes I will write $f^{-1}(S)$ as $[f(\omega) \in S]$ or even $[f \in S]$.

Lemma 6.1.2 Let $f: \Omega \rightarrow(-\infty, \infty]$ where $\mathscr{F}$ is a $\sigma$ algebra of subsets of $\Omega$. The following are equivalent.

$$
\begin{gathered}
f^{-1}((d, \infty]) \in \mathscr{F} \text { for all finite } d, \\
f^{-1}((-\infty, d)) \in \mathscr{F} \text { for all finite } d, \\
f^{-1}([d, \infty]) \in \mathscr{F} \text { for all finite } d, \\
f^{-1}((-\infty, d]) \in \mathscr{F} \text { for all finite } d, \\
f^{-1}((a, b)) \in \mathscr{F} \text { for all } a<b,-\infty<a<b<\infty .
\end{gathered}
$$

Proof: First note that the first and the third are equivalent. To see this, observe

$$
f^{-1}([d, \infty])=\cap_{n=1}^{\infty} f^{-1}((d-1 / n, \infty])
$$

and so if the first condition holds, then so does the third.

$$
f^{-1}((d, \infty])=\cup_{n=1}^{\infty} f^{-1}([d+1 / n, \infty])
$$

and so if the third condition holds, so does the first.

Similarly, the second and fourth conditions are equivalent. Now

$$
f^{-1}((-\infty, d])=\left(f^{-1}((d, \infty])\right)^{C}
$$

so the first and fourth conditions are equivalent. Thus the first four conditions are equivalent and if any of them hold, then for $-\infty<a<b<\infty$,

$$
f^{-1}((a, b))=f^{-1}((-\infty, b)) \cap f^{-1}((a, \infty]) \in \mathscr{F} .
$$

Finally, if the last condition holds,

$$
f^{-1}([d, \infty])=\left(\cup_{k=1}^{\infty} f^{-1}((-k+d, d))\right)^{C} \in \mathscr{F}
$$

and so the third condition holds. Therefore, all five conditions are equivalent.
Definition 6.1.3 When a function satisfies any of these equivalent conditions, we say the function is measuragle .

From this, it is easy to verify that pointwise limits of a sequence of measurable functions are measurable.

Corollary 6.1.4 If $f_{n}(\omega) \rightarrow f(\omega)$ where all functions have values in $(-\infty, \infty]$, then if each $f_{n}$ is measurable, so is $f$.

Proof: Note the following which holds for any $c \in \mathbb{R}$ :

$$
f^{-1}((c, \infty])=\cup_{k=1}^{\infty} \cap_{n \geq k} f_{n}^{-1}((c, \infty]) \subseteq f^{-1}([c, \infty])
$$

This follows from the definition of the limit. Therefore,

$$
\begin{aligned}
f^{-1}((b, \infty]) & =\cup_{l=1}^{\infty} f^{-1}\left(\left(b+\frac{1}{l}, \infty\right]\right)=\cup_{l=1}^{\infty} \cup_{k=1}^{\infty} \cap_{n \geq k} f_{n}^{-1}\left(\left(b+\frac{1}{l}, \infty\right]\right) \\
& \subseteq \cup_{l=1}^{\infty} f^{-1}\left(\left[b+\frac{1}{l}, \infty\right]\right)=f^{-1}((b, \infty])
\end{aligned}
$$

The messy term on the middle is measurable because it consists of countable unions and intersections of measurable sets. It equals $f^{-1}((b, \infty])$ and so this is also measurable. By Lemma 6.1.2, $f$ is measurable.

Observation 6.1.5 If $f: \Omega \rightarrow \mathbb{R}$ then the above definition of measurability holds with no change. In this case, $f$ never achieves the value $\infty$. This is actually the case of most interest.

The following theorem is of major significance. I will use this whenever it is convenient. Let $(\Omega, \mathscr{F})$ be a measurable space.

Theorem 6.1.6 Suppose $f: \Omega \rightarrow \mathbb{R}$ is measurable and $g: \mathbb{R} \rightarrow \mathbb{R}$ is continuous. Then $g \circ f$ is measurable. Also, if $f, g$ are measurable real valued functions, then their sum is also measurable and real valued as are all linear combinations of these functions.

Proof: $(g \circ f)^{-1}((a, \infty))=f^{-1}\left(g^{-1}((a, \infty))\right)$ and by continuity of $g$, it follows that $g^{-1}((a, \infty))$ is an open set. Thus it is the disjoint union of countably many open intervals by Theorem 2.10.8. It follows that $f^{-1}\left(g^{-1}((a, \infty))\right)$ is the countable union of measurable sets and is therefore measurable.

Why is $f+g$ measurable when $f, g$ are real valued measurable functions? This is a little trickier. Let the rational numbers be $\left\{r_{n}\right\}_{n=1}^{\infty}$.

$$
(f+g)^{-1}((a, \infty))=\cup_{n=1}^{\infty} g^{-1}\left(r_{n}, \infty\right) \cap f^{-1}\left(a-r_{n}, \infty\right)
$$

It is clear that the expression on the right is contained in $(f+g)^{-1}(a, \infty)$. Why are they actually equal? Suppose $\omega \in(f+g)^{-1}(a, \infty)$. Then $f(\omega)+g(\omega)>a$ and there exists $r_{n}$ a rational number smaller than $g(\omega)$ such that $f(\omega)+r_{n}>a$. Therefore, $\omega \in g^{-1}\left(r_{n}, \infty\right) \cap$ $f^{-1}\left(a-r_{n}\right)$ and so the two sets are actually equal as claimed. Now by the first part, if $f$ is measurable and $a$ is a real number, then $a f$ is measurable also. Thus linear combinations of measurable functions are measurable.

The above is now generalized to give a theorem about measurability of a continuous combination of measurable functions. First note the following.

Lemma 6.1.7 Let $(\Omega, \mathscr{F})$ be a measurable space. Then $f: \Omega \rightarrow \mathbb{R}$ is measurable if and only if $f^{-1}(U) \in \mathscr{F}$ whenever $U$ is an open set.

Proof: First suppose $f$ is measurable. From Theorem 2.4.2, $U=\cup_{k=1}^{\infty}\left(a_{k}, b_{k}\right)$ for suitable open intervals. Hence $f^{-1}(U)=\cup_{k=1}^{\infty} f^{-1}\left(a_{k}, b_{k}\right)$ and each term in the union is measurable. Conversely, if $f^{-1}(U) \in \mathscr{F}$ for every $U$ open, then this is true for $U=(a, b)$ and so by Lemma 6.1.2, $f$ is measurable.

Proposition 6.1.8 Let $f_{i}: \Omega \rightarrow \mathbb{R}$ be measurable, $(\Omega, \mathscr{F})$ a measurable space, and let $g: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be continuous. If $\mathbf{f}(\omega)=\left(\begin{array}{lll}f_{1}(\omega) & \cdots & f_{n}(\omega)\end{array}\right)^{T}$, then $g \circ \mathbf{f}$ is measurable.

Proof: From the above lemma, it suffices to verify that $(g \circ \mathbf{f})^{-1}(U)$ is measurable whenever $U$ is open. However,

$$
(g \circ \mathbf{f})^{-1}(U)=\mathbf{f}^{-1}\left(g^{-1}(U)\right)
$$

Since $g$ is continuous, it follows from Proposition 2.5.19 that $g^{-1}(U)$ is an open set in $\mathbb{R}^{n}$. By Proposition 2.4.5 there are countably many open sets $B_{i}=B\left(\mathbf{x}_{i}, r_{i}\right)$ whose union is $g^{-1}(U)$. We will use the norm $\|\cdot\|_{\infty}$ so that these $B_{i}$ are of the form $B_{i}=\prod_{k=1}^{n}\left(a_{k}^{i}, b_{k}^{i}\right)$. Thus

$$
\mathbf{f}^{-1}\left(g^{-1}(U)\right)=\mathbf{f}^{-1}\left(\cup_{i=1}^{\infty} B_{i}\right)=\cup_{i=1}^{\infty} \mathbf{f}^{-1}\left(B_{i}\right)=\cup_{i=1}^{\infty} \cap_{k=1}^{n} f_{k}^{-1}\left(\left(a_{k}^{i}, b_{k}^{i}\right)\right) \in \mathscr{F}
$$

Note that this includes all of Theorem 6.1.6 as a special case.
There is a fundamental theorem about the relationship of simple functions to measurable functions given in the next theorem.

Definition 6.1.9 Let $E \in \mathscr{F}$ for $\mathscr{F}$ a $\sigma$ algebra. Then

$$
\mathscr{X}_{E}(\omega) \equiv\left\{\begin{array}{l}
1 \text { if } \omega \in E \\
0 \text { if } \omega \notin E
\end{array}\right.
$$

This is called the indicator function of the set $E$. Let $s:(\Omega, \mathscr{F}) \rightarrow \mathbb{R}$. Then $s$ is a simple function if it is of the form

$$
s(\omega)=\sum_{i=1}^{n} c_{i} \mathscr{X}_{E_{i}}(\omega)
$$

where $E_{i} \in \mathscr{F}$ and $c_{i} \in \mathbb{R}$, the $E_{i}$ being disjoint. Thus simple functions have finitely many values and are measurable. In the next theorem, it will also be assumed that each $c_{i} \geq 0$.

Each simple function is measurable. This is easily seen as follows. First of all, you can assume the $c_{i}$ are distinct because if not, you could just replace those $E_{i}$ which correspond to a single value with their union. Then if you have any open interval $(a, b)$,

$$
s^{-1}((a, b))=\cup\left\{E_{i}: c_{i} \in(a, b)\right\}
$$

and this is measurable because it is the finite union of measurable sets.
Theorem 6.1.10 Let $f \geq 0$ be measurable. Then there exists a sequence of nonnegative simple functions $\left\{s_{n}\right\}$ satisfying

$$
\begin{gather*}
0 \leq s_{n}(\omega)  \tag{6.1}\\
\cdots s_{n}(\omega) \leq s_{n+1}(\omega) \cdots \\
f(\omega)=\lim _{n \rightarrow \infty} s_{n}(\omega) \text { for all } \omega \in \Omega \tag{6.2}
\end{gather*}
$$

If $f$ is bounded, the convergence is actually uniform. Conversely, if $f$ is nonnegative and is the pointwise limit of such simple functions, then $f$ is measurable.

Proof: Letting $I \equiv\{\omega: f(\omega)=\infty\}$, define

$$
t_{n}(\omega)=\sum_{k=0}^{2^{n}} \frac{k}{n} \mathscr{X}_{f^{-1}\left(\left[\frac{k}{n}, \frac{k+1}{n}\right)\right)}(\omega)+2^{n} \mathscr{X}_{I}(\omega)
$$

Then $t_{n}(\omega) \leq f(\omega)$ for all $\omega$ and $\lim _{n \rightarrow \infty} t_{n}(\omega)=f(\omega)$ for all $\omega$. This is because $t_{n}(\omega)=$ $2^{n}$ for $\omega \in I$ and if $f(\omega) \in\left[0, \frac{2^{n}+1}{n}\right)$, then

$$
\begin{equation*}
0 \leq f(\omega)-t_{n}(\omega) \leq \frac{1}{n} \tag{6.3}
\end{equation*}
$$

Thus whenever $\omega \notin I$, the above inequality will hold for all $n$ large enough. Let

$$
s_{1}=t_{1}, s_{2}=\max \left(t_{1}, t_{2}\right), s_{3}=\max \left(t_{1}, t_{2}, t_{3}\right), \cdots
$$

Then the sequence $\left\{s_{n}\right\}$ satisfies 6.1-6.2. Also each $s_{n}$ has finitely many values and is measurable. To see this, note that

$$
s_{n}^{-1}((a, \infty])=\cup_{k=1}^{n} t_{k}^{-1}((a, \infty]) \in \mathscr{F}
$$

To verify the last claim, note that in this case the term $2^{n} \mathscr{X}_{I}(\omega)$ is not present and for $n$ large enough, $2^{n} / n$ is larger than all values of $f$. Therefore, for all $n$ large enough, 6.3 holds for all $\omega$. Thus the convergence is uniform.

Now consider the converse assertion. Why is $f$ measurable if it is the pointwise limit of an increasing sequence simple functions?

$$
f^{-1}((a, \infty])=\cup_{n=1}^{\infty} s_{n}^{-1}((a, \infty])
$$

because $\omega \in f^{-1}((a, \infty])$ if and only if $\omega \in s_{n}^{-1}((a, \infty])$ for all $n$ sufficiently large.

### 6.2 Measures and Their Properties

First, what is meant by a measure? These are defined on measurable spaces $(\Omega, \mathscr{F})$ as follows.

## Definition 6.2.1 Let $(\Omega, \mathscr{F})$ be a measurable space. Here $\mathscr{F}$ is a $\sigma$ algebra of sets

 of $\Omega$. Then $\mu: \mathscr{F} \rightarrow[0, \infty]$ is called a measure if whenever $\left\{F_{i}\right\}_{i=1}^{\infty}$ is a sequence of disjoint sets of $\mathscr{F}$, it follows that$$
\mu\left(\cup_{i=1}^{\infty} F_{i}\right)=\sum_{i=1}^{\infty} \mu\left(E_{i}\right)
$$

Note that the series could equal $\infty$. If $\mu(\Omega)<\infty$, then $\mu$ is called a finite measure. An important case is when $\mu(\Omega)=1$ when $\mu$ is called a probability measure.

Note that $\mu(\emptyset)=\mu(\emptyset \cup \emptyset)=\mu(\emptyset)+\mu(\emptyset)$ and so $\mu(\emptyset)=0$.
Example 6.2.2 You could have $\mathscr{P}(\mathbb{N})=\mathscr{F}$ and you could define $\mu(S)$ to be the number of elements of $S$. This is called counting measure. It is left as an exercise to show that this is a measure.

Example 6.2.3 Here is a well known pathological example. Let $\Omega$ be uncountable and $\mathscr{F}$ will be those sets which have the property that either the set is countable or its complement is countable. Let $\mu(E)=0$ if $E$ is countable and $\mu(E)=1$ if $E$ is uncountable. It is left as an exercise to show that this is a measure.

Of course the most important measure in this book will be Lebesgue measure which gives the "volume" of a subset of $\mathbb{R}^{n}$. However, this requires a lot more work. First is a fundamental result about general measures.

Lemma 6.2.4 If $\mu$ is a measure and $F_{i} \in \mathscr{F}$, then

1. $\mu\left(\cup_{i=1}^{\infty} F_{i}\right) \leq \sum_{i=1}^{\infty} \mu\left(F_{i}\right)$.
2. If $F_{n} \in \mathscr{F}$ and $F_{n} \subseteq F_{n+1}$ for all $n$, then if $F=\cup_{n} F_{n}, \mu(F)=\lim _{n \rightarrow \infty} \mu\left(F_{n}\right)$. Symbolically, if $F_{n} \uparrow F$, then $\mu\left(F_{n}\right) \uparrow \mu(F)$.
3. If $F_{n} \supseteq F_{n+1}$ for all $n$, then if $\mu\left(F_{1}\right)<\infty$ and $F=\cap_{n} F_{n}$, then $\mu(F)=\lim _{n \rightarrow \infty} \mu\left(F_{n}\right)$. Symbolically, if $\mu\left(F_{1}\right)<\infty$ and $F_{n} \downarrow F$, then $\mu\left(F_{n}\right) \downarrow \mu(F)$.

Proof: 1.) Let $G_{1}=F_{1}$ and if $G_{1}, \cdots, G_{n}$ have been chosen disjoint, let

$$
G_{n+1} \equiv F_{n+1} \backslash \cup_{i=1}^{n} G_{i}
$$

Thus the $G_{i}$ are disjoint. In addition, these are all measurable sets. Now

$$
\mu\left(G_{n+1}\right)+\mu\left(F_{n+1} \cap\left(\cup_{i=1}^{n} G_{i}\right)\right)=\mu\left(F_{n+1}\right)
$$

and so $\mu\left(G_{n}\right) \leq \mu\left(F_{n}\right)$. Therefore, $\mu\left(\cup_{i=1}^{\infty} G_{i}\right)=\sum_{i} \mu\left(G_{i}\right) \leq \sum_{i} \mu\left(F_{i}\right)$.
2.) Now consider the increasing sequence of $F_{n} \in \mathscr{F}$. If $F \subseteq G$ and these are sets of $\mathscr{F}, \mu(G)=\mu(F)+\mu(G \backslash F)$ so $\mu(G) \geq \mu(F)$. Also

$$
F=\cup_{i=1}^{\infty}\left(F_{i+1} \backslash F_{i}\right)+F_{1}
$$

Then $\mu(F)=\sum_{i=1}^{\infty} \mu\left(F_{i+1} \backslash F_{i}\right)+\mu\left(F_{1}\right)$. Now $\mu\left(F_{i+1} \backslash F_{i}\right)+\mu\left(F_{i}\right)=\mu\left(F_{i+1}\right)$. If any $\mu\left(F_{i}\right)=\infty$, there is nothing to prove. Assume then that these are all finite. Then

$$
\begin{aligned}
\mu\left(F_{i+1} \backslash F_{i}\right) & =\mu\left(F_{i+1}\right)-\mu\left(F_{i}\right) \text { since } \\
\mu\left(F_{i+1} \backslash F_{i}\right)+\mu\left(F_{i}\right) & =\mu\left(F_{i+1}\right)
\end{aligned}
$$

and so

$$
\begin{aligned}
\mu(F) & =\sum_{i=1}^{\infty} \mu\left(F_{i+1}\right)-\mu\left(F_{i}\right)+\mu\left(F_{1}\right) \\
& =\lim _{n \rightarrow \infty} \sum_{i=1}^{n} \mu\left(F_{i+1}\right)-\mu\left(F_{i}\right)+\mu\left(F_{1}\right)=\lim _{n \rightarrow \infty} \mu\left(F_{n+1}\right)
\end{aligned}
$$

3.) Next suppose $\mu\left(F_{1}\right)<\infty$ and $\left\{F_{n}\right\}$ is a decreasing sequence. Then $F_{1} \backslash F_{n}$ is increasing to $F_{1} \backslash F$ and so by the first part,

$$
\begin{aligned}
\mu\left(F_{1}\right)-\mu(F) & =\mu\left(F_{1} \backslash F\right)=\lim _{n \rightarrow \infty} \mu\left(F_{1} \backslash F_{n}\right) \\
& =\lim _{n \rightarrow \infty}\left(\mu\left(F_{1}\right)-\mu\left(F_{n}\right)\right)=\mu\left(F_{1}\right)-\lim _{n \rightarrow \infty} \mu\left(F_{n}\right)
\end{aligned}
$$

so $\lim _{n \rightarrow \infty} \mu\left(F_{n}\right)=\mu(F)$.

### 6.3 Dynkin's Lemma

Dynkin's lemma is a very useful result. It is used quite a bit in books on probability. First note that if $\mathscr{K}$ is any collection of subsets of $\Omega$ which contains $\emptyset$ and $\Omega$, one can take the intersection of all $\sigma$ algebras which contain $\mathscr{K}$, one such being $\mathscr{P}(\Omega)$. This intersection is also a $\sigma$ algebra and is denoted as $\sigma(\mathscr{K})$ and is the smallest $\sigma$ algebra containing $\mathscr{K}$.

Definition 6.3.1 Let $\Omega$ be a set and let $\mathscr{K}$ be a collection of subsets of $\Omega$. Then $\mathscr{K}$ is called a $\pi$ system if $\emptyset, \Omega \in \mathscr{K}$ and whenever $A, B \in \mathscr{K}$, it follows $A \cap B \in \mathscr{K} . \sigma(\mathscr{K})$ will denote the smallest $\sigma$ algebra which contains $\mathscr{K}$. More precisely, the intersection of all $\sigma$ algebras which contain $\mathscr{K}$.

The following is the fundamental lemma which shows these $\pi$ systems are useful. This is due to Dynkin.

Lemma 6.3.2 Let $\mathscr{K}$ be a $\pi$ system of subsets of $\Omega$, a non empty set. Also let $\mathscr{G}$ be a collection of subsets of $\Omega$ which satisfies the following three properties.

1. $\mathscr{K} \subseteq \mathscr{G}$
2. If $A \in \mathscr{G}$, then $A^{C} \in \mathscr{G}$
3. If $\left\{A_{i}\right\}_{i=1}^{\infty}$ is a sequence of disjoint sets from $\mathscr{G}$ then $\cup_{i=1}^{\infty} A_{i} \in \mathscr{G}$.

Then $\mathscr{G} \supseteq \sigma(\mathscr{K})$, where $\sigma(\mathscr{K})$ is the smallest $\sigma$ algebra which contains $\mathscr{K}$.
Proof: First note that if $\mathscr{H} \equiv\{\mathscr{G}: 1-3$ all hold $\}$ then $\cap \mathscr{H}$ yields a collection of sets which also satisfies 1-3. Therefore, I will assume in the argument that $\mathscr{G}$ is the smallest collection satisfying 1-3. Let $A \in \mathscr{K}$ and define

$$
\mathscr{G}_{A} \equiv\{B \in \mathscr{G}: A \cap B \in \mathscr{G}\}
$$

I want to show $\mathscr{G}_{A}$ satisfies $1-3$ because then it must equal $\mathscr{G}$ since $\mathscr{G}$ is the smallest collection of subsets of $\Omega$ which satisfies $1-3$. This will give the conclusion that for $A \in \mathscr{K}$ and $B \in \mathscr{G}, A \cap B \in \mathscr{G}$. This information will then be used to show that if $A, B \in \mathscr{G}$ then $A \cap B \in \mathscr{G}$. From this it will follow very easily that $\mathscr{G}$ is a $\sigma$ algebra which will imply it contains $\sigma(\mathscr{K})$. Now here are the details of the argument.

Since $\mathscr{K}$ is given to be a $\pi$ system, $\mathscr{K} \subseteq \mathscr{G}_{A}$. Property 3 is obvious because if $\left\{B_{i}\right\}$ is a sequence of disjoint sets in $\mathscr{G}_{A}$, then

$$
A \cap \cup_{i=1}^{\infty} B_{i}=\cup_{i=1}^{\infty} A \cap B_{i} \in \mathscr{G}
$$

because $A \cap B_{i} \in \mathscr{G}$ and the property 3 of $\mathscr{G}$.
It remains to verify Property 2 so let $B \in \mathscr{G}_{A}$. I need to verify that $B^{C} \in \mathscr{G}_{A}$. In other words, I need to show that $A \cap B^{C} \in \mathscr{G}$. However, $\left(A^{C} \cup(A \cap B)\right)^{C}=A \cap\left(A^{C} \cup B\right)=A \cap B^{C}$ and so

$$
A \cap B^{C}=(\begin{array}{l}
A^{C} \cup
\end{array} \overbrace{A \cap B}^{\in \mathscr{G}}))^{C} \in \mathscr{G}
$$

Here is why. Since $B \in \mathscr{G}_{A}, A \cap B \in \mathscr{G}$ and since $A \in \mathscr{K} \subseteq \mathscr{G}$ it follows $A^{C} \in \mathscr{G}$ by assumption 2. It follows from assumption 3 the union of the disjoint sets, $A^{C}$ and $(A \cap B)$ is in $\mathscr{G}$ and then from 2 the complement of their union is in $\mathscr{G}$. Thus $\mathscr{G}_{A}$ satisfies 1-3 and this implies since $\mathscr{G}$ is the smallest such, that $\mathscr{G}_{A} \supseteq \mathscr{G}$. However, $\mathscr{G}_{A}$ is constructed as a subset of $\mathscr{G}$. This proves that for every $B \in \mathscr{G}$ and $A \in \mathscr{K}, A \cap B \in \mathscr{G}$. Now pick $B \in \mathscr{G}$ and consider $\mathscr{G}_{B} \equiv\{A \in \mathscr{G}: A \cap B \in \mathscr{G}\}$. I just proved $\mathscr{K} \subseteq \mathscr{G}_{B}$. The other arguments are identical to show $\mathscr{G}_{B}$ satisfies 1-3 and is therefore equal to $\mathscr{G}$. This shows that whenever $A, B \in \mathscr{G}$ it follows $A \cap B \in \mathscr{G}$.

This implies $\mathscr{G}$ is a $\sigma$ algebra. To show this, all that is left is to verify $\mathscr{G}$ is closed under countable unions because then it follows $\mathscr{G}$ is a $\sigma$ algebra. Let $\left\{A_{i}\right\} \subseteq \mathscr{G}$. Then let $A_{1}^{\prime}=A_{1}$ and

$$
A_{n+1}^{\prime} \equiv A_{n+1} \backslash\left(\cup_{i=1}^{n} A_{i}\right)=A_{n+1} \cap\left(\cap_{i=1}^{n} A_{i}^{C}\right)=\cap_{i=1}^{n}\left(A_{n+1} \cap A_{i}^{C}\right) \in \mathscr{G}
$$

because finite intersections of sets of $\mathscr{G}$ are in $\mathscr{G}$. Since the $A_{i}^{\prime}$ are disjoint, it follows

$$
\cup_{i=1}^{\infty} A_{i}=\cup_{i=1}^{\infty} A_{i}^{\prime} \in \mathscr{G}
$$

Therefore, $\mathscr{G} \supseteq \sigma(\mathscr{K})$.
Example 6.3.3 Suppose you have $(U, \mathscr{F})$ and $(V, \mathscr{S})$, two measurable spaces. Let $\mathscr{K} \subseteq$ $U \times V$ consist of all sets of the form $A \times B$ where $A \in \mathscr{F}$ and $B \in \mathscr{S}$. This is easily seen to be a $\pi$ system. When this is done, $\sigma(\mathscr{K})$ is denoted as $\mathscr{F} \times \mathscr{S}$.

An important example of a $\sigma$ algebra is the Borel sets.
Definition 6.3.4 The Borel sets on $\mathbb{R}^{p}$, denoted by $\mathscr{B}\left(\mathbb{R}^{p}\right)$ consists of the smallest $\sigma$ algebra containing the open sets.

Don't ever try to describe a generic Borel set. Always work with the definition that it is the smallest $\sigma$ algebra containing the open sets. Attempts to give an explicit description of a "typical" Borel set tend to lead nowhere because there are so many things which can be done. You can take countable unions and complements and then countable intersections of
what you get and then another countable union followed by complements and on and on. You just can't get a good useable description in this way. However, it is easy to see that something like $\left(\cap_{i=1}^{\infty} \cup_{j=i}^{\infty} E_{j}\right)^{C}$ is a Borel set if the $E_{j}$ are. This is useful. This said, you might consider the book by Hewitt and Stromberg [22] who do essentially describe them in their argument that there are more Lebesgue measurable sets than Borel measurable ones but it isn't easy.

An important example of the above is the case of a random vector and its distribution measure.

Definition 6.3.5 A measurable function $\mathbf{X}:(\Omega, \mathscr{F}, \mu) \rightarrow \mathbb{R}^{p}$ is called a random variable when $\mu(\Omega)=1$. For such a random variable, one can define a distribution measure $\lambda_{\mathbf{X}}$ on the Borel sets of $\mathbb{R}^{p}$ as follows. $\lambda_{\mathbf{X}}(G) \equiv \mu\left(\mathbf{X}^{-1}(G)\right)$. This is a well defined measure on the Borel sets of $Z$ because it makes sense for every $G$ open and $\mathscr{G} \equiv$ $\left\{G \subseteq \mathbb{R}^{p}: \mathbf{X}^{-1}(G) \in \mathscr{F}\right\}$ is a $\sigma$ algebra which contains the open sets, hence the Borel sets. Such a random variable is also called a random vector.

### 6.4 Measures and Outer Measures

There is also something called an outer measure which is defined on the set of all subsets.
Definition 6.4.1 Let $\Omega$ be a nonempty set and let $\lambda: \mathscr{P}(\Omega) \rightarrow[0, \infty)$ satisfy the following:

1. $\lambda(\emptyset)=0$
2. If $A \subseteq B$, then $\lambda(A) \leq \lambda(B)$
3. $\lambda\left(\cup_{i=1}^{\infty} E_{i}\right) \leq \sum_{i=1}^{\infty} \lambda\left(E_{i}\right)$

Then $\lambda$ is called an outer measure.
Every measure determines an outer measure. For example, suppose that $\mu$ is a measure on $\mathscr{F}$ a $\sigma$ algebra of subsets of $\Omega$. Then define

$$
\hat{\mu}(S) \equiv \inf \{\mu(E): E \supseteq S, E \in \mathscr{F}\}
$$

This is easily seen to be an outer measure. Also, we have the following Proposition.
Proposition 6.4.2 Let $\mu$ be a measure as just described. Then $\hat{\mu}$ as defined above, is an outer measure and also, if $E \in \mathscr{F}$, then $\hat{\mu}(E)=\mu(E)$.

Proof: The first two properties of an outer measure are obvious. What of the third? If any $\hat{\mu}\left(E_{i}\right)=\infty$, then there is nothing to show so suppose each of these is finite. Let $F_{i} \supseteq E_{i}$ such that $F_{i} \in \mathscr{F}$ and $\hat{\mu}\left(E_{i}\right)+\frac{\varepsilon}{2^{i}}>\mu\left(F_{i}\right)$. Then

$$
\begin{aligned}
\hat{\mu}\left(\cup_{i=1}^{\infty} E_{i}\right) & \leq \mu\left(\cup_{i=1}^{\infty} F_{i}\right) \leq \sum_{i=1}^{\infty} \mu\left(F_{i}\right) \\
& <\sum_{i=1}^{\infty}\left(\hat{\mu}\left(E_{i}\right)+\frac{\varepsilon}{2^{i}}\right)=\sum_{i=1}^{\infty} \hat{\mu}\left(E_{i}\right)+\varepsilon
\end{aligned}
$$

Since $\varepsilon$ is arbitrary, this establishes the third condition. Finally, if $E \in \mathscr{F}$, then by definition, $\hat{\mu}(E) \leq \mu(E)$ because $E \supseteq E$. Also, $\mu(E) \leq \mu(F)$ for all $F \in \mathscr{F}$ such that $F \supseteq E$. It follows that $\mu(E)$ is a lower bound of all such $\mu(F)$ and so $\hat{\mu}(E) \geq \mu(E)$.

### 6.5 An Outer Measure on $\mathscr{P}(\mathbb{R})$

Next is an outer measure which includes the usual concept of length. Recall $\mathscr{P}(S)$ denotes the set of all subsets of $S$.

Theorem 6.5.1 There exists a function $m: \mathscr{P}(\mathbb{R}) \rightarrow[0, \infty]$ which satisfies the following properties.

1. If $A \subseteq B$, then $0 \leq m(A) \leq m(B), m(\emptyset)=0$.
2. $m\left(\cup_{k=1}^{\infty} A_{i}\right) \leq \sum_{i=1}^{\infty} m\left(A_{i}\right)$
3. $m([a, b])=b-a=m((a, b))$.

Proof: First it is necessary to define the function $m$. This is contained in the following definition.

## Definition 6.5.2 For $A \subseteq \mathbb{R}, m(A)=\inf \left\{\sum_{i=1}^{\infty}\left(b_{i}-a_{i}\right): A \subseteq \cup_{i=1}^{\infty}\left(a_{i}, b_{i}\right)\right\}$

In words, you look at all coverings of $A$ with open intervals. For each of these open coverings, you add the lengths of the individual open intervals and you take the infimum of all such numbers obtained.

Then 1.) is obvious because if a countable collection of open intervals covers $B$, then it also covers $A$. Thus the set of numbers obtained for $B$ is smaller than the set of numbers for $A$. Why is $m(\emptyset)=0$ ? Then $\emptyset \subseteq(a-\delta, a+\delta)$ and so $m(\emptyset) \leq 2 \delta$ for every $\delta>0$. Letting $\delta \rightarrow 0$, it follows that $m(\emptyset)=0$.

Consider 2.). If any $m\left(A_{i}\right)=\infty$, there is nothing to prove. The assertion simply is $\infty \leq \infty$. Assume then that $m\left(A_{i}\right)<\infty$ for all $i$. Then for each $m \in \mathbb{N}$ there exists a countable set of open intervals, $\left\{\left(a_{i}^{m}, b_{i}^{m}\right)\right\}_{i=1}^{\infty}$ such that

$$
m\left(A_{m}\right)+\frac{\varepsilon}{2^{m}}>\sum_{i=1}^{\infty}\left(b_{i}^{m}-a_{i}^{m}\right) .
$$

Then using Theorem 1.11.3 on Page 28,

$$
\begin{aligned}
m\left(\cup_{m=1}^{\infty} A_{m}\right) & \leq \sum_{i, m}\left(b_{i}^{m}-a_{i}^{m}\right)=\sum_{m=1}^{\infty} \sum_{i=1}^{\infty}\left(b_{i}^{m}-a_{i}^{m}\right) \\
& \leq \sum_{m=1}^{\infty} m\left(A_{m}\right)+\frac{\varepsilon}{2^{m}}=\sum_{m=1}^{\infty} m\left(A_{m}\right)+\varepsilon
\end{aligned}
$$

and since $\varepsilon$ is arbitrary, this establishes 2.).
Next consider 3.). By definition, there exists a sequence of open intervals, $\left\{\left(a_{i}, b_{i}\right)\right\}_{i=1}^{\infty}$ whose union contains $[a, b]$ such that

$$
m([a, b])+\varepsilon \geq \sum_{i=1}^{\infty}\left(b_{i}-a_{i}\right)
$$

Since $[a, b]$ is compact, finitely many of these intervals also cover $[a, b]$. It follows there exist finitely many of these intervals, denoted as $\left\{\left(a_{i}, b_{i}\right)\right\}_{i=1}^{n}$, which overlap, such that $a \in\left(a_{1}, b_{1}\right), b_{1} \in\left(a_{2}, b_{2}\right), \cdots, b \in\left(a_{n}, b_{n}\right)$. Therefore, $m([a, b]) \leq \sum_{i=1}^{n}\left(b_{i}-a_{i}\right)$. It follows

$$
\sum_{i=1}^{n}\left(b_{i}-a_{i}\right) \geq m([a, b]) \geq \sum_{i=1}^{n}\left(b_{i}-a_{i}\right)-\varepsilon \geq(b-a)-\varepsilon
$$

Therefore, since $\left(a-\frac{\varepsilon}{2}, b+\frac{\varepsilon}{2}\right) \supseteq[a, b],(b-a)+\varepsilon \geq m([a, b]) \geq(b-a)-\varepsilon$. Since $\varepsilon$ is arbitrary, $(b-a)=m([a, b])$. From what was just shown,

$$
(b-a)-2 \delta=m([a+\delta, b-\delta]) \leq m((a, b)) \leq b-a
$$

and so, since this holds for every $\delta,(b-a) \leq m((a, b)) \leq m([a, b]) \leq(b-a)$. This shows 3.)

### 6.6 Measures from Outer Measures

Theorem 6.5.1, exibits an outer measure on $\mathscr{P}(\mathbb{R})$. This can be used to obtain a measure defined on $\mathbb{R}$. However, the procedure for doing so is a special case of a general approach due to Caratheodory in about 1918.
Definition 6.6.1 Let $\Omega$ be a nonempty set and let $\mu: \mathscr{P}(\Omega) \rightarrow[0, \infty]$ be an outer measure. For $E \subseteq \Omega, E$ is $\mu$ measurable if for all $S \subseteq \Omega$,

$$
\begin{equation*}
\mu(S)=\mu(S \backslash E)+\mu(S \cap E) \tag{6.4}
\end{equation*}
$$

To help in remembering 6.4, think of a measurable set $E$, as a process which divides a given set into two pieces, the part in $E$ and the part not in $E$ as in 6.4. In the Bible, there are several incidents recorded in which a process of division resulted in more stuff than was originally present. ${ }^{1}$ Measurable sets are exactly those which are incapable of such a miracle. You might think of the measurable sets as the non-miraculous sets. The idea is to show that they form a $\sigma$ algebra on which the outer measure $\mu$ is a measure.

First here is a definition and a lemma.
Definition 6.6.2 $(\mu\lfloor S)(A) \equiv \mu(S \cap A)$ for all $A \subseteq \Omega$. Thus $\mu\lfloor S$ is the name of $a$ new outer measure, called $\mu$ restricted to $S$.

The next lemma indicates that the property of measurability is not lost by considering this restricted measure.
Lemma 6.6.3 If $A$ is $\mu$ measurable, then $A$ is $\mu\lfloor S$ measurable.
Proof: Suppose $A$ is $\mu$ measurable. It is desired to to show that for all $T \subseteq \Omega$,

$$
(\mu\lfloor S)(T)=(\mu\lfloor S)(T \cap A)+(\mu\lfloor S)(T \backslash A)
$$

Thus it is desired to show

$$
\begin{equation*}
\mu(S \cap T)=\mu(T \cap A \cap S)+\mu\left(T \cap S \cap A^{C}\right) \tag{6.5}
\end{equation*}
$$

But 6.5 holds because $A$ is $\mu$ measurable. Apply Definition 6.6.1 to $S \cap T$ instead of $S$.
If $A$ is $\mu\lfloor S$ measurable, it does not follow that $A$ is $\mu$ measurable. Indeed, if you believe in the existence of non measurable sets, you could let $A=S$ for such a $\mu$ non measurable set and verify that $S$ is $\mu\lfloor S$ measurable.

The next theorem is the main result on outer measures which shows that starting with an outer measure you can obtain a measure.

[^2]Theorem 6.6.4 Let $\Omega$ be a set and let $\mu$ be an outer measure on $\mathscr{P}(\Omega)$. The collection of $\mu$ measurable sets $\mathscr{S}$, forms a $\sigma$ algebra and

$$
\begin{equation*}
\text { If } F_{i} \in \mathscr{S}, F_{i} \cap F_{j}=\emptyset, \text { then } \mu\left(\cup_{i=1}^{\infty} F_{i}\right)=\sum_{i=1}^{\infty} \mu\left(F_{i}\right) \tag{6.6}
\end{equation*}
$$

If $\cdots F_{n} \subseteq F_{n+1} \subseteq \cdots$, then if $F=\cup_{n=1}^{\infty} F_{n}$ and $F_{n} \in \mathscr{S}$, it follows that

$$
\begin{equation*}
\mu(F)=\lim _{n \rightarrow \infty} \mu\left(F_{n}\right) \tag{6.7}
\end{equation*}
$$

If $\cdots F_{n} \supseteq F_{n+1} \supseteq \cdots$, and if $F=\cap_{n=1}^{\infty} F_{n}$ for $F_{n} \in \mathscr{S}$ then if $\mu\left(F_{1}\right)<\infty$,

$$
\begin{equation*}
\mu(F)=\lim _{n \rightarrow \infty} \mu\left(F_{n}\right) \tag{6.8}
\end{equation*}
$$

This measure space is also complete which means that if $\mu(F)=0$ for some $F \in \mathscr{S}$ then if $G \subseteq F$, it follows $G \in \mathscr{S}$ also.

Proof: First note that $\emptyset$ and $\Omega$ are obviously in $\mathscr{S}$. Now suppose $A, B \in \mathscr{S}$. I will show $A \backslash B \equiv A \cap B^{C}$ is in $\mathscr{S}$. To do so, consider the following picture.


It is required to show that

$$
\mu(S)=\mu(S \backslash(A \backslash B))+\mu(S \cap(A \backslash B))
$$

First consider $S \backslash(A \backslash B)$. From the picture, it equals

$$
\left(S \cap A^{C} \cap B^{C}\right) \cup(S \cap A \cap B) \cup\left(S \cap A^{C} \cap B\right)
$$

Therefore,

$$
\begin{aligned}
& \quad \mu(S) \leq \mu(S \backslash(A \backslash B))+\mu(S \cap(A \backslash B)) \\
& \leq \mu\left(S \cap A^{C} \cap B^{C}\right)+\mu(S \cap A \cap B)+\mu\left(S \cap A^{C} \cap B\right)+\mu(S \cap(A \backslash B)) \\
& =\mu\left(S \cap A^{C} \cap B^{C}\right)+\mu(S \cap A \cap B)+\mu\left(S \cap A^{C} \cap B\right)+\mu\left(S \cap A \cap B^{C}\right) \\
& =\mu\left(S \cap A^{C} \cap B^{C}\right)+\mu\left(S \cap A \cap B^{C}\right)+\mu(S \cap A \cap B)+\mu\left(S \cap A^{C} \cap B\right) \\
& =\mu\left(S \cap B^{C}\right)+\mu(S \cap B)=\mu(S)
\end{aligned}
$$

and so this shows that $A \backslash B \in \mathscr{S}$ whenever $A, B \in \mathscr{S}$.
Since $\Omega \in \mathscr{S}$, this shows that $A \in \mathscr{S}$ if and only if $A^{C} \in \mathscr{S}$. Now if $A, B \in \mathscr{S}, A \cup B=$ $\left(A^{C} \cap B^{C}\right)^{C}=\left(A^{C} \backslash B\right)^{C} \in \mathscr{S}$. By induction, if $A_{1}, \cdots, A_{n} \in \mathscr{S}$, then so is $\cup_{i=1}^{n} A_{i}$. If $A, B \in \mathscr{S}$, with $A \cap B=\emptyset$,

$$
\mu(A \cup B)=\mu((A \cup B) \cap A)+\mu((A \cup B) \backslash A)=\mu(A)+\mu(B)
$$

By induction, if $A_{i} \cap A_{j}=\emptyset$ and $A_{i} \in \mathscr{S}$,

$$
\begin{equation*}
\mu\left(\cup_{i=1}^{n} A_{i}\right)=\sum_{i=1}^{n} \mu\left(A_{i}\right) \tag{6.9}
\end{equation*}
$$

Now let $A=\cup_{i=1}^{\infty} A_{i}$ where $A_{i} \cap A_{j}=\emptyset$ for $i \neq j$.

$$
\sum_{i=1}^{\infty} \mu\left(A_{i}\right) \geq \mu(A) \geq \mu\left(\cup_{i=1}^{n} A_{i}\right)=\sum_{i=1}^{n} \mu\left(A_{i}\right)
$$

Since this holds for all $n$, you can take the limit as $n \rightarrow \infty$ and conclude that $\sum_{i=1}^{\infty} \mu\left(A_{i}\right)=$ $\mu(A)$. which establishes 6.6.

Consider part 6.7. Without loss of generality $\mu\left(F_{k}\right)<\infty$ for all $k$ since otherwise there is nothing to show. Suppose $\left\{F_{k}\right\}$ is an increasing sequence of sets of $\mathscr{S}$. Then letting $F_{0} \equiv \emptyset,\left\{F_{k+1} \backslash F_{k}\right\}_{k=0}^{\infty}$ is a sequence of disjoint sets of $\mathscr{S}$ since it was shown above that the difference of two sets of $\mathscr{S}$ is in $\mathscr{S}$. Also note that from 6.9

$$
\mu\left(F_{k+1} \backslash F_{k}\right)+\mu\left(F_{k}\right)=\mu\left(F_{k+1}\right)
$$

and so if $\mu\left(F_{k}\right)<\infty$, then

$$
\mu\left(F_{k+1} \backslash F_{k}\right)=\mu\left(F_{k+1}\right)-\mu\left(F_{k}\right)
$$

Therefore, letting $F \equiv \cup_{k=1}^{\infty} F_{k}$ which also equals $\cup_{k=1}^{\infty}\left(F_{k+1} \backslash F_{k}\right)$, it follows from part 6.6 just shown that

$$
\begin{aligned}
\mu(F) & =\sum_{k=0}^{\infty} \mu\left(F_{k+1} \backslash F_{k}\right)=\lim _{n \rightarrow \infty} \sum_{k=0}^{n} \mu\left(F_{k+1} \backslash F_{k}\right) \\
& =\lim _{n \rightarrow \infty} \sum_{k=0}^{n} \mu\left(F_{k+1}\right)-\mu\left(F_{k}\right)=\lim _{n \rightarrow \infty} \mu\left(F_{n+1}\right)
\end{aligned}
$$

In order to establish 6.8 , let the $F_{n}$ be as given there. Then, since $\left(F_{1} \backslash F_{n}\right)$ increases to $\left(F_{1} \backslash F\right), 6.7$ implies

$$
\lim _{n \rightarrow \infty}\left(\mu\left(F_{1}\right)-\mu\left(F_{n}\right)\right)=\mu\left(F_{1} \backslash F\right)
$$

The problem is, I don't know $F \in \mathscr{S}$ and so it is not clear that $\mu\left(F_{1} \backslash F\right)=\mu\left(F_{1}\right)-\mu(F)$. However, $\mu\left(F_{1} \backslash F\right)+\mu(F) \geq \mu\left(F_{1}\right)$ and so $\mu\left(F_{1} \backslash F\right) \geq \mu\left(F_{1}\right)-\mu(F)$. Hence

$$
\lim _{n \rightarrow \infty}\left(\mu\left(F_{1}\right)-\mu\left(F_{n}\right)\right)=\mu\left(F_{1} \backslash F\right) \geq \mu\left(F_{1}\right)-\mu(F)
$$

which implies $\lim _{n \rightarrow \infty} \mu\left(F_{n}\right) \leq \mu(F)$. But since $F \subseteq F_{n}, \mu(F) \leq \lim _{n \rightarrow \infty} \mu\left(F_{n}\right)$ and this establishes 6.8. Note that it was assumed $\mu\left(F_{1}\right)<\infty$ because $\mu\left(F_{1}\right)$ was subtracted from both sides.

It remains to show $\mathscr{S}$ is closed under countable unions. Recall that if $A \in \mathscr{S}$, then $A^{C} \in \mathscr{S}$ and $\mathscr{S}$ is closed under finite unions. Let $A_{i} \in \mathscr{S}, A=\cup_{i=1}^{\infty} A_{i}, B_{n}=\cup_{i=1}^{n} A_{i}$. Then

$$
\begin{align*}
\mu(S) & =\mu\left(S \cap B_{n}\right)+\mu\left(S \backslash B_{n}\right)  \tag{6.10}\\
& =\left(\mu\lfloor S)\left(B_{n}\right)+\left(\mu\lfloor S)\left(B_{n}^{C}\right) .\right.\right.
\end{align*}
$$

By Lemma 6.6.3 $B_{n}$ is $\left(\mu\lfloor S)\right.$ measurable and so is $B_{n}^{C}$. I want to show $\mu(S) \geq \mu(S \backslash A)+$ $\mu(S \cap A)$. If $\mu(S)=\infty$, there is nothing to prove. Assume $\mu(S)<\infty$. Then apply Parts 6.8 and 6.7 to the outer measure $\mu\left\lfloor S\right.$ in 6.10 and let $n \rightarrow \infty$. Thus $B_{n} \uparrow A, B_{n}^{C} \downarrow A^{C}$ and this yields $\mu(S)=\left(\mu\lfloor S)(A)+\left(\mu\lfloor S)\left(A^{C}\right)=\mu(S \cap A)+\mu(S \backslash A)\right.\right.$.

Therefore $A \in \mathscr{S}$ and this proves Parts 6.6, 6.7, and 6.8.
It only remains to verify the assertion about completeness. Letting $G$ and $F$ be as described above, let $S \subseteq \Omega$. I need to verify $\mu(S) \geq \mu(S \cap G)+\mu(S \backslash G)$. However,

$$
\begin{aligned}
\mu(S \cap G)+\mu(S \backslash G) & \leq \mu(S \cap F)+\mu(S \backslash F)+\mu(F \backslash G) \\
& =\mu(S \cap F)+\mu(S \backslash F)=\mu(S)
\end{aligned}
$$

because by assumption, $\mu(F \backslash G) \leq \mu(F)=0$.
The measure $m$ which results from the outer measure of Theorem 6.5.1 is called Lebesgue measure. The following is a general result about completion of a measure space.

Proposition 6.6.5 Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. Also let $\hat{\mu}$ be the outer measure defined by

$$
\hat{\mu}(F) \equiv \inf \{\mu(E): E \supseteq F \text { and } E \in \mathscr{F}\}
$$

Then $\hat{\mu}$ is an outer measure which is a measure on $\hat{\mathscr{F}}$, the set of $\hat{\mu}$ measurable sets. Also $\hat{\mu}(E)=\mu(E)$ for $E \in \mathscr{F}$ and $\mathscr{F} \subseteq \hat{\mathscr{F}}$. If $(\Omega, \mathscr{F}, \mu)$ is already complete, then no new sets are obtained from this process and $\mathscr{F}=\hat{\mathscr{F}}$.

Proof: The first part of this follows from Proposition 6.4.2. It only remains to verify that $\mathscr{F} \subseteq \hat{\mathscr{F}}$. Let $S$ be a set and let $E \in \mathscr{F}, E_{S} \supseteq S, E_{S} \in \mathscr{F}$. Then

$$
\mu\left(E_{S}\right)=\mu\left(E_{S} \backslash E\right)+\mu\left(E_{S} \cap E\right)
$$

due to the fact that $\mu$ is a measure. As usual, if $\hat{\mu}(S)=\infty$, it is obvious that $\hat{\mu}(S) \geq$ $\hat{\mu}(S \backslash E)+\hat{\mu}(S \cap E)$. Therefore, assume this is not $\infty$. Then let $\hat{\mu}(S)>\mu\left(E_{S}\right)-\varepsilon$. Then from the above,

$$
\varepsilon+\hat{\mu}(S) \geq \mu\left(E_{S} \backslash E\right)+\mu\left(E_{S} \cap E\right) \geq \mu(S \backslash E)+\mu(S \cap E)
$$

Since $\varepsilon$ is arbitrary, this shows that $E \in \hat{\mathscr{F}}$. Thus $\mathscr{F} \subseteq \hat{\mathscr{F}}$.
Why are these two $\sigma$ algebras equal if $(\Omega, \mathscr{F}, \mu)$ is complete? Suppose now that $(\Omega, \mathscr{F}, \mu)$ is complete. Let $F \in \hat{\mathscr{F}}$. Then there exists $E \supseteq F$ such that $\mu(E)=\hat{\mu}(F)$. This is obvious if $\hat{\mu}(F)=\infty$. Otherwise, let $E_{n} \supseteq F, \hat{\mu}(F)+\frac{1}{n}>\mu\left(E_{n}\right)$. Just let $E=\cap_{n} E_{n}$. Now $\hat{\mu}(E \backslash F)=0$. Now also, there exists a set of $\mathscr{F}$ called $W$ such that $\mu(W)=0$ and $W \supseteq E \backslash F$. Thus $E \backslash F \subseteq W$, a set of measure zero. Hence by completeness of $(\Omega, \mathscr{F}, \mu)$, it must be the case that $E \backslash F=E \cap F^{C}=G \in \mathscr{F}$. Then taking complements of both sides, $E^{C} \cup F=G^{C} \in \mathscr{F}$. Now take intersections with $E . F \in E \cap G^{C} \in \mathscr{F}$.

### 6.7 When is a Measure a Borel Measure?

You have an outer measure defined on the set of all subsets of $\mathbb{R}^{p}$. How can you tell that the $\sigma$ algebra of measurable sets includes the Borel sets? This is what is discussed here.

## Definition 6.7.1 For two sets, $A, B$ we define

$$
\operatorname{dist}(A, B) \equiv \inf \{d(x, y): x \in A, y \in B\}
$$

Theorem 6.7.2 Let $\mu$ be an outer measure on the subsets of $(X, d)$, a metric space. If

$$
\mu(A \cup B)=\mu(A)+\mu(B)
$$

whenever $\operatorname{dist}(A, B)>0$, then the $\sigma$ algebra of measurable sets $\mathscr{S}$ contains the Borel sets.
Proof: It suffices to show that closed sets are in $\mathscr{S}$, the $\sigma$-algebra of measurable sets, because then the open sets are also in $\mathscr{S}$ and consequently $\mathscr{S}$ contains the Borel sets. Let $K$ be closed and let $S$ be a subset of $\Omega$. Is $\mu(S) \geq \mu(S \cap K)+\mu(S \backslash K)$ ? It suffices to assume $\mu(S)<\infty$. Let

$$
K_{n} \equiv\left\{x: \operatorname{dist}(x, K) \leq \frac{1}{n}\right\}
$$

By Lemma 2.4.8 on Page $44, x \rightarrow \operatorname{dist}(x, K)$ is continuous and so $K_{n}$ is closed. By the assumption of the theorem,

$$
\begin{equation*}
\mu(S) \geq \mu\left((S \cap K) \cup\left(S \backslash K_{n}\right)\right)=\mu(S \cap K)+\mu\left(S \backslash K_{n}\right) \tag{6.11}
\end{equation*}
$$

since $S \cap K$ and $S \backslash K_{n}$ are a positive distance apart. Now

$$
\begin{equation*}
\mu\left(S \backslash K_{n}\right) \leq \mu(S \backslash K) \leq \mu\left(S \backslash K_{n}\right)+\mu\left(\left(K_{n} \backslash K\right) \cap S\right) \tag{6.12}
\end{equation*}
$$

If $\lim _{n \rightarrow \infty} \mu\left(\left(K_{n} \backslash K\right) \cap S\right)=0$ then the theorem will be proved because this limit along with 6.12 implies $\lim _{n \rightarrow \infty} \mu\left(S \backslash K_{n}\right)=\mu(S \backslash K)$ and then taking a limit in $6.11, \mu(S) \geq$ $\mu(S \cap K)+\mu(S \backslash K)$ as desired. Therefore, it suffices to establish this limit.

Since $K$ is closed, a point, $x \notin K$ must be at a positive distance from $K$ and so

$$
K_{n} \backslash K=\cup_{k=n}^{\infty} K_{k} \backslash K_{k+1}
$$

Therefore

$$
\begin{equation*}
\mu\left(S \cap\left(K_{n} \backslash K\right)\right) \leq \sum_{k=n}^{\infty} \mu\left(S \cap\left(K_{k} \backslash K_{k+1}\right)\right) \tag{6.13}
\end{equation*}
$$

If

$$
\begin{equation*}
\sum_{k=1}^{\infty} \mu\left(S \cap\left(K_{k} \backslash K_{k+1}\right)\right)<\infty \tag{6.14}
\end{equation*}
$$

then $\mu\left(S \cap\left(K_{n} \backslash K\right)\right) \rightarrow 0$ because it is dominated by the tail of a convergent series so it suffices to show 6.14.

$$
\begin{gather*}
\sum_{k=1}^{M} \mu\left(S \cap\left(K_{k} \backslash K_{k+1}\right)\right)= \\
\sum_{\text {keven, }, k \leq M} \mu\left(S \cap\left(K_{k} \backslash K_{k+1}\right)\right)+\sum_{k \text { odd }, k \leq M} \mu\left(S \cap\left(K_{k} \backslash K_{k+1}\right)\right) . \tag{6.15}
\end{gather*}
$$

By the construction, the distance between any pair of sets, $S \cap\left(K_{k} \backslash K_{k+1}\right)$ for different even values of $k$ is positive and the distance between any pair of sets, $S \cap\left(K_{k} \backslash K_{k+1}\right)$ for different odd values of $k$ is positive. Therefore,

$$
\begin{gathered}
\sum_{k \text { even }, k \leq M} \mu\left(S \cap\left(K_{k} \backslash K_{k+1}\right)\right)+\sum_{k \text { odd }, k \leq M} \mu\left(S \cap\left(K_{k} \backslash K_{k+1}\right)\right) \leq \\
\mu\left(\bigcup_{\text {keven }, k \leq M}\left(S \cap\left(K_{k} \backslash K_{k+1}\right)\right)\right)+\mu\left(\bigcup_{k o d d, k \leq M}\left(S \cap\left(K_{k} \backslash K_{k+1}\right)\right)\right) \\
\leq \mu(S)+\mu(S)=2 \mu(S)
\end{gathered}
$$

and so for all $M, \sum_{k=1}^{M} \mu\left(S \cap\left(K_{k} \backslash K_{k+1}\right)\right) \leq 2 \mu(S)$ showing 6.14.

### 6.8 One Dimensional Lebesgue Measure

Now with these major results about measures, it is time to specialize to the outer measure of Theorem 6.5.1. The next theorem describes some fundamental properties of Lebesgue measure on $\mathbb{R}$. The conditions 6.16 and 6.17 given below are known respectively as inner and outer regularity.

Lemma 6.8.1 Let $\mathscr{F}$ denote the $\sigma$ algebra of Theorem 6.6.4, associated with the outer measure $\mu$ in Theorem 6.5.1, on which $\mu$ is a measure. Then $\mathscr{F} \supseteq \mathscr{B}(\mathbb{R})$.

Proof: Suppose $\operatorname{dist}(A, B)=\delta>0$. Is it the case that $\mu(A \cup B)=\mu(A)+\mu(B)$ ? If either on the right are $\infty$ then there is nothing to show so assume both $\mu(A), \mu(B)<$ $\infty$. Let $\left\{J_{j}\right\}_{j=1}^{\infty}$ be open intervals such that $\mu(A \cup B)+\varepsilon>\sum_{j=1}^{\infty} \mu\left(J_{j}\right)$. Without loss of generality we assume that every $J_{j}$ intersects either $A$ or $B$ otherwise, the interval could be discarded. Suppose some $J_{j}$ intersects both $A$ and $B$. Say $a \in A \cap J_{j}$. There are at most two open intervals comprising $J_{j} \backslash\left[a-\frac{2}{3} \delta, a+\frac{2}{3} \delta\right]$ and this closed interval has no points of $B$. Neither can intersect both $A$ and $B$ because they are spaced apart by $\frac{4}{3} \delta$. Let the new $J_{j}$ be the one which intersects $B$. In this way, we can assume none of the $J_{j}$ intersects both $A$ and $B$. Let $\mathfrak{A}$ be those $i$ for which $J_{i}$ intersects $A$ and let $\mathfrak{B}$ be those $i$ for which $J_{i}$ intersects $B$. Then

$$
\mu(A \cup B)+\varepsilon>\sum_{j=1}^{\infty} \mu\left(J_{j}\right)=\sum_{j \in \mathfrak{A}} \mu\left(J_{j}\right)+\sum_{j \in \mathfrak{B}} \mu\left(J_{j}\right) \geq \mu(A)+\mu(B)
$$

and since $\varepsilon$ is arbitrary and $\mu(A \cup B) \leq \mu(A)+\mu(B)$, it follows that $\mu(A)+\mu(B)=$ $\mu(A \cup B)$. By Theorem 6.7.2, $\mathscr{F} \supseteq \mathscr{B}(\mathbb{R})$.

Theorem 6.8.2 Let $\mathscr{F}$ denote the $\sigma$ algebra of Theorem 6.6.4, associated with the outer measure $\mu$ in Theorem 6.5.1, on which $\mu$ is a measure. Then every open interval is in $\mathscr{F}$. So are all open and closed sets. Furthermore, if $E$ is any set in $\mathscr{F}$

$$
\begin{gather*}
\mu(E)=\sup \{\mu(K): K \text { compact, } K \subseteq E\}  \tag{6.16}\\
\mu(E)=\inf \{\mu(V): V \text { is an open set } V \supseteq E\} \tag{6.17}
\end{gather*}
$$

Proof: By Lemma 6.8.1, $\mathscr{F} \supseteq \mathscr{B}(\mathbb{R})$.

Now consider the assertion of outer regularity. The assertion of outer regularity is not hard to get. Letting $E$ be any set $\mu(E)<\infty$, there exist open intervals covering $E$ denoted by $\left\{\left(a_{i}, b_{i}\right)\right\}_{i=1}^{\infty}$ such that

$$
\mu(E)+\varepsilon>\sum_{i=1}^{\infty} b_{i}-a_{i}=\sum_{i=1}^{\infty} \mu\left(a_{i}, b_{i}\right) \geq \mu(V)
$$

where $V$ is the union of the open intervals just mentioned. Thus

$$
\mu(E) \leq \mu(V) \leq \mu(E)+\varepsilon
$$

This shows outer regularity. If $\mu(E)=\infty$, there is nothing to show.
Now consider the assertion of inner regularity 6.16. Suppose $I$ is a closed and bounded interval and $E \subseteq I$ with $E \in \mathscr{F}$. By outer regularity, there exists open $V$ containing $I \cap E^{C}$ such that

$$
\mu\left(I \cap E^{C}\right)+\varepsilon>\mu(V)
$$

Then since $\mu$ is additive on $\mathscr{F}$, it follows that $\mu\left(V \backslash\left(I \cap E^{C}\right)\right)<\varepsilon$. Then $K \equiv V^{C} \cap I$ is a compact subset of $E$. This is because $V \supseteq I \cap E^{C}$ so $V^{C} \subseteq I^{C} \cup E$ and so

$$
V^{C} \cap I \subseteq\left(I^{C} \cup E\right) \cap I=E \cap I=E
$$

Also,

$$
E \backslash\left(V^{C} \cap I\right)=E \cap V=V \backslash E^{C} \subseteq V \backslash\left(I \cap E^{C}\right),
$$

a set of measure less than $\varepsilon$. Therefore,

$$
\mu\left(V^{C} \cap I\right)+\varepsilon \geq \mu\left(V^{C} \cap I\right)+\mu\left(E \backslash\left(V^{C} \cap I\right)\right)=\mu(E)
$$

so the desired conclusion holds in the case where $E$ is contained in a compact interval.
Now suppose $E$ is arbitrary and let $l<\mu(E)$. Then choosing $\varepsilon$ small enough, $l+\varepsilon<$ $\mu(E)$ also. Letting $E_{n} \equiv E \cap[-n, n]$, it follows from Lemma 6.2.4 that for $n$ large enough, $\mu\left(E_{n}\right)>l+\varepsilon$. Now from what was just shown, there exists $K \subseteq E_{n}$ such that $\mu(K)+\varepsilon>$ $\mu\left(E_{n}\right)$. Hence $\mu(K)>l$. This shows 6.16.

Definition 6.8.3 The countable union of closed sets is called an $F_{\sigma}$ set and the countable union of open sets is called $a G_{\delta}$ set. These are Borel sets.

Proposition 6.8.4 For $m$ Lebesgue measure, $m([a, b])=m((a, b))=b-a$. Also $m$ is translation invariant in the sense that if $E$ is any Lebesgue measurable set, then $m(x+E)=$ $m(E)$.

Proof: Let $\mathscr{K}$ consist of the open intervals including $\mathbb{R}$ and $\emptyset$. Then $\mathscr{K}$ is a $\pi$ system. Also $m(x+I)=m(I)$ is obvious for any $I \in \mathscr{K}$. Let $\mathscr{G}$ denote those Borel sets $E$ such that for all $x, m(x+E \cap(-n, n))=m(E \cap(-n, n))$. Thus $\mathscr{K} \subseteq \mathscr{G}$. If $E_{i}$ are disjoint sets in $\mathscr{G}$, $x+\cup_{i}\left(E_{i} \cap(-n, n)\right)=\cup\left(x+E_{i} \cap(-n, n)\right)$ and so

$$
\begin{aligned}
m\left(x+\cup_{i} E_{i} \cap(-n, n)\right) & =m\left(\cup\left(x+E_{i} \cap(-n, n)\right)\right)=\sum_{i} m\left(x+E_{i} \cap(-n, n)\right) \\
& =\sum_{i} m\left(E_{i} \cap(-n, n)\right)=m\left(\cup_{i} E_{i} \cap(-n, n)\right)
\end{aligned}
$$

so $\mathscr{G}$ is closed with respect to countable disjoint unions. If $E \in \mathscr{G}$, then

$$
\begin{aligned}
& m(x+E \cap(-n, n))+m\left(x+E^{C} \cap(-n, n)\right) \\
= & m(x+(-n, n))=m(-n, n)=m(E \cap(-n, n))+m\left(E^{C} \cap(-n, n)\right)
\end{aligned}
$$

and $m(x+E \cap(-n, n))=m(E \cap(-n, n))$ so subtracting this from both sides shows that $\mathscr{G}$ is closed with respect to complements also. Therefore, $\mathscr{G}=\sigma(\mathscr{K})=\mathscr{B}(\mathbb{R})$ by Dynkin's lemma. Thus the measure is translation invariant on all Borel sets because you can let $n \rightarrow \infty$. Now let $E$ be an arbitrary measurable set. By Theorem 6.8 .2 there is an increasing sequence of compact sets $\left\{K_{n}\right\}$ and a decreasing sequence of open sets $\left\{V_{n}\right\}$ such that for $F \equiv \cup_{n} F_{n}$ and $G \equiv \cap_{n} V_{n}, m(F)=m(E)=m(G)$ and $F \subseteq E \subseteq G$. Then

$$
m(F)=m(x+F) \leq m(x+E) \leq m(x+G)=m(G)=m(E)=m(F)
$$

To see $x+E$ is measurable, assume first that $E$ is bounded. Then $x+E$ is between $x+F$ and $x+G$ and $m(x+G \backslash(x+F))=m(x+(G \backslash F))=m(G \backslash F)=0$ so since $x+G, x+F$ are both Borel sets, completeness shows that $x+E$ is also measurable. For an arbitrary measurable set $E, x+E$ equals $\cup_{n}(x+E \cap(-n, n))$.
Definition 6.8.5 There is an important idea which is often seen in the context of measures. Something happens a.e. (almost everywhere) means that it happens off a set of measure zero.

### 6.9 Exercises

1. Show carefully that if $\mathfrak{S}$ is a set whose elements are $\sigma$ algebras which are subsets of $\mathscr{P}(\Omega)$, then $\cap \mathfrak{S}$ is also a $\sigma$ algebra. Now let $\mathscr{G} \subseteq \mathscr{P}(\Omega)$ satisfy property $P$ if $\mathscr{G}$ is closed with respect to complements and countable disjoint unions as in Dynkin's lemma, and contains $\emptyset$ and $\Omega$. If $\mathfrak{H} \subseteq \mathscr{G}$ is any set whose elements are subsets of $\mathscr{P}(\Omega)$ which satisfies property $P$, then $\cap \mathfrak{H}$ also satisfies property $P$. Thus there is a smallest subset of $\mathscr{G}$ satisfying $P$.
2. Show $\mathscr{B}\left(\mathbb{R}^{p}\right)=\sigma(\mathscr{P})$ where $\mathscr{P}$ consists of the half open rectangles which are of the form $\prod_{i=1}^{p}\left[a_{i}, b_{i}\right)$.
3. Recall that $f:(\Omega, \mathscr{F}) \rightarrow \mathbb{R}$ is measurable means $f^{-1}$ (open) $\in \mathscr{F}$. Show that if $E$ is any set in $\mathscr{B}(\mathbb{R})$, then $f^{-1}(E) \in \mathscr{F}$. Thus, inverse images of Borel sets are measurable. Next consider $f:(\Omega, \mathscr{F}) \rightarrow \mathbb{R}$ being measurable and $g: \mathbb{R} \rightarrow \mathbb{R}$ is Borel measurable, meaning that $g^{-1}$ (open $) \in \mathscr{B}(\mathbb{R})$. Explain why $g \circ f$ is measurable.
Hint: You know that $(g \circ f)^{-1}(U)=f^{-1}\left(g^{-1}(U)\right)$. For your information, it does not work the other way around. That is, measurable composed with Borel measurable is not necessarily measurable. In fact examples exist which show that if $g$ is measurable and $f$ is continuous, then $g \circ f$ may fail to be measurable.
4. Let $X_{i} \equiv \mathbb{R}^{n_{i}}$ and let $X=\prod_{i=1}^{n} X_{i}$ and let the distance between two points in $X$ be given by

$$
\|\mathbf{x}-\mathbf{y}\| \equiv \max \left\{\left\|\mathbf{x}_{i}-\mathbf{y}_{i}\right\|, i=1,2, \cdots, n\right\}
$$

Show that any set of the form

$$
\prod_{i=1}^{n} E_{i}, E_{i} \in \mathscr{B}\left(X_{i}\right)
$$

is a Borel set. That is, the product of Borel sets is Borel. Hint: You might consider the continuous functions $\pi_{i}: \prod_{j=1}^{n} X_{j} \rightarrow X_{i}$ which are the projection maps. Thus $\pi_{i}(\mathbf{x}) \equiv x_{i}$. Then $\pi_{i}^{-1}\left(E_{i}\right)$ would have to be Borel measurable whenever $E_{i} \in \mathscr{B}\left(X_{i}\right)$. Explain why. You know $\pi_{i}$ is continuous. Why would $\pi_{i}^{-1}$ (Borel) be a Borel set? Then you might argue that $\prod_{i=1}^{n} E_{i}=\cap_{i=1}^{n} \pi_{i}^{-1}\left(E_{i}\right)$.
5. You have two finite measures defined on $\mathscr{B}(X) \mu, v$. Suppose these are equal on every open set. Show that these must be equal on every Borel set. Hint: You should use Dynkin's lemma to show this very easily.
6. Show that $(\mathbb{N}, \mathscr{P}(\mathbb{N}), \mu)$ is a measure space where $\mu(S)$ equals the number of elements of $S$. You need to verify that if the sets $E_{i}$ are disjoint, then $\mu\left(\cup_{i=1}^{\infty} E_{i}\right)=$ $\sum_{i=1}^{\infty} \mu\left(E_{i}\right)$.
7. Let $\Omega$ be an uncountable set and let $\mathscr{F}$ denote those subsets of $\Omega, F$ such that either $F$ or $F^{C}$ is countable. Show that this is a $\sigma$ algebra. Next define the following measure. $\mu(A)=1$ if $A$ is uncountable and $\mu(A)=0$ if $A$ is countable. Show that $\mu$ is a measure.
8. Let $\mu(E)=1$ if $0 \in E$ and $\mu(E)=0$ if $0 \notin E$. Show this is a measure on $\mathscr{P}(\mathbb{R})$.
9. Give an example of a measure $\mu$ and a measure space and a decreasing sequence of measurable sets $\left\{E_{i}\right\}$ such that $\lim _{n \rightarrow \infty} \mu\left(E_{n}\right) \neq \mu\left(\cap_{i=1}^{\infty} E_{i}\right)$.
10. If you have a finite measure $\mu$ on $\mathscr{B}\left(\mathbb{R}^{p}\right)$, and if $F \in \mathscr{B}\left(\mathbb{R}^{p}\right)$, show that there exist sets $E, G$ such that $G$ is a countable intersection of open sets and $E$ is a countable union of closed sets such that $E \subseteq F \subseteq G$ and $\mu(G \backslash E)=0$.
11. You have a measure space $(\Omega, \mathscr{F}, P)$ where $P$ is a probability measure on $\mathscr{F}$. Then you also have a measurable function $X: \Omega \rightarrow \mathbb{R}^{n}$. Thus $X^{-1}(U) \in \mathscr{F}$ whenever $U$ is open. Now define a measure on $\mathscr{B}\left(\mathbb{R}^{n}\right)$ denoted by $\lambda_{X}$ and defined by $\lambda_{X}(E)=$ $P(\{\omega: X(\omega) \in E\})$. Explain why this yields a well defined probability measure on $\mathscr{B}\left(\mathbb{R}^{n}\right)$. This is called the distribution measure.
12. Let $K \subseteq V$ where $K$ is closed and $V$ is open. Consider the following function.

$$
f(x)=\frac{\operatorname{dist}\left(x, V^{C}\right)}{\operatorname{dist}(x, K)+\operatorname{dist}\left(x, V^{C}\right)}
$$

Explain why this function is continuous, equals 0 off $V$ and equals 1 on $K$.
13. Let $(\Omega, \mathscr{F})$ be a measurable space and let $f: \Omega \rightarrow \mathbb{R}^{n}$ be a measurable function. Then $\sigma(f)$ denotes the smallest $\sigma$ algebra such that $f$ is measurable with respect to this $\sigma$ algebra. Show that $\sigma(f)=\left\{f^{-1}(E): E \in \mathscr{B}\left(\mathbb{R}^{n}\right)\right\}$. More generally, you have a whole set of measurable functions $\mathscr{S}$ and $\sigma(\mathscr{S})$ denotes the smallest $\sigma$ algebra such that each function in $\mathscr{S}$ is measurable. If you have an increasing list $\mathscr{S}_{t}$ for $t \in[0, \infty)$, then $\sigma\left(\mathscr{S}_{t}\right)$ will be what is called a filtration. You have a $\sigma$ algebra for each $t \in[0, \infty)$ and as $t$ increases, these $\sigma$ algebras get larger. This is an essential part of the construction which is used to show that Wiener process is a martingale. In fact the whole subject of martingales has to do with filtrations.
14. There is a monumentally important theorem called the Borel Cantelli lemma. It says the following. If you have a measure space $(\Omega, \mathscr{F}, \mu)$ and if $\left\{E_{i}\right\} \subseteq \mathscr{F}$ is such that $\sum_{i=1}^{\infty} \mu\left(E_{i}\right)<\infty$, then there exists a set $N$ of measure $0(\mu(N)=0)$ such that if $\omega \notin N$, then $\omega$ is in only finitely many of the $E_{i}$. Hint: You might look at the set of all $\omega$ which are in infinitely many of the $E_{i}$. First explain why this set is of the form $\cap_{n=1}^{\infty} \cup_{k \geq n} E_{k}$.
15. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. A sequence of functions $\left\{f_{n}\right\}$ is said to converge in measure to a measurable function $f$ if and only if for each $\varepsilon>0$,

$$
\lim _{n \rightarrow \infty} \mu\left(\left\{\omega:\left|f_{n}(\omega)-f(\omega)\right|>\varepsilon\right\}\right)=0
$$

Show that if this happens, then there exists a subsequence $\left\{f_{n_{k}}\right\}$ and a set of measure $N$ such that if $\omega \notin N$, then

$$
\lim _{k \rightarrow \infty} f_{n_{k}}(\omega)=f(\omega)
$$

Also show that if $\mu$ is finite and $\lim _{n \rightarrow \infty} f_{n}(\omega)=f(\omega)$, then $f_{n}$ converges in measure to $f$.
16. Let $\mathbb{N}$ be the positive integers and let $\mathscr{F}$ denote the set of all subsets of $\mathbb{N}$. Explain why $\mathbb{N}$ is a $\sigma$ algebra. You could let $\mu(S)$ be the number of elements of $S$. This is called counting measure. Explain why $\mu$ is a measure.
17. Show $f: \Omega \rightarrow \mathbb{R}$ is measurable if and only if $f^{-1}(U)$ is measurable whenever $U$ is an open set. Hint: This is pretty easy if you recall that every open set is the disjoint union of countably many connected components.
18. The smallest $\sigma$ algebra on $\mathbb{R}$ which contains the open intervals, denoted by $\mathscr{B}$ is called the Borel sets. Show that $\mathscr{B}$ contains all open sets and is also the smallest $\sigma$ algebra which contains all open sets. Show that all continuous functions $g: \mathbb{R} \rightarrow \mathbb{R}$ are $\mathscr{B}$ measurable. A word of advice pertaining to Borel sets: Don't try to describe a typical Borel set. Instead, use the definition that it is a set in the smallest $\sigma$ algebra containing the open sets.
19. Show that $f: \Omega \rightarrow \mathbb{R}$ is measurable if and only if $f^{-1}(B)$ is measurable for every Borel $\mathscr{B}$. Recall $\mathscr{B}$ is the smallest $\sigma$ algebra which contains the open sets. Hint: Let $\mathscr{G}$ be those sets $B$ such that $f^{-1}(B)$ is measurable. Argue it is a $\sigma$ algebra.
20. Now suppose $f: \Omega \rightarrow \mathbb{R}$ where $(\Omega, \mathscr{F})$ is a measureable space. Suppose $g: \mathbb{R} \rightarrow \mathbb{R}$ is $\mathscr{B}$ measurable. Explain why $g \circ f$ is $\mathscr{F}$ measurable.
21. The open sets in $\mathbb{R}^{n}$ are defined to be all sets $U$ which are unions of open rectangles of the form $R=\prod_{i=1}^{n}\left(a_{i}, b_{i}\right)$ Show that all open sets in $\mathbb{R}^{n}$ are a countable union of such open rectangles. If a pi system $\mathscr{K}$ consists of products of open intervals like the above, show that $\sigma(\mathscr{K})$ is $\mathscr{B}$ the Borel sets. Hint: There are countably many open rectangles of the form $\prod_{i=1}^{n}(p, q), q, p \in \mathbb{Q}$ Show that an arbitrary open rectangle is the union of open rectangles of this sort having rational end points.
22. $\uparrow$ Show that a set of the form $\prod_{i=1}^{n} B_{i}$ is a Borel set in $\mathbb{R}^{n}$ if each $B_{i}$ is a Borel set in $\mathbb{R}$. The Borel sets in $\mathbb{R}^{n}$ are the smallest $\sigma$ algebra which contains the open sets. Hint: You might let $f_{i}: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be the projection map. Explain why $f_{i}^{-1}(B)$ is a Borel set when $B$ is a Borel set in $\mathbb{R}$. You know $f_{i}$ is continuous and that it follows that it is Borel measurable. Now consider intersections of sets like this.
23. Lebesgue measure was discussed. Recall that $m((a, b))=b-a$ and it is defined on a $\sigma$ algebra which contains the Borel sets. It comes from an outer measure defined on $\mathscr{P}(\mathbb{R})$. Also recall that $m$ is translation invariant. Let $x \sim y$ if and only if $x-y \in \mathbb{Q}$. Show this is an equivalence relation. Now let $W$ be a set of positive measure which is contained in $(0,1)$. For $x \in W$, let $[x]$ denote those $y \in W$ such that $x \sim y$. Thus the equivalence classes partition $W$. Use axiom of choice to obtain a set $S \subseteq W$ such that $S$ consists of exactly one element from each equivalence class. Let $\mathbb{T}$ denote the rational numbers in $[-1,1]$. Consider $\mathbb{T}+S \subseteq[-1,2]$. Explain why $\mathbb{T}+S \supseteq$ $W$. For $\mathbb{T} \equiv\left\{r_{j}\right\}$, explain why the sets $\left\{r_{j}+S\right\}_{j}$ are disjoint. Now suppose $S$ is measurable. Then show that you have a contradiction if $m(S)=0$ since $m(W)>0$ and you also have a contradiction if $m(S)>0$ because $\mathbb{T}+S$ consists of countably many disjoint sets. Explain why $S$ cannot be measurable. Thus there exists $T \subseteq \mathbb{R}$ such that $m(T)<m(T \cap S)+m\left(T \cap S^{C}\right)$. Is there an open interval $(a, b)$ such that if $T=(a, b)$, then the above inequality holds?
24. Consider the following nested sequence of compact sets, $\left\{P_{n}\right\}$.Let $P_{1}=[0,1], P_{2}=$ $\left[0, \frac{1}{3}\right] \cup\left[\frac{2}{3}, 1\right]$, etc. To go from $P_{n}$ to $P_{n+1}$, delete the open interval which is the middle third of each closed interval in $P_{n}$. Let $P=\cap_{n=1}^{\infty} P_{n}$. By the finite intersection property of compact sets, $P \neq \emptyset$. Show $m(P)=0$. If you feel ambitious also show there is a one to one onto mapping of $[0,1]$ to $P$. The set $P$ is called the Cantor set. Thus, although $P$ has measure zero, it has the same number of points in it as $[0,1]$ in the sense that there is a one to one and onto mapping from one to the other. Hint: There are various ways of doing this last part but the most enlightenment is obtained by exploiting the topological properties of the Cantor set rather than some silly representation in terms of sums of powers of two and three. All you need to do is use the Schroder Bernstein theorem and show there is an onto map from the Cantor set to $[0,1]$.
25. Consider the sequence of functions defined in the following way. Let $f_{1}(x)=x$ on $[0,1]$. To get from $f_{n}$ to $f_{n+1}$, let $f_{n+1}=f_{n}$ on all intervals where $f_{n}$ is constant. If $f_{n}$ is nonconstant on $[a, b]$, let $f_{n+1}(a)=f_{n}(a), f_{n+1}(b)=f_{n}(b), f_{n+1}$ is piecewise linear and equal to $\frac{1}{2}\left(f_{n}(a)+f_{n}(b)\right)$ on the middle third of $[a, b]$. Sketch a few of these and you will see the pattern. The process of modifying a nonconstant section of the graph of this function is illustrated in the following picture.


Show $\left\{f_{n}\right\}$ converges uniformly on $[0,1]$. If $f(x)=\lim _{n \rightarrow \infty} f_{n}(x)$, show that $f(0)=$ $0, f(1)=1, f$ is continuous, and $f^{\prime}(x)=0$ for all $x \notin P$ where $P$ is the Cantor set of Problem 24. This function is called the Cantor function.It is a very important example to remember. Note it has derivative equal to zero a.e. and yet it succeeds in climbing from 0 to 1 . Explain why this interesting function is not absolutely continuous although it is continuous. Hint: This isn't too hard if you focus on getting a careful estimate on the difference between two successive functions in the list considering only a typical small interval in which the change takes place. The above picture should be helpful.
26. $\uparrow$ This problem gives a very interesting example found in the book by McShane [34]. Let $g(x)=x+f(x)$ where $f$ is the strange function of Problem 25. Let $P$ be the

Cantor set of Problem 24. Let $[0,1] \backslash P=\cup_{j=1}^{\infty} I_{j}$ where $I_{j}$ is open and $I_{j} \cap I_{k}=\emptyset$ if $j \neq k$. These intervals are the connected components of the complement of the Cantor set. Show $m\left(g\left(I_{j}\right)\right)=m\left(I_{j}\right)$ so

$$
m\left(g\left(\cup_{j=1}^{\infty} I_{j}\right)\right)=\sum_{j=1}^{\infty} m\left(g\left(I_{j}\right)\right)=\sum_{j=1}^{\infty} m\left(I_{j}\right)=1
$$

Thus $m(g(P))=1$ because $g([0,1])=[0,2]$. By Problem 23 there exists a set, $A \subseteq g(P)$ which is non measurable. Define $\phi(x)=\mathscr{X}_{A}(g(x))$. Thus $\phi(x)=0$ unless $x \in P$. Tell why $\phi$ is measurable. (Recall $m(P)=0$ and Lebesgue measure is complete.) Now show that $\mathscr{X}_{A}(y)=\phi\left(g^{-1}(y)\right)$ for $y \in[0,2]$. Tell why $g^{-1}$ is continuous but $\phi \circ g^{-1}$ is not measurable. (This is an example of measurable $\circ$ continuous $\neq$ measurable.) Show there exist Lebesgue measurable sets which are not Borel measurable. Hint: The function, $\phi$ is Lebesgue measurable. Now recall that Borel $\circ$ measurable $=$ measurable.
27. For $\mathbf{x} \in \mathbb{R}^{p}$ to be in $\prod_{i=1}^{p} A_{i}$, it means that the $i^{\text {th }}$ component of $\mathbf{x}, x_{i}$ is in $A_{i}$ for each $i$. Now for $\prod_{i=1}^{p}\left(a_{i}, b_{i}\right) \equiv R$, let $V(R)=\prod_{i=1}^{p}\left(b_{i}-a_{i}\right)$. Next, for $A \in \mathscr{P}\left(\mathbb{R}^{p}\right)$ let

$$
\mu(A) \equiv \inf \left\{\sum_{k} V\left(R^{k}\right): A \subseteq \cup_{k} R^{k}\right\}
$$

This is just like one dimensional Lebesgue measure except that instead of open intervals, we are using open boxes $R^{k}$. Show the following.
(a) $\mu$ is an outer measure.
(b) $\mu\left(\prod_{i=1}^{p}\left[a_{i}, b_{i}\right]\right)=\prod_{i=1}^{p}\left(b_{i}-a_{i}\right)=\mu\left(\prod_{i=1}^{p}\left(a_{i}, b_{i}\right)\right)$.
(c) If $\operatorname{dist}(A, B)>0$, then $\mu(A)+\mu(B)=\mu(A \cup B)$ so $\mathscr{B}\left(\mathbb{R}^{p}\right) \subseteq \mathscr{F}$ the set of sets measurable with respect to this outer measure $\mu$.

This is Lebesgue measure on $\mathbb{R}^{p}$. Hint: Suppose for some $j, b_{j}-a_{j}<\varepsilon$. Show that $\mu\left(\prod_{i=1}^{p}\left(a_{i}, b_{i}\right)\right) \leq \varepsilon \prod_{i \neq j}\left(b_{i}-a_{i}\right)$. Now use this to show that if you have a covering by finitely many open boxes, such that the sum of their volumes is less than some number, you can replace with a covering of open boxes which also has the sum of their volumes less than that number but which has each box with sides less than $\delta$. To do this, you might consider replacing each box in the covering with $2^{m p}$ open boxes obtained by bisecting each side $m$ times where $m$ is small enough that each little box has sides smaller than $\delta / 2$ in each of the finitely many boxes in the cover and then fatten each of these just a little to cover up what got left out and retain the sum of the volumes of the little boxes to still be less than the number you had.
28. $\uparrow$ Show that Lebesgue measure defined in the above problem is both inner and outer regular and is translation invariant.
29. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and let $s(\omega)=\sum_{i=0}^{n} c_{i} \mathscr{X}_{E_{i}}(\omega)$ where the $E_{i}$ are distinct measurable sets but the $c_{i}$ might not be. Thus the $c_{i}$ are the finitely many values of $s$. Say each $c_{i} \geq 0$ and $c_{0}=0$. Define $\int s d \mu$ as $\sum_{i} c_{i} \mu\left(E_{i}\right)$. Show that this is well defined and that if you have $s(\omega)=\sum_{i=1}^{n} c_{i} \mathscr{X}_{E_{i}}(\omega), t(\omega)=\sum_{j=1}^{m} d_{j} \mathscr{X}_{F_{j}}(\omega)$, then for $a, b$ nonnegative numbers, as $(\omega)+b t(\omega)$ can be written also in this form and that $\int(a s+b t) d \mu=a \int s d \mu+b \int t d \mu$. Hint: $s(\omega)=\sum_{i} \sum_{j} c_{i} \mathscr{X}_{E_{i} \cap F_{j}}(\omega)=$ $\sum_{j} \sum_{i} c_{i} \mathscr{X}_{E_{i} \cap F_{j}}(\omega)$ and $(a s+b t)(\omega)=\sum_{j} \sum_{i}\left(a c_{i}+b d_{j}\right) \mathscr{X}_{E_{i} \cap F_{j}}(\omega)$.
30. $\uparrow$ Having defined the integral of nonnegative simple functions in the above problem, letting $f$ be nonnegative and measurable. Define

$$
\int f d \mu \equiv \sup \left\{\int s d \mu: 0 \leq s \leq f, s \text { simple }\right\}
$$

Show that if $f_{n}$ is nonnegative and measurable and $n \rightarrow f_{n}(\omega)$ is increasing, show that for $f(\omega)=\lim _{n \rightarrow \infty} f_{n}(\omega)$, it follows that $\int f d \mu=\lim _{n \rightarrow \infty} \int f_{n} d \mu$. Hint: Show $\int f_{n} d \mu$ is increasing to something $\alpha \leq \infty$. Explain why $\int f d \mu \geq \alpha$. Now pick a nonnegative simple function $s \leq f$. For $r \in(0,1),\left[f_{n}>r s\right] \equiv E_{n}$ is increasing in $n$ and $\cup_{n} E_{n}=\Omega$. Tell why $\int f_{n} d \mu \geq \int \mathscr{X}_{E_{n}} f_{n} d \mu \geq r \int s d \mu$. Let $n \rightarrow \infty$ and show that $\alpha \geq r \int s d \mu$. Now explain why $\alpha \geq r \int f d \mu$. Since $r$ is arbitrary, $\alpha \geq \int f d \mu \geq \alpha$.
31. $\uparrow$ Show that if $f, g$ are nonnegative and measurable and $a, b \geq 0$, then

$$
\int(a f+b g) d \mu=a \int f d \mu+b \int g d \mu
$$

## Chapter 7

## The Abstract Lebesgue Integral

The general Lebesgue integral requires a measure space, $(\Omega, \mathscr{F}, \mu)$ and, to begin with, a nonnegative measurable function. I will use Lemma 1.11.2 about interchanging two supremums frequently. Also, I will use the observation that if $\left\{a_{n}\right\}$ is an increasing sequence of points of $[0, \infty]$, then $\sup _{n} a_{n}=\lim _{n \rightarrow \infty} a_{n}$ which is obvious from the definition of sup. Lebesgue integration is a theory which depends on absolute convergence. Thus we understand things in terms of nonnegative functions. For complex valued functions, we consider positive and negative parts of real and imaginary parts. Thus one typically discusses nonnegative functions in statements of the main theorems.

### 7.1 Nonnegative Measurable Functions

### 7.1.1 Riemann Integrals For Decreasing Functions

First of all, the notation $[g<f]$ means $\{\omega \in \Omega: g(\omega)<f(\omega)\}$ with other variants of this notation being similar. Also, the convention, $0 \cdot \infty=0$ will be used to simplify the presentation whenever it is convenient to do so. The notation $a \wedge b$ means the minimum of $a$ and $b$.

Definition 7.1.1 Let $f:[a, b] \rightarrow[0, \infty]$ be decreasing. Note that $\infty$ is a possible value. Define

$$
\int_{a}^{b} f(\lambda) d \lambda \equiv \lim _{M \rightarrow \infty} \int_{a}^{b} M \wedge f(\lambda) d \lambda=\sup _{M} \int_{a}^{b} M \wedge f(\lambda) d \lambda
$$

where $a \wedge b$ means the minimum of $a$ and $b$. Note that for $f$ bounded,

$$
\sup _{M} \int_{a}^{b} M \wedge f(\lambda) d \lambda=\int_{a}^{b} f(\lambda) d \lambda
$$

where the integral on the right is the usual Riemann integral because eventually $M>f$. For $f$ a nonnegative decreasing function defined on $[0, \infty)$,

$$
\int_{0}^{\infty} f d \lambda \equiv \lim _{R \rightarrow \infty} \int_{0}^{R} f d \lambda=\sup _{R>1} \int_{0}^{R} f d \lambda=\sup _{R} \sup _{M>0} \int_{0}^{R} f \wedge M d \lambda
$$

Since decreasing bounded functions are Riemann integrable, the above definition is well defined. See Theorem 5.1.11. Now here is an obvious property.
Lemma 7.1.2 Let $f$ be a decreasing nonnegative function defined on an interval $[a, b]$. Then if $[a, b]=\cup_{k=1}^{m} I_{k}$ where $I_{k} \equiv\left[a_{k}, b_{k}\right]$ and the intervals $I_{k}$ are non overlapping, it follows

$$
\int_{a}^{b} f d \lambda=\sum_{k=1}^{m} \int_{a_{k}}^{b_{k}} f d \lambda
$$

Proof: This follows from Theorems 5.1.7 and 5.1.11 along with the computation,

$$
\int_{a}^{b} f d \lambda \equiv \lim _{M \rightarrow \infty} \int_{a}^{b} f \wedge M d \lambda=\lim _{M \rightarrow \infty} \sum_{k=1}^{m} \int_{a_{k}}^{b_{k}} f \wedge M d \lambda=\sum_{k=1}^{m} \int_{a_{k}}^{b_{k}} f d \lambda
$$

Note both sides could equal $+\infty$.
In all considerations below, we assume $h$ is fairly small, certainly much smaller than $R$. Thus $R-h>0$.

Lemma 7.1.3 Let $g$ be a decreasing nonnegative function defined on an interval $[0, R]$. Then

$$
\int_{0}^{R} g \wedge M d \lambda=\sup _{h>0} \sum_{i=1}^{m(R, h)}(g(i h) \wedge M) h
$$

where $m(h, R) \in \mathbb{N}$ satisfies $R-h<h m(h, R) \leq R$.
Proof: Since $g \wedge M$ is a decreasing bounded function the lower sums converge to the integral as $h \rightarrow 0$. Thus

$$
\int_{0}^{R} g \wedge M d \lambda=\lim _{h \rightarrow 0}\left(\sum_{i=1}^{m(R, h)}(g(i h) \wedge M) h+(g(R) \wedge M)(R-h m(h, R))\right)
$$

Now the last term in the above is no more than $M h$ and so the above is

$$
\lim _{h \rightarrow 0}\left(\sum_{i=1}^{m(R, h)}(g(i h) \wedge M) h\right)=\sup _{h>0}\left(\sum_{i=1}^{m(R, h)}(g(i h) \wedge M) h\right) .
$$

### 7.1.2 The Lebesgue Integral for Nonnegative Functions

Here is the definition of the Lebesgue integral of a function which is measurable and has values in $[0, \infty]$.

Definition 7.1.4 Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and suppose $f: \Omega \rightarrow[0, \infty]$ is measurable. Then define

$$
\int f d \mu \equiv \int_{0}^{\infty} \mu([f>\lambda]) d \lambda
$$

which makes sense because $\lambda \rightarrow \mu([f>\lambda])$ is nonnegative and decreasing.
Note that if $f \leq g$, then $\int f d \mu \leq \int g d \mu$ because $\mu([f>\lambda]) \leq \mu([g>\lambda])$.
For convenience $\sum_{i=1}^{0} a_{i} \equiv 0$.
Lemma 7.1.5 In the situation of the above definition,

$$
\int f d \mu=\sup _{h>0} \sum_{i=1}^{\infty} \mu([f>h i]) h
$$

Proof: Let $m(h, R) \in \mathbb{N}$ satisfy $R-h<h m(h, R) \leq R$. Then

$$
\lim _{R \rightarrow \infty} m(h, R)=\infty
$$

and so from Lemma 7.1.3,

$$
\begin{aligned}
\int f d \mu & \equiv \int_{0}^{\infty} \mu([f>\lambda]) d \lambda=\sup _{M} \sup _{R} \int_{0}^{R} \mu([f>\lambda]) \wedge M d \lambda \\
& =\sup _{M} \sup _{R>0} \sup _{h>0} \sum_{k=1}^{m(h, R)}(\mu([f>k h]) \wedge M) h
\end{aligned}
$$

Hence, switching the order of the sups, this equals

$$
\begin{aligned}
& \sup _{R>0} \sup _{h>0} \sup _{M} \sum_{k=1}^{m(h, R)}(\mu([f>k h]) \wedge M) h=\sup _{R>0} \sup _{h>0} \lim _{M \rightarrow \infty} \sum_{k=1}^{m(h, R)}(\mu([f>k h]) \wedge M) h \\
& =\sup _{h>0} \sup _{R} \sum_{k=1}^{m(R, h)}(\mu([f>k h])) h=\sup _{h>0} \sum_{k=1}^{\infty}(\mu([f>k h])) h .
\end{aligned}
$$

### 7.2 Integration of Nonnegative Simple Functions

To begin with, here is a useful lemma.
Lemma 7.2.1 If $f(\lambda)=0$ for all $\lambda>a$, where $f$ is a decreasing nonnegative function, then

$$
\int_{0}^{\infty} f(\lambda) d \lambda=\int_{0}^{a} f(\lambda) d \lambda
$$

Proof: From the definition,

$$
\begin{aligned}
& \int_{0}^{\infty} f(\lambda) d \lambda=\lim _{R \rightarrow \infty} \int_{0}^{R} f(\lambda) d \lambda=\sup _{R>1} \int_{0}^{R} f(\lambda) d \lambda \\
&= \sup _{R>1} \sup _{M} \int_{0}^{R} f(\lambda) \wedge M d \lambda=\sup _{M} \sup \int_{R>1}^{R} f(\lambda) \wedge M d \lambda \\
&=\sup _{M} \sup _{R>1} \int_{0}^{a} f(\lambda) \wedge M d \lambda=\sup _{M} \int_{0}^{a} f(\lambda) \wedge M d \lambda \equiv \int_{0}^{a} f(\lambda) d \lambda . \mid
\end{aligned}
$$

Now the Lebesgue integral for a nonnegative function has been defined, what does it do to a nonnegative simple function? Recall a nonnegative simple function is one which has finitely many nonnegative real values which it assumes on measurable sets. Thus a simple function can be written in the form

$$
s(\omega)=\sum_{i=1}^{n} c_{i} \mathscr{X}_{E_{i}}(\omega)
$$

where the $c_{i}$ are each nonnegative, the distinct values of $s$.
Lemma 7.2.2 Let $s(\omega)=\sum_{i=1}^{p} a_{i} \mathscr{X}_{E_{i}}(\omega)$ be a nonnegative simple function where the $E_{i}$ are distinct but the $a_{i}$ might not be. Thus the values of s are the $a_{i}$. Then

$$
\begin{equation*}
\int s d \mu=\sum_{i=1}^{p} a_{i} \mu\left(E_{i}\right) \tag{7.1}
\end{equation*}
$$

Proof: Without loss of generality, assume $0 \equiv a_{0}<a_{1} \leq a_{2} \leq \cdots \leq a_{p}$ and that $\mu\left(E_{i}\right)<$ $\infty, i>0$. Here is why. If $\mu\left(E_{i}\right)=\infty$, then letting $a \in\left(a_{i-1}, a_{i}\right)$, by Lemma 7.2.1, the left side is

$$
\begin{aligned}
\int_{0}^{a_{p}} \mu([s>\lambda]) d \lambda & \geq \int_{a_{0}}^{a_{i}} \mu([s>\lambda]) d \lambda \\
& \equiv \sup _{M} \int_{0}^{a_{i}} \mu([s>\lambda]) \wedge M d \lambda \geq \sup _{M} M a_{i}=\infty
\end{aligned}
$$

and so both sides of 7.1 are equal to $\infty$. Thus assume for each $i>0, \mu\left(E_{i}\right)<\infty$. Then it follows from Lemma 7.2.1 and Lemma 7.1.2,

$$
\begin{aligned}
& \int_{0}^{\infty} \mu([s>\lambda]) d \lambda=\int_{0}^{a_{p}} \mu([s>\lambda]) d \lambda=\sum_{k=1}^{p} \int_{a_{k-1}}^{a_{k}} \mu([s>\lambda]) d \lambda \\
= & \sum_{k=1}^{p}\left(a_{k}-a_{k-1}\right) \sum_{i=k}^{p} \mu\left(E_{i}\right)=\sum_{i=1}^{p} \mu\left(E_{i}\right) \sum_{k=1}^{i}\left(a_{k}-a_{k-1}\right)=\sum_{i=1}^{p} a_{i} \mu\left(E_{i}\right)
\end{aligned}
$$

Note that this is the same result as in Problem 29 on Page 157 but here there is no question about the definition of the integral of a simple function being well defined.

Lemma 7.2.3 If $a, b \geq 0$ and if $s$ and $t$ are nonnegative simple functions, then

$$
\int a s+b t d \mu=a \int s d \mu+b \int t d \mu
$$

Proof: Let $s(\omega)=\sum_{i=1}^{n} \alpha_{i} \mathscr{X}_{A_{i}}(\omega), t(\omega)=\sum_{i=1}^{m} \beta_{j} \mathscr{X}_{B_{j}}(\omega)$ where $\alpha_{i}$ are the distinct values of $s$ and the $\beta_{j}$ are the distinct values of $t$. Clearly $a s+b t$ is a nonnegative simple function because it has finitely many values on measurable sets. In fact,

$$
(a s+b t)(\omega)=\sum_{j=1}^{m} \sum_{i=1}^{n}\left(a \alpha_{i}+b \beta_{j}\right) \mathscr{X}_{A_{i} \cap B_{j}}(\omega)
$$

where the sets $A_{i} \cap B_{j}$ are disjoint and measurable. By Lemma 7.2.2,

$$
\begin{aligned}
\int a s+b t d \mu & =\sum_{j=1}^{m} \sum_{i=1}^{n}\left(a \alpha_{i}+b \beta_{j}\right) \mu\left(A_{i} \cap B_{j}\right) \\
& =\sum_{i=1}^{n} a \sum_{j=1}^{m} \alpha_{i} \mu\left(A_{i} \cap B_{j}\right)+b \sum_{j=1}^{m} \sum_{i=1}^{n} \beta_{j} \mu\left(A_{i} \cap B_{j}\right) \\
& =a \sum_{i=1}^{n} \alpha_{i} \mu\left(A_{i}\right)+b \sum_{j=1}^{m} \beta_{j} \mu\left(B_{j}\right)=a \int s d \mu+b \int t d \mu .
\end{aligned}
$$

### 7.3 The Monotone Convergence Theorem

The following is called the monotone convergence theorem. This theorem and related convergence theorems are the reason for using the Lebesgue integral. If $\lim _{n \rightarrow \infty} f_{n}(\omega)=$ $f(\omega)$ and $f_{n}$ is increasing in $n$, then clearly $f$ is also measurable because of Corollary 6.1.4. Also

$$
f^{-1}((a, \infty])=\cup_{k=1}^{\infty} f_{k}^{-1}((a, \infty]) \in \mathscr{F}
$$

For a different approach to this, see Problem 29 on Page 157.
Theorem 7.3.1 (Monotone Convergence theorem) Suppose that the function $f$ has all values in $[0, \infty]$ and suppose $\left\{f_{n}\right\}$ is a sequence of nonnegative measurable functions having values in $[0, \infty]$ and satisfying

$$
\begin{gathered}
\lim _{n \rightarrow \infty} f_{n}(\omega)=f(\omega) \text { for each } \omega \\
\cdots f_{n}(\omega) \leq f_{n+1}(\omega) \cdots
\end{gathered}
$$

Then $f$ is measurable and

$$
\int f d \mu=\lim _{n \rightarrow \infty} \int f_{n} d \mu
$$

Proof: By Lemma 7.1.5

$$
\begin{gathered}
\lim _{n \rightarrow \infty} \int f_{n} d \mu=\sup _{n} \int f_{n} d \mu \\
=\sup _{n} \sup _{h>0} \sum_{k=1}^{\infty} \mu\left(\left[f_{n}>k h\right]\right) h=\sup _{h>0} \sup _{N} \sup _{n} \sum_{k=1}^{N} \mu\left(\left[f_{n}>k h\right]\right) h \\
=\sup _{h>0} \sup _{N} \sum_{k=1}^{N} \mu([f>k h]) h=\sup _{h>0} \sum_{k=1}^{\infty} \mu([f>k h]) h=\int f d \mu .
\end{gathered}
$$

Note how it was important to have $\int_{0}^{\infty}[f>\lambda] d \lambda$ in the definition of the integral and not $[f \geq \lambda]$. You need to have $\left[f_{n}>k h\right] \uparrow[f>k h]$ so $\mu\left(\left[f_{n}>k h\right]\right) \rightarrow \mu([f>k h])$. To illustrate what goes wrong without the Lebesgue integral, consider the following example.

Example 7.3.2 Let $\left\{r_{n}\right\}$ denote the rational numbers in $[0,1]$ and let

$$
f_{n}(t) \equiv\left\{\begin{array}{l}
1 \text { if } t \notin\left\{r_{1}, \cdots, r_{n}\right\} \\
0 \text { otherwise }
\end{array}\right.
$$

Then $f_{n}(t) \uparrow f(t)$ where $f$ is the function which is one on the rationals and zero on the irrationals. Each $f_{n}$ is Riemann integrable (why?) but $f$ is not Riemann integrable because it is everywhere discontinuous. Also, there is a gap between all upper sums and lower sums. Therefore, you can't write $\int f d x=\lim _{n \rightarrow \infty} \int f_{n} d x$.

An observation which is typically true related to this type of example is this. If you can choose your functions, you don't need the Lebesgue integral. The Riemann Darboux integral is just fine. It is when you can't choose your functions and they come to you as pointwise limits that you really need the superior Lebesgue integral or at least something more general than the Riemann integral. The Riemann integral is entirely adequate for evaluating the seemingly endless lists of boring problems found in calculus books. It is shown later that the two integrals coincide when the Lebesgue integral is taken with respect to Lebesgue measure and the function being integrated is continuous.

### 7.4 Other Definitions

To review and summarize the above, if a nonnegative function $f$ is measurable,

$$
\begin{equation*}
\int f d \mu \equiv \int_{0}^{\infty} \mu([f>\lambda]) d \lambda \tag{7.2}
\end{equation*}
$$

another way to get the same thing for $\int f d \mu$ is to take an increasing sequence of nonnegative simple functions, $\left\{s_{n}\right\}$ with $s_{n}(\omega) \rightarrow f(\omega)$ and then by monotone convergence theorem,

$$
\int f d \mu=\lim _{n \rightarrow \infty} \int s_{n}
$$

where if $s_{n}(\omega)=\sum_{j=1}^{m} c_{i} \mathscr{X}_{E_{i}}(\omega), \int s_{n} d \mu=\sum_{i=1}^{m} c_{i} \mu\left(E_{i}\right)$. Similarly this also shows that for such a nonnegative measurable function,

$$
\int f d \mu=\sup \left\{\int s: 0 \leq s \leq f, s \text { simple }\right\}
$$

This is in Problem 29 on Page 157. Here is an equivalent definition of the integral of a nonnegative measurable function. The fact it is well defined has been discussed above. I think that the following definition is more standard than the above one involving the distribution function. One can begin with this one instead. An outline of a different proof of the monotone convergence theorem is in Problem 30 on Page 158.
Definition 7.4.1 For sa nonnegative simple function,

$$
s(\omega)=\sum_{k=1}^{n} c_{k} \mathscr{X}_{E_{k}}(\omega), \int s \equiv \sum_{k=1}^{n} c_{k} \mu\left(E_{k}\right) .
$$

For $f$ a nonnegative measurable function,

$$
\int f d \mu=\sup \left\{\int s: 0 \leq s \leq f, s \text { simple }\right\}
$$

### 7.5 Fatou's Lemma

The next theorem, known as Fatou's lemma is another important theorem which justifies the use of the Lebesgue integral.

Theorem 7.5.1 (Fatou's lemma) Let $f_{n}$ be a nonnegative measurable function. Let $g(\omega)=\liminf _{n \rightarrow \infty} f_{n}(\omega)$. Then $g$ is measurable and

$$
\int g d \mu \leq \lim \inf _{n \rightarrow \infty} \int f_{n} d \mu
$$

In other words,

$$
\int\left(\lim \inf _{n \rightarrow \infty} f_{n}\right) d \mu \leq \lim \inf _{n \rightarrow \infty} \int f_{n} d \mu
$$

Proof: Let

$$
g_{n}(\omega)=\inf \left\{f_{k}(\omega): k \geq n\right\}
$$

Then $g_{n}^{-1}([a, \infty])=\cap_{k=n}^{\infty} f_{k}^{-1}([a, \infty]) \in \mathscr{F}$. Thus $g_{n}$ is measurable by Lemma 6.1.2. Now the functions $g_{n}$ form an increasing sequence of nonnegative measurable functions. Thus $g^{-1}((a, \infty))=\cup_{n=1}^{\infty} g_{n}^{-1}((a, \infty)) \in \mathscr{F}$ so $g$ is measurable also. By monotone convergence theorem,

$$
\int g d \mu=\lim _{n \rightarrow \infty} \int g_{n} d \mu \leq \lim \inf _{n \rightarrow \infty} \int f_{n} d \mu
$$

The last inequality holding because $\int g_{n} d \mu \leq \int f_{n} d \mu$. (Note that it is not known whether $\lim _{n \rightarrow \infty} \int f_{n} d \mu$ exists.)

### 7.6 The Integral's Righteous Algebraic Desires

The monotone convergence theorem shows that the integral wants to be linear. This is the essential content of the next theorem.
Theorem 7.6.1 Let $f, g$ be nonnegative measurable functions and let $a, b$ be nonnegative numbers. Then af $+b g$ is measurable and

$$
\begin{equation*}
\int(a f+b g) d \mu=a \int f d \mu+b \int g d \mu \tag{7.3}
\end{equation*}
$$

Proof: By Theorem 6.1.10 on Page 140 there exist increasing sequences of nonnegative simple functions, $s_{n} \rightarrow f$ and $t_{n} \rightarrow g$. Then $a f+b g$, being the pointwise limit of the simple functions $a s_{n}+b t_{n}$, is measurable. Now by the monotone convergence theorem and Lemma 7.2.3,

$$
\begin{aligned}
\int(a f+b g) d \mu & =\lim _{n \rightarrow \infty} \int a s_{n}+b t_{n} d \mu \\
& =\lim _{n \rightarrow \infty}\left(a \int s_{n} d \mu+b \int t_{n} d \mu\right) \\
& =a \int f d \mu+b \int g d \mu
\end{aligned}
$$

As long as you are allowing functions to take the value $+\infty$, you cannot consider something like $f+(-g)$ and so you can't expect a satisfactory statement about the integral being linear until you restrict yourself to functions which have values in a vector space. To be linear, a function must be defined on a vector space. This is discussed next.

### 7.7 The Lebesgue Integral, $L^{1}$

The functions considered here have values in $\mathbb{C}$, which is a vector space. A function $f$ with values in $\mathbb{C}$ is of the form $f=\operatorname{Re} f+i \operatorname{Im} f$ where $\operatorname{Re} f$ and $\operatorname{Im} f$ are real valued functions. In fact

$$
\operatorname{Re} f=\frac{f+\bar{f}}{2}, \operatorname{Im} f=\frac{f-\bar{f}}{2 i}
$$

We first define the integral of real valued functions and then the integral of a complex valued function will be of the form

$$
\int f d \mu=\int \operatorname{Re}(f) d \mu+i \int \operatorname{Im}(f) d \mu
$$

Definition 7.7.1 Let $(\Omega, \mathscr{S}, \mu)$ be a measure space and suppose $f: \Omega \rightarrow \mathbb{C}$. Then $f$ is said to be measurable if both $\operatorname{Re} f$ and $\operatorname{Im} f$ are measurable real valued functions.

As is always the case for complex numbers, $|z|^{2}=(\operatorname{Re} z)^{2}+(\operatorname{Im} z)^{2}$. Also, for $g$ a real valued function, one can consider its positive and negative parts defined respectively as

$$
g^{+}(x) \equiv \frac{g(x)+|g(x)|}{2}, g^{-}(x)=\frac{|g(x)|-g(x)}{2}
$$

Thus $|g|=g^{+}+g^{-}$and $g=g^{+}-g^{-}$and both $g^{+}$and $g^{-}$are measurable nonnegative functions if $g$ is measurable. This follows because of Theorem 6.1.6. The mappings $x \rightarrow$ $x^{+}, x \rightarrow x^{-}$are clearly continuous. Thus $g^{+}$is the composition of a continuous function with a measurable function.

Then the following is the definition of what it means for a complex valued function $f$ to be in $L^{1}(\Omega)$.

Definition 7.7.2 $\operatorname{Let}(\Omega, \mathscr{F}, \mu)$ be a measure space. Then a complex valued measurable function $f$ is in $L^{1}(\Omega)$ if

$$
\int|f| d \mu<\infty
$$

For a function in $L^{1}(\Omega)$, the integral is defined as follows.

$$
\int f d \mu \equiv \int(\operatorname{Re} f)^{+} d \mu-\int(\operatorname{Re} f)^{-} d \mu+i\left[\int(\operatorname{Im} f)^{+} d \mu-\int(\operatorname{Im} f)^{-} d \mu\right]
$$

I will show that with this definition, the integral is linear and well defined. First note that it is clearly well defined because all the above integrals are of nonnegative functions and are each equal to a nonnegative real number because for $h$ equal to any of the functions, $|h| \leq|f|$ and $\int|f| d \mu<\infty$.

Here is a lemma which will make it possible to show the integral is linear.
Lemma 7.7.3 Let $g, h, g^{\prime}, h^{\prime}$ be nonnegative measurable functions in $L^{1}(\Omega)$ and suppose that

$$
g-h=g^{\prime}-h^{\prime}
$$

Then

$$
\int g d \mu-\int h d \mu=\int g^{\prime} d \mu-\int h^{\prime} d \mu
$$

Proof: By assumption, $g+h^{\prime}=g^{\prime}+h$. Then from the Lebesgue integral's righteous algebraic desires, Theorem 7.6.1,

$$
\int g d \mu+\int h^{\prime} d \mu=\int g^{\prime} d \mu+\int h d \mu
$$

which implies the claimed result.
Lemma 7.7.4 Let $\operatorname{Re}\left(L^{1}(\Omega)\right)$ denote the vector space of real valued functions in $L^{1}(\Omega)$ where the field of scalars is the real numbers. Then $\int d \mu$ is linear on $\operatorname{Re}\left(L^{1}(\Omega)\right)$, the scalars being real numbers.

Proof: First observe that from the definition of the positive and negative parts of a function,

$$
(f+g)^{+}-(f+g)^{-}=f^{+}+g^{+}-\left(f^{-}+g^{-}\right)
$$

because both sides equal $f+g$. Therefore from Lemma 7.7.3 and the definition, it follows from Theorem 7.6.1 that

$$
\begin{aligned}
\int f+g d \mu & \equiv \int(f+g)^{+}-(f+g)^{-} d \mu=\int f^{+}+g^{+} d \mu-\int f^{-}+g^{-} d \mu \\
& =\int f^{+} d \mu+\int g^{+} d \mu-\left(\int f^{-} d \mu+\int g^{-} d \mu\right)=\int f d \mu+\int g d \mu
\end{aligned}
$$

what about taking out scalars? First note that if $a$ is real and nonnegative, then $(a f)^{+}=a f^{+}$ and $(a f)^{-}=a f^{-}$while if $a<0$, then $(a f)^{+}=-a f^{-}$and $(a f)^{-}=-a f^{+}$. These claims follow immediately from the above definitions of positive and negative parts of a function. Thus if $a<0$ and $f \in L^{1}(\Omega)$, it follows from Theorem 7.6.1 that

$$
\begin{aligned}
\int a f d \mu & \equiv \int(a f)^{+} d \mu-\int(a f)^{-} d \mu=\int(-a) f^{-} d \mu-\int(-a) f^{+} d \mu \\
& =-a \int f^{-} d \mu+a \int f^{+} d \mu=a\left(\int f^{+} d \mu-\int f^{-} d \mu\right) \equiv a \int f d \mu
\end{aligned}
$$

The case where $a \geq 0$ works out similarly but easier.
Now here is the main result.

Theorem 7.7.5 $\int d \mu$ is linear on $L^{1}(\Omega)$ and $L^{1}(\Omega)$ is a complex vector space. If $f \in L^{1}(\Omega)$, then $\operatorname{Re} f, \operatorname{Im} f$, and $|f|$ are all in $L^{1}(\Omega)$. Also, for $f \in L^{1}(\Omega)$,

$$
\begin{aligned}
\int f d \mu & \equiv \int(\operatorname{Re} f)^{+} d \mu-\int(\operatorname{Re} f)^{-} d \mu+i\left[\int(\operatorname{Im} f)^{+} d \mu-\int(\operatorname{Im} f)^{-} d \mu\right] \\
& \equiv \int \operatorname{Re} f d \mu+i \int \operatorname{Im} f d \mu
\end{aligned}
$$

and the triangle inequality holds,

$$
\begin{equation*}
\left|\int f d \mu\right| \leq \int|f| d \mu \tag{7.4}
\end{equation*}
$$

Also, for every $f \in L^{1}(\Omega)$ it follows that for every $\varepsilon>0$ there exists a simple function s such that $|s| \leq|f|$ and

$$
\int|f-s| d \mu<\varepsilon
$$

Proof: First consider the claim that the integral is linear. It was shown above that the integral is linear on $\operatorname{Re}\left(L^{1}(\Omega)\right)$. Then letting $a+i b, c+i d$ be scalars and $f, g$ functions in $L^{1}(\Omega)$,

$$
\begin{gathered}
(a+i b) f+(c+i d) g=(a+i b)(\operatorname{Re} f+i \operatorname{Im} f)+(c+i d)(\operatorname{Re} g+i \operatorname{Im} g) \\
=c \operatorname{Re}(g)-b \operatorname{Im}(f)-d \operatorname{Im}(g)+a \operatorname{Re}(f)+i(b \operatorname{Re}(f)+c \operatorname{Im}(g)+a \operatorname{Im}(f)+d \operatorname{Re}(g))
\end{gathered}
$$

It follows from the definition that

$$
\begin{gather*}
\int(a+i b) f+(c+i d) g d \mu=\int(c \operatorname{Re}(g)-b \operatorname{Im}(f)-d \operatorname{Im}(g)+a \operatorname{Re}(f)) d \mu \\
+i \int(b \operatorname{Re}(f)+c \operatorname{Im}(g)+a \operatorname{Im}(f)+d \operatorname{Re}(g)) \tag{7.5}
\end{gather*}
$$

Also, from the definition,

$$
\begin{aligned}
(a+i b) \int f d \mu+(c+i d) \int g d \mu= & (a+i b)\left(\int \operatorname{Re} f d \mu+i \int \operatorname{Im} f d \mu\right) \\
& +(c+i d)\left(\int \operatorname{Re} g d \mu+i \int \operatorname{Im} g d \mu\right)
\end{aligned}
$$

which equals

$$
\begin{aligned}
= & a \int \operatorname{Re} f d \mu-b \int \operatorname{Im} f d \mu+i b \int \operatorname{Re} f d \mu+i a \int \operatorname{Im} f d \mu \\
& +c \int \operatorname{Re} g d \mu-d \int \operatorname{Im} g d \mu+i d \int \operatorname{Re} g d \mu-d \int \operatorname{Im} g d \mu .
\end{aligned}
$$

Using Lemma 7.7.4 and collecting terms, it follows that this reduces to 7.5. Thus the integral is linear as claimed.

Consider the claim about approximation with a simple function. Letting $h$ equal any of

$$
\begin{equation*}
(\operatorname{Re} f)^{+},(\operatorname{Re} f)^{-},(\operatorname{Im} f)^{+},(\operatorname{Im} f)^{-}, \tag{7.6}
\end{equation*}
$$

It follows from the monotone convergence theorem and Theorem 6.1.10 on Page 140 there exists a nonnegative simple function $s \leq h$ such that

$$
\int|h-s| d \mu<\frac{\varepsilon}{4}
$$

Therefore, letting $s_{1}, s_{2}, s_{3}, s_{4}$ be such simple functions, approximating respectively the functions listed in 7.6, and $s \equiv s_{1}-s_{2}+i\left(s_{3}-s_{4}\right)$,

$$
\begin{gathered}
\int|f-s| d \mu \leq \int\left|(\operatorname{Re} f)^{+}-s_{1}\right| d \mu+\int\left|(\operatorname{Re} f)^{-}-s_{2}\right| d \mu \\
+\int\left|(\operatorname{Im} f)^{+}-s_{3}\right| d \mu+\int\left|(\operatorname{Im} f)^{-}-s_{4}\right| d \mu<\varepsilon
\end{gathered}
$$

It is clear from the construction that $|s| \leq|f|$.
What about 7.4? Let $\theta \in \mathbb{C}$ be such that $|\theta|=1$ and $\theta \int f d \mu=\left|\int f d \mu\right|$. Then from what was shown above about the integral being linear,

$$
\left|\int f d \mu\right|=\theta \int f d \mu=\int \theta f d \mu=\int \operatorname{Re}(\theta f) d \mu \leq \int|f| d \mu
$$

It is routine to verify that for $f, g$ measurable, meaning real and imaginary parts are measurable, then any complex linear combination is also measurable. This follows right away from Theorem 6.1.6 and looking at the real and imaginary parts of this complex linear combination. Also

$$
\int|a f+b g| d \mu \leq \int|a||f|+|b||g| d \mu<\infty
$$

The following corollary follows from this. The conditions of this corollary are sometimes taken as a definition of what it means for a function $f$ to be in $L^{1}(\Omega)$.

Corollary 7.7.6 $f \in L^{1}(\Omega)$ if and only if there exists a sequence of complex simple functions, $\left\{s_{n}\right\}$ such that

$$
\begin{gather*}
s_{n}(\omega) \rightarrow f(\omega) \text { for all } \omega \in \Omega \\
\lim _{m, n \rightarrow \infty} \int\left(\left|s_{n}-s_{m}\right|\right)=0 \tag{7.7}
\end{gather*}
$$

When $f \in L^{1}(\Omega)$,

$$
\begin{equation*}
\int f d \mu \equiv \lim _{n \rightarrow \infty} \int s_{n} \tag{7.8}
\end{equation*}
$$

Proof: $\Rightarrow$ From the above theorem, if $f \in L^{1}$ there exists a sequence of simple functions $\left\{s_{n}\right\}$ such that

$$
\int\left|f-s_{n}\right| d \mu<1 / n, s_{n}(\omega) \rightarrow f(\omega) \text { for all } \omega
$$

Then

$$
\int\left|s_{n}-s_{m}\right| d \mu \leq \int\left|s_{n}-f\right| d \mu+\int\left|f-s_{m}\right| d \mu \leq \frac{1}{n}+\frac{1}{m}
$$

$\Leftarrow$ Next suppose the existence of the approximating sequence of simple functions. Then $f$ is measurable because its real and imaginary parts are the limit of measurable functions. By Fatou's lemma,

$$
\int|f| d \mu \leq \lim \inf _{n \rightarrow \infty} \int\left|s_{n}\right| d \mu<\infty
$$

because

$$
\left|\int\right| s_{n}\left|d \mu-\int\right| s_{m}|d \mu| \leq \int\left|s_{n}-s_{m}\right| d \mu
$$

which is given to converge to 0 . Hence $\left\{\int\left|s_{n}\right| d \mu\right\}$ is a Cauchy sequence and is therefore, bounded.

In case $f \in L^{1}(\Omega)$, letting $\left\{s_{n}\right\}$ be the approximating sequence, Fatou's lemma implies

$$
\left|\int f d \mu-\int s_{n} d \mu\right| \leq \int\left|f-s_{n}\right| d \mu \leq \lim \inf _{m \rightarrow \infty} \int\left|s_{m}-s_{n}\right| d \mu<\varepsilon
$$

provided $n$ is large enough. Hence 7.8 follows.
This is a good time to observe the following fundamental observation which follows from a repeat of the above arguments.

Theorem 7.7.7 Suppose $\Lambda(f) \in[0, \infty]$ for all nonnegative measurable functions and suppose that for $a, b \geq 0$ and $f, g$ nonnegative measurable functions,

$$
\Lambda(a f+b g)=a \Lambda(f)+b \Lambda(g)
$$

In other words, $\Lambda$ wants to be linear. Then $\Lambda$ has a unique linear extension to the set of measurable functions

$$
\{f \text { measurable }: \Lambda(|f|)<\infty\},
$$

this set being a vector space.
If you want, you could say the same thing replacing measurable with continuous.
Notation 7.7.8 If $E$ is a measurable set and $f$ is a measurable nonnegative function or one in $L^{1}$, the integral $\int \mathscr{X}_{E} f d \mu$ is often denoted as $\int_{E} f d \mu$.

### 7.8 The Dominated Convergence Theorem

One of the major theorems in this theory is the dominated convergence theorem. Before presenting it, here is a technical lemma about limsup and liminf which is really pretty obvious from the definition.

Lemma 7.8.1 Let $\left\{a_{n}\right\}$ be a sequence in $[-\infty, \infty]$. Then $\lim _{n \rightarrow \infty} a_{n}$ exists if and only if

$$
\lim _{n \rightarrow \infty} \inf _{n \rightarrow \infty} a_{n}=\lim \sup _{n \rightarrow \infty} a_{n}
$$

and in this case, the limit equals the common value of these two numbers.
Proof: Suppose first $\lim _{n \rightarrow \infty} a_{n}=a \in \mathbb{R}$. Then, let $\varepsilon>0$ be given, $a_{n} \in(a-\varepsilon, a+\varepsilon)$ for all $n$ large enough, say $n \geq N$. Therefore, both $\inf \left\{a_{k}: k \geq n\right\}$ and $\sup \left\{a_{k}: k \geq n\right\}$ are contained in $[a-\varepsilon, a+\varepsilon]$ whenever $n \geq N$. It follows $\limsup _{n \rightarrow \infty} a_{n}$ and $\liminf _{n \rightarrow \infty} a_{n}$ are both in $[a-\varepsilon, a+\varepsilon]$, showing

$$
\left|\lim \inf _{n \rightarrow \infty} a_{n}-\lim \sup _{n \rightarrow \infty} a_{n}\right|<2 \varepsilon
$$

Since $\varepsilon$ is arbitrary, the two must be equal and they both must equal $a$. Next suppose $\lim _{n \rightarrow \infty} a_{n}=\infty$. Then if $l \in \mathbb{R}$, there exists $N$ such that for $n \geq N, l \leq a_{n}$ and therefore, for such $n$,

$$
l \leq \inf \left\{a_{k}: k \geq n\right\} \leq \sup \left\{a_{k}: k \geq n\right\}
$$

and this shows, since $l$ is arbitrary that $\liminf _{n \rightarrow \infty} a_{n}=\limsup _{n \rightarrow \infty} a_{n}=\infty$. The case for $-\infty$ is similar.

Conversely, suppose $\liminf _{n \rightarrow \infty} a_{n}=\limsup \sup _{n \rightarrow \infty} a_{n}=a$. Suppose first that $a \in \mathbb{R}$. Then, letting $\varepsilon>0$ be given, there exists $N$ such that if $n \geq N$,

$$
\sup \left\{a_{k}: k \geq n\right\}-\inf \left\{a_{k}: k \geq n\right\}<\varepsilon
$$

therefore, if $k, m>N$, and $a_{k}>a_{m}$,

$$
\left|a_{k}-a_{m}\right|=a_{k}-a_{m} \leq \sup \left\{a_{k}: k \geq n\right\}-\inf \left\{a_{k}: k \geq n\right\}<\varepsilon
$$

showing that $\left\{a_{n}\right\}$ is a Cauchy sequence. Therefore, it converges to $a \in \mathbb{R}$, and as in the first part, the liminf and limsup both equal $a$. If $\liminf _{n \rightarrow \infty} a_{n}=\limsup _{n \rightarrow \infty} a_{n}=\infty$, then given $l \in \mathbb{R}$, there exists $N$ such that for $n \geq N, \inf _{n>N} a_{n}>l$. Therefore, $\lim _{n \rightarrow \infty} a_{n}=\infty$. The case for $-\infty$ is similar.

Here is the dominated convergence theorem.
Theorem 7.8.2 (Dominated Convergence theorem) Let $f_{n} \in L^{1}(\Omega)$ and suppose

$$
f(\omega)=\lim _{n \rightarrow \infty} f_{n}(\omega)
$$

and there exists a measurable function $g$, with values in $[0, \infty],{ }^{1}$ such that

$$
\left|f_{n}(\omega)\right| \leq g(\omega) \text { and } \int g(\omega) d \mu<\infty .
$$

Then $f \in L^{1}(\Omega)$ and

$$
0=\lim _{n \rightarrow \infty} \int\left|f_{n}-f\right| d \mu=\lim _{n \rightarrow \infty}\left|\int f d \mu-\int f_{n} d \mu\right|
$$

Proof: $f$ is measurable by Corollary 6.1.4 applied to real and imaginary parts. Since $|f| \leq g$, it follows that

$$
f \in L^{1}(\Omega) \text { and }\left|f-f_{n}\right| \leq 2 g .
$$

By Fatou's lemma (Theorem 7.5.1),

$$
\begin{aligned}
\int 2 g d \mu & \leq \lim _{n \rightarrow \infty} \int 2 g-\left|f-f_{n}\right| d \mu \\
& =\int 2 g d \mu-\lim \sup _{n \rightarrow \infty} \int\left|f-f_{n}\right| d \mu
\end{aligned}
$$

Subtracting $\int 2 g d \mu, 0 \leq-\limsup \sin _{n \rightarrow \infty} \int\left|f-f_{n}\right| d \mu$. Hence

$$
\begin{aligned}
0 & \geq \lim \sup _{n \rightarrow \infty}\left(\int\left|f-f_{n}\right| d \mu\right) \\
& \geq \lim \inf _{n \rightarrow \infty}\left(\int\left|f-f_{n}\right| d \mu\right) \geq \lim \inf _{n \rightarrow \infty}\left|\int f d \mu-\int f_{n} d \mu\right| \geq 0 .
\end{aligned}
$$

This proves the theorem by Lemma 7.8 .1 because the limsup and liminf are equal.

[^3]Corollary 7.8.3 Suppose $f_{n} \in L^{1}(\Omega)$ and $f(\omega)=\lim _{n \rightarrow \infty} f_{n}(\omega)$. Suppose also there exist measurable functions, $g_{n}$, $g$ with values in $[0, \infty]$ such that

$$
\lim _{n \rightarrow \infty} \int g_{n} d \mu=\int g d \mu, g_{n}(\omega) \rightarrow g(\omega) \mu \text { a.e. }
$$

and both $\int g_{n} d \mu$ and $\int g d \mu$ are finite. Also suppose $\left|f_{n}(\omega)\right| \leq g_{n}(\omega)$. Then

$$
\lim _{n \rightarrow \infty} \int\left|f-f_{n}\right| d \mu=0
$$

Proof: It is just like the above. This time $g+g_{n}-\left|f-f_{n}\right| \geq 0$ and so by Fatou's lemma,

$$
\begin{gathered}
\int 2 g d \mu-\lim \sup _{n \rightarrow \infty} \int\left|f-f_{n}\right| d \mu=\lim _{n \rightarrow \infty} \int\left(g_{n}+g\right) d \mu-\lim \sup _{n \rightarrow \infty} \int\left|f-f_{n}\right| d \mu \\
=\lim _{n \rightarrow \infty} \int\left(g_{n}+g\right) d \mu-\lim \sup _{n \rightarrow \infty} \int\left|f-f_{n}\right| d \mu \\
=\lim _{n \rightarrow \infty} \int\left(\left(g_{n}+g\right)-\left|f-f_{n}\right|\right) d \mu \geq \int 2 g d \mu
\end{gathered}
$$

and so $-\limsup \operatorname{sum}_{n \rightarrow \infty} \int\left|f-f_{n}\right| d \mu \geq 0$. Thus

$$
\begin{aligned}
0 & \geq \lim \sup _{n \rightarrow \infty}\left(\int\left|f-f_{n}\right| d \mu\right) \\
& \geq \lim \inf _{n \rightarrow \infty}\left(\int\left|f-f_{n}\right| d \mu\right) \geq\left|\int f d \mu-\int f_{n} d \mu\right| \geq 0
\end{aligned}
$$

## Definition 7.8.4 Let $E$ be a measurable subset of $\Omega$.

$$
\int_{E} f d \mu \equiv \int f \mathscr{X}_{E} d \mu
$$

If $L^{1}(E)$ is written, the $\sigma$ algebra is defined as $\{E \cap A: A \in \mathscr{F}\}$ and the measure is $\mu$ restricted to this smaller $\sigma$ algebra. Clearly, if $f \in L^{1}(\Omega)$, then $f \mathscr{X}_{E} \in L^{1}(E)$ and if $f \in L^{1}(E)$, then letting $\tilde{f}$ be the 0 extension of $f$ off of $E$, it follows $\tilde{f} \in L^{1}(\Omega)$.

What about something ordinary, the integral of a continuous function?
Theorem 7.8.5 Let $f$ be continuous on $[a, b]$. Then

$$
\int_{a}^{b} f(x) d x=\int_{[a, b]} f d m
$$

where the integral on the left is the usual Riemann integral and the integral on the right is the Lebesgue integral.

Proof: From Theorems 6.5.1 and 6.8.2 $f \mathscr{X}_{[a, b]}$ is Lebesgue measurable. Assume for the sake of simplicity that $f(x) \geq 0$. If not, apply what is about to be shown to $f^{+}$and $f^{-}$. Let $s_{n}(x)$ be a step function and let this converge uniformly to $f(x)$ on $[a, b]$ with $s_{n}(x)=0$ for $x \notin[a, b]$. For example, let

$$
s_{n}(x) \equiv \sum_{j=1}^{n} f\left(x_{j-1}\right) \mathscr{X}_{\left[x_{j-1}, x_{j}\right)}(x)
$$

Then $\int_{a}^{b} s_{n}(x) d x=\int_{[a, b]} s_{n} d m$ thanks to Theorem 6.5.1 which gives the measure of intervals. Then one can apply the definition of the Riemann integral to obtain the left side converging to $\int_{a}^{b} f(x) d x$ because $f$ is continuous and $\int_{a}^{b} s_{n}(x) d x$ is nothing more than a left sum. Then apply the dominated convergence theorem on the right to obtain the claim of the theorem. Indeed $f$ is bounded and so there exists $M \geq f(x) \geq 0$ for all $x \in[a, b]$.

This shows that for reasonable functions, there is nothing new in the Lebesgue integral. The big difference is that now you have limit theorems which may be applied and you can integrate more functions. In fact, every Riemann integrable function on an interval is Lebesgue integrable. See Problem 13 on Page 183.

### 7.9 Product Measures

First of all is a definition.
Definition 7.9.1 Let $(X, \mathscr{F}, \mu)$ be a measure space. Then it is called $\sigma$ finite if there exists an increasing sequence of sets $R_{n} \in \mathscr{F}$ such that $\mu\left(R_{n}\right)<\infty$ for all $n$ and also $X=\cup_{n=1}^{\infty} R_{n}$.

Now I will show how to define a measure on $\prod_{i=1}^{p} X_{i}$ given that $\left(X_{i}, \mathscr{F}_{i}, \mu_{i}\right)$ is a $\sigma$ finite measure space.

Let $\mathscr{K}$ denote all subsets of $\mathbf{X} \equiv \prod_{i=1}^{p} X_{i}$ which are the form $\prod_{i=1}^{p} E_{i}$ where $E_{i} \in \mathscr{F}_{i}$. These are called measurable rectangles. Let $\left\{R_{i}^{n}\right\}_{n=1}^{\infty}$ be the sequence of sets in $\mathscr{F}_{i}$ whose union is all of $X_{i}, R_{i}^{n} \subseteq R_{i}^{n+1}$, and $\mu_{i}\left(R_{i}^{n}\right)<\infty$. Thus if $\mathbf{R}^{n} \equiv \prod_{i=1}^{p} R_{i}^{n}$, and $\mathbf{E} \equiv \prod_{i=1}^{p} E_{i}$, then

$$
\mathbf{R}^{n} \cap \mathbf{E}=\prod_{i=1}^{p} R_{i}^{n} \cap E_{i}
$$

Let $\mathbf{I} \equiv\left(i_{1}, \cdots, i_{p}\right)$ where $\left(i_{1}, \cdots, i_{p}\right)$ is a permutation of $\{1, \cdots, p\}$. Also, to save on space, denote the iterated integral

$$
\int_{X_{1_{1}}} \cdots \int_{X_{i_{p}}} \mathscr{X}_{\mathbf{F}}\left(x_{1}, \cdots, x_{p}\right) d \mu_{i_{1}} \cdots d \mu_{i_{p}}
$$

as $\int_{\mathbf{I}} \mathscr{X}_{\mathbf{F}}\left(x_{1}, \cdots, x_{p}\right) d \mu_{\mathbf{I}}$. Then define $\mathscr{G}$ as follows. $\mathscr{G}$ will consist of all $\mathbf{F} \subseteq \mathbf{X}$ satisfying the following condition.

$$
\left\{\text { For all } n, \int_{\mathbf{I}} \mathscr{X}_{\mathbf{F} \cap \mathbf{R}^{n}}\left(x_{1}, \cdots, x_{p}\right) d \mu_{\mathbf{I}} \text { makes sense independent of } \mathbf{I}\right\}
$$

The iterated integral means what the symbols indicate. Integrate $\mathscr{X}_{\mathbf{F}}\left(x_{1}, \cdots, x_{p}\right)$ with respect to $d \mu_{i_{1}}$ and then you have a function of the other variables other than $x_{i_{1}}$. Then integrate what is left with respect to $x_{i_{2}}$ and so forth. This is just like what was done with iterated integrals in calculus. In order for this to make sense, every function encountered must be measurable with respect to the appropriate $\sigma$ algebra. Now obviously $\mathscr{K} \subseteq \mathscr{G}$. In fact, if $\mathbf{F} \in \mathscr{K}$, then $\int_{\mathbf{I}} \mathscr{X}_{\mathbf{F} \cap \mathbf{R}^{n}}\left(x_{1}, \cdots, x_{p}\right) d \mu_{\mathbf{I}}=\prod_{i=1}^{p} \mu_{i}\left(F_{i} \cap R_{i}^{n}\right)$ for any choice of $n$.

Proposition 7.9.2 Let $\mathscr{K}$ and $\mathscr{G}$ be as just defined, then $\mathscr{G} \supseteq \sigma(\mathscr{K})$. We define $\sigma(\mathscr{K})$ as $\mathscr{F}^{p}$, better denoted as $\mathscr{F}_{1} \times \cdots \times \mathscr{F}_{p}$. Then if

$$
\vec{\mu}(\mathbf{F}) \equiv \lim _{n \rightarrow \infty} \int_{\mathbf{I}} \mathscr{X}_{\mathbf{F} \cap \mathbf{R}^{n}}\left(x_{1}, \cdots, x_{p}\right) d \mu_{\mathbf{I}}
$$

then $\vec{\mu}$ is a measure which does not depend on $\mathbf{I}$ the particular permutation chosen for the order of integration. $\vec{\mu}$ often denoted as $\mu_{1} \times \cdots \times \mu_{p}$ is called product measure. $f: \mathbf{X} \rightarrow[0, \infty)$ is measurable with respect to $\mathscr{F}^{p}$ then for any permutation $\left(i_{1}, \cdots, i_{p}\right)$ of $\{1, \cdots, p\}$ it follows

$$
\begin{equation*}
\int f d \vec{\mu}=\int \cdots \int f\left(x_{1}, \cdots, x_{p}\right) d \mu_{i_{1}} \cdots d \mu_{i_{p}} \tag{7.9}
\end{equation*}
$$

Proof: I will show that $\mathscr{G}$ is closed with respect to complements and countable disjoint unions. Then the result will follow. Now suppose $\left\{\mathbf{F}^{k}\right\}_{k=1}^{\infty}$ are disjoint, each in $\mathscr{G}$. Then if $\mathbf{F} \equiv \cup_{k=1}^{\infty} \mathbf{F}^{k}$,

$$
\mathbf{F} \cap \mathbf{R}^{n}=\cup_{k=1}^{\infty} \mathbf{F}^{k} \cap \mathbf{R}^{n}
$$

and since these sets are disjoint, $\mathscr{X}_{\mathbf{F} \cap \mathbf{R}^{n}}=\sum_{k=1}^{\infty} \mathscr{X}_{\mathbf{F}^{k} \cap \mathbf{R}^{n}}$. Therefore, applying the monotone convergence theorem repeatedly for the iterated integrals and using the fact that measurability is not lost on taking limits, then for two permutations $\left(i_{1}, \cdots, i_{p}\right),\left(j_{1}, \cdots, j_{p}\right)$,

$$
\begin{aligned}
& \int \cdots \int \mathscr{X}_{\mathbf{F} \cap \mathbf{R}^{n}}\left(x_{1}, \cdots, x_{p}\right) d \mu_{i_{1}} \cdots d \mu_{i_{p}} \\
= & \int \cdots \int \lim _{N \rightarrow \infty} \sum_{k=1}^{N} \mathscr{X}_{\mathbf{F}^{k} \cap \mathbf{R}^{n}}\left(x_{1}, \cdots, x_{p}\right) d \mu_{i_{1}} \cdots d \mu_{i_{p}} \\
= & \lim _{N \rightarrow \infty} \int \cdots \int \sum_{k=1}^{N} \mathscr{X}_{\mathbf{F}^{k} \cap \mathbf{R}^{n}}\left(x_{1}, \cdots, x_{p}\right) d \mu_{i_{1}} \cdots d \mu_{i_{p}} \\
= & \lim _{N \rightarrow \infty} \sum_{k=1}^{N} \int \cdots \int \mathscr{X}_{\mathbf{F}^{k} \cap \mathbf{R}^{n}}\left(x_{1}, \cdots, x_{p}\right) d \mu_{i_{1}} \cdots d \mu_{i_{p}} \\
= & \lim _{N \rightarrow \infty} \sum_{k=1}^{N} \int \cdots \int \mathscr{X}_{\mathbf{F}^{k} \cap \mathbf{R}^{n}}\left(x_{1}, \cdots, x_{p}\right) d \mu_{j_{1}} \cdots d \mu_{j_{p}} \\
= & \lim _{N \rightarrow \infty} \int \cdots \int \sum_{k=1}^{N} \mathscr{X}_{\mathbf{F}^{k} \cap \mathbf{R}^{n}}\left(x_{1}, \cdots, x_{p}\right) d \mu_{j_{1}} \cdots d \mu_{j_{p}} \\
= & \int \cdots \int \lim _{N \rightarrow \infty} \sum_{k=1}^{N} \mathscr{X}_{\mathbf{F}^{k} \cap \mathbf{R}^{n}}\left(x_{1}, \cdots, x_{p}\right) d \mu_{j_{1}} \cdots d \mu_{j_{p}} \\
= & \int \cdots \int \mathscr{X}_{\mathbf{F} \cap \mathbf{R}^{n}}\left(x_{1}, \cdots, x_{p}\right) d \mu_{j_{1}} \cdots d \mu_{j_{p}}
\end{aligned}
$$

Thus $\mathscr{G}$ is closed with respect to countable disjoint unions. So suppose $\mathbf{F} \in \mathscr{G}$. Then $\mathscr{X}_{\mathbf{F}^{C} \cap \mathbf{R}^{n}}=\mathscr{X}_{\mathbf{R}^{n}}-\mathscr{X}_{\mathbf{F} \cap \mathbf{R}^{n}}$. Everything works for both terms on the right and in addition, $\int_{\mathbf{I}} \mathscr{X}_{\mathbf{R}^{n}} d \mu_{\mathbf{I}}$ is finite and independent of $\mathbf{I}$. Therefore, everything works as it should for the function on the left using similar arguments to the above. You simply verify that all makes sense for each integral at a time and apply monotone convergence theorem as needed. Therefore, $\mathscr{G}$ is indeed closed with respect to complements. It follows that $\mathscr{G} \supseteq \sigma(\mathscr{K})$ by Dynkin's lemma, Lemma 6.3.2. Now define for $\mathbf{F} \in \sigma(\mathscr{K})$,

$$
\vec{\mu}(\mathbf{F}) \equiv \lim _{n \rightarrow \infty} \int_{\mathbf{I}} \mathscr{X}_{\mathbf{F} \cap \mathbf{R}^{n}}\left(x_{1}, \cdots, x_{p}\right) d \mu_{\mathbf{I}}
$$

By definition of $\mathscr{G}$ this definition of $\vec{\mu}$ does not depend on $\mathbf{I}$. If you have $\left\{\mathbf{F}^{k}\right\}_{k=1}^{\infty}$ is a sequence of disjoint sets in $\mathscr{G}$, then if $\mathbf{F}$ is their union,

$$
\vec{\mu}(\mathbf{F}) \equiv \lim _{n \rightarrow \infty} \int_{\mathbf{I}} \sum_{k=1}^{\infty} \mathscr{X}_{\mathbf{F}^{k} \cap \mathbf{R}^{n}}\left(x_{1}, \cdots, x_{p}\right) d \mu_{\mathbf{I}}
$$

and one can apply the monotone convergence theorem one integral at a time and obtain that this is

$$
\lim _{n \rightarrow \infty} \sum_{k=1}^{\infty} \int_{\mathbf{I}} \mathscr{X}_{\mathbf{F}^{k} \cap \mathbf{R}^{n}}\left(x_{1}, \cdots, x_{p}\right) d \mu_{\mathbf{I}}
$$

Now applying the monotone convergence theorem again, this time for the Lebesgue integral given by a sum with counting measure, the above is

$$
\sum_{k=1}^{\infty} \lim _{n \rightarrow \infty} \int_{\mathbf{I}} \mathscr{X}_{\mathbf{F}^{k} \cap \mathbf{R}^{n}}\left(x_{1}, \cdots, x_{p}\right) d \mu_{\mathbf{I}} \equiv \sum_{k=1}^{\infty} \vec{\mu}\left(\mathbf{F}^{k}\right)
$$

which shows that $\vec{\mu}$ is indeed a measure. Also from the construction, it follows that this measure does not depend on the particular permutation of the iterated integrals used to compute it.

The claim about the integral 7.9 follows right away from the monotone convergence theorem applied in the right side one iterated integral at a time and approximation with simple functions as in Theorem 6.1.10. The result holds for each of an increasing sequence of simple functions from linearity of integrals and the definition of $\vec{\mu}$. Then you apply the monotone convergence theorem to obtain the claim of the theorem.

### 7.10 Some Important General Theorems

### 7.10.1 Eggoroff's Theorem

Eggoroff's theorem says that if a sequence converges pointwise, then it almost converges uniformly in a certain sense.

Theorem 7.10.1 (Egoroff) Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space,

$$
(\mu(\Omega)<\infty)
$$

and let $f_{n}, f$ be complex valued functions such that $\operatorname{Re} f_{n}, \operatorname{Im} f_{n}$ are all measurable and

$$
\lim _{n \rightarrow \infty} f_{n}(\omega)=f(\omega)
$$

for all $\omega \notin E$ where $\mu(E)=0$. Then for every $\varepsilon>0$, there exists a set,

$$
F \supseteq E, \mu(F)<\varepsilon,
$$

such that $f_{n}$ converges uniformly to $f$ on $F^{C}$.
Proof: First suppose $E=\emptyset$ so that convergence is pointwise everywhere. It follows then that $\operatorname{Re} f$ and $\operatorname{Im} f$ are pointwise limits of measurable functions and are therefore measurable. Let $E_{k m}=\left\{\omega \in \Omega:\left|f_{n}(\omega)-f(\omega)\right| \geq 1 / m\right.$ for some $\left.n>k\right\}$. Note that

$$
\left|f_{n}(\omega)-f(\omega)\right|=\sqrt{\left(\operatorname{Re} f_{n}(\omega)-\operatorname{Re} f(\omega)\right)^{2}+\left(\operatorname{Im} f_{n}(\omega)-\operatorname{Im} f(\omega)\right)^{2}}
$$

and so,

$$
\left[\left|f_{n}-f\right| \geq \frac{1}{m}\right]
$$

is measurable. Hence $E_{k m}$ is measurable because

$$
E_{k m}=\cup_{n=k+1}^{\infty}\left[\left|f_{n}-f\right| \geq \frac{1}{m}\right]
$$

For fixed $m, \cap_{k=1}^{\infty} E_{k m}=\emptyset$ because $f_{n}$ converges to $f$. Therefore, if $\omega \in \Omega$ there exists $k$ such that if $n>k,\left|f_{n}(\omega)-f(\omega)\right|<\frac{1}{m}$ which means $\omega \notin E_{k m}$. Note also that

$$
E_{k m} \supseteq E_{(k+1) m} .
$$

Since $\mu\left(E_{1 m}\right)<\infty$, Theorem 6.2.4 on Page 141 implies

$$
0=\mu\left(\cap_{k=1}^{\infty} E_{k m}\right)=\lim _{k \rightarrow \infty} \mu\left(E_{k m}\right)
$$

Let $k(m)$ be chosen such that $\mu\left(E_{k(m) m}\right)<\varepsilon 2^{-m}$ and let

$$
F=\bigcup_{m=1}^{\infty} E_{k(m) m}
$$

Then $\mu(F)<\varepsilon$ because

$$
\mu(F) \leq \sum_{m=1}^{\infty} \mu\left(E_{k(m) m}\right)<\sum_{m=1}^{\infty} \varepsilon 2^{-m}=\varepsilon
$$

Now let $\eta>0$ be given and pick $m_{0}$ such that $m_{0}^{-1}<\eta$. If $\omega \in F^{C}$, then

$$
\omega \in \bigcap_{m=1}^{\infty} E_{k(m) m}^{C} .
$$

Hence $\omega \in E_{k\left(m_{0}\right) m_{0}}^{C}$ so

$$
\left|f_{n}(\omega)-f(\omega)\right|<1 / m_{0}<\eta
$$

for all $n>k\left(m_{0}\right)$. This holds for all $\omega \in F^{C}$ and so $f_{n}$ converges uniformly to $f$ on $F^{C}$.
Now if $E \neq \emptyset$, consider $\left\{\mathscr{X}_{E^{C}} f_{n}\right\}_{n=1}^{\infty}$. Each $\mathscr{X}_{E^{C}} f_{n}$ has real and imaginary parts measurable and the sequence converges pointwise to $\mathscr{X}_{E} f$ everywhere. Therefore, from the first part, there exists a set of measure less than $\varepsilon, F$ such that on $F^{C},\left\{\mathscr{X}_{E^{C}} f_{n}\right\}$ converges uniformly to $\mathscr{X}_{E^{C}} f$. Therefore, on $(E \cup F)^{C},\left\{f_{n}\right\}$ converges uniformly to $f$.

### 7.10.2 The Vitali Convergence Theorem

The Vitali convergence theorem is a convergence theorem which in the case of a finite measure space is superior to the dominated convergence theorem.
Definition 7.10.2 Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and let $\mathfrak{S} \subseteq L^{1}(\Omega)$. $\mathfrak{S}$ is uniformly integrable if for every $\varepsilon>0$ there exists $\delta>0$ such that for all $f \in \mathfrak{S}$

$$
\left|\int_{E} f d \mu\right|<\varepsilon \text { whenever } \mu(E)<\delta
$$

Lemma 7.10.3 If $\mathfrak{S}$ is uniformly integrable, then $|\mathfrak{S}| \equiv\{|f|: f \in \mathfrak{S}\}$ is uniformly integrable. Also $\mathfrak{S}$ is uniformly integrable if $\mathfrak{S}$ is finite.

Proof: Let $\varepsilon>0$ be given and suppose $\mathfrak{S}$ is uniformly integrable. First suppose the functions are real valued. Let $\delta$ be such that if $\mu(E)<\delta$, then

$$
\left|\int_{E} f d \mu\right|<\frac{\varepsilon}{2}
$$

for all $f \in \mathfrak{S}$. Let $\mu(E)<\delta$. Then if $f \in \mathfrak{S}$,

$$
\begin{aligned}
\int_{E}|f| d \mu & \leq \int_{E \cap[f \leq 0]}(-f) d \mu+\int_{E \cap[f>0]} f d \mu \\
& =\left|\int_{E \cap[f \leq 0]} f d \mu\right|+\left|\int_{E \cap[f>0]} f d \mu\right|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon .
\end{aligned}
$$

In general, if $\mathfrak{S}$ is a uniformly integrable set of complex valued functions, the inequalities,

$$
\left|\int_{E} \operatorname{Re} f d \mu\right| \leq\left|\int_{E} f d \mu\right|,\left|\int_{E} \operatorname{Im} f d \mu\right| \leq\left|\int_{E} f d \mu\right|,
$$

imply $\operatorname{Re} \mathfrak{S} \equiv\{\operatorname{Re} f: f \in \mathfrak{S}\}$ and $\operatorname{Im} \mathfrak{S} \equiv\{\operatorname{Im} f: f \in \mathfrak{S}\}$ are also uniformly integrable. Therefore, applying the above result for real valued functions to these sets of functions, it follows $|\mathfrak{S}|$ is uniformly integrable also.

For the last part, is suffices to verify a single function in $L^{1}(\Omega)$ is uniformly integrable. To do so, note that from the dominated convergence theorem,

$$
\lim _{R \rightarrow \infty} \int_{\||f|>R]}|f| d \mu=0 .
$$

Let $\varepsilon>0$ be given and choose $R$ large enough that $\int_{[|f|>R]}|f| d \mu<\frac{\varepsilon}{2}$. Now let $\mu(E)<\frac{\varepsilon}{2 R}$. Then

$$
\begin{aligned}
\int_{E}|f| d \mu & =\int_{E \cap[|f| \leq R]}|f| d \mu+\int_{E \cap[|f|>R]}|f| d \mu \\
& <R \mu(E)+\frac{\varepsilon}{2}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon .
\end{aligned}
$$

This proves the lemma.
The following gives a nice way to identify a uniformly integrable set of functions.
Lemma 7.10.4 Let $\mathfrak{S}$ be a subset of $L^{1}(\Omega, \mu)$ where $\mu(\Omega)<\infty$. Let $t \rightarrow h(t)$ be a continuous function which satisfies

$$
\lim _{t \rightarrow \infty} \frac{h(t)}{t}=\infty
$$

Then $\mathfrak{S}$ is uniformly integrable and bounded in $L^{1}(\Omega)$ if

$$
\sup \left\{\int_{\Omega} h(|f|) d \mu: f \in \mathfrak{S}\right\}=N<\infty .
$$

Proof: First I show $\mathfrak{S}$ is bounded in $L^{1}(\Omega ; \mu)$ which means there exists a constant $M$ such that for all $f \in \mathfrak{S}$,

$$
\int_{\Omega}|f| d \mu \leq M
$$

From the properties of $h$, there exists $R_{n}$ such that if $t \geq R_{n}$, then $h(t) \geq n t$. Therefore,

$$
\int_{\Omega}|f| d \mu=\int_{\left[|f| \geq R_{n}\right]}|f| d \mu+\int_{\left[|f|<R_{n}\right]}|f| d \mu
$$

Letting $n=1$,

$$
\begin{aligned}
\int_{\Omega}|f| d \mu & \leq \int_{\left[|f| \geq R_{1}\right]} h(|f|) d \mu+R_{1} \mu\left(\left[|f|<R_{1}\right]\right) \\
& \leq N+R_{1} \mu(\Omega) \equiv M
\end{aligned}
$$

Next let $E$ be a measurable set. Then for every $f \in \mathfrak{S}$,

$$
\begin{gathered}
\int_{E}|f| d \mu=\int_{\left[|f| \geq R_{n}\right] \cap E}|f| d \mu+\int_{\left[|f|<R_{n}\right] \cap E}|f| d \mu \\
\quad \leq \frac{1}{n} \int_{\Omega}|f| d \mu+R_{n} \mu(E) \leq \frac{N}{n}+R_{n} \mu(E)
\end{gathered}
$$

and letting $n$ be large enough, this is less than $\varepsilon / 2+R_{n} \mu(E)$. Now if $\mu(E)<\varepsilon / 2 R_{n}$, it follows that for all $f \in \mathfrak{S}, \int_{E}|f| d \mu<\varepsilon$. This proves the lemma.

Letting $h(t)=t^{2}$, it follows that if all the functions in $\mathfrak{S}$ are bounded, then the collection of functions is uniformly integrable.

The following theorem is Vitali's convergence theorem.
Theorem 7.10.5 Let $\left\{f_{n}\right\}$ be a uniformly integrable set of complex valued functions, $\mu(\Omega)<\infty$, and $f_{n}(x) \rightarrow f(x)$ a.e. where $f$ is a measurable complex valued function. Then $f \in L^{1}(\Omega)$ and

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \int_{\Omega}\left|f_{n}-f\right| d \mu=0 \tag{7.10}
\end{equation*}
$$

Proof: First it will be shown that $f \in L^{1}(\Omega)$. By uniform integrability, there exists $\delta>0$ such that if $\mu(E)<\delta$, then $\int_{E}\left|f_{n}\right| d \mu<1$ for all $n$. By Egoroff's theorem, there exists a set, $E$ of measure less than $\delta$ such that on $E^{C},\left\{f_{n}\right\}$ converges uniformly. Therefore, for $p$ large enough, and $n>p, \int_{E^{C}}\left|f_{p}-f_{n}\right| d \mu<1$ which implies

$$
\int_{E^{C}}\left|f_{n}\right| d \mu<1+\int_{\Omega}\left|f_{p}\right| d \mu
$$

Then since there are only finitely many functions, $f_{n}$ with $n \leq p$, there exists a constant, $M_{1}$ such that for all $n, \int_{E^{C}}\left|f_{n}\right| d \mu<M_{1}$. But also,

$$
\int_{\Omega}\left|f_{m}\right| d \mu=\int_{E^{C}}\left|f_{m}\right| d \mu+\int_{E}\left|f_{m}\right| \leq M_{1}+1 \equiv M
$$

Therefore, by Fatou's lemma, $\int_{\Omega}|f| d \mu \leq \liminf _{n \rightarrow \infty} \int\left|f_{n}\right| d \mu \leq M$, showing that $f \in L^{1}$ as hoped.

Now $\mathfrak{S} \cup\{f\}$ is uniformly integrable so there exists $\boldsymbol{\delta}_{1}>0$ such that if $\mu(E)<\delta_{1}$, then $\int_{E}|g| d \mu<\varepsilon / 3$ for all $g \in \mathfrak{S} \cup\{f\}$.

By Egoroff's theorem, there exists a set, $F$ with $\mu(F)<\delta_{1}$ such that $f_{n}$ converges uniformly to $f$ on $F^{C}$. Therefore, there exists $N$ such that if $n>N$, then

$$
\int_{F^{C}}\left|f-f_{n}\right| d \mu<\frac{\varepsilon}{3}
$$

It follows that for $n>N$,

$$
\begin{aligned}
\int_{\Omega}\left|f-f_{n}\right| d \mu & \leq \int_{F^{C}}\left|f-f_{n}\right| d \mu+\int_{F}|f| d \mu+\int_{F}\left|f_{n}\right| d \mu \\
& <\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon
\end{aligned}
$$

which verifies 7.10.

### 7.11 Radon Nikodym Theorem

Let $\mu, \nu$ be two finite measures on the measurable space $(\Omega, \mathscr{F})$ and let $\alpha \geq 0$. Let $\lambda \equiv$ $v-\alpha \mu$. Then it is clear that if $\left\{E_{i}\right\}_{i=1}^{\infty}$ are disjoint sets of $\mathscr{F}$, then $\lambda\left(\cup_{i} E_{i}\right)=\sum_{i=1}^{\infty} \lambda\left(E_{i}\right)$ and that the series converges. The next proposition is fairly obvious.

Proposition 7.11.1 Let $(\Omega, \mathscr{F}, \lambda)$ be a measure space and let $\lambda: \mathscr{F} \rightarrow[0, \infty)$ be a measure. Then $\lambda$ is a finite measure.

Proof: Since $\lambda(\Omega)<\infty$ this is a finite measure.
Definition 7.11.2 Let $(\Omega, \mathscr{F})$ be a measurable space and let $\lambda: \mathscr{F} \rightarrow \mathbb{R}$ satisfy: If $\left\{E_{i}\right\}_{i=1}^{\infty}$ are disjoint sets of $\mathscr{F}$, then $\lambda\left(\cup_{i} E_{i}\right)=\sum_{i=1}^{\infty} \lambda\left(E_{i}\right)$ and the series converges. Such a real valued function is called a signed measure. In this context, a set $E \in \mathscr{F}$ is called positive if whenever $F$ is a measurable subset of $E$, it follows $\lambda(F) \geq 0$. A negative set is defined similarly. Note that this requires $\lambda(\Omega) \in \mathbb{R}$.

Lemma 7.11.3 The countable union of disjoint positive sets is positive.
Proof: Let $E_{i}$ be positive and consider $E \equiv \cup_{i=1}^{\infty} E_{i}$. If $A \subseteq E$ with $A$ measurable, then $A \cap E_{i} \subseteq E_{i}$ and so $\lambda\left(A \cap E_{i}\right) \geq 0$. Hence $\lambda(A)=\sum_{i} \lambda\left(A \cap E_{i}\right) \geq 0$.

Lemma 7.11.4 Let $\lambda$ be a signed measure on $(\Omega, \mathscr{F})$. If $E \in \mathscr{F}$ with $0<\lambda(E)$, then $E$ has a measurable subset which is positive.

Proof: If every measurable subset $F$ of $E$ has $\lambda(F) \geq 0$, then $E$ is positive and we are done. Otherwise there exists measurable $F \subseteq E$ with $\lambda(F)<0$. Let the elements of $\mathfrak{F}$ consist of sets of disjoint sets of measurable subsets of $E$ each of which has measure less than 0 . Partially order $\mathfrak{F}$ by set inclusion. By the Hausdorff maximal theorem, there is a maximal chain $\mathscr{C}$. Then $\cup \mathscr{C}$ is a set consisting of disjoint measurable sets $F \in \mathscr{F}$ such that $\lambda(F)<0$. Since each set in $\cup \mathscr{C}$ has measure strictly less than 0 , it follows that $\cup \mathscr{C}$ is a countable set, $\left\{F_{i}\right\}_{i=1}^{\infty}$. Otherwise, there would exist an infinite subset of $\cup \mathscr{C}$ with each set having measure less than $-\frac{1}{n}$ for some $n \in \mathbb{N}$ so $\lambda$ would not be real valued. Letting $F=\cup_{i} F_{i}$, then $E \backslash F$ has no measurable subsets $S$ for which $\lambda(S)<0$ since, if it did, $\mathscr{C}$ would not have been maximal. Thus $E \backslash F$ is positive.

A major result is the following, called a Hahn decomposition.

Theorem 7.11.5 Let $\lambda$ be a signed measure on a measurable space $(\Omega, \mathscr{F})$. Then there are disjoint measurable sets $P, N$ such that $P$ is a positive set, $N$ is a negative set, and $P \cup N=\Omega$.

Proof: If $\Omega$ is either positive or negative, there is nothing to show, so suppose $\Omega$ is neither positive nor negative. $\mathfrak{F}$ will consist of collections of disjoint measurable sets $F$ such that $\lambda(F)>0$. Thus each element of $\mathfrak{F}$ is necessarily countable. Partially order $\mathfrak{F}$ by set inclusion and use the Hausdorff maximal theorem to get $\mathscr{C}$ a maximal chain. Then, as in the above lemma, $\cup \mathscr{C}$ is countable, say $\left\{P_{i}\right\}_{i=1}^{\infty}$ because $\lambda(F)>0$ for each $F \in \cup \mathscr{C}$ and $\lambda$ has values in $\mathbb{R}$. The sets in $\cup \mathscr{C}$ are disjoint because if $A, B$ are two of them, then they are both in a single element of $\mathscr{C}$. Letting $P \equiv \cup_{i} P_{i}$, and $N=P^{C}$, it follows from Lemma 7.11.3 that $P$ is positive. It is also the case that $N$ must be negative because otherwise, $\mathscr{C}$ would not be maximal.

Clearly a Hahn decomposition is not unique. For example, you could have obtained a different Hahn decomposition if you had considered disjoint negative sets $F$ for which $\lambda(F)<0$ in the above argument .

Let $k \in \mathbb{N},\left\{\alpha_{n}^{k}\right\}_{n=0}^{\infty}$ be equally spaced points $\alpha_{n}^{k}=2^{-k} n$. Then $\alpha_{2 n}^{k}=2^{-k}(2 n)=$ $2^{-(k-1)} n \equiv \alpha_{n}^{k-1}$ and $\alpha_{2 n}^{k+1} \equiv 2^{-(k+1)} 2 n=\alpha_{n}^{k}$. Similarly $N_{2 n}^{k+1}=N_{n}^{k}$ because these depend on the $\alpha_{n}^{k}$. Also let $\left(P_{n}^{k}, N_{n}^{k}\right)$ be a Hahn decomposition for the signed measure $v-\alpha_{n}^{k} \mu$ where $v, \mu$ are two finite measures. Now from the definition, $N_{n+1}^{k} \backslash N_{n}^{k}=N_{n+1}^{k} \cap P_{n}^{k}$. Also, $N_{n} \subseteq N_{n+1}$ for each $n$ and we can take $N_{0}=\emptyset$. then $\left\{N_{n+1}^{k} \backslash N_{n}^{k}\right\}_{n=0}^{\infty}$ covers all of $\Omega$ except possibly for a set of $\mu$ measure 0 .

Lemma 7.11.6 Let $S \equiv \Omega \backslash\left(\cup_{n} N_{n}^{k}\right)=\Omega \backslash\left(\cup_{n} N_{n}^{l}\right)$ for anyl. Then $\mu(S)=0$.
Proof: $S=\cap_{n} P_{n}^{k}$ so for all $n, v(S)-\alpha_{n}^{k} \mu(S) \geq 0$. But letting $n \rightarrow \infty$, it must be that $\mu(S)=0$.

As just noted, if $E \subseteq N_{n+1}^{k} \backslash N_{n}^{k}$, then

$$
\begin{equation*}
v(E)-\alpha_{n}^{k} \mu(E) \geq 0 \geq v(E)-\alpha_{n+1}^{k} \mu(E), \text { so } \alpha_{n+1}^{k} \mu(E) \geq v(E) \geq \alpha_{n}^{k} \mu(E) \tag{7.11}
\end{equation*}
$$

$$
\begin{gathered}
N_{n+1}^{k} \\
\alpha_{n+1}^{k} \mu(E) \geq v(E) \geq \alpha_{n}^{k} \mu(E) \\
N_{n}^{k} \\
\hline
\end{gathered}
$$

Then define $f^{k}(\omega) \equiv \sum_{n=0}^{\infty} \alpha_{n}^{k} \mathscr{X}_{\Delta_{n}^{k}}(\omega)$ where $\Delta_{m}^{k} \equiv N_{m+1}^{k} \backslash N_{m}^{k}$. Thus,

$$
\begin{gather*}
f^{k}=\sum_{n=0}^{\infty} \alpha_{2 n}^{k+1} \mathscr{X}_{\left(N_{2 n+2}^{k+1} \backslash N_{2 n}^{k+1}\right)}=\sum_{n=0}^{\infty} \alpha_{2 n}^{k+1} \mathscr{X}_{\Delta_{2 n+1}^{k+1}}+\sum_{n=0}^{\infty} \alpha_{2 n}^{k+1} \mathscr{X}_{\Delta_{2 n}^{k+1}} \\
\leq \sum_{n=0}^{\infty} \alpha_{2 n+1}^{k+1} \mathscr{X}_{\Delta_{2 n+1}^{k+1}}+\sum_{n=0}^{\infty} \alpha_{2 n}^{k+1} \mathscr{X}_{\Delta_{2 n}^{k+1}}=f^{k+1} \tag{7.12}
\end{gather*}
$$

Thus $k \rightarrow f^{k}(\omega)$ is increasing. Let $f(\omega) \equiv \lim _{k \rightarrow \infty} f(\omega)$. Also, from the above and 7.11, for $E \subseteq S^{C}$ so $E \subseteq \cup_{n}\left(N_{n+1}^{k} \backslash N_{n}^{k}\right)$,

$$
\int \mathscr{X}_{E} f^{k} d \mu \leq \sum_{n=0}^{\infty} \alpha_{n+1}^{k} \mu\left(E \cap \Delta_{n}^{k}\right) \leq \sum_{n=0}^{\infty} \alpha_{n}^{k} \mu\left(E \cap \Delta_{n}^{k}\right)+\sum_{n=0}^{\infty} 2^{-k} \mu\left(E \cap \Delta_{n}^{k}\right)
$$

$$
\begin{equation*}
\leq \sum_{n=0}^{\infty} v\left(E \cap \Delta_{n}^{k}\right)+2^{-k} \mu(E)=v(E)+2^{-k} \mu(E) \leq \int \mathscr{X}_{E} f^{k} d \mu+2^{-k} \mu(E) \tag{7.13}
\end{equation*}
$$

From the monotone convergence theorem it follows $v(E)=\int \mathscr{X}_{E} f d \mu$. This is summarized as follows.

Lemma 7.11.7 There exists $f$ nonnegative and measurable such that if $E \subseteq S^{C}$, then $\int \mathscr{X}_{E} f d \mu=v(E)$.

This proves most of the following theorem which is the Radon Nikodym theorem. First is a definition.

Definition 7.11.8 Let $\mu, v$ be finite measures on $(\Omega, \mathscr{F})$. Then $v \ll \mu$ means that whenever $\mu(E)=0$, it follows that $v(E)=0$.

Theorem 7.11.9 Let $v$ and $\mu$ be finite measures defined on a measurable space $(\Omega, \mathscr{F})$. Then there exists a set of $\mu$ measure zero $S$ and a real valued, measurable function $\omega \rightarrow f(\omega)$ such that if $E \subseteq S^{C}, E \in \mathscr{F}$, then $v(E)=\int_{E} f d \mu$. If $v \ll \mu, v(E)=\int_{E} f d \mu$ for any measurable $E$. In any case, $v(E) \geq \int_{E} f d \mu$. This function $f \in L^{1}(\Omega)$. If $f, \hat{f}$ both work, then $f=\hat{f} \mu$ a.e.

Proof: Let $S$ be defined in Lemma 7.11.6 so $S \equiv \Omega \backslash\left(\cup_{n} N_{n}^{k}\right)$ and $\mu(S)=0$. If $E \in \mathscr{F}$, and $f$ as described above,

$$
v(E)=v\left(E \cap S^{C}\right)+v(E \cap S)=\int_{E \cap S C} f d \mu+v(E \cap S)=\int_{E} f d \mu+v(E \cap S)
$$

Thus if $E \subseteq S^{C}$, we have $v(E)=\int_{E} f d \mu$. If $v \ll \mu$, then in the above, $v(E \cap S)=0$ so $\int_{E \cap S^{C}} f d \mu=\int_{E} f d \mu=v(E)$. In any case, $v(E) \geq \int_{E} f d \mu$, strict inequality holding if $v(E \cap S)>0$.

Sometimes people write $f=\frac{d \lambda}{d \mu}$, in the case $v \ll \mu$ and this is called the Radon Nikodym derivative.

## Definition 7.11.10 Let $S$ be in the above theorem. Then

$$
v_{\|}(E) \equiv v\left(E \cap S^{C}\right)=\int_{E \cap S^{C}} f d \mu=\int_{E} f d \mu
$$

while $v_{\perp}(E) \equiv v(E \cap S)$. Thus $v_{\|} \ll \mu$ and $v_{\perp}$ is nonzero only on sets which are contained in $S$ which has $\mu$ measure 0 .

This decomposition of a measure $v$ into the sum of two measures, one absolutely continuous with respect to $\mu$ and the other supported on a set of $\mu$ measure zero is called the Lebesgue decomposition.

Definition 7.11.11 A measure space $(\Omega, \mathscr{F}, \mu)$ is $\sigma$ finite if there are countably many measurable sets $\left\{\Omega_{n}\right\}$ such that $\mu$ is finite on measurable subsets of $\Omega_{n}$.

There is a routine corollary of the above theorem.

Corollary 7.11.12 Suppose $\mu, v$ are both $\sigma$ finite measures defined on $(\Omega, \mathscr{F})$. Then the same conclusion in the above theorem can be obtained.

$$
\begin{equation*}
v(E)=v\left(E \cap S^{C}\right)+v(E \cap S)=\int_{E} f d \mu+v(E \cap S), \mu(S)=0 \tag{7.14}
\end{equation*}
$$

In particular, if $v \ll \mu$ then there is a real valued function $f$ such that $v(E)=\int_{E} f d \mu$ for all $E \in \mathscr{F}$. Also, if $v(\hat{\Omega}), \mu(\hat{\Omega})<\infty$, then $f \in L^{1}(\hat{\Omega})$. This $f$ is unique up to a set of $\mu$ measure zero.

Proof: Since both $\mu, v$ are $\sigma$ finite, there are $\left\{\tilde{\Omega}_{k}\right\}_{k=1}^{\infty}$ such that $v\left(\tilde{\Omega}_{k}\right), \mu\left(\tilde{\Omega}_{k}\right)$ are finite. Letting $\Omega_{0}=\emptyset$ and $\Omega_{k} \equiv \tilde{\Omega}_{k} \backslash\left(\cup_{j=0}^{k-1} \tilde{\Omega}_{j}\right)$ so that $\mu, \nu$ are finite on $\Omega_{k}$ and the $\Omega_{k}$ are disjoint. Let $\mathscr{F}_{k}$ be the measurable subsets of $\Omega_{k}$, equivalently the intersections with $\Omega_{k}$ with sets of $\mathscr{F}$. Now let $v_{k}(E) \equiv v\left(E \cap \Omega_{k}\right)$, similar for $\mu_{k}$. By Theorem 7.11.9, there exists $S_{k} \subseteq \Omega_{k}$, and $f_{k}$ as described there. Thus $\mu_{k}\left(S_{k}\right)=0$ and

$$
v_{k}(E)=v_{k}\left(E \cap S_{k}^{C}\right)+v_{k}\left(E \cap S_{k}\right)=\int_{E \cap \Omega_{k}} f_{k} d \mu_{k}+v_{k}\left(E \cap S_{k}\right)
$$

Now let $f(\omega) \equiv f_{k}(\omega)$ for $\omega \in \Omega_{k}$. Thus

$$
\begin{equation*}
v\left(E \cap \Omega_{k}\right)=v\left(E \cap\left(\Omega_{k} \backslash S_{k}\right)\right)+v\left(E \cap S_{k}\right)=\int_{E \cap \Omega_{k}} f d \mu+v\left(E \cap S_{k}\right) \tag{7.15}
\end{equation*}
$$

Summing over all $k$, and letting $S \equiv \cup_{k} S_{k}$, it follows $\mu(S)=0$ and that for $S_{k}$ as above, a subset of $\Omega_{k}$ where the $\Omega_{k}$ are disjoint, $\Omega \backslash S=\cup_{k}\left(\Omega_{k} \backslash S_{k}\right)$. Thus, summing on $k$ in 7.15, $v(E)=v\left(E \cap S^{C}\right)+v(E \cap S)=\int_{E} f d \mu+v(E \cap S)$. In particular, if $v \ll \mu$, then $v(E \cap S)=0$ and so $v(E)=\int_{E} f d \mu$. The last claim is obvious from 7.14.

Corollary 7.11.13 In the above situation, let $\lambda$ be a signed measure and let $\lambda \ll \mu$ meaning that if $\mu(E)=0 \Rightarrow \lambda(E)=0$. Here assume that $\mu$ is a finite measure. Then there exists $h \in L^{1}$ such that $\lambda(E)=\int_{E} h d \mu$.

Proof: Let $P \cup N$ be a Hahn decomposition of $\lambda$. Let

$$
\lambda_{+}(E) \equiv \lambda(E \cap P), \quad \lambda_{-}(E) \equiv-\lambda(E \cap N)
$$

Then both $\lambda_{+}$and $\lambda_{-}$are absolutely continuous measures and so there are nonnegative $h_{+}$ and $h_{-}$with $\lambda_{-}(E)=\int_{E} h_{-} d \mu$ and a similar equation for $\lambda_{+}$. Then $0 \leq-\lambda(\Omega \cap N) \leq$ $\lambda_{-}(\Omega)<\infty$, similar for $\lambda_{+}$so both of these measures are necessarily finite. Hence both $h_{-}$and $h_{+}$are in $L^{1}$ so $h \equiv h_{+} h_{-}$is also in $L^{1}$ and $\lambda(E)=\lambda_{+}(E)-\lambda_{-}(E)=$ $\int_{E}\left(h_{+}-h_{-}\right) d \mu$.

### 7.12 Exercises

1. Let $\Omega=\mathbb{N}=\{1,2, \cdots\}$. Let $\mathscr{F}=\mathscr{P}(\mathbb{N})$, the set of all subsets of $\mathbb{N}$, and let $\mu(S)=$ number of elements in $S$. Thus $\mu(\{1\})=1=\mu(\{2\}), \mu(\{1,2\})=2$, etc. In this case, all functions are measurable. For a nonnegative function, $f$ defined on $\mathbb{N}$, show $\int_{\mathbb{N}} f d \mu=\sum_{k=1}^{\infty} f(k)$. What do the monotone convergence and dominated convergence theorems say about this example?
2. For the measure space of Problem 1, give an example of a sequence of nonnegative measurable functions $\left\{f_{n}\right\}$ converging pointwise to a function $f$, such that inequality is obtained in Fatou's lemma.
3. If $(\Omega, \mathscr{F}, \mu)$ is a measure space and $f, g \geq 0$ is measurable, show that if $g(\omega)=f(\omega)$ a.e. $\omega$, then $\int g d \mu=\int f d \mu$. Show that if $f, g \in L^{1}(\Omega)$ and $g(\omega)=f(\omega)$ a.e. then $\int g d \mu=\int f d \mu$.
4. Let $\left\{f_{n}\right\}, f$ be measurable functions with values in $\mathbb{C}$. $\left\{f_{n}\right\}$ converges in measure if

$$
\lim _{n \rightarrow \infty} \mu\left(x \in \Omega:\left|f(x)-f_{n}(x)\right| \geq \varepsilon\right)=0
$$

for each fixed $\varepsilon>0$. Prove the theorem of F. Riesz. If $f_{n}$ converges to $f$ in measure, then there exists a subsequence $\left\{f_{n_{k}}\right\}$ which converges to $f$ a.e. In case $\mu$ is a probability measure, this is called convergence in probability. It does not imply pointwise convergence but does imply that there is a subsequence which converges pointwise off a set of measure zero. Hint: Choose $n_{1}$ such that $\mu\left(x:\left|f(x)-f_{n_{1}}(x)\right| \geq 1\right)<1 / 2$. Choose $n_{2}>n_{1}$ such that $\mu\left(x:\left|f(x)-f_{n_{2}}(x)\right| \geq 1 / 2\right)<1 / 2^{2}, n_{3}>n_{2}$ such that $\mu\left(x:\left|f(x)-f_{n_{3}}(x)\right| \geq 1 / 3\right)<1 / 2^{3}$, etc. Now consider what it means for $f_{n_{k}}(x)$ to fail to converge to $f(x)$. Use the Borel Cantelli lemma of Problem 14 on Page 155.
5. Suppose $(\Omega, \mu)$ is a finite measure space $(\mu(\Omega)<\infty)$ and $\subseteq \subseteq L^{1}(\Omega)$. Then $\mathfrak{S}$ is said to be uniformly integrable if for every $\varepsilon>0$ there exists $\delta>0$ such that if $E$ is a measurable set satisfying $\mu(E)<\delta$, then $\int_{E}|f| d \mu<\varepsilon$ for all $f \in \mathfrak{S}$. Show $\mathfrak{S}$ is uniformly integrable and bounded in $L^{1}(\Omega)$ if there exists an increasing function $h$ which satisfies

$$
\lim _{t \rightarrow \infty} \frac{h(t)}{t}=\infty, \sup \left\{\int_{\Omega} h(|f|) d \mu: f \in \mathfrak{S}\right\}<\infty
$$

$\mathfrak{S}$ is bounded if there is some number, $M$ such that $\int|f| d \mu \leq M$ for all $f \in \mathfrak{S}$. This is in the chapter but write it down in your own words.
6. A collection $\mathfrak{S} \subseteq L^{1}(\Omega),(\Omega, \mathscr{F}, \mu)$ a finite measure space, is called equiintegrable if for every $\varepsilon>0$ there exists $\lambda>0$ such that $\int_{[|f| \geq \lambda]}|f| d \mu<\varepsilon$ for all $f \in \mathfrak{S}$. Show that $\mathfrak{S}$ is equiintegrable, if and only if it is uniformly integrable and bounded. The equiintegrable condition is pretty popular in probability.
7. Product measure is described in the chapter. Go through the construction in detail for two measure spaces as follows.

$$
(X, \mathscr{F}, \mu),(Y, \mathscr{G}, v)
$$

Let $\mathscr{K}$ be the $\pi$ system of measurable rectangles $A \times B$ where $A \in \mathscr{F}$ and $B \in \mathscr{G}$. Explain why this is really a $\pi$ system. Now let $\mathscr{F} \times \mathscr{G}$ denote the smallest $\sigma$ algebra which contains $\mathscr{K}$. Let

$$
\mathfrak{P} \equiv\left\{A \in \mathscr{F} \times \mathscr{G}: \int_{X} \int_{Y} \mathscr{X}_{A} d v d \mu=\int_{Y} \int_{X} \mathscr{X}_{A} d \mu d v\right\}
$$

where both integrals make sense and are equal. Then show that $\mathfrak{P}$ is closed with respect to complements and countable disjoint unions. By Dynkin's lemma, $\mathfrak{P}=$
$\mathscr{F} \times \mathscr{G}$. Then define a measure $\mu \times v$ as follows. For $A \in \mathscr{F} \times \mathscr{G}, \mu \times v(A) \equiv$ $\int_{X} \int_{Y} \mathscr{X}_{A} d v d \mu$. Explain why this is a measure and why if $f$ is $\mathscr{F} \times \mathscr{G}$ measurable and nonnegative, then

$$
\int_{X \times Y} f d(\mu \times v)=\int_{X} \int_{Y} \mathscr{X}_{A} d v d \mu=\int_{Y} \int_{X} \mathscr{X}_{A} d \mu d v
$$

Hint: Pay special attention to the way the monotone convergence theorem is used.
8. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and suppose $f, g: \Omega \rightarrow(-\infty, \infty]$ are measurable. Prove the sets $\{\omega: f(\omega)<g(\omega)\}$ and $\{\omega: f(\omega)=g(\omega)\}$ are measurable. Hint: The easy way to do this is to write $\{\omega: f(\omega)<g(\omega)\}=\cup_{r \in \mathbb{Q}}[f<r] \cap[g>r]$. Note that $l(x, y)=x-y$ is not continuous on $(-\infty, \infty]$ so the obvious idea doesn't work. Here $[g>r]$ signifies $\{\omega: g(\omega)>r\}$.
9. Let $\left\{f_{n}\right\}$ be a sequence of real or complex valued measurable functions. Let $S=$ $\left\{\omega:\left\{f_{n}(\omega)\right\}\right.$ converges $\}$. Show $S$ is measurable. Hint: You might try to exhibit the set where $f_{n}$ converges in terms of countable unions and intersections using the definition of a Cauchy sequence.
10. Suppose $u_{n}(t)$ is a differentiable function for $t \in(a, b)$ and suppose that for $t \in(a, b)$, $\left|u_{n}(t)\right|,\left|u_{n}^{\prime}(t)\right|<K_{n}$ where $\sum_{n=1}^{\infty} K_{n}<\infty$. Show $\left(\sum_{n=1}^{\infty} u_{n}(t)\right)^{\prime}=\sum_{n=1}^{\infty} u_{n}^{\prime}(t)$.
Hint: This is an exercise in the use of the dominated convergence theorem and the mean value theorem.
11. Suppose $\left\{f_{n}\right\}$ is a sequence of nonnegative measurable functions defined on a measure space, $(\Omega, \mathscr{S}, \mu)$. Show that $\int \sum_{k=1}^{\infty} f_{k} d \mu=\sum_{k=1}^{\infty} \int f_{k} d \mu$. Hint: Use the monotone convergence theorem along with the fact the integral is linear.
12. Show $\lim _{n \rightarrow \infty} \frac{n}{2^{n}} \sum_{k=1}^{n} \frac{2^{k}}{k}=2$. This problem was shown to me by Shane Tang, a former student. It is a nice exercise in dominated convergence theorem if you massage it a little. Hint:

$$
\begin{aligned}
\frac{n}{2^{n}} \sum_{k=1}^{n} \frac{2^{k}}{k} & =\sum_{k=1}^{n} 2^{k-n} \frac{n}{k}=\sum_{l=0}^{n-1} 2^{-l} \frac{n}{n-l} \\
& =\sum_{l=0}^{n-1} 2^{-l}\left(1+\frac{l}{n-l}\right) \leq \sum_{l}^{n-1} 2^{-l}(1+l)
\end{aligned}
$$

13. If $f$ is nonnegative and Riemann integrable on $[a, b]$, show that there is an increasing sequence of lower sums and a decreasing sequence of upper sums which converge to $\int_{a}^{b} f d x$. These come from step functions. Show we can assume these step functions corresponding to the lower sums are increasing and those from the upper sums are decreasing. Now the Riemann integral and Lebesgue integral are the same for a step function. Tell why. Now let $g$ be the limit of the step functions corresponding to the lower sums and let $h$ be the limit of the step functions corresponding to the upper sums. Show these are both Borel measurable and

$$
\int_{[a, b]} h d m=\int_{a}^{b} f d x=\int_{a}^{b} g d m
$$

and so $f=g$ off a set of measure zero with $g$ a Borel measurable function. Explain why by completeness of the measure that $f$ is Lebesgue measurable and $\int_{[a, b]} f d \mu=$ $\int_{a}^{b} f d x$. Explain why this shows that Riemann integrable implies Lebesgue integrable for functions having values in $\mathbb{C}$.
14. Show the Vitali Convergence theorem implies the Dominated Convergence theorem for finite measure spaces but there exist examples where the Vitali convergence theorem works and the dominated convergence theorem does not.
15. Suppose $v \ll \mu$ where these are finite measures so there exists $h \geq 0$ and measurable such that $v(E)=\int_{E} h d \mu$ by the Radon Nikodym theorem. Show that if $f$ is measurable and non-negative, then $\int f d \nu=\int f h d \mu$. Hint: It holds if $f$ is $\chi_{E}$ and so it holds for a simple function. Now consider a sequence of simple functions increasing to $f$ and use the monotone convergence theorem.
16. Consider the $p$ dimensional Lebesgue measure of Problem 27 on Page 157 denoted as $m_{p}$. Explain why $m_{p}=m \times m \times \cdots \times m \equiv m^{p}$ the product measure of this chapter on all the Borel sets of $\mathbb{R}^{p}$. Hint: The two are equal on open boxes and half open boxes of the form $\prod_{i=1}^{p}\left[a_{i}, b_{i}\right)$. Recall how $m$ was defined in Theorem 6.5.1.

## Chapter 8

## Positive Linear Functionals

In this chapter is a standard way to obtain many examples of measures from extending positive linear functionals. I will consider positive linear functionals defined on $C_{c}(X)$ where $(X, d)$ is a metric space. This can all be generalized to $X$ a locally compact Hausdorff space, but I don't have many examples which need this level of generality. Also, we will always assume that the closure of balls in $(X, d)$ are compact. Thus you see that the main example is $\mathbb{R}^{p}$ or some closed subset of $\mathbb{R}^{p}$ like a $m$ dimensional surface in $\mathbb{R}^{p}$ where $m<p$. This approach will not work for finding measures on infinite dimensional Banach spaces for example, which appears to limit its applications to probability but it is a very general approach which gives outstanding results very quickly. To see more generality including the locally compact Hausdorff spaces, see Rudin [39] which is where I first saw this, actually in an earlier version of this book. Another source is in Hewitt and Stromberg [22].

Lemma 8.0.1 Let $f \in C([a, b])$. Then $\int_{a}^{b} f d x=\int_{[a, b]} f d m_{1}$. The Riemann integral from calculus equals the Lebesgue integral.

Proof: One can reduce to the case where $f(x) \geq 0$ by looking at positive and negative parts of real and imaginary parts. Let $a=x_{0}<\cdots<x_{n}=b$ and consider the step functions $f_{n}(x) \equiv \sum_{k=1}^{n} f\left(x_{k-1}\right) \mathscr{X}_{\left[x_{k-1}, x_{k}\right)}(x)$. This converges uniformly to $f$ and each is a simple function. By the fact that $m_{1}(I)$ is just the length of $I$ for $I$ an interval, we have $\int_{a}^{b} f_{n}(s) d x=\int_{[a, b]} f_{n} d m_{1}$. Now let $n \rightarrow \infty$ and use the uniform convergence to conclude $\int_{a}^{b} f d x=\int_{[a, b]} f d m_{1}$.

This is based on extending functionals. The most obvious functional is as follows:

$$
\begin{equation*}
L f \equiv \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f\left(x_{1}, \cdots, x_{p}\right) d x_{p} d x_{p-1} \cdots d x_{1} \tag{8.1}
\end{equation*}
$$

the iterated integral in which $f \in C_{c}\left(\mathbb{R}^{p}\right)$. You do exactly what the notation says. First integrate with respect to $x_{p}$ then with respect to $x_{p-1}$ and so forth. This makes perfect sense whenever $f \in C_{c}\left(\mathbb{R}^{p}\right)$ and we can consider each iterated integral as either a Riemann integral from Calculus or a Lebesgue integral with respect to $d m_{1}$ since the above lemma shows these are the same.

Lemma 8.0.2 The functional $L$ makes sense for $f \in C_{c}\left(\mathbb{R}^{p}\right)$.
Proof: Let $f$ be zero off $[-R, R]^{p}$ a compact set. Then by uniform continuity of $f$ on this compact set, if $\left|\hat{x}_{p-1}-x_{p-1}\right|$ is small enough,

$$
\left|f\left(x_{1}, \cdots, \hat{x}_{p-1}, x_{p}\right)-f\left(x_{1}, \cdots, x_{p-1}, x_{p}\right)\right|<\varepsilon / 2 R
$$

Therefore, for $\left|\hat{x}_{p-1}-x_{p-1}\right|$ this small,

$$
\begin{aligned}
& \left|\int_{-\infty}^{\infty} f\left(x_{1}, \cdots, x_{p-1}, x_{p}\right) d x_{p}-\int_{-\infty}^{\infty} f\left(x_{1}, \cdots, \hat{x}_{p-1}, x_{p}\right) d x_{p}\right| \\
= & \left|\int_{-R}^{R} f\left(x_{1}, \cdots, x_{p-1}, x_{p}\right) d x_{p}-\int_{-R}^{R} f\left(x_{1}, \cdots, \hat{x}_{p-1}, x_{p}\right) d x_{p}\right|
\end{aligned}
$$

$$
\int_{-R}^{R}\left|f\left(x_{1}, \cdots, x_{p-1}, x_{p}\right)-f\left(x_{1}, \cdots, \hat{x}_{p-1}, x_{p}\right)\right| d x_{p}<\frac{\varepsilon}{2 R} 2 R=\varepsilon
$$

and so $x_{p-1} \rightarrow \int_{-\infty}^{\infty} f\left(x_{1}, \cdots, x_{p-1}, x_{p}\right) d x_{p}$ is continuous and zero off some interval and so it is integrable. Continuing this way shows that the functional defined above makes perfect sense. You can keep doing the iterated integrals.

The idea of the following Lemma is in Problem 6. You could use the result of that problem for transpositions obtain the conclusion of the following lemma by considering the product of transpositions.

Lemma 8.0.3 If $L f$ is given in 8.1 and if $\left(i_{1}, \cdots, i_{p}\right)$ is any permutation of $(1, \cdots, p)$ with $\sigma$ being the name of this permutation, then defining

$$
L_{\sigma} f \equiv \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f\left(x_{1}, \cdots, x_{p}\right) d x_{i_{p}} d x_{i_{p-1}} \cdots d x_{i_{1}}
$$

it follows that $L_{\sigma}=L_{\alpha}$ on $C_{c}\left(\mathbb{R}^{p}\right)$ where $\alpha$ is any other permutation.
Proof: Let $\mathscr{T}_{n}$ denote a tiling of $\mathbb{R}^{p}$ into disjoint half open rectangles, each of diameter $1 / 2^{n}$. Let $\prod_{i=1}^{p}\left[0,2^{-n}\right)$ be in $\mathscr{T}_{n}$ to be specific, thus forcing each $Q$ in $\mathscr{T}_{n}$ to be the union of the $Q$ in $\mathscr{T}_{n+1}$. Also denote by $Q_{\mathbf{r}}=\prod_{i=1}^{p}\left[r_{i}, r_{i}+2^{-n}\right)$ one of the half open rectangles so described and letting $\mathscr{V}$ be the set of such vertices $\mathbf{r} \equiv\left(r_{1}, \cdots, r_{p}\right)$. Then using the mean value theorem for one dimensional integrals in the successive iterated integrals, (See Problem 5 on Page 131) it follows that

$$
L_{\sigma} f \equiv \sum_{\mathbf{r} \in \mathscr{V}} \int_{r_{i_{1}}}^{r_{i_{1}}+2^{-n}} \cdots \int_{r_{i_{p}}}^{r_{i_{p}}+2^{-n}} f\left(x_{1}, \cdots, x_{p}\right) d x_{i_{p}} d x_{i_{p-1}} \cdots d x_{i_{1}}=\sum_{\mathbf{r} \in \mathscr{V}}\left(2^{-n}\right)^{p} f\left(\mathbf{x}_{\mathbf{r} \sigma}\right)
$$

there being only finitely many terms in the above sum and $\mathbf{x}_{\mathbf{r} \sigma}$ is a point of $Q_{\mathbf{r}}$. Since $f$ has compact support, there is a positive integer $m$ such that the support of $f$ is contained in $\prod_{i=1}^{p}[-m, m)$. Thus the only $\mathbf{r}$ in the above sum are those for which $r_{i}=k 2^{-n}$ for $k$ an integer in $\left[-m 2^{n}, m 2^{n}\right)$.

By uniform continuity of $f$ there is $\delta$ such that if $|\mathbf{x}-\mathbf{y}|<\delta$, then $|f(\mathbf{x})-f(\mathbf{y})|<$ $\varepsilon /(2 m p)^{p}$. Then by choosing $n$ large enough so that each $Q_{\mathbf{r}}$ has diameter less than $\delta$, if follows that

$$
\left|L_{\sigma} f-L_{\alpha} f\right|<\sum_{k_{1}=-m 2^{n}}^{m 2^{n}} \ldots \sum_{k_{p}=-m 2^{n}}^{m 2^{n}}\left(2^{-n}\right)^{p} \varepsilon=\left(2 m p 2^{n}\right)^{p}\left(2^{-n}\right)^{p} \frac{\varepsilon}{(2 m p)^{p}}=\varepsilon
$$

Therefore, since $\varepsilon$ is arbitrary, $L_{\sigma}=L_{\alpha}$ for any two permutations $\sigma, \alpha$.

### 8.1 Partitions of Unity

The support of a function $f$, denoted as $\operatorname{spt}(f)$, is the closure of the set on which the function is nonzeo.

Definition 8.1.1 Define $C_{c}(X)$ to be the functions which have complex values and compact support. This means $\operatorname{spt}(f) \equiv \overline{\{x \in X: f(x) \neq 0\}}$ is a compact set. Then $L$ : $C_{c}(X) \rightarrow \mathbb{C}$ is called a positive linear functional if it is linear and if, whenever $f \geq 0$, then $L(f) \geq 0$ also. When $f$ is a continuous function and $\operatorname{spt}(f) \subseteq V$ an open set, we say $f \in C_{c}(V)$. Here $X$ is some metric space.

The following definition gives some notation.
Definition 8.1.2 If $K$ is a compact subset of an open set, $V$, then $K \prec \phi \prec V$ if

$$
\phi \in C_{c}(V), \phi(K)=\{1\}, \phi(X) \subseteq[0,1]
$$

where $X$ denotes the whole metric space. Also for $\phi \in C_{c}(X), K \prec \phi$ if

$$
\phi(X) \subseteq[0,1] \text { and } \phi(K)=1
$$

and $\phi \prec V$ if

$$
\phi(X) \subseteq[0,1] \text { and } \operatorname{spt}(\phi) \subseteq V
$$

Next is a useful theorem. Recall from Theorem 2.4.8, $x \rightarrow \operatorname{dist}(x, S)$ is continuous.
Theorem 8.1.3 Let $H$ be a compact subset of an open set $U$ in $X$ where $(X, d)$ is a metric space in which the closures of balls are compact. Then there exists an open set $V$ such that

$$
H \subseteq V \subseteq \bar{V} \subseteq U
$$

with $\bar{V}$ compact. There also exists $\psi$ such that $H \prec \psi \prec V$, meaning that $\psi=1$ on $H$ and $\operatorname{spt}(\psi) \subseteq \bar{V}$. If $U$ is an open subset of $\mathbb{R}^{p}$, then there is an increasing sequence of continuous functions $\psi_{n} \in C_{c}(U)$ such that $\lim _{n \rightarrow \infty} \psi_{n}(\mathbf{x})=\mathscr{X}_{U}(\mathbf{x})$.

Proof: Consider $h \rightarrow \operatorname{dist}\left(h, U^{C}\right)$. This continuous function achieves its minimum at some $h_{0} \in H$ because $H$ is compact. Let $\delta \equiv \frac{1}{2} \operatorname{dist}\left(h_{0}, U^{C}\right)$. The distance is positive because $U^{C}$ is closed. Now $H \subseteq \cup_{h \in H} B(h, \delta)$. Since $H$ is compact, there are finitely many of these balls which cover $H$. Say $H \subseteq \cup_{i=1}^{k} B\left(h_{i}, \delta\right) \equiv V$. Then, since there are finitely many of these balls, let

$$
\bar{V} \equiv \cup_{i=1}^{k} \overline{B\left(h_{i}, \delta\right)}, V \equiv \cup_{i=1}^{k} B\left(h_{i}, \delta\right)
$$

$\bar{V}$ is a compact set since it is a finite union of compact sets.
To obtain $\psi$, let

$$
\psi(x) \equiv \frac{\operatorname{dist}\left(x, V^{C}\right)}{\operatorname{dist}\left(x, V^{C}\right)+\operatorname{dist}(x, H)}
$$

Then $\psi(x) \leq 1$ and if $x \in H$, its distance to $V^{C}$ is positive and $\operatorname{dist}(x, H)=0$ so $\psi(x)=1$. If $x \in V^{C}$, then its distance to $H$ is positive and so $\psi(x)=0$. It is obviously continuous because the denominator is a continuous function and never vanishes since both $V^{C}$ and $H$ are closed so if either $\operatorname{dist}\left(x, V^{C}\right)$ or $\operatorname{dist}(x, H)$ is 0 , then $x$ is in either $V^{C}$ or $H$. Thus, if one of dist $\left(x, V^{C}\right)$, dist $(x, H)$ is 0 , the other isn't. Thus $H \prec \psi \prec V$.

For the last claim, Let $C_{n} \equiv\left\{\mathbf{x} \in U: \operatorname{dist}\left(\mathbf{x}, U^{C}\right) \geq 1 / n\right\}$ and let $H_{n} \equiv C_{n} \cap \overline{B(\mathbf{0}, n)}$ for $n \in \mathbb{N}$. Then $H_{n}$ is compact, the $H_{n}$ are increasing in $n$, and $\cup_{n} H_{n}=U$. Now for some $m$, $H_{m} \neq \emptyset$, let $H_{m} \prec \phi_{m} \prec U$ from the first part Let $\psi_{1} \equiv \phi_{m}$. If $\psi_{1}, \ldots, \psi_{n}$ have been chosen, let $\psi_{n+1}=\max \left(\psi_{1}, \ldots, \psi_{n}, \phi_{n+1+m}\right)$. Then eventually, if $\mathbf{x} \in U$, for all $n$ large enough, $\psi_{n}(\mathbf{x})=1=\mathscr{X}_{U}(\mathbf{x})$ and if $\mathbf{x} \notin U$, then all $\psi_{n}(\mathbf{x})=0$.
Theorem 8.1.4 (Partition of unity) Let $K$ be a compact subset of $X$ and suppose

$$
K \subseteq V=\cup_{i=1}^{n} V_{i}, V_{i} \text { open }
$$

Then there exist $\psi_{i} \prec V_{i}$ with $\sum_{i=1}^{n} \psi_{i}(x)=1$ for all $x \in K$. If $H$ is a compact subset of $V_{i}$ for some $V_{i}$, there exists a partition of unity such that $\psi_{i}(x)=1$ for all $x \in H$

Proof: Let $K_{1}=K \backslash \cup_{i=2}^{n} V_{i}$. Thus $K_{1}$ is compact and $K_{1} \subseteq V_{1}$. Let $K_{1} \subseteq W_{1} \subseteq \bar{W}_{1} \subseteq$ $V_{1}$ with $\bar{W}_{1}$ compact. To obtain $W_{1}$, use Theorem 8.1.3 to get $f$ such that $K_{1} \prec f \prec V_{1}$ and let $W_{1} \equiv\{x: f(x) \neq 0\}$. Thus $W_{1}, V_{2}, \cdots V_{n}$ covers $K$ and $\bar{W}_{1} \subseteq V_{1}$. Let $K_{2}=K \backslash\left(\cup_{i=3}^{n} V_{i} \cup W_{1}\right)$. Then $K_{2}$ is compact and $K_{2} \subseteq V_{2}$. Let $K_{2} \subseteq W_{2} \subseteq \bar{W}_{2} \subseteq V_{2} \bar{W}_{2}$ compact. Continue this way finally obtaining $W_{1}, \cdots, W_{n}, K \subseteq W_{1} \cup \cdots \cup W_{n}$, and $\bar{W}_{i} \subseteq V_{i} \bar{W}_{i}$ compact. Now let $\bar{W}_{i} \subseteq U_{i} \subseteq \bar{U}_{i} \subseteq V_{i}, \bar{U}_{i}$ compact.


By Theorem 8.1.3, let $\bar{U}_{i} \prec \phi_{i} \prec V_{i}, \cup_{i=1}^{n} \bar{W}_{i} \prec \gamma \prec \cup_{i=1}^{n} U_{i}$. Define

$$
\psi_{i}(x)=\left\{\begin{array}{l}
\gamma(x) \phi_{i}(x) / \sum_{j=1}^{n} \phi_{j}(x) \text { if } \sum_{j=1}^{n} \phi_{j}(x) \neq 0 \\
0 \text { if } \sum_{j=1}^{n} \phi_{j}(x)=0
\end{array}\right.
$$

If $x$ is such that $\sum_{j=1}^{n} \phi_{j}(x)=0$, then $x \notin \cup_{i=1}^{n} \bar{U}_{i}$. Consequently $\gamma(y)=0$ for all $y$ near $x$ and so $\psi_{i}(y)=0$ for all $y$ near $x$. Hence $\psi_{i}$ is continuous at such $x$. If $\sum_{j=1}^{n} \phi_{j}(x) \neq 0$, this situation persists near $x$ and so $\psi_{i}$ is continuous at such points from the top description of $\psi_{i}$. Therefore $\psi_{i}$ is continuous. If $x \in K$, then $\gamma(x)=1$ and so $\sum_{j=1}^{n} \psi_{j}(x)=1$. Clearly $0 \leq \psi_{i}(x) \leq 1$ and $\operatorname{spt}\left(\psi_{j}\right) \subseteq V_{j}$. As to the last claim, keep $V_{i}$ the same but replace $V_{j}, j \neq i$ with $\widetilde{V}_{j} \equiv V_{j} \backslash H$. Now in the proof above, applied to this modified collection of open sets, if $j \neq i, \phi_{j}(x)=0$ whenever $x \in H$. Therefore, $\psi_{i}(x)=1$ on $H$.

### 8.2 Positive Linear Functionals and Measures

Now with this preparation, here is the main result called the Riesz representation theorem for positive linear functionals. I am presenting this for a metric space, but in this book, we will typically have $X=\mathbb{R}^{p}$.

Theorem 8.2.1 (Riesz representation theorem) Let $L$ be a positive linear functional on $C_{c}(X)$ where $(X, d)$ is a metric space having closed balls compact. Thus $L f \in \mathbb{C}$ if $f \in$ $C_{c}(X)$. Then there exists a $\sigma$ algebra $\mathscr{F}$ containing the Borel sets and a unique measure $\mu$, defined on $\mathscr{F}$, such that

$$
\begin{gather*}
\mu \text { is complete, }  \tag{8.2}\\
\mu(K)<\infty \text { for all } K \text { compact },  \tag{8.3}\\
\mu(F)=\sup \{\mu(K): K \subseteq F, K \text { compact }\}, \tag{8.4}
\end{gather*}
$$

for all $F \in \mathscr{F}$,

$$
\begin{equation*}
\mu(F)=\inf \{\mu(V): V \supseteq F, V \text { open }\} \tag{8.5}
\end{equation*}
$$

for all $F \in \mathscr{F}$, and

$$
\begin{equation*}
\int f d \mu=L f \text { for all } f \in C_{c}(X) \tag{8.6}
\end{equation*}
$$

This extends the functional $L$ because the integral will be defined for all $f \in L^{1}(X)$ in general, a much larger set than $C_{C}(X)$. The two assertions 8.4 and 8.5 are called respectively
inner and outer regularity. A measure $\mu$ satisfying the above conditions is called a Radon measure.

The plan is to define an outer measure and then to show that it, together with the $\sigma$ algebra of sets measurable in the sense of Caratheodory, satisfies the conclusions of the theorem. Always, $K$ will be a compact set and $V$ will be an open set.

Definition 8.2.2 $\mu(V) \equiv \sup \{L f: f \prec V\}$ for $V$ open, $\mu(\emptyset)=0$. $\mu(E) \equiv \inf \{\mu(V)$ : $V \supseteq E\}$ for arbitrary sets $E$.

Lemma 8.2.3 $\mu$ is a well-defined outer measure.
Proof: First it is necessary to verify $\mu$ is well defined because there are two descriptions of it on open sets. Suppose then that $\mu_{1}(V) \equiv \inf \{\mu(U): U \supseteq V$ and $U$ is open $\}$. It is required to verify that $\mu_{1}(V)=\mu(V)$ where $\mu$ is given as $\sup \{L f: f \prec V\}$. If $U \supseteq V$, then $\mu(U) \geq \mu(V)$ directly from the definition. Hence from the definition of $\mu_{1}$, it follows $\mu_{1}(V) \geq \mu(V)$. On the other hand, $V \supseteq V$ and so $\mu_{1}(V) \leq \mu(V)$. This verifies $\mu$ is well defined.

It remains to show that $\mu$ is an outer measure. First I show that it acts like and outer measure on open sets. Let $V=\cup_{i=1}^{\infty} V_{i}$ and let $f \prec V$. Then $\operatorname{spt}(f) \subseteq \cup_{i=1}^{n} V_{i}$ for some $n$. Let $\psi_{i} \prec V_{i}, \sum_{i=1}^{n} \psi_{i}=1$ on $\operatorname{spt}(f)$.

$$
L f=\sum_{i=1}^{n} L\left(f \psi_{i}\right) \leq \sum_{i=1}^{n} \mu\left(V_{i}\right) \leq \sum_{i=1}^{\infty} \mu\left(V_{i}\right)
$$

Hence $\mu(V) \leq \sum_{i=1}^{\infty} \mu\left(V_{i}\right)$ since $f \prec V$ is arbitrary.
Now let $E=\cup_{i=1}^{\infty} E_{i}$. Is $\mu(E) \leq \sum_{i=1}^{\infty} \mu\left(E_{i}\right)$ ? Without loss of generality, it can be assumed $\mu\left(E_{i}\right)<\infty$ for each $i$ since if not so, there is nothing to prove. Let $V_{i} \supseteq E_{i}$ with $\mu\left(E_{i}\right)+\varepsilon 2^{-i}>\mu\left(V_{i}\right)$.

$$
\mu(E) \leq \mu\left(\cup_{i=1}^{\infty} V_{i}\right) \leq \sum_{i=1}^{\infty} \mu\left(V_{i}\right) \leq \varepsilon+\sum_{i=1}^{\infty} \mu\left(E_{i}\right)
$$

Since $\varepsilon$ was arbitrary, $\mu(E) \leq \sum_{i=1}^{\infty} \mu\left(E_{i}\right)$ which proves the lemma.
Lemma 8.2.4 Let $K$ be compact, $g \geq 0, g \in C_{c}(X)$, and $g=1$ on $K$. Then $\mu(K) \leq L g$. Also $\mu(K)<\infty$ whenever $K$ is compact.

Proof: Let $V_{\alpha} \equiv\{x: g(x)>1-\alpha\}$ where $\alpha$ is small. I want to compare $\mu\left(V_{\alpha}\right)$ with $\mu(K)$. Thus let $h \prec V_{\alpha}$.


Then $h \leq 1$ on $V_{\alpha}$ while $g(1-\alpha)^{-1} \geq 1$ on $V_{\alpha}$ and so $g(1-\alpha)^{-1} \geq h$ which implies $L\left(g(1-\alpha)^{-1}\right) \geq L h$ and that therefore, since $L$ is linear,

$$
L g \geq(1-\alpha) L h
$$

Taking sup for all such $h$, it follows that $L g \geq(1-\alpha) \mu\left(V_{\alpha}\right) \geq(1-\alpha) \mu(K)$. Letting $\alpha \rightarrow 0$ yields $L g \geq \mu(K)$. This proves the first part of the lemma. The second assertion follows from this and Theorem 8.1.3. If $K$ is given, let $K \prec g \prec X$ and so from what was just shown, $\mu(K) \leq L g<\infty$.

For two sets $A, B$, recall dist $(A, B) \equiv \inf \{|a-b|: a \in A, b \in B\}$.
Lemma 8.2.5 If $A$ and $B$ are disjoint subsets of $X$, with $\operatorname{dist}(A, B)>0$ then $\mu(A \cup B)=$ $\mu(A)+\mu(B)$.

Proof: There is nothing to show if $\mu(A \cup B)=\infty$ so assume $\mu(A \cup B)<\infty$. Let $\delta \equiv$ $\operatorname{dist}(A, B)>0$. Then let $U_{1} \equiv \cup_{a \in A} B\left(a, \frac{\delta}{3}\right), V_{1} \equiv \cup_{b \in B} B\left(b, \frac{\delta}{3}\right)$. It follows that these two open sets have empty intersection. Also, there exists $W \supseteq A \cup B$ such that $\mu(W)-\varepsilon<$ $\mu(A \cup B)$. let $U \equiv U_{1} \cap W, V \equiv V_{1} \cap W$. Then

$$
\mu(A \cup B)+\varepsilon>\mu(W) \geq \mu(U \cup V)
$$

Now let $f \prec U, g \prec V$ such that $L f+\varepsilon>\mu(U), L g+\varepsilon>\mu(V)$. Then

$$
\begin{aligned}
\mu(U \cup V) & \geq L(f+g)=L(f)+L(g) \\
& >\mu(U)-\varepsilon+(\mu(V)-\varepsilon) \\
& \geq \mu(A)+\mu(B)-2 \varepsilon
\end{aligned}
$$

It follows that

$$
\mu(A \cup B)+\varepsilon>\mu(A)+\mu(B)-2 \varepsilon
$$

and since $\varepsilon$ is arbitrary, $\mu(A \cup B) \geq \mu(A)+\mu(B) \geq \mu(A \cup B)$.
It follows from Theorem 6.7 .2 that the $\sigma$ algebra of measurable sets $\mathscr{F}$ determined by this outer measure $\mu$ contains the Borel $\sigma$ algebra $\mathscr{B}(X)$. Since closures of balls are compact, it follows from Lemma 8.2.4 that $\mu$ is finite on every ball.

From the definition, for any $E \in \mathscr{F}$,

$$
\mu(E)=\inf \{\mu(V): V \supseteq E, V \text { open }\}
$$

Lemma 8.2.6 If $\mu$ is outer regular and $F$ is measurable and contained in a closed ball $B$, then

$$
\mu(F)=\sup \{\mu(K): K \subseteq F, K \text { compact }\}
$$

Proof: By outer regularity, there exists $V$ open with $V \supseteq B \cap F^{C}$ and

$$
\mu(V \backslash(B \backslash F))<\varepsilon
$$

Thus $V^{C} \subseteq B^{C} \cup F$ and $V^{C} \cap B \subseteq\left(B^{C} \cup F\right) \cap B=B \cap F$.


In the picture $B \backslash F$ is pink between the solid circle and the solid ellipse and $F$ is in green. The open set $V$ is between the two dashed lines.

Then $B \cap V^{C}$ is a compact subset of $F$ and

$$
\mu\left(F \backslash\left(B \cap V^{C}\right)\right) \leq \mu(V \backslash(B \backslash F))<\varepsilon
$$

and this shows that $\mu(F)-\varepsilon \leq \mu\left(B \cap V^{C}\right) \leq \mu(F)$ and so $\mu$ is inner regular for $F$.
If $F$ is not necessarily contained in a closed ball, let $B_{n}$ be a sequence of closed balls having increasing radii and let $F_{n}=B_{n} \cap F$. Then if $l<\mu(F)$, it follows that $\mu\left(F_{n}\right)>l$ for all large enough $n$. Then picking one of these, it follows from what was just shown that there is a compact set $K \subseteq F_{n}$ such that also $\mu(K)>l$.

Thus $\mathscr{F}$ contains the Borel sets and $\mu$ is inner regular on all sets of $\mathscr{F}$, outer regular by definition.

It remains to show $\mu$ satisfies 8.6.
Lemma 8.2.7 $\int f d \mu=L f$ for all $f \in C_{c}(X)$.
Proof: Let $f \in C_{c}(X), f$ real-valued, and suppose $f(X) \subseteq[a, b]$. Choose $t_{0}<a$ and let $t_{0}<t_{1}<\cdots<t_{n}=b, t_{i}-t_{i-1}<\varepsilon$. Let

$$
\begin{equation*}
E_{i}=f^{-1}\left(\left(t_{i-1}, t_{i}\right]\right) \cap \operatorname{spt}(f) \tag{8.7}
\end{equation*}
$$

Note that $\cup_{i=1}^{n} E_{i}$ is a closed set equal to $\operatorname{spt}(f)$.

$$
\begin{equation*}
\cup_{i=1}^{n} E_{i}=\operatorname{spt}(f) \tag{8.8}
\end{equation*}
$$

Since $X=\cup_{i=1}^{n} f^{-1}\left(\left(t_{i-1}, t_{i}\right]\right)$. Let $V_{i} \supseteq E_{i}, V_{i}$ is open and let $V_{i}$ satisfy

$$
\begin{equation*}
f(x)<t_{i}+\varepsilon \text { for all } x \in V_{i}, \mu\left(V_{i} \backslash E_{i}\right)<\varepsilon / n \tag{8.9}
\end{equation*}
$$

By Theorem 8.1.4, there exists $h_{i} \in C_{c}(X)$ such that

$$
h_{i} \prec V_{i}, \quad \sum_{i=1}^{n} h_{i}(x)=1 \text { on } \operatorname{spt}(f) \text {. }
$$

Now note that for each $i$,

$$
f(x) h_{i}(x) \leq h_{i}(x)\left(t_{i}+\varepsilon\right)
$$

If $x \notin V_{i}$ both sides equal 0 .) Therefore,

$$
\begin{aligned}
L f & =L\left(\sum_{i=1}^{n} f h_{i}\right) \leq L\left(\sum_{i=1}^{n} h_{i}\left(t_{i}+\varepsilon\right)\right)=\sum_{i=1}^{n}\left(t_{i}+\varepsilon\right) L\left(h_{i}\right) \\
& =\sum_{i=1}^{n}\left(\left|t_{0}\right|+t_{i}+\varepsilon\right) L\left(h_{i}\right)-\left|t_{0}\right| L\left(\sum_{i=1}^{n} h_{i}\right) .
\end{aligned}
$$

Now note that $\left|t_{0}\right|+t_{i}+\varepsilon \geq 0$ and so from the definition of $\mu$ and Lemma 8.2.4, this is no larger than

$$
\sum_{i=1}^{n}\left(\left|t_{0}\right|+t_{i}+\varepsilon\right) \mu\left(V_{i}\right)-\left|t_{0}\right| \mu(\operatorname{spt}(f))
$$

$$
\begin{aligned}
& \leq \sum_{i=1}^{n}\left(\left|t_{0}\right|+t_{i}+\varepsilon\right)\left(\mu\left(E_{i}\right)+\varepsilon / n\right)-\left|t_{0}\right| \mu(\operatorname{spt}(f)) \\
& \leq \quad\left|t_{0}\right| \sum_{i=1}^{n} \mu\left(E_{i}\right)+\frac{\varepsilon}{n} n\left|t_{0}\right|+\sum_{i} t_{i} \mu\left(E_{i}\right) \\
& \\
& +\sum_{\sum_{i} t_{i} \frac{\varepsilon}{n}+\sum_{i} \varepsilon \mu\left(E_{i}\right)+\frac{\varepsilon^{2}}{n}-\left|t_{0}\right| \mu(\operatorname{spt}(f))}^{\leq \quad \varepsilon\left|t_{0}\right|+\varepsilon\left(\left|t_{0}\right|+|b|\right)+\varepsilon \mu(\operatorname{spt}(f))+\varepsilon^{2}+\sum_{i} t_{i} \mu\left(E_{i}\right)} \\
& \leq \quad \varepsilon\left|t_{0}\right|+\varepsilon\left(\left|t_{0}\right|+|b|\right)+2 \varepsilon \mu(\operatorname{spt}(f))+\varepsilon^{2}+\sum_{i=1}^{n} t_{i-1} \mu\left(E_{i}\right) \\
& \leq \\
& \quad \leq \varepsilon\left(2\left|t_{0}\right|+|b|+2 \mu(\operatorname{spt}(f))+\varepsilon\right)+\int f d \mu
\end{aligned}
$$

Since $\varepsilon>0$ is arbitrary, $L f \leq \int f d \mu$ for all $f \in C_{c}(X), f$ real. Hence equality holds because

$$
-L(f)=L(-f) \leq \int(-f) d \mu=-\int f d \mu
$$

so $L(f) \geq \int f d \mu$. Thus $L f=\int f d \mu$ for all $f \in C_{c}(X)$. Just apply the result for real functions to the real and imaginary parts of $f$.

In fact the two outer measures are equal on all sets. Thus the measurable sets are exactly the same and so they have the same $\sigma$ algebra of measurable sets and are equal on this $\sigma$ algebra. Of course, if you were willing to consider $\sigma$ algebras for which the measures are not complete, then you might have different $\sigma$ algebras, but note that if you define the measurable sets in terms of Caratheodory as done above, the $\sigma$ algebras are also unique.

As a special case of the above,
Corollary 8.2.8 Let $\mu$ be a Borel measure. Also let $\mu(B)<\infty$ for every ball $B$ contained in a metric space $X$ which has closed balls compact. Then $\mu$ must be regular.

Proof: This follows right away from using the Riesz representation theorem above on the functional

$$
L f \equiv \int f d \mu
$$

for all $f \in C_{c}(X)$.
Here is another interesting result.
Corollary 8.2.9 Let $\mathbf{X}$ be a random variable with values in $\mathbb{R}^{p}$. Then $\lambda_{\mathbf{X}}$ is an inner and outer regular measure defined on $\mathscr{B}\left(\mathbb{R}^{p}\right)$.

This is obvious when you recall the definition of the distribution measure for the random vector $\mathbf{X} . \lambda_{\mathbf{X}}(E) \equiv P(\omega: \mathbf{X}(\omega) \in E)$ where this means the probability that $\mathbf{X}$ is in $E$ a Borel set of $\mathbb{R}^{p}$. It is a finite Borel measure and so it is regular.

There is an interesting application of regularity to approximation of a measurable function with one that is continuous.

Lemma 8.2.10 Suppose $f: \mathbb{R}^{p} \rightarrow[0, \infty)$ is measurable where $\mu$ is a regular measure, both inner and outer regular, as in the Riesz representation theorem for positive linear functionals, finite on balls. Then there is a set of measure zero $N$ and a sequence of functions $\left\{h_{n}\right\}, h_{n}: \mathbb{R}^{p} \rightarrow[0, \infty)$ each in $C_{c}\left(\mathbb{R}^{p}\right)$ such that for all $\mathbf{x} \in \mathbb{R}^{p} \backslash N, h_{n}(\mathbf{x}) \rightarrow f(\mathbf{x})$. Also, for $\mathbf{x} \notin N, h_{n}(\mathbf{x}) \leq f(\mathbf{x})$ for all $n$ large enough.

Proof: Consider $f_{n}(\mathbf{x}) \equiv \mathscr{X}_{B_{n}}(\mathbf{x}) \min (f(\mathbf{x}), n)$ where $B_{n}$ is a ball centered at $\mathbf{x}_{0}$ which has radius $n$. Thus $f_{n}(\mathbf{x})$ is an increasing sequence and converges to $f(\mathbf{x})$ for each $\mathbf{x}$. Also by Corollary 6.1.10, there exists a simple function $s_{n}$ such that

$$
s_{n}(\mathbf{x}) \leq f_{n}(\mathbf{x}), \sup _{\mathbf{x} \in \mathbb{R}^{p}}\left|f_{n}(\mathbf{x})-s_{n}(\mathbf{x})\right|<\frac{1}{2^{n}}
$$

Let

$$
s_{n}(\mathbf{x})=\sum_{k=1}^{m_{n}} c_{k}^{n} \mathscr{X}_{E_{k}^{n}}(\mathbf{x}), c_{k}^{n}>0
$$

Then it must be the case that $\mu\left(E_{k}^{n}\right)<\infty$ because $\int f_{n} d \mu<\infty$.
By regularity, there exists a compact set $K_{k}^{n}$ and an open set $V_{k}^{n}$ such that

$$
K_{k}^{n} \subseteq E_{k}^{n} \subseteq V_{k}^{n}, \sum_{k=1}^{m_{n}} \mu\left(V_{k}^{n} \backslash K_{k}^{n}\right)<\frac{1}{2^{n}}
$$

Now let $K_{k}^{n} \prec \psi_{k}^{n} \prec V_{k}^{n}$ and let

$$
h_{n}(\mathbf{x}) \equiv \sum_{k=1}^{m_{n}} c_{k}^{n} \psi_{k}^{n}(\mathbf{x})
$$

Thus for $N_{n}=\cup_{k=1}^{m_{n}} V_{k}^{n} \backslash K_{k}^{n}$, it follows $\mu\left(N_{n}\right)<1 / 2^{n}$ and

$$
\sup _{\mathbf{x} \notin N_{n}}\left|f_{n}(\mathbf{x})-h_{n}(\mathbf{x})\right|<\frac{1}{2^{n}}
$$

If $h_{n}(\mathbf{x})$ fails to converge to $f(\mathbf{x})$, then $\mathbf{x}$ must be in infinitely many of the $N_{n}$. That is,

$$
\mathbf{x} \in \cap_{n=1}^{\infty} \cup_{k \geq n} N_{k} \equiv N
$$

However, this set $N$ is contained in

$$
\cup_{k=n}^{\infty} N_{k}, \mu\left(\cup_{k=n}^{\infty} N_{k}\right) \leq \sum_{k=n}^{\infty} \mu\left(N_{k}\right)<\frac{1}{2^{n-1}}
$$

and so $\mu(N)=0$. If $\mathbf{x}$ is not in $N$, then eventually $\mathbf{x}$ fails to be in $N_{n}$ and also $\mathbf{x} \in B_{n}$ so $h_{n}(\mathbf{x})=s_{n}(\mathbf{x})$ for all $n$ large enough. Now $f_{n}(\mathbf{x}) \rightarrow f(\mathbf{x})$ and $\left|s_{n}(\mathbf{x})-f_{n}(\mathbf{x})\right|<1 / 2^{n}$ so also $s_{n}(\mathbf{x})=h_{n}(\mathbf{x}) \rightarrow f(\mathbf{x})$.

Note that each $N_{k}$ is an open set and so, $N$ is a Borel set. Thus the above lemma leads to the following corollary.

Corollary 8.2.11 Let $f$ be measurable in the context of a regular measure space. Then there exists a Borel measurable function $g$ and a Borel set of measure zero $N$ such that $f(\mathbf{x})=g(\mathbf{x})$ for all $\mathbf{x} \notin N$. In fact, if $\mathbf{x} \notin N, f(\mathbf{x})=\lim _{n \rightarrow \infty} h_{n}(\mathbf{x})$ where $h_{n}$ is continuous and $\left|h_{n}(\mathbf{x})\right| \leq|f(\mathbf{x})|$ for all $n$ large enough.

Proof: Apply the above lemma to the positive and negative parts of the real and imaginary parts of $f$. Let $N$ be the union of the exceptional Borel sets which result. Thus, $f \mathscr{X}_{N^{C}}$ is the limit of a sequence $h_{n} \mathscr{X}_{N^{C}}$ where $h_{n}$ is continuous and for large enough $n,\left|h_{n}(\mathbf{x})\right| \leq|f(\mathbf{x})|$ for $\mathbf{x} \notin N$. Thus $h_{n} \mathscr{X}_{N^{C}}$ is Borel and it follows that $f \mathscr{X}_{N^{C}}$ is Borel measurable. Let $g=f \mathscr{X}_{N^{C}}$.

Here is an interesting lemma which is very easy to prove with the above representation theorem.

Lemma 8.2.12 Suppose $\mu$ is a measure defined on the Borel sets of $X$ which is finite on compact sets. Assume closed balls in $X$ are compact. Then there exists a unique Radon measure, $\bar{\mu}$ which equals $\mu$ on the Borel sets. In particular $\mu$ must be both inner and outer regular on all Borel sets.

Proof: Define a positive linear functional, $\Lambda(f)=\int f d \mu$. Let $\bar{\mu}$ be the Radon measure which comes from the Riesz representation theorem for positive linear functionals. Thus for all $f \in C_{c}(X), \int f d \mu=\int f d \bar{\mu}$. If $V$ is an open set, let $\left\{f_{n}\right\}$ be a sequence of continuous functions in $C_{c}(X)$ which is increasing and converges to $\mathscr{X}_{V}$ pointwise. Then applying the monotone convergence theorem,

$$
\int \mathscr{X}_{V} d \mu=\mu(V)=\int \mathscr{X}_{V} d \bar{\mu}=\bar{\mu}(V)
$$

and so the two measures coincide on all open sets. Every compact set is a countable intersection of open sets and so the two measures coincide on all compact sets. Now let $B(a, n)$ be a ball of radius $n$ and let $E$ be a Borel set contained in this ball. Then by regularity of $\bar{\mu}$ there exist sets $F, G$ such that $G$ is a countable intersection of open sets and $F$ is a countable union of compact sets such that $F \subseteq E \subseteq G$ and $\bar{\mu}(G \backslash F)=0$. Now $\mu(G)=\bar{\mu}(G)$ and $\mu(F)=\bar{\mu}(F)$. Thus

$$
\bar{\mu}(G \backslash F)+\bar{\mu}(F)=\bar{\mu}(G)=\mu(G)=\mu(G \backslash F)+\mu(F)
$$

and so $\mu(G \backslash F)=\bar{\mu}(G \backslash F)=0$. It follows $\mu(E)=\mu(F)=\bar{\mu}(F)=\bar{\mu}(G)=\bar{\mu}(E)$. If $E$ is an arbitrary Borel set, then $\mu(E \cap B(a, n))=\bar{\mu}(E \cap B(a, n))$ and letting $n \rightarrow \infty$, this yields $\mu(E)=\bar{\mu}(E)$.

### 8.3 Approximation with $G_{\delta}$ and $F_{\sigma}$

The inner and outer regularity results imply an important Proposition which is partly alluded to in the above.

Definition 8.3.1 A countable union of closed sets is called an $F_{\sigma}$ set and a countable intersection of open sets is called a $G_{\delta}$ set. Obviously these sets are Borel sets.

Proposition 8.3.2 Let $\mu$ be the Radon measure from Theorem 8.2.1 coming from a positive linear functional on $C_{c}(X)$ for $X$ a metric space in which closed balls are compact, and let $\mathscr{F}$ be the $\sigma$ algebra obtained there. Then if $E \in \mathscr{F}$, there exists $F$ an $F_{\sigma}$ set and $G a G_{\delta}$ set such that $F \subseteq E \subseteq G$ and $\mu(F)=\mu(E)=\mu(G)$. It can also be assumed that $\mu(G \backslash F)=0$. If $f \in L^{1}(X, \mathscr{F}, \mu)$, then there exists $g \in L^{1}(X, \mathscr{B}(X), \mu)$ such that $|g(x)| \leq|f(x)|$ and $g(x)=f(x)$ off a Borel set of measure zero.

Proof: From Corollary 8.2.8, if $E \in \mathscr{F}$, there exists $\left\{K_{n}\right\}$ be an increasing sequence of compact sets such that

$$
\mu(E)=\lim _{n \rightarrow \infty} \mu\left(K_{n}\right) .
$$

Then if $F \equiv \cup_{n} K_{n}$, it follows that $F$ is an $F_{\sigma}$ set and $\mu(F)=\lim _{n \rightarrow \infty} \mu\left(K_{n}\right)=\mu(E)$. Thus, in particular $F$ is a Borel set. Actually, one can say a little more. Note that, by assumption, $\mu(B)<\infty$ for any ball since $\mu(K)<\infty$ for any compact set and $\bar{B}$ is compact. Let $x$ be given in $X$ and let $B_{n} \equiv B(x, n)$. Let $E_{n} \equiv E \cap B_{n}$ so $\mu\left(E_{n}\right)<\infty$. From what was just shown, there exists an $F_{\sigma}$ set $F_{n} \subseteq E_{n}$ such that $\mu\left(E_{n}\right)=\mu\left(F_{n}\right)$. Since the measures are finite, $\mu\left(E_{n} \backslash F_{n}\right)=0$. Then letting $F \equiv \cup_{n=1}^{\infty} F_{n}$, it follows that this new $F$ is an $F_{\sigma}$ set and

$$
\begin{aligned}
\mu(E \backslash F) & =\mu\left(\cup_{n} E_{n} \backslash \cup_{n} F_{n}\right) \leq \mu\left(\cup_{n}\left(E_{n} \backslash F_{n}\right)\right) \\
& \leq \sum_{n} \mu\left(E_{n} \backslash F_{n}\right)=0
\end{aligned}
$$

Let $E, E_{n}$ be as above. Using outer regularity, there is an open set $V_{n}$ containing $E_{n}$ such that $\mu\left(V_{n} \backslash E_{n}\right)<\varepsilon 2^{-n}$. Let $W_{\varepsilon} \equiv \cup_{n} V_{n}$. Thus $\mu\left(W_{\varepsilon} \backslash E\right) \leq \mu\left(\cup_{k=1}^{\infty}\left(V_{k} \backslash E_{k}\right)\right) \leq \varepsilon$ and $W_{\varepsilon}$ is open and

$$
\mu\left(W_{\varepsilon}\right)<\varepsilon+\mu(E), \mu\left(W_{\varepsilon} \backslash E\right)<\varepsilon
$$

It follows there exists a decreasing sequence of open sets $W_{n}$ each containing $E$ such that $\mu\left(W_{n}\right)<2^{-n}+\mu(E)$, and $\mu\left(W_{n} \backslash E\right)<2^{-n}$. Let $G \equiv \cap_{n} W_{n}$. Then $G$ is a $G_{\delta}$ set containing $E$ and for each $n$,

$$
\mu(G \backslash E) \leq \mu\left(W_{n} \backslash E\right)<2^{-n}
$$

and so $\mu(G \backslash E)=0$ which implies $\mu(G)=\mu(E)$. Now $\mu(G)=\mu(E)=\mu(F)$. Also, the $F_{\sigma}$ set $F$ from the first part with $\mu(E \backslash F)=0$,

$$
\mu(G \backslash F)=\mu(G \backslash E)+\mu(E \backslash F)=0
$$

This proves the first part.
For the remaining part, it suffices to consider only $f(x) \geq 0$ because you can reduce to positive and negative parts of real and imaginary parts of $f$. By Theorem 6.1.10, there is an increasing sequence of simple functions $s_{k}$ such that for all $x, s_{k}(x) \uparrow f(x)$. Now for $s_{k}(x)=$ $\sum_{i=1}^{m_{k}} a_{i}^{k} \mathscr{X}_{E_{i}^{k}},\left(a_{i}^{k}>0\right)$ replace each $E_{i}^{k}$ with $F_{i}^{k}$ an $F_{\sigma}$ set with $\mu\left(E_{i}^{k} \backslash F_{i}^{k}\right)=0, F_{i}^{k} \subseteq E_{i}^{k}$. Let $N_{k} \equiv \cup_{i=1}^{m_{k}}\left(E_{i}^{k} \backslash F_{i}^{k}\right)$ and let $\hat{N} \equiv \cup_{k} N_{k}$ a set of measure zero. Thus there exists a Borel set $N \supseteq \hat{N}$ which also has measure zero, this by the first part. In fact, we can take $N$ to be a $G_{\delta}$ set. Let $\hat{s}_{k}(x)=s_{k}(x)$ off $N$ and let $\hat{s}(x)=0$ on $N$. Thus $\hat{s}_{k}(x)$ is an increasing function which converges to $f(x)$ off the set of measure zero $N$ and converges to 0 on $N$. Each $\hat{s}_{k}$ is Borel measurable and so letting $g$ be the pointwise limit, it follows from Corollary 6.1.4 that $g$ is Borel measurable and $0 \leq g \leq f$.

The above approximation result applies to any of the measures from Theorem 8.2.1. Next is a specialization to Lebesgue measure on $\mathbb{R}^{p}$.

### 8.4 Lebesgue Measure

Now we define Lebesgue measure in terms of a functional from beginning calculus.
Definition 8.4.1 Lebesgue measure, called $m_{p}$ is obtained from using the above Riesz representation theorem for positive linear functionals on the functional

$$
L f \equiv \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f\left(x_{1}, \cdots, x_{p}\right) d x_{p} d x_{p-1} \cdots d x_{1}
$$

where $f \in C_{c}\left(\mathbb{R}^{p}\right)$. Thus for such $f, L f=\int f d m_{p}$ but $m_{p}$ is a complete Borel measure which is also regular. One dimensional Lebesgue measure has already been discussed. I am writing $d x_{i_{k}} \equiv d m_{1}\left(x_{i_{k}}\right)$.

Then from Lemma 8.0.3 this functional $L$ and $L_{\sigma}$ give the same Borel measure $m_{p}$. Here $\sigma$ is the permutation which yields $\left(i_{1}, \cdots, i_{p}\right)$. Now let $U$ be an open set. Then from Theorem 8.1.3, and letting $\psi_{n}$ be the increasing sequence of functions in $C_{c}(U)$ converging pointwise to $\mathscr{X}_{U}$, we obtain the following from the monotone convergence theorem applied to the indicated succession of iterated integrals,

$$
\begin{gathered}
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathscr{X}_{U} d x_{p} d x_{p-1} \cdots d x_{1}=\lim _{n \rightarrow \infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \psi_{n} d x_{p} d x_{p-1} \cdots d x_{1} \\
=\lim _{n \rightarrow \infty} \int \psi_{n} d m_{p}=\int \mathscr{X}_{U} d m_{p} \\
=\lim _{n \rightarrow \infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \psi_{n} d x_{i_{p}} d x_{i_{p-1}} \cdots d x_{i_{1}}=\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathscr{X}_{U} d x_{i_{p}} d x_{i_{p-1}} \cdots d x_{i_{1}}
\end{gathered}
$$

This has proved part of the following result.
Lemma 8.4.2 For any $E$ Borel and $\left(i_{1}, \cdots, i_{p}\right)$ a permutation,

$$
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathscr{X}_{E} d x_{i_{p}} d x_{i_{p-1}} \cdots d x_{i_{1}}=\int \mathscr{X}_{E} d m_{p} *
$$

and all iterated integrals make sense. I am writing $d x_{i_{k}}$ for $d m_{1}\left(x_{i_{k}}\right)$.
Proof: Let $\mathscr{S}$ consist of the Borel sets $E$ such that $*$ holds for $E \cap(-R, R)^{p}$. Then $\mathscr{S}$ contains the open sets by what was just argued. The open sets are a $\pi$ system because they are closed with respect to finite intersections. Also, $\mathscr{S}$ is closed with respect to countable disjoint unions by an application of the monotone convergence theorem on each iterated integral. If $E \in \mathscr{S}$, does it follow that $E^{C} \in \mathscr{S}$ ? First note that each iterated integral in $\mathscr{X}_{E^{C} \cap(-R, R)^{p}}$ makes sense because the corresponding integrals for $\mathscr{X}_{E \cap(-R, R)^{p}}$ and $\mathscr{X}_{(-R, R)^{p}}$ make sense and $\mathscr{X}_{E^{C} \cap(-R, R)^{p}}=\mathscr{X}_{(-R, R)^{p}}-\mathscr{X}_{E \cap(-R, R)^{p}}$.

$$
\begin{aligned}
& \int_{(-R, R)^{p}} d m_{p}= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathscr{X}_{E \cap(-R, R)^{p}} d x_{i_{p}} d x_{i_{p-1}} \cdots d x_{i_{1}} \\
&+\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathscr{X}_{E^{C} \cap(-R, R)^{p}} d x_{i_{p}} d x_{i_{p-1}} \cdots d x_{i_{1}} \\
&=\int \mathscr{X}_{E \cap(-R, R)^{p}} d m_{p}+\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathscr{X}_{E^{C} \cap(-R, R)^{p}} d x_{i_{p}} d x_{i_{p-1}} \cdots d x_{i_{1}}
\end{aligned}
$$

Therefore,

$$
\int_{(-R, R)^{p} \cap E^{C}} d m_{p}=\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathscr{X}_{E^{C} \cap(-R, R)^{p}} d x_{i_{p}} d x_{i_{p-1}} \cdots d x_{i_{1}}
$$

and so by the lemma on $\pi$ systems, $\mathscr{S}$ consists of the Borel sets because $\mathscr{S}$ contains the open sets and the smallest $\sigma$ algebra containing the open sets. Thus for any $E$ Borel,

$$
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathscr{X}_{(-R, R)^{p} \cap E} d x_{i_{p}} d x_{i_{p-1}} \cdots d x_{i_{1}}=\int \mathscr{X}_{(-R, R)^{p} \cap E} d m_{p}
$$

Let $R \rightarrow \infty$ and use the monotone convergence theorem as needed to obtain that the iterated integrals all make sense and that the equality is preserved with $E$ in place of $(-R, R)^{p} \cap E$.

Theorem 8.4.3 Let $f \geq 0$ and be Borel measurable. Then for any permutation $\left(i_{1}, \cdots, i_{p}\right)$,

$$
\begin{equation*}
\int f d m_{p}=\int \cdots \int f\left(x_{1}, \cdots, x_{p}\right) d m_{1}\left(x_{i_{1}}\right) \cdots d m_{1}\left(x_{i_{p}}\right) \tag{8.10}
\end{equation*}
$$

Proof: By Theorem 6.1.10 there is an increasing sequence of simple, Borel measurable functions $\left\{s_{n}\right\}$ which converges pointwise to $f$. Since each is a finite linear combination of indicator functions of Borel sets,

$$
\int s_{n} d m_{p}=\int \cdots \int s_{n}\left(x_{1}, \cdots, x_{p}\right) d m_{1}\left(x_{i_{1}}\right) \cdots d m_{1}\left(x_{i_{p}}\right)
$$

Now apply the monotone convergence theorem to the succession of iterated integrals on the right and to the single integral on the left to obtain 8.10.

Corollary 8.4.4 Suppose $f \in L^{1}\left(\mathbb{R}^{p}, m_{p}\right)$ and $f$ is Borel measurable. Then 8.10 holds for $f$.

Proof: This is obvious from applying Theorem 8.4.3 to the positive and negative parts of the real and imaginary parts of $f$.

Another thing should probably be noted. You can use Fubini's theorem even if the function is not Borel measurable. This depends on Corollary 8.2.11. Say $f \in L^{1}\left(\mathbb{R}^{p}, m_{p}\right)$ so it is Lebesgue measurable but possibly not Borel measurable. Then from this corollary, there is a set of measure zero $N$ such that for $\mathbf{x} \notin N, f(\mathbf{x})=g(\mathbf{x})$ where $g$ is Borel measurable. By regularity, we can also assume $N$ is Borel measurable. Then

$$
\begin{aligned}
\int f d m_{p} & =\int \mathscr{X}_{N^{c}} g d m_{p}+\overbrace{\int \mathscr{X}_{N} f d m_{p}}^{=0} \\
& =\int \cdots \int \mathscr{X}_{N^{c}} g\left(x_{1}, \cdots, x_{p}\right) d m_{1}\left(x_{i_{1}}\right) \cdots d m_{1}\left(x_{i_{p}}\right) \\
& =\int \cdots \int g\left(x_{1}, \cdots, x_{p}\right) d m_{1}\left(x_{i_{1}}\right) \cdots d m_{1}\left(x_{i_{p}}\right)
\end{aligned}
$$

Since $g=f$ in $L^{1}\left(\mathbb{R}^{p}\right)$, you can typically use $g$ as a representative of $f$ when using any sort of computation involving iterated integrals. The thing you want is $\int f d m_{p}$ the iterated integral is a tool for finding it. Therefore, no harm is done in using $g$ rather than $f$.

### 8.5 Translation Invariance Lebesgue Measure

A very important property of Lebesgue measure is that it is translation invariant.
Definition 8.5.1 For $E$ a set, $E+\mathbf{x}$ will be $\{\mathbf{y}+\mathbf{x}: \mathbf{y} \in E\}$.
Theorem 8.5.2 Let $E \in \mathscr{F}_{p}$. Then $m_{p}(E)=m_{p}(E+\mathbf{z})$.
Proof: Let $\mathbf{z}=\left(z_{1}, \cdots, z_{p}\right)$. The conclusion is obvious if $E$ is an open rectangle

$$
E=\prod_{i=1}^{p}\left(a_{i}, b_{i}\right) .
$$

So let $\mathscr{K}$ be the set of open rectangles along with $\emptyset$ and $\mathbb{R}^{p}$ and let $\mathscr{G}$ consist of all measurable sets such that $m_{p}\left(E \cap R_{n}\right)=m_{p}\left(E \cap R_{n}+\mathbf{z}\right)$ with $E \cap R_{n}+\mathbf{z}$ measurable. Here $R_{n} \equiv \prod_{i=1}^{p}(-n, n)$. Then, similar to the proof of Lemma 8.4.2, you show that $\mathscr{G}$ is closed with respect to countable disjoint unions and complements. Then you use Dynkin's lemma to conclude that $\mathscr{G}$ contains the Borel sets. Next let $n \rightarrow \infty$ to obtain the conclusion for any $E$ Borel. Now suppose $E$ is just an arbitrary measurable set in $\mathscr{F}_{p}$. Apply Proposition 8.3.2 to get $F, G$ as described there, an $F_{\sigma}$ and $G_{\delta}$ set with $F \subseteq E \subseteq G$ and all three having the same Lebesgue measure. Thus, from the first part,

$$
m_{p}(E)=m_{p}(G)=m_{p}(G+\mathbf{z}) \geq m_{p}(E+\mathbf{z}) \geq m_{p}(F+\mathbf{z})=m_{p}(F)=m_{p}(E)
$$

and so all inequalities are equal signs.

### 8.6 The Vitali Covering Theorems

These theorems are remarkable and fantastically useful. They are covering theorems because they have to do with covering sets with balls. These balls may be open, closed, or neither open nor closed.
Lemma 8.6.1 In a normed linear space, $\overline{B(\mathbf{x}, r)}=\{\mathbf{y}:\|\mathbf{y}-\mathbf{x}\| \leq r\}$
Proof: $\mathbf{y} \rightarrow\|\mathbf{y}-\mathbf{x}\|$ is continuous and so $\{\mathbf{y}:\|\mathbf{y}-\mathbf{x}\| \leq r\}$ is a closed set which contains $B(\mathbf{x}, r)$. Therefore,

$$
\begin{equation*}
\overline{B(\mathbf{x}, r)} \subseteq\{\mathbf{y}:\|\mathbf{y}-\mathbf{x}\| \leq r\} \tag{8.11}
\end{equation*}
$$

Now let $\mathbf{y}$ be in the right side. It suffices to consider $\mathbf{y}$ such that $\|\mathbf{y}-\mathbf{x}\|=1$. Consider $\mathbf{x}+\frac{n-1}{n}(\mathbf{y}-\mathbf{x}) \equiv \mathbf{x}_{n}$. Then

$$
\left\|\mathbf{x}_{n}-\mathbf{y}\right\|=\left\|\mathbf{x}+\frac{n-1}{n}(\mathbf{y}-\mathbf{x})-\mathbf{y}\right\|=\frac{1}{n}\|\mathbf{x}-\mathbf{y}\|
$$

and so $\mathbf{y}$ is a limit point of $B(\mathbf{x}, t)$ and is therefore in $B(\mathbf{x}, r)$ so the two sets in 8.11 are equal.

Thus the usual way we think about the closure of a ball is completely correct in a normed linear space. This lemma is not always true in the context of a metric space. Recall the discrete metric for example in which the distance between different points is 1 and distance between a point and itself is 0 . In what follows we will use the result of this lemma without comment. Balls will be either open, closed or neither. I am going to use the Hausdorff maximality theorem because it yields a very simple argument.

Recall the following definition of a partially ordered set. A nonempty set is partially ordered if there exists a partial order, $\prec$, satisfying $x \prec x$ and if $x \prec y$ and $y \prec z$ then $x \prec z$. An example of a partially ordered set is the set of all subsets of a given set and $\prec$ is defined as $\subseteq$. Note that two elements in a partially ordered set may not be related. In other words, just because $x, y$ are in the partially ordered set, it does not follow that either $x \prec y$ or $y \prec x$. A subset of a partially ordered set $\mathscr{C}$, is called a chain if $x, y \in \mathscr{C}$ implies that either $x \prec y$ or $y \prec x$. If either $x \prec y$ or $y \prec x$ then $x$ and $y$ are described as being comparable. A chain is also called a totally ordered set. $\mathscr{C}$ is a maximal chain if whenever $\tilde{\mathscr{C}}$ is a chain containing $\mathscr{C}$, it follows the two chains are equal. In other words $\mathscr{C}$ is a maximal chain if there is no strictly larger chain. It turns out that every nonempty partially ordered set has a maximal chain. This is the Hausdorff maximal theorem discussed in Section 1.4. I will need to use this major result a few other times, so this might be a good place to introduce it.

Lemma 8.6.2 Let $\mathscr{F}$ be a collection of balls satisfying

$$
\infty>M \equiv \sup \{r: B(\mathbf{p}, r) \in \mathscr{F}\}>0
$$

and let $k \in(0, \infty)$. Then there exists $\mathscr{G} \subseteq \mathscr{F}$ such that

$$
\begin{gather*}
\text { If } B(\mathbf{p}, r) \in \mathscr{G} \text { then } r>k,  \tag{8.12}\\
\text { If } B_{1}, B_{2} \in \mathscr{G} \text { then } \overline{B_{1}} \cap \overline{B_{2}}=\emptyset,  \tag{8.13}\\
\mathscr{G} \text { is maximal with respect to } 8.12 \text { and } 8.13 \text {. } \tag{8.14}
\end{gather*}
$$

By this is meant that if $\mathscr{H}$ is a collection of balls satisfying 8.12 and 8.13 , then $\mathscr{H}$ cannot properly contain $\mathscr{G}$.

Proof: Let $\mathfrak{S}$ denote a subset of $\mathscr{F}$ such that 8.12 and 8.13 are satisfied. Then if $\mathfrak{S}=\emptyset$, it means there is no ball having radius larger than $k$. Otherwise, $\mathfrak{S} \neq \emptyset$. Partially order $\mathfrak{S}$ with respect to set inclusion. Thus $\mathscr{A} \prec \mathscr{B}$ for $\mathscr{A}, \mathscr{B}$ in $\mathfrak{S}$ means that $\mathscr{A} \subseteq \mathscr{B}$. By the Hausdorff maximal theorem, there is a maximal chain in $\mathfrak{S}$ denoted by $\mathscr{C}$. Then let $\mathscr{G}$ be $\cup \mathscr{C}$. If $B_{1}, B_{2}$ are in $\mathscr{C}$, then since $\mathscr{C}$ is a chain, both $B_{1}, B_{2}$ are in some element of $\mathscr{C}$ and so $\overline{B_{1}} \cap \overline{B_{2}}=\emptyset$. The maximality of $\mathscr{C}$ is violated if there is any other element of $\mathfrak{S}$ which properly contains $\mathscr{G}$.

Proposition 8.6.3 Let $\mathscr{F}$ be a collection of balls, and let $A \equiv \cup\{B: B \in \mathscr{F}\}$. Suppose

$$
\infty>M \equiv \sup \{r: B(\mathbf{p}, r) \in \mathscr{F}\}>0 .
$$

Then there exists $\mathscr{G} \subseteq \mathscr{F}$ such that $\mathscr{G}$ consists of balls whose closures are disjoint and

$$
A \subseteq \cup\{\widehat{B}: B \in \mathscr{G}\}
$$

where for $B=B(\mathbf{x}, r)$ a ball, $\widehat{B}$ denotes the open ball $B(\mathbf{x}, 5 r)$.
Proof: Let $\mathscr{G}_{1}$ satisfy $8.12,8.13,8.14$ for $k=\frac{2 M}{3}$.
Suppose $\mathscr{G}_{1}, \cdots, \mathscr{G}_{m-1}$ have been chosen for $m \geq 2$. Let $\overline{\mathscr{G}_{i}}$ denote the collection of closures of the balls of $\mathscr{G}_{i}$. Then let $\mathscr{F}_{m}$, be those balls of $\mathscr{F}$, such that if $B$ is one of these balls, $\bar{B}$ has empty intersection with every closed ball of $\overline{\mathscr{G}}_{i}$ for each $i \leq m-1$. Then using Lemma 8.6.2, let $\mathscr{G}_{m}$ be a maximal collection of balls from $\mathscr{F}_{m}$ with the property that each ball has radius larger than $\left(\frac{2}{3}\right)^{m} M$ and their closures are disjoint. Let $\mathscr{G} \equiv \cup_{k=1}^{\infty} \mathscr{G}_{k}$. Thus the closures of balls in $\mathscr{G}$ are disjoint. Let $\mathbf{x} \in B(\mathbf{p}, r) \in \mathscr{F} \backslash \mathscr{G}$. Choose $m$ such that

$$
\left(\frac{2}{3}\right)^{m} M<r \leq\left(\frac{2}{3}\right)^{m-1} M
$$

Then $\overline{B(\mathbf{p}, r)}$ must have nonempty intersection with the closure of some ball from $\mathscr{G}_{1} \cup \cdots \cup$ $\mathscr{G}_{m}$ because if it didn't, then $\mathscr{G}_{m}$ would fail to be maximal. Denote by $B\left(\mathbf{p}_{0}, r_{0}\right)$ a ball in $\mathscr{G}_{1} \cup \cdots \cup \mathscr{G}_{m}$ whose closure has nonempty intersection with $\overline{B(\mathbf{p}, r)}$. Thus $r_{0}, r>\left(\frac{2}{3}\right)^{m} M$. Consider the picture, in which $\mathbf{w} \in \overline{B\left(\mathbf{p}_{0}, r_{0}\right)} \cap \overline{B(\mathbf{p}, r)}$.


Then for $\mathbf{x} \in \overline{B(\mathbf{p}, r)}$,

$$
\begin{gathered}
\mid\left\|\mathbf{x}-\mathbf{p}_{0}\right\| \leq\|\mathbf{x}-\mathbf{p}\|+\|\mathbf{p}-\mathbf{w}\|+\overbrace{\left\|\mathbf{w}-\mathbf{p}_{0}\right\|}^{\leq r_{0}} \\
\leq r+r+r_{0} \leq 2 \overbrace{\left(\frac{2}{3}\right)^{m-1} M+}^{<\frac{3}{2} r_{0}}+r_{0} \leq 2\left(\frac{3}{2}\right) \overbrace{\left(\frac{2}{3}\right)^{m} M}^{<r_{0}}+r_{0} \leq 4 r_{0}
\end{gathered}
$$

Thus $B(\mathbf{p}, r)$ is contained in $\overline{B\left(\mathbf{p}_{0}, 4 r_{0}\right)}$. It follows that the closures of the balls of $\mathscr{G}$ are disjoint and the set $\{\hat{B}: B \in \mathscr{G}\}$ covers $A$.

Next is a version of the Vitali covering theorem which involves covering with disjoint closed balls. Here is the concept of a Vitali covering.

Definition 8.6.4 Let $S$ be a set and let $\mathscr{C}$ be a covering of $S$ meaning that every point of $S$ is contained in a set of $\mathscr{C}$. This covering is said to be a Vitali covering iffor each $\varepsilon>0$ and $\mathbf{x} \in S$, there exists a set $B \in \mathscr{C}$ containing $\mathbf{x}$, the diameter of $B$ is less than $\varepsilon$, and there exists an upper bound to the set of diameters of sets of $\mathscr{C}$.

The following corollary is a consequence of the above Vitali covering theorem.
Corollary 8.6.5 Let $F$ be a bounded set and let $\mathscr{C}$ be a Vitali covering of $F$ consisting of closed balls. Let $r(B)$ denote the radius of one of these balls. Then assume also that $\sup \{r(B): B \in \mathscr{C}\}=M<\infty$. Then there is a countable subset of $\mathscr{C}$ denoted by $\left\{B_{i}\right\}$ such that $\bar{m}_{p}\left(F \backslash \cup_{i=1}^{N} B_{i}\right)=0$ for $N \leq \infty$, and $B_{i} \cap B_{j}=\emptyset$ whenever $i \neq j$.

Proof: Let $U$ be a bounded open set containing $F$ such that $U$ approximates $F$ so well that

$$
m_{p}(U) \leq r \bar{m}_{p}(F), r>1 \text { and very close to } 1, r-5^{-p} \equiv \hat{\theta}_{p}<1
$$

Since this is a Vitali covering, for each $\mathbf{x} \in F$, there is one of these balls $B$ containing $\mathbf{x}$ such that $\hat{B} \subseteq U$. Let $\hat{\mathscr{C}}$ denote those balls of $\mathscr{C}$ such that $\hat{B} \subseteq U$ also. Thus, this is also a cover of $F$. By the Vitali covering theorem above, there are disjoint balls from $\mathscr{C},\left\{B_{i}\right\}$ such that $\left\{\hat{B}_{i}\right\}$ covers $F$. Thus

$$
\begin{aligned}
\bar{m}_{p}\left(F \backslash \cup_{j=1}^{\infty} B_{j}\right) & \leq m_{p}\left(U \backslash \cup_{j=1}^{\infty} B_{j}\right)=m_{p}(U)-\sum_{j=1}^{\infty} m_{p}\left(B_{j}\right) \\
& \leq r \bar{m}_{p}(F)-5^{-p} \sum_{j=1}^{\infty} m_{p}\left(\widehat{B}_{j}\right) \\
& \leq r \bar{m}_{p}(F)-5^{-p} \bar{m}_{p}(F) \\
& \equiv\left(r-5^{-p}\right) \bar{m}_{p}(F) \equiv \hat{\theta}_{p} \bar{m}_{p}(F)
\end{aligned}
$$

Now if $n_{1}$ is large enough and $\theta_{p}$ is chosen such that $1>\theta_{p}>\hat{\boldsymbol{\theta}}_{p}$, then

$$
\bar{m}_{p}\left(F \backslash \cup_{j=1}^{n_{1}} B_{j}\right) \leq m_{p}\left(U \backslash \cup_{j=1}^{n_{1}} B_{j}\right) \leq \theta_{p} \bar{m}_{p}(F)
$$

If $\bar{m}\left(F \backslash \cup_{j=1}^{n_{1}} B_{j}\right)=0$, stop. Otherwise, do for $F \backslash \cup_{j=1}^{n_{1}} B_{j}$ exactly the same thing that was done for $F$. Since $\cup_{j=1}^{n_{1}} B_{j}$ is closed, you can arrange to have the approximating open
set be contained in the open set $\left(\cup_{j=1}^{n_{1}} B_{j}\right)^{C}$. It follows there exist disjoint closed balls from $\mathscr{C}$ called $B_{n_{1}+1}, \cdots, B_{n_{2}}$ such that

$$
\bar{m}\left(\left(F \backslash \cup_{j=1}^{n_{1}} B_{j}\right) \backslash \cup_{j=n_{1+1}}^{n_{2}} B_{j}\right)<\theta_{p} \bar{m}\left(F \backslash \cup_{j=1}^{n_{1}} B_{j}\right)<\theta_{p}^{2} \bar{m}(F)
$$

continuing this way and noting that $\lim _{n \rightarrow \infty} \theta_{p}^{n}=0$ while $\bar{m}(F)<\infty$, this shows the desired result. Either the process stops because $\bar{m}\left(F \backslash \cup_{j=1}^{n_{k}} B_{j}\right)=0$ or else you obtain $\bar{m}\left(F \backslash \cup_{j=1}^{\infty} B_{j}\right)=0$.

The conclusion holds for arbitrary balls, open or closed or neither. This follows from observing that the measure of the boundary of a ball is 0 . Indeed, let

$$
S(\mathbf{x}, r) \equiv\{\mathbf{y}:|\mathbf{y}-\mathbf{x}|=r\}
$$

Then for each $\varepsilon<r$,

$$
\begin{aligned}
m_{p}(S(\mathbf{x}, r)) & \subseteq m_{p}(B(\mathbf{x}, r+\varepsilon))-m_{p}(B(\mathbf{x}, r-\varepsilon)) \\
& =m_{p}(B(\mathbf{0}, r+\varepsilon))-m_{p}(B(\mathbf{0}, r-\varepsilon)) \\
& =\left(\left(\frac{r+\varepsilon}{r}\right)^{p}-\left(\frac{r-\varepsilon}{r}\right)^{p}\right)\left(m_{p}(B(\mathbf{0}, r))\right)
\end{aligned}
$$

Hence $m_{p}(S(\mathbf{x}, r))=0$.
Thus you can simply omit the boundaries or part of the boundary of the closed balls and there is no change in the conclusion. Just first apply the above corollary to the Vitali cover consisting of closures of the balls before omitting part or all of the boundaries. The following theorem is also obtained. You don't need to assume the set is bounded.
Theorem 8.6.6 Let E be a bounded set and let $\mathscr{C}$ be a Vitali covering of $E$ consisting of balls, open, closed, or neither. Let $r(B)$ denote the radius of one of these balls. Then assume also that $\sup \{r(B): B \in \mathscr{C}\}=M<\infty$. Then there is a countable subset of $\mathscr{C}$ denoted by $\left\{B_{i}\right\}$ such that $\bar{m}_{p}\left(E \backslash \cup_{i=1}^{N} B_{i}\right)=0, N \leq \infty$, and $B_{i} \cap B_{j}=\emptyset$ whenever $i \neq j$. Here $\bar{m}_{p}$ denotes the outer measure determined by $m_{p}$. The same conclusion follows if you omit the assumption that $E$ is bounded.

Proof: It remains to consider the last claim. Consider the balls

$$
B(\mathbf{0}, 1), B(\mathbf{0}, 2), B(\mathbf{0}, 3), \cdots
$$

If $E$ is some set, let $E_{r}$ denote that part of $E$ which is between $B(\mathbf{0}, r-1)$ and $B(\mathbf{0}, r)$ but not on the boundary of either of these balls, where $B(\mathbf{0},-1) \equiv \emptyset$. Then $\cup_{r=0}^{\infty} E_{r}$ differs from $E$ by a set of measure zero and so you can apply the first part of the theorem to each $E_{r}$ keeping all balls between $B(\mathbf{0}, r-1)$ and $B(\mathbf{0}, r)$ allowing for no intersection with any of the boundaries. Then the union of the disjoint balls associated with $E_{r}$ gives the desired cover.

### 8.7 Hard Topology Theorems

### 8.7.1 The Brouwer Fixed Point Theorem

I found this proof of the Brouwer fixed point theorem in Evans [17] and Dunford and Schwartz [15]. The main idea which makes proofs like this work is Lemma 4.7.2 which is stated next for convenience.

Lemma 8.7.1 Let $\mathbf{g}: U \rightarrow \mathbb{R}^{p}$ be $C^{2}$ where $U$ is an open subset of $\mathbb{R}^{p}$. Then

$$
\sum_{j=1}^{p} \operatorname{cof}(D \mathbf{g})_{i j, j}=0
$$

where here $(D \mathbf{g})_{i j} \equiv g_{i, j} \equiv \frac{\partial g_{i}}{\partial x_{j}}$. Also, $\operatorname{cof}(D \mathbf{g})_{i j}=\frac{\partial \operatorname{det}(D \mathbf{g})}{\partial g_{i, j}}$.
Definition 8.7.2 Let $\mathbf{h}$ be a function defined on an open set, $U \subseteq \mathbb{R}^{p}$. Then $\mathbf{h} \in$ $C^{k}(\bar{U})$ if there exists a function $\mathbf{g}$ defined on an open set, $W$ containng $\bar{U}$ such that $\mathbf{g}=\mathbf{h}$ on $U$ and $\mathbf{g}$ is $C^{k}(W)$.

Lemma 8.7.3 There does not exist $\mathbf{h} \in C^{2}(\overline{B(\mathbf{0}, R)})$ such that $\mathbf{h}: \overline{B(\mathbf{0}, R)} \rightarrow \partial B(\mathbf{0}, R)$ which also has the property that $\mathbf{h}(\mathbf{x})=\mathbf{x}$ for all $\mathbf{x} \in \partial B(\mathbf{0}, R) \equiv\{\mathbf{x}:|\mathbf{x}|=R\}$ Such $a$ function is called a retract.

Proof: If $\mathbf{h}$ is such a retract, then for all $\mathbf{x} \in B(\mathbf{0}, R), \operatorname{det}(D \mathbf{h}(\mathbf{x}))=0$. This is because if $\operatorname{det}(D \mathbf{h}(\mathbf{x})) \neq 0$ for some such $\mathbf{x}$, then by the inverse function theorem, $\mathbf{h}(B(\mathbf{x}, \boldsymbol{\delta}))$ is an open set for small enough $\delta$ but this would require that this open set is a subset of $\partial B(\mathbf{0}, R)$ which is impossible because no open ball is contained in $\partial B(\mathbf{0}, R)$. Here and below, let $B_{R}$ denote $\overline{B(\mathbf{0}, R)}$.

Now suppose such an $\mathbf{h}$ exists. Let $\lambda \in[0,1]$ and let

$$
\mathbf{p}_{\lambda}(\mathbf{x}) \equiv \mathbf{x}+\lambda(\mathbf{h}(\mathbf{x})-\mathbf{x})
$$

This function, $\mathbf{p}_{\lambda}$ is a homotopy of the identity map and the retract $\mathbf{h}$. Let

$$
I(\lambda) \equiv \int_{B(\mathbf{0}, R)} \operatorname{det}\left(D \mathbf{p}_{\lambda}(\mathbf{x})\right) d x
$$

Then using the dominated convergence theorem,

$$
\begin{aligned}
I^{\prime}(\lambda) & =\int_{B(\mathbf{0}, R)} \sum_{i . j} \frac{\partial \operatorname{det}\left(D \mathbf{p}_{\lambda}(\mathbf{x})\right)}{\partial p_{\lambda i, j}} \frac{\partial p_{\lambda i j}(\mathbf{x})}{\partial \lambda} d x \\
& =\int_{B(\mathbf{0}, R)} \sum_{i} \sum_{j} \frac{\partial \operatorname{det}\left(D \mathbf{p}_{\lambda}(\mathbf{x})\right)}{\partial p_{\lambda i, j}}\left(h_{i}(\mathbf{x})-x_{i}\right)_{, j} d x \\
& =\int_{B(\mathbf{0}, R)} \sum_{i} \sum_{j} \operatorname{cof}\left(D \mathbf{p}_{\lambda}(\mathbf{x})\right)_{i j}\left(h_{i}(\mathbf{x})-x_{i}\right)_{, j} d x
\end{aligned}
$$

Now by assumption, $h_{i}(\mathbf{x})=x_{i}$ on $\partial B(\mathbf{0}, R)$ and so one can integrate by parts, in the iterated integrals used to compute $\int_{B(\mathbf{0}, R)}$ and write

$$
I^{\prime}(\lambda)=-\sum_{i} \int_{B(\mathbf{0}, R)} \sum_{j} \operatorname{cof}\left(D \mathbf{p}_{\lambda}(\mathbf{x})\right)_{i j, j}\left(h_{i}(\mathbf{x})-x_{i}\right) d x=0 .
$$

Therefore, $I(\boldsymbol{\lambda})$ equals a constant. However, $I(0)=m_{p}(B(\mathbf{0}, R)) \neq 0$ and as pointed out above, $I(1)=0$.

The following is the Brouwer fixed point theorem for $C^{2}$ maps.

Lemma 8.7.4 If $\mathbf{h} \in C^{2}(\overline{B(\mathbf{0}, R)})$ and $\mathbf{h}: \overline{B(\mathbf{0}, R)} \rightarrow \overline{B(\mathbf{0}, R)}$, then $\mathbf{h}$ has a fixed point, $\mathbf{x}$ such that $\mathbf{h}(\mathbf{x})=\mathbf{x}$.

Proof: Suppose the lemma is not true. Then for all $\mathbf{x},|\mathbf{x}-\mathbf{h}(\mathbf{x})| \neq 0$ for all $\mathbf{x} \in \overline{B(\mathbf{0}, R)}$. Then define

$$
\mathbf{g}(\mathbf{x})=\mathbf{h}(\mathbf{x})+(\mathbf{x}-\mathbf{h}(\mathbf{x})) t(\mathbf{x})
$$

where $t(\mathbf{x})$ is nonnegative and is chosen such that $\mathbf{g}(\mathbf{x}) \in \partial B(\mathbf{0}, R)$.
This mapping is illustrated in the following picture.


If $\mathbf{x} \rightarrow t(\mathbf{x})$ is $C^{2}$ near $\overline{B(\mathbf{0}, R)}$, it will follow $\mathbf{g}$ is a $C^{2}$ retract onto $\partial B(\mathbf{0}, R)$ contrary to Lemma 8.7.3. Thus $t(\mathbf{x})$ is the nonnegative solution $t$ to

$$
\begin{equation*}
|\mathbf{h}(\mathbf{x})+(\mathbf{x}-\mathbf{h}(\mathbf{x})) t(\mathbf{x})|^{2}=|\mathbf{h}(\mathbf{x})|^{2}+2(\mathbf{h}(\mathbf{x}), \mathbf{x}-\mathbf{h}(\mathbf{x})) t+t^{2}=R^{2} \tag{8.15}
\end{equation*}
$$

then by the quadratic formula,

$$
t(\mathbf{x})=-(\mathbf{h}(\mathbf{x}), \mathbf{x}-\mathbf{h}(\mathbf{x}))+\sqrt{(\mathbf{h}(\mathbf{x}), \mathbf{x}-\mathbf{h}(\mathbf{x}))^{2}+\left(R^{2}-|\mathbf{h}(\mathbf{x})|^{2}\right)}
$$

Is $\mathbf{x} \rightarrow t(\mathbf{x})$ a function in $C^{2}$ ? If what is under the radical is positive, then this is so because $s \rightarrow \sqrt{s}$ is smooth for $s>0$. In fact, this is the case here. The inside of the radical is positive if $R>|\mathbf{h}(\mathbf{x})|$. If $|\mathbf{h}(\mathbf{x})|=R$, it is still positive because in this case, the angle between $\mathbf{h}(\mathbf{x})$ and $\mathbf{x}-\mathbf{h}(\mathbf{x})$ cannot be $\pi / 2$. This shows that $\mathbf{x} \rightarrow t(\mathbf{x})$ is the composition of $C^{2}$ functions and is therefore $C^{2}$. Thus this $\mathbf{g}(\mathbf{x})$ is a $C^{2}$ retract and by the above lemma, there isn't one.

Now it is easy to prove the Brouwer fixed point theorem. The following theorem is the Brouwer fixed point theorem for a ball.
Theorem 8.7.5 Let $B_{R}$ be the above closed ball and let $\mathbf{f}: B_{R} \rightarrow B_{R}$ be continuous. Then there exists $\mathbf{x} \in B_{R}$ such that $\mathbf{f}(\mathbf{x})=\mathbf{x}$.

Proof: Let $\mathbf{f}_{k}(\mathbf{x}) \equiv \frac{\mathbf{f}(\mathbf{x})}{1+k^{-1}}$. Thus

$$
\begin{aligned}
\left\|\mathbf{f}_{k}-\mathbf{f}\right\| & =\max _{\mathbf{x} \in B_{R}}\left\{\left|\frac{\mathbf{f}(\mathbf{x})}{1+(1 / k)}-\mathbf{f}(\mathbf{x})\right|\right\}=\max _{\mathbf{x} \in B_{R}}\left\{\left|\frac{\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{x})(1+(1 / k))}{1+(1 / k)}\right|\right\} \\
& =\max _{\mathbf{x} \in B_{R}}\left\{\left|\frac{\mathbf{f}(\mathbf{x})(1 / k)}{1+(1 / k)}\right|\right\} \leq \frac{R}{1+k}
\end{aligned}
$$

Letting $\|\mathbf{h}\| \equiv \max \left\{|\mathbf{h}(\mathbf{x})|: \mathbf{x} \in B_{R}\right\}$, It follows from the Weierstrass approximation theorem, that there exists a function whose components are polynomials $\mathbf{g}_{k}$ such that

$$
\left\|\mathbf{g}_{k}-\mathbf{f}_{k}\right\|<\frac{R}{k+1}
$$

Then if $\mathbf{x} \in B_{R}$, it follows

$$
\begin{aligned}
\left|\mathbf{g}_{k}(\mathbf{x})\right| & \leq\left|\mathbf{g}_{k}(\mathbf{x})-\mathbf{f}_{k}(\mathbf{x})\right|+\left|\mathbf{f}_{k}(\mathbf{x})\right| \\
& <\frac{R}{1+k}+\frac{k R}{1+k}=R
\end{aligned}
$$

and so $\mathbf{g}_{k}$ maps $B_{R}$ to $B_{R}$. By Lemma 8.7.4 each of these $\mathbf{g}_{k}$ has a fixed point $\mathbf{x}_{k}$ such that $\mathbf{g}_{k}\left(\mathbf{x}_{k}\right)=\mathbf{x}_{k}$. The sequence of points $\left\{\mathbf{x}_{k}\right\}$ is contained in the compact set $B_{R}$ and so there exists a convergent subsequence still denoted by $\left\{\mathbf{x}_{k}\right\}$ which converges to a point $\mathbf{x} \in B_{R}$. Then from uniform convergence of $\mathbf{g}_{k}$ to $\mathbf{f}$,

$$
\mathbf{f}(\mathbf{x})=\lim _{k \rightarrow \infty} \mathbf{f}\left(\mathbf{x}_{k}\right)=\lim _{k \rightarrow \infty} \mathbf{g}_{k}\left(\mathbf{x}_{k}\right)=\lim _{k \rightarrow \infty} \mathbf{x}_{k}=\mathbf{x} \square
$$

It is not surprising that the ball does not need to be centered at $\mathbf{0}$.
Corollary 8.7.6 Let $\mathbf{f}: \overline{B(\mathbf{a}, R)} \rightarrow \overline{B(\mathbf{a}, R)}$ be continuous. Then there exists $\mathbf{x} \in \overline{B(\mathbf{a}, R)}$ such that $\mathbf{f}(\mathbf{x})=\mathbf{x}$.

Proof: Let $\mathbf{g}: B_{R} \rightarrow B_{R}$ be defined by $\mathbf{g}(\mathbf{y}) \equiv \mathbf{f}(\mathbf{y}+\mathbf{a})-\mathbf{a}$. Then $\mathbf{g}$ is a continuous map from $B_{R}$ to $B_{R}$. Therefore, there exists $\mathbf{y} \in B_{R}$ such that $\mathbf{g}(\mathbf{y})=\mathbf{y}$. Therefore, $\mathbf{f}(\mathbf{y}+\mathbf{a})-$ $\mathbf{a}=\mathbf{y}$ and so letting $\mathbf{x}=\mathbf{y}+\mathbf{a}, \mathbf{f}$ also has a fixed point as claimed.

Definition 8.7.7 $A$ set $A$ is a retract of a set $B$ if $A \subseteq B$, and there is a continuous map $\mathbf{h}: B \rightarrow A$ such that $\mathbf{h}(\mathbf{x})=\mathbf{x}$ for all $\mathbf{x} \in A$ and $\mathbf{h}$ is onto. $B$ has the fixed point property means that whenever $\mathbf{g}$ is continuous and $\mathbf{g}: B \rightarrow B$, it follows that $\mathbf{g}$ has a fixed point.

Proposition 8.7.8 Let A be a retract of B and suppose B has the fixed point property. Then so does $A$.

Proof: Suppose $\mathbf{f}: A \rightarrow A$. Let $\mathbf{h}$ be the retract of $B$ onto $A$. Then $\mathbf{f} \circ \mathbf{h}: B \rightarrow B$ is continuous. Thus, it has a fixed point $\mathbf{x} \in B$ so $\mathbf{f}(\mathbf{h}(\mathbf{x}))=\mathbf{x}$. However, $\mathbf{h}(\mathbf{x}) \in A$ and $\mathbf{f}: A \rightarrow A$ so in fact, $\mathbf{x} \in A$. Now $h(\mathbf{x})=\mathbf{x}$ and so $\mathbf{f}(\mathbf{x})=\mathbf{x}$.

Recall that every convex compact subset $K$ of $\mathbb{R}^{p}$ is a retract of all of $\mathbb{R}^{p}$ obtained by using the projection map. See Problems beginning with 22 on Page 77. In particular, $K$ is a retract of a large closed ball containing $K$, which ball has the fixed point property. Therefore, $K$ also has the fixed point property. This shows the following which is a convenient formulation of the Brouwer fixed point theorem. However, Proposition 8.7.8 is more general. You can probably imagine lots of sets which are retracts of some larger ball.

Theorem 8.7.9 Every convex closed and bounded subset of $\mathbb{R}^{p}$ has the fixed point property.

### 8.7.2 Invariance of Domain

As an application of the inverse function theorem is a simple proof of the important invariance of domain theorem which says that continuous and one to one functions defined on an open set in $\mathbb{R}^{n}$ with values in $\mathbb{R}^{n}$ take open sets to open sets. You know that this is true for functions of one variable because a one to one continuous function must be either strictly increasing or strictly decreasing. This will be used when considering orientations of curves later. However, the $n$ dimensional version isn't at all obvious but is just as important if you
want to consider manifolds with boundary for example. The need for this theorem occurs in many other places as well in addition to being extremely interesting for its own sake. The inverse function theorem gives conditions under which a differentiable function maps open sets to open sets. The following lemma, depending on the Brouwer fixed point theorem is the thing which will allow this to be extended to continuous one to one functions. It says roughly that if a continuous function does not move points near $\mathbf{p}$ very far, then the image of a ball centered at $\mathbf{p}$ contains an open set.

Lemma 8.7.10 Let $\mathbf{f}$ be continuous and map $\overline{B(\mathbf{p}, r)} \subseteq \mathbb{R}^{n}$ to $\mathbb{R}^{n}$. Suppose that for all $\mathbf{x} \in \overline{B(\mathbf{p}, r)},|\mathbf{f}(\mathbf{x})-\mathbf{x}|<\varepsilon r$ Then it follows that $\mathbf{f}(\overline{B(\mathbf{p}, r)}) \supseteq B(\mathbf{p},(1-\varepsilon) r)$

Proof: This is from the Brouwer fixed point theorem, Theorem 8.7.9. Consider for $\mathbf{y} \in$ $B(\mathbf{p},(1-\varepsilon) r)$ the function $\mathbf{h}(\mathbf{x}) \equiv \mathbf{x}-\mathbf{f}(\mathbf{x})+\mathbf{y}$ Then $\mathbf{h}$ is continuous and for $\mathbf{x} \in \overline{B(\mathbf{p}, r),}$

$$
|\mathbf{h}(\mathbf{x})-\mathbf{p}|=|\mathbf{x}-\mathbf{f}(\mathbf{x})+\mathbf{y}-\mathbf{p}|<\varepsilon r+|\mathbf{y}-\mathbf{p}|<\varepsilon r+(1-\varepsilon) r=r
$$

Hence $\mathbf{h}: \overline{B(\mathbf{p}, r)} \rightarrow \overline{B(\mathbf{p}, r)}$ and so it has a fixed point $\mathbf{x}$ by Theorem 8.7.9. Thus $\mathbf{x}-\mathbf{f}(\mathbf{x})+$ $\mathbf{y}=\mathbf{x}$ so $\mathbf{f}(\mathbf{x})=\mathbf{y}$.

The notation $\|\mathbf{f}\|_{K}$ will mean $\sup _{\mathbf{x} \in K}|\mathbf{f}(\mathbf{x})|$. If you have a continuous function $\mathbf{h}$ defined on a compact set $K$, then the Stone Weierstrass theorem implies you can uniformly approximate it with a polynomial $\mathbf{g}$. That is $\|\mathbf{h}-\mathbf{g}\|_{K}$ is small. The following lemma says that you can also have $\mathbf{g}(\mathbf{z})=\mathbf{h}(\mathbf{z})$ and $D \mathbf{g}(\mathbf{z})^{-1}$ exists so that near $\mathbf{z}$, the function $\mathbf{g}$ will map open sets to open sets as claimed by the inverse function theorem. First is a little observation about approximating.

Lemma 8.7.11 Let $K$ be a compact set in $\mathbb{R}^{n}$ and let $\mathbf{h}: K \rightarrow \mathbb{R}^{n}$ be continuous, $\mathbf{z} \in K$ is fixed. Let $\delta>0$. Then there exists a polynomial $\mathbf{g}$ (each component a polynomial) such that

$$
\|\mathbf{g}-\mathbf{h}\|_{K}<\delta, \mathbf{g}(\mathbf{z})=\mathbf{h}(\mathbf{z}), D \mathbf{g}(\mathbf{z})^{-1} \text { exists }
$$

Proof: By the Weierstrass approximation theorem, Theorem 3.2.4, there exists a polynomial $\hat{\mathbf{g}}$ such that $\|\hat{\mathbf{g}}-\mathbf{h}\|_{K}<\frac{\delta}{3}$. Then define for $\mathbf{y} \in K$

$$
\mathbf{g}(\mathbf{y}) \equiv \hat{\mathbf{g}}(\mathbf{y})+\mathbf{h}(\mathbf{z})-\hat{\mathbf{g}}(\mathbf{z})
$$

Then $\mathbf{g}(\mathbf{z})=\hat{\mathbf{g}}(\mathbf{z})+\mathbf{h}(\mathbf{z})-\hat{\mathbf{g}}(\mathbf{z})=\mathbf{h}(\mathbf{z})$. Also

$$
\begin{aligned}
|\mathbf{g}(\mathbf{y})-\mathbf{h}(\mathbf{y})| & \leq|(\hat{\mathbf{g}}(\mathbf{y})+\mathbf{h}(\mathbf{z})-\hat{\mathbf{g}}(\mathbf{z}))-\mathbf{h}(\mathbf{y})| \\
& \leq|\hat{\mathbf{g}}(\mathbf{y})-\mathbf{h}(\mathbf{y})|+|\mathbf{h}(\mathbf{z})-\hat{\mathbf{g}}(\mathbf{z})|<\frac{2 \delta}{3}
\end{aligned}
$$

and so since $\mathbf{y}$ was arbitrary, $\|\mathbf{g}-\mathbf{h}\|_{K} \leq \frac{2 \delta}{3}<\delta$. If $D \mathbf{g}(\mathbf{z})^{-1}$ exists, then this is what is wanted. If not, use Lemma 4.7.1 and note that for all $\eta$ small enough, you could replace $\mathbf{g}$ with $\mathbf{y} \rightarrow \mathbf{g}(\mathbf{y})+\eta(\mathbf{y}-\mathbf{z})$ and it will still be the case that $\|\mathbf{g}-\mathbf{h}\|_{K}<\delta$ along with $\mathbf{g}(\mathbf{z})=\mathbf{h}(\mathbf{z})$ but now $D \mathbf{g}(\mathbf{z})^{-1}$ exists. Simply use the modified $\mathbf{g}$.

The main result is essentially the following lemma which combines the conclusions of the above.

Lemma 8.7.12 Let $\mathbf{f}: \overline{B(\mathbf{p}, r)} \rightarrow \mathbb{R}^{n}$ where the ball is also in $\mathbb{R}^{n}$. Let $\mathbf{f}$ be one to one, $\mathbf{f}$ continuous. Then there exists $\delta>0$ such that

$$
\mathbf{f}(\overline{B(\mathbf{p}, r)}) \supseteq B(\mathbf{f}(\mathbf{p}), \delta)
$$

In other words, $\mathbf{f}(\mathbf{p})$ is an interior point of $\mathbf{f}(\overline{B(\mathbf{p}, r)})$.
Proof: Since $\mathbf{f}(\overline{B(\mathbf{p}, r)})$ is compact, it follows that $\mathbf{f}^{-1}: \mathbf{f}(\overline{B(\mathbf{p}, r)}) \rightarrow \overline{B(\mathbf{p}, r)}$ is continuous. By Lemma 8.7.11, there exists a polynomial $\mathbf{g}: \mathbf{f}(\overline{B(\mathbf{p}, r)}) \rightarrow \mathbb{R}^{n}$ such that $\left\|\mathbf{g}-\mathbf{f}^{-1}\right\|_{\mathbf{f}(\overline{B(\mathbf{p}, r)})}<\varepsilon r, \varepsilon<1$,

$$
D \mathbf{g}(\mathbf{f}(\mathbf{p}))^{-1} \text { exists, and } \mathbf{g}(\mathbf{f}(\mathbf{p}))=\mathbf{f}^{-1}(\mathbf{f}(\mathbf{p}))=\mathbf{p}
$$

From the first inequality in the above,

$$
|\mathbf{g}(\mathbf{f}(\mathbf{x}))-\mathbf{x}|=\left|\mathbf{g}(\mathbf{f}(\mathbf{x}))-\mathbf{f}^{-1}(\mathbf{f}(\mathbf{x}))\right| \leq\left\|\mathbf{g}-\mathbf{f}^{-1}\right\|_{\mathbf{f}(\overline{B(\mathbf{p}, r)})}<\varepsilon r
$$

By Lemma 8.7.10,

$$
\mathbf{g} \circ \mathbf{f}(\overline{B(\mathbf{p}, r)}) \supseteq B(\mathbf{p},(1-\varepsilon) r)=B(\mathbf{g}(\mathbf{f}(\mathbf{p})),(1-\varepsilon) r)
$$

Since $D \mathbf{g}(\mathbf{f}(\mathbf{p}))^{-1}$ exists, it follows from the inverse function theorem that $\mathbf{g}^{-1}$ also exists and that $\mathbf{g}, \mathbf{g}^{-1}$ are open maps on small open sets containing $\mathbf{f}(\mathbf{p})$ and $\mathbf{p}$ respectively. Thus there exists $\eta<(1-\boldsymbol{\varepsilon}) r$ such that $\mathbf{g}^{-1}$ is an open map on $B(\mathbf{p}, \eta) \subseteq B(\mathbf{p},(1-\varepsilon) r)$. Thus

$$
\mathbf{g} \circ \mathbf{f}(\overline{B(\mathbf{p}, r)}) \supseteq B(\mathbf{p},(1-\varepsilon) r) \supseteq B(\mathbf{p}, \eta)
$$

So do $\mathbf{g}^{-1^{‘}}$ to both ends. Then you have $\mathbf{g}^{-1}(\mathbf{p})=\mathbf{f}(\mathbf{p})$ is in the open set $\mathbf{g}^{-1}(B(\mathbf{p}, \eta))$. Thus

$$
\mathbf{f}(\overline{B(\mathbf{p}, r)}) \supseteq \mathbf{g}^{-1}(B(\mathbf{p}, \eta)) \supseteq B\left(\mathbf{g}^{-1}(\mathbf{p}), \delta\right)=B(\mathbf{f}(\mathbf{p}), \delta)
$$



With this lemma, the invariance of domain theorem comes right away. This remarkable theorem states that if $\mathbf{f}: U \rightarrow \mathbb{R}^{n}$ for $U$ an open set in $\mathbb{R}^{n}$ and if $\mathbf{f}$ is one to one and continuous, then $\mathbf{f}(U)$ is also an open set in $\mathbb{R}^{n}$.

Theorem 8.7.13 Let $U$ be an open set in $\mathbb{R}^{n}$ and let $\mathbf{f}: U \rightarrow \mathbb{R}^{n}$ be one to one and continuous. Then $\mathbf{f}(U)$ is also an open subset in $\mathbb{R}^{n}$.

Proof: It suffices to show that if $\mathbf{p} \in U$ then $\mathbf{f}(\mathbf{p})$ is an interior point of $\mathbf{f}(U)$. Let $\overline{B(\mathbf{p}, r)} \subseteq U$. By Lemma 8.7.12, $\mathbf{f}(U) \supseteq \mathbf{f}(\overline{B(\mathbf{p}, r)}) \supseteq B(\mathbf{f}(\mathbf{p}), \delta)$ so $\mathbf{f}(\mathbf{p})$ is indeed an interior point of $\mathbf{f}(U)$.

### 8.7.3 Jordan Curve Theorem

This treatment of the Jordan curve theorem based on the Brouwer fixed point theorem is the shortest and most direct proof I have seen. It is from [33]. Any errors are mine. Here $J \subseteq \mathbb{R}^{2}$ will denote a Jordan curve, defined as the homeomorphic image of the unit circle meaning that $J$ is $\gamma\left(S^{1}\right)$ where $\gamma$ is one to one and continuous.

To begin with, note that there is exactly one unbounded component of the complement of a Jordan curve or more generally the complement of a compact set. If $U, V$ are both unbounded components, they both contain the circle $\partial B(\mathbf{0}, R)$ for large enough $R$ that $J \subseteq$ $B(\mathbf{0}, R)$, and so there is a continuous curve joining any point in $U$ with a point in $V$ so that the two must be the same component. Also note that for any nonempty set $S$ it has the same diameter as its closure. Recall that an arc, also called a simple curve, is the one to one continuous image of a closed interval.

Lemma 8.7.14 Let $U$ be an open set. Then if $\partial U$ denotes those points $\mathbf{p}$ such that for each $r>0, B(\mathbf{p}, r)$ contains points of $U$ and points of $U^{C}$, then $\partial U=\bar{U} \backslash U$.

Proof: If $\mathbf{p} \in \partial U$, then $\mathbf{p} \notin U$ because $U$ is open. If $\mathbf{p} \notin \bar{U}$ then there would be a ball containing $\mathbf{p}$ which has no points of $U$ and so $\mathbf{p}$ would not be in $\partial U$. Therefore, $\mathbf{p} \in \bar{U} \backslash U$ and so $\partial U \subseteq \bar{U} \backslash U$. If $\mathbf{p} \in \bar{U} \backslash U$, then $\mathbf{p}$ is a limit point of $U$ and so $B(\mathbf{p}, r)$ contains points of $U$ for every $r>0$. Since $\mathbf{p} \notin U$, every $B(\mathbf{p}, r)$ contains points of $U^{C}$ and so $\mathbf{p} \in \partial U$.

In the following $U$ will be a connected component of $J^{C}$.
Lemma 8.7.15 Let $J=\gamma^{*}$ where $\gamma: S^{1} \rightarrow \mathbb{R}^{2}$ is one to one and onto and continuous. If $U$ is a connected component of $J^{C}$ then $\bar{U} \backslash U \subseteq J$.

Proof: Suppose $\mathbf{x} \in \bar{U} \backslash U$. I want to show that $\mathbf{x} \in J$. If $\mathbf{x} \notin J$, then, since $\mathbf{x}$ is not in $U$, it must be in a different component $V \neq U$. But then $\mathbf{x}$ cannot be a limit point of $U$ so $\mathbf{x} \in J$ as desired. Thus $\bar{U} \backslash U \subseteq J$.

Lemma 8.7.16 Let $\gamma:[a, b] \rightarrow \gamma^{*}=\gamma([a, b])$ be one to one and continuous. Then there exists $\mathbf{r}: \mathbb{R}^{2} \rightarrow \gamma^{*}$ such that $\mathbf{r}(\mathbf{x})=\mathbf{x}$ for all $\mathbf{x} \in \gamma^{*}$. Also if $J$ is a simple closed curve and $K$ is a proper compact subset of $J$, then there exists a simple curve $A \subseteq J$ such that $K \subseteq A \subseteq J$.

Proof: By the Tietze extention theorem, there is an extention of $\gamma^{-1}$ denoted as $\hat{\mathbf{r}}$ which maps $\mathbb{R}^{2}$ onto $[a, b]$. Consider $\gamma \circ \hat{\mathbf{r}} \equiv \mathbf{r}$. Then $\mathbf{r}(\gamma(t)) \equiv \gamma(\hat{\mathbf{r}}(\gamma(t)))=\gamma\left(\gamma^{-1}(\gamma(t))\right)=\gamma(t)$ so $\mathbf{r}$ does what it should. It fixes all the points of $\gamma^{*}$.

Now consider the second claim. Since $K$ is a compact proper subset of $J$, you can start with a point of $J$ which is not in $K$ and following $J$ in either of the two orientations, there is a first point of $K$ called $\mathbf{a}$ and a last point $\mathbf{b}$. Then the simple curve $\mathbf{a b}$ denoting motion from $\mathbf{a}$ to $\mathbf{b}$ is a simple curve which contains $K$. This is $A$.

With the above observation that $\bar{U} \backslash U \subseteq J$, it remains to show that in fact $\bar{U} \backslash U=J$. First this is shown under an assumption that $J^{C}$ has at least two components, in particular a bounded component.


Lemma 8.7.17 Suppose $J$ is a simple closed curve and suppose there exists a bounded component of $J^{C}\left(J^{C}\right.$ has at least two components). Then for any component $U$ of $J^{C}$, $\bar{U} \backslash U=J$.

Proof: Suppose $U$ is a bounded component of $J^{C}$ and $\bar{U} \backslash U$ is a proper subset of $J$. Let $A \subseteq J$ be the simple curve which contains $\bar{U} \backslash U$ which is mentioned in Lemma 8.7.16. Let $\mathbf{w} \in U$.

Let $\mathbf{r}: \mathbb{R}^{2} \rightarrow A$ be the map of Lemma 8.7.16 which fixes all the points of $A$. Let

$$
\mathbf{f}(\mathbf{x}) \equiv\left\{\begin{array}{c}
\mathbf{r}(\mathbf{x}) \text { if } \mathbf{x} \in \bar{U} \\
\mathbf{x} \text { if } \mathbf{x} \in U^{C}
\end{array}\right.
$$

Thus this defines continuous $\mathbf{f}$ on all of $\mathbb{R}^{2}$. It is continuous at each point of $\bar{U}$ and at each point of $U^{C}$. If $\mathbf{x}$ is in $\bar{U} \cap U^{C}$, then $\mathbf{r}(\mathbf{x})=\mathbf{x}$ and so this is indeed continuous on all of $\mathbb{R}^{2}$. Also, $\mathbf{f}(\mathbf{x}) \neq \mathbf{w}$ for any $\mathbf{x}$ because $\mathbf{x} \in U$ so $\mathbf{x} \notin A$. Now consider $\mathbf{w}-R \frac{\mathbf{f}(\mathbf{x})-\mathbf{w}}{|\mathbf{f}(\mathbf{x})-\mathbf{w}|}$ where $R$ is chosen very large, larger than $\operatorname{diam}(\bar{U})$. In case $\mathbf{f}(\mathbf{x})=\mathbf{r}(\mathbf{x})$, it is not possible that $\mathbf{w}-R \frac{\mathbf{f}(\mathbf{x})-\mathbf{w}}{|\mathbf{f ( x )}-\mathbf{w}|}=\mathbf{x}$ because if so, you would have $\mathbf{x} \in \bar{U}$ and $-R \frac{\mathbf{f}(\mathbf{x})-\mathbf{w}}{\mid \mathbf{f ( \mathbf { x } ) - \mathbf { w } |}}=\mathbf{x}-\mathbf{w}$ and the right side is smaller in magnitude than the left. In case $\mathbf{f}(\mathbf{x})=\mathbf{x}$, you cannot have $\mathbf{w}-R \frac{\mathbf{f}(\mathbf{x})-\mathbf{w}}{|\mathbf{f}(\mathbf{x})-\mathbf{w}|}=\mathbf{x}$ either because if so, then $-R \frac{\mathbf{x}-\mathbf{w}}{|\mathbf{x}-\mathbf{w}|}=\mathbf{x}-\mathbf{w}$ and the two vectors point in different directions. However, $\mathbf{x} \rightarrow \mathbf{w}-R \frac{\mathbf{f}(\mathbf{x})-\mathbf{w}}{\mid \mathbf{f} \mathbf{( \mathbf { x } ) - \mathbf { w } |}}$ is continuous and maps $\overline{B(\mathbf{w}, R)}$ to $\overline{B(\mathbf{w}, R)}$ and so this would contradict the Brouwer fixed point theorem. Hence $\bar{U} \backslash U=J$ as claimed.

Next suppose $U$ is unbounded and let $V$ be a bounded component of $J^{C}$ and let $\mathbf{w} \in V$. This time let

$$
\mathbf{f}(\mathbf{x}) \equiv\left\{\begin{array}{c}
\mathbf{r}(\mathbf{x}) \text { if } \mathbf{x} \in U^{C} \\
\mathbf{x} \text { if } \mathbf{x} \in \bar{U}
\end{array}\right.
$$

As before, this defines continuous $\mathbf{f}$ on all of $\mathbb{R}^{2}$. Since $\mathbf{w} \in V, \mathbf{w}$ is not in $J$ and so $\mathbf{f}(\mathbf{x}) \neq \mathbf{w}$ for all $\mathbf{x}$. For $R$ large enough, $B(\mathbf{w}, R) \supseteq U^{C}$ because $B(\mathbf{w}, R)^{C} \subseteq U$. If $\mathbf{w}-R \frac{\mathbf{f}(\mathbf{x})-\mathbf{w}}{\mid \mathbf{f ( \mathbf { x } ) - \mathbf { w } |}}=\mathbf{x}$ for $\mathbf{x} \in U^{C}$, you would have $-R \frac{\mathbf{r}(\mathbf{x})-\mathbf{w}}{|\mathbf{r}(\mathbf{x})-\mathbf{w}|}=\mathbf{x}-\mathbf{w}$ and this is impossible because $|\mathbf{x}-\mathbf{w}|$ is less than $R$ while the left side has magnitude $R$. If $\mathbf{x} \in \bar{U}$ it is also impossible that $\mathbf{w}-R \frac{\mathbf{f}(\mathbf{x})-\mathbf{w}}{|\mathbf{f ( x )}-\mathbf{w}|}=$
 directions. Thus $\mathbf{x} \rightarrow \mathbf{w}-R \frac{\mathbf{f}(\mathbf{x})-\mathbf{w}}{\mid \mathbf{f ( \mathbf { x } ) - \mathbf { w } |}}$ maps $\overline{B(\mathbf{w}, R)}$ to itself and is continuous but has no fixed point contradicting the Brouwer fixed point theorem. It follows that $\bar{U} \backslash U=J$ as claimed.

Let $\mathbf{x}(t) \equiv(u(t), v(t))$ for $t \in[-1,1]$ and let $\mathbf{y}(t) \equiv(f(t), g(t)), t \in[-1,1]$ and suppose $u([-1,1])=[a, b]$ and $g([-1,1])=[c, d]$ where $\mathbf{x}, \mathbf{y}$ are both one to one. The following picture represents these two curves which lie in the rectangle $[a, b] \times[c, d]$ as shown in the picture. Then the conclusion of the following lemma says these two simple curves intersect.


Lemma 8.7.18 Let $t \rightarrow(u(t), v(t))$ and $t \rightarrow(f(t), g(t))$ for $t \in[-1,1]$ be parametrizations of two curves in which lie in $[a, b] \times[c, d]$ such that $u(-1)=a, u(1)=b$, and $g(-1)=$ $c, g(1)=d$. Second component of second curve goes from $c$ to $d$ and first component of
first curve from a to $b$. Then there exists a point of intersection of these two paths. That is, there is $s \in[-1,1], t \in[-1,1]$ such that $(u(s), v(s))=(f(t), g(t))$.

Proof: Suppose this is not true, then $\mathbf{G}:[-1,1] \times[-1,1] \rightarrow[-1,1] \times[-1,1]$ is continuous where $\mathbf{G}(s, t) \equiv$

$$
\equiv\left(\frac{f(t)-u(s)}{\max \{|f(t)-u(s)|,|v(s)-g(t)|\}}, \frac{v(s)-g(t)}{\max \{|f(t)-u(s)|,|v(s)-g(t)|\}}\right)
$$

Both components are in $[-1,1]$ and one of them is $\pm 1$.Thus $\mathbf{G}$ maps $[-1,1] \times[-1,1]$ to $\partial([-1,1] \times[-1,1])$, and is continuous. The only fixed points possible are of the form $( \pm 1, t),(s, \pm 1)$. Then letting $(\leq 0, \hat{t})$ denote an ordered pair in which the first component is non-positive and other uses of this notation similar,

$$
\begin{aligned}
& \mathbf{G}(1, t)=(\leq 0, \hat{t}) \neq(1, t), \mathbf{G}(-1, t)=(\geq 0, \hat{t}) \neq(-1, t) \\
& \mathbf{G}(s, 1)=(\hat{s}, \leq 0) \neq(s, 1), \mathbf{G}(s,-1)=(\hat{s}, \geq 0) \neq(s,-1)
\end{aligned}
$$

Thus $\mathbf{G}$ has no fixed point contrary to Brouwer fixed point theorem. It follows that $G(s, t)=$ 0 for some $(s, t)$ and this says there is a point of intersection of these two curves.

A Jordan arc will be the continuous one to one image of a closed interval. Then the conclusion of the above lemma implies the following easier to use proposition.

Corollary 8.7.19 Let $J_{1}, J_{2}$ be two oriented Jordan arcs which lie in $[a, b] \times[c, d]$ and suppose the first component of $J_{1}$ includes both $a$ and $b$ and the second component of $J_{2}$ includes both $c$ and $d$, then there is a point on the intersection of these Jordan arcs.

Proof: This follows from the above lemma by changing the parametrization $\gamma^{i}$ to have $\gamma_{1}^{1}(-1)=a, \gamma_{1}^{1}(1)=b, \gamma_{2}^{2}(-1)=c, \gamma_{2}^{2}(1)=d$. Then apply the above Lemma to these functions restricted to $[-1,1]$.

Proposition 8.7.20 Let $y \rightarrow \alpha(y), \beta(y)$ be non-negative continuous functions and let $U$ consist of $(x, y)$ such that $a-\alpha(y) \leq x \leq b+\beta(y)$ and $y \in[c, d]$. Let $J_{1}$ and $J_{2}$ be two oriented Jordan arcs such that some first component of $J_{1}$ equals $a-a(y)$ for some $y$ and some first component of $J_{1}$ equals $b+\beta(y)$ for some $y$ and some second component of points on $J_{2}$ equals $c$ while some second component of $J_{2}$ equals $d$. Then the two Jordan arcs intersect.

Proof: Let $\sigma$ be the midpoint of $[a, b]$. Let $f:[a, b] \times[c, d] \rightarrow U$ be defined as follows. For $x \geq \sigma, f(x, y) \equiv\left(x+\left(\frac{x-\sigma}{b-\sigma}\right) \beta(y), y\right)$ and for $x \leq \sigma, f(x, y) \equiv\left(x-\left(\frac{\sigma-x}{\sigma-a}\right) \alpha(y), y\right)$. Then $f$ is one to one onto and continuous. Also the left side of $[a, b] \times[c, d]$ is mapped to the left side of $U$ while the right side of $[a, b] \times[c, d]$ is mapped to the right side of $U$. Now if $J_{i}$ are as described, then $f^{-1}\left(J_{i}\right)$ satisfy the conditions of the above corollary and so these intersect at some point $\left(x_{0}, y_{0}\right)$. Then $f\left(x_{0}, y_{0}\right)$ is a point of intersection of $J_{1}$ and $J_{2}$.

Theorem 8.7.21 Let $J$ be a Jordan curve in the plane, $J=\gamma\left(S^{1}\right)$ where $\gamma$ is continuous and one to one. Then $J^{C}$ consists of a bounded component $U_{i}$ called the inside, and an unbounded component $U_{o}$ called the outside and $J=\partial U_{i}=\partial U_{o}$. That is, $J$ is the common boundary of both $U_{i}$ and $U_{o}$.

Proof: The following picture is to illustrate the proof of the Jordan curve theorem which follows.


Now let $J$ be a Jordan curve. Thus $J=\gamma\left(S^{1}\right)$ where $S^{1}$ is the unit circle. Since $J$ is compact, there exists $a \equiv \inf \{x:(x, y) \in J\}$ and $b \equiv \sup \{x:(x, y) \in J\}>a$. Denote these points on $J$ by $\left(a, l^{\prime}\right) \equiv \mathbf{a},\left(b, r^{\prime}\right) \equiv \mathbf{b}$. Thus the first is a point on $J$ farthest to the left and the second a point farthest to the right. There are also top and bottom points $\hat{\mathbf{d}}$ and $\hat{\mathbf{c}}$ respectively with second components $\hat{d}$ and $\hat{c}$ respectively. Let $c<\hat{c}, d>\hat{d}$ as shown. Thus $J$ is contained in $[a, b] \times[c, d]$ as shown. Let $C$ be the boundary of this box. Let $L$ denote the vertical $x=\frac{a+b}{2}$ from d to $\mathbf{c}$. There are two Jordan arcs whose union is $J$ joined at the points $\mathbf{a}, \mathbf{b}$. By Proposition 8.7.20, the line $L$ intersects each of these Jordan arcs. Let $J_{t}$ be the first of these arcs intersected by this line $L$ at $\mathbf{r}$ in moving from top to bottom and let $J_{b}$ be the other one. Let $\mathbf{q}$ be the smallest point of $L \cap J$. I claim that $\mathbf{q}$ is in $J_{b}$. If not, then $\mathbf{q}$ is in $J_{t}$ and is neither $\mathbf{a}$ nor $\mathbf{b}$ and neither is $\mathbf{r}$ since both are on $L$. Thus the part of $J_{t}$ which goes from $\mathbf{r}$ to $\mathbf{q}$ does not include the endpoints of $J_{t}, \mathbf{a}, \mathbf{b}$. Then Proposition 8.7.20 applied to $[a, b] \times[q, r]$ where $q, r$ are the second components of $\mathbf{q}, \mathbf{r}$ respectively, would imply that this part of $J_{t}$ between $\mathbf{r}, \mathbf{q}$ must intersect $J_{b}$ which is impossible because neither a nor $\mathbf{b}$ are on this part of $J_{t}$. Such an intersection would mean $J$ is not a simple closed curve.

Let $\mathbf{p}$ be the top point of $L \cap J_{b}$ which must be below $\mathbf{r}$. Let $\mathbf{l}$ be the bottom point of $J_{t} \cap L$ which is above $\mathbf{p}$. This point must exist since otherwise there would be $\mathbf{l}_{n} \in J_{t}, \mathbf{l}_{n} \rightarrow \mathbf{p}$ so $\mathbf{p} \in J_{t} \cap J_{b}=\{\mathbf{a}, \mathbf{b}\}$ which is impossible on $L$. Also let $\mathbf{q}$ be the last point of $J$ encountered. Thus $\mathbf{q}$ is on $J_{b}$ as mentioned earlier. Let $\mathbf{z}$ be the midpoint of $\mathbf{l}$ and $\mathbf{p}$. Then $\mathbf{z} \notin J_{t}$ and $\mathbf{z} \notin J_{b}$ so $\mathbf{z}$ is in some component of $J^{C}$.

I want to argue that this component which contains $\mathbf{z}$ is a bounded component. When this is done, I will show that it is the only bounded component.

If $\mathbf{z}$ is in the unbounded component of $J^{C}$, then there exists a continuous curve $\eta$ from $\mathbf{z}$ to a point $\mathbf{w}$ on $C$ which does not intersect $J$. Letting $l$ be the straight line between a and $\mathbf{b}$, if $\mathbf{w}$ is above $l$ you could modify $\eta$ by placing $\mathbf{w}$ on the top line of $C$ and if $\mathbf{w}$ is below $l$ we could modify $\eta$ to place $\mathbf{w}$ on the bottom line of $C$. This involves going from $\mathbf{z}$ to the first point of the bounding box and then out to one of $L_{r t}, L_{r b}, L_{l b}, L_{l t}$ and along one of those slanted lines to a point on the top or bottom of $C$. (The reason for these slanted lines is that
there could be points of $J$ other than $\mathbf{a}, \mathbf{b}$ on the vertical sides of $C$.) In the first case where $\mathbf{w}$ is above $l$, go from $\mathbf{c}$ to $\mathbf{q}$ along $L$ and then along $J_{b}$ to $\mathbf{p}$ then to $\mathbf{z}$ and along $\eta$ to the top line of the box. (Note that neither $\mathbf{q}$ nor $\mathbf{p}$ can be in $\{\mathbf{a}, \mathbf{b}\}$ so this cannot intersect $J_{t}$.) This curve does not intersect $J_{t}$ in contradiction to Proposition 8.7.20 since $J_{t}$ has points with first components $a, b$ on the extreme left and right and this new curve through $\mathbf{z}$ has some second components equal to $c, d$. If $\mathbf{w}$ is below $l$, use the curve along $L$ from $\mathbf{d}$ to $\mathbf{l}$ to $\mathbf{z}$ and then along $\eta$ to the bottom line of $C$. This fails to intersect $J_{b}$ and so contradicts Proposition 8.7.20 because this curve has some second components equal to $c, d$ and the curve $J_{b}$ has some first components equal to $a, b$. Thus the component which contains $\mathbf{z}$ is a bounded component. Hence $J^{C}$ has at least two components.

Suppose $V$ is another bounded component and that $U$ is the bounded component just described containing $\mathbf{z}$. Thus $V$ is contained in the inside of the box $C$. Moving up on $L$ let $\mathbf{r}$ be the last point of $J$ encountered. Thus by definition, $\mathbf{r} \in J_{t}$. Moving down on $L$ from $\mathbf{r}$ let $\mathbf{q}$ be the last point of $J$ encountered. It is in $J_{b}$ as explained earlier. Now go from $\mathbf{r}$ to $\mathbf{l}$ on $J_{t}$. Neither of $\mathbf{r}, \mathbf{l}$ is an endpoint of $J_{t}$. Then go from $\mathbf{l}$ to $\mathbf{p}$ along the segment in $U$ and from $\mathbf{p}$ to $\mathbf{q}$ on $J_{b}$ avoiding the end points $\mathbf{a}, \mathbf{b}$. Including a ray from $\mathbf{r}$ pointing up and a ray from $\mathbf{q}$ pointing down, this set of points $B$ contains no points of $V$ because the segment between $\mathbf{l}, \mathbf{p}$ is in $U$. Also a and $\mathbf{b}$ are in different components of $B^{C}$. Now for $\delta$ small enough, $B(\mathbf{a}, \boldsymbol{\delta})$ and $B(\mathbf{b}, \boldsymbol{\delta})$ contain no points of $B$ and by Lemma 8.7.17, there is a point $\mathbf{a}_{1}$ of $V$ in $B(\mathbf{a}, \boldsymbol{\delta})$ and a point $\mathbf{b}_{1}$ of $V$ in $B(\mathbf{b}, \delta)$ and so $V$ fails to be connected after all.

### 8.8 Exercises

1. Let $(X, \mathscr{F}, \mu)$ be a regular measure space. For example, it could be $\mathbb{R}$ with Lebesgue measure. Why do we care about a measure space being regular? This problem will show why. Suppose that closures of balls are compact as in the case of $\mathbb{R}$.
(a) Let $\mu(E)<\infty$. By regularity, there exists $K \subseteq E \subseteq V$ where $K$ is compact and $V$ is open such that $\mu(V \backslash K)<\varepsilon$. Show there exists $W$ open such that $K \subseteq \bar{W} \subseteq V$ and $\bar{W}$ is compact. Now show there exists a function $h$ such that $h$ has values in $[0,1], h(x)=1$ for $x \in K$, and $h(x)$ equals 0 off $W$. Hint: You might consider Problem 12 on Page 154.
(b) Show that $\int\left|\mathscr{X}_{E}-h\right| d \mu<\varepsilon$
(c) Next suppose $s=\sum_{i=1}^{n} c_{i} \mathscr{X}_{E_{i}}$ is a nonnegative simple function where each $\mu\left(E_{i}\right)<\infty$. Show there exists a continuous nonnegative function $h$ which equals zero off some compact set such that $\int|s-h| d \mu<\varepsilon$
(d) Now suppose $f \geq 0$ and $f \in L^{1}(\Omega)$. Show that there exists $h \geq 0$ which is continuous and equals zero off a compact set such that

$$
\int|f-h| d \mu<\varepsilon
$$

(e) If $f \in L^{1}(\Omega)$ with complex values, show the conclusion in the above part of this problem is the same. That is, $C_{c}\left(\mathbb{R}^{p}\right)$ is dense in $L^{1}\left(\mathbb{R}^{p}\right)$.
2. Let $F$ be an increasing function defined on $\mathbb{R}$. For $f$ a continuous function having compact support in $\mathbb{R}$, consider the functional $L f \equiv \int f d F$ where here the integral signifies $\int_{a}^{b} f d F$ where $\operatorname{spt}(f) \subseteq[a, b]$ and the integral is the ordinary Rieman Stieltjes integral. For a discussion of these, see my single variable advanced calculus
book. If $\mu$ is the measure which results, show that $\mu((\alpha, \beta))=F(\beta-)-F(\alpha+)$ and $\mu((\alpha, \beta])=F(\beta+)-F(\alpha+), \mu([\alpha, \beta])=F(\beta+)-F(\alpha-)$. Here $F(x+) \equiv$ $\lim _{y \rightarrow x, y>x} F(y), F(x-) \equiv \lim _{y \rightarrow x, y<x} F(y)$ Explain why the measure $\mu$ is a regular complete measure. It is easy from Theorem 8.2.1.
3. Let $\delta_{z}(f)=f(z)$ for $f \in C_{c}(\mathbb{R})$. Describe the resulting measure for which $\delta_{z}=L$.
4. Let $L f \equiv \sum_{k=1}^{\infty} f(k)$. Show this is a positive linear functional on $C_{c}(\mathbb{R})$ and describe the resulting Radon measure.
5. Consider the two functionals $L f \equiv \int f(x) d x$ and $L_{z} f \equiv \int f(x-z) d x$ both defined on $C_{c}(\mathbb{R})$. Explain, using beginning calculus, why these functionals are the same. Explain why whenever $f$ is measurable and nonnegative,

$$
\int f(x) d m_{1}(x)=\int f(x-y) d m_{1}(x)
$$

Obtain continuity of translation of Lebesgue measure right away directly from the Riesz representation theorem. Generalize to $\mathbb{R}^{p}$.
6. Show that Lemma 8.2.10 works for metric space, not just $\mathbb{R}^{p}$.
7. If you have a nonempty open set $V$ in $\mathbb{R}^{p}$, show that there is an increasing sequence of open sets $\left\{W_{n}\right\}, \overline{W_{n}} \subseteq W_{n+1}$, and $\cup_{n} W_{n}=V$. Next show that you can also arrange to have $\overline{W_{n}}$ compact. Hint: You might consider using dist $\left(\mathbf{x}, V^{C}\right)$ and its properties.
8. Suppose $\mathbf{h}$ is continuous on an open set $U$. Using Problem 7, verify that $\mathbf{h}(U)$ is a Borel set.
9. Let $N$ be a set of measure zero with respect to Lebesgue measure. Also let $\mathbf{h}$ be a Lipschitz continuous function meaning that for some $K,\|\mathbf{h}(\mathbf{x})-\mathbf{h}(\mathbf{y})\| \leq K\|\mathbf{x}-\mathbf{y}\|$ and $\mathbf{h}$ is defined near $N$. Show that $\mathbf{h}(N)$ also has measure zero. Follow the steps and fill in needed details.
(a) Let $\varepsilon>0$. There is $V$ open such that $m_{p}(V)<\varepsilon$ and $V \supseteq N$.
(b) For each $\mathbf{x} \in N$, there is a ball $B_{\mathbf{x}}$ centered at $\mathbf{x}$ with $\hat{B}_{\mathbf{x}}$ contained in $V$. Go ahead and let the ball be taken with respect to the norm $\|\mathbf{x}\| \equiv \max \left\{\left|x_{i}\right|, i \leq p\right\}$. Thus these $B_{\mathbf{x}}$ are open cubes.
(c) You know from Problem 8 that $\mathbf{h}\left(B_{\mathbf{x}}\right)$ is measurable. Obtain countably many disjoint balls $\left\{B_{\mathbf{x}_{i}}\right\}_{i=1}^{\infty}$ such that $\left\{\hat{B}_{\mathbf{x}_{i}}\right\}$ covers $N$.
(d) Explain why $\mathbf{h}(N)$ is covered by $\left\{\mathbf{h}\left(\hat{B}_{\mathbf{x}_{i}}\right)\right\}$. Now fill in the details of the following estimate. $m_{p}(\mathbf{h}(N)) \leq \sum_{i=1}^{\infty} m_{p}\left(\mathbf{h}\left(\hat{B}_{\mathbf{x}_{i}}\right)\right)$ $\leq \sum_{i=1}^{\infty} K^{p} m_{p}\left(\hat{B}_{\mathbf{x}_{i}}\right)=\sum_{i=1}^{\infty} K^{p} 5^{p} m_{p}\left(B_{\mathbf{x}_{i}}\right)=(5 K)^{p} \varepsilon$.
(e) Now explain why this shows that $m_{p}(\mathbf{h}(N))=0$. Thus Lipschitz mappings take sets of measure zero to sets of measure zero.
10. Use this and Proposition 8.3 .2 to show that if $\mathbf{h}$ is a Lipschitz function, then if $E$ is Lebesgue measurable, so is $\mathbf{h}(E)$. Hint: This will involve completeness of the measure and Problem 9. You could first show that it suffices to assume that $E$ is contained in some ball to begin with if this would make it any easier.
11. Show that the continuous functions with compact support are dense in $L^{1}\left(\mathbb{R}^{p}\right)$ with respect to Lebesgue measure. Will this work for a general Radon measure? Hint: You should show that the simple functions are dense in $L^{1}\left(\mathbb{R}^{p}\right)$ using the norm in $L^{1}\left(\mathbb{R}^{p}\right)$ and then consider regularity of the measure.
12. Suppose $A \subseteq \mathbb{R}^{p}$ is covered by a finite collection of Balls, $\mathscr{F}$. Show that then there exists a disjoint collection of these balls, $\left\{B_{i}\right\}_{i=1}^{m}$, such that $A \subseteq \cup_{i=1}^{m} \widehat{B}_{i}$ where $\widehat{B}_{i}$ has the same center as $B_{i}$ but 3 times the radius. Hint: Since the collection of balls is finite, they can be arranged in order of decreasing radius.
13. This problem will help to understand that a certain kind of function exists. $f(x)=$ $\left\{\begin{array}{l}e^{-1 / x^{2}} \text { if } x \neq 0 \\ 0 \text { if } x=0\end{array}\right.$ show that $f$ is infinitely differentiable. Note that you only need to be concerned with what happens at 0 . There is no question elsewhere. This is a little fussy but is not too hard.
14. $\uparrow$ Let $f(x)$ be as given above. Now let $\hat{f}(x) \equiv\left\{\begin{array}{l}f(x) \text { if } x \leq 0 \\ 0 \text { if } x>0\end{array}\right.$. Show that $\hat{f}(x)$ is also infinitely differentiable. Let $r>0$ and define $g(x) \equiv \hat{f}(-(x-r)) \hat{f}(x+r)$. Show that $g$ is infinitely differentiable and vanishes for $|x| \geq r$. Let

$$
\psi(\mathbf{x})=\prod_{k=1}^{n} g\left(x_{k}\right)
$$

For $U=B(\mathbf{0}, 2 r)$ with the norm given by $\|\mathbf{x}\|=\max \left\{\left|x_{k}\right|, k \leq n\right\}$, show that $\psi \in$ $C_{c}^{\infty}(U)$.
15. $\uparrow$ Using the above problem, let $\psi \in C_{c}^{\infty}(B(\mathbf{0}, 1))$. Also let $\psi \geq 0$ as in the above problem. Show there exists $\psi \geq 0$ such that $\psi \in C_{c}^{\infty}(B(\mathbf{0}, 1))$ and $\int \psi d m_{n}=1$. Now define $\psi_{k}(\mathbf{x}) \equiv k^{n} \psi(k \mathbf{x})$. Show that $\psi_{k}$ equals zero off a compact subset of $B\left(\mathbf{0}, \frac{1}{k}\right)$ and $\int \psi_{k} d m_{n}=1$. We say that $\operatorname{spt}\left(\psi_{k}\right) \subseteq B\left(\mathbf{0}, \frac{1}{k}\right) \cdot \operatorname{spt}(f)$ is defined as the closure of the set on which $f$ is not equal to 0 . Such a sequence of functions as just defined $\left\{\psi_{k}\right\}$ where $\int \psi_{k} d m_{n}=1$ and $\psi_{k} \geq 0$ and $\operatorname{spt}\left(\psi_{k}\right) \subseteq B\left(\mathbf{0}, \frac{1}{k}\right)$ is called a mollifier.
16. $\uparrow$ It is important to be able to approximate functions with those which are infinitely differentiable. Suppose $f \in L^{1}\left(\mathbb{R}^{p}\right)$ and let $\left\{\psi_{k}\right\}$ be a mollifier as above. We define the convolution as follows.

$$
f * \psi_{k}(\mathbf{x}) \equiv \int f(\mathbf{x}-\mathbf{y}) \psi_{k}(\mathbf{y}) d m_{n}(\mathbf{y})
$$

Here the notation means that the variable of integration is $\mathbf{y}$. Show that $f * \psi_{k}(\mathbf{x})$ exists and equals $\int \psi_{k}(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d m_{n}(\mathbf{y})$. Now show using the dominated convergence theorem that $f * \psi_{k}$ is infinitely differentiable. Next show that

$$
\lim _{k \rightarrow \infty} \int\left|f(\mathbf{x})-f * \psi_{k}(\mathbf{x})\right| d m_{n}=0
$$

Thus, in terms of being close in $L^{1}\left(\mathbb{R}^{p}\right)$, every function in $L^{1}\left(\mathbb{R}^{p}\right)$ is close to one which is infinitely differentiable.
17. $\uparrow$ From Problem 1 above and $f \in L^{1}\left(\mathbb{R}^{p}\right)$, there exists $h \in C_{c}\left(\mathbb{R}^{p}\right)$, continuous and $\operatorname{spt}(h)$ a compact set, such that $\int|f-h| d m_{n}<\varepsilon$. Now consider $h * \psi_{k}$. Show that this function is in $C_{c}^{\infty}\left(\operatorname{spt}(h)+B\left(\mathbf{0}, \frac{2}{k}\right)\right)$. The notation means you start with the compact set $\operatorname{spt}(h)$ and fatten it up by adding the set $B\left(\mathbf{0}, \frac{1}{k}\right)$. It means $\mathbf{x}+\mathbf{y}$ such that $\mathbf{x} \in \operatorname{spt}(h)$ and $\mathbf{y} \in B\left(\mathbf{0}, \frac{1}{k}\right)$. Show the following. For all $k$ large enough,

$$
\int\left|f-h * \psi_{k}\right| d m_{n}<\varepsilon
$$

so one can approximate with a function which is infinitely differentiable and also has compact support. Also show that $h * \psi_{k}$ converges uniformly to $h$. If $h$ is a function in $C^{k}\left(\mathbb{R}^{k}\right)$ in addition to being continuous with compact support, show that for each $|\alpha| \leq k, D^{\alpha}\left(h * \psi_{k}\right) \rightarrow D^{\alpha} h$ uniformly. Hint: If you do this for a single partial derivative, you will see how it works in general.
18. $\uparrow$ Let $f \in L^{1}(\mathbb{R})$. Show that $\lim _{k \rightarrow \infty} \int f(x) \sin (k x) d m=0$ Hint: Use the result of the above problem to obtain $g \in C_{c}^{\infty}(\mathbb{R})$, continuous and zero off a compact set, such that $\int|f-g| d m<\varepsilon$. Then show that

$$
\lim _{k \rightarrow \infty} \int g(x) \sin (k x) d m(x)=0
$$

You can do this by integration by parts. Then consider this.

$$
\begin{array}{r}
\left|\int f(x) \sin (k x) d m\right|= \\
+\left|\int f(x) \sin (k x) d m-\int g(x) \sin (k x) d m\right| \\
\leq \int|f-g| d m+\left|\int g(x) \sin (k x) d m\right|
\end{array}
$$

This is the celebrated Riemann Lebesgue lemma which is the basis for all theorems about pointwise convergence of Fourier series.
19. As another application, here is a very important result. Suppose $f \in L^{1}\left(\mathbb{R}^{p}\right)$ and for every $\psi \in C_{c}^{\infty}\left(\mathbb{R}^{p}\right), \int f \psi d m_{n}=0$. Show that then it follows that $f(\mathbf{x})=0$ for a.e. $\mathbf{x}$. That is, there is a set of measure zero such that off this set $f$ equals 0 . Hint: What you can do is to let $E$ be a measurable which is bounded and let $K_{k} \subseteq E \subseteq V_{k}$ where $m_{n}\left(V_{k} \backslash K_{k}\right)<2^{-k}$. Here $K_{k}$ is compact and $V_{k}$ is open. By an earlier exercise, Problem 12 on Page 154, there exists a function $\phi_{k}$ which is continuous, has values in $[0,1]$ equals 1 on $K_{k}$ and $\operatorname{spt}\left(\phi_{k}\right) \subseteq V$. To get this last part, show there exists $W_{k}$ open such that $\overline{W_{k}} \subseteq V_{k}$ and $W_{k}$ contains $K_{k}$. Then you use the problem to get $\operatorname{spt}\left(\phi_{k}\right) \subseteq \overline{W_{k}}$. Now you form $\eta_{k}=\phi_{k} * \psi_{l}$ where $\left\{\psi_{l}\right\}$ is a mollifier. Show that for $l$ large enough, $\eta_{k}$ has values in $[0,1], \operatorname{spt}\left(\eta_{k}\right) \subseteq V_{k}$ and $\eta_{k} \in C_{c}^{\infty}\left(V_{k}\right)$. Now explain why $\eta_{k} \rightarrow \mathscr{X}_{E}$ off a set of measure zero. Then

$$
\begin{aligned}
\left|\int f \mathscr{X}_{E} d m_{n}\right| & =\left|\int f\left(\mathscr{X}_{E}-\eta_{k}\right) d m_{n}\right|+\left|\int f \eta_{k} d m_{n}\right| \\
& =\left|\int f\left(\mathscr{X}_{E}-\eta_{k}\right) d m_{n}\right|
\end{aligned}
$$

Now explain why this converges to 0 on the right. This will involve the dominated convergence theorem. Conclude that $\int f \mathscr{X}_{E} d m_{n}=0$ for every bounded measurable set $E$. Show that this implies that $\int f \mathscr{X}_{E} d m_{n}=0$ for every measurable $E$. Explain why this requires $f=0$ a.e. The result which gets used over and over in all of this is the dominated convergence theorem.
20. Let $F(x)=\left(\int_{0}^{x} e^{-t^{2}} d t\right)^{2}$, so

$$
F^{\prime}(x)=2 e^{-x^{2}}\left(\int_{0}^{x} e^{-t^{2}}\right)=2 x e^{-x^{2}}\left(\int_{0}^{1} e^{-(u x)^{2}} d u\right)
$$

Now integrate by parts to get the following.

$$
F(x)=e(x)+1+\int_{0}^{x} e^{-t^{2}} \int_{0}^{1}\left(-2 t u^{2} e^{-t^{2} u^{2}}\right) d u d t, \lim _{x \rightarrow \infty} e(x)=0
$$

Now change the order of integration in this integral to get

$$
F(x)=e(x)+1-\int_{0}^{1} u^{2} \int_{0}^{x} 2 t e^{-t^{2}\left(1+u^{2}\right)} d t d u
$$

Modifying $e(x)$ as needed, obtain

$$
F(x)=e(x)+1-\int_{0}^{1} \frac{u^{2}}{1+u^{2}}=e(x)+\int_{0}^{1} \frac{1}{1+u^{2}} d u=e(x)+\frac{\pi}{4}
$$

Show $\int_{0}^{\infty} e^{-t^{2}} d t=\frac{\sqrt{\pi}}{2}$. Justify all the steps in the above using whatever theorems are applicable.
21. The Dini derivates are as follows. In these formulas, $f$ is a real valued function defined on $\mathbb{R}$ and $\Delta f(+)$ will be $\frac{f(x+h)-f(x)}{h}$ for $h>0$ and $\Delta f(-)$ will be $\frac{f(x)-f(x-h)}{h}$ for $h>0$.

$$
\begin{aligned}
D^{+} f(x) & \equiv \lim \sup _{h \rightarrow 0+} \Delta f(+), D_{+} f(x) \equiv \lim \inf _{h \rightarrow 0+} \Delta f(+) \\
D^{-} f(x) & \equiv \lim \sup _{h \rightarrow 0+} \Delta f(-), D_{-} f(x) \equiv \lim \inf _{h \rightarrow 0+} \Delta f(-)
\end{aligned}
$$

Thus when these are all equal, the function has a derivative. Now suppose $f$ is an increasing function. Let

$$
N_{a b}=\left\{x: D^{+} f(x)>b>a>D_{+} f(x)\right\}, a \geq 0
$$

Let $V$ be an open set which contains $N_{a b} \cap(-r, r) \equiv N_{a b}^{r}$ such that

$$
m\left(V \backslash\left(N_{a b} \cap(-r, r)\right)\right)<\varepsilon
$$

Then explain why there exist disjoint intervals $\left[a_{i}, b_{i}\right]$ such that

$$
m\left(N_{a b}^{r} \backslash \cup_{i}\left[a_{i}, b_{i}\right]\right)=m\left(N_{a b}^{r} \backslash \cup_{i}\left(a_{i}, b_{i}\right)\right)=0
$$

and

$$
f\left(b_{i}\right)-f\left(a_{i}\right) \leq a m\left(a_{i}, b_{i}\right)
$$

each interval being contained in $V \cap(-r, r)$. Thus you have

$$
m\left(N_{a b}^{r}\right)=m\left(\cup_{i} N_{a b}^{r} \cap\left(a_{i}, b_{i}\right)\right)
$$

Next show there exist disjoint intervals $\left(a_{j}, b_{j}\right)$ such that each of these is contained in some $\left(a_{i}, b_{i}\right)$, the $\left(a_{j}, b_{j}\right)$ are disjoint, $f\left(b_{j}\right)-f\left(a_{j}\right) \geq b m\left(a_{j}, b_{j}\right)$, and

$$
\sum_{j} m\left(N_{a b}^{r} \cap\left(a_{j}, b_{j}\right)\right)=m\left(N_{a b}^{r}\right)
$$

. Then you have the following thanks to the fact that $f$ is increasing.

$$
\begin{aligned}
a\left(m\left(N_{a b}^{r}\right)+\varepsilon\right) & >a m(V) \geq a \sum_{i}\left(b_{i}-a_{i}\right)>\sum_{i} f\left(b_{i}\right)-f\left(a_{i}\right) \\
& \geq \sum_{j} f\left(b_{j}\right)-f\left(a_{j}\right) \geq b \sum_{j} b_{j}-a_{j} \\
& \geq b \sum_{j} m\left(N_{a b}^{r} \cap\left(a_{j}, b_{j}\right)\right)=b m\left(N_{a b}^{r}\right)
\end{aligned}
$$

and since $\varepsilon>0$,

$$
\operatorname{am}\left(N_{a b}^{r}\right) \geq b m\left(N_{a b}^{r}\right)
$$

showing that $m\left(N_{a b}^{r}\right)=0$. This is for any $r$ and so $m\left(N_{a b}\right)=0$. Thus the derivative from the right exists for a.e. $x$ by taking the complement of the union of the $N_{a b}$ for $a, b$ nonnegative rational numbers. Now do the same thing to show that the derivative from the left exists a.e. and finally, show that $D_{-} f(x)=D^{+} f(x)$ for almost a.e. $x$. Off the union of these three exceptional sets of measure zero all the derivates are the same and so the derivative of $f$ exists a.e. In other words, an increasing function has a derivative a.e.
22. This problem is on Eggoroff's theorem. This was presented earlier in the book. The idea is for you to review this by going through a proof. Suppose you have a measure space $(\Omega, \mathscr{F}, \mu)$ where $\mu(\Omega)<\infty$. Also suppose that $\left\{f_{k}\right\}$ is a sequence of measurable, complex valued functions which converge to $f$ pointwise. Then Eggoroff's theorem says that for any $\varepsilon>0$ there is a set $N$ with $\mu(N)<\varepsilon$ and convergence is uniform on $N^{C}$.
(a) Define $E_{m k} \equiv \cup_{r=m}^{\infty}\left\{\omega:\left|f(\omega)-f_{r}(\omega)\right|>\frac{1}{k}\right\}$. Show $E_{m k} \supseteq E_{(m+1) k}$ for all $m$ and that $\cap_{m} E_{m k}=\emptyset$
(b) Show that there exists $m(k)$ such that $\mu\left(E_{m(k) k}\right)<\varepsilon 2^{-k}$.
(c) Let $N \equiv \cup_{k=1}^{\infty} E_{m(k) k}$. Explain why $\mu(N)<\varepsilon$ and that for all $\omega \notin N^{C}$, if $r>$ $m(k)$, then $\left|f(\omega)-f_{r}(\omega)\right| \leq \frac{1}{k}$. Thus uniform convergence takes place on $N^{C}$.
23. Suppose you have a sequence $\left\{f_{n}\right\}$ which converges uniformly on each of finitely many sets $A_{1}, \cdots, A_{n}$. Why does the sequence converge uniformly on $\cup_{i=1}^{n} A_{i}$ ?
24. $\uparrow$ Now suppose you have $\mu$ is a finite Radon measure on $\mathbb{R}^{p}$. For example, you could have Lebesgue measure. Suppose you have $f$ has nonnegative real values for all $\mathbf{x}$ and is measurable. Then Lusin's theorem says that for every $\varepsilon>0$, there exists an open set $V$ with measure less than $\varepsilon$ and a continuous function defined on $\mathbb{R}^{p}$ such that $f(\mathbf{x})=g(\mathbf{x})$ for all $\mathbf{x} \notin V$. That is, off an open set of small measure, the function is equal to a continuous function.
(a) By Lemma 8.2.10, there exists a sequence $\left\{f_{n}\right\} \subseteq C_{c}(\Omega)$ which converges to $f$ off a set $N$ of measure zero. Use Eggoroff's theorem to enlarge $N$ to $\hat{N}$ such that $\mu(\hat{N})<\frac{\varepsilon}{2}$ and convergence is uniform off $\hat{N}$.
(b) Next use outer regularity to obtain open $V \supseteq \hat{N}$ having measure less than $\varepsilon$. Thus $\left\{f_{n}\right\}$ converges uniformly on $V^{C}$. Therefore, that which it converges to is continuous on $V^{C}$ a closed set. Now use the Tietze extension theorem.
25. Let $A: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be continuous and let $\mathbf{f} \in \mathbb{R}^{n}$. Also let $(\cdot, \cdot)$ denote the standard inner product in $\mathbb{R}^{n}$. Letting $K$ be a closed and bounded and convex set, show that there exists $\mathbf{x} \in K$ such that for all $\mathbf{y} \in K,(\mathbf{f}-A \mathbf{x}, \mathbf{y}-\mathbf{x}) \leq 0$. Hint: Show that this is the same as saying $P(\mathbf{f}-A \mathbf{x}+\mathbf{x})=\mathbf{x}$ for some $\mathbf{x} \in K$ where here $P$ is the projection map discussed above in the problems beginning with Problem 22 on Page 77. Now use the Brouwer fixed point theorem. This little observation is called Browder's lemma. It is a fundamental result in nonlinear analysis.
26. $\uparrow$ In the above problem, suppose that you have a coercivity result which is

$$
\lim _{\|\mathbf{x}\| \rightarrow \infty} \frac{(A \mathbf{x}, \mathbf{x})}{\|\mathbf{x}\|}=\infty
$$

Show that if you have this, then you don't need to assume the convex closed set is bounded. In case $K=\mathbb{R}^{n}$, and this coercivity holds, show that $A$ maps onto $\mathbb{R}^{n}$.
27. Suppose $\mathbf{f}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ is one to one and continuous. Suppose

$$
\lim _{\|\mathbf{x}\| \rightarrow \infty}\|\mathbf{f}(\mathbf{x})\|=\infty
$$

Show that $\mathbf{f}$ must also be onto. Hint: By invariance of domain, $\mathbf{f}\left(\mathbb{R}^{n}\right)$ is open. Show that $\mathbb{R}^{n} \backslash \mathbf{f}\left(\mathbb{R}^{n}\right)$ is also open. Since $\mathbf{f}\left(\mathbb{R}^{n}\right)$ is connected (by theorems on connected sets), one of these open sets is empty.
28. Explain why, if $J$ is a simple closed curve, it has empty interior. Hint: If $J$ contains a ball, then would some point of $J$ fail to be a limit point of the components of $J^{C}$ ?
29. A simple square curve is one which is a simple curve and consists of finitely many horizontal and vertical segments arranged end to end. If $J$ is a simple closed curve and $U_{i}$ is its inside, then a simple square curve contained in $U_{i}$ does not separate $U_{i}$. Suppose not. Then let $C$ be a simple square curve which does separate $U_{i}$ and suppose every curve from $\mathbf{x}$ to $\mathbf{y}$ must intersect $C$ for $\mathbf{x}, \mathbf{y}$ not on $C$. Let $\delta$ be a positive number less than $1 / 4$ the length of any of the horizontal and vertical segments and also let $\delta<\frac{1}{4} \min (\operatorname{dist}(\mathbf{x}, C), \operatorname{dist}(\mathbf{y}, C), \operatorname{dist}(C, J))$ and also less than $1 / 4$ the distance between the end points of $C$. Where for convenience, $\|(x, y)\| \equiv \max (|x|,|y|)$. Now consider $C+\overline{B(\mathbf{0}, \delta)} \equiv\{\mathbf{u}+\mathbf{v}: \mathbf{u} \in C, \mathbf{v} \in \overline{B(\mathbf{0}, \boldsymbol{\delta})}\}$. The boundary $\hat{J}$ is a simple closed curve which contains $C$ inside its bounded component with $\mathbf{x}$ and $\mathbf{y}$ in the unbounded component. For illustration, see the following picture.


Then if $B$ is a curve from $\mathbf{x}$ to $\mathbf{y}$, it must, by assumption, intersect $C$ and so it must intersect $\hat{J}$. Let a be the first point of intersection of $\hat{J}$ and $\mathbf{b}$ its last. Then the curve xaby where ab goes along $\hat{J}$ avoids $C$ and so $C$ does not separate $U_{i}$ after all. Explain all this.
30. Let $J$ be a simple closed curve in the plane with the interior component $U_{i}$. Let $\mathbf{z}$ be a point on $J$ and let $\mathbf{x}$ be some point of $U_{i}$. Show there exists a simple curve joining $\mathbf{z}$ and $\mathbf{x}$. Hint: Fill in the details. Let $\mathbf{x}_{n} \rightarrow \mathbf{z}$ where $\mathbf{x}_{n} \in U_{i}$ the $\mathbf{x}_{k}$ being distinct points, $k=0,1,2,3 \ldots$ and $\mathbf{x}_{0}=\mathbf{x}$. Let $a_{n}$ denote a strictly increasing sequence of positive numbers increasing to 1 with $a_{0}=0$. Then let $\gamma_{n}:\left[a_{n-1}, a_{n}\right] \rightarrow U_{i}$ such that $\gamma_{n}\left(a_{n-1}\right)=\mathbf{x}_{n-1}, \gamma_{n}\left(a_{n}\right)=\mathbf{x}_{n}$ and $\gamma_{n}^{*} \cap \gamma_{k}^{*}=\emptyset$ if $|n-k|>1$ while $\gamma_{n}^{*} \cap \gamma_{n-1}^{*}=\mathbf{x}_{n-1}$. Let $\gamma(t) \equiv \gamma_{n}(t)$ for $t \in\left[a_{n-1}, a_{n}\right]$ and $\gamma(1) \equiv \mathbf{z}$. You could let these $\gamma_{n}$ be square curves and use the result of the above problem.

## Chapter 9

## Basic Function Spaces

In this chapter is an introduction to some of the most important vector spaces of functions. First of all, recall from linear algebra that if you have any nonempty set $S$ and $V$ is the set of all functions defined on $S$ having values in $\mathbb{F}$ or more generally some vector space, then defining

$$
\begin{gathered}
(f+g)(x) \equiv f(x)+g(x) \\
(\alpha g)(x) \equiv \alpha g(x)
\end{gathered}
$$

this defines vector addition and scalar multiplication of functions. You should check that all the axioms of a vector space hold for this situation. Note also that the usual situation in linear algebra $\mathbb{F}^{n}$ where vectors are ordered lists of numbers is a special case. There you are considering functions mapping $\{1, \cdots, n\}$ to $\mathbb{F}$ so the set $S$ consists of the first $n$ natural numbers. This was a finite dimensional vector space, but if $S$ is the unit interval and $V$ consists of functions defined on $S$, then this will not be finite dimensional because for each $x \in S$, you could consider $f_{x}(x) \equiv 1$ and $f_{x}(y)=0$ for $y \neq x$ and you would have infinitely many vectors such that every finite subset of them is linearly independent.

There are two kinds of function spaces discussed here, the space of bounded continuous functions and the $L^{p}$ spaces. First I will consider the space of bounded continuous functions.

### 9.1 Bounded Continuous Functions

As before, $\mathbb{F}$ will denote either $\mathbb{R}$ or $\mathbb{C}$.
Definition 9.1.1 Let $T$ be a subset of some $\mathbb{F}^{m}$, possibly all of $\mathbb{F}^{m}$. Let $B C\left(T ; \mathbb{F}^{n}\right)$ denote the bounded continuous functions defined on $T .{ }^{1}$ Then this is a vector space (linear space) with respect to the usual operations of addition and scalar multiplication of functions. Also, define a norm as follows:

$$
\|\mathbf{f}\| \equiv \sup _{t \in T}|\mathbf{f}(t)|<\infty
$$

This is a norm because it satisfies the axioms of a norm which are as follows:

$$
\begin{gathered}
\|\mathbf{f}+\mathbf{g}\| \leq\|\mathbf{f}\|+\|\mathbf{g}\|,\|\alpha \mathbf{f}\|=|\alpha|\|\mathbf{f}\| \\
\|\mathbf{f}\| \geq 0 \text { and equals } 0 \text { if and only if } \mathbf{f}=\mathbf{0}
\end{gathered}
$$

A sequence $\left\{\mathbf{f}_{n}\right\}$ in $B C\left(T ; \mathbb{F}^{n}\right)$ is a Cauchy sequence iffor every $\varepsilon>0$ there exists $M_{\varepsilon}$ such that if $m, n \geq M_{\mathcal{E}}$, then

$$
\left\|\mathbf{f}_{n}-\mathbf{f}_{m}\right\|<\varepsilon
$$

Such a normed linear space is called complete if every Cauchy sequence converges. Such a complete normed linear space is called a Banach space. This norm is often denoted as $\|\cdot\|_{\infty}$.

I am letting $T$ be a subset of $\mathbb{F}^{n}$ just to keep things in familiar territory. $T$ can be an arbitrary metric space or even a general topological space.

Now consider the general case where $T$ is just some set.

[^4]Lemma 9.1.2 The collection of functions $B C\left(T ; \mathbb{F}^{n}\right)$ is a normed linear space (vector space) and it is also complete which means by definition that every Cauchy sequence converges.

Proof: Showing that this is a normed linear space is entirely similar to the argument in the above for $\gamma=0$ and $T=[a, b]$.

Let $\left\{\mathbf{f}_{n}\right\}$ be a Cauchy sequence. Then for each $\mathbf{t} \in T,\left\{\mathbf{f}_{n}(\mathbf{t})\right\}$ is a Cauchy sequence in $\mathbb{F}^{n}$. By completeness of $\mathbb{F}^{n}$ this converges to some $\mathbf{g}(\mathbf{t}) \in \mathbb{F}^{n}$. We need to verify that $\left\|\mathbf{g}-\mathbf{f}_{n}\right\| \rightarrow 0$ and that $\mathbf{g} \in B C\left(T ; \mathbb{F}^{n}\right)$. Let $\varepsilon>0$ be given. There exists $M_{\varepsilon}$ such that if $m, n \geq M_{\mathcal{\varepsilon}}$, then $\left\|\mathbf{f}_{n}-\mathbf{f}_{m}\right\|<\frac{\varepsilon}{4}$. Let $n>M_{\mathcal{\varepsilon}}$. By Lemma 1.11 .2 which says you can switch supremums,

$$
\begin{align*}
\sup _{\mathbf{t} \in T}\left|\mathbf{g}(\mathbf{t})-\mathbf{f}_{n}(\mathbf{t})\right| & \leq \sup _{\mathbf{t} \in T} \sup _{k \geq M_{\varepsilon}}\left|\mathbf{f}_{k}(\mathbf{t})-\mathbf{f}_{n}(\mathbf{t})\right| \\
& =\sup _{k \geq M_{\varepsilon}} \sup _{\mathbf{t} \in T}\left|\mathbf{f}_{k}(\mathbf{t})-\mathbf{f}_{n}(\mathbf{t})\right|=\sup _{k \geq M_{\varepsilon}}\left\|\mathbf{f}_{k}-\mathbf{f}_{n}\right\| \leq \frac{\varepsilon}{4} \tag{*}
\end{align*}
$$

Therefore,

$$
\sup _{\mathbf{t} \in T}\left(|\mathbf{g}(\mathbf{t})|-\left|\mathbf{f}_{n}(\mathbf{t})\right|\right) \leq \sup _{\mathbf{t} \in T}\left|\mathbf{g}(\mathbf{t})-\mathbf{f}_{n}(\mathbf{t})\right| \leq \frac{\varepsilon}{4}
$$

Hence

$$
\begin{gathered}
\frac{\varepsilon}{4} \geq \sup _{\mathbf{t} \in T}\left(|\mathbf{g}(\mathbf{t})|-\left|\mathbf{f}_{n}(\mathbf{t})\right|\right)=\sup _{\mathbf{t} \in T}|\mathbf{g}(\mathbf{t})|-\inf _{\mathbf{t} \in T}\left|\mathbf{f}_{n}(\mathbf{t})\right| \geq \sup _{\mathbf{t} \in T}|\mathbf{g}(\mathbf{t})|-\left\|\mathbf{f}_{n}\right\| \\
\sup _{\mathbf{t} \in T}|\mathbf{g}(\mathbf{t})| \leq \frac{\varepsilon}{4}+\left\|\mathbf{f}_{n}\right\|<\infty
\end{gathered}
$$

so in fact $\mathbf{g}$ is bounded. Now by the fact that $\mathbf{f}_{n}$ is continuous, there exists $\delta>0$ such that if $|\mathbf{t}-\mathbf{s}|<\delta$, then $\left|\mathbf{f}_{n}(\mathbf{t})-\mathbf{f}_{n}(\mathbf{s})\right|<\frac{\varepsilon}{3}$. It follows that

$$
|\mathbf{g}(\mathbf{t})-\mathbf{g}(\mathbf{s})| \leq\left|\mathbf{g}(\mathbf{t})-\mathbf{f}_{n}(\mathbf{t})\right|+\left|\mathbf{f}_{n}(\mathbf{t})-\mathbf{f}_{n}(\mathbf{s})\right|+\left|\mathbf{f}_{n}(\mathbf{s})-\mathbf{g}(\mathbf{s})\right| \leq \frac{\varepsilon}{4}+\frac{\varepsilon}{3}+\frac{\varepsilon}{4}<\varepsilon
$$

Therefore, $\mathbf{g}$ is continuous at $\mathbf{t}$. Since $\mathbf{t}$ is arbitrary, this shows that $\mathbf{g}$ is continuous on $T$. Thus $\mathbf{g} \in B C\left(T ; \mathbb{F}^{n}\right)$. By $*,\left\|\mathbf{f}_{n}-\mathbf{g}\right\|<\varepsilon$ when $n$ is large enough so $\lim _{n \rightarrow \infty}\left\|\mathbf{f}_{n}-\mathbf{g}\right\|=0$.

Definition 9.1.3 When $\lim _{n \rightarrow \infty}\left\|\mathbf{f}_{n}-\mathbf{f}\right\|=0$, we say that $\mathbf{f}_{n}$ converges uniformly to $\mathbf{f}$ and speak of uniform convergence. This norm is also called the uniform norm.

Note that uniform convergence of continuous functions imparts continuity to the limit function. This is not true of pointwise convergence, that the sequence converges for each $t$, as can be seen by consideration of $f_{n}(t)=t^{n}$ for $t \in[0,1]$. The limit function is discontinuous on this interval and is 0 on $[0,1)$ and 1 at 1 .

Now here is a major theorem called the Banach fixed point theorem. This theorem lives on complete normed linear spaces, more generally on complete metric spaces.

Theorem 9.1.4 Let $(X,\|\cdot\|)$ be a complete (Cauchy sequences converge.) normed linear space and let $F: X \rightarrow X$ be a contraction map. That is,

$$
\|F x-F y\| \leq r\|x-y\|, \quad 0 \leq r<0
$$

Then $F$ has a unique fixed point, that is a point $x \in X$ such that $F x=x$. In addition to this, if $\left\|F x_{0}-x_{0}\right\|<R\left(\underline{(1-r) \text { and } F}\right.$ is only defined on $\overline{B\left(x_{0}, R\right)}$ then $F$ has a unique fixed point in this ball. Here $\overline{B\left(x_{0}, R\right)}$ signifies the set of all $x$ such that $\left\|x-x_{0}\right\| \leq R$. Also, the sequence $\left\{F^{n} x_{0}\right\}$ converges.

Proof: Pick any $x_{0} \in X$. Consider the sequence $\left\{F^{n} x_{0}\right\}$. I will argue that this is a Cauchy sequence. To see this, suppose $n, m \geq M$ with $n>m$ and consider the following which comes from the triangle inequality for the norm, $\|x+y\| \leq\|x\|+\|y\|$.

$$
\left\|F^{n} x_{0}-F^{m} x_{0}\right\| \leq \sum_{k=m}^{n-1}\left\|F^{k+1} x_{0}-F^{k} x_{0}\right\|
$$

Now $\left\|F^{k+1} x_{0}-F^{k} x_{0}\right\| \leq$

$$
r\left\|F^{k} x_{0}-F^{k-1} x_{0}\right\| \leq r^{2}\left\|F^{k-1} x_{0}-F^{k-2} x_{0}\right\| \cdots \leq r^{k}\left\|F x_{0}-x_{0}\right\| .
$$

Using this in the above, $\left\|F^{n} x_{0}-F^{m} x_{0}\right\| \leq$

$$
\begin{equation*}
\sum_{k=m}^{n-1}\left\|F^{k+1} x_{0}-F^{k} x_{0}\right\| \leq \sum_{k=m}^{n-1} r^{k}\left\|F x_{0}-x_{0}\right\| \leq \frac{r^{m}}{1-r}\left\|F x_{0}-x_{0}\right\| \tag{9.1}
\end{equation*}
$$

since $r<1$, this is a Cauchy sequence. Hence it converges to some $x$. Therefore,

$$
x=\lim _{n \rightarrow \infty} F^{n} x_{0}=\lim _{n \rightarrow \infty} F^{n+1} x_{0}=F \lim _{n \rightarrow \infty} F^{n} x_{0}=F x
$$

The third equality is a consequence of the following consideration. If $z_{n} \rightarrow z$, then

$$
\left\|F z_{n}-F z\right\| \leq r\left\|z_{n}-z\right\|
$$

so also $F z_{n} \rightarrow F z$. In the above, $F^{n} x_{0}$ plays the role of $z_{n}$ and its limit plays the role of $z$.
The fixed point is unique because if you had two of them, $x, \hat{x}$, then

$$
\|x-\hat{x}\|=\|F x-F \hat{x}\| \leq r\|x-\hat{x}\|
$$

and so $x=\hat{x}$.
In the second case, let $m=0$ in 9.1 and you get the estimate

$$
\left\|F^{n} x_{0}-x_{0}\right\| \leq \frac{1}{1-r}\left\|F x_{0}-x_{0}\right\|<R
$$

It is still the case that the sequence $\left\{F^{n} x_{0}\right\}$ is a Cauchy sequence and must therefore converge to some $x \in \overline{B\left(x_{0}, R\right)}$ which is a fixed point as before. The fixed point is unique because of the same argument as before.

Now there is another norm which works just as well in the case where $T \equiv[a, b]$, an interval. This is described in the following definition.

Definition 9.1.5 For $\mathbf{f} \in B C\left([a, b] ; \mathbb{F}^{n}\right)$, let $c \in[a, b], \gamma$ a real number. Then

$$
\|\mathbf{f}\|_{\gamma} \equiv \sup _{t \in[a, b]}\left|\mathbf{f}(t) e^{-|\gamma(t-c)|}\right|
$$

Then this is a norm. The above Definition 9.1.1 corresponds to $\gamma=0$.

Lemma 9.1.6 $\|\cdot\|_{\gamma}$ is a norm for $B C\left([a, b] ; \mathbb{F}^{n}\right)$ and $B C\left([a, b] ; \mathbb{F}^{n}\right)$ is a complete normed linear space. Also, a sequence is Cauchy in $\|\cdot\|_{\gamma}$ if and only if it is Cauchy in $\|\cdot\|$.

Proof: First consider the claim about $\|\cdot\|_{\gamma}$ being a norm. That $\|\cdot\|_{\gamma}$ is a norm follows directly from the definition. The claim that $B C\left([a, b] ; \mathbb{F}^{n}\right)$ is complete with respect to this norm follows from the observation that the two norms $\|\cdot\|,\|\cdot\|_{\gamma}$ are equivalent so they have the same Cauchy sequences. This follows from:

$$
\begin{aligned}
\|\mathbf{f}\| & \equiv \sup _{t \in T}|\mathbf{f}(t)|=\sup _{t \in T}\left|\mathbf{f}(t) e^{-|\gamma(t-c)|} e^{|\gamma(t-c)|}\right| \leq e^{|\gamma(b-a)|}\|\mathbf{f}\|_{\gamma} \\
& \equiv e^{|\gamma(b-a)|} \sup _{t \in T}\left|\mathbf{f}(t) e^{-|\gamma(t-c)|}\right| \leq e^{|\gamma||b-a|} \sup _{t \in T}|\mathbf{f}(t)|=e^{|\gamma||b-a|}\|\mathbf{f}\|
\end{aligned}
$$

Why do we care about complete normed linear spaces? The following is a fundamental existence theorem for ordinary differential equations. It is one of those things which, incredibly, is not presented in ordinary differential equations courses. However, this is the mathematically interesting thing. The initial value problem is to find $t \rightarrow \mathbf{x}(t)$ on $[a, b]$ such that

$$
\mathbf{x}^{\prime}(t)=\mathbf{f}(t, \mathbf{x}(t)), \quad \mathbf{x}(c)=\mathbf{x}_{0}
$$

Assuming $(t, \mathbf{x}) \rightarrow \mathbf{f}(t, \mathbf{x})$ is continuous, this is obviously equivalent to the single integral equation

$$
\mathbf{x}(t)=\mathbf{x}_{0}+\int_{c}^{t} \mathbf{f}(s, \mathbf{x}(s)) d s
$$

Indeed, if $\mathbf{x}(\cdot)$ is a solution to the initial value problem, then you can integrate and obtain the above. Conversely, if you find a solution to the above, integral equation, then you can use the fundamental theorem of calculus to differentiate and find that it is a solution to the initial value problem.

## Theorem 9.1.7 Let f satisfy the Lipschitz condition

$$
\begin{equation*}
|\mathbf{f}(t, \mathbf{x})-\mathbf{f}(t, \mathbf{y})| \leq K|\mathbf{x}-\mathbf{y}| \tag{9.2}
\end{equation*}
$$

and the continuity condition

$$
\begin{equation*}
(t, \mathbf{x}) \rightarrow \mathbf{f}(t, \mathbf{x}) \text { is continuous. } \tag{9.3}
\end{equation*}
$$

Then there exists a unique solution to the initial value problem,

$$
\begin{equation*}
\mathbf{x}(t)=\mathbf{x}_{0}+\int_{c}^{t} \mathbf{f}(s, \mathbf{x}(s)) d s, c \in[a, b] \tag{9.4}
\end{equation*}
$$

on $[a, b]$.
Proof: It is necessary to find a solution to the integral equation

$$
\mathbf{x}(t)=\mathbf{x}_{0}+\int_{c}^{t} \mathbf{f}(s, \mathbf{x}(s)) d s, t \in[a, b]
$$

Let $a, b$ be finite but given and completely arbitrary, $c \in[a, b]$. Let

$$
F \mathbf{x}(t) \equiv \mathbf{x}_{0}+\int_{c}^{t} \mathbf{f}(s, \mathbf{x}(s)) d s
$$

Thus $F: B C\left([a, b], \mathbb{F}^{n}\right) \rightarrow B C\left([a, b], \mathbb{F}^{n}\right)$ Let $\|\cdot\|_{\gamma}$ be the new norm on $B C\left([a, b], \mathbb{F}^{n}\right)$.

$$
\|\mathbf{f}\|_{\gamma} \equiv \sup _{t \in[a, b]}\left|\mathbf{f}(t) e^{-\gamma(t-a)}\right|, \gamma>0
$$

Note $|\mathbf{x}(s)-\mathbf{y}(s)|=e^{|\gamma(s-a)|} e^{-|\gamma(s-a)|}|\mathbf{x}(s)-\mathbf{y}(s)| \leq e^{\gamma(s-a)}\|\mathbf{x}-\mathbf{y}\|_{\gamma}$. Then for $t \in[a, b]$,

$$
\begin{align*}
\mid F \mathbf{x}(t)- & F \mathbf{y}(t)\left|\leq\left|\int_{c}^{t}\right| \mathbf{f}(s, \mathbf{x}(s))-\mathbf{f}(s, \mathbf{y}(s))\right| d s\left|\leq\left|\int_{c}^{t} K\right| \mathbf{x}(s)-\mathbf{y}(s)\right| d s \mid \\
& \leq K\left|\int_{c}^{t} e^{\gamma(s-a)}\|\mathbf{x}-\mathbf{y}\|_{\gamma} d s\right|=K\|\mathbf{x}-\mathbf{y}\|_{\gamma}\left|\int_{c}^{t} e^{\gamma(s-a)} d s\right| \tag{*}
\end{align*}
$$

Now the right end is no more than

$$
\left.K\|\mathbf{x}-\mathbf{y}\|_{\gamma} \frac{e^{\gamma(s-a)}}{\gamma}\right|_{c} ^{t} \leq K\|\mathbf{x}-\mathbf{y}\|_{\gamma}\left(\frac{e^{\gamma(t-a)}}{\gamma}\right)
$$

and so $|F \mathbf{x}(t)-F \mathbf{y}(t)| e^{-\gamma(t-a)} \leq \frac{K}{\gamma}\|\mathbf{x}-\mathbf{y}\|_{\gamma}$ so, $\|F \mathbf{x}-F \mathbf{y}\|_{\gamma} \leq \frac{K}{\gamma}\|\mathbf{x}-\mathbf{y}\|_{\gamma}<\frac{1}{2}\|\mathbf{x}-\mathbf{y}\|_{\gamma}$ if $\gamma>2 K$ and this shows that $F$ is a contraction map on $B C\left([a, b] ; \mathbb{F}^{n}\right)$ with respect to $\|\cdot\|_{\gamma}$.

Thus there is a unique solution to the above integral equation 9.4 on $[a, b]$.
Definition 9.1.8 For the integral equation, $\mathbf{x}(t)=\mathbf{x}_{0}+\int_{c}^{t} \mathbf{f}(s, \mathbf{x}(s)) d s$ one considers the Picard iterates. These are given as follows. $\mathbf{x}_{0}(t) \equiv \mathbf{x}_{0}$ and

$$
\mathbf{x}_{n+1}(t) \equiv \mathbf{x}_{0}+\int_{c}^{t} \mathbf{f}\left(s, \mathbf{x}_{n}(s)\right) d s
$$

Thus letting $F \mathbf{x}(t) \equiv \mathbf{x}_{0}+\int_{c}^{t} \mathbf{f}(s, \mathbf{x}(s)) d s$, the Picard iterates are of the form $F \mathbf{x}_{n}=\mathbf{x}_{n+1}$.
By Theorem 9.1.4, the Picard iterates converge in $B C\left([a, b], \mathbb{F}^{n}\right)$ with respect to $\|\cdot\|_{\gamma}$ and so they also converge in $B C\left([a, b], \mathbb{F}^{n}\right)$ with respect to the usual norm $\|\cdot\|$ by Lemma 9.1.6.

### 9.2 Compactness in $C\left(K, \mathbb{R}^{n}\right)$

Let $K$ be a nonempty compact set in $\mathbb{R}^{m}$ and consider all the continuous functions defined on this set having values in $\mathbb{R}^{n}$. It is desired to give conditions which will show that a subset of $C\left(K, \mathbb{R}^{n}\right)$ is compact. First is an important observation about compact sets.

Proposition 9.2.1 Let $K$ be a nonempty compact subset of $\mathbb{R}^{m}$. Then for each $\varepsilon>0$ there is a finite set of points $\left\{\mathbf{x}_{i}\right\}_{i=1}^{r}$ such that $K \subseteq \cup_{i} B\left(\mathbf{x}_{i}, \varepsilon\right)$. This finite set of points is called an $\varepsilon$ net. If $D_{1 / k}$ is this finite set of points corresponding to $\varepsilon=1 / k$, then $\cup_{k} D_{1 / k}$ is a dense countable subset of $K$.

Proof: The last claim is obvious. Indeed, if $B(\mathbf{x}, r) \equiv\{\mathbf{y} \in K:|\mathbf{y}-\mathbf{x}|<r\}$, then consider $D_{1 / k}$ where $\frac{1}{k}<\frac{1}{3} r$. Then the given ball must contain a point of $D_{1 / k}$ since its center is within $1 / k$ of some point of $D_{k}$. Now consider the first claim about the $\varepsilon$ net. Pick $\mathbf{x}_{1} \in K$. If $B\left(\mathbf{x}_{1}, \varepsilon\right) \supseteq K$, stop. You have your $\varepsilon$ net. Otherwise pick $\mathbf{x}_{2} \notin B\left(\mathbf{x}_{1}, \varepsilon\right)$. If $K \subseteq B\left(\mathbf{x}_{1}, \varepsilon\right) \cup B\left(\mathbf{x}_{2}, \varepsilon\right)$, stop. You have found your $\varepsilon$ net. Continue this way. Eventually,
the process must stop since otherwise, you would have an infinite sequence of points with not limit point because they are all $\varepsilon$ apart. This contradicts the compactness of $K$.

Recall Lemma 9.1.2. If you consider $C\left(K, \mathbb{R}^{n}\right)$ it is automatically equal to $B C\left(K, \mathbb{R}^{n}\right)$ because of the extreme value theorem applied to $\mathbf{x} \rightarrow|\mathbf{f}(\mathbf{x})|$ for $\mathbf{x} \in K$. Therefore, the space $C\left(K, \mathbb{R}^{n}\right)$ is complete with respect to the norm defined there.
Definition 9.2.2 Let $\mathscr{A}$ be a set of functions in $C\left(K, \mathbb{R}^{n}\right)$. It is called equicontinuous if for every $\varepsilon>0$ there exists $\delta>0$ such that if $|\mathbf{x}-\mathbf{y}|<\delta$, then $|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})|<\varepsilon$ for all $\mathbf{f} \in \mathscr{A}$. In words, the functions in $\mathscr{A}$ are uniformly continuous for all $\mathbf{f}$ at once. A set $\mathscr{A} \subseteq C\left(K, \mathbb{R}^{n}\right)$ is uniformly bounded if there is a large enough positive number $M$ such that $\max \{|\mathbf{f}(\mathbf{x})|: \mathbf{x} \in K, \mathbf{f} \in \mathscr{A}\}<M$.

The significant property of an equicontinuous set of functions is the following.
Lemma 9.2.3 If $\left\{\mathbf{g}_{k}\right\}_{k=1}^{\infty}$ is equicontinuous and converges pointwise to $\mathbf{g}$ on a compact set $K$, then the sequence converges uniformly on $K$.

Proof of claim: Let $\varepsilon>0$ be given and let $\delta$ go with $\varepsilon / 4$ in the definition of equicontinuous. By compactness and Proposition 9.2.1, there are finitely many points of $K$, denoted as $\left\{\mathbf{x}_{1}, \cdots, \mathbf{x}_{s}\right\}$ such that $K \subseteq \cup_{i=1}^{s} B\left(\mathbf{x}_{i}, \boldsymbol{\delta}\right)$. There exists $N_{i}$ such that if $k, l \geq N_{i}$, then $\left|\mathbf{g}_{l}\left(\mathbf{x}_{i}\right)-\mathbf{g}_{k}\left(\mathbf{x}_{i}\right)\right|<\frac{\varepsilon}{4}$. Thus if $N \geq \max \left\{N_{i}, i=1, \cdots, s\right\}$, then for all $\mathbf{x}_{i},\left|\mathbf{g}_{l}\left(\mathbf{x}_{i}\right)-\mathbf{g}_{k}\left(\mathbf{x}_{i}\right)\right|<$ $\frac{\varepsilon}{4}$ if $k \geq N$. Then for $k, l \geq N$, and $\mathbf{x}$ arbitrary, let $\mathbf{x} \in B\left(\mathbf{x}_{i}, \boldsymbol{\delta}\right)$. Then

$$
\begin{aligned}
\left|\mathbf{g}_{l}(\mathbf{x})-\mathbf{g}_{k}(\mathbf{x})\right| & \leq\left|\mathbf{g}_{l}(\mathbf{x})-\mathbf{g}_{l}\left(\mathbf{x}_{i}\right)\right|+\left|\mathbf{g}_{l}\left(\mathbf{x}_{i}\right)-\mathbf{g}_{k}\left(\mathbf{x}_{i}\right)\right|+\left|\mathbf{g}_{k}\left(\mathbf{x}_{i}\right)-\mathbf{g}_{k}(\mathbf{x})\right| \\
& <\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}
\end{aligned}
$$

Thus for $k, l \geq N,\left\|\mathbf{g}_{l}-\mathbf{g}_{k}\right\|_{\infty}<\frac{3 \varepsilon}{4}<\varepsilon$. This shows $\left\{\mathbf{g}_{k}\right\}$ is a Cauchy sequence in $C\left(K, \mathbb{R}^{n}\right)$ which is complete. Thus this sequence converges uniformly to some $\mathbf{g} \in C\left(K, \mathbb{R}^{n}\right)$.

The following is the Arzela Ascoli theorem . Actually, the converse is also true but I will only give the direction of most use in applications.

Theorem 9.2.4 Let $\mathscr{A} \subseteq C\left(K, \mathbb{R}^{n}\right)$ be both equicontinuous and uniformly bounded. Then every sequence in $\mathscr{A}$ has a convergent subsequence converging to some $\mathbf{g} \in C\left(K, \mathbb{R}^{n}\right)$, the convergence taking place with respect to $\|\cdot\|_{\infty}$, the uniform norm.

Proof: Let $\left\{\mathbf{f}_{j}\right\}_{j=1}^{\infty}$ be a sequence of functions in $\mathscr{A}$. Let $D$ be a countable dense subset of $K$. Say $D \equiv\left\{\mathbf{d}_{k}\right\}_{k=1}^{\infty}$. Then $\left\{\mathbf{f}_{j}\left(d_{1}\right)\right\}_{j=1}^{\infty}$ is a bounded set of points in $\mathbb{R}^{n}$. By the Heine Borel theorem, there is a subsequence, denoted by $\left\{\mathbf{f}_{(j, 1)}\left(\mathbf{d}_{1}\right)\right\}_{j=1}^{\infty}$ which converges. Now apply what was just done with $\left\{\mathbf{f}_{j}\right\}$ to $\left\{\mathbf{f}_{(j, 1)}\right\}$ and feature $\mathbf{d}_{2}$ instead of $\mathbf{d}_{1}$. Thus $\left\{\mathbf{f}_{(j, 2)}\right\}_{j=1}^{\infty}$ is a subsequence of $\left\{\mathbf{f}_{(j, 1)}\right\}$ which converges at $\mathbf{d}_{2}$. This new subsequence still converges at $\mathbf{d}_{1}$ thanks to Theorem 2.2.10. Continue this way. Thus we get the following

| $\mathbf{f}_{(1,1)}$ | $\mathbf{f}_{(2,1)}$ | $\mathbf{f}_{(3,1)}$ | $\cdots$ | converges at $\mathbf{d}_{1}$ |
| :---: | :---: | :---: | :---: | :---: |
| $\mathbf{f}_{(1,2)}$ | $\mathbf{f}_{(2,2)}$ | $\mathbf{f}_{(3,2)}$ | $\cdots$ | converges at $\mathbf{d}_{1}, \mathbf{d}_{2}$ |
| $\mathbf{f}_{(1,3)}$ | $\mathbf{f}_{(2,3)}$ | $\mathbf{f}_{(3,3)}$ | $\cdots$ | converges at $\mathbf{d}_{1}, \mathbf{d}_{2}, \mathbf{d}_{3}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |  | $\vdots$ |
| $\mathbf{f}_{(1, l)}$ | $\mathbf{f}_{(2, l)}$ | $\mathbf{f}_{(3, l)}$ | $\cdots$ | converges at $\mathbf{d}_{j}, j \leq l$ |
| $\vdots$ | $\vdots$ | $\vdots$ |  | $\vdots$ |

Each subsequence $\left\{\mathbf{f}_{(j, r)}\right\}_{j=1}^{\infty}$ is a subsequence of the one above it $\left\{\mathbf{f}_{(j, r-1)}\right\}_{j=1}^{\infty}$ and converges at $\mathbf{d}_{j}$ for $j \leq r$. Consider the subsequence $\left\{\mathbf{f}_{(r, r)}\right\}_{r=1}^{\infty}$ the diagonal subsequence. Then $\left\{\mathbf{f}_{(r, r)}\right\}_{r=j}^{\infty}$ is a subsequence of $\left\{\mathbf{f}_{(i, j)}\right\}_{i=1}^{\infty}$ and so $\left\{\mathbf{f}_{(r, r)}\left(\mathbf{d}_{i}\right)\right\}_{r=j}^{\infty}$ converges for each $i \leq j$. Since $j$ is arbitrary, this shows that $\left\{\mathbf{f}_{(r, r)}\right\}_{r=1}^{\infty}$ converges at every point of $D$ as $r \rightarrow \infty$.

From now on, denote this subsequence of the original sequence as $\left\{\mathbf{g}_{k}\right\}_{k=1}^{\infty}$. It has the property that it converges at every point of $D$.

Claim: $\left\{\mathbf{g}_{k}\right\}_{k=1}^{\infty}$ converges at every $\mathbf{x} \in K$.
Proof of claim: Let $\varepsilon>0$ be given. Let $\delta$ go with $\varepsilon / 4$ in the definition of equicontinuity. Then pick $\mathbf{d} \in D$ such that $|\mathbf{d}-\mathbf{x}|<\delta$. Then

$$
\begin{aligned}
\left|\mathbf{g}_{k}(\mathbf{x})-\mathbf{g}_{l}(\mathbf{x})\right| & \leq\left|\mathbf{g}_{k}(\mathbf{x})-\mathbf{g}_{k}(\mathbf{d})\right|+\left|\mathbf{g}_{k}(\mathbf{d})-\mathbf{g}_{l}(\mathbf{d})\right|+\left|\mathbf{g}_{l}(\mathbf{d})-\mathbf{g}_{l}(\mathbf{x})\right| \\
& <\frac{\varepsilon}{4}+\left|\mathbf{g}_{k}(\mathbf{d})-\mathbf{g}_{l}(\mathbf{d})\right|+\frac{\varepsilon}{4}
\end{aligned}
$$

There exists $N$ such that if $k, l \geq N$, then $\left|\mathbf{g}_{k}(\mathbf{d})-\mathbf{g}_{l}(\mathbf{d})\right|<\frac{\varepsilon}{3}$. Thus, if $k, l \geq N$,

$$
\left|\mathbf{g}_{k}(\mathbf{x})-\mathbf{g}_{l}(\mathbf{x})\right|<\frac{\varepsilon}{4}+\frac{\varepsilon}{3}+\frac{\varepsilon}{4}<\varepsilon
$$

which shows that, since $\varepsilon$ is arbitrary, $\left\{\mathbf{g}_{k}(\mathbf{x})\right\}_{k=1}^{\infty}$ is a Cauchy sequence and so it converges to some $\mathbf{g}(\mathbf{x})$. This shows the claim. Now from the Lemma 9.2.3, this $\mathbf{g}$ is in $C\left(K, \mathbb{R}^{n}\right)$ and $\left\|\mathbf{g}_{k}-\mathbf{g}\right\|_{\infty} \rightarrow 0$.

### 9.3 The $L^{p}$ Spaces

Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. Recall that the space $L^{1}(\Omega)$ consists of functions $f$ : $\Omega \rightarrow \mathbb{F}$ such that $f$ is measurable and $\int_{\Omega}|f(\omega)| d \mu<\infty$. The $L^{p}$ spaces are defined as follows.

Definition 9.3.1 Let $(\Omega, \mathbb{F}, \mu)$ be a measure space. Then $L^{p}(\Omega)$ consists of those measurable functions $f$ such that $\int_{\Omega}|f|^{p} d \mu<\infty$. Here it is assumed that $p>1$. Also define the conjugate exponent $q$ as satisfying $\frac{1}{p}+\frac{1}{q}=1$ In case $p=1$, we let $q=\infty$ and give a special meaning to $L^{\infty}(\Omega)$ discussed later.

Here we assume $p>1$. There is an essential inequality which makes possible the study of $L^{p}(\Omega)$.

Proposition 9.3.2 Let $0 \leq a, b$. Then for $p>1$

$$
\begin{equation*}
\frac{a^{p}}{p}+\frac{b^{q}}{q} \geq a b \tag{9.5}
\end{equation*}
$$

Proof: Let $b \geq 0$ be fixed and let $f(a) \equiv \frac{a^{p}}{p}+\frac{b^{q}}{q}-a b$. Then $f(0)=\frac{b^{q}}{q} \geq 0$ and $f^{\prime}(a)=a^{p-1}-b$. If $b=0$ the desired inequality is obvious. If $b>0$, then $f^{\prime}(a)<0$ for $a$ close to 0 and $f^{\prime}(a)>0$ if $a^{p-1}>b$. Thus $f$ has a minimum at the point where $a^{p-1}=b$. But $p-1=p / q$ and so, at this point $a^{p}=b^{q}$. Therefore, at this point, $f(a)=$ $\frac{a^{p}}{p}+\frac{a^{p}}{q}-a a^{p-1}=a^{p}-a^{p}=0$. Therefore, $f(a) \geq 0$ for all $a \geq 0$ and it equals 0 exactly when $a^{p}=b^{q}$.

This implies the following major result, Holder's inequality.

Theorem 9.3.3 Let $f, g$ be measurable and nonnegative functions. Then

$$
\int_{\Omega} f g d \mu \leq\left(\int_{\Omega} f^{p} d \mu\right)^{1 / p}\left(\int_{\Omega} g^{q} d \mu\right)^{1 / q}
$$

Proof: If either $\left(\int_{\Omega} f^{p} d \mu\right)^{1 / p}$ or $\left(\int_{\Omega} g^{q} d \mu\right)^{1 / q}$ is 0 , then there is nothing to show because if $\left(\int_{\Omega} f^{p} d \mu\right)^{1 / p}=0$, then $\int_{\Omega} f^{p} d \mu=0$ and you could let $A_{n} \equiv\left\{\omega: f^{p}(\omega) \geq 1 / n\right\}$. Then

$$
0=\int_{\Omega} f^{p} d \mu \geq \int_{A_{n}} f^{p} d \mu \geq(1 / n) \mu\left(A_{n}\right)
$$

and so $\mu\left(A_{n}\right)=0$. Therefore, $\{\omega: f(\omega) \neq 0\}=\cup_{n=1}^{\infty} A_{n}$ and each of these sets in the union has measure zero. It follows that $\{\omega: f(\omega) \neq 0\}$ has measure zero. Therefore, $\int_{\Omega} f g d \mu=$ 0 and so indeed, there is nothing left to show. The situation is the same if $\left(\int_{\Omega} g^{q} d \mu\right)^{1 / q}=0$. Thus assume both of the factors on the right in the inequality are nonzero. Then let $A \equiv$ $\left(\int_{\Omega} f^{p} d \mu\right)^{1 / p}, B \equiv\left(\int_{\Omega} g^{q} d \mu\right)^{1 / q}$. Proposition 9.3.2,

$$
\begin{aligned}
\int_{\Omega} \frac{f}{A} \frac{g}{B} d \mu & \leq \int_{\Omega} \frac{f^{p}}{A^{p} p} d \mu+\int_{\Omega} \frac{g^{q}}{B^{q} q} d \mu \\
& =\frac{1}{p} \frac{\int_{\Omega} f^{p} d \mu}{A^{p}}+\frac{1}{q} \frac{\int_{\Omega} g^{q} d \mu}{B^{q}}=\frac{1}{p}+\frac{1}{q}=1
\end{aligned}
$$

Therefore, $\int_{\Omega} f g d \mu \leq A B=\left(\int_{\Omega} f^{p} d \mu\right)^{1 / p}\left(\int_{\Omega} g^{q} d \mu\right)^{1 / q}$
This makes it easy to prove the Minkowski inequality for the sum of two functions.
Theorem 9.3.4 Let $f, g$ be two measurable functions with values in $\mathbb{F}$. Then

$$
\begin{equation*}
\left(\int_{\Omega}|f+g|^{p} d \mu\right)^{1 / p} \leq\left(\int_{\Omega}|f|^{p} d \mu\right)^{1 / p}+\left(\int_{\Omega}|g|^{p} d \mu\right)^{1 / p} \tag{9.6}
\end{equation*}
$$

Proof: First of all,

$$
|f+g|^{p}=|f+g|^{p-1}|f+g| \leq|f+g|^{p-1}(|f|+|g|)
$$

Recall that $p-1=p / q$. Then, using Theorem 9.3.3,

$$
\begin{aligned}
& \int_{\Omega}|f+g|^{p} d \mu \leq \int_{\Omega}|f+g|^{p / q}|f| d \mu+\int_{\Omega}|f+g|^{p / q}|g| d \mu \\
\leq & \left(\int_{\Omega}|f+g|^{p} d \mu\right)^{1 / q}\left(\int|f|^{p} d \mu\right)^{1 / p}+\left(\int_{\Omega}|f+g|^{p} d \mu\right)^{1 / q}\left(\int_{\Omega}|g|^{p} d \mu\right)^{1 / p} \\
= & \left(\int_{\Omega}|f+g|^{p} d \mu\right)^{1 / q}\left(\left(\int|f|^{p} d \mu\right)^{1 / p}+\left(\int_{\Omega}|g|^{p} d \mu\right)^{1 / p}\right)
\end{aligned}
$$

If $\int_{\Omega}|f+g|^{p} d \mu=0$, then 9.6 is obvious. If $\int_{\Omega}|f+g|^{p} d \mu=\infty$, then

$$
\infty \leq \int_{\Omega} 2^{p-1}\left(|f|^{p}+|g|^{p}\right) d \mu
$$

and so one of the terms on the right side in 9.6 is $\infty$. Therefore again 9.6 is obvious. Otherwise divide $\left(\int_{\Omega}|f+g|^{p} d \mu\right)^{1 / q}$ on both sides to obtain 9.6.

By induction, you have

$$
\left(\int_{\Omega}\left|\sum_{k=1}^{n} f_{i}\right|^{p} d \mu\right)^{1 / p} \leq \sum_{k=1}^{n}\left(\int_{\Omega}\left|f_{i}\right|^{p} d \mu\right)^{1 / p}
$$

Observation 9.3.5 If $f, g$ are in $L^{p}$ and $\alpha, \beta$ are scalars, then $\alpha f+\beta g \in L^{p}$ also.
To see this, note that $\alpha f+\beta g$ is measurable thanks to Proposition 6.1.8. Is $|\alpha f+\beta g|^{p}$ in $L^{1}$ ?

$$
\begin{aligned}
\left(\int_{\Omega}|\alpha f+\beta g|^{p} d \mu\right)^{1 / p} & \leq\left(\int_{\Omega}|\alpha f|^{p} d \mu\right)^{1 / p}+\left(\int_{\Omega}|\beta g|^{p} d \mu\right)^{1 / p} \\
& =|\alpha|\left(\int_{\Omega}|f|^{p} d \mu\right)^{1 / p}+|\beta|\left(\int_{\Omega}|g|^{p} d \mu\right)^{1 / p}<\infty
\end{aligned}
$$

and so it follows that $L^{p}(\Omega)$ is a vector space of functions. If $\|f\|_{p} \equiv\left(\int_{\Omega}|f|^{p} d \mu\right)^{1 / p}$, then the above computation shows that $\|\cdot\|_{p}$ is a norm except for one problem. If $\|f\|_{p}=0$, it does not follow that $f=0$. What can be concluded if $\|f\|_{p}=0$ ? From the first part of the argument in Theorem 9.3.3, it follows that if $\|f\|_{p}=0$, then $f(\omega)=0$ for a.e. $\omega$.

## Definition 9.3.6 $L^{p}(\Omega)$ is a normed vector space (normed linear space) if we agree

 to identify any two functions in $L^{p}(\Omega)$ which are equal off a set of measure zero and let $\|f\|_{p} \equiv\left(\int_{\Omega}|f|^{p}\right)^{1 / p}$. More precisely, $L^{p}(\Omega)$ consists of a vector space of equivalence classes of functions, the equivalence relation being that the functions are equal a.e.The big result about $L^{p}(\Omega)$ is that it is a complete space. Recall that this means that every Cauchy sequence converges. Recall Theorem 2.3.3 which said that if a subsequence of a Cauchy sequence in $\mathbb{R}^{p}$ converges then the original Cauchy sequence converges. Have a look a that theorem and notice that the specific context is completely irrelevant. The same argument shows that in an arbitrary normed linear space, if a subsequence of a Cauchy sequence converges, then the original Cauchy sequence converges. Also note that Theorem 2.3.2 which said that Cauchy sequences are bounded also does not depend on the context. It holds for an arbitrary normed linear space.

To show $L^{p}(\Omega)$ is complete, I will show that a Cauchy sequence has a subsequence which converges for a.e. $\omega$. Then an appeal to limit theorems will show $L^{p}(\Omega)$ is complete.

Theorem 9.3.7 Let $\left\{f_{n}\right\}_{n=1}^{\infty}$ be a Cauchy sequence in $L^{p}(\Omega)$. Then there exists $g \in$ $L^{p}(\Omega)$ and $\left\{f_{n_{k}}\right\}_{k=1}^{\infty}$ such that $f_{n_{k}}(\omega) \rightarrow g(\omega)$ a.e. $\omega$ and

$$
\lim _{n \rightarrow \infty}\left\|f_{n}-g\right\|_{p}=0
$$

Proof: First note that there exists $M$ such that $\left\|f_{n_{k}}\right\|_{p}^{p}<M$ by Theorem 2.3.2 (Cauchy sequences are bounded.) applied to this normed linear space. (Same argument) Select a subsequence $\left\{f_{n_{k}}\right\}$ such that if $m \geq n_{k},\left\|f_{n_{k}}-f_{m}\right\|_{p}^{p}<4^{-k}$. Let

$$
B_{k} \equiv\left\{\omega:\left|f_{n_{k+1}}(\omega)-f_{n_{k}}(\omega)\right|^{p}>2^{-k}\right\}
$$

Then

$$
2^{-k} \mu\left(B_{k}\right) \leq \int_{B_{k}}\left|f_{n_{k+1}}(\omega)-f_{n_{k}}(\omega)\right|^{p} d \mu<4^{-k}
$$

and so $\mu\left(B_{k}\right)<2^{-k}$. Now if $f_{n_{k}}(\omega)$ fails to be a Cauchy sequence, then $\omega \in B_{k}$ for infinitely many $k$. In other words, $\omega \in \cap_{n=1}^{\infty} \cup_{k \geq n} B_{k} \equiv B$. This measurable set $B$ has measure zero because

$$
\mu(B) \leq \mu\left(\cup_{k \geq n} B_{k}\right) \leq \sum_{k=n}^{\infty} \mu\left(B_{k}\right)<\frac{1}{2^{n-1}} \text { for every } n \in \mathbb{N}
$$

Therefore, for $\omega \notin B,\left\{f_{n_{k}}\right\}_{k=1}^{\infty}$ is a Cauchy sequence. Let $g(\omega) \equiv 0$ on $B$ and let $g(\omega) \equiv$ $\lim _{k \rightarrow \infty} f_{n_{k}}(\omega)$ if $\omega \notin B$. Why is $g \in L^{p}$ and why does $f_{n}$ converge to $g$ in $L^{p} ? g$ is the limit of the measurable functions $f_{n_{k}} \mathscr{X}_{B^{C}}$ and so it is measurable. By Fatou's lemma,

$$
\int_{\Omega}|g(\omega)|^{p} d \mu \leq \lim \inf _{k \rightarrow \infty} \int_{\Omega}\left|f_{n_{k}}(\omega)\right|^{p} d \mu \leq M
$$

and so $g \in L^{p}(\Omega)$. Now by construction, $\left\|f_{n_{k}}-f_{n_{k+1}}\right\|_{p}<4^{-k / p}$ therefore,

$$
\begin{aligned}
\left(\int_{\Omega}\left|f_{n_{k}}-f_{n_{k+m}}\right|^{p} d \mu\right)^{1 / p} & \equiv\left\|f_{n_{k}}-f_{n_{k+m}}\right\|_{p} \leq \sum_{j=0}^{m-1}\left\|f_{n_{k+j}}-f_{n_{k+\bar{j}+1}}\right\|_{p} \\
& \leq \sum_{j=k}^{\infty}\left(4^{-1 / p}\right)^{j}=\frac{\left(4^{-1 / p}\right)^{k-1}}{1-4^{-1 / p}}
\end{aligned}
$$

Now use Fatou's lemma to obtain, as $m \rightarrow \infty$,

$$
\left(\int_{\Omega}\left|f_{n_{k}}-g\right|^{p} d \mu\right)^{1 / p} \leq \frac{\left(4^{-1 / p}\right)^{k-1}}{1-4^{-1 / p}}
$$

The expression on the right converges to 0 as $k \rightarrow \infty$ and so $\lim _{k \rightarrow \infty}\left\|f_{n_{k}}-g\right\|_{p}=0$. It follows from Theorem 2.3.3 (If the sequence is Cauchy then if a subsequence converges, so does the original sequence.) applied to this normed linear space that

$$
\lim _{n \rightarrow \infty}\left\|f_{n}-g\right\|_{p}=0
$$

What about $L^{\infty}(\Omega)$, the case conjugate to $p=1$ ? How is the norm defined for $L^{\infty}(\Omega)$ ? What does it mean to be in $L^{\infty}(\Omega)$ ?

Definition 9.3.8 A function $f$ is in $L^{\infty}(\Omega)$ if it is measurable and if there is some constant $M$ such that off a set of measure zero, $|f(\omega)| \leq M$. Then $\|f\|_{\infty}$ is defined to be the infimum of all such constants $M$. Such a function is said to be "essentially bounded".

Obviously $L^{\infty}(\Omega)$ is a vector space. The next task is to verify that $\|\cdot\|_{\infty}$ is a norm under the convention that any two functions which are equal off a set of measure zero are the same.

Proposition 9.3.9 Let $f \in L^{\infty}(\Omega)$. Then

$$
\mu\left(\left\{\omega:|f(\omega)|>\|f\|_{\infty}\right\}\right)=0
$$

and if $\lambda<\|f\|_{\infty}$, then $\mu(\{\omega:|f(\omega)|>\lambda\})>0$.

Proof: The second claim follows right away from the definition of $\|f\|_{\infty}$. If it is not so, then $\mu(\{\omega:|f(\omega)|>\lambda\})=0$ and so $\lambda$ would be one of those constants $M$ in the description of $\|f\|_{\infty}$ and $\|f\|_{\infty}$ would not really be the infimum of these numbers. Consider the other claim. By definition,

$$
\mu\left(\left\{\omega:|f(\omega)|>\frac{1}{n}+\|f\|_{\infty}\right\}\right)=0
$$

This is because there must be some essential upper bound $M$ between $\|f\|_{\infty}$ and $\|f\|_{\infty}+\frac{1}{n}$ since otherwise, $\|f\|_{\infty}$ would not be the infimum. But

$$
\mu\left(\left\{\omega:|f(\omega)|>\|f\|_{\infty}\right\}\right)=\cup_{n=1}^{\infty} \mu\left(\left\{\omega:|f(\omega)|>\frac{1}{n}+\|f\|_{\infty}\right\}\right)
$$

and each of the sets in the union has measure zero.
Note that this implies that if $\|f\|_{\infty}=0$, then $f=0$ a.e. so $f$ is regarded as 0 . Thus, to say that $\|f-g\|_{\infty}=0$ is to say that the two functions inside the norm are equal except for a set of measure zero, and the convention is that when this happens, we regard them as the same function. If $\alpha=0$ then $\|\alpha f\|_{\infty}=0=0\|f\|_{\infty}$. If $\alpha \neq 0$, then $M \geq|f(\omega)|$ implies $|\alpha| M \geq|\alpha f(\omega)|$. In particular, $|\alpha|\left(\|f\|_{\infty}+\frac{1}{n}\right) \geq|\alpha f(\omega)|$ for all $\omega$ not in the union of the sets of measure zero corresponding to each $\|f\|_{\infty}+\frac{1}{n}$. Thus there is a set of measure zero $N$ such that for $\omega \notin N$,

$$
|\alpha|\left(\|f\|_{\infty}+\frac{1}{n}\right) \geq|\alpha f(\omega)| \text { for all } n
$$

Therefore, for $\omega \notin N,|\alpha|\|f\|_{\infty} \geq\|\alpha f\|_{\infty}$. This implies that $\|f\|_{\infty}=\left\|\frac{1}{\alpha} \alpha f\right\|_{\infty} \leq \frac{1}{|\alpha|}\|\alpha f\|_{\infty}$ and so $\|\alpha f\|_{\infty} \geq|\alpha|\|f\|_{\infty}$ also. This shows that this acts like a norm relative to multiplication by scalars. What of the triangle inequality? Let $M_{n} \downarrow\|f\|_{\infty}$ and $N_{n} \downarrow\|g\|_{\infty}$. Thus for each $n$, there is an exceptional set of measure zero such that off this set $M_{n} \geq|f(\omega)|$ and a similar condition holding for $g$ and $N_{n}$. Let $N$ be the union of all the exceptional sets for $f$ and $g$ for each $n$. Then for $\omega \notin N$, the following holds for all $\omega \notin N$

$$
M_{n}+N_{n} \geq|f(\omega)|+|g(\omega)| \geq|f(\omega)+g(\omega)|
$$

So take a limit of both sides and find that $\|f\|_{\infty}+\|g\|_{\infty} \geq|f(\omega)+g(\omega)|$ for all $\omega$ off a set of measure zero. Therefore,

$$
\|f\|_{\infty}+\|g\|_{\infty} \geq\|f+g\|_{\infty}
$$

## Theorem 9.3.10 $L^{\infty}(\Omega)$ is complete.

Proof: Let $\left\{f_{n}\right\}$ be a Cauchy sequence. Let $N$ be the union of all sets where it is not the case that $\left|f_{n}(\omega)-f_{m}(\omega)\right| \leq\left\|f_{n}-f_{m}\right\|_{\infty}$. By Proposition 9.3.9, there is such an exceptional set $M_{m n}$ for each choice of $m, n$. Thus $N$ is the countable union of these sets of measure zero. Therefore, for $\omega \notin N,\left\{f_{n}(\omega)\right\}_{n=1}^{\infty}$ is a Cauchy sequence and so we let $g(\omega)=0$ if $\omega \notin N$ and $g(\omega) \equiv \lim _{n \rightarrow \infty} f_{n}(\omega)$. Thus $g$ is measurable. Also, for $\omega \notin N$,

$$
\left|g(\omega)-f_{n}(\omega)\right|=\lim _{m \rightarrow \infty}\left|f_{m}(\omega)-f_{n}(\omega)\right| \leq \lim \sup _{m \rightarrow \infty}\left\|f_{m}-f_{n}\right\|_{\infty}<\varepsilon
$$

provided $n$ is sufficiently large. This shows $|g(\omega)| \leq\left\|f_{n}\right\|_{\infty}+\varepsilon, \omega \notin N$ so $\|g\|_{\infty}<\infty$. Also it shows that there is a set of measure zero $N$ such that for all $\omega \notin N$, for any $\varepsilon>0$, $\left|g(\omega)-f_{n}(\omega)\right|<\varepsilon$ which means $\left\|g-f_{n}\right\|_{\infty} \leq \varepsilon$. Since $\varepsilon$ is arbitrary, this shows that $\lim _{n \rightarrow \infty}\left\|g-f_{n}\right\|_{\infty}=0$.

### 9.4 Approximation Theorems

First is a significant result on approximating with simple functions in $L^{p}$.
Theorem 9.4.1 Let $f \in L^{p}(\Omega)$ for $p \geq 1$. Then for each $\varepsilon>0$ there is a simple function s such that $\|f-s\|_{p} \leq \varepsilon$.

Proof: It suffices to consider the case where $f \geq 0$ because you can then apply what is shown to the positive and negative parts of the real and imaginary parts of $f$ to get the general case. Thus, suppose $f \geq 0$ and in $L^{p}(\Omega)$. By Theorem 6.1.10, there exists a sequence of simple functions increasing to $f$. Then $\left|f(\omega)-s_{n}(\omega)\right|^{p} \leq|f(\omega)|^{p}$. This is a suitable dominating function. Then by the dominated convergence theorem, $0=\lim _{n \rightarrow \infty} \int_{\Omega}\left|f(\omega)-s_{n}(\omega)\right|^{p} d \mu$ which establishes the desired conclusion unless $p=\infty$.

Use Proposition 9.3.9 to get a set of measure zero $N$ such that off this set, $|f(\omega)| \leq$ $\|f\|_{\infty}$. Then consider $f \mathscr{X}_{N^{c}}$. It is a measurable and bounded function so by Theorem 6.1.10, there is an increasing sequence of simple functions $\left\{s_{n}\right\}$ converging uniformly to this function. Hence, for $n$ large enough, $\left\|f-s_{n}\right\|_{\infty}<\varepsilon$.

Theorem 9.4.2 Let $\mu$ be a regular Borel measure on $\mathbb{R}^{n}$ and $f \in L^{p}\left(\mathbb{R}^{n}\right)$. Then for each $p \geq 1, p \neq \infty$, there exists $g$ a continuous function which is zero off a compact set such that $\|f-g\|_{p}<\varepsilon$.

Proof: Without loss of generality, assume $f \geq 0$. First suppose that $f$ is 0 off some ball $B(\mathbf{0}, R)$. There exists a simple function $0 \leq s \leq f$ such that $\int|f-s|^{p} d \mu<(\varepsilon / 2)^{p}$. Thus it suffices to show the existence of a continuous function $h$ which is zero off a compact set which satisfies $\left(\int|h-s|^{p} d \mu\right)^{1 / p}<\varepsilon / 2$. Let

$$
s(\mathbf{x})=\sum_{i=1}^{m} c_{i} \mathscr{X}_{E_{i}}(\mathbf{x}), E_{i} \subseteq B(\mathbf{0}, R)
$$

where $E_{i}$ is in $\mathscr{F}_{p}$. Thus each $E_{i}$ is bounded. By regularity, there exist compact sets $K_{i}$ and open sets $V_{i}$ with $K_{i} \subseteq E_{i} \subseteq V_{i} \subseteq B(\mathbf{0}, R)$ and $\sum_{i=1}^{m}\left(c_{i}^{p} \mu\left(V_{i} \backslash K_{i}\right)\right)^{1 / p}<\varepsilon / 2$.

Now define $h_{i}(\mathbf{x}) \equiv \frac{\operatorname{dist}\left(\mathbf{x}, V^{C}\right)}{\operatorname{dist}(\mathbf{x}, K)+\operatorname{dist}\left(\mathbf{x}, V^{C}\right)}$. Thus $h_{i}$ equals zero off a compact set and it equals 1 on $K_{i}$ and 0 off $V_{i}$. Let $h \equiv \sum_{i=1}^{m} c_{i} h_{i}$. Thus $0 \leq h \leq \max \left\{c_{i}, i=1, \cdots, m\right\}$. Then

$$
\int\left|c_{i} h_{i}-c_{i} \mathscr{X}_{E_{i}}\right|^{p} d \mu \leq c_{i}^{p} \mathscr{X}_{V_{i}-K_{i}} \leq c_{i}^{p} \mu\left(V_{i} \backslash K_{i}\right)
$$

It follows that, from the Minkowski inequality,

$$
\begin{gathered}
\left(\int\left|f-\sum_{i} c_{i} h_{i}\right|^{p} d \mu\right)^{1 / p} \leq\left(\int|f-s|^{p} d \mu\right)^{1 / p}+\left(\int\left|s-\sum_{i} c_{i} h_{i}\right|^{p} d \mu\right)^{1 / p} \\
\leq \frac{\varepsilon}{2}+\left(\int\left(\sum_{i}\left|c_{i} \mathscr{X}_{E_{i}}-c_{i} h_{i}\right|\right)^{p} d \mu\right)^{1 / p} \leq \frac{\varepsilon}{2}+\sum_{i}\left(\int\left|c_{i} \mathscr{X}_{E_{i}}-c_{i} h_{i}\right|^{p} d \mu\right)^{1 / p} \\
\leq \frac{\varepsilon}{2}+\sum_{i}\left(c_{i}^{p} \mu\left(V_{i} \backslash K_{i}\right)\right)^{1 / p}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon
\end{gathered}
$$

This shows that if $f$ is zero off some ball, then it can be approximated with a continuous function which is zero off a compact set.

Now consider the general case. Then let $f_{n}=f \mathscr{X}_{B(\mathbf{0}, n)}$. Then by dominated convergence theorem, for $n$ large enough, $\left(\int\left|f-f_{n}\right|^{p} d \mu\right)^{1 / p}<\frac{\varepsilon}{2}$ and now from what was just shown, there exists $h$ continuous, zero off some compact set, such that $\left(\int\left|f_{n}-h\right|^{p} d \mu\right)^{1 / p}<$ $\frac{\varepsilon}{2}$. Thus from the triangle inequality,

$$
\left(\int|f-h|^{p} d \mu\right)^{1 / p}<\left(\int\left|f-f_{n}\right|^{p} d \mu\right)^{1 / p}+\left(\int\left|f_{n}-h\right|^{p} d \mu\right)^{1 / p}<\varepsilon
$$

### 9.5 Fundamental Theorem of Calculus

In this section the Vitali covering theorem, Proposition 8.6 .3 will be used to give a generalization of the fundamental theorem of calculus. Let $f$ be in $L^{1}\left(\mathbb{R}^{p}\right)$ where the measure is Lebesgue measure as discussed above.

Let $M f: \mathbb{R}^{p} \rightarrow[0, \infty]$ by

$$
M f(\mathbf{x}) \equiv \sup _{r \leq 1} \frac{1}{m_{p}(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f| d m_{p} \text { if } \mathbf{x} \notin Z .
$$

We denote as $\|f\|_{1}$ the integral $\int_{\Omega}|f| d m_{p}$.
The special points described in the following theorem are called Lebesgue points. Also $\overline{m_{p}}$ will denote the outer measure determined by Lebesgue measure. See Proposition 6.4.2. $\overline{m_{p}}(E) \equiv \inf \left\{\overline{m_{p}}(F): F\right.$ is measurable and $\left.F \supseteq E\right\}$.

Theorem 9.5.1 Let $m_{p}$ be $p$ dimensional Lebesgue measure measure and let $f \in$ $L^{1}\left(\mathbb{R}^{p}, m_{p}\right) \cdot\left(\int_{\Omega}|f| d m_{p}<\infty\right)$. Then for $m_{p}$ a.e. $\mathbf{x}$,

$$
\lim _{r \rightarrow 0} \frac{1}{m_{p}(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d m_{p}(y)=0
$$

Proof: First consider the following claim which is called a weak type estimate.
Claim 1: The following inequality holds for $N_{p}$ the constant of the Vitali covering theorem, Proposition 8.6.3.

$$
\overline{m_{p}}([M f>\varepsilon]) \leq 5^{p} \varepsilon^{-1}\|f\|_{1}
$$

Proof: For each $\mathbf{x} \in[M f>\varepsilon]$ there exists a ball $B_{\mathbf{x}}=B\left(\mathbf{x}, r_{\mathbf{x}}\right)$ with $0<r_{\mathbf{x}} \leq 1$ and

$$
\begin{equation*}
m_{p}\left(B_{\mathbf{x}}\right)^{-1} \int_{B\left(\mathbf{x}, r_{\mathbf{x}}\right)}|f| d m_{p}>\varepsilon \tag{9.7}
\end{equation*}
$$

Let $\mathscr{F}$ be this collection of balls. By the Vitali covering theorem, there is a collection of disjoint balls $\mathscr{G}$ such that if each ball in $\mathscr{G}$ is enlarged making the center the same but the radius 5 times as large, then the corresponding collection of enlarged balls covers $[M f>\varepsilon]$. By separability, $\mathscr{G}$ is countable, say $\left\{B_{i}\right\}_{i=1}^{\infty}$ and the enlarged balls will be denoted as $\hat{B}_{i}$. Then from 9.7,

$$
\overline{m_{p}}([M f>\varepsilon]) \leq \sum_{i} m_{p}\left(\hat{B}_{i}\right) \leq 5^{p} \sum_{i} m_{p}\left(B_{i}\right) \leq \frac{5^{p}}{\varepsilon} \sum_{i} \int_{B_{i}}|f| d m_{p} \leq 5^{p} \varepsilon^{-1}\|f\|_{1}
$$

This proves claim 1.
Claim 2: If $g \in C_{c}\left(\mathbb{R}^{p}\right)$, then

$$
\lim _{r \rightarrow 0} \frac{1}{m_{p}(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|g(\mathbf{y})-g(\mathbf{x})| d m_{p}(y)=0
$$

Proof: Since $g$ is continuous at $\mathbf{x}$, whenever $r$ is small enough,

$$
\frac{1}{m_{p}(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|g(\mathbf{y})-g(\mathbf{x})| d m_{p}(y) \leq \frac{1}{m_{p}(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} \varepsilon d m_{p}(y)=\varepsilon
$$

This proves the claim.
Now let $g \in C_{c}\left(\mathbb{R}^{p}\right)$. Then from the above observations about continuous functions in Claim 2,

$$
\begin{align*}
& \overline{m_{p}}\left(\left[\mathbf{x}: \lim _{r \rightarrow 0} \sup _{r \rightarrow 0} \frac{1}{m_{p}(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d m_{p}(y)>\varepsilon\right]\right)  \tag{9.8}\\
& \leq \quad \overline{m_{p}}\left(\left[\mathbf{x}: \limsup _{r \rightarrow 0} \frac{1}{m_{p}(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-g(\mathbf{y})| d m_{p}(y)>\frac{\varepsilon}{2}\right]\right) \\
& \quad+\overline{m_{p}}\left(\left[\mathbf{x}: \limsup _{r \rightarrow 0} \frac{1}{m_{p}(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|g(\mathbf{y})-g(\mathbf{x})| d m_{p}(y)>\frac{\varepsilon}{2}\right]\right) \\
& +\overline{m_{p}}\left(\left[\mathbf{x}:|g(\mathbf{x})-f(\mathbf{x})|>\frac{\varepsilon}{2}\right]\right) . \\
& \quad \leq \overline{m_{p}}\left(\left[M(f-g)>\frac{\varepsilon}{2}\right]\right)+m_{p}\left(\left[|f-g|>\frac{\varepsilon}{2}\right]\right) \tag{9.9}
\end{align*}
$$

Now

$$
\|f-g\|_{1} \geq \int_{\left[|f-g|>\frac{\varepsilon}{2}\right]}|f-g| d m_{p} \geq \frac{\varepsilon}{2} m_{p}\left(\left[|f-g|>\frac{\varepsilon}{2}\right]\right)
$$

and so using Claim 1 and 9.9, 9.8 is dominated by

$$
\left(\frac{2}{\varepsilon}+\frac{5^{p}}{\varepsilon}\right) \int|f-g| d m_{p}
$$

But by Theorem 9.4.2, $g$ can be chosen to make the above as small as desired. Hence 9.8 is 0 .

$$
\begin{aligned}
& \overline{m_{p}}\left(\left[\lim \sup _{r \rightarrow 0} \frac{1}{m_{p}(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d m_{p}(y)>0\right]\right) \\
\leq & \sum_{k=1}^{\infty} \overline{m_{p}}\left(\left[\lim \sup _{r \rightarrow 0} \frac{1}{m_{p}(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d m_{p}(y)>\frac{1}{k}\right]\right)=0
\end{aligned}
$$

By completeness of $m_{p}$ this implies

$$
\left[\limsup \sup _{r \rightarrow 0} \frac{1}{m_{p}(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d m_{p}(y)>0\right]
$$

is a set of $m_{p}$ measure zero.
The following corollary is the main result referred to as the Lebesgue Differentiation theorem.

Definition 9.5.2 $f \in L_{\text {loc }}^{1}\left(\mathbb{R}^{p}, m_{p}\right)$ means $f \mathscr{X}_{B}$ is in $L^{1}\left(\mathbb{R}^{n}, m_{p}\right)$ whenever $B$ is a ball.

Corollary 9.5.3 If $f \in L_{\text {loc }}^{1}\left(\mathbb{R}^{p}, m_{p}\right)$, then for a.e. $\mathbf{x}$,

$$
\begin{equation*}
\lim _{r \rightarrow 0} \frac{1}{m_{p}(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d m_{p}(y)=0 . \tag{9.10}
\end{equation*}
$$

In particular, for a.e. $\mathbf{x}$,

$$
\lim _{r \rightarrow 0} \frac{1}{m_{p}(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} f(\mathbf{y}) d m_{p}(y)=f(\mathbf{x})
$$

Proof: If $f$ is replaced by $f \mathscr{X}_{B(0, k)}$ then the conclusion 9.10 holds for all $\mathbf{x} \notin F_{k}$ where $F_{k}$ is a set of $m_{p}$ measure 0 . Letting $k=1,2, \cdots$, and $F \equiv \cup_{k=1}^{\infty} F_{k}$, it follows that $F$ is a set of measure zero and for any $\mathbf{x} \notin F$, and $k \in\{1,2, \cdots\}, 9.10$ holds if $f$ is replaced by $f \mathscr{X}_{B(0, k)}$. Picking any such $\mathbf{x}$, and letting $k>|\mathbf{x}|+1$, this shows

$$
\begin{gathered}
\lim _{r \rightarrow 0} \frac{1}{m_{p}(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d m_{p}(y) \\
=\lim _{r \rightarrow 0} \frac{1}{m_{p}(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}\left|f \mathscr{X}_{B(\mathbf{0}, k)}(\mathbf{y})-f \mathscr{X}_{B(\mathbf{0}, k)}(\mathbf{x})\right| d m_{p}(y)=0 .
\end{gathered}
$$

The last claim holds because

$$
\left|f(\mathbf{x})-\frac{1}{m_{p}(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} f(\mathbf{y}) d m_{p}(y)\right| \leq \frac{1}{m_{p}(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)}|f(\mathbf{y})-f(\mathbf{x})| d m_{p}(y)
$$

Definition 9.5.4 Let $E$ be a measurable set. Then $\mathbf{x} \in E$ is called a point of density if

$$
\lim _{r \rightarrow 0} \frac{m_{p}(B(\mathbf{x}, r) \cap E)}{m_{p}(B(\mathbf{x}, r))}=1
$$

Proposition 9.5.5 Let $E$ be a measurable set. Then $m_{p}$ a.e. $\mathbf{x} \in E$ is a point of density.
Proof: This follows from letting $f(\mathbf{x})=\mathscr{X}_{E}(\mathbf{x})$ in Corollary 9.5.3.

### 9.6 A Useful Inequality

There is an extremely useful inequality. To prove this theorem first consider a special case of it in which technical considerations which shed no light on the proof are excluded.

Lemma 9.6.1 Let $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{F}, \lambda)$ be finite measure spaces and let $f$ be $\mu \times \lambda$ measurable and uniformly bounded. Then the following inequality is valid for $p \geq 1$.

$$
\begin{equation*}
\int_{X}\left(\int_{Y}|f(x, y)|^{p} d \lambda\right)^{\frac{1}{p}} d \mu \geq\left(\int_{Y}\left(\int_{X}|f(x, y)| d \mu\right)^{p} d \lambda\right)^{\frac{1}{p}} . \tag{9.11}
\end{equation*}
$$

Proof: Since $f$ is bounded and $\mu(X), \lambda(Y)<\infty,\left(\int_{Y}\left(\int_{X}|f(x, y)| d \mu\right)^{p} d \lambda\right)^{\frac{1}{p}}<\infty$. Let $J(y)=\int_{X}|f(x, y)| d \mu$. Note there is no problem in writing this for a.e. $y$ because $f$ is product measurable. Then by Fubini's theorem,

$$
\begin{aligned}
\int_{Y}\left(\int_{X}|f(x, y)| d \mu\right)^{p} d \lambda & =\int_{Y} J(y)^{p-1} \int_{X}|f(x, y)| d \mu d \lambda \\
& =\int_{X} \int_{Y} J(y)^{p-1}|f(x, y)| d \lambda d \mu
\end{aligned}
$$

Now apply Holder's inequality in the last integral above and recall $p-1=\frac{p}{q}$. This yields

$$
\begin{gather*}
\quad \int_{Y}\left(\int_{X}|f(x, y)| d \mu\right)^{p} d \lambda \leq \int_{X}\left(\int_{Y} J(y)^{p} d \lambda\right)^{\frac{1}{q}}\left(\int_{Y}|f(x, y)|^{p} d \lambda\right)^{\frac{1}{p}} d \mu \\
=\left(\int_{Y} J(y)^{p} d \lambda\right)^{\frac{1}{q}} \int_{X}\left(\int_{Y}|f(x, y)|^{p} d \lambda\right)^{\frac{1}{p}} d \mu \\
=\left(\int_{Y}\left(\int_{X}|f(x, y)| d \mu\right)^{p} d \lambda\right)^{\frac{1}{q}} \int_{X}\left(\int_{Y}|f(x, y)|^{p} d \lambda\right)^{\frac{1}{p}} d \mu \tag{9.12}
\end{gather*}
$$

Therefore, dividing both sides by the first factor in the above expression,

$$
\begin{equation*}
\left(\int_{Y}\left(\int_{X}|f(x, y)| d \mu\right)^{p} d \lambda\right)^{\frac{1}{p}} \leq \int_{X}\left(\int_{Y}|f(x, y)|^{p} d \lambda\right)^{\frac{1}{p}} d \mu \tag{9.13}
\end{equation*}
$$

Note that 9.13 holds even if the first factor of 9.12 equals zero.
Now consider the case where $f$ is not assumed to be bounded and where the measure spaces are $\sigma$ finite.
Theorem 9.6.2 Let $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{F}, \lambda)$ be $\sigma$-finite measure spaces and let $f$ be product measurable. Then the following inequality is valid for $p \geq 1$.

$$
\begin{equation*}
\int_{X}\left(\int_{Y}|f(x, y)|^{p} d \lambda\right)^{\frac{1}{p}} d \mu \geq\left(\int_{Y}\left(\int_{X}|f(x, y)| d \mu\right)^{p} d \lambda\right)^{\frac{1}{p}} \tag{9.14}
\end{equation*}
$$

Proof: Since the two measure spaces are $\sigma$ finite, there exist measurable sets, $X_{m}$ and $Y_{k}$ such that $X_{m} \subseteq X_{m+1}$ for all $m, Y_{k} \subseteq Y_{k+1}$ for all $k$, and also $\mu\left(X_{m}\right), \lambda\left(Y_{k}\right)<\infty$. Now define

$$
f_{n}(x, y) \equiv\left\{\begin{array}{l}
f(x, y) \text { if }|f(x, y)| \leq n \\
n \text { if }|f(x, y)|>n
\end{array}\right.
$$

Thus $f_{n}$ is uniformly bounded and product measurable. By the above lemma,

$$
\begin{equation*}
\int_{X_{m}}\left(\int_{Y_{k}}\left|f_{n}(x, y)\right|^{p} d \lambda\right)^{\frac{1}{p}} d \mu \geq\left(\int_{Y_{k}}\left(\int_{X_{m}}\left|f_{n}(x, y)\right| d \mu\right)^{p} d \lambda\right)^{\frac{1}{p}} \tag{9.15}
\end{equation*}
$$

Now observe that $\left|f_{n}(x, y)\right|$ increases in $n$ and the pointwise limit is $|f(x, y)|$. Therefore, using the monotone convergence theorem in 9.15 yields the same inequality with $f$ replacing $f_{n}$. Next let $k \rightarrow \infty$ and use the monotone convergence theorem again to replace $Y_{k}$ with $Y$. Finally let $m \rightarrow \infty$ in what is left to obtain 9.14 .

Note that the proof of this theorem depends on two manipulations, the interchange of the order of integration and Holder's inequality. Note that there is nothing to check in the case of double sums. Thus if $a_{i j} \geq 0$, it is always the case that

$$
\left(\sum_{j}\left(\sum_{i} a_{i j}\right)^{p}\right)^{1 / p} \leq \sum_{i}\left(\sum_{j} a_{i j}^{p}\right)^{1 / p}
$$

because the integrals in this case are just sums and $(i, j) \rightarrow a_{i j}$ is measurable.

### 9.7 Exercises

1. Establish the inequality $\|f g\|_{r} \leq\|f\|_{p}\|g\|_{q}$ whenever $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$.
2. Let $(\Omega, \mathscr{S}, \mu)$ be counting measure on $\mathbb{N}$. Thus $\Omega=\mathbb{N}$ and $\mathscr{S}=\mathscr{P}(\mathbb{N})$ with $\mu(S)=$ number of things in $S$. Let $1 \leq p \leq q$. Show that in this case,

$$
L^{1}(\mathbb{N}) \subseteq L^{p}(\mathbb{N}) \subseteq L^{q}(\mathbb{N})
$$

Hint: This is real easy if you consider what $\int_{\Omega} f d \mu$ equals. How are the norms related?
3. Consider the function, $f(x, y)=\frac{x^{p-1}}{p y}+\frac{y^{q-1}}{q x}$ for $x, y>0$ and $\frac{1}{p}+\frac{1}{q}=1$. Show directly that $f(x, y) \geq 1$ for all such $x, y$ and show this implies $x y \leq \frac{x^{p}}{p}+\frac{y^{q}}{q}$.
4. Give an example of a sequence of functions in $L^{p}(\mathbb{R})$ which converges to zero in $L^{p}$ but does not converge pointwise to 0 . Does this contradict the proof of the theorem that $L^{p}$ is complete?
5. Let $\phi: \mathbb{R} \rightarrow \mathbb{R}$ be convex. This means $\phi(\lambda x+(1-\lambda) y) \leq \lambda \phi(x)+(1-\lambda) \phi(y)$ whenever $\lambda \in[0,1]$. Verify that if $x<y<z$, then $\frac{\phi(y)-\phi(x)}{y-x} \leq \frac{\phi(z)-\phi(y)}{z-y}$ and that $\frac{\phi(z)-\phi(x)}{z-x} \leq \frac{\phi(z)-\phi(y)}{z-y}$. Show if $s \in \mathbb{R}$ there exists $\lambda$ such that $\phi(s) \leq \phi(t)+\lambda(s-t)$ for all $t$. Show that if $\phi$ is convex, then $\phi$ is continuous.
6. $\uparrow$ Prove Jensen's inequality. If $\phi: \mathbb{R} \rightarrow \mathbb{R}$ is convex, $\mu(\Omega)=1$, and $f: \Omega \rightarrow \mathbb{R}$ is in $L^{1}(\Omega)$, then $\phi\left(\int_{\Omega} f d u\right) \leq \int_{\Omega} \phi(f) d \mu$. Hint: Let $s=\int_{\Omega} f d \mu$ and use Problem 5.
7. $B(p, q)=\int_{0}^{1} x^{p-1}(1-x)^{q-1} d x, \Gamma(p)=\int_{0}^{\infty} e^{-t} t^{p-1} d t$ for $p, q>0$. The first of these is called the beta function, while the second is the gamma function. Show a.) $\Gamma(p+$ $1)=p \Gamma(p) ;$ b.) $\Gamma(p) \Gamma(q)=B(p, q) \Gamma(p+q)$.
8. Let $f \in C_{c}(0, \infty)$ and define $F(x)=\frac{1}{x} \int_{0}^{x} f(t) d t$. Show

$$
\|F\|_{L^{p}(0, \infty)} \leq \frac{p}{p-1}\|f\|_{L^{p}(0, \infty)}
$$

whenever $p>1$. Hint: Argue there is no loss of generality in assuming $f \geq 0$ and then assume this is so. Integrate $\int_{0}^{\infty}|F(x)|^{p} d x$ by parts as follows: $\int_{0}^{\infty} F^{p} d x=$ show $=0$
$\overbrace{\left.x F^{p}\right|_{0} ^{\infty}}^{\infty}-p \int_{0}^{\infty} x F^{p-1} F^{\prime} d x$. Now show $x F^{\prime}=f-F$ and use this in the last integral. Complete the argument by using Holder's inequality and $p-1=p / q$. The measure is one dimensional Lebesgue measure in this problem.
9. $\uparrow$ Now suppose $f \in L^{p}(0, \infty), p>1$, and $f$ not necessarily in $C_{c}(0, \infty)$. Show that $F(x)=\frac{1}{x} \int_{0}^{x} f(t) d t$ still makes sense for each $x>0$. Show the inequality of Problem 8 is still valid. This inequality is called Hardy's inequality. Hint: To show this, use the above inequality along with the density of $C_{c}(0, \infty)$ in $L^{p}(0, \infty)$.
10. Suppose $f, g \geq 0$. When does equality hold in Holder's inequality?
11. Let $\alpha \in(0,1]$. We define, for $X$ a compact subset of $\mathbb{R}^{p}$,

$$
C^{\alpha}\left(X ; \mathbb{R}^{n}\right) \equiv\left\{\mathbf{f} \in C\left(X ; \mathbb{R}^{n}\right): \rho_{\alpha}(\mathbf{f})+\|\mathbf{f}\| \equiv\|\mathbf{f}\|_{\alpha}<\infty\right\}
$$

where $\|\mathbf{f}\| \equiv \sup \{|\mathbf{f}(\mathbf{x})|: \mathbf{x} \in X\}$ and

$$
\rho_{\alpha}(\mathbf{f}) \equiv \sup \left\{\frac{|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{\alpha}}: \mathbf{x}, \mathbf{y} \in X, \mathbf{x} \neq \mathbf{y}\right\}
$$

Show that $\left(C^{\alpha}\left(X ; \mathbb{R}^{n}\right),\|\cdot\|_{\alpha}\right)$ is a complete normed linear space. This is called a Holder space. What would this space consist of if $\alpha>1$ ?
12. Let $\left\{\mathbf{f}_{n}\right\}_{n=1}^{\infty} \subseteq C^{\alpha}\left(X ; \mathbb{R}^{n}\right)$ where $X$ is a compact subset of $\mathbb{R}^{p}$ and suppose $\left\|\mathbf{f}_{n}\right\|_{\alpha} \leq M$ for all $n$. Show there exists a subsequence, $n_{k}$, such that $\mathbf{f}_{n_{k}}$ converges in $C\left(X ; \mathbb{R}^{n}\right)$. We say the given sequence is precompact when this happens. (This also shows the embedding of $C^{\alpha}\left(X ; \mathbb{R}^{n}\right)$ into $C\left(X ; \mathbb{R}^{n}\right)$ is a compact embedding.) Hint: You might want to use the Ascoli Arzela theorem, Theorem 9.2.4.
13. Let $\mathbf{f}: \mathbb{R} \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be continuous and bounded and let $\mathbf{x}_{0} \in \mathbb{R}^{n}$. If $\mathbf{x}:[0, T] \rightarrow \mathbb{R}^{n}$ and $h>0$, let

$$
\tau_{h} \mathbf{x}(s) \equiv\left\{\begin{array}{l}
\mathbf{x}_{0} \text { if } s \leq h, \\
\mathbf{x}(s-h), \text { if } s>h
\end{array}\right.
$$

For $t \in[0, T]$, let $\mathbf{x}_{h}(t)=\mathbf{x}_{0}+\int_{0}^{t} \mathbf{f}\left(s, \tau_{h} \mathbf{x}_{h}(s)\right) d s$. Show using the Ascoli Arzela theorem that there exists a sequence $h \rightarrow 0$ such that $\mathbf{x}_{h} \rightarrow \mathbf{x}$ in $C\left([0, T] ; \mathbb{R}^{n}\right)$. Next argue $\mathbf{x}(t)=\mathbf{x}_{0}+\int_{0}^{t} \mathbf{f}(s, \mathbf{x}(s)) d s$ and conclude the following theorem. If $\mathbf{f}: \mathbb{R} \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ is continuous and bounded, and if $\mathbf{x}_{0} \in \mathbb{R}^{n}$ is given, there exists a solution to the following initial value problem.

$$
\mathbf{x}^{\prime}=\mathbf{f}(t, \mathbf{x}), t \in[0, T], \mathbf{x}(0)=\mathbf{x}_{0}
$$

This is the Peano existence theorem for ordinary differential equations.
14. Suppose $f \in L^{\infty} \cap L^{1}$. Show $\lim _{p \rightarrow \infty}\|f\|_{L^{p}}=\|f\|_{\infty}$. Hint:

$$
\begin{gathered}
\left(\|f\|_{\infty}-\varepsilon\right)^{p} \mu\left(\left[|f|>\|f\|_{\infty}-\varepsilon\right]\right) \leq \int_{\left[|f|>\|f\|_{\infty}-\varepsilon\right]}|f|^{p} d \mu \leq \\
\int|f|^{p} d \mu=\int|f|^{p-1}|f| d \mu \leq\|f\|_{\infty}^{p-1} \int|f| d \mu
\end{gathered}
$$

Now raise both ends to the $1 / p$ power and take liminf and limsup as $p \rightarrow \infty$. You should get $\|f\|_{\infty}-\varepsilon \leq \liminf \|f\|_{p} \leq \limsup \|f\|_{p} \leq\|f\|_{\infty}$
15. Suppose $\mu(\Omega)<\infty$. Show that if $1 \leq p<q$, then $L^{q}(\Omega) \subseteq L^{p}(\Omega)$. Hint Use Holder's inequality.
16. Show $L^{1}(\mathbb{R}) \nsubseteq L^{2}(\mathbb{R})$ and $L^{2}(\mathbb{R}) \nsubseteq L^{1}(\mathbb{R})$ if Lebesgue measure is used. Hint: Consider $1 / \sqrt{x}$ and $1 / x$.
17. Suppose that $\theta \in[0,1]$ and $r, s, q>0$ with $\frac{1}{q}=\frac{\theta}{r}+\frac{1-\theta}{s}$. show that

$$
\left(\int|f|^{q} d \mu\right)^{1 / q} \leq\left(\left(\int|f|^{r} d \mu\right)^{1 / r}\right)^{\theta}\left(\left(\int|f|^{s} d \mu\right)^{1 / s}\right)^{1-\theta}
$$

If $q, r, s \geq 1$ this says that $\|f\|_{q} \leq\|f\|_{r}^{\theta}\|f\|_{s}^{1-\theta}$. Using this, show that

$$
\ln \left(\|f\|_{q}\right) \leq \theta \ln \left(\|f\|_{r}\right)+(1-\theta) \ln \left(\|f\|_{s}\right)
$$

Hint: $\int|f|^{q} d \mu=\int|f|^{q \theta}|f|^{q(1-\theta)} d \mu$. Now note that $1=\frac{\theta q}{r}+\frac{q(1-\theta)}{s}$ and then use Holder's inequality.
18. Suppose $f$ is a function in $L^{1}(\mathbb{R})$ and $f$ is infinitely differentiable. Is $f^{\prime} \in L^{1}(\mathbb{R})$ ? Hint: What if $\phi \in C_{c}^{\infty}(0,1)$ and $f(x)=\phi\left(2^{n}(x-n)\right)$ for $x \in(n, n+1), f(x)=0$ if $x<0$ ?
19. Let $T$ be a real number, $T<1$. Let $A_{0}=0, A_{n+1}=A_{n}+\frac{1}{2}\left(T-A_{n}^{2}\right)$. Show that $A_{n} \in\left[0, \frac{1+T}{2}\right]$. Use the mean value theorem to show that $f(x) \equiv x+\frac{1}{2}\left(T-x^{2}\right)$ maps $\left[0, \frac{1+T}{2}\right]$ to $\left[0, \frac{1+T}{2}\right]$ and is a contraction map. Obtain a unique square root for $T$ as a fixed point.

## Chapter 10

## Change of Variables

Lemma 10.0.1 Every open set in $\mathbb{R}^{p}$ is the countable disjoint union of half open boxes of the form

$$
\prod_{i=1}^{p}\left(a_{i}, a_{i}+2^{-k}\right]
$$

where $a_{i}=l 2^{-k}$ for some integers, $l, k$ where $k \geq m$. If $\mathscr{B}_{m}$ denotes this collection of half open boxes, then every box of $\mathscr{B}_{m+1}$ is contained in a box of $\mathscr{B}_{m}$ or equals a box of $\mathscr{B}_{m}$.

Proof: Let $m \in \mathbb{N}$ be given and let $k \geq m$. Let $\mathscr{C}_{k}$ denote all half open boxes of the form $\prod_{i=1}^{p}\left(a_{i}, a_{i}+2^{-k}\right]$ where $a_{i}=l 2^{-k}$ for some integer $l$. Thus $\mathscr{C}_{k}$ consists of a countable disjoint collection of boxes whose union is $\mathbb{R}^{p}$. This is sometimes called a tiling of $\mathbb{R}^{p}$. Think of tiles on the floor of a bathroom and you will get the idea. Note that each box has Euclidean diameter no larger than $2^{-k} \sqrt{p}$. This is because if we have two points, $\mathbf{x}, \mathbf{y} \in$ $\prod_{i=1}^{p}\left(a_{i}, a_{i}+2^{-k}\right]$, then $\left|x_{i}-y_{i}\right| \leq 2^{-k}$. Therefore, $|\mathbf{x}-\mathbf{y}| \leq\left(\sum_{i=1}^{p}\left(2^{-k}\right)^{2}\right)^{1 / 2}=2^{-k} \sqrt{p}$. Also, a box of $\mathscr{C}_{k+1}$ is either contained in a box of $\mathscr{C}_{k}$ or it has empty intersection with this box of $\mathscr{C}_{k}$.

Let $U$ be open and let $\mathscr{B}_{1} \equiv$ all sets of $\mathscr{C}_{1}$ which are contained in $U$. If $\mathscr{B}_{1}, \cdots, \mathscr{B}_{k}$ have been chosen, $\mathscr{B}_{k+1} \equiv$ all sets of $\mathscr{C}_{k+1}$ contained in

$$
U \backslash \cup\left(\cup_{i=1}^{k} \mathscr{B}_{i}\right)
$$

Let $\mathscr{B}_{\infty}=\cup_{i=1}^{\infty} \mathscr{B}_{i}$. I claim $\cup \mathscr{B}_{\infty}=U$. Clearly $\cup \mathscr{B}_{\infty} \subseteq U$ because every box of every $\mathscr{B}_{i}$ is contained in $U$. If $p \in U$, let $k$ be the smallest integer such that $p$ is contained in a box from $\mathscr{C}_{k}$ which is also a subset of $U$. Thus $p \in \cup \mathscr{B}_{k} \subseteq \cup \mathscr{B}_{\infty}$. Hence $\mathscr{B}_{\infty}$ is the desired countable disjoint collection of half open boxes whose union is $U$. The last claim follows from the construction.

### 10.1 Linear Transformations

Lemma 10.1.1 Let $A: \mathbb{R}^{p} \rightarrow \mathbb{R}^{p}$ be linear and invertible. Then $A$ maps open sets to open sets.

Proof: This follows from the observation that if $B$ is any linear transformation, then $B$ is continuous. Indeed, it is realized by matrix multiplication and so it is clear that if $\mathbf{x}_{n} \rightarrow \mathbf{x}$, then $B \mathbf{x}_{n} \rightarrow B \mathbf{x}$. Then for $U$ open, $A(U)=\left(A^{-1}\right)^{-1}(U)$ which is open because $A^{-1}$ is continuous.

First is a general result.
Proposition 10.1.2 Let $\mathbf{h}: U \rightarrow \mathbb{R}^{p}$ is continuous where $U$ is an open subset of $\mathbb{R}^{p}$. Also suppose $\mathbf{h}$ is differentiable on $H \subseteq U$ where $H$ is Lebesgue measurable. Then if $E$ is a Lebesgue measurable set contained in $H$, then $\mathbf{h}(E)$ is also Lebesgue measurable. Also if $N \subseteq H$ is a set of measure zero, then $\mathbf{h}(N)$ is a set of measure zero. In particular, a linear function A maps measurable sets to measurable sets.

Proof: Consider the second claim first. Let $N$ be a set of measure zero contained in $H$ and let

$$
N_{k} \equiv\{\mathbf{x} \in N:\|D \mathbf{h}(\mathbf{x})\| \leq k\}
$$

There is an open set $V \supseteq N_{k}$ such that $m_{p}(V)<\varepsilon$. For each $\mathbf{x} \in N_{k}$, there is a ball $B_{\mathbf{x}}$ centered at $\mathbf{x}$ with radius $5 r_{\mathbf{x}}<1$ such that $\hat{B}_{\mathbf{x}} \subseteq V$, where $B_{\mathbf{x}}=B\left(\mathbf{x}, r_{\mathbf{x}}\right), \hat{B}_{\mathbf{x}}=B\left(\mathbf{x}, 5 r_{\mathbf{x}}\right)$ and for $\mathbf{y} \in \hat{B}_{\mathbf{x}}$,

$$
\begin{aligned}
\mathbf{h}(\mathbf{y}) & \in \mathbf{h}(\mathbf{x})+D \mathbf{h}(\mathbf{x}) B\left(\mathbf{0}, 5 r_{\mathbf{x}}\right)+B\left(\mathbf{0}, \varepsilon 5 r_{\mathbf{x}}\right) \\
& \subseteq \mathbf{h}(\mathbf{x})+B\left(\mathbf{0},\|D \mathbf{h}(\mathbf{x})\| 5 r_{\mathbf{x}}\right)+B\left(\mathbf{0}, \varepsilon 5 r_{\mathbf{x}}\right) \\
& \leq B\left(\mathbf{h}(\mathbf{x}),(k+\varepsilon) 5 r_{\mathbf{x}}\right)
\end{aligned}
$$

$\operatorname{So} \mathbf{h}\left(B\left(\mathbf{x}, 5 r_{\mathbf{x}}\right)\right) \leq B\left(\mathbf{h}(\mathbf{x}),(k+\varepsilon) 5 r_{\mathbf{x}}\right)$ and so

$$
\overline{m_{p}}\left(\mathbf{h}\left(\hat{B}_{\mathbf{x}}\right)\right) \leq(k+\varepsilon)^{p} m_{p}\left(B\left(\mathbf{x}, 5 r_{\mathbf{x}}\right)\right)
$$

Then, the balls $B\left(\mathbf{x}, r_{\mathbf{x}}\right)$ for $\mathbf{x} \in N_{k}$, cover $N_{k}$ and so by the Vitali covering theorem, there are disjoint balls $B_{i}=B\left(\mathbf{x}_{i}, r_{\mathbf{x}_{i}}\right)$ such that for $\hat{B}_{i}$ the ball with same center and 5 times the radius as $B_{i}, N_{k} \subseteq \cup_{k} \hat{B}_{k}$. Thus

$$
\begin{aligned}
\overline{m_{p}}\left(\mathbf{h}\left(N_{k}\right)\right) & \subseteq \overline{m_{p}}\left(\cup_{k}\left(\mathbf{h}\left(\hat{B}_{k}\right)\right)\right) \leq \sum_{k} \overline{m_{p}}\left(\mathbf{h}\left(\hat{B}_{k}\right)\right) \\
& \leq \sum_{k}(k+\varepsilon)^{p} m_{p}\left(\hat{B}_{k}\right)=\sum_{k}(k+\varepsilon)^{p} 5^{p} m_{p}\left(B_{k}\right) \\
& \leq 5^{p}(k+\varepsilon)^{p} m_{p}(V)<\varepsilon 5^{p}(k+\varepsilon)^{p}
\end{aligned}
$$

Since $\varepsilon>0$ is arbitrary, it follows that $\overline{m_{p}}\left(\mathbf{h}\left(N_{k}\right)\right)=0$ and so $\mathbf{h}\left(N_{k}\right)$ is measurable and has measure zero. Now let $k \rightarrow \infty$ to conclude that $m_{p}(\mathbf{h}(N))=0$.

Now the other claim is shown as follows. By Proposition 8.3.2, if $E$ is Lebesgue measurable, $E \subseteq H$, there is an $F_{\sigma}$ set $F \subseteq E$ such that $m_{p}(E \backslash F)=0$. Then $\mathbf{h}(F)$ is clearly measurable because $\mathbf{h}$ is continuous and $F$ is a countable union of compact sets. Thus $\mathbf{h}(E)=\mathbf{h}(F) \cup \mathbf{h}(E \backslash F)$ and the second was just shown measurable while the first is an $F_{\sigma}$ set so it is actually a Borel set.

From Linear Algebra,(My Elementary Linear Algebra book has the necessary theorems carefully proved.) if $A$ is an invertible linear transformation, it is the composition of finitely many invertible linear transformations which are of the following form.

$$
\begin{aligned}
&\left(\begin{array}{ccccccc}
x_{1} & \cdots & x_{r} & \cdots & x_{s} & \cdots & x_{p}
\end{array}\right)^{T} \rightarrow\left(\begin{array}{lllllll}
x_{1} & \cdots & x_{r} & \cdots & x_{s} & \cdots & x_{p}
\end{array}\right)^{T} \\
&\left(\begin{array}{lllll}
x_{1} & \cdots & x_{r} & \cdots & x_{p}
\end{array}\right)^{T} \rightarrow\left(\begin{array}{llllll}
x_{1} & \cdots & c x_{r} & \cdots & x_{p}
\end{array}\right)^{T}, c \neq 0 \\
&\left(\begin{array}{lllllll}
x_{1} & \cdots & x_{r} & \cdots & x_{S} & \cdots & x_{p}
\end{array}\right)^{T} \\
& \rightarrow\left(\begin{array}{lllllll}
x_{1} & \cdots & x_{r} & \cdots & x_{s}+x_{r} & \cdots & x_{p}
\end{array}\right)^{T}
\end{aligned}
$$

where these are the actions obtained by multiplication by elementary matrices. Denote these special linear transformations by $E(r \leftrightarrow s), E(c r), E(s \rightarrow s+r)$.

Let $R=\prod_{i=1}^{p}\left(a_{i}, b_{i}\right)$. Then it is easily seen that

$$
m_{p}(E(r \leftrightarrow s)(R))=m_{p}(R)=|\operatorname{det}(E(r \leftrightarrow s))| m_{p}(R)
$$

since this transformation just switches two sides of $R$.

$$
m_{p}(E(c r)(R))=|c| m_{p}(R)=|\operatorname{det}(E(c r))| m_{p}(R)
$$

since this transformation just magnifies one side, multiplying it by $c$.
The other linear transformation which represents a sheer is a little harder. However,

$$
\begin{aligned}
m_{p}(E(s \rightarrow s+r)(R)) & =\int_{E(s \rightarrow s+r)(R)} d m_{p} \\
& =\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathscr{X}_{E(s \rightarrow s+r)(R)} d x_{s} d x_{r} d x_{p_{1}} \cdots d x_{p_{p-2}}
\end{aligned}
$$

Now recall Theorem 7.8 .5 which says you can integrate using the usual Riemann integral when the function involved is Borel. Thus the above becomes

$$
\begin{aligned}
& \int_{a_{p_{p-2}}}^{b_{p_{p-2}}} \cdots \int_{a_{p_{1}}}^{b_{p_{1}}} \int_{a_{r}}^{b_{r}} \int_{a_{s}+x_{r}}^{b_{s}+x_{r}} d x_{s} d x_{r} d x_{p_{1}} \cdots d x_{p_{p-2}} \\
= & m_{p}(R)=|\operatorname{det}(E(s \rightarrow s+r))| m_{p}(R)
\end{aligned}
$$

Recall that when a row (column) is added to another row (column), the determinant of the resulting matrix is unchanged.

Lemma 10.1.3 Let L be any of the above elementary linear transformations. Then

$$
m_{p}(L(F))=|\operatorname{det}(L)| m_{p}(F)
$$

for any Borel set $F$. Also $L(F)$ is Lebesgue measurable if $F$ is Lebesgue measurable. If $F$ is Borel, then so is $L(F)$.

Proof: Let $R_{k}=\prod_{i=1}^{p}(-k, k)$. Let $\mathscr{G}$ be those Borel sets $F$ such that $L(F)$ is Borel and

$$
\begin{equation*}
m_{p}\left(L\left(F \cap R_{k}\right)\right)=|\operatorname{det}(L)| m_{p}\left(F \cap R_{k}\right) \tag{10.1}
\end{equation*}
$$

Letting $\mathscr{K}$ be the open rectangles, it follows from the above discussion that the pi system $\mathscr{K}$ is in $\mathscr{G}$. It is also obvious that if $F_{i} \in \mathscr{G}$ the $F_{i}$ being disjoint, then since $L$ is one to one,

$$
\begin{aligned}
m_{p}\left(L\left(\cup_{i=1}^{\infty} F_{i} \cap R_{k}\right)\right) & =\sum_{i=1}^{\infty} m_{p}\left(L\left(F_{i} \cap R_{k}\right)\right)=|\operatorname{det}(L)| \sum_{i=1}^{\infty} m_{p}\left(F_{i} \cap R_{k}\right) \\
& =|\operatorname{det}(L)| m_{p}\left(\cup_{i=1}^{\infty} F_{i} \cap R_{k}\right)
\end{aligned}
$$

Thus $\mathscr{G}$ is closed with respect to countable disjoint unions. If $F \in \mathscr{G}$ then

$$
\begin{gathered}
m_{p}\left(L\left(F^{C} \cap R_{k}\right)\right)+m_{p}\left(L\left(F \cap R_{k}\right)\right)=m_{p}\left(L\left(R_{k}\right)\right) \\
m_{p}\left(L\left(F^{C} \cap R_{k}\right)\right)+|\operatorname{det}(L)| m_{p}\left(F \cap R_{k}\right)=|\operatorname{det}(L)| m_{p}\left(R_{k}\right) \\
m_{p}\left(L\left(F^{C} \cap R_{k}\right)\right)=|\operatorname{det}(L)| m_{p}\left(R_{k}\right)-|\operatorname{det}(L)| m_{p}\left(F \cap R_{k}\right) \\
=|\operatorname{det}(L)| m_{p}\left(F^{C} \cap R_{k}\right)
\end{gathered}
$$

It follows that $\mathscr{G}$ is closed with respect to complements also. Therefore, $\mathscr{G}=\sigma(\mathscr{K})=$ $\mathscr{B}\left(\mathbb{R}^{p}\right)$. Now let $k \rightarrow \infty$ in 10.1 to obtain the desired conclusion.

Theorem 10.1.4 Let $L$ be a linear transformation which is invertible. Then for any Borel F, $L(F)$ is Borel and

$$
m_{p}(L(F))=|\operatorname{det}(L)| m_{p}(F)
$$

More generally, if $L$ is an arbitrary linear transformation, then for any $F \in \mathscr{F}_{p}$,

$$
L(F) \in \mathscr{F}_{p}
$$

and the above formula holds.
Proof: From linear algebra, there are $L_{i}$ each elementary such that $L=L_{1} \circ L_{2} \circ \cdots \circ L_{s}$. By Proposition 10.1.2, each $L_{i}$ maps Borel sets to Borel sets. Hence, using Lemma 10.1.3,

$$
\begin{aligned}
m_{p}(L(F)) & =\left|\operatorname{det}\left(L_{1}\right)\right| m_{p}\left(L_{2} \circ \cdots \circ L_{s}(F)\right) \\
& =\left|\operatorname{det}\left(L_{1}\right)\right|\left|\operatorname{det}\left(L_{2}\right)\right| m_{p}\left(L_{3} \circ \cdots \circ L_{s}(F)\right) \\
& =\cdots=\prod_{i=1}^{s}\left|\operatorname{det}\left(L_{i}\right)\right| m_{p}(F)=|\operatorname{det}(L)| m_{p}(F)
\end{aligned}
$$

the last claim from properties of the determinant.
Next consider the general case. First I clam that if $N$ has measure 0 then so does $L(N)$ and if $F \in \mathscr{F}_{p}$, then so is $L(F) \in \mathscr{F}_{p}$ for any linear $L$. This follows from Proposition 10.1.2 since $L$ is differentiable.

By Proposition 8.3.2, if $E \in \mathscr{F}_{p}$, then for $L$ invertible, there is an $F_{\sigma}$ set $F$ and a $G_{\delta}$ set $G$ such that $m_{p}(G \backslash F)=0$ and $F \subseteq E \subseteq G$. Then for $L$ invertible,

$$
m_{p}(L(F)) \leq m_{p}(L(E)) \leq m_{p}(L(G))
$$

and so, since $F, G$ are Borel,

$$
\begin{aligned}
|\operatorname{det}(L)| m_{p}(F) & \leq m_{p}(L(E)) \leq|\operatorname{det}(L)| m_{p}(G) \\
& =|\operatorname{det}(L)| m_{p}(F)=|\operatorname{det}(L)| m_{p}(E)
\end{aligned}
$$

and so all the inequalities are equal signs. Hence, $m_{p}(L(E))=|\operatorname{det}(L)| m_{p}(E)$.
If $L^{-1}$ does not exist and $E \in \mathscr{F}_{p}$, then there are elementary matrices $L_{k}$ such that $L_{1} \circ$ $L_{2} \circ \cdots \circ L_{m} \circ L$ maps $\mathbb{R}^{p}$ to $\left\{\mathbf{x} \in \mathbb{R}^{p}: x_{p}=0\right\}$, a set of $m_{p}$ measure zero. By completeness of Lebesgue measure, $L_{1} \circ L_{2} \circ \cdots \circ L_{m} \circ L(E)$ and $L(E)$ are both measurable and

$$
\prod_{i=1}^{m}\left|\operatorname{det}\left(L_{i}\right)\right| m_{p}(L(E))=m_{p}\left(L_{1} \circ L_{2} \circ \cdots \circ L_{m} \circ L(E)\right)=0
$$

so in this case, $m_{p}(L(E))=0=|\operatorname{det}(L)| m_{p}(0)$. Thus the formula holds regardless.
For $A, B$ nonempty sets in $\mathbb{R}^{p}, A+B$ denotes all vectors of the form $a+b$ where $a \in A$ and $b \in B$. Thus if $Q$ is a linear transformation,

$$
Q(A+B)=Q A+Q B
$$

The following proposition uses standard linear algebra to obtain an interesting estimate on the measure of a set. It is illustrated by the following picture.


In the above picture, the slanted set is of the form $B+D$ where $B$ is a ball and the un-slanted version is obtained by doing the linear transformation $Q$ to the slanted set. The reason the two look the same is that the $Q$ used will preserve all distances. It will be an orthogonal linear transformation.

Proposition 10.1.5 Let the norm be the standard Euclidean norm and let $V$ be a $k$ dimensional subspace of $\mathbb{R}^{p}$ where $k<p$. Suppose $D$ is a $F_{\sigma}$ subset of $V$ which has diameter d. Then

$$
m_{p}(D+B(\mathbf{0}, r)) \leq 2^{p}(d+r)^{p-1} r
$$

Proof: Let $\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{k}\right\}$ be an orthonormal basis for $V$. Enlarge to an orthonormal basis of all of $\mathbb{R}^{p}$ using the Gram Schmidt process to obtain

$$
\left\{\mathbf{v}_{1}, \cdots, \mathbf{v}_{k}, \mathbf{v}_{k+1}, \cdots, \mathbf{v}_{p}\right\} .
$$

Now define an orthogonal transformation $Q$ by $Q \mathbf{v}_{i}=\mathbf{e}_{i}$. Thus $Q^{T} Q=I$ and $Q$ preserves all lengths. Thus also $\operatorname{det}(Q)=1$. Then

$$
Q(D+B(\mathbf{0}, r))=Q D+B(\mathbf{0}, r)
$$

where the diameter of $Q D$ is the same as the diameter of $D$ and $Q B(\mathbf{0}, r)=B(\mathbf{0}, r)$ because $Q$ preserves lengths in the Euclidean norm. This is why we use this norm rather than some other. Therefore, from the definition of the Lebesgue measure and the above result on the magnification factor,

$$
m_{p}(D+B(\mathbf{0}, r))=\operatorname{det}(Q) m_{p}(D+B(\mathbf{0}, r))=m_{p}(Q D+B(\mathbf{0}, r))
$$

and this last is no larger than $(2 d+2 r)^{p-1} 2 r=2^{p}(d+r)^{p-1} r$.

### 10.2 Change of Variables Nonlinear Maps

The very interesting approach given here follows Rudin [40].
Now recall Lemma 8.7.10 which is stated here for convenience.
Lemma 10.2.1 Let $\mathbf{g}$ be continuous and map $\overline{B(\mathbf{p}, r)} \subseteq \mathbb{R}^{n}$ to $\mathbb{R}^{n}$. Suppose that for all $\mathbf{x} \in \overline{B(\mathbf{p}, r)}$,

$$
|\mathbf{g}(\mathbf{x})-\mathbf{x}|<\varepsilon r
$$

Then it follows that

$$
\mathbf{g}(\overline{B(\mathbf{p}, r)}) \supseteq B(\mathbf{p},(1-\varepsilon) r)
$$

Now suppose $U \subseteq \mathbb{R}^{p}$ is open, $\mathbf{h}: U \rightarrow \mathbb{R}^{p}$ is continuous, and

$$
m_{p}(\mathbf{h}(U \backslash H))=0
$$

where $H \subseteq U$ and $H$ is Borel measurable. Suppose also that $\mathbf{h}$ is one to one and differentiable on $H$. Define for Lebesgue measurable $E \subseteq U$

$$
\lambda(E) \equiv m_{p}(\mathbf{h}(E \cap H))
$$

Then it is clear that $\lambda$ is indeed a measure on the $\sigma$ algebra of Lebesgue measurable subsets of $U$. Note that

$$
\begin{aligned}
& m_{p}(\mathbf{h}(E))-m_{p}(\mathbf{h}(E \cap H)) \\
\leq & m_{p}(\mathbf{h}(E \cap H))+m_{p}(\mathbf{h}(E \backslash H))-m_{p}(\mathbf{h}(E \cap H)) \\
\leq & m_{p}(\mathbf{h}(U \backslash H))=0
\end{aligned}
$$

Thus one could just as well let $\lambda(E) \equiv m_{p}(\mathbf{h}(E))$.
Since $\mathbf{h}$ is one to one on $H$, this along with Proposition 10.1.2 implies that $\lambda \ll m_{p}$ since if $m_{p}(E)=0$, then $\mathbf{h}(E \cap H)$ also has measure zero. Also $\lambda$ and $m_{p}$ are finite on closed balls so both are $\sigma$ finite.

Therefore, for measurable $E \subseteq U$, it follows from the Radon Nikodym theorem Corollary 7.11.12 that there is a real valued, nonnegative, measurable function $f$ in $L^{1}(K)$ for any compact set $K$ such that

$$
\begin{equation*}
\lambda(E)=m_{p}(\mathbf{h}(E \cap H))=\int_{U} \mathscr{X}_{E} f(\mathbf{x}) d m_{p}=\int \mathscr{X}_{E} f(\mathbf{x}) d m_{p} \tag{10.2}
\end{equation*}
$$

So what is $f(\mathbf{x})$ ? To begin with, assume $D \mathbf{h}(\mathbf{x})^{-1}$ exists. By differentiability, and using $D \mathbf{h}(\mathbf{x})^{-1}$ exists as needed,

$$
\begin{aligned}
\mathbf{h}(B(\mathbf{x}, r))-\mathbf{h}(\mathbf{x}) & \subseteq D \mathbf{h}(\mathbf{x}) B(\mathbf{0}, r)+D \mathbf{h}(\mathbf{x}) B(\mathbf{0}, \varepsilon r) \\
& \subseteq D \mathbf{h}(\mathbf{x})(B(\mathbf{0}, r(1+\varepsilon)))
\end{aligned}
$$

for all $r$ small enough. Therefore, by translation invariance of Lebesgue measure,

$$
\begin{aligned}
m_{p}(\mathbf{h}(B(\mathbf{x}, r))) & \leq m_{p}(D \mathbf{h}(\mathbf{x})(B(\mathbf{0}, r(1+\boldsymbol{\varepsilon})))) \\
& =|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| m_{p}(B(\mathbf{0}, r(1+\boldsymbol{\varepsilon})))
\end{aligned}
$$

Also, for $|\mathbf{v}|<r$ small enough,

$$
\begin{aligned}
\mathbf{h}(\mathbf{x}+\mathbf{v})-\mathbf{h}(\mathbf{x}) & =D \mathbf{h}(\mathbf{x}) \mathbf{v}+\mathbf{o}(\mathbf{v}) \\
D \mathbf{h}(\mathbf{x})^{-1}(\mathbf{h}(\mathbf{x}+\mathbf{v})-\mathbf{h}(\mathbf{x})) & =\mathbf{v}+\mathbf{o}(\mathbf{v})
\end{aligned}
$$

and so if $\mathbf{g}(\mathbf{v}) \equiv D \mathbf{h}(\mathbf{x})^{-1}(\mathbf{h}(\mathbf{x}+\mathbf{v})-\mathbf{h}(\mathbf{x})),|\mathbf{g}(\mathbf{v})-\mathbf{v}|<\varepsilon r$ provided $r$ is small enough. Therefore, from Lemma 10.2.1, for small enough $r$,

$$
D \mathbf{h}(\mathbf{x})^{-1}(\mathbf{h}(\mathbf{x}+B(\mathbf{0}, r))-\mathbf{h}(\mathbf{x})) \supseteq B(\mathbf{0},(1-\varepsilon) r)
$$

Thus

$$
\mathbf{h}(B(\mathbf{x}, r)) \supseteq \mathbf{h}(\mathbf{x})+D \mathbf{h}(\mathbf{x}) B(\mathbf{0},(1-\varepsilon) r)
$$

and so, using Theorem 10.1.4 again,

$$
m_{p}(\mathbf{h}(B(\mathbf{x}, r))) \geq|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| m_{p}(B(\mathbf{0}, r(1-\varepsilon)))
$$

for $r$ small enough. Thus, since

$$
m_{p}(\mathbf{h}(B(\mathbf{x}, r)))=m_{p}(\mathbf{h}(B(\mathbf{x}, r) \cap H))=\lambda(B(\mathbf{x}, r))=\int_{B(\mathbf{x}, r)} f(\mathbf{x}) d m_{p}
$$

it follows that

$$
\begin{aligned}
|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| m_{p}(B(\mathbf{0}, r(1-\varepsilon))) & \leq \int_{B(\mathbf{x}, r)} f(\mathbf{x}) d m_{p} \\
& \leq|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| m_{p}(B(\mathbf{0}, r(1+\varepsilon)))
\end{aligned}
$$

for $r$ small enough. Now divide by $m_{p}(B(\mathbf{x}, r))$ and use the fundamental theorem of calculus Corollary 9.5.3 to find that for $\varepsilon$ small $\varepsilon>0$,

$$
|\operatorname{det}(D \mathbf{h}(\mathbf{x}))|(1-\varepsilon)^{p} \leq f(\mathbf{x}) \leq|\operatorname{det}(D \mathbf{h}(\mathbf{x}))|(1+\varepsilon)^{p} \text { a.e. }
$$

Letting $\varepsilon_{k} \rightarrow 0$ and picking a set of measure zero for each $\varepsilon_{k}$, it follows that off a set of measure zero $f(\mathbf{x})=|\operatorname{det}(D \mathbf{h}(\mathbf{x}))|$.

If $D \mathbf{h}(\mathbf{x})^{-1}$ does not exist, then you have

$$
\mathbf{h}(B(\mathbf{x}, r))-\mathbf{h}(\mathbf{x}) \subseteq D \mathbf{h}(\mathbf{x}) B(\mathbf{0}, r)+B(\mathbf{0}, \varepsilon r)
$$

and $D \mathbf{h}(\mathbf{x})$ maps into a bounded subset of a $p-1$ dimensional subspace. Therefore, using Proposition 10.1.5, the right side has measure no more than an expression of the form $\mathrm{Cr}^{p} \varepsilon$, $C$ depending on $D \mathbf{h}(\mathbf{x})$. Therefore, in this case,

$$
\lim _{r \rightarrow 0} \frac{1}{m_{p}(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} f(\mathbf{x}) d m_{p}=\lim _{r \rightarrow 0} \frac{m_{p}(\mathbf{h}(B(\mathbf{x}, r)))}{m_{p}(B(\mathbf{x}, r))} \leq \varepsilon \frac{C r^{p}}{\alpha_{p} r^{p}}
$$

and since $\varepsilon$ is arbitrary, this shows that for a.e. $\mathbf{x}$, such that $D \mathbf{h}(\mathbf{x})^{-1}$ does not exist, $f(\mathbf{x})=$ $0=|\operatorname{det}(D \mathbf{h}(\mathbf{x}))|$ in this case also. Therefore, whenever $E$ is a Lebesgue measurable set $E \subseteq H$,

$$
\begin{aligned}
m_{p}(\mathbf{h}(E)) & =\int_{\mathbf{h}(H)} \mathscr{X}_{\mathbf{h}(E)}(\mathbf{y}) d m_{p}=\int_{U} \mathscr{X}_{E}(\mathbf{x})|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d m_{p} \\
& =\int_{H} \mathscr{X}_{E}(\mathbf{x})|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d m_{p}
\end{aligned}
$$

The difficulty here is that the inverse image of a Lebesgue measurable set might not be measurable. However, there is no problem with the inverse image of a Borel set. Let $F$ be a Borel subset of the measurable set $\mathbf{h}(H)$. Then $\mathbf{h}^{-1}(F)$ is measurable. Indeed, $\mathbf{h}^{-1}(F)$ is open if $F$ is open. If $\mathscr{B}$ consists of the sets $\mathbf{h}^{-1}(F)$ were $F$ is Borel, $\mathscr{B}$ is a $\sigma$ algebra which contains the open sets. Thus $\mathscr{B}$ contains the Borel sets. Then from what was just shown,

$$
\int_{\mathbf{h}(H)} \mathscr{X}_{\mathbf{h}\left(\mathbf{h}^{-1}(F)\right)}(\mathbf{y}) d m_{p}=\int_{H} \mathscr{X}_{\mathbf{h}^{-1}(F)}(\mathbf{x})|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d m_{p}
$$

and rewriting this gives

$$
\int_{\mathbf{h}(H)} \mathscr{X}_{F}(\mathbf{y}) d m_{p}=\int_{H} \mathscr{X}_{F}(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d m_{p}
$$

and all needed measurability holds. Now for $E$ an arbitrary measurable subset of $H$ let $F \subseteq E \subseteq G$ where $F$ is $F_{\sigma}$ and $G$ is $G_{\delta}$ and $m_{p}(G \backslash F)=0$. Without loss of generality, assume $G \subseteq U$. Then

$$
\int_{H}\left(\mathscr{X}_{G}(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))|-\mathscr{X}_{F}(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))|\right) d m_{p}=0
$$

because the integral of each function in the difference equals

$$
\int_{\mathbf{h}(H)} \mathscr{X}_{E}(\mathbf{y}) d m_{p}=\int_{\mathbf{h}(H)} \mathscr{X}_{F}(\mathbf{y}) d m_{p}=\int_{\mathbf{h}(H)} \mathscr{X}_{G}(\mathbf{y}) d m_{p}
$$

and so the integrands are equal off a set of measure zero. Furthermore,

$$
\mathscr{X}_{E}(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| \in\left[\mathscr{X}_{F}(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))|, \mathscr{X}_{G}(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))|\right]
$$

and so $\mathscr{X}_{E}(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))|$ equals a measurable function a.e. By completeness of the measure $m_{p}$ it follows that $\mathbf{x} \rightarrow \mathscr{X}_{E}(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))|$ is also measurable. I am not saying that $\mathbf{x} \rightarrow \mathscr{X}_{E}(\mathbf{h}(\mathbf{x}))$ is measurable. In fact this might not be so because it is $\mathscr{X}_{\mathbf{h}^{-1}(E)}(\mathbf{x})$ and the inverse image of a measurable set is not necessarily measurable. For a well known example, see Problem 26 on Page 156. The thing which is measurable is the product in the integrand. Therefore, for $E$ a measurable subset of $H$

$$
\begin{equation*}
\int_{\mathbf{h}(H)} \mathscr{X}_{E}(\mathbf{y}) d m_{p}=\int_{H} \mathscr{X}_{E}(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d m_{p} \tag{10.3}
\end{equation*}
$$

The following theorem gives a change of variables formula.
Theorem 10.2.2 Let $U \subseteq \mathbb{R}^{p}$ be open, $\mathrm{h}: U \rightarrow \mathbb{R}^{p}$ continuous, and

$$
m_{p}(\mathbf{h}(U \backslash H))=0
$$

where $H \subseteq U$ and $H$ is Lebesgue measurable. Suppose also that $\mathbf{h}$ is differentiable on $H$ and is one to one on $H$. Then $\mathbf{h}(H)$ is Lebesgue measurable and if $g \geq 0$ is Lebesgue measurable, then

$$
\begin{equation*}
\int_{\mathbf{h}(H)} g(\mathbf{y}) d m_{p}=\int_{H} g(\mathbf{h}(\mathbf{x}))|\operatorname{det}(D \mathbf{h}(\mathbf{x}))| d m_{p} \tag{10.4}
\end{equation*}
$$

and all needed measurability holds.
Proof: Formula 10.3 implies that 10.4 holds for any nonnegative simple function $s$. Then for $g$ nonnegative and measurable, it is the pointwise increasing limit of such simple functions. Therefore, 10.4 follows from the monotone convergence theorem.

Note that the above theorem holds if $H=U$. One might wonder why the fuss over having a separate $H$ on which $\mathbf{h}$ is differentiable. One reason for this is Rademacher's theorem which states that every Lipshitz continuous function is differentiable a.e. Thus if you have a Lipshitz function defined on $U$ an open set, then if you let $H$ be the set where this function is differentiable, it will follow that $U \backslash H$ has measure zero and so, by Problem 9 on Page 212, you also have $\mathbf{h}(U \backslash H)$ has measure zero. Thus the above theorem is at least as good as what is needed to give a change of variables formula for transformations which are only Lipschitz continuous.

Next is a significant result called Sard's lemma. In the proof, it does not matter which norm you use in defining balls but it may be easiest to consider the norm

$$
\|\mathbf{x}\| \equiv \max \left\{\left|x_{i}\right|, i=1, \cdots, p\right\}
$$

Lemma 10.2.3 (Sard) Let $U$ be an open set in $\mathbb{R}^{p}$ and let $\mathbf{h}: U \rightarrow \mathbb{R}^{p}$ be differentiable. Let

$$
Z \equiv\{\mathbf{x} \in U: \operatorname{det} D \mathbf{h}(\mathbf{x})=0\}
$$

Then $m_{p}(\mathbf{h}(Z))=0$.
Proof: For convenience, assume the balls in the following argument come from $\|\cdot\|_{\infty}$. First note that $Z$ is a Borel set because $\mathbf{h}$ is continuous and so the component functions of the Jacobian matrix are each Borel measurable. Hence the determinant is also Borel measurable.

Suppose that $U$ is a bounded open set. Let $\varepsilon>0$ be given. Also let $V \supseteq Z$ with $V \subseteq U$ open, and

$$
m_{p}(Z)+\varepsilon>m_{p}(V)
$$

Now let $\mathbf{x} \in Z$. Then since $\mathbf{h}$ is differentiable at $\mathbf{x}$, there exists $\delta_{\mathbf{x}}>0$ such that if $r<\delta_{\mathbf{x}}$, then $B(\mathbf{x}, r) \subseteq V$ and also,

$$
\mathbf{h}(B(\mathbf{x}, r)) \subseteq \mathbf{h}(\mathbf{x})+D \mathbf{h}(\mathbf{x})(B(\mathbf{0}, r))+B(\mathbf{0}, r \eta), \eta<1
$$

Regard $D \mathbf{h}(\mathbf{x})$ as an $n \times n$ matrix, the matrix of the linear transformation $D \mathbf{h}(\mathbf{x})$ with respect to the usual coordinates. Since $\mathbf{x} \in Z$, it follows that there exists an invertible matrix $A$ such that $A D \mathbf{h}(\mathbf{x})$ is in row reduced echelon form with a row of zeros on the bottom. Therefore,

$$
\begin{equation*}
m_{p}(A(\mathbf{h}(B(\mathbf{x}, r)))) \leq m_{p}(A D \mathbf{h}(\mathbf{x})(B(\mathbf{0}, r))+A B(\mathbf{0}, r \eta)) \tag{10.5}
\end{equation*}
$$

The diameter of $A D \mathbf{h}(\mathbf{x})(B(\mathbf{0}, r))$ is no larger than $\|A\|\|D \mathbf{h}(\mathbf{x})\| 2 r$ and it lies in $\mathbb{R}^{p-1} \times$ $\{0\}$. The diameter of $A B(\mathbf{0}, r \eta)$ is no more than $\|A\|(2 r \eta)$. Therefore, the measure of the right side in 10.5 is no more than

$$
\begin{aligned}
& {[(\|A\|\|D \mathbf{h}(\mathbf{x})\| 2 r+\|A\|(2 \eta)) r]^{p-1}(r \eta) } \\
\leq & C(\|A\|,\|D \mathbf{h}(\mathbf{x})\|)(2 r)^{p} \eta
\end{aligned}
$$

Hence from the change of variables formula for linear maps,

$$
m_{p}(\mathbf{h}(B(\mathbf{x}, r))) \leq \eta \frac{C(\|A\|,\|D \mathbf{h}(\mathbf{x})\|)}{|\operatorname{det}(A)|} m_{p}(B(\mathbf{x}, r))
$$

Then letting $\delta_{\mathbf{x}}$ be still smaller if necessary, corresponding to sufficiently small $\eta$,

$$
m_{p}(\mathbf{h}(B(\mathbf{x}, r))) \leq \varepsilon m_{p}(B(\mathbf{x}, r))
$$

The balls of this form constitute a Vitali cover of $Z$. Hence, by the Vitali covering theorem Theorem 8.6.6, there exists $\left\{B_{i}\right\}_{i=1}^{\infty}, B_{i}=B_{i}\left(\mathbf{x}_{i}, r_{i}\right)$, a collection of disjoint balls, each of which is contained in $V$, such that $m_{p}\left(\mathbf{h}\left(B_{i}\right)\right) \leq \varepsilon m_{p}\left(B_{i}\right)$ and $m_{p}\left(Z \backslash \cup_{i} B_{i}\right)=0$. Hence from Lemma 10.1.2,

$$
m_{p}\left(\mathbf{h}(Z) \backslash \cup_{i} \mathbf{h}\left(B_{i}\right)\right) \leq m_{p}\left(\mathbf{h}\left(Z \backslash \cup_{i} B_{i}\right)\right)=0
$$

Therefore,

$$
\begin{aligned}
m_{p}(\mathbf{h}(Z)) & \leq \sum_{i} m_{p}\left(\mathbf{h}\left(B_{i}\right)\right) \leq \varepsilon \sum_{i} m_{p}\left(B_{i}\right) \\
& \leq \varepsilon\left(m_{p}(V)\right) \leq \varepsilon\left(m_{p}(Z)+\varepsilon\right)
\end{aligned}
$$

Since $\varepsilon$ is arbitrary, this shows $m_{p}(\mathbf{h}(Z))=0$. What if $U$ is not bounded? Then consider $Z_{n}=Z \cap B(\mathbf{0}, n)$. From what was just shown, $\mathbf{h}\left(Z_{n}\right)$ has measure 0 and so it follows that $\mathbf{h}(Z)$ also does, being the countable union of sets of measure zero.

### 10.3 Mappings Which are Not One to One

Now suppose $\mathbf{h}: U \rightarrow V=\mathbf{h}(U)$ and $\mathbf{h}$ is only $C^{1}$, not necessarily one to one. Note that I am using $C^{1}$, not just differentiable. This makes it convenient to use the inverse function theorem. You can get more generality if you work harder. For

$$
U_{+} \equiv\{\mathbf{x} \in U:|\operatorname{det} D \mathbf{h}(x)|>0\}
$$

and $Z$ the set where $|\operatorname{det} D \mathbf{h}(\mathbf{x})|=0$, Lemma 10.2.3 implies $m_{p}(\mathbf{h}(Z))=0$. For $\mathbf{x} \in U_{+}$, the inverse function theorem implies there exists an open set $B_{\mathbf{x}} \subseteq U_{+}$, such that $\mathbf{h}$ is one to one on $B_{\mathbf{x}}$.

Let $\left\{B_{i}\right\}$ be a countable subset of $\left\{B_{\mathbf{x}}\right\}_{\mathbf{x} \in U_{+}}$such that $U_{+}=\cup_{i=1}^{\infty} B_{i}$. Let $E_{1}=B_{1}$. If $E_{1}, \cdots, E_{k}$ have been chosen, $E_{k+1}=B_{k+1} \backslash \cup_{i=1}^{k} E_{i}$. Thus

$$
\cup_{i=1}^{\infty} E_{i}=U_{+}, \mathbf{h} \text { is one to one on } E_{i}, E_{i} \cap E_{j}=\emptyset
$$

and each $E_{i}$ is a Borel set contained in the open set $B_{i}$. Now define

$$
n(\mathbf{y}) \equiv \sum_{i=1}^{\infty} \mathscr{X}_{\mathbf{h}\left(E_{i}\right)}(\mathbf{y})+\mathscr{X}_{\mathbf{h}(Z)}(\mathbf{y})
$$

The sets $\mathbf{h}\left(E_{i}\right), \mathbf{h}(Z)$ are measurable by Proposition 10.1.2. Thus $n(\cdot)$ is measurable.
Lemma 10.3.1 Let $F \subseteq \mathbf{h}(U)$ be measurable. Then

$$
\int_{\mathbf{h}(U)} n(\mathbf{y}) \mathscr{X}_{F}(\mathbf{y}) d m_{p}=\int_{U} \mathscr{X}_{F}(\mathbf{h}(\mathbf{x}))|\operatorname{det} D \mathbf{h}(\mathbf{x})| d m_{p}
$$

Proof: Using Lemma 10.2.3 and the Monotone Convergence Theorem

$$
\begin{gathered}
\int_{\mathbf{h}(U)} n(\mathbf{y}) \mathscr{X}_{F}(\mathbf{y}) d m_{p}=\int_{\mathbf{h}(U)}(\sum_{i=1}^{\infty} \mathscr{X}_{\mathbf{h}\left(E_{i}\right)}(\mathbf{y})+\overbrace{\mathscr{X}_{\mathbf{h}(Z)}(\mathbf{y})}^{m_{p}(\mathbf{h}(Z))=0}) \mathscr{X}_{F}(\mathbf{y}) d m_{p} \\
=\sum_{i=1}^{\infty} \int_{\mathbf{h}(U)} \mathscr{X}_{\mathbf{h}\left(E_{i}\right)}(\mathbf{y}) \mathscr{X}_{F}(\mathbf{y}) d m_{p} \\
=\sum_{i=1}^{\infty} \int_{\mathbf{h}\left(B_{i}\right)} \mathscr{X}_{\mathbf{h}\left(E_{i}\right)}(\mathbf{y}) \mathscr{X}_{F}(\mathbf{y}) d m_{p}=\sum_{i=1}^{\infty} \int_{B_{i}} \mathscr{X}_{E_{i}}(\mathbf{x}) \mathscr{X}_{F}(\mathbf{h}(\mathbf{x}))|\operatorname{det} D \mathbf{h}(\mathbf{x})| d m_{p} \\
=\sum_{i=1}^{\infty} \int_{U} \mathscr{X}_{E_{i}(\mathbf{x})} \mathscr{X}_{F}(\mathbf{h}(\mathbf{x}))|\operatorname{det} D \mathbf{h}(\mathbf{x})| d m_{p} \\
=\int_{U} \sum_{i=1}^{\infty} \mathscr{X}_{E_{i}}(\mathbf{x}) \mathscr{X}_{F}(\mathbf{h}(\mathbf{x}))|\operatorname{det} D \mathbf{h}(\mathbf{x})| d m_{p} \\
=\int_{U_{+}} \mathscr{X}_{F}(\mathbf{h}(\mathbf{x}))|\operatorname{det} D \mathbf{h}(\mathbf{x})| d m_{p}=\int_{U} \mathscr{X}_{F}(\mathbf{h}(\mathbf{x}))|\operatorname{det} D \mathbf{h}(\mathbf{x})| d m_{p} .
\end{gathered}
$$

Definition 10.3.2 For $\mathbf{y} \in \mathbf{h}(U)$, define a function, \#, according to the formula $\#(\mathbf{y}) \equiv$ number of elements in $\mathbf{h}^{-1}(\mathbf{y})$.

Observe that

$$
\begin{equation*}
\#(\mathbf{y})=n(\mathbf{y}) \quad \text { a.e. } \tag{10.6}
\end{equation*}
$$

because $n(\mathbf{y})=\#(\mathbf{y})$ if $\mathbf{y} \notin \mathbf{h}(Z)$, a set of measure 0 . Therefore, \# is a measurable function because of completeness of Lebesgue measure.

Theorem 10.3.3 Let $g \geq 0$, $g$ measurable, and let $\mathbf{h}$ be $C^{1}(U)$. Then

$$
\begin{equation*}
\int_{\mathbf{h}(U)} \#(\mathbf{y}) g(\mathbf{y}) d m_{p}=\int_{U} g(\mathbf{h}(\mathbf{x}))|\operatorname{det} D \mathbf{h}(\mathbf{x})| d m_{p} \tag{10.7}
\end{equation*}
$$

In fact, you can have $E$ some Borel measurable subset of $U$ and conclude that

$$
\int_{\mathbf{h}(E)} \#(\mathbf{y}) g(\mathbf{y}) d m_{p}=\int_{E} g(\mathbf{h}(\mathbf{x}))|\operatorname{det} D \mathbf{h}(\mathbf{x})| d m_{p}
$$

Proof: From 10.6 and Lemma 10.3.1, 10.7 holds for all $g$, a nonnegative simple function. Approximating an arbitrary measurable nonnegative function $g$, with an increasing pointwise convergent sequence of simple functions and using the monotone convergence theorem, yields 10.7 for an arbitrary nonnegative measurable function $g$. To get the last claim, simply replace $g$ with $g \mathscr{X}_{\mathbf{h}(E)}$ in the first formula.

### 10.4 Spherical Coordinates in $p$ Dimensions

Sometimes there is a need to deal with spherical coordinates in more than three dimensions. In this section, this concept is defined and formulas are derived for these coordinate systems. Recall polar coordinates are of the form

$$
\begin{aligned}
& y_{1}=\rho \cos \theta \\
& y_{2}=\rho \sin \theta
\end{aligned}
$$

where $\rho>0$ and $\theta \in \mathbb{R}$. Thus these transformation equations are not one to one but they are one to one on $(0, \infty) \times[0,2 \pi)$. Here I am writing $\rho$ in place of $r$ to emphasize a pattern which is about to emerge. I will consider polar coordinates as spherical coordinates in two dimensions. I will also simply refer to such coordinate systems as polar coordinates regardless of the dimension. This is also the reason I am writing $y_{1}$ and $y_{2}$ instead of the more usual $x$ and $y$. Now consider what happens when you go to three dimensions. The situation is depicted in the following picture.


From this picture, you see that $y_{3}=\rho \cos \phi_{1}$. Also the distance between $\left(y_{1}, y_{2}\right)$ and $(0,0)$ is $\rho \sin \left(\phi_{1}\right)$. Therefore, using polar coordinates to write $\left(y_{1}, y_{2}\right)$ in terms of $\theta$ and this distance,

$$
\begin{aligned}
& y_{1}=\rho \sin \phi_{1} \cos \theta \\
& y_{2}=\rho \sin \phi_{1} \sin \theta \\
& y_{3}=\rho \cos \phi_{1}
\end{aligned}
$$

where $\phi_{1} \in \mathbb{R}$ and the transformations are one to one if $\phi_{1}$ is restricted to be in $[0, \pi]$. What was done is to replace $\rho$ with $\rho \sin \phi_{1}$ and then to add in $y_{3}=\rho \cos \phi_{1}$. Having done this, there is no reason to stop with three dimensions. Consider the following picture:


From this picture, you see that $y_{4}=\rho \cos \phi_{2}$. Also the distance from $\left(y_{1}, y_{2}, y_{3}\right)$ to $(0,0,0)$ is $\rho \sin \left(\phi_{2}\right)$. Therefore, using polar coordinates to write $\left(y_{1}, y_{2}, y_{3}\right)$ in terms of $\theta, \phi_{1}$, and this distance,

$$
\begin{aligned}
& y_{1}=\rho \sin \phi_{2} \sin \phi_{1} \cos \theta \\
& y_{2}=\rho \sin \phi_{2} \sin \phi_{1} \sin \theta \\
& y_{3}=\rho \sin \phi_{2} \cos \phi_{1} \\
& y_{4}=\rho \cos \phi_{2}
\end{aligned}
$$

where $\phi_{2} \in \mathbb{R}$ and the transformations will be one to one if

$$
\phi_{2}, \phi_{1} \in(0, \pi), \theta \in(0,2 \pi), \rho \in(0, \infty)
$$

Continuing this way, given spherical coordinates in $\mathbb{R}^{p}$, to get the spherical coordinates in $\mathbb{R}^{p+1}$, you let $y_{p+1}=\rho \cos \phi_{p-1}$ and then replace every occurance of $\rho$ with $\rho \sin \phi_{p-1}$ to obtain $y_{1}, \cdots, y_{p}$ in terms of $\phi_{1}, \phi_{2}, \cdots, \phi_{p-1}, \theta$, and $\rho$.

It is always the case that $\rho$ measures the distance from the point in $\mathbb{R}^{p}$ to the origin in $\mathbb{R}^{p}, \mathbf{0}$. Each $\phi_{i} \in \mathbb{R}$ and the transformations will be one to one if each $\phi_{i} \in(0, \pi)$, and $\theta \in(0,2 \pi)$. Denote by $\mathbf{h}_{p}(\rho, \vec{\phi}, \theta)$ the above transformation.

It can be shown using math induction and geometric reasoning that these coordinates map $\prod_{i=1}^{p-2}(0, \pi) \times(0,2 \pi) \times(0, \infty)$ one to one onto an open subset of $\mathbb{R}^{p}$ which is everything except for the set of measure zero $\Psi_{p}(N)$ where $N$ results from having some $\phi_{i}$ equal to 0 or $\pi$ or for $\rho=0$ or for $\theta$ equal to either $2 \pi$ or 0 . Each of these are sets of Lebesgue measure zero and so their union is also a set of measure zero. You can see that $\mathbf{h}_{p}\left(\prod_{i=1}^{p-2}(0, \pi) \times(0,2 \pi) \times(0, \infty)\right)$ omits the union of the coordinate axes except for maybe one of them. This is not important to the integral because it is just a set of measure zero.

Theorem 10.4.1 Let $\mathbf{y}=\mathbf{h}_{p}(\vec{\phi}, \theta, \rho)$ be the spherical coordinate transformations in $\mathbb{R}^{p}$. Then letting $A=\prod_{i=1}^{p-2}(0, \pi) \times(0,2 \pi)$, it follows $\mathbf{h}$ maps $A \times(0, \infty)$ one to one onto all of $\mathbb{R}^{p}$ except a set of measure zero given by $\mathbf{h}_{p}(N)$ where $N$ is the set of measure zero

$$
(\bar{A} \times[0, \infty)) \backslash(A \times(0, \infty))
$$

Also $\left|\operatorname{det} D \mathbf{h}_{p}(\vec{\phi}, \theta, \rho)\right|$ will always be of the form

$$
\begin{equation*}
\left|\operatorname{det} D \mathbf{h}_{p}(\vec{\phi}, \theta, \rho)\right|=\rho^{p-1} \Phi(\vec{\phi}, \theta) \tag{10.8}
\end{equation*}
$$

where $\Phi$ is a continuous function of $\vec{\phi}$ and $\theta .{ }^{1}$ Then if $f$ is nonnegative and Lebesgue measurable,

$$
\begin{equation*}
\int_{\mathbb{R}^{p}} f(\mathbf{y}) d m_{p}=\int_{\mathbf{h}_{p}(A)} f(\mathbf{y}) d m_{p}=\int_{A} f\left(\mathbf{h}_{p}(\vec{\phi}, \theta, \rho)\right) \rho^{p-1} \Phi(\vec{\phi}, \theta) d m_{p} \tag{10.9}
\end{equation*}
$$

Furthermore whenever $f$ is Borel measurable and nonnegative, one can apply Fubini's theorem and write

$$
\begin{equation*}
\int_{\mathbb{R}^{p}} f(\mathbf{y}) d y=\int_{0}^{\infty} \rho^{p-1} \int_{A} f(\mathbf{h}(\vec{\phi}, \theta, \rho)) \Phi(\vec{\phi}, \theta) d \vec{\phi} d \theta d \rho \tag{10.10}
\end{equation*}
$$

where here $d \vec{\phi} d \theta$ denotes $d m_{p-1}$ on $A$. The same formulas hold if $f \in L^{1}\left(\mathbb{R}^{p}\right)$.
Proof: Formula 10.8 is obvious from the definition of the spherical coordinates because in the matrix of the derivative, there will be a $\rho$ in $p-1$ columns. The first claim is also clear from the definition and math induction or from the geometry of the above description. It remains to verify 10.9 and 10.10 . It is clear $\mathbf{h}_{p}$ maps $\bar{A} \times[0, \infty)$ onto $\mathbb{R}^{p}$. Since $\mathbf{h}_{p}$ is differentiable, it maps sets of measure zero to sets of measure zero. Then

$$
\mathbb{R}^{p}=\mathbf{h}_{p}(N \cup A \times(0, \infty))=\mathbf{h}_{p}(N) \cup \mathbf{h}_{p}(A \times(0, \infty)),
$$

the union of a set of measure zero with $\mathbf{h}_{p}(A \times(0, \infty))$. Therefore, from the change of variables formula,

$$
\begin{aligned}
\int_{\mathbb{R}^{p}} f(\mathbf{y}) d m_{p} & =\int_{\mathbf{h}_{p}(A \times(0, \infty))} f(\mathbf{y}) d m_{p} \\
& =\int_{A \times(0, \infty)} f\left(\mathbf{h}_{p}(\vec{\phi}, \theta, \rho)\right) \rho^{p-1} \Phi(\vec{\phi}, \theta) d m_{p}
\end{aligned}
$$

which proves 10.9. This formula continues to hold if $f$ is in $L^{1}\left(\mathbb{R}^{p}\right)$ by consideration of positive and negative parts of real and imaginary parts.

Finally, if $f \geq 0$ or in $L^{1}\left(\mathbb{R}^{n}\right)$ and is Borel measurable, the Borel sets denoted as $\mathscr{B}\left(\mathbb{R}^{p}\right)$ then one can write the following. From the definition of $m_{p}$

$$
\begin{aligned}
& \int_{A \times(0, \infty)} f\left(\mathbf{h}_{p}(\vec{\phi}, \theta, \rho)\right) \rho^{p-1} \Phi(\vec{\phi}, \theta) d m_{p} \\
= & \int_{(0, \infty)} \int_{A} f\left(\mathbf{h}_{p}(\vec{\phi}, \theta, \rho)\right) \rho^{p-1} \Phi(\vec{\phi}, \theta) d m_{p-1} d m \\
= & \int_{(0, \infty)} \rho^{p-1} \int_{A} f\left(\mathbf{h}_{p}(\vec{\phi}, \theta, \rho)\right) \Phi(\vec{\phi}, \theta) d m_{p-1} d m
\end{aligned}
$$

Now the claim about $f \in L^{1}$ follows routinely from considering the positive and negative parts of the real and imaginary parts of $f$ in the usual way.

Note that the above equals $\int_{\bar{A} \times[0, \infty)} f\left(\mathbf{h}_{p}(\vec{\phi}, \theta, \rho)\right) \rho^{p-1} \Phi(\vec{\phi}, \theta) d m_{p} \quad$ and the iterated integral is also equal to

$$
\int_{[0, \infty)} \rho^{p-1} \int_{\bar{A}} f\left(\mathbf{h}_{p}(\vec{\phi}, \theta, \rho)\right) \Phi(\vec{\phi}, \theta) d m_{p-1} d m
$$

because the difference is just a set of measure zero.

[^5]Notation 10.4.2 Often this is written differently. Note that from the spherical coordinate formulas, $f(\mathbf{h}(\vec{\phi}, \theta, \rho))=f(\rho \omega)$ where $|\omega|=1$. Letting $S^{p-1}$ denote the unit sphere, $\left\{\omega \in \mathbb{R}^{p}:|\omega|=1\right\}$, the inside integral in the above formula is sometimes written as

$$
\int_{S^{p-1}} f(\rho \omega) d \sigma
$$

where $\sigma$ is a measure on $S^{p-1}$. See [27] for another description of this measure. It isn't an important issue here. Either 10.10 or the formula

$$
\int_{0}^{\infty} \rho^{p-1}\left(\int_{S^{p-1}} f(\rho \omega) d \sigma\right) d \rho
$$

will be referred to as polar coordinates and is very useful in establishing estimates. Here $\sigma\left(S^{p-1}\right) \equiv \int_{A} \Phi(\vec{\phi}, \theta) d m_{p-1}$.

Example 10.4.3 For what values of $s$ is the integral $\int_{B(\mathbf{0}, R)}\left(1+|\mathbf{x}|^{2}\right)^{s}$ dy bounded independent of $R$ ? Here $B(\mathbf{0}, R)$ is the ball, $\left\{\mathbf{x} \in \mathbb{R}^{p}:|\mathbf{x}| \leq R\right\}$.

I think you can see immediately that $s$ must be negative but exactly how negative? It turns out it depends on $p$ and using polar coordinates, you can find just exactly what is needed. From the polar coordinates formula above,

$$
\begin{aligned}
\int_{B(\mathbf{0}, R)}\left(1+|\mathbf{x}|^{2}\right)^{s} d y & =\int_{0}^{R} \int_{S^{p-1}}\left(1+\rho^{2}\right)^{s} \rho^{p-1} d \sigma d \rho \\
& =C_{p} \int_{0}^{R}\left(1+\rho^{2}\right)^{s} \rho^{p-1} d \rho
\end{aligned}
$$

Now the very hard problem has been reduced to considering an easy one variable problem of finding when $\int_{0}^{R} \rho^{p-1}\left(1+\rho^{2}\right)^{s} d \rho$ is bounded independent of $R$. You need $2 s+$ $(p-1)<-1$ so you need $s<-p / 2$.

### 10.5 Approximation with Smooth Functions

It is very important to be able to approximate measurable and integrable functions with continuous functions having compact support. Recall Theorem 9.4.2. This implies the following.
Theorem 10.5.1 Let $f \geq 0$ be $\mathscr{F}_{n}$ measurable and let $\int f d m_{n}<\infty$. Then there exists a sequence of continuous functions $\left\{h_{n}\right\}$ which are zero off a compact set such that $\lim _{n \rightarrow \infty} \int\left|f-h_{n}\right|^{p} d m_{n}=0$.

Definition 10.5.2 Let $U$ be an open subset of $\mathbb{R}^{n} . C_{c}^{\infty}(U)$ is the vector space of all infinitely differentiable functions which equal zero for all $\mathbf{x}$ outside of some compact set contained in $U$. Similarly, $C_{c}^{m}(U)$ is the vector space of all functions which are $m$ times continuously differentiable and whose support is a compact subset of $U$.

Corollary 10.5.3 Let $U$ be a nonempty open set in $\mathbb{R}^{n}$ and let $f \in L^{p}\left(U, m_{n}\right)$. Then there exists $g \in C_{c}(U)$ such that

$$
\int|f-g|^{p} d m_{n}<\varepsilon
$$

Proof: If $f \geq 0$ then extend it to be 0 off $U$. In the above argument, let all functions involved, the simple functions and the continuous functions be zero off $U$. Simply intersect all $V_{i}$ with $U$ and no harm is done. Now to extend to $L^{p}(U)$, simply apply Theorem 10.5.1 to the positive and negative parts of real and imaginary parts of $f$.

Example 10.5.4 Let $U=B(\mathbf{z}, 2 r)$

$$
\psi(\mathbf{x})=\left\{\begin{array}{l}
\exp \left[\left(|\mathbf{x}-\mathbf{z}|^{2}-r^{2}\right)^{-1}\right] \text { if }|\mathbf{x}-\mathbf{z}|<r \\
0 \text { if }|\mathbf{x}-\mathbf{z}| \geq r
\end{array}\right.
$$

Then a little work shows $\psi \in C_{c}^{\infty}(U)$. The following also is easily obtained.
Lemma 10.5.5 Let $U$ be any open set. Then $C_{c}^{\infty}(U) \neq \emptyset$.
Proof: Pick $\mathbf{z} \in U$ and let $r$ be small enough that $B(\mathbf{z}, 2 r) \subseteq U$. Then let

$$
\psi \in C_{c}^{\infty}(B(\mathbf{z}, 2 r)) \subseteq C_{c}^{\infty}(U)
$$

be the function of the above example.
For a different approach see Problem 13 on Page 213.
This leads to a really remarkable result about approximation with smooth functions.
Definition 10.5.6 Let $U=\left\{\mathbf{x} \in \mathbb{R}^{n}:|\mathbf{x}|<1\right\}$. A sequence $\left\{\psi_{m}\right\} \subseteq C_{c}^{\infty}(U)$ is called a mollifier (This is sometimes called an approximate identity if the differentiability is not included.) if

$$
\psi_{m}(\mathbf{x}) \geq 0, \psi_{m}(\mathbf{x})=0, \text { if }|\mathbf{x}| \geq \frac{1}{m}
$$

and $\int \psi_{m}(\mathbf{x})=1$. Sometimes it may be written as $\left\{\psi_{\varepsilon}\right\}$ where $\psi_{\varepsilon}$ satisfies the above conditions except $\psi_{\varepsilon}(\mathbf{x})=0$ if $|\mathbf{x}| \geq \varepsilon$. In other words, $\varepsilon$ takes the place of $1 / m$. There certainly exist mollifiers. Let $\psi \in C_{c}^{\infty}(B(\mathbf{0}, 1)), \psi(\mathbf{x}) \geq 0, \int \psi(\mathbf{x}) d m_{n}=1$. Then let

$$
\psi_{m}(\mathbf{x}) \equiv c_{m} \psi(m \mathbf{x})
$$

where $c_{m}$ is chosen to make $\int c_{m} \psi(m \mathbf{x}) d m_{n}=1$. Thus $\psi_{m}$ is 0 off $B\left(\mathbf{0}, \frac{1}{m}\right)$.
The notation $\int f(\mathbf{x}, \mathbf{y}) d \mu(\mathbf{y})$ will mean $\mathbf{x}$ is fixed and the function $\mathbf{y} \rightarrow f(\mathbf{x}, \mathbf{y})$ is being integrated. To make the notation more familiar, $d x$ is written instead of $d m_{n}(x)$.

Lemma 10.5.7 Let $g \in C_{c}(U)$ then there exists $h \in C_{c}^{\infty}(U)$ such that

$$
\int|g-h|^{p} d m_{n}<\varepsilon
$$

Proof: Let $\psi_{m}$ be a mollifier. Consider

$$
h_{m}(\mathbf{x}) \equiv \int g(\mathbf{x}-\mathbf{y}) \psi_{m}(\mathbf{y}) d m_{n}(y)
$$

Then since the integral of $\psi_{m}$ is 1 , it follows that

$$
h_{m}(\mathbf{x})-g(\mathbf{x})=\int(g(\mathbf{x}-\mathbf{y})-g(\mathbf{x})) \psi_{m}(\mathbf{y}) d m_{n}(y)
$$

Since $g$ is zero off a compact set, it follows that $g$ is uniformly continuous and so there is $\delta>0$ such that if $|\mathbf{x}-\hat{\mathbf{x}}|<\delta$, then $|g(\mathbf{x})-g(\hat{\mathbf{x}})|<\varepsilon$. Choose $m$ such that $1 / m<\delta$. Then

$$
\begin{aligned}
\left|h_{m}(\mathbf{x})-g(\mathbf{x})\right| & =\left|\int(g(\mathbf{x}-\mathbf{y})-g(\mathbf{x})) \psi_{m}(\mathbf{y}) d m_{n}\right| \\
& \leq \int_{B(\mathbf{0}, 1 / m)}|g(\mathbf{x}-\mathbf{y})-g(\mathbf{x})| \psi_{m}(\mathbf{y}) d m_{n}(y) \\
& <\int_{B(\mathbf{0}, 1 / m)} \varepsilon \psi_{m}(\mathbf{y}) d m_{n}(y)=\varepsilon
\end{aligned}
$$

This is true for all $\mathbf{x}$. Now note that $h_{m}$ is only nonzero if $\mathbf{x} \in K+B(\mathbf{0}, 1 / m)$ where $K$ is defined as the compact set off which $g$ equals 0 . Since $K$ is contained in $U$, it follows that $K+B(\mathbf{0}, 1 / m) \subseteq U$ for all $m$ small enough. In fact, $K+\overline{B(\mathbf{0}, 1 / m)}$ is a compact set contained in $U$ off which $h_{m}$ is zero for all $m$ large enough because $\overline{B(\mathbf{0}, 1 /(m+1))}$ $\subseteq \overline{B(\mathbf{0}, 1 / m)}$. Thus $h_{m}$ is zero off a compact subset of $U$. In addition to this, $h_{m}$ is infinitely differentiable. To see this last claim, note that

$$
h_{m}(\mathbf{x})=\int g(\mathbf{x}-\mathbf{y}) \psi_{m}(\mathbf{y}) d m_{n}(y)=\int g(\mathbf{y}) \psi_{m}(\mathbf{x}-\mathbf{y}) d m_{n}(y)
$$

This follows from the change of variables formulas presented above.
To see the function is differentiable,

$$
\frac{h_{m}\left(\mathbf{x}+h \mathbf{e}_{i}\right)-h(\mathbf{x})}{h}=\int g(\mathbf{y}) \frac{\psi_{m}\left(\mathbf{x}+h \mathbf{e}_{i}-\mathbf{y}\right)-\psi_{m}(\mathbf{x}-\mathbf{y})}{h} d m_{n}(y)
$$

and now, since $\psi_{m}$ is zero off a compact set, it and its partial derivatives of all order are uniformly continuous. Hence, one can pass to a limit and obtain

$$
h_{x_{i}}(\mathbf{x})=\int g(\mathbf{y}) \frac{\partial \psi_{m}(\mathbf{x})}{\partial x_{i}} d m_{n}(y)
$$

Repeat the same argument using the partial derivative of $\psi_{m}$ in place of $\psi_{m}$. Continuing this way, one obtains the existence of all partial derivatives at any $\mathbf{x}$. Thus $h_{m} \in C_{c}^{\infty}(U)$ for all $m$ large enough and $\int\left|h_{m}-g\right|^{p} d m_{n}<\varepsilon$ for all $m$ large enough.

Note that this would have worked for $\mu$ an arbitrary regular measure.
Now it is obvious that the functions in $C_{c}^{\infty}(U)$ are dense in $L^{p}(U), p \geq 1$. Pick $f \in$ $L^{p}(U)$. Then there exists $g \in C_{c}(U)$ such that

$$
\left(\int_{U}|f-g|^{p} d m_{n}\right)^{1 / p}<\varepsilon / 2
$$

and there is $h \in C_{c}^{\infty}(U)$ such that

$$
\left(\int_{U}|h-g|^{p} d m_{n}\right)^{1 / p}<\varepsilon / 2
$$

Then

$$
\begin{aligned}
& \left(\int_{U}|f-h|^{p} d m_{n}\right)^{1 / p} \leq\left(\int_{U}|f-g|^{p} d m_{n}\right)^{1 / p} \\
+ & \left(\int_{U}|h-g|^{p} d m_{n}\right)^{1 / p}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon \boldsymbol{\square}
\end{aligned}
$$

Theorem 10.5.8 Let $U$ be an open set in $\mathbb{R}^{n}$ and let $f \in L^{p}(U)$. Then there exists $h \in C_{c}^{\infty}(U)$ such that

$$
\left(\int_{U}|f-h|^{p} d m_{n}\right)^{1 / p}<\varepsilon
$$

In words, $C_{c}^{\infty}(U)$ is dense in $L^{p}(U)$.
Functions which vanish off a compact set are said to have "compact support". Note that all of this would work for any regular measure $\mu$. Now what follows will be dependent on the measure being Lebesgue measure or something like it.

### 10.6 Continuity of Translation

This is a property which directly exploits density of continuous functions with compact support and the translation invariance of Lebesgue measure.
Definition 10.6.1 Let $f$ be a function defined on $U \subseteq \mathbb{R}^{n}$ and let $\mathbf{w} \in \mathbb{R}^{n}$. Then $f_{\mathbf{w}}$ will be the function defined on $\mathbf{w}+U$ by

$$
f_{\mathbf{w}}(\mathbf{x})=f(\mathbf{x}-\mathbf{w}) .
$$

We will write $\operatorname{spt}(g)$ to indicate the closure of the set on which $g$ is nonzero. This is called the support of the function.
Theorem 10.6.2 (Continuity of translation in $\left.L^{p}\right)$ Let $f \in L^{p}\left(\mathbb{R}^{n}\right)$ with the measure being Lebesgue measure. Then

$$
\lim _{\|\mathbf{w}\| \rightarrow 0}\left\|f_{\mathbf{w}}-f\right\|_{p}=0
$$

Proof: Let $\varepsilon>0$ be given and let $g \in C_{c}\left(\mathbb{R}^{n}\right)$ with $\|g-f\|_{p}<\frac{\varepsilon}{3}$. Since Lebesgue measure is translation invariant $\left(m_{n}(\mathbf{w}+E)=m_{n}(E)\right)$,

$$
\left\|g_{\mathbf{w}}-f_{\mathbf{w}}\right\|_{p}=\|g-f\|_{p}<\frac{\varepsilon}{3} .
$$

You can see this from looking at simple functions and passing to the limit or you could use the change of variables formula to verify it.

Therefore

$$
\begin{align*}
\left\|f-f_{\mathbf{w}}\right\|_{p} & \leq\|f-g\|_{p}+\left\|g-g_{\mathbf{w}}\right\|_{p}+\left\|g_{\mathbf{w}}-f_{\mathbf{w}}\right\| \\
& <\frac{2 \varepsilon}{3}+\left\|g-g_{\mathbf{w}}\right\|_{p} \tag{10.11}
\end{align*}
$$

But $\lim _{|\mathbf{w}| \rightarrow 0} g_{\mathbf{w}}(\mathbf{x})=g(\mathbf{x})$ uniformly in $\mathbf{x}$ because $g$ is uniformly continuous. Now let $B$ be a large ball containing spt $(g)$ and let $\delta_{1}$ be small enough that $B(\mathbf{x}, \boldsymbol{\delta}) \subseteq B$ whenever $\mathbf{x} \in \operatorname{spt}(g)$. If $\varepsilon>0$ is given there exists $\delta<\delta_{1}$ such that if $|\mathbf{w}|<\delta$, it follows that $|g(\mathbf{x}-\mathbf{w})-g(\mathbf{x})|<\varepsilon / 3\left(1+m_{n}(B)^{1 / p}\right)$. Therefore,

$$
\left\|g-g_{\mathbf{w}}\right\|_{p}=\left(\int_{B}|g(\mathbf{x})-g(\mathbf{x}-\mathbf{w})|^{p} d m_{n}\right)^{1 / p} \leq \varepsilon \frac{m_{n}(B)^{1 / p}}{3\left(1+m_{n}(B)^{1 / p}\right)}<\frac{\varepsilon}{3} .
$$

Therefore, whenever $|\mathbf{w}|<\delta$, it follows $\left\|g-g_{\mathbf{w}}\right\|_{p}<\frac{\varepsilon}{3}$ and so from $10.11\left\|f-f_{\mathbf{w}}\right\|_{p}<\varepsilon$.

### 10.7 Separability

When dealing with a Radon measure, (complete, Borel, regular, and finite on compact sets) one can assert that the $L^{p}$ spaces are separable. Recall this means that they have a countable dense subset.

Theorem 10.7.1 For $p \geq 1$ and $\mu$ a Radon measure, $L^{p}\left(\mathbb{R}^{n}, \mu\right)$ is separable. Recall this means there exists a countable set, $\mathscr{D}$, such that if $f \in L^{p}\left(\mathbb{R}^{n}, \mu\right)$ and $\varepsilon>0$, there exists $g \in \mathscr{D}$ such that $\|f-g\|_{p}<\varepsilon$.

Proof: Let $Q$ be all functions of the form $c \mathscr{X}_{[\mathbf{a}, \mathbf{b})}$ where $[\mathbf{a}, \mathbf{b}) \equiv\left[a_{1}, b_{1}\right) \times\left[a_{2}, b_{2}\right) \times$ $\cdots \times\left[a_{n}, b_{n}\right)$, and both $a_{i}, b_{i}$ are rational, while $c$ has rational real and imaginary parts. Let $\mathscr{D}$ be the set of all finite sums of functions in $Q$. Thus, $\mathscr{D}$ is countable. In fact $\mathscr{D}$ is dense in $L^{p}\left(\mathbb{R}^{n}, \mu\right)$. To prove this it is necessary to show that for every $f \in L^{p}\left(\mathbb{R}^{n}, \mu\right)$, there exists an element of $\mathscr{D}, s$ such that $\|s-f\|_{p}<\varepsilon$. If it can be shown that for every $g \in C_{c}\left(\mathbb{R}^{n}\right)$ there exists $h \in \mathscr{D}$ such that $\|g-h\|_{p}<\varepsilon$, then this will suffice because if $f \in L^{p}\left(\mathbb{R}^{n}\right)$ is arbitrary, Theorem 9.4.2 implies there exists $g \in C_{c}\left(\mathbb{R}^{n}\right)$ such that $\|f-g\|_{p} \leq \frac{\varepsilon}{2}$ and then there would exist $h \in C_{c}\left(\mathbb{R}^{n}\right)$ such that $\|h-g\|_{p}<\frac{\varepsilon}{2}$. By the triangle inequality,

$$
\|f-h\|_{p} \leq\|h-g\|_{p}+\|g-f\|_{p}<\varepsilon .
$$

Therefore, assume at the outset that $f \in C_{c}\left(\mathbb{R}^{n}\right)$.
Let $\mathscr{P}_{m}$ consist of all sets of the form $[\mathbf{a}, \mathbf{b}) \equiv \prod_{i=1}^{n}\left[a_{i}, b_{i}\right)$ where $a_{i}=j 2^{-m}$ and $b_{i}=(j+$ 1) $2^{-m}$ for $j$ an integer. Thus $\mathscr{P}_{m}$ consists of a tiling of $\mathbb{R}^{n}$ into half open rectangles having diameters $2^{-m} n^{\frac{1}{2}}$. There are countably many of these rectangles; so, let $\mathscr{P}_{m}=\left\{\left[\mathbf{a}_{i}, \mathbf{b}_{i}\right)\right\}$ for $i \geq 1$, and $\mathbb{R}^{n}=\cup_{i=1}^{\infty}\left[\mathbf{a}_{i}, \mathbf{b}_{i}\right)$. Let $c_{i}^{m}$ be complex numbers with rational real and imaginary parts satisfying

$$
\begin{equation*}
\left|f\left(\mathbf{a}_{i}\right)-c_{i}^{m}\right|<2^{-m}, \quad\left|c_{i}^{m}\right| \leq\left|f\left(\mathbf{a}_{i}\right)\right| . \tag{10.12}
\end{equation*}
$$

Let $s_{m}(\mathbf{x})=\sum_{i=1}^{\infty} c_{i}^{m} \mathscr{X}_{\left[\mathbf{a}_{i}, \mathbf{b}_{i}\right)}(\mathbf{x})$. Since $f\left(\mathbf{a}_{i}\right)=0$ except for finitely many values of $i$, the above is a finite sum. Then 10.12 implies $s_{m} \in \mathscr{D}$. If $s_{m}$ converges uniformly to $f$ then it follows $\left\|s_{m}-f\right\|_{p} \rightarrow 0$ because $\left|s_{m}\right| \leq|f|$ and so

$$
\begin{aligned}
\left\|s_{m}-f\right\|_{p} & =\left(\int\left|s_{m}-f\right|^{p} d \mu\right)^{1 / p}=\left(\int_{\operatorname{spt}(f)}\left|s_{m}-f\right|^{p} d \mu\right)^{1 / p} \\
& \leq\left[\varepsilon m_{n}(\operatorname{spt}(f))\right]^{1 / p}
\end{aligned}
$$

whenever $m$ is large enough.
Since $f \in C_{c}\left(\mathbb{R}^{n}\right)$ it follows that $f$ is uniformly continuous and so given $\varepsilon>0$ there exists $\delta>0$ such that if $|\mathbf{x}-\mathbf{y}|<\delta,|f(\mathbf{x})-f(\mathbf{y})|<\varepsilon / 2$. Now let $m$ be large enough that every box in $\mathscr{P}_{m}$ has diameter less than $\delta$ and also that $2^{-m}<\varepsilon / 2$. Then if $\left[\mathbf{a}_{i}, \mathbf{b}_{i}\right)$ is one of these boxes of $\mathscr{P}_{m}$, and $\mathbf{x} \in\left[\mathbf{a}_{i}, \mathbf{b}_{i}\right)$,

$$
\left|f(\mathbf{x})-f\left(\mathbf{a}_{i}\right)\right|<\varepsilon / 2
$$

and

$$
\left|f\left(\mathbf{a}_{i}\right)-c_{i}^{m}\right|<2^{-m}<\varepsilon / 2
$$

Therefore, using the triangle inequality, it follows that

$$
\left|f(\mathbf{x})-c_{i}^{m}\right|=\left|s_{m}(\mathbf{x})-f(\mathbf{x})\right|<\varepsilon
$$

and since $\mathbf{x}$ is arbitrary, this establishes uniform convergence.
Here is an easier proof if you know the Weierstrass approximation theorem.
Theorem 10.7.2 For $p \geq 1$ and $\mu$ a Radon measure, $L^{p}\left(\mathbb{R}^{n}, \mu\right)$ is separable. Recall this means there exists a countable set, $\mathscr{D}$, such that if $f \in L^{p}\left(\mathbb{R}^{n}, \mu\right)$ and $\varepsilon>0$, there exists $g \in \mathscr{D}$ such that $\|f-g\|_{p}<\varepsilon$.

Proof: Let $\mathscr{P}$ denote the set of all polynomials which have rational coefficients. Then $\mathscr{P}$ is countable. Let $\tau_{k} \in C_{c}\left((-(k+1),(k+1))^{n}\right)$ such that

$$
\overline{[-k, k]^{n}} \prec \tau_{k} \prec(-(k+1),(k+1))^{n} .
$$

The notation means that $\tau_{k}$ is one on $\overline{[-k, k]^{n}}$, has values between 0 and 1 and vanishes off a compact subset of $(-(k+1),(k+1))^{n}$. Let $\mathscr{D}_{k}$ denote the functions which are of the form, $p \tau_{k}$ where $p \in \mathscr{P}$. Thus $\mathscr{D}_{k}$ is also countable. Let $\mathscr{D} \equiv \cup_{k=1}^{\infty} \mathscr{D}_{k}$. It follows each function in $\mathscr{D}$ is in $C_{c}\left(\mathbb{R}^{n}\right)$ and so it in $L^{p}\left(\mathbb{R}^{n}, \mu\right)$. Let $f \in L^{p}\left(\mathbb{R}^{n}, \mu\right)$. By regularity of $\mu$ there exists $g \in C_{c}\left(\mathbb{R}^{n}\right)$ such that $\|f-g\|_{L^{p}\left(\mathbb{R}^{n}, \mu\right)}<\frac{\varepsilon}{3}$. Let $k$ be such that $\operatorname{spt}(g) \subseteq(-k, k)^{n}$. Now by the Weierstrass approximation theorem there exists a polynomial $q$ such that

$$
\begin{aligned}
\|g-q\|_{[-(k+1), k+1]^{n}} & \equiv \sup \left\{|g(\mathbf{x})-q(\mathbf{x})|: \mathbf{x} \in[-(k+1),(k+1)]^{n}\right\} \\
& <\frac{\varepsilon}{3 \mu\left((-(k+1), k+1)^{n}\right)} .
\end{aligned}
$$

It follows

$$
\begin{aligned}
\left\|g-\tau_{k} q\right\|_{[-(k+1), k+1]^{n}} & =\left\|\tau_{k} g-\tau_{k} q\right\|_{[-(k+1), k+1]^{n}} \\
& <\frac{\varepsilon}{3 \mu\left((-(k+1), k+1)^{n}\right)} .
\end{aligned}
$$

Without loss of generality, it can be assumed this polynomial has all rational coefficients. Therefore, $\tau_{k} q \in \mathscr{D}$.

$$
\begin{aligned}
\left\|g-\tau_{k} q\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}^{p} & =\int_{(-(k+1), k+1)^{n}}\left|g(\mathbf{x})-\tau_{k}(\mathbf{x}) q(\mathbf{x})\right|^{p} d \mu \\
& \leq\left(\frac{\varepsilon}{3 \mu\left((-(k+1), k+1)^{n}\right)}\right)^{p} \mu\left((-(k+1), k+1)^{n}\right)<\left(\frac{\varepsilon}{3}\right)^{p}
\end{aligned}
$$

It follows

$$
\left\|f-\tau_{k} q\right\|_{L^{p}\left(\mathbb{R}^{n}, \mu\right)} \leq\|f-g\|_{L^{p}\left(\mathbb{R}^{n}, \mu\right)}+\left\|g-\tau_{k} q\right\|_{L^{p}\left(\mathbb{R}^{n}, \mu\right)}<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}<\varepsilon
$$

Corollary 10.7.3 Let $\Omega$ be any $\mu$ measurable subset of $\mathbb{R}^{n}$ and let $\mu$ be a Radon measure. Then $L^{p}(\Omega, \mu)$ is separable. Here the $\sigma$ algebra of measurable sets will consist of all intersections of measurable sets with $\Omega$ and the measure will be $\mu$ restricted to these sets.

Proof: Let $\widetilde{\mathscr{D}}$ be the restrictions of $\mathscr{D}$ to $\Omega$. If $f \in L^{p}(\Omega)$, let $F$ be the zero extension of $f$ to all of $\mathbb{R}^{n}$. Let $\varepsilon>0$ be given. By Theorem 10.7.1 or 10.7.2 there exists $s \in \mathscr{D}$ such that $\|F-s\|_{p}<\varepsilon$. Thus

$$
\|s-f\|_{L^{p}(\Omega, \mu)} \leq\|s-F\|_{L^{p}\left(\mathbb{R}^{n}, \mu\right)}<\varepsilon
$$

and so the countable set $\widetilde{\mathscr{D}}$ is dense in $L^{p}(\Omega)$.

### 10.8 Green's Theorem

It will always be assumed in this section that the bounding curves are piecewise $C^{1}$ meaning that there is a parametrization $t \rightarrow \mathbf{R}(t) \equiv(x(t), y(t))$ for $t \in[a, b]$ and a partition of $[a, b],\left\{x_{0}, \cdots, x_{n}\right\}$ such that $x, y$ are $C^{1}\left(\left[x_{i-1}, x_{i}\right]\right)$.

Definition 10.8.1 Here and elsewhere, an open connected set will be called a region unless another use for this term is specified.

Green's theorem is an important theorem which relates line integrals to integrals over a surface in the plane. It can be used to establish Stoke's theorem but is interesting for it's own sake. Historically, something like it was important in the development of complex analysis. I will first establish Green's theorem for regions of a particular sort and then show that the theorem holds for many other regions also. Suppose a region is of the form indicated in the following picture in which

$$
\begin{aligned}
U & =\{(x, y): x \in(a, b) \text { and } y \in(b(x), t(x))\} \\
& =\{(x, y): y \in(c, d) \text { and } x \in(l(y), r(y))\} .
\end{aligned}
$$



I will refer to such a region as being convex in both the $x$ and $y$ directions. For sufficiently simple regions like those just described, it is easy to see what is meant by counter clockwise motion over the pieces where $\mathbf{R}$ has derivatives which are continuous on each $\left[x_{k-1}, x_{k}\right]$ and $\mathbf{R}$ is continuous on $[a, b]$. Thus these curves are of bounded variation thanks to Lemma 5.2.2. One can then compute the line integrals by adding together the integrals over the sub-intervals thanks to Lemma 5.2.9. In particular, one writes for one of these integrals

$$
\int_{x_{k-1}}^{x_{k}} \mathbf{F}(\mathbf{R}(t)) \cdot \mathbf{R}^{\prime}(t) d t=\int_{\mathbf{R}\left(\left[x_{k-1}, x_{k}\right]\right)} \mathbf{F} \cdot d \mathbf{R}
$$

Lemma 10.8.2 Let $\mathbf{F}(x, y) \equiv(P(x, y), Q(x, y))$ be a $C^{1}$ vector field defined near $U$ where $U$ is a region of the sort indicated in the above picture which is convex in both the $x$ and $y$ directions. Suppose also that the functions, $r, l, t$, and $b$ in the above picture are all $C^{1}$ functions and denote by $\partial U$ the boundary of $U$ oriented such that the direction of motion is counter clockwise. (As you walk around $U$ on $\partial U$, the points of $U$ are on your left.) Then

$$
\begin{equation*}
\int_{\partial U} P d x+Q d y \equiv \int_{\partial U} \mathbf{F} \cdot d \mathbf{R}=\int_{U}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) d A \tag{10.13}
\end{equation*}
$$

Proof: First consider the right side of 10.13 .

$$
\int_{U}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) d A=\int_{c}^{d} \int_{l(y)}^{r(y)} \frac{\partial Q}{\partial x} d x d y-\int_{a}^{b} \int_{b(x)}^{t(x)} \frac{\partial P}{\partial y} d y d x
$$

$$
\begin{equation*}
=\int_{c}^{d}(Q(r(y), y)-Q(l(y), y)) d y+\int_{a}^{b}(P(x, b(x)))-P(x, t(x)) d x \tag{10.14}
\end{equation*}
$$

Now consider the left side of 10.13 . Denote by $V$ the vertical parts of $\partial U$ and by $H$ the horizontal parts.

$$
\begin{gathered}
\int_{\partial U} \mathbf{F} \cdot d \mathbf{R}=\int_{\partial U}((0, Q)+(P, 0)) \cdot d \mathbf{R} \\
=\int_{c}^{d}(0, Q(r(s), s)) \cdot\left(r^{\prime}(s), 1\right) d s+\int_{H}(0, Q(r(s), s)) \cdot( \pm 1,0) d s \\
\quad-\int_{c}^{d}(0, Q(l(s), s)) \cdot\left(l^{\prime}(s), 1\right) d s+\int_{a}^{b}(P(s, b(s)), 0) \cdot\left(1, b^{\prime}(s)\right) d s \\
\quad+\int_{V}(P(s, b(s)), 0) \cdot(0, \pm 1) d s-\int_{a}^{b}(P(s, t(s)), 0) \cdot\left(1, t^{\prime}(s)\right) d s \\
=\int_{c}^{d} Q(r(s), s) d s-\int_{c}^{d} Q(l(s), s) d s+\int_{a}^{b} P(s, b(s)) d s-\int_{a}^{b} P(s, t(s)) d s
\end{gathered}
$$

which coincides with 10.14 .

Corollary 10.8.3 Let everything be the same as in Lemma 10.8.2 but only assume the functions $r, l, t$, and $b$ are continuous and piecewise $C^{1}$ functions. Then the conclusion this lemma is still valid.

Proof: The details are left for you. All you have to do is to break up the various line integrals into the sum of integrals over sub intervals on which the function of interest is $C^{1}$.

From this corollary, it follows 10.13 is valid for any triangle for example.
Now suppose 10.13 holds for $U_{1}, U_{2}, \cdots, U_{m}$ and the open sets, $U_{k}$ have the property that no two have nonempty intersection and their boundaries intersect only in a finite number of piecewise smooth curves. Then 10.13 must hold for $U \equiv \cup_{i=1}^{m} U_{i}$, the union of these sets. This is because

$$
\int_{U}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) d m_{2}=\sum_{k=1}^{m} \int_{U_{k}}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) d m_{2}=\sum_{k=1}^{m} \int_{\partial U_{k}} \mathbf{F} \cdot d \mathbf{R}=\int_{\partial U} \mathbf{F} \cdot d \mathbf{R}
$$

because if $\Gamma=\partial U_{k} \cap \partial U_{j}$, then its orientation as a part of $\partial U_{k}$ is opposite to its orientation as a part of $\partial U_{j}$ and consequently the line integrals over $\Gamma$ will cancel, points of $\Gamma$ also not being in $\partial U$. It is obvious from the definition of Lebesgue measure given earlier that the intersection of two of these in a smooth curve has measure zero. Thus adding an integral with respect to $m_{2}$ over such a curve yields 0 . I am not trying to be completely general here. I am just noting that when you paste together simple shapes like triangles and rectangles, this kind of cancelation will take place. As part of the development of a general Green's theorem given in an appendix, it is shown that whenever you have a curve of bounded variation, it will have two dimensional Lebesgue measure zero.

As an illustration, consider the following picture for two such $U_{k}$.


You can see that you could paste together many such simple regions composed of triangles or rectangles and obtain a region on which Green's theorem will hold even though it is not convex in each direction. This approach is developed much more in the book by Spivak [44] who pastes together boxes as part of his treatment of a general Stokes theorem.

Roughly speaking, you can drill holes in a region for which 10.13, Green's theorem, holds and get another region for which this continues to hold provided 10.13 holds for the holes.

Corollary 10.8.4 If $U \subseteq V$ and if also $\partial U \subseteq V$ and both $U$ and $V$ are open sets for which 10.13 holds, then the open set, $V \backslash(U \cup \partial U)$ consisting of what is left in $V$ after deleting $U$ along with its boundary also satisfies 10.13.

Proof: Consider the following picture which typifies the situation just described.


Then $\int_{\partial V} \mathbf{F} \cdot d \mathbf{R}=$

$$
\begin{aligned}
\int_{V}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) & d A=\int_{U}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) d A+\int_{V \backslash U}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) d A \\
= & \int_{\partial U} \mathbf{F} \cdot d \mathbf{R}+\int_{V \backslash U}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) d A
\end{aligned}
$$

and so $\int_{V \backslash U}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) d A=\int_{\partial V} \mathbf{F} \cdot d \mathbf{R}-\int_{\partial U} \mathbf{F} \cdot d \mathbf{R}$ which equals $\int_{\partial(V \backslash U)} \mathbf{F} \cdot d \mathbf{R}$ where $\partial V$ is oriented as shown in the picture. (If you walk around the region, $V \backslash U$ with the area on the left, you get the indicated orientation for this curve.)

You can see that 10.13 is valid quite generally. Let the $u$ and $v$ axes be in the same relation as the $x$ and $y$ axes. That is, the following picture holds. The positive $x$ and $u$ axes both point to the right and the positive $y$ and $v$ axes point up. This will be understood in the following.



Theorem 10.8.5 Let $U$ be a region in the uv plane ${ }^{2}$ for which Green's theorem holds such that $\partial U$ is oriented counter clockwise around this curve such that Green's theorem holds. Also let $\mathbf{r}(u, v)=(x, y)^{T}$ be any $C^{2}(\bar{U})$ map with $V=\mathbf{r}(U)$ and suppose $\operatorname{det}(D \mathbf{r}(u, v)) \geq 0$. Then Green's theorem holds for $V$ also.

Proof: Let $(x, y) \rightarrow P(x, y), Q(x, y)$ be $C^{1}(\bar{V})$ functions. Then by the change of variables formula,

$$
\begin{gather*}
\int_{V}\left(Q_{x}(x, y)-P_{y}(x, y)\right) d m_{2}(x, y)= \\
\int_{U}\left(Q_{x}(x(u, v), y(u, v))-P_{y}(x(u, v), y(u, v))\right)\left|\begin{array}{ll}
x_{u} & x_{v} \\
y_{u} & y_{v}
\end{array}\right| d m_{2}(u, v) \tag{10.15}
\end{gather*}
$$

Now consider the integrand. It is

$$
\begin{equation*}
\left(Q_{x}-P_{y}\right)\left(x_{u} y_{v}-x_{v} y_{u}\right)=Q_{x} x_{u} y_{v}-Q_{x} x_{v} y_{u}-P_{y} x_{u} y_{v}+P_{y} x_{v} y_{u} \tag{10.16}
\end{equation*}
$$

Let $\mathbf{F}(x, y)=(P(x, y), Q(x, y))$.

$$
\begin{gathered}
\left(x_{v}, y_{v}\right) \cdot(\mathbf{F} \circ \mathbf{r})_{u}-\left(x_{u}, y_{u}\right) \cdot(\mathbf{F} \circ \mathbf{r})_{v}=x_{v}\left(P_{x} x_{u}+P_{y} y_{u}\right)+y_{v}\left(Q_{x} x_{u}+Q_{y} y_{u}\right) \\
-\left[x_{u}\left(P_{x} x_{v}+P_{y} y_{v}\right)+y_{u}\left(Q_{x} x_{v}+Q_{y} y_{v}\right)\right]=Q_{x} x_{u} y_{v}+P_{y} x_{v} y_{u}-\left(P_{y} y_{v} x_{u}+Q_{x} x_{v} y_{u}\right)
\end{gathered}
$$

This is the same thing as 10.16 . Thus 10.15 reduces to

$$
\begin{equation*}
\int_{U}\left(\mathbf{r}_{v} \cdot \mathbf{F}_{u}-\mathbf{r}_{u} \cdot \mathbf{F}_{v}\right) d m_{2}(u, v) \tag{10.17}
\end{equation*}
$$

where $\mathbf{F}=\mathbf{F} \circ \mathbf{r}$ to save notation. This integrand is of the form

$$
\left(\mathbf{r}_{v} \cdot \mathbf{F}\right)_{u}-\mathbf{r}_{v u} \cdot \mathbf{F}-\left(\left(\mathbf{r}_{u} \cdot \mathbf{F}\right)_{v}-\mathbf{r}_{u v} \cdot \mathbf{F}\right)=\left(\mathbf{r}_{v} \cdot \mathbf{F}\right)_{u}-\left(\mathbf{r}_{u} \cdot \mathbf{F}\right)_{v}
$$

by equality of mixed partial derivatives. Thus 10.17 equals

$$
\begin{gathered}
\int_{U}\left(\mathbf{r}_{v} \cdot \mathbf{F}\right)_{u}-\left(\mathbf{r}_{u} \cdot \mathbf{F}\right)_{v} d m_{2}(u, v)=\int_{\partial U} \mathbf{r}_{u} \cdot \mathbf{F} d u+\mathbf{r}_{v} \cdot \mathbf{F} d v \\
=\int_{\partial U}(\mathbf{F} \circ \mathbf{r}) \cdot\left(\frac{d \mathbf{r}}{d t}\right) d t=\int_{\partial V} \mathbf{F} \cdot d \mathbf{r}=\int_{\partial V} P(x, y) d x+\int Q(x, y) d y
\end{gathered}
$$

By Green's theorem applied to $\left(\mathbf{r}_{u} \cdot \mathbf{F}, \mathbf{r}_{v} \cdot \mathbf{F}\right)=\left(\mathbf{r}_{u} \cdot \mathbf{F} \circ \mathbf{r}, \mathbf{r}_{v} \cdot \mathbf{F} \circ \mathbf{r}\right)$. Recall motion around $\partial U$ is counter clockwise with the $u, v$ axes oriented as shown above. Now the curve $\partial U$ is piecewise smooth and a typical smooth piece is $t \rightarrow(u(t), v(t))$. Then on $\partial V$ we have $t \rightarrow \mathbf{r}(u(t), v(t))=(x, y)$ and $\frac{d \mathbf{r}}{d t}=\mathbf{r}_{u} u^{\prime}+\mathbf{r}_{v} v^{\prime}$ which is the explanation of the last line.

The above is a reasonably good theorem and is enough for most applications to complex analysis but it requires a map which takes $\bar{U}$ to $\bar{V}$ and the best version of this theorem only requires a map from $S^{1}$ to $\partial U$. It is in an appendix. This will also include the information that the Green's theorem specifies an orientation over $\partial V$. Anyway, the main message of the above theorem is that Green's theorem holds for very general situations. In applications, one can usually see that the theorem will hold based on the considerations discussed above.

[^6]
### 10.9 Exercises

1. Explain why for each $t>0, x \rightarrow e^{-t x}$ is a function in $L^{1}(\mathbb{R})$ and $\int_{0}^{\infty} e^{-t x} d x=\frac{1}{t}$. Thus

$$
\int_{0}^{R} \frac{\sin (t)}{t} d t=\int_{0}^{R} \int_{0}^{\infty} \sin (t) e^{-t x} d x d t
$$

Now explain why you can change the order of integration in the above iterated integral. Then compute what you get. Next pass to a limit as $R \rightarrow \infty$ and show $\int_{0}^{\infty} \frac{\sin (t)}{t} d t=\frac{1}{2} \pi$. This is a very important integral. Note that the thing on the left is an improper integral. $\sin (t) / t$ is not Lebesgue integrable because it is not absolutely integrable. That is $\int_{0}^{\infty}\left|\frac{\sin t}{t}\right| d m=\infty$. It is important to understand that the Lebesgue theory of integration only applies to nonnegative functions and those which are absolutely integrable.
2. Polar coordinates are $x=r \cos (\theta), y=r \sin (\theta)$. These transformation equations map $[0,2 \pi) \times[0, \infty)$ onto $\mathbb{R}^{2}$. It is the restriction to this set of the same transformation defined on the open set $(-1,2 \pi) \times(-1, \infty)$ and so from Theorem 10.3.3, if $g$ is Lebesgue measurable and zero off $[0,2 \pi) \times[0, \infty)$,

$$
\int_{\mathbb{R}^{2}} \#(\mathbf{x}) g(\mathbf{x}) d m_{2}=\int_{0}^{2 \pi} \int_{0}^{\infty} g(r \cos \theta, r \sin \theta) r d r d \theta
$$

where $\#(\mathbf{x})$ is 1 except for a set of measure zero consisting of either $\theta=0$ or $r=0$. This set, has its image also of measure zero. Hence one can simply write

$$
\int_{\mathbb{R}^{2}} g(\mathbf{x}) d m_{2}=\int_{0}^{2 \pi} \int_{0}^{\infty} g(r \cos \theta, r \sin \theta) r d r d \theta
$$

Similar considerations apply to the general case of spherical coordinates as explained above. Use this change of variables for polar coordinates to show $\int_{-\infty}^{\infty} e^{-x^{2}} d x=\sqrt{\pi}$. Hint: Let $I=\int_{-\infty}^{\infty} e^{-x^{2}} d x$ and explain why $I^{2}=\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\left(x^{2}+y^{2}\right)} d x d y$. Now use polar coordinates.
3. Let $E$ be a Lebesgue measurable set in $\mathbb{R}$. Suppose $m(E)>0$. Consider the set

$$
E-E=\{x-y: x \in E, y \in E\} .
$$

Show that $E-E$ contains an interval. Hint: Let $f(x)=\int \mathscr{X}_{E}(t) \mathscr{X}_{E}(x+t) d t$. Show $f$ is continuous at 0 and $f(0)>0$ and use continuity of translation in $L^{p}$.
4. Let $K$ be a bounded subset of $L^{p}\left(\mathbb{R}^{n}\right)$ and suppose that there exists $G$ such that $\bar{G}$ is compact with $\int_{\mathbb{R}^{n} \backslash \bar{G}}|u(\mathbf{x})|^{p} d x<\varepsilon^{p}$ and for all $\varepsilon>0$, there exist a $\delta>0$ and such that if $|\mathbf{h}|<\delta$, then $\int|u(\mathbf{x}+\mathbf{h})-u(\mathbf{x})|^{p} d x<\varepsilon^{p}$ for all $u \in K$. Show that $K$ is precompact in $L^{p}\left(\mathbb{R}^{n}\right)$. Hint: Let $\phi_{k}$ be a mollifier and consider $K_{k} \equiv\left\{u * \phi_{k}: u \in K\right\}$. The notation means the following:

$$
u * \phi_{k}(\mathbf{x}) \equiv \int u(\mathbf{x}-\mathbf{y}) \phi_{k}(\mathbf{y}) d m_{n}(y)=\int u(\mathbf{y}) \phi_{k}(\mathbf{x}-\mathbf{y}) d m_{n}(y)
$$

It is called the convolution. Verify the conditions of the Ascoli Arzela theorem Theorem 9.2.4 for these functions defined on $\bar{G}$ and show there is an $\varepsilon$ net for each $\varepsilon>0$. Can you modify this to let an arbitrary open set take the place of $\mathbb{R}^{n}$ ?
5. Let $\phi_{m} \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right), \phi_{m}(\mathbf{x}) \geq 0$, and $\int_{\mathbb{R}^{n}} \phi_{m}(\mathbf{y}) d y=1$ with

$$
\lim _{m \rightarrow \infty} \sup \left\{|\mathbf{x}|: \mathbf{x} \in \operatorname{spt}\left(\phi_{m}\right)\right\}=0
$$

Show if $f \in L^{p}\left(\mathbb{R}^{n}\right), \lim _{m \rightarrow \infty} f * \phi_{m}=f$ in $L^{p}\left(\mathbb{R}^{n}\right)$. Hint: Use Minkowski's inequality for integrals to get a short proof of this fact.
6. Let $\frac{1}{p}+\frac{1}{p^{\prime}}=1, p>1$, let $f \in L^{p}\left(\mathbb{R}^{n}\right), g \in L^{p^{\prime}}\left(\mathbb{R}^{n}\right)$. Show $f * g$ is uniformly continuous on $\mathbb{R}$ and $|(f * g)(\mathbf{x})| \leq\|f\|_{L^{p}}\|g\|_{L^{p^{\prime}}} . f * g(\mathbf{x}) \equiv \int_{\mathbb{R}^{n}} f(\mathbf{x}-\mathbf{y}) g(\mathbf{y}) d m_{n}$. Hint: You need to consider why $f * g$ exists and then this follows from the definition of convolution and continuity of translation of Lebesgue measure.
7. Suppose $f$ is a strictly decreasing nonnegative function defined on $[0, \infty)$.

Let $f^{-1}(y) \equiv\{x$ : such that $y \in[f(x+), f(x-)]\}$. Show that

$$
\int_{0}^{\infty} f(t) d t=\int_{0}^{f(0)} f^{-1}(y) d y
$$

Hint: Try to show that $f^{-1}(y)=m([f>y])$.
8. Let $f(y)=g(y)=|y|^{-1 / 2}$ if $y \in(-1,0) \cup(0,1)$ and $f(y)=g(y)=0$ outside of this set. For which values of $x$ does it make sense to write the integral

$$
\int_{\mathbb{R}} f(x-y) g(y) d y \equiv f * g(x)
$$

This is asking for you to find where the convolution of $f$ and $g$ makes sense.
9. Let $f \in L^{1}\left(\mathbb{R}^{p}\right)$ and let $g \in L^{1}\left(\mathbb{R}^{p}\right)$. Define the convolution of $f$ and $g$ as follows. It equals

$$
f * g(\mathbf{x}) \equiv \int f(\mathbf{x}-\mathbf{y}) g(\mathbf{y}) d m_{p}
$$

Show that the above integral makes sense for a.e. $\mathbf{x}$ that is, for all $\mathbf{x}$ off a set of measure zero. If $f * g$ is defined to equal 0 at points where the above integral does not make sense, show that $\|f * g\|_{1} \leq\|f\|_{1}\|g\|_{1}$ where $\|h\|_{1} \equiv \int|h| d m_{p}$.
10. Consider $\mathscr{D} \equiv\left\{p\left(e^{-\alpha t}\right)\right\}$ where $p(t)$ is some real polynomial having zero constant term and $\alpha$ is some positive number at least as large as a given $\alpha_{0}>0$. Show that $\mathscr{D}$ is an algebra and is dense in $C_{0}([0, \infty))$ with respect to the norm $\|f\|_{\infty} \equiv$ $\max \{|f(x)|: x \in[0, \infty)\}$.
11. $\uparrow$ Suppose $f \in L^{1}([0, \infty))$ and $\int_{0}^{\infty} f(t) g(t) d m=0$ for all $g \in \mathscr{D}$ in the above problem. Explain why $f(t)=0$ a.e. $t$. Hint: You can assume $f$ is real since if not, you could look at the real and imaginary parts. You can also assume that $f$ is nonnegative. Show density of $C_{c}([0, \infty))$ in $L^{1}([0, \infty))$. Then show there is a sequence of things in $\mathscr{D}$ which converges in $L^{1}$ to $f$. Finally, go over why you can get a further subsequence which converges to $f$ a.e. Then use Fatou's lemma.
12. A measurable function $f$ defined on $[0, \infty)$ has exponential growth if $f(t) \leq C e^{r t}$ for some real $r$. Suppose you have $f$ measurable with exponential growth. Show $L f(s) \equiv \int_{0}^{\infty} e^{-s t} f(t) d t$ the Laplace transform, is well defined for all $s$ large enough. Now show that if $L f(s)=0$ for all $s$ large enough, then $f(t)=0$ for a.e. $t$. This shows that if two measurable functions with exponential growth have the same Laplace transform for large $s$, then they are a.e. the same function.
13. Stirling's formula from elementary calculus says that for $n \in \mathbb{N}$,

$$
\lim _{n \rightarrow \infty} \frac{e^{n} n!}{n^{n+(1 / 2)}}=\sqrt{2 \pi}
$$

Show $\Gamma(x) \equiv \int_{0}^{\infty} e^{-t} t^{x-1} d t$ exists whenever $x>0$. Also show that $\Gamma(x+1)=x \Gamma(x)$. Then show that for $n \in \mathbb{N}, \Gamma(n+1)=n$ !. This function is discussed in the next chapter.
14. For $n \in \mathbb{N}$, Stirling's formula says $\lim _{n \rightarrow \infty} \frac{\Gamma(n+1) e^{n}}{n^{n+(1 / 2)}}=\sqrt{2 \pi}$. Here $\Gamma(n+1)=n$ !. The idea here is to show that you get the same result if you replace $n$ with $x \in(0, \infty)$. To do this, show
(a) $n \rightarrow \frac{\Gamma(n+1) e^{n}}{n^{n+(1 / 2)}}$ is decreasing on the positive integers. This follows from the properties of the Gamma function and a little work.
(b) Show that $x \rightarrow \frac{\Gamma(x+1) e^{x}}{x^{x+(1 / 2)}}$ is decreasing on $(m, m+1)$ for $m \in \mathbb{N}$. This is a little harder.

Hint: For $x \in(m, m+1), \ln \left(\frac{\Gamma(x+1) e^{x}}{x^{x+(1 / 2)}}\right)=x+\ln \Gamma(x+1)-\left(x+\frac{1}{2}\right) \ln x$

$$
\begin{aligned}
& =x+\ln (x(x-1)(x-2) \cdots(x-m+1) \Gamma(x-m))-\left(x+\frac{1}{2}\right) \ln x \\
& =x+\sum_{k=0}^{m-1} \ln (x-k)+\ln (\Gamma(x-m))-\left(x+\frac{1}{2}\right) \ln x
\end{aligned}
$$

Now differentiate and try to show that the derivative is negative for $x \in(m, m+1)$. Thus the desired derivative is

$$
\left(\sum_{k=0}^{m-1} \frac{1}{x-k}-\ln x\right)+\frac{1}{\Gamma(x-m)} \int_{0}^{\infty} \ln (t) t^{x-(m+1)} e^{-t} d t-\frac{1}{2 x}
$$

The first term is negative from the definition of $\ln (x)$. The derivative being negative will be shown if it is shown that the integral term is negative. Do an integration by parts and split the integrals to obtain

$$
\begin{aligned}
\int_{0}^{\infty} \ln (t) t^{x-(m+1)} e^{-t} d t= & -\int_{0}^{1} t^{\sigma} e^{-t} d t+\int_{0}^{1}(t-\sigma) e^{-t} t^{\sigma} \ln (t) \\
& +\int_{1}^{\infty} t^{\sigma} e^{-t}(1-(t-\sigma) \ln (t)) d t
\end{aligned}
$$

where $\sigma=(x-m)-1 \in(-1,0)$ so $-\sigma>0$. The last integral is negative because $(t-\sigma)=t+(-\sigma)>1$. The other two are obviously negative.

## Chapter 11

## Fundamental Transforms

### 11.1 Gamma Function

With the Lebesgue integral, it becomes easy to consider the Gamma function and the theory of Laplace transforms. I will use the standard notation for the integral used in calculus, but remember that all integrals will be Lebesgue integrals taken with respect to one dimensional Lebesgue measure. First is a very important function defined in terms of an integral. Problem 13 on Page 183 shows that in the case of a continuous function, the Riemann integral and the Lebesgue integral are exactly the same. Thus all the standard calculus manipulations are valid for the Lebesgue integral provided the functions integrated are continuous. This also implies immediately that the two integrals coincide whenever the function is piecewise continuous on a finite interval. Recall that the value of the Riemann integral does not depend on the value of the function at single points and the same is true of the Lebesgue integral because single points have zero measure.
Definition 11.1.1 The gamma function $\alpha \rightarrow \Gamma(\alpha)$ is defined as

$$
\Gamma(\alpha) \equiv \int_{0}^{\infty} e^{-t} t^{\alpha-1} d t
$$

whenever $\alpha>0$.
Lemma 11.1.2 The integral is finite for each $\alpha>0$.
Proof: By the monotone convergence theorem, for $n \in \mathbb{N}$

$$
\begin{aligned}
\Gamma(\alpha) & =\lim _{n \rightarrow \infty} \int_{1 / n}^{n} e^{-t} t^{\alpha-1} \leq \lim _{n \rightarrow \infty}\left(\int_{1 / n}^{1} t^{\alpha-1} d t+\int_{1}^{n} C e^{-t / 2}\right) \\
& \leq \frac{1}{\alpha}+\lim _{n \rightarrow \infty}\left(-2 C e^{-\frac{1}{2} n}+2 C e^{-\frac{1}{2}}\right)<\infty
\end{aligned}
$$

The explanation for the constant is as follows. Letting $m$ be a positive integer larger than $\alpha-1$, for $t \geq 1, e^{-t} t^{\alpha-1}<e^{-t} t^{m} \leq C e^{-t / 2}$ for suitable $C$.
Proposition 11.1.3 For $n$ a positive integer, $n!=\Gamma(n+1)$. In general, the following fundamental identity holds. $\Gamma(1)=1, \Gamma(\alpha+1)=\alpha \Gamma(\alpha)$

Proof: First of all, $\left.\Gamma(1)=\lim _{\delta \rightarrow 0} \int_{\delta}^{\delta^{-1}} e^{-t} d t=\lim _{\delta \rightarrow 0}\left(e^{-\delta}-e^{-\left(\delta^{-1}\right.}\right)\right)=1$. Next, for $\alpha>0$,

$$
\begin{gathered}
\Gamma(\alpha+1)=\lim _{\delta \rightarrow 0} \int_{\delta}^{\delta^{-1}} e^{-t} t^{\alpha} d t=\lim _{\delta \rightarrow 0}\left[-\left.e^{-t} t^{\alpha}\right|_{\delta} ^{\delta^{-1}}+\alpha \int_{\delta}^{\delta^{-1}} e^{-t} t^{\alpha-1} d t\right] \\
=\lim _{\delta \rightarrow 0}\left(e^{-\delta} \delta^{\alpha}-\frac{1}{\delta^{\alpha} e^{1 / \delta}}+\alpha \int_{\delta}^{\delta^{-1}} e^{-t} t^{\alpha-1} d t\right)=\alpha \Gamma(\alpha)
\end{gathered}
$$

Note that $\lim _{\delta \rightarrow 0+} \ln \left(\frac{1}{\delta^{\alpha} e^{1 / \delta}}\right)=\lim _{\delta \rightarrow 0+}\left(\alpha \ln \delta+\frac{1}{\delta}\right)=-\infty$ so $\lim _{\delta \rightarrow 0+} \frac{1}{\delta^{\alpha} e^{1 / \delta}}=0$. Now it is defined that $0!=1$ and so $\Gamma(1)=0$ !. Suppose that $\Gamma(n+1)=n!$, what of $\Gamma(n+2)$ ? Is it $(n+1)!?$ if so, then by induction, the proposition is established. From what was just shown, $\Gamma(n+2)=\Gamma(n+1)(n+1)=n!(n+1)=(n+1)!$ and so this proves the proposition.

### 11.2 Laplace Transform

Here is the definition of a Laplace transform.
Definition 11.2.1 A function $\phi$ has exponential growth on $[0, \infty)$ if there are positive constants $\lambda, C$ such that $|\phi(t)| \leq C e^{\lambda t}$ for all $t$. Then for $s>\lambda$, one defines the Laplace transform $\mathscr{L} \phi(s) \equiv \int_{0}^{\infty} \phi(t) e^{-s t} d t$.

In general, a function of a complex variable $z$ has a derivative exactly when

$$
\lim _{h \rightarrow 0} \frac{f(z+h)-f(h)}{z}
$$

exists and this is defined as $f^{\prime}(z)$ as in the case where $z$ is a real variable.
Theorem 11.2.2 Let $f(s)=\int_{0}^{\infty} e^{-s t} \phi(t) d t$ where $t \rightarrow \phi(t) e^{-s t}$ is in $L^{1}([0, \infty))$ for all s large enough and $\phi$ has exponential growth. Then for slarge enough, $f^{(k)}(s)$ exists and equals $\int_{0}^{\infty}(-t)^{k} e^{-s t} \phi(t) d t$. In fact if $s$ is a complex number and $\operatorname{Re} s>\lambda$ where $|\phi(t)| \leq C e^{\lambda t}$, then for $\operatorname{Re} s>\lambda$,

$$
\lim _{h \rightarrow 0} \frac{f(s+h)-f(s)}{h} \equiv f^{\prime}(s)=\int_{0}^{\infty}(-t) e^{-s t} \phi(t) d t
$$

Proof: First consider the real case. Suppose true for some $k \geq 0$. By definition it is so for $k=0$. Then always assuming $s>\lambda,|h|<s-\lambda$, where $|\phi(t)| \leq C e^{\lambda t}, \lambda \geq 0$,

$$
\frac{f^{(k)}(s+h)-f^{(k)}(s)}{h}=\int_{0}^{\infty}(-t)^{k} \frac{e^{-(s+h) t}-e^{-s t}}{h} \phi(t) d t
$$

Using the mean value theorem, the integrand satisfies

$$
\left|(-t)^{k} \frac{e^{-(s+h) t}-e^{-s t}}{h} \phi(t)\right| \leq t^{k}|\phi(t)|\left|-t e^{-\hat{s} t}\right|
$$

where $\hat{s} \in(s, s+h)$ or $(s+h, s)$ if $h<0$. In case $h>0$, the integrand is less than

$$
t^{k+1}|\phi(t)| e^{-s t} \leq t^{k+1} C e^{(\lambda-s) t}
$$

a function in $L^{1}$ since $s>\lambda$. In the other case, for $|h|$ small enough, the integrand is dominated by $t^{k+1} C e^{(\lambda-(s+|h|)) t}$. Letting $|h|<\varepsilon$ where $s-\lambda<\varepsilon$, the integrand is dominated by $t^{k+1} C e^{(\lambda-(s+\varepsilon)) t}<C t^{k+1} e^{-\varepsilon t}$, also a function in $L^{1}$. By the dominated convergence theorem, one can pass to the limit and obtain

$$
f^{(k+1)}(s)=\int_{0}^{\infty}(-t)^{k+1} e^{-s t} \phi(t) d t
$$

Let $\operatorname{Re} s>\lambda$. However, $s$ will be complex as will $h$. From the properties of the complex exponential, Section 1.5.2, $\int_{0}^{\infty} \frac{e^{-(s+h) t}-e^{-s t}}{h} \phi(t) d t=$

$$
\begin{equation*}
\int_{0}^{\infty}-t e^{-s t} \frac{e^{-h t}-1}{-t h} \phi(t) d t=\int_{0}^{\infty} e^{-s t} \frac{e^{-h t}-1}{h} \phi(t) d t \tag{11.1}
\end{equation*}
$$

Now it follows from that section that the integrand converges to $-t e^{-s t} \phi(t)$. Therefore, it only is required to obtain an estimate and use the dominated convergence theorem. Then $e^{-s t} \frac{e^{-h t}-1}{h} \phi(t)=$

$$
\left|e^{-(s+h) t} \frac{1-e^{h t}}{h} \phi(t)\right| \leq C e^{\lambda t}\left|\frac{1-e^{h t}}{h}\right| e^{-(s+h)} \leq C t e^{-(s+h-\lambda) t}\left|\frac{1-e^{h t}}{t h}\right|
$$

Now $\left|e^{-(s+h-\lambda) t}\right|=e^{-(\operatorname{Re}(s+h)-\lambda) t} \leq e^{-\varepsilon t}$ whenever $|h|$ is small enough because of the assumption that $\operatorname{Re} s>\lambda . e^{h t}-1=\int_{0}^{t} h e^{h u} d u$ and so the last expression is

$$
\left|\frac{\int_{0}^{t} h e^{h u} d u}{t h}\right| \leq \frac{1}{t}\left|\int_{0}^{t} e^{h u} d u\right| \leq \frac{1}{t} \int_{0}^{t} e^{|h| u} d u \leq e^{t|h|}
$$

and so for $|h|$ small enough, say smaller than $\varepsilon / 2$, the integrand in 11.1 is no larger than $C t e^{-\varepsilon t} e^{\frac{\varepsilon}{2} t}=C t e^{-\frac{\varepsilon}{2} t}$ which is in $L^{1}$ and so the dominated convergence theorem applies and it follows that $f^{\prime}(s)=\int_{0}^{\infty}(-t) e^{-s t} \phi(t) d t$ whenever $\operatorname{Re}(s)>\lambda$. Continuing similarly, yields all the derivatives. However, the existence of all the derivatives will follow from general results on analytic functions presented later.

The whole approach for Laplace transforms in differential equations is based on the assertion that if $\mathscr{L}(f)=\mathscr{L}(g)$, then $f=g$. However, this is not even true because if you change the function on a set of measure zero, you don't change the transform. However, if $f, g$ are continuous, then it will be true. Actually, it is shown here that if $\mathscr{L}(f)=0$, and $f$ is continuous, then $f=0$. The approach here is based on the Weierstrass approximation theorem or rather a case of it.

Lemma 11.2.3 Suppose $q$ is a continuous function defined on $[0,1]$. Also suppose that for all $n=0,1,2, \cdots$, that $\int_{0}^{1} q(x) x^{n} d x=0$. Then it follows that $q=0$.

Proof: By assumption, for $p(x)$ any polynomial, $\int_{0}^{1} q(x) p(x) d x=0$. Now let $\left\{p_{n}(x)\right\}$ be a sequence of polynomials which converge uniformly to $q(x)$ by Corollary 3.1.3. Say $\max _{x \in[0,1]}\left|q(x)-p_{n}(x)\right|<\frac{1}{n}$. Then from this uniform convergence, of $p_{n}$ to $q$,

$$
\int_{0}^{1} q^{2}(x) d x=\lim _{n \rightarrow \infty} \int_{0}^{1} q(x) p_{n}(x) d x=0
$$

By continuity, it must be the case that $q(x)=0$ for all $x$ since otherwise, there would be a small interval on which $q^{2}(x)$ is positive and so the integral could not have been 0 after all.

Lemma 11.2.4 Suppose $|f(t)| \leq C e^{-\delta t}$ for some $\delta>0$ and all $t>0$ and also that $f$ is continuous. Suppose that $\int_{0}^{\infty} e^{-s t} f(t) d t=0$ for all $s>0$. Then $f=0$.

Proof: First note that $\lim _{t \rightarrow \infty}|f(t)|=0$. Next change the variable letting $x=e^{-t}$ and so $x \in[0,1]$. Then this reduces to $\int_{0}^{1} x^{s-1} f(-\ln (x)) d x$. Now if you let $q(x)=f(-\ln (x))$, it is not defined when $x=0$, but $x=0$ corresponds to $t \rightarrow \infty$. Thus $\lim _{x \rightarrow 0+} q(x)=0$. Defining $q(0) \equiv 0$, it follows that it is continuous and letting $s-1$ be various integers, for all $n=0,1,2, \cdots, \int_{0}^{1} x^{n} q(x) d x=0$ and so $q(x)=0$ for all $x$ from Lemma 11.2.3. Thus $f(-\ln (x))=0$ for all $x \in(0,1]$ and so $f(t)=0$ for all $t \geq 0$.

Now suppose only that $|f(t)| \leq C e^{r t}$ so $f$ has exponential growth and that for all $s$ sufficiently large, $\mathscr{L}(f)=0$. Does it follow that $f=0$ ? Say this holds for all $s \geq s_{0}$ where also $s_{0}>r$. Then consider $\hat{f}(t) \equiv e^{-s_{0} t} f(t) \cdot|\hat{f}(t)| \leq e^{-s_{0} t} C e^{r t}=C e^{-\left(s_{0}-r\right) t}$. Then if $s>0$,

$$
\int_{0}^{\infty} e^{-s t} \hat{f}(t) d t=\int_{0}^{\infty} e^{-s t} e^{-s_{0} t} f(t) d t=\int_{0}^{\infty} e^{-\left(s+s_{0}\right) t} f(t) d t=0
$$

because $s+s_{0}$ is large enough for this to happen. It follows from Lemma 11.2.4 that $\hat{f}=0$. But this implies that $f=0$ also. This proves the following fundamental theorem.

Theorem 11.2.5 Suppose $f$ has exponential growth and is continuous on $[0, \infty)$. Suppose also that for all s large enough, $\mathscr{L}(f)(s)=0$. Then $f=0$.

Now this will be extended to more general functions.
Corollary 11.2.6 Suppose $|f(t)|,|g(t)|$ have exponential growth and are functions in $L^{1}(0, \infty)$. Then if $\mathscr{L}(f)(s)=\mathscr{L}(g)(s)$ for all s large enough, it follows that $f=g$ a.e.

Proof: Say $|f(t)|,|g(t)| \leq C e^{\lambda t}$ and $\mathscr{L}(f)(s)=\mathscr{L}(g)(s)$ for all $s>\lambda$. Then by definition and Fubini's theorem, for $h=f-g$, picking Borel measurable representatives for $f, g$,

$$
\begin{aligned}
\int_{0}^{\infty}\left(\int_{0}^{t} h(u) d u\right) e^{-s t} d t & =\int_{0}^{\infty} h(u) \int_{u}^{\infty} e^{-s t} d t d u \\
& =\int_{0}^{\infty} h(u) \frac{1}{s} e^{-u s}=\frac{1}{s} \mathscr{L}(h)(s)=0
\end{aligned}
$$

Thus from Theorem 11.2.5, $\int_{0}^{t} h(u) d u=0$. By fundamental theorem of calculus, $h(t)=0$ a.e.

### 11.3 Fourier Transform

Definition 11.3.1 The Fourier transform is defined as follows for $f \in L^{1}(\mathbb{R})$.

$$
F f(t) \equiv \frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} e^{-i t x} f(x) d x
$$

where here I am using the usual notation from calculus to denote the Lebesgue integral in which, to be more precise, you would put dm $m_{1}$ in place of $d x$. The inverse Fourier transform is defined the same way except you delete the minus sign in the complex exponential.

$$
F^{-1} f(t) \equiv \frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} e^{i t x} f(x) d x
$$

Does it deserve to be called the "inverse" Fourier transform? This question will be explored somewhat below.

In studying the Fourier transform, I will use some improper integrals.
Definition 11.3.2 Define $\int_{a}^{\infty} f(t) d t \equiv \lim _{r \rightarrow \infty} \int_{a}^{r} f(t) d t$. This coincides with the Lebesgue integral when $f \in L^{1}(a, \infty)$. However, situations will be considered below in which $f$ is not in $L^{1}$.

With this convention, there is a very important improper integral involving $\sin (x) / x$. You can show with a little estimating that $x \rightarrow \sin (x) / x$ is not in $L^{1}(0, \infty)$. Nevertheless, a lot can be said about improper integrals involving this function.

## Theorem 11.3.3 The following hold

1. $\int_{0}^{\infty} \frac{\sin u}{u} d u=\frac{\pi}{2}$
2. $\lim _{r \rightarrow \infty} \int_{\delta}^{\infty} \frac{\sin (r u)}{u} d u=0$ whenever $\delta>0$.
3. If $f \in L^{1}(\mathbb{R})$, then $\lim _{r \rightarrow \infty} \int_{\mathbb{R}} \sin (r u) f(u) d u=0$. This is called the Riemann Lebesgue lemma.

Proof: You know $\frac{1}{u}=\int_{0}^{\infty} e^{-u t} d t$. Therefore, using Fubini's theorem,

$$
\int_{0}^{r} \frac{\sin u}{u} d u=\int_{0}^{r} \sin (u) \int_{0}^{\infty} e^{-u t} d t d u=\int_{0}^{\infty} \int_{0}^{r} e^{-u t} \sin (u) d u d t
$$

Now you integrate that inside integral by parts to obtain

$$
\int_{0}^{\infty}\left(\frac{1}{t^{2}+1}-e^{-t r} \frac{\cos (r)+t \sin (r)}{1+t^{2}}\right) d t
$$

This integrand converges to $\frac{1}{t^{2}+1}$ as $r \rightarrow \infty$ for each $t>0$. I would like to use the dominated convergence theorem. That second term is of the form

$$
e^{-t r} \frac{\sqrt{1+t^{2}} \cos (r-\phi(t, r))}{1+t^{2}} \leq \frac{1}{\left(1+t^{2}\right)^{1 / 2}} e^{-t r}
$$

For $r>1$, this is no larger than $\frac{1}{\left(1+t^{2}\right)^{1 / 2}} e^{-t}$ which is obviously in $L^{1}$ and so one can apply the dominated convergence theorem and conclude that

$$
\lim _{r \rightarrow \infty} \int_{0}^{r} \frac{\sin u}{u} d u=\int_{0}^{\infty} \frac{1}{1+t^{2}} d t=\frac{\pi}{2}
$$

This shows part 1.
Now consider $\int_{\delta}^{\infty} \frac{\sin (r u)}{u} d u$. It equals $\int_{0}^{\infty} \frac{\sin (r u)}{u} d u-\int_{0}^{\delta} \frac{\sin (r u)}{u} d u$ which can be seen from the definition of what the improper integral means. Let $r u=t$ so $r d u=d t$ and

$$
\int_{\delta}^{\infty} \frac{\sin (r u)}{u} d u=\int_{0}^{\infty} \frac{\sin (t)}{t} r \frac{1}{r} d t-\int_{0}^{r \delta} \frac{\sin (t)}{t} d t=\frac{\pi}{2}-\int_{0}^{r \delta} \frac{\sin (t)}{t} d t
$$

so $\lim _{r \rightarrow \infty} \int_{\delta}^{\infty} \frac{\sin (r u)}{u} d u=\lim _{r \rightarrow \infty}\left(\frac{\pi}{2}-\int_{0}^{r \delta} \frac{\sin (t)}{t} d t\right)=0$ from the first part.
Now consider the Riemann Lebesgue lemma. Let $h \in C_{c}^{\infty}(\mathbb{R})$ such that $\int_{\mathbb{R}}|f-h| d m_{p}<$ $\varepsilon$. Then

$$
\begin{aligned}
\left|\int_{\mathbb{R}} \sin (r u) f(u) d u\right| & \leq\left|\int_{\mathbb{R}} \sin (r u)(f(u)-h(u)) d u\right|+\left|\int_{\mathbb{R}} \sin (r u) h(u)\right| \\
& \leq \varepsilon+\left|\int_{\mathbb{R}} \sin (r u) h(u)\right|
\end{aligned}
$$

with this last integral, do an integration by parts. Since $h$ vanishes off some interval,

$$
\int_{\mathbb{R}} \sin (r u) h(u) d m_{p}=-\frac{1}{r} \int_{\mathbb{R}} \cos (r u) h^{\prime}(u) d m_{p}
$$

Thus this last integral is dominated by $\frac{C}{r}$ so it converges to 0 . For $r$ large enough, it follows that $\left|\int_{\mathbb{R}} \sin (r u) f(u) d u\right| \leq 2 \varepsilon$ and since $\varepsilon$ is arbitrary, this establishes the claim.

Definition 11.3.4 The following notation will be used assuming the limits exist.

$$
\lim _{r \rightarrow 0+} g(x+r) \equiv g(x+), \lim _{r \rightarrow 0+} g(x-r) \equiv g(x-)
$$

Theorem 11.3.5 Suppose that $g \in L^{1}(\mathbb{R})$ and that at some $x$, $g$ is locally Holder continuous from the right and from the left. This means there exist constants $K, \delta>0$ and $r \in(0,1]$ such that for $|x-y|<\delta$,

$$
\begin{equation*}
|g(x+)-g(y)|<K|x-y|^{r} \tag{11.2}
\end{equation*}
$$

for $y>x$ and

$$
\begin{equation*}
|g(x-)-g(y)|<K|x-y|^{r} \tag{11.3}
\end{equation*}
$$

for $y<x$. Then

$$
\begin{gathered}
\lim _{r \rightarrow \infty} \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin (u r)}{u}\left(\frac{g(x-u)+g(x+u)}{2}\right) d u \\
=\frac{g(x+)+g(x-)}{2}
\end{gathered}
$$

Proof: As in the proof of Theorem 11.3.3, changing variables shows that for large positive $r$,

$$
\frac{2}{\pi} \int_{0}^{\infty} \frac{\sin (r u)}{u} d u=1
$$

Therefore,

$$
\begin{align*}
& \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin (u r)}{u}\left(\frac{g(x-u)+g(x+u)}{2}\right) d u-\frac{g(x+)+g(x-)}{2} \\
= & \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin (u r)}{u}\left(\frac{g(x-u)-g(x-)+g(x+u)-g(x+)}{2}\right) d u \\
= & \frac{2}{\pi} \int_{0}^{\delta} \sin (u r)\left(\frac{g(x-u)-g(x-)}{2 u}+\frac{g(x+u)-g(x+)}{2 u}\right) d u \\
& +\frac{2}{\pi} \int_{\delta}^{\infty} \frac{\sin (u r)}{u}\left(\frac{g(x-u)-g(x-)}{2}+\frac{g(x+u)-g(x+)}{2}\right) d u \tag{11.4}
\end{align*}
$$

Second Integral: It equals

$$
\frac{2}{\pi} \int_{\delta}^{\infty} \frac{\sin (u r)}{u}\left(\frac{g(x-u)+g(x+u)}{2}-\frac{g(x-)+g(x+)}{2}\right) d u
$$

$$
\begin{align*}
= & \frac{2}{\pi} \int_{\delta}^{\infty} \frac{\sin (u r)}{u}\left(\frac{g(x-u)+g(x+u)}{2}\right) \\
& -\frac{2}{\pi} \int_{\delta}^{\infty} \frac{\sin (u r)}{u}\left(\frac{g(x-)+g(x+)}{2}\right) \tag{11.5}
\end{align*}
$$

From part 2 of Theorem 11.3.3,

$$
\lim _{r \rightarrow \infty} \frac{2}{\pi} \int_{\delta}^{\infty} \frac{\sin (u r)}{u} \frac{g(x-)+g(x+)}{2} d u=0
$$

Thus consider the first integral in 11.4.

$$
\left|\frac{g(x-u)+g(x+u)}{2 u}\right| \leq \frac{1}{2 \delta}(|g(x-u)|+|g(x+u)|)
$$

and so $u \rightarrow\left|\frac{g(x-u)+g(x+u)}{2 u}\right|$ is in $L^{1}(\mathbb{R})$. Then by the Riemann Lebesgue theorem of Theorem 11.3.3, this integral also converges to 0 as $r \rightarrow \infty$.

First Integral in 11.4: This converges to 0 as $r \rightarrow \infty$ because of the Riemann Lebesgue lemma. Indeed, for $0 \leq u \leq \delta$,

$$
\left|\frac{g(x-u)-g(x-)}{2 u}\right| \leq K \frac{1}{u^{1-r}}
$$

which is integrable on $[0, \delta]$. The other quotient also is integrable by similar reasoning.
The next theorem justifies the terminology above which defines $F^{-1}$ and calls it the inverse Fourier transform. Roughly it says that the inverse Fourier transform of the Fourier transform equals the mid point of the jump. Thus if the original function is continuous, it restores the original value of this function. Surely this is what you would want by calling something the inverse Fourier transform. However, note that in this theorem, it is defined in terms of an improper integral. This is because there is no guarantee that the Fourier transform will end up being in $L^{1}$. Thus instead of $\int_{-\infty}^{\infty}$ we write $\lim _{R \rightarrow \infty} \int_{-R}^{R}$. Of course, IF the Fourier transform ends up being in $L^{1}$, then this amounts to the same thing. The interesting thing is that even if this is not the case, the formula still works provided you consider an improper integral.

Now for certain special kinds of functions, the Fourier transform is indeed in $L^{1}$ and one can show that it maps this special kind of function to another function of the same sort and this will be discussed later. This can be used as the basis for a general theory of Fourier transforms. However, the following does indeed give adequate justification for the terminology that $F^{-1}$ is called the inverse Fourier transform.

Theorem 11.3.6 Let $g \in L^{1}(\mathbb{R})$ and suppose $g$ is locally Holder continuous from the right and from the left at $x$ as in 11.2 and 11.3. Then

$$
\lim _{R \rightarrow \infty} \frac{1}{2 \pi} \int_{-R}^{R} e^{i x t} \int_{-\infty}^{\infty} e^{-i t y} g(y) d y d t=\frac{g(x+)+g(x-)}{2}
$$

Proof: Consider the following manipulations. $\frac{1}{2 \pi} \int_{-R}^{R} e^{i x t} \int_{-\infty}^{\infty} e^{-i t y} g(y) d y d t=$

$$
\frac{1}{2 \pi} \int_{-\infty}^{\infty} \int_{-R}^{R} e^{i x t} e^{-i t y} g(y) d t d y=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \int_{-R}^{R} e^{i(x-y) t} g(y) d t d y
$$

$$
\begin{gathered}
=\frac{1}{2 \pi} \int_{-\infty}^{\infty} g(y)\left(\int_{0}^{R} e^{i(x-y) t} d t+\int_{0}^{R} e^{-i(x-y) t} d t\right) d y \\
=\frac{1}{2 \pi} \int_{-\infty}^{\infty} g(y)\left(\int_{0}^{R} 2 \cos ((x-y) t) d t\right) d y \\
=\frac{1}{\pi} \int_{-\infty}^{\infty} g(y) \frac{\sin R(x-y)}{x-y} d y=\frac{1}{\pi} \int_{-\infty}^{\infty} g(x-y) \frac{\sin R y}{y} d y \\
=\frac{1}{\pi} \int_{0}^{\infty}(g(x-y)+g(x+y)) \frac{\sin R y}{y} d y \\
=\frac{2}{\pi} \int_{0}^{\infty}\left(\frac{g(x-y)+g(x+y)}{2}\right) \frac{\sin R y}{y} d y
\end{gathered}
$$

From Theorem 11.3.5,

$$
\begin{aligned}
& \lim _{R \rightarrow \infty} \frac{1}{2 \pi} \int_{-R}^{R} e^{i x t} \int_{-\infty}^{\infty} e^{-i t y} g(y) d y d t \\
= & \lim _{R \rightarrow \infty} \frac{2}{\pi} \int_{0}^{\infty}\left(\frac{g(x-y)+g(x+y)}{2}\right) \frac{\sin R y}{y} d y \\
= & \frac{g(x+)+g(x-)}{2}
\end{aligned}
$$

Observation 11.3.7 If $t \rightarrow \int_{-\infty}^{\infty} e^{-i t y} g(y) d y$ is itself in $L^{1}(\mathbb{R})$, then you don't need to do the inversion in terms of a principal value integral as above in which

$$
\lim _{R \rightarrow \infty} \int_{-R}^{R} e^{i x t} \int_{-\infty}^{\infty} e^{-i t y} g(y) d y d t
$$

was considered. Instead, you simply get

$$
\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{i x t} \int_{-\infty}^{\infty} e^{-i t y} g(y) d y d t=\frac{g(x+)+g(x-)}{2}
$$

Does this situation ever occur? Yes, it does. This is discussed a little later.

### 11.4 Inversion of Laplace Transforms

How does the Fourier transform relate to the Laplace transform? This is considered next. Recall that from Theorem 11.2.2 if $g$ has exponential growth $|g(t)| \leq C e^{\eta t}$, then if $\operatorname{Re}(s)>$ $\eta$, one can define $\mathscr{L} g(s)$ as $\mathscr{L} g(s) \equiv \int_{0}^{\infty} e^{-s u} g(u) d u$ and also $s \rightarrow \mathscr{L} g(s)$ is differentiable on $\operatorname{Re}(s)>\eta$ in the sense that if $h \in \mathbb{C}$ and $G(s) \equiv \mathscr{L} g(s)$, then

$$
\lim _{h \rightarrow 0} \frac{G(s+h)-G(s)}{h}=G^{\prime}(s)=-\int_{0}^{\infty} u e^{-s u} g(u) d u
$$

This is an example of an analytic function of the complex variable $s$. The next theorem shows how to invert the Laplace transform. One can prove similar theorems about Fourier series. See my single variable analysis book on the web site for this.

Theorem 11.4.1 Let $g$ be a measurable function defined on $(0, \infty)$ which has exponential growth

$$
|g(t)| \leq C e^{\eta t} \text { for some real } \eta
$$

and is Holder continuous from the right and left as in 11.2 and 11.3. For $\operatorname{Re}(s)>\eta$

$$
\mathscr{L} g(s) \equiv \int_{0}^{\infty} e^{-s u} g(u) d u
$$

Then for any $\gamma>\eta$, and $t>0$,

$$
\begin{equation*}
\lim _{R \rightarrow \infty} \frac{1}{2 \pi} \int_{-R}^{R} e^{(\gamma+i y) t} \mathscr{L} g(\gamma+i y) d y=\frac{g(t+)+g(t-)}{2} \tag{11.6}
\end{equation*}
$$

In case of $t=0$, you would only assume the Holder continuity from the right and the above result would be $g(0+) / 2$.

Proof: This follows from plugging in the formula for the Laplace transform of $g$ and then using the above. Thus

$$
\begin{gathered}
\frac{1}{2 \pi} \int_{-R}^{R} e^{(\gamma+i y) t} \mathscr{L} g(\gamma+i y) d y= \\
\frac{1}{2 \pi} \int_{-R}^{R} e^{(\gamma+i y) t} \int_{0}^{\infty} e^{-(\gamma+i y) u} g(u) d u d y=\frac{1}{2 \pi} \int_{-R}^{R} e^{\gamma t} e^{i y t} \int_{0}^{\infty} e^{-(\gamma+i y) u} g(u) d u d y \\
=e^{\gamma t} \frac{1}{2 \pi} \int_{-R}^{R} e^{i y t} \int_{0}^{\infty} e^{-i y u} e^{-\gamma u} g(u) d u d y
\end{gathered}
$$

Let $\hat{g}(u)=0$ for all $u \leq 0$ so this becomes

$$
=e^{\gamma t} \frac{1}{2 \pi} \int_{-R}^{R} e^{i y t} \int_{-\infty}^{\infty} e^{-i y u} e^{-\gamma u} \hat{g}(u) d u d y
$$

Now apply Theorem 11.3 .6 which said that for $g$ in $L^{1}$ having the Holder condition at $t$,

$$
\lim _{R \rightarrow \infty} \frac{1}{2 \pi} \int_{-R}^{R} e^{i y t} \int_{-\infty}^{\infty} e^{-i y u} g(u) d u d y=\frac{g(t+)+g(t-)}{2}
$$

to conclude that if $t>0$,

$$
\begin{aligned}
& \lim _{R \rightarrow \infty} \frac{1}{2 \pi} \int_{-R}^{R} e^{(\gamma+i y) t} \mathscr{L} g(\gamma+i y) d y \\
= & e^{\gamma t} \lim _{R \rightarrow \infty} \frac{1}{2 \pi} \int_{-R}^{R} e^{i y t} \int_{-\infty}^{\infty} e^{-i y u} e^{-\gamma u} \hat{g}(u) d u d y \\
= & e^{\gamma t} \frac{\hat{g}(t+) e^{-\gamma t+}+\hat{g}(t-) e^{-\gamma t-}}{2}=\frac{g(t+)+g(t-)}{2} .
\end{aligned}
$$

If $t=0$ you would have $\hat{g}(0-)=0$ so you would end up finding $\frac{1}{2} g(0+)$.
In particular, this shows that if $\mathscr{L} g(s)=\mathscr{L} h(s)$ for all $s$ large enough, both $g, h$ having exponential growth, then these must be equal except for jumps and in fact, at any point where they are both Holder continuous from right and left, the mid point of their jumps is the same. This gives an alternate proof of Corollary 11.2.6 in the case of points of continuity.

### 11.5 Fourier Transforms in $\mathbb{R}^{n}$

In this section is a general treatment of Fourier transforms. It turns out you can take the Fourier transform of almost anything you like. First is a definition of a very specialized set of functions. Here the measure space will be $\left(\mathbb{R}^{n}, m_{n}, \mathscr{F}_{n}\right), m_{n}$ Lebesgue measure on $\mathbb{R}^{n}$.

First is the definition of a polynomial in many variables.
Definition 11.5.1 $\alpha=\left(\alpha_{1}, \cdots, \alpha_{n}\right)$ for $\alpha_{1} \cdots \alpha_{n}$ nonnegative integers is called $a$ multi-index. For $\alpha$ a multi-index, $|\alpha| \equiv \alpha_{1}+\cdots+\alpha_{n}$ and if $\mathbf{x} \in \mathbb{R}^{n}, \mathbf{x}=\left(x_{1}, \cdots, x_{n}\right)$, and $f$ a function, define $\mathbf{x}^{\alpha} \equiv x_{1}^{\alpha_{1}} x_{2}^{\alpha_{2}} \cdots x_{n}^{\alpha_{n}}$. A polynomial in $n$ variables of degree $m$ is a function of the form

$$
p(\mathbf{x})=\sum_{|\alpha| \leq m} a_{\alpha} \mathbf{x}^{\alpha}
$$

Here $\alpha$ is a multi-index as just described and $a_{\alpha} \in \mathbb{C}$. Also define for $\alpha=\left(\alpha_{1}, \cdots, \alpha_{n}\right)$ a multi-index

$$
D^{\alpha} f(\mathbf{x}) \equiv \frac{\partial^{|\alpha|} f}{\partial x_{1}^{\alpha_{1}} \partial x_{2}^{\alpha_{2}} \cdots \partial x_{n}^{\alpha_{n}}}
$$

Definition 11.5.2 Define $\mathscr{G}_{1}$ to be the functions of the form $p(\mathbf{x}) e^{-a|\mathbf{x}|^{2}}$ where $a>0$ is rational and $p(\mathbf{x})$ is a polynomial having all rational coefficients, $a_{\alpha}$ being "rational" if it is of the form $a+$ ib for $a, b \in \mathbb{Q}$. Let $\mathscr{G}$ be all finite sums of functions in $\mathscr{G}_{1}$. Thus $\mathscr{G}$ is an algebra of functions which has the property that if $f \in \mathscr{G}$ then $\bar{f} \in \mathscr{G}$.

Thus there are countably many functions in $\mathscr{G}_{1}$. This is because, for each $m$, there are countably many choices for $a_{\alpha}$ for $|\alpha| \leq m$ since there are finitely many $\alpha$ for $|\alpha| \leq m$ and for each such $\alpha$, there are countably many choices for $a_{\alpha}$ since $\mathbb{Q}+i \mathbb{Q}$ is countable. (Why?) Thus there are countably many polynomials having degree no more than $m$. This is true for each $m$ and so the number of different polynomials is a countable union of countable sets which is countable. Now there are countably many choices of $e^{-\alpha|\mathbf{x}|^{2}}$ and so there are countably many in $\mathscr{G}_{1}$ because the Cartesian product of countable sets is countable.

Now $\mathscr{G}$ consists of finite sums of functions in $\mathscr{G}_{1}$. Therefore, it is countable because for each $m \in \mathbb{N}$, there are countably many such sums which are possible.

I will show now that $\mathscr{G}$ is dense in $L^{p}\left(\mathbb{R}^{n}\right)$ but first, here is a lemma which follows from the Stone Weierstrass theorem.

Lemma 11.5.3 $\mathscr{G}$ is dense in $C_{0}\left(\mathbb{R}^{n}\right)$ with respect to the norm,

$$
\|f\|_{\infty} \equiv \sup \left\{|f(\mathbf{x})|: \mathbf{x} \in \mathbb{R}^{n}\right\}
$$

Proof: By the Weierstrass approximation theorem, it suffices to show $\mathscr{G}$ separates the points and annihilates no point. It was already observed in the above definition that $\bar{f} \in \mathscr{G}$ whenever $f \in \mathscr{G}$. If $\mathbf{y}_{1} \neq \mathbf{y}_{2}$ suppose first that $\left|\mathbf{y}_{1}\right| \neq\left|\mathbf{y}_{2}\right|$. Then in this case, you can let $f(\mathbf{x}) \equiv e^{-|\mathbf{x}|^{2}}$. Then $f \in \mathscr{G}$ and $f\left(\mathbf{y}_{1}\right) \neq f\left(\mathbf{y}_{2}\right)$. If $\left|\mathbf{y}_{1}\right|=\left|\mathbf{y}_{2}\right|$, then suppose $y_{1 k} \neq y_{2 k}$. This must happen for some $k$ because $\mathbf{y}_{1} \neq \mathbf{y}_{2}$. Then let $f(\mathbf{x}) \equiv x_{k} e^{-|\mathbf{x}|^{2}}$. Thus $\mathscr{G}$ separates points. Now $e^{-|\mathbf{x}|^{2}}$ is never equal to zero and so $\mathscr{G}$ annihilates no point of $\mathbb{R}^{n}$.

These functions are clearly quite specialized. Therefore, the following theorem is somewhat surprising.

Theorem 11.5.4 For each $p \geq 1, p<\infty, \mathscr{G}$ is dense in $L^{p}\left(\mathbb{R}^{n}\right)$. Since $\mathscr{G}$ is countable, this shows that $L^{p}\left(\mathbb{R}^{n}\right)$ is separable.

Proof: Let $f \in L^{p}\left(\mathbb{R}^{n}\right)$. Then there exists $g \in C_{c}\left(\mathbb{R}^{n}\right)$ such that $\|f-g\|_{p}<\varepsilon$. Now let $b>0$ be large enough that $\int_{\mathbb{R}^{n}}\left(e^{-b|\mathbf{x}|^{2}}\right)^{p} d x<\varepsilon^{p}$. Then $\mathbf{x} \rightarrow g(\mathbf{x}) e^{b|\mathbf{x}|^{2}}$ is in $C_{c}\left(\mathbb{R}^{n}\right) \subseteq$ $C_{0}\left(\mathbb{R}^{n}\right)$. Therefore, from Lemma 11.5.3 there exists $\psi \in \mathscr{G}$ such that $\left\|g e^{b|\cdot|^{2}}-\psi\right\|_{\infty}<1$. Therefore, letting $\phi(\mathbf{x}) \equiv \psi(\mathbf{x})$ it follows that $\phi \in \mathscr{G}$ and for all $\mathbf{x} \in \mathbb{R}^{n},|g(\mathbf{x})-\phi(\mathbf{x})|<$ $e^{-b|\mathbf{x}|^{2}}$ Therefore,

$$
\left(\int_{\mathbb{R}^{n}}|g(\mathbf{x})-\phi(\mathbf{x})|^{p} d x\right)^{1 / p} \leq\left(\int_{\mathbb{R}^{n}}\left(e^{-b|\mathbf{x}|^{2}}\right)^{p} d x\right)^{1 / p}<\varepsilon
$$

It follows $\|f-\phi\|_{p} \leq\|f-g\|_{p}+\|g-\phi\|_{p}<2 \varepsilon$.
From now on, drop the restriction that the coefficients of the polynomials in $\mathscr{G}$ are rational. Also drop the restriction that $a$ is rational. Thus $\mathscr{G}$ will be finite sums of functions which are of the form $p(\mathbf{x}) e^{-a|\mathbf{x}|^{2}}$ where the coefficients of $p$ are complex and $a>0$.

The following lemma is also interesting even if it is obvious.
Lemma 11.5.5 For $\psi \in \mathscr{G}$, $p$ a polynomial, and $\alpha, \beta$ multi-indices, $D^{\alpha} \psi \in \mathscr{G}$ and $p \psi \in \mathscr{G}$. Also

$$
\sup \left\{\left|\mathbf{x}^{\beta} D^{\alpha} \psi(\mathbf{x})\right|: \mathbf{x} \in \mathbb{R}^{n}\right\}<\infty
$$

Thus these special functions are infinitely differentiable (smooth). They also have the property that they and all their partial derivatives vanish as $|\mathbf{x}| \rightarrow \infty$. This is because every mixed partial derivative of one of these will be a finite sum of polynomials multiplied by $e^{-b|\mathbf{x}|^{2}}$ for some positive $b$.

The idea is to first understand the Fourier transform on these very specialized functions in $\mathscr{G}$.
Definition 11.5.6 For $\psi \in \mathscr{G}$, define the Fourier transform $F$ and the inverse Fourier transform $F^{-1}$ by

$$
\begin{aligned}
& F \psi(\mathbf{t}) \equiv(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \cdot \mathbf{x}} \psi(\mathbf{x}) d x \\
& F^{-1} \psi(\mathbf{t}) \equiv(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{i \mathbf{t} \cdot \mathbf{x}} \psi(\mathbf{x}) d x
\end{aligned}
$$

where $\mathbf{t} \cdot \mathbf{x} \equiv \sum_{i=1}^{n} t_{i} x_{i}$. Note there is no problem with this definition because $\psi$ is in $L^{1}\left(\mathbb{R}^{n}\right)$ and therefore, $\left|e^{i \mathbf{t} \cdot \mathbf{x}} \psi(\mathbf{x})\right| \leq|\psi(\mathbf{x})|$, an integrable function.

One reason for using the functions $\mathscr{G}$ is that it is very easy to compute the Fourier transform of these functions. The first thing to do is to verify $F$ and $F^{-1}$ map $\mathscr{G}$ to $\mathscr{G}$ and that $F^{-1} \circ F(\psi)=\psi$.

Lemma 11.5.7 The following holds. $(c>0)$

$$
\begin{gather*}
\left(\frac{1}{2 \pi}\right)^{n / 2} \int_{\mathbb{R}^{n}} e^{-c|\mathbf{t}|^{2}} e^{-i \mathbf{s} \cdot \mathbf{t}} d t=\left(\frac{1}{2 \pi}\right)^{n / 2} \int_{\mathbb{R}^{n}} e^{-c|\mathbf{t}|^{2}} e^{i \mathbf{s} \cdot \mathbf{t}} d t \\
=\left(\frac{1}{2 \pi}\right)^{n / 2} e^{-\frac{|\mathbf{s}|^{2}}{4 c}}\left(\frac{\sqrt{\pi}}{\sqrt{c}}\right)^{n}=\left(\frac{1}{2 c}\right)^{n / 2} e^{-\frac{1}{4 c}|s|^{2}} \tag{11.7}
\end{gather*}
$$

Proof: Consider first the case of one dimension. Let $H(s)$ be given by

$$
H(s) \equiv \int_{\mathbb{R}} e^{-c t^{2}} e^{-i s t} d t=\int_{\mathbb{R}} e^{-c t^{2}} \cos (s t) d t
$$

Then using the dominated convergence theorem to differentiate,

$$
\begin{aligned}
H^{\prime}(s) & =\int_{\mathbb{R}}\left(-e^{-c t^{2}}\right) t \sin (s t) d t \\
& =\left(\left.\frac{e^{-c t^{2}}}{2 c} \sin (s t)\right|_{-\infty} ^{\infty}-\frac{s}{2 c} \int_{\mathbb{R}} e^{-c t^{2}} \cos (s t) d t\right)=-\frac{s}{2 c} H(s)
\end{aligned}
$$

Also $H(0)=\int_{\mathbb{R}} e^{-c t^{2}} d t$. Thus $H(0)=\int_{\mathbb{R}} e^{-c x^{2}} d x \equiv I$ and so

$$
I^{2}=\int_{\mathbb{R}^{2}} e^{-c\left(x^{2}+y^{2}\right)} d x d y=\int_{0}^{\infty} \int_{0}^{2 \pi} e^{-c r^{2}} r d \theta d r=\frac{\pi}{c}
$$

For another proof of this which does not use change of variables and polar coordinates, see Problems 4, 5 below. Hence

$$
H^{\prime}(s)+\frac{s}{2 c} H(s)=0, H(0)=\sqrt{\frac{\pi}{c}}
$$

It follows that $H(s)=e^{-\frac{s^{2}}{4 c}} \frac{\sqrt{\pi}}{\sqrt{c}}$. Hence $\frac{1}{\sqrt{2 \pi}} \int_{\mathbb{R}} e^{-c t^{2}} e^{-i s t} d t=\sqrt{\frac{\pi}{c}} \frac{1}{\sqrt{2 \pi}} e^{-\frac{s^{2}}{4 c}}=\left(\frac{1}{2 c}\right)^{1 / 2} e^{-\frac{s^{2}}{4 c}}$. This proves the formula in the case of one dimension. The case of the inverse Fourier transform is similar. The $n$ dimensional formula follows from Fubini's theorem.

With these formulas, it is easy to verify $F, F^{-1} \operatorname{map} \mathscr{G}$ to $\mathscr{G}$ and $F \circ F^{-1}=F^{-1} \circ F=i d$.
Theorem 11.5.8 Each of $F$ and $F^{-1} \operatorname{map} \mathscr{G}$ to $\mathscr{G}$. Also for $\psi \in \mathscr{G}, F^{-1} \circ F(\psi)=\psi$ and $F \circ F^{-1}(\psi)=\psi$.

Proof: To make the notation simpler, $\int$ will symbolize $\frac{1}{(2 \pi)^{n / 2}} \int_{\mathbb{R}^{n}}$. Also, $f_{b}(\mathbf{x}) \equiv$ $e^{-b|\mathbf{x}|^{2}}$. Then from the above, $F f_{b}=(2 b)^{-n / 2} f_{(4 b)^{-1}}$ The first claim will be shown if it is shown that $F \psi \in \mathscr{G}$ for $\psi(\mathbf{x}) \equiv \mathbf{x}^{\alpha} e^{-b|\mathbf{x}|^{2}}$ because an arbitrary function of $\mathscr{G}$ is a finite sum of scalar multiples of functions such as $\psi$. Using Lemma 11.5.7,

$$
\begin{aligned}
F \psi(\mathbf{t}) & \equiv \int e^{-i \mathbf{t} \cdot \mathbf{x}} \mathbf{x}^{\alpha} e^{-b|\mathbf{x}|^{2}} d x \\
& =(-i)^{-|\alpha|} D_{t}^{\alpha}\left(\int e^{-i \mathbf{t} \cdot \mathbf{x}} e^{-b|\mathbf{x}|^{2}} d x\right), \text { (Differentiating under integral) } \\
& =(-i)^{-|\alpha|} D_{t}^{\alpha}\left(e^{-\frac{|t|^{2}}{4 b}}\left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^{n}\right) \text { by Lemma 11.5.7 }
\end{aligned}
$$

and this is clearly in $\mathscr{G}$ because it equals a polynomial times $e^{-\frac{|t|^{2}}{4 b}}$. Similarly, $F^{-1}: \mathscr{G} \rightarrow \mathscr{G}$. Now consider $F^{-1} \circ F(\psi)(\mathbf{s})$ where $\psi=\mathbf{x}^{\alpha} e^{-b|\mathbf{x}|^{2}}$ was just used. From the above, and integrating by parts,

$$
\begin{aligned}
F^{-1} \circ F(\psi)(\mathbf{s}) & =(-i)^{-|\alpha|} \int e^{i \mathbf{s} \cdot \mathbf{t}} D_{t}^{\alpha}\left(\int e^{-i \mathbf{t} \cdot \mathbf{x}} e^{-b|\mathbf{x}|^{2}} d x\right) d t \\
& =(-i)^{-|\alpha|}(-i)^{|\alpha|} \mathbf{s}^{\alpha} \int e^{i \mathbf{s} \cdot \mathbf{t}}\left(\int e^{-i \mathbf{t} \cdot \mathbf{x}} e^{-b|\mathbf{x}|^{2}} d x\right) d t \\
& =\mathbf{s}^{\alpha} F^{-1}\left(F\left(f_{b}\right)\right)(\mathbf{s})
\end{aligned}
$$

From Lemma 11.5.7,

$$
\begin{aligned}
F^{-1}\left(F\left(f_{b}\right)\right)(\mathbf{s}) & =F^{-1}\left((2 b)^{-n / 2} f_{(4 b)^{-1}}\right)(\mathbf{s})=(2 b)^{-n / 2} F^{-1}\left(f_{(4 b)^{-1}}\right)(\mathbf{s}) \\
& =(2 b)^{-n / 2}\left(2(4 b)^{-1}\right)^{-n / 2} f_{\left(4(4 b)^{-1}\right)^{-1}(\mathbf{s})=f_{b}(\mathbf{s})}
\end{aligned}
$$

Hence $F^{-1} \circ F(\psi)(\mathbf{s})=\mathbf{s}^{\alpha} f_{b}(\mathbf{s})=\psi(\mathbf{s})$.
Another way to see this is to use Observation 11.3.7. If $n=1$ then $F^{-1} \circ F(\psi)(\mathbf{s})=$ $\psi(\mathbf{s})$ from Observation 11.3.7 and Theorem 11.3.6. So suppose $F^{-1} \circ F(\psi)(\mathbf{s})=\psi(\mathbf{s})$ on $\mathbb{R}^{n-1}$. From the definition and the observation and Fubini's theorem, if $\psi \in \mathscr{G}$ on $\mathbb{R}^{n}$, then from the special form of $\psi$ and neglecting the $(1 / 2 \pi)^{n / 2}$ to make it simpler to write,

$$
\begin{gathered}
F^{-1} \circ F(\psi)(\mathbf{s}) \equiv \\
\int_{\mathbb{R}} e^{i s_{n} t_{n}} \int_{\mathbb{R}} e^{-i t_{n} x_{n}} \int_{\mathbb{R}^{n-1}} e^{i \hat{\mathbf{s}}_{n} \cdot \hat{\mathbf{t}}_{n}} \int_{\mathbb{R}^{n-1}} e^{-\hat{\mathbf{t}}_{n} \cdot \hat{\mathbf{x}}_{n}} \psi\left(\hat{\mathbf{x}}_{n}, x_{n}\right) d \hat{x}_{n} d \hat{t}_{n} d x_{n} d t_{n}
\end{gathered}
$$

Now by induction and Theorem 11.3.6, this is

$$
\int_{\mathbb{R}} e^{i s_{n} t_{n}} \int_{\mathbb{R}} e^{-i t_{n} x_{n}} \psi\left(\hat{\mathbf{s}}_{n}, x_{n}\right) d x_{n} d t_{n}=\psi\left(\hat{\mathbf{s}}_{n}, s_{n}\right)=\psi(\mathbf{s})
$$

### 11.6 Fourier Transforms of Just About Anything

### 11.6.1 Fourier Transforms in $\mathscr{G}^{*}$

It turns out you can make sense of the Fourier transform of any linear map defined on $\mathscr{G}$. This is a very abstract way to look at things but if you want ultimate generality, you must do something like this. Part of the problem is that it is desired to take Fourier transforms of functions which are not in $L^{1}\left(\mathbb{R}^{n}\right)$. Thus the integral which defines the Fourier transform in the above will not make sense. You run into this problem as soon as you try to take the Fourier transform of a function in $L^{2}$ because such functions might not be in $L^{1}$ if they are defined on $\mathbb{R}$ or $\mathbb{R}^{n}$. However, it was realized long ago that if a function is in $L^{1} \cap L^{2}$, then the $L^{2}$ norm of the function is equal to the $L^{2}$ norm of the Fourier transform of the function. Thus there is an obvious question about whether you can get a definition which will allow you to directly deal with the Fourier transform on $L^{2}$. If you solve this, perhaps by using density of $L^{1} \cap L^{2}$ in $L^{2}$, you are still faced with the problem of taking the Fourier transform of an arbitrary function in $L^{p}$. The method developed here removes all these difficulties at once.

Definition 11.6.1 Let $\mathscr{G}^{*}$ denote the vector space of linear functions defined on $\mathscr{G}$ which have values in $\mathbb{C}$. Thus $T \in \mathscr{G}^{*}$ means $T: \mathscr{G} \rightarrow \mathbb{C}$ and $T$ is linear,

$$
T(a \psi+b \phi)=a T(\psi)+b T(\phi) \text { for all } a, b \in \mathbb{C}, \psi, \phi \in \mathscr{G}
$$

Let $\psi \in \mathscr{G}$. Then we can regard $\psi$ as an element of $\mathscr{G}^{*}$ by defining

$$
\psi(\phi) \equiv \int_{\mathbb{R}^{n}} \psi(\mathbf{x}) \phi(\mathbf{x}) d x
$$

This implies the following important lemma.

Lemma 11.6.2 The following is obtained for all $\phi, \psi \in \mathscr{G}$.

$$
F \psi(\phi)=\psi(F \phi), F^{-1} \psi(\phi)=\psi\left(F^{-1} \phi\right)
$$

Also if $\psi \in \mathscr{G}$ and $\psi=0$ in $\mathscr{G}^{*}$ so that $\psi(\phi)=0$ for all $\phi \in \mathscr{G}$, then $\psi=0$ as a function.
Proof:

$$
\begin{aligned}
F \psi(\phi) & \equiv \int_{\mathbb{R}^{n}} F \psi(\mathbf{t}) \phi(\mathbf{t}) d t=\int_{\mathbb{R}^{n}}\left(\frac{1}{2 \pi}\right)^{n / 2} \int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \mathbf{x}} \psi(\mathbf{x}) d x \phi(\mathbf{t}) d t \\
& =\int_{\mathbb{R}^{n}} \psi(\mathbf{x})\left(\frac{1}{2 \pi}\right)^{n / 2} \int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \mathbf{x}} \phi(\mathbf{t}) d t d x \\
& =\int_{\mathbb{R}^{n}} \psi(\mathbf{x}) F \phi(\mathbf{x}) d x \equiv \psi(F \phi)
\end{aligned}
$$

The other claim is similar.
Suppose now $\psi(\phi)=0$ for all $\phi \in \mathscr{G}$. Then $\int_{\mathbb{R}^{n}} \psi \phi d x=0$ for all $\phi \in \mathscr{G}$. Therefore, this is true for $\phi=\psi$ and so $\psi=0$.

This lemma suggests a way to define the Fourier transform of something in $\mathscr{G}^{*}$.
Definition 11.6.3 For $T \in \mathscr{G}^{*}$, define $F T, F^{-1} T \in \mathscr{G}^{*}$ by

$$
F T(\phi) \equiv T(F \phi), F^{-1} T(\phi) \equiv T\left(F^{-1} \phi\right)
$$

Lemma 11.6.4 $F$ and $F^{-1}$ are both one to one, onto, and are inverses of each other.
Proof: First note $F$ and $F^{-1}$ are both linear. This follows directly from the definition. Suppose now $F T=0$. Then $F T(\phi) \equiv T(F \phi)=0$ for all $\phi \in \mathscr{G}$. But $F$ and $F^{-1}$ map $\mathscr{G}$ onto $\mathscr{G}$ because if $\psi \in \mathscr{G}$, then as shown above, $\psi=F\left(F^{-1}(\psi)\right)$. Therefore, $T=0$ and so $F$ is one to one. Similarly $F^{-1}$ is one to one. Now $F^{-1}(F T)(\phi) \equiv(F T)\left(F^{-1} \phi\right) \equiv$ $T\left(F\left(F^{-1}(\phi)\right)\right)=T \phi$. Therefore, $F^{-1} \circ F(T)=T$. Similarly, $F \circ F^{-1}(T)=T$. Thus both $F$ and $F^{-1}$ are one to one and onto and are inverses of each other as suggested by the notation.

Probably the most interesting things in $\mathscr{G}^{*}$ are functions of various kinds. The following lemma will be useful in considering this situation.

Definition 11.6.5 a function $f$ defined on $\mathbb{R}^{n}$ is in $L_{\text {loc }}^{1}\left(\mathbb{R}^{n}\right)$ if $f \mathscr{X}_{B} \in L^{1}\left(\mathbb{R}^{n}\right)$ for every ball B. Such functions are termed locally integrable.

Lemma 11.6.6 If $f \in L_{l o c}^{1}\left(\mathbb{R}^{n}\right)$ and $\int_{\mathbb{R}^{n}} f \phi d x=0$ for all $\phi \in C_{c}\left(\mathbb{R}^{n}\right)$, then $f=0$ a.e.
Proof: Let $E$ be bounded and Lebesgue measurable. By regularity, there exists a compact set $K_{k} \subseteq E$ and an open set $V_{k} \supseteq E$ such that $m_{n}\left(V_{k} \backslash K_{k}\right)<2^{-k}$. Let $h_{k}$ equal 1 on $K_{k}$, vanish on $V_{k}^{C}$, and take values between 0 and 1 . Then $h_{k}$ converges to $\mathscr{X}_{E}$ off $\cap_{k=1}^{\infty} \cup_{l=k}^{\infty}\left(V_{l} \backslash K_{l}\right)$, a set of measure zero. Hence, by the dominated convergence theorem,

$$
\int f \mathscr{X}_{E} d m_{n}=\lim _{k \rightarrow \infty} \int f h_{k} d m_{n}=0 .
$$

It follows that for $E$ an arbitrary Lebesgue measurable set, $\int f \mathscr{X}_{B(\mathbf{0}, R)} \mathscr{X}_{E} d m_{n}=0$. Let

$$
\operatorname{sgn} f=\left\{\begin{array}{l}
\frac{\bar{f}}{|f|} \text { if }|f| \neq 0 \\
0 \text { if }|f|=0
\end{array}\right.
$$

By Corollary 7.7.6, there exists $\left\{s_{k}\right\}$, a sequence of simple functions converging pointwise to $\operatorname{sgn} f \mathscr{X}_{B(\mathbf{0}, R)}$ such that $\left|s_{k}\right| \leq 1$. Then by the dominated convergence theorem again,

$$
\int|f| \mathscr{X}_{B(\mathbf{0}, R)} d m_{n}=\lim _{k \rightarrow \infty} \int f \mathscr{X}_{B(\mathbf{0}, R)}^{s_{k}} d m_{n}=0
$$

Since $R$ is arbitrary, $|f|=0$ a.e.
Corollary 11.6.7 Let $f \in L^{1}\left(\mathbb{R}^{n}\right)$ and suppose $\int_{\mathbb{R}^{n}} f(\mathbf{x}) \phi(\mathbf{x}) d x=0$ for all $\phi \in \mathscr{G}$. Then $f=0$ a.e.

Proof: Let $\psi \in C_{c}\left(\mathbb{R}^{n}\right)$. Then by the Stone Weierstrass approximation theorem, there exists a sequence of functions, $\left\{\phi_{k}\right\} \subseteq \mathscr{G}$ such that $\phi_{k} \rightarrow \psi$ uniformly. Then by the dominated convergence theorem, $\int f \psi d x=\lim _{k \rightarrow \infty} \int f \phi_{k} d x=0$. By Lemma 11.6.6 $f=0$.

The next theorem is the main result of this sort.
Theorem 11.6.8 Let $f \in L^{p}\left(\mathbb{R}^{n}\right)$, $p \geq 1$, or suppose $f$ is measurable and has polynomial growth, defined as $|f(\mathbf{x})| \leq K\left(1+|\mathbf{x}|^{2}\right)^{m}$ for some $K$ and $m$. Then if $\int f \psi d x=0$ for all $\psi \in \mathscr{G}$, then it follows $f=0$.

Proof: First note that if $f \in L^{p}\left(\mathbb{R}^{n}\right)$ or has polynomial growth, then it makes sense to write the integral $\int f \psi d x$ described above. This is obvious in the case of polynomial growth. In the case where $f \in L^{p}\left(\mathbb{R}^{n}\right)$ it also makes sense because

$$
\int|f||\psi| d x \leq\left(\int|f|^{p} d x\right)^{1 / p}\left(\int|\psi|^{p^{\prime}} d x\right)^{1 / p^{\prime}}<\infty
$$

due to the fact mentioned above that all these functions in $\mathscr{G}$ are in $L^{p}\left(\mathbb{R}^{n}\right)$ for every $p \geq 1$. Suppose now that $f \in L^{p}, p \geq 1$. The case where $f \in L^{1}\left(\mathbb{R}^{n}\right)$ was dealt with in Corollary 11.6.7. Suppose $f \in L^{p}\left(\mathbb{R}^{n}\right)$ for $p>1$. Then

$$
|f|^{p-2} \bar{f} \in L^{p^{\prime}}\left(\mathbb{R}^{n}\right),\left(p^{\prime}=q, \frac{1}{p}+\frac{1}{q}=1\right)
$$

and by density of $\mathscr{G}$ in $L^{p^{\prime}}\left(\mathbb{R}^{n}\right)$ (Theorem 11.5.4), there exists a sequence $\left\{g_{k}\right\} \subseteq \mathscr{G}$ such that

$$
\left\|g_{k}-|f|^{p-2} \bar{f}\right\|_{p^{\prime}} \rightarrow 0
$$

Then

$$
\begin{aligned}
\int_{\mathbb{R}^{n}}|f|^{p} d x & =\int_{\mathbb{R}^{n}} f\left(|f|^{p-2} \bar{f}-g_{k}\right) d x+\int_{\mathbb{R}^{n}} f g_{k} d x \\
& =\int_{\mathbb{R}^{n}} f\left(|f|^{p-2} \bar{f}-g_{k}\right) d x \leq\|f\|_{L^{p}}\left\|g_{k}-|f|^{p-2} \bar{f}\right\|_{p^{\prime}}
\end{aligned}
$$

which converges to 0 . Hence $f=0$.

It remains to consider the case where $f$ has polynomial growth. Thus $\mathbf{x} \rightarrow f(\mathbf{x}) e^{-|\mathbf{x}|^{2}} \in$ $L^{1}\left(\mathbb{R}^{n}\right)$. Therefore, for all $\psi \in \mathscr{G}$,

$$
0=\int f(\mathbf{x}) e^{-|\mathbf{x}|^{2}} \psi(\mathbf{x}) d x
$$

because $e^{-|\mathbf{x}|^{2}} \psi(\mathbf{x}) \in \mathscr{G}$. Therefore, by the first part, $f(\mathbf{x}) e^{-|\mathbf{x}|^{2}}=0$ a.e.
Note that "polynomial growth" could be replaced with a condition of the form

$$
|f(\mathbf{x})| \leq K\left(1+|\mathbf{x}|^{2}\right)^{m} e^{k|x|^{\alpha}}, \alpha<2
$$

and the same proof would yield that these functions are in $\mathscr{G}^{*}$. The main thing to observe is that almost all functions of interest are in $\mathscr{G}^{*}$.

Theorem 11.6.9 Let $f$ be a measurable function with polynomial growth,

$$
|f(\mathbf{x})| \leq C\left(1+|\mathbf{x}|^{2}\right)^{N} \text { for some } N
$$

or let $f \in L^{p}\left(\mathbb{R}^{n}\right)$ for some $p \in[1, \infty]$. Then $f \in \mathscr{G}^{*}$ if $f(\phi) \equiv \int f \phi d x$.
Proof: Let $f$ have polynomial growth first. Then the above integral is clearly well defined and so in this case, $f \in \mathscr{G}^{*}$.

Next suppose $f \in L^{p}\left(\mathbb{R}^{n}\right)$ with $\infty>p \geq 1$. Then it is clear again that the above integral is well defined because of the fact that $\phi$ is a sum of polynomials times exponentials of the form $e^{-c|\mathbf{x}|^{2}}$ and these are in $L^{p^{\prime}}\left(\mathbb{R}^{n}\right)$. Also $\phi \rightarrow f(\phi)$ is clearly linear in both cases.

This has shown that for nearly any reasonable function, you can define its Fourier transform as described above. You could also define the Fourier transform of a finite Borel measure $\mu$ because for such a measure $\psi \rightarrow \int_{\mathbb{R}^{n}} \psi d \mu$ is a linear functional on $\mathscr{G}$. This includes the very important case of probability distribution measures.

### 11.6.2 Fourier Transforms of Functions $\operatorname{In} L^{1}\left(\mathbb{R}^{n}\right)$

First suppose $f \in L^{1}\left(\mathbb{R}^{n}\right)$. As mentioned, you can think of it as being in $\mathscr{G}^{*}$ and so one can take its Fourier transform as described above. However, since it is in $L^{1}\left(\mathbb{R}^{n}\right)$, there is a natural way to define its Fourier transform. Do the two give the same thing?

Theorem 11.6.10 Let $f \in L^{1}\left(\mathbb{R}^{n}\right)$. Then $F f(\phi)=\int_{\mathbb{R}^{n}} g \phi d t$ where

$$
g(\mathbf{t})=\left(\frac{1}{2 \pi}\right)^{n / 2} \int_{\mathbb{R}^{n}} e^{-i \boldsymbol{t} \cdot \mathbf{x}} f(\mathbf{x}) d x
$$

and $F^{-1} f(\phi)=\int_{\mathbb{R}^{n}} g \phi d t$ where $g(\mathbf{t})=\left(\frac{1}{2 \pi}\right)^{n / 2} \int_{\mathbb{R}^{n}} e^{i \mathbf{t} \cdot \mathbf{x}} f(\mathbf{x}) d x$. In short,

$$
\begin{aligned}
& F f(\mathbf{t}) \equiv(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \cdot \mathbf{x}} f(\mathbf{x}) d x \\
& F^{-1} f(\mathbf{t}) \equiv(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{i \mathbf{t} \cdot \mathbf{x}} f(\mathbf{x}) d x
\end{aligned}
$$

Proof: From the definition and Fubini's theorem,

$$
\begin{aligned}
F f(\phi) & \equiv \int_{\mathbb{R}^{n}} f(\mathbf{t}) F \phi(\mathbf{t}) d t=\int_{\mathbb{R}^{n}} f(\mathbf{t})\left(\frac{1}{2 \pi}\right)^{n / 2} \int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \cdot \mathbf{x}} \phi(\mathbf{x}) d x d t \\
& =\int_{\mathbb{R}^{n}}\left(\left(\frac{1}{2 \pi}\right)^{n / 2} \int_{\mathbb{R}^{n}} f(\mathbf{t}) e^{-i \mathbf{t} \cdot \mathbf{x}} d t\right) \phi(\mathbf{x}) d x
\end{aligned}
$$

Since $\phi \in \mathscr{G}$ is arbitrary, it follows from Theorem 11.6.8 that $F f(\mathbf{x})$ is given by the claimed formula. The case of $F^{-1}$ is identical.

Here are interesting properties of these Fourier transforms of functions in $L^{1}$.
Theorem 11.6.11 If $f \in L^{1}\left(\mathbb{R}^{n}\right)$ and $\left\|f_{k}-f\right\|_{1} \rightarrow 0$, then $F f_{k}$ and $F^{-1} f_{k}$ converge uniformly to $F f$ and $F^{-1} f$ respectively. If $f \in L^{1}\left(\mathbb{R}^{n}\right)$, then $F^{-1} f$ and $F f$ are both continuous and bounded. Also,

$$
\begin{equation*}
\lim _{|\mathbf{x}| \rightarrow \infty} F^{-1} f(\mathbf{x})=\lim _{|\mathbf{x}| \rightarrow \infty} F f(\mathbf{x})=0 \tag{11.8}
\end{equation*}
$$

Furthermore, for $f \in L^{1}\left(\mathbb{R}^{n}\right)$ both $F f$ and $F^{-1} f$ are uniformly continuous.
Proof: The first claim follows from the following inequality.

$$
\begin{aligned}
\left|F f_{k}(\mathbf{t})-F f(\mathbf{t})\right| & \leq(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}}\left|e^{-i \mathbf{t} \cdot \mathbf{x}} f_{k}(\mathbf{x})-e^{-i \mathbf{t} \cdot \mathbf{x}} f(\mathbf{x})\right| d x \\
& =(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}}\left|f_{k}(\mathbf{x})-f(\mathbf{x})\right| d x=(2 \pi)^{-n / 2}\left\|f-f_{k}\right\|_{1}
\end{aligned}
$$

which a similar argument holding for $F^{-1}$.
Now consider the second claim of the theorem.

$$
\left|F f(\mathbf{t})-F f\left(\mathbf{t}^{\prime}\right)\right| \leq(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}}\left|e^{-i \mathbf{t} \cdot \mathbf{x}}-e^{-i \mathbf{t}^{\prime} \cdot \mathbf{x}}\right||f(\mathbf{x})| d x
$$

The integrand is bounded by $2|f(\mathbf{x})|$, a function in $L^{1}\left(\mathbb{R}^{n}\right)$ and converges to 0 as $\mathbf{t}^{\prime} \rightarrow \mathbf{t}$ and so the dominated convergence theorem implies $F f$ is continuous. To see $F f(\mathbf{t})$ is uniformly bounded,

$$
|F f(\mathbf{t})| \leq(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}}|f(\mathbf{x})| d x<\infty
$$

A similar argument gives the same conclusions for $F^{-1}$.
Let $\left\|\mathbf{t}_{k}\right\|_{\infty} \rightarrow \infty$. Then for $g \in \mathscr{G}$, we see that for some $i_{k}$ where, $\left|i_{k}\right|=\left\|\mathbf{t}_{k}\right\|_{\infty}$

$$
\mathbf{t}_{k} \equiv\left(t_{1}^{1}, \cdots, t_{k}^{n}\right), \lim _{k \rightarrow \infty}\left|t_{k}^{i_{k}}\right|=\infty
$$

since otherwise we could not have $\left\|\mathbf{t}_{k}\right\|_{\infty} \rightarrow \infty$. Then integrating by parts,

$$
\left|F g\left(\mathbf{t}_{k}\right)\right|=\left|(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{-i \mathbf{t}_{k} \cdot \mathbf{x}} g(\mathbf{x}) d x\right| \leq C_{n} \int_{\mathbb{R}^{n}} \frac{1}{\left|t_{k}^{i_{k}}\right|}\left|D_{t_{i}} g(\mathbf{x})\right| d x
$$

which converges to 0 as $k \rightarrow \infty$. This shows 11.8 in case the function is in $\mathscr{G}$. Now in the general case, let $\|g-f\|_{L^{1}\left(\mathbb{R}^{n}\right)}<\varepsilon$ where $g \in \mathscr{G}$ and $f \in L^{1}\left(\mathbb{R}^{n}\right)$. Then it follows that $|F(f)(\mathbf{t})-F g(\mathbf{t})| \leq C_{n}\|f-g\|_{L^{1}\left(\mathbb{R}^{n}\right)}$ and so if $\left\|\mathbf{t}_{k}\right\|_{\infty} \rightarrow \infty$,

$$
\lim \sup _{k \rightarrow \infty}\left|F(f)\left(\mathbf{t}_{k}\right)\right| \leq \lim \sup _{k \rightarrow \infty}\left|F(f)\left(\mathbf{t}_{k}\right)-F g\left(\mathbf{t}_{k}\right)\right|+\lim _{k \rightarrow \infty}\left|F g\left(\mathbf{t}_{k}\right)\right| \leq \varepsilon
$$

if $g$ is chosen to make $C_{n}\|f-g\|_{L^{1}\left(\mathbb{R}^{n}\right)}<\varepsilon$. Since $\varepsilon$ is arbitrary, this shows 11.8 in general.
It remains to verify the claim that $F f$ and $F^{-1} f$ are uniformly continuous. Let $\varepsilon>0$ be given. Then there exists $R$ such that if $\|\mathbf{t}\|_{\infty}>R$, then $|F f(\mathbf{t})|<\frac{\varepsilon}{2}$. Since $F f$ is continuous, it is uniformly continuous on the compact set $[-R-1, R+1]^{n}$. Therefore, there exists $\delta_{1}$ such that if $\left\|\mathbf{t}-\mathbf{t}^{\prime}\right\|_{\infty}<\delta_{1}$ for $\mathbf{t}^{\prime}, \mathbf{t} \in[-R-1, R+1]^{n}$, then

$$
\begin{equation*}
\left|F f(\mathbf{t})-F f\left(\mathbf{t}^{\prime}\right)\right|<\varepsilon / 2 \tag{11.9}
\end{equation*}
$$

Now let $0<\boldsymbol{\delta}<\min \left(\delta_{1}, 1\right)$ and suppose $\left\|\mathbf{t}-\mathbf{t}^{\prime}\right\|_{\infty}<\boldsymbol{\delta}$. If both $\mathbf{t}, \mathbf{t}^{\prime}$ are contained in $[-R, R]^{n}$, then 11.9 holds. If $\mathbf{t} \in[-R, R]^{n}$ and $\mathbf{t}^{\prime} \notin[-R, R]^{n}$, then both are contained in $[-R-1, R+1]^{n}$ and so this verifies 11.9 in this case. The other case is that neither point is in $[-R, R]^{n}$ and in this case,

$$
\left|F f(\mathbf{t})-F f\left(\mathbf{t}^{\prime}\right)\right| \leq|F f(\mathbf{t})|+\left|F f\left(\mathbf{t}^{\prime}\right)\right|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon
$$

### 11.6.3 Convolutions in $L^{1}\left(\mathbb{R}^{n}\right)$

There is a very interesting relation between the Fourier transform and convolutions. Recall

$$
f * g(\mathbf{x}) \equiv \int_{\mathbb{R}^{n}} f(\mathbf{x}-\mathbf{y}) g(\mathbf{y}) d y
$$

and part of the problem is in showing that this even makes sense. This is dealt with in the following theorem.
Theorem 11.6.12 Suppose that $f, g \in L^{1}\left(\mathbb{R}^{n}\right)$. Then also $f * g \in L^{1}$ and it follows that $F(f * g)=(2 \pi)^{n / 2} F f F g$.

Proof: Assume both $f$ and $g$ are Borel measurable representatives. Consider

$$
\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}}|f(\mathbf{x}-\mathbf{y}) g(\mathbf{y})| d y d x
$$

By Fubini's theorem,

$$
\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}}|f(\mathbf{x}-\mathbf{y}) g(\mathbf{y})| d y d x=\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}}|f(\mathbf{x}-\mathbf{y}) g(\mathbf{y})| d x d y=\|f\|_{1}\|g\|_{1}<\infty .
$$

It follows that for a.e. $\mathbf{x}, \int_{\mathbb{R}^{n}}|f(\mathbf{x}-\mathbf{y}) g(\mathbf{y})| d y<\infty$ and for each of these values of $\mathbf{x}$, it follows that $\int_{\mathbb{R}^{n}} f(\mathbf{x}-\mathbf{y}) g(\mathbf{y}) d y$ exists and equals a function of $\mathbf{x}$ which is in $L^{1}\left(\mathbb{R}^{n}\right), f *$ $g(\mathbf{x})$. Now

$$
\begin{aligned}
& F(f * g)(\mathbf{t}) \equiv(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \cdot \mathbf{x}} f * g(\mathbf{x}) d x \\
= & (2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \cdot \mathbf{x}} \int_{\mathbb{R}^{n}} f(\mathbf{x}-\mathbf{y}) g(\mathbf{y}) d y d x \\
= & (2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \cdot \mathbf{y}} g(\mathbf{y}) \int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \cdot \mathbf{x}-\mathbf{y})} f(\mathbf{x}-\mathbf{y}) d x d y \\
= & (2 \pi)^{n / 2} F f(\mathbf{t}) F g(\mathbf{t}) .
\end{aligned}
$$

There are many other considerations involving Fourier transforms of functions which are in $L^{1}\left(\mathbb{R}^{n}\right)$. Some others are in the exercises.

### 11.6.4 Fourier Transforms of Functions $\operatorname{In} L^{2}\left(\mathbb{R}^{n}\right)$

Consider $F f$ and $F^{-1} f$ for $f \in L^{2}\left(\mathbb{R}^{n}\right)$. First note that the formula given for $F f$ and $F^{-1} f$ when $f \in L^{1}\left(\mathbb{R}^{n}\right)$ will not work for $f \in L^{2}\left(\mathbb{R}^{n}\right)$ unless $f$ is also in $L^{1}\left(\mathbb{R}^{n}\right)$. However, there is no problem for functions in $\mathscr{G}$.

Theorem 11.6.13 For $\phi \in \mathscr{G},\|F \phi\|_{2}=\left\|F^{-1} \phi\right\|_{2}=\|\phi\|_{2}$.
Proof: First note that for $\psi \in \mathscr{G}$,

$$
\begin{equation*}
F(\bar{\psi})=\overline{F^{-1}(\psi)}, F^{-1}(\bar{\psi})=\overline{F(\psi)} \tag{11.10}
\end{equation*}
$$

This follows from the definition. For example,

$$
F \bar{\psi}(\mathbf{t})=(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \cdot \mathbf{x}} \bar{\psi}(\mathbf{x}) d x=\overline{(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{i t \cdot \mathbf{x}} \psi(\mathbf{x}) d x}=\overline{F(\psi)}(\mathbf{t})
$$

Let $\phi, \psi \in \mathscr{G}$. It was shown above that $\int_{\mathbb{R}^{n}}(F \phi) \psi(\mathbf{t}) d t=\int_{\mathbb{R}^{n}} \phi(F \psi) d x$. Similarly,

$$
\begin{equation*}
\int_{\mathbb{R}^{n}} \phi\left(F^{-1} \psi\right) d x=\int_{\mathbb{R}^{n}}\left(F^{-1} \phi\right) \psi d t \tag{11.11}
\end{equation*}
$$

Now, 11.10-11.11 imply

$$
\begin{aligned}
\int_{\mathbb{R}^{n}}|\phi|^{2} d x & =\int_{\mathbb{R}^{n}} \phi \bar{\phi} d x=\int_{\mathbb{R}^{n}} \phi \overline{F^{-1}(F \phi)} d x=\int_{\mathbb{R}^{n}} \phi F(\overline{F \phi}) d x \\
& =\int_{\mathbb{R}^{n}} F \phi(\overline{F \phi}) d x=\int_{\mathbb{R}^{n}}|F \phi|^{2} d x .
\end{aligned}
$$

Similarly $\|\phi\|_{2}=\left\|F^{-1} \phi\right\|_{2}$.
Lemma 11.6.14 Let $f \in L^{2}\left(\mathbb{R}^{n}\right)$ and let $\phi_{k} \rightarrow f$ in $L^{2}\left(\mathbb{R}^{n}\right)$ where $\phi_{k} \in \mathscr{G}$. (Such a sequence exists because of density of $\mathscr{G}$ in $L^{2}\left(\mathbb{R}^{n}\right)$. Theorem 11.5.4) Then $F f$ and $F^{-1} f$ are both in $L^{2}\left(\mathbb{R}^{n}\right)$ and the following limits take place in $L^{2}$.

$$
\lim _{k \rightarrow \infty} F\left(\phi_{k}\right)=F(f), \lim _{k \rightarrow \infty} F^{-1}\left(\phi_{k}\right)=F^{-1}(f)
$$

Proof: Let $\phi_{k} \rightarrow f$ in $L^{2}\left(\mathbb{R}^{n}\right)$. Then if $\psi \in \mathscr{G}$, by the Cauchy Schwarz inequality,

$$
\left|\int_{\mathbb{R}^{n}} \phi_{k} F \psi d x-\int_{\mathbb{R}^{n}} \phi_{m} F \psi d x\right| \leq\left\|\phi_{k}-\phi_{m}\right\|_{L^{2}}\|F \psi\|_{L^{2}}
$$

and so $\lim _{k \rightarrow \infty} \int_{\mathbb{R}^{n}} \phi_{k}(\mathbf{x}) F \psi(\mathbf{x}) d x$ exists. Now

$$
\begin{aligned}
F f(\psi) & \equiv f(F \psi) \equiv \int_{\mathbb{R}^{n}} f(\mathbf{x}) F \psi(\mathbf{x}) d x \\
& =\lim _{k \rightarrow \infty} \int_{\mathbb{R}^{n}} \phi_{k}(\mathbf{x}) F \psi(\mathbf{x}) d x=\lim _{k \rightarrow \infty} \int_{\mathbb{R}^{n}} F \phi_{k}(\mathbf{x}) \psi(\mathbf{x}) d x
\end{aligned}
$$

Also by Theorem 11.6.13 $\left\{F \phi_{k}\right\}_{k=1}^{\infty}$ is Cauchy in $L^{2}\left(\mathbb{R}^{n}\right)$ and so it converges to some $h \in L^{2}\left(\mathbb{R}^{n}\right)$. Therefore, from the above,

$$
F f(\psi)=\int_{\mathbb{R}^{n}} h(\mathbf{x}) \psi(\mathbf{x})
$$

which shows that $F(f) \in L^{2}\left(\mathbb{R}^{n}\right)$ and $h=F(f)$. The case of $F^{-1}$ is entirely similar.
Since $F f$ and $F^{-1} f$ are in $L^{2}\left(\mathbb{R}^{n}\right)$, this also proves the following theorem.
Theorem 11.6.15 If $f \in L^{2}\left(\mathbb{R}^{n}\right), F f$ and $F^{-1} f$ are the unique elements of $L^{2}\left(\mathbb{R}^{n}\right)$ such that for all $\phi \in \mathscr{G}$,

$$
\begin{align*}
\int_{\mathbb{R}^{n}} F f(\mathbf{x}) \phi(\mathbf{x}) d x & =\int_{\mathbb{R}^{n}} f(\mathbf{x}) F \phi(\mathbf{x}) d x  \tag{11.12}\\
\int_{\mathbb{R}^{n}} F^{-1} f(\mathbf{x}) \phi(\mathbf{x}) d x & =\int_{\mathbb{R}^{n}} f(\mathbf{x}) F^{-1} \phi(\mathbf{x}) d x \tag{11.13}
\end{align*}
$$

## Theorem 11.6.16 (Plancherel)

$$
\begin{equation*}
\|f\|_{2}=\|F f\|_{2}=\left\|F^{-1} f\right\|_{2} \tag{11.14}
\end{equation*}
$$

Proof: Use the density of $\mathscr{G}$ in $L^{2}\left(\mathbb{R}^{n}\right)$ to obtain a sequence, $\left\{\phi_{k}\right\}$ converging to $f$ in $L^{2}\left(\mathbb{R}^{n}\right)$. Then by Lemma 11.6 .14

$$
\|F f\|_{2}=\lim _{k \rightarrow \infty}\left\|F \phi_{k}\right\|_{2}=\lim _{k \rightarrow \infty}\left\|\phi_{k}\right\|_{2}=\|f\|_{2} .
$$

Similarly, $\|f\|_{2}=\left\|F^{-1} f\right\|_{2}$.
The following corollary is a simple generalization of this. To prove this corollary, use the following simple lemma which comes as a consequence of the Cauchy Schwarz inequality.

Lemma 11.6.17 Suppose $f_{k} \rightarrow f$ in $L^{2}\left(\mathbb{R}^{n}\right)$ and $g_{k} \rightarrow g$ in $L^{2}\left(\mathbb{R}^{n}\right)$. Then

$$
\lim _{k \rightarrow \infty} \int_{\mathbb{R}^{n}} f_{k} g_{k} d x=\int_{\mathbb{R}^{n}} f g d x
$$

Proof:

$$
\begin{gathered}
\left|\int_{\mathbb{R}^{n}} f_{k} g_{k} d x-\int_{\mathbb{R}^{n}} f g d x\right| \leq\left|\int_{\mathbb{R}^{n}} f_{k} g_{k} d x-\int_{\mathbb{R}^{n}} f_{k} g d x\right|+ \\
\left|\int_{\mathbb{R}^{n}} f_{k} g d x-\int_{\mathbb{R}^{n}} f g d x\right| \\
\leq\left\|f_{k}\right\|_{2}\left\|g-g_{k}\right\|_{2}+\|g\|_{2}\left\|f_{k}-f\right\|_{2} .
\end{gathered}
$$

Now $\left\|f_{k}\right\|_{2}$ is a Cauchy sequence and so it is bounded independent of $k$. Therefore, the above expression is smaller than $\varepsilon$ whenever $k$ is large enough.

Corollary 11.6.18 For $f, g \in L^{2}\left(\mathbb{R}^{n}\right)$,

$$
\int_{\mathbb{R}^{n}} f \bar{g} d x=\int_{\mathbb{R}^{n}} F f \overline{F g} d x=\int_{\mathbb{R}^{n}} F^{-1} f \overline{F^{-1} g} d x
$$

Proof: First note the above formula is obvious if $f, g \in \mathscr{G}$. To see this, use 11.10 to write

$$
\int_{\mathbb{R}^{n}} F f \overline{F g} d x=\int_{\mathbb{R}^{n}} F f F^{-1}(\bar{g}) d x=\int_{\mathbb{R}^{n}} F^{-1}(F f)(\bar{g}) d x=\int_{\mathbb{R}^{n}} f \bar{g} d x
$$

The formula with $F^{-1}$ is exactly similar.
Now to verify the corollary, let $\phi_{k} \rightarrow f$ in $L^{2}\left(\mathbb{R}^{n}\right)$ and let $\psi_{k} \rightarrow g$ in $L^{2}\left(\mathbb{R}^{n}\right)$. Then by Lemma 11.6.14 $F \phi_{k} \rightarrow F f$ and $F \psi_{k} \rightarrow F g$ and so

$$
\int_{\mathbb{R}^{n}} F f \overline{F g} d x=\lim _{k \rightarrow \infty} \int_{\mathbb{R}^{n}} F \phi_{k} \overline{F \psi_{k}} d x=\lim _{k \rightarrow \infty} \int_{\mathbb{R}^{n}} \phi_{k} \overline{\psi_{k}} d x=\int_{\mathbb{R}^{n}} f \bar{g} d x
$$

A similar argument holds for $F^{-1}$.
How does one compute $F f$ and $F^{-1} f$ ?
Theorem 11.6.19 For $f \in L^{2}\left(\mathbb{R}^{n}\right)$, let $f_{r}=f \mathscr{X}_{E_{r}}$ where $E_{r}$ is a bounded measurable set with $E_{r} \uparrow \mathbb{R}^{n}$. Then the following limits hold in $L^{2}\left(\mathbb{R}^{n}\right)$.

$$
F f=\lim _{r \rightarrow \infty} F f_{r}, F^{-1} f=\lim _{r \rightarrow \infty} F^{-1} f_{r}
$$

Proof: $\left\|f-f_{r}\right\|_{2} \rightarrow 0$ and so $\left\|F f-F f_{r}\right\|_{2} \rightarrow 0$ and $\left\|F^{-1} f-F^{-1} f_{r}\right\|_{2} \rightarrow 0$ by Plancherel's Theorem.

What are $F f_{r}$ and $F^{-1} f_{r}$ ? Let $\phi \in \mathscr{G}$

$$
\begin{aligned}
\int_{\mathbb{R}^{n}} F f_{r} \phi d x & =\int_{\mathbb{R}^{n}} f_{r} F \phi d x \\
& =(2 \pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} f_{r}(\mathbf{x}) e^{-i \mathbf{x} \cdot \mathbf{y}} \phi(\mathbf{y}) d y d x \\
& =\int_{\mathbb{R}^{n}}\left[(2 \pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{n}} f_{r}(\mathbf{x}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x\right] \phi(\mathbf{y}) d y
\end{aligned}
$$

Since this holds for all $\phi \in \mathscr{G}$, a dense subset of $L^{2}\left(\mathbb{R}^{n}\right)$, it follows that

$$
F f_{r}(\mathbf{y})=(2 \pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{n}} f_{r}(\mathbf{x}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x
$$

Similarly

$$
F^{-1} f_{r}(\mathbf{y})=(2 \pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{n}} f_{r}(\mathbf{x}) e^{i \mathbf{x} \cdot \mathbf{y}} d x
$$

This shows that to take the Fourier transform of a function in $L^{2}\left(\mathbb{R}^{n}\right)$, it suffices to take the limit as $r \rightarrow \infty$ in $L^{2}\left(\mathbb{R}^{n}\right)$ of $(2 \pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{n}} f_{r}(\mathbf{x}) e^{-i \mathbf{x} \cdot \mathbf{y}} d x$. A similar procedure works for the inverse Fourier transform.

Note this reduces to the earlier definition in case $f \in L^{1}\left(\mathbb{R}^{n}\right)$. Now consider the convolution of a function in $L^{2}$ with one in $L^{1}$.

Theorem 11.6.20 Let $h \in L^{2}\left(\mathbb{R}^{n}\right)$ and let $f \in L^{1}\left(\mathbb{R}^{n}\right)$. Then $h * f \in L^{2}\left(\mathbb{R}^{n}\right)$,

$$
\begin{gathered}
F^{-1}(h * f)=(2 \pi)^{n / 2} F^{-1} h F^{-1} f, \\
F(h * f)=(2 \pi)^{n / 2} F h F f,
\end{gathered}
$$

and

$$
\begin{equation*}
\|h * f\|_{2} \leq\|h\|_{2}\|f\|_{1} \tag{11.15}
\end{equation*}
$$

Proof: An application of Minkowski's inequality to Borel representatives yields

$$
\begin{equation*}
\left(\int_{\mathbb{R}^{n}}\left(\int_{\mathbb{R}^{n}}|h(\mathbf{x}-\mathbf{y})||f(\mathbf{y})| d y\right)^{2} d x\right)^{1 / 2} \leq\|f\|_{1}\|h\|_{2} . \tag{11.16}
\end{equation*}
$$

Hence $\int|h(\mathbf{x}-\mathbf{y})||f(\mathbf{y})| d y<\infty$ a.e. $\mathbf{x}$ and $\mathbf{x} \rightarrow \int h(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d y$ is in $L^{2}\left(\mathbb{R}^{n}\right)$. Let $E_{r} \uparrow$ $\mathbb{R}^{n}, m_{n}\left(E_{r}\right)<\infty$. Thus,

$$
h_{r} \equiv \mathscr{X}_{E_{r}} h \in L^{2}\left(\mathbb{R}^{n}\right) \cap L^{1}\left(\mathbb{R}^{n}\right)
$$

and letting $\phi \in \mathscr{G}, \int F\left(h_{r} * f\right)(\phi) d x$

$$
\begin{aligned}
& \equiv \int\left(h_{r} * f\right)(F \phi) d x \\
& =(2 \pi)^{-n / 2} \iiint h_{r}(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) e^{-i \mathbf{x} \cdot \mathbf{t}} \phi(\mathbf{t}) d t d y d x \\
& =(2 \pi)^{-n / 2} \iint\left(\int h_{r}(\mathbf{x}-\mathbf{y}) e^{-i(\mathbf{x}-\mathbf{y}) \cdot \mathbf{t}} d x\right) f(\mathbf{y}) e^{-i \mathbf{y} \cdot \mathbf{t}} d y \phi(\mathbf{t}) d t \\
& =\int(2 \pi)^{n / 2} F h_{r}(\mathbf{t}) F f(\mathbf{t}) \phi(\mathbf{t}) d t
\end{aligned}
$$

Since $\phi$ is arbitrary and $\mathscr{G}$ is dense in $L^{2}\left(\mathbb{R}^{n}\right)$,

$$
F\left(h_{r} * f\right)=(2 \pi)^{n / 2} F h_{r} F f
$$

Now by Minkowski's Inequality, $h_{r} * f \rightarrow h * f$ in $L^{2}\left(\mathbb{R}^{n}\right)$ and also it is clear that $h_{r} \rightarrow h$ in $L^{2}\left(\mathbb{R}^{n}\right)$; so, by Plancherel's theorem, you may take the limit in the above and conclude $F(h * f)=(2 \pi)^{n / 2} F h F f$. The assertion for $F^{-1}$ is similar and 11.15 follows from 11.16.

### 11.6.5 The Schwartz Class

The problem with $\mathscr{G}$ is that it does not contain $C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$. I have used it in presenting the Fourier transform because the functions in $\mathscr{G}$ have a very specific form which made some technical details work out easier than in any other approach I have seen. The Schwartz class is a larger class of functions which does contain $C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$ and also has the same nice properties as $\mathscr{G}$. The functions in the Schwartz class are infinitely differentiable and they vanish very rapidly as $|\mathbf{x}| \rightarrow \infty$ along with all their partial derivatives. This is the description of these functions, not a specific form involving polynomials times $e^{-\alpha|\mathbf{x}|^{2}}$. To describe this precisely requires some notation.
Definition 11.6.21 $f \in \mathfrak{S}$, the Schwartz class, if $f \in C^{\infty}\left(\mathbb{R}^{n}\right)$ and for all positive integers $N, \rho_{N}(f)<\infty$ where

$$
\rho_{N}(f)=\sup \left\{\left(1+|\mathbf{x}|^{2}\right)^{N}\left|D^{\alpha} f(\mathbf{x})\right|: \mathbf{x} \in \mathbb{R}^{n},|\alpha| \leq N\right\}
$$

Thus $f \in \mathfrak{S}$ if and only if $f \in C^{\infty}\left(\mathbb{R}^{n}\right)$ and

$$
\begin{equation*}
\sup \left\{\left|\mathbf{x}^{\beta} D^{\alpha} f(\mathbf{x})\right|: \mathbf{x} \in \mathbb{R}^{n}\right\}<\infty \tag{11.17}
\end{equation*}
$$

for all multi indices $\alpha$ and $\beta$.

Also note that if $f \in \mathfrak{S}$, then $p \circ f \in \mathfrak{S}$ for any polynomial $p$ with $p(0)=0$ and that

$$
\mathfrak{S} \subseteq L^{p}\left(\mathbb{R}^{n}\right) \cap L^{\infty}\left(\mathbb{R}^{n}\right)
$$

for any $p \geq 1$. To see this assertion about the $p(f)$, it suffices to consider the case of the product of two elements of the Schwartz class. If $f, g \in \mathfrak{S}$, then $D^{\alpha}(f g)$ is a finite sum of derivatives of $f$ times derivatives of $g$. Therefore, $\rho_{N}(f g)<\infty$ for all $N$. You may wonder about examples of things in $\mathfrak{S}$. Clearly any function in $C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$ is in $\mathfrak{S}$. However there are other functions in $\mathfrak{S}$. For example $e^{-|\mathbf{x}|^{2}}$ is in $\mathfrak{S}$ as you can verify for yourself and so is any function from $\mathscr{G}$. Note also that the density of $C_{c}\left(\mathbb{R}^{n}\right)$ in $L^{p}\left(\mathbb{R}^{n}\right)$ shows that $\mathfrak{S}$ is dense in $L^{p}\left(\mathbb{R}^{n}\right)$ for every $p$.

Recall the Fourier transform of a function in $L^{1}\left(\mathbb{R}^{n}\right)$ is given by

$$
F f(\mathbf{t}) \equiv(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{-i \mathbf{t} \cdot \mathbf{x}} f(\mathbf{x}) d x
$$

Therefore, this gives the Fourier transform for $f \in \mathfrak{S}$. The nice property which $\mathfrak{S}$ has in common with $\mathscr{G}$ is that the Fourier transform and its inverse map $\mathfrak{S}$ one to one onto $\mathfrak{S}$. This means I could have presented the whole of the above theory in terms of $\mathfrak{S}$ rather than in terms of $\mathscr{G}$. However, it is more technical.

Theorem 11.6.22 If $f \in \mathfrak{S}$, then $F f$ and $F^{-1} f$ are also in $\mathfrak{S}$.
Proof: To begin with, let $\alpha=\mathbf{e}_{j}=(0,0, \cdots, 1,0, \cdots, 0)$, the 1 in the $j^{t h}$ slot.

$$
\begin{equation*}
\frac{F^{-1} f\left(\mathbf{t}+h \mathbf{e}_{j}\right)-F^{-1} f(\mathbf{t})}{h}=(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{i \mathbf{t} \cdot \mathbf{x}} f(\mathbf{x})\left(\frac{e^{i h x_{j}}-1}{h}\right) d x \tag{11.18}
\end{equation*}
$$

Consider the integrand in 11.18 .

$$
\begin{aligned}
\left|e^{i \mathbf{t} \cdot \mathbf{x}} f(\mathbf{x})\left(\frac{e^{i h x_{j}}-1}{h}\right)\right| & =|f(\mathbf{x})|\left|\left(\frac{e^{i(h / 2) x_{j}}-e^{-i(h / 2) x_{j}}}{h}\right)\right| \\
& =|f(\mathbf{x})|\left|\frac{i \sin \left((h / 2) x_{j}\right)}{(h / 2)}\right| \leq|f(\mathbf{x})|\left|x_{j}\right|
\end{aligned}
$$

and this is a function in $L^{1}\left(\mathbb{R}^{n}\right)$ because $f \in \mathfrak{S}$. Therefore by the Dominated Convergence Theorem,

$$
\frac{\partial F^{-1} f(\mathbf{t})}{\partial t_{j}}=(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{i \mathbf{t} \cdot \mathbf{x}} i x_{j} f(\mathbf{x}) d x=i(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{i \mathbf{t} \cdot \mathbf{x}} \mathbf{x}^{\mathbf{e}_{j}} f(\mathbf{x}) d x
$$

Now $\mathbf{x}^{\mathbf{e}_{j}} f(\mathbf{x}) \in \mathfrak{S}$ and so one can continue in this way and take derivatives indefinitely. Thus $F^{-1} f \in C^{\infty}\left(\mathbb{R}^{n}\right)$ and from the above argument,

$$
D^{\alpha} F^{-1} f(\mathbf{t})=(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{i \mathbf{t} \cdot \mathbf{x}}(i \mathbf{x})^{\alpha} f(\mathbf{x}) d x
$$

To complete showing $F^{-1} f \in \mathfrak{S}$,

$$
\begin{aligned}
\mathbf{t}^{\beta} D^{\alpha} F^{-1} f(\mathbf{t}) & =(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} e^{i \mathbf{t} \cdot \mathbf{x}} \mathbf{t}^{\beta}(i \mathbf{x})^{a} f(\mathbf{x}) d x \\
& =(2 \pi)^{-n / 2} \int_{\mathbb{R}^{n}} i^{|\beta|} e^{i \mathbf{t} \cdot \mathbf{x}} D^{\beta}\left((i \mathbf{x})^{a} f(\mathbf{x})\right) d x
\end{aligned}
$$

The second equal sign following from integration by parts on the last integral. Now use the fact that $\left|e^{i a}\right|=1$ to conclude

$$
\left|\mathbf{t}^{\beta} D^{\alpha} F^{-1} f(\mathbf{t})\right| \leq C \int_{\mathbb{R}^{n}}\left|D^{\beta}\left((i \mathbf{x})^{a} f(\mathbf{x})\right)\right| d x<\infty
$$

It follows $F^{-1} f \in \mathfrak{S}$. Similarly $F f \in \mathfrak{S}$ whenever $f \in \mathfrak{S}$.
Of course $\mathfrak{S}$ can be considered a subset of $\mathscr{G}^{*}$ as follows. For $\psi \in \mathfrak{S}$,

$$
\psi(\phi) \equiv \int_{\mathbb{R}^{n}} \psi \phi d x
$$

Theorem 11.6.23 Let $\psi \in \mathfrak{S}$. Then $\left(F \circ F^{-1}\right)(\psi)=\psi$ and $\left(F^{-1} \circ F\right)(\psi)=\psi$ whenever $\psi \in \mathfrak{S}$. Also $F$ and $F^{-1}$ map $\mathfrak{S}$ one to one and onto $\mathfrak{S}$.

Proof: The first claim follows from the fact that $F$ and $F^{-1}$ are inverses of each other on $\mathscr{G}^{*}$ which was established above. For the second, let $\psi \in \mathfrak{S}$. Then $\psi=F\left(F^{-1} \psi\right)$. Thus $F$ maps $\mathfrak{S}$ onto $\mathfrak{S}$. If $F \psi=0$, then do $F^{-1}$ to both sides to conclude $\psi=0$. Thus $F$ is one to one and onto. Similarly, $F^{-1}$ is one to one and onto.

### 11.6.6 Convolution

To begin with it is necessary to discuss the meaning of $\phi f$ where $f \in \mathscr{G}^{*}$ and $\phi \in \mathscr{G}$. What should it mean? First suppose $f \in L^{p}\left(\mathbb{R}^{n}\right)$ or measurable with polynomial growth. Then $\phi f$ also has these properties. Hence, it should be the case that $\phi f(\psi)=\int_{\mathbb{R}^{n}} \phi f \psi d x=$ $\int_{\mathbb{R}^{n}} f(\phi \psi) d x$. This motivates the following definition.
Definition 11.6.24 Let $T \in \mathscr{G}^{*}$ and let $\phi \in \mathscr{G}$. Then $\phi T \equiv T \phi \in \mathscr{G}^{*}$ will be defined by

$$
\phi T(\psi) \equiv T(\phi \psi)
$$

The next topic is that of convolution. It was just shown that

$$
F(f * \phi)=(2 \pi)^{n / 2} F \phi F f, F^{-1}(f * \phi)=(2 \pi)^{n / 2} F^{-1} \phi F^{-1} f
$$

whenever $f \in L^{2}\left(\mathbb{R}^{n}\right)$ and $\phi \in \mathscr{G}$ so the same definition is retained in the general case because it makes perfect sense and agrees with the earlier definition.

Definition 11.6.25 Let $f \in \mathscr{G}^{*}$ and let $\phi \in \mathscr{G}$. Then define the convolution of $f$ with an element of $\mathscr{G}$ as follows.

$$
f * \phi \equiv(2 \pi)^{n / 2} F^{-1}(F \phi F f) \in \mathscr{G}^{*}
$$

There is an obvious question. With this definition, is it true that

$$
F^{-1}(f * \phi)=(2 \pi)^{n / 2} F^{-1} \phi F^{-1} f
$$

as it was earlier?
Theorem 11.6.26 Let $f \in \mathscr{G}^{*}$ and let $\phi \in \mathscr{G}$.

$$
\begin{gather*}
F(f * \phi)=(2 \pi)^{n / 2} F \phi F f,  \tag{11.19}\\
F^{-1}(f * \phi)=(2 \pi)^{n / 2} F^{-1} \phi F^{-1} f . \tag{11.20}
\end{gather*}
$$

Proof: Note that 11.19 follows from Definition 11.6.25 and both assertions hold for $f \in \mathscr{G}$. Consider 11.20. Here is a simple formula involving a pair of functions in $\mathscr{G}$.

$$
\begin{aligned}
& \left(\psi * F^{-1} F^{-1} \phi\right)(\mathbf{x}) \\
& =\left(\iiint \psi(\mathbf{x}-\mathbf{y}) e^{i \mathbf{y} \cdot \mathbf{y}_{1}} e^{i \mathbf{y}_{1} \cdot \mathbf{z}} \phi(\mathbf{z}) d z d y_{1} d y\right)(2 \pi)^{n} \\
& =\left(\iiint \psi(\mathbf{x}-\mathbf{y}) e^{-i \mathbf{y} \cdot \tilde{\mathbf{y}}_{1}} e^{-i \tilde{\mathbf{y}}_{1} \cdot \mathbf{z}} \phi(\mathbf{z}) d z d \tilde{y}_{1} d y\right)(2 \pi)^{n} \\
& =(\psi * F F \phi)(\mathbf{x}) .
\end{aligned}
$$

Now for $\psi \in \mathscr{G}$,

$$
\begin{gather*}
(2 \pi)^{n / 2} F\left(F^{-1} \phi F^{-1} f\right)(\psi) \equiv(2 \pi)^{n / 2}\left(F^{-1} \phi F^{-1} f\right)(F \psi) \equiv \\
(2 \pi)^{n / 2} F^{-1} f\left(F^{-1} \phi F \psi\right) \equiv(2 \pi)^{n / 2} f\left(F^{-1}\left(F^{-1} \phi F \psi\right)\right)= \\
f\left((2 \pi)^{n / 2} F^{-1}\left(\left(F F^{-1} F^{-1} \phi\right)(F \psi)\right)\right) \equiv \\
f\left(\psi * F^{-1} F^{-1} \phi\right)=f(\psi * F F \phi) \tag{11.21}
\end{gather*}
$$

Also

$$
\begin{gather*}
(2 \pi)^{n / 2} F^{-1}(F \phi F f)(\psi) \equiv(2 \pi)^{n / 2}(F \phi F f)\left(F^{-1} \psi\right) \equiv \\
(2 \pi)^{n / 2} F f\left(F \phi F^{-1} \psi\right) \equiv(2 \pi)^{n / 2} f\left(F\left(F \phi F^{-1} \psi\right)\right)= \\
=f\left(F\left((2 \pi)^{n / 2}\left(F \phi F^{-1} \psi\right)\right)\right) \\
=f\left(F\left((2 \pi)^{n / 2}\left(F^{-1} F F \phi F^{-1} \psi\right)\right)\right)=f\left(F\left(F^{-1}(F F \phi * \psi)\right)\right) \\
f(F F \phi * \psi)=f(\psi * F F \phi) \tag{11.22}
\end{gather*}
$$

The last line follows from the following.

$$
\begin{aligned}
\int F F \phi(\mathbf{x}-\mathbf{y}) \psi(\mathbf{y}) d y & =\int F \phi(\mathbf{x}-\mathbf{y}) F \psi(\mathbf{y}) d y=\int F \psi(\mathbf{x}-\mathbf{y}) F \phi(\mathbf{y}) d y \\
& =\int \psi(\mathbf{x}-\mathbf{y}) F F \phi(y) d y
\end{aligned}
$$

From 11.22 and 11.21 , since $\psi$ was arbitrary,

$$
(2 \pi)^{n / 2} F\left(F^{-1} \phi F^{-1} f\right)=(2 \pi)^{n / 2} F^{-1}(F \phi F f) \equiv f * \phi
$$

which shows 11.20.

### 11.7 Exercises

1. For $f \in L^{1}\left(\mathbb{R}^{n}\right)$, show that if $F^{-1} f \in L^{1}$ or $F f \in L^{1}$, then $f$ equals a continuous bounded function a.e. This is Theorem 11.6.11 but review it.
2. Suppose $f, g \in L^{1}(\mathbb{R})$ and $F f=F g$. Show $f=g$ a.e.
3. $\uparrow$ Suppose $f * f=f$ or $f * f=0$ and $f \in L^{1}(\mathbb{R})$. Show $f=0$.
4. Let $h(x)=\left(\int_{0}^{x} e^{-t^{2}} d t\right)^{2}+\left(\int_{0}^{1} \frac{e^{-x^{2}\left(1+t^{2}\right)}}{1+t^{2}} d t\right)$. Show that $h^{\prime}(x)=0$ and $h(0)=\pi / 4$. Then let $x \rightarrow \infty$ to conclude that $\int_{0}^{\infty} e^{-t^{2}} d t=\sqrt{\pi} / 2$. Show that $\int_{-\infty}^{\infty} e^{-t^{2}} d t=\sqrt{\pi}$ and that $\int_{-\infty}^{\infty} e^{-c t^{2}} d t=\frac{\sqrt{\pi}}{\sqrt{c}}$.
5. Let $h(x)=\left(\int_{0}^{x} e^{-t^{2}} d t\right)^{2}$. Then

$$
h^{\prime}(x)=2\left(\int_{0}^{x} e^{-t^{2}} d t\right) e^{-x^{2}}=2 x e^{-x^{2}}\left(\int_{0}^{1} e^{-(x u)^{2}} d u\right)
$$

Now $h(x)=\int_{0}^{x} h^{\prime}(t) d t$. Do integration by parts to obtain

$$
\begin{aligned}
& -\left.e^{-t^{2}} \int_{0}^{1} e^{-(t u)^{2}} d u\right|_{0} ^{x}-\int_{0}^{x} e^{-t^{2}} \int_{0}^{1} e^{-(t u)^{2}} 2 t u^{2} d u d t \\
= & -e^{-x^{2}} \int_{0}^{1} e^{-(x u)^{2}} d u+1-\int_{0}^{x} \int_{0}^{1} e^{-t^{2}\left(1+u^{2}\right)} 2 t u^{2} d u d t \\
= & -e^{-x^{2}} \int_{0}^{1} e^{-(x u)^{2}} d u+1-\int_{0}^{1} u^{2} \int_{0}^{x} e^{-t^{2}\left(1+u^{2}\right)} 2 t d t d u \\
= & e(x)+1-\int_{0}^{1} u^{2}\left(-\frac{e^{-x^{2}\left(1+u^{2}\right)}-1}{1+u^{2}}\right) d u \\
= & e(x)+1-\int_{0}^{1} \frac{u^{2}}{1+u^{2}} d u
\end{aligned}
$$

where $\lim _{x \rightarrow \infty} e(x)=0$. Now explain why $1-\int_{0}^{1} \frac{u^{2}}{1+u^{2}} d u=\frac{1}{4} \pi$. Hence $\int_{0}^{\infty} e^{-t^{2}} d t=$ $\frac{\sqrt{\pi}}{2}$.
6. Recall that for $f$ a function, $f_{\mathbf{y}}(\mathbf{x})=f(\mathbf{x}-\mathbf{y})$. Find a relationship between $F f_{\mathbf{y}}(\mathbf{t})$ and $F f(\mathbf{t})$ given that $f \in L^{1}\left(\mathbb{R}^{n}\right)$.
7. For $f \in L^{1}\left(\mathbb{R}^{n}\right)$, simplify $F f(\mathbf{t}+\mathbf{y})$.
8. For $f \in L^{1}\left(\mathbb{R}^{n}\right)$ and $c$ a nonzero real number, show $F f(c \mathbf{t})=F g(\mathbf{t})$ where $g(\mathbf{x})=$ $f\left(\frac{\mathbf{x}}{c}\right)$.
9. Suppose that $f \in L^{1}(\mathbb{R})$ and that $\int|x||f(x)| d x<\infty$. Find a way to use the Fourier transform of $f$ to compute $\int x f(x) d x$.
10. Suppose $f \in \mathscr{G}$. Go over why $F\left(f_{x_{j}}\right)(\mathbf{t})=i t_{j} F f(\mathbf{t})$.
11. Let $f \in \mathscr{G}$ and let $k$ be a positive integer.

$$
\|f\|_{k, 2} \equiv\left(\|f\|_{2}^{2}+\sum_{|\alpha| \leq k}\left\|D^{\alpha} f\right\|_{2}^{2}\right)^{1 / 2}
$$

One could also define

$$
\left\||f \||_{k, 2} \equiv\left(\int_{R^{n}}|F f(\mathbf{x})|^{2}\left(1+|\mathbf{x}|^{2}\right)^{k} d x\right)^{1 / 2}\right.
$$

Show both $\left\|\|_{k, 2}\right.$ and $\|\left\|\left\|\|_{k, 2}\right.\right.$ are norms on $\mathscr{G}$ and that they are equivalent. These are Sobolev space norms. For which values of $k$ does the second norm make sense? How about the first norm? Since they are equivalent norms, we usually just use $\left\|\|_{k, 2}\right.$ or $\left\|\|_{H^{k}\left(\mathbb{R}^{n}\right)}\right.$.
12. $\uparrow$ Define $H^{k}\left(\mathbb{R}^{n}\right), k \geq 0$ by $f \in L^{2}\left(\mathbb{R}^{n}\right)$ such that

$$
\begin{gathered}
\left(\int|F f(\mathbf{x})|^{2}\left(1+|\mathbf{x}|^{2}\right)^{k} d x\right)^{\frac{1}{2}}<\infty \\
\|\mid f\| \|_{k, 2} \equiv\left(\int|F f(\mathbf{x})|^{2}\left(1+|\mathbf{x}|^{2}\right)^{k} d x\right)^{\frac{1}{2}}
\end{gathered}
$$

Show $H^{k}\left(\mathbb{R}^{n}\right)$ is a Banach space, and that if $k$ is a positive integer, $H^{k}\left(\mathbb{R}^{n}\right)=\{$ $f \in L^{2}\left(\mathbb{R}^{n}\right)$ : there exists $\left\{u_{j}\right\} \subseteq \mathscr{G}$ with $\left\|u_{j}-f\right\|_{2} \rightarrow 0$ and $\left\{u_{j}\right\}$ is a Cauchy sequence in $\left\|\|_{k, 2}\right.$ of Problem 11\}. This is one way to define Sobolev Spaces. Hint: One way to do the second part of this is to define a new measure $\mu$ by $\mu(E) \equiv$ $\int_{E}\left(1+|\mathbf{x}|^{2}\right)^{k} d x$. Then show $\mu$ is a Borel measure which is inner and outer regular and show there exists $\left\{g_{m}\right\}$ such that $g_{m} \in \mathscr{G}$ and $g_{m} \rightarrow F f$ in $L^{2}(\mu)$. Thus $g_{m}=F f_{m}, f_{m} \in \mathscr{G}$ because $F$ maps $\mathscr{G}$ onto $\mathscr{G}$. Then by Problem 11, $\left\{f_{m}\right\}$ is Cauchy in the norm $\left\|\|_{k, 2}\right.$. By using the countable version of $\mathscr{G}$ in which the polynomials all have rational coefficients and in $e^{-a|\mathbf{x}|^{2}}$ the $a$ is a positive rational, show that $H^{k}\left(\mathbb{R}^{n}\right)$ is separable.
13. $\uparrow$ If $2 k>n$, show that if $f \in H^{k}\left(\mathbb{R}^{n}\right)$, then $f$ equals a bounded continuous function a.e. Hint: Show that for $k$ this large, $F f \in L^{1}\left(\mathbb{R}^{n}\right)$, and then use Problem 1 or Theorem 11.6.11. To do this, write

$$
|F f(\mathbf{x})|=|F f(\mathbf{x})|\left(1+|\mathbf{x}|^{2}\right)^{\frac{k}{2}}\left(1+|\mathbf{x}|^{2}\right)^{\frac{-k}{2}}
$$

So

$$
\int|F f(\mathbf{x})| d x=\int|F f(\mathbf{x})|\left(1+|\mathbf{x}|^{2}\right)^{\frac{k}{2}}\left(1+|\mathbf{x}|^{2}\right)^{\frac{-k}{2}} d x
$$

Use the Cauchy Schwarz inequality. This is an example of a Sobolev imbedding Theorem.
14. For $u \in \mathscr{G}$, define $\gamma u\left(\mathbf{x}^{\prime}\right) \equiv u\left(\mathbf{x}^{\prime}, 0\right)$. Show that there is a constant $C$ independent of $u$ such that

$$
\int_{\mathbb{R}^{n-1}}\left|\gamma u\left(\mathbf{x}^{\prime}\right)\right|^{2} d x^{\prime} \leq C^{2}\|u\|_{1,2}^{2}
$$

where this is the Sobolev norm described in Problem 11. Explain how this implies that one can give a meaningful description of the value of $u$ on an $n-1$ dimensional
subspace even though this $n-1$ dimensional subspace has measure zero, whenever of $u \in H^{1}\left(\mathbb{R}^{n}\right)$. Hint: $u\left(\mathbf{x}^{\prime}, 0\right)=\int_{0}^{\infty} D_{n}\left(-e^{-s^{2}} u\left(\mathbf{x}^{\prime}, s\right)\right) d s=\int_{0}^{\infty}-2 s e^{-s^{2}} u\left(\mathbf{x}^{\prime}, s\right)-$ $e^{-s^{2}} D_{n} u\left(\mathbf{x}^{\prime}, s\right) d s$.
Thus

$$
\left|\gamma u\left(\mathbf{x}^{\prime}\right)\right|^{2} \leq C\left(\int_{0}^{\infty} 2 s e^{-s^{2}}\left|u\left(\mathbf{x}^{\prime}, s\right)\right|^{2} d s+\int_{0}^{\infty} e^{-s^{2}}\left|D_{n} u\left(\mathbf{x}^{\prime}, s\right)\right|^{2} d s\right)
$$

Now do $\int_{\mathbb{R}^{n-1}}$ to both sides.
15. For $u \in \mathscr{G}$, let $\gamma u\left(\mathbf{x}^{\prime}\right) \equiv u\left(\mathbf{x}^{\prime}, 0\right)$. Justify the following arguments. $F^{\prime}$ refers to the Fourier transform with respect to $\mathbf{x}^{\prime},\left(x_{1}, \ldots, x_{n-1}\right)$. This and the next problem are on a more refined version of Problem 14.

$$
\begin{aligned}
& \int_{\mathbb{R}} F u\left(\mathbf{x}^{\prime}, x_{n}\right) d x_{n}=\lim _{\varepsilon \rightarrow 0} \int_{\mathbb{R}} e^{-\left(\varepsilon x_{n}\right)^{2}} F u\left(\mathbf{x}^{\prime}, x_{n}\right) d x_{n} \\
&= \lim _{\varepsilon \rightarrow 0}\left(\frac{1}{2 \pi}\right)^{n / 2} \int_{\mathbb{R}^{n}} u\left(\mathbf{y}^{\prime}, y_{n}\right) e^{-i \mathbf{x}^{\prime} \cdot \mathbf{y}^{\prime}} \int_{\mathbb{R}} e^{-\left(\varepsilon x_{n}\right)^{2}} e^{-i x_{n} y_{n}} d x_{n} d y^{\prime} d y_{n} \\
&= \lim _{\varepsilon \rightarrow 0} K_{n} \int_{\mathbb{R}^{n}} u\left(\mathbf{y}^{\prime}, y_{n}\right) e^{-i \mathbf{x}^{\prime} \cdot \mathbf{y}^{\prime}} e^{-\varepsilon^{2} \frac{y_{n}^{2}}{4}} \int_{\mathbb{R}} e^{-\varepsilon^{2}\left(x_{n}+\frac{i y_{n}}{2}\right)^{2}} d x_{n} d y^{\prime} d y_{n} \\
&= \lim _{\varepsilon \rightarrow 0} K_{n} \int_{\mathbb{R}^{n}} u\left(\mathbf{y}^{\prime}, y_{n}\right) e^{-i \mathbf{x}^{\prime} \cdot \mathbf{y}^{\prime}} e^{-\varepsilon^{2} \frac{y_{n}^{2}}{4}} \frac{1}{\varepsilon} d y^{\prime} d y_{n} \\
&= \hat{K}_{n} \int_{\mathbb{R}^{n}} u\left(\mathbf{y}^{\prime}, 0\right) e^{-i \mathbf{x}^{\prime} \cdot \mathbf{y}^{\prime}} d y^{\prime} \\
& F^{\prime}(\gamma u)\left(\mathbf{x}^{\prime}\right)=C_{n} \int_{\mathbb{R}} F u\left(\mathbf{x}^{\prime}, x_{n}\right) d x_{n} .
\end{aligned}
$$

16. $\uparrow$ First show that if $a>0$ and $t>1 / 2$, then $\int_{\mathbb{R}}\left(a^{2}+x^{2}\right)^{-t} d x<C_{t} a^{1-2 t}$. Next explain the following steps where $K_{n}$ is a constant depending on $n$

$$
\begin{gathered}
\int_{\mathbb{R}^{n-1}}\left(1+\left|\mathbf{y}^{\prime}\right|^{2}\right)^{s}\left|F \gamma u\left(\mathbf{y}^{\prime}\right)\right|^{2} d y^{\prime} \\
=C_{n} \int_{\mathbb{R}^{n-1}}\left(1+\left|\mathbf{y}^{\prime}\right|^{2}\right)^{s}\left|\int_{\mathbb{R}} F u\left(\mathbf{y}^{\prime}, y_{n}\right) d y_{n}\right|^{2} d y^{\prime} \\
=C_{n} \int_{\mathbb{R}^{n-1}}\left(1+\left|\mathbf{y}^{\prime}\right|^{2}\right)^{s}\left|\int_{\mathbb{R}} F u\left(\mathbf{y}^{\prime}, y_{n}\right)\left(1+|\mathbf{y}|^{2}\right)^{t / 2}\left(1+|\mathbf{y}|^{2}\right)^{-t / 2} d y_{n}\right|^{2} d y^{\prime}
\end{gathered}
$$

Now apply the Cauchy Schwarz inequality to get:

$$
\begin{aligned}
\leq & C_{n} \int_{\mathbb{R}^{n-1}}\left(1+\left|\mathbf{y}^{\prime}\right|^{2}\right)^{s} \int_{\mathbb{R}}\left|F u\left(\mathbf{y}^{\prime}, y_{n}\right)\right|^{2}\left(1+|\mathbf{y}|^{2}\right)^{t} d y_{n} \\
& \cdot \int_{\mathbb{R}}\left(1+|\mathbf{y}|^{2}\right)^{-t} d y_{n} d y^{\prime}
\end{aligned}
$$

Now use the first part with $a^{2}=1+\left|\mathbf{y}^{\prime}\right|^{2}$. Obtain

$$
\begin{aligned}
\leq & C_{n} \int_{\mathbb{R}^{n-1}}\left(1+\left|\mathbf{y}^{\prime}\right|^{2}\right)^{s}\left(1+\left|\mathbf{y}^{\prime}\right|^{2}\right)^{(1-2 t) / 2} \\
& \cdot \int_{\mathbb{R}}\left|F u\left(\mathbf{y}^{\prime}, y_{n}\right)\right|^{2}\left(1+|\mathbf{y}|^{2}\right)^{t} d y_{n} d y^{\prime}
\end{aligned}
$$

Conclude that if $s+\frac{1-2 t}{2}=s+\frac{1}{2}-t \leq 0$ and $t>1 / 2$, then $\|\gamma(u)\|_{s, 2}^{2} \leq C_{n}\|u\|_{t, 2}^{2}$. In particular, if $t=1$ and $s=1 / 2$, you have $\|\gamma(u)\|_{(1 / 2), 2}^{2} \leq C_{n}\|u\|_{1,2}^{2}$. This exhibits the phenomenon of the loss of $1 / 2$ derivative when considering $\gamma u$ on an $n-1$ dimensional subspace.
17. In dealing with Sobolev spaces, the following interpolation inequality is very useful. Let $0 \leq r<s<t$. Then if $u \in H^{t}\left(\mathbb{R}^{n}\right)$,

$$
\|u\|_{H^{s}\left(\mathbb{R}^{n}\right)} \leq\|u\|_{H^{r}\left(\mathbb{R}^{n}\right)}^{\theta}\|u\|_{H^{t}\left(\mathbb{R}^{n}\right)}^{1-\theta}
$$

where $\theta \in(0,1)$ such that $\theta r+(1-\theta) t=s$. Hint:

$$
\|u\|_{H^{s}\left(\mathbb{R}^{n}\right)}=\left(\int_{\mathbb{R}^{n}}\left(1+|\mathbf{x}|^{2}\right)^{\theta r}\left(1+|\mathbf{x}|^{2}\right)^{(1-\theta) t}|F u(\mathbf{x})|^{2} d x\right)^{1 / 2}
$$

Regard $|F u(\mathbf{x})|^{2} d x=d \mu$ as a measure and use Holder's inequality.

$$
\begin{aligned}
& \left(\int_{\mathbb{R}^{n}}|F u(\mathbf{x})|^{2}\left(1+|\mathbf{x}|^{2}\right)^{s} d x\right)^{1 / 2} \\
= & \left(\int_{\mathbb{R}^{n}}\left(1+|\mathbf{x}|^{2}\right)^{\theta r}\left(1+|\mathbf{x}|^{2}\right)^{(1-\theta) t}|F u(\mathbf{x})|^{2} d x\right)^{1 / 2} \\
\leq & \binom{\left(\int_{\mathbb{R}^{n}}\left(1+|\mathbf{x}|^{2}\right)^{r}|F u(\mathbf{x})|^{2} d x\right)^{\theta} \cdot}{\left(\int_{\mathbb{R}^{n}}\left(1+|\mathbf{x}|^{2}\right)^{t}|F u(\mathbf{x})|^{2} d x\right)^{1-\theta}}^{1 / 2} \\
= & \|u\|_{H^{r}\left(\mathbb{R}^{n}\right)}^{\theta}\|u\|_{H^{t}\left(\mathbb{R}^{n}\right)}^{1-\theta}
\end{aligned}
$$

18. If $\varepsilon>0$ and if $\theta r+(1-\theta) t=s, 0 \leq r<s<t$, show that there exists a constant $C_{\varepsilon}$ such that

$$
\|u\|_{H^{s}\left(\mathbb{R}^{n}\right)} \leq \varepsilon\|u\|_{H^{t}\left(\mathbb{R}^{n}\right)}+C_{\varepsilon}\|u\|_{H^{r}\left(\mathbb{R}^{n}\right)}
$$

Hint: For $r$ large and positive,

$$
\begin{aligned}
\|u\|_{H^{s}\left(\mathbb{R}^{n}\right)} & \leq\left(r\|u\|_{H^{r}\left(\mathbb{R}^{n}\right)}\right)^{\theta} \frac{1}{r^{\theta}}\|u\|_{H^{t}\left(\mathbb{R}^{n}\right)}^{1-\theta} \\
& =\left(r\|u\|_{H^{r}\left(\mathbb{R}^{n}\right)}\right)^{\theta}\left(\left(\frac{1}{r^{\theta}}\right)^{1 /(1-\theta)}\|u\|_{H^{t}\left(\mathbb{R}^{n}\right)}\right)^{1-\theta} .
\end{aligned}
$$

Now recall Proposition 9.3 .2 and then pick $r$ sufficiently large. This is very useful for checking conditions needed in non-linear partial differential equations and inclusions.

## Part III

## Abstract Analysis

## Chapter 12

## Banach Spaces

### 12.1 Theorems Based on Baire Category

Banach spaces are complete normed linear spaces. These have been mentioned throughout the book so far. In this chapter are the most significant theorems relative to Banach spaces and more generally normed linear spaces. The main theorems to be presented here are the uniform boundedness theorem, the open mapping theorem, the closed graph theorem, and the Hahn Banach Theorem. The first three of these theorems come from the Baire category theorem which is about to be presented. They are topological in nature. The Hahn Banach theorem has nothing to do with topology. First some definitions and a review of the notion of a Banach space. Always we are considering a complete normed linear space of the sort discussed earlier. As noted earlier, Banach spaces are all examples of metric space so all the theory of metric space applies. In particular,

Proposition 12.1.1 The open ball $B(z, r)$ is an open set. Also $\overline{B(z, r)}=D(z, r) \equiv$ $\{x:\|x-z\| \leq r\}$. In an arbitrary metric space, $\overline{B(z, r)} \subseteq D(z, r)$.

Proof: First note that $D(z, r)$ is closed because $x \rightarrow\|x-z\|$ is continuous by Theorem 2.4.8 for example, so inverse images of closed sets are closed. Thus $D(z, r)$ is a closed set containing $B(z, r)$ so $\overline{B(z, r)} \subseteq D(z, r)$. In normed linear space these are equal because if $\|z-x\|=r$, then for $n \in \mathbb{N}$, consider $x_{n}=x+\frac{n-1}{n}(z-x) .\left\|x_{n}-x\right\|=\frac{n-1}{n} r<r$ so $x_{n} \in$ $B(z, r)$ but $\left\|x_{n}-z\right\|=\left\|x+\frac{n-1}{n}(z-x)-z\right\|=\frac{1}{n}\|x-z\|$ so $z$, being the limit of a sequence of points of $B(z, r)$ is in $\overline{B(z, r)}$.

Note here that equality does not work in general metric space because you could have an infinite set with the metric $d(x, y)=0$ if $x=y$ and 1 if $x \neq y$. Then $B(x, 1)$ would consist of only $x$ while $D(x, 1)$ would yield the whole set.

Recall also that $|\|z\|-\|w\|| \leq\|z-w\|$, note $\|z\|=\|z-w+w\| \leq\|z-w\|+\|w\|$ which implies $\|z\|-\|w\| \leq\|z-w\|$ and now switching $z$ and $w$, yields $\|w\|-\|z\| \leq\|z-w\|$ which implies $|\|w\|-\|z\|| \leq\|w-z\|$. This was done earlier. It is just another version of the triangle inequality.

Also recall the definition of a Cauchy sequence.
Definition 12.1.2 $\left\{x_{n}\right\}$ is called a Cauchy sequence if for every $\varepsilon>0$ there exists $N$ such that if $m, n \geq N$, then $\left\|x_{n}-x_{m}\right\|<\varepsilon$.

As discussed earlier in Section 2.9,
Definition 12.1.3 $\mathscr{L}(X, Y)$ is the space of continuous linear maps from $X$ to $Y$. Recall also that if $L \in \mathscr{L}(X, Y),\|L\| \equiv \sup \{\|L x\|:\|x\| \leq 1\}$ and that this is well defined and $\|L \circ M\| \leq\|L\|\|M\|$.

As noted earlier, in Section 2.9, whenever you have $L \in \mathscr{L}\left(\mathbb{R}^{p}, \mathbb{R}^{q}\right), L$ is automatically continuous. However, in infinite dimensional settings, this might not hold. Here is a simple example.

Example 12.1.4 Let $V$ denote all linear combinations of functions of the form $e^{-\alpha x^{2}}$ for $\alpha>0$. Thus typical elements of $V$ are of the form $\sum_{k=1}^{n} \beta_{k} e^{-\alpha_{k} x^{2}}$. Let $L: V \rightarrow \mathbb{C}$ be given
by $L f \equiv \int_{\mathbb{R}} f(x) d x$ and for a norm on $V,\|f\| \equiv \max \{|f(x)|: x \in \mathbb{R}\}$. Of course $V$ is not complete, but it is a normed linear space. Recall that

$$
\int_{-\infty}^{\infty} e^{-x^{2}} d x=\int_{-\infty}^{\infty} \frac{1}{n} e^{-\left(x^{2} / n^{2}\right)}=\sqrt{\pi}
$$

where here $n \in \mathbb{N}$. Consider the sequence of functions $f_{n}(x) \equiv \frac{1}{n} e^{-\left(x^{2} / n^{2}\right)}$. Its maximum value is $1 / n$ and so $\left\|f_{n}\right\| \rightarrow 0$ but $L f_{n}$ fails to converge to the 0 function. Thus $L$ is not continuous although it is linear.

### 12.1.1 Bair Category Theorem

The following remarkable result is called the Baire category theorem. To get an idea of its meaning, imagine you draw a line in the plane. The complement of this line is an open set and is dense because every point, even those on the line, are limit points of this open set. Now draw another line. The complement of the two lines is still open and dense. Keep drawing lines and looking at the complements of the union of these lines. You always have an open set which is dense. Now what if there were countably many lines? The Baire category theorem implies the complement of the union of these lines is dense. In particular it is nonempty. Thus you cannot write the plane as a countable union of lines. This is a rather rough description of this very important theorem. The precise statement and proof follow. These theorems work more generally for a complete metric space so I am stating them for this case.

Theorem 12.1.5 Let $X$ be a complete metric space and let $\left\{U_{n}\right\}_{n=1}^{\infty}$ be a sequence of open subsets of $X$ satisfying $\overline{U_{n}}=X$ ( $U_{n}$ is dense). Then $D \equiv \cap_{n=1}^{\infty} U_{n}$ is a dense subset of $X$.

Proof: Let $p \in X$ and let $r_{0}>0$. I need to show $D \cap B\left(p, r_{0}\right) \neq \emptyset$. Since $U_{1}$ is dense, there exists $p_{1} \in U_{1} \cap B\left(p, r_{0}\right)$, an open set. Let $p_{1} \in B\left(p_{1}, r_{1}\right) \subseteq \overline{B\left(p_{1}, r_{1}\right)} \subseteq U_{1} \cap B\left(p, r_{0}\right)$ and $r_{1}<2^{-1}$. This is possible because $U_{1} \cap B\left(p, r_{0}\right)$ is an open set and so there exists $r_{1}$ such that $B\left(p_{1}, 2 r_{1}\right) \subseteq U_{1} \cap B\left(p, r_{0}\right)$. But $B\left(p_{1}, r_{1}\right) \subseteq \overline{B\left(p_{1}, r_{1}\right)} \subseteq D\left(p_{1}, r_{1}\right) \subseteq B\left(p_{1}, 2 r_{1}\right)$ by Proposition 12.1.1.


There exists $p_{2} \in U_{2} \cap B\left(p_{1}, r_{1}\right)$ because $U_{2}$ is dense. Let

$$
p_{2} \in B\left(p_{2}, r_{2}\right) \subseteq \overline{B\left(p_{2}, r_{2}\right)} \subseteq U_{2} \cap B\left(p_{1}, r_{1}\right) \subseteq U_{1} \cap U_{2} \cap B\left(p, r_{0}\right)
$$

and let $r_{2}<2^{-2}$. Continue in this way. Thus $r_{n}<2^{-n}$,

$$
\begin{gathered}
\overline{B\left(p_{n}, r_{n}\right)} \subseteq U_{1} \cap U_{2} \cap \ldots \cap U_{n} \cap B\left(p, r_{0}\right) \\
\overline{B\left(p_{n}, r_{n}\right)} \subseteq B\left(p_{n-1}, r_{n-1}\right)
\end{gathered}
$$

The sequence, $\left\{p_{n}\right\}$ is a Cauchy sequence because all terms of $\left\{p_{k}\right\}$ for $k \geq n$ are contained in $B\left(p_{n}, r_{n}\right)$, a set whose diameter is no larger than $2^{-n}$. Since $X$ is complete,
there exists $p_{\infty}$ such that $\lim _{n \rightarrow \infty} p_{n}=p_{\infty}$. Since all but finitely many terms of $\left\{p_{n}\right\}$ are in $\overline{B\left(p_{m}, r_{m}\right)}$, it follows that $p_{\infty} \in \overline{B\left(p_{m}, r_{m}\right)}$ for each $m$. Therefore, $p_{\infty} \in \cap_{m=1}^{\infty} \overline{B\left(p_{m}, r_{m}\right)} \subseteq$ $\cap_{i=1}^{\infty} U_{i} \cap B\left(p, r_{0}\right)$.

The following corollary is also called the Baire category theorem.
Corollary 12.1.6 Let $X$ be a complete metric space and suppose $X=\cup_{i=1}^{\infty} F_{i}$ where each $F_{i}$ is a closed set. Then for some $i$, interior $F_{i} \neq \emptyset$.

Proof: If all $F_{i}$ has empty interior, then $F_{i}^{C}$ would be a dense open set. Therefore, from Theorem 12.1.5, it would follow that

$$
\emptyset=\left(\cup_{i=1}^{\infty} F_{i}\right)^{C}=\cap_{i=1}^{\infty} F_{i}^{C} \neq \emptyset .
$$

The set $D$ of Theorem 12.1.5 is called a $G_{\delta}$ set because it is the countable intersection of open sets. Thus $D$ is a dense $G_{\delta}$ set.

### 12.1.2 Uniform Boundedness Theorem

The next big result is sometimes called the Uniform Boundedness theorem, or the BanachSteinhaus theorem. This is a very surprising theorem which implies that for a collection of bounded linear operators, if they are bounded pointwise, then they are also bounded uniformly. As an example of a situation in which pointwise bounded does not imply uniformly bounded, consider the functions $f_{\alpha}(x) \equiv \mathscr{X}_{(\alpha, 1)}(x) x^{-1}$ for $\alpha \in(0,1)$. Clearly each function is bounded and the collection of functions is bounded at each point of $(0,1)$, but there is no bound for all these functions taken together. One problem is that $(0,1)$ is not a Banach space. Therefore, the functions cannot be linear. Since the theorem is about linear functions, it only applies to linear spaces.

Theorem 12.1.7 Let $X$ be a Banach space and let $Y$ be a normed linear space. Let $\left\{L_{\alpha}\right\}_{\alpha \in \Lambda}$ be a collection of elements of $\mathscr{L}(X, Y)$. Then one of the following happens.
a.) $\sup \left\{\left\|L_{\alpha}\right\|: \alpha \in \Lambda\right\}<\infty$
b.) There exists a dense $G_{\delta}$ set, $D$, such that for all $x \in D$,

$$
\sup \left\{\left\|L_{\alpha} x\right\| \alpha \in \Lambda\right\}=\infty
$$

Proof: For each $n \in \mathbb{N}$, define $U_{n}=\left\{x \in X: \sup \left\{\left\|L_{\alpha} x\right\|: \alpha \in \Lambda\right\}>n\right\}$. Then $U_{n}$ is an open set because if $x \in U_{n}$, then there exists $\alpha \in \Lambda$ such that $\left\|L_{\alpha} x\right\|>n$. But then, since $L_{\alpha}$ is continuous, this situation persists for all $y$ sufficiently close to $x$, say for all $y \in B(x, \delta)$. Then $B(x, \delta) \subseteq U_{n}$ which shows $U_{n}$ is open.

Case b.) is obtained from Theorem 12.1.5 if each $U_{n}$ is dense.
The other case is that for some $n, U_{n}$ is not dense. If this occurs, there exists $x_{0}$ and $r>0$ such that for all $x \in B\left(x_{0}, r\right),\left\|L_{\alpha} x\right\| \leq n$ for all $\alpha$. Now if $y \in B(0, r), x_{0}+y \in B\left(x_{0}, r\right)$. Consequently, for all such $y,\left\|L_{\alpha}\left(x_{0}+y\right)\right\| \leq n$. This implies that for all $\alpha \in \Lambda$ and $\|y\|<r$,

$$
\left\|L_{\alpha} y\right\| \leq n+\left\|L_{\alpha}\left(x_{0}\right)\right\| \leq 2 n
$$

Therefore, if $\|y\| \leq 1,\left\|\frac{r}{2} y\right\|<r$ and so for all $\alpha,\left\|L_{\alpha}\left(\frac{r}{2} y\right)\right\| \leq 2 n$. Now multiplying by $r / 2$ it follows that whenever $\|y\| \leq 1,\left\|L_{\alpha}(y)\right\| \leq 4 n / r$. Hence case a.) holds.

### 12.1.3 Open Mapping Theorem

Another remarkable theorem which depends on the Baire category theorem is the open mapping theorem. Unlike Theorem 12.1.7 it requires both $X$ and $Y$ to be Banach spaces.

Theorem 12.1.8 Let $X$ and $Y$ be Banach spaces, let $L \in \mathscr{L}(X, Y)$, and suppose $L$ is onto. Then L maps open sets onto open sets.

To aid in the proof, here is a lemma.
Lemma 12.1.9 Let $a$ and $b$ be positive constants and suppose

$$
B(0, a) \subseteq \overline{L(B(0, b))}
$$

Then

$$
\overline{L(B(0, b))} \subseteq L(B(0,2 b))
$$

Proof of Lemma 12.1.9: Let $y \in \overline{L(B(0, b))}$. There exists $x_{1} \in B(0, b)$ such that

$$
\left\|y-L x_{1}\right\|<\frac{a}{2}
$$

Now this implies

$$
2 y-2 L x_{1} \in B(0, a) \subseteq \overline{L(B(0, b))}
$$

Therefore, there exists $x_{2} \in B(0, b)$ such that $\left\|2 y-2 L x_{1}-L x_{2}\right\|<a / 2$. Hence

$$
\left\|4 y-4 L x_{1}-2 L x_{2}\right\|<a
$$

and there exists $x_{3} \in B(0, b)$ such that

$$
\left\|4 y-4 L x_{1}-2 L x_{2}-L x_{3}\right\|<a / 2
$$

Continuing in this way, there exist $x_{1}, x_{2}, x_{3}, x_{4}, \ldots$ in $B(0, b)$ such that

$$
\left\|2^{n} y-\sum_{i=1}^{n} 2^{n-(i-1)} L\left(x_{i}\right)\right\|<a
$$

which implies

$$
\begin{equation*}
\left\|y-\sum_{i=1}^{n} 2^{-(i-1)} L\left(x_{i}\right)\right\|=\left\|y-L\left(\sum_{i=1}^{n} 2^{-(i-1)}\left(x_{i}\right)\right)\right\|<2^{-n} a \tag{12.1}
\end{equation*}
$$

Now consider the partial sums of the series, $\sum_{i=1}^{\infty} 2^{-(i-1)} x_{i}$.

$$
\left\|\sum_{i=m}^{n} 2^{-(i-1)} x_{i}\right\| \leq b \sum_{i=m}^{\infty} 2^{-(i-1)}=b 2^{-m+2}
$$

Therefore, these partial sums form a Cauchy sequence and so since $X$ is complete, there exists $x=\sum_{i=1}^{\infty} 2^{-(i-1)} x_{i}$. Letting $n \rightarrow \infty$ in 12.1 yields $\|y-L x\|=0$. Now

$$
\|x\|=\lim _{n \rightarrow \infty}\left\|\sum_{i=1}^{n} 2^{-(i-1)} x_{i}\right\|
$$

$$
\leq \lim _{n \rightarrow \infty} \sum_{i=1}^{n} 2^{-(i-1)}\left\|x_{i}\right\|<\lim _{n \rightarrow \infty} \sum_{i=1}^{n} 2^{-(i-1)} b=2 b
$$

This proves the lemma.
Proof of Theorem 12.1.8: $Y=\cup_{n=1}^{\infty} \overline{L(B(0, n))}$. Thus $Y$ is a countable union of closed sets. By Corollary 12.1.6, $\overline{L\left(B\left(0, n_{0}\right)\right)}$ has nonempty interior for some $n_{0}$. Thus

$$
B(y, r) \subseteq \overline{L\left(B\left(0, n_{0}\right)\right)}
$$

for some $y$ and some $r>0$. Since $L$ is linear $B(-y, r) \subseteq \overline{L\left(B\left(0, n_{0}\right)\right)}$ also. Here is why. If $z \in B(-y, r)$, then $-z \in B(y, r)$ and so there exists $x_{n} \in B\left(0, n_{0}\right)$ such that $L x_{n} \rightarrow-z$. Therefore, $L\left(-x_{n}\right) \rightarrow z$ and $-x_{n} \in B\left(0, n_{0}\right)$ also. Therefore $z \in \overline{L\left(B\left(0, n_{0}\right)\right)}$. Then it follows that

$$
\begin{aligned}
B(0, r) & \subseteq B(y, r)+B(-y, r) \\
& \equiv\left\{y_{1}+y_{2}: y_{1} \in B(y, r) \text { and } y_{2} \in B(-y, r)\right\} \\
& \subseteq \frac{L\left(B\left(0,2 n_{0}\right)\right)}{}
\end{aligned}
$$

The reason for the last inclusion is that from the above, if $y_{1} \in B(y, r)$ and $y_{2} \in B(-y, r)$, there exists $x_{n}, z_{n} \in B\left(0, n_{0}\right)$ such that

$$
L x_{n} \rightarrow y_{1}, L z_{n} \rightarrow y_{2} .
$$

Therefore, $\left\|x_{n}+z_{n}\right\| \leq 2 n_{0}$ and so $\left(y_{1}+y_{2}\right) \in \overline{L\left(B\left(0,2 n_{0}\right)\right)}$.
By Lemma 12.1.9, $\overline{L\left(B\left(0,2 n_{0}\right)\right)} \subseteq L\left(B\left(0,4 n_{0}\right)\right)$ which shows

$$
B(0, r) \subseteq L\left(B\left(0,4 n_{0}\right)\right)
$$

Letting $a=r\left(4 n_{0}\right)^{-1}$, it follows, since $L$ is linear, that $B(0, a) \subseteq L(B(0,1))$. It follows since $L$ is linear,

$$
\begin{equation*}
L(B(0, r)) \supseteq B(0, a r) \tag{12.2}
\end{equation*}
$$

Now let $U$ be open in $X$ and let $x+B(0, r)=B(x, r) \subseteq U$. Using 12.2,

$$
\begin{gathered}
L(U) \supseteq L(x+B(0, r)) \\
=L x+L(B(0, r)) \supseteq L x+B(0, a r)=B(L x, a r) .
\end{gathered}
$$

Hence $L x \in B(L x, a r) \subseteq L(U)$ which shows that every point, $L x \in L U$, is an interior point of $L U$ and so $L U$ is open.

This theorem is surprising because it implies that if $|\cdot|$ and $\|\cdot\|$ are two norms with respect to which a vector space $X$ is a Banach space such that $|\cdot| \leq K\|\cdot\|$, then there exists a constant $k$, such that $\|\cdot\| \leq k|\cdot|$. This can be useful because sometimes it is not clear how to compute $k$ when all that is needed is its existence. To see the open mapping theorem implies this, consider the identity map id $x=x$. Then id : $(X,\|\cdot\|) \rightarrow(X,|\cdot|)$ is continuous and onto. Hence id is an open map which implies id ${ }^{-1}$ is continuous. Theorem 2.9.1 gives the existence of the constant $k$.

### 12.1.4 Closed Graph Theorem

Definition 12.1.10 Let $f: D \rightarrow E$. The graph of $f$ consists of the set of all ordered pairs of the form $\{(x, f(x)): x \in D\}$.

Definition 12.1.11 If $X$ and $Y$ are normed linear spaces, make $X \times Y$ into a normed linear space by using the norm $\|(x, y)\|=\max (\|x\|,\|y\|)$ along with componentwise addition and scalar multiplication. Thus $a(x, y)+b(z, w) \equiv(a x+b z, a y+b w)$.

There are other ways to give a norm for $X \times Y$. For example, you could define $\|(x, y)\|=$ $\|x\|+\|y\|$

Lemma 12.1.12 The norm defined in Definition 12.1.11 on $X \times Y$ along with the definition of addition and scalar multiplication given there make $X \times Y$ into a normed linear space.

Proof: The only axiom for a norm which is not obvious is the triangle inequality. Therefore, consider

$$
\begin{aligned}
&\left\|\left(x_{1}, y_{1}\right)+\left(x_{2}, y_{2}\right)\right\|=\left\|\left(x_{1}+x_{2}, y_{1}+y_{2}\right)\right\| \\
&=\max \left(\left\|x_{1}+x_{2}\right\|,\left\|y_{1}+y_{2}\right\|\right) \\
& \leq \max \left(\left\|x_{1}\right\|+\left\|x_{2}\right\|,\left\|y_{1}\right\|+\left\|y_{2}\right\|\right) \\
& \leq \max \left(\left\|x_{1}\right\|,\left\|y_{1}\right\|\right)+\max \left(\left\|x_{2}\right\|,\left\|y_{2}\right\|\right) \\
&=\left\|\left(x_{1}, y_{1}\right)\right\|+\left\|\left(x_{2}, y_{2}\right)\right\| .
\end{aligned}
$$

It is obvious $X \times Y$ is a vector space from the above definition.
Lemma 12.1.13 If $X$ and $Y$ are Banach spaces, then $X \times Y$ with the norm and vector space operations defined in Definition 12.1.11 is also a Banach space.

Proof: The only thing left to check is that the space is complete. But this follows from the simple observation that $\left\{\left(x_{n}, y_{n}\right)\right\}$ is a Cauchy sequence in $X \times Y$ if and only if $\left\{x_{n}\right\}$ and $\left\{y_{n}\right\}$ are Cauchy sequences in $X$ and $Y$ respectively. Thus if $\left\{\left(x_{n}, y_{n}\right)\right\}$ is a Cauchy sequence in $X \times Y$, it follows there exist $x$ and $y$ such that $x_{n} \rightarrow x$ and $y_{n} \rightarrow y$. But then from the definition of the norm, $\left(x_{n}, y_{n}\right) \rightarrow(x, y)$.

Lemma 12.1.14 Every closed subspace of a Banach space is a Banach space.

Proof: If $F \subseteq X$ where $X$ is a Banach space and $\left\{x_{n}\right\}$ is a Cauchy sequence in $F$, then since $X$ is complete, there exists a unique $x \in X$ such that $x_{n} \rightarrow x$. However this means $x \in \bar{F}=F$ since $F$ is closed.

Definition 12.1.15 Let $X$ and $Y$ be Banach spaces and let $D \subseteq X$ be a subspace. A linear map $L: D \rightarrow Y$ is said to be closed if its graph is a closed subspace of $X \times Y$. Equivalently, $L$ is closed if $x_{n} \rightarrow x$ and $L x_{n} \rightarrow y$ implies $x \in D$ and $y=L x$.

Note the distinction between closed and continuous. If the operator is closed the assertion that $y=L x$ only follows if it is known that the sequence $\left\{L x_{n}\right\}$ converges. In the case of a continuous operator, the convergence of $\left\{L x_{n}\right\}$ follows from the assumption that $x_{n} \rightarrow x$. It is not always the case that a mapping which is closed is necessarily continuous. Consider the function $f(x)=\tan (x)$ if $x$ is not an odd multiple of $\frac{\pi}{2}$ and $f(x) \equiv 0$ at every odd multiple of $\frac{\pi}{2}$. Then the graph is closed and the function is defined on $\mathbb{R}$ but it clearly fails to be continuous. Of course this function is not linear. You could also consider the map,

$$
\frac{d}{d x}:\left\{y \in C^{1}([0,1]): y(0)=0\right\} \equiv D \rightarrow C([0,1])
$$

where the norm is the uniform norm on $C([0,1]),\|y\|_{\infty}$. If $y \in D$, then $y(x)=\int_{0}^{x} y^{\prime}(t) d t$. Therefore, if $\frac{d y_{n}}{d x} \rightarrow f \in C([0,1])$ and if $y_{n} \rightarrow y$ in $C([0,1])$ it follows that

$$
\begin{array}{cl}
y_{n}(x) & =\int_{0}^{x} \frac{d y_{n}(t)}{d x} d t \\
\downarrow & \downarrow \\
y(x) & =\int_{0}^{x} f(t) d t
\end{array}
$$

and so by the fundamental theorem of calculus $f(x)=y^{\prime}(x)$ and so the mapping is closed. It is obviously not continuous because it takes $y(x)$ and $y(x)+\frac{1}{n} \sin (n x)$ to two functions which are far from each other even though these two functions are very close in $C([0,1])$. Furthermore, it is not defined on the whole space, $C([0,1])$.

The next theorem, the closed graph theorem, gives conditions under which closed implies continuous.

Theorem 12.1.16 Let $X$ and $Y$ be Banach spaces and suppose $L: X \rightarrow Y$ is closed and linear. Then $L$ is continuous.

Proof: Let $G$ be the graph of $L . G=\{(x, L x): x \in X\}$. By Lemma 12.1.14 it follows that $G$ is a Banach space. Define $P: G \rightarrow X$ by $P(x, L x)=x$. $P$ maps the Banach space $G$ onto the Banach space $X$ and is continuous and linear. By the open mapping theorem, $P$ maps open sets onto open sets. Since $P$ is also one to one, this says that $P^{-1}$ is continuous. Thus $\left\|P^{-1} x\right\| \leq K\|x\|$. Hence

$$
\|L x\| \leq \max (\|x\|,\|L x\|) \leq K\|x\|
$$

By Theorem 2.9.1 on Page 68, this shows $L$ is continuous.
The following corollary is quite useful. It shows how to obtain a new norm on the domain of a closed operator such that the domain with this new norm becomes a Banach space.

Corollary 12.1.17 Let $L: D \subseteq X \rightarrow Y$ where $X, Y$ are a Banach spaces, and $L$ is a closed operator. Then define a new norm on $D$ by $\|x\|_{D} \equiv\|x\|_{X}+\|L x\|_{Y}$. Then $D$ with this new norm is a Banach space.

Proof: If $\left\{x_{n}\right\}$ is a Cauchy sequence in $D$ with this new norm, it follows both $\left\{x_{n}\right\}$ and $\left\{L x_{n}\right\}$ are Cauchy sequences and therefore, they converge. Since $L$ is closed, $x_{n} \rightarrow x$ and $L x_{n} \rightarrow L x$ for some $x \in D$. Thus $\left\|x_{n}-x\right\|_{D} \rightarrow 0$.

### 12.2 Basic Theory of Hilbert Spaces

The norm in a Hilbert space acts just like the absolute value in $\mathbb{R}$ or $\mathbb{C}$ both algebraically and geometrically so I will often use $|\cdot|$ to denote the norm in a Hilbert space to emphasize this fact.

Lemma 12.2.1 For $x \in H$, an inner product space,

$$
\begin{equation*}
\|x\|=\sup _{\|y\| \leq 1}|(x, y)| \tag{12.3}
\end{equation*}
$$

Proof: By the Cauchy Schwarz inequality, if $x \neq 0,\|x\| \geq \sup _{\|y\| \leq 1}|(x, y)| \geq\left(x, \frac{x}{\|x\|}\right)=$ $\|x\|$. It is obvious that 12.3 holds in the case that $x=0$.

In Hilbert space, one can define a projection map onto closed convex nonempty sets.
Definition 12.2.2 $A$ set, $K$, is convex if whenever $\lambda \in[0,1]$ and $x, y \in K, \lambda x+(1-$ $\lambda) y \in K$.

Theorem 12.2.3 Let $K$ be a closed convex nonempty subset of a Hilbert space, $H$, and let $x \in H$. Then there exists a unique point $P x \in K$ such that $\|P x-x\| \leq\|y-x\|$ for all $y \in K$.

Proof: Consider uniqueness. Suppose that $z_{1}$ and $z_{2}$ are two elements of $K$ such that for $i=1,2$,

$$
\begin{equation*}
\left\|z_{i}-x\right\| \leq\|y-x\| \tag{12.4}
\end{equation*}
$$

for all $y \in K$. Also, note that since $K$ is convex, $\frac{z_{1}+z_{2}}{2} \in K$. Therefore, by the parallelogram identity, Proposition 1.7.2 on Page 17,

$$
\begin{aligned}
\left\|z_{1}-x\right\|^{2} & \leq\left\|\frac{z_{1}+z_{2}}{2}-x\right\|^{2}=\left\|\frac{z_{1}-x}{2}+\frac{z_{2}-x}{2}\right\|^{2} \\
& =2\left(\left\|\frac{z_{1}-x}{2}\right\|^{2}+\left\|\frac{z_{2}-x}{2}\right\|^{2}\right)-\left\|\frac{z_{1}-z_{2}}{2}\right\|^{2} \\
& =\frac{1}{2}\left\|z_{1}-x\right\|^{2}+\frac{1}{2}\left\|z_{2}-x\right\|^{2}-\left\|\frac{z_{1}-z_{2}}{2}\right\|^{2} \\
& \leq\left\|z_{1}-x\right\|^{2}-\left\|\frac{z_{1}-z_{2}}{2}\right\|^{2},
\end{aligned}
$$

where the last inequality holds because of 12.4 letting $z_{i}=z_{2}$ and $y=z_{1}$. Hence $z_{1}=z_{2}$ and this shows uniqueness.

Now let $\lambda=\inf \{\|x-y\|: y \in K\}$ and let $y_{n}$ be a minimizing sequence. This means $\left\{y_{n}\right\} \subseteq K$ satisfies $\lim _{n \rightarrow \infty}\left\|x-y_{n}\right\|=\lambda$. Now the following follows from properties of the norm.

$$
\left\|y_{n}-x+y_{m}-x\right\|^{2}=4\left(\left\|\frac{y_{n}+y_{m}}{2}-x\right\|^{2}\right)
$$

Then by the parallelogram identity, and convexity of $K, \frac{y_{n}+y_{m}}{2} \in K$, and so

$$
\begin{aligned}
\left\|\left(y_{n}-x\right)-\left(y_{m}-x\right)\right\|^{2} & =2\left(\left\|y_{n}-x\right\|^{2}+\left\|y_{m}-x\right\|^{2}\right)-\overbrace{4\left(\left\|\frac{y_{n}+y_{m}}{2}-x\right\|^{2}\right.}^{=\left\|y_{n}-x+y_{m}-x\right\|^{2}} \\
& \leq 2\left(\left\|y_{n}-x\right\|^{2}+\left\|y_{m}-x\right\|^{2}\right)-4 \lambda^{2}
\end{aligned}
$$

Since $\left\|x-y_{n}\right\| \rightarrow \lambda$, this shows $\left\{y_{n}-x\right\}$ is a Cauchy sequence. Thus also $\left\{y_{n}\right\}$ is a Cauchy sequence. Since $H$ is complete, $y_{n} \rightarrow y$ for some $y \in H$ which must be in $K$ because $K$ is closed. Therefore $\|x-y\|=\lim _{n \rightarrow \infty}\left\|x-y_{n}\right\|=\lambda$. Let $P x=y$.

Corollary 12.2.4 Let $K$ be a closed, convex, nonempty subset of a Hilbert space, $H$, and let $x \in H$. Then for $z \in K, z=P x$ if and only if

$$
\begin{equation*}
\operatorname{Re}(x-z, y-z) \leq 0 \tag{12.5}
\end{equation*}
$$

for all $y \in K$.
Before proving this, consider what it says in the case where the Hilbert space is $\mathbb{R}^{n}$. Condition 12.5 says the angle, $\theta$, shown in the diagram is always obtuse. Remember from calculus,
 the sign of $\mathbf{x} \cdot \mathbf{y}$ is the same as the sign of the cosine of the included angle between $\mathbf{x}$ and $\mathbf{y}$. Thus, in finite dimensions, the conclusion of this corollary says that $z=P x$ exactly when the angle of the indicated angle is obtuse. Surely the picture suggests this is reasonable.
The inequality 12.5 is an example of a variational inequality and this corollary characterizes the projection of $x$ onto $K$ as the solution of this variational inequality.

Proof of Corollary: Let $z \in K$ and let $y \in K$ also. Since $K$ is convex, it follows that if $t \in[0,1], z+t(y-z)=(1-t) z+t y \in K$. Furthermore, every point of $K$ can be written in this way. (Let $t=1$ and $y \in K$.) Therefore, $z=P x$ if and only if for all $y \in K$ and $t \in[0,1]$,

$$
\|x-(z+t(y-z))\|^{2}=\|(x-z)-t(y-z)\|^{2} \geq\|x-z\|^{2}
$$

for all $t \in[0,1]$ and $y \in K$ if and only if for all $t \in[0,1]$ and $y \in K$

$$
\begin{equation*}
\|x-z\|^{2}+t^{2}\|y-z\|^{2}-2 t \operatorname{Re}(x-z, y-z) \geq\|x-z\|^{2} \tag{12.6}
\end{equation*}
$$

If and only if for all $t \in[0,1], t^{2}\|y-z\|^{2}-2 t \operatorname{Re}(x-z, y-z) \geq 0$. Now this is equivalent to 12.6 holding for all $t \in(0,1)$. Therefore, dividing by $t \in(0,1), 12.6$ is equivalent to $t\|y-z\|^{2}-2 \operatorname{Re}(x-z, y-z) \geq 0$ for all $t \in(0,1)$ which is equivalent to 12.5 .

Corollary 12.2.5 Let $K$ be a nonempty convex closed subset of a Hilbert space, $H$. Then the projection map, $P$ is continuous. In fact, $|P x-P y| \leq|x-y|$.

Proof: Let $x, x^{\prime} \in H$. Then by Corollary 12.2.4,

$$
\operatorname{Re}\left(x^{\prime}-P x^{\prime}, P x-P x^{\prime}\right) \leq 0, \operatorname{Re}\left(x-P x, P x^{\prime}-P x\right) \leq 0
$$

Hence

$$
\begin{aligned}
0 & \leq \operatorname{Re}\left(x-P x, P x-P x^{\prime}\right)-\operatorname{Re}\left(x^{\prime}-P x^{\prime}, P x-P x^{\prime}\right) \\
& =\operatorname{Re}\left(x-x^{\prime}, P x-P x^{\prime}\right)-\left|P x-P x^{\prime}\right|^{2}
\end{aligned}
$$

and so $\left|P x-P x^{\prime}\right|^{2} \leq\left|x-x^{\prime}\right|\left|P x-P x^{\prime}\right|$.
The next corollary is a more general form for the Brouwer fixed point theorem. This was discussed in exercises and elsewhere earlier. However, here is a complete proof.

Corollary 12.2.6 Let $\mathbf{f}: K \rightarrow K$ where $K$ is a convex compact subset of $\mathbb{R}^{n}$. Then $\mathbf{f}$ has a fixed point.

Proof: Let $K \subseteq \overline{B(\mathbf{0}, R)}$ and let $P$ be the projection map onto $K$. Then consider the map $\mathbf{f} \circ P$ which maps $\overline{B(\mathbf{0}, R)}$ to $\overline{B(\mathbf{0}, R)}$ and is continuous. By the Brouwer fixed point theorem for balls, this map has a fixed point. Thus there exists $\mathbf{x}$ such that $\mathbf{f} \circ P(\mathbf{x})=\mathbf{x}$. Now the equation also requires $\mathbf{x} \in K$ and so $P(\mathbf{x})=\mathbf{x}$. Hence $\mathbf{f}(\mathbf{x})=\mathbf{x}$.

Definition 12.2.7 Let $H$ be a vector space and let $U$ and $V$ be subspaces. $U \oplus V=$ $H$ if every element of $H$ can be written as a sum of an element of $U$ and an element of $V$ in a unique way.

The case where the closed convex set is a closed subspace is of special importance and in this case the above corollary implies the following.

Corollary 12.2.8 Let $K$ be a closed subspace of a Hilbert space, $H$, and let $x \in H$. Then for $z \in K, z=P x$ if and only if

$$
\begin{equation*}
(x-z, y)=0 \tag{12.7}
\end{equation*}
$$

for all $y \in K$. Furthermore, $H=K \oplus K^{\perp}$ where

$$
K^{\perp} \equiv\{x \in H:(x, k)=0 \text { for all } k \in K\}
$$

and

$$
\begin{equation*}
\|x\|^{2}=\|x-P x\|^{2}+\|P x\|^{2} . \tag{12.8}
\end{equation*}
$$

Proof: Since $K$ is a subspace, the condition 12.5 implies $\operatorname{Re}(x-z, y) \leq 0$ for all $y \in K$. Replacing $y$ with $-y$, it follows $\operatorname{Re}(x-z,-y) \leq 0$ which implies $\operatorname{Re}(x-z, y) \geq 0$ for all $y$. Therefore, $\operatorname{Re}(x-z, y)=0$ for all $y \in K$. Now let $|\alpha|=1$ and $\alpha(x-z, y)=|(x-z, y)|$. Since $K$ is a subspace, it follows $\bar{\alpha} y \in K$ for all $y \in K$. Therefore,

$$
0=\operatorname{Re}(x-z, \bar{\alpha} y)=(x-z, \bar{\alpha} y)=\alpha(x-z, y)=|(x-z, y)| .
$$

This shows that $z=P x$, if and only if 12.7.
For $x \in H, x=x-P x+P x$ and from what was just shown, $x-P x \in K^{\perp}$ and $P x \in K$. This shows that $K^{\perp}+K=H$. Is there only one way to write a given element of $H$ as a sum of a vector in $K$ with a vector in $K^{\perp}$ ? Suppose $y+z=y_{1}+z_{1}$ where $z, z_{1} \in K^{\perp}$ and $y, y_{1} \in K$. Then $\left(y-y_{1}\right)=\left(z_{1}-z\right)$ and so from what was just shown, $\left(y-y_{1}, y-y_{1}\right)=$ $\left(y-y_{1}, z_{1}-z\right)=0$ which shows $y_{1}=y$ and consequently $z_{1}=z$. Finally, letting $z=P x$,

$$
\begin{aligned}
\|x\|^{2} & =(x-z+z, x-z+z)=\|x-z\|^{2}+(x-z, z)+(z, x-z)+\|z\|^{2} \\
& =\|x-z\|^{2}+\|z\|^{2} ■
\end{aligned}
$$

The following theorem is called the Riesz representation theorem for the dual of a Hilbert space. If $z \in H$ then define an element $f \in H^{\prime}$ by the rule $(x, z) \equiv f(x)$. It follows from the Cauchy Schwarz inequality and the properties of the inner product that $f \in H^{\prime}$. The Riesz representation theorem says that all elements of $H^{\prime}$ are of this form.

Theorem 12.2.9 Let $H$ be a Hilbert space and let $f \in H^{\prime}$. Then there exists a unique $z \in H$ such that $f(x)=(x, z)$ for all $x \in H$.

Proof: Letting $y, w \in H$ the assumption that $f$ is linear implies

$$
f(y f(w)-f(y) w)=f(w) f(y)-f(y) f(w)=0
$$

which shows that $y f(w)-f(y) w \in f^{-1}(0)$, which is a closed subspace of $H$ since $f$ is continuous. If $f^{-1}(0)=H$, then $f$ is the zero map and $z=0$ is the unique element of $H$ which satisfies $f(x)=(x, z)$.

If $f^{-1}(0) \neq H$, pick $u \notin f^{-1}(0)$ and let $w \equiv u-P u \neq 0$. Thus Corollary 12.2.8 implies $(y, w)=0$ for all $y \in f^{-1}(0)$. In particular, let $y=x f(w)-f(x) w$ where $x \in H$ is arbitrary. Therefore, $0=(f(w) x-f(x) w, w)=f(w)(x, w)-f(x)\|w\|^{2}$. Thus, solving for $f(x)$ and using the properties of the inner product, $f(x)=\left(x, \frac{\overline{f(w) w}}{\|w\|^{2}}\right)$. Let $z=\overline{f(w)} w /\|w\|^{2}$. This proves the existence of $z$. If $f(x)=\left(x, z_{i}\right) i=1,2$, for all $x \in H$, then for all $x \in H$, then $\left(x, z_{1}-z_{2}\right)=0$ which implies, upon taking $x=z_{1}-z_{2}$ that $z_{1}=z_{2}$.

If $R: H \rightarrow H^{\prime}$ is defined by $R x(y) \equiv(y, x)$, the Riesz representation theorem above states this map is onto. This map is called the Riesz map. It is routine to show $R$ is conjugate linear and $\|R x\|=\|x\|$. In fact,

$$
\begin{aligned}
R(\alpha x+\beta y)(u) & \equiv(u, \alpha x+\beta y)=\bar{\alpha}(u, x)+\bar{\beta}(u, y) \\
& \equiv \bar{\alpha} R x(u)+\bar{\beta} R y(u)=(\bar{\alpha} R x+\bar{\beta} R y)(u)
\end{aligned}
$$

so it is conjugate linear meaning it goes across plus signs and you factor out conjugates.

$$
\|R x\| \equiv \sup _{\|y\| \leq 1}|R x(y)| \equiv \sup _{\|y\| \leq 1}|(y, x)|=\|x\|
$$

### 12.2.1 Partially Ordered Sets

Recall the notion of a partially ordered set.
Definition 12.2.10 Let $\mathscr{F}$ be a nonempty set. $\mathscr{F}$ is called a partially ordered set if there is a relation, denoted here by $\leq$, such that

$$
\begin{gathered}
x \leq x \text { for all } x \in \mathscr{F} \\
\text { If } x \leq y \text { and } y \leq z \text { then } x \leq z
\end{gathered}
$$

$\mathscr{C} \subseteq \mathscr{F}$ is said to be a chain if every two elements of $\mathscr{C}$ are related. This means that if $x, y \in \mathscr{C}$, then either $x \leq y$ or $y \leq x$. Sometimes a chain is called a totally ordered set. $\mathscr{C}$ is said to be a maximal chain if whenever $\mathscr{D}$ is a chain containing $\mathscr{C}, \mathscr{D}=\mathscr{C}$.

For a discussion of the next theorem, see Theorem 1.4.2 on Page 9.
Theorem 12.2.11 (Hausdorff Maximal Principle) Let $\mathscr{F}$ be a nonempty partially ordered set. Then there exists a maximal chain.

### 12.2.2 Maximal Orthonormal Sets in Hilbert Space

Definition 12.2.12 Let $H$ be a Hilbert space and let $D \subseteq H$. Then $D$ is called an orthonormal set if whenever $x, y \in D$, then $(x, y)=0$ if $x \neq y$ and $(x, x)=1$. The orthonormal set $D$ is called maximal if whenever $\hat{D}$ is an orthonormal set containing $D$, it follows that $\hat{D}=D$.

Then with this definition, the fundamental result about orthonormal sets in Hilbert space is the following.

Theorem 12.2.13 Let $H$ be a nonempty Hilbert space and let $D_{0}$ be an orthonormal set. Then

1. There exists a maximal orthonormal set $D \supseteq D_{0}$.
2. $\overline{\operatorname{span}(D)}=H$.
3. If $H$ is separable, then $D$ is countable.
4. In this case where $H$ is separable and $D=\left\{d_{n}\right\}_{n=1}^{\infty}$, it follows that for any $x \in H, x=$ $\sum_{n=1}^{\infty}\left(x, d_{n}\right) d_{n}$ meaning the series converges in $H$ to $x$. This is called the Fourier series of $x$.
5. Also for $x \in H,|x|^{2}=\sum_{n=1}^{\infty}\left|\left(x, b_{n}\right)\right|^{2}$.

Proof: 1.) First note that there exists such an orthonormal set $D_{0}$. It could be $x \in H$ where $|x|=1$. Let $\mathscr{F}$ consist of orthonormal sets containing $D_{0}$ and partially order these sets by set inclusion. By the Hausdorff maximal theorem, there is a maximal chain $\mathscr{C}$. Let $D=\cup \mathscr{C}$. Then $D$ is orthonormal because $\mathscr{C}$ is a chain and any pair of vectors in $D$ must be in a single element of $\mathscr{C}$. $D$ contains $D_{0}$ and if $D$ is not maximal, there must be some $z \notin D$ such that $D \cup\{z\}$ is also an orthonormal set. But then this contradicts the maximality of $\mathscr{C}$ because $\mathscr{C} \cup\{D \cup\{z\}\}$ would be larger chain.
2.) If there exists $y \in H \backslash \overline{\operatorname{span}(D)}$, note that $\overline{\operatorname{span}(D)}$ is a closed subspace of $H$. Therefore, $y-P y$ is not 0 and $z \equiv(y-P y) /|y-P y|$ satisfies $(z, x)=0$ for all $x \in \overline{\operatorname{span}(D)}$. In particular, $(z, x)=0$ for all $x \in D$. Therefore, $D$ is not maximal after all. Thus $\overline{\operatorname{span}(D)}=H$.
3.) Now suppose $H$ is separable. If $x \neq y, x, y \in D$, then $|x-y|^{2}=|x|^{2}+|y|^{2}=2$ and so $|x-y|=\sqrt{2}$. Thus the balls $B(x, 1)$ for $x \in D$ are disjoint. Since $H$ is separable, there are only countably many.
4.) Let $D=\left\{d_{n}\right\}_{n=1}^{\infty}$. Let $V_{k} \equiv \operatorname{span}\left\{d_{1}, \ldots, d_{k}\right\}$. Now consider the problem of choosing $\alpha_{k}$ to minimize $\left|x-\sum_{n=1}^{k} \alpha_{n} d_{n}\right|^{2}$. This expression to minimize equals

$$
\left|x-\sum_{n=1}^{k}\left(x, d_{n}\right) d_{n}+\sum_{n=1}^{k}\left(\left(x, d_{n}\right)-\alpha_{n}\right) d_{n}\right|^{2}
$$

and some algebra using the $d_{n}$ are orthonormal shows this equals

$$
\begin{aligned}
& \left|x-\sum_{n=1}^{k}\left(x, d_{n}\right) d_{n}\right|^{2}+\sum_{n=1}^{k}\left|\left(x, d_{n}\right)-\alpha_{n}\right|^{2}+2 \operatorname{Re} \sum_{n=1}^{k}\left(\overline{\left(x, d_{n}\right)-\alpha_{n}}\right)\left(x, d_{n}\right) \\
& -2 \operatorname{Re} \sum_{n=1}^{k}\left(x, d_{n}\right)\left(\overline{\left(x, d_{n}\right)-\alpha_{n}}\right)
\end{aligned}
$$

Thus the solution to the minimization problem has $\alpha_{n}=\left(x, d_{n}\right)$. Since $\overline{\operatorname{span}(D)}=H$, it follows that $x=\lim _{k \rightarrow \infty} \sum_{n=1}^{k}\left(x, d_{n}\right) d_{n}$
5.) From 4.) and algebra,

$$
\begin{aligned}
0 & =\lim _{k \rightarrow \infty}\left|x-\sum_{n=1}^{k}\left(x, d_{n}\right) d_{n}\right|^{2}=\lim _{k \rightarrow \infty}\left(|x|^{2}+\sum_{n=1}^{k}\left|\left(x, d_{n}\right)\right|^{2}-2 \sum_{n=1}^{k}\left|\left(x, d_{n}\right)\right|^{2}\right) \\
& =\lim _{k \rightarrow \infty}\left(|x|^{2}-\sum_{n=1}^{k}\left|\left(x, d_{n}\right)\right|^{2}\right)
\end{aligned}
$$

### 12.3 Hahn Banach Theorem

The closed graph, open mapping, and uniform boundedness theorems are the three major topological theorems in functional analysis. The other major theorem is the Hahn-Banach theorem which has nothing to do with topology.

### 12.3.1 Gauge Functions and Hahn Banach Theorem

Definition 12.3.1 Let $X$ be a real vector space $\rho: X \rightarrow \mathbb{R}$ is called a gauge function if

$$
\begin{equation*}
\rho(x+y) \leq \rho(x)+\rho(y), \quad \rho(a x)=a \rho(x) \text { if } a \geq 0 . \tag{12.9}
\end{equation*}
$$

Suppose $M$ is a subspace of $X$ and $z \notin M$. Suppose also that $f$ is a linear real-valued function having the property that $f(x) \leq \rho(x)$ for all $x \in M$. Consider the problem of extending $f$ to $M \oplus \mathbb{R} z$ such that if $F$ is the extended function, $F(y) \leq \rho(y)$ for all $y \in$ $M \oplus \mathbb{R} z$ and $F$ is linear. Since $F$ is to be linear, it suffices to determine how to define $F(z)$. Letting $a>0$, it is required to define $F(z)$ such that the following hold for all $x, y \in M$.

$$
\begin{align*}
& \overbrace{F(x)}^{f(x)}+a F(z)=F(x+a z) \leq \rho(x+a z), \\
& \overbrace{F(y)}^{f(y)}-a F(z)=F(y-a z) \leq \rho(y-a z) . \tag{12.10}
\end{align*}
$$

Therefore, multiplying by $a^{-1} 12.9$ implies that what is needed is to choose $F(z)$ such that for all $x, y \in M$,

$$
f(x)+F(z) \leq \rho(x+z), f(y)-\rho(y-z) \leq F(z)
$$

and that if $F(z)$ can be chosen in this way, this will satisfy 12.10 for all $x, y$ and the problem of extending $f$ will be solved. Hence it is necessary to choose $F(z)$ such that for all $x, y \in M$

$$
\begin{equation*}
f(y)-\rho(y-z) \leq F(z) \leq \rho(x+z)-f(x) \tag{12.11}
\end{equation*}
$$

Is there any such number between $f(y)-\rho(y-z)$ and $\rho(x+z)-f(x)$ for every pair $x, y \in$ $M$ ? This is where $f(x) \leq \rho(x)$ on $M$ and that $f$ is linear is used. For $x, y \in M$,

$$
\begin{gathered}
\rho(x+z)-f(x)-[f(y)-\rho(y-z)] \\
=\rho(x+z)+\rho(y-z)-(f(x)+f(y)) \geq \rho(x+y)-f(x+y) \geq 0 .
\end{gathered}
$$

Therefore there exists a number between the following two numbers

$$
\sup \{f(y)-\rho(y-z): y \in M\}, \inf \{\rho(x+z)-f(x): x \in M\}
$$

Choose $F(z)$ to satisfy 12.11 . This has proved the following lemma.

Lemma 12.3.2 Let $M$ be a subspace of $X$, a real linear space, and let $\rho$ be a gauge function on $X$. Suppose $f: M \rightarrow \mathbb{R}$ is linear, $z \notin M$, and $f(x) \leq \rho(x)$ for all $x \in M$. Then $f$ can be extended to $M \oplus \mathbb{R} z$ such that, if $F$ is the extended function, then $F$ is linear and $F(x) \leq \rho(x)$ for all $x \in M \oplus \mathbb{R} z$.

With this lemma, the Hahn Banach theorem is easy to show.
Theorem 12.3.3 (Hahn Banach theorem) Let $X$ be a real vector space, let $M$ be a subspace of $X$, let $f: M \rightarrow \mathbb{R}$ be linear, let $\rho$ be a gauge function on $X$, and suppose $f(x) \leq \rho(x)$ for all $x \in M$. Then there exists a linear function, $F: X \rightarrow \mathbb{R}$, such that
a.) $F(x)=f(x)$ for all $x \in M$
b.) $F(x) \leq \rho(x)$ for all $x \in X$.

Proof: Let $\mathscr{F}=\{(V, g): V \supseteq M, V$ is a subspace of $X, g: V \rightarrow \mathbb{R}$ is linear, $g(x)=f(x)$ for all $x \in M$, and $g(x) \leq \rho(x)$ for $x \in V\}$. Then $(M, f) \in \mathscr{F}$ so $\mathscr{F} \neq \emptyset$. Define a partial order by the following rule. $(V, g) \leq(W, h)$ means $V \subseteq W$ and $h(x)=g(x)$ if $x \in V$. By Theorem 12.2.11, there exists a maximal chain, $\mathscr{C} \subseteq \mathscr{\mathscr { F }}$. Let $Y=\cup\{V:(V, g) \in \mathscr{C}\}$ and let $h: Y \rightarrow \mathbb{R}$ be defined by $h(x)=g(x)$ where $x \in V$ and $(V, g) \in \mathscr{C}$. This is well defined because if $x \in V_{1}$ and $V_{2}$ where $\left(V_{1}, g_{1}\right)$ and $\left(V_{2}, g_{2}\right)$ are both in the chain, then since $\mathscr{C}$ is a chain, the two element related. Therefore, $g_{1}(x)=g_{2}(x)$. Also $h$ is linear because if $a x+b y \in Y$, then $x \in V_{1}$ and $y \in V_{2}$ where $\left(V_{1}, g_{1}\right)$ and $\left(V_{2}, g_{2}\right)$ are elements of $\mathscr{C}$. Therefore, letting $V$ denote the larger of the two $V_{i}$, and $g$ be the function that goes with $V$, it follows $a x+b y \in V$ where $(V, g) \in \mathscr{C}$. Therefore,

$$
\begin{aligned}
h(a x+b y) & =g(a x+b y)=a g(x)+b g(y) \\
& =a h(x)+b h(y) .
\end{aligned}
$$

Also, $h(x)=g(x) \leq \rho(x)$ for any $x \in Y$ because for such $x, x \in V$ where $(V, g) \in \mathscr{C}$.
Is $Y=X$ ? If not, there exists $z \in X \backslash Y$ and there exists an extension of $h$ to $Y \oplus \mathbb{R} z$ using Lemma 12.3.2. Letting $\bar{h}$ denote this extended function, contradicts the maximality of $\mathscr{C}$. Indeed, $\mathscr{C} \cup\{(Y \oplus \mathbb{R} z, \bar{h})\}$ would be a longer chain.

### 12.3.2 The Complex Hahn Banach Theorem

This is the original version of the theorem. There is also a version of this theorem for complex vector spaces which is based on a trick. First note that linear $f$ satisfying $f(x) \leq$ $\|x\|$ means $f(-x) \leq K\|-x\|$ means $-f(x) \leq K\|x\|$ so $f(x) \geq-K\|x\|$ so $|f(x)| \leq K\|x\|$. To say $f(x) \leq K\|x\|$ is the same as saying $|f(x)| \leq K\|x\|$. Of course $x \rightarrow K\|x\|$ is an example a gauge function $\rho(x)$.

Note that if $F$ is linear on $V$ a complex normed linear space, then

$$
\begin{aligned}
\operatorname{Re} F(i x)+i \operatorname{Im} F(i x) & =F(i x)=i F(x)=i(\operatorname{Re} F(x)+i \operatorname{Im} F(x)) \\
& =-\operatorname{Im} F(x)+i \operatorname{Re} F(x)
\end{aligned}
$$

and so $\operatorname{Im} F(x)=-\operatorname{Re} F(i x)$ so $F(x)=\operatorname{Re} F(x)-i \operatorname{Re} F(i x)$. Also note that $\operatorname{Re} F(x+y)=$ $\operatorname{Re} F(x)+\operatorname{Re} F(y)$ and that if $a$ is real, then $a \operatorname{Re} F(x)=\operatorname{Re} F(a x)$. Conversely, if $f$ is real linear we can produce a complex linear function from it as explained in the following Lemma.

Lemma 12.3.4 Suppose $V$ is a complex normed linear space and $f: V \rightarrow \mathbb{R}$ satisfies af $(x)=f(a x)$ for all a real and $f(x+y)=f(x)+f(y)$. Define $F(x) \equiv f(x)-i f(i x)$. Then $F$ is linear on $V$ with field of scalars equal to $\mathbb{C}$.

Proof: Obviously $F(x+y)=F(x)+F(y)$. Now

$$
\begin{aligned}
F((a+i b) x) & \equiv f((a+i b) x)-i f(i(a+i b) x)=f(a x+i b x)-i f(-b x+i a x) \\
& =a f(x)+b f(i x)+b i f(x)-a i f(i a x) \\
(a+i b) F(x) & \equiv(a+i b)(f(x)-i f(i x)=a f(x)-i a f(i x))+i b f(x)+b f(i x)
\end{aligned}
$$

Thus $F$ is linear as claimed.
Corollary 12.3.5 (Hahn Banach) Let $M$ be a subspace of a complex normed linear space $X$, and suppose $f: M \rightarrow \mathbb{C}$ is linear and satisfies $|f(x)| \leq K\|x\|$ for all $x \in M$. Then there exists a linear function $F$, defined on all of $X$ such that $F(x)=f(x)$ for all $x \in M$ and $|F(x)| \leq K\|x\|$ for all $x$.

Proof: Since $|f(x)| \leq K\|x\|$ for all $x \in M$, then $|\operatorname{Re} f(x)| \leq K\|x\|$ on $M$ and so, since $\operatorname{Re} f$ is real and real linear on $M, \operatorname{Re} f(x) \leq K\|x\| \equiv \rho(x)$. By the Hahn Banach theorem, let $h$ be a real valued linear extension of $\operatorname{Re} f$ satisfying $h(x) \leq K\|x\|$ on $X$. Now let $F(x) \equiv h(x)-i h(i x)$ so $F$ is complex linear on $X$. For a given $x \in X$, there is $\alpha \in \mathbb{C}$, $|\alpha|=1$ such that $\alpha F(x)=|F(x)|$. Then

$$
|F(x)|=\alpha F(x)=F(\alpha x)=h(\alpha x)-\overbrace{i h(i \alpha x)}^{=0}=h(\alpha x) \leq K\|\alpha x\|=K\|x\|
$$

### 12.3.3 The Dual Space and Adjoint Operators

Definition 12.3.6 Let $X$ be a Banach space. Denote by $X^{\prime}$ the space of continuous linear functions which map $X$ to the field of scalars. Thus $X^{\prime}=\mathscr{L}(X, \mathbb{F})$. By Theorem 2.9.4 on Page $69, X^{\prime}$ is a Banach space. Remember with the norm defined on $\mathscr{L}(X, \mathbb{F})$, $\|f\|=\sup \{|f(x)|:\|x\| \leq 1\} . X^{\prime}$ is called the dual space.

Definition 12.3.7 Let $X$ and $Y$ be Banach spaces and $L \in \mathscr{L}(X, Y)$. Then define the adjoint map in $\mathscr{L}\left(Y^{\prime}, X^{\prime}\right)$, denoted by $L^{*}$, by $L^{*} y^{*}(x) \equiv y^{*}(L x)$ for all $y^{*} \in Y^{\prime}$.

The following diagram is a good one to help remember this definition.


This is a generalization of the adjoint of a linear transformation on an inner product space, the conjugate transpose. Recall $(A x, y)=\left(x, A^{*} y\right)$. What is being done here is to generalize this algebraic concept to arbitrary Banach spaces. There are some issues which need to be discussed relative to the above definition. First of all, it must be shown that $L^{*} y^{*} \in X^{\prime}$. Also, it will be useful to have the following lemma which is a useful application of the Hahn Banach theorem.

Lemma 12.3.8 Let $X$ be a normed linear space and let $x \in X \backslash V$ where $V$ is a closed subspace of $X$. Then there exists $x^{*} \in X^{\prime}$ such that $x^{*}(x)=\|x\|, x^{*}(V)=\{0\}$, and

$$
\left\|x^{*}\right\| \leq \frac{1}{\operatorname{dist}(x, V)}
$$

In the case that $V=\{0\},\left\|x^{*}\right\|=1$.
Proof: Let $f: \mathbb{F} x+V \rightarrow \mathbb{F}$ be defined by $f(\alpha x+v)=\alpha\|x\|$. First it is necessary to show $f$ is well defined and continuous. If $\alpha_{1} x+v_{1}=\alpha_{2} x+v_{2}$ then if $\alpha_{1} \neq \alpha_{2}$, then $x \in V$ which is assumed not to happen so $f$ is well defined. It remains to show $f$ is continuous. Suppose then that $\alpha_{n} x+v_{n} \rightarrow 0$. It is necessary to show $\alpha_{n} \rightarrow 0$. If this does not happen, then there exists a subsequence, still denoted by $\alpha_{n}$ such that $\left|\alpha_{n}\right| \geq \delta>0$. Then $\frac{1}{\left|\alpha_{n}\right|} \leq \frac{1}{\delta}$. But then $x+\left(1 / \alpha_{n}\right) v_{n} \rightarrow 0$ so $x$ is a limit of points of $V$ which is closed, and this means $x \in V$ which is not so. Hence $f$ is continuous on $\mathbb{F} x+V$. Now $\operatorname{dist}(x, V) \equiv \inf \{\|x+v\|: v \in V\}$. Thus if it is required that $|\alpha|\|x+(v / \alpha)\| \leq 1$, to make $|\alpha|$ as large as possible one would make $\|x+(v / \alpha)\|$ as small as possible. Hence, $\sup _{|\alpha|\|x+(v / \alpha)\| \leq 1}|\alpha|=\frac{1}{\operatorname{dist}(x, V)}$. Therefore,

$$
\|f\|=\sup _{\|\alpha x+v\| \leq 1}|f(\alpha x+v)|=\sup _{|\alpha|\|x+(v / \alpha)\| \leq 1}|\alpha|\|x\|=\frac{1}{\operatorname{dist}(x, V)}\|x\|
$$

By the Hahn Banach theorem, there exists $x^{*} \in X^{\prime}$ such that $x^{*}=f$ on $\mathbb{F} x+V$. Thus $x^{*}(x)=$ $\|x\|$ and also $\left\|x^{*}\right\| \leq\|f\|=\frac{1}{\operatorname{dist}(x, V)}$.

In case $V=\{0\}$, the result follows from the above or alternatively,

$$
\|f\| \equiv \sup _{\|\alpha x\| \leq 1}|f(\alpha x)|=\sup _{|\alpha| \leq 1 /\|x\|}|\alpha|\|x\|=1
$$

and so, in this case, $\left\|x^{*}\right\| \leq\|f\|=1$. Since $x^{*}(x)=\|x\|$ it follows

$$
\left\|x^{*}\right\| \geq\left|x^{*}\left(\frac{x}{\|x\|}\right)\right|=\frac{\|x\|}{\|x\|}=1 .
$$

Thus $\left\|x^{*}\right\|=1$.
Note that this says that if $x \neq y$, then there exists $x^{*} \in X^{\prime}$ with $x^{*}(x-y)=\|x-y\|$ and so $x^{*}(x) \neq x^{*}(y)$. This proves

Proposition 12.3.9 If $x \neq y$, there exists $x^{*} \in X^{\prime}$ such that $x^{*}(x) \neq x^{*}(y)$.
Theorem 12.3.10 Let $L \in \mathscr{L}(X, Y)$ where $X$ and $Y$ are Banach spaces. Then
a.) $L^{*} \in \mathscr{L}\left(Y^{\prime}, X^{\prime}\right)$ as claimed and $\left\|L^{*}\right\|=\|L\|$.
b.) If $L$ maps one to one onto a closed subspace of $Y$, then $L^{*}$ is onto.
c.) If $L$ maps onto a dense subset of $Y$, then $L^{*}$ is one to one.

Proof: It is routine to verify $L^{*} y^{*}$ and $L^{*}$ are both linear. This follows immediately from the definition. As usual, the interesting thing concerns continuity.

$$
\left\|L^{*} y^{*}\right\|=\sup _{\|x\| \leq 1}\left|L^{*} y^{*}(x)\right|=\sup _{\|x\| \leq 1}\left|y^{*}(L x)\right| \leq\left\|y^{*}\right\|\|L\|
$$

Thus $L^{*}$ is continuous as claimed and $\left\|L^{*}\right\| \leq\|L\|$.

By Lemma 12.3.8, there exists $y_{x}^{*} \in Y^{\prime}$ such that $\left\|y_{x}^{*}\right\|=1$ and $y_{x}^{*}(L x)=\|L x\|$. Therefore,

$$
\begin{aligned}
\left\|L^{*}\right\| & =\sup _{\left\|y^{*}\right\| \leq 1}\left\|L^{*} y^{*}\right\|=\sup _{\left\|y^{*}\right\| \leq 1} \sup _{\|x\| \leq 1}\left|L^{*} y^{*}(x)\right| \\
& =\sup _{\left\|y^{*}\right\| \leq 1\|x\| \leq 1} \sup \left|y^{*}(L x)\right|=\sup _{\|x\| \leq 1\left\|y^{*}\right\| \leq 1} \sup \left|y^{*}(L x)\right| \\
& \geq \sup _{\|x\| \leq 1}\left|y_{x}^{*}(L x)\right|=\sup _{\|x\| \leq 1}\|L x\|=\|L\|
\end{aligned}
$$

showing that $\left\|L^{*}\right\| \geq\|L\|$ and this shows part a.).
If $L$ is one to one and onto a closed subset of $Y$, then $L(X)$ being a closed subspace of a Banach space, is itself a Banach space and so the open mapping theorem implies $L^{-1}: L(X) \rightarrow X$ is continuous. Hence $\|x\|=\left\|L^{-1} L x\right\| \leq\left\|L^{-1}\right\|\|L x\|$. Now let $x^{*} \in X^{\prime}$ be given. Define $f \in \mathscr{L}(L(X), \mathbb{C})$ by $f(L x)=x^{*}(x)$. The function, $f$ is well defined because if $L x_{1}=L x_{2}$, then since $L$ is one to one, it follows $x_{1}=x_{2}$ and so $f\left(L\left(x_{1}\right)\right)=x^{*}\left(x_{1}\right)=$ $x^{*}\left(x_{2}\right)=f\left(L\left(x_{1}\right)\right)$. Also, $f$ is linear because

$$
\begin{gathered}
f\left(a L\left(x_{1}\right)+b L\left(x_{2}\right)\right)=f\left(L\left(a x_{1}+b x_{2}\right)\right) \equiv x^{*}\left(a x_{1}+b x_{2}\right) \\
=a x^{*}\left(x_{1}\right)+b x^{*}\left(x_{2}\right)=a f\left(L\left(x_{1}\right)\right)+b f\left(L\left(x_{2}\right)\right) .
\end{gathered}
$$

In addition to this, $|f(L x)|=\left|x^{*}(x)\right| \leq\left\|x^{*}\right\|$ and also $\|x\| \leq\left\|x^{*}\right\|\left\|L^{-1}\right\|\|L x\|$ and so the norm of $f$ on $L(X)$ is no larger than $\left\|x^{*}\right\|\left\|L^{-1}\right\|$. By the Hahn Banach theorem, there exists an extension of $f$ to an element $y^{*} \in Y^{\prime}$ such that $\left\|y^{*}\right\| \leq\left\|x^{*}\right\|\left\|L^{-1}\right\|$. Then $L^{*} y^{*}(x)=$ $y^{*}(L x)=f(L x)=x^{*}(x)$ so $L^{*} y^{*}=x^{*}$ because this holds for all $x$. Since $x^{*}$ was arbitrary, this shows $L^{*}$ is onto and proves b.).

Consider the last assertion. Suppose $L^{*} y^{*}=0$. Is $y^{*}=0$ ? In other words is $y^{*}(y)=0$ for all $y \in Y$ ? Pick $y \in Y$. Since $L(X)$ is dense in $Y$, there exists a sequence, $\left\{L x_{n}\right\}$ such that $L x_{n} \rightarrow y$. But then by continuity of $y^{*}, y^{*}(y)=\lim _{n \rightarrow \infty} y^{*}\left(L x_{n}\right)=\lim _{n \rightarrow \infty} L^{*} y^{*}\left(x_{n}\right)=0$. Since $y^{*}(y)=0$ for all $y$, this implies $y^{*}=0$ and so $L^{*}$ is one to one.

Corollary 12.3.11 Suppose $X$ and $Y$ are Banach spaces, $L \in \mathscr{L}(X, Y)$, and $L$ is one to one and onto. Then $L^{*}$ is also one to one and onto.

There exists a natural mapping, called the James map from a normed linear space, $X$, to the dual of the dual space which is described in the following definition.
Definition 12.3.12 Define $J: X \rightarrow X^{\prime \prime}$ by $J(x)\left(x^{*}\right)=x^{*}(x)$.
Theorem 12.3.13 The map, J, has the following properties.
a.) $J$ is one to one and linear.
b.) $\|J x\|=\|x\|$ and $\|J\|=1$.
c.) $J(X)$ is a closed subspace of $X^{\prime \prime}$ if $X$ is complete.

Also if $x^{*} \in X^{\prime}$,

$$
\left\|x^{*}\right\|=\sup \left\{\left|x^{* *}\left(x^{*}\right)\right|:\left\|x^{* *}\right\| \leq 1, x^{* *} \in X^{\prime \prime}\right\}
$$

Proof: First note that from the definition,

$$
J(a x+b y)\left(x^{*}\right) \equiv x^{*}(a x+b y)=a x^{*}(x)+b x^{*}(y)=(a J(x)+b J(y))\left(x^{*}\right)
$$

Since this holds for all $x^{*} \in X^{\prime}$, it follows that $J(a x+b y)=a J(x)+b J(y)$ and so $J$ is linear. If $J x=0$, then by Lemma 12.3.8 there exists $x^{*}$ such that $x^{*}(x)=\|x\|$ and $\left\|x^{*}\right\|=1$. Then $0=J(x)\left(x^{*}\right)=x^{*}(x)=\|x\|$. This shows a.).

To show b.), let $x \in X$ and use Lemma 12.3.8 to obtain $x^{*} \in X^{\prime}$ such that $x^{*}(x)=\|x\|$ with $\left\|x^{*}\right\|=1$. Then

$$
\begin{aligned}
\|x\| & \geq \sup \left\{\left|y^{*}(x)\right|:\left\|y^{*}\right\| \leq 1\right\}=\sup \left\{\left|J(x)\left(y^{*}\right)\right|:\left\|y^{*}\right\| \leq 1\right\}=\|J x\| \\
& \geq\left|J(x)\left(x^{*}\right)\right|=\left|x^{*}(x)\right|=\|x\|
\end{aligned}
$$

Therefore, $\|J x\|=\|x\|$ as claimed. Therefore,

$$
\|J\|=\sup \{\|J x\|:\|x\| \leq 1\}=\sup \{\|x\|:\|x\| \leq 1\}=1
$$

This shows b.).
To verify c.), use b.). If $J x_{n} \rightarrow y^{* *} \in X^{\prime \prime}$ then by b.), $x_{n}$ is a Cauchy sequence converging to some $x \in X$ because $\left\|x_{n}-x_{m}\right\|=\left\|J x_{n}-J x_{m}\right\|$ and $\left\{J x_{n}\right\}$ is a Cauchy sequence. Then $J x=\lim _{n \rightarrow \infty} J x_{n}=y^{* *}$.

Finally, to show the assertion about the norm of $x^{*}$, use what was just shown applied to the James map from $X^{\prime}$ to $X^{\prime \prime \prime}$ still referred to as $J$.

$$
\begin{gathered}
\left\|x^{*}\right\|=\sup \left\{\left|x^{*}(x)\right|:\|x\| \leq 1\right\}=\sup \left\{\left|J(x)\left(x^{*}\right)\right|:\|J x\| \leq 1\right\} \\
\leq \sup \left\{\left|x^{* *}\left(x^{*}\right)\right|:\left\|x^{* *}\right\| \leq 1\right\}=\sup \left\{\left|J\left(x^{*}\right)\left(x^{* *}\right)\right|:\left\|x^{* *}\right\| \leq 1\right\} \\
\equiv\left\|J x^{*}\right\|=\left\|x^{*}\right\| .
\end{gathered}
$$

Definition 12.3.14 When $J$ maps $X$ onto $X^{\prime \prime}, X$ is called reflexive.

### 12.4 Exercises

1. Is $\mathbb{N}$ a $G_{\delta}$ set? What about $\mathbb{Q}$ ? What about $\mathbb{R} \backslash \mathbb{Q}$ ?
2. $\uparrow$ Let $f: \mathbb{R} \rightarrow \mathbb{C}$ be a function. Define the oscillation of a function in $B(x, r)$ by $\omega_{r} f(x)=\sup \{|f(z)-f(y)|: y, z \in B(x, r)\}$. Define the oscillation of the function at the point, $x$ by $\omega f(x)=\lim _{r \rightarrow 0} \omega_{r} f(x)$. Show $f$ is continuous at $x$ if and only if $\omega f(x)=0$. Then show the set of points where $f$ is continuous is a $G_{\delta}$ set (try $\left.U_{n}=\left\{x: \omega f(x)<\frac{1}{n}\right\}\right)$. Does there exist a function continuous at only the rational numbers? Does there exist a function continuous at every irrational and discontinuous elsewhere? Hint: Suppose $D$ is any countable set, $D=\left\{d_{i}\right\}_{i=1}^{\infty}$, and define the function, $f_{n}(x)$ to equal zero for every $x \notin\left\{d_{1}, \cdots, d_{n}\right\}$ and $2^{-n}$ for $x$ in this finite set. Then consider $g(x) \equiv \sum_{n=1}^{\infty} f_{n}(x)$. Show that this series converges uniformly.
3. Let $f \in C([0,1])$ and suppose $f^{\prime}(x)$ exists. Show there exists a constant, $K$, such that $|f(x)-f(y)| \leq K|x-y|$ for all $y \in[0,1]$. Let $U_{n}=\{f \in C([0,1])$ such that for each $x \in[0,1]$ there exists $y \in[0,1]$ such that $|f(x)-f(y)|>n|x-y|\}$. Show that $U_{n}$ is open and dense in $C([0,1])$ where for $f \in C([0,1]),\|f\| \equiv \sup \{|f(x)|: x \in[0,1]\}$. Show that $\cap_{n} U_{n}$ is a dense $G_{\delta}$ set of nowhere differentiable continuous functions. Thus every continuous function is uniformly close to one which is nowhere differentiable.
4. $\uparrow$ Suppose $f(x)=\sum_{k=1}^{\infty} u_{k}(x)$ where the convergence is uniform and each $u_{k}$ is a polynomial. Is it reasonable to conclude that $f^{\prime}(x)=\sum_{k=1}^{\infty} u_{k}^{\prime}(x)$ ? The answer is no. Use Problem 3 and the Weierstrass approximation theorem to show this.
5. Let $X$ be a normed linear space. $A \subseteq X$ is "weakly bounded" if for each $x^{*} \in$ $X^{\prime}, \sup \left\{\left|x^{*}(x)\right|: x \in A\right\}<\infty$, while $A$ is bounded if $\sup \{\|x\|: x \in A\}<\infty$. Show $A$ is weakly bounded if and only if it is bounded.
6. $\uparrow$ It turns out that the Fourier series sometimes converges to the function pointwise. Suppose $f$ is $2 \pi$ periodic and Holder continuous. That is $|f(x)-f(y)| \leq K|x-y|^{\theta}$ where $\theta \in(0,1]$. Show that if $f$ is like this, then the Fourier series converges to $f$ at every point. Next modify your argument to show that if at every point, $x$, $|f(x+)-f(y)| \leq K|x-y|^{\theta}$ for $y$ close enough to $x$ and larger than $x$ and

$$
|f(x-)-f(y)| \leq K|x-y|^{\theta}
$$

for every $y$ close enough to $x$ and smaller than $x$, then $S_{n} f(x) \rightarrow \frac{f(x+)+f(x-)}{2}$, the midpoint of the jump of the function. Hint: Use the Riemann Lebesgue lemma.
7. $\uparrow$ Let $Y=\{f$ such that $f$ is continuous, defined on $\mathbb{R}$, and $2 \pi$ periodic $\}$. Define $\|f\|_{Y}=\sup \{|f(x)|: x \in[-\pi, \pi]\}$. Show that $\left(Y,\| \|_{Y}\right)$ is a Banach space. Let $x \in \mathbb{R}$ and define $L_{n}(f)=S_{n} f(x)$. Show $L_{n} \in Y^{\prime}$ but $\lim _{n \rightarrow \infty}\left\|L_{n}\right\|=\infty$. Show that for each $x \in \mathbb{R}$, there exists a dense $G_{\delta}$ subset of $Y$ such that for $f$ in this set, $\left|S_{n} f(x)\right|$ is unbounded. Finally, show there is a dense $G_{\delta}$ subset of $Y$ having the property that $\left|S_{n} f(x)\right|$ is unbounded on the rational numbers. Hint: To do the first part, let $f(y)$ approximate $\operatorname{sgn}\left(D_{n}(x-y)\right)$. Here $\operatorname{sgn} r=1$ if $r>0,-1$ if $r<0$ and 0 if $r=0$. This rules out one possibility of the uniform boundedness principle. After this, show the countable intersection of dense $G_{\delta}$ sets must also be a dense $G_{\delta}$ set.
8. Let $\alpha \in(0,1]$. Define, for $X$ a compact subset of $\mathbb{R}^{p}$,

$$
C^{\alpha}\left(X ; \mathbb{R}^{n}\right) \equiv\left\{\mathbf{f} \in C\left(X ; \mathbb{R}^{n}\right): \rho_{\alpha}(\mathbf{f})+\|\mathbf{f}\| \equiv\|\mathbf{f}\|_{\alpha}<\infty\right\}
$$

where $\|\mathbf{f}\| \equiv \sup \{|\mathbf{f}(\mathbf{x})|: \mathbf{x} \in X\}$ and

$$
\rho_{\alpha}(\mathbf{f}) \equiv \sup \left\{\frac{|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{\alpha}}: \mathbf{x}, \mathbf{y} \in X, \mathbf{x} \neq \mathbf{y}\right\}
$$

Show that $\left(C^{\alpha}\left(X ; \mathbb{R}^{n}\right),\|\cdot\|_{\alpha}\right)$ is a complete normed linear space. This is called a Holder space. What would this space consist of if $\alpha>1$ ?
9. $\uparrow$ Let $X$ be the Holder functions which are periodic of period $2 \pi$. Define $L_{n} f(x)=$ $S_{n} f(x)$ where $L_{n}: X \rightarrow Y$ for $Y$ given in Problem 7. Show $\left\|L_{n}\right\|$ is bounded independent of $n$. Conclude that $L_{n} f \rightarrow f$ in $Y$ for all $f \in X$. In other words, for the Holder continuous and $2 \pi$ periodic functions, the Fourier series converges to the function uniformly. Hint: $L_{n} f(x)$ is given by

$$
L_{n} f(x)=\int_{-\pi}^{\pi} D_{n}(y) f(x-y) d y
$$

where $f(x-y)=f(x)+g(x, y)$ where $|g(x, y)| \leq C|y|^{\alpha}$. Use the fact the Dirichlet kernel integrates to one to write

$$
\begin{array}{r}
\left|\int_{-\pi}^{\pi} D_{n}(y) f(x-y) d y\right| \leq \overbrace{\left|\int_{-\pi}^{\pi} D_{n}(y) f(x) d y\right|}^{=|f(x)|} \\
+C\left|\int_{-\pi}^{\pi} \sin \left(\left(n+\frac{1}{2}\right) y\right)(g(x, y) / \sin (y / 2)) d y\right|
\end{array}
$$

Show the functions, $y \rightarrow g(x, y) / \sin (y / 2)$ are bounded in $L^{1}$ independent of $x$ and get a uniform bound on $\left\|L_{n}\right\|$. Now use a similar argument to show $\left\{L_{n} f\right\}$ is equicontinuous in addition to being uniformly bounded. In doing this you might proceed as follows. Show $\left|L_{n} f(x)-L_{n} f\left(x^{\prime}\right)\right| \leq$

$$
\begin{gathered}
\left|\int_{-\pi}^{\pi} D_{n}(y)\left(f(x-y)-f\left(x^{\prime}-y\right)\right) d y\right| \leq\|f\|_{\alpha}\left|x-x^{\prime}\right|^{\alpha} \\
+\left|\int_{-\pi}^{\pi} \sin \left(\left(n+\frac{1}{2}\right) y\right)\left(\frac{f(x-y)-f(x)-\left(f\left(x^{\prime}-y\right)-f\left(x^{\prime}\right)\right)}{\sin \left(\frac{y}{2}\right)}\right) d y\right|
\end{gathered}
$$

Then split this last integral into two cases, one for $|y|<\eta$ and one where $|y| \geq \eta$. If $L_{n} f$ fails to converge to $f$ uniformly, then there exists $\varepsilon>0$ and a subsequence, $n_{k}$ such that $\left\|L_{n_{k}} f-f\right\|_{\infty} \geq \varepsilon$ where this is the norm in $Y$ or equivalently the sup norm on $[-\pi, \pi]$. By the Arzela Ascoli theorem, there is a further subsequence, $L_{n_{k_{l}}} f$ which converges uniformly on $[-\pi, \pi]$. But by Problem $6 L_{n} f(x) \rightarrow f(x)$.
10. Let $X$ be a normed linear space and let $M$ be a convex open set containing 0 . Define $\rho(x)=\inf \left\{t>0: \frac{x}{t} \in M\right\}$. Show $\rho$ is a gauge function defined on $X$. This particular example is called a Minkowski functional. It is of fundamental importance in the study of locally convex topological vector spaces. A set $M$, is convex if $\lambda x+(1-$ $\lambda) y \in M$ whenever $\lambda \in[0,1]$ and $x, y \in M$.
11. $\uparrow$ The Hahn Banach theorem can be used to establish separation theorems. Let $M$ be an open convex set containing 0 . Let $x \notin M$. Show there exists $x^{*} \in X^{\prime}$ such that $\operatorname{Re} x^{*}(x) \geq 1>\operatorname{Re} x^{*}(y)$ for all $y \in M$. Hint: If $y \in M, \rho(y)<1$. Show this. If $x \notin M, \rho(x) \geq 1$. Try $f(\alpha x)=\alpha \rho(x)$ for $\alpha \in \mathbb{R}$. Then extend $f$ to the whole space using the Hahn Banach theorem and call the result $F$, show $F$ is continuous, then fix it so $F$ is the real part of $x^{*} \in X^{\prime}$.
12. A Banach space is said to be strictly convex if whenever $\|x\|=\|y\|$ and $x \neq y$, then $\left\|\frac{x+y}{2}\right\|<\|x\| . F: X \rightarrow X^{\prime}$ is said to be a duality map if it satisfies the following: a.) $\|F(x)\|=\|x\|$. b.) $F(x)(x)=\|x\|^{2}$. Show that if $X^{\prime}$ is strictly convex, then such a duality map exists. The duality map is an attempt to duplicate some of the features of the Riesz map in Hilbert space. This Riesz map $R$ is the map which takes a Hilbert space to its dual defined as follows: $R(x)(y)=(y, x)$. The Riesz representation theorem for Hilbert space says this map is onto. Hint: For an arbitrary Banach space, let

$$
F(x) \equiv\left\{x^{*}:\left\|x^{*}\right\| \leq\|x\| \text { and } x^{*}(x)=\|x\|^{2}\right\}
$$

Show $F(x) \neq \emptyset$ by using the Hahn Banach theorem on $f(\alpha x)=\alpha\|x\|^{2}$. Next show $F(x)$ is closed and convex. Finally show that you can replace the inequality in the definition of $F(x)$ with an equal sign. Now use strict convexity to show there is only one element in $F(x)$.
13. Prove the following theorem which is an improved version of the open mapping theorem, [14]. Let $X$ and $Y$ be Banach spaces and let $A \in \mathscr{L}(X, Y)$. Then the following are equivalent: $A X=Y, A$ is an open map. Note this gives the equivalence between $A$ being onto and $A$ being an open map. The open mapping theorem says that if $A$ is onto then it is open.
14. Suppose $D \subseteq X$ and $D$ is dense in $X$. Suppose $L: D \rightarrow Y$ is linear and $\|L x\| \leq K\|x\|$ for all $x \in D$. Show there is a unique extension of $L, \widetilde{L}$, defined on all of $X$ with $\|\widetilde{L} x\| \leq K\|x\|$ and $\widetilde{L}$ is linear. You do not get uniqueness when you use the Hahn Banach theorem. Therefore, in the situation of this problem, it is better to use this result.
15. $\uparrow$ A Banach space is uniformly convex if whenever $\left\|x_{n}\right\|,\left\|y_{n}\right\| \leq 1$ and $\left\|x_{n}+y_{n}\right\| \rightarrow 2$, it follows that $\left\|x_{n}-y_{n}\right\| \rightarrow 0$. Show uniform convexity implies strict convexity (See Problem 12). Hint: Suppose it is not strictly convex. Then there exist $\|x\|$ and $\|y\|$ both equal to 1 and $\left\|\frac{x_{n}+y_{n}}{2}\right\|=1$ consider $x_{n} \equiv x$ and $y_{n} \equiv y$, and use the conditions for uniform convexity to get a contradiction.
16. Show that a closed subspace of a reflexive Banach space is reflexive.
17. $x_{n}$ converges weakly to $x$ if for every $x^{*} \in X^{\prime}, x^{*}\left(x_{n}\right) \rightarrow x^{*}(x)$. $x_{n} \rightharpoonup x$ denotes weak convergence. Show that if $\left\|x_{n}-x\right\| \rightarrow 0$, then $x_{n} \rightharpoonup x$.
18. $\uparrow$ Show that if $X$ is uniformly convex, then if $x_{n} \rightharpoonup x$ and $\left\|x_{n}\right\| \rightarrow\|x\|$, it follows $\left\|x_{n}-x\right\| \rightarrow 0$. Hint: Use Lemma 12.3.8 to obtain $f \in X^{\prime}$ with $\|f\|=1$ and $f(x)=$ $\|x\|$. See Problem 15 for the definition of uniform convexity. Now by the weak convergence, you can argue that if $x \neq 0, f\left(x_{n} /\left\|x_{n}\right\|\right) \rightarrow f(x /\|x\|)$. You also might try to show this in the special case where $\left\|x_{n}\right\|=\|x\|=1$.
19. Suppose $L \in \mathscr{L}(X, Y)$ and $M \in \mathscr{L}(Y, Z)$. Show $M L \in \mathscr{L}(X, Z)$ and that $(M L)^{*}=$ $L^{*} M^{*}$.
20. This problem presents the Radon Nikodym theorem. Suppose $(\Omega, \mathscr{F})$ is a measurable space and that $\mu$ and $\lambda$ are two finite measures defined on $\mathscr{F}$. Suppose also that $\lambda \ll \mu$ which means that if $\mu(E)=0$ then $\lambda(E)=0$. Now define $\Lambda \in L^{2}(\Omega, \mu+\lambda)$ as $\Lambda(f) \equiv \int_{\Omega} f d \lambda$. Verify that this is really a bounded linear transformation on $L^{2}(\Omega, \mu+\lambda)$. Then by the Riesz representation theorem, Theorem 12.2.9, there exists $h \in L^{2}(\Omega, \mu+\lambda)$ such that for all $f \in L^{2}(\Omega, \mu+\lambda), \int_{\Omega} f d \lambda=\int_{\Omega} h f d(\lambda+\mu)$. Verify that $h$ has almost all values real and contained in $[0,1)$. This will use $\lambda \ll \mu$. Then note the following: $\int_{\Omega} f(1-h) d \lambda=\int_{\Omega} h f d \mu$. Now for $E \in \mathscr{F}$, let $f_{n}=$ $\mathscr{X}_{E} \sum_{k=0}^{n-1} h^{k}$. Thus $\int_{E}\left(1-h^{n}\right) d \lambda=\int_{E} \sum_{k=1}^{n} h^{k} d \mu$. Use monotone convergence to show that $\lambda(E)=\int_{E} g d \mu, g=\sum_{k=1}^{\infty} h^{k}$. Show that $g \in L^{1}(\Omega, \mu)$. Formulate this as a theorem. It is called the Radon Nikodym theorem. This elegant approach is due to Von Neumann.
21. Let $H$ be a separable Hilbert space and let $D=\left\{d_{n}\right\}_{n=1}^{\infty}$ be a orthonormal set such that $\bar{D}=H$. Show using Theorem 12.2.13 that $(x, y)=\sum_{k=1}^{\infty}\left(x, d_{k}\right) \overline{\left(y, d_{k}\right)}$.
22. Given an inner product space $H$ show that every orthonormal set of vectors is linearly independent. Now suppose $V$ is a finite dimensional subspace of $H$. Show that $V$ is span $\left(d_{1}, \ldots, d_{n}\right)$ where $\left\{d_{k}\right\}_{k=1}^{n}$ is a maximal orthonormal set of vectors contained in $V$ and that $n$ is the dimension of $V$. Show that $P x \equiv \sum_{k=1}^{n}\left(x, d_{k}\right) d_{k}$ is the projection of $x$ to $V$. That is $\left|x-\sum_{k=1}^{n}\left(x, d_{k}\right) d_{k}\right| \leq\left|x-\sum_{k=1}^{n} a_{k} d_{k}\right|$ for every choice of $a_{k}$. Also show that this projection map is unique.
23. Show that if $g \in C([-\pi, \pi])$ with $g(-\pi)=g(\pi)$ then for $x$ the point on the unit circle $S^{1}$ determined in the usual way by $x$, then $g: S^{1} \rightarrow \mathbb{C}$ is continuous. Letting $\mathscr{A}$ be the algebra which consists of all products and linear combinations of the functions $\left\{e^{i n x}\right\}_{n \in \mathbb{Z}}$, show that $\mathscr{A}$ is dense in $D$ consisting of the set of functions of $C([-\pi, \pi])$ with $g(-\pi)=g(\pi)$. Now if $f \in L^{2}(-\pi, \pi)$, show there is $g \in D$ such that $\|f-g\|_{L^{2}(-\pi, \pi)}<\varepsilon$. Show the functions $x \rightarrow \frac{1}{\sqrt{2 \pi}} e^{i n x}$ for $n \in \mathbb{Z}$ are an orthonormal set of functions for the inner product $(f, g) \equiv \int_{-\pi}^{\pi} f \bar{g} d x$ and that therefore, $S_{n} f(x)$ gives the best approximation to $f$ in $L^{2}(-\pi, \pi)$ out of all linear combinations of $e^{i k x}$ for $|k| \leq n$. Conclude from this that $\left\|f-S_{n} f\right\|_{L^{2}(-\pi, \pi)} \rightarrow 0$. Also explain why $\left\{e^{i n x}\right\}_{n \in \mathbb{Z}}$ is a maximal orthonormal set.

## Chapter 13

## Representation Theorems

### 13.1 Radon Nikodym Theorem

This chapter is on various representation theorems. The first theorem, the Radon Nikodym Theorem, is a representation theorem for one measure in terms of another. This important theorem represents one measure in terms of another. It is Theorem 7.11.9 on Page 180 or Problem 19 on Page 317, this problem utilizing the approach of Von Neumann which is also featured in [40].
Definition 13.1.1 Let $\mu$ and $\lambda$ be two measures defined on a $\sigma$-algebra $\mathscr{S}$, of subsets of a set, $\Omega$. $\lambda$ is absolutely continuous with respect to $\mu$, written as $\lambda \ll \mu$, if $\lambda(E)=0$ whenever $\mu(E)=0$. A complex measure $\lambda$ defined on a $\sigma$-algebra $\mathscr{S}$ is one which has the property that if the $E_{i}$ are distinct and measurable, then $\lambda\left(\cup_{i} E_{i}\right)=\sum_{i} \lambda\left(E_{i}\right) \in \mathbb{C}$.

Recall Corollary7.11.13 on Page 181. I am stating it next for convenience.
Corollary 13.1.2 Let $\lambda$ be a signed $\sigma$ finite measure and let $\lambda \ll \mu$ meaning that if $\mu(E)=0 \Rightarrow \lambda(E)=0$. Here assume that $\mu$ is a finite measure. Then there exists $h \in L^{1}$ such that $\lambda(E)=\int_{E} h d \mu$.

There is an easy corollary to this.
Corollary 13.1.3 Let $\lambda$ be a complex measure and $\lambda \ll \mu$ for $\mu$ a finite measure. Then there exists $h \in L^{1}$ such that $\lambda(E)=\int_{E} h d \mu$.

Proof: Let $(\operatorname{Re} \lambda)(E)=\operatorname{Re}(\lambda(E))$ with $\operatorname{Im} \lambda$ defined similarly. Then these are signed measures and so there are functions $f_{1}, f_{2}$ in $L^{1}$ such that $\operatorname{Re} \lambda(E)=\int_{E} f_{1} d \mu, \operatorname{Im} \lambda(E)=$ $\int_{E} f_{2} d \mu$. Then $h \equiv f_{1}+i f_{2}$ satisfies the necessary condition.

More general versions are available. To see one of these, one can read the treatment in Hewitt and Stromberg [22]. This involves the notion of decomposable measure spaces, a generalization of $\sigma$ finite.

### 13.2 Vector Measures

The next topic will use the Radon Nikodym theorem. It is the topic of vector and complex measures. The main interest is in complex measures although a vector measure can have values in any topological vector space. Whole books have been written on this subject. See for example the book by Diestal and Uhl [13] titled Vector measures.

## Definition 13.2.1 Let $(V,\|\cdot\|)$ be a normed linear space and let $(\Omega, \mathscr{S})$ be a mea-

 sure space. A function $\mu: \mathscr{S} \rightarrow V$ is a vector measure if $\mu$ is countably additive. That is, if $\left\{E_{i}\right\}_{i=1}^{\infty}$ is a sequence of disjoint sets of $\mathscr{S}, \mu\left(\cup_{i=1}^{\infty} E_{i}\right)=\sum_{i=1}^{\infty} \mu\left(E_{i}\right)$.Note that it makes sense to take finite sums because it is given that $\mu$ has values in a vector space in which vectors can be summed. In the above, $\mu\left(E_{i}\right)$ is a vector. It might be a point in $\mathbb{R}^{n}$ or in any other vector space. In many of the most important applications, it is a vector in some sort of function space which may be infinite dimensional. The infinite sum has the usual meaning. That is $\sum_{i=1}^{\infty} \mu\left(E_{i}\right)=\lim _{n \rightarrow \infty} \sum_{i=1}^{n} \mu\left(E_{i}\right)$ where the limit takes place relative to the norm on $V$.

Definition 13.2.2 Let $(\Omega, \mathscr{S})$ be a measure space and let $\mu$ be a vector measure defined on $\mathscr{S}$. A subset, $\pi(E)$, of $\mathscr{S}$ is called a partition of $E$ if $\pi(E)$ consists of finitely many disjoint sets of $\mathscr{S}$ and $\cup \pi(E)=E$. Let

$$
|\mu|(E)=\sup \left\{\sum_{F \in \pi(E)}\|\mu(F)\|: \pi(E) \text { is a partition of } E\right\} .
$$

$|\mu|$ is called the total variation of $\mu$.
The next theorem may seem a little surprising. It states that, if finite, the total variation is a nonnegative measure.

Theorem 13.2.3 If $|\mu|(\Omega)<\infty$, then $|\mu|$ is a measure on $\mathscr{S}$. Even if $|\mu|(\Omega)=$ $\infty,|\mu|\left(\cup_{i=1}^{\infty} E_{i}\right) \leq \sum_{i=1}^{\infty}|\mu|\left(E_{i}\right)$. That is $|\mu|$ is subadditive and $|\mu|(A) \leq|\mu|(B)$ whenever $A, B \in \mathscr{S}$ with $A \subseteq B$.

Proof: Consider the last claim. Let $a<|\mu|(A)$ and let $\pi(A)$ be a partition of $A$ such that

$$
a<\sum_{F \in \pi(A)}\|\mu(F)\| .
$$

Then $\pi(A) \cup\{B \backslash A\}$ is a partition of $B$ and

$$
|\mu|(B) \geq \sum_{F \in \pi(A)}\|\mu(F)\|+\|\mu(B \backslash A)\|>a .
$$

Since this is true for all such $a$, it follows $|\mu|(B) \geq|\mu|(A)$ as claimed.
Let $\left\{E_{j}\right\}_{j=1}^{\infty}$ be a sequence of disjoint sets of $\mathscr{S}$ and let $E_{\infty}=\cup_{j=1}^{\infty} E_{j}$. Then letting $a<|\mu|\left(E_{\infty}\right)$, it follows from the definition of total variation there exists a partition of $E_{\infty}$, $\pi\left(E_{\infty}\right)=\left\{A_{1}, \cdots, A_{n}\right\}$ such that $a<\sum_{i=1}^{n}\left\|\mu\left(A_{i}\right)\right\|$. Also,

$$
A_{i}=\cup_{j=1}^{\infty} A_{i} \cap E_{j}
$$

and so by the triangle inequality, $\left\|\mu\left(A_{i}\right)\right\| \leq \sum_{j=1}^{\infty}\left\|\mu\left(A_{i} \cap E_{j}\right)\right\|$. Therefore, by the above, and either Fubini's theorem or Lemma 1.11.3 on Page 28

$$
a<\sum_{i=1}^{n} \overbrace{\sum_{j=1}^{\infty}\left\|\mu\left(A_{i} \cap E_{j}\right)\right\|}^{\geq\left\|\mu\left(A_{i}\right)\right\|}=\sum_{j=1}^{\infty} \sum_{i=1}^{n}\left\|\mu\left(A_{i} \cap E_{j}\right)\right\| \leq \sum_{j=1}^{\infty}|\mu|\left(E_{j}\right)
$$

because $\left\{A_{i} \cap E_{j}\right\}_{i=1}^{n}$ is a partition of $E_{j}$.
Since $a$ is arbitrary, this shows $|\mu|\left(\cup_{j=1}^{\infty} E_{j}\right) \leq \sum_{j=1}^{\infty}|\mu|\left(E_{j}\right)$. If the sets, $E_{j}$ are not disjoint, let $F_{1}=E_{1}$ and if $F_{n}$ has been chosen, let $F_{n+1} \equiv E_{n+1} \backslash \cup_{i=1}^{n} E_{i}$. Thus the sets, $F_{i}$ are disjoint and $\cup_{i=1}^{\infty} F_{i}=\cup_{i=1}^{\infty} E_{i}$. Therefore,

$$
|\mu|\left(\cup_{j=1}^{\infty} E_{j}\right)=|\mu|\left(\cup_{j=1}^{\infty} F_{j}\right) \leq \sum_{j=1}^{\infty}|\mu|\left(F_{j}\right) \leq \sum_{j=1}^{\infty}|\mu|\left(E_{j}\right)
$$

and proves $|\mu|$ is always subadditive as claimed regardless of whether $|\mu|(\Omega)<\infty$.

Now suppose $|\mu|(\Omega)<\infty$ and let $E_{1}$ and $E_{2}$ be sets of $\mathscr{S}$ such that $E_{1} \cap E_{2}=\emptyset$ and let $\left\{A_{1}^{i} \cdots A_{n_{i}}^{i}\right\}=\pi\left(E_{i}\right)$, a partition of $E_{i}$ which is chosen such that

$$
|\mu|\left(E_{i}\right)-\varepsilon<\sum_{j=1}^{n_{i}}\left\|\mu\left(A_{j}^{i}\right)\right\| i=1,2
$$

Such a partition exists because of the definition of the total variation. Consider the sets which are contained in either of $\pi\left(E_{1}\right)$ or $\pi\left(E_{2}\right)$, it follows this collection of sets is a partition of $E_{1} \cup E_{2}$ denoted by $\pi\left(E_{1} \cup E_{2}\right)$. Then by the above inequality and the definition of total variation,

$$
|\mu|\left(E_{1} \cup E_{2}\right) \geq \sum_{F \in \pi\left(E_{1} \cup E_{2}\right)}\|\mu(F)\|>|\mu|\left(E_{1}\right)+|\mu|\left(E_{2}\right)-2 \varepsilon,
$$

which shows that since $\varepsilon>0$ was arbitrary,

$$
\begin{equation*}
|\mu|\left(E_{1} \cup E_{2}\right) \geq|\mu|\left(E_{1}\right)+|\mu|\left(E_{2}\right) \tag{13.1}
\end{equation*}
$$

Then 13.1 implies that whenever the $E_{i}$ are disjoint, $|\mu|\left(\cup_{j=1}^{n} E_{j}\right) \geq \sum_{j=1}^{n}|\mu|\left(E_{j}\right)$. Therefore,

$$
\sum_{j=1}^{\infty}|\mu|\left(E_{j}\right) \geq|\mu|\left(\cup_{j=1}^{\infty} E_{j}\right) \geq|\mu|\left(\cup_{j=1}^{n} E_{j}\right) \geq \sum_{j=1}^{n}|\mu|\left(E_{j}\right)
$$

Since $n$ is arbitrary, $|\mu|\left(\cup_{j=1}^{\infty} E_{j}\right)=\sum_{j=1}^{\infty}|\mu|\left(E_{j}\right)$ which shows that $|\mu|$ is a measure as claimed.

In the case that $\mu$ is a complex measure, it is always the case that $|\mu|(\Omega)<\infty$ this is shown soon. However, first is an interesting corollary. It concerns the case that $\mu$ is only finitely additive.

Corollary 13.2.4 Suppose $(\Omega, \mathscr{F})$ is a set with a $\sigma$ algebra of subsets $\mathscr{F}$ and suppose $\mu: \mathscr{F} \rightarrow \mathbb{C}$ is only finitely additive. That is, $\mu\left(\cup_{i=1}^{n} E_{i}\right)=\sum_{i=1}^{n} \mu\left(E_{i}\right)$ whenever the $E_{i}$ are disjoint. Then $|\mu|$, defined in the same way as above, is also finitely additive provided $|\mu|$ is finite.

Proof: Say $E \cap F=\emptyset$ for $E, F \in \mathscr{F}$. Let $\pi(E), \pi(F)$ suitable partitions for which the following holds.

$$
|\mu|(E \cup F) \geq \sum_{A \in \pi(E)}|\mu(A)|+\sum_{B \in \pi(F)}|\mu(B)| \geq|\mu|(E)+|\mu|(F)-2 \varepsilon .
$$

Since $\varepsilon$ is arbitrary, $|\mu|(E \cap F) \geq|\mu|(E)+|\mu|(F)$. Similar considerations apply to any finite union of disjoint sets. That is, if the $E_{i}$ are disjoint, then $|\mu|\left(\cup_{i=1}^{n} E_{i}\right) \geq \sum_{i=1}^{n}|\mu|\left(E_{i}\right)$.

Now let $E=\cup_{i=1}^{n} E_{i}$ where the $E_{i}$ are disjoint. Then letting $\pi(E)$ be a suitable partition of $E$,

$$
|\mu|(E)-\varepsilon \leq \sum_{F \in \pi(E)}|\mu(F)|,
$$

it follows that

$$
|\mu|(E) \leq \varepsilon+\sum_{F \in \pi(E)}|\mu(F)|=\varepsilon+\sum_{F \in \pi(E)}\left|\sum_{i=1}^{n} \mu\left(F \cap E_{i}\right)\right|
$$

$$
\leq \varepsilon+\sum_{i=1}^{n} \sum_{F \in \pi(E)}\left|\mu\left(F \cap E_{i}\right)\right| \leq \varepsilon+\sum_{i=1}^{n}|\mu|\left(E_{i}\right)
$$

Since $\varepsilon$ is arbitrary, this shows $|\mu|\left(\cup_{i=1}^{n} E_{i}\right) \leq \sum_{i=1}^{n}|\mu|\left(E_{i}\right)$. Thus $|\mu|$ is finitely additive.
In the case that $\mu$ is a complex measure, it is always the case that $|\mu|(\Omega)<\infty$. First is a lemma.

Lemma 13.2.5 Suppose $\mu$ is a real valued measure (signed measure by Definition 7.11.2). Then $|\mu|$ is a finite measure.

Proof: Suppose $\mu: \mathscr{F} \rightarrow \mathbb{R}$ is a vector measure (signed measure by Definition 7.11.2). By the Hahn decomposition, Theorem 7.11.5 on Page $179, \Omega=P \cup N$ where $P$ is a positive set and $N$ is a negative one. Then on $N,-\mu$ is a measure and if $A \subseteq B$ and $A, B$ measurable subsets of $N$, then $-\mu(A) \leq-\mu(B)$. Similarly $\mu$ is a measure on $P$.

$$
\begin{gathered}
\sum_{F \in \pi(\Omega)}|\mu(F)| \leq \sum_{F \in \pi(\Omega)}(|\mu(F \cap P)|+|\mu(F \cap N)|) \\
=\sum_{F \in \pi(\Omega)} \mu(F \cap P)+\sum_{F \in \pi(\Omega)}-\mu(F \cap N) \\
=\mu\left(\left(\cup_{F \in \pi(\Omega)} F\right) \cap P\right)+-\mu\left(\left(\cup_{F \in \pi(\Omega)} F\right) \cap N\right) \leq \mu(P)+|\mu(N)|
\end{gathered}
$$

It follows that $|\mu|(\Omega)<\mu(P)+|\mu(N)|$ and so $|\mu|$ has finite total variation.
Theorem 13.2.6 Suppose $\mu$ is a complex measure on $(\Omega, \mathscr{S})$ where $\mathscr{S}$ is a $\sigma$ algebra of subsets of $\Omega$. That is, whenever $\left\{E_{i}\right\}$ is a sequence of disjoint sets of $\mathscr{S}$,

$$
\mu\left(\cup_{i=1}^{\infty} E_{i}\right)=\sum_{i=1}^{\infty} \mu\left(E_{i}\right)
$$

Then $|\mu|(\Omega)<\infty$.
Proof: If $\mu$ is a vector measure with values in $\mathbb{C}, \operatorname{Re} \mu$ and $\operatorname{Im} \mu$ have values in $\mathbb{R}$. Then

$$
\begin{aligned}
\sum_{F \in \pi(\Omega)}|\mu(F)| & \leq \sum_{F \in \pi(\Omega)}|\operatorname{Re} \mu(F)|+|\operatorname{Im} \mu(F)| \\
& =\sum_{F \in \pi(\Omega)}|\operatorname{Re} \mu(F)|+\sum_{F \in \pi(\Omega)}|\operatorname{Im} \mu(F)| \\
& \leq|\operatorname{Re} \mu|(\Omega)+|\operatorname{Im} \mu|(\Omega)<\infty
\end{aligned}
$$

thanks to Lemma 13.2.5.
Theorem 13.2.7 Let $(\Omega, \mathscr{S})$ be a measure space and let $\lambda: \mathscr{S} \rightarrow \mathbb{C}$ be a complex vector measure. Thus $|\lambda|(\Omega)<\infty$. Let $\mu: \mathscr{S} \rightarrow[0, \mu(\Omega)]$ be a finite measure such that $\lambda \ll \mu$. Then there exists a unique $f \in L^{1}(\Omega)$ such that for all $E \in \mathscr{S}$,

$$
\int_{E} f d \mu=\lambda(E)
$$

Proof: It is clear that $\operatorname{Re} \lambda$ and $\operatorname{Im} \lambda$ are real-valued vector measures on $\mathscr{S}$. Since $|\lambda|(\Omega)<\infty$, it follows easily that $|\operatorname{Re} \lambda|(\Omega)$ and $|\operatorname{Im} \lambda|(\Omega)<\infty$. This is clear because

$$
|\lambda(E)| \geq|\operatorname{Re} \lambda(E)|,|\operatorname{Im} \lambda(E)|
$$

Therefore, each of

$$
\frac{|\operatorname{Re} \lambda|+\operatorname{Re} \lambda}{2}, \frac{|\operatorname{Re} \lambda|-\operatorname{Re}(\lambda)}{2}, \frac{|\operatorname{Im} \lambda|+\operatorname{Im} \lambda}{2}, \text { and } \frac{|\operatorname{Im} \lambda|-\operatorname{Im}(\lambda)}{2}
$$

are finite measures on $\mathscr{S}$. It is also clear that each of these finite measures are absolutely continuous with respect to $\mu$ and so there exist unique nonnegative functions in $L^{1}(\Omega), f_{1}, f_{2}, g_{1}, g_{2}$ such that for all $E \in \mathscr{S}$,

$$
\begin{aligned}
\frac{1}{2}(|\operatorname{Re} \lambda|+\operatorname{Re} \lambda)(E) & =\int_{E} f_{1} d \mu \\
\frac{1}{2}(|\operatorname{Re} \lambda|-\operatorname{Re} \lambda)(E) & =\int_{E} f_{2} d \mu \\
\frac{1}{2}(|\operatorname{Im} \lambda|+\operatorname{Im} \lambda)(E) & =\int_{E} g_{1} d \mu \\
\frac{1}{2}(|\operatorname{Im} \lambda|-\operatorname{Im} \lambda)(E) & =\int_{E} g_{2} d \mu
\end{aligned}
$$

Now let $f=f_{1}-f_{2}+i\left(g_{1}-g_{2}\right)$.
Theorem 13.2.8 The following hold where $\lambda$ will be a complex measure so $|\lambda|$ is finite and $\mu$ will be a finite measure, both defined on $\mathscr{S}$.

1. If $\mu$ is a finite nonnegative measure on $\mathscr{S}$, and $\lambda(E) \equiv \int_{E} h d \mu$ for $h \in L^{1}(\Omega, \mu)$, then $|\lambda|(E)=\int_{E}|h| d \mu$.
2. If $\left|\int_{E} f d \mu\right| \leq \mu(E)$ for all $E \in \mathscr{S}$, then $|f| \leq 1$ a.e. If $\left|\int_{E} f d \mu\right|=\mu(E)$ for all $E \in \mathscr{S}$, then $|f|=1 \mu$ a.e.
3. Letting $g$ be such that $\lambda(E)=\int_{E} g d|\lambda|$, it follows that $|g|=1$ for $|\lambda|$ a.e. If also $\lambda(E)=\int_{E} h d \mu$ for $h \in L^{1}(\Omega, \mu)$, then $|h|=\bar{g} h \mu$ a.e.

Proof: 1.) Letting $\pi(E)=\left\{F_{1}, \ldots, F_{n}\right\}$,

$$
\sum_{k=1}^{n}\left|\lambda\left(F_{k}\right)\right|=\sum_{k=1}^{n}\left|\int_{F_{k}} h d \mu\right| \leq \sum_{k=1}^{n} \int_{F_{k}}|h| d \mu=\int_{E}|h| d \mu
$$

and so, taking the sup for all such partitions, $|\lambda|(E) \leq \int_{E}|h| d \mu$. Let simple functions $s_{n} \rightarrow \operatorname{sgn}(h)$ where $|\operatorname{sgn}(h)|=1$ and $\operatorname{sgn}(h) h=|h|$. We can assume also that $\left|s_{n}\right| \leq 1$. Say $s_{n}=\sum_{i=1}^{m_{n}} c_{i}^{n} \mathscr{X}_{F_{i}^{n}}$ where the $F_{i}^{n}$ are disjoint, $\left\{F_{i}^{n}\right\}_{i=1}^{m_{n}}$ a partition of $E$. Then $|\lambda|(E) \leq$

$$
\begin{gathered}
\int_{E}|h| d \mu=\int_{E} \operatorname{sgn}(h) h d \mu=\lim _{n \rightarrow \infty} \int_{E} s_{n} h d \mu=\lim _{n \rightarrow \infty} \sum_{i=1}^{m_{n}} \int_{F_{i}^{n}} c_{i}^{n} h d \mu \\
\leq \lim _{n \rightarrow \infty}\left|\sum_{i=1}^{m_{n}} \int_{F_{i}^{n}} c_{i}^{n} h d \mu\right| \leq \lim \inf _{n \rightarrow \infty} \sum_{i=1}^{m_{n}}\left|\int_{F_{i}^{n}} h d \mu\right| \leq \sup _{n} \sum_{i=1}^{m_{n}}\left|\lambda\left(F_{i}^{n}\right)\right| \leq|\lambda|(E)
\end{gathered}
$$

so $|\lambda|(E)=\int_{E}|h| d \mu$.
2.) Now let $\left|\int_{E} f d \mu\right| \leq \mu(E)$ for all $E$. Consider the following picture where

$$
B(p, r) \cap B(0,1)=\emptyset
$$



Let $E=f^{-1}(B(p, r))$. In fact $\mu(E)=0$. If $\mu(E) \neq 0$ then

$$
\left|\frac{1}{\mu(E)} \int_{E} f d \mu-p\right|=\left|\frac{1}{\mu(E)} \int_{E}(f-p) d \mu\right| \leq \frac{1}{\mu(E)} \int_{E}|f-p| d \mu<r
$$

because on $E,|f(\omega)-p|<r$. Hence $\frac{1}{\mu(E)} \int_{E} f d \mu$ is closer to $p$ than $r$ and so

$$
\left|\frac{1}{\mu(E)} \int_{E} f d \mu\right|>1
$$

Refer to the picture. However, this contradicts the assumption of the lemma. It follows $\mu(E)=0$. Since the set of complex numbers $z$ such that $|z|>1$ is an open set, it equals the union of countably many balls, $\left\{B_{i}\right\}_{i=1}^{\infty}$. Therefore,

$$
\mu\left(f^{-1}(\{z \in \mathbb{C}:|z|>1\})=\mu\left(\cup_{k=1}^{\infty} f^{-1}\left(B_{k}\right)\right) \leq \sum_{k=1}^{\infty} \mu\left(f^{-1}\left(B_{k}\right)\right)=0\right.
$$

Thus $|f(\omega)| \leq 1$ a.e. as claimed. If $\left|\int_{E} f d \mu\right|=\mu(E)$ for all $E$ then from Part 1.), $\mu(E)=$ $|\mu|(E)=\left|\int_{E} f d \mu\right|=\int_{E}|f| d \mu$ and so $|f|=1$ a.e.
3.) Clearly $\lambda \ll|\lambda|$ so there exists a unique $g$ in $L^{1}(\Omega, \lambda)$ such that $\lambda(E)=\int_{E} g d|\lambda|$. From Part 1.), $|\lambda|(E)=\int_{E}|g| d|\lambda|$ for all $E$ and so $|g|=1$ a.e. Now if also $\lambda(E)=$ $\int_{E} g d|\lambda|=\int_{E} h d \mu$, let $s_{n}$ be simple functions converging pointwise to $\bar{g}$. Then $\int_{E} g s_{n} d|\lambda|=$ $\int_{E} s_{n} h d \mu$. From the dominated convergence theorem, $\int_{E} d|\lambda|=\int_{E} \bar{g} h d \mu$. Thus $\bar{g} h \geq 0$ $\mu$ a.e. and $|\bar{g}|=1$. Therefore, $|h|=|\bar{g} h|=\bar{g} h$. More formally, it is assumed $g d|\lambda|=$ $h d \mu,|g|=1$ so $d|\lambda|=\bar{g} h d \mu$ and so we must have $\bar{g} h \geq 0$ hence equal to $|h|$.

### 13.3 The Dual Space of $L^{p}(\Omega)$

This is on representation of the dual space of $L^{p}(\Omega)$.
Theorem 13.3.1 (Riesz representation theorem) Let $\infty>p>1$ and let $(\Omega, \mathscr{S}, \mu)$ be a finite measure space. If $\Lambda \in\left(L^{p}(\Omega)\right)^{\prime}$, then there exists a unique $h \in L^{q}(\Omega)\left(\frac{1}{p}+\frac{1}{q}=1\right)$ such that

$$
\Lambda f=\int_{\Omega} h f d \mu
$$

This function satisfies $\|h\|_{q}=\|\Lambda\|$ where $\|\Lambda\|$ is the operator norm of $\Lambda$.

Proof: (Uniqueness) If $h_{1}$ and $h_{2}$ in $L^{q}$ both represent $\Lambda$, consider

$$
f=\left|h_{1}-h_{2}\right|^{q-2}\left(\overline{h_{1}}-\overline{h_{2}}\right),
$$

where $\bar{h}$ denotes complex conjugation. By Holder's inequality, it is easy to see that $f \in$ $L^{p}(\Omega)$. Thus
$0=\Lambda f-\Lambda f=\int h_{1}\left|h_{1}-h_{2}\right|^{q-2}\left(\overline{h_{1}}-\overline{h_{2}}\right)-h_{2}\left|h_{1}-h_{2}\right|^{q-2}\left(\overline{h_{1}}-\overline{h_{2}}\right) d \mu=\int\left|h_{1}-h_{2}\right|^{q} d \mu$.
Therefore $h_{1}=h_{2}$ and this proves uniqueness.
Now let $\lambda(E)=\Lambda\left(\mathscr{X}_{E}\right)$. Since this is a finite measure space, $\mathscr{X}_{E}$ is an element of $L^{p}(\Omega)$ and so it makes sense to write $\Lambda\left(\mathscr{X}_{E}\right)$. Is $\lambda$ a complex measure?

If $\left\{E_{i}\right\}_{i=1}^{\infty}$ is a sequence of disjoint sets of $\mathscr{S}$, let $F_{n}=\cup_{i=1}^{n} E_{i}, F=\cup_{i=1}^{\infty} E_{i}$. Then by the Dominated Convergence theorem, $\left\|\mathscr{X}_{F_{n}}-\mathscr{X}_{F}\right\|_{p} \rightarrow 0$. Therefore, by continuity of $\Lambda$,

$$
\lambda(F) \equiv \Lambda\left(\mathscr{X}_{F}\right)=\lim _{n \rightarrow \infty} \Lambda\left(\mathscr{X}_{F_{n}}\right)=\lim _{n \rightarrow \infty} \sum_{k=1}^{n} \Lambda\left(\mathscr{X}_{E_{k}}\right)=\sum_{k=1}^{\infty} \lambda\left(E_{k}\right) .
$$

This shows $\lambda$ is a complex measure.
It is also clear from the definition of $\lambda$ that $\lambda \ll \mu$. Therefore, by the Radon Nikodym theorem, there exists $h \in L^{1}(\Omega)$ with $\lambda(E)=\int_{E} h d \mu=\Lambda\left(\mathscr{X}_{E}\right)$. Actually $h \in L^{q}$ and satisfies the other conditions above. This is shown next.

Let $s=\sum_{i=1}^{m} c_{i} \mathscr{X}_{E_{i}}$ be a simple function. Then since $\Lambda$ is linear,

$$
\begin{equation*}
\Lambda(s)=\sum_{i=1}^{m} c_{i} \Lambda\left(\mathscr{X}_{E_{i}}\right)=\sum_{i=1}^{m} c_{i} \int_{E_{i}} h d \mu=\int h s d \mu . \tag{13.2}
\end{equation*}
$$

Claim: If $f$ is uniformly bounded and measurable, then $\Lambda(f)=\int h f d \mu$.
Proof of claim: Since $f$ is bounded and measurable, there exists a sequence of simple functions, $\left\{s_{n}\right\}$ which converges to $f$ pointwise and in $L^{p}(\Omega),\left|s_{n}\right| \leq|f|$. This follows from Theorem 6.1.10 on Page 140 upon breaking $f$ up into positive and negative parts of real and complex parts. In fact this theorem gives uniform convergence. Then

$$
\Lambda(f)=\lim _{n \rightarrow \infty} \Lambda\left(s_{n}\right)=\lim _{n \rightarrow \infty} \int h s_{n} d \mu=\int h f d \mu
$$

the first equality holding because of continuity of $\Lambda$, the second following from 13.2 and the third holding by the dominated convergence theorem.

This is a very nice formula but it still has not been shown that $h \in L^{q}(\Omega)$.
Let $E_{n}=\{x:|h(x)| \leq n\}$. Thus $\left|h \mathscr{X}_{E_{n}}\right| \leq n$. Then $\left|h \mathscr{X}_{E_{n}}\right|^{q-2}\left(\bar{h} \mathscr{X}_{E_{n}}\right) \in L^{p}(\Omega)$. By the claim, it follows that

$$
\begin{gathered}
\left\|h \mathscr{X}_{E_{n}}\right\|_{q}^{q}=\int h\left|h \mathscr{X}_{E_{n}}\right|^{q-2}\left(\bar{h} \mathscr{X}_{E_{n}}\right) d \mu=\Lambda\left(\left|h \mathscr{X}_{E_{n}}\right|^{q-2}\left(\bar{h} \mathscr{X}_{E_{n}}\right)\right) \\
\leq\|\Lambda\|\left\|\left|h \mathscr{X}_{E_{n}}\right|^{q-2}\left(\bar{h} \mathscr{X}_{E_{n}}\right)\right\|_{p}=\left(\int\left|h \mathscr{X}_{E_{n}}\right|^{q} d \mu\right)^{1 / p}=\|\Lambda\|\left\|h \mathscr{X}_{E_{n}}\right\|_{q}^{\frac{q}{p}},
\end{gathered}
$$

because $q-1=q / p$ and so it follows that $\left\|h \mathscr{X}_{E_{n}}\right\|_{q} \leq\|\Lambda\|$. Letting $n \rightarrow \infty$, the monotone convergence theorem implies $\|h\|_{q} \leq\|\Lambda\|$.

Now that $h$ is in $L^{q}(\Omega)$, it follows from the density of the simple functions, Theorem 9.4.1 on Page 230 , that $\Lambda f=\int h f d \mu$ for all $f \in L^{p}(\Omega)$. It only remains to verify the last claim that $\|h\|_{q}=\|\Lambda\|$. However, from the definition and Holder's inequality and $\|h\|_{q} \leq\|\Lambda\|,\|\Lambda\| \equiv \sup \left\{\int h f:\|f\|_{p} \leq 1\right\} \leq\|h\|_{q} \leq\|\Lambda\|$.

Next consider the case of $L^{1}(\Omega)$. What is its dual space? I will assume here that $(\Omega, \mu)$ is a finite measure space. The argument will be a little different than the one given above for $L^{p}$ with $p>1$.
Theorem 13.3.2 (Riesz representation theorem) Let $(\Omega, \mathscr{S}, \mu)$ be a finite measure space. If $\Lambda \in\left(L^{1}(\Omega)\right)^{\prime}$, then there exists a unique $h \in L^{\infty}(\Omega)$ such that

$$
\Lambda(f)=\int_{\Omega} h f d \mu
$$

for all $f \in L^{1}(\Omega)$. If $h$ is the function in $L^{\infty}(\Omega)$ representing $\Lambda \in\left(L^{1}(\Omega)\right)^{\prime}$, then $\|h\|_{\infty}=$ $\|\Lambda\|$.

Proof: For measurable $E$, it follows that $\mathscr{X}_{E} \in L^{1}(\Omega, \mu)$. Define a measure $\lambda(E) \equiv$ $\Lambda\left(\mathscr{X}_{E}\right)$. This is a complex measure as in the proof of Theorem 13.3.1. Then it follows from Corollary 13.1.3 that there exists a unique $h \in L^{1}(\Omega, \mu)$ such that

$$
\begin{equation*}
\lambda(E) \equiv \Lambda\left(\mathscr{X}_{E}\right)=\int_{E} h d \mu \tag{13.3}
\end{equation*}
$$

I will show that $h \in L^{\infty}(\Omega, \mu)$ and that $\Lambda(f)=\int h f d \mu$ for all $f \in L^{1}(\Omega)$. First of all, 13.3 implies that for all simple functions $s, \Lambda(s)=\int \operatorname{shd} \mu$. Let $\left\{s_{n}\right\}$ be a sequence of simple functions which satisfies $\left|s_{n}(\omega)\right| \leq 1$ and $s_{n} \rightarrow \operatorname{sgn} h$ in $L^{1}(\Omega)$ where $\operatorname{sgn}(h) h=|h|$ so $|\operatorname{sgn}(h)|=1$. Since this is a finite measure space, $\operatorname{sgn}(h)$ is in $L^{1}$ and $s_{n} \rightarrow \operatorname{sgn}(h)$ in $L^{1}(\Omega)$. Also for $E$ a measurable set, $\mathscr{X}_{E} s_{n} \rightarrow \mathscr{X}_{E} \operatorname{sgn}(h)$ pointwise and in $L^{1}$. Then, using the dominated convergence theorem and continuity of $\Lambda$,

$$
\begin{aligned}
\int_{E}|h| d \mu & =\lim _{n \rightarrow \infty} \int_{E} s_{n} h d \mu=\lim _{n \rightarrow \infty} \Lambda\left(s_{n} \mathscr{X}_{E}\right)=\Lambda\left(\mathscr{X}_{E} \operatorname{sgn}(h)\right) \\
& \leq\|\Lambda\| \int \mathscr{X}_{E}|\operatorname{sgn} h| d \mu \leq\|\Lambda\| \mu(E)
\end{aligned}
$$

Thus, whenever $E$ is measurable, $\frac{1}{\mu(E)} \int_{E}|h| d \mu \leq\|\Lambda\|, \int_{E} \frac{|h|}{\|\Lambda\|} d \mu \leq \mu(E)$. By Theorem 13.2.8

$$
\begin{equation*}
\frac{|h|}{\|\Lambda\|} \leq 1 \text { a.e. and so }|h(\omega)| \leq\|\Lambda\| \text { a.e. } \omega \tag{13.4}
\end{equation*}
$$

This shows $h \in L^{\infty}$ and the density of the simple functions in $L^{1}$ implies that for any $f \in$ $L^{1}, \Lambda(f)=\int h f d \mu$. It remains to verify that in fact $\|h\|_{\infty}=\|\Lambda\|$.

$$
|\Lambda(f)|=\left|\int h f d \mu\right| \leq\|h\|_{\infty} \int|f| d \mu=\|h\|_{\infty}\|f\|_{1}
$$

and so $\|\Lambda\| \leq\|h\|_{\infty}$. With 13.4, this shows the two are equal.
A more geometric treatment of the case where $\infty>p>1$ is in Hewitt and Stromberg [22]. It is also included in my Real and Abstract Analysis book on my web site. I have been assured that this other way is the right way to look at it because of its link to geometry and I think that those who say this are right. However, I have never needed this representation theorem for any measure space which is not $\sigma$ finite and it is shorter to do what is being done here. Next these results are extended to the $\sigma$ finite case through the use of a trick.

Lemma 13.3.3 Let $(\Omega, \mathscr{S}, \mu)$ be a measure space and suppose there exists a measurable function, $r$ such that $r(x)>0$ for all $x$, there exists $M$ such that $|r(x)|<M$ for all $x$, and $\int r d \mu<\infty$. Then for $\Lambda \in\left(L^{p}(\Omega, \mu)\right)^{\prime}, p \geq 1$, there exists $h \in L^{q}(\Omega, \mu), L^{\infty}(\Omega, \mu)$ if $p=1$ such that $\Lambda f=\int h f d \mu$. Also $\|h\|=\|\Lambda\| .\left(\|h\|=\|h\|_{q}\right.$ if $p>1,\|h\|_{\infty}$ if $\left.p=1\right)$. Here $\frac{1}{p}+\frac{1}{q}=1$.

Proof: Define a new measure $\widetilde{\mu}$, according to the rule

$$
\begin{equation*}
\widetilde{\mu}(E) \equiv \int_{E} r d \mu . \tag{13.5}
\end{equation*}
$$

Thus $\tilde{\mu}$ is a finite measure on $\mathscr{S}$. For

$$
\Lambda \in\left(L^{p}(\mu)\right)^{\prime}, \Lambda(f)=\Lambda\left(r^{1 / p}\left(r^{-1 / p} f\right)\right)=\tilde{\Lambda}\left(r^{-1 / p} f\right)
$$

where $\tilde{\Lambda}(g) \equiv \Lambda\left(r^{1 / p} g\right)$. Now $\tilde{\Lambda}$ is in $L^{p}(\tilde{\mu})^{\prime}$ because

$$
\begin{aligned}
|\widetilde{\Lambda}(g)| & \equiv\left|\Lambda\left(r^{1 / p} g\right)\right| \leq\|\Lambda\|\left(\int_{\Omega}\left|r^{1 / p} g\right|^{p} d \mu\right)^{1 / p} \\
& =\|\Lambda\|(\int_{\Omega}|g|^{p} \overbrace{r d \mu}^{d \tilde{\mu}})^{1 / p}=\|\Lambda\|\|g\|_{L^{p}(\tilde{\mu})}
\end{aligned}
$$

Therefore, by Theorems 13.3.2 and 13.3.1 there exists a unique $h \in L^{q}(\widetilde{\mu})$ which represents $\widetilde{\Lambda}$. Here $q=\infty$ if $p=1$ and satisfies $1 / q+1 / p=1$ otherwise. Then

$$
\Lambda(f)=\tilde{\Lambda}\left(r^{-1 / p} f\right)=\int_{\Omega} h f r^{-1 / p} r d \mu=\int_{\Omega} f\left(h r^{1 / q}\right) d \mu
$$

Now $h r^{1 / q} \equiv \tilde{h} \in L^{q}(\mu)$ since $h \in L^{q}(\tilde{\mu})$. In case $p=1, L^{q}(\widetilde{\mu})$ and $L^{q}(\mu)$ are exactly the same. In this case you have $\Lambda(f)=\tilde{\Lambda}\left(r^{-1} f\right)=\int_{\Omega} h f r^{-1} r d \mu=\int_{\Omega} f h d \mu$ Thus the desired representation holds. Then in any case, $|\Lambda(f)| \leq\|\tilde{h}\|_{L^{q}}\|f\|_{L^{p}}$ so $\|\Lambda\| \leq\|\tilde{h}\|_{L^{q}}$. Also, as before,

$$
\begin{aligned}
\|\tilde{h}\|_{L^{q}(\mu)}^{q} & =\left.\left|\int_{\Omega} \tilde{h}\right| \tilde{h}\right|^{q-2} \bar{h} d \mu\left|=\left|\Lambda\left(|\tilde{h}|^{q-2} \tilde{\tilde{h}}\right)\right| \leq\|\Lambda\|\left(\left.\int_{\Omega}|\tilde{h}|^{q-2} \tilde{\tilde{h}}\right|^{p} d \mu\right)^{1 / p}\right. \\
& =\|\Lambda\|\left(\int_{\Omega}\left(|\tilde{h}|^{q / p}\right)^{p}\right)^{1 / p}=\|\Lambda\|\|h\|^{q / p}
\end{aligned}
$$

and so $\|\tilde{h}\|_{L^{q}(\mu)} \leq\|\Lambda\| \leq\|\tilde{h}\|_{L^{q}(\mu)}$. It works the same for $p=1$. Thus $\|\tilde{h}\|_{L^{q}(\mu)}=\|\Lambda\|$.
A situation in which the conditions of the lemma are satisfied is the case where the measure space is $\sigma$ finite. In fact, you should show this is the only case in which the conditions of the above lemma hold.

Theorem 13.3.4 (Riesz representation theorem) Let $(\Omega, \mathscr{S}, \mu)$ be $\sigma$ finite and let $\Lambda \in\left(L^{p}(\Omega, \mu)\right)^{\prime}, p \geq 1$. Then there exists a unique $h \in L^{q}(\Omega, \mu), L^{\infty}(\Omega, \mu)$ if $p=1$ such that $\Lambda f=\int h f d \mu$. Also $\|h\|=\|\Lambda\| .\left(\|h\|=\|h\|_{q}\right.$ if $p>1,\|h\|_{\infty}$ if $\left.p=1\right)$. Here $\frac{1}{p}+\frac{1}{q}=1$.

Proof: Without loss of generality, assume $\mu(\Omega)=\infty$. By Proposition 7.11.1, either $\mu$ is a finite measure or $\mu(\Omega)=\infty$. These are the only two cases. Then let $\left\{\Omega_{n}\right\}$ be a sequence of disjoint elements of $\mathscr{S}$ having the property that $1<\mu\left(\Omega_{n}\right)<\infty, \cup_{n=1}^{\infty} \Omega_{n}=\Omega$. Define $r(x)=\sum_{n=1}^{\infty} \frac{1}{n^{2}} \mathscr{X}_{\Omega_{n}}(x) \mu\left(\Omega_{n}\right)^{-1}, \widetilde{\mu}(E)=\int_{E} r d \mu$. Thus $\int_{\Omega} r d \mu=\widetilde{\mu}(\Omega)=\sum_{n=1}^{\infty} \frac{1}{n^{2}}<\infty$ so $\tilde{\mu}$ is a finite measure. The above lemma gives the existence part of the conclusion of the theorem. Uniqueness is done as before.

### 13.4 The Dual Space of $L^{\infty}(\Omega)$

What about the dual space of $L^{\infty}(\Omega)$ ? This will involve the following Lemma. Also recall the notion of total variation defined in Definition 13.2.2.

Lemma 13.4.1 Let $(\Omega, \mathscr{F})$ be a measure space. Denote by $B V(\Omega)$ the space of finitely additive complex measures $v$ such that $|v|(\Omega)<\infty$. Then defining $\|v\| \equiv|v|(\Omega)$, it follows that $B V(\Omega)$ is a Banach space.

Proof: It is obvious that $B V(\Omega)$ is a vector space with the obvious conventions involving scalar multiplication. Why is $\|\cdot\|$ a norm? All the axioms are obvious except for the triangle inequality. However, this is not too hard either.

$$
\begin{aligned}
\|\mu+v\| & \equiv|\mu+v|(\Omega)=\sup _{\pi(\Omega)}\left\{\sum_{A \in \pi(\Omega)}|\mu(A)+v(A)|\right\} \\
& \leq \sup _{\pi(\Omega)}\left\{\sum_{A \in \pi(\Omega)}|\mu(A)|\right\}+\sup _{\pi(\Omega)}\left\{\sum_{A \in \pi(\Omega)}|v(A)|\right\} \\
& \equiv|\mu|(\Omega)+|v|(\Omega)=\|v\|+\|\mu\|
\end{aligned}
$$

Suppose now that $\left\{v_{n}\right\}$ is a Cauchy sequence. For each $E \in \mathscr{F}$,

$$
\left|v_{n}(E)-v_{m}(E)\right| \leq\left\|v_{n}-v_{m}\right\|
$$

and so the sequence of complex numbers $v_{n}(E)$ converges. That to which it converges is called $v(E)$. Then it is obvious that $v(E)$ is finitely additive. Why is $|v|$ finite? Since $\|\cdot\|$ is a norm, it follows that there exists a constant $C$ such that for all $n,\left|v_{n}\right|(\Omega)<C$. Let $\pi(\Omega)$ be any partition. Then

$$
\sum_{A \in \pi(\Omega)}|v(A)|=\lim _{n \rightarrow \infty} \sum_{A \in \pi(\Omega)}\left|v_{n}(A)\right| \leq C .
$$

Hence $v \in B V(\Omega)$. Let $\varepsilon>0$ be given and let $N$ be such that if $n, m>N$, then $\left\|v_{n}-v_{m}\right\|<$ $\varepsilon / 2$. Pick any such $n$. Then choose $\pi(\Omega)$ such that

$$
\begin{gathered}
\left|v-v_{n}\right|(\Omega)-\varepsilon / 2<\sum_{A \in \pi(\Omega)}\left|v(A)-v_{n}(A)\right| \\
=\lim _{m \rightarrow \infty} \sum_{A \in \pi(\Omega)}\left|v_{m}(A)-v_{n}(A)\right|<\lim _{m \rightarrow \infty} \inf _{m \rightarrow}\left|v_{n}-v_{m}\right|(\Omega) \leq \varepsilon / 2
\end{gathered}
$$

It follows that $\lim _{n \rightarrow \infty}\left\|v-v_{n}\right\|=0$.

Corollary 13.4.2 Suppose $(\Omega, \mathscr{F})$ is a measure space as above and suppose $\mu$ is a measure defined on $\mathscr{F}$. Denote by $B V(\Omega ; \mu)$ those finitely additive measures of $B V(\Omega) v$ such that $v \ll \mu$ in the usual sense that if $\mu(E)=0$, then $v(E)=0$. Then $B V(\Omega ; \mu)$ is a closed subspace of $B V(\Omega)$.

Proof: It is clear that it is a subspace. Is it closed? Suppose $v_{n} \rightarrow v$ and each $v_{n}$ is in $B V(\Omega ; \mu)$. Then if $\mu(E)=0$, it follows that $v_{n}(E)=0$ and so $v(E)=0$ also, being the limit of 0 .
Definition 13.4.3 For s a simple function $s(\omega)=\sum_{k=1}^{n} c_{k} \mathscr{X}_{E_{k}}(\omega)$ and $v \in B V(\Omega)$, define an "integral" with respect to $v$ as follows.

$$
\int s d v \equiv \sum_{k=1}^{n} c_{k} v\left(E_{k}\right) .
$$

For $f$ function which is in $L^{\infty}(\Omega ; \mu)$, define $\int f d v$ as follows. Applying Theorem 6.1.10, to the positive and negative parts of real and imaginary parts of $f$, there exists a sequence of simple functions $\left\{s_{n}\right\}$ which converges uniformly to $f$ off a set of $\mu$ measure zero. Then

$$
\int f d v \equiv \lim _{n \rightarrow \infty} \int s_{n} d v
$$

Lemma 13.4.4 The above definition of the integral with respect to a finitely additive measure in $B V(\Omega ; \mu)$ is well defined.

Proof: First consider the claim about the integral being well defined on the simple functions. This is clearly true if it is required that the $c_{k}$ are disjoint and the $E_{k}$ also disjoint having union equal to $\Omega$. Thus define the integral of a simple function in this manner. First write the simple function as $\sum_{k=1}^{n} c_{k} \mathscr{X}_{E_{k}}$ where the $c_{k}$ are the values of the simple function. Then use the above formula to define the integral. Next suppose the $E_{k}$ are disjoint but the $c_{k}$ are not necessarily distinct. Let the distinct values of the $c_{k}$ be $a_{1}, \cdots, a_{m}$

$$
\begin{aligned}
\sum_{k} c_{k} \mathscr{X}_{E_{k}} & =\sum_{j} a_{j}\left(\sum_{i: c_{i}=a_{j}} \mathscr{X}_{E_{i}}\right)=\sum_{j} a_{j} v\left(\cup_{i: c_{i}=a_{j}} E_{i}\right) \\
& =\sum_{j} a_{j} \sum_{i: c_{i}=a_{j}} v\left(E_{i}\right)=\sum_{k} c_{k} v\left(E_{k}\right)
\end{aligned}
$$

and so the same formula for the integral of a simple function is obtained in this case also. Now consider two simple functions

$$
s=\sum_{k=1}^{n} a_{k} \mathscr{X}_{E_{k}}, t=\sum_{j=1}^{m} b_{j} \mathscr{X}_{F_{j}}
$$

where the $a_{k}$ and $b_{j}$ are the distinct values of the simple functions. Then from what was just shown,

$$
\begin{aligned}
\int(\alpha s+\beta t) d v & =\int\left(\sum_{k=1}^{n} \sum_{j=1}^{m} \alpha a_{k} \mathscr{X}_{E_{k} \cap F_{j}}+\sum_{j=1}^{m} \sum_{k=1}^{n} \beta b_{j} \mathscr{X}_{E_{k} \cap F_{j}}\right) d v \\
& =\int\left(\sum_{j, k} \alpha a_{k} \mathscr{\mathscr { X }}_{E_{k} \cap F_{j}}+\beta b_{j} \mathscr{X}_{E_{k} \cap F_{j}}\right) d v \\
& =\sum_{j, k}\left(\alpha a_{k}+\beta b_{j}\right) v\left(E_{k} \cap F_{j}\right)
\end{aligned}
$$

$$
\begin{aligned}
& =\sum_{k=1}^{n} \sum_{j=1}^{m} \alpha a_{k} v\left(E_{k} \cap F_{j}\right)+\sum_{j=1}^{m} \sum_{k=1}^{n} \beta b_{j} v\left(E_{k} \cap F_{j}\right) \\
& =\sum_{k=1}^{n} \alpha a_{k} v\left(E_{k}\right)+\sum_{j=1}^{m} \beta b_{j} v\left(F_{j}\right)=\alpha \int s d v+\beta \int t d v
\end{aligned}
$$

Thus the integral is linear on simple functions so, in particular, the formula given in the above definition is well defined regardless.

So what about the definition for $f \in L^{\infty}(\Omega ; \mu)$ ? Since $f \in L^{\infty}$, there is a set of $\mu$ measure zero $N$ such that on $N^{C}$ there exists a sequence of simple functions which converges uniformly to $f$ on $N^{C}$. Consider $s_{n}$ and $s_{m}$. As in the above, they can be written as $\sum_{k=1}^{p} c_{k}^{n} \mathscr{X}_{E_{k}}, \sum_{k=1}^{p} c_{k}^{m} \mathscr{X}_{E_{k}}$ respectively, where the $E_{k}$ are disjoint having union equal to $\Omega$. Then by uniform convergence, if $m, n$ are sufficiently large, $\left|c_{k}^{n}-c_{k}^{m}\right|<\varepsilon$ or else the corresponding $E_{k}$ is contained in $N^{C}$ a set of $v$ measure 0 thanks to $v \ll \mu$. Hence

$$
\begin{aligned}
\left|\int s_{n} d v-\int s_{m} d v\right| & =\left|\sum_{k=1}^{p}\left(c_{k}^{n}-c_{k}^{m}\right) v\left(E_{k}\right)\right| \\
& \leq \sum_{k=1}^{p}\left|c_{k}^{n}-c_{k}^{m}\right|\left|v\left(E_{k}\right)\right| \leq \varepsilon\|v\|
\end{aligned}
$$

and so the integrals of these simple functions converge. Similar reasoning shows that the definition is not dependent on the choice of approximating sequence.

Note also that for $s$ simple,

$$
\left|\int s d v\right| \leq\|s\|_{L^{\infty}}|v|(\Omega)=\|s\|_{L^{\infty}}\|v\|
$$

Next the dual space of $L^{\infty}(\Omega ; \mu)$ will be identified with $B V(\Omega ; \mu)$. First here is a simple observation. Let $v \in B V(\Omega ; \mu)$. Then define the following for $f \in L^{\infty}(\Omega ; \mu) \cdot T_{V}(f) \equiv$ $\int f d v$

## Lemma 13.4.5 For $T_{v}$ just defined, $\left|T_{v} f\right| \leq\|f\|_{L^{\infty}}\|v\|$

Proof: As noted above, the conclusion true if $f$ is simple. Now if $f$ is in $L^{\infty}$, then it is the uniform limit of simple functions off a set of $\mu$ measure zero. Therefore, by the definition of the $T_{v}$,

$$
\left|T_{v} f\right|=\lim _{n \rightarrow \infty}\left|T_{V} s_{n}\right| \leq \lim \inf _{n \rightarrow \infty}\left\|s_{n}\right\|_{L^{\infty}}\|v\|=\|f\|_{L^{\infty}}\|v\| .
$$

Thus each $T_{v}$ is in $\left(L^{\infty}(\Omega ; \mu)\right)^{\prime}$
Here is the representation theorem, due to Kantorovitch, for the dual of $L^{\infty}(\Omega ; \mu)$.
Theorem 13.4.6 Let $\theta: B V(\Omega ; \mu) \rightarrow\left(L^{\infty}(\Omega ; \mu)\right)^{\prime}$ be given by $\theta(v) \equiv T_{v}$. Then $\theta$ is one to one, onto and preserves norms.

Proof: It was shown in the above lemma that $\theta$ maps into $\left(L^{\infty}(\Omega ; \mu)\right)^{\prime}$. It is obvious that $\theta$ is linear. Why does it preserve norms? From the above lemma,

$$
\|\theta v\| \equiv \sup _{\|f\|_{\infty} \leq 1}\left|T_{v} f\right| \leq\|v\|
$$

It remains to turn the inequality around. Let $\pi(\Omega)$ be a partition. Then

$$
\sum_{A \in \pi(\Omega)}|v(A)|=\sum_{A \in \pi(\Omega)} \operatorname{sgn}(v(A)) v(A) \equiv \int f d v
$$

where $\operatorname{sgn}(v(A))$ is defined to be a complex number of modulus 1 such that

$$
\operatorname{sgn}(v(A)) v(A)=|v(A)|
$$

and

$$
f(\omega)=\sum_{A \in \pi(\Omega)} \operatorname{sgn}(v(A)) \mathscr{X}_{A}(\omega)
$$

Therefore, choosing $\pi(\Omega)$ suitably, since $\|f\|_{\infty} \leq 1$,

$$
\begin{aligned}
\|v\|-\varepsilon & =|v|(\Omega)-\varepsilon \leq \sum_{A \in \pi(\Omega)}|v(A)|=T_{v}(f) \\
& =\left|T_{v}(f)\right|=|\theta(v)(f)| \leq\|\theta(v)\| \leq\|v\|
\end{aligned}
$$

Thus $\theta$ preserves norms. Hence it is one to one also. Why is $\theta$ onto?
Let $\Lambda \in\left(L^{\infty}(\Omega ; \mu)\right)^{\prime}$. Then define

$$
\begin{equation*}
v(E) \equiv \Lambda\left(\mathscr{X}_{E}\right) \tag{13.6}
\end{equation*}
$$

This is obviously finitely additive because $\Lambda$ is linear. Also, if $\mu(E)=0$, then $\mathscr{X}_{E}=0$ in $L^{\infty}$ and so $\Lambda\left(\mathscr{X}_{E}\right)=0$. If $\pi(\Omega)$ is any partition of $\Omega$, then

$$
\begin{aligned}
\sum_{A \in \pi(\Omega)}|v(A)| & =\sum_{A \in \pi(\Omega)}\left|\Lambda\left(\mathscr{X}_{A}\right)\right|=\sum_{A \in \pi(\Omega)} \operatorname{sgn}\left(\Lambda\left(\mathscr{X}_{A}\right)\right) \Lambda\left(\mathscr{X}_{A}\right) \\
& =\Lambda\left(\sum_{A \in \pi(\Omega)} \operatorname{sgn}\left(\Lambda\left(\mathscr{X}_{A}\right)\right) \mathscr{X}_{A}\right) \leq\|\Lambda\|
\end{aligned}
$$

and so $\|v\| \leq\|\Lambda\|$ showing that $v \in B V(\Omega ; \mu)$. Also from 13.6, if $s=\sum_{k=1}^{n} c_{k} \mathscr{X}_{E_{k}}$ is a simple function,

$$
\int s d v=\sum_{k=1}^{n} c_{k} v\left(E_{k}\right)=\sum_{k=1}^{n} c_{k} \Lambda\left(\mathscr{X}_{E_{k}}\right)=\Lambda\left(\sum_{k=1}^{n} c_{k} \mathscr{X}_{E_{k}}\right)=\Lambda(s)
$$

Then letting $f \in L^{\infty}(\Omega ; \mu)$, there exists a sequence of simple functions converging to $f$ uniformly off a set of $\mu$ measure zero and so passing to a limit in the above with $s$ replaced with $s_{n}$ it follows that $\Lambda(f)=\int f d \nu$ and so $\theta$ is onto.

### 13.5 The Dual Space of $C_{0}(X)$

Consider the dual space of $C_{0}(X)$ where $X$ is a Polish space in which the balls have compact closure. It will turn out to be a space of measures. To show this, the following lemma will be convenient. Recall $C_{0}(X)$ is defined as follows.
Definition 13.5.1 $f \in C_{0}(X)$ means that for every $\varepsilon>0$ there exists a compact set $K$ such that $|f(x)|<\varepsilon$ whenever $x \notin K$. Recall the norm on this space is

$$
\|f\|_{\infty} \equiv\|f\| \equiv \sup \{|f(x)|: x \in X\}
$$

One should note right away that $C_{c}(X)$ is dense in $C_{0}(X)$ because if $f \in C_{0}(X)$, then $|f(x)|<\varepsilon$ off some compact set $K$. Let $V \supseteq K$ where $V$ is open and let $K \prec h \prec V$. Now consider $h f \in C_{C}(X) .\|f-h f\|_{\infty}<\varepsilon$.

The next lemma has to do with finding a functional which wants to be linear on the space of positive continuous functions. This will be like the abstract Lebesgue integral. Then the linear extension, as with the Lebesgue integral, will be set up to use the big theorem on positive linear functionals.

Let $L \in C_{0}(X)^{\prime}$. Also denote by $C_{0}^{+}(X)$ the set of nonnegative continuous functions in $C_{0}(X)$ defined on $X$.

Definition 13.5.2 Letting $L \in C_{0}(X)^{\prime}$, define for $f \in C_{0}^{+}(X)$

$$
\lambda(f)=\sup \left\{|L g|:|g| \leq f, g \in C_{0}^{+}(X)\right\} .
$$

Note that $\lambda(f)<\infty$ because $|L g| \leq\|L\|\|g\| \leq\|L\|\|f\|$ for $|g| \leq f$. Isn't this a lot like the total variation of a vector measure? Indeed it is, and the proof that $\lambda$ wants to be linear is also similar to the proof that the total variation is a measure. This is the content of the following lemma.

Lemma 13.5.3 If $c \geq 0, \lambda(c f)=c \lambda(f), f_{1} \leq f_{2}$ implies $\lambda\left(f_{1}\right) \leq \lambda\left(f_{2}\right)$, and

$$
\lambda\left(f_{1}+f_{2}\right)=\lambda\left(f_{1}\right)+\lambda\left(f_{2}\right) .
$$

Also

$$
0 \leq \lambda(f) \leq\|L\|\|f\|_{\infty}
$$

Proof: The first two assertions are easy to see so consider the third. For $i=1,2$ and $f_{i} \in C_{0}^{+}(X)$, let

$$
\left|g_{i}\right| \leq f_{i}, \lambda\left(f_{i}\right) \leq\left|L g_{i}\right|+\varepsilon
$$

Then let $\left|\omega_{i}\right|=1$ and $\omega_{i} L\left(g_{i}\right)=\left|L\left(g_{i}\right)\right|$ so that

$$
\begin{aligned}
\left|L\left(g_{1}\right)\right|+\left|L\left(g_{2}\right)\right| & =\omega_{1} L\left(g_{1}\right)+\omega_{2} L\left(g_{2}\right) \\
& =L\left(\omega_{1} g_{1}+\omega_{2} g_{2}\right)=\left|L\left(\omega_{1} g_{1}+\omega_{2} g_{2}\right)\right|
\end{aligned}
$$

Then

$$
\begin{gathered}
\lambda\left(f_{1}\right)+\lambda\left(f_{2}\right) \leq\left|L\left(g_{1}\right)\right|+\left|L\left(g_{2}\right)\right|+2 \varepsilon \\
=\omega_{1} L\left(g_{1}\right)+\omega_{2} L\left(g_{2}\right)+2 \varepsilon=L\left(\omega_{1} g_{1}\right)+L\left(\omega_{2} g_{2}\right)+2 \varepsilon \\
=L\left(\omega_{1} g_{1}+\omega_{2} g_{2}\right)+2 \varepsilon=\left|L\left(\omega_{1} g_{1}+\omega_{2} g_{2}\right)\right|+2 \varepsilon
\end{gathered}
$$

where $\left|g_{i}\right| \leq f_{i}$ and now $\left|\omega_{1} g_{1}+\omega_{2} g_{2}\right| \leq\left|g_{1}\right|+\left|g_{2}\right| \leq f_{1}+f_{2}$ and so the above shows

$$
\lambda\left(f_{1}\right)+\lambda\left(f_{2}\right) \leq \lambda\left(f_{1}+f_{2}\right)+2 \varepsilon .
$$

Since $\varepsilon$ is arbitrary, $\lambda\left(f_{1}\right)+\lambda\left(f_{2}\right) \leq \lambda\left(f_{1}+f_{2}\right)$. It remains to verify the other inequality.
Let $|g| \leq f_{1}+f_{2},|L g| \geq \lambda\left(f_{1}+f_{2}\right)-\varepsilon$. Let

$$
h_{i}(x)=\left\{\begin{array}{l}
\frac{f_{i}(x) g(x)}{f_{1}(x)+f_{2}(x)} \text { if } f_{1}(x)+f_{2}(x)>0 \\
0 \text { if } f_{1}(x)+f_{2}(x)=0
\end{array}\right.
$$

Then $h_{i}$ is continuous and

$$
h_{1}(x)+h_{2}(x)=g(x),\left|h_{i}\right| \leq f_{i} .
$$

The function $h_{i}$ is clearly continuous at points $x$ where $f_{1}(x)+f_{2}(x)>0$. The reason it is continuous at a point where $f_{1}(x)+f_{2}(x)=0$ is that at every point $y$ where $f_{1}(y)+f_{2}(y)>$ 0 , the top description of $h_{i}$ gives

$$
\left|h_{i}(y)\right|=\left|\frac{f_{i}(y) g(y)}{f_{1}(y)+f_{2}(y)}\right| \leq|g(y)| \leq f_{1}(y)+f_{2}(y)
$$

so if $|y-x|$ is small enough, $\left|h_{i}(y)\right|$ is also small. Then it follows from this definition of the $h_{i}$ that

$$
\begin{aligned}
-\varepsilon+\lambda\left(f_{1}+f_{2}\right) & \leq|L g|=\left|L h_{1}+L h_{2}\right| \leq\left|L h_{1}\right|+\left|L h_{2}\right| \\
& \leq \lambda\left(f_{1}\right)+\lambda\left(f_{2}\right) .
\end{aligned}
$$

Since $\varepsilon>0$ is arbitrary, this shows that

$$
\lambda\left(f_{1}+f_{2}\right) \leq \lambda\left(f_{1}\right)+\lambda\left(f_{2}\right) \leq \lambda\left(f_{1}+f_{2}\right)
$$

The last assertion follows from the observation that if $|g| \leq f$, then $\|g\|_{\infty} \leq\|f\|_{\infty}$ so

$$
\lambda(f)=\sup \{|L g|:|g| \leq f\} \leq \sup _{\|g\|_{\infty} \leq\|f\|_{\infty}}\|L\|\|g\|_{\infty} \leq\|L\|\|f\|_{\infty}
$$

Let $\Lambda$ be the unique linear extension of Theorem 7.7.7 for which $\Lambda f=\lambda(f)$ when $f \geq$ 0 . It is just like defining the integral for functions when you understand it for nonnegative functions. As with integrals $\Lambda(f) \leq \Lambda(|f|)=\lambda(|f|)$. Then from the above lemma,

$$
\begin{equation*}
|\Lambda f| \leq \lambda(|f|) \leq\|L\|\|f\|_{\infty} . \tag{13.7}
\end{equation*}
$$

Also, if $f \geq 0, \Lambda f=\lambda(f) \geq 0$. Therefore, $\Lambda$ is a positive linear functional on $C_{0}(X)$. In particular, it is a positive linear functional on $C_{c}(X)$. Thus there are now two linear continuous mappings $L, \Lambda$ which are defined on $C_{0}(X)$ with the norm $\|\cdot\|_{\infty}$. The above 13.7 shows that in fact $\|\Lambda\| \leq\|L\|$. Also, from the definition of $\Lambda$

$$
|L g| \leq \lambda(|g|)=\Lambda(|g|) \leq\|\Lambda\|\| \| \|_{\infty}
$$

so in fact, $\|L\| \leq\|\Lambda\|$ showing that these two have the same operator norms, $\|L\|=\|\Lambda\|$.
By Theorem 8.2.1 on Page 188, since $\Lambda$ is a positive linear functional on $C_{c}(X)$, there exists a unique measure $\mu$ such that $\Lambda f=\int_{X} f d \mu$ for all $f \in C_{c}(X)$. This measure is regular. In fact, it is actually a finite measure. First note that by density of $C_{c}(X)$ in $C_{0}(X)$

$$
\begin{aligned}
\|\Lambda\| & =\sup \left\{\Lambda f: f \in C_{c}(X),\|f\|_{\infty} \leq 1\right\}=\sup \left\{\Lambda f: 0 \leq f \leq 1, f \in C_{c}(X)\right\} \\
& =\sup \left\{\int f d \mu: 0 \leq f \leq 1, f \in C_{c}(X)\right\}=\mu(X)
\end{aligned}
$$

This is stated in the following lemma.
Lemma 13.5.4 Let $L \in C_{0}(X)^{\prime}$ as above. Then letting $\mu$ be the Radon measure just described, it follows $\mu$ is finite and $\mu(X)=\|\Lambda\|=\|L\|$.

What follows is the Riesz representation theorem for $C_{0}(X)^{\prime}$.
Theorem 13.5.5 Let $L \in\left(C_{0}(X)\right)^{\prime}$. Then there exists a finite Radon measure $\mu$ and a function $\sigma \in L^{\infty}(X, \mu)$ such that for all $f \in C_{0}(X)$,

$$
L(f)=\int_{X} f \sigma d \mu
$$

Furthermore, $\mu(X)=\|L\|,|\sigma|=1$ a.e. and if $v(E) \equiv \int_{E} \sigma d \mu$ then $\mu=|v|$.
Proof: From the above there exists a unique Radon measure $\mu$ such that for all $f \in$ $C_{c}(X), \Lambda f=\int_{X} f d \mu$. Then for $f \in C_{c}(X)$,

$$
|L f| \leq \lambda(|f|)=\Lambda(|f|)=\int_{X}|f| d \mu=\|f\|_{L^{1}(\mu)}
$$

Since $\mu$ is both inner and outer regular, $C_{c}(X)$ is dense in $L^{1}(X, \mu)$. (See Theorem 9.4.2) Therefore $L$ extends uniquely to an element of $\left(L^{1}(X, \mu)\right)^{\prime}, \widetilde{L}$. By the Riesz representation theorem for $L^{1}$ for finite measure spaces, there exists a unique $\sigma \in L^{\infty}(X, \mu)$ such that for all $f \in L^{1}(X, \mu), \widetilde{L} f=\int_{X} f \sigma d \mu$. In particular, for all $f \in C_{0}(X), L f=\int_{X} f \sigma d \mu$ and it follows from Lemma 13.5.4, $\mu(X)=\|L\|$.

It remains to verify $|\sigma|=1$ a.e. For any continuous $f \geq 0, \Lambda f \equiv \int_{X} f d \mu \geq|L f|=$ $\left|\int_{X} f \sigma d \mu\right|$. Now if $E$ is measurable, the regularity of $\mu$ implies that there exists a sequence of nonnegative bounded functions $f_{n} \in C_{c}(X)$ such that $f_{n}(x) \rightarrow \mathscr{X}_{E}(x)$ a.e. and in $L^{1}(\mu)$. Then using the dominated convergence theorem in the above,

$$
\begin{aligned}
\int_{E} d \mu & =\lim _{n \rightarrow \infty} \int_{X} f_{n} d \mu=\lim _{n \rightarrow \infty} \Lambda\left(f_{n}\right) \geq \lim _{n \rightarrow \infty}\left|L f_{n}\right| \\
& =\lim _{n \rightarrow \infty}\left|\int_{X} f_{n} \sigma d \mu\right|=\left|\int_{E} \sigma d \mu\right|
\end{aligned}
$$

and so if $\mu(E)>0,1 \geq\left|\frac{1}{\mu(E)} \int_{E} \sigma d \mu\right|$ which shows from Theorem 13.2.8 that $|\sigma| \leq 1 \mu$ a.e. But also, from Theorem 13.2.8, if $\|f\|_{\infty} \leq 1$,

$$
\begin{aligned}
|\mu|(X) & =\mu(X)=\|L\|=\sup _{\|f\|_{\infty} \leq 1}\left|\int_{X} f \sigma d \mu\right| \leq \int_{X}|f||\sigma| d \mu \\
& \leq \int_{X}|\sigma| d \mu \leq \int_{X} d \mu=\mu(X)
\end{aligned}
$$

and so $|\sigma|=1$ a.e. since $\mu(X)=\int_{X}|\sigma| d \mu=\mu(X)$ and it is known that $|\sigma| \leq 1$. If $|\sigma|$ were less than 1 on a set of positive measure, this could not hold.

It only remains to verify $\mu=|v|$. Recall $v(E) \equiv \int_{E} \sigma d \mu$. By Theorem 13.2.8, $|v|(E)=$ $\int_{E}|\sigma| d \mu=\int_{E} 1 d \mu=\mu(E)$ and so $\mu=|v|$.

Sometimes people write $\int_{X} f d \nu \equiv \int_{X} f \sigma d|v|$ where $\sigma d|v|$ is the polar decomposition of the complex measure $v$. Then with this convention, the above representation is

$$
L(f)=\int_{X} f d v,|v|(X)=\|L\|
$$

Also note that at most one $v$ can represent $L$. If there were two of them $v_{i}, i=1,2$, then $v_{1}-v_{2}$ would represent 0 and so $\left|v_{1}-v_{2}\right|(X)=0$. Hence $v_{1}=v_{2}$.

The following is a rather important application of the above theory along with the material on Fourier transforms presented earlier. It has to do with the fact that the characteristic function of a probability measure is unique so if two such measures have the same characteristic function, then they are the same measure. This can actually be shown rather easily. You don't have to accept this kind of thing on faith and speculations based on special cases.

Example 13.5.6 Let $\mu$ and $v$ be two Radon probability measures on the Borel sets of $\mathbb{R}^{p}$. The typical situation is that these are probability distribution functions for two random variables. Then the characteristic function of $\mu$ is $(2 \pi)^{p / 2}$ times the inverse Fourier transform of $\mu$

$$
\phi_{\mu}(\mathbf{t}) \equiv \int_{X} e^{i \cdot \mathbf{x} \cdot \mathbf{x}} d \mu(\mathbf{x})
$$

then a very important theorem from probability says that if $\phi_{\mu}(\mathbf{t})=\phi_{v}(\mathbf{t})$, then the two measures are equal. This is very easy at this point, but not so easy if you don't have the general treatment of Fourier transforms presented above.

We have $F^{-1}(\mu)=F^{-1}(v)$ in $\mathscr{G}^{*}$. Therefore, $\mu=v$ in $\mathscr{G}^{*}$ and by definition, $\int_{X} \psi d \mu=$ $\int_{X} \psi d v$ for all $\psi \in \mathscr{G}$. But by the Stone Weierstrass theorem, $\mathscr{G}$ is dense in $C_{0}\left(\mathbb{R}^{p}\right)$ and so the equation holds for all $\psi \in C_{0}\left(\mathbb{R}^{p}\right)$. Now $C_{c}\left(\mathbb{R}^{p}\right) \subseteq C_{0}\left(\mathbb{R}^{p}\right)$ and so the equation also holds for all $\psi \in C_{c}\left(\mathbb{R}^{p}\right)$. By uniqueness in the Riesz representation theorem, it follows that $\mu=v$. You could also use Theorem 13.5.5.

### 13.6 Exercises

1. Suppose $\mu$ is a vector measure having values in $\mathbb{R}^{n}$ or $\mathbb{C}^{n}$. Can you show that $|\mu|$ must be finite? Hint: You might define for each $\mathbf{e}_{i}$, one of the standard basis vectors, the real or complex measure, $\mu_{\mathbf{e}_{i}}$ given by $\mu_{\mathbf{e}_{i}}(E) \equiv \mathbf{e}_{i} \cdot \mu(E)$. Why would this approach not yield anything for an infinite dimensional normed linear space in place of $\mathbb{R}^{n}$ ?
2. The Riesz representation theorem of the $L^{p}$ spaces can be used to prove a very interesting inequality. Let $r, p, q \in(1, \infty)$ satisfy

$$
\frac{1}{r}=\frac{1}{p}+\frac{1}{q}-1
$$

Then

$$
\frac{1}{q}=1+\frac{1}{r}-\frac{1}{p}>\frac{1}{r}
$$

and so $r>q$. Let $\theta \in(0,1)$ be chosen so that $\theta r=q$. Then also

$$
\frac{1}{r}=(\overbrace{1-\frac{1}{p^{\prime}}}^{1 / p+1 / p^{\prime}=1})+\frac{1}{q}-1=\frac{1}{q}-\frac{1}{p^{\prime}}
$$

and so

$$
\frac{\theta}{q}=\frac{1}{q}-\frac{1}{p^{\prime}}
$$

which implies $p^{\prime}(1-\theta)=q$. Now let $f \in L^{p}\left(\mathbb{R}^{n}\right), g \in L^{q}\left(\mathbb{R}^{n}\right), f, g \geq 0$. Justify the steps in the following argument using what was just shown that $\theta r=q$ and $p^{\prime}(1-\theta)=q$. Let

$$
\begin{align*}
& h \in L^{r^{\prime}}\left(\mathbb{R}^{n}\right) \cdot\left(\frac{1}{r}+\frac{1}{r^{\prime}}=1\right) \\
& \left|\int f * g(\mathbf{x}) h(\mathbf{x}) d x\right|=\left|\iint f(\mathbf{y}) g(\mathbf{x}-\mathbf{y}) h(\mathbf{x}) d x d y\right| \text {. } \\
& \leq \iint|f(\mathbf{y})||g(\mathbf{x}-\mathbf{y})|^{\theta}|g(\mathbf{x}-\mathbf{y})|^{1-\theta}|h(\mathbf{x})| d y d x \\
& \leq \int\left(\int\left(|g(\mathbf{x}-\mathbf{y})|^{1-\theta}|h(\mathbf{x})|\right)^{r^{\prime}} d x\right)^{1 / r^{\prime}} . \\
& \left(\int\left(|f(\mathbf{y})||g(\mathbf{x}-\mathbf{y})|^{\theta}\right)^{r} d x\right)^{1 / r} d y \\
& \leq\left[\int\left(\int\left(|g(\mathbf{x}-\mathbf{y})|^{1-\theta}|h(\mathbf{x})|\right)^{r^{\prime}} d x\right)^{p^{\prime} / r^{\prime}} d y\right]^{1 / p^{\prime}} . \\
& {\left[\int\left(\int\left(|f(\mathbf{y})||g(\mathbf{x}-\mathbf{y})|^{\theta}\right)^{r} d x\right)^{p / r} d y\right]^{1 / p}} \\
& \leq\left[\int\left(\int\left(|g(\mathbf{x}-\mathbf{y})|^{1-\theta}|h(\mathbf{x})|\right)^{p^{\prime}} d y\right)^{r^{\prime} / p^{\prime}} d x\right]^{1 / r^{\prime}} . \\
& {\left[\int|f(\mathbf{y})|^{p}\left(\int|g(\mathbf{x}-\mathbf{y})|^{\theta r} d x\right)^{p / r} d y\right]^{1 / p}} \\
& =\left[\int|h(\mathbf{x})|^{r^{\prime}}\left(\int|g(\mathbf{x}-\mathbf{y})|^{(1-\theta) p^{\prime}} d y\right)^{r^{\prime} / p^{\prime}} d x\right]^{1 / r^{\prime}}\|g\|_{q}^{q / r}\|f\|_{p} \\
& =\|g\|_{q}^{q / r}\|g\|_{q}^{q / p^{\prime}}\|f\|_{p}\|h\|_{r^{\prime}}=\|g\|_{q}\|f\|_{p}\|h\|_{r^{\prime}} . \tag{13.8}
\end{align*}
$$

Young's inequality says that

$$
\begin{equation*}
\|f * g\|_{r} \leq\|g\|_{q}\|f\|_{p} \tag{13.9}
\end{equation*}
$$

Therefore $\|f * g\|_{r} \leq\|g\|_{q}\|f\|_{p}$. How does this inequality follow from the above computation? Does 13.8 continue to hold if $r, p, q$ are only assumed to be in $[1, \infty]$ ? Explain. Does 13.9 hold even if $r, p$, and $q$ are only assumed to lie in $[1, \infty]$ ?
3. Suppose $(\Omega, \mu, \mathscr{S})$ is a finite measure space and that $\left\{f_{n}\right\}$ is a sequence of functions which converge weakly to 0 in $L^{p}(\Omega)$. This means that

$$
\int_{\Omega} f_{n} g d \mu \rightarrow 0
$$

for every $g \in L^{p^{\prime}}(\Omega)$. Suppose also that $f_{n}(x) \rightarrow 0$ a.e. Show that then $f_{n} \rightarrow 0$ in $L^{p-\varepsilon}(\Omega)$ for every $\varepsilon>0$ such that $p-\varepsilon>1$.

Hint: The weak convergence implies $f_{n}$ is bounded in $L^{p}(\Omega)$ where $p>p-\varepsilon$. (Consider the uniform boundedness theorem.) Show $\left\{f_{n}^{p-\varepsilon}\right\}$ is uniformly integrable. Then consider the Vitali convergence theorem.
4. Give an example of a sequence of functions in $L^{\infty}(-\pi, \pi)$ which converges weak $*$ to zero but which does not converge pointwise a.e. to zero. Convergence weak $*$ to 0 means that for every $g \in L^{1}(-\pi, \pi), \int_{-\pi}^{\pi} g(t) f_{n}(t) d t \rightarrow 0$. Hint: First consider $g \in C_{c}^{\infty}(-\pi, \pi)$ and maybe try something like $f_{n}(t)=\sin (n t)$. Do integration by parts. Recall the Riemann Lebesgue lemma.
5. Let $\lambda$ be a real vector measure on the measure space $(\Omega, \mathscr{F})$. That is $\lambda$ has values in $\mathbb{R}$. The Hahn decomposition says there exist measurable sets $P, N$ such that

$$
P \cup N=\Omega, P \cap N=\emptyset,
$$

and for each $F \subseteq P, \lambda(F) \geq 0$ and for each $F \subseteq N, \lambda(F) \leq 0$. These sets $P, N$ are called the positive set and the negative set respectively. Show from the polar decomposition of a vector measure the existence of the Hahn decomposition. Also explain how this decomposition is unique in the sense that if $P^{\prime}, N^{\prime}$ is another Hahn decomposition, then $\left(P \backslash P^{\prime}\right) \cup\left(P^{\prime} \backslash P\right)$ has measure zero, a similar formula holding for $N, N^{\prime}$. When you have the Hahn decomposition, as just described, you define $\lambda^{+}(E) \equiv \lambda(E \cap P), \lambda^{-}(E) \equiv \lambda(E \cap N)$. This is sometimes called the Hahn Jordan decomposition.
6. The Hahn decomposition holds for measures which have values in $(-\infty, \infty]$. Let $\lambda$ be such a measure which is defined on a $\sigma$ algebra of sets $\mathscr{F}$. This is not a vector measure because the set on which it has values is not a vector space. Thus this case is not included in the above discussion. $N \in \mathscr{F}$ is called a negative set if $\lambda(B) \leq 0$ for all $B \subseteq N . P \in \mathscr{F}$ is called a positive set if for all $F \subseteq P, \lambda(F) \geq 0$. (Here it is always assumed you are only considering sets of $\mathscr{F}$.) Show that if $\lambda(A) \leq 0$, then there exists $N \subseteq A$ such that $N$ is a negative set and $\lambda(N) \leq \lambda(A)$. Hint: This is done by subtracting off disjoint sets having positive measure. Let $A \equiv N_{0}$ and suppose $N_{n} \subseteq A$ has been obtained. Tell why $t_{n} \equiv \sup \left\{\lambda(E): E \subseteq N_{n}\right\} \geq 0$. Let $B_{n} \subseteq N_{n}$ such that

$$
\lambda\left(B_{n}\right)>\frac{t_{n}}{2}
$$

Then $N_{n+1} \equiv N_{n} \backslash B_{n}$. Thus the $N_{n}$ are decreasing in $n$ and the $B_{n}$ are disjoint. Explain why $\lambda\left(N_{n}\right) \leq \lambda\left(N_{0}\right)$. Let $N=\cap N_{n}$. Argue $t_{n}$ must converge to 0 since otherwise $\lambda(N)=-\infty$. Explain why this requires $N$ to be a negative set in $A$ which has measure no larger than that of $A$.
7. Using Problem 6 complete the Hahn decomposition for $\lambda$ having values in $(-\infty, \infty]$. Now the Hahn Jordan decomposition for the measure $\lambda$ is

$$
\lambda^{+}(E) \equiv \lambda(E \cap P), \lambda^{-}(E) \equiv-\lambda(E \cap N)
$$

Explain why $\lambda^{-}$is a finite measure. Hint: From the above problem, if $\lambda(A) \leq 0$, there is a negative set $N$ contained in $A$. If $\Omega$ is positive, then you are done. If not, you could consider a maximal disjoint union of negative sets. This will be countable. Then take complement of its union which must be positive.
8. What if $\lambda$ has values in $[-\infty, \infty)$. Prove there exists a Hahn decomposition for $\lambda$ as in the above problem. Why do we not allow $\lambda$ to have values in $[-\infty, \infty]$ ? Hint: You might want to consider $-\lambda$.
9. Suppose $X$ is a Banach space and let $X^{\prime}$ denote its dual space. A sequence $\left\{x_{n}^{*}\right\}_{n=1}^{\infty}$ in $X^{\prime}$ is said to converge weak $*$ to $x^{*} \in X^{\prime}$ if for every $x \in X, \lim _{n \rightarrow \infty} x_{n}^{*}(x)=x^{*}(x)$. Let $\left\{\phi_{n}\right\}$ be a mollifier defined on $\mathbb{R}^{p}$. Also let $\delta$ be the measure defined by

$$
\delta(E)=1 \text { if } 0 \in E \text { and } 0 \text { if } 1 \notin E .
$$

Explain how $\phi_{n} \rightarrow \delta$ weak $*$ in the dual space of $C_{0}\left(\mathbb{R}^{p}\right)$.
10. Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $X: \Omega \rightarrow \mathbb{R}^{n}$ be a random variable. This means $X^{-1}$ (open set) $\in \mathscr{F}$. Define a measure $\lambda_{X}$ on the Borel sets of $\mathbb{R}^{n}$ as follows. For $E$ a Borel set, $\lambda_{X}(E) \equiv P\left(X^{-1}(E)\right)$ Explain why this is well defined. Next explain why $\lambda_{X}$ can be considered a Radon probability measure by completion. Explain why $\lambda_{X} \in \mathscr{G}^{*}$ if

$$
\lambda_{X}(\psi) \equiv \int_{\mathbb{R}^{n}} \psi d \lambda_{X}
$$

where $\mathscr{G}$ is the collection of functions used to define the Fourier transform.
11. Using the above problem, the characteristic function of this measure (random variable) is

$$
\phi_{X}(\mathbf{y}) \equiv \int_{\mathbb{R}^{n}} e^{i \mathbf{x} \cdot \mathbf{y}} d \lambda_{X}
$$

Show this always exists for any such random variable and is continuous. Next show that for two random variables $X, Y, \lambda_{X}=\lambda_{Y}$ if and only if $\phi_{X}(\mathbf{y})=\phi_{Y}(\mathbf{y})$ for all $\mathbf{y}$. In other words, show the distribution measures are the same if and only if the characteristic functions are the same. A lot more can be concluded by looking at characteristic functions of this sort. The important thing about these characteristic functions is that they always exist, unlike moment generating functions. Note that this is a specific version of Example 13.5.6.
12. Let $B$ be the ball $\left\{f \in L^{p}(\Omega):\|f\|_{p} \leq M\right\}$ where $\Omega$ is a measurable subset of $\mathbb{R}^{p}$ and the measure is a Radon measure, and suppose you have a sequence $\left\{f_{k}\right\} \subseteq B$. Show that there exists a subsequence, still denoted as $\left\{f_{k}\right\}$ and $f \in B$ such that for all $g \in L^{q}(\Omega)$,

$$
\lim _{k \rightarrow \infty} \int f_{k} g d \mu=\int f g d \mu
$$

That is, show that $B$ is weakly sequentially compact. This is the term for what you will show and it is the most important case of the Eberlein Smulian theorem on weak compactness. It serves as the basis for may existence theorems in non linear analysis. Hint: First use the fact that $L^{q}(\Omega)$ is separable, $\frac{1}{q}+\frac{1}{p}=1$. See Corollary 10.7.3. Then use the Cantor diagonalization process which was used earlier in the proof of the Arzela Ascoli theorem, Theorem 9.2.4 to obtain a subsequence, still denoted as $\left\{f_{k}\right\}$ such that for each $g \in D$ for $D$ the countable dense subset of $L^{q}(\Omega)$, the sequence of complex numbers $\left\{\int f_{k} g d \mu\right\}$ converges. Now show that this converges for every $g \in L^{q}(\Omega)$. Let $F(g) \equiv \lim _{k \rightarrow \infty} \int f_{k} g d \mu$. Show that $F \in\left(L^{q}(\Omega)\right)^{\prime}$. Now use the Riesz representation theorem to get $f \in L^{p}(\Omega)$ representing $F$. Observe that this works.

## Part IV

## Complex Analysis

## Chapter 14

## Fundamentals

### 14.1 Banach Spaces

I am going to present the most basic theorems in complex analysis for the case where the functions have values in a Banach space. See Chapter 12 above for a short discussion of the main properties of these spaces. There are good reasons for allowing functions to have values in a complex Banach space. In particular, when $X$ is a Banach space, so is $\mathscr{L}(X, X)$. The presentation of fundamental topics will include the case of $\lambda \rightarrow(\lambda I-A)^{-1}$ for $A \in \mathscr{L}(X, X)$. This will make possible an easy discussion of some very important theorems. Thus this extra generality is not just generalization for the sake of generalization. It is here for a real reason, to make the transition to spectral theory of linear operators believable. However, if the reader is convinced that these functional analysis topics will never be of use to them, then replace $X$ with $\mathbb{C}$ and a standard treatment will be obtained. After presenting the fundamental concepts of this subject, in this general setting, I will specialize to the usual case in which the functions have complex values. In fact, this is the main thrust of this book. I am just trying not to neglect the other application.

It is useful to recall Problem 26 on Page 78 about when a function of a single real variable has derivative equal to 0 the function is constant even if it has values in a Banach space.

There is a fundamental theorem about intersections. As before, a set $F$ is closed if its complement is open or equivalently if it contains all of its limit points. The proof is exactly as done earlier.

In the following, for $S$ a nonempty set, $\operatorname{diam}(S) \equiv \sup \{\|x-y\|: x, y \in S\}$.
Theorem 14.1.1 Let $F_{n} \supseteq F_{n+1} \cdots$ where each $F_{n}$ is a closed set in $X$ a Banach space and suppose that the diameter of $F_{n}$ converges to 0 as $n \rightarrow \infty$. Then there exists a unique point in $\cap_{n=1}^{\infty} F_{n}$.

Proof: Obviously there can be no more than one point in the intersection because if $x, y$ are two points, then $\|x-y\|>\delta>0$ for some $\delta$ but eventually both points would be in $F_{n}$ where $n$ is so large that the diameter of $F_{n}$ is less than $\delta$. As to existence of the point in the intersection, pick $p_{n} \in F_{n}$. It is a Cauchy sequence since $\operatorname{diam}\left(F_{n}\right) \rightarrow 0$ and so $\left\{p_{n}\right\}$ converges to $p$. Now the $p_{k}$ for $k \geq n$ is in $F_{n}$, a closed set so by Corollary 2.2.8 applied to $X$ rather than $\mathbb{F}^{p}$, it follows that $p \in F_{n}$, this for each $n$. Hence $p \in \cap_{n=1}^{\infty} F_{n}$.

### 14.2 The Cauchy Riemann Equations

These fundamental equations pertain to a complex valued function of a complex variable. Recall the complex numbers should be considered as points in the plane. Thus a complex number is of the form $x+i y$ where $i^{2}=-1$. Recall that the complex conjugate is defined by $\overline{x+i y} \equiv x-i y$ and for $z$ a complex number, $|z| \equiv(z \bar{z})^{1 / 2}=\sqrt{x^{2}+y^{2}}$. Thus when $x+i y$ is considered an ordered pair $(x, y) \in \mathbb{R}^{2}$ the magnitude of a complex number is nothing more than the usual norm of the ordered pair. Also for $z=x+i y, w=u+i v,|z-w|=$ $\sqrt{(x-u)^{2}+(y-v)^{2}}$ so in terms of all topological considerations, $\mathbb{R}^{2}$ is the same as $\mathbb{C}$. Thus to say $z \rightarrow f(z)$ is continuous, is the same as saying $(x, y) \rightarrow u(x, y),(x, y) \rightarrow v(x, y)$ are continuous where $f(z) \equiv u(x, y)+i v(x, y)$ with $u$ and $v$ being called the real and imaginary parts of $f$. The only new thing is that writing an ordered pair $(x, y)$ as $x+i y$ with the
convention $i^{2}=-1$ makes $\mathbb{C}$ into a field all this is earlier in the book. Now here is the definition of what it means for a function to be analytic.

## Definition 14.2.1 Let $U$ be an open subset of $\mathbb{C},\left(\mathbb{R}^{2}\right)$ and let $f: U \rightarrow X$ where $X$

 is a complex Banach space. Then $f$ is said to be analytic on $U$ if for every $z \in U$,$$
\lim _{h \rightarrow 0} \frac{f(z+h)-f(z)}{h} \equiv f^{\prime}(z)
$$

exists where $z \rightarrow f^{\prime}(z)$ is a continuous function of $z \in U$. For a function having values in $\mathbb{C}$ denote by $u(x, y)$ the real part of $f$ and $v(x, y)$ the imaginary part. Both $u$ and $v$ have real values and

$$
f(x+i y) \equiv f(z) \equiv u(x, y)+i v(x, y)
$$

As earlier, the above definition of the derivative is equivalent to saying

$$
f(z+h)=f(z)+f^{\prime}(z) h+o(h),
$$

But here we insist that $z \rightarrow f^{\prime}(z)$ be continuous
All the usual rules of differentiation hold from using the same proofs.
First are some simple results in the case that $f$ has values in $\mathbb{C}$.
Proposition 14.2.2 Let $U$ be an open subset of $\mathbb{C}$. Then $f: U \rightarrow \mathbb{C}$ is analytic if and only iffor $f(x+i y) \equiv u(x, y)+i v(x, y), u(x, y), v(x, y)$ being the real and imaginary parts of $f$, it follows

$$
u_{x}(x, y)=v_{y}(x, y), u_{y}(x, y)=-v_{x}(x, y)
$$

and all these partial derivatives, $u_{x}, u_{y}, v_{x}, v_{y}$ are continuous on $U$. (The above equations are called the Cauchy Riemann equations.)

Proof: If $f^{\prime}(z)$ exists, then clearly $\operatorname{Re}(f)^{\prime}$ and $\operatorname{Im}(f)^{\prime}$ exist and so in particular, all the partial derivatives of $u, v$ exist. Thus $f^{\prime}(z)$ exists if and only if for $\Delta z=h+i k$,

$$
\begin{gather*}
f(z+\Delta z)-f(z)=u(x+h, y+k)+i v(x+h, y+k)-(u(x, y)+i v(x, y)) \\
=u_{x}(x, y) h+u_{y}(x, y) k+i\left(v_{x}(x, y) h+v_{y}(x, y) k\right)+o((h, k)) \\
=u_{x}(x, y) h+u_{y}(x, y) k+i\left(v_{x}(x, y) h+v_{y}(x, y) k\right)+o(\Delta z) \tag{14.1}
\end{gather*}
$$

Then from the above,

$$
\begin{aligned}
f^{\prime}(z) & =\lim _{k \rightarrow 0}(f(z+i k)-f(z)) \frac{1}{i k}=-u_{y}(x, y) i+v_{y}(x, y) \\
f^{\prime}(z) & =\lim _{h \rightarrow 0}(f(z+h)-f(z)) \frac{1}{h}=u_{x}(x, y)+i v_{x}(x, y)
\end{aligned}
$$

and so $u_{x}(x, y)=v_{y}(x, y), u_{y}(x, y)=-v_{x}(x, y)$ the Cauchy Riemann equations. Thus these Cauchy Riemann equations hold if $f^{\prime}(z)$ exists. Also, it follows from Theorem 4.5.2 on Page 96 that if $u, v$ are $C^{1}$ then 14.1 holds and so $f^{\prime}(z)$ exists and is continuous.

Example 14.2.3 What if $f: \mathbb{C} \rightarrow \mathbb{R}$ is analytic?

It turns out that this is not very interesting. Letting $u, v$ be as above, it follows that $v=0$ and so $u_{x}=0_{y}=0$ and $u_{y}=-0_{x}=0$ so $\nabla u=0$ and so $u$ is a constant. Therefore, $f(z)=c \in \mathbb{R}$.

Example 14.2.4 $f(z)=z^{2}$ is analytic. $f(z)=\left|z^{2}\right|$ is not.
For functions of a real variable, it is perfectly possible for the derivative to exist and not be continuous. For example, consider

$$
f(x) \equiv\left\{\begin{array}{l}
x^{2} \sin \left(\frac{1}{x}\right) \text { if } x \neq 0 \\
0 \text { if } x=0
\end{array}\right.
$$

You can verify that $f^{\prime}(x)$ exists for all $x$ but at 0 this derivative is not continuous. This will NEVER happen with functions of a complex variable. This is shown later when it is more convenient. For now make continuity of $f^{\prime}$ part of the requirement for $f$ to be analytic.

### 14.3 The Logarithm

One of the most important examples of an analytic function is the logarithm. Earlier the function $e^{z}$ was discussed. See Section 1.5.2. Recall $e^{x+i y} \equiv e^{x}(\cos y+i \sin y)$ and this will be an analytic function because the real and imaginary parts are $C^{1}$ and the Cauchy Riemann equations hold. This is one way to see this. Now I want to consider $\log (z)$ which is in some sense the inverse of $e^{z}$.

You want to have $e^{\log (z)}=z=|z|(\cos \theta+i \sin \theta)$ where $\theta$ is the angle of $z$. Now $\log (z)$ should be a complex number and so it will have a real and imaginary part. Thus

$$
\begin{equation*}
e^{\operatorname{Re}(\log (z))+i \operatorname{Im}(\log (z))}=|z|(\cos \theta+i \sin \theta)=z \tag{14.2}
\end{equation*}
$$

where $\theta$ is the angle of $z$. The magnitude of the left side needs to equal the magnitude of the right side. Hence, $e^{\operatorname{Re}(\log (z))}=|z|$ and so it is clear that $\operatorname{Re}(\log (z))=\ln |z|$. Note that we must exclude $z=0$ just as in the real case. What about $\operatorname{Im}(\log (z))$ ? Having found $\operatorname{Re}(\log (z)), 14.2$ is

$$
\begin{equation*}
|z|(\cos (\operatorname{Im}(\log z))+i \sin (\operatorname{Im}(\log z)))=|z|(\cos \theta+i \sin \theta) \tag{14.3}
\end{equation*}
$$

which happens if and only if

$$
\begin{equation*}
\operatorname{Im}(\log z)=\theta+2 k \pi \tag{14.4}
\end{equation*}
$$

for $k$ an integer. Thus there are many solutions for $\operatorname{Im}(\log z)$ to the above problem. A branch of the logarithm is determined by picking one of them. The idea is that there is only one possible solution for $\operatorname{Im}(\log z)$ in any open interval of length $2 \pi$ because if you have two different $k$ in 14.4, the two values of $\operatorname{Im}(\log z)$ would differ by at least $2 \pi$ so they could not both be in an open interval of length $2 \pi$.

What is done is to consider $e^{z}$ where if $z=|z| e^{i \theta}$, then $\theta \in(a-\pi, a+\pi)$ for some $a$. In other words, you consider the ray coming from 0 in the complex plane and including 0 which has angle $a$. Then regard $e^{z}$ as being defined for all of $\mathbb{C}$ other than this ray.


This involves restricting the domain of the function to an open set so that it has an inverse. It is like what was done for arctan and other trig. functions in calculus, except here we are careful to have the domain be an open set. Then if this restriction is made, there is exactly one solution $\operatorname{Im}(\log z)$ to 14.3 . The most common assignment of $a$ is $\pi$, so we leave out the negative real axis. However, one could leave out any other ray. If the usual one is left out, this shows that we need to have $\log (z)=\ln (|z|)+i \arg (z)$ where $\arg (z)$ is the angle in the polar form of $z$ which is in $(-\pi, \pi)$. It is called the principal branch of the logarithm when this is done. If you left out some other ray, then $\arg (z)$ would refer to an angle in some other open interval of length $2 \pi$.

Now the above geometric description shows that any branch of $z \rightarrow \log (z)$ is continuous. Indeed, if $z_{n} \rightarrow z$, then by the triangle inequality, $\left|\left|z_{n}\right|-|z|\right| \leq\left|z_{n}-z\right|$ and so by continuity of $\ln$, you get $\ln \left(\left|z_{n}\right|\right) \rightarrow \ln (|z|)$. As to convergence of $\arg \left(z_{n}\right)$ to $\arg (z)$, just note that saying one is close to another is the same as saying that $\arg \left(z_{n}\right)$ is in any open set determined by two rays emanating from 0 which include $z$. This happens if $z_{n} \rightarrow z$. Is $z \rightarrow \log (z)$ differentiable? First note that, from Proposition 14.2.2, the Cauchy Riemann equations, and the definition of $e^{z} \equiv e^{x}(\cos y+i \sin y)$, it follows that $\left(e^{z}\right)^{\prime}=e^{z}$ and so

$$
\begin{equation*}
h=e^{\log (z+h)}-e^{\log (z)}=e^{\log (z)}(\log (z+h)-\log (z))+o(\log (z+h)-\log (z)) \tag{14.5}
\end{equation*}
$$

Then for $z \neq 0$,

$$
\begin{equation*}
\frac{h}{z}=\log (z+h)-\log (z)+o(\log (z+h)-\log (z)) \tag{14.6}
\end{equation*}
$$

By continuity, if $h$ is small enough,

$$
|o(\log (z+h)-\log (z))|<\frac{1}{2}|\log (z+h)-\log (z)| .
$$

Hence $\left|\frac{h}{z}\right| \geq \frac{1}{2}|\log (z+h)-\log (z)|$ This shows that $\frac{|\log (z+h)-\log (z)|}{|h|} \leq \frac{2}{|z|}$ for $|h|$ small enough. Now

$$
\frac{|o(|\log (z+h)-\log (z)|)|}{|h|}=\frac{o(|\log (z+h)-\log (z)|)}{|\log (z+h)-\log (z)|} \frac{|\log (z+h)-\log (z)|}{|h|}
$$

and the second term on the right is bounded while the first converges to 0 as $h \rightarrow 0$. Therefore, $o(\log (z+h)-\log (z))=o(h)$ and so it follows from 14.6, $\log (z+h)-\log (z)=$ $\left(\frac{1}{z}\right) h+o(h)$ which shows that, just as in the real variable case $\log ^{\prime}(z)=\frac{1}{z}$.

Note that by the same arguments used for functions of a real variable $z \rightarrow \frac{1}{z}$ is continuous on $|z| \neq 0$.
Definition 14.3.1 For $a \in \mathbb{R}$, let $l$ be the ray from 0 in the complex plane which includes 0 and consider all complex numbers $D_{a}$ whose angle is in $(a-\pi, a+\pi)$ and not 0.

$$
\log (z)=\ln (|z|)+i \arg (z)
$$

where $\arg (z)$ is the angle for $z$ which is in $(a-\pi, a+\pi)$. This function is one to one and analytic on $D_{a}$ and $e^{\log (z)}=z$. This is called a branch of the logarithm. It is called the principal branch if the ray defining $D_{a}$ is 0 along with the negative real axis.

Note that $\log \left(D_{a}\right)$, is the open set in $\mathbb{C}$ defined by $\operatorname{Im} z \in(a-\pi, a+\pi)$. Thus there is a one to one and onto analytic map which maps $D_{a}$ onto

$$
\{z \in \mathbb{C}: \operatorname{Im} z \in(a-\pi, a+\pi)\}
$$

This book is not about a detailed study of such conformal maps, (analytic functions with values in $\mathbb{C}$ are called this) but this is an interesting example. Some people find these kind of mappings very useful and they are certainly beautiful when you keep track of level curves of real and imaginary parts. You can have lots of fun by having Matlab graph real and imaginary parts.

### 14.4 Contour Integrals

In the theory of functions of a complex variable, the most important results are those involving contour integration. I will use contour integration on curves of bounded variation as in [10], [31], [23] and referred to in [15]. This is more general than piecewise $C^{1}$ curves but most results can be obtained from only considering the special case. The most important tools in complex analysis are Cauchy's theorem in some form and Cauchy's formula for an analytic function. This section will give some of the very best versions of these theorems. They all involve something called a contour integral. Now a contour integral is just a sort of line integral as will be shown later. As earlier, $\gamma^{*}$ will denote the set of points and $\gamma$ will denote a parametrization. Here is the definition. It should look familiar and resemble a corresponding definition for line integrals presented earlier.

Definition 14.4.1 Let $\gamma:[a, b] \rightarrow \mathbb{C}$ be continuous and of bounded variation and let $f: \gamma^{*} \rightarrow X$ where $X$ is a complex Banach space, usually $\mathbb{C}$. Letting $P \equiv\left\{t_{0}, \cdots, t_{n}\right\}$ where $a=t_{0}<t_{1}<\cdots<t_{n}=b$, define

$$
\|P\| \equiv \max \left\{\left|t_{j}-t_{j-1}\right|: j=1, \cdots, n\right\}
$$

and the Riemann Stieltjes sum by $S(P) \equiv \sum_{j=1}^{n} f\left(\gamma\left(\tau_{j}\right)\right)\left(\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right)$ where $\tau_{j} \in$ $\left[t_{j-1}, t_{j}\right]$. (Note this notation is a little sloppy because it does not identify the specific point $\tau_{j}$ used. It is understood that this point is arbitrary.) Define $\int_{\gamma} f(z) d z$ as the unique number which satisfies the following condition. For all $\varepsilon>0$ there exists a $\delta>0$ such that if $\|P\| \leq \delta$, then $\left|\int_{\gamma} f(z) d z-S(P)\right|<\varepsilon$. Sometimes this is written as $\int_{\gamma} f(z) d z \equiv$ $\lim _{\|P\| \rightarrow 0} S(P)$.

You note that this is essentially the same definition given earlier for the line integral only this time the function has values in $\mathbb{C}$ (more generally $X$ ) rather than $\mathbb{R}^{n}$ and there is no dot product involved. Instead, you multiply by the complex number $\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)$ in the Riemann Stieltjes sum.

Since the contour integral is defined in terms of limits of sums, it follows that the contour integral is linear because sums are linear. This is just like what was done earlier for line integrals.

The fundamental result in this subject is the following theorem. It is just like the earlier material on line integrals. The proof is included for convenience.

Lemma 14.4.2 If $\gamma$ is $C^{1}$ on $[a, b]$ having values in $\mathbb{C}$, then $\int_{a}^{b} \gamma^{\prime}(s) d s=\gamma(b)-\gamma(a)$ and also the triangle inequality holds $\left|\int_{a}^{b} \gamma^{\prime}(s) d s\right| \leq \int_{a}^{b}\left|\gamma^{\prime}(s)\right| d s$.

Proof: The first claim is the fundamental theorem of calculus applied to real and imaginary parts. Consider the second. There exists $\omega \in \mathbb{C}$ with $|\omega|=1$ and $\left|\int_{a}^{b} \gamma^{\prime}(s) d s\right|=$
$\omega \int_{a}^{b} \gamma^{\prime}(s) d s$. Therefore,

$$
\left|\int_{a}^{b} \gamma^{\prime}(s) d s\right|=\int_{a}^{b} \omega \gamma^{\prime}(s) d s=\int_{a}^{b} \operatorname{Re}\left(\omega \gamma^{\prime}(t)\right) d s \leq \int_{a}^{b}\left|\gamma^{\prime}(t)\right| d t
$$

As before, the total variation is denoted by $V(\gamma,[a, b])$. The reason I am going through the details again and the argument may seem a little complicated is that the function does not have values in $\mathbb{R}^{p}$ but in some Banach space.

Theorem 14.4.3 Let $f: \gamma^{*} \rightarrow X$ be continuous and let $\gamma:[a, b] \rightarrow \mathbb{C}$ be continuous and of bounded variation. Then $\int_{\gamma} f d z$ exists. Also letting $\delta_{m}>0$ be such that $|t-s|<\delta_{m}$ implies $\|f(\gamma(t))-f(\gamma(s))\|<\frac{1}{m}$,

$$
\left\|\int_{\gamma} f d z-S(P)\right\| \leq \frac{2 V(\gamma,[a, b])}{m}
$$

whenever $\|P\|<\delta_{m}$. In addition to this, if $\gamma$ has a continuous derivative on $[a, b]$, the derivative taken from left or right at the endpoints, then

$$
\begin{equation*}
\int_{\gamma} f d z=\int_{a}^{b} f(\gamma(t)) \gamma^{\prime}(t) d t \tag{14.7}
\end{equation*}
$$

In general, if $\phi \in X^{\prime}$, then $\phi\left(\int_{\gamma} f d z\right)=\int_{\gamma} \phi(f) d z$.
Proof: The function, $f \circ \gamma$, is uniformly continuous because it is defined on a compact set. Therefore, there exists a decreasing sequence of positive numbers, $\left\{\boldsymbol{\delta}_{m}\right\}$ such that if $|s-t|<\delta_{m}$, then $\|f(\gamma(t))-f(\gamma(s))\|<\frac{1}{m}$. Let

$$
F_{m} \equiv \overline{\left\{S(P):\|P\|<\delta_{m}\right\}}
$$

Thus $F_{m}$ is a closed set. (The symbol, $S(P)$ in the above definition, means to include all sums corresponding to $P$ for any choice of $\tau_{j}$.) It is shown that

$$
\begin{equation*}
\operatorname{diam}\left(F_{m}\right) \leq \frac{2 V(\gamma,[a, b])}{m} \tag{14.8}
\end{equation*}
$$

and then it will follow there exists a unique point, $I \in \cap_{m=1}^{\infty} F_{m}$. This is because $X$, the space where $f$ has its values is complete and Theorem 14.1.1. It will then follow $I=\int_{\gamma} f d z$. To verify 14.8 , it suffices to verify that whenever $P$ and $Q$ are partitions satisfying $\|P\|<\delta_{m}$ and $\|Q\|<\delta_{m}$,

$$
\begin{equation*}
\|S(P)-S(Q)\| \leq \frac{2}{m} V(\gamma,[a, b]) \tag{14.9}
\end{equation*}
$$

Suppose $\|P\|<\delta_{m}$ and $Q \supseteq P$. Then also $\|Q\|<\delta_{m}$. To begin with, suppose that $P \equiv\left\{t_{0}, \cdots, t_{p}, \cdots, t_{n}\right\}$ and $Q \equiv\left\{t_{0}, \cdots, t_{p-1}, t^{*}, t_{p}, \cdots, t_{n}\right\}$. Thus $Q$ contains only one more point than $P$. Letting $S(Q)$ and $S(P)$ be Riemann Steiltjes sums,

$$
S(Q) \equiv \sum_{j=1}^{p-1} f\left(\gamma\left(\sigma_{j}\right)\right)\left(\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right)+f\left(\gamma\left(\sigma_{*}\right)\right)\left(\gamma\left(t^{*}\right)-\gamma\left(t_{p-1}\right)\right)
$$

$$
\begin{aligned}
& +f\left(\gamma\left(\sigma^{*}\right)\right)\left(\gamma\left(t_{p}\right)-\gamma\left(t^{*}\right)\right)+\sum_{j=p+1}^{n} f\left(\gamma\left(\sigma_{j}\right)\right)\left(\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right) \\
& S(P) \equiv \sum_{j=1}^{p-1} f\left(\gamma\left(\tau_{j}\right)\right)\left(\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right)+ \\
& \overbrace{f\left(\gamma\left(\tau_{p}\right)\right)\left(\gamma\left(t^{*}\right)-\gamma\left(t_{p-1}\right)\right)+f\left(\gamma\left(\tau_{p}\right)\right)\left(\gamma\left(t_{p}\right)-\gamma\left(t^{*}\right)\right)}^{\left.=f\left(\tau_{p}\right)\right)\left(\gamma\left(t_{p}\right)-\gamma\left(t_{p-1}\right)\right)} \\
& \quad+\sum_{j=p+1}^{n} f\left(\gamma\left(\tau_{j}\right)\right)\left(\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right) .
\end{aligned}
$$

Therefore,

$$
\begin{gather*}
\|S(P)-S(Q)\| \leq \sum_{j=1}^{p-1} \frac{1}{m}\left|\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right|+\frac{1}{m}\left|\gamma\left(t^{*}\right)-\gamma\left(t_{p-1}\right)\right|+ \\
\frac{1}{m}\left|\gamma\left(t_{p}\right)-\gamma\left(t^{*}\right)\right|+\sum_{j=p+1}^{n} \frac{1}{m}\left|\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right| \leq \frac{1}{m} V(\gamma,[a, b]) \tag{14.10}
\end{gather*}
$$

Clearly the extreme inequalities would be valid in 14.10 if $Q$ had more than one extra point. You simply do the above trick more than one time. Let $S(P)$ and $S(Q)$ be Riemann Steiltjes sums for which $\|P\|$ and $\|Q\|$ are less than $\delta_{m}$ and let $R \equiv P \cup Q$. Then from what was just observed,

$$
\|S(P)-S(Q)\| \leq\|S(P)-S(R)\|+\|S(R)-S(Q)\| \leq \frac{2}{m} V(\gamma,[a, b])
$$

and this shows 14.9 which proves 14.8 . Therefore, there exists a unique point, $I \in \cap_{m=1}^{\infty} F_{m}$ which satisfies the definition of $\int_{\gamma} f d z$.

Now consider the claim about $C^{1}$ contours. First, why is $\gamma$ of bounded variation if it is $C^{1}$ ? For $P \equiv\left\{t_{0}, t_{1}, \ldots, t_{n}\right\}$ a partition of $[a, b]$,

$$
\begin{aligned}
\sum_{k=1}^{n}\left|\gamma\left(t_{k}\right)-\gamma\left(t_{k-1}\right)\right| & =\sum_{k=1}^{n}\left|\int_{t_{k-1}}^{t_{k}} \gamma^{\prime}(s) d s\right| \leq \sum_{k=1}^{n} \int_{t_{k-1}}^{t_{k}}\left|\gamma^{\prime}(s)\right| d s \\
& \leq M \sum_{k=1}^{n}\left(t_{k}-t_{k-1}\right)=M(b-a)
\end{aligned}
$$

where $M \geq \max \left\{\left|\gamma^{\prime}(t)\right|: t \in[a, b]\right\}$. Thus $\int_{\gamma} f d z$ exists.
Let $P=\left\{t_{0}, \cdots, t_{n}\right\}$. Let $\gamma=\gamma_{1}+i \gamma_{2}, \gamma_{j}$ real. Then using the mean value theorem,

$$
\begin{aligned}
S(P) & =\sum_{k=1}^{n} f\left(\gamma\left(\sigma_{k}\right)\right)\left(\gamma\left(t_{k}\right)-\gamma\left(t_{k-1}\right)\right) \\
& =\sum_{k=1}^{n} f\left(\gamma\left(\sigma_{k}\right)\right)\left(\gamma_{1}\left(t_{k}\right)-\gamma_{1}\left(t_{k-1}\right)\right)+i \sum_{k=1}^{n} f\left(\gamma\left(\sigma_{k}\right)\right)\left(\gamma_{2}\left(t_{k}\right)-\gamma_{2}\left(t_{k-1}\right)\right) \\
& =\sum_{k=1}^{n} f\left(\gamma\left(\sigma_{k}\right)\right) \gamma_{1}^{\prime}\left(\tau_{k}\right)\left(t_{k}-t_{k-1}\right)+i \sum_{k=1}^{n} f\left(\gamma\left(\sigma_{k}\right)\right) \gamma_{2}^{\prime}\left(\xi_{k}\right)\left(t_{k}-t_{k-1}\right)
\end{aligned}
$$

where $\tau_{k}, \xi_{k} \in\left(t_{k-1}, t_{k}\right)$. By uniform continuity of $\gamma_{j}^{\prime}$, and continuity of $f$ on $\gamma^{*}$ which implies $\|f\|$ is bounded, the above equals

$$
=\sum_{k=1}^{n} f\left(\gamma\left(\sigma_{k}\right)\right) \gamma_{1}^{\prime}\left(\sigma_{k}\right)\left(t_{k}-t_{k-1}\right)+i \sum_{k=1}^{n} f\left(\gamma\left(\sigma_{k}\right)\right) \gamma_{2}^{\prime}\left(\sigma_{k}\right)\left(t_{k}-t_{k-1}\right)+e(\|P\|)
$$

where $\lim _{\|P\| \rightarrow 0} e(\|P\|)=0$. Therefore, passing to a limit gives

$$
\lim _{\|P\| \rightarrow 0} S(P)=\int_{a}^{b} f(\gamma(t))\left(\gamma_{1}^{\prime}(t)+i \gamma_{2}^{\prime}(t)\right) d t=\int_{a}^{b} f(\gamma(t)) \gamma^{\prime}(t) d t
$$

For the last claim, $\phi(f)$ is continuous and so for $\left\|P_{n}\right\| \rightarrow 0$,

$$
\phi\left(\int_{\gamma} f d z\right)=\lim _{n \rightarrow \infty} \phi\left(S\left(f, P_{n}\right)\right)=\lim _{n \rightarrow \infty} S\left(\phi(f), P_{n}\right)=\int_{\gamma} \phi(f) d z
$$

In the case that $f$ is specialized to have complex values, it is reasonable to ask for the real and imaginary parts of the contour integral. It turns out these are just line integrals. Let $z=x+i y$ and let $f(z) \equiv u(x, y)+i v(x, y)$. Also let the parametrization be $\gamma(t) \equiv$ $x(t)+i y(t)$. Then a term in the approximating sum is of the form

$$
\begin{aligned}
& (u+i v)\left(x\left(t_{k}\right)+i y\left(t_{k}\right)-\left(x\left(t_{k-1}\right)+i y\left(t_{k-1}\right)\right)\right) \\
= & (u+i v)\left(\left(x\left(t_{k}\right)-x\left(t_{k-1}\right)\right)+i\left(y\left(t_{k}\right)-y\left(t_{k-1}\right)\right)\right) \\
= & u\left(x\left(t_{k}\right)-x\left(t_{k-1}\right)\right)-v\left(y\left(t_{k}\right)-y\left(t_{k-1}\right)\right) \\
& +i\left[v\left(x\left(t_{k}\right)-x\left(t_{k-1}\right)\right)+u\left(y\left(t_{k}\right)-y\left(t_{k-1}\right)\right)\right]
\end{aligned}
$$

Thus in the limit, one obtains the contour integral is the sum of two line integrals

$$
\int_{\gamma}(u(x, y),-v(x, y)) \cdot d \mathbf{r}+i \int_{\gamma}(v(x, y), u(x, y)) \cdot d \mathbf{r}
$$

where $\mathbf{r}(t) \equiv(x(t), y(t))$. Also, if $F^{\prime}(z)=f(z)=(u+i v)$, then by the Cauchy Riemann equations, $u=\operatorname{Re}(F)_{x}=\operatorname{Im}(F)_{y}, v=\operatorname{Im}(F)_{x}=-\operatorname{Re}(F)_{y} .\left(F^{\prime}(z)=(\operatorname{Re} F)_{x}+i(\operatorname{Im} F)_{x}=\right.$ $\left.(\operatorname{Im} F)_{y}+i(-\operatorname{Re} F)_{y}\right)$.

Proposition 14.4.4 Suppose $f: \gamma^{*} \rightarrow \mathbb{C}$ is continuous for $\gamma:[a, b] \rightarrow \mathbb{C}$ bounded variation and continuous. Then if $\mathbf{r}(t) \equiv(\operatorname{Re} \gamma(t), \operatorname{Im} \gamma(t))$

$$
\int_{\gamma} f d z=\int_{\gamma}(u(x, y),-v(x, y)) \cdot d \mathbf{r}+i \int_{\gamma}(v(x, y), u(x, y)) \cdot d \mathbf{r}
$$

Also, if $F^{\prime}(z)=f(z)$, then

$$
\int_{\gamma} f d z=\int_{\gamma}\left(\operatorname{Re}(F)_{x}, \operatorname{Re}(F)_{y}\right) \cdot d \mathbf{r}+i \int_{\gamma}\left(\operatorname{Im}(F)_{x}, \operatorname{Im}(F)_{y}\right) \cdot d \mathbf{r}
$$

and so

$$
\begin{aligned}
\int_{\gamma} f d z & =\operatorname{Re}(F(\gamma(b)))-\operatorname{Re}(F(\gamma(a)))+i(\operatorname{Im}(F(\gamma(b)))-\operatorname{Im}(F(\gamma(a)))) \\
& \equiv F(\gamma(b))-F(\gamma(a))
\end{aligned}
$$

The last identity holds if $F$ has values in a complex Banach space.

Proof: This follows from the above observation which comes from the Cauchy Riemann equations and Theorem 5.3.1 the earlier result about conservative vector fields.

For the last claim, let $\phi \in X^{\prime}$. Then $\phi\left(\int_{\gamma} f d z\right)=\int_{\gamma} \phi(f) d z=\phi(F(\gamma(b))-F(\gamma(a)))$ and since $X^{\prime}$ separates the points of $X, \int_{\gamma} f d z=F(\gamma(b))-F(\gamma(a))$.

Definition 14.4.5 $A$ function $F$ such that $F^{\prime}=f$ is called a primitive of $f$.
As in the case of line integrals, these contour integrals are independent of parametrization in the sense that if $\gamma(t)=\eta(s)$ where $t=t(s)$ with $s \rightarrow t(s)$ an increasing continuous function, then $\int_{\gamma} f d z=\int_{\eta} f d w$.

Definition 14.4.6 If one reverses the order in which points of $\gamma^{*}$ are encountered, then one replaces $\gamma$ with $-\gamma$ in which, for $\gamma:[a, b] \rightarrow \mathbb{C},-\gamma(t)$ encounters the points of $\gamma^{*}$ in the opposite order, the definition of the contour integral shows that $-\int_{\gamma} f d z=\int_{-\gamma} f d z$. You could get a parametrization for $-\gamma$ as $-\gamma(t) \equiv \gamma(b-t)$ for $t \in[0, b-a]$ or if you wanted to use the same interval, define $-\gamma:[a, b] \rightarrow \mathbb{C}$ by $-\gamma(t) \equiv \gamma(b+a-t)$.

The following theorem follows easily from the above definitions and theorem. One can also see from the definition that something like the triangle inequality will hold. This is contained in the next theorem.

Theorem 14.4.7 Let $f$ be continuous on $\gamma^{*}$ having values in a complex Banach space $X$, writen as $f \in C\left(\gamma^{*}, X\right)$ and let $\gamma:[a, b] \rightarrow \mathbb{C}$ be of bounded variation and continuous. Let

$$
\begin{equation*}
M \geq \max \{\|f \circ \gamma(t)\|: t \in[a, b]\} . \tag{14.11}
\end{equation*}
$$

Then

$$
\begin{equation*}
\left\|\int_{\gamma} f d z\right\| \leq M V(\gamma,[a, b]) \tag{14.12}
\end{equation*}
$$

Also if $\left\{f_{n}\right\}$ is a sequence of functions of $C\left(\gamma^{*}, X\right)$ which is converging uniformly to the function $f$ on $\gamma^{*}$, then

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \int_{\gamma} f_{n} d z=\int_{\gamma} f d z \tag{14.13}
\end{equation*}
$$

Proof: Let 14.11 hold. From Theorem 14.4.3, when $\|P\|<\delta_{m}$,

$$
\left\|\int_{\gamma} f d z-S(P)\right\| \leq \frac{2}{m} V(\gamma,[a, b])
$$

and so

$$
\begin{aligned}
&\left\|\int_{\gamma} f d z\right\| \leq\|S(P)\|+ \frac{2}{m} V(\gamma,[a, b]) \leq \sum_{j=1}^{n} M\left|\gamma\left(t_{j}\right)-\gamma\left(t_{j-1}\right)\right|+\frac{2}{m} V(\gamma,[a, b]) \\
& \leq M V(\gamma,[a, b])+\frac{2}{m} V(\gamma,[a, b])
\end{aligned}
$$

This proves 14.12 since $m$ is arbitrary. To verify 14.13 use the above inequality to write

$$
\left\|\int_{\gamma} f d z-\int_{\gamma} f_{n} d z\right\|=\left\|\int_{\gamma}\left(f-f_{n}\right) d z\right\|
$$

$$
\leq \max \left\{\left\|f \circ \gamma(t)-f_{n} \circ \gamma(t)\right\|: t \in[a, b]\right\} V(\gamma,[a, b])
$$

Since the convergence is assumed to be uniform, this proves 14.13.
Now suppose $\gamma$ is continuous and bounded variation and $f: \gamma^{*} \times K \rightarrow \mathbb{C}$ is continuous where $K$ is compact. Then consider the function $F(w) \equiv \int_{\gamma} f(z, w) d z$

Lemma 14.4.8 The function $F$ just defined is continuous.
Proof: The function $f$ is uniformly continuous since it is continuous on a compact set. Therefore, there exists $\delta>0$ such that if $\left|\left(z_{1}, w_{1}\right)-\left(z_{2}, w_{2}\right)\right|<\delta$, then

$$
\left\|f\left(z_{1}, w_{1}\right)-f\left(z_{2}, w_{2}\right)\right\|<\varepsilon
$$

It follows that if $\left|w_{1}-w_{2}\right|<\boldsymbol{\delta}$, then from Theorem 14.4.7

$$
\left\|F\left(w_{1}\right)-F\left(w_{2}\right)\right\|=\left\|\int_{\gamma}\left(f\left(z, w_{1}\right)-f\left(z, w_{2}\right)\right) d z\right\| \leq \varepsilon V(\gamma,[a, b])
$$

Since $\varepsilon$ is arbitrary, this proves the lemma.
With this lemma, it becomes easy to give a version of Fubini's theorem.
Theorem 14.4.9 Let $\gamma_{i}$ be continuous and bounded variation. Let $f$ be continuous on $\gamma_{1}^{*} \times \gamma_{2}^{*}$ having values in $X$ a complex complete normed linear space. Then

$$
\int_{\gamma_{1}} \int_{\gamma_{2}} f(z, w) d w d z=\int_{\gamma_{2}} \int_{\gamma_{1}} f(z, w) d z d w
$$

Proof: This follows quickly from the above lemma and the definition of the contour integral. Say $\gamma_{i}$ is defined on $\left[a_{i}, b_{i}\right]$. Let a partition of $\left[a_{1}, b_{1}\right]$ be denoted by $\left\{t_{0}, t_{1}, \cdots, t_{n}\right\}=$ $P_{1}$ and a partition of $\left[a_{2}, b_{2}\right]$ be denoted by $\left\{s_{0}, s_{1}, \cdots, s_{m}\right\}=P_{2}$.

$$
\begin{gathered}
\int_{\gamma_{1}} \int_{\gamma_{2}} f(z, w) d w d z=\sum_{i=1}^{n} \int_{\gamma_{1}\left(\left[t_{i-1}, t_{i}\right]\right)} \int_{\gamma_{2}} f(z, w) d w d z \\
=\sum_{i=1}^{n} \sum_{j=1}^{m} \int_{\gamma_{1}\left(\left[t_{i-1}, t_{i}\right]\right)} \int_{\gamma_{2}\left(\left[s_{j-1}, s_{j}\right]\right)} f(z, w) d w d z
\end{gathered}
$$

To save room, denote $\gamma_{1}\left(\left[t_{i-1}, t_{i}\right]\right)$ by $\gamma_{1 i}$ and $\gamma_{2}\left(\left[s_{j-1}, s_{j}\right]\right)$ by $\gamma_{2 j}$ Then if $\left\|P_{i}\right\|, i=1,2$ is small enough, Theorem 14.4.7 implies

$$
\begin{gather*}
\left\|\int_{\gamma_{1 i}} \int_{\gamma_{2 j}} f(z, w) d w d z-\int_{\gamma_{1 i}} \int_{\gamma_{2 j}} f\left(\gamma_{1}\left(t_{i}\right), \gamma_{2}\left(s_{j}\right)\right) d w d z\right\| \\
=\left\|\int_{\gamma_{1 i}} \int_{\gamma_{2 j}}\left(f(z, w)-f\left(\gamma_{1}\left(t_{i}\right), \gamma_{2}\left(s_{j}\right)\right)\right) d w d z\right\| \leq \\
\max \left(\left\|\int_{\gamma_{2 j}}\left(f(z, w)-f\left(\gamma_{1}\left(t_{i}\right), \gamma_{2}\left(s_{j}\right)\right)\right) d w\right\|\right) V\left(\gamma_{1},\left[t_{i-1}, t_{i}\right]\right) \\
\leq \varepsilon V\left(\gamma_{2},\left[s_{j-1}, s_{j}\right]\right) V\left(\gamma_{1},\left[t_{i-1}, t_{i}\right]\right) \tag{14.14}
\end{gather*}
$$

Also from this theorem,

$$
\begin{gather*}
\left\|\int_{\gamma_{2 j}} \int_{\gamma_{1 i}} f(z, w) d z d w-\int_{\gamma_{2 j}} \int_{\gamma_{1 i}} f\left(\gamma_{1}\left(t_{i}\right), \gamma_{2}\left(s_{j}\right)\right) d z d w\right\| \\
\leq \max \left(\left\|\int_{\gamma_{1 i}}\left(f(z, w)-f\left(\gamma_{1}\left(t_{i}\right), \gamma_{2}\left(s_{j}\right)\right)\right) d z\right\|\right) V\left(\gamma_{2},\left[s_{j-1}, s_{j}\right]\right) \\
\leq \varepsilon V\left(\gamma_{2},\left[s_{j-1}, s_{j}\right]\right) V\left(\gamma_{1},\left[t_{i-1}, t_{i}\right]\right) \tag{14.15}
\end{gather*}
$$

Now approximating with sums and $\int_{\gamma_{1 i}} d z=\gamma_{1}\left(t_{j}\right)-\gamma_{1}\left(t_{j-1}\right)$, (Note that a primitive of 1 is $F(z)=z$ )

$$
\begin{align*}
& \quad \int_{\gamma_{2 j}} \int_{\gamma_{1 i}} f\left(\gamma_{1}\left(t_{i}\right), \gamma_{2}\left(s_{j}\right)\right) d z d w=f\left(\gamma_{1}\left(t_{i}\right), \gamma_{2}\left(s_{j}\right)\right) \int_{\gamma_{2 j}} \int_{\gamma_{1 i}} d z d w \\
& =f\left(\gamma_{1}\left(t_{i}\right), \gamma_{2}\left(s_{j}\right)\right) \int_{\gamma_{1 i}} \int_{\gamma_{2 j}} d w d z=\int_{\gamma_{1 i}} \int_{\gamma_{2 j}} f\left(\gamma_{1}\left(t_{i}\right), \gamma_{2}\left(s_{j}\right)\right) d w d z \tag{14.16}
\end{align*}
$$

Therefore,

$$
\begin{aligned}
& \left\|\int_{\gamma_{1}} \int_{\gamma_{2}} f(z, w) d w d z-\int_{\gamma_{2}} \int_{\gamma_{1}} f(z, w) d z d w\right\| \leq \\
& \left\|\begin{array}{c}
\sum_{i=1}^{n} \sum_{j=1}^{m} \int_{\gamma_{1 i}} \int_{\gamma_{2 j}} f(z, w) d w d z \\
-\sum_{i=1}^{n} \sum_{j=1}^{m} \int_{\gamma_{1 i}} \int_{\gamma_{2 j}} f\left(\gamma_{1}\left(t_{i}\right), \gamma_{2}\left(s_{j}\right)\right) d w d z
\end{array}\right\| \\
& +\left\|\begin{array}{c}
\sum_{i=1}^{n} \sum_{j=1}^{m} \int_{\gamma_{1 i}} \int_{\gamma_{2 j}} f\left(\gamma_{1}\left(t_{i}\right), \gamma_{2}\left(s_{j}\right)\right) d w d z \\
-\sum_{i=1}^{n} \sum_{j=1}^{m} \int_{\gamma_{2 j}} \int_{\gamma_{1 i}} f\left(\gamma_{1}\left(t_{i}\right), \gamma_{2}\left(s_{j}\right)\right) d z d w
\end{array}\right\| \\
& +\left\|\begin{array}{c}
\sum_{i=1}^{n} \sum_{j=1}^{m} \int_{\gamma_{2 j}} \int_{\gamma_{1 i}} f\left(\gamma_{1}\left(t_{i}\right), \gamma_{2}\left(s_{j}\right)\right) d z d w \\
-\sum_{i=1}^{n} \sum_{j=1}^{m} \int_{\gamma_{2 j}} \int_{\gamma_{1 i}} f(z, w) d z d w
\end{array}\right\|
\end{aligned}
$$

From 14.16 the middle term is 0 . Thus, from the estimates 14.15 and 14.14 ,

$$
\begin{aligned}
& \left\|\int_{\gamma_{1}} \int_{\gamma_{2}} f(z, w) d w d z-\int_{\gamma_{2}} \int_{\gamma_{1}} f(z, w) d z d w\right\| \\
\leq & 2 \varepsilon V\left(\gamma_{2},\left[a_{2}, b_{2}\right]\right) V\left(\gamma_{1},\left[a_{1}, b_{1}\right]\right)
\end{aligned}
$$

Since $\varepsilon$ is arbitrary, the two integrals are equal.

### 14.5 Primitives and Cauchy Goursat Theorem

In beginning calculus, the notion of an antiderivative was very important. It is similar for functions of complex variables. The role of a primitive is also a lot like a potential in computing line integrals. Recall that a primitive of $f$ is a function $F$ such that $F^{\prime}(z)=$ $f(z)$. In calculus, in the context of a function of one real variable, this is often called an antiderivative and every continuous function has one thanks to the fundamental theorem of calculus. However, it will be shown below that the situation is not at all the same for functions of a complex variable.

So what if a function has a primitive? Say $F^{\prime}(z)=f(z)$ where $f$ is continuous with values in $X$ a complex Banach space.

Theorem 14.5.1 Suppose $\gamma$ is continuous and also of bounded variation, and that $\gamma$ is a parametrization of $\Gamma$ where $t \in[a, b]$ and the orientation is from $t=a$ to $t=b$. Suppose $f: \Gamma \rightarrow X$ is continuous and has a primitive $F$. Thus $F^{\prime}(z)=f(z)$ for some open $\Omega \supseteq \Gamma$. Then $\int_{\gamma} f(z) d z=F(\gamma(b))-F(\gamma(a))$.

Proof: First suppose $f$ has values in $\mathbb{C}$. Then the desired conclusion is from Proposition 14.4.4. It holds in general because if $\phi \in X^{\prime}$, then $\phi(F)^{\prime}=\phi(f)$ and so from Theorem 14.4.3, $\phi(F(b)-F(a))=\phi(F(b))-\phi(F(a))=\int_{\gamma} \phi(f(z)) d z=\phi\left(\int_{\gamma} f(z) d z\right)$ By Proposition 12.3.9 about how the elements of $X^{\prime}$ separate points, $F(b)-F(a)=\int_{\gamma} f(z) d z$ as claimed.

Probably the most fundamental result in the subject is Cauchy's theorem which says that the contour integral of an analytic function over a simple closed curve equals 0 . The following is like what was first done by Cauchy back in the early 1800's. Recall that $f^{\prime}$ continuous is part of the definition of "analytic".

Theorem 14.5.2 Let $\gamma^{*}$ be a simple closed curve with parametrization $\gamma(t)$ having finite length, ( $\gamma$ has finite total variation). Letting $U$ be the inside, assume $U$ is a region for which Green's theorem holds. Let $f$ be analytic near $U \cup \gamma^{*}$ and have complex values. Then $\int_{\gamma} f d z=0$. The same conclusion holds if $f$ has values in a complex Banach space.

Proof: From Observation 14.4.4,

$$
\int_{\gamma} f d z=\int_{C}(u(x, y),-v(x, y)) \cdot d \mathbf{r}+i \int_{C}(v(x, y), u(x, y)) \cdot d \mathbf{r}
$$

where $C$ is the oriented simple closed curve in the plane resulting from $\gamma^{*}$. Now by Green's theorem, this equals $\int_{U_{i}}\left((-v)_{x}-u_{y}\right) d m_{2}+i \int_{U_{i}}\left(u_{x}-v_{y}\right) d m_{2}=0$ because of the Cauchy Riemann equations, Proposition 14.2.2. To obtain the last claim, by Theorem 14.4.3, $\phi\left(\int_{\gamma} f d z\right)=\int_{\gamma} \phi(f) d z=0$. By Proposition 12.3.9 about how the elements of $X^{\prime}$ separate points, it follows that $\int_{\gamma} f d z=0$.

The following picture illustrates the theorem which follows in which we punch holes in $U$ above. Assume the closures of these holes do not intersect, as illustrated in the picture.


Assume the curves $\gamma_{k}$ illustrated above bound ellipses or circular disks or more generally regions for which Green's theorem holds. The main interest is in circular disks. Orient these clockwise as shown. Then using Corollary 10.8.4, it follows that for $U$ the inside of $\gamma_{1}$ and $U_{j}$ the inside of $\bar{\gamma}_{j}, U \backslash\left(\cup_{j=1}^{n} \bar{U}_{j}\right)$ also satisfies Green's theorem. In this picture $n=3$. The following is a generalization of the above theorem.

Theorem 14.5.3 Let $\gamma^{*}$ be a simple closed curve with parametrization $\gamma(t)$ having counter clockwise orientation and finite length, ( $\gamma$ has finite total variation). Letting $U$ be the inside, assume $U$ is a region for which Green's theorem holds with the orientation of $\gamma$. Let the $U_{j}$ be as described above for $j=1,2, \ldots, n$ and suppose $f$ is analytic near the closed
set $\left(U \cup \gamma^{*}\right) \backslash \cup_{j=1}^{n} U_{j}$ on an open set $\Omega$. (Maybe $f$ is not analytic at the $z_{j}$ for example.) Then $\int_{\gamma} f(w) d w+\sum_{j=1}^{n} \int_{\bar{\gamma}_{j}} f(w) d w=0$. Letting $\gamma_{j}$ be the opposite orientation which is counter clockwise, $\int_{\gamma} f(w) d w=\sum_{j=1}^{n} \int_{\gamma_{j}} f(w) d w$.

Proof: This follows from Corollary 10.8.4 and a repeat of the above proof which uses the Cauchy Riemann equations and Green's theorem.

If you use the general Green's theorem of the appendix, you can generalize the assumptions of this theorem. Green's theorem holds automatically if $\gamma^{*}$ has finite length and it suffices to have $f$ continuous on $U \cup \gamma^{*}$ and analytic on $U$.

The following Cauchy Goursat theorem will provide the needed generalization which involves not assuming that $z \rightarrow f^{\prime}(z)$ is continuous.

If you have two points in $\mathbb{C}, z_{1}$ and $z_{2}$, you can consider $\gamma(t) \equiv z_{1}+t\left(z_{2}-z_{1}\right)$ for $t \in[0,1]$ to obtain a continuous bounded variation curve from $z_{1}$ to $z_{2}$. More generally, if $z_{1}, \cdots, z_{m}$ are points in $\mathbb{C}$ you can obtain a continuous bounded variation curve from $z_{1}$ to $z_{m}$ which consists of first going from $z_{1}$ to $z_{2}$ and then from $z_{2}$ to $z_{3}$ and so on, till in the end one goes from $z_{m-1}$ to $z_{m}$. Denote this piecewise linear curve as $\gamma\left(z_{1}, \cdots, z_{m}\right)$. Now let $T$ be a triangle with vertices $z_{1}, z_{2}$ and $z_{3}$ encountered in the counter clockwise direction as shown.


Denote by $\int_{\partial T} f(z) d z$, the expression, $\int_{\gamma\left(z_{1}, z_{2}, z_{3}, z_{1}\right)} f(z) d z$. Consider the following picture.


Thus

$$
\begin{equation*}
\int_{\partial T} f(z) d z=\sum_{k=1}^{4} \int_{\partial T_{k}^{1}} f(z) d z \tag{14.17}
\end{equation*}
$$

On the "inside lines" the integrals cancel because there are two integrals going in opposite directions for each of these inside lines. Recall Theorem 14.4.3 which tells how to evaluate a line integral with a $C^{1}$ parametrization.

Theorem 14.5.4 (Cauchy Goursat) Let $f: \Omega \rightarrow X$, where $\Omega$ is an open subset of $\mathbb{C}$ and $X$ is a complex Banach space, have the property that $f^{\prime}(z)$ exists for all $z \in \Omega$ and let $T$ be a triangle contained in $\Omega$,meaning that the triangle and its inside is contained in $\Omega$. Then $\int_{\partial T} f(w) d w=0$.

Proof: Suppose not. Then $\left\|\int_{\partial T} f(w) d w\right\|=\alpha \neq 0$.From 14.17 it follows

$$
\alpha \leq \sum_{k=1}^{4}\left\|\int_{\partial T_{k}^{1}} f(w) d w\right\|
$$

and so for at least one of these $T_{k}^{1}$, denoted from now on as $T_{1},\left\|\int_{\partial T_{1}} f(w) d w\right\| \geq \frac{\alpha}{4}$. Now
let $T_{1}$ play the same role as $T$. Subdivide as in the above picture, and obtain $T_{2}$ such that

$$
\left\|\int_{\partial T_{2}} f(w) d w\right\| \geq \frac{\alpha}{4^{2}} .
$$

Continue in this way, obtaining a sequence of triangles,

$$
T_{k} \supseteq T_{k+1}, \operatorname{diam}\left(T_{k}\right) \leq \operatorname{diam}(T) 2^{-k}
$$

and $\left\|\int_{\partial T_{k}} f(w) d w\right\| \geq \frac{\alpha}{4^{k}}$. Then let $z \in \cap_{k=1}^{\infty} T_{k}$ and note that by assumption, $f^{\prime}(z)$ exists. Therefore, for all $k$ large enough,

$$
\int_{\partial T_{k}} f(w) d w=\int_{\partial T_{k}}\left(f(z)+f^{\prime}(z)(w-z)+g(w)\right) d w
$$

where $|g(w)|<\varepsilon|w-z|$. Now observe that $w \rightarrow f(z)+f^{\prime}(z)(w-z)$ has a primitive, namely,

$$
F(w)=f(z) w+f^{\prime}(z)(w-z)^{2} / 2
$$

Therefore, by Theorem 14.5.1, $\int_{\partial T_{k}} f(w) d w=\int_{\partial T_{k}} g(w) d w$. From Theorem 14.4.7,

$$
\begin{aligned}
\frac{\alpha}{4^{k}} & \leq\left\|\int_{\partial T_{k}} g(w) d w\right\| \leq \varepsilon \operatorname{diam}\left(T_{k}\right)\left(\text { length of } \partial T_{k}\right) \\
& \leq \varepsilon 2^{-k}(\text { length of } T) \operatorname{diam}(T) 2^{-k}
\end{aligned}
$$

and so $\alpha \leq \varepsilon$ (length of $T$ ) $\operatorname{diam}(T)$. Since $\varepsilon$ is arbitrary, this shows $\alpha=0$, a contradiction. Thus $\int_{\partial T} f(w) d w=0$ as claimed.

Note that no assumption of continuity of $z \rightarrow f^{\prime}(z)$ was needed.
Obviously, there is a version of the above Cauchy Goursat theorem which is valid for a rectangle. Indeed, apply the Cauchy Goursat theorem for the triangles obtained from a diagonal of the rectangle. The diagonal will be oriented two different ways depending on which triangle it is a part of.


Corollary 14.5.5 Let $\Omega$ be an open set on which $f^{\prime}(z)$ exists. Here $f$ has values in a complex Banach space. Then if $R$ is a rectangle contained in $\Omega$ along with its inside, then $\int_{R} f(z) d z=0$.

### 14.6 Primitives for Differentiable Functions

This section is on the existence of primitives.
Theorem 14.6.1 (Morera ${ }^{1}$ ) Let $\Omega$ be an open set and let $f^{\prime}(z)$ exist for all $z \in \Omega$, where $f: \Omega \rightarrow X$ a complex Banach space. Let $D \equiv \overline{B\left(z_{0}, r\right)} \subseteq \Omega$. Then there exists $\varepsilon>0$ such that $f$ has a primitive on $B\left(z_{0}, r+\varepsilon\right)$.

[^7]Proof: Choose $\varepsilon>0$ small enough that $B\left(z_{0}, r+\varepsilon\right) \subseteq \Omega$. Then for $w \in B\left(z_{0}, r+\varepsilon\right)$, define $F(w) \equiv \int_{\gamma\left(z_{0}, w\right)} f(u) d u$. Then by the Cauchy Goursat theorem, and $w \in B\left(z_{0}, r+\varepsilon\right)$, it follows that for $|h|$ small enough,

$$
\frac{F(w+h)-F(w)}{h}=\frac{1}{h} \int_{\gamma(w, w+h)} f(u) d u=\frac{1}{h} \int_{0}^{1} f(w+t h) h d t=\int_{0}^{1} f(w+t h) d t
$$

which converges to $f(w)$ due to the continuity of $f$ at $w$.

## Definition 14.6.2 A star shaped open set $\Omega$ has a special point $p$ called a star

 center such that $\gamma(p, z)$ is contained in $\Omega$ for every $z \in \Omega$.Proposition 14.6.3 Let $f^{\prime}(z)$ exist for all $z \in \Omega$ a star shaped open set. Then $f$ has a primitive on $\Omega$.

Proof: Define the primitive as $F(w) \equiv \int_{\gamma(p, w)} f(u) d u$ where $p$ is the star center.

### 14.7 The Winding Number

First I will give a heuristic description which gives a useful way to see what the winding number is for simple enough contours. The winding number of a circle oriented counter clockwise is 1 . You simply parametrize the circle in the counter clockwise direction and compute the contour integral or use the Cauchy integral formula for a circle with $f(w)=1$. However, more generally, it is helpful to use the analytic function $\log$. Now $\log (w-z)=$ $\ln |w-z|+i \arg (w-z)$ is a primitive for $\frac{1}{w-z}$ for $w-z$ not on the negative real axis. That is, for $w-z$ a complex number not a negative real number. Thus the integral is $\ln (z-\operatorname{Re} z)+$ $i \pi-(\ln (z-\operatorname{Re} z)+i(-\pi))=2 \pi i$ and so the winding number is 1 . Actually, you would let $\varepsilon \rightarrow 0$ in the following where the angles between the horizontal line beginning at $z$ and the other two lines are both $\varepsilon$ and obtain the integral as

$$
\lim _{\varepsilon \rightarrow 0} \ln \left|w_{\varepsilon}^{+}\right|+i(\pi+\varepsilon)-\left(\ln \left|w_{\varepsilon}^{-}\right|+i(-\pi-\varepsilon)\right)
$$

where $w_{\varepsilon}^{+} \rightarrow z-\operatorname{Re} z$ and $w_{\varepsilon}^{-} \rightarrow z-\operatorname{Re} z$. It is illustrated in the case of a circle.


Thus the definition of a winding number is $n(z, \gamma) \equiv \frac{1}{2 \pi i} \int_{\gamma} \frac{1}{w-z} d w$, and as just described, it gives $\pm 1$ depending on the orientation of the curve. It was shown for a circle in the above, but the same would result if you had other sufficiently simple closed curve with rectifiable boundary having $z$ on its inside. This is because you would have $|z-w|$ bounded away from 0 and so the extra pieces of the line integral disappear in the limit and you simply pick up the jump in $\arg (z)$ which is $2 \pi$ or $-2 \pi$ depending on the direction of motion. This is how we define positive and negative directions on a closed curve in the plane.

However, if you have a simple closed curve $\Gamma$ and $z$ is not on its inside, then you could obtain a branch of the logarithm by sending a ray from $z$ away from the simple closed curve
and letting $\log (w-z)$ be determined for $w$ not on this ray, a star shaped region. Thus the entire simple closed curve would be in a star shaped open set on which $w \rightarrow \log (w-z)$ is analytic and so $\int_{\gamma} \frac{1}{w-z} d w=0$ because the integrand has a primitive.

But what if you have a simple closed curve with $z$ on its inside/outside but every ray from $z$ intersects the curve in many points? It becomes increasingly unclear that the above simple argument is satisfactory. It is fine for things like circles and rectangles and ellipses and in fact is clear for the curves we typically encounter in complex analysis. However, the winding number of a general situation still is an integer which is shown next.

Note that the complement of a compact set has exactly one unbounded connected component. This is because if $K$ is the compact set, then $K \subseteq B(0, r)$ for large $r$ and the connected set $B(0, r)^{C} \subseteq K^{C}$. Thus any other components are contained in $B(0, r)$.

Proposition 14.7.1 Let $\gamma:[a, b] \rightarrow \mathbb{C}$ be a parametrization of a closed curve $\gamma^{*}$ of finite length and $\gamma(a)=\gamma(b)$. Then there is an integer $m$ such that

$$
m=n(\gamma, z)=\frac{1}{2 \pi i} \int_{\gamma} \frac{1}{w-z} d w, z \notin \gamma^{*}
$$

Then also $z \rightarrow n(\gamma, z)$ is continuous and so it is a constant on every component of $\gamma^{* C}$. It equals 0 on the unbounded component of $\gamma^{* C}$.

Proof: For convenience let the interval be $[0,2 \pi]$ and $\gamma(0)=\gamma(2 \pi)$ and first assume $\gamma$ is a $C^{1}$ function. Let $F(t) \equiv \int_{0}^{t} \frac{\gamma^{\prime}(s)}{\gamma(s)-z} d s$. Then $F(2 \pi)=\int_{\gamma} \frac{1}{w-z} d w$ and

$$
\left(e^{-F(t)}(\gamma(t)-z)\right)^{\prime}=\frac{-\gamma^{\prime}(t)}{\gamma(t)-z} e^{-F(t)}(\gamma(t)-z)+e^{-F(t)} \gamma^{\prime}(t)=0
$$

and so $e^{-F(2 \pi)}(\gamma(2 \pi)-z)=(\gamma(0)-z)$. Since $\gamma(2 \pi)=\gamma(0), e^{F(2 \pi)}=1$ and so $F(2 \pi)=$ $m 2 \pi i$ for some integer $m$. Hence $\frac{1}{2 \pi i} \int_{\gamma} \frac{1}{w-z} d w=m$.

In case $\gamma$ is only of bounded variation, let $\gamma_{n} \rightarrow \gamma$ uniformly on $[0,2 \pi]$ and

$$
\frac{1}{2 \pi i} \int_{\gamma_{n}} \frac{1}{w-z} d w \rightarrow \frac{1}{2 \pi i} \int_{\gamma} \frac{1}{w-z} d w
$$

where each $\gamma_{n}$ is $C^{1}$ and coincides with $\gamma$ at the end points of the interval. This is from Theorem 5.2.3. Since $n\left(\gamma_{n}, z\right)$ is an integer, eventually for large $n, n\left(\gamma_{n}, z\right)=m$ for some integer $m$ and so $\frac{1}{2 \pi i} \int_{\gamma} \frac{1}{w-z} d w=m$.

As to continuity, $\left|\int_{\gamma} \frac{1}{w-z} d w-\int_{\gamma} \frac{1}{w-\hat{z}} d w\right|=\left|\int_{\gamma} \frac{z-\hat{z}}{(w-z)(w-\hat{\hat{z}})} d w\right| \leq|z-\hat{z}| \frac{1}{\delta^{2}}|\gamma|$ where $|\gamma|$ is the length of $\gamma$ and $0<\delta \leq \min \left(\operatorname{dist}\left(z, \gamma^{*}\right)\right.$, $\left.\operatorname{dist}\left(\hat{z}, \gamma^{*}\right)\right)$. Thus $z \rightarrow n(\gamma, z)$ is continuous and integer valued so by Corollary 2.10 .13 , it must be constant on every component of $\gamma^{* C}$. Thus $n(\gamma, z)=0$ on the unbounded component because you can let $|z| \rightarrow \infty$ in the formula for the winding number. The winding number is constant, but as $|z| \rightarrow \infty, n(\gamma, z) \rightarrow 0$.

### 14.8 The Cauchy Formula

The issue which must be considered next is the continuity of the derivative. Recall that we have theorems about analytic functions and we have theorems about differentiable functions. It turns out that the derivative will end up being continuous. Thus one can use either analytic or differentiable in what follows.

Consider the following picture where you have a large circle of radius $R$ and a small circle of radius $r$ centered at $z$, a point on the inside of $\gamma_{R}$. The Cauchy integral formula gives $f(z)$ in terms of the values of $f$ on the large circle.

Theorem 14.8.1 If $f$ is differentiable on $\Omega$ an open set, then $f^{\prime}$ is continuous and in fact, $f(z)=\frac{1}{2 \pi i} \int_{\gamma_{r}} \frac{f(w)}{w-z} d w$ where $\overline{B\left(z_{0}, r\right)} \subseteq \Omega, \gamma_{r}$ the counter clockwise boundary of the ball.

Proof: Let $B\left(z_{0}, r+\boldsymbol{\varepsilon}\right) \subseteq \Omega$. Let $z \in B\left(z_{0}, r\right)$ and let $\gamma_{r}$ be the counter clockwise orientation of the circle of radius $r$ which is the boundary of $B\left(z_{0}, r\right)$. Pick $\delta>0$ such that the following picture is applicable, the small circle having radius $\delta$.


There are two closed curves $\Gamma_{1}, \Gamma_{2}$ which intersect in vertical lines in the above picture, these contours each contained in a star shaped region on which the derivative of $f$ exists. Therefore, there exists a primitive of $w \rightarrow \frac{f(w)-f(z)}{w-z}$ valid on those two star shaped regions. Then adding the contour integrals which are each 0 ,

$$
\int_{\gamma_{r}} \frac{f(w)-f(z)}{w-z} d w=\int_{\gamma_{\delta}} \frac{f(w)-f(z)}{w-z} d w
$$

Since $f^{\prime}(z)$ exists, the integral on the right converges as $\delta \rightarrow 0$ to 0 . Then it follows that

$$
\int_{\gamma_{r}} \frac{f(w)}{w-z} d w=\int_{\gamma_{r}} \frac{f(z)}{w-z} d w=n(f, z)(2 \pi i) f(z)
$$

However, from the above argument about the winding number, $n(f, z)=1$. Therefore, $f(z)=\frac{1}{2 \pi i} \int_{\gamma_{r}} \frac{f(w)}{w-z} d w$. Then using the above representation

$$
\frac{f(z+h)-f(z)}{h}=\frac{1}{2 \pi i} \int_{\gamma_{r}} f(w)\left(-\frac{1}{(w-z)(h-w+z)}\right) d w
$$

Passing to a limit as $h \rightarrow 0$, this yields $f^{\prime}(z)=\frac{1}{2 \pi i} \int_{\gamma_{r}} \frac{f(w)}{(w-z)^{2}} d w$. If desired, you could continue taking derivatives and so $f^{\prime}$ is continuous and in fact, $f$ is infinitely differentiable.

Corollary 14.8.2 The same Cauchy integral formula holds if $f$ is only differentiable on the inside of the large circle and continuous on its closure.

Proof: Letting $z$ be on the inside of the large circle, you could shrink the large circle by decreasing its radius to $\hat{r}$ and be in the situation of the above theorem. Thus

$$
f(z)=\frac{1}{2 \pi i} \int_{\gamma_{\hat{r}}} \frac{f(w)}{w-z} d w=\frac{1}{2 \pi} \int_{0}^{2 \pi} \frac{f\left(z_{0}+\hat{r} e^{i t}\right)}{z_{0}+\hat{r} e^{i t}-z} \hat{r} e^{i t} d t
$$

and now take the limit as $\hat{r} \rightarrow r$ and use uniform continuity of $f$ on $\overline{B\left(z_{0}, r\right)}$ to obtain $f(z)=$ $\frac{1}{2 \pi i} \int_{\gamma_{r}} \frac{f(w)}{w-z} d w$.

This is the Cauchy integral formula for a disk. This remarkable formula is sufficient to show that if a function has a derivative, then it has infinitely many and in addition to this, the function can be represented as a power series. Let $z_{0}$ be the center of the large circle.

In the situation of Theorem 14.8.1,

$$
f(z)=\frac{1}{2 \pi i} \int_{\gamma_{R}} \frac{f(w)}{w-z_{0}-\left(z-z_{0}\right)} d w=\frac{1}{2 \pi i} \int_{\gamma_{R}} \frac{1}{w-z_{0}} \frac{f(w)}{1-\frac{z-z_{0}}{w-z_{0}}} d w
$$

Now $\left|\frac{z-z_{0}}{w-z_{0}}\right|=\frac{\left|z-z_{0}\right|}{R}<1$ for all $w \in \gamma_{R}^{*}$. Therefore, the above equals

$$
\frac{1}{2 \pi i} \int_{\gamma_{R}} \sum_{k=0}^{\infty} \frac{f(w)\left(z-z_{0}\right)^{k}}{\left(w-z_{0}\right)^{k+1}} d w=\frac{1}{2 \pi i} \int_{\gamma_{R}}\left(\sum_{k=0}^{\infty} \frac{\left(z-z_{0}\right)^{k}}{\left(w-z_{0}\right)^{k+1}}\right) f(w) d w
$$

Since $f$ is continuous, one can apply the Weierstrass $M$ test Theorem 2.5.42 to conclude that the above series converges uniformly on $\gamma_{R}^{*}$. Then the above reduces to

$$
f(z)=\frac{1}{2 \pi i} \sum_{k=0}^{\infty}\left(\int_{\gamma_{R}} \frac{f(w)}{\left(w-z_{0}\right)^{k+1}} d w\right)\left(z-z_{0}\right)^{k}
$$

This proves part of the next theorem which says, among other things, that when $f$ has one derivative on the interior of a circle, then it must have all derivatives. Note that the function has values in a complex Banach space.

Theorem 14.8.3 Suppose $z_{0} \in U$, an open set in $\mathbb{C}$ and $f: U \rightarrow X$ has a derivative for each $z \in U$. Then if $B\left(z_{0}, R\right) \subseteq U$, then for each $z \in B\left(z_{0}, R\right)$,

$$
\begin{equation*}
f(z)=\sum_{n=0}^{\infty} a_{n}\left(z-z_{0}\right)^{n} \tag{14.18}
\end{equation*}
$$

where $a_{n} \equiv \frac{1}{2 \pi i} \int_{\gamma_{R}} \frac{1}{\left(w-z_{0}\right)^{n+1}} f(w) d w \in X$ and $\gamma_{R}$ is a positively oriented parametrization for the circle bounding $B\left(z_{0}, R\right)$. Then

$$
\begin{gather*}
f^{(k)}\left(z_{0}\right)=k!a_{k}  \tag{14.19}\\
\lim _{n \rightarrow \infty}\left\|a_{n}\right\|^{1 / n}\left|z-z_{0}\right|<1  \tag{14.20}\\
f^{(k)}(z)=\sum_{n=k}^{\infty} n(n-1) \cdots(n-k+1) a_{n}\left(z-z_{0}\right)^{n-k}, \tag{14.21}
\end{gather*}
$$

Proof: 14.18 follows from the above argument. Now consider 14.20. The above argument based on the Cauchy integral formula for a disk shows that if $R>\left|\hat{z}-z_{0}\right|>\left|z-z_{0}\right|$, then $f(\hat{z})=\sum_{n=0}^{\infty} a_{n}\left(\hat{z}-z_{0}\right)^{n}$ and so, by the root test, Theorem 1.12.1,

$$
1 \geq \lim \sup _{n \rightarrow \infty}\left\|a_{n}\right\|^{1 / n}\left|\hat{z}-z_{0}\right|>\lim \sup _{n \rightarrow \infty}\left\|a_{n}\right\|^{1 / n}\left|z-z_{0}\right|
$$

Consider 14.21 which involves identifying the $a_{n}$ in terms of the derivatives of $f$. This is obvious if $k=0$. Suppose it is true for $k$. Then for small $h \in \mathbb{C}$,

$$
\begin{gathered}
\quad \frac{1}{h}\left(f^{(k)}(z+h)-f^{(k)}(z)\right) \\
=\frac{1}{h} \sum_{n=k}^{\infty} n(n-1) \cdots(n-k+1) a_{n}\left(\left(z+h-z_{0}\right)^{n-k}-\left(z-z_{0}\right)^{n-k}\right) \\
=\frac{1}{h} \sum_{n=k+1}^{\infty} n(n-1) \cdots(n-k+1) a_{n}\left(\begin{array}{c}
\sum_{j=0}^{n-k}\left(\begin{array}{c}
n-k \\
j \\
-\left(z-z_{0}\right)^{n-k}
\end{array}\right) h^{j}\left(z-z_{0}\right)^{(n-k)-j} \\
=\sum_{n=k+1}^{\infty} n(n-1) \cdots(n-k+1) a_{n}\left(\sum_{j=1}^{n-k}\binom{n-k}{j} h^{j-1}\left(z-z_{0}\right)^{(n-k)-j}\right) \\
= \\
+h \sum_{n=k+1}^{\infty} n(n-1) \cdots(n-k+1)(n-k) a_{n}\left(z-z_{0}\right)^{(n-k)-1} n(n-1) \cdots(n-k+1) a_{n}\left(\sum_{j=2}^{n-k}\binom{n-k}{j} h^{j-2}\left(z-z_{0}\right)^{(n-k)-j}\right)
\end{array}\right.
\end{gathered}
$$

By what was shown earlier, $\lim \sup _{n \rightarrow \infty}\left\|a_{n}\right\|^{1 / n}\left|z-z_{0}\right|<1$. Consider the norm of the part in the above which multiplies $h,|h|<1$.

$$
\begin{gathered}
\left\|n(n-1) \cdots(n-k+1) a_{n}\left(\sum_{j=2}^{n-k}\binom{n-k}{j} h^{j-2}\left(z-z_{0}\right)^{(n-k)-j}\right)\right\| \\
\leq n(n-1) \cdots(n-k+1)\left\|a_{n}\right\|\left|z-z_{0}\right|^{(n-k)-2} \sum_{j=2}^{n-k}\binom{n-k}{j}\left(\frac{|h|}{\left|z-z_{0}\right|}\right)^{j-2} \\
\binom{m}{j}=\frac{m(m-1) \cdots(m-j+1)}{j!} \\
=\frac{m(m-1) \cdots(m-j+3)(m-j+2)(m-j+1)}{j(j-1)(j-2)!} \\
=\binom{m}{j-2} \frac{(m-j+2)(m-j+1)}{j(j-1)} \leq\binom{ m}{j-2} \frac{(m)(m-1)}{2}
\end{gathered}
$$

Thus the above is no more than

$$
\begin{aligned}
& \leq n(n-1) \cdots(n-k+1)\left\|a_{n}\right\|\left|z-z_{0}\right|^{(n-k)-2} \\
& \quad \frac{(n-k)((n-k)-1)}{2} \sum_{j=2}^{n-k}\binom{n-k}{j-2}\left(\frac{h}{\left|z-z_{0}\right|}\right)^{j-2} \\
& \leq n(n-1) \cdots(n-k+1)(n-k)((n-k)-1)\left\|a_{n}\right\|\left(1+\frac{|h|}{\left|z-z_{0}\right|}\right)^{n-k}
\end{aligned}
$$

and

$$
\begin{aligned}
& \limsup _{n \rightarrow \infty}\binom{n(n-1) \cdots(n-k+1)(n-k)((n-k)-1) \cdot}{\left\|a_{n}\right\|\left(1+\frac{|h|}{\left|z-z_{0}\right|}\right)^{n-k}}^{1 / n} \\
= & \lim \sup _{n \rightarrow \infty}\left\|a_{n}\right\|^{1 / n}\left|z-z_{0}\right|\left(1+\frac{|h|}{\left|z-z_{0}\right|}\right)<1 \text { if } h \text { is small enough. }
\end{aligned}
$$

Thus, by the root test, the infinite series which multiplies $h$ converges for small $h$ and is decreasing in $h$ and so that entire term converges to 0 as $h \rightarrow 0$. This leaves

$$
f^{(k+1)}(z)=\sum_{n=k+1}^{\infty} n(n-1) \cdots(n-k+1)(n-k) a_{n}\left(z-z_{0}\right)^{n-(k+1)}
$$

Corollary 14.8.4 Suppose $f$ is continuous on $\partial B\left(z_{0}, r\right)$ and suppose that for all $z \in$ B $\left(z_{0}, r\right)$,

$$
f(z) \equiv \frac{1}{2 \pi i} \int_{\gamma} \frac{f(w)}{w-z} d w
$$

where $\gamma(t) \equiv z_{0}+r e^{i t}, t \in[0,2 \pi]$. Then $f$ is analytic on $B\left(z_{0}, r\right)$ and in fact has infinitely many derivatives on $B\left(z_{0}, r\right)$.

Proof: This is just a repeat of the above arguments. You show that $f(z)$ is given by a power series for $\left|z-z_{0}\right|<r$ and from this, the result follows.

Also, the following illustrates a difference from what is expected in real analysis. It says that uniform convergence tends to take with it differentiability.

Lemma 14.8.5 Let $\gamma(t)=z_{0}+r e^{i t}$, for $t \in[0,2 \pi]$. Let $f_{n} \rightarrow f$ uniformly on $\overline{B\left(z_{0}, r\right)}$ and suppose $f_{n}(z)=\frac{1}{2 \pi i} \int_{\gamma} \frac{f_{n}(w)}{w-z} d w$ for $z \in B\left(z_{0}, r\right)$. Then it follows that $f(z)=\frac{1}{2 \pi i} \int_{\gamma} \frac{f(w)}{w-z} d w$, implying that $f$ is analytic on $B\left(z_{0}, r\right)$.

Proof: From the formula for $f_{n}$ and the uniform convergence of $f_{n}$ to $f$ on $\gamma^{*}$, the integrals converge to $\frac{1}{2 \pi i} \int_{\gamma} \frac{f(w)}{w-z} d w$. Therefore, $f(z)=\frac{1}{2 \pi i} \int_{\gamma} \frac{f(w)}{w-z} d w$.

Note that this shows you can't expect anything like the Weierstrass approximation theorem to hold. That theorem allows the uniform appoximation of a continuous nowhere differentiable function with polynomials.

Because of Theorem 14.8.3, from now on, the term analytic will be used interchangeably with "has a derivative". This has shown that if the function has one derivative on an open set, then it has all of them. Now here is another version of Morera's theorem.

Corollary 14.8.6 Let $\Omega$ be an open set. Suppose that whenever $\gamma\left(z_{1}, z_{2}, z_{3}, z_{1}\right)$ is a closed curve bounding a triangle $T$, which is contained in $\Omega$, and $f$ is a continuous function defined on $\Omega$, it follows that $\int_{\gamma\left(z_{1}, z_{2}, z_{3}, z_{1}\right)} f(z) d z=0$, then $f$ is analytic on $\Omega$.

Proof: As in the proof of Morera's theorem, let $\overline{B\left(z_{0}, r\right)} \subseteq \Omega$ and use the given condition to construct a primitive, $F$ for $f$ on $B\left(z_{0}, r\right)$. Then $F$ is analytic and so by Theorem 14.8.3, it follows that $F$ and hence $f$ have infinitely many derivatives, implying that $f$ is analytic on $B\left(z_{0}, r\right)$. Since $z_{0}$ is arbitrary, this shows $f$ is analytic on $\Omega$.

The following observation is useful to keep in mind.

Observation 14.8.7 Suppose $\sum_{n=0}^{\infty} a_{n} h^{n}$ converges for $|h|<r$. Then it follows that $\lim _{h \rightarrow 0} \frac{1}{h^{k}} \sum_{n=k+1}^{\infty} a_{n} h^{n}=0$. To see this, note the expression is $h \sum_{n=k+1}^{\infty} a_{n} h^{n-(k+1)}$. Now the sum of the absolute values is $\sum_{n=k+1}^{\infty}\left\|a_{n}\right\||h|^{n-(k+1)}$ and it converges because there exists $\hat{h}$, such that $r>|\hat{h}|>|h|$. By the root test, Theorem 1.12.1, $\limsup _{n \rightarrow \infty}\left\|a_{n}\right\|^{1 / n}|\hat{h}| \leq 1$ so $\lim \sup _{n \rightarrow \infty}\left\|a_{n}\right\|^{1 / n}|h|<1$ Now applying this to the sum in question,

$$
\lim \sup _{n \rightarrow \infty}\left\|a_{n}\right\|^{1 / n}|h|^{\frac{n-(k+1)}{n}}=\lim \sup _{n \rightarrow \infty}\left\|a_{n}\right\|^{1 / n}|h|<1
$$

Also the sum decreases in $|h|$ and so

$$
\lim _{h \rightarrow 0}\left\|\left|h \sum_{n=k+1}^{\infty} a_{n} h^{n-(k+1)}\left\|\leq \lim _{h \rightarrow 0}|h| \sum_{n=k+1}^{\infty}\right\| a_{n} \||h|^{n-(k+1)}=0\right.\right.
$$

The tail of the series just described is sometimes referred to as "higher order terms".

### 14.9 Zeros of Analytic Functions

The following is a remarkable result about the zeros of an analytic function on a connected open set. It turns out that if the set of zeros have a limit point, then the function ends up being constant. Note how radically different this is from the theory of functions of a real variable. Consider, for example the function

$$
f(x) \equiv\left\{\begin{array}{l}
x^{2} \sin \left(\frac{1}{x}\right) \text { if } x \neq 0 \\
0 \text { if } x=0
\end{array}\right.
$$

which has a derivative for all $x \in \mathbb{R}$ and for which 0 is a limit point of the set of zeros $Z$, even though $f$ is not identically equal to zero.

Definition 14.9.1 Suppose $f$ is an analytic function defined near a point $\alpha$ where $f(\alpha)=0$. Thus $\alpha$ is a zero of the function $f$. The zero is of order $m$ if $f(z)=(z-\alpha)^{m} g(z)$ where $g$ is an analytic function which is not equal to zero at $\alpha$.

Theorem 14.9.2 Let $\Omega$ be a connected open set (region) and let $f: \Omega \rightarrow X$ be analytic. Then the following are equivalent.

1. $f(z)=0$ for all $z \in \Omega$
2. There exists $z_{0} \in \Omega$ such that $f^{(n)}\left(z_{0}\right)=0$ for all $n$.
3. There exists $z_{0} \in \Omega$ which is a limit point of the set, $Z \equiv\{z \in \Omega: f(z)=0\}$.

Proof: It is clear the first condition implies the second two. Suppose the third holds. Then for $z$ near $z_{0}, f(z)=\sum_{n=k}^{\infty} \frac{f^{(n)}\left(z_{0}\right)}{n!}\left(z-z_{0}\right)^{n}$ where $k \geq 1$ since $z_{0}$ is a zero of $f$. Suppose $k<\infty$. Then $f(z)=\left(z-z_{0}\right)^{k} g(z)$ where $g\left(z_{0}\right) \neq 0$. Letting $z_{n} \rightarrow z_{0}$ where $z_{n} \in Z, z_{n} \neq z_{0}$, it follows $0=\left(z_{n}-z_{0}\right)^{k} g\left(z_{n}\right)$ which implies $g\left(z_{n}\right)=0$. Then by continuity of $g$, we see that $g\left(z_{0}\right)=0$ also, contrary to the choice of $k$. Therefore, $k$ cannot be less than $\infty$ and so $z_{0}$ is a point satisfying the second condition.

Now suppose the second condition and let $S \equiv\left\{z \in \Omega: f^{(n)}(z)=0\right.$ for all $\left.n\right\}$. Then $\Omega \backslash S$ is an open set because if $f^{(n)}(z) \neq 0$ then this situation persists for $w$ near $z$. However, $S$ is also open. To see this, let $z \in S$. Then for all $w$ close enough to $z, f(w)=$ $\sum_{k=0}^{\infty} \frac{f^{(k)}(z)}{k!}(w-z)^{k}=0$. Thus $f$ is identically equal to zero near $z \in S$. Therefore, all points near $z$ are contained in $S$ also, showing that $S$ is an open set. Now $\Omega=S \cup(\Omega \backslash S)$, the union of two disjoint open sets, $S$ being nonempty. It follows the other open set, $\Omega \backslash S$, must be empty because $\Omega$ is connected. Therefore, the first condition is verified.
(See the following diagram.)


Recall that

$$
\begin{equation*}
e^{z} \equiv e^{x}(\cos (y)+i \sin (y)) \tag{14.22}
\end{equation*}
$$

Is it also true that $e^{z}=\sum_{k=0}^{\infty} \frac{z^{k}}{k!}$ ?

## Theorem 14.9.3 (Euler's Formula) Let $z=x+i y$. Then $e^{z}=\sum_{k=0}^{\infty} \frac{z^{k}}{k!}$.

Proof: The Cauchy Riemann equations show that $e^{z}$ given by 14.22 is analytic. So is $\exp (z) \equiv \sum_{k=0}^{\infty} \frac{z^{k}}{k!}$. In fact the power series converges for all $z \in \mathbb{C}$. Furthermore the two functions, $e^{z}$ and $\exp (z)$ agree on the real line which is a set which contains a limit point. Therefore, they agree for all values of $z \in \mathbb{C}$.

There is an amazing theorem which counts the number of zeroes on the inside of a closed curve provided these are counted according to multiplicity. $z \in \mathbb{C}$ is a zero of $f$ of multiplicity $m \in \mathbb{N}$ means that for $w$ near $z, f(w)=g(w)(w-z)^{m}$ where $g(z) \neq 0$. The counting zeros theorem is as follows:

Theorem 14.9.4 Let $f$ be analytic with values in $\mathbb{C}$ in an open set containing the closed disk

$$
D\left(z_{0}, r\right) \equiv\left\{z:\left|z-z_{0}\right| \leq r\right\}
$$

and suppose $f$ has no zeros on the circle $C\left(z_{0}, r\right)$, the boundary of $D\left(z_{0}, r\right)$. Then the number of zeros of $f$ counted according to multiplicity which are contained in $D\left(z_{0}, r\right)$ is $\frac{1}{2 \pi i} \int_{C\left(z_{0}, r\right)} \frac{f^{\prime}(z)}{f(z)} d z$ where $C\left(z_{0}, r\right)$ is oriented in the counter clockwise direction.

Proof: There are only finitely many zeros in $D\left(z_{0}, r\right)$. Otherwise, there would exist a limit point of the set of zeros $z$. If $z$ is in $B\left(z_{0}, r\right)$, then by Theorem 14.5.3, $f=0$ on $D\left(z_{0}, r\right)$. If it is on $C\left(z_{0}, r\right)$, this would contradict having no zeros on the boundary.

Let these zeros be $\left\{z_{1}, \ldots, z_{p}\right\}$ listed according to multiplicity. Consider $z_{k}$ and suppose it is a zero of order $m_{k}$ so $f(z)=\left(z-z_{k}\right)^{m_{k}} g(z)$ where $g\left(z_{k}\right) \neq 0, m \geq 1$. Then enclose each $z_{k}$ with a small circle $C\left(z_{k}, r_{k}\right)$ bounding $D\left(z_{k}, r\right)$ such that the $\overline{D\left(z_{k}, r_{k}\right)}$ do not intersect. Thus $\frac{f^{\prime}(z)}{f(z)}$ is analytic on the inside of $C\left(z_{0}, r\right)$ except for these zeroes.

$$
\frac{f^{\prime}(z)}{f(z)}=\frac{m\left(z-z_{k}\right)^{m-1} g_{k}(z)+\left(z-z_{k}\right)^{m} g_{k}^{\prime}(z)}{\left(z-z_{k}\right)^{m} g_{k}(z)}=\frac{m_{k}}{\left(z-z_{k}\right)}+\frac{g_{k}^{\prime}(z)}{g_{k}(z)}
$$

By Theorem 14.5.3,

$$
\begin{aligned}
\frac{1}{2 \pi i} \int_{C\left(z_{0}, r\right)} \frac{f^{\prime}(z)}{f(z)} d z & =\sum_{k=1}^{p} \frac{1}{2 \pi i} \int_{C\left(z_{k}, r_{k}\right)} \frac{m_{k}}{\left(z-z_{k}\right)} \\
+\frac{1}{2 \pi i} \int_{C\left(z_{k}, r_{k}\right)} \frac{g_{k}^{\prime}(z)}{g_{k}(z)} & =\sum_{k=1}^{p} m_{k}=p
\end{aligned}
$$

### 14.10 The Gamma and Zeta Functions

Up to now we have considered the exponential function, logarithms, and polynomials. Two other very important examples of analytic functions are the gamma and Zeta functions. It turns out these two functions are related. I am giving the obvious generalization of the Gamma function from the real variable version. However, more complete discussions employ infinite products to describe the gamma function, a topic not emphasized in this book. See [10] for a more complete discussion of these two functions.
Definition 14.10.1 For $\operatorname{Re}(z)>0$ define $\Gamma(z) \equiv \int_{0}^{\infty} e^{-t} t^{z-1} d z$.
Theorem 14.10.2 For $f,|f(t)| \leq C e^{-r t}, r>0, G(z) \equiv \int_{0}^{\infty} f(t) t^{z-1} d t$ is analytic for $\operatorname{Re}(z)>0$. In case $f(t)=e^{-t}$ so $G=\Gamma$, the fundamental identity $\Gamma(z+1)=z \Gamma(z)$ holds for all $\operatorname{Re} z>0$.

Proof: Formally differentiating under the integral sign, one would expect that $G^{\prime}(z)=$ $\int_{0}^{\infty} f(t) \ln (t) t^{z-1} d t$. This just needs to be shown.

Is $z \rightarrow \int_{0}^{\infty} f(t) \ln (t) t^{z-1} d t$ continuous for $\operatorname{Re} z>0$ ? Let $0<\delta<\operatorname{Re}(z)<\Delta$. If $\operatorname{Re}(\hat{z}) \in$ $(\delta, \Delta)$, then $\left|f(t) \ln (t) t^{\hat{z}-1}\right| \leq g(t)$ where

$$
g(t) \equiv\left\{\begin{array}{c}
|f(t)||\ln (t)| t^{\Delta-1} \text { for } t \geq 1 \\
|f(t)||\ln (t)| t^{\delta-1} \text { for } 0<t<1
\end{array}\right.
$$

a function in $L^{1}$. That this is in $L^{1}$ is shown using integration by parts. Now if $z_{n} \rightarrow z$, then eventually $\operatorname{Re} z_{n} \in(\delta, \Delta)$ and so $f(t) \ln (t) t^{z_{n}-1} \rightarrow f(t) \ln (t) t^{z-1}$ and by the dominated convergence theorem, the integrals also converge. Thus $z \rightarrow \int_{0}^{\infty} f(t) \ln (t) t^{z-1} d t$ is continuous.

Now for $\operatorname{Re}(z)>0,(G(z+h)-G(z)) h^{-1}=$

$$
\begin{gathered}
=\int_{0}^{\infty} f(t) h^{-1}\left(t^{z+h-1}-t^{z-1}\right) d t=\int_{0}^{\infty} f(t) h^{-1}\left(\int_{0}^{1} h \ln (t) t^{z+s h-1} d s\right) d t \\
=\int_{0}^{\infty} \int_{0}^{1} f(t) \ln (t) t^{z+s h-1} d s d t
\end{gathered}
$$

The integrand is absolutely integrable and so we can use Fubini's theorem to obtain

$$
(G(z+h)-G(z)) h^{-1}=\int_{0}^{1} \int_{0}^{\infty} f(t) \ln (t) t^{z+s h-1} d t d s
$$

By continuity, $\lim _{h \rightarrow 0} \int_{0}^{\infty} f(t) \ln (t) t^{z+s h-1} d t=\int_{0}^{\infty} f(t) \ln (t) t^{z-1} d t$ and so it follows that $\lim _{h \rightarrow 0}(G(z+h)-G(z)) h^{-1}=\int_{0}^{\infty} f(t) \ln (t) t^{z-1} d t$.

The last claim about the identity follows from Theorem 14.9.2 applied to the analytic function $\Gamma(z+1)-z \Gamma(z)$. It is valid on $(0, \infty)$ so the identity continues to hold for $\operatorname{Re}(z)>$ 0 . All these positive real numbers are limit points of the set where $\Gamma(z+1)-z \Gamma(z)=0$. You could probably prove this through a computation also.

Corollary 14.10.3 Let $\operatorname{Re} z, \operatorname{Re} w>0$. Then $B(z, w)=\int_{0}^{1} z^{z-1}(1-t)^{w-1} d t$ exists and for fixed $w, z \rightarrow B(z, w)$ is analytic and similarly $w \rightarrow B(z, w)$ is analytic. Also $\Gamma(z) \Gamma(w)=$ $\Gamma(w+z) B(z, w)$. In addition to this, $\Gamma(z) \neq 0$ for all $\operatorname{Re} z>0$ and $B(z, w) \neq 0$ for $\operatorname{Re} z>$ $0, \operatorname{Re} w>0$.

Proof: The existence of the integral follows from consideration of the real and imaginary parts. For $u$ positive, $\left|u^{\alpha}\right| \equiv\left|e^{\ln (u)(\operatorname{Re}(\alpha)+i \operatorname{Im}(\alpha))}\right|=u^{\operatorname{Re}(\alpha)}$. The integral is a an analytic function of $z, w$ also from similar arguments given above.

Now consider the identity. It is true if both $z, w$ are positive real numbers. This is by Problem 7 on Page 235. It is an exercise in Fubini's theorem. Thus by Theorem 14.9.2 the identity is true whenever $\operatorname{Re} z>0$ and $w$ is real and positive. Then by the same theorem again, the identity is true for $\operatorname{Re} w>0$ and $\operatorname{Re} z>0$.

Let $z_{n}$ be positive, distinct real numbers and $z_{n} \rightarrow 0$. Suppose $\Gamma(w)=0, \operatorname{Re} w>0$. Then, by the identity, $\Gamma\left(z_{n}\right) \Gamma\left(w-z_{n}\right)=\Gamma(w) B\left(z_{n}, w-z_{n}\right)$, a simple computation shows that $\Gamma\left(z_{n}\right) \rightarrow \infty$ so eventually $\Gamma\left(z_{n}\right) \neq 0$. But then, since the right side is $0, \lim _{n \rightarrow \infty} \Gamma\left(w-z_{n}\right)=$ 0 and so $w$ is a limit point in the set of zeros of $\Gamma$ so by Theorem 14.9.2, $\Gamma(w)=0$ for all $\operatorname{Re} w>0$ which is not true because $\Gamma(n)=(n-1)$ ! for $n$ a positive integer. It follows that $\Gamma(w) \neq 0$ for all $\operatorname{Re} w>0$. Then, by the identity again, $B(z, w) \neq 0$ for $\operatorname{Re} z>0, \operatorname{Re} w>0$.

Next is the zeta function $\zeta(z)$.
Definition 14.10.4 For $\operatorname{Re}(z)>1, \zeta(z) \equiv \sum_{k=1}^{\infty} \frac{1}{k^{2}}$. This series will converge for such $z=x+$ iy because $\left|k^{z}\right| \equiv\left|e^{\ln (k)(x+i y)}\right|=k^{x}$ so if $x>1$, the series converges absolutely.

In the formula for $\Gamma(z)$, change the variable letting $t=n s$ for $n \in \mathbb{N}$.

$$
\begin{gathered}
\Gamma(z) \equiv \int_{0}^{\infty} e^{-t} t^{z-1} d z=\int_{0}^{\infty} e^{-n s}(n s)^{z-1} n d s=n^{z} \int_{0}^{\infty} e^{-n s} s^{z-1} d s \\
\frac{1}{n^{z}} \Gamma(z)=\int_{0}^{\infty} e^{-n s} s^{z-1} d s
\end{gathered}
$$

Now, assuming $\operatorname{Re} z>1$ so the series for $\zeta(z)$ is defined, you could write

$$
\zeta(z) \Gamma(z)=\sum_{n=1}^{\infty} \int_{0}^{\infty} e^{-n s} s^{z-1} d s
$$

It is routine to see that $\sum_{n=1}^{\infty} \int_{0}^{\infty}\left|e^{-n s} s^{z-1}\right| d s<\infty$ so Fubini's theorem applies. Indeed, for $z=x+i y, x>1$,

$$
\begin{aligned}
\sum_{n=1}^{\infty} \int_{0}^{\infty}\left|e^{-n s} s^{z-1}\right| d s & =\sum_{n=1}^{\infty} \int_{0}^{\infty}\left|e^{-n s} s^{x-1}(\cos (y \ln (n s)))\right| d s \\
& \leq \sum_{n=1}^{\infty} \int_{0}^{\infty}\left|e^{-n s} s^{x-1}\right| d s=\sum_{n=1}^{\infty} \frac{\Gamma(x)}{n^{x}}<\infty
\end{aligned}
$$

Think of the sum as a Lebesgue integral with respect to counting measure. Therefore, from the formula for the sum of a geometric series, the above is of the form

$$
\zeta(z) \Gamma(z)=\int_{0}^{\infty}\left(\sum_{n=1}^{\infty} e^{-n s}\right) s^{z-1} d s=\int_{0}^{\infty}\left(e^{s}-1\right)^{-1} s^{z-1} d s
$$

Now $z \rightarrow \frac{1}{k^{z}}$ is analytic and so $z \rightarrow \sum_{k=1}^{n} \frac{1}{k^{z}}$ is also analytic. Since the series converges uniformly for $\operatorname{Re} z \geq \frac{1}{l}+1$, it follows that $z \rightarrow \sum_{k=1}^{\infty} \frac{1}{k^{z}}$ is also analytic on $\operatorname{Re} z>1$. This applies Lemma 14.8.5 to a small circle centered at $z$ with $\operatorname{Re} z>1$, the circle lying in $\operatorname{Re} z \geq 1+\frac{1}{l}$ for some $l \in \mathbb{N}$. The uniform convergence implies the limit of the functions $\sum_{k=1}^{n} \frac{1}{k^{2}}$ as $n \rightarrow \infty$ is analytic.

Proposition 14.10.5 $z \rightarrow \zeta(z)$ is analytic on $\operatorname{Re} z>1$ and on this region the following identity holds. $\zeta(z) \Gamma(z)=\int_{0}^{\infty}\left(e^{s}-1\right)^{-1} s^{z-1} d s$. In particular, $z \rightarrow \int_{0}^{\infty}\left(e^{s}-1\right)^{-1} s^{z-1} d s$ is analytic on $\operatorname{Re} z>1$.

Consider $\frac{1}{e^{z}-1}$. This is undefined when $z=0$. Writing it in terms of a power series in the bottom, it equals

$$
\frac{1}{z+\frac{z^{2}}{2}+\hat{p}(z)}=\frac{1}{z} \frac{1}{1+\frac{z}{2}+p(z)}
$$

where $p(z)$ is a power series beginning with exponent 2 . Then for small $z$ this is of the form

$$
\frac{1}{z}\left(1-\left(\frac{z}{2}+p(z)\right)+\left(\frac{z}{2}+p(z)\right)^{2}-\left(\frac{z}{2}+p(z)\right)^{3}+\cdots\right)
$$

and this equals $1 / z-1 / 2+q(z)$ where $q(z)$ is some power series which begins with exponent 1 . Thus $\frac{1}{e^{z}-1}-\frac{1}{z}$ equals a power series which will converge near $z=0$.

Now consider

$$
\int_{0}^{1}\left(\frac{1}{e^{t}-1}-\frac{1}{t}\right) t^{z-1} d t
$$

The function $\frac{1}{e^{z}-1}-\frac{1}{z}$ is unbounded near $2 k \pi i$ for $k$ a nonzero integer but elsewhere is analytic. In particular, there is a power series $p(z)$ whose first nonzero term is of degree 2 such that

$$
\frac{1}{e^{z}-1}-\frac{1}{z}=\frac{1}{z+p(z)}-\frac{1}{z}=\frac{-p(z)}{(z+p(z)) z}
$$

which can be redefined at $z=0$ to make it analytic. Thus if $|z| \leq 1$, this function $\frac{1}{e^{z}-1}-\frac{1}{z}$ will be bounded. By Theorem 14.10.2, applied to $f(t)$ the zero extension of $\frac{1}{e^{t}-1}-\frac{1}{t}$ off $[0,1], z \rightarrow \int_{0}^{1}\left(\frac{1}{e^{t}-1}-\frac{1}{t}\right) t^{z-1} d t$ is analytic. It follows that for all $\operatorname{Re} z>1$,

$$
\begin{aligned}
\zeta(z) \Gamma(z) & =\int_{0}^{\infty}\left(e^{t}-1\right)^{-1} t^{z-1} d t \\
& =\int_{0}^{1}\left(\frac{1}{e^{t}-1}-\frac{1}{t}\right) t^{z-1} d t+\int_{0}^{1} \frac{1}{t} t^{z-1} d t+\int_{1}^{\infty}\left(e^{t}-1\right)^{-1} t^{z-1} d t \\
& =\overbrace{\int_{0}^{1}\left(\frac{1}{e^{t}-1}-\frac{1}{t}\right) t^{z-1} d t}^{\text {analytic on Re } z>0}+\overbrace{\frac{1}{z-1}}^{\text {analytic at } z \neq 1}+\overbrace{\int_{1}^{\infty}\left(e^{t}-1\right)^{-1} t^{z-1} d t}^{\text {analytic on Re } z>0}
\end{aligned}
$$

That last integral is analytic because you can let $f(t)$ be 0 on $[0,1]$ and $\frac{1}{e^{t}-1} \leq 3 e^{-t}$. Indeed, it is clear that $\frac{e^{t}}{e^{t}-1} \leq 3$ for $t \geq 1$. Now apply Theorem 14.10.2. Thus the right side extends the product $\zeta(z) \Gamma(z)$ to an analytic function valid for $\operatorname{Re} z \in(0,1)$. By Corollary 14.10.3, $z \rightarrow \zeta(z)$ can be extended to an analytic function on $0<\operatorname{Re} z<1$ given by

$$
\zeta(z)=\frac{1}{\Gamma(z)}\left(\int_{0}^{1}\left(\frac{1}{e^{t}-1}-\frac{1}{t}\right) t^{z-1} d t+\frac{1}{z-1}+\int_{1}^{\infty} \frac{t^{z-1}}{e^{t}-1} d t\right)
$$

No one knows whether all zeros of $\zeta$ are on the line $\operatorname{Re} z=1 / 2$. The hypothesis that these zeros are all on this line is Riemann's hypothesis and it is one of the most famous of all unsolved problems.

### 14.11 The General Cauchy Integral Formula

In general, closed curves can be very wriggly and it may not be entirely clear about what is meant by counter clockwise orientation. The following is the general Cauchy integral formula. Note that for most contours encountered, $n(\gamma, z)$ is easy to see. It is 1 if motion is counter clockwise and -1 if motion is clockwise. This comes from using an appropriate branch of the logarithm to determine the winding number, as explained earlier.

Theorem 14.11.1 Let $\Gamma$ be a simple closed rectifable curve and let $\gamma$ be an oriented parametrization for $\Gamma$ which has the inside $U_{i}$, an open set for which Green's theorem holds. ${ }^{2}$ Also let $f$ be analytic near $U_{i} \cup \Gamma$ with values in $X$ a complex Banach space. Then if $z \in U_{i}, n(\gamma, z) f(z)=\frac{1}{2 \pi i} \int_{\gamma} \frac{f(w)}{w-z} d w$.

Proof: Consider the function

$$
g(w) \equiv\left\{\begin{array}{l}
\frac{f(w)-f(z)}{w-z} \text { if } w \neq z  \tag{14.23}\\
f^{\prime}(z) \text { if } w=z
\end{array}\right.
$$

This is clearly continuous on $U_{i} \cup \Gamma$. It is also clear that $g^{\prime}(w)$ exists if $w \neq z$. It remains to consider whether $g^{\prime}(z)$ exists. However, $f$ has all derivatives near $z$ and so

$$
g(z+h)=\frac{f(z+h)-f(z)}{h}=\frac{f^{\prime}(z)(h)+\frac{1}{2} f^{\prime \prime}(z) h^{2}+o\left(h^{2}\right)}{h} .
$$

Therefore, $\lim _{h \rightarrow 0} \frac{1}{h}(g(z+h)-g(z))=\frac{1}{2} f^{\prime \prime}(z)$. It follows that

$$
\begin{aligned}
0 & =\frac{1}{2 \pi i} \int_{\gamma} g(w) d w=\frac{1}{2 \pi i} \int_{\gamma} \frac{f(w)}{w-z} d w-\frac{1}{2 \pi i} \int_{\gamma} \frac{f(z)}{w-z} d w \\
& =\frac{1}{2 \pi i} \int_{\gamma} \frac{f(w)}{w-z} d w-f(z) n(\gamma, z)
\end{aligned}
$$

It will be the case that $n(\gamma, z)= \pm 1$. This is shown later in the appendix after the proof of the general Green's theorem, but for most ordinary situations, this is easily seen. The following is a spectacular application. It is Liouville's theorem.

Theorem 14.11.2 Suppose $f$ is analytic on $\mathbb{C}$, having values in $X$ a Banach space, and that $\|f(z)\|$ is bounded for $z \in \mathbb{C}$. Then $f$ is constant.

Proof: It was shown above that if $\gamma_{r}$ is a counter clockwise oriented parametrization for the circle of radius $r$ centered at $z$, then

$$
f^{\prime}(z)=\frac{1}{2 \pi i} \int_{\gamma_{r}} \frac{f(w)}{(w-z)^{2}} d w \text { if }|z|<r
$$

[^8]and so $\left\|f^{\prime}(z)\right\| \leq \frac{1}{2 \pi} C 2 \pi r \frac{1}{r^{2}}$ where $\|f(z)\|<C$ for all $z$ and this is true for any $r$ so let $r \rightarrow \infty$ and you can conclude that $f^{\prime}(z)=0$ for all $z \in \mathbb{C}$. However, continuing this argument shows that $f^{(k)}(z)=0$ for all $z$ and for each $k \geq 1$. Thus the power series for $f(z)$, which exists by Theorem 14.8.3, is $f(z)=f(0)+\sum_{k=1}^{\infty} \frac{f^{(k)}(0)}{k!} z^{k}=f(0)$.

Alternatively, consider the line segment between $z$ and $w$ and use Problem 26 on Page 78 on $t \rightarrow f(z+t(w-z))$.

This leads right away to the shortest proof of the fundamental theorem of algebra.
Theorem 14.11.3 Let $p(z)$ be a non constant polynomial with complex coeffcients. Then $p(z)=0$ for some $z \in \mathbb{C}$. That is, $p(z)$ has a root in $\mathbb{C}$.

Proof: Suppose not. Then $1 / p(z)$ is analytic on $\mathbb{C}$. Also, the leading term dominates the others and so $1 / p(z)$ must be bounded. Indeed, $\lim _{|z| \rightarrow \infty}(1 /|p(z)|)=0$ and the continuous function $z \rightarrow 1 /|p(z)|$ achieves a maximum on any bounded closed ball centered at 0 . By Liouville's theorem, this quotient must be constant. However, by assumption, $1 / p(z)$ is not constant. Hence there is a root of $p(z)$.

### 14.12 Simply Connected Regions

The Riemann sphere is a useful way to present the concept of the extended complex plane. Consider the unit sphere, $S^{2}$ given by $(z-1)^{2}+y^{2}+x^{2}=1$. Define a map from the complex plane to the surface of this sphere as follows. Extend a line from the point, $p$ in the complex plane to the point $(0,0,2)$ on the top of this sphere and let $\theta(p)$ denote the point of this sphere which the line intersects. Think of $\infty$ as a point which, when added in to $\mathbb{C}$ gives an "extended complex plane" $\widehat{\mathbb{C}}$ in such a way that $\infty$ corresponds to $\theta(\infty) \equiv(0,0,2)$.


Then $\theta^{-1}$ is sometimes called sterographic projection.
Definition 14.12.1 Let $\widehat{\mathbb{C}} \equiv \mathbb{C} \cup\{\infty\}$ be a metric space as follows. $d(z, w) \equiv \rho(\theta z, \theta w)$ where $\rho$ is the distance in $\mathbb{R}^{3}$.

Lemma 14.12.2 The above does make $\widehat{\mathbb{C}}$ into a metric space. $z_{n} \rightarrow \infty$ is the same as $\lim _{n \rightarrow \infty}\left|z_{n}\right|=\infty$. In fact $\widehat{\mathbb{C}}$ is a compact metric space.

Proof: It is clear that $d(z, w) \geq 0$ and equals 0 if and only if $z=w$ because $\theta$ is one to one. The main issue is the triangle inequality. However, $d(z, w)+d(w, u) \equiv \rho(\theta z, \theta w)+$ $\rho(\theta w, \theta u) \geq \rho(\theta z, \theta u) \equiv d(z, u)$. Thus it is a metric space. To say that $z_{n} \rightarrow \infty$ is to say that $\rho\left(\theta z_{n}, \theta \infty\right) \rightarrow 0$. From the diagram, this means $\lim _{n \rightarrow \infty}\left|z_{n}\right|=\infty$.

Suppose you have $\left\{z_{n}\right\}$ a sequence in $\widehat{\mathbb{C}}$. If the sequence has no subsequence which converges to $\infty$, then there exists $\varepsilon>0$ such that for $n$ large enough, $\rho\left(\theta z_{n}, \theta \infty\right) \geq \varepsilon$ which corresponds to $\left\{z_{n}\right\}$ remaining in a compact subset of $\mathbb{C}$ for all $n$ large enough. Hence there is a convergent subsequence which converges to some point of $\mathbb{C}$.

The mapping $\theta$ is clearly continuous because it takes converging sequences, to converging sequences. Furthermore, it is clear that $\theta^{-1}$ is also continuous. The notion of connected sets is the same as earlier. The usual definition of being connected if it is not separated will be retained in this extended complex plane.

Example 14.12.3 Consider the open set $S \equiv\{z \in \mathbb{C}$ such that $\operatorname{Im}(z)>0\}$. Let $S \cup\{\infty\} \equiv$ $\widehat{S}$. This is connected in $\widehat{\mathbb{C}}$.

It is clear that $\theta(S \cup\{\infty\})$ is half of $S^{2}$ including the top point, obviously an arcwise connected set. Thus $\widehat{S}$ is connected.
Definition 14.12.4 a connected set $S \subseteq \mathbb{C}$ is said to be simply connected if $S$ is connected and also $\widehat{\mathbb{C}} \backslash S$ is connected in $\widehat{\mathbb{C}}$. Equivalently $\theta(S)$ and $\theta(\widehat{\mathbb{C}} \backslash S)$ are connected subsets of the Riemann sphere.

Example 14.12.5 Consider the set $S \equiv\{z \in \mathbb{C}$ such that $|z|>1\}$. This is a connected set, but it is not simply connected.

Consider $\theta(S)$. This is connected, but $\theta(\widehat{\mathbb{C}} \backslash S)$ is not.
Is there an easy way to see that an open connected set is NOT simply connected? The answer is yes, using the Jordan curve theorem. Suppose you have an open connected set $\Omega$ and a simple closed curve contained in $\Omega$ called $\Gamma$. If there is a point $z \in U_{\Gamma}$ the inside region of $\Gamma$ which is not in $\Omega$, then $\Omega$ cannot be simply connected because, letting $V_{\Gamma}$ be the unbounded component of $\Gamma^{C}, \widehat{\mathbb{C}} \backslash \Omega \subseteq \hat{V}_{\Gamma} \cup U_{\Gamma}$ disjoint open sets in $\widehat{\mathbb{C}}$ and there are points of $\widehat{\mathbb{C}} \backslash \Omega$ in each of these disjoint open sets. Therefore, $\widehat{\mathbb{C}} \backslash \Omega$ cannot be connected. Here $\hat{V}_{\Gamma} \equiv V_{\Gamma} \cup\{\infty\}$.
Example 14.12.6 Consider the set $S \equiv\{z \in \mathbb{C}$ such that $|z|>1\}$. This is a connected set, but it is not simply connected. Indeed, $\{z:|z|=1.5\}$ is in $S$ but its inside isn't. From the original definition, $S^{C}=\{\infty\} \cup\{z:|z| \leq 1\}$ which is clearly not connected in $\widehat{\mathbb{C}}$.
Example 14.12.7 Consider $S \equiv\{z \in \mathbb{C}$ such that $|z| \leq 1\}$. This connected set is simply connected because $\widehat{\mathbb{C}} \backslash S$ corresponds to a connected set on $S^{2}$.

### 14.13 Exercises

1. Suppose $U$ is an open subset of $\mathbb{C}$ and $f: U \rightarrow \mathbb{R}$ is analytic. Describe $f$ on the connected components of $U$.
2. Suppose $f: \mathbb{C} \rightarrow \mathbb{C}$ is analytic. Let $g(z) \equiv f(\bar{z})$. Show that $g^{\prime}(z)$ does not exist unless $f^{\prime}(\bar{z})=0$.
3. Suppose $f$ is an entire function (analytic on $\mathbb{C}$ ) and suppose $\operatorname{Re} f$ is never 0 . Show that $f$ must be constant. Hint: Consider

$$
U=\{(x, y): \operatorname{Re} f(x, y)>0\}, V=\{(x, y): \operatorname{Re} f(x, y)<0\}
$$

These are open and disjoint so one must be empty. If $V$ is empty, consider $1 / e^{f(z)}$. Use Liouville's theorem.
4. Suppose $f: \mathbb{C} \rightarrow \mathbb{C}$ is analytic. Suppose also there is an estimate

$$
|f(z)| \leq M\left(1+|z|^{\alpha}\right), \alpha>0
$$

Show that $f$ must be a polynomial. Hint: Consider the formula for the derivative in which $\gamma_{r}$ is positively oriented and a circle or radius $r$ for $r$ very large centered at 0 , $f^{(n)}(z)=\frac{n!}{2 \pi i} \int_{\gamma_{r}} \frac{f(w)}{(w-z)^{n+1}} d w$ and pick large $n$. Then let $r \rightarrow \infty$.
5. Define for $z \in \mathbb{C}, \sin z \equiv \sum_{k=0}^{\infty}(-1)^{n} \frac{z^{2 n+1}}{(2 n+1)!}$. That is, you just replace $x$ with $z$. Give a similar definition for $\cos z$, and $e^{z}$. Show that the series converges for $\sin z$ and that a corresponding series converges for $\cos z$. Then show that $\sin z=\frac{e^{i z}-e^{-i z}}{2 i}, \cos z=$ $\frac{e^{i z}+e^{-i z}}{2}$. Show that it is no longer true that the functions $\sin z, \cos z$ must be bounded in absolute value by 1 . Hint: This is a very easy problem if you use the theorem about the zeros of an analytic function, Theorem 14.9.2.
6. Verify the identities $\cos (z-w)=\cos z \cos w+\sin z \sin w$ and similar identities. Hint: This is a very easy problem if you use the theorem about the zeros of an analytic function, Theorem 14.9.2.
7. Consider the following contour in which the large semicircle has radius $R$ and the small one has radius $r \equiv 1 / R$.


The function $z \rightarrow \frac{e^{i z}}{z}$ is analytic on the curve and on its inside. Therefore, the contour integral with respect to the given orientation is 0 . Use this contour and the Cauchy integral theorem to verify that $\int_{0}^{\infty} \frac{\sin z}{z} d z=\pi / 2$ where this improper integral is defined as $\lim _{R \rightarrow \infty} \int_{-1 / R}^{R} \frac{\sin z}{z} d z$. The function is actually not absolutely integrable and so the precise description of its meaning just given is important. You can use dominated convergence theorem to simplify some of the limits if you like but it is possible to establish the needed estimates through elementary means. I recommend using the dominated convergence theorem. To do this, show that the integral over the large circle of $\int_{C_{R}} \frac{e^{-z}}{z} d z \rightarrow 0$ as $R \rightarrow \infty$ and verify that you get something else like $-\pi$ for the integral over the small integral as $r \rightarrow 0$.
8. Suppose $f(z)=u(x, y)+i v(x, u)$ is analytic. Show that both $u, v$ satisfy Laplace's equation, $u_{x x}+u_{y y}=0$.
9. Suppose you have two complex numbers $z=a+i b$ and $w=x+i y$. Show that the dot product of the two vectors $(a, b) \cdot(x, y)$ is $\operatorname{Re}((a+i b)(x-i y))=\operatorname{Re}(z \bar{w})$.
10. $\uparrow$ Suppose you have two curves $t \rightarrow z(t)$ and $s \rightarrow w(s)$ which intersect at some point $z_{0}$ corresponding to $t=t_{0}$ and $s=s_{0}$. Show that the cosine of the angle $\theta$ between these two curves at this point is $\cos (\theta)=\frac{\operatorname{Re}\left(z^{\prime}\left(t_{0}\right) \overline{w^{\prime}\left(s_{0}\right)}\right)}{\left|z^{\prime}\left(t_{0}\right)\right|\left|w^{\prime}\left(s_{0}\right)\right|}$. Now suppose $z \rightarrow f(z)$ is analytic. Thus there are two curves $t \rightarrow f(z(t))$ and $s \rightarrow f(w(s))$ which intersect when $t=t_{0}$ and $s=s_{0}$. Show that the angle between these two new curves at their point of intersection is also $\theta$. This shows that analytic mappings preserve the angles between curves.
11. Suppose $z=x+i y$ and $f(z)=u(x, y)+i v(x, y)$ where $f$ is analytic. Explain how level curves of $u$ and $v$ intersect in right angles.
12. Suppose $\Gamma$ is a simple closed rectifiable curve and that $\gamma$ is an oriented parametrization for $\Gamma$ which is oriented positively, $n(\gamma, z)=1$ for all $z$ on the inside of $\Gamma$. Now let $\hat{\Gamma}$ be a simple closed rectifiable curve on the inside of $\Gamma$ and let $\hat{\gamma}$ be an orientation of $\hat{\Gamma}$ also oriented positively. Explain why, if $z$ is on the inside of $\hat{\Gamma}$ and $f$ is analytic
on the inside $U_{i}$ of $\Gamma$, continuous on $U_{i} \cup \Gamma$, then $\int_{\hat{\gamma}} \frac{f(w)}{w-z} d w=\int_{\gamma} \frac{f(w)}{w-z} d w$ and if $z$ is on the inside of $\Gamma$ but outside of $\hat{\Gamma}$, Then $\int_{\hat{\gamma}} \frac{f(w)}{w-z} d w=0$ while $\int_{\gamma} \frac{f(w)}{w-z} d w=f(z)$ and if $z$ is outside of $\Gamma$ then both integrals are 0 .
13. Give another very short proof of the fundamental theorem of algebra using the result of Theorem 14.9.4. In fact, show directly that if $p(z)$ is a polynomial of degree $n$ then it has $n$ roots counted according to multiplicity. Hint: Let $p(z)$ be a polynomial. Then by the Euclidean algorithm, Lemma 1.8.3, you can see that there can be no more than $n$ roots of the polynomial $p(z)$ having complex coefficients. Otherwise the polynomial could not have degree $n$. You should show this. Now there must exist $\Gamma_{R}$, a circle centered at 0 of radius $R$ which encloses all roots of $p(z)$. Letting $m$ be the number of roots, $m=\frac{1}{2 \pi i} \int_{\gamma_{R}} \frac{p^{\prime}(z)}{p(z)} d z$. Now write down in terms of an integral on $[0,2 \pi]$ and let $R \rightarrow \infty$ to get $n$ in the limit on the right. Hence $n=m$.
14. Suppose now you have a rectifiable simple closed curve $\Gamma$ and on $\Gamma^{*},|f(z)|>|g(z)|$ where $f, g$ are analytic on an open set containing $\Gamma^{*}$. Suppose also that $f$ has no zeros on $\Gamma^{*}$. In particular, $f$ is not identically 0 . Let $\lambda \in[0,1]$.
(a) Verify that for $\lambda \in[0,1], f+\lambda g$ has no zeros on $\Gamma^{*}$.
(b) Verify that on $\Gamma^{*},\left|\frac{\left(f^{\prime}(z)+\lambda g^{\prime}(z)\right)}{f(z)+\lambda g(z)}-\frac{f^{\prime}(z)+\mu g^{\prime}(z)}{f(z)+\mu g(z)}\right| \leq C|\mu-\lambda|$.
(c) Use Theorem 14.4.7 to show that for $\gamma$ a positively oriented parametrization of $\Gamma, \lambda \rightarrow \frac{1}{2 \pi i} \int_{\gamma} \frac{f^{\prime}(z)+\lambda g^{\prime}(z)}{f(z)+\lambda g(z)} d z$ is continuous.
(d) Now explain why this shows that the number of zeros of $f+\lambda g$ on the inside of $\Gamma$ is the same as the number of zeros of $f$ on the inside of $\Gamma$. This is a version of Rouche's theorem.
15. Give an extremely easy proof of the fundamental theorem of algebra as follows. Let $\gamma_{R}$ be a parametrization of the circle centered at 0 having radius $R$ which has positive orientation so $n(\gamma, z)=1$. Let $p(z)$ be a polynomial $a_{n} z^{n}+a_{n-1} z^{n-1}+\cdots+a_{1} z+a_{0}$. Now explain why you can choose $R$ so large that $\left|a_{n} z^{n}\right|>\left|a_{n-1} z^{n-1}+\cdots+a_{1} z+a_{0}\right|$ for all $|z| \geq R$. Using Problem 14 above explain why all zeros of $p(z)$ are inside $\gamma_{R}^{*}$ and why there are exactly $n$ of them counted according to multiplicity.
16. The polynomial $z^{5}+z^{4}-z^{3}-3 z^{2}-5 z+1=p(z)$ has no rational roots. You can check this by applying the rational root theorem from algebra. However, it has five complex roots. Also $\left|z^{4}-z^{3}-3 z^{2}-5 z+1\right| \leq|z|^{4}+|z|^{3}+3|z|^{2}+5|z|+1$. By graphing, observe that $x^{5}-\left(x^{4}+x^{3}+3 x^{2}+5 x+1\right)>0$ for all $x \geq 2.4$. Explain why the roots of $p(z)$ are inside the circle $|z|=2.4$.
17. Let $f$ be analytic on $U$ and let $B(z, r) \subseteq U$. Let $\gamma_{r}$ be the positively oriented boundary of $B(z, r)$. Explain, using the Cauchy integral formula why

$$
|f(z)| \leq \max \left\{|f(w)|: w \in \gamma_{r}^{*}\right\} \equiv m_{r} .
$$

Show that if equality is achieved, then $|f(w)|$ must be constantly equal to $m_{r}$ on $\gamma_{r}^{*}$.
18. Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be analytic with $f^{\prime}(z) \neq 0$ for all $z$. Say $f(x+i y)=u(x, y)+i v(x, y)$. Thus the mapping $(x, y) \rightarrow\binom{u(x, y)}{v(x, y)}$ is a $C^{1}$ mapping of $\mathbb{R}^{2}$ to $\mathbb{R}^{2}$. Show that at any point $\left|\begin{array}{ll}u_{x} & u_{y} \\ v_{x} & v_{y}\end{array}\right| \neq 0$. Therefore, by the inverse function theorem, Theorem 4.8.7, this mapping is locally one to one. However, the function does not need to be globally one to one. Give an easy example which shows this to be the case. Hint: You might want to consider something involving the exponential function.
19. Let $\Gamma$ be a simple closed rectifiable curve and let $\left\{f_{n}\right\}$ be a sequence of functions which are analytic on $U_{i}$, the inside of $\Gamma$ and continuous on $\Gamma^{*}$. Then if $\gamma$ is a parametrization of $\Gamma$ with $n(\gamma, z)=1$ for $z \in U_{i}$, then $f_{n}(z)=\frac{1}{2 \pi i} \int_{\gamma} \frac{f_{n}(w)}{w-z} d w$. This is by the Cauchy integral formula presented above. Suppose $f_{n}$ converges uniformly on $\Gamma^{*}$ to a continuous function $f$. Show that then, for $z \in U_{i}$, and $f(z)$ defined as $f(z) \equiv \frac{1}{2 \pi i} \int_{\gamma} \frac{f(w)}{w-z} d w$. It follows that $f_{n}(z) \rightarrow f(z)$ for each $z \in U_{i}$ and also $f$ is analytic on $U_{i}$. Hint: You might use Theorem 14.4.7. This is very different than what happens with functions of a real variable in which uniform convergence of polynomials $p_{n}$ to $f$ does not necessarily confer differentiability on $f$. For example, to approximate $f$, a continuous function having no derivatives or even a very easy function like $f(x)=|x-(1 / 2)|$ for $x \in[0,1]$.
20. Sketch an example of two differentiable functions defined on $[0,1]$ such that their product is 0 but neither function is 0 . Explain why this never happens for the set of analytic functions defined on an open connected set. In other words, if you have $f g=0$ where $f, g$ are analytic on $D$ an open connected set, then either $f=0$ or $g=0$. For those who like to classify algebraically, this says that the set of analytic functions defined on an open connected set is an integral domain. It is clear that this set of functions is a ring with the usual operations. The extra ingredient is this observation that there are no nonzero zero divisors. Hint: To show this, consider $D \backslash f^{-1}(0)$ an open set. If $f^{-1}(0)=D$, then you are done. Otherwise, you have $g$ is 0 on an open set. Now use Theorem 14.9.2.
21. For $D \equiv\{z \in \mathbb{C}:|z|<1\}$, consider the function $\sin \left(\frac{1}{1-z}\right)$. Show that this function has infinitely many zeros in $D$. Thus there is a limit point to the set of zeros, but its limit point is not in $D$. It is good to keep this example in mind when considering Theorem 14.9.2.
22. Suppose $\Omega$ is an open connected subset of $\mathbb{C}$ with the property that if $\Gamma$ is a simple closed curve contained in $\Omega$ and $U_{\Gamma}, V_{\Gamma}$ respectively the bounded and unbounded components of $\Gamma^{C}$, then $U_{\Gamma} \subseteq \Omega$. Show that $\Omega$ is simply connected. Hint: If $\widehat{\mathbb{C}} \backslash \Omega=$ $A \cup B$ where $A, B$ separate $\widehat{\mathbb{C}} \backslash \Omega$ with $\infty \in B$, argue that $B \backslash\{\infty\}$ is closed and $A$ is compact. Show $\operatorname{dist}(A, B \backslash\{\infty\})=\delta>0$. Then make a grating of finitely many horizontal and vertical lines equally spaced with distance between successive lines no more than $\delta / 10$ such that the small squares cover up $A$. Now consider the vertices which are contained in $\Omega$. Each has four lines or maybe two lines emanating from it. Start at such a vertex $p$ and travel over sides of squares, never going back the way you came along one of these sides till you return to the point $p$ at which you started. Delete all paths of the form $\hat{p}, \hat{p}_{1}, \ldots, \hat{p}$ where $\hat{p} \neq p$. The result must be a simple closed curve $\Gamma$ contained in $\Omega, \infty \notin U_{\Gamma}$. By assumption $U_{\Gamma}$ does not contain
any points of $A$. Since you could have started at any point of this grating, and there are finitely many vertices, $A$ must be empty. A more systematic way of doing this is in the material on cycles in next chapter showing that the points of $A$ must be included.

## Chapter 15

## Isolated Singularities

### 15.1 Open Mapping Theorem

The open mapping theorem is for an analytic function with values in $\mathbb{C}$. It is an even more surprising result than the theorem about the zeros of an analytic function. The following proof of this important theorem uses an interesting local representation of the analytic function. First is a useful lemma.

Lemma 15.1.1 Suppose $V$ is open and $\phi: V \rightarrow B(0, \delta) \subseteq \mathbb{C}$ is one to one and onto and $\phi^{-1}$ is continuous. Then if $\phi$ is analytic with $\phi^{\prime}(z) \neq 0$ for all $z \in V$, then $\phi^{-1}$ is also analytic and the usual formula for $\left(\phi^{-1}\right)^{\prime}$ is valid.

Proof: Let $z \in V$. Say $w=\phi(z)$. By definition,

$$
\begin{align*}
h= & \phi\left(\phi^{-1}(w+h)\right)-\phi\left(\phi^{-1}(w)\right)=\phi^{\prime}\left(\phi^{-1}(w)\right)\left(\phi^{-1}(w+h)-\phi^{-1}(w)\right) \\
& +o\left(\phi^{-1}(w+h)-\phi^{-1}(w)\right) \tag{15.1}
\end{align*}
$$

Then

$$
\left|o\left(\phi^{-1}(w+h)-\phi^{-1}(w)\right)\right|<\frac{\left|\phi^{\prime}(z)\right|}{2}\left|\phi^{-1}(w+h)-\phi^{-1}(w)\right| \text { where } z \equiv \phi^{-1}(w)
$$

whenever $|h|$ is small enough. Thus, from 15.1, for small enough $h$,

$$
|h|>\frac{\left|\phi^{\prime}(z)\right|}{2}\left|\phi^{-1}(w+h)-\phi^{-1}(w)\right|,\left|\frac{\phi^{-1}(w+h)-\phi^{-1}(w)}{h}\right|<\frac{2}{\left|\phi^{\prime}(z)\right|}
$$

Then, for such small $h$,

$$
\begin{aligned}
\left|\frac{o\left(\phi^{-1}(w+h)-\phi^{-1}(w)\right)}{h}\right| & =\left|\frac{o\left(\phi^{-1}(w+h)-\phi^{-1}(w)\right)}{\phi^{-1}(w+h)-\phi^{-1}(w)}\right|\left|\frac{\phi^{-1}(w+h)-\phi^{-1}(w)}{h}\right| \\
& \leq\left|\frac{o\left(\phi^{-1}(w+h)-\phi^{-1}(w)\right)}{\phi^{-1}(w+h)-\phi^{-1}(w)}\right| \frac{2}{\left|\phi^{\prime}(z)\right|}
\end{aligned}
$$

and so, by continuity of $\phi^{-1}, o\left(\phi^{-1}(w+h)-\phi^{-1}(w)\right)=o(h)$. From 15.1, it follows that $\frac{1}{\phi^{\prime}(z)} h+o(h)=\phi^{-1}(w+h)-\phi^{-1}(w)$ showing that $\left(\phi^{-1}\right)^{\prime}(w)=\frac{1}{\phi^{\prime}(z)}$ where $\phi(z)=w$.

Theorem 15.1.2 (Open mapping theorem) Let $\Omega$ be a region (open connected set) in $\mathbb{C}$ and suppose $f: \Omega \rightarrow \mathbb{C}$ is analytic. Then $f(\Omega)$ is either a point or a region. In the case where $f(\Omega)$ is a region, it follows that for each $z_{0} \in \Omega$, there exists an open set $V$ containing $z_{0}$ and $m \in \mathbb{N}$ such that for all $z \in V$,

$$
\begin{equation*}
f(z)=f\left(z_{0}\right)+\phi(z)^{m} \tag{15.2}
\end{equation*}
$$

where $\phi: V \rightarrow B(0, \delta)$ is one to one, analytic and onto, $\phi\left(z_{0}\right)=0, \phi^{\prime}(z) \neq 0$ on $V$ and $\phi^{-1}$ analytic on $B(0, \delta)$. Thus $f\left(z_{0}\right)+B\left(0, \delta^{m}\right) \subseteq f(V)$. If $f$ is one to one then $m=1$ for each $z_{0}$ and $f^{-1}: f(\Omega) \rightarrow \Omega$ is analytic.

Proof: Suppose $f(\Omega)$ is not a point. Then for $z_{0} \in \Omega$ it follows there exists $r>0$ such that $f(z) \neq f\left(z_{0}\right)$ for all $z \in B\left(z_{0}, r\right) \backslash\left\{z_{0}\right\}$. Otherwise, $z_{0}$ would be a limit point of the set,

$$
\left\{z \in \Omega: f(z)-f\left(z_{0}\right)=0\right\}
$$

which would imply from Theorem 14.9 .2 that $f(z)=f\left(z_{0}\right)$ for all $z \in \Omega$. Therefore, making $r$ smaller if necessary, and using the power series of $f$,

$$
f(z)=f\left(z_{0}\right)+\left(z-z_{0}\right)^{m} g(z) \stackrel{?}{=}\left(f\left(z_{0}\right)+\left(\left(z-z_{0}\right) g(z)^{1 / m}\right)^{m}\right)
$$

where $g$ is analytic near $z_{0}$ and $g\left(z_{0}\right) \neq 0$. Does an analytic $g(z)^{1 / m}$ exist? By continuity, $g\left(B\left(z_{0}, r\right)\right) \subseteq B\left(g\left(z_{0}\right), \varepsilon\right)$ where $\varepsilon$ is small enough that $0 \notin B\left(g\left(z_{0}\right), \varepsilon\right)$, so there exists a branch of the logarithm on $\mathbb{C} \backslash B\left(g\left(z_{0}\right), \varepsilon\right)$, Definition 14.3.1. Call it log even though it might not be the principle branch. Then consider $e^{(1 / m) \log (g(z))} \equiv g(z)^{1 / m}$ and so we can obtain an analytic function denoted by $g(z)^{1 / m}$ as in the above formula. Let $\phi(z)=$ $\left(z-z_{0}\right) g(z)^{1 / m}$. Then $\phi\left(z_{0}\right)=0$ and

$$
\phi^{\prime}(z)=e^{(1 / m) \log (g(z))}+\left(z-z_{0}\right) e^{(1 / m) \log (g(z))} \frac{1}{g(z)} g^{\prime}(z)
$$

so $\phi^{\prime}\left(z_{0}\right)=e^{(1 / m) \log \left(g\left(z_{0}\right)\right)} \neq 0$. Shrinking $r$ some more if necessary, assume $\phi^{\prime}(z) \neq 0$ for all $z \in B\left(z_{0}, r\right)$. The representation

$$
f(z)=f\left(z_{0}\right)+\phi(z)^{m}, z \in B\left(z_{0}, r\right)
$$

where $\phi^{\prime}(z) \neq 0$ for all $z \in B\left(z_{0}, r\right)$ and $\phi\left(z_{0}\right)=0$ has been obtained.
1.) Let $\phi(z)=u(x, y)+i v(x, y)$ where $z=x+i y$. Consider the mapping

$$
\binom{x}{y} \rightarrow\binom{u(x, y)}{v(x, y)}
$$

where $u, v$ are $C^{1}$ because $\phi$ is given to be analytic. The Jacobian of this map at $(x, y) \in$ $B\left(z_{0}, r\right)$ is

$$
\begin{gathered}
\left|\begin{array}{cc}
u_{x}(x, y) & u_{y}(x, y) \\
v_{x}(x, y) & v_{y}(x, y)
\end{array}\right|=\left|\begin{array}{cc}
u_{x}(x, y) & -v_{x}(x, y) \\
v_{x}(x, y) & u_{x}(x, y)
\end{array}\right| \\
=u_{x}(x, y)^{2}+v_{x}(x, y)^{2}=\left|\phi^{\prime}(z)\right|^{2} \neq 0 .
\end{gathered}
$$

This follows from a use of the Cauchy Riemann equations. Also

$$
\binom{u\left(x_{0}, y_{0}\right)}{v\left(x_{0}, y_{0}\right)}=\binom{0}{0}
$$

Therefore, by the inverse function theorem there exists an open set $V$, containing $z_{0}$ and $\delta>0$ such that $(u, v)^{T}$ maps $V \subseteq B\left(z_{0}, r\right)$ one to one onto $B(0, \delta)$ with $\phi^{\prime}(z) \neq 0$ on $V, \phi$ maps open subsets of $V$ to open sets, and by Lemma 15.1.1, $\phi^{-1}$ is analytic.

It also follows that $\phi^{m}$ maps $V$ onto $B\left(0, \delta^{m}\right)$. Indeed, $|\phi(z)|^{m}=\left|\phi(z)^{m}\right|$. Therefore, the formula 15.2 implies that $f$ maps the open set $V$, containing $z_{0}$ to an open set. This shows $f(\Omega)$ is an open set because $z_{0}$ was arbitrary. $f(\Omega)$ is connected because $f$ is continuous and $\Omega$ is connected. Thus $f(\Omega)$ is a region (open and connected).
2.) Alternatively, let $\delta$ be small enough that the only zero of $\phi(z)-\phi\left(z_{0}\right)$ is $z_{0}$ in $\overline{B\left(z_{0}, \delta\right)}$. If no such small positive $\delta$ exists, then the zeroes of $\phi(z)-\phi\left(z_{0}\right)$ would have a limit point and so $\phi$ would be a constant. This would force $f$ to be constant also. Then $\phi\left(z_{0}\right) \notin \phi\left(C\left(z_{0}, \delta\right)\right)$ and so if $\left|w-\phi\left(z_{0}\right)\right|$ is small enough, then $w \notin \phi\left(C\left(z_{0}, \delta\right)\right)$ either. Thus there is $\varepsilon>0$ with $B\left(\phi\left(z_{0}\right), \varepsilon\right) \cap \phi\left(C\left(z_{0}, \delta\right)\right)=\emptyset$. Consider for $w \in B\left(\phi\left(z_{0}\right), \varepsilon\right)=$ $B(0, \varepsilon)$ the formula for counting zeroes.

$$
\frac{1}{2 \pi i} \int_{C\left(z_{0}, \delta\right)} \frac{\phi^{\prime}(z)}{\phi(z)-w} d z
$$

It is a continuous function of $w$ and equals 1 at $0=\phi\left(z_{0}\right)$ so, since it is integer valued, it equals 1 on all of $B(0, \varepsilon)$, but this is the number of zeroes of $\phi(z)-w$. Thus $\phi\left(B\left(z_{0}, \delta\right)\right)=$ $B(0, \varepsilon)$. Hence, $\phi^{m}\left(B\left(z_{0}, \delta\right)\right)=B\left(0, \varepsilon^{m}\right)$. It follows that

$$
f\left(B\left(z_{0}, \delta\right)\right)=f\left(z_{0}\right)+B\left(0, \varepsilon^{m}\right)=B\left(f\left(z_{0}\right), \varepsilon^{m}\right)
$$

and so this shows that $f$ maps small open balls to open balls. Thus $f(\Omega)$ is a connected open set.

It only remains to verify the assertion about the case where $f$ is one to one. If $m>1$, then $e^{\frac{2 \pi i}{m}} \neq 1$ and so for $z_{1} \in V$,

$$
\begin{equation*}
e^{\frac{2 \pi i}{m}} \phi\left(z_{1}\right) \neq \phi\left(z_{1}\right) \tag{15.3}
\end{equation*}
$$

But $e^{\frac{2 \pi i}{m}} \phi\left(z_{1}\right) \in B(0, \delta)$ and so there exists $z_{2} \neq z_{1}$ (since $\phi$ is one to one) such that $\phi\left(z_{2}\right)=$ $e^{\frac{2 \pi i}{m}} \phi\left(z_{1}\right)$. But then

$$
\phi\left(z_{2}\right)^{m}=\left(e^{\frac{2 \pi i}{m}} \phi\left(z_{1}\right)\right)^{m}=e^{2 \pi i} \phi\left(z_{1}\right)^{m}=\phi\left(z_{1}\right)^{m}
$$

implying $f\left(z_{2}\right)=f\left(z_{1}\right)$ contradicting an assumption that $f$ is one to one. Thus $m=1$ and $f^{\prime}(z)=\phi^{\prime}(z) \neq 0$ on $V$. Since $f$ maps open sets to open sets, it follows that $f^{-1}$ is continuous and so by Lemma 15.1.1 again, $f^{-1}$ is analytic.

One does not have to look very far to find that this sort of thing does not hold for functions mapping $\mathbb{R}$ to $\mathbb{R}$. Take for example, the function $f(x)=x^{2}$. Then $f(\mathbb{R})$ is neither a point nor a region. In fact $f(\mathbb{R})$ fails to be open.

Corollary 15.1.3 Suppose in the situation of Theorem 15.1.2 $m>1$ for the local representation of $f$ given in this theorem. Then there exists $\boldsymbol{\delta}>0$ such that if $w \in B\left(f\left(z_{0}\right), \boldsymbol{\delta}\right)=$ $f(V)$ for $V$ an open set containing $z_{0}$, then $f^{-1}(w)$ consists of $m$ distinct points in $V .(f$ is $m$ to one on $V$ )

Proof: Let $w \in B\left(f\left(z_{0}\right), \delta^{m}\right)$. Then $w=f(\widehat{z})$ where $\widehat{z} \in V$. Thus $f(\widehat{z})=f\left(z_{0}\right)+$ $\phi(\widehat{z})^{m}$. Consider the $m$ distinct numbers, $\left\{e^{\frac{2 k \pi i}{m}} \phi(\widehat{z})\right\}_{k=1}^{m}$. Then each of these numbers is in $B(0, \delta)$ and so since $\phi$ maps $V$ one to one onto $B(0, \delta)$, there are $m$ distinct numbers in $V,\left\{z_{k}\right\}_{k=1}^{m}$ such that $\phi\left(z_{k}\right)=e^{\frac{2 k \pi i}{m}} \phi(\widehat{z})$. Then

$$
\begin{aligned}
f\left(z_{k}\right) & =f\left(z_{0}\right)+\phi\left(z_{k}\right)^{m}=f\left(z_{0}\right)+\left(e^{\frac{2 k \pi i}{m}} \phi(\widehat{z})\right)^{m} \\
& =f\left(z_{0}\right)+e^{2 k \pi i} \phi(\widehat{z})^{m}=f\left(z_{0}\right)+\phi(\widehat{z})^{m}=f(\widehat{z})=w
\end{aligned}
$$

Nothing remotely resembling this happens for functions of a real variable. This is yet another manifestation of the fact that analytic functions are really glorified polynomials. With the open mapping theorem, the maximum modulus theorem is fairly easy.

Theorem 15.1.4 Let $\Omega$ be an open connected, bounded set in $\mathbb{C}$ and let $f: \Omega \rightarrow \mathbb{C}$ be analytic. Let $\partial \Omega \equiv \bar{\Omega} \backslash \Omega$. Then

$$
\max \{|f(z)|: z \in \bar{\Omega}\}=\max \{|f(z)|: z \in \partial \Omega\}
$$

and if the maximum of $|f(z)|$ is achieved at a point of $\Omega$, then $f$ is a constant.
Proof: Suppose $f(\Omega)$ is not a single point. That is, $f$ is not constant. Then by the open mapping theorem, $f(\Omega)$ is an open connected subset of $\mathbb{C}$ and so $z \rightarrow|f(z)|$ has no maximum. Indeed, if $w \in f(\Omega)$, then for some $r>0, B(w, r) \subseteq f(\Omega)$ and so there are points of $f(\Omega)$ farther from 0 than $w$. Thus the maximum of $|f(z)|$ for $z \in \bar{\Omega}$ is on $\partial \Omega$. If $f(\Omega)$ is a single point, then the equation still holds.

### 15.2 Functions Analytic on an Annulus

First consider the definition of an annulus.
Definition 15.2.1 Define ann $(a, r, R) \equiv\{z: r<|z-a|<R\}$.
Thus ann $(a, 0, R)$ would denote the punctured ball, $B(a, R) \backslash\{a\}$ and when $r>0$, the annulus looks like the following.


The annulus consists of the points between the two circles.
In the following picture, let there be two parametrizations, $\gamma_{R}$ for the large circle and $\hat{\gamma}_{r}$ for the small one with clockwise orientation as shown. I will let $\gamma_{r}$ be the same circle oriented counter clockwise. There are also two line segments oriented as shown which miss a particular $z \in \operatorname{ann}\left(z_{0}, r, R\right)$ and constitute the intersection of the two simple closed curves $\Gamma_{1}, \Gamma_{2}$. These two simple closed curves are oriented as shown, both counter clockwise. Thus $n\left(\Gamma_{1}, z\right)=1$ and $n\left(\Gamma_{2}, z\right)=0$. Let $f$ be continuous on $\overline{\text { ann }\left(z_{0}, r, R\right)}$ and be analytic on ann $\left(z_{0}, r, R\right)$.


First suppose $f$ is analytic near $\operatorname{ann}\left(z_{0}, r, R\right)$. It follows from Theorem 14.11.1, that for $z$ in the annulus,

$$
f(z)=\frac{1}{2 \pi i} \int_{\Gamma_{1}} \frac{f(w)}{w-z} d w+\frac{1}{2 \pi i} \int_{\Gamma_{2}} \frac{f(w)}{w-z} d w=\int_{\gamma_{R}} \frac{f(w)}{w-z} d w-\int_{\gamma_{r}} \frac{f(w)}{w-z} d w
$$

since the contributions to the line integrals along those straight lines is 0 .
The same formula holds in case $f$ is only analytic on ann $\left(z_{0}, r, R\right)$ and continuous on $\overline{\operatorname{ann}\left(z_{0}, r, R\right)}$. To see this, replace $R$ with $\hat{R}<R$ but close to $R$ and $r$ with $\hat{r}>r$ but close to
$r$, and obtain the above formula valid for $\gamma_{\hat{R}}$ and $\gamma_{\hat{r}}$. Then pass to a limit using continuity of $f$ on $\overline{\operatorname{ann}\left(z_{0}, r, R\right)}$ to replace $\hat{R}$ and $\hat{r}$ with $R$ and $r$. The details are routine and left for you. Thus,

$$
\begin{aligned}
f(z) & =\frac{1}{2 \pi i}\left[\int_{\gamma_{R}} \frac{f(w)}{w-z_{0}-\left(z-z_{0}\right)} d w+\int_{\gamma_{r}} \frac{f(w)}{\left(z-z_{0}\right)-\left(w-z_{0}\right)} d w\right] \\
& =\frac{1}{2 \pi i}\left[\int_{\gamma_{R}} \frac{1}{w-z_{0}} \frac{f(w)}{1-\frac{z-z_{0}}{w-z_{0}}} d w+\int_{\gamma_{r}} \frac{1}{z-z_{0}} \frac{f(w)}{1-\frac{w-z_{0}}{z-z_{0}}} d w\right]
\end{aligned}
$$

Now note that for $z$ in the annulus between the two circles and $w \in \gamma_{R}^{*},\left|\frac{z-z_{0}}{w-z_{0}}\right|<1$, and for $w \in \gamma_{r}^{*},\left|\frac{w-z_{0}}{z-z_{0}}\right|<1$. In fact, in each case, there is $b<1$ such that

$$
\begin{equation*}
w \in \gamma_{R}^{*},\left|\frac{z-z_{0}}{w-z_{0}}\right|<b<1, w \in \gamma_{r}^{*},\left|\frac{w-z_{0}}{z-z_{0}}\right|<b<1 \tag{15.4}
\end{equation*}
$$

Thus you can use the formula for the sum of an infinite geometric series and conclude

$$
f(z)=\frac{1}{2 \pi i}\left[\begin{array}{c}
\int_{\gamma_{R}} f(w) \frac{1}{w-z_{0}} \sum_{n=0}^{\infty}\left(\frac{z-z_{0}}{w-z_{0}}\right)^{n} d w \\
+\int_{\gamma_{r}} f(w) \frac{1}{\left(z-z_{0}\right)} \sum_{n=0}^{\infty}\left(\frac{w-z_{0}}{z-z_{0}}\right)^{n} d w
\end{array}\right]
$$

Then from the uniform estimates of 15.4 and the Weierstrass M test, Theorem 2.5.42, one can conclude uniform convergence of the partial sums for $w \in \gamma_{R}^{*}$ or $\gamma_{r}^{*}$ and so one can interchange the summation with the integral and write

$$
\begin{aligned}
f(z)= & \sum_{n=0}^{\infty}\left(\frac{1}{2 \pi i} \int_{\gamma_{R}} f(w) \frac{1}{\left(w-z_{0}\right)^{n+1}} d w\right)\left(z-z_{0}\right)^{n} \\
& +\sum_{n=0}^{\infty}\left(\frac{1}{2 \pi i} \int_{\gamma_{r}} f(w)\left(w-z_{0}\right)^{n} d w\right) \frac{1}{\left(z-z_{0}\right)^{n+1}}
\end{aligned}
$$

both series converging absolutely. Thus there are $a_{n}, b_{n} \in X$ such that

$$
f(z)=\sum_{n=0}^{\infty} a_{n}\left(z-z_{0}\right)^{n}+\sum_{n=1}^{\infty} b_{n}\left(z-z_{0}\right)^{-n}
$$

This proves the following theorem.
Theorem 15.2.2 Let $z \in \operatorname{ann}\left(z_{0}, r, R\right)$ and let $f: \operatorname{ann}\left(z_{0}, r, R\right) \rightarrow X$ be analytic on ann $\left(z_{0}, r, R\right)$ and continuous on $\overline{\operatorname{ann}\left(z_{0}, r, R\right)}$. Then for any $z \in \operatorname{ann}\left(z_{0}, r, R\right)$,

$$
\begin{equation*}
f(z)=\sum_{n=0}^{\infty} a_{n}\left(z-z_{0}\right)^{n}+\sum_{n=1}^{\infty} b_{n}\left(z-z_{0}\right)^{-n} \tag{15.5}
\end{equation*}
$$

where

$$
a_{n}=\frac{1}{2 \pi i} \int_{\gamma_{R}} f(w) \frac{1}{\left(w-z_{0}\right)^{n+1}} d w, b_{n}=\frac{1}{2 \pi i} \int_{\gamma_{r}} f(w)\left(w-z_{0}\right)^{n-1} d w
$$

and both of these series in 15.5 converge absolutely. If $r<\hat{r}<\hat{R}<R$, then convergence of both series is absolute and uniform for $z \in \operatorname{ann}\left(z_{0}, \hat{r}, \hat{R}\right)$.

Proof: Let $|f(w)| \leq M$ on the closure of the annulus. Consider the sum with the negative exponents. The other case is similar.

$$
\sum_{n=1}^{\infty} b_{n}\left(z-z_{0}\right)^{-n}, b_{n}=\left(\frac{1}{2 \pi i} \int_{\gamma_{r}} f(w)\left(w-z_{0}\right)^{n} d w\right)
$$

Therefore, $\left\|b_{n}\right\| \leq 2 \pi r M r^{n}$ and $\left|z-z_{0}\right| \geq \hat{r}>r$. Thus

$$
\sum_{n=p}^{q}\left\|b_{n}\right\|\left|z-z_{0}\right|^{-n} \leq \sum_{n=p}^{q} 2 \pi \hat{r} M \frac{r^{n}}{\hat{r}^{n}}<\varepsilon
$$

if $p$ is large enough. Therefore, the partial sums are a uniformly Cauchy sequence so the sum converges absolutely and uniformly on the set $\left\{z: \hat{r} \leq\left|z-z_{0}\right| \leq \hat{R}\right\}$.

Note that for arbitrary $\alpha$ with $r \leq \alpha \leq R$,

$$
\begin{equation*}
\frac{1}{2 \pi i} \int_{\gamma_{R}} f(w) \frac{1}{\left(w-z_{0}\right)^{n+1}} d w=\frac{1}{2 \pi i} \int_{\gamma_{\alpha}} f(w) \frac{1}{\left(w-z_{0}\right)^{n+1}} d w \tag{15.6}
\end{equation*}
$$

Here $\gamma_{\alpha}$ is oriented counter clockwise like $\Gamma_{R}$ and $\gamma_{\alpha}$ is oriented counter clockwise. This is a simple application of the Cauchy integral theorem, Theorem 14.5.3 applied to the union of two simple closed curves of the sort used to prove Theorem 15.2.2. You consider the annulus ann $\left(z_{0}, \alpha, R\right)$ and the following diagram.


The integrand is analytic on the inside of the two simple closed curves $\Gamma_{1}$ and $\Gamma_{2}$. Letting $\gamma_{1}$ and $\gamma_{2}$ be oriented parametrizations for these and using the argument that the integrals over the straight lines cancel, this yields

$$
\frac{1}{2 \pi i} \int_{\gamma_{R}} f(w) \frac{1}{\left(w-z_{0}\right)^{n+1}} d w+\frac{1}{2 \pi i} \int_{\hat{\gamma}_{\alpha}} f(w) \frac{1}{\left(w-z_{0}\right)^{n+1}} d w=0
$$

and so, letting $\gamma_{\alpha} \equiv-\hat{\gamma}_{\alpha}$, yields formula 15.6.
Similar considerations apply to $b_{n}=\frac{1}{2 \pi i} \int_{\gamma_{\alpha}} f(w)\left(w-z_{0}\right)^{n-1} d w$.
Corollary 15.2.3 The $a_{n}$ and $b_{n}$ are uniquely determined. Specifically,

$$
b_{n}=\frac{1}{2 \pi i} \int_{\gamma_{\alpha}} f(w)\left(w-z_{0}\right)^{n-1} d w, a_{n}=\frac{1}{2 \pi i} \int_{\gamma_{\alpha}} f(w) \frac{1}{\left(w-z_{0}\right)^{n+1}} d w, \quad \alpha \in(r, R)
$$

Also, $\int_{\gamma_{\alpha}} f(z) d z=2 \pi i b_{1}$.
Proof: Let $\alpha \in(r, R)$ and let $\gamma_{\alpha}$ be a parametrization of the circle centered at $z_{0}$ of radius $\alpha$ which is counterclockwise. Then $f(w)=\sum_{n=0}^{\infty} a_{n}\left(w-z_{0}\right)^{n}+\sum_{n=1}^{\infty} b_{n}\left(w-z_{0}\right)^{-n}$ for $w$ in the annulus. For $k \geq 1$,

$$
f(w)\left(w-z_{0}\right)^{k-1}=\sum_{n=0}^{\infty} a_{n}\left(w-z_{0}\right)^{n+k-1}+\sum_{n=1}^{\infty} b_{n}\left(w-z_{0}\right)^{-n+k-1}
$$

By uniform convergence,

$$
\int_{\gamma_{\alpha}} f(w)\left(w-z_{0}\right)^{k-1} d w=\sum_{n=0}^{\infty} a_{n} \int_{\gamma_{\alpha}}\left(w-z_{0}\right)^{n+k-1} d w+\sum_{n=1}^{\infty} b_{n} \int_{\gamma_{\alpha}}\left(w-z_{0}\right)^{-n+k-1} d w
$$

Now in the sums, all integrals are 0 except the one when $n=k$ in the second sum. Therefore,

$$
\int_{\gamma_{\alpha}} f(w)\left(w-z_{0}\right)^{k-1} d w=b_{k} \int_{\gamma_{\alpha}}\left(w-z_{0}\right)^{-1} d w=2 \pi i b_{k}
$$

This shows that for any $\alpha, r<\alpha<R, b_{k}=\frac{1}{2 \pi i} \int_{\gamma_{\alpha}} f(w)\left(w-z_{0}\right)^{k-1} d w$. Similar reasoning gives $a_{n}=\frac{1}{2 \pi i} \int_{\gamma_{\alpha}} f(w) \frac{1}{\left(w-z_{0}\right)^{n+1}} d w$ and as explained above, nothing changes when $\alpha$ is changed.

The last claim follows from observing that all the terms in the Laurent series have primitives except the one corresponding to $b_{1}=\frac{1}{2 \pi i} \int_{\gamma_{\alpha}} f(w) d w$.
Definition 15.2.4 For $f$ continuous on the closure of an annulus as just described and analytic on the annulus, it follows that on the annulus, $f$ can be written as the sum of a power series and a series involving $\left(z-z_{0}\right)$ raised to negative powers. This is called the Laurent series. The series involving negative powers of $\left(z-z_{0}\right)$ is called the principal part of the Laurent series. $b_{1}$ is called the residue at $z_{0}$ denoted as res $\left(f, z_{0}\right)$.

Note that if $f$ is analytic near $z_{0}$, but possibly not at $z_{0}$ then the $r$ in $\gamma_{r}$ can be taken as small as desired.

### 15.3 Cauchy Integral Formula for a Cycle

There is a much more elaborate formulation of the Cauchy integral formula which involves cycles and an additional assumption that the function of interest is analytic on an open set which contains the contours over which the integrals are taken. Thus we abandon the generality which allows the weaker assumption that the function is continuous on the contours of interest and is only known to be analytic on the inside.

The reason for this is that a key part of the argument uses Liouville's theorem. This, and the notion of winding number are what makes possible the Cauchy integral theorem for a cycle. Recall that the winding number is defined as $n(\gamma, z) \equiv \frac{1}{2 \pi i} \int_{\gamma} \frac{1}{w-z} d w$ where $\gamma$ is a parametrization of an oriented curve $\gamma^{*}$ and $z \notin \gamma^{*}$. This makes perfect sense for any bounded variation curve $\gamma^{*}$. Recall that $z \rightarrow n(\gamma, z)$ is continuous on the complement of $\gamma^{*}$.

Consider a situation typified by the following picture in which $\Omega$ is the open set between the dotted curves and $\gamma_{j}$ are closed rectifiable curves in $\Omega$.


The open set is between the dotted lines and also excludes the points $z_{2}, z_{3}, z_{4}$. Note how if you pick any $z \notin \Omega$, including the $z_{k}$, then $\sum_{k=1}^{4} n\left(\gamma_{k}, z\right)=0$. This open set is not simply connected as described in Definition 14.12.4. Its complement relative to the extended complex plane is not connected. Note how this is manifested by the points left out.

Definition 15.3.1 Let $\left\{\gamma_{k}\right\}_{k=1}^{n}$ be a set of parametrizations which are continuous and of bounded variation. Then $\left\{\gamma_{k}\right\}_{k=1}^{n}$ is called a cycle if whenever, $z \notin \cup_{k=1}^{n} \gamma_{k}^{*}$, $\sum_{k=1}^{n} n\left(\gamma_{k}, z\right)$ is an integer. Note that, unlike the above picture, there is no reason to believe the $\gamma_{k}^{*}$ are closed curves.

Now the following is the general Cauchy integral formula. In the theorem, it is only assumed that the sum of the winding numbers is an integer. Thus, it is probably, but not necessarily the case that you have in mind each contour being a closed curve.

Theorem 15.3.2 Let $\Omega$ be an open subset of the plane (not necessarily simply connected) and let $f: \Omega \rightarrow X$ be analytic. If $\gamma_{k}:\left[a_{k}, b_{k}\right] \rightarrow \Omega, k=1, \cdots, m$ are continuous curves having bounded variation such that for all $z \notin \cup_{k=1}^{m} \gamma_{k}^{*}$,

$$
\sum_{k=1}^{m} n\left(\gamma_{k}, z\right) \text { equals an integer }
$$

and for all $z \notin \Omega, \sum_{k=1}^{m} n\left(\gamma_{k}, z\right)=0$. Then for all $z \in \Omega \backslash \cup_{k=1}^{m} \gamma_{k}^{*}$,

$$
f(z) \sum_{k=1}^{m} n\left(\gamma_{k}, z\right)=\sum_{k=1}^{m} \frac{1}{2 \pi i} \int_{\gamma_{k}} \frac{f(w)}{w-z} d w .
$$

Proof: Let $\phi$ be defined on $\Omega \times \Omega$ by

$$
\phi(z, w) \equiv\left\{\begin{array}{l}
\frac{f(w)-f(z)}{} \text { if } w \neq z . \\
f^{\prime}(z-z) \text { if } w=z
\end{array} .\right.
$$

Then $\phi$ is analytic as a function of $z$ and analytic as a function of $w$ and is continuous in $\Omega \times \Omega$. This follows from the argument given in Theorem 14.11.1, resulting from the fact that if a function has one derivative on an open set, then it has them all. Indeed, it is obvious that $z \rightarrow \phi(z, w)$ and $w \rightarrow \phi(z, w)$ are analytic if $z \neq w$. In case $z=w$, for small $h$,

$$
\begin{aligned}
\frac{\phi(z+h, z)-\phi(z, z)}{h} & =\left(\frac{f(z)-\sum_{k=0}^{\infty} \frac{f^{(k)}(z)}{k!} h^{k}}{-h}-f^{\prime}(z)\right) \frac{1}{h} \\
& =\frac{1}{h}\left(\frac{-f^{\prime}(z) h}{-h}+o(h)-f^{\prime}(z)\right) \rightarrow 0
\end{aligned}
$$

The case of $w \rightarrow \phi(z, w)$ is similar.
Define $h(z) \equiv \frac{1}{2 \pi i} \sum_{k=1}^{m} \int_{\gamma_{k}} \phi(z, w) d w$. Is $h$ analytic on $\Omega$ ? To show this is the case, verify $\int_{\partial T} h(z) d z=0$ for every triangle $T$, such that the triangle and its inside are contained in $\Omega$ and apply the Corollary 14.8 .6 . This is an application of the Fubini theorem of The-
orem 14.4.9. By Theorem 14.4.9, $\int_{\partial T} \int_{\gamma_{k}} \phi(z, w) d w d z=\int_{\gamma_{k}} \overbrace{\int_{\partial T} \phi(z, w) d z} d w=0$ because $z \rightarrow \phi(z, w)$ is analytic. By Corollary 14.8.6, $h$ is analytic on $\Omega$ as claimed.

Now recall that by assumption, $\sum_{k=1}^{m} n\left(\gamma_{k}, z\right)=0$ for $z \in \Omega^{C}$. Let $H \supseteq \Omega^{C}$,

$$
\begin{aligned}
H & \equiv\left\{z \in \mathbb{C} \backslash \cup_{k=1}^{m} \gamma_{k}^{*}: \sum_{k=1}^{m} n\left(\gamma_{k}, z\right)=0\right\} \\
& =\left\{z \in \mathbb{C} \backslash \cup_{k=1}^{m} \gamma_{k}^{*}: \sum_{k=1}^{m} n\left(\gamma_{k}, z\right) \in(-1 / 2,1 / 2)\right\}
\end{aligned}
$$

the second equality holding because it is given that the sum of these is integer valued. Thus $H$ is an open set because $z \rightarrow \sum_{k=1}^{m} n\left(\gamma_{k}, z\right)$ is continuous. Also, $\Omega \cup H=\mathbb{C}$ because by assumption, $\Omega^{C} \subseteq H$. Extend $h(z)$ to a function $g(z)$ defined on all of $\mathbb{C}$ as follows:

$$
g(z) \equiv\left\{\begin{array}{l}
h(z) \equiv \frac{1}{2 \pi i} \sum_{k=1}^{m} \int_{\gamma_{k}} \phi(z, w) d w \text { if } z \in \Omega  \tag{15.7}\\
\frac{1}{2 \pi i} \sum_{k=1}^{m} \int_{\gamma_{k}} \frac{f(w)}{w-z} d w \text { if } z \in H \text { if } \sum_{k=1}^{m} n\left(\gamma_{k}, z\right)=0
\end{array}\right.
$$

Why is $g(z)$ well defined? On $\Omega \cap H, z \notin \cup_{k=1}^{m} \gamma_{k}^{*}$ and so

$$
\begin{aligned}
g(z) & =\frac{1}{2 \pi i} \sum_{k=1}^{m} \int_{\gamma_{k}} \phi(z, w) d w=\frac{1}{2 \pi i} \sum_{k=1}^{m} \int_{\gamma_{k}} \frac{f(w)-f(z)}{w-z} d w \\
& =\frac{1}{2 \pi i} \sum_{k=1}^{m} \int_{\gamma_{k}} \frac{f(w)}{w-z} d w-\frac{1}{2 \pi i} \sum_{k=1}^{m} \int_{\gamma_{k}} \frac{f(z)}{w-z} d w=\frac{1}{2 \pi i} \sum_{k=1}^{m} \int_{\gamma_{k}} \frac{f(w)}{w-z} d w
\end{aligned}
$$

because $z \in H$ so $\sum_{k=1}^{m} n\left(\gamma_{k}, z\right)=0$. This shows $g(z)$ is well defined. Also, $g$ is analytic on $\Omega$ because it equals $h$ there. It is routine to verify that $g$ is analytic on $H$ also because of the second line of 15.7 which is an analytic function of $z$. (See discussion at the end if this is not clear. )

Therefore, $g$ is an entire function, meaning that it is analytic on all of $\mathbb{C}$.
Now note that $\sum_{k=1}^{m} n\left(\gamma_{k}, z\right)=0$ for all $z$ contained in the unbounded component of $\mathbb{C} \backslash \cup_{k=1}^{m} \gamma_{k}^{*}$ which component contains $B(0, r)^{C}$ for $r$ large enough. It follows that for $|z|>r$, it must be the case that $z \in H$ and so for such $z$, the bottom description of $g(z)$ found in 15.7 is valid. Therefore, it follows $\lim _{|z| \rightarrow \infty}\|g(z)\|=0$ and so $g$ is bounded and analytic on all of $\mathbb{C}$. By Liouville's theorem, $g$ is a constant. Hence, the constant can only equal zero.

For $z \in \Omega \backslash \cup_{k=1}^{m} \gamma_{k}^{*}$, since it was just shown that $h(z)=g(z)=0$ on $\Omega$

$$
\begin{aligned}
0=h(z)= & \frac{1}{2 \pi i} \sum_{k=1}^{m} \int_{\gamma_{k}} \phi(z, w) d w=\frac{1}{2 \pi i} \sum_{k=1}^{m} \int_{\gamma_{k}} \frac{f(w)-f(z)}{w-z} d w= \\
& \frac{1}{2 \pi i} \sum_{k=1}^{m} \int_{\gamma_{k}} \frac{f(w)}{w-z} d w-f(z) \sum_{k=1}^{m} n\left(\gamma_{k}, z\right) .
\end{aligned}
$$

In case it is not obvious why $g$ is analytic on $H$, use the formula. For $z \in H, z \notin \gamma_{k}^{*}$ for any $k$. The issue reduces to showing that $z \rightarrow \sum_{k=1}^{m} \int_{\gamma_{k}} \frac{f(w)}{w-z} d w$ is analytic. You can show this by taking a limit of a difference quotient and argue that the limit can be taken inside the integral. Taking a difference quotient and simplifying a little, one obtains

$$
\frac{\int_{\gamma_{k}} \frac{f(w)}{w-(z+h)} d w-\int_{\gamma_{k}} \frac{f(w)}{w-z} d w}{h}=\int_{\gamma_{k}} \frac{f(w)}{(w-z)(w-(z+h))} d w
$$

considering only small $h$, the denominator is bounded below by some $\delta>0$ and also $f(w)$ is bounded on the compact set $\gamma_{k}^{*},|f(w)| \leq M$. Then for such small $h$,

$$
\begin{gathered}
\left|\frac{f(w)}{(w-z)(w-(z+h))}-\frac{f(w)}{(w-z)^{2}}\right| \\
=\left|\frac{1}{w-z}\left(\frac{1}{(w-(z+h))}-\frac{1}{(w-z)}\right) f(w)\right| \leq\left|\frac{1}{w-z}\right| \frac{1}{\delta} h M
\end{gathered}
$$

it follows that one obtains uniform convergence as $h \rightarrow 0$ of the integrand to $\frac{f(w)}{(w-z)^{2}}$ for any sequence $h \rightarrow 0$ and by Theorem 14.4.7, the integral converges to $\int_{\gamma_{k}} \frac{f(w)}{(w-z)^{2}} d w$.

The following is an interesting and fairly easy corollary.
Corollary 15.3.3 Let $\Omega$ be an open set (note that $\Omega$ might not be simply connected) and let $\gamma_{k}:\left[a_{k}, b_{k}\right] \rightarrow \Omega, k=1, \cdots, m$, be closed, continuous and of bounded variation. Suppose also that $\sum_{k=1}^{m} n\left(\gamma_{k}, z\right)=0$ for all $z \notin \Omega$ and $\sum_{k=1}^{m} n\left(\gamma_{k}, z\right)$ is an integer for $z \in$ $\cap_{k=1}^{m}\left(\Omega \backslash \gamma_{k}^{*}\right)$. Then if $f: \Omega \rightarrow X$ is analytic, $\sum_{k=1}^{m} \int_{\gamma_{k}} f(w) d w=0$.

Proof: This follows from Theorem 15.3.2 as follows. Let $g(w)=f(w)(w-z)$ where $z \in \Omega \backslash \cup_{k=1}^{m} \gamma_{k}\left(\left[a_{k}, b_{k}\right]\right)$. Then by this theorem,

$$
0=0 \sum_{k=1}^{m} n\left(\gamma_{k}, z\right)=g(z) \sum_{k=1}^{m} n\left(\gamma_{k}, z\right)=\sum_{k=1}^{m} \frac{1}{2 \pi i} \int_{\gamma_{k}} \frac{g(w)}{w-z} d w=\frac{1}{2 \pi i} \sum_{k=1}^{m} \int_{\gamma_{k}} f(w) d w .
$$

What if $\Omega$ is simply connected? Can one assert something interesting in this case beyond what is said above? Yes, and this is a very important result. Recall what it meant for an open set $\Omega \subseteq \mathbb{C}$ to be simply connected. It meant that $\Omega$ is connected and $\Omega^{C}$ is connected in the extended complex plane $\widehat{\mathbb{C}}$.

Corollary 15.3.4 Let $\gamma:[a, b] \rightarrow \Omega$ be a continuous closed curve of bounded variation where $\Omega$ is a simply connected region contained in $\mathbb{C}$ (Thus $\Omega$ does not contain $\infty$.) and let $f: \Omega \rightarrow X$ be analytic. Then $\int_{\gamma} f(w) d w=0$.

Proof: Let $D$ denote the unbounded component of $\widehat{\mathbb{C}} \backslash \gamma^{*}$. Thus $\infty \in \widehat{\mathbb{C}} \backslash \gamma^{*}$. Then the connected set, $\widehat{\mathbb{C}} \backslash \Omega$ is contained in $D$ since $\infty$ is contained in both $\widehat{\mathbb{C}} \backslash \Omega$ and $D$. It follows that $n(\gamma, \cdot)$ must be constant on $\widehat{\mathbb{C}} \backslash \Omega$, its value being its value on $D$. However, for $z \in D, n(\gamma, z)=\frac{1}{2 \pi i} \int_{\gamma} \frac{1}{w-z} d w$ and so $\lim _{|z| \rightarrow \infty} n(\gamma, z)=0$ showing $n(\gamma, z)=0$ on $D$. Therefore this verifies the hypothesis of Theorem 15.3.2. Let $z \in \Omega \cap D$ and define $g(w) \equiv$ $f(w)(w-z)$. Thus $g$ is analytic on $\Omega$ and by Theorem 15.3.2,

$$
0=n(z, \gamma) g(z)=\frac{1}{2 \pi i} \int_{\gamma} \frac{g(w)}{w-z} d w=\frac{1}{2 \pi i} \int_{\gamma} f(w) d w .
$$

The following is a very significant result which will be used later. It is a fairly large jump to go from star shaped to simply connected which is what this corollary does.
Corollary 15.3.5 Suppose $\Omega$ is a simply connected open set and $f: \Omega \rightarrow X$ is analytic. Then $f$ has a primitive $F$, on $\Omega$. Recall this means there exists $F$ such that $F^{\prime}(z)=f(z)$ for all $z \in \Omega$.

Proof: Pick a point, $z_{0} \in \Omega$ and let $V$ denote those points $z$ of $\Omega$ for which there exists a curve, $\gamma:[a, b] \rightarrow \Omega$ such that $\gamma$ is continuous, of bounded variation, $\gamma(a)=z_{0}$, and $\gamma(b)=$ $z$. Then it is easy to verify that $V$ is both open and closed in $\Omega$ and therefore, $V=\Omega$ because $\Omega$ is connected. Denote by $\gamma_{z_{0}, z}$ such a curve from $z_{0}$ to $z$ and define $F(z) \equiv \int_{\gamma_{z_{0}, z}} f(w) d w$. Then $F$ is well defined because if $\gamma_{j}, j=1,2$ are two such curves, it follows from Corollary 15.3.4 that $\int_{\gamma_{1}} f(w) d w+\int_{-\gamma_{2}} f(w) d w=0$, implying that $\int_{\gamma_{1}} f(w) d w=\int_{\gamma_{2}} f(w) d w$.Now this function $F$ is a primitive because, thanks to Corollary 15.3.4

$$
(F(z+h)-F(z)) h^{-1}=\frac{1}{h} \int_{\gamma_{z, z+h}} f(w) d w=\frac{1}{h} \int_{0}^{1} f(z+t h) h d t
$$

and so, taking the limit as $h \rightarrow 0, F^{\prime}(z)=f(z)$.
Next is a technical result about finding suitable cycles which seems to often be taken for granted, but it is not at all obvious. It is about getting closed curves which enclose each of finitely many compact sets such that there is no intersection between curves, and each goes around the corresponding compact set in the positive direction.

### 15.4 An Example of a Cycle

The next theorem deals with the existence of a cycle with nice properties. Basically, you go around the compact subset of an open set with suitable contours while staying in the open set. The method involves the following simple concept. If a cycle $\Gamma$ consists of oriented curves $\left\{\gamma_{1}, \cdots, \gamma_{r}\right\}$, define for $p \notin \Gamma^{*}, n(\Gamma, p) \equiv \sum_{i=1}^{r} n\left(\gamma_{i}, p\right)$. Also, $\int_{\Gamma} f(\lambda) d \lambda \equiv \sum_{i=1}^{r} \int_{\gamma_{i}} f(\lambda) d \lambda$.

## Definition 15.4.1 A tiling of $\mathbb{R}^{2}=\mathbb{C}$ is the union of infinitely many equally spaced

 vertical and horizontal lines. You can think of the small squares which result as tiles. To tile the plane or $\mathbb{R}^{2}=\mathbb{C}$ means to consider such a union of horizontal and vertical lines. It is like graph paper. See the picture below for a representation of part of a tiling of $\mathbb{C}$.

Here is something which is clear. If you have a square parametrized by $\gamma$ oriented in the counter clockwise direction, then for $z$ on the inside of this square, $n(z, \gamma)=1$ and if $z$ is on the outside of this square, then $n(z, \gamma)=0$. This observation is used in the following.

Theorem 15.4.2 Let $K_{1}, K_{2}, \cdots, K_{m}$ be disjoint compact subsets of an open set $\Omega$ in $\mathbb{C}$. Then there exist continuous, closed, bounded cycles $\left\{\Gamma_{j}\right\}_{j=1}^{m}$ for which $\Gamma_{j}^{*} \cap K_{k}=\emptyset$ for each $k, j, \Gamma_{j}^{*} \cap \Gamma_{k}^{*}=\emptyset, \Gamma_{j}^{*} \subseteq \Omega$. Also, if $p \in K_{k}$ and $j \neq k, n\left(\Gamma_{k}, p\right)=1, n\left(\Gamma_{j}, p\right)=0$ so if $p$ is in some $K_{k}, \sum_{j=1}^{m} n\left(\Gamma_{j}, p\right)=1$ each $\Gamma_{j}$ being the union of oriented simple closed curves, while for all $z \notin \Omega, \sum_{k=1}^{m} n\left(\Gamma_{k}, z\right)=0$. Also, if $p \in \Gamma_{j}^{*}$, then for $i \neq j, n\left(\Gamma_{i}, p\right)=0$.

Proof: Consider $z \rightarrow \operatorname{dist}\left(z, \cup_{k \neq j} K_{k} \cup \Omega^{C}\right)$. From Lemma 2.4.8 this is a continuous function. Thus it has a minimum on $K_{j}$. Let this be $\delta_{j}$. Then $\delta_{j}>0$ because the $K_{j}$ are disjoint. Let $0<\delta<\min \left\{\delta_{j}, j=1, \cdots, m\right\}$. Now tile the plane with squares, each of which has diameter less than $\delta / 8$. Thus none of these squares can intersect more than one $K_{j}$. Orient the boundaries of each square counter clockwise. Thus a direction of motion is specified along all the edges of each square in the tiling.

Let $F_{j}$ denote the oriented boundaries of squares from the tiling which intersect $K_{j}$. Also for $z \notin \gamma^{*}, \gamma \in F_{i}$, let $n\left(F_{i}, z\right)$ denote the sum of the winding numbers $n(\gamma, z)$ for $\gamma \in F_{i}$. Thus, for such $z, n\left(F_{j}, z\right)=1$ if $z \in K_{j}$ and $n\left(F_{j}, z\right)=0$ if $z \in K_{i}$ for $i \neq j$ or for $z \in \Omega^{C}$. When an edge of some $\gamma \in F_{j}$ intersects $K_{j}$, delete this edge retaining the orientations of the line segments which were not deleted.


Let $F_{j}$ be this new collection of oriented simple closed curves. It is still the case that $n\left(F_{j}, z\right)=1$ if $z \in K_{j}$ and $n\left(F_{j}, z\right)=0$ if $z \in K_{i}$ for $i \neq j$ or for $z \in \Omega^{C}$ because you can
add together the integrals over the small oriented squares considered in obtaining $\Gamma_{j}$ to get $n\left(F_{j}, z\right)$. If $z \in K_{j}, n\left(z, \Gamma_{j}\right)=1$ if $z$ is not on a removed edge. If it is on an edge which was removed, then continuity of $z \rightarrow n\left(z, \Gamma_{j}\right)$ gives the same result. The construction is illustrated in the following picture.


Then as explained above, if $p$ is in some $K_{j}$ then $n\left(F_{j}, z\right)=1$ and if $z$ is in $K_{i}, i \neq j$, then $n\left(F_{j}, z\right)=0$. Each step in the process results in $F_{j}$ which is a finite set of simple closed curves. However, if this is not clear, consider the following which likely could be used as another way to prove the theorem.

Each orientation on an edge corresponds to a direction of motion over that edge. Call such a motion over the edge a route. Initially, every vertex, (corner of $\gamma \in F_{j}$ ) has the property that there are the same number of routes to and from that vertex. When an edge containing a point of $K$ is deleted, every vertex either remains unchanged as to the number of routes to and from that vertex or it loses both a route away and a route to. Thus the property of having the same number of routes to and from each vertex is preserved by deleting these edges. It follows that you can begin at any of the remaining vertices and follow the routes leading out from this and successive vertices according to orientation and eventually return to that vertex from which you started. Otherwise, there would be a vertex which would have only one route leading to it which does not happen. Now if you have used all the routes out of this vertex, pick another vertex and do the same process. Otherwise, pick an unused route out of the vertex and follow it to return. Continue this way till all routes are used exactly once, resulting in closed oriented curves, $\Gamma_{k}$.

### 15.5 Isolated Singularities

This is about the situation where the Laurent series of $f$ has nonzero principal part. When this occurs, we say that $z_{0}$ is a singularity. The singularities are isolated if each is the center of a ball such that $f$ is analytic except for the center of the ball.

## Definition 15.5.1 Let $B^{\prime}(a, r) \equiv\{z \in \mathbb{C}$ such that $0<|z-a|<r\}$. Thus this is the

 usual ball without the center. A function is said to have an isolated singularity at the point $a \in \mathbb{C}$ if $f$ is analytic on $B^{\prime}(a, r)$ for some $r>0$.It turns out isolated singularities can be neatly classified into three types, removable singularities, poles, and essential singularities. The next theorem deals with the case of a removable singularity.

Definition 15.5.2 An isolated singularity of $f$ is said to be removable if there exists an analytic function $g$ analytic at $a$ and near a such that $f=g$ at all points near $a$.

Theorem 15.5.3 Let $f: B^{\prime}(a, r) \rightarrow X$ be analytic. Thus $f$ has an isolated singularity at $a$. Then $a$ is a removable singularity if and only if

$$
\lim _{z \rightarrow a} f(z)(z-a)=0
$$

Thus the above limit occurs if and only if there exists a unique analytic function, $g$ : $B(a, r) \rightarrow X$ such that $g=f$ on $B^{\prime}(a, r)$. In other words, you can re define $f$ at a so that the resulting function is analytic.

Proof: $\Rightarrow$ Let $h(z) \equiv(z-a)^{2} f(z), h(a) \equiv 0$. Then $h$ is analytic on $B(a, r)$ because it is easy to see that

$$
\begin{equation*}
h^{\prime}(a)=\lim _{z \rightarrow a} \frac{h(z)-h(a)}{(z-a)}=\lim _{z \rightarrow a} \frac{(z-a)^{2} f(z)}{z-a}=\lim _{z \rightarrow a}(z-a) f(z)=0 \tag{15.8}
\end{equation*}
$$

Thus $h(z)=\sum_{k=2}^{\infty} a_{k}(z-a)^{k}$ where $a_{0}=a_{1}=0$ because of 15.8 , that $h^{\prime}(a)=h(a)=0$. It follows that for $|z-a|>0, f(z)=\sum_{k=2}^{\infty} a_{k}(z-a)^{k-2} \equiv g(z)$.
$\Leftarrow$ The converse is obvious.
In general, we have the following definition for an isolated singularity.

## Definition 15.5.4 Let a be an isolated singularity of $a X$ valued function $f$. When

$$
\begin{equation*}
f(z)=g(z)+\sum_{k=1}^{M} \frac{b_{k}}{(z-a)^{k}} \tag{15.9}
\end{equation*}
$$

for some finite $M$ for $z$ near a then a is called a pole. The order of the pole in 15.9 is $M$. Essential singularities are those which have infinitely many nonzero terms in the principal part of the Laurent series. When a function $f$ is analytic except for isolated singularities and the isolated singularities are all poles, and there are only finitely many of these poles in every compact set, the function is called meromorphic.

Analytic functions are the analog of polynomials in algebra. Meromorphic functions are the appropriate generalization of rational functions. To further understand the case of a non-removable singularity, we have the amazing Casorati Weierstrass theorem. This theorem pertains to the special case where $f$ has values in $\mathbb{C}$ rather than a complex Banach space.

Theorem 15.5.5 (Casorati Weierstrass) Let a be an isolated singularity and suppose for some $r>0, f\left(B^{\prime}(a, r)\right)$ is not dense in $\mathbb{C}$. Then either a is a removable singularity or there exist finitely many $b_{1}, \cdots, b_{M}$ for some finite number $M$ such that for $z$ near $a$, where $g(z)$ is analytic near a.Thus either $f$ equals an analytic function near $a, f$ has a pole at a or $f\left(B^{\prime}(a, r)\right)$ is dense in $\mathbb{C}$ for each $r>0$.

Proof: Suppose $B\left(z_{0}, \delta\right)$ has no points of $f\left(B^{\prime}(a, r)\right)$. Such a ball must exist if the set of points $f\left(B^{\prime}(a, r)\right)$ is not dense. Then for $z \in B^{\prime}(a, r),\left|f(z)-z_{0}\right| \geq \delta>0$. It follows from Theorem 15.5.3 that $\frac{1}{f(z)-z_{0}}$ has a removable singularity at $a$ since $\lim _{z \rightarrow a} \frac{1}{f(z)-z_{0}}(z-a)=$ 0 . Hence

$$
\frac{1}{f(z)-z_{0}}=\sum_{k=0}^{\infty} a_{k}(z-a)^{k}
$$

If $a_{k}=0$ for $k=0, \ldots, m-1$, then $\frac{1}{\left(f(z)-z_{0}\right)(z-a)^{m}}=g(z)$ where $g(a) \neq 0$ and so

$$
\left(f(z)-z_{0}\right)(z-a)^{m}=h(z)
$$

for $h$ analytic near $a$ so $f(x)$ has a pole of order $m$ at $a$. If $m=0$ so $a_{0} \neq 0$, then $f$ is analytic near $a$. Thus either $f$ equals an analytic function or it has a pole or $f\left(B^{\prime}(a, r)\right)$ is dense in $\mathbb{C}$ for all $r>0$.

Thus the case where $f\left(B^{\prime}(a, r)\right)$ is dense corresponds to the principal part of the Laurent series being infinite. For more about essential singularities see Conway [10] about the Picard theorems.

Actually, if you insist only that the singularities are isolated and poles, then you can prove that there are finitely many in any compact set so part of the above definition is actually redundant, but this will be shown later. What follows is the definition of something called a residue. This pertains to a singularity which has a pole at an isolated singularity.
Definition 15.5.6 The residue of $f$ at an isolated singularity $\alpha$ which is a pole, written res $(f, \alpha)$ is the coefficient of $(z-\alpha)^{-1}$ where

$$
f(z)=g(z)+\sum_{k=1}^{m} \frac{b_{k}}{(z-\alpha)^{k}} .
$$

Thus $\operatorname{res}(f, \alpha)=b_{1}$ in the above. Here it suffices to assume that $f$ has values in $X$ a Banach space.

### 15.6 The Residue Theorem

The following is the residue theorem. To illustrate the situation, here is a picture.


Recall the definition of the residue at a pole.
Definition 15.6.1 Let $f(z)=\sum_{n=0}^{\infty} a_{n}\left(z-z_{0}\right)^{n}+\sum_{n=1}^{\infty} b_{n}\left(z-z_{0}\right)^{-n}$ be the Laurent series at the pole $z_{0}$. Then res $\left(z_{0}\right)$ is defined to be $b_{1}$.

Theorem 15.6.2 Let $\Gamma$ be a simple closed curve which divides the plane into two regions, a bounded region, the inside, and an unbounded region, the outside. Suppose also that $\Gamma$ is of finite length and oriented so that Green's theorem holds for $U$ the inside of $\Gamma$. Suppose $f$ is a function analytic on an open set containing $U \cup \Gamma$ except for finitely many singularities $\left\{a_{k}\right\}_{k=1}^{n}$ which are all either removable or poles. Then $\int_{\Gamma} f(z) d z=$ $2 \pi i \sum_{k=1}^{n} \operatorname{res}\left(f, a_{k}\right)$.

Proof: It was shown that $f^{\prime}$ is continuous on the open set $U \backslash \cup\left\{a_{k}\right\}_{k=1}^{n}$. Therefore, $f$ is analytic on this set and from Theorem 14.5.3, if $\gamma_{k}$ is a circle oriented counter clockwise centered at $a_{k}$, such that the closed disks bounded by the $\gamma_{k}$ are disjoint, then it follows $\int_{\Gamma} f(w) d w=\sum_{j=1}^{n} \int_{\gamma_{j}} f(w) d w$. However, from Corollary 15.2.3, at $a_{k}, \int_{\gamma_{k}} f(z) d z=$ $2 \pi i b_{1} \equiv 2 \pi i \operatorname{res}\left(f, a_{k}\right)$.

In words, the contour integral is $2 \pi i$ times the sum of the residues. So is there a way to find the residues? The answer is yes.

## Procedure 15.6.3 Say you want to find $\operatorname{res}(f, a)=b_{1}$ in

$$
f(z)=g(z)+\sum_{n=1}^{M} \frac{b_{n}}{(z-a)^{n}}, g \text { analytic }
$$

This is the case where you have a pole of order $M$ at $a$. You would multiply by $(z-a)^{M}$. This would give

$$
f(z)(z-a)^{M}=g(z)(z-a)^{M}+\sum_{n=1}^{M} b_{n}(z-a)^{M-n}
$$

Then you would take $M-1$ derivatives and then take the limit as $z \rightarrow a$. This would give $(M-1)!b_{1}$.

You can see from the formula that this will work and so there is no question that the limit exists. Because of this, you could use L'Hospitals rule to formally find this limit. This rule pertains only to real functions of a real variable. However, since you know the limit exists in this case from the existence of the Laurent series and that $a$ is a pole so the principal part is finite. Thus you can pick a one dimensional direction and apply L'Hospital to the real and imaginary parts to identify the limit which is typically what needs to be done.

### 15.7 Evaluation of Improper Integrals

Letting $p(x), q(x)$ be polynomials, you can use the above method of residues to evaluate obnoxious integrals of the form $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} d x \equiv \lim _{R \rightarrow \infty} \int_{-R}^{R} \frac{p(x)}{q(x)} d x$ provided the degree of $p(x)$ is two less than degree of $q(x)$ and the zeros of $q(z)$ involve $\operatorname{Im}(z)>0$. These integrals are called Cauchy principal value integrals. The contour to use for such problems is $\gamma_{R}$ which goes from $(-R, 0)$ to $(R, 0)$ along the real line and then on the semicircle of radius $R$ from $(R, 0)$ to $(-R, 0)$.


Letting $C_{R}$ be the circular part of this contour, for large $R,\left|\int_{C_{R}} \frac{p(z)}{q(z)} d z\right| \leq \pi R \frac{C R^{k}}{R^{k+2}}$ which converges to 0 as $R \rightarrow \infty$. Therefore, it is only a matter of taking large enough $R$ to enclose all the roots of $q(z)$ which are in the upper half plane, finding the residues at these points and then computing the contour integral. Then you would let $R \rightarrow \infty$ and the part of the contour on the semicircle will disappear leaving the Cauchy principal value integral which is desired. There are other situations which will work just as well. You simply need to have the case where the integral over the curved part of the contour converges to 0 as $R \rightarrow \infty$.

Here is an easy example.

## Example 15.7.1 Find $\int_{-\infty}^{\infty} \frac{1}{x^{2}+1} d x$

You know from calculus that the answer is $\pi$. Lets use the method of residues to find this. The function $\frac{1}{z^{2}+1}$ has poles at $i$ and $-i$. We don't need to consider $-i$. It
seems clear that the pole at $i$ is of order 1 and so all we have to do is take $\lim _{z \rightarrow i} \frac{x-i}{1+x^{2}}=$ $\frac{1}{(x-i)(x+i)}(x-i)=\frac{1}{2 i}$. Then the integral equals $2 \pi i\left(\frac{1}{2 i}\right)=\pi$.

That one is easy. Now here is a genuinely obnoxious integral.
Example 15.7.2 Find $\int_{-\infty}^{\infty} \frac{1}{1+x^{4}} d x$
It will have poles at the roots of $1+x^{4}$. These roots are

$$
\left(\frac{1}{2}-\frac{1}{2} i\right) \sqrt{2},-\left(\frac{1}{2}+\frac{1}{2} i\right) \sqrt{2},-\left(\frac{1}{2}-\frac{1}{2} i\right) \sqrt{2},\left(\frac{1}{2}+\frac{1}{2} i\right) \sqrt{2}
$$

Using the above contour, we only need consider $-\left(\frac{1}{2}-\frac{1}{2} i\right) \sqrt{2},\left(\frac{1}{2}+\frac{1}{2} i\right) \sqrt{2}$. Since they are all distinct, the poles at these two will be of order 1 . To find the residues at these points, you would need

$$
\lim _{z \rightarrow-\left(\frac{1}{2}+\frac{1}{2} i\right) \sqrt{2}} \frac{\left(z-\left(-\left(\frac{1}{2}-\frac{1}{2} i\right) \sqrt{2}\right)\right)}{1+z^{4}}, \lim _{z \rightarrow\left(\frac{1}{2}+\frac{1}{2} i\right) \sqrt{2}} \frac{\left(z-\left(\left(\frac{1}{2}+\frac{1}{2} i\right) \sqrt{2}\right)\right)}{1+z^{4}}
$$

As noted above, you could use L'Hospital's rule to find these limits.

$$
\lim _{z \rightarrow-\left(\frac{1}{2}+\frac{1}{2} i\right) \sqrt{2}} \frac{1}{4 z^{3}}, \lim _{z \rightarrow\left(\frac{1}{2}+\frac{1}{2} i\right) \sqrt{2}} \frac{1}{4 z^{3}}
$$

and these are $\frac{1}{4\left(-\left(\frac{1}{2}+\frac{1}{2} i\right) \sqrt{2}\right)^{3}}$ and $\frac{1}{4\left(\left(\frac{1}{2}+\frac{1}{2} i\right) \sqrt{2}\right)^{3}}$ which are $\frac{1}{8} \sqrt{2}+\frac{1}{8} i \sqrt{2}$ and $-\frac{1}{8} \sqrt{2}-\frac{1}{8} i \sqrt{2}$.
Then the contour integral is

$$
2 \pi i\left(\left(\frac{1}{8}-\frac{1}{8} i\right) \sqrt{2}\right)+2 \pi i\left(-\left(\frac{1}{8}+\frac{1}{8} i\right) \sqrt{2}\right)=\frac{1}{2} \sqrt{2} \pi
$$

You might observe that this is a lot easier than doing the usual partial fractions and trig substitutions etc. Now here is another tedious example.

Example 15.7.3 Find $\int_{-\infty}^{\infty} \frac{x+2}{\left(x^{2}+1\right)\left(x^{2}+4\right)^{2}} d x$
The poles of interest are located at $i, 2 i$. The pole at $2 i$ is of order 2 and the one at $i$ is of order 1. In this case, the partial fractions expansion is

$$
\frac{\frac{1}{9} x+\frac{2}{9}}{x^{2}+1}-\frac{\frac{1}{3} x+\frac{2}{3}}{\left(x^{2}+4\right)^{2}}-\frac{\frac{1}{9} x+\frac{2}{9}}{x^{2}+4}
$$

and you could use this to find the integral or the residues. However, lets use what was described above. At $2 i$,

$$
\begin{aligned}
& \lim _{z \rightarrow 2 i} \frac{d}{d z}\left(\frac{(z-2 i)^{2}(z+2)}{\left(z^{2}+1\right)\left(z^{2}+4\right)^{2}}\right)=\lim _{z \rightarrow 2 i} \frac{d}{d z}\left(\frac{(z+2)}{\left(z^{2}+1\right)(z+2 i)^{2}}\right) \\
= & \lim _{z \rightarrow 2 i}\left(-\frac{3 z^{3}+2 i z^{2}+z-2 i+8 z^{2}+8 i z+4}{\left(z^{2}+1\right)^{2}(z+2 i)^{3}}\right)=\left(-\frac{1}{18}+\frac{11}{144} i\right)
\end{aligned}
$$

The pole at $i$ would be $\lim _{z \rightarrow i} \frac{\left(\frac{1}{9} z+\frac{2}{9}\right)(z-i)}{(z+i)(z-i)}=\frac{\left(\frac{1}{9} i+\frac{2}{9}\right)}{(i+i)}=\frac{1}{18}-\frac{1}{9} i$ Thus the integral is

$$
2 \pi i\left(\frac{1}{18}-\frac{1}{9} i\right)+2 \pi i\left(-\frac{1}{18}+\frac{11}{144} i\right)=\frac{5}{72} \pi
$$

Sometimes you don't blow up the curves and take limits. Sometimes the problem of interest reduces directly to a complex integral over a closed curve. Here is an example of this.

Example 15.7.4 The integral is $\int_{0}^{2 \pi} \frac{\sin \theta}{2+\sin \theta} d \theta$.
For $z$ on the unit circle, $z=e^{i \theta}, \bar{z}=\frac{1}{z}$ and therefore,

$$
\cos \theta=\frac{1}{2}\left(z+\frac{1}{z}\right), \sin \theta=\frac{1}{2 i}\left(z-\frac{1}{z}\right)
$$

Thus $d z=i e^{i \theta} d \theta$ and so $d \theta=\frac{d z}{i z}$. Note that this is done in order to get a complex integral which reduces to the one of interest. It follows that a contour integral which reduces to the integral of interest is, for $\gamma$ the positive orientation of the unit circle, the integral is

$$
\int_{\gamma} \frac{\frac{1}{2 i}\left(z-\frac{1}{z}\right)}{2+\frac{1}{2 i}\left(z-\frac{1}{z}\right)} \frac{d z}{i z}=\int_{\gamma} \frac{z^{2}-1}{z\left(-4 z+i z^{2}-i\right)} d z
$$

The poles are $z=0, z=-2 i+i \sqrt{3}, z=-2 i-i \sqrt{3}$. The first two are inside the circle $\gamma^{*}$ and the third is not. The residues are

$$
\begin{aligned}
\lim _{z \rightarrow 0} z \frac{z^{2}-1}{z\left(-4 z+i z^{2}-i\right)} & =-i, \\
\lim _{z \rightarrow(-2 i+i \sqrt{3})}(z-(-2 i+i \sqrt{3})) \frac{z^{2}-1}{z\left(-4 z+i z^{2}-i\right)} & =\frac{2}{3} i \sqrt{3} .
\end{aligned}
$$

It follows that the original integral equals $2 \pi i\left(-i+\frac{2}{3} i \sqrt{3}\right)=2 \pi-\frac{4}{3} \pi \sqrt{3}$. Other rational functions of the trig functions will work out by this method also, provided you integrate on $[0,2 \pi]$. These may have a pole at 0 so you would not want the contour to pass through this point.

Sometimes we have to be clever about which version of an analytic function should be used. The following is such an example.

Example 15.7.5 The integral here is $\int_{0}^{\infty} \frac{\ln x}{1+x^{4}} d x$.
It is natural to try and use the contour in the following picture in which the small circle has radius $r$ and the large one has radius $R$.


However, this will create problems with the log since the usual version of the log is not defined on the negative real axis. This difficulty may be eliminated by simply using another branch of the logarithm. Leave out the ray from 0 along the negative $y$ axis and use this example to define $L(z)$ on this set. Thus $L(z)=\ln |z|+i \arg _{1}(z)$ where $\arg _{1}(z)$ will be the angle $\theta$, between $-\frac{\pi}{2}$ and $\frac{3 \pi}{2}$ such that $z=|z| e^{i \theta}$. Then the function used is $f(z) \equiv \frac{L(z)}{1+z^{4}}$. Now the only singularities contained in this contour are $\frac{1}{2} \sqrt{2}+\frac{1}{2} i \sqrt{2},-\frac{1}{2} \sqrt{2}+\frac{1}{2} i \sqrt{2}$ and the integrand $f$ has simple poles at these points. Thus res $\left(f, \frac{1}{2} \sqrt{2}+\frac{1}{2} i \sqrt{2}\right)=$

$$
\begin{gathered}
\lim _{z \rightarrow \frac{1}{2} \sqrt{2}+\frac{1}{2} i \sqrt{2}} \frac{\left(z-\left(\frac{1}{2} \sqrt{2}+\frac{1}{2} i \sqrt{2}\right)\right)\left(\ln |z|+i \arg _{1}(z)\right)}{1+z^{4}} \\
=\lim _{z \rightarrow \frac{1}{2} \sqrt{2}+\frac{1}{2} i \sqrt{2}} \frac{\left(\ln |z|+i \arg _{1}(z)\right)+\left(z-\left(\frac{1}{2} \sqrt{2}+\frac{1}{2} i \sqrt{2}\right)\right)(1 / z)}{4 z^{3}} \\
=\frac{\ln \left(\sqrt{\frac{1}{2}+\frac{1}{2}}\right)+i \frac{\pi}{4}}{4\left(\frac{1}{2} \sqrt{2}+\frac{1}{2} i \sqrt{2}\right)^{3}}=\left(\frac{1}{32}-\frac{1}{32} i\right) \sqrt{2} \pi
\end{gathered}
$$

Similarly res $\left(f, \frac{-1}{2} \sqrt{2}+\frac{1}{2} i \sqrt{2}\right)=\frac{3}{32} \sqrt{2} \pi+\frac{3}{32} i \sqrt{2} \pi$. Of course it is necessary to consider the integral along the small semicircle of radius $r$. This reduces to $\int_{\pi}^{0} \frac{\ln |r|+i t}{1+\left(r e^{i t}\right)^{4}}\left(r i e^{i t}\right) d t$ which clearly converges to zero as $r \rightarrow 0$ because $r \ln r \rightarrow 0$. Therefore, taking the limit as $r \rightarrow 0$,

$$
\begin{gathered}
\int_{\text {large semicircle }} \frac{L(z)}{1+z^{4}} d z+\lim _{r \rightarrow 0+} \int_{-R}^{-r} \frac{\ln (-t)+i \pi}{1+t^{4}} d t+ \\
\lim _{r \rightarrow 0+} \int_{r}^{R} \frac{\ln t}{1+t^{4}} d t=2 \pi i\left(\frac{3}{32} \sqrt{2} \pi+\frac{3}{32} i \sqrt{2} \pi+\frac{1}{32} \sqrt{2} \pi-\frac{1}{32} i \sqrt{2} \pi\right) .
\end{gathered}
$$

Observing that $\int_{\text {large semicircle }} \frac{L(z)}{1+z^{4}} d z \rightarrow 0$ as $R \rightarrow \infty$,

$$
e(R)+2 \lim _{r \rightarrow 0+} \int_{r}^{R} \frac{\ln t}{1+t^{4}} d t+i \pi \int_{-\infty}^{0} \frac{1}{1+t^{4}} d t=\left(-\frac{1}{8}+\frac{1}{4} i\right) \pi^{2} \sqrt{2}
$$

where $e(R) \rightarrow 0$ as $R \rightarrow \infty$. From an Example 15.7.2, this becomes

$$
e(R)+2 \lim _{r \rightarrow 0+} \int_{r}^{R} \frac{\ln t}{1+t^{4}} d t+i \pi\left(\frac{\sqrt{2}}{4} \pi\right)=\left(-\frac{1}{8}+\frac{1}{4} i\right) \pi^{2} \sqrt{2}
$$

Now letting $r \rightarrow 0+$ and $R \rightarrow \infty$,

$$
2 \int_{0}^{\infty} \frac{\ln t}{1+t^{4}} d t=\left(-\frac{1}{8}+\frac{1}{4} i\right) \pi^{2} \sqrt{2}-i \pi\left(\frac{\sqrt{2}}{4} \pi\right)=-\frac{1}{8} \sqrt{2} \pi^{2}
$$

and so $\int_{0}^{\infty} \frac{\ln t}{1+t^{4}} d t=-\frac{1}{16} \sqrt{2} \pi^{2}$, which is probably not the first thing you would thing of. You might try to imagine how this could be obtained using elementary techniques. Showing the integral exists is routine, but I think that finding it might prove impossible. This process is not always routine.

Example 15.7.6 Let $\alpha \in(0,1)$. Find $\int_{0}^{\infty} \frac{x^{\alpha}}{1+x^{2}} d x$.
Note that $z \rightarrow z^{\alpha}$ is analytic for $z \neq 0$. In fact, using the branch of the logarithm used above, it is $e^{\ln (|z|) \alpha+i \arg _{1}(z) \alpha}$. Now consider $\int_{\Gamma_{r, R}} \frac{z^{\alpha}}{1+z^{2}} d z$ where $\Gamma_{r, R}$ is the contour of the above problem including the large semi-circle and the small semi-circle. Then it is routine to see that the integrals over the small and large semi-circles converge to 0 as $R \rightarrow \infty$ and $r \rightarrow 0$. There is only one residue at $i$ and it is $\lim _{z \rightarrow i}(z-i) \frac{z^{\alpha}}{1+z^{2}}=\frac{1}{2} \sin \frac{1}{2} \pi \alpha-\frac{1}{2} i \cos \frac{1}{2} \pi \alpha$. Thus in the limit,

$$
\int_{-\infty}^{0} \frac{|x|^{\alpha} e^{i \pi \alpha}}{1+x^{2}} d x+\int_{0}^{\infty} \frac{x^{\alpha}}{1+x^{2}} d x=2 \pi i\left(\frac{1}{2} \sin \frac{1}{2} \pi \alpha-\frac{1}{2} i \cos \frac{1}{2} \pi \alpha\right)
$$

and so $\left(\int_{0}^{\infty} \frac{x^{\alpha}}{1+x^{2}} d x\right)\left(e^{i \pi \alpha}+1\right)=\pi\left(e^{i \frac{1}{2} \pi \alpha}\right)$.Then simplifying, you get the amazing formula $\int_{0}^{\infty} \frac{x^{\alpha}}{1+x^{2}} d x=\frac{\pi}{2 \cos \frac{1}{2} \pi \alpha}$.

Sometimes one must be "creative" about which contour to use. In the next case, $\cos \left(z^{2}\right)$ is not bounded and so integrals which involve a contour over a large semicircle like the above, are not likely to be helpful.

Example 15.7.7 The Fresnel integrals are $\int_{0}^{\infty} \cos x^{2} d x, \int_{0}^{\infty} \sin x^{2} d x$.
To evaluate these integrals we will consider $f(z)=e^{i z^{2}}$ on the curve which goes from the origin to the point $r$ on the $x$ axis and from this point to the point $r\left(\frac{1+i}{\sqrt{2}}\right)$ along a circle of radius $r$, and from there back to the origin as illustrated in the following picture.


Thus the curve is shaped like a slice of pie. The angle is $45^{\circ}$. Denote by $\gamma_{r}$ the curved part. Since $f$ is analytic,

$$
\begin{align*}
0 & =\int_{\gamma_{r}} e^{i z^{2}} d z+\int_{0}^{r} e^{i x^{2}} d x-\int_{0}^{r} e^{i\left(t\left(\frac{1+i}{\sqrt{2}}\right)\right)^{2}}\left(\frac{1+i}{\sqrt{2}}\right) d t \\
& =\int_{\gamma_{r}} e^{i z^{2}} d z+\int_{0}^{r} e^{i x^{2}} d x-\int_{0}^{r} e^{-t^{2}}\left(\frac{1+i}{\sqrt{2}}\right) d t \\
& =\int_{\gamma_{r}} e^{i z^{2}} d z+\int_{0}^{r} e^{i x^{2}} d x-\frac{\sqrt{\pi}}{2}\left(\frac{1+i}{\sqrt{2}}\right)+e(r) \tag{15.10}
\end{align*}
$$

where $e(r) \rightarrow 0$ as $r \rightarrow \infty$. This used $\int_{0}^{\infty} e^{-t^{2}} d t=\frac{\sqrt{\pi}}{2}$. Now examine the first of these integrals.

$$
\begin{aligned}
\left|\int_{\gamma_{r}} e^{i z^{2}} d z\right| & =\left|\int_{0}^{\frac{\pi}{4}} e^{i\left(r e^{i t}\right)^{2}} r i e^{i t} d t\right| \leq r \int_{0}^{\frac{\pi}{4}} e^{-r^{2} \sin 2 t} d t=\frac{r}{2} \int_{0}^{1} \frac{e^{-r^{2} u}}{\sqrt{1-u^{2}}} d u \\
& =\frac{r}{2} \int_{0}^{r^{-(3 / 2)}} \frac{1}{\sqrt{1-u^{2}}} d u+\frac{r}{2}\left(\int_{0}^{1} \frac{1}{\sqrt{1-u^{2}}}\right) e^{-\left(r^{1 / 2}\right)}
\end{aligned}
$$

which converges to zero as $r \rightarrow \infty$. Therefore, taking the limit as $r \rightarrow \infty$, in 15.10,

$$
\frac{\sqrt{\pi}}{2}\left(\frac{1+i}{\sqrt{2}}\right)=\int_{0}^{\infty} e^{i x^{2}} d x
$$

and so the Fresnel integrals are given by $\int_{0}^{\infty} \sin x^{2} d x=\frac{\sqrt{\pi}}{2 \sqrt{2}}=\int_{0}^{\infty} \cos x^{2} d x$.
The following example is one of the most interesting. By an auspicious choice of the contour it is possible to obtain a very interesting formula for $\cot \pi z$ known as the Mittag Leffler expansion of $\cot \pi z$.

Example 15.7.8 Let $\gamma_{N}$ be the contour which goes from $-N-\frac{1}{2}-$ Ni horizontally to $N+$ $\frac{1}{2}-N i$ and from there, vertically to $N+\frac{1}{2}+N i$ and then horizontally to $-N-\frac{1}{2}+N i$ and finally vertically to $-N-\frac{1}{2}-N i$. Thus the contour is a large rectangle and the direction of integration is in the counter clockwise direction.


Consider the following integral. $I_{N} \equiv \int_{\gamma_{N}} \frac{\pi \cos \pi z}{\left(\alpha^{2}-z^{2}\right) \sin \pi z}$, where $\alpha$ is not an integer. This will be used to verify the formula of Mittag Leffler,

$$
\begin{equation*}
\frac{1}{\alpha^{2}}+\sum_{n=1}^{\infty} \frac{2}{\alpha^{2}-n^{2}}=\frac{\pi \cot \pi \alpha}{\alpha} \tag{15.11}
\end{equation*}
$$

It is left as an exercise to verify that $\cot \pi z$ is bounded on this contour and that therefore, $I_{N} \rightarrow 0$ as $N \rightarrow \infty$. Now compute the residues of the integrand at $\pm \alpha$ and at $n$ where $|n|<N+\frac{1}{2}$ for $n$ an integer. These are the only singularities of the integrand in this contour and therefore, $I_{N}$ can be obtained by using these. First consider the residue at $\pm \alpha$. These are obviously poles of order 1 and so to get the one at $\alpha$, you take

$$
\lim _{z \rightarrow \alpha} \frac{(z-\alpha) \pi \cos \pi z}{\left(\alpha^{2}-z^{2}\right) \sin \pi z}=\lim _{z \rightarrow \alpha} \frac{-\pi \cos \pi z}{(\alpha+z) \sin \pi z}=\frac{-\pi \cos \pi \alpha}{2 \alpha \sin \pi \alpha}
$$

You get the same thing at $-\alpha$. Next consider the residue at $n$. If you consider the power series, you will see that this should also be a pole of order 1 . Thus it is

$$
\begin{aligned}
\lim _{z \rightarrow n} \frac{(z-n) \pi \cos \pi z}{\left(\alpha^{2}-z^{2}\right) \sin \pi z} & =\lim _{z \rightarrow n} \frac{\pi \cos \pi z-(z-n) \pi^{2} \sin (\pi z)}{-2 z \sin \pi z+\left(\alpha^{2}-z^{2}\right) \pi \cos (\pi z)} \\
& =\frac{\pi(-1)^{n}}{\left(\alpha^{2}-n^{2}\right) \pi(-1)^{n}}=\frac{1}{\alpha^{2}-n^{2}}
\end{aligned}
$$

Therefore, $0=\lim _{N \rightarrow \infty} I_{N}=\lim _{N \rightarrow \infty} 2 \pi i\left[\sum_{n=-N}^{N} \frac{1}{\alpha^{2}-n^{2}}-\frac{\pi \cot \pi \alpha}{\alpha}\right]$ which establishes the following formula of Mittag Leffler. $\lim _{N \rightarrow \infty} \sum_{n=-N}^{N} \frac{1}{\alpha^{2}-n^{2}}=\frac{\pi \cot \pi \alpha}{\alpha}$. Writing this in a slightly nicer form, we obtain 15.11 .

The next example illustrates the technique of a branch cut. Note that a branch of the logarithm is determined by cutting out the nonnegative real axis and defining a logarithm on what is left.

Example 15.7.9 For $p \in(0,1)$, find $\int_{0}^{\infty} \frac{x^{p-1}}{1+x} d x$. This example illustrates the use of something called a branch cut. The idea is you need to pick a single determination of $z^{p-1}$ which converges to $x^{p-1}$ for $x$ real and $z$ getting close to $x$. It will make use of the following contour. In this contour, the radius of the large circle is $R$ and the radius of the small one is $r$. The angle between the straight lines and the $x$ axis is $\varepsilon$. Denote this contour by $\gamma_{R, r, \varepsilon}$.


Choose a branch of the logarithm of the form $\log (z)=\ln |z|+i A(z)$ where $A(z)$ is the angle of $z$ in $(0,2 \pi)$. Thus $z^{p-1}=e^{(p-1)(\ln |z|+i A(z))}$

The straight line on the top is parametrized by $r e^{i \varepsilon}+t\left(R e^{i \varepsilon}\right)=z, t \in[0,1]$. The contour integral along this line is $\int_{0}^{1} \frac{\left|r e^{i \varepsilon}+t\left(R e^{i \varepsilon}\right)\right|^{p-1} e^{(p-1) i \varepsilon}}{1+r e^{i \varepsilon}+t\left(R e^{i \varepsilon}\right)} R e^{i \varepsilon} d t$. Along the bottom of the two straight lines, you get the parametrization $r e^{i(2 \pi-\varepsilon)}+t\left(\operatorname{Re} e^{i(2 \pi-\varepsilon)}\right)=z, t \in[0,1]$ and contour integral

$$
-\int_{0}^{1} \frac{\left|r e^{i(2 \pi-\varepsilon)}+t\left(R e^{i(2 \pi-\varepsilon)}\right)\right|^{p-1} e^{(p-1) i(2 \pi-\varepsilon)}}{1+r e^{i(2 \pi-\varepsilon)}+t\left(\operatorname{Re}^{i(2 \pi-\varepsilon)}\right)} R e^{i(2 \pi-\varepsilon)} d t
$$

The contour integral over the small circle: $z=r e^{i t}, t \in[\varepsilon, 2 \pi-\varepsilon]$ is

$$
-\int_{\varepsilon}^{2 \pi-\varepsilon} \frac{r^{p-1} e^{(p-1) i t}}{1+r e^{i t}} r i e^{i t} d t
$$

This integral is dominated by $4 \pi r^{p}$ provided $|r|<1 / 2$ which converges to 0 as $r \rightarrow 0$ uniformly in $\varepsilon$. The integral over the large circle: $z=R e^{i t}, t \in[\varepsilon, 2 \pi-\varepsilon]$ is similar to this but with $r$ replaced with $R$. This one is dominated by $2 \pi R^{p} /(1+R)$ which converges to 0 as $R \rightarrow \infty$. Thus $\int_{\gamma_{R, r, \varepsilon}} \frac{z^{p-1}}{1+z} d z=$

$$
\begin{aligned}
& \int_{0}^{1} \frac{\left|r e^{i \varepsilon}+t\left(R e^{i \varepsilon}\right)\right|^{p-1} e^{(p-1) i \varepsilon}}{1+r e^{i \varepsilon}+t\left(R e^{i \varepsilon}\right)} R e^{i \varepsilon} d t \\
& -\int_{0}^{1} \frac{\left|r e^{i(2 \pi-\varepsilon)}+t\left(R e^{i(2 \pi-\varepsilon)}\right)\right|^{p-1} e^{(p-1) i(2 \pi-\varepsilon)}}{1+r e^{i(2 \pi-\varepsilon)}+t\left(R e^{i(2 \pi-\varepsilon)}\right)} R e^{i(2 \pi-\varepsilon)} d t \\
& +e(R, \varepsilon)+e(r, \varepsilon)
\end{aligned}
$$

where the last two terms converge to 0 uniformly in $\varepsilon$ as $r \rightarrow 0$ and $R \rightarrow \infty$. Let $\varepsilon \rightarrow 0+$ and this yields an expression of the form

$$
\int_{0}^{1} \frac{|r+t R|^{p-1}}{1+r+t R} R d t-\int_{0}^{1} \frac{|r+t R|^{p-1} e^{(p-1) i(2 \pi)}}{1+r+t R} R d t+e(R)+e(r)
$$

where the last two terms converge to 0 as $r \rightarrow 0, R \rightarrow \infty$. Now let $x=r+t R$ and this all reduces to

$$
\int_{r}^{R} \frac{x^{p-1}}{1+x} d x-e^{(p-1) i(2 \pi)} \int_{r}^{R} \frac{x^{p-1}}{1+x} d x+e(R)+e(r)=\lim _{\varepsilon \rightarrow 0} \int_{\gamma_{R, r, \varepsilon}} \frac{z^{p-1}}{1+z} d z
$$

and this last integral can be computed using the method of residues. It has a residue at -1 and since the pole is of order 1 , this residue is

$$
\lim _{z \rightarrow-1}(z+1) \frac{z^{p-1}}{z+1}=\lim _{z \rightarrow-1} e^{(p-1)(\ln |z|+i A(z))}=e^{(p-1) i(\pi)}
$$

Thus, letting $r=1 / R$ and letting $R \rightarrow \infty, \int_{0}^{\infty} \frac{x^{p-1}}{1+x} d x\left(1-e^{(p-1) i(2 \pi)}\right)=2 \pi i e^{(p-1) i(\pi)}$ which shows that

$$
\begin{aligned}
\int_{0}^{\infty} \frac{x^{p-1}}{1+x} d x & =\frac{2 \pi i e^{(p-1) i(\pi)}}{1-e^{(p-1) i(2 \pi)}}=\frac{2 \pi i}{e^{-(p-1) i(\pi)}-e^{(p-1) i(\pi)}} \\
& =\frac{2 \pi i}{-i 2 \sin ((p-1) \pi)}=-\frac{\pi}{\sin ((p-1) \pi)}=\frac{\pi}{\sin (p \pi)}
\end{aligned}
$$

Isn't this an amazing formula?
Actually, people typically are a little more informal in the consideration of such integrals. They regard the bottom side of the line $x \geq 0$ as being associated with $\theta=2 \pi$ and the top side being associated with $\theta=0$ and leave out the fuss with taking limits as $\varepsilon \rightarrow 0$ and so forth. Recall, for example, the simple idea of the winding number in terms of a branch of the logarithm which was discussed earlier. It is a little like that. This kind of integral is a case of a general concept called a Mellen transformation. These are of the form $\int_{0}^{\infty} x^{p-1} f(x) d x$. The zeta function was obtained in this way a little earlier.

### 15.8 The Inversion of Laplace Transforms

Recall Theorem 11.4.1 about the inversion of the Laplace transform.
Theorem 15.8.1 Let $g$ be a measurable function defined on $(0, \infty)$ which has exponential growth

$$
|g(t)| \leq C e^{\eta t} \text { for some real } \eta
$$

and is Holder continuous from the right and left as in 11.2 and 11.3. For $\operatorname{Re}(s)>\eta$

$$
\mathscr{L} g(s) \equiv \int_{0}^{\infty} e^{-s u} g(u) d u
$$

Then for any $c>\eta$,

$$
\begin{equation*}
\lim _{R \rightarrow \infty} \frac{1}{2 \pi} \int_{-R}^{R} e^{(c+i y) t} \mathscr{L} g(c+i y) d y=\frac{g(t+)+g(t-)}{2} \tag{15.12}
\end{equation*}
$$

The idea is to find a way to evaluate that Cauchy principal value integral on the left, at least for simple cases. Write the integral on the left as a contour integral. Thus $z=c+i y$ and $d z=i d y$ and this is just the contour integral $\frac{1}{2 \pi i} \int_{c-i R}^{c+i R} e^{u t} \mathscr{L} g(u) d u$ where the contour is the straight line from $c-i R$ to $c+i R$. Indeed, if you parametrize this contour as $z=c+i y$
and use the procedures for evaluation of contour integrals, you get the integral in 15.12. Then taking the limit as $R \rightarrow \infty$ it is customary to write this limit as $\frac{1}{2 \pi i} \int_{c-i \infty}^{c+i \infty} e^{u t} \mathscr{L} g(u) d u$. This is called the Bromwich integral and as shown earlier it recovers the mid point of the jump of $g$ at $t$ for every point $t$ where $g$ is Holder continuous from the right and from the left. Remember $t \geq 0$. Now $u \rightarrow e^{u t} \mathscr{L} g(u)$ is analytic for $\operatorname{Re}(u)>\eta$ and in particular for $\operatorname{Re}(u) \geq c$ therefore, all of the poles of $u \rightarrow \mathscr{L} g(u)$ are contained in the set $\operatorname{Re}(u)<c$. Indeed, in practice, $u \rightarrow \mathscr{L} g(u)$ ends up being represented by a formula which is clearly a meromorphic function, one which is analytic except for isolated poles.

So how do you compute this Bromwich integral? This is where the method of residues is very useful. Consider the following contour.


Let $\gamma_{R}$ be the above contour oriented as shown. The radius of the circular part is $R$. Let $C_{R}$ be this curved part. Then one can show that under suitable assumptions

$$
\begin{equation*}
\lim _{R \rightarrow \infty} \frac{1}{2 \pi i} \int_{C_{R}} e^{u t} F(u) d u=0 \tag{15.13}
\end{equation*}
$$

and this is the case we will consider because it is very easy to compute. We assume $c>0$.
Lemma 15.8.2 Let the contour be as shown and assume 15.13 for meromorphic $F(u)$ such that $F(u)$ also has only finitely many poles whose real parts are less than $\eta<c$, a positive numer, $F(u)$ being analytic if $\operatorname{Re}(u) \geq c$. Then $f(t)$, given by the Bromwich integral, has exponential growth $|f(t)| \leq C e^{c t}$ Lipschitz continuous near every point $t>0$ and its Laplace transform is $F(s)$.

Proof: Since $F$ has only finitely many poles It only remains to verify Lipschitz continuity near a point $t>0$. Let $R$ be so large that the above contour $\gamma_{R}^{*}$ encloses all poles of $F$. Then for such large $R$, the contour integrals are not changing because all the poles are enclosed. Thus, letting

$$
\begin{gathered}
\hat{t} \in(t-\delta, t+\delta), t-\delta>0 \\
f(\hat{t})=\lim _{R \rightarrow \infty} \frac{1}{2 \pi i} \int_{\gamma_{R}} e^{u \hat{t}} F(u) d u=\frac{1}{2 \pi i} \int_{\gamma_{R}} e^{u \hat{t}} F(u) d u
\end{gathered}
$$

Therefore,

$$
\begin{aligned}
& |f(\hat{t})-f(t)|=\left|\frac{1}{2 \pi i} \int_{\gamma_{R}} e^{u \hat{t}} F(u) d u-\frac{1}{2 \pi i} \int_{\gamma_{R}} e^{u t} F(u) d u\right| \\
= & \frac{1}{2 \pi}\left|\int_{\gamma_{R}}\left(e^{u \hat{t}}-e^{u t}\right) F(u) d u\right|=\frac{1}{2 \pi}\left|\int_{\gamma_{R}}\left(\int_{t}^{\hat{t}} s e^{u s} d s\right) F(u) d u\right|
\end{aligned}
$$

The contour contains no poles and $F$ is continuous, so $|F(u)|$ is bounded by some number $M$ on $\gamma_{R}$. Then the above is no more than

$$
\leq M \frac{1}{2 \pi} 2 \pi R\left|\int_{t}^{\hat{t}} s e^{u s} d s\right|=R M \hat{t} e^{c(\hat{t}+t)}|\hat{t}-t| \leq R M(t+\delta) e^{c(\hat{t}+t)}|\hat{t}-t|
$$

The claim of exponential growth follows from observing that the residue at each pole $z$ is of the form $e^{z t} b_{1}$ and $\left|e^{z t} b_{1}\right|<\left|b_{1}\right| e^{c t}$.

A sufficient condition for 15.13 is that for all $|z|$ large enough,

$$
\begin{equation*}
|F(z)| \leq \frac{C}{|z|^{\alpha}}, \text { some } \alpha>0 \tag{15.14}
\end{equation*}
$$

Note that this assumption implies there are finitely many poles for $F(z)$ because if $w$ is a pole, you have $\lim _{z \rightarrow w}|F(z)|=\infty$.
Lemma 15.8.3 Let the contour be as shown and assume 15.14. Then the above limit in 15.13 exists for $t>0$.

Proof: Assume $c \geq 0$ as shown and let $\theta$ be the angle between the positive $x$ axis and a point on $C_{R}$. Let $0<\bar{\beta}<\alpha$. Then the contour integral over $C_{R}$ will be broken up into three pieces, two pieces around the $y$ axis

$$
\begin{aligned}
\theta \in & {\left[\frac{\pi}{2}-\arcsin \left(\frac{c}{R}\right), \frac{\pi}{2}+\arcsin \left(\frac{c}{R^{1-\beta}}\right)\right] } \\
& {\left[\frac{3 \pi}{2}-\arcsin \left(\frac{c}{R^{1-\beta}}\right), \frac{3 \pi}{2}+\arcsin \left(\frac{c}{R}\right)\right], }
\end{aligned}
$$

and the third having $\theta \in\left(\frac{\pi}{2}+\arcsin \left(\frac{c}{R^{1-\beta}}\right), \frac{3 \pi}{2}-\arcsin \left(\frac{c}{R^{1-\beta}}\right)\right)$. Then,

$$
\begin{align*}
\int_{C_{R}} e^{t z} F(z) d z= & \int_{\frac{\pi}{2}+\arcsin \left(\frac{c}{R^{1-\beta}}\right)}^{\frac{3 \pi}{2}-\arcsin \left(\frac{c}{R^{1-\beta}}\right)} e^{(R \cos \theta+i R \sin \theta) t} F\left(R e^{i \theta}\right) R i e^{i \theta} d \theta+  \tag{15.15}\\
& +\int_{\frac{\pi}{2}-\arcsin \left(\frac{c}{R}\right)}^{\frac{\pi}{2}+\arcsin \left(\frac{c}{R^{1-\beta}}\right)} e^{(R \cos \theta+i R \sin \theta) t} F\left(R^{i \theta}\right) R i e^{i \theta} d \theta \\
& +\int_{\frac{3 \pi}{2}-\arcsin \left(\frac{c}{R^{1-\beta}}\right)}^{\frac{3 \pi}{2}+\arcsin \left(\frac{c}{R}\right)} e^{(R \cos \theta+i R \sin \theta) t} F\left(R e^{i \theta}\right) R i e^{i \theta} d \theta
\end{align*}
$$

Consider the last two integrals first. For large $|z|$, with $z \in C_{R}^{*}$, the sum of the absolute values of these is no more than

$$
\begin{aligned}
& \left|\int_{\frac{\pi}{2}-\arcsin \left(\frac{c}{R}\right)}^{\frac{\pi}{2}+\arcsin \left(\frac{c}{R^{1-\beta}}\right)} e^{R(\cos \theta) t} \frac{C}{R^{\alpha}} R d \theta\right|+\left|\int_{\frac{3 \pi}{2}-\arcsin \left(\frac{c}{R^{1-\beta}}\right)}^{\frac{3 \pi}{R}+\arcsin \left(\frac{c}{R}\right)} e^{R(\cos \theta) t} \frac{C}{R^{\alpha}} R d \theta\right| \\
\leq & C e^{R\left(\cos \left(\frac{\pi}{2}-\arcsin \left(\frac{c}{R}\right)\right)\right) t}\left(\arcsin \left(\frac{c}{R^{1-\beta}}\right)+\arcsin \left(\frac{c}{R}\right)\right) R^{1-\alpha} \\
& +C e^{R\left(\cos \left(\frac{3 \pi}{2}+\arcsin \left(\frac{c}{R}\right)\right)\right) t}\left(\arcsin \left(\frac{c}{R^{1-\beta}}\right)+\arcsin \left(\frac{c}{R}\right)\right) R^{1-\alpha}
\end{aligned}
$$

Now from trig. identities, $\cos \left(\frac{\pi}{2}-\arcsin (\theta)\right)=\theta, \cos \left(\frac{3 \pi}{2}+\arcsin (\theta)\right)=\theta$, and so the above reduces to $2 C e^{c t}\left(\arcsin \left(\frac{c}{R^{1-\beta}}\right)+\arcsin \left(\frac{c}{R}\right)\right) R^{1-\alpha}$ which converges to 0 as $R \rightarrow \infty$. Recall $0<\beta<\alpha$. It remains to consider the integral in 15.15. For large $|z|$, the absolute value of this integral is no more than

$$
\int_{\frac{\pi}{2}+\arcsin \left(\frac{c}{R}\right)}^{\frac{3 \pi}{2}-\arcsin \left(\frac{c}{R}\right)} e^{R(\cos \theta) t} \frac{C}{R^{\alpha}} R d \theta \leq C \pi e^{R t \cos \left(\frac{\pi}{2}+\arcsin \left(\frac{c}{R^{1-\beta}}\right)\right)} R^{1-\alpha}=C \pi R^{1-\alpha} e^{-c t R^{\beta}}
$$

which converges to 0 as $R \rightarrow \infty$.

Proposition 15.8.4 If $\operatorname{Re} p<c$ for all $p$ a pole of $F(s)$ and if $F(s)$ is meromorphic and satisfies the growth condition 15.14, and if $f(t)$ is defined by the Bromwich integral, then $F(s)$ is the Laplace transform of $f(t)$ for large $s$.

Proof: This follows from Lemmas 15.8 .2 and 15.8.3. These lemmas say that if $F(s)$ satisfies the growth condition, then we can define a locally Lipschitz function $f(t)$ in terms of that Bromwich integral or equivalently the contour integral. Then consider $\hat{F}(s)$ the Laplace transform of $f(t)$ for large $s$. Then Bromwich integral makes $F(s)$ into $f(t)$ by definition and by the earlier theory, it makes $\hat{F}(s)$ into $f(t)$ and so $F(s)=\hat{F}(s)$.

From this proposition, we have the following procedure.
Procedure 15.8.5 Suppose $F(s)$ is a Laplace transform and is meromorphic on $\mathbb{C}$ and satisfies 15.14. (This situation is quite typical) Then to compute the Holder continuous function of $t, f(t)$ whose Laplace transform gives $F(s)$, do the following. Find the sum of the residues of $e^{z t} F(z)$ for $\operatorname{Re} z<c$ where all poles have real part smaller than $c$.

Example 15.8.6 Suppose $F(s)=\frac{s}{\left(s^{2}+1\right)^{2}}$. Find $f(t)$ such that $F(s)$ is the Laplace transform of $f(t)$.

There are two residues of this function, one at $i$ and one at $-i$. At both of these points the poles are of order two and so we find the residue at $i$ by res $(f, i)=\lim _{s \rightarrow i} \frac{d}{d s}\left(\frac{e^{t s} s(s-i)^{2}}{\left(s^{2}+1\right)^{2}}\right)=$ $\frac{-i t e^{i t}}{4}$ and the residue at $-i$ is res $(f,-i)=\lim _{s \rightarrow-i} \frac{d}{d s}\left(\frac{e^{t s} s(s+i)^{2}}{\left(s^{2}+1\right)^{2}}\right)=\frac{i t e^{-i t}}{4}$. From the above procedure, the function $f(t)$ is the sum of these.

$$
\frac{i t e^{-i t}}{4}+\frac{-i t e^{i t}}{4}=\frac{1}{4} i t\left(e^{-i t}-e^{i t}\right)=\frac{1}{4} i t(\cos (t)-i \sin t-(\cos t+i \sin t))=\frac{1}{2} t \sin t
$$

You should verify that this actually works giving $\mathscr{L}(f)=\frac{s}{\left(s^{2}+1\right)^{2}}$.
Example 15.8.7 Find $f(t)$ if $F(s)$, the Laplace transform is $e^{-s} / s$.
You need to compute the residues of $\frac{e^{s t} e^{-s}}{s}$. The function equals $\frac{1}{s} \sum_{k=0}^{\infty} \frac{(-1)^{k}(t-1)^{k} s^{k}}{k!}$. Thus the residue is 1 . However, this fails to be the function whose Laplace transform is $F(s)$. What is wrong? The problem with this is the failure of the estimate on $F(s)$ to hold for large $s$. Indeed, if $s=-n$, you would have $e^{n} / n$ but it would need to be less than $C / n^{\alpha}$ which is not possible. The estimate requires $F(s) \rightarrow 0$ as $|s| \rightarrow \infty$ and this does not happen here. You can verify directly that the function which works is $u_{1}(t)$ which is 0 for $t<1$ and 1 for $t \geq 1$. Thus if the estimate does not hold, the procedure does not necessarily hold either.

### 15.9 Exercises

1. Suppose $f$ has a pole at $z$. Show that $\lim _{w \rightarrow z} f(w)=\infty$. Recall that this means that $\lim _{w \rightarrow z}|f(w)|=\infty$.
2. Find the following improper integral. $\int_{-\infty}^{\infty} \frac{\cos x}{1+x^{4}} d x$ Hint: Use upper semicircle contour and consider instead $\int_{-\infty}^{\infty} \frac{e^{i x}}{1+x^{4}} d x$. This is because the integral over the semicircle
will converge to 0 as $R \rightarrow \infty$ if you have $e^{i z}$ but this won't happen if you use $\cos z$ because $\cos z$ will be unbounded. Just write down and check and you will see why this happens. Thus you should use $\frac{e^{i z}}{1+z^{4}}$ and take real part. I think the standard calculus techniques will not work for this horrible integral.
3. Find $\int_{-\infty}^{\infty} \frac{\cos (x)}{\left(1+x^{2}\right)^{2}} d x$. Hint: Do the same as above replacing $\cos x$ with $e^{i x}$.
4. Let $\alpha \in(0,1)$. Find $\int_{0}^{\infty} \frac{x^{2 \alpha-1}}{1+x^{2}} d x$. Hint: Use the contour $\Gamma_{R, r}$ of Example 15.7.5.
5. Consider the following contour.


The small semicircle has radius $r$ and is centered at $(1,0)$. The large semicircle has radius $R$ and is centered at $(0,0)$. Use the method of residues to compute

$$
\lim _{r \rightarrow 0}\left(\lim _{R \rightarrow \infty} \int_{r}^{R} \frac{x}{1-x^{3}} d x+\int_{-R}^{r} \frac{x}{1-x^{3}} d x\right)
$$

This is called the Cauchy principal value for $\int_{-\infty}^{\infty} \frac{x}{1-x^{3}} d x$. The integral makes no sense in terms of a real honest integral. The function has a pole on the $x$ axis. Another instance of this was in Problem 7 on Page 369 where $\int_{0}^{\infty} \sin (x) / x d x$ was determined similarly. However, you can define such a Cauchy principal value. Rather than belabor this issue, I will illustrate with this example. These principal value integrals occur because of cancelation. They depend on a particular way of taking a limit. They are not mathematically respectable but are certainly interesting. They are in that general area of finding something by taking a certain kind of symmetric limit. Such problems include the Lebesgue fundamental theorem of calculus with the symmetric derivative.
6. Find $\int_{0}^{2 \pi} \frac{\cos (\theta)}{1+\sin ^{2}(\theta)} d \theta$.
7. Find $\int_{0}^{2 \pi} \frac{d \theta}{2-\sin \theta}$.
8. Find $\int_{-\pi / 2}^{\pi / 2} \frac{d \theta}{2-\sin \theta}$.
9. Suppose you have a function $f(z)$ which is the quotient of two polynomials in which the degree of the top is two less than the degree of the bottom and you consider the contour.


Then define $\int_{\gamma_{R}} f(z) e^{i s z} d z$. in which $s$ is real and positive. Explain why the integral makes sense and why the part of it on the semicircle converges to 0 as $R \rightarrow \infty$. Use this to find $\int_{-\infty}^{\infty} \frac{e^{i s x}}{k^{2}+x^{2}} d x, k>0$.
10. Show using methods from real analysis that for $b \geq 0, \int_{0}^{\infty} e^{-x^{2}} \cos (2 b x) d x=\frac{\sqrt{\pi}}{2} e^{-b^{2}}$. Hint: Let $F(b) \equiv \int_{0}^{\infty} e^{-x^{2}} \cos (2 b x) d x-\frac{\sqrt{\pi}}{2} e^{-b^{2}}$. Then from Problem 2 on Page 262, $F(0)=0$. Using the mean value theorem on difference quotients and the dominated convergence theorem, explain why

$$
\begin{aligned}
& F^{\prime}(b)=\int_{0}^{\infty}-2 x e^{-x^{2}} \sin (2 b x) d x+2 b \frac{\sqrt{\pi}}{2} e^{-b^{2}} \\
& F^{\prime}(b)=2 b\left(\int_{0}^{\infty} e^{-x^{2}} \cos (2 b x) d x+\frac{\sqrt{\pi}}{2} e^{-b^{2}}\right) \\
&=2 b\left(F(b)+\frac{\sqrt{\pi}}{2} e^{-b^{2}}+\frac{\sqrt{\pi}}{2} e^{-b^{2}}\right)=2 b F(b)+\sqrt{\pi} 2 b e^{-b^{2}}
\end{aligned}
$$

Now use the integrating factor method for solving linear differential equations from beginning differential equations to solve the ordinary differential equation.

$$
\frac{d}{d b}\left(e^{-b^{2}} F(b)\right)=\sqrt{\pi} 2 b e^{-2 b^{2}}
$$

Then $e^{-b^{2}} F(b)-0=-\frac{1}{2} e^{-2 b^{2}} \sqrt{\pi}+\frac{1}{2} \sqrt{\pi}, F(b)=-\frac{1}{2} e^{-b^{2}}+\frac{1}{2} \sqrt{\pi} e^{-b^{2}}=0$. You fill in the details. This is meant to be a review of real variable techniques.
11. You can do the same problem as above using contour integration. For $b=0$ it follows from Problem 2 on Page 262. For $b>0$, use the contour which goes from $-a$ to $a$ to $a+i b$ to $-a+i b$ to $-a$. Then let $a \rightarrow \infty$ and show that the integral of $e^{-z^{2}}$ over the vertical parts of this contour converge to 0 . Hint: You know from the earlier problem what happens on the bottom part of the contour. Also for $z=x+i b, e^{-z^{2}}=$ $e^{-\left(x^{2}-b^{2}+2 i x b\right)}=e^{b^{2}} e^{-x^{2}}(\cos (2 x b)+i \sin (2 x b))$.
12. Consider the circle of radius 1 oriented counter clockwise. Evaluate $\int_{\gamma} z^{-6} \cos (z) d z$
13. Consider the circle of radius 1 oriented counter clockwise. Evaluate $\int_{\gamma} z^{-7} \cos (z) d z$
14. Find $\int_{0}^{\infty} \frac{2+x^{2}}{1+x^{4}} d x$.
15. Suppose $f$ is an entire function and that it has no zeros. Show there must exist an entire function $g$ such that $f(z)=e^{g(z)}$. Hint: Letting $\gamma(0, z)$ be the line segment which goes from 0 to $z$, let $\hat{g}(z) \equiv \int_{\gamma(0, z)} \frac{f^{\prime}(w)}{f(w)} d w$. Then show that $\hat{g}^{\prime}(z)=\frac{f^{\prime}(z)}{f(z)}$. Then $\left(e^{-\hat{g}(z)} f(z)\right)^{\prime}=e^{-\hat{g}(z)} \frac{-f^{\prime}(z)}{f(z)} f(z)+e^{-\hat{g}(z)} f^{\prime}(t)=0$. Now when you have an entire function whose derivative is 0 , it must be a constant. Modify $\hat{g}(z)$ to make $f(z)=$ $e^{g(z)}$.
16. Let $f$ be an entire function with zeros $\left\{\alpha_{1}, \cdots, \alpha_{n}\right\}$ listed according to multiplicity. Thus you might have repeats in this list. Show that there is an analytic function $g(z)$ such that for all $z \in \mathbb{C}, f(z)=\prod_{k=1}^{n}\left(z-\alpha_{k}\right) e^{g(z)}$ Hint: You know $f(z)=$ $\prod_{k=1}^{n}\left(z-\alpha_{k}\right) h(z)$ where $h(z)$ has no zeros. To see this, note that near $\alpha_{1}, f(z)=$ $a_{1}\left(z-\alpha_{1}\right)+a_{2}\left(z-\alpha_{1}\right)^{2}+\cdots$ and so $f(z)=\left(z-\alpha_{1}\right) f_{1}(z)$ where $f_{1}(z) \neq 0$ at $\alpha_{1}$. Now do the same for $f_{1}$ and continue till $f_{n}=h$. Now use the above problem.
17. Let $F(s)=\frac{2}{(s-1)^{2}+4}$ so it is the Laplace transform of some $f(t)$. Use the method of residues to determine $f(t)$.
18. This problem is about finding the fundamental matrix for a system of ordinary differential equations $\Phi^{\prime}(t)=A \Phi(t), \Phi(0)=I$ having constant coefficients. Here $A$ is an $n \times n$ matrix and $I$ is the identity matrix. A matrix, $\Phi(t)$ satisfying the above is called a fundamental matrix for $A$. In the following, $s$ will be large, larger than all poles of $(s I-A)^{-1}$.
(a) Show that $\mathscr{L}\left(\int_{0}^{(\cdot)} f(u) d u\right)(s)=\frac{1}{s} F(s)$ where $F(s) \equiv \mathscr{L}(f)(s)$
(b) Show that $\mathscr{L}(I)=\frac{1}{s} I$ where $I$ is the identity matrix.
(c) Show that there exists an $n \times n$ matrix $\Phi(t)$ such that $\mathscr{L}(\Phi)(s)=(s I-A)^{-1}$.

Hint: From linear algebra

$$
\left((s I-A)^{-1}\right)_{i j}=\frac{\operatorname{cof}(s I-A)_{j i}}{\operatorname{det}(s I-A)}
$$

Show that the $i j^{\text {th }}$ entry of $(s I-A)^{-1}$ satisfies the conditions of Proposition 15.8.4 and so there exists $\Phi(t)$ such that $\mathscr{L}(\Phi)(s)=(s I-A)^{-1}$. By Lemma 15.8.2, this $t \rightarrow \Phi(t)$ is continuous.
(d) Thus $(s I-A) \mathscr{L}(\Phi)(s)=I$. Then explain why $\left(I-\frac{1}{s} A\right) \mathscr{L}(\Phi)(s)=\frac{1}{s} I=$ $\mathscr{L}(I)$ and $\mathscr{L}(\Phi)(s)-\frac{1}{s} \mathscr{L}(A \Phi)(s)=\mathscr{L}(I)$

$$
\mathscr{L}(\Phi)-\mathscr{L}\left(\int_{0}^{(\cdot)} A \Phi(u) d u\right)=\mathscr{L}(I)
$$

so $\Phi(t)-\int_{0}^{t} A \Phi(u) d u=I$ and so $\Phi$ is a fundamental matrix.
(e) Next explain why $\Phi$ must be unique by showing that if $\Phi(t)$ is a fundamental matrix, then its Laplace transform must be $(s I-A)^{-1}$ and use the theorem which says that if the two continuous functions have the same Laplace transform, then they are the same function.
19. In the situation of the above problem, show that there is one and only one solution to the initial value problem

$$
\mathbf{x}^{\prime}(t)=A \mathbf{x}(t)+\mathbf{f}(t), \mathbf{x}(0)=\mathbf{x}_{0}, t \geq 0
$$

and it is given by $\mathbf{x}(t)=\Phi(t) \mathbf{x}_{0}+\int_{0}^{t} \Phi(t-u) \mathbf{f}(u) d u$ Hint: Verify that

$$
\mathscr{L}\left(\int_{0}^{(\cdot)} \Phi(t-u) \mathbf{f}(u) d u\right)(s)=\mathscr{L}(\Phi)(s) \mathscr{L}(\mathbf{f})(s)
$$

Thus if $\mathbf{x}$ is given by the variation of constants formula just listed, then

$$
\begin{aligned}
\mathscr{L}(\mathbf{x})(s) & =(s I-A)^{-1} \mathbf{x}_{0}+(s I-A)^{-1} \mathscr{L}(\mathbf{f})(s) \\
(s I-A) \mathscr{L}(\mathbf{x})(s) & =\mathbf{x}_{0}+\mathscr{L}(f)
\end{aligned}
$$

Now divide by $s$ and verify $\mathbf{x}(t)=\mathbf{x}_{0}+\int_{0}^{t} A \mathbf{x}(u) d u+\int_{0}^{t} \mathbf{f}(u) d u$. You could also simply differentiate the variation of constants formula using chain rule and verify it works. This completes most of the mathematical substance of an entire undergraduate ordinary differential equations course. When you have a way to find the fundamental matrix and the variation of constants formula, there really isn't much else left, at least in terms of finding solutions.
20. Find the fundamental matrix for the system of ordinary differential equations $\mathbf{x}^{\prime}=$ $A \mathbf{x}$ where $A=\left(\begin{array}{ccc}2 & 2 & -1 \\ -1 & 1 & -1 \\ -1 & 2 & -2\end{array}\right)$. Hint: As above, $\Phi(t)$ is the inverse Laplace transform of $(s I-A)^{-1}$ where $I$ is the identity. For your convenience, it follows from the linear algebra formula for the inverse in terms of the transpose of the cofactor matrix that

$$
(s I-A)^{-1}=\left(\begin{array}{ccc}
\frac{s}{s^{2}-2 s+1} & \frac{2}{s^{2}-2 s+1} & -\frac{1}{s^{2}-2 s+1} \\
-\frac{1}{s^{2}-2 s+1} & \frac{s^{2}-5}{(s+1)(s-1)^{2}} & \frac{s-3}{-s^{3}+s^{2}+s-1} \\
\frac{s+1}{-s^{3}+s^{2}+s-1} & -\frac{2 s-6}{-s^{3}+s^{2}+s-1} & -\frac{s^{2}-3 s+4}{-s^{3}+s^{2}+s-1}
\end{array}\right)
$$

Now use the procedure for finding residues of $e^{s t}(s I-A)^{-1}$. The sum of these residues being the inverse Laplace transform.
21. Find the fundamental matrix for the system of ordinary differential equations $\mathbf{x}^{\prime}=A \mathbf{x}$ where $A=\left(\begin{array}{ccc}-2 & -2 & -2 \\ 2 & 2 & 1 \\ 3 & 2 & 3\end{array}\right)$. Then it is routine from the formula for the inverse in terms of the transpose of the cofactor matrix that

$$
(s I-A)^{-1}=\left(\begin{array}{ccc}
\frac{s-4}{s^{2}-2 s+2} & -\frac{2}{s^{2}-2 s+2} & -\frac{2}{s^{2}-2 s+2} \\
-\frac{2 s-3}{-s^{3}+3 s^{2}-4 s+2} & \frac{s}{s^{2}-2 s+2} & -\frac{s-2}{-s^{3}+3 s^{2}-4 s+2} \\
\frac{1}{2 s-2} \frac{3 s-2}{\frac{1}{2} s^{2}-s+1} & \frac{1}{\frac{1}{2} s^{2}-s+1} & \frac{s^{2}}{(2 s-2)\left(\frac{1}{2} s^{2}-s+1\right)}
\end{array}\right)
$$

Use this to find the fundamental matrix.
22. The Schwarz lemma is as follows: Suppose $F: B(0,1) \rightarrow B(0,1), F$ is analytic, and $F(0)=0$. Then for all $z \in B(0,1),|F(z)| \leq|z|$, and $\left|F^{\prime}(0)\right| \leq 1$.
If $\left|F^{\prime}(0)\right|=1$, then there exists $\lambda \in \mathbb{C}$ with $|\lambda|=1$ and $F(z)=\lambda z$. Prove the Schwarz lemma. Hint: Since $F$ has a power series of the form $\sum_{k=1}^{\infty} a_{k} z^{k}$, it follows that $F(z) / z$ equals an analytic function $g(z)$ for all $z \in B(0,1)$. By the maximum modulus theorem, applied to $g(z)$, if $|z|<r<1,\left|\frac{F(z)}{z}\right| \leq \max _{t \in[0,2 \pi]} \frac{\left|F\left(r e^{i t}\right)\right|}{r} \leq \frac{1}{r}$. Explain why this implies $|g(z)|=\left|\frac{F(z)}{z}\right| \leq 1$. Now explain why $\lim _{z \rightarrow 0} \frac{F(z)}{z}=F^{\prime}(0)=g(0)$ and so $\left|F^{\prime}(0)\right| \leq 1$. It only remains to verify that if $\left|F^{\prime}(0)\right|=1$, then $F(z)$ is just a rotation as described. If $\left|F^{\prime}(0)\right|=1$, then the analytic function $g(z)$ has the property that it achieves its maximum at an interior point. Apply the maximum modulus theorem to conclude that $g(z)$ must be a constant. Explain why this requires $\left|\frac{F(z)}{z}\right|=1$ for all $z$. Use this to conclude the proof.

## Chapter 16

## Mapping Theorems

In this chapter the functions will have values in $\mathbb{C}$.

### 16.1 Meromorphic Functions

First is a review of the definition of a meromorphic function. Just as polynomials are generalized by analytic functions, rational functions are generalized by meromorphic functions. These are designed to rule out the case where $f\left(B^{\prime}(\alpha, r)\right)$ is dense in $\mathbb{C}$. They have poles and removable singularities and that is all. Also all singularities are isolated.

Definition 16.1.1 A function $f$ is meromorphic on an open set $\Omega$, written as $f \in$ $\mathscr{M}(\Omega)$ for $\Omega$ an open set, means $f$ is analytic in

$$
B^{\prime}(\alpha, r) \equiv\{z: 0<|z-\alpha|<r\}
$$

for some $r>0$ for every $\alpha \in \Omega$ and at each $\alpha \in \Omega$ either $\lim _{z \rightarrow \alpha}(z-\alpha) f(z)=0$ so $\alpha$ is removable or $\lim _{z \rightarrow \alpha}|f(z)|=\infty$.

Example 16.1.2 Every rational function is meromorphic. This is because of the fundamental theorem of algebra and the partial fractions theorem presented much earlier along with the next lemma which says that if $f$ is analytic on $B^{\prime}(\alpha, r), \lim _{z \rightarrow \alpha}|f(z)|=\infty$ if and only if $\alpha$ is a pole.

Lemma 16.1.3 Let $f \in \mathscr{M}(\Omega)$. Then the poles are those $\alpha$ where $\lim _{z \rightarrow \alpha}|f(z)|=\infty$ The set of poles of a meromorphic function can't have a limit point in $\Omega$. There are at most countably many poles in $\Omega$. Thus all singularities are removable or poles.

Proof: Let $\alpha \in \Omega$. Since $f$ is analytic on $B^{\prime}(\alpha, r), f(z)=g(z)+\sum_{k=1}^{M} \frac{b_{k}}{(z-\alpha)^{k}}$ where $M \leq \infty$ and $g$ is analytic. If $\alpha$ is not removable, then the principal part of the Laurent series is nonzero. If $M<\infty$, then $\alpha$ is a pole and $f(z)=h(z)+\sum_{k=1}^{m} \frac{b_{k}}{(z-\alpha)^{k}}, b_{m} \neq 0$ and so

$$
\begin{aligned}
|f(z)||z-\alpha|^{m} & \geq\left|b_{m}\right|-\left(|h(z)||z-\alpha|^{m}+\sum_{k=1}^{m-1} b_{k}|z-\alpha|^{m-k}\right) \\
& >\frac{\left|b_{m}\right|}{2} \text { if }|z-\alpha| \text { small enough }
\end{aligned}
$$

Conversely, if $\lim _{z \rightarrow \alpha}|f(z)|=\infty$, you can't have $M=\infty$ because by the Casorati Weierstrass theorem, Theorem 15.5.5, about an isolated singularity $\alpha, f\left(B^{\prime}(\alpha, r)\right)$ is dense in $\mathbb{C}$ for all small $r>0$. Thus poles are exactly those $\alpha$ where $\lim _{z \rightarrow \alpha}|f(z)|=\infty$.

Now if $\alpha_{k}$ is a pole and $\alpha_{k} \rightarrow \alpha \in \Omega$, for $\alpha$ not equal to any $\alpha_{k}$, then it follows that $f$ is not analytic on $B^{\prime}(\alpha, r)$ for some $r>0$. Thus the poles can't have a limit point in $\Omega$. Observe that

$$
\Omega=\cup_{k=1}^{\infty}\left\{z: \operatorname{dist}\left(z, \Omega^{C}\right) \leq \frac{1}{k}\right\} \cap \overline{B(0, k)} \equiv \cup_{k=1}^{\infty} K_{k}
$$

where $K_{k}$ is compact. If $\Omega=\mathbb{C}$, let $K_{k}=\overline{B(0, k)}$. Then by what was just shown, there are finitely many poles in $K_{k}$ and so the number of poles is at most countable.

The fact that the poles cannot have a limit point in $\Omega$ is fairly significant. It shows that if $P$ is the set of poles and if $\Omega$ is connected, then $\Omega \backslash P$ is an open connected set if $\Omega$ is.

Lemma 16.1.4 Suppose $\Omega$ is an open connected set and suppose $P$ is a set of points contained in $\Omega$ such that $P$ has no limit point in $\Omega$. Then $\Omega \backslash P$ is a connected open set.

Proof: Suppose $\Omega \backslash P=A \cup B$ where $A, B$ have empty intersection and $A$ has no limit points of $B$ while $B$ has no limit points of $A$ and neither is empty. For $p \in P$, we know that $p$ is not a limit point of $P$ and so $B(p, r)$ contains points of $A \cup B$. For $r$ sufficiently small, all points in $B^{\prime}(p, r)$ are in $A \cup B$ because $p$ is not a limit point of $P$ but, since $\Omega$ is open, $B(p, r) \subseteq \Omega$ for $r$ small enough. Also, each $A, B$ is an open set since $\Omega \backslash P$ is open. Now it is obvious that $B^{\prime}(p, r)$ is a connected open set contained in $A \cup B$ and so $B^{\prime}(p, r)$ must be contained in either $A$ or in $B$. Let $P_{A}$ be those points of $P$ such that $B^{\prime}(p, r) \subseteq A$ for all $r$ small enough and let $P_{B}$ be defined similarly. Then $\Omega=\left(A \cup P_{A}\right) \cup\left(B \cup P_{B}\right)$ since this on right involves adding in $P$ to $\Omega \backslash P$. If $p \in P_{A}$, then $B^{\prime}(p, r) \subseteq A$ and so $p$ is not a limit point of $B$. From what was shown above, $p$ is not a limit point of $P_{B}$ either. If $x \in A$, then $x$ is not a limit point of $B$. Neither is it a limit point of $P_{B}$ because $P_{B}$ has no limit points in $\Omega$ as shown in Lemma 16.1.3. Similarly, $B \cup P_{B}$ has no limit points of $A \cup P_{A}$. This separates $\Omega$ and is a contradiction to $\Omega$ being connected.

The following is a useful lemma. It is about subtracting off all the singular parts of a function in $\mathscr{M}(\Omega)$ and getting one which is analytic.

Lemma 16.1.5 Let $f \in \mathscr{M}(\Omega)$ and suppose $P_{f}$ the set of poles consists of $\left\{\alpha_{1}, \cdots, \alpha_{n}\right\}$. Then there exists a function $g$ analytic on $\Omega$ such that for all $z \notin P_{f}, f(z)-\sum_{i=1}^{n} S_{i}(z)=g(z)$ where $S_{i}(z)$ is the singular part corresponding to $\alpha_{i}$. That is, for $z$ near $\alpha_{i}$,

$$
\begin{equation*}
f(z)=h_{i}(z)+\sum_{k=1}^{m_{i}} \frac{b_{k}}{\left(z-\alpha_{i}\right)^{k}}=h_{i}(z)+S_{i}(z), h_{i} \text { analytic near } \alpha_{i} \tag{16.1}
\end{equation*}
$$

Proof: Note that $f(z)-\sum_{i=1}^{n} S_{i}(z)$ is meromorphic on $\Omega$. However, it has no poles. Indeed, if $\alpha$ is a pole, then it must be one of the $\alpha_{i}$ since all the $S_{i}$ would be analytic at $\alpha$ if this were not the case. But $\lim _{z \rightarrow \alpha_{i}}\left(z-\alpha_{i}\right)\left(f(z)-\sum_{i=1}^{n} S_{i}(z)\right)=0$ and so each $\alpha_{i}$ is a removable singularity. Thus one can re-define at each $\alpha_{i}$ and so there is an analytic $g(z)=f(z)-\sum_{i=1}^{n} S_{i}(z)$.

Because of this lemma, it is all right to be a little sloppy and simply write $f(z)-$ $\sum_{i=1}^{n} S_{i}(z)$ equals an analytic function.

Proposition 16.1.6 Let $\Omega$ be a connected open set. Then $\mathscr{M}(\Omega)$ is a field with the usual conventions about summation and multiplication of functions.

Proof: It is almost obvious that $\mathscr{M}(\Omega)$ is a ring. The part of this which is not entirely obvious is whether the product of two meromorphic functions is meromorphic. It is clear that $f g$ is analytic on $B^{\prime}(\alpha, r)$ for small enough $r$. The only difficulty arises when $\alpha$ is a zero for $f$ but a pole for $g$. Thus $f(z)=\sum_{k=r}^{\infty} a_{k}(z-\alpha)^{k}, g(z)=h(z)+\sum_{k=1}^{m} \frac{b_{k}}{(z-\alpha)^{k}}$ for $h$ analytic. But this means $f(z) g(z)=f(z) h(z)+f(z) \sum_{k=1}^{m} \frac{b_{k}}{(z-\alpha)^{k}}$. If $r$ is as large as $m$, then $f g$ has a removable singularity. Otherwise, $f g$ will have a pole. Thus the product of meromorphic functions is indeed meromorphic. It is clear that the sum of two of these meromorphic functions is meromorphic.

As usual, the main issue is the existence of multiplicative inverses. So suppose $f \in$ $\mathscr{M}(\Omega)$ and $f \neq 0$. Then it is analytic on the connected set $\Omega \backslash P$ where $P$ is the set of poles. Thus the set of zeros has no limit point. If it did, then $f$ would be 0 on the connected set
$\Omega \backslash P$. Consider $1 / f$. It is analytic in $B^{\prime}(\alpha, r)$ for $r$ small enough. If $\alpha$ is a zero of $f$, then $\lim _{z \rightarrow \alpha} \frac{1}{|f(z)|}=\infty$ and $1 / f$ is analytic near $\alpha$ so $\alpha$ is a pole for $1 / f$. If $\alpha$ is not a zero of $f$, then $\lim _{z \rightarrow \alpha} \frac{1}{f(z)}(z-\alpha)=0$ so $\alpha$ is a removable singularity. Thus $1 / f \in \mathscr{M}(\Omega)$.

### 16.2 Meromorphic on Extended Complex Plane

I won't pay a lot of attention to this topic, but sometimes people like to consider functions which are called meromorphic on the extended complex plane. It turns out this forces the function to have only finitely many poles and in fact the function ends up being a rational function. Thus this is a fancy way to say that the function is a rational function.

Definition 16.2.1 We say $f \in \mathscr{M}(\widehat{\mathbb{C}})$ if it is meromorphic on $\mathbb{C}$ which means that either $\lim _{z \rightarrow 0} z f\left(\frac{1}{z}\right)=0$ when $f$ is said to have a removable singularity at $\infty$ or $\lim _{|z| \rightarrow 0}\left|f\left(\frac{1}{z}\right)\right|=\infty$ and there are no poles $\alpha,|\alpha|>r$ for some $r$ when we say $f$ has a pole at $\infty$.

It turns out from Problem 1 on Page 415 that this is just a fancy way of saying that the function is a rational function.

### 16.3 Rouche's Theorem

Rouche's theorem counts the number of poles and zeros of a meromorphic function $f$ $\in \mathscr{M}(\mathbb{C})$ inside a simple closed curve $\Gamma$. There are only finitely many of these poles and zeroes in $U$ the connected inside of a simple closed curve $\Gamma$. Indeed, there are finitely many poles in $\Gamma \cup U$ as explained above. If $\Gamma$ contains no poles and no zeroes, then this means there are finitely many poles in $U$. Then $f(z)-\sum_{i=1}^{m} S_{i}(z)$ is analytic in $U$ where the $S_{i}$ are the singular parts corresponding to poles $z_{i}$. If there are infinitely many zeroes, then there is a limit point which can only be in $U$ and so $f-\sum_{i} S_{i}$ would be identically zero which is impossible if there are any poles. In this case, $f$ would be identically 0 on $U$ and hence zero on points of $\Gamma$, contrary to assumption.

Not surprisingly, there are more general formulations of Rouche's theorem. I am specializing to the case which is usually of most interest.

Theorem 16.3.1 Let $f \in \mathscr{M}(U)$ where $U$ is the inside of a bounded variation simple closed curve $\gamma^{*}$ such that Green's theorem holds ${ }^{1}$. Also suppose $\gamma^{*}$ contains none of the poles nor any of the zeros of $f$, and let $\gamma$ be positively oriented so that for $z$ on the inside of $\gamma^{*}, n(\gamma, z)=1$. Now let $\left\{p_{1}, \cdots, p_{m}\right\}$ and $\left\{z_{1}, \cdots, z_{n}\right\}$ be respectively the poles and zeros of $f$ which are on the inside of $\gamma^{*}$. Let $z_{k}$ be a zero of multiplicity $r_{k}$ and let $p_{k}$ be a pole of multiplicity $l_{k}$. Then

$$
\frac{1}{2 \pi i} \int_{\gamma} \frac{f^{\prime}(z)}{f(z)} d z=\sum_{k=1}^{n} r_{k}-\sum_{k=1}^{m} l_{k}
$$

Thus the zeros and poles are counted according to multiplicity.
Proof: This theorem follows from computing the residues of $f^{\prime} / f$ which has residues only at poles and zeros of $f$. I will do this now. First suppose $f$ has a pole of multiplicity $p$

[^9]at $\alpha$. Then $f$ has the form given in 16.1. Therefore,
$$
\frac{f^{\prime}(z)}{f(z)}=\frac{h^{\prime}(z)-\sum_{k=1}^{p} \frac{k b_{k}}{(z-\alpha)^{k+1}}}{h(z)+\sum_{k=1}^{p} \frac{b_{k}}{(z-\alpha)^{k}}}=\frac{-p \frac{b_{p}}{(z-\alpha)}+s(z)}{b_{p}+r(z)}
$$
where $\lim _{z \rightarrow \alpha} s(z)=\lim _{z \rightarrow \alpha} r(z)=0$. Thus
$$
\lim _{z \rightarrow \alpha}(z-\alpha) \frac{f^{\prime}(z)}{f(z)}=-p=\operatorname{res}\left(\frac{f^{\prime}}{f}, \alpha\right)
$$
where $p$ is the multiplicity of the pole.
Next suppose $f$ has a zero of multiplicity $p$ at $\alpha$. Then
$$
\lim _{z \rightarrow \alpha}(z-\alpha) \frac{f^{\prime}(z)}{f(z)}=\lim _{z \rightarrow \alpha} \frac{\sum_{k=p}^{\infty} a_{k} k(z-\alpha)^{k}}{\sum_{k=p}^{\infty} a_{k}(z-\alpha)^{k}}=\lim _{z \rightarrow \alpha} \frac{\sum_{k=p}^{\infty} a_{k} k(z-\alpha)^{k-p}}{\sum_{k=p}^{\infty} a_{k}(z-\alpha)^{k-p}}=p
$$
and from this, res $\left(f^{\prime} / f\right)=p$, the multiplicity of the zero. The conclusion of this theorem now follows from the residue theorem, Theorem 15.6.2.

### 16.4 Fractional Linear Transformations

These mappings map lines and circles to either lines or circles.
Definition 16.4.1 a fractional linear transformation is a function of the form

$$
\begin{equation*}
f(z)=\frac{a z+b}{c z+d} \tag{16.2}
\end{equation*}
$$

where $a d-b c \neq 0$.
Note that if $c=0$, this reduces to a linear transformation $(a / d) z+(b / d)$. Special cases of these are defined as follows.

$$
\begin{aligned}
& \text { dilations: } z \rightarrow \delta z, \delta \neq 0, \text { inversions: } z \rightarrow \frac{1}{z} \\
& \text { translations: } z \rightarrow z+\rho
\end{aligned}
$$

The next lemma is the key to understanding fractional linear transformations.
Lemma 16.4.2 The fractional linear transformation, 16.2 can be written as a finite composition of dilations, inversions, and translations.

Proof: If $d=0$ then $c \neq 0$ and 16.2 reduces to $\frac{a}{c}+\frac{b}{c}\left(\frac{1}{z}\right)$ which is recovered as

$$
z \rightarrow \frac{1}{z} \rightarrow \frac{b}{c}\left(\frac{1}{z}\right) \rightarrow \frac{b}{c}\left(\frac{1}{z}\right)+\frac{a}{c}
$$

So assume $d \neq 0$. Then, using the special transformations, consider

$$
\begin{aligned}
z & \rightarrow \frac{1}{z} \rightarrow \frac{d}{z} \rightarrow \frac{d}{z}+c=\frac{c z+d}{z} \rightarrow \frac{z}{c z+d} \\
& \rightarrow \frac{\alpha z}{c z+d} \rightarrow \frac{\alpha z}{c z+d}+p=\frac{(\alpha+p c) z+d p}{c z+d}
\end{aligned}
$$

Now let $d p=b$ and $\alpha+p c=a$. Thus $p=\frac{b}{d}, \alpha=a-\frac{b c}{d}=\frac{a d-b c}{d} \neq 0$. Thus 16.2 is a composition of the special transformations.

This lemma implies the following corollary.
Corollary 16.4.3 Fractional linear transformations map circles and lines to circles or lines.

Proof: It is obvious that dilations and translations map circles to circles and lines to lines. What of inversions? If inversions have this property, the above lemma implies a general fractional linear transformation has this property as well.

Note that all circles and lines may be put in the form

$$
\alpha\left(x^{2}+y^{2}\right)+a x+b y=r
$$

where $\alpha=1$ gives a circle centered at $(a, b)$ with radius $r$ and $\alpha=0$ gives a line. In terms of complex variables you may therefore consider all possible circles and lines in the form

$$
\begin{gather*}
\alpha z \bar{z}+a \operatorname{Re}(z)+b \operatorname{Im}(z)+\gamma=0 \\
\alpha z \bar{z}+a\left(\frac{z+\bar{z}}{2}\right)+b\left(\frac{z-\bar{z}}{2 i}\right)+\gamma=0 \\
\alpha z z \bar{z}+\left(\frac{a}{2}+\frac{b}{2 i}\right) z+\left(\frac{a}{2}-\frac{b}{2 i}\right) \bar{z}+\gamma=0 \\
\alpha z \bar{z}+\left(\frac{a}{2}-\frac{b}{2} i\right) z+\left(\frac{a}{2}+\frac{b}{2} i\right) \bar{z}+\gamma=0 \\
\alpha z \bar{z}+\beta z+\bar{\beta} \bar{z}+\gamma=0 \tag{16.3}
\end{gather*}
$$

Note that even if $\alpha$ is not 0 or 1 the expression still corresponds to either a circle or a line because you can divide by $\alpha$ if $\alpha \neq 0$. Now I verify that replacing $z$ with $\frac{1}{z}$ results in an expression of the form in 16.3. Thus, let $w=\frac{1}{z}$ where $z$ satisfies 16.3. Then

$$
(\alpha+\beta \bar{w}+\bar{\beta} w+\gamma w \bar{w})=\frac{1}{z \bar{z}}(\alpha z \bar{z}+\beta z+\bar{\beta} \bar{z}+\gamma)=0
$$

and so $w$ also satisfies a relation like 16.3. One simply switches $\alpha$ with $\gamma$ and $\beta$ with $\bar{\beta}$. Note the situation is slightly different than with dilations and translations. These obviously take circles and lines to circles and lines. In the case of an inversion, a circle becomes either a line or a circle and similarly, a line becomes either a circle or a line.

The next example is quite important. It takes a line to a circle.
Example 16.4.4 Consider the fractional linear transformation, $w=\frac{z-i}{z+i}$. This maps the upper half plane to the unit disk centered at 0 .

The upper half plane is composed of points of the form $x+i y$ where $y>0$. Substituting in to the transformation,

$$
w=\frac{x+i(y-1)}{x+i(y+1)}
$$

which is seen to be a point on the interior of the unit disk because $|y-1|<|y+1|$ which implies $|x+i(y+1)|>|x+i(y-1)|$. Therefore, this transformation maps the upper half plane to the interior of the unit disk. If $y=0,|x+i|=|x-i|$ so the line $x=x+i 0$ is mapped to the boundary of the unit circle.

One might wonder whether the mapping is one to one and onto. The mapping is clearly one to one because it has an inverse, $z=-i \frac{w+1}{w-1}$ for all $w$ in the interior of the unit disk. Also, a short computation verifies that $z$ so defined is in the upper half plane. Therefore, this transformation maps $\{z \in \mathbb{C}$ such that $\operatorname{Im} z>0\}$ one to one and onto the unit disk $\{z \in \mathbb{C}$ such that $|z|<1\}$. Note that this transformation is analytic near $\operatorname{Im}(z) \geq 0$.

### 16.5 Some Examples

There is a simple procedure for determining a fractional linear transformation which maps a given set of three points to another set of three points. The problem is as follows: There are three distinct points in the complex plane, $z_{1}, z_{2}$, and $z_{3}$ and it is desired to find a fractional linear transformation such that $z_{i} \rightarrow w_{i}$ for $i=1,2,3$ where here $w_{1}, w_{2}$, and $w_{3}$ are three distinct points in the complex plane. Then the procedure says that to find the desired fractional linear transformation solve the following equation for $w$.

$$
\frac{w-w_{1}}{w-w_{3}} \cdot \frac{w_{2}-w_{3}}{w_{2}-w_{1}}=\frac{z-z_{1}}{z-z_{3}} \cdot \frac{z_{2}-z_{3}}{z_{2}-z_{1}}
$$

The result will be a fractional linear transformation with the desired properties.
Why should this procedure work? First note that it will be a fractional linear transformation because it involves solving for $w$ in $\frac{w-w_{1}}{w-w_{3}} \cdot a=\frac{z-z_{1}}{z-z_{3}} \cdot b$ which will turn out as it should. Here is a heuristic argument to indicate why you would expect this to map the points as desired rather than a rigorous proof. The reader may want to tighten the argument to give a proof. First suppose $z=z_{1}$. Then the right side equals zero and so the left side also must equal zero. However, this requires $w=w_{1}$. Next suppose $z=z_{2}$. Then the right side equals 1 . To get a 1 on the left, you need $w=w_{2}$. Finally $z_{3}$ is a pole on the right so to have a pole on the left at $w_{3}$ you need $w=w_{3}$.

Example 16.5.1 Let $z_{1}=0, z_{2}=1$, and $z_{3}=2$ and let $w_{1}=0, w_{2}=i$, and $w_{3}=2 i$.
Then the equation to solve is $\frac{w}{w-2 i} \cdot \frac{-i}{i}=\frac{z}{z-2} \cdot \frac{-1}{1}$. Solving this yields $w=i z$ which clearly works.

Example 16.5.2 Let $\xi \in \mathbb{C}$ and suppose $\operatorname{Im}(\xi)>0$. This is a more general example than 16.4.4 where $\xi=$ i. It will have similar mapping properties. As in 16.4.4, the pole is in the lower half plane. Define

$$
f(z) \equiv \frac{z-\xi}{z-\bar{\xi}}
$$

Then $f(\mathbb{C} \backslash\{\bar{\xi}\})=\mathbb{C} \backslash\{1\}$ and $f$ maps the upper half plane to the unit ball centered at 0 .

Let $U \equiv \mathbb{C} \backslash\{\bar{\xi}\}$. This is clearly an open connected set. Also let $V \equiv \mathbb{C} \backslash\{1\}$. Then $f$ maps $U$ one to one and onto $V$. Indeed, if $w \in V$, then solve $w=\frac{z-\xi}{z-\bar{\xi}}$ for $z$. This yields $z=$ $\frac{\bar{\xi} w-\xi}{w-1}$. As long as $w \neq 1$, this gives a solution because $z \neq \bar{\xi} .(\bar{\xi}(w-1) \neq \bar{\xi} w-\xi)$. Thus $f$
is onto. If you have $\frac{z-\xi}{z-\bar{\xi}}=\frac{\hat{z}-\xi}{\hat{z}-\bar{\xi}}$, then you would have $z \hat{z}-\xi \hat{z}-\bar{\xi} z-\xi \bar{\xi}=z \hat{z}-\xi z-\bar{\xi} \hat{z}-\xi \bar{\xi}$ and so you would need to have $\xi \hat{z}+\bar{\xi} z=\xi z+\bar{\xi} \hat{z}, \quad(\xi-\bar{\xi}) \hat{z}=(\xi-\bar{\xi}) z$ which requires $\hat{z}=z$. The function is one to one and analytic on $U$. Therefore, $f^{-1}$ is also continuous by the open mapping theorem.

What about $f(\mathbb{R}) ? f(\mathbb{R}) \subseteq S^{1}$, the unit circle $\{z:|z|=1\}$ because $|x-\bar{\xi}|=|x-\xi|$. But $f(\mathbb{R})$ is missing the point 1 . Therefore, $f(\mathbb{R})=S^{1} \backslash\{1\}$ since otherwise, $f(\mathbb{R})$ would not be connected. In other words, $f(\mathbb{R})$ cannot miss any other points.

What does $f$ do to the upper half plane $U_{+} \equiv\{z: \operatorname{Im}(z)>0\}$ ? Say $\xi=x+i y, y>0$. Then for $a, b \in \mathbb{R}$, with $b>0$ so $a+i b$ a typical point of $U_{+}$,

$$
f(a+i b)=\frac{(a+b i)-(x+i y)}{(a+b i)-(x-i y)}=\frac{(a-x)+i(b-y)}{(a-x)+i(b+y)}, \text { so }|f(a+i b)|<1
$$

Thus $f\left(U_{+}\right) \subseteq B$. If $|w|<1$, then as above, $w=f(z)$ where $z=\frac{\bar{\xi}_{w-\xi}}{w-1} \in U_{+}$. Thus $f\left(U_{+}\right)=$ $B$ just like Example 16.4.4. Alternatively, for $L \equiv$ the lower half plane, $B=\left(B \cap f\left(U_{+}\right)\right) \cup$ $(B \cap f(L \backslash\{\bar{\xi}\}))$, two disjoint open sets. Thus one is empty and it can only be the second so $f\left(U_{+}\right)=B$.

Thus this is another example of an analytic function defined on a connected open set, in this case, the upper half plane such that the image of this analytic function is the unit ball. This begs the question of which connected open sets can be mapped one to one by an analytic function onto the unit ball. It turns out that every simply connected open set will have this property. Recall that an open connected set is simply connected if its complement is connected in $\widehat{\mathbb{C}}$. Also recall Corollary 15.3 .5 which says that an analytic function on a simply connected region has a primitive.

Lemma 16.5.3 For $\alpha \in B(0,1)$, let $\phi_{\alpha}(z) \equiv \frac{z-\alpha}{1-\bar{\alpha} z}$. Then $\phi_{\alpha}$ is analytic on $B(0,1), \phi_{\alpha}$ maps $B(0,1)$ one to one and onto $B(0,1), \phi_{\alpha}^{-1}=\phi_{-\alpha}$, and $\phi_{\alpha}^{\prime}(\alpha)=\frac{1}{1-|\alpha|^{2}}$.

Proof: Notice that $\phi_{\alpha}$ is analytic on $B(0,1)$ because the only possible singularity is a pole at $z=1 / \bar{\alpha}$ which is not in $B(0,1)$. For $|z|<1 /|\alpha|$,

$$
\phi_{\alpha} \circ \phi_{-\alpha}(z) \equiv \frac{\left(\frac{z+\alpha}{1+\bar{\alpha} z}\right)-\alpha}{1-\bar{\alpha}\left(\frac{z+\alpha}{1+\bar{\alpha} z}\right)}=\frac{(z+\alpha)-\alpha(1+\bar{\alpha} z)}{(1+\bar{\alpha} z)-\bar{\alpha}(z+\alpha)}=\frac{z-|\alpha|^{2} z}{1-|\alpha|^{2}}=z
$$

If I show that $\phi_{\alpha}$ maps $B(0,1)$ to $B(0,1)$ for all $|\alpha|<1$, this will have shown that $\phi_{\alpha}$ is one to one and onto $B(0,1)$. Note that geometric considerations or a simple computation shows $\left|\frac{1-z}{1-\bar{z}}\right|=1$.

Consider $\left|\phi_{\alpha}\left(e^{i \theta}\right)\right|$. This yields $\left|\frac{e^{i \theta}-\alpha}{1-\bar{\alpha} e^{i \theta}}\right|=\left|\frac{1-\alpha e^{-i \theta}}{1-\bar{\alpha} e^{i \theta}}\right|=1$ where the first equality is obtained by multiplying by $\left|e^{-i \theta}\right|=1$. Therefore, $\phi_{\alpha}$ maps $\partial B(0,1)$ one to one and onto $\partial B(0,1)$. By the maximum modulus theorem, Theorem 15.1.4, it follows $\left|\phi_{\alpha}(z)\right|<1$ whenever $|z|<1$. The same is true of $\phi_{-\alpha}$.

It only remains to verify the assertion about the derivative. Long division gives $\phi_{\alpha}(z)=$ $(-\bar{\alpha})^{-1}+\left(\frac{-\alpha+(\bar{\alpha})^{-1}}{1-\bar{\alpha} z}\right)$ and so $\phi_{\alpha}^{\prime}(z)=(-1)(1-\bar{\alpha} z)^{-2}\left(-\alpha+(\bar{\alpha})^{-1}\right)(-\bar{\alpha})$

$$
=\bar{\alpha}(1-\bar{\alpha} z)^{-2}\left(-\alpha+(\bar{\alpha})^{-1}\right)=(1-\bar{\alpha} z)^{-2}\left(1-|\alpha|^{2}\right)
$$

and so $\phi_{\alpha}^{\prime}(\alpha)=\frac{1}{1-|\alpha|^{2}}$.
The next lemma is called the Schwarz lemma. It was presented earlier in the exercises.
Lemma 16.5.4 Suppose $F: B(0,1) \rightarrow B(0,1), F$ is analytic, and $F(0)=0$. Then
1.) for all $z \in B(0,1),|F(z)| \leq|z|$ and $\left|F^{\prime}(0)\right| \leq 1$.
2.) $\left|F^{\prime}(0)\right|=1$ if and only if $\bar{F}(z)=\lambda z$ where $|\bar{\lambda}|=1$.

Proof: 1.) $G(z)=\frac{F(z)}{z}$ if $z \neq 0, F^{\prime}(0)$ if $z=0$. Then $G(z)$ is analytic. Then by the maximum modulus theorem, for $|z| \leq r<1,\left|\frac{F(z)}{z}\right| \leq \max \left\{|G(z)|=\left|\frac{F(z)}{z}\right|:|z|=r\right\} \leq \frac{1}{r}$ and $|G(0)|=\left|F^{\prime}(0)\right| \leq 1 / r$. Thus $|F(z)| \leq|z| / r$ for each $r<1$ and so $|F(z)| \leq|z|$ and $\left|F^{\prime}(0)\right| \leq 1$.
2.) If $F(z)=\lambda z,|\lambda|=1$, then $\left|F^{\prime}(0)\right|=1$. Conversely, if $\left|F^{\prime}(0)\right|=1,|G(z)|$ achieves its maximum at 0 , an interior point, and so $G(z)$ is therefore constant by the open mapping theorem. Thus $G(z)=\lambda$ for some $|\lambda|=1$ which says $F(z)=\lambda z$.

Rudin [40] gives a memorable description of what this lemma says: If an analytic function maps the unit ball to itself, keeping 0 fixed, then it must do one of two things, either be a rotation or move all points closer to 0 . Note that if $|F(z)|=|z|$ for any $z \in B(0,1)$, then $|F(z)|$ is a constant because the analytic function $F(z) / z$ has maximum modulus at an interior point.

In the next section, the problem of considering which regions can be mapped onto the unit ball by a one to one analytic function will be considered. Some can and some can't. The main result in this subject is the Riemann mapping theorem which says that any simply connected open set $\Omega, \Omega \neq \mathbb{C}$ can be so mapped onto the unit ball by a one to one analytic function. A key result in showing this is the fact that such regions have something called the square root property.

## Definition 16.5.5 A region, $\Omega$ has the square root property if whenever $f, \frac{1}{f}: \Omega \rightarrow$

 $\mathbb{C}$ are both analytic, it follows there exists $\phi: \Omega \rightarrow \mathbb{C}$ such that $\phi$ is analytic and $f(z)=$ $\phi^{2}(z)$.The following lemma says that every simply connected region has the square root property. This holds because analytic functions have primitives on simply connected regions.

Lemma 16.5.6 Let $\Omega$ be a simply connected region properly contained in $\mathbb{C}$. Then $\Omega$ has the square root property.

Proof: Let $f$ and $\frac{1}{f}$ both be analytic on $\Omega$. Then $\frac{f^{\prime}}{f}$ is analytic on $\Omega$ so by Corollary 15.3.5, there exists $\widetilde{F}$, analytic on $\Omega$ such that $\widetilde{F}^{\prime}=\frac{f^{\prime}}{f}$ on $\Omega$. Then $\left(f e^{-\widetilde{F}}\right)^{\prime}=0$ and so $f(z)=C e^{\widetilde{F}}=e^{a+i b} e^{\widetilde{F}}$. Now let $F \equiv \widetilde{F}+a+i b$. Then $F$ is still a primitive of $f^{\prime} / f$ and $f(z)=e^{F(z)}$. Now let $\phi(z) \equiv e^{\frac{1}{2} F(z)}$. Then $\phi$ is the desired square root and so $\Omega$ has the square root property.

### 16.6 Riemann Mapping Theorem

From the open mapping theorem, analytic functions map regions to other regions or else to single points. The Riemann mapping theorem states that for every simply connected region $\Omega$ which is not equal to all of $\mathbb{C}$ there exists an analytic function, $f$ such that $f(\Omega)=B(0,1)$
and in addition to this, $f$ is one to one. It involves several ideas which have been developed up to now. An important part of the proof is based on the following theorem, a case of Montel's theorem. Before beginning, note that the Riemann mapping theorem is a classic example of a major existence theorem. In mathematics there are two sorts of questions, those related to whether something exists and those involving methods for finding it. I am afraid that the latter is typically all that gets studied by undergraduate students but the real questions are related to existence.

There is a long and involved history for proofs of this theorem. The first proofs were based on the Dirichlet principle and turned out to be incorrect, thanks to Weierstrass who pointed out the errors. For more on the history of this theorem, see Hille [23].

The following is about the existence of a subsequence having certain salubrious properties. It is this wonderful result which will give the existence of the mapping desired. The other parts of the argument are technical details to set things up and use this theorem. See Conway [10] for a more general version. The theorem is a lot like the Arzela Ascoli theorem and is a compactness result.

### 16.6.1 Montel's Theorem

Theorem 16.6.1 Let $\Omega$ be an open set in $\mathbb{C}$ and let $\mathscr{F}$ denote a set of analytic functions mapping $\Omega$ to $B(0, M) \subseteq \mathbb{C}$. Then there exists a sequence of functions from $\mathscr{F}$, $\left\{f_{n}\right\}_{n=1}^{\infty}$ and an analytic function $f$ such that for each $k \in \mathbb{N}, f_{n}^{(k)}$ converges uniformly to $f^{(k)}$ on every compact subset of $\Omega$. Here $f^{(k)}$ denotes the $k^{\text {th }}$ derivative.

Proof: First note there exists a sequence of compact sets $K_{n}, K_{n} \subseteq \operatorname{int} K_{n+1} \subseteq \Omega$ for all $n$ where here int $K$ denotes the interior of the set $K$, the union of all open sets contained in $K$ and $\cup_{n=1}^{\infty} K_{n}=\Omega$. In fact, you can verify that $\overline{B(0, n)} \cap\left\{z \in \Omega\right.$ : dist $\left.\left(z, \Omega^{C}\right) \leq \frac{1}{n}\right\}$ works for $K_{n}$. Then there exist positive numbers, $\delta_{n}$ such that if $z \in K_{n}$, then $\overline{B\left(z, \delta_{n}\right)} \subseteq$ int $K_{n+1}$. Now denote by $\mathscr{F}_{n}$ the set of restrictions of functions of $\mathscr{F}$ to $K_{n}$. Then let $z \in K_{n}$ and let $\gamma(t) \equiv z+\delta_{n} e^{i t}, t \in[0,2 \pi]$. It follows that for $z_{1} \in B\left(z, \boldsymbol{\delta}_{n}\right)$, and $f \in \mathscr{F}$,

$$
\begin{aligned}
\left|f(z)-f\left(z_{1}\right)\right| & =\left|\frac{1}{2 \pi i} \int_{\gamma} f(w)\left(\frac{1}{w-z}-\frac{1}{w-z_{1}}\right) d w\right| \\
& \leq \frac{1}{2 \pi}\left|\int_{\gamma} f(w) \frac{z-z_{1}}{(w-z)\left(w-z_{1}\right)} d w\right|
\end{aligned}
$$

Letting $\left|z_{1}-z\right|<\frac{\delta_{n}}{2},\left|f(z)-f\left(z_{1}\right)\right| \leq \frac{M}{2 \pi} 2 \pi \delta_{n} \frac{\left|z-z_{1}\right|}{\delta_{n}^{2} / 2} \leq 2 M \frac{\left|z-z_{1}\right|}{\delta_{n}}$. If $\varepsilon>0$ is given and if $\left|z-z_{1}\right|<\varepsilon \delta_{n} / 2 M$ for $z, z_{1} \in K_{n}$, then $\left|f(z)-f\left(z_{1}\right)\right|<\varepsilon$. It follows that $\mathscr{F}_{n}$ is equicontinuous and uniformly bounded, so by the Arzela Ascoli theorem, Theorem 9.2.4 there exists a sequence, $\left\{f_{n k}\right\}_{k=1}^{\infty} \subseteq \mathscr{F}$ which converges uniformly on $K_{n}$. Let $\left\{f_{1 k}\right\}_{k=1}^{\infty}$ converge uniformly on $K_{1}$. Then use the Arzela Ascoli theorem applied to this sequence to get a subsequence, denoted by $\left\{f_{2 k}\right\}_{k=1}^{\infty}$ which also converges uniformly on $K_{2}$. Continue in this way to obtain $\left\{f_{n k}\right\}_{k=1}^{\infty}$ which converges uniformly on $K_{1}, \cdots, K_{n}$. Now the diagonal sequence $\left\{f_{n n}\right\}_{n=m}^{\infty}$ is a subsequence of $\left\{f_{m k}\right\}_{k=1}^{\infty}$ and so it converges uniformly on $K_{m}$ for all $m$. Denoting $f_{n n}$ by $f_{n}$ for short, this is the sequence of functions promised by the theorem. It is clear $\left\{f_{n}\right\}_{n=1}^{\infty}$ converges uniformly on every compact subset of $\Omega$ because every such set is contained in $K_{m}$ for all $m$ large enough. (Why?) Let $f(z)$ be the
point to which $f_{n}(z)$ converges. Then $f$ is a continuous function defined on $\Omega$. Is $f$ analytic? Yes it is by Lemma 14.8.5. Alternatively, you could let $T \subseteq \Omega$ be a triangle. Then $\int_{\partial T} f(z) d z=\lim _{n \rightarrow \infty} \int_{\partial T} f_{n}(z) d z=0$. Therefore, by Morera's theorem, $f$ is analytic.

As for the uniform convergence of the derivatives of $f$, recall Theorem 15.4.2 about the existence of a cycle. Let $K$ be a compact subset of $\Omega$. Then for some $n, K$ is a compact subset of int $\left(K_{n}\right)$ and let $\left\{\gamma_{k}\right\}_{k=1}^{m}$ be closed oriented curves contained $\gamma_{k}^{*} \subseteq \operatorname{int}\left(K_{n}\right) \backslash K$ such that $\sum_{k=1}^{m} n\left(\gamma_{k}, z\right)=1$ for every $z \in K$. Also let $\eta$ denote the distance between $\cup_{j} \gamma_{j}^{*}$ and $K, \eta \equiv \inf \left\{|z-w|: z \in K, w \in \cup_{j} \gamma_{j}^{*}\right\}$ It follows that $\eta>0$. (Why? In general, two disjoint compact sets are at a positive distance from each other. ) Then for $z \in K$,

$$
\begin{aligned}
\left|f^{(k)}(z)-f_{n}^{(k)}(z)\right| & =\left|\frac{k!}{2 \pi i} \sum_{j=1}^{m} \int_{\gamma_{j}} \frac{f(w)-f_{n}(w)}{(w-z)^{k+1}} d w\right| \\
& \leq \frac{k!}{2 \pi}\left\|f_{k}-f\right\|_{K_{n}} \sum_{j=1}^{m}\left(\text { length of } \gamma_{k}\right) \frac{1}{\eta^{k+1}}
\end{aligned}
$$

where here $\left\|f_{k}-f\right\|_{K_{n}} \equiv \max \left\{\left|f_{k}(z)-f(z)\right|: z \in K_{n}\right\}$. Thus you get uniform convergence of the derivatives on each compact subset of $\Omega$.

Another surprising consequence of this theorem is that the property of being one to one is preserved if the target function is known to not be a constant.

Lemma 16.6.2 Suppose $h_{n}$ is one to one, analytic on $\Omega$, a connected open set (region) and converges uniformly to $h$ on compact subsets of $\Omega$ along with all derivatives. Then if $h$ is not a constant, it follows that $h$ is also one to one.

Proof: Pick $z_{1} \in \Omega$ and suppose $z_{2}$ is another point of $\Omega$. As shown above, $h$ is analytic. Thus, if the zeros of $h-h\left(z_{1}\right)$ have a limit point in $\Omega$, then $h(z)$ is a constant which is assumed to not be the case. Since the zeros of $h-h\left(z_{1}\right)$ have no limit point, there exists a circular contour bounding a circle which has $z_{2}$ on the inside of this circle but not $z_{1}$ such that $\gamma^{*}$ contains no zeros of $h-h\left(z_{1}\right)$. Taking a subsequence if necessary, it can be assumed $\gamma^{*}$ contains no zeros of $h_{n}-h_{n}\left(z_{1}\right)$ either.


Using the theorem on counting zeros, Theorem 16.3.1, and the fact that $h_{n}$ is one to one, we know that $h_{n}-h_{n}\left(z_{1}\right)$ has no zeros inside this circle and so

$$
0=\lim _{n \rightarrow \infty} \frac{1}{2 \pi i} \int_{\gamma} \frac{h_{n}^{\prime}(w)}{h_{n}(w)-h_{n}\left(z_{1}\right)} d w=\frac{1}{2 \pi i} \int_{\gamma} \frac{h^{\prime}(w)}{h(w)-h\left(z_{1}\right)} d w
$$

which shows that $h-h\left(z_{1}\right)$ has no zeros in $B\left(z_{2}, r\right)$. In particular $z_{2}$ is not a zero of $h-$ $h\left(z_{1}\right)$. This shows that $h$ is one to one since $z_{2} \neq z_{1}$ was arbitrary.

Theorem 16.6.1 is an example of a normal family of functions.
Definition 16.6.3 Let $\mathscr{F}$ denote a collection of functions which are analytic on $\Omega$, a region (open and connected). Then $\mathscr{F}$ is normal if every sequence contained in $\mathscr{F}$ has a subsequence which converges uniformly on compact subsets of $\Omega$.

### 16.6.2 The Proof of Riemann Mapping Theorem

The existence part of the Riemann mapping theorem is from Montel's theorem, Theorem 16.6.1 and showing that what is obtaned is actually what is desired comes from the Schwarz lemma, Lemma 16.5.4 and the remarkable properties of the special fractional linear transformation of Lemma 16.5.3.

This approach is in Rudin [40] and Conway [10]. I will present it in a sequence of lemmas, each of which is interesting for its own sake. All that is needed for $\Omega \nsubseteq \mathbb{C}$ is that it is a region with the square root property. Recall that simply connected $\Omega$ implies square root property, Lemma 16.5.6. In fact the other direction also holds but I won't go into that here.

Lemma 16.6.4 Let $\Omega$ have square root property, contain 0 and define $\mathscr{F}$ to be the set of functions $f$ such that $f: \Omega \rightarrow B(0,1)$ is one to one and analytic. Suppose $\mathscr{F}$ is nonempty. Letting $\eta \equiv \sup \left\{\left|\psi^{\prime}(0)\right|: \psi \in \mathscr{F}\right\}$, suppose there exists $h \in \mathscr{F}$ with $h^{\prime}(0)=\eta, h(0)=0$. Then $h$ is onto $B(0,1)$.

Proof: Suppose $\alpha \in B(0,1) \backslash h(\Omega)$. Then both $\phi_{\alpha} \circ h, 1 / \phi_{\alpha} \circ h$ are analytic so since
 Lemma 16.5.3, $\phi_{\alpha}(z) \equiv \frac{z-\alpha}{1-\overline{\alpha z}}$ and $\phi_{\alpha}(\alpha)=0$. Let

$$
\begin{equation*}
\psi \equiv \phi_{\sqrt{\phi_{\alpha} \circ h(0)}} \circ \sqrt{\phi_{\alpha} \circ h} \tag{16.4}
\end{equation*}
$$

Thus $\psi(0)=\phi \sqrt{\phi_{\alpha^{\circ}(0)}} \circ \sqrt{\phi_{\alpha} \circ h(0)}=0$ and $\psi$ is a one to one mapping of $\Omega$ into $B(0,1)$ so $\psi$ is also in $\mathscr{F}$. Therefore,

$$
\begin{equation*}
\left|\psi^{\prime}(0)\right| \leq \eta,\left|\left(\sqrt{\phi_{\alpha} \circ h}\right)^{\prime}(0)\right| \leq \eta \tag{16.5}
\end{equation*}
$$

Define $s(w) \equiv w^{2}$. Then using Lemma 16.5.3, in particular, the description of $\phi_{\alpha}^{-1}=\phi_{-\alpha}$, you can solve 16.4 for $h$ to obtain

$$
\begin{equation*}
h(z)=\phi_{-\alpha} \circ s \circ \phi_{-\sqrt{\phi_{\alpha} \circ h(0)}} \circ \psi=(\overbrace{\phi_{-\alpha} \circ S \circ \phi_{-\sqrt{\phi_{\alpha} \circ h(0)}}}^{\equiv F} \circ \psi)(z)=(F \circ \psi)(z) \tag{16.6}
\end{equation*}
$$

Now $F(0)=\phi_{-\alpha} \circ s \circ \phi_{-\sqrt{\phi_{\alpha} \circ h(0)}}(0)=\phi_{\alpha}^{-1}\left(\phi_{\alpha} \circ h(0)\right)=h(0)=0$ and $F \operatorname{maps} B(0,1)$ into $B(0,1)$ because it is the composition of functions which map onto $B(0,1)$. Also, $F$ is not one to one because, by Lemma 16.5.3, it maps $B(0,1)$ onto $B(0,1)$ and has $s$ in its definition. Indeed, there exists $z_{1}, z_{2} \in B(0,1)$ such that

$$
\phi_{-\sqrt{\phi_{\alpha^{\circ}} h(0)}}\left(z_{1}\right)=-\frac{1}{2}, \phi_{-\sqrt{\phi_{\alpha^{\prime}} h(0)}}\left(z_{2}\right)=\frac{1}{2} .
$$

Since $\phi_{-\sqrt{\phi_{\alpha} \circ h(0)}}$ is one to one, $z_{1} \neq z_{2}$ but, since $s(z)=z^{2}, F\left(z_{1}\right)=F\left(z_{2}\right)$.
Since $F(0)=h(0)=0$, you can apply the Schwarz lemma to $F$. Since $F$ is not one to one, it can't be true that $F(z)=\lambda z$ for $|\lambda|=1$ and so by the Schwarz lemma it must be the case that $\left|F^{\prime}(0)\right|<1$. But this implies from 16.6 and 16.5 that

$$
\eta=\left|h^{\prime}(0)\right|=\left|F^{\prime}(\psi(0))\right|\left|\psi^{\prime}(0)\right|=\left|F^{\prime}(0)\right|\left|\psi^{\prime}(0)\right|<\left|\psi^{\prime}(0)\right| \leq \eta
$$

a contradiction. Thus $h$ is onto after all.

Lemma 16.6.5 $\mathscr{F}$ in the above lemma is nonempty and $\eta$ is a positive real number.
Proof: Since $\Omega \neq \mathbb{C}$ it follows there exists $\xi \notin \Omega$.Thus $\frac{1}{z-\xi}, z-\xi$ are both analytic and since $\Omega$ has the square root property, there exists analytic $\phi$ with $\phi(z)=\sqrt{z-\xi}$ and $\phi$ is not constant, so $\phi(\Omega)$ is an open connected set, not a single point. Also $\phi$ is one to one. Pick $a \in \phi(\Omega), a \neq 0$ and $z_{a}$ such that $a=\sqrt{z_{a}-\xi}$. Then consider $0<r \equiv$ $\inf \{|\sqrt{z-\xi}+a|\}$. Is $r>0$ ? If so, we can let $\psi(z)=\frac{r}{\sqrt{z-\xi}+a}$ and obtain $\psi$ maps $\Omega$ to $B(0,1)$. If this is not so, there exists $z_{n}, \sqrt{z_{n}-\xi}+a \rightarrow 0$ and so

$$
z_{n}-\xi+2 a \sqrt{z_{n}-\xi}+a^{2} \rightarrow 0 \text { so } z_{n}-\xi-2 a^{2}+a^{2} \rightarrow 0
$$

and so there exists $z=\lim _{n \rightarrow \infty} z_{n}$ and $z-\xi=a^{2}=z_{a}-\xi$ so $z=z_{a}$. But this is impossible because it requires that $\sqrt{z_{a}-\xi}+a=0$ so $\sqrt{z_{a}-\xi}=-a \neq a$. Note that $\psi$ just defined, is one to one. Thus $\mathscr{F}$ is nonempty.

For $\psi \in \mathscr{F}$, let $\gamma$ be a small circular contour of radius $r$ about 0 and $B(0,1) \subseteq \Omega$.

$$
\psi^{\prime}(0)=\frac{1}{2 \pi i} \int_{\gamma} \frac{\psi(w)}{w^{2}} d w,\left|\psi^{\prime}(0)\right| \leq(1 / 2 \pi) 2 \pi r\left(1 / r^{2}\right)=1 / r
$$

thus $\eta<\infty$. Consider the special $\psi$ of Claim 1. $\psi(z)=\frac{r}{\sqrt{z-\xi}+a}$. Then

$$
\psi(z)(\sqrt{z-\xi}+a)=r
$$

and so

$$
\psi^{\prime}(z)(\sqrt{z-\xi}+a)+\psi(z)\left(\frac{1}{2 \sqrt{z-\xi}}\right)=0
$$

and so $\psi^{\prime}(0)(\sqrt{-\xi}+a)=-\psi(0)\left(\frac{1}{2 \sqrt{-\xi}}\right)$. Now from the construction, $\psi(0) \neq 0$ and also $|\sqrt{-\xi}+a| \geq r>0$ so $\psi^{\prime}(0) \neq 0$ which shows $\eta>0$.
Lemma 16.6.6 There is an analytic function $h \in \mathscr{F}$ such that $\left|h^{\prime}(0)\right|=h^{\prime}(0)=\eta$. Also $h(0)=0$. Thus if $0 \in \Omega \nsubseteq \mathbb{C}$, for $\Omega$ having the square root property, $h(\Omega)=B(0,1)$.

Proof: By Theorem 16.6.1, there exists a sequence, $\left\{\psi_{n}\right\}$, of functions in $\mathscr{F}$ and an analytic function $h$, such that $\left|\psi_{n}^{\prime}(0)\right| \rightarrow \eta$ and $\psi_{n} \rightarrow h, \psi_{n}^{\prime} \rightarrow h^{\prime}$, uniformly on each compact subset of $\Omega$. It follows

$$
\begin{equation*}
\left|h^{\prime}(0)\right|=\lim _{n \rightarrow \infty}\left|\psi_{n}^{\prime}(0)\right|=\eta>0 \tag{16.7}
\end{equation*}
$$

Now let $|\omega|=1$ and let $\omega h^{\prime}(0)=\left|h^{\prime}(0)\right|$. Thus $\left\{\omega \psi_{n}\right\}$ could be used in place of $\left\{\psi_{n}\right\}$ and we can assume $h^{\prime}(0)=\left|h^{\prime}(0)\right|=\eta$ and for all $z \in \Omega$,

$$
\begin{equation*}
|h(z)|=\lim _{n \rightarrow \infty}\left|\psi_{n}(z)\right| \leq 1 . \tag{16.8}
\end{equation*}
$$

By $16.7, h$ is not a constant. Therefore, in fact, $|h(z)|<1$ for all $z \in \Omega$ in 16.8 by the open mapping theorem because $h(\Omega)$ is a region (open and connected).

It follows from Lemma 16.6.2 that $h$ is one to one. In particular $h^{-1}$ is analytic on $h(\Omega)$ by the open mapping theorem. Why is $h(0)=0$ ?

If $h(0) \neq 0$, then $h(0)<1$ and you can consider $\phi_{h(0)} \circ h$ where $\phi_{\alpha}(z) \equiv \frac{z-\alpha}{1-\bar{\alpha} z}$ is the fractional linear transformation defined in Lemma 16.5.3. By this lemma it follows $\phi_{h(0)} \circ h \in \mathscr{F}$. Now using the chain rule and this lemma,

$$
\begin{aligned}
\left|\left(\phi_{h(0)} \circ h\right)^{\prime}(0)\right| & =\left|\phi_{h(0)}^{\prime}(h(0))\right|\left|h^{\prime}(0)\right| \\
& =\left|\frac{1}{1-|h(0)|^{2}}\right|\left|h^{\prime}(0)\right|=\left|\frac{1}{1-|h(0)|^{2}}\right| \eta>\eta
\end{aligned}
$$

Contradicting the definition of $\eta$. The last claim is from the above lemmas.
Theorem 16.6.7 (Riemann mapping theorem) Let $\Omega \neq \mathbb{C}$ for $\Omega$ a region (connected open set) and suppose $\Omega$ has the square root property. Then for $z_{0} \in \Omega$ there exists a unique $h: \Omega \rightarrow B(0,1)$ such that $h$ is one to one, onto, analytic, $h^{-1}$ is analytic, $h(\Omega)=B(0,1)$, $h^{\prime}\left(z_{0}\right)>0$, and $h\left(z_{0}\right)=0$. In particular, a unique such $h$ exists whenever $\Omega$ is a simply connected proper subset of $\mathbb{C}$.

Proof: This follows from the above lemma. Consider $\hat{\Omega} \equiv \Omega-z_{0}$ so $0 \in \hat{\Omega}$. Then let $\hat{h}: \hat{\Omega} \rightarrow B(0,1)$ with all the right properties in the lemma and let $h(z)=\hat{h}\left(z-z_{0}\right)$.

Suppose $g$ also has these same properties for $z_{0}=0$. Suppose $g^{\prime}(0) \leq h^{\prime}(0)$. Otherwise turn around the following argument. Let $F \equiv h \circ g^{-1}$. Then $F$ maps $B(0,1)$ onto $B(0,1), F(0)=0$, and $F$ one to one. Also $F^{\prime}(0)=h^{\prime}(0)\left(1 / g^{\prime}(0)\right) \geq 1$. By The Schwarz Lemma, Lemma 16.5.3, $F(z)=\lambda z=h\left(g^{-1}(z)\right)$. For $w=g^{-1}(z), h(w)=\lambda z=\lambda g(w)$ where $|\lambda|=1$. Therefore, $h^{\prime}(w)=\lambda g^{\prime}(w)$. However, when $w=0$, both $h^{\prime}(w), g^{\prime}(w)$ are positive and so $\lambda=1$. It follows $h=g$. This works the same way with arbitrary $z_{0}$. The last claim follows from Lemma 16.5 .6 which says that simply connected sets have the square root property.

### 16.7 Exercises

1. Suppose $f \in \mathscr{M}(\widehat{\mathbb{C}})$. Then $f$ is a rational function.
(a) First show there are finitely many poles of $f\left\{\alpha_{1}, \cdots, \alpha_{n}\right\}$ so $f$ is analytic for $|z|>r$.
(b) Suppose $f$ has a removable singularity at $\infty$. That is $\lim _{z \rightarrow 0} z f\left(\frac{1}{z}\right)=0$. First of all, let $S_{i}(z)$ be the singular part of the Laurent series expanded about $\alpha_{i}$. Explain why $f(z)-\sum_{i=1}^{n} S_{i}(z) \equiv f_{n}(z), f_{n}$ an entire function. Explain why $f_{n}(z)=\sum_{k=0}^{\infty} a_{k}\left(\frac{1}{z^{k}}\right)$ and so $f_{n}(z)$ is bounded for large $|z|$. Now explain why $f_{n}$ is bounded and use Liouville's theorem. Conclude that the function is a rational function.
(c) Next case is when $f$ has a pole at $\infty$, meaning $\lim _{|z| \rightarrow 0}|f(1 / z)|=\infty$. Show that in this case also, $f$ is a rational function.
2. Explain why any rational function is in $\mathscr{M}(\widehat{\mathbb{C}})$. Thus, with the preceeding problem, $\mathscr{M}(\widehat{\mathbb{C}})$ equals the rational functions.
3. Suppose you have $f \in \mathscr{M}(\mathbb{C})$, not $\mathscr{M}(\widehat{\mathbb{C}})$. Show that if $f$ has finitely many zeros $\left\{\alpha_{1}, \cdots, \alpha_{n}\right\}$ and poles $\left\{\beta_{1}, \cdots, \beta_{m}\right\}$, then there is an entire function $g(z)$ such that
 order of the zero at $\alpha_{i}$. Hint: First show $f(z)=\prod_{k=1}^{n}\left(z-\alpha_{k}\right)^{r_{k}} h(z)$ where $h(z)$ is meromorphic but has no zeros. Then $h(z)$ has the same poles with the same orders as $f(z)$. Then $h(z)-\sum_{i=1}^{m} S_{i}(z)=l(z)$ where $l(z)$ is entire and the $S_{i}(z)$ are the principal parts $h(z)$ corresponding to $\beta_{i}$. Argue now that

$$
f(z)=\frac{\prod_{k=1}^{n}\left(z-\alpha_{k}\right)^{r_{k}}}{\prod_{k=1}^{m}\left(z-\beta_{k}\right)^{p_{k}}}(q(z))
$$

where $q(z)$ is analytic on $\mathbb{C}$ and can't have any zeros. Next use Problem 15 on Page 399.
4. Let $w_{1}, w_{2}, w_{3}$ be independent periods for a meromorphic function $f(z)$. This means that if $\sum_{i=1}^{3} a_{i} w_{i}=0$ for each $a_{i}$ an integer, then each $a_{i}=0$. Hint: At some point you may want to use Lemma 16.1.4.
(a) Show that if $a_{i}$ is an integer, then $\sum_{i=1}^{3} a_{i} w_{i}$ is also a period of $f(z)$.
(b) Let $P_{N}$ be periods of the form $\sum_{i=1}^{3} a_{i} w_{i}$ for $a_{i}$ an integer with $\left|a_{i}\right| \leq N$. Show there are $(2 N+1)^{3}$ such periods.
(c) Show $P_{N} \subseteq\left[-N\left(\sum_{i=1}^{3}\left|w_{i}\right|\right), N\left(\sum_{i=1}^{3}\left|w_{i}\right|\right)\right]^{2} \equiv Q_{N}$.
(d) Between the cubes of any two successive positive integers, there is the square of a positive integer. Thus $(2 N)^{3}<M^{2}<(2 N+1)^{3}$. Show this is so. It is easy to verify if you show that $(x+1)^{3 / 2}-x^{3 / 2}>2$ for all $x \geq 2$ showing that there is an integer $m$ between $(n+1)^{3 / 2}$ and $n^{3 / 2}$. Then squaring things, you get the result.
(e) Partition $Q_{N}$ into $M^{2}$ small squares. If $Q$ is one of these, show its sides are no longer than

$$
\left(\frac{\left(2 N\left(\sum_{i=1}^{3}\left|w_{i}\right|\right)\right)^{2}}{M^{2}}\right)^{1 / 2} \leq\left(\frac{\left(N\left(\sum_{i=1}^{3}\left|w_{i}\right|\right)\right)^{2}}{(2 N)^{3}}\right)^{1 / 2} \leq \frac{C}{N^{1 / 2}}
$$

(f) You have $(2 N+1)^{3}$ points which are contained in $M^{2}$ squares where $M^{2}$ is smaller than $(2 N+1)^{3}$. Explain why one of these squares must contain two different periods of $P_{N}$.
(g) Suppose the two periods are $\sum_{i=1}^{3} a_{i} w_{i}$ and $\sum_{i=1}^{3} \hat{a}_{i} w_{i}$, both in $Q$ which has sides of length no more than $C / N^{1 / 2}$. Thus the distance between these two periods has length no more than $\sqrt{2} C / \sqrt{N}$. Explain why this shows that there is a sequence of periods of $f$ which converges to 0 . Explain why this requires $f$ to be a constant.

This result, that there are at most two independent periods is due to Jacobi from around 1835. In fact, there are nonconstant functions which have two independent periods but they can't be bounded.
5. Suppose you have $f$ is analytic and has two independent periods. Show that $f$ is a constant. Hint: Consider a parallelogram determined by the two periods and apply Liouville's theorem. Functions having two independent periods which are analytic except for poles are known as elliptic functions.
6. Suppose $f$ is an entire function, analytic on $\mathbb{C}$, and that it has two periods $w_{1}, w_{2}$. That is $f\left(z+w_{1}\right)=f(z)$ and $f\left(z+w_{2}\right)=f(z)$. Suppose also that the ratio of these two periods is not a real number so vectors, $w_{1}$ and $w_{2}$ are not parallel. Show, using Liouville's theorem, that $f(z)$ equals a constant. Hint: Consider the parallelogram determined by the two vectors $w_{1}, w_{2}$ and tile $\mathbb{C}$ with similar parallelograms. Elliptic functions are those which have two periods like this and are analytic except for poles. These are points where $|f(z)|$ becomes unbounded. Thus the only analytic elliptic functions are constants.
7. You can show that if $r$ is a real irrational number the expressions of the form $m+n r$ for $m, n$ integers are dense in $\mathbb{R}$. See my single variable advanced calculus book or modify the argument in Problem 4. This is due to Dirichlet also in the 1830s. (Let $P_{N}$ be everything of the form $m+n r$ where $|m|,|n| \leq N$. Thus there are $(2 N+1)^{2}$ such numbers contained in $[-N(1+|r|), N(1+|r|)] \equiv I$. Let $M$ be an integer, $(2 N)^{2}<$ $M<(2 N+1)^{2}$ and partition $I$ into $M$ equal intervals. Now argue some interval has two of these numbers in $P_{N}$ etc.) In particular, $|m+n r|$ can be made as small as desired. Now suppose $f$ is a non constant meromorphic function and it is periodic having periods $w_{1}, w_{2}$ where if, for $m, n$ integers, $m w_{1}+n w_{2}=0$ then both $m, n$ are zero. Show that $w_{1} / w_{2}$ cannot be real. This was also done by Jacobi.
8. Suppose you have a nonconstant meromorphic function $f$ which has two periods $w_{1}, w_{2}$ such that if $m w_{1}+n w_{2}=0$ for $m, n$ integers, then $m=n=0$. Let $P_{a}$ be a parallelogram with lower left vertex at $a$ and sides determined by $w_{1}$ and $w_{2}$ such that no pole of $f$ is on any of the sides. Show that the sum of the residues of $f$ found inside $P_{a}$ must be zero.
9. Let $f(z)=\frac{a z+b}{c z+d}$ and let $g(z)=\frac{a_{1} z+b_{1}}{c_{1} z+d_{1}}$. Show that $f \circ g(z)$ equals the quotient of two expressions, the numerator being the top entry in the vector

$$
\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)\left(\begin{array}{ll}
a_{1} & b_{1} \\
c_{1} & d_{1}
\end{array}\right)\binom{z}{1}
$$

and the denominator being the bottom entry. Show that if you define

$$
\phi\left(\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)\right) \equiv \frac{a z+b}{c z+d}
$$

then $\phi(A B)=\phi(A) \circ \phi(B)$. Find an easy way to find the inverse of $f(z)=\frac{a z+b}{c z+d}$ and give a condition on the $a, b, c, d$ which insures this function has an inverse.
10. The modular group ${ }^{2}$ is the set of fractional linear transformations, $\frac{a z+b}{c z+d}$ such that $a, b, c, d$ are integers and $a d-b c=1$. Using Problem 9 or brute force show this modular group is really a group with the group operation being composition. Also show the inverse of $\frac{a z+b}{c z+d}$ is $\frac{d z-b}{-c z+a}$.

[^10]11. The next few problems are about approximating an analytic function with rational functions. For $K$ compact and $f, g$ continuous on $K$, let
$$
\|f-g\|_{K, \infty} \equiv \max \{\|f(z)-g(z)\|: z \in K\}
$$

Let $R$ be a rational function (Laurent series is finite) which has a pole only at $a \in V$, a component of $\mathbb{C} \backslash K$. Suppose $b \in V$. Then for $\varepsilon>0$ given, there exists a rational function $Q$, having a pole only at $b$ such that $\|R-Q\|_{K, \infty}<\varepsilon$. If it happens that $V$ is unbounded, then there exists a polynomial, $P$ such that $\|R-P\|_{K, \infty}<\varepsilon$. Hint: Say that $b \in V$ satisfies $\mathscr{P}$ if for all $\varepsilon>0$ there exists a rational function, $Q_{b}$, having a pole only at $b$ such that $\left\|R-Q_{b}\right\|_{K, \infty}<\varepsilon$. Now define a set, $S \equiv\{b \in V: b$ satisfies $\mathscr{P}\}$. Observe that $S \neq \emptyset$ because $a \in S$. Now set about showing that $S$ is open and also contains all its limit points in the connected sets $V$. Conclude that $S=V$ since V is connected.
12. $\uparrow$ Let $K$ be a compact subset of an open set, $\Omega$ and let $f$ be analytic on $\Omega$. Then there exists a rational function, $Q$ whose poles are not in $K$ such that $\|Q-f\|_{K, \infty}<\varepsilon$. Hint: By Theorem 15.4.2 there are oriented curves, $\gamma_{k}$ described there such that for all $z \in K, f(z)=\frac{1}{2 \pi i} \sum_{k=1}^{p} \int_{\gamma_{k}} \frac{f(w)}{w-z} d w$. Now approximate these contour integrals with Riemann sums.
13. $\uparrow$ Use the above problems to verify Runge's theorem: Let $K$ be a compact subset of an open set, $\Omega$ and let $\left\{b_{j}\right\}$ be a set which consists of one point from each component of $\widehat{\mathbb{C}} \backslash K$. Let $f$ be analytic on $\Omega$. Then for each $\varepsilon>0$, there exists a rational function, $Q$ whose poles are all contained in the set, $\left\{b_{j}\right\}$ such that $\|Q-f\|_{K, \infty}<\varepsilon$. If $\widehat{\mathbb{C}} \backslash K$ has only one component, then $Q$ may be taken to be a polynomial.
14. $\uparrow$ Generalize the above Runge theorem to this version of Runge's theorem: Let $\Omega$ be an open set, and let $A$ be a set which has one point in each component of $\widehat{\mathbb{C}} \backslash \Omega$ and let $f$ be analytic on $\Omega$. Then there exists a sequence of rational functions, $\left\{R_{n}\right\}$ having poles only in $A$ such that $R_{n}$ converges uniformly to $f$ on compact subsets of $\Omega$. Hint: Show that there is a sequence of compact sets $K_{n}$ such that $\Omega=\cup_{k=1}^{\infty} K_{n}, \cdots, K_{n} \subseteq$ int $K_{n+1} \cdots$, and use the result of the above problem.

## Chapter 17

## Spectral Theory of Linear Maps

This chapter provides a short introduction to the spectral theory of linear maps defined on a Banach space. It is only an introduction. You should see Dunford and Schwarz [15] for a complete treatment of these topics.

### 17.1 The Resolvent and Spectral Radius

The idea is that you have $A \in \mathscr{L}(X, X)$ where $X$ is a complex Banch space. We eliminate from consideration the stupid case that $X$ is only the 0 vector. To begin with, here is a fundamental lemma which will be used whenever convenient. It is about taking a continuous linear transformation through the integral sign.

Lemma 17.1.1 Let $f: \gamma^{*} \rightarrow \mathscr{L}(X, X)$ be continuous where $\gamma:[a, b] \rightarrow \mathbb{C}$ has finite total variation and $X$ is a Banach space. Let $A \in \mathscr{L}(X, X)$. Then

$$
A \int_{\gamma} f(z) d z=\int_{\gamma} A f(z) d z,\left(\int_{\gamma} f(z) d z\right) A=\int_{\gamma} f(z) A d z
$$

When we write $A B$ for $A, B \in \mathscr{L}(X, X)$, it means $A \circ B$. That is, $A \circ B(x)=A(B(x))$
Proof: This follows from the definition of the integral, see Theorem 14.4.3 on Page 346. Let $P$ denote a sequence of partitions such that $\|P\| \rightarrow 0$.

$$
\int_{\gamma} f(z) d z \equiv \lim _{\|P\| \rightarrow 0} \sum_{P} f\left(\gamma\left(\tau_{i}\right)\right)\left(\gamma\left(t_{i}\right)-\gamma\left(t_{i-1}\right)\right)
$$

Now multiplication of an element of $\mathscr{L}(X, X)$ by $A \in \mathscr{L}(X, X)$ is continuous because

$$
\left\|A B_{1}-A B_{2}\right\| \leq\|A\|\left\|B_{1}-B_{2}\right\|, \quad\left\|B_{1} A-B_{2} A\right\| \leq\left\|B_{1}-B_{2}\right\|\|A\|
$$

Therefore,

$$
\begin{aligned}
A \int_{\gamma} f(z) d z & \equiv A \lim _{\|P\| \rightarrow 0} \sum_{P} f\left(\gamma\left(\tau_{i}\right)\right)\left(\gamma\left(t_{i}\right)-\gamma\left(t_{i-1}\right)\right) \\
& =\lim _{\|P\| \rightarrow 0} A \sum_{P} f\left(\gamma\left(\tau_{i}\right)\right)\left(\gamma\left(t_{i}\right)-\gamma\left(t_{i-1}\right)\right) \\
& =\lim _{\|P\| \rightarrow 0} \sum_{P} A f\left(\gamma\left(\tau_{i}\right)\right)\left(\gamma\left(t_{i}\right)-\gamma\left(t_{i-1}\right)\right) \equiv \int_{\gamma} A f(z) d z
\end{aligned}
$$

There are no issues regarding existence of the various quantities because the functions are continuous and the curve is of bounded variation. The other claim is completely similar.

Corresponding to $A$ there are two sets defined next.
Definition 17.1.2 The resolvent set, denoted as $\rho(A)$ is defined as

$$
\left\{\lambda \in \mathbb{C}:(\lambda I-A)^{-1} \in \mathscr{L}(X, X)\right\}
$$

The spectrum of $A$, denoted as $\sigma(A)$ is $\mathbb{C} \backslash \rho(A)$. When $\lambda \in \rho(A)$, we call $(\lambda I-A)^{-1}$ the resolvent. Thus, in particular, when $\lambda$ is in $\rho(A), \lambda I-A$ is one to one and onto.

There is a fundamental identity involving the resolvent which should be noted first. In order to remember this identity, simply write $(\lambda I-A)^{-1} \approx \frac{1}{\lambda-A}$ and proceed formally.

$$
\frac{1}{\lambda-A}-\frac{1}{\mu-A}=\frac{\mu-\lambda}{(\lambda-A)(\mu-A)}
$$

This suggests that for $\mu, \lambda \in \rho(A)$,

$$
\begin{align*}
(\lambda I-A)^{-1}-(\mu I-A)^{-1} & =(\mu-\lambda)(\lambda I-A)^{-1}(\mu I-A)^{-1} \\
& =(\mu-\lambda)(\mu I-A)^{-1}(\lambda I-A)^{-1} \tag{17.1}
\end{align*}
$$

Now since $(\lambda I-A),(\mu I-A)$ are both one to one and onto, we observe by multiplying on the left by $(\mu I-A)$ and on the right by $(\lambda I-A)$ that the result on both sides are the same. Thus these are indeed the same.

$$
\begin{aligned}
& (\mu I-A)(\lambda I-A)^{-1}-I \text { and }(\mu-\lambda)(\lambda I-A)^{-1} \\
& (\mu I-A)-(\lambda I-A) \text { and }(\mu-\lambda)
\end{aligned}
$$

which are the same. Similarly, the second line of 17.1 holds.
Proposition 17.1.3 For $A \in \mathscr{L}(X, X)$ and $\lambda, \mu \in \rho(A)$,

$$
\begin{aligned}
(\lambda I-A)^{-1}-(\mu I-A)^{-1} & =(\mu-\lambda)(\lambda I-A)^{-1}(\mu I-A)^{-1} \\
& =(\mu-\lambda)(\mu I-A)^{-1}(\lambda I-A)^{-1}
\end{aligned}
$$

Next is a useful lemma.
Lemma 17.1.4 Let $B \in \mathscr{L}(X, X)$ and suppose $\|B\|<1$. Then $(I-B)^{-1}$ exists and is given by the series $(I-B)^{-1}=\sum_{k=0}^{\infty} B^{k}$. Also, $\left\|(I-B)^{-1}\right\| \leq \frac{1}{1-\|B\|}$. The series converges in $\mathscr{L}(X, X)$.

Proof: The series converges by the root test, Theorem 1.12.1 generalized to the case where $\mathbb{F}^{p}$ is replaced by $X$ as in Problem 5 on Page 75. Indeed,

$$
\left\|B^{n}\right\|^{1 / n} \leq\left(\|B\|^{n}\right)^{1 / n}=\|B\|<1
$$

so $\limsup \mathrm{m}_{n \rightarrow \infty}\left\|B^{n}\right\|^{1 / n} \leq\|B\|<1$. Now also $(I-B) \sum_{k=0}^{n} B^{k}=I-B^{n+1}$ where $\left\|B^{n+1}\right\| \leq$ $\|B\|^{n+1}$ and converges to 0 . Thus,

$$
(I-B) \sum_{k=0}^{\infty} B^{k}=\lim _{n \rightarrow \infty}(I-B) \sum_{k=0}^{n} B^{k}=\lim _{n \rightarrow \infty}\left(I-B^{n+1}\right)=I
$$

Similarly the infinite sum is the left inverse of $I-B$. To see this, note that if $\left\|A_{n}-A\right\| \rightarrow 0$, then $A_{n} C \rightarrow A C$ because

$$
\left\|A_{n} C-A C\right\| \equiv \sup _{\|x\| \leq 1}\left\|\left(A_{n}-A\right) C x\right\| \leq\left\|A_{n}-A\right\|\|C\|
$$

Therefore, $\sum_{k=0}^{\infty} B^{k}(I-B)=\lim _{n \rightarrow \infty} \sum_{k=0}^{n} B^{k}(I-B)=\lim _{n \rightarrow \infty}\left(I-B^{n+1}\right)=I$. As to the estimate, if $\|x\| \leq 1$,

$$
\begin{aligned}
\left\|(I-B)^{-1} x\right\| & =\left\|\left(\sum_{k=0}^{\infty} B^{k}\right)(x)\right\|=\left\|\sum_{k=0}^{\infty} B^{k}(x)\right\| \\
& \leq \sum_{k=0}^{\infty}\left\|B^{k}\right\| \leq \sum_{k=0}^{\infty}\|B\|^{k}=\frac{1}{1-\|B\|}
\end{aligned}
$$

The major result about resolvents is the following proposition. Note that the resolvent has values in $\mathscr{L}(X, X)$ which is a Banach space.

Proposition 17.1.5 Let $A \in \mathscr{L}(X, X)$ for $X$ a Banach space. Then the following hold.

1. $\rho(A)$ is open
2. $\lambda \rightarrow(\lambda I-A)^{-1}$ is continuous
3. $\lambda \rightarrow(\lambda I-A)^{-1}$ is analytic
4. For $|\lambda|>\|A\|,\left\|(\lambda I-A)^{-1}\right\| \leq \frac{1}{|\lambda|-\|A\|}$ and $(\lambda I-A)^{-1}=\sum_{k=0}^{\infty} \frac{A^{k}}{\lambda^{k+1}}$

Proof: 1.) Let $\lambda \in \rho(A)$. Let $|\mu-\lambda|<\left\|(\lambda I-A)^{-1}\right\|^{-1}$. Then

$$
\begin{equation*}
\mu I-A=(\mu-\lambda) I+\lambda I-A=(\lambda I-A)\left[I-(\lambda-\mu)(\lambda I-A)^{-1}\right] \tag{17.2}
\end{equation*}
$$

Now $\left\|(\lambda-\mu)(\lambda I-A)^{-1}\right\|=|\lambda-\mu|\left\|(\lambda I-A)^{-1}\right\|<1$ from the assumed estimate. Thus,

$$
\left[I-(\lambda-\mu)(\lambda I-A)^{-1}\right]^{-1}=\sum_{k=0}^{\infty}(\lambda-\mu)^{k}\left((\lambda I-A)^{-1}\right)^{k}
$$

and so from 17.2,

$$
\begin{align*}
(\mu I-A)^{-1} & =\left[I-(\lambda-\mu)(\lambda I-A)^{-1}\right]^{-1}(\lambda I-A)^{-1}  \tag{17.3}\\
& =\sum_{k=0}^{\infty}(\lambda-\mu)^{k}\left((\lambda I-A)^{-1}\right)^{k+1}
\end{align*}
$$

This shows $\rho(A)$ is open.
2.) Next consider continuity. This follows from Proposition 17.1.3.
3.) From Theorem 14.8.1, it suffices to show that the function is differentiable. This follows from the continuity and the resolvent equation.

$$
\begin{aligned}
\frac{(\lambda I-A)^{-1}-(\mu I-A)^{-1}}{\lambda-\mu} & =\frac{(\mu-\lambda)(\mu I-A)^{-1}(\lambda I-A)^{-1}}{\lambda-\mu} \\
& =-(\mu I-A)^{-1}(\lambda I-A)^{-1}
\end{aligned}
$$

so taking the limit as $\lambda \rightarrow \mu$ one obtains the derivative at $\mu$ is $-\left((\mu I-A)^{-1}\right)^{2}$.
4.) Finally consider the estimate. For $|\lambda|>\|A\|,(\lambda I-A)=\lambda\left(I-\frac{A}{\lambda}\right)$ and so from Lemma 17.1.4,

$$
(\lambda I-A)^{-1}=\frac{1}{\lambda}\left(I-\frac{A}{\lambda}\right)^{-1}=\sum_{k=0}^{\infty} \frac{A^{k}}{\lambda^{k+1}}
$$

and $\left\|(\lambda I-A)^{-1}\right\| \leq \frac{1}{|\lambda|} \frac{1}{1-\frac{|A| \mid}{|\lambda|}}=\frac{1}{|\lambda|-||A||}$.
The notion of spectrum is just a generalization of the concept of eigenvalues in linear algebra. In linear algebra, everything is finite dimensional and the spectrum is just the set of eigenvalues. You can get them by looking for solutions to the characteristic equation which involves a determinant and they exist because of the fundamental theorem of algebra. No such thing is available in a general Banach space. However, many of the same results still hold.
Proposition 17.1.6 For $A \in \mathscr{L}(X, X), \sigma(A) \neq \emptyset$ and $\sigma(A)$ is a compact set. If $\lambda \in$ $\sigma(A)$, then for all $n \in \mathbb{N}$, it follows that $\lambda^{n} \in \sigma\left(A^{n}\right)$.

Proof: Suppose first that $\sigma(A)=\emptyset$. Then you would have $\lambda \rightarrow(\lambda I-A)^{-1}$ is analytic and in fact from estimate 4 above,

$$
\begin{equation*}
\lim _{|\lambda| \rightarrow \infty}\left\|(\lambda I-A)^{-1}\right\|=0 \tag{17.4}
\end{equation*}
$$

Thus there exists $r$ such that if $|\lambda|<r$, then $\left\|(\lambda I-A)^{-1}\right\|<1$. However, $\left\|(\lambda I-A)^{-1}\right\|$ is bounded for $|\lambda| \leq r$ and so $\lambda \rightarrow(\lambda I-A)^{-1}$ is analytic on all of $\mathbb{C}$ (entire) and is bounded. Therefore, it is constant thanks to Liouville's theorem, Theorem 14.11.2. But the constant can only be 0 thanks to 17.4. Therefore, $(\lambda I-A)^{-1}=0$ which is nonsense since we are not considering $X=\{0\}$. Indeed, if this is so, then $I=(\lambda I-A)(\lambda I-A)^{-1}$ would be the zero map.
$\sigma(A)$ is the complement of the open set $\rho(A)$ and so $\sigma(A)$ is closed. It is bounded thanks to 4 of Proposition 17.1.5. Therefore, it is compact by the Heine Borel theorem.

It remains to verify the last claim.

$$
\begin{align*}
(\lambda I-A) \sum_{k=0}^{n-1} \lambda^{k} A^{(n-1)-k} & =\sum_{k=0}^{n-1} \lambda^{k+1} A^{(n-1)-k}-\sum_{k=0}^{n-1} \lambda^{k} A^{n-k} \\
& =\sum_{k=1}^{n} \lambda^{k} A^{n-k}-\sum_{k=0}^{n-1} \lambda^{k} A^{n-k}=\lambda^{n} I-A^{n}  \tag{17.5}\\
& =\sum_{k=0}^{n-1} \lambda^{k} A^{(n-1)-k}(\lambda I-A)
\end{align*}
$$

If $\lambda \in \sigma(A)$, then this means $(\lambda I-A)^{-1}$ does not exist. Thanks to the open mapping theorem, this is equivalent to either $\lambda I-A$ not being one to one or not onto because by this theorem, if it is both, then it is continuous and has continuous inverse because it maps open sets to open sets. Now consider the identity 17.5. If $\lambda I-A$ is not one to one, then the bottom line and the middle line shows that $\lambda^{n} I-A$ is not one to one. If $(\lambda I-A)$ is not onto, then the top line and the middle line show that $\lambda^{n} I-A^{n}$ is not onto. Thus $\lambda^{n} \in \sigma\left(A^{n}\right)$.

Definition 17.1.7 $\sigma(A)$ is closed and bounded. Let $r(A)$ denote the spectral radius defined as $\max \{|\lambda|: \lambda \in \sigma(A)\}$

It is very important to have a description of this spectral radius. The following famous result is due to Gelfand.

Theorem 17.1.8 Let $A \in \mathscr{L}(X, X)$. Then $r(A)=\lim _{n \rightarrow \infty}\left\|A^{n}\right\|^{1 / n}$.
Proof: Let $\lambda \in \sigma(A)$. By Proposition 17.1.6 $\lambda^{n} \in \sigma\left(A^{n}\right)$ and so by Proposition 17.1.5, $\left|\lambda^{n}\right|=|\lambda|^{n} \leq\left\|A^{n}\right\|$. Thus for $\lambda \in \sigma(A)$,

$$
\begin{equation*}
|\lambda| \leq \lim \inf _{n \rightarrow \infty}\left\|A^{n}\right\|^{1 / n} \text { so } r(A) \leq \lim \inf _{n \rightarrow \infty}\left\|A^{n}\right\|^{1 / n} \tag{17.6}
\end{equation*}
$$

But also, from Proposition 17.1.5, for $|\lambda|>\|A\|,(\lambda I-A)^{-1}=\sum_{k=0}^{\infty} \frac{A^{k}}{\lambda^{k+1}}$. By Corollary 15.2.3 about uniqueness of the Laurent series, the above formula for the Laurent series must hold for all $|\lambda|>r(A)$. By the root test, Theorem 1.12.1, it must be the case that if $|\lambda|>r(A)$, then $\lim \sup _{n \rightarrow \infty}\left\|\frac{A^{n}}{\lambda^{n+1}}\right\|^{1 / n}=\frac{1}{|\lambda|} \limsup _{n \rightarrow \infty}\left\|A^{n}\right\|^{1 / n} \leq 1$. Therefore, if $|\lambda|>$ $r(A)$, it follows that $\limsup _{n \rightarrow \infty}\left\|A^{n}\right\|^{1 / n} \leq|\lambda|$. This being true for all such $\lambda$ implies with 17.6 that $r(A) \geq \lim \sup _{n \rightarrow \infty}\left\|A^{n}\right\|^{1 / n} \geq \liminf _{n \rightarrow \infty}\left\|A^{n}\right\|^{1 / n} \geq r(A)$.

### 17.2 Functions of Linear Transformations

This will all be based on the theorems about existence of cycles, stated here for convenience.

Theorem 17.2.1 Let $K_{1}, K_{2}, \cdots, K_{m}$ be disjoint compact subsets of an open set $\Omega$ in $\mathbb{C}$. Then there exist continuous, closed, bounded cycles $\left\{\Gamma_{j}\right\}_{j=1}^{m}$ for which $\Gamma_{j}^{*} \cap K_{k}=\emptyset$ for each $k, j, \Gamma_{j}^{*} \cap \Gamma_{k}^{*}=\emptyset, \Gamma_{j}^{*} \subseteq \Omega$. Also, if $p \in K_{k}$ and $j \neq k, n\left(\Gamma_{k}, p\right)=1, n\left(\Gamma_{j}, p\right)=0$ so if $p$ is in some $K_{k}, \sum_{j=1}^{m} n\left(\Gamma_{j}, p\right)=1$ each $\Gamma_{j}$ being the union of oriented simple closed curves, while for all $z \notin \Omega, \sum_{k=1}^{m} n\left(\Gamma_{k}, z\right)=0$. Also, if $p \in \Gamma_{j}^{*}$, then for $i \neq j, n\left(\Gamma_{i}, p\right)=0$.

One can add in the compact sets $\Gamma_{k}^{*}$ to the list of disjoint compact sets and obtain the following corollary. Essentially, what this does is to change each $\Gamma_{k}$ a little bit, getting $\hat{\Gamma}_{k}$ with the same properties but also $n\left(\hat{\Gamma}_{k}, p\right)=0$ if $p \in \Gamma_{k}^{*}$.

Corollary 17.2.2 In the context of Theorem 15.4.2, there exist continuous, closed and bounded cycles $\left\{\hat{\Gamma}_{j}\right\}_{j=1}^{m}$ having exactly the same properties as the $\Gamma_{j}$ above but also with the property that, $n\left(\hat{\Gamma}_{j}, z\right)=0$ if $z$ is in any of the $\Gamma_{i}^{*}$ even if $i=j$. Thus as before, if $p$ is in some $K_{k}, \sum_{j=1}^{m} n\left(\hat{\Gamma}_{j}, p\right)=1$ each $\hat{\Gamma}_{j}$ being the union of oriented simple closed curves, while for all $z \notin \Omega, \sum_{k=1}^{m} n\left(\hat{\Gamma}_{k}, z\right)=0$.

How to think of this? The $\Gamma_{k}$ go counter clockwise around the $K_{k}$ and so do the $\hat{\Gamma}_{k}$ but these are "inside" the $\Gamma_{k}$. All that given above in terms of winding numbers is just the precise way to express this idea.

Let $A \in \mathscr{L}(X, X)$ and suppose $\sigma(A)=\cup_{k=1}^{n} K_{k}$ where the $K_{k}$ are disjoint compact sets. Let $\delta$ be small enough that no point of any $K_{k}$ is within $\delta$ of any other $K_{j}$ as in the proof of Theorem 15.4.2. Let the open set containing $K_{j}$ be given by $U_{j} \equiv K_{j}+B(0, \delta / 2)$, defined as all numbers in $\mathbb{C}$ of the form $k+d$ where $|d|<\frac{\delta}{2}$ and $k \in K_{j}$. Let and $\Gamma_{j}^{*} \subseteq U_{j}$ also where $\Gamma_{j}$ is an oriented contour for which $n\left(\Gamma_{j}, z\right)=1$ for all $z \in K_{j}$. In addition to this, the
open sets just described are disjoint. Now let $f_{k}(z)=1$ on $U_{k}$ and let it be 0 everywhere else. Thus $f_{k}$ is not analytic on $\mathbb{C}$ but it is analytic on $\cup_{j} U_{j} \equiv \Omega$ and $\Omega$ contains $\sigma(A)$.

Now $f_{k}(\lambda)(\lambda I-A)^{-1}$ is not analytic on $\Omega$ because $\Omega$ contains $\sigma(A)$.
There is a general notion of finding functions of linear operators. First recall the following corollary of the Cauchy integral formula.

Corollary 17.2.3 Let $\Omega$ be an open set (note that $\Omega$ might not be simply connected or even connected) and let $\gamma_{k}:\left[a_{k}, b_{k}\right] \rightarrow \Omega, k=1, \cdots, m$, be closed, continuous and of bounded variation. Suppose also that $\sum_{k=1}^{m} n\left(\gamma_{k}, z\right)=0$ for all $z \notin \Omega$ and $\sum_{k=1}^{m} n\left(\gamma_{k}, z\right)$ is an integer for $z \in \Omega$. Then if $f: \Omega \rightarrow X$ is analytic, $\sum_{k=1}^{m} \int_{\gamma_{k}} f(w) d w=0$. Thus if $\Gamma$ is the sum of these oriented curves, $\int_{\Gamma} f(w) d w=0$.
Definition 17.2.4 Let $A \in \mathscr{L}(X, X)$ and let $\Omega$ be an open set which contains $\sigma(A)$. Let $\Gamma$ be a cycle which has $\Gamma^{*} \subseteq \Omega \cap \sigma(A)^{C}$, and suppose that

1. $n(\Gamma, z)=1$ if $z \in \sigma(A)$.
2. $n(\Gamma, z)$ is an integer if $z \in \Omega$.
3. $n(\Gamma, z)=0$ if $z \notin \Omega$.

Then if $f$ is analytic on $\Omega$, define $f(A) \equiv \frac{1}{2 \pi i} \int_{\Gamma} f(\lambda)(\lambda I-A)^{-1} d \lambda$.
First of all, why does this make sense for things which have another meaning? In particular, why does it make sense if $\lambda^{n}=f(\lambda)$ where $n \geq 0$ ? Is $A^{n}$ correctly given by this formula? If not, then this isn't a very good way to define $f(A)$. This involves the following theorem which says that if you look at $f(A) g(A)$ defined above, it gives the same thing as the above applied to $f(\boldsymbol{\lambda}) g(\boldsymbol{\lambda})$.

Theorem 17.2.5 Suppose $\Omega \supseteq \sigma(A)$ where $\Omega$ is an open set. Let $\Gamma$ be an oriented cycle, the union of oriented simple closed curves such that $n(\Gamma, z)=1$ if $z \in \sigma(A), n(\Gamma, z)$ is an integer if $z \in \Omega$, and $n(\Gamma, z)=0$ if $z \notin \Omega$. Then for $f, g$ analytic on $\Omega$,

$$
f(A) \equiv \frac{1}{2 \pi i} \int_{\Gamma} f(\lambda)(\lambda I-A)^{-1} d \lambda, g(A) \equiv \frac{1}{2 \pi i} \int_{\Gamma} g(\lambda)(\lambda I-A)^{-1} d \lambda
$$

It follows that $f(A) g(A)=\frac{1}{2 \pi i} \int_{\Gamma} f(\lambda) g(\lambda)(\lambda I-A)^{-1} d \lambda$.
Proof: From Corollary 17.2 .2 , let $\hat{\Gamma}$ be such that for $\lambda \in \Gamma, n(\hat{\Gamma}, \lambda)=0$ but the definition of $g(A)$ works as well for $\hat{\Gamma}$. This is an application of Corollary 17.2.3. You get the same thing for $g(A), f(A)$ with either cycle $\hat{\Gamma}$ or $\Gamma$. Then using Lemma 17.1.1 as needed,

$$
\begin{aligned}
-4 \pi^{2} f(A) g(A) & =\left(\int_{\Gamma} f(\lambda)(\lambda I-A)^{-1} d \lambda\right)\left(\int_{\hat{\Gamma}} g(\mu)(\mu I-A)^{-1} d \mu\right) \\
& =\int_{\Gamma} \int_{\hat{\Gamma}} f(\lambda) g(\mu)(\lambda I-A)^{-1}(\mu I-A)^{-1} d \mu d \lambda
\end{aligned}
$$

Using the resolvent identity 17.1 , this equals

$$
\int_{\Gamma} \int_{\hat{\Gamma}} f(\lambda) g(\mu) \frac{1}{\mu-\lambda}\left[(\lambda I-A)^{-1}-(\mu I-A)^{-1}\right] d \mu d \lambda
$$

Now using the Fubini theorem, Theorem 14.4.9,

$$
=\int_{\Gamma} f(\lambda)(\lambda I-A)^{-1} \int_{\hat{\Gamma}} g(\mu) \frac{1}{\mu-\lambda} d \mu d \lambda+\int_{\hat{\Gamma}} g(\mu)(\mu I-A)^{-1} \int_{\Gamma} f(\lambda) \frac{1}{\lambda-\mu} d \lambda d \mu
$$

The first term is 0 from Cauchy's integral formula, Theorem 14.11.1 applied to the individual simple closed curves whose union is $\Gamma$ because the winding number $n(\hat{\Gamma}, \lambda)=0$. Thus the inside integral vanishes. From the Cauchy integral formula, the second term is $2 \pi i \int_{\hat{\Gamma}} g(\mu)(\mu I-A)^{-1} f(\mu) d \mu$ and so

$$
f(A) g(A)=\frac{1}{2 \pi i} \int_{\hat{\Gamma}} f(\mu) g(\mu)(\mu I-A)^{-1} d \mu=\frac{1}{2 \pi i} \int_{\Gamma} f(\lambda) g(\lambda)(\lambda I-A)^{-1} d \lambda
$$

Now consider the case that $f(\lambda)=\lambda$. Is $f(A)$, defined in terms of integrals as above, equal to $A$ ? If so, then from the theorem just shown, $\lambda^{n}$ used in the integral formula does lead to $A^{n}$. Thus one considers $\frac{1}{2 \pi i} \int_{\Gamma} \lambda(\lambda I-A)^{-1} d \lambda$ where $\Gamma$ is a cycle such that $n(\Gamma, z)=$ 1 if $z \in \sigma(A), n(\Gamma, z)$ is an integer if $z \in \Omega$, and $n(\Gamma, z)=0$ if $z \notin \Omega, \Gamma^{*} \cap \sigma(A)=\emptyset$.

Let $\hat{\Omega} \equiv \sigma(A)^{C}$ and let $\hat{\gamma}_{R}$ be a large circle of radius $R>\|A\|$ oriented clockwise, which includes $\sigma(A) \cup \Gamma^{*}$ on its inside.

Consider the following picture in which $\sigma(A)$ is the union of the two compact sets, $K_{1}, K_{2}$ which are contained in the closed curves shown and $\Gamma$ is the union of the oriented cycles $\Gamma_{i}$. A similar picture would apply if there were more than two $K_{i}$. All that is of interest here is that there is a cycle $\Gamma$ oriented such that for all $z \in \sigma(A), n(\Gamma, z)=1$, $n(\Gamma, z)$ is an integer if $z \in \hat{\Omega}$, and $\hat{\gamma}_{R}$ is a large circle oriented clockwise as shown, $R>\|A\|$.


Then in this case, $f(\lambda)=\lambda$ is analytic everywhere and $\hat{\Omega} \equiv \sigma(A)^{C}$. Let $\gamma_{R} \equiv-\hat{\gamma}_{R}$. Thus, by Corollary 17.2.3, $\frac{1}{2 \pi i} \int_{\Gamma} f(\lambda)(\lambda I-A)^{-1} d \lambda+\frac{1}{2 \pi i} \int_{\hat{\gamma}_{R}} f(\lambda)(\lambda I-A)^{-1} d \lambda=0$ and so, $f(A) \equiv \frac{1}{2 \pi i} \int_{\Gamma} f(\lambda)(\lambda I-A)^{-1} d \lambda=\frac{1}{2 \pi i} \int_{\gamma_{R}} f(\lambda)(\lambda I-A)^{-1} d \lambda$. The integrand in the integral on the right is $\lambda \sum_{k=0}^{\infty} \frac{A^{k}}{\lambda^{k+1}}$ for $\lambda \in \gamma_{R}^{*}$ and convergence is uniform on $\gamma_{R}^{*}$. Then all terms vanish except the one when $k=1$ because all the other terms have primitives. The uniform convergence implies that the integral of the sum is the sum of the integrals and there is only one which survives. Therefore,

$$
f(A) \equiv \frac{1}{2 \pi i} \int_{\gamma_{R}} \frac{A}{\lambda} d \lambda=A
$$

It follows from Theorem 17.2.5 that if $f(\lambda)=\lambda^{n}$, then $f(A)=A^{n}$. This shows that it is not unreasonable to make this definition. Similar reasoning yields

$$
\begin{equation*}
f(A)=I \text { if } f(\lambda)=\lambda^{0}=1 \tag{17.7}
\end{equation*}
$$

We may not be able to explicitly evaluate $f(A)$ in case $f(\lambda)$ is not analytic off $\sigma(A)$ but at least the definition is reasonable and seems to work as it should in the special case when $f(A)$ is known by another method, say a power series. These details about a power series are left to the reader.

With Theorem 17.2.5, one can prove the spectral mapping theorem.
Theorem 17.2.6 Suppose $\Omega \supseteq \sigma(A)$ where $\Omega$ is an open set and let $f$ be analytic on $\Omega$. Let $\Gamma$ be an oriented cycle, the union of oriented simple closed curves such that $n(\Gamma, z)=1$ if $z \in \sigma(A), n(\Gamma, z)$ is an integer if $z \in \Omega$, and $n(\Gamma, z)=0$ if $z \notin \Omega$. Then for

$$
f(A) \equiv \frac{1}{2 \pi i} \int_{\Gamma} f(\lambda)(\lambda I-A)^{-1} d \lambda
$$

it follows that $\sigma(f(A))=f(\sigma(A))$
Proof: Let $\mu \in \Omega . \mu \rightarrow \frac{f(\lambda)-f(\mu)}{\lambda-\mu}$ has a removable singularity at $\lambda$ so it is equal to an analytic function of $\mu$ called $g(\mu)$. Recall why this is. Since $f$ is analytic at $\lambda$, we know that for $\mu$ near $\lambda$

$$
f(\mu)=\sum_{k=0}^{\infty} a_{k}(\mu-\lambda)^{k}
$$

and so, when this is substituted into the difference quotient, one obtains the power series for an analytic function called $g(\mu)$.

Then $f(\lambda)-f(\mu)=g(\mu)(\lambda-\mu)$. From Theorem 17.2.5,

$$
\begin{aligned}
f(A)-f(\lambda) I & =\frac{1}{2 \pi i} \int_{\Gamma}(f(\mu)-f(\lambda))(\mu I-A)^{-1} d \mu \\
& =\frac{1}{2 \pi i} \int_{\Gamma} g(\mu)(\mu-\lambda)(\mu I-A)^{-1} d \mu \\
& =g(A)(A-\lambda I)
\end{aligned}
$$

If $\lambda \in \sigma(A)$ then either $(A-\lambda I)$ fails to be one to one or it fails to be onto. If it fails to be one to one, then $f(A)-f(\lambda) I$ also fails to be one to one and so $f(\lambda) \in \sigma(f(A))$. If $(A-\lambda I)$ fails to be onto, then $f(A)-f(\lambda) I$ also fails to be onto and so $f(\lambda) \in \sigma(f(A))$. In other words, $f(\sigma(A)) \subseteq \sigma(f(A))$.

Now suppose $v \in \sigma(f(A))$. Is $v=f(\lambda)$ for some $\lambda \in \sigma(A)$ ? If not, then $(f(\lambda)-v)^{-1}$ is analytic function of $\lambda \in \sigma(A)$ and by continuity, this must hold in an open set $\hat{\Omega}$ which contains $\sigma(A)$. Therefore, using this open set in the above considerations, it follows from Theorem 17.2.5 that for $\Gamma$ pertaining to this new open set as above,

$$
(f(A)-v)^{-1}(f(A)-v)=\frac{1}{2 \pi i} \int_{\Gamma}(f(\mu)-v)^{-1}(f(\mu)-v)(\mu I-A)^{-1} d \mu=I
$$

and so $v$ was not really in $\sigma(f(A))$ after all. Hence the equality holds.

### 17.3 Invariant Subspaces

This is where we need the machinery of Theorem 17.2.1. Up till now, we could have done most things by simply considering large circles containing $\sigma(A)$. Here the idea is to consider pieces of $\sigma(A)$. Let $\sigma(A)=\cup_{i=1}^{n} K_{i}$ where the $K_{i}$ are disjoint and compact. Let

$$
\delta_{i}=\min \left\{\operatorname{dist}\left(z, \cup_{j \neq i} K_{j}\right): z \in K_{i}\right\}>0
$$

and let $0<\boldsymbol{\delta}<\min \left\{\boldsymbol{\delta}_{i}, i=1, \cdots, n\right\}$. Then letting $U_{i} \equiv K_{i}+B(0, \delta)$, it follows that the $U_{i}$ are disjoint open sets. Let $\Gamma_{j}$ be a oriented cycle such that for $z \in K_{j}, n\left(\Gamma_{j}, z\right)=1$ and for $z \in K_{i}, i \neq j, n\left(\Gamma_{j}, z\right)=0$. and if $z \in \Gamma_{j}^{*}$, then $n\left(\Gamma_{i}, z\right)=0$. Let $\Gamma$ be the union of these oriented cycles. Thus $n(\Gamma, z)=0$ if $z \notin \Omega \equiv \cup_{i=1}^{n} U_{i}$ and $U_{i} \supseteq \Gamma_{i}^{*}$. Define

$$
f_{i}(\lambda) \equiv\left\{\begin{array}{l}
1 \text { on } U_{i} \\
0 \text { on } U_{j} \text { for } j \neq i
\end{array}\right.
$$

Thus $f_{i}$ is analytic on $\Omega$. Then $f_{i}(A) \equiv P_{i} \equiv \frac{1}{2 \pi i} \int_{\Gamma} f_{i}(\lambda)(\lambda I-A)^{-1} d \lambda$. By the spectral mapping theorem, Theorem 17.2.6,

$$
\begin{equation*}
\sigma\left(f_{i}(A)\right)=f_{i}(\sigma(A))=\{0,1\} \tag{17.8}
\end{equation*}
$$

Note that for $\lambda \in \rho(A), A(\lambda I-A)^{-1}=(\lambda I-A)^{-1} A$ as can be seen by multiplying both sides by $(\lambda I-A)$ and observing that the result is $A$ on both sides. Then since $(\lambda I-A)$ is one to one, the identity follows. Now let $P_{k} \in \mathscr{L}(X, X)$ be the linear transformation given by $P_{k}=\frac{1}{2 \pi i} \int_{\Gamma_{k}}(\lambda I-A)^{-1} d \lambda$.

From Lemma 17.1.1,

$$
\begin{align*}
& A P_{k}=A \frac{1}{2 \pi i} \int_{\Gamma_{k}}(\lambda I-A)^{-1} d \lambda=\frac{1}{2 \pi i} \int_{\Gamma_{k}} A(\lambda I-A)^{-1} d \lambda \\
= & \frac{1}{2 \pi i} \int_{\Gamma_{k}}(\lambda I-A)^{-1} A d \lambda=\frac{1}{2 \pi i}\left(\int_{\Gamma_{k}}(\lambda I-A)^{-1} d \lambda\right) A=P_{k} A \tag{17.9}
\end{align*}
$$

With these introductory observations, the following is the main result about invariant subspaces. First is some notation.

Definition 17.3.1 Let $X$ be a vector space and let $X_{k}$ be a subspace. Then $X=$ $\sum_{k=1}^{n} X_{k}$ means that every $x \in X$ can be written in the form $x=\sum_{k=1}^{n} x_{k}, x_{k} \in X_{k}$. We write $X=\bigoplus_{k=1}^{n} X_{k}$ if whenever $0=\sum_{k} x_{k}$, it follows that each $x_{k}=0$. In other words, we use the new notation when there is a unique way to write each vector in $X$ as a sum of vectors in the $X_{k}$. When this uniqueness holds, the sum is called a direct sum. In case $A X_{k} \subseteq X_{k}$, we say that $X_{k}$ is A invariant and $X_{k}$ is an invariant subspace.

Theorem 17.3.2 Let $\sigma(A)=\cup_{k=1}^{n} K_{k}$ where $K_{j} \cap K_{i}=\emptyset$, each $K_{j}$ being compact. There exist $P_{k} \in \mathscr{L}(X, X)$ for each $k=1, \cdots, n$ such that

1. $I=\sum_{k=1}^{n} P_{k}$
2. $P_{i} P_{j}=0$ if $i \neq j$
3. $P_{i}^{2}=P_{i}$ for each $i$
4. $X=\bigoplus_{k=1}^{n} X_{k}$ where $X_{k}=P_{k} X$ and each $X_{k}$ is a Banach space.
5. $A X_{k} \subseteq X_{k}$ which says that $X_{k}$ is $A$ invariant.
6. $P_{k} x=x$ if $x \in X_{k}$. If $x \in X_{j}$, then $P_{k} x=0$ if $k \neq j$.

Proof: Consider 1.

$$
\begin{aligned}
\sum_{k=1}^{n} P_{k} & =\frac{1}{2 \pi i} \int_{\Gamma} \sum_{k=1}^{n} f_{k}(\lambda)(\lambda I-A)^{-1} d \lambda \\
& =\frac{1}{2 \pi i} \int_{\Gamma} 1(\lambda-I)^{-1} d \lambda=I
\end{aligned}
$$

from 17.7. Consider 2. Let $\lambda$ be on $\Gamma_{i}$ and $\mu$ on $\Gamma_{j}$. Then from Theorem 17.2.5

$$
P_{i} P_{j}=\frac{1}{2 \pi i} \int_{\Gamma} f_{i}(\lambda) f_{j}(\lambda)(\lambda I-A)^{-1} d \lambda=\frac{1}{2 \pi i} \int_{\Gamma} 0 d \lambda=0
$$

Now from 1., $I=\sum_{i=1}^{n} P_{i}$ and so, multiplying both sides by $P_{i}, P_{i}=P_{i}^{2}$. This shows 3 .
Consider 4. Note that from 1. $X=\sum_{k=1}^{n} X_{k}$ where $X_{k} \equiv P_{k} X$. However, this is a direct sum because if $0=\sum_{k} P_{k} x_{k}$, then doing $P_{j}$ to both sides and using part $2 ., 0=P_{j}^{2} x_{j}=P_{j} x_{j}$ and so the summands are all 0 since $j$ is arbitrary. As to $X_{k}$ being a Banach space, suppose $P_{k} x_{n} \rightarrow y$. Is $y \in X_{k}$ ? $P_{k} x_{n}=P_{k}\left(P_{k} x_{n}\right)$ and so, by continuity of $P_{k}$, this converges to $P_{k} y \in X_{k}$. Thus $P_{k} x_{n} \rightarrow y$ and $P_{k} x_{n} \rightarrow P_{k} y$ so $y=P_{k} y$ and $y \in X_{k}$. Thus $X_{k}$ is a closed subspace of a Banach space and must therefore be a Banach space itself.
5. follows from 17.9. If $P_{k} x \in X_{k}$, then $A P_{k} x=P_{k} A x \in X_{k}$. Hence $A: X_{k} \rightarrow X_{k}$.

Finally consider 6. Suppose $x \in X_{k}$. Then $x=P_{k} y$. Then $P_{k} x=P_{k}^{2} y=P_{k} y=x$ so for $x \in X_{k}, P_{k} x=x$ and $P_{k}$ restricted to $X_{k}$ is just the identity. If $P_{j} x$ is a vector of $X_{j}$, and $k \neq j$, then $P_{k} P_{j} x=0 x=0$.

From the spectral mapping theorem, Theorem 17.2.6, $\sigma\left(P_{k}\right)=\sigma\left(f_{k}(A)\right)=\{0,1\}$ because $f_{k}(\lambda)$ has only these two values. Then the following is also obtained

Theorem 17.3.3 Let $n>1, \sigma(A)=\cup_{k=1}^{n} K_{k}$ where the $K_{k}$ are compact and disjoint. Let $P_{k}$ be the projection map defined above and $X_{k} \equiv P_{k} X$. Then define $A_{k} \equiv A P_{k}$. The following hold

1. $A_{k}: X_{k} \rightarrow X_{k}, A_{k} x=A x$ for all $x \in X_{k}$ so $A_{k}$ is just the restriction of $A$ to $X_{k}$.
2. $\sigma\left(A_{k}\right)=\left\{0, K_{k}\right\}$.
3. $A=\sum_{k=1}^{n} A_{k}$.
4. If we regard $A_{k}$ as a mapping $A: X_{k} \rightarrow X_{k}$, then $\sigma\left(A_{k}\right)=K_{k}$.

Proof: Letting $f_{k}(\lambda)$ be the function in the above theorem equal to 1 on $U_{k}$,

$$
g(\lambda) \equiv \lambda, A_{k}=\frac{1}{2 \pi i} \int_{\Gamma} f_{k}(\lambda) g(\lambda)(\lambda I-A)^{-1} d \lambda
$$

and so, by the spectral mapping theorem,

$$
\sigma\left(A_{k}\right)=\sigma\left(f_{k}(A) g(A)\right)=f_{k}(\sigma(A)) g(\sigma(A))=\left\{0, K_{k}\right\}
$$

because the possible values for $f_{k}(\lambda) g(\lambda)$ for $\lambda \in \sigma(A)$ are $\left\{0, K_{k}\right\}$. This shows 2. Part 1. is obvious from Theorem 17.3.2. So is Part 3. Consider the last claim about $A_{k}$.

If $\mu \notin K_{k}$ then in all of the above, $U_{k}$ could have excluded $\mu$. Assume this is the case. Thus $\lambda \rightarrow \frac{1}{\mu-\lambda}$ is analytic on $U_{k}$. Therefore, using Theorem 17.2.5 applied to the Banach space $X_{k}$,

$$
(\mu I-A)^{-1}=\frac{1}{2 \pi i} \int_{\Gamma_{k}} \frac{1}{\mu-\lambda}(\lambda I-A)^{-1} d \lambda
$$

and so $\mu \notin \sigma\left(A_{k}\right)$. Therefore, $K_{k} \supseteq \sigma\left(A_{k}\right)$. If $\mu I-A$ fails to be onto, then this must be the case for some $A_{k}$. Here is why. If $y \notin(\mu I-A)(X)$, then there is no $x$ such that $(\mu I-A) x=y$. However, $y=\sum_{k} P_{k} y$. If each $A_{k}$ is onto, then there would be $x_{k} \in X_{k}$ such that $\left(\mu I-A_{k}\right) x_{k}=y_{k}$. Recall that $P_{k} x_{k}=x_{k}$. Therefore, $\left(\mu P_{k}-A P_{k}\right) x_{k}=P_{k} y$, and also $(\mu I-A)\left(P_{k} x_{k}\right)=P_{k} y$. Summing this on $k,(\mu I-A) \sum_{k} x_{k}=\sum_{k} P_{k} y=y$ and it would be the case that $(\mu I-A)$ is onto. Thus if $(\mu I-A)$ is not onto, then $\mu I-A_{k}: X_{k} \rightarrow X_{k}$ is not onto for some $k$. If $\mu I-A$ fails to be one to one, then there exists $x \neq 0$ such that $(\mu I-A) x=0$. However, $x=\sum_{k} x_{k}$ where $x_{k} \in X_{k}$. Then, since $A_{k}$ is just the restriction of $A$ to $X_{k}$ and $P_{k}$ is the restriction of $I$ to $X_{k}, \sum_{k}\left(\mu P_{k}-A_{k}\right) x_{k}=0$ where $I$ refers to $X_{k}$. Now recall that this is a direct sum. Hence $\left(\mu P_{k}-A_{k}\right) x_{k}=\left(\mu I-A_{k}\right) x_{k}=0$. If each $\mu I-A_{k}$ is one to one, then each $x_{k}=0$ and so it follows that $x=0$ also, it being the sum of the $x_{k}$. It follows that $\sigma(A) \subseteq \cup_{k} \sigma\left(A_{k}\right)$ and so $\sigma(A) \subseteq \cup_{k} \sigma\left(A_{k}\right) \subseteq \cup_{k} K_{k}=\sigma(A), \sigma\left(A_{k}\right) \subseteq K_{k}$, and so you cannot have $\sigma\left(A_{k}\right) \varsubsetneqq K_{k}$, proper inclusion, for any $k$ since otherwise, the above could not hold.

It might be interesting to compare this with the algebraic approach to the same problem in Linear Algebra. That approach in Linear Algebra has the advantage of dealing with arbitrary fields of scalars and is based on polynomials, the division algorithm, and the minimum polynomial where this in this chapter is limited to the field of complex numbers. However, the approach in this chapter based on complex analysis applies to arbitrary Banach spaces whereas the algebraic methods only apply to finite dimensional spaces. Isn't it interesting how two totally different points of view lead to essentially the same result about a direct sum of invariant subspaces?

Another thing to note is that asside from having $\sigma(A)$ a compact set, it was not all that important to know that $A$ is a bounded linear operator. Everything was done in terms of the resolvent $(\lambda I-A)^{-1}$. This suggests that all of the above theory generalizes to unbounded closed operators of various kinds. A case of this is considered next.

### 17.4 Sectorial Operators and Analytic Semigroups

In solving ordinary differential equations, the main result involves the fundamental matrix $\Phi(t)$ where $\Phi^{\prime}(t)=A \Phi(t), \Phi(0)=I$, or $\Phi^{\prime}(t)+A \Phi(t)=0, \Phi(0)=I$ and the variation of constants formula. Recall that $\Phi(t+s)=\Phi(t) \Phi(s)$. This was discussed starting with Problem 18 on Page 400. This idea generalizes to the situation where $A$ is a closed densely defined operator defined on $D(A) \subseteq X$, a Banach space under some conditions which are sufficiently general to include what was done above with $A$ an $n \times n$ matrix as a special case. The identity $\Phi(t) \Phi(s)=\Phi(t+s)$ holds for any $t, s \in \mathbb{R}$ and so is called a group of transformations. However, in the more general case, the identity only holds for $t, s \geq 0$ which is why it is called a semigroup. In this more general setting, I will call it $S(t)$. I am mostly following the presentation in Henry [21] in this short introduction. In what follows $H$ will be a Banach space unless specified to be a Hilbert space. This new material differs in letting $A$ be only a closed densely defined operator. It might not be a bounded operator. Thus $A: D(A) \subseteq H \rightarrow H$ where $A$ is closed.

These semigroups are useful in considering various partial differential equations which can be considered just like they were ordinary differential equations in the form $u^{\prime}+A u=$ $f(u)$. The semigroups discussed here, when applied to actual examples, have the property of allowing one to begin with a very un-smooth initial condition, something in $H$, and making $S(t) x$ in $D(A)$ for all $t>0$. When applied to partial differential equations, this typically has the effect of making a solution $t \rightarrow S(t) x$ smoother for positive $t$ than the
initial condition.
One can show that $\lambda \rightarrow(\lambda I-A)^{-1}$ is analytic on its so called resolvent set defined as those $\lambda$ for which $(\lambda I-A)^{-1} \in \mathscr{L}(H, H)$. This follows from two things, the resolvant identity

$$
(\lambda I-A)^{-1}(\mu I-A)^{-1}=(\mu-\lambda)^{-1}\left((\lambda I-A)^{-1}-(\mu I-A)^{-1}\right)
$$

which follows from an observation that $(\mu I-A),(\lambda I-A)$ are onto so the identity holds if and only if

$$
(\lambda I-A)^{-1}(\mu I-A)^{-1}(\mu I-A)=(\mu-\lambda)^{-1}\left((\lambda I-A)^{-1}-(\mu I-A)^{-1}\right)(\mu I-A)
$$

if and only if

$$
\begin{aligned}
(\lambda I-A)^{-1} & =(\mu-\lambda)^{-1}\left((\lambda I-A)^{-1}(\mu I-A)-I\right) \\
& =(\mu-\lambda)^{-1}\left((\lambda I-A)^{-1}((\mu-\lambda) I+(\lambda I-A))-I\right)
\end{aligned}
$$

if and only if

$$
\begin{aligned}
(\mu-\lambda)(\lambda I-A)^{-1} & =(\lambda I-A)^{-1}((\mu-\lambda) I+(\lambda I-A))-I \\
& =(\mu-\lambda)(\lambda I-A)^{-1}+I-I
\end{aligned}
$$

and an assumption that for each $x \in H, \sup _{\lambda}\left\|(\lambda I-A)^{-1} x\right\|<\infty$ for all $\lambda$ near $\mu$ which by the uniform boundedness theorem implies $\left\|(\lambda I-A)^{-1}\right\|$ is bounded for $\lambda$ near $\mu$.

Thus I will always assume this resolvent $\lambda \rightarrow(\lambda I-A)^{-1}$ is analytic for $\lambda$ on its resolvent set. As to the resolvent set, the following describes it in this case of sectorial operators.

Definition 17.4.1 Let $\phi<\pi / 2$ and for $a \in \mathbb{R}$, let $S_{a \phi}$ denote the sector in the complex plane

$$
\{z \in \mathbb{C} \backslash\{a\}:|\arg (z-a)| \leq \pi-\phi\}
$$

This sector is as shown below.


A closed, densely defined linear operator $A$ is called sectorial if for some sector as described above, it follows that for all $\lambda \in S_{a \phi}$,

$$
\begin{equation*}
(\lambda I-A)^{-1} \in \mathscr{L}(H, H) \tag{17.10}
\end{equation*}
$$

and for some $M$

$$
\begin{equation*}
\left\|(\lambda I-A)^{-1}\right\| \leq \frac{M}{|\lambda-a|} \tag{17.11}
\end{equation*}
$$

The following perturbation theorem is very useful for sectorial operators. I won't use it here, but in applications of this theory, it is useful. First note that for $\lambda \in S_{a \phi}$,

$$
\begin{equation*}
A(\lambda I-A)^{-1}=-I+\lambda(\lambda I-A)^{-1} \tag{17.12}
\end{equation*}
$$

Also, if $x \in D(A)$,

$$
\begin{equation*}
(\lambda-A)^{-1} A x=-x+\lambda(\lambda I-A)^{-1} x \tag{17.13}
\end{equation*}
$$

This follows from algebra and noting that $\lambda I-A$ maps $D(A)$ onto $H$ because $(\lambda I-A)^{-1} \in$ $\mathscr{L}(H, H)$. Thus the above is true if and only if

$$
A=\left(-I+\lambda(\lambda I-A)^{-1}\right)(\lambda I-A)
$$

which is obviously true. 17.13 is similar. Thus from 17.12,

$$
\begin{equation*}
\left\|A(\lambda I-A)^{-1}\right\| \leq 1+|\lambda|\left\|(\lambda I-A)^{-1}\right\| \leq 1+|\lambda| \frac{M}{|\lambda-a|} \leq C \tag{17.14}
\end{equation*}
$$

for some constant $C$ whenever $|\lambda|$ is large enough and in $S_{a \phi}$.
Proposition 17.4.2 Suppose $A$ is a sectorial operator as defined above so it is a densely defined closed operator on $D(A) \subseteq H$ which satisfies

$$
\begin{equation*}
\left\|A(\lambda I-A)^{-1}\right\| \leq C \tag{17.15}
\end{equation*}
$$

whenever $|\lambda|, \lambda \in S_{a \phi}$, is sufficiently large and suppose $B$ is a densely defined closed operator such that $D(B) \supseteq D(A)$ and for all $x \in D(A)$,

$$
\begin{equation*}
\|B x\| \leq \varepsilon\|A x\|+K\|x\| \tag{17.16}
\end{equation*}
$$

for some $K$, where $\varepsilon C<1$. Then $A+B$ is also sectorial.
Proof: I need to consider $(\lambda I-(A+B))^{-1}$. This equals

$$
\begin{equation*}
\left(\left(I-B(\lambda I-A)^{-1}\right)(\lambda I-A)\right)^{-1} \tag{17.17}
\end{equation*}
$$

The issue is whether this makes any sense for all $\lambda \in S_{b \phi}$ for some $b \in \mathbb{R}$. Let $b>a$ be very large so that if $\lambda \in S_{b \phi}$, then 17.15 holds. Then from 17.16, it follows that for $\|x\| \leq 1$,

$$
\begin{aligned}
\left\|B(\lambda I-A)^{-1} x\right\| & \leq \varepsilon\left\|A(\lambda I-A)^{-1} x\right\|+K\left\|(\lambda I-A)^{-1} x\right\| \\
& \leq \varepsilon C+K /|\lambda-a|
\end{aligned}
$$

and so if $b$ is made sufficiently large and $\lambda \in S_{b \phi}$, then for all $\|x\| \leq 1$,

$$
\left\|B(\lambda I-A)^{-1} x\right\| \leq \varepsilon C+K /|\lambda-a|<r<1
$$

Therefore, for such $b,\left(I-B(\lambda I-A)^{-1}\right)^{-1}=\sum_{k=0}^{\infty}\left(B(\lambda I-A)^{-1}\right)^{k}$ exists and so for such $b$, the expression in 17.17 makes sense and equals $(\lambda I-A)^{-1}\left(I-B(\lambda I-A)^{-1}\right)^{-1}$ and furthermore,

$$
\left\|(\lambda I-A)^{-1}\left(I-B(\lambda I-A)^{-1}\right)^{-1}\right\| \leq \frac{M}{|\lambda-a|} \frac{1}{1-r} \leq \frac{M^{\prime}}{|\lambda-b|}
$$

by adjusting the constants because $\frac{M}{|\lambda-a|} \frac{|\lambda-b|}{1-r}$ is bounded for $\lambda \in S_{b \phi}$.
In finite dimensions, this kind of thing just shown always holds. There you have $D(A)$ is the whole space typically and $B$ will satisfy such an inequality in 17.16. The following example shows that all the bounded operators are sectorial.

Example 17.4.3 If $A \in \mathscr{L}(H, H)$, then $A$ is sectorial.
The spectrum $\sigma(A)$ is bounded by $\|A\|$ and so there is clearly a sector of the above form contained in the resolvent set of $A$. As to the estimate 17.11 , let $a$ be larger than $2\|A\|$ and let $S_{a \phi}$ be contained in the resolvent set. Then for $\lambda \in S_{a \phi},|\lambda|>2\|A\|$ and so

$$
\left\|(\lambda I-A)^{-1}\right\|=|\lambda|^{-1}\left\|\left(I-\frac{A}{\lambda}\right)^{-1}\right\| \leq|\lambda|^{-1}\left\|\sum_{k=0}^{\infty}\left(\frac{A}{\lambda}\right)^{k}\right\| \leq|\lambda|^{-1} 2
$$

Now for $\lambda \in S_{a \phi},\left|\frac{\lambda-a}{\lambda}\right| \leq M$ for some constant $M$ and so $\left\|(\lambda I-A)^{-1}\right\| \leq \frac{2 M}{|\lambda-a|}$ Thus this theory includes the case of ordinary differential equations in $\mathbb{R}^{p}$. You might note that both $A$ and $-A$ will be sectorial in this case.
Definition 17.4.4 For a sectorial operator as defined above, let the contour $\gamma$ be as shown next where the orientation is also as shown by the arrow, a being the center of the two circles in the second picture and the center of the small circle in the first.


The radius of the little circle is not important because the functions to be integrated will be analytic on $S_{a \phi}$. Now we will define a semigroup for $t$ in the open sector described in the following picture. The angle between the dotted lines and the solid lines emanating up and down from 0 will be a right angle. Thus the angle between the horizontal $x$ axis and either of the dotted lines is $\pi / 2-\phi$. The interest will be in $t$ in the sector $S_{r}$ between the two solid lines.

$$
S_{r} \equiv\left\{t: 0 \leq|\arg (t)| \leq r<\frac{\pi}{2}-\phi\right\}
$$



Definition 17.4.5 ${ }_{\text {For } t} \in S_{r}$ define

$$
\begin{equation*}
S(t) \equiv \frac{1}{2 \pi i} \int_{\gamma} e^{\lambda t}(\lambda I-A)^{-1} d \lambda \tag{17.18}
\end{equation*}
$$

Also define $S(0) \equiv I$.
Lemma 17.4.6 For $\Gamma_{R}$ described above, and $\operatorname{Re}(t)>0, \arg (t)<\frac{\pi}{2}-\phi$, and $f(\boldsymbol{\lambda})$ bounded, $|f(\lambda)| \leq M$, and continuous,

$$
\lim _{R \rightarrow \infty} \int_{\Gamma_{R}} e^{\lambda t} f(\lambda) d \lambda=0
$$

Proof: On $\Gamma_{R}, \lambda=a+R e^{i \theta}, \theta \in[\pi-\phi, \pi+\phi]$ and the integrand is dominated by

$$
M e^{\operatorname{Re}(t \lambda)}=M e^{\operatorname{Re} t a} e^{\operatorname{Re}(t(R \cos (\theta)+i \sin (\theta)))} \leq M e^{|t| a} e^{|t| R \cos (\pi-\phi)}=M e^{|t| a} e^{-R|t| \sin (\phi)}
$$

and so $\left|\int_{\Gamma_{R}} e^{\lambda t} f(\lambda) d \lambda\right|<M e^{|t| a} e^{-R|t| \sin (\phi)} \pi R$ which clearly converges to 0 .
Lemma 17.4.7 For $t \in S_{r}$, and $\lambda$ on the either of the straight sides of $\gamma$, there exists $\delta_{r}>0$ such that

$$
\begin{equation*}
\left|e^{\lambda t}\right| \leq\left|e^{a t}\right| e^{-\delta_{r}|\lambda-a||t|} \tag{17.19}
\end{equation*}
$$

Proof: On either of these lines, $\lambda=y w+a$ where $|w|=1, y=|\lambda-a|$ and $\arg (w)$ is either $\pi-\phi$ or $\frac{3 \pi}{2}-\phi$. Therefore,

$$
\left|e^{\lambda t}\right|=\left|e^{a t}\right| e^{\operatorname{Re}(t y w)}=\left|e^{a t}\right| e^{|y||t| \cos (\arg (t w))}=\left|e^{a t}\right| e^{|y||t| \cos (\arg (t)+\arg (w))}
$$

Now $\arg (t w)=\arg (t)+\arg (w) \in(\pi-\phi)+[-r, r] \in\left(\frac{\pi}{2}, \frac{3 \pi}{2}\right)$ on the top line and $\arg (t w) \in$ $\frac{3 \pi}{2}-\phi+[-r, r] \in\left(\frac{\pi}{2}, \frac{3 \pi}{2}\right)$ on the bottom line. Either way, there exists $\delta_{r}>0$ such that $\cos (\arg (t w)) \leq-\delta_{r}$. This shows 17.19.

Lemma 17.4.8 Let $f(\lambda), A f(y)$ be bounded, $|f(\lambda)|,|A f(\lambda)|<M$ and continuous on $\gamma_{\varepsilon, \phi}^{*}$ and have values in $D(A)$. Then $A \int_{\gamma_{\varepsilon, \phi}} e^{\lambda t} f(\lambda) d \lambda=\int_{\gamma_{\varepsilon, \phi}} e^{\lambda t} A f(\lambda) d \lambda$ if $t \in S_{0(\phi+\pi / 2)}^{0}$.

Proof: Now consider $\int_{\gamma} e^{\lambda t} g(\lambda) d \lambda$ in case $g$ is bounded by $M$ and continuous. Does it exist?

On one of the straight lines making up the contour, we have $\lambda=a+y w$ where $|w|=$ $1, y \geq 0$. Then by Lemma 17.4.7, $\left|e^{\lambda t}\right| \leq\left|e^{a t}\right| e^{-\delta_{r}|\lambda-a||t|}$ so by the dominated convergence theorem, the integrals over the straight lines exist. There is no difficulty in integrating over the small circular part of the following contour where $a$ is the center of both circles.


Let $\gamma_{R}$ denote the part of $\gamma$ which extends only till the large circular part in the above picture which consists of $\Gamma_{R}$. Then using Riemann sums to approximate the integrals, and the fact that $A$ is closed, one can write

$$
A\left(\int_{\gamma_{\varepsilon, \phi, R}} e^{\lambda t} f(\lambda) d \lambda+\int_{\Gamma_{R}} e^{\lambda t} f(\lambda) d \lambda\right)=\left(\int_{\gamma_{\varepsilon, \phi, R}} e^{\lambda t} A f(\lambda) d \lambda+\int_{\Gamma_{R}} e^{\lambda t} A f(\lambda) d \lambda\right)
$$

and now, passing to a limit as $R \rightarrow \infty$ and using Lemma 17.4.6 one obtains the conclusion of the lemma.

Because of this lemma, I will move $A$ into and out of the integrals which occur in what follows. Also, as just shown, it is possible to approximate contour integrals over $\gamma$ with closed contours and use the Cauchy integral formula.

Next is consideration of the above definition along with estimates.
Lemma 17.4.9 The above Definition 17.4 .5 is well defined for $t \in S_{r}$. Also there is a constant $M_{r}$ such that

$$
\begin{equation*}
\|S(t)\| \leq M_{r}\left|e^{a t}\right| \tag{17.20}
\end{equation*}
$$

for every $t \in S_{r}$ such that $|\arg t| \leq r<\left(\frac{\pi}{2}-\phi\right)$. If $S_{r}$ is the sector $j u s t$ described, $t$ such that $|\arg t| \leq r<\left(\frac{\pi}{2}-\phi\right)$, then for any $x \in H$,

$$
\begin{equation*}
\lim _{t \rightarrow 0, t \in S_{r}} S(t) x=x \tag{17.21}
\end{equation*}
$$

Also, for $|\arg t| \leq r<\left(\frac{\pi}{2}-\phi\right)$

$$
\begin{equation*}
\|A S(t)\| \leq M_{r}\left|e^{a t}\right| \frac{1}{|t|}+N_{r}\left|e^{a t}\right||a| \tag{17.22}
\end{equation*}
$$

Proof: In the definition of $S(t), S(t) \equiv \frac{1}{2 \pi i} \int_{\gamma_{\varepsilon, \phi}} e^{\lambda t}(\lambda I-A)^{-1} d \lambda$. Since $S(t)$ does not depend on $\varepsilon>0$, we can take $\varepsilon=1 /|t|$. Then the circular part of the contour is $\lambda=$ $a+\frac{1}{|t|} e^{i \theta}, d \lambda=\frac{i}{|t|} e^{i \theta} d \theta$. Then $e^{\lambda t}=e^{\left(a+\frac{1}{|t|} e^{i \theta}\right)\left(|t| \mid\left(e^{i \arg t}\right)\right)}=e^{a t} e^{i(\theta+\arg (t))}$. Then on the circle which is part of $\gamma$ the contour integral equals

$$
\begin{array}{r}
\frac{1}{2 \pi} \int_{\phi-\pi}^{\pi-\phi} e^{a t} e^{i(\theta+\arg (t))}\left(\left(a+\frac{1}{|t|} e^{i \theta}\right) I-A\right)^{-1} \frac{1}{|t|} e^{i \theta} d \theta \\
\leq \frac{e}{2 \pi} \int_{\phi-\pi}^{\pi-\phi}\left|e^{a t}\right| \frac{M}{\left|\frac{1}{|t|} e^{i \theta}\right|} \frac{1}{|t|} d \theta=\frac{M e}{2 \pi} \int_{\phi-\pi}^{\pi-\phi}\left|e^{a t}\right| d \theta<M e\left|e^{a t}\right| \equiv \hat{M}\left|e^{a t}\right| \tag{17.23}
\end{array}
$$

where $\hat{M}$ does not depend on $t$. The estimate 17.11 was used to obtain the inequality. What about the rest of the contour defining $S(t)$ ? Letting $\arg (w)$ be chosen as either $\pi-\phi$ or $\frac{3 \pi}{2}-\phi, \lambda=y w+a$ where $y=|\lambda-a|$. Then on either straight segment we have

$$
\frac{1}{2 \pi i} \int_{1 /|t|}^{\infty} e^{(y w+a) t}((y w+a) I-A)^{-1} w d y,|w|=1
$$

and the bottom is similar. Thus, from Lemma 17.4.7 and the resolvent estimate 17.11 we can dominate these two by an expression of the form

$$
\begin{aligned}
\frac{1}{\pi} \int_{1 /|t|}^{\infty}\left|e^{a t}\right| \frac{M}{y} e^{-\delta_{r} y|t|} d y & =\frac{M}{\pi}\left|e^{a t}\right| \frac{1}{|t|} \int_{1}^{\infty} e^{-\delta_{r} u}|t| d u \\
& =\frac{M}{\pi}\left|e^{a t}\right| \int_{1}^{\infty} e^{-\delta_{r} u} d u \leq \frac{M}{\pi} \frac{1}{\delta_{r}}\left|e^{a t}\right|
\end{aligned}
$$

Taking $u=y|t|$. This together with 17.23 gives 17.20. In particular, $\|S(t)\| e^{-a t}$ is bounded for $t \in[0, \infty)$. As noted in Lemma 17.4.8. It was important that $|\arg t| \leq r<\left(\frac{\pi}{2}-\phi\right)$.

Now let $x \in D(A)$. From 17.13,

$$
\begin{equation*}
\frac{e^{\lambda t}}{\lambda}(\lambda-A)^{-1} A x+\frac{e^{\lambda t}}{\lambda} x=e^{\lambda t}(\lambda I-A)^{-1} x \tag{17.24}
\end{equation*}
$$

On the circular part of the contour, $\lambda=a+\frac{1}{|t|} e^{i \theta}$. Consider the first term on the left in the above equation. The contour integral is of the form

$$
\int_{\phi-\pi}^{\pi-\phi} e^{a t} e^{e^{i(\theta+\arg (t))}} \frac{1}{a+\frac{1}{|t|} e^{i \theta}}\left(\left(a+\frac{1}{|t|} e^{i \theta}\right) I-A\right)^{-1} A x \frac{i}{|t|} e^{i \theta} d \theta
$$

which is dominated by

$$
\begin{aligned}
e\left|e^{a t}\right| \int_{\phi-\pi}^{\pi-\phi} \frac{1}{\left|a+\frac{1}{|t|} e^{i \theta}\right|} \frac{M}{\left|\frac{1}{|t|} e^{i \theta}\right|}\|A x\| \frac{1}{|t|} & \leq e^{a t} \hat{M}\|A x\| \int_{\phi-\pi}^{\pi-\phi} \frac{|t|}{|a| t\left|+e^{i \theta}\right|} d \theta \\
& \leq e^{a t} \hat{M}\|A x\| \int_{\phi-\pi}^{\pi-\phi} \frac{|t|}{1-|a||t|} d \theta
\end{aligned}
$$

which converges to 0 as $t \rightarrow 0$. On the other part of the contour, $\lambda=y w+a$ where $|w|=1$, $\arg (w)=\pi-\phi, y>1 /|t|$, either of the straight segments are of the form

$$
\frac{e^{a t}}{2 \pi i} \int_{1 /|t|}^{\infty} e^{y w t} \frac{1}{y w+a}((y w+a) I-A)^{-1} w A x d y
$$

and from Lemma 17.4.7 these two sides are dominated by

$$
\frac{\left|e^{a t}\right|}{\pi} \int_{1 /|t|}^{\infty} e^{-\delta_{r} y|t|} \frac{1}{|y w+a|} \frac{M}{y} d y\|A x\|
$$

Now letting $u=|t| y$ this equals

$$
\frac{\left|e^{a t}\right|}{\pi} \int_{1}^{\infty} e^{-\delta_{r} u} \frac{1}{\left|\frac{u}{|t|} w+a\right|} \frac{|t| M}{u} \frac{1}{|t|} d u\|A x\|=\frac{\left|e^{a t}\right|}{\pi} \int_{1}^{\infty} e^{-\delta_{r} u} \frac{|t|}{|u w+|t| a|} \frac{M}{u} d u\|A x\|
$$

Which converges to 0 as $t \rightarrow 0$ in the sector $|\arg t| \leq r<\left(\frac{\pi}{2}-\phi\right)$. Thus from 17.24

$$
\begin{equation*}
S(t) x=\varepsilon(t)+\frac{1}{2 \pi i} \int_{\gamma} \frac{e^{\lambda t}}{\lambda} x d \lambda, \lim _{t \rightarrow 0+} \varepsilon(t)=0 \tag{17.25}
\end{equation*}
$$

Now approximate $\gamma$ with a closed contour having a large circular arc of radius $R$ such that the resulting bounded contour $\gamma_{R}$ has 0 on its inside and

$$
\|\frac{1}{2 \pi i} \int_{\gamma} \frac{e^{\lambda t}}{\lambda} x d \lambda-\overbrace{\frac{1}{2 \pi i} \int_{\gamma_{R}} \frac{e^{\lambda t}}{\lambda} x d \lambda}^{=x}\|<\eta(R)
$$

where $\lim _{R \rightarrow \infty} \eta(R)=0$. This is possible from Lemma 17.4.6. By the Cauchy integral formula, $\frac{1}{2 \pi i} \int_{\gamma_{R}} \frac{e^{\lambda t}}{\lambda} x d \lambda=x$ and so, from this, the above, and 17.25,

$$
\|S(t) x-x\| \leq\left\|\varepsilon(t)+\frac{1}{2 \pi i} \int_{\gamma} \frac{e^{\lambda t}}{\lambda} x d \lambda-x\right\| \leq\|\varepsilon(t)\|+\eta(R)
$$

Let $R \rightarrow \infty$ and then it follows $\lim _{t \rightarrow \infty}|S(t) x-x|=0$. By the first part, $\|S(t)\|$ is bounded for small $t$ in that $S_{r}$ so it follows that, since $D(A)$ is dense, then for any $x \in H$, It follows that $\lim _{t \rightarrow 0, t \in S_{r}} S(t) x=x$ where $t$ is in the sector $S_{r}$ given by $|\arg t| \leq r<\left(\frac{\pi}{2}-\phi\right)$.

Now for $|\arg t| \leq r<\left(\frac{\pi}{2}-\phi\right), A S(t)=\frac{1}{2 \pi i} \int_{\gamma_{\varepsilon, \phi}} e^{\lambda t} A(\lambda I-A)^{-1} d \lambda$. From 17.12 this is

$$
\frac{1}{2 \pi i} \int_{\gamma} e^{\lambda t}\left(-I+\lambda(\lambda I-A)^{-1}\right) d \lambda
$$

On the circle, $\lambda=a+\frac{1}{|t|} e^{i \theta}$ and as above, this is

$$
\int_{\phi-\pi}^{\pi-\phi} e^{a t} e^{e^{i(\theta+\arg (t))}}\left(-I+\left(a+\frac{1}{|t|} e^{i \theta}\right)\left(\left(a+\frac{1}{|t|} e^{i \theta}\right) I-A\right)^{-1}\right) \frac{i}{|t|} e^{i \theta} d \theta
$$

and by the estimates and letting $M>1$, this is dominated by

$$
\begin{align*}
& e\left|e^{a t}\right| \int_{\phi-\pi}^{\pi-\phi}\left(1+M \frac{\left|a+\frac{1}{|t|} e^{i \theta}\right|}{1 /|t|}\right) \frac{1}{|t|} d \theta \leq e\left|e^{a t}\right| M \int_{\phi-\pi}^{\pi-\phi}\left(1+|a| t\left|+e^{i \theta \mid}\right|\right) \frac{1}{|t|} d \theta \\
& \left.\quad \leq e\left|e^{a t \mid}\right| M \int_{\phi-\pi}^{\pi-\phi}(2+a|t|) \frac{1}{|t|} d \theta \leq e\left|e^{a t}\right| M \frac{1}{|t|} 4 \pi+M 2 \pi e\left|e^{a t \mid}\right| a \right\rvert\, \tag{17.26}
\end{align*}
$$

Now consider one of the straight lines. On either of these $\lambda=a+w y$ where $|w|=1$ and $y \geq 1 /|t|$. Then the contour integral is

$$
\frac{e^{a t}}{2 \pi i} \int_{1 /|t|}^{\infty} e^{y w t}\left(-I+(a+w y)((a+w y) I-A)^{-1}\right) w d y
$$

As earlier, the norm of this is dominated by $\frac{\left|e^{a t}\right|}{2 \pi} \int_{1 /|t|}^{\infty} e^{-y|t| \delta_{r}}\left(1+M \frac{|a+w y|}{|w y|}\right) d y=$

$$
\begin{gathered}
=\frac{\left|e^{a t}\right|}{2 \pi} \int_{1}^{\infty} e^{-x c(r)}\left(1+M \frac{|a+w(x /|t|)|}{|w(x /|t|)|}\right) \frac{1}{|t|} d x \\
=\frac{\left|e^{a t}\right|}{2 \pi} \int_{1}^{\infty} e^{-x \delta_{r}}\left(1+M \frac{|a| t|+w x|}{|x|}\right) \frac{1}{|t|} d x \leq \frac{\left|e^{a t}\right|}{2 \pi}\left(M_{r} \frac{1}{|t|}\right)+N_{r}|a| \frac{\left|e^{a t}\right|}{2 \pi}
\end{gathered}
$$

Combining this with 17.26 and adjusting constants, $\|A S(t)\| \leq M_{r}\left|e^{a t}\right| \frac{1}{|t|}+N_{r}\left|e^{a t}\right||a|$
Also note that if the contour is shifted to the right slightly, the integral over the shifted contour, $\gamma^{\prime}$ coincides with the integral over $\gamma$ thanks to the Cauchy integral formula and Lemmas 17.4.8, 17.4.6which allows the approximation of the above integrals with one on a closed contour. The following is the main result. Recall that the radius of the small circle in $\gamma$ is not important.
Theorem 17.4.10 Let A be a sectorial operator as defined in Definition 17.4. 1 for the sector $S_{a, \phi}$. Then there exists a semigroup $S(t)$ for $t \in|\arg z| \leq r<\left(\frac{\pi}{2}-\phi\right)$ which satisfies the following conditions.

1. Then $S(t)$ given above in 17.18 is analytic for $t \in S_{r}$.
2. For any $x \in H$ and $t \in S_{r}$, then for $n$ a positive integer, $S^{(n)}(t) x=A^{n} S(t) x$
3. $S$ is a semigroup on the open sector, $S_{r}$. That is, for all $t, s \in S_{r}$,

$$
S(t+s)=S(t) S(s)
$$

4. $\lim _{t \rightarrow 0, t \in S_{r}} S(t) x=x$ for all $x \in H$ where $|\arg t| \leq r<\left(\frac{\pi}{2}-\phi\right)$
5. For some constants $M, N$, ift is positive and real, then it follows that $\|S(t)\| \leq M e^{a t}$, $\|A S(t)\| \leq M e^{a t} \frac{1}{|t|}+N\left|e^{a t}\right||a|$

Proof: Consider the first claim. This follows right away from the formula: $S(t) \equiv$ $\frac{1}{2 \pi i} \int_{\gamma} e^{\lambda t}(\lambda I-A)^{-1} d \lambda$. One can differentiate under the integral sign using the dominated convergence theorem and estimates from Lemma 17.4.8 to obtain

$$
\begin{gathered}
S^{\prime}(t) \equiv \frac{1}{2 \pi i} \int_{\gamma} \lambda e^{\lambda t}(\lambda I-A)^{-1} d \lambda=\frac{1}{2 \pi i} \int_{\gamma} e^{\lambda t}\left(I+A(\lambda I-A)^{-1}\right) d \lambda \\
=\frac{1}{2 \pi i} \int_{\gamma} e^{\lambda t} A(\lambda I-A)^{-1} d \lambda
\end{gathered}
$$

because of Lemma 17.4.8 the Cauchy integral theorem, and approximating $\gamma_{\varepsilon, \phi}$ with closed contours.

Now from Lemma 17.4.8 one can take $A$ out of the integral and


$$
S^{\prime}(t)=A\left(\frac{1}{2 \pi i} \int_{\gamma} e^{\lambda t}(\lambda I-A)^{-1} d \lambda\right)=A S(t)
$$

To get the higher derivatives, note $S(t)$ has infinitely many derivatives due to $t$ being a complex variable. Therefore,

$$
S^{\prime \prime}(t)=\lim _{h \rightarrow 0} \frac{S^{\prime}(t+h)-S^{\prime}(t)}{h}=\lim _{h \rightarrow 0} A \frac{S(t+h)-S(t)}{h}
$$

and $\frac{S(t+h)-S(t)}{h} \rightarrow A S(t)$ and so since $A$ is closed, $A S(t) \in D(A)$ and the above becomes $A^{2} S(t)$. Continuing this way yields the claims 1.) and 2.). Note this also implies $S(t) x \in$ $D(A)$ for each $t \in S_{r}$ which says more than $S(t) x \in H$. In practice this has the effect of regularizing the solution to an initial value problem.

Next consider the semigroup property. Let $s, t \in S_{r}$. As described above let $\gamma^{\prime}$ denote the contour shifted slightly to the right. Then

$$
\begin{equation*}
S(t) S(s)=\left(\frac{1}{2 \pi i}\right)^{2} \int_{\gamma} \int_{\gamma^{\prime}} e^{\lambda t}(\lambda I-A)^{-1} e^{\mu s}(\mu I-A)^{-1} d \mu d \lambda \tag{17.27}
\end{equation*}
$$

Using the resolvent identity,

$$
(\lambda I-A)^{-1}(\mu I-A)^{-1}=(\mu-\lambda)^{-1}\left((\lambda I-A)^{-1}-(\mu I-A)^{-1}\right)
$$

then substituting this resolvent identity in 17.27, it equals

$$
\left(\frac{1}{2 \pi i}\right)^{2} \int_{\gamma} \int_{\gamma^{\prime}} e^{\mu s} e^{\lambda t}\left((\mu-\lambda)^{-1}\left((\lambda I-A)^{-1}-(\mu I-A)^{-1}\right)\right) d \mu d \lambda
$$

$$
\begin{aligned}
= & -\left(\frac{1}{2 \pi i}\right)^{2} \int_{\gamma} e^{\lambda t} \int_{\gamma^{\prime}} e^{\mu s}(\mu-\lambda)^{-1}(\mu I-A)^{-1} d \mu d \lambda \\
& +\left(\frac{1}{2 \pi i}\right)^{2} \int_{\gamma} \int_{\gamma^{\prime}} e^{\mu s} e^{\lambda t}(\mu-\lambda)^{-1}(\lambda I-A)^{-1} d \mu d \lambda
\end{aligned}
$$

The order of integration can be interchanged because of the absolute convergence and Fubini's theorem. Then this reduces to

$$
\begin{aligned}
= & -\left(\frac{1}{2 \pi i}\right)^{2} \int_{\gamma^{\prime}}(\mu I-A)^{-1} e^{\mu s} \int_{\gamma} e^{\lambda t}(\mu-\lambda)^{-1} d \lambda d \mu \\
& +\left(\frac{1}{2 \pi i}\right)^{2} \int_{\gamma}(\lambda I-A)^{-1} e^{\lambda t} \int_{\gamma^{\prime}} e^{\mu s}(\mu-\lambda)^{-1} d \mu d \lambda
\end{aligned}
$$

Now the following diagram might help in drawing some interesting conclusions.


The first iterated integral equals 0 . This can be seen from the above picture. By Lemma 17.4.8, the inner integral taken over $\gamma$ is essentially equal to the integral over the closed contour in the above picture provided the radius of the part of the large circle in the above closed contour is large enough. This closed contour integral equals 0 by the Cauchy integral theorem. Passing to a limit, the integral on $\gamma$ is 0 . The second iterated integral equals

$$
\frac{1}{2 \pi i} \int_{\gamma}(\lambda I-A)^{-1} e^{\lambda t} e^{\lambda s} d \lambda=S(t+s)
$$

from the Cauchy integral formula. This verifies the semigroup identity.
4.) is done in Lemma 17.4.9 which also includes 5 .) when you let $t$ be positive and real.

### 17.5 The Numerical Range

In Hilbert space, there is a useful easy to check criterion which implies an operator is sectorial.

Definition 17.5.1 Let A be a closed densely defined operator $A: D(A) \rightarrow H$ for $H$ a Hilbert space. The numerical range is the following set.

$$
\{(A u, u): u \in D(A)\}
$$

Also recall the resolvent set $\rho(A)$, those $\lambda \in \mathbb{C}$ such that $(\lambda I-A)^{-1} \in \mathscr{L}(H, H)$. Thus, to be in this set $\lambda I-A$ is one to one and onto with continuous inverse.

Proposition 17.5.2 Suppose the numerical range of $A$, a closed densely defined operator $A: D(A) \rightarrow H$ for $H$ a Hilbert space is contained in the set

$$
\{z \in \mathbb{C}:|\arg (z)| \geq \pi-\phi\}
$$

where $0<\phi<\pi / 2$ and suppose $A^{-1} \in \mathscr{L}(H, H),(0 \in \rho(A))$. Then $A$ is sectorial with the sector

$$
S_{0, \phi^{\prime}} \equiv\left\{\lambda \neq 0:|\arg (\lambda)| \leq \pi-\phi^{\prime}\right\}
$$

where $\pi / 2>\phi^{\prime}>\phi$. Here $\arg (z)$ is the angle which is between $-\pi$ and $\pi$.
Proof: Here is a picture of the situation along with details used to motivate the proof.


In the picture the angle which is a little larger than $\phi$ is $\phi^{\prime}$. Let $\lambda$ be as shown with $|\arg \lambda| \leq \pi-\phi^{\prime}$. Then from the picture and trigonometry, if $u \in D(A)$,

$$
|\lambda| \sin \left(\phi^{\prime}-\phi\right)<\left|\lambda-\left(A \frac{u}{|u|}, \frac{u}{|u|}\right)\right|
$$

and so $|u||\lambda| \sin \left(\phi^{\prime}-\phi\right)<\left|\left(\lambda u-A u, \frac{u}{|u|}\right)\right| \leq\|(\lambda I-A) u\|$. Hence for all $\lambda$ such that $|\arg \lambda| \leq \pi-\phi^{\prime}$ and $u \in D(A)$,

$$
|u|<\left(\frac{1}{\sin \left(\phi^{\prime}-\phi\right)}\right) \frac{1}{|\lambda|}|(\lambda I-A) u| \equiv \frac{M}{|\lambda|}|(\lambda I-A) u|
$$

Thus $(\lambda I-A)$ is one to one on $S_{0, \phi^{\prime}}$ and if $\lambda \in \rho(A)$, then

$$
\left\|(\lambda I-A)^{-1}\right\|<\frac{M}{|\lambda|}
$$

By assumption $0 \in \rho(A)$ so $A$ is onto and $A^{-1}$ exists. Now if $|\mu|$ is small, $(\mu I-A)^{-1}$ must exist because it equals $\left(\left(\mu A^{-1}-I\right) A\right)^{-1}$ and for $|\mu|<\left\|A^{-1}\right\|$,

$$
\left(\mu A^{-1}-I\right)^{-1} \in \mathscr{L}(H, H)
$$

since the infinite series

$$
\sum_{k=0}^{\infty}(-1)^{k}\left(\mu A^{-1}\right)^{k}
$$

converges and must equal to $\left(\mu A^{-1}-I\right)^{-1}$. Therefore, there exists $\mu \in S_{0, \phi^{\prime}}$ such that $\mu \neq 0$ and $\mu \in \rho(A)$. Also if $\mu \neq 0$ and $\mu \in S_{0, \phi^{\prime}}$, then if $|\lambda-\mu|<\frac{|\mu|}{M},(\lambda I-A)^{-1}$ must exist because

$$
(\lambda I-A)^{-1}=\left[\left((\lambda-\mu)(\mu I-A)^{-1}-I\right)(\mu I-A)\right]^{-1}
$$

where $\left((\lambda-\mu)(\mu I-A)^{-1}-I\right)^{-1}$ exists because

$$
\left\|(\lambda-\mu)(\mu I-A)^{-1}\right\|=|\lambda-\mu|\left\|(\mu I-A)^{-1}\right\|<\frac{|\mu|}{M} \cdot \frac{M}{|\mu|}=1
$$

It follows that if $S \equiv\left\{\lambda \in S_{0, \phi^{\prime}}: \lambda \in \rho(A)\right\}$, then $S$ is open in $S_{0, \phi}$. However, $S$ is also closed because if $\lambda=\lim _{n \rightarrow \infty} \lambda_{n}$ where $\lambda_{n} \in S$, then if $\lambda=0$, it is given $\lambda \in S$. If $\lambda \neq 0$, then for large enough $n,\left|\lambda-\lambda_{n}\right|<\frac{\left|\lambda_{n}\right|}{M}$ and so $\lambda \in S$. Since $S_{0, \phi^{\prime}}$ is connected, it follows $S=S_{0, \phi^{\prime}}$.

Corollary 17.5.3 If for some $a \in \mathbb{R}$, the numerical values of $-a I+A$ are in the set $\{\lambda:|\lambda| \geq \pi-\phi\}$ where $0<\phi<\pi / 2$, and $a \in \rho(A)$ then $A$ is sectorial.

Proof: By assumption, $0 \in \rho(-a I+A)$ and also from Proposition 17.5.2, for $\mu \in S_{0, \phi^{\prime}}$ where $\pi / 2>\phi^{\prime}>\phi$,

$$
((-a I+A)-\mu I)^{-1} \in \mathscr{L}(H, H),\left\|((-a I+A)-\mu I)^{-1}\right\| \leq \frac{M}{|\mu|}
$$

Therefore, for $\mu \in S_{0, \phi^{\prime}}, \mu+a \in \rho(A)$. Therefore, if $\lambda \in S_{a, \phi^{\prime}}, \lambda-a \in S_{0, \phi^{\prime}}$

$$
\left\|(A-\lambda I)^{-1}\right\|=\left\|(A-a I-(\lambda-a) I)^{-1}\right\| \leq \frac{M}{|\lambda-a|}
$$

Can you consider fractional powers of sectorial operators? See Henry [21] for more on these topics along with fractional powers of these operators. It turns out to be useful in defining intermediate Banach spaces.

## Appendix A

## Green's Theorem for a Jordan Curve

This chapter contains a more general version of Green's theorem which applies to a Jordan Curve which has finite length along with its inside. This is more general than what was done earlier because you just start with a Jordan curve having finite length then the inside is automatically a region for which Green's theorem applies. If you want, you could generalize all those theorems in the chapters on complex analysis where it is assumed Green's theorem holds. See Theorem 8.7.21 presented earlier on the Jordan Curve Theorem.

The top and bottom points of $J$ determine two simple curves joined at these two points. Consider the horizontal line $y=\frac{y_{1}+y_{2}}{2}$ in moving from left to right, $J_{l}$ will be the first of these two simple curves encountered. As in the proof of the Jordan curve theorem, the last one encountered must be the other simple curve denoted as $J_{r}$. Otherwise, as explained there in the proof of Theorem 8.7.21 the two simple curves would have to intersect at points other than $\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$. Now as in the proof of the Jordan curve theorem, consider the horizontal line $y=\frac{y_{1}+y_{2}}{2}$. In moving along this line from left to right, let the first point of $J_{r}$ be $\beta$ and let the last point of $J_{l}$ encountered before $\beta$ be $\alpha$. As in the proof of the Jordan curve theorem, the open segment is contained on the inside component of $J^{C}$. Thus there are now two simple curves joined at $\{\alpha, \beta\}$, namely the one which goes from $\alpha$ to $\beta$ to $\left(x_{1}, y_{1}\right)$ and back to $\alpha$, also specifying an orientation, and the one which goes from $\beta$ to $\alpha$ to $\left(x_{2}, y_{2}\right)$ and back to $\beta$. Each has height at least $\left(y_{1}-y_{2}\right) / 2$ and the insides of the two new simple closed curves also have height at least $\left(y_{1}-y_{2}\right) / 2$.


Lemma A.0.1 Let J be a simple closed rectifiable curve. Also let $\delta>0$ be given such that $2 \delta$ is smaller than both the height and width of $J$. Then there exist finitely many non overlapping regions $\left\{R_{k}\right\}_{k=1}^{n}$ consisting of simple closed rectifiable curves along with their insides whose union equals $U_{i} \cup J$. These regions consist of two kinds, those contained in $U_{i}$ and those with nonempty intersection with $J$. These latter regions are called "border" regions. The boundary of a border region consists of straight line segments parallel to the coordinate axes which are of the form $x=m\left(\frac{\delta}{4}\right)$ or $y=k\left(\frac{\delta}{4}\right)$ for $m, k$ integers along with arcs from J. The non border regions consist of rectangles. Thus all of these regions have boundaries which are rectifiable simple closed curves. Also each region is contained in a square having sides of length no more than $\delta$. The construction also yields an orientation for $J$ and for all these regions, the orientations for any segment shared by two regions are opposite.

Proof: Let $0<2 \delta<\min$ (height, width) and for $y_{0}=m\left(\frac{\delta}{4}\right)$ and $l$ the line $y=m \delta / 4$ where $m \in \mathbb{Z}$ is chosen to make $m\left(\frac{\delta}{4}\right)$ as close as possible to the average of the second
components of the top and bottom points of $J\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$. Thus $m\left(\frac{\delta}{4}\right)$ is within $\frac{\delta}{4}$ of this average. Here $0<2 \delta<\min$ (height, width) of $J$. In the argument, $\delta$ is fixed. Let $\gamma_{0}^{*}$ be the open segment with endpoints $\alpha, \beta$ described above which divides $J$ into two simple closed curves both at least as high as $\frac{1}{2}\left(y_{1}-y_{2}\right)$.

You can orient $J$ by letting the direction of motion along $\gamma^{*}$ be from left to right for the simple closed curve which contains $\left(x_{1}, y_{1}\right)$. Thus the orientation is left end point of segment, right end point followed by $\left(x_{1}, y_{1}\right)$ Thus this specifies a direction of motion on part of $J$ including three points and therefore on all of $J$. This is the orientation to be used from now on. Now you have two simple closed curves joined along a line segment. Do the same process as the above on each of these and continue doing it till all regions have height no more than $\delta$.

Each time you do this process, you get two simple closed curves in place of one. If the height of one of the curves is $h=\left(y_{1}-y_{2}\right)>\boldsymbol{\delta}$, then both new simple closed curves have length at least $\frac{1}{8} \delta$. To see this, note that, by the definition of $m \delta / 4$ with a new $m$ so $m \delta / 4$ approximates the mid point as well as possible, the new simple closed curve $\hat{J}$ has

$$
\operatorname{height}(\hat{J}) \geq y_{1}-\frac{m \delta}{4} \geq y_{1}-\left(\left(\frac{y_{1}+y_{2}}{2}\right)+\frac{\delta}{4}\right)=\frac{h}{2}-\frac{\delta}{4}>\frac{\delta}{8}
$$

When a region has height no more than $\delta$, don't split into two regions. This must eventually take place because if not, then there would be a sequence of simple closed curves each with height larger than $\frac{\delta}{8}$ stacked on top of each other. This would violate the assumption that $J$ has finite length.

Use the orientation on $J$ obtained earlier to orient the resulting horizontal line segments. Thus, from this proposition, each segment has opposite orientation as part of the two simple closed curves resulting from its inclusion in this process.

Now follow the same process just described on each of the non overlapping "short" regions just obtained using vertical rather than horizontal lines, letting the orientation of the vertical edges be determined from the orientation already obtained for $J$, but this time feature width instead of height and let the lines be vertical of the form $x=k\left(\frac{\delta}{4}\right)$ where $k$ is an integer.

It follows that each of the resulting regions has sides of length no more than $\delta$ but at least $\delta / 8$. Next is an estimate of how many of these regions can contain points of $J$. I am making absolutely no effort to get any kind of best estimate or even one which looks nice. In general, for a curve $C$, let $|C|$ denote the length of $C$.

Lemma A.0.2 Suppose $\overline{U_{i}}$ is covered with non-overlapping boxes having sides at least $\eta$. Then there are no more than $4\left(\frac{4|J|}{\eta}+1\right)$ of these boxes which intersect J. In particular, in the above construction, there are no more than $4+\frac{128|J|}{\delta}$ border regions, meaning those which have nonempty intersection with $J$. Also $m_{2}(J)=0$.

Proof: Decompose $J$ into $N$ arcs of length $\left(\frac{\eta}{4}\right)$ with maybe one having length less than $\left(\frac{\eta}{4}\right)$. Thus $N-1 \leq \frac{|J|}{\left(\frac{\eta}{4}\right)}$ and so $N \leq \frac{4|J|}{\eta}+1$. The resulting arcs are each contained in a box having sides of length no more than $\eta$. Each of these $N$ arcs can't intersect any more than four of the rectangles which have sides of length at least $\eta$. Therefore, at most $4 N$ boxes can intersect $J$. Thus there are no more than $4\left(\frac{4|J|}{\eta}+1\right)$ border regions consisting of those regions which intersect $J$. In particular, for the above construction, there are no more than
$4\left(\frac{4|J|}{\delta / 8}+1\right)=4+\frac{128|J|}{\delta}$. This proves the lemma. This also shows that, since each region has sides of length no more than $\delta, J$ is contained, up to a set of measure zero in a Borel set of total volume no more than $\left(4+\frac{128|J|}{\delta}\right) \delta^{2}$ which is as small as desired by making $\delta$ smaller.

The following is a proof of Green's theorem based on the above. Included in this formula is a way to analytically define the orientation of a simple closed curve in the plane. Recall that there were two orientations of a simple closed curve depending on the fact that there are two orientations for a circle in the plane. Green's theorem can distinguish between these two orientations. First is a simple lemma.

Lemma A.0.3 Let $R=[a, b] \times[c, d]$ be a rectangle and let $P, Q$ be functions which are $C^{1}$ in some open set containing $R$. Orient the boundary of $R$ as shown in the following picture. This is called the counter clockwise direction or the positive orientation


Then letting $\gamma$ denote the oriented boundary of $R$ as shown,

$$
\int_{R}\left(Q_{x}(x, y)-P_{y}(x, y)\right) d m_{2}=\int_{\gamma} \mathbf{f} \cdot d \gamma
$$

where $\mathbf{f}(x, y) \equiv(P(x, y), Q(x, y))$. In this context the line integral is usually written using the notation $\int_{\partial R} P d x+Q d y$. If the bounding curve were oriented in the opposite direction, then the area integral would be $\int_{R}\left(P_{y}(x, y)-Q_{x}(x, y)\right) d m_{2}$.

Proof: This follows from direct computation. It also follows from Lemma 10.8.2. Writing as an integral with respect to $m_{2}$ is just Fubini's theorem.

With this lemma, it is possible to prove Green's theorem and also give an analytic criterion which will distinguish between different orientations of a simple closed rectifiable curve. First here is a discussion which amounts to a computation.

Let $J$ be a rectifiable simple closed curve with inside $U_{i}$ and outside $U_{o}$. Let $\left\{R_{k}\right\}_{k=1}^{n_{\delta}}$ denote the non overlapping regions of Lemma A.0.1 all oriented as explained there and let $J$ also be oriented as explained there. Since the shared edges of the horizontal and vertical lines have opposite orientations, all these regions which are on the inside of $J$ are rectangles and have the same orientation, counter clockwise.

Let $\mathscr{B}_{\delta}$ be the set of border regions and let $\mathscr{I}_{\delta}$ be the rectangles contained in $U_{i}$. Thus in taking the sum of the line integrals over the boundaries of the interior rectangles, the integrals over the "interior edges" cancel out and you are left with a line integral over the exterior edges of a polygon which is composed of the union of the squares in $\mathscr{I}_{\delta}$.

Now let $\mathbf{f}(x, y)=(P(x, y), Q(x, y))$ be a vector field which is continuous on $\overline{U_{i}}$ and $C^{1}$ on $U_{i}$, and suppose also that both $P_{y}$ and $Q_{x}$ are in $L^{1}\left(U_{i}\right)$ (Absolutely integrable) and that $P, Q$ are continuous on $U_{i} \cup J$. (An easy way to get all this to happen is to let $P, Q$ be in $C^{1}\left(\overline{U_{i} \cup J}\right)$, restrictions to $U_{i} \cup J$ of functions which are $C^{1}$ on some open set containing $U_{i} \cup J$, but what is assumed here is a lot more general.) Note that $\cup_{\delta>0}\left\{R: R \in \mathscr{I}_{\delta}\right\}=U_{i}$ and that for $I_{\delta} \equiv \cup\left\{R: R \in \mathscr{I}_{\delta}\right\}$, the following pointwise convergence holds for $\delta$ denoting a sequence converging to 0 .

$$
\lim _{\delta \rightarrow 0} \mathscr{X}_{I_{\delta}}(\mathbf{x})=\mathscr{X}_{U_{i}}(\mathbf{x})
$$

By the dominated convergence theorem,

$$
\lim _{\delta \rightarrow 0} \int_{I_{\delta}}\left(Q_{x}-P_{y}\right) d m_{2}=\int_{U_{i}}\left(Q_{x}-P_{y}\right) d m_{2}
$$

where $m_{2}$ denotes two dimensional Lebesgue measure discussed earlier. Let $\partial R$ denote the boundary of $R$ for $R$ one of these regions of Lemma A.0.1 oriented as described. Let $w_{\delta}(R)^{2}$ denote

$$
\begin{aligned}
& (\max \{Q(\mathbf{x}): \mathbf{x} \in \partial R\}-\min \{Q(\mathbf{x}): \mathbf{x} \in \partial R\})^{2} \\
& +(\max \{P(\mathbf{x}): \mathbf{x} \in \partial R\}-\min \{P(\mathbf{x}): \mathbf{x} \in \partial R\})^{2}
\end{aligned}
$$

By uniform continuity of $P, Q$ on the compact set $U_{i} \cup J$, if $\delta$ is small enough, $w_{\delta}(R)<\varepsilon$ for all $R \in \mathscr{B}_{\delta}$. Then for $R \in \mathscr{B}_{\delta}$, it follows from Theorem 5.2.1, for $|\partial R|$ denoting the length of $\partial R$,

$$
\begin{equation*}
\left|\int_{\partial R} \mathbf{f} \cdot d \gamma\right| \leq\left(\frac{1}{2}\right) w_{\delta}(R)(|\partial R|)<\varepsilon(|\partial R|) \tag{1.1}
\end{equation*}
$$

whenever $\delta$ is small enough. Always let $\delta$ be this small.
Also since the line integrals cancel on shared edges

$$
\begin{equation*}
\sum_{R \in \mathscr{I}_{\delta}} \int_{\partial R} \mathbf{f} \cdot d \gamma+\sum_{R \in \mathscr{B}_{\delta}} \int_{\partial R} \mathbf{f} \cdot d \gamma=\int_{J} \mathbf{f} \cdot d \gamma \tag{1.2}
\end{equation*}
$$

Consider the second sum on the left. From 1.1,

$$
\left|\sum_{R \in \mathscr{B}_{\delta}} \int_{\partial R} \mathbf{f} \cdot d \gamma\right| \leq \sum_{R \in \mathscr{B}_{\delta}}\left|\int_{\partial R} \mathbf{f} \cdot d \gamma\right| \leq \varepsilon \sum_{R \in \mathscr{B}_{\delta}}(|\partial R|)
$$

Denote by $J_{R}$ the part of $J$ which is contained in $R \in \mathscr{B}_{\delta}$. Then the above sum equals

$$
\varepsilon\left(\sum_{R \in \mathscr{B}_{\delta}}\left(\left|J_{R}\right|+\left|\partial R_{\delta}\right|\right)\right)=\left(\varepsilon|J|+\varepsilon \sum_{R \in \mathscr{B}_{\delta}}\left|\partial R_{\delta}\right|\right)
$$

where $\left|\partial R_{\delta}\right|$ is the sum of the lengths of the straight edges of $R_{\delta}$. This last sum is easy to estimate. Recall from A.0.1 there are no more than $4+\frac{128|J|}{\delta}$ of these border regions. Furthermore, the sum of the lengths of all four edges of one of these is no more than $4 \delta$ and so

$$
\sum_{R \in \mathscr{B}_{\delta}}\left|\partial R_{\delta}\right| \leq 4\left(4+\frac{128|J|}{\delta}\right) 4 \delta=1024|J|+64 \delta
$$

Thus $\left|\sum_{R \in \mathscr{B}_{\delta}} \int_{\partial R} \mathbf{f} \cdot d \gamma\right| \leq \varepsilon(1025|J|+64 \delta)$ Let $\varepsilon_{n} \rightarrow 0$ and let $\delta_{n}$ be the corresponding sequence of $\delta$ such that $\delta_{n} \rightarrow 0$ also. Hence

$$
\lim _{n \rightarrow \infty}\left|\sum_{R \in \mathscr{B}_{\delta_{n}}} \int_{\partial R} \mathbf{f} \cdot d \gamma\right|=0
$$

Then using Green's theorem proved above for squares,

$$
\int_{J}^{\mathbf{f}} \cdot d \gamma=\lim _{n \rightarrow \infty} \sum_{R \in \mathscr{I}_{\delta_{n}}} \int_{\partial R} \mathbf{f} \cdot d \gamma+\lim _{n \rightarrow \infty} \sum_{R \in \mathscr{B}_{\delta_{n}}} \int_{\partial R} \mathbf{f} \cdot d \gamma
$$

$$
=\lim _{n \rightarrow \infty} \sum_{R \in \mathscr{I}_{\delta_{n}}} \int_{\partial R} \mathbf{f} \cdot d \gamma=\lim _{n \rightarrow \infty} \int_{I_{\delta_{n}}} \pm\left(Q_{x}-P_{y}\right) d m_{2}=\int_{U_{i}} \pm\left(Q_{x}-P_{y}\right) d m_{2}
$$

where the $\pm$ adjusts for whether the interior rectangles are all oriented positively (counter clockwise) or all oriented negatively (clockwise). It was assumed these rectangles are oriented counter clockwise and so the + sign would be used.

This has proved the general form of Green's theorem which is stated in the following theorem.

Theorem A.0. 4 Let $J$ be a rectifiable simple closed curve in $\mathbb{R}^{2}$ having inside $U_{i}$ and outside $U_{o}$. Let $P, Q$ be functions with the property that $Q_{x}, P_{y} \in L^{1}\left(U_{i}\right)$ and $P, Q$ are $C^{1}$ on $U_{i}$. Assume also $P, Q$ are continuous on $J \cup U_{i}$. Then there exists an orientation for $J$ (Remember there are only two.) such that for

$$
\mathbf{f}(x, y)=(P(x, y), Q(x, y)), \int_{J}^{\mathbf{f}} \cdot d \gamma=\int_{U_{i}}\left(Q_{x}-P_{y}\right) d m_{2}
$$

Proof: In the construction of the regions, an orientation was imparted to $J$. The above computation shows $\int_{J} \mathbf{f} \cdot d \gamma=\int_{U_{i}}\left(Q_{x}-P_{y}\right) d m_{2}$

Recall the winding number gives an analytical description of the number of times a curve winds around a point. When this is 1 it is the positive direction. How does this relate to Green's theorem?

Corollary A.0.5 Let $\gamma:[a, b] \rightarrow \mathbb{C}$ be one to one on $[a, b)$ and $\gamma(a)=\gamma(b)$. Then if $z$ is a point on the inside of $\gamma^{*}$ it follows that $n(\gamma, z)= \pm 1$ and if $z$ is in the unbounded component of $\gamma^{* C}$, the outside, then $n(\gamma, z)=0$.

Proof: Letting $U_{i}$ denote the inside of $\gamma^{*}$, let the regions describe in the above be such that the inside point $z$, is on the inside of one of the interior regions. Choose the regions in such a way that $z$ is on the inside of an interior region. Then the winding number of $z$ with respect to the specified interior region is $\pm 1$ and the winding number of $z$ with respect to all the other regions is 0 . Then, since the orientations of all edges of intersecting regions are opposite, one adds the line integrals about the boundaries of all these regions and the whole sum reduces to $\frac{1}{2 \pi i} \int_{\gamma} \frac{1}{w-z} d w= \pm 1$ depending on the orientation of the simple closed curve. If $z$ is not on the inside of $\gamma^{*}$, then the winding number about $z$ on each small region is clearly 0 provided we take the regions with small enough diameter. Indeed, a branch of the logarithm can be obtained to use as a primitive of $\frac{1}{z-w}$.

## A. 1 Exercises

1. Consider the following diagram.


In this diagram there is a polygonal curve from $\mathbf{z}_{1}$ to $\mathbf{z}_{2}$ in the inside component of $J^{C}, U_{i}$ in which there are no horizontal segments. Here $\mathbf{z}_{1}, \mathbf{z}_{2}$ are very close to the top and bottom points respectively. This intersects a horizontal line shown in finitely many points and for one of the segments, it crosses the horizontal line an odd number of times. Pick the part of $J$ the simple closed curve which corresponds to that segment. Explain why this divides $J$ into two simple closed curves and that if the horizontal line is $y=c$, one Jordan curve has height at least $\left|y_{1}-c\right|$ and the other having height at least $\left|c-y_{2}\right|$. For the existence of the curves from $\mathbf{z}_{1}$ to $\left(x_{1}, y_{1}\right)$ and from $\mathbf{z}_{2}$ to $\left(x_{2}, y_{2}\right)$ shown in the picture, consult Problem 30 on Page 218. This approach to splitting up the simple closed curve into smaller regions is in Apostol [2].

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[^0]:    ${ }^{1}$ Actually, it is only necessary to assume one of the series converges and the other converges absolutely. This is known as Merten's theorem and may be read in the 1974 book by Apostol listed in the bibliography.

[^1]:    ${ }^{1}$ In the Bible, there was a battle between Ephraimites and Gilleadites during the time of Jepthah, the judge who sacrificed his daughter to Jehovah, one of several instances of human sacrifice in the Bible. The cause of this battle was very strange. However, the Ephramites lost and when they tried to cross a river to get back home, they had to say shibboleth. If they said "sibboleth" they were killed because their inability to pronounce the "sh" sound identified them as Ephramites. They usually don't tell this story in Sunday school. The word has come to denote something which is arbitrary and no longer important.

[^2]:    ${ }^{1} 1$ Kings 17, 2 Kings 4, Mathew 14, and Mathew 15 all contain such descriptions. The stuff involved was either oil, bread, flour or fish. In mathematics such things have also been done with sets. In the book by Bruckner Bruckner and Thompson there is an interesting discussion of the Banach Tarski paradox which says it is possible to divide a ball in $\mathbb{R}^{3}$ into five disjoint pieces and assemble the pieces to form two disjoint balls of the same size as the first. The details can be found in: The Banach Tarski Paradox by Wagon, Cambridge University press. 1985. It is known that all such examples must involve the axiom of choice.

[^3]:    ${ }^{1}$ Note that, since $g$ is allowed to have the value $\infty$, it is not known that $g \in L^{1}(\Omega)$.

[^4]:    ${ }^{1}$ In fact, they will be automatically bounded if the set T is a closed interval like $[0, \mathrm{~T}]$, but the considerations presented here will work even when a compact set is not being considered.

[^5]:    ${ }^{1}$ Actually it is only a function of the first but this is not important in what follows.

[^6]:    ${ }^{2}$ For a general version see the advanced calculus book by Apostol. This is presented in the appendix also. The general versions involve the concept of a rectifiable Jordan curve. You need to be able to take the area integral and to take the line integral around the boundary.

[^7]:    ${ }^{1}$ Giancinto Morera 1856-1909. This theorem or one like it dates from around 1886

[^8]:    ${ }^{2}$ From the more general Green's theorem in the appendix, it is automatic that Green's theorem holds. Also, it suffices to have $f$ continuous on $\Gamma$.

[^9]:    ${ }^{1}$ This is always the case from the general version of Green's theorem in the appendix.

[^10]:    ${ }^{2}$ This is the terminology used in Rudin's book Real and Complex Analysis.

